UNIVERSITY OF ALBERTA

ATTENDING IN MATHEMATICS: A DYNAMIC VIEW ABOUT STUDENTS' THINKING

by

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It is a Dynamical Landscape Laid in Living: A Space Enlarged in Doing.

Dedication

To my Father Elifazi Sekalo and my Sister Eva Samalie Kakooza

Abstract

This study is an inquiry into the nature of mathematical thinking. Particularly I explore the embodied, embedded and extended nature of mathematical attentiveness. The writing is both theoretical and empirical.

I examine the ways in which students enact mathematical words in which they are not only invited to think mathematically but also where it makes deep sense for them to do so. I draw upon enactivist and complexity research to understand inner level aspects of students' engagement on tasks.

Through close observations of the diversity and subtleties among students' interactions, my research suggests that students enact what they attend to. This bringing forth of worlds is not only conscious and formal. Mathematical attentiveness spans other layers of signification including the bodily, the social collective, the cultural and the material. To understand how this could be, I explore ecological and geographical metaphors of complexity. Emergence shows how mathematical attentiveness is a global, relational property. Mutual feedback illustrates how there is multi-threaded interaction among the many aspects. Dynamical attractors illustrate how subtle differences may actually give rise to larger differences in objects attended to.

This study is hermeneutic. Interpretations are organized around interpretive moments. I use the metaphor of a landscape transformed in living to share the insights gained. These include: good mathematical problems might not only evoke adequate mathematical behaviors, but they may also structure students' attention. Observing students' attentiveness involves a layering of the structural gaze with the cognitive gaze that are in turn successively layered by the subjective, inter-subjective, inter-objective

and inter-domain gazes. Mathematically adequate actions like writing appear central; they, in addition to originating, sustain mathematical attentiveness. During mathematical activity students act and interact themselves into thinking mathematically, concepts are represented, *re*-presented and presented.

The phenomenology of learning aside, mathematics educators have yet to problematize radically the ontology of mathematical concepts. Thinking mathematically is more than perceiving, interpreting and experiencing mathematically; it is seen as acting and being in ways that expand the living landscape of what is mathematically possible. In sensing, perceiving and observing mathematics we enact mathematical objects of attention and ways of attending.

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List of Abbreviations

CS-Consecutive Terms

CT-Chessboard Squares

BG-Bee Genealogy

PA-Pirates Aboard

1. INTRODUCTION

1.1 A Classroom Vignette

Let me begin with a vignette from a junior high classroom in which I participated as a research assistant. The teacher had been engaging the grade seven students in a range of mathematical experiences to promote the learning of fractional concepts. In preceding lessons, the students had done paper-folding activities, like many recent reform initiatives that appeared to have the potential to generate varied experiences with fractions. In the particular lesson from which I draw the vignette the teacher introduced the fraction kit[1]:

> She gave each student a fraction kit; an envelope containing different colored pieces. After explaining that the kits had been assembled by cutting different colored sheets of paper into specific sizes, she took a white piece and two red pieces out of her kit. Holding the two red pieces against the white piece she asserted, "If this white piece is a whole, then each of these two reds will be ...?" "A half," the students responded in chorus.
>
> The teacher then affirmed, "If this *white* piece is a *whole* and it takes *two reds* to cover it, then *a red* is a *half.*" Next the teacher asked the students to find the sizes of the rest of the pieces in the kit. All students appeared to be engaged in the task. Some of them occasionally talked to a nearby student about their findings, but most students worked independently. (A few students put up their hands up, mainly asking for technical help. For example, one student noted, "I am missing a pink!")
>
> After the teacher had circulated around the class observing what the students were doing she reminded them to record their results. Most of their records were in the form: White—whole (1), Red—halves (½), Orange—thirds (⅓), and so on.
>
> A brief glance around the room indicated that the students seemed to be working

[1] A fraction kit is a manipulative activity that was developed by Dr. T. E. Kieren at the University of Alberta to provide learners with a range of experiences with fractions. It is constructed by cutting sheets of different colored paper into halves, thirds, fourths, sixths, eighths, twelfths and twenty-fourths (Kieren, Davis & Mason, 1996).

on the task in somewhat different ways. In addition to the approach directly prompted by the teacher—that of placing the smaller pieces against the whole—another approach was evident. Some members of the class were neatly arranging the pieces of a given color in a stack without covering the whole piece.

I began to wonder about the ways in which students were able to figure out the sizes of the pieces without covering the whole. How were the students who stacked pieces reasoning and visualizing? Although the *stacking* approach differed from the one that the teacher demonstrated, most of the students who used it finished ahead of the other students. A third approach was evident among a few students who neither stacked nor covered the whole piece. Instead of covering the whole white piece, they covered smaller pieces, that is to say they had quarters covering halves, eighths covering quarters, and so on. This seemed close to the teacher's approach. However, this method appeared to be more complex since it involved reasoning in terms of parts of parts—a recursive approach to covering.

There could have been a fourth approach, one of finding out how many pieces of a given set make up a whole (by assembling wholes of different colors rather than covering a white whole), but this was not noticed by the teacher or myself, the research assistant. However I have observed this approach in a workshop with schoolteachers.

After the students were done with figuring out the size of the pieces, the teacher asked them to explore ways of covering a half piece. "What do you mean by covering? Should we use same color pieces?" a few students asked anxiously as they shifted in their chairs. The teacher explained individually to those who asked while the rest of the class proceeded to generate combinations for a half. Some students created a half piece from 2 one-quarter pieces or 1 one-quarter piece plus 2 one-eighth pieces. Others produced more complicated combinations such as a half covered by $[(\frac{1}{2} \text{ of } \frac{1}{3}) + \frac{1}{4} + \frac{1}{8} + \frac{1}{24}]$ *minus something*.

In a conversation after the lesson the teacher commented, "I noticed that the students who used the stacking approach rather than the intended—*covering*—approach on the first task were the ones who found difficulty with understanding the second task."

The relevance of this vignette for me is centered on the following questions: What did the students attend to when the teacher held two red pieces against a white piece? What did they see, or not see? What was mathematically relevant for the students to attend to in the fraction kit task? Or more specifically, what distinctions did the students need to make in order to perceive each of the reds as a half, each of the oranges as thirds, each of the yellows as quarters, etc.?

By holding two red pieces, the only red pieces in the kit, against the white sheet, how was the teacher inviting the students to perceive the Fraction Kit? The teacher had included in the kit one whole, two halves, three thirds, and so on. What possibilities and constraints would such a manipulative offer? In which mathematical world(s) does the statement, "If this *white* piece is a *whole* and it takes *two reds* to cover it, then *a red* piece is a *half* of a whole" make sense?

Whereas the mathematical task posed by the teacher was unitary, it appears to have triggered different responses from different students (or should we say different tasks?) two of which are distinct: one of stacking to count the pieces in a set, and the other of covering a whole to figure out the portion of the whole covered by a piece of a given color. Given the distinct approaches and the responses of the students who stacked on the second task, it is tempting to ponder about whether there were radical differences in what the students were attending to. I reflect on the ways in which these students, whether they stacked or covered, perceived the wholes, halves, thirds, and so on. More generally I ask, "How did students think mathematically when they engaged in this task?"

This study investigates how students think mathematically by closely looking at their experiences in mathematical activities. I take an ontological stance. The phenomenology of learning aside, mathematics education researchers have yet to problematize radically the ontology of phenomena, events and tasks experienced. While thinking mathematically could be viewed as perceiving, interpreting and experiencing mathematically, to enactivist researchers such as Maturana and Varela (1987/1992) and Towers (1998) it is acting and being in ways that enact mathematical worlds. Perceived objects and thoughts arise with *acts of observation* and *thinking*. What we know constrains what we perceive and what we perceive suddenly influences what we know. I am, therefore, interested in the dynamics of what students attend to in mathematical tasks and how they attend. I am hoping that by examining the ways students pay attention as they engage in mathematical tasks, researchers, educators and teachers will be able to examine the conditions under which students produce, or to use Maturana and Varela's (1987/1992) term, *bring forth* mathematical worlds.

In the lesson described above, different students brought forth a variety of aspects. The pieces were of different sizes, quantity, colors, areas and perimeters, but it appears specific attributes had to be made salient for the students to manipulate the paper pieces as fractional amounts. The students had to enact a fractional world, a world in which paper pieces were defined in relation to a whole piece of paper. This world included more than the explicit attributes of the pieces. There seem to be basic conditions of possibilities, however implicit, that are necessary for *relational properties* such as fractions to be emphasized. It might be the case that the grade seven students, depending on the approach they used, enacted distinct mathematical properties.

It is common in mathematics classes, as we will see, for students to respond to tasks differently, and at times they enact radically different tasks. In the fraction kit lesson, there arose a time-saving parallel approach that worked equally well, albeit only for this particular kit. For instance, as long as there were four one-quarter, yellow size pieces in the kit, the students who used this approach were able to figure out that each yellow was $\frac{1}{4}$ without laying yellow pieces down to cover or assemble a whole. Had the fraction kit had more or fewer pieces of each size this approach would have still worked, but not in the same way.

In most mathematics classrooms, students' alternative approaches, especially when they are valenced and novel, pass unnoticed by many teachers. At other times the teacher may notice varied approaches, but amidst the flow of classroom activity he or she might interpret them as mere alternatives to the desired approach. Indeed they might be just alternatives. In unfortunate situations, they might be assessed simply as "wrong" approaches and thus not examined further. Yet the emergent approaches could be windows to the distinct mathematical or even non-mathematical embodiments that students bring forth. Thus it appears significant to ask: In what ways do the students think mathematically as they engage in mathematical tasks? How do they attend?

The Fraction Kit vignette is an empirical example of the phenomena that engage me. In Chapter 5, I will return to this vignette to tease out the nuances of what students attended to and how they attended.

1.2 The Research Interest

During my first two years of graduate studies, I came to embrace *mathematical thinking* as my area of research interest. In part, I can trace this interest to my own experience, first as a student and later on as a teacher of school mathematics. My interest in how students make sense of mathematical tasks was mainly cultivated in one-to-one experiences I had as a teacher tutoring students (mostly *weaker* students) in mathematics.

In my teaching and tutoring experiences, I began to develop a hunch that whereas explaining mathematics content or telling it to students helped, it was not all that was required for students to make sense of mathematics. It appeared there was more to teaching or tutoring mathematics than explaining. During those tutoring moments I felt uncomfortable when some of my "best" explanations did not translate into deeper understandings for the students. So I began to develop the idea that what I thought were pre-given mathematical structures to be explored are not always given. Or are they?

Later on, the courses I took and the discussions I participated in during my early graduate work furthered my interest in gaining insight into the sense making that co-emerges with students' everyday *mathematical behavior*. I began to explore what it meant for students to bring forth their own mathematical worlds in which the mathematics content and procedures make perfect sense.

In this study I broadly ask questions such as: What is the nature of the actions and thinking that mathematics requires from students? Does it arise with doing mathematics, or is it a prerequisite to adequate mathematical behavior? Perhaps addressing these questions will offer insight into ways of creating space for students to bring forth *worlds of mathematical significance* in which it makes deep sense for them to do so.

1.3 Statement of the Problem

In the mathematics education community there have been many efforts to reform learning in ways that make mathematics more meaningful. Notions of mathematical cognitive processes, mathematical conceptualization, mathematical understanding, mathematical meaning making, mathematical power and mathematical knowledge have been constructed by researchers who seek to understand how students learn mathematics. In most of the research the terms *conceptualizations* and *thinking* are understood to be individual psychological processes.

In a few studies, ways other than the individual psychological processes have been explored to understand mathematical cognition. Some studies promote a more discursive understanding of mathematics learning by using terms like *mathematical knowing* as opposed to *mathematical knowledge*, and *thinking mathematically* rather than *mathematical thinking*. In my study I have chosen to use *mathematical thinking* as a synonym for all efforts that seek to study how students learn mathematics. I extend the term *thinking* beyond ways that assume the thinker to play a physically passive, mentally mechanical and solely internal, all in the head, conscious role.

In the vast amount of literature on mathematical thinking, three questions remain challenging: (a) What is the nature of mathematical thinking? (b)To what extent or in what ways is it an individual, a social or a contextual phenomenon? And (c) In what ways do teachers or researchers observe and occasion students to think mathematically?

Recent post-structural, and ecological and complexity frameworks such as enactivism attempt to move beyond the debate of whether mathematical thinking is psychological or social, native or learnt, and whether its potential is quasi-universal or a talent. I draw from enactivism and complexity research to explore the nature of students' mathematical behavior.

In the case of the Fraction Kit activity, taking this non traditional view of mathematical thinking I am interested in the: (a) perturbations and regularities caused by the nature of the fraction kit and other environmental factors; (b) initial conditions and internal dynamics that inclined some students to stack and yet others to cover; (c) nature of influence that the social environment including the teacher had on what students attended to; (d) other elements that correlated with the Fraction Kit manipulative to enact fractional worlds; and (e) ways to occasion students to enact adequate mathematical fractional worlds. My study involves both theoretical and empirical research.

1.4 Overview of the Research Narrative

This study is an inquiry into mathematical thinking, particularly the dynamics of mathematical attentiveness. The dissertation is outlined in seven main parts with eleven chapters altogether.

Dissertation Landscape

Attending in Mathematics
A Dynamic View

- Part I
  Introduction
- Part II
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  - Chapter 9
    Theoretical Dynamics of Attention
- Part VII
  Questions Answered and Problems Raised
  - Chapter 10
    Orienting Mathematical Attentiveness
  - Chapter 11
    Dynamics of Mathematical Attentiveness

Chapter 1 is the first part, in which I introduce the study. I situate the study and survey the literature on mathematical thinking in two chapters in the second part: Chapter 2 is a general review of research on mathematical thinking classified by paradigm, and Chapter 3 is an analysis of insights gained from the reviewed studies. The third part consists of an elaboration of the research design and methodologies in two chapters (chapters 4 and 5). Chapter 4 is a description of the design and context of the study. In Chapter 5, I explore the research methodologies of this study. In the fourth part, Chapter 6, I return to elaborate on enactivism and on metaphors I draw from complexity science to investigate students' mathematical thinking. The fifth part, chapters 7 and 8, traces how the research focus and question have evolved during the study. In Chapter 7, I pay particular attention to how I attended to students' mathematical attentiveness. In Chapter 8, I trace the layering of my research attention to explore the shifts in what I attended to. The sixth part is Chapter 9, which is consists of further theoretical explorations. In this chapter, I explore a dynamic view of sensation, perception and observing. This appearance of theory far into the dissertation is appropriate for a study that seeks to offer an alternative view of the role of sensation, perception and observation in mathematical thinking. In the seventh part, the last two chapters return to the questions that motivated the study—questions answered and problems raised—and to envisioning new possibilities. Chapter 10 is an exploration of how to orient systemically students' mathematical attentiveness. In Chapter 11, I conclude by first summarizing the ways of thinking about mathematical thinking and lastly by reflecting on the implications and consequences of study.

The writing relates my experiences. At many times I use the first person to narrate the landscape that has been formed during the study. It is a fractal landscape with many self-similar micro-theoretical and -empirical inquiries popping up within the parts, chapters and sections. Also, interwoven in all the parts are explications of my own understanding and expectations at different times during the study, reflections on the transformations I underwent, and comments on the surprises I encountered in the process. I believe by sharing the explications of my observations there is potential to transform the reader's landscape and hence the community mind to a novel understanding of the nature of students' mathematical thinking. Also, as in Chapter 1, I interweave research vignettes from the classroom or research sessions in all chapters, except chapters 2, 3 and 10. It is a multiple threaded writing with theoretical and methodological research brought into conversation with my experiences in the research projects. As well it is multi-layered with my observations in the projects, my initial interpretations and elaborations enacted from attending to my earlier interpretations.

I began this writing by introducing my research interest and stating the research problem. In the remaining sections of this chapter, I outline the purpose of the research, the research questions, necessary definitions and the significance of the study.

1.5 The Research Purpose

This study explores the dynamics of what students attend to in mathematical tasks and seeks a deeper understanding of the *embodied*, *embedded* and *extended* nature of students' mathematical thinking. I seek to engage in a conversation about thinking as complex human phenomenon. This conversation aims at triggering new ways of thinking, talking and acting about students' learning. The emphasis on *emergent structures* offers insights for studying the dynamics that afford individual and collective learning and knowledge systems, coherence and novelty without collapsing them into one. This study is informed by ecological and complexity—ecological complexity—frameworks, including enactivist theory, that consider most learning as a complex adaptive and emergent phenomenon.

1.6 The Research Questions

My study seeks to engage in the emerging conversation that foregrounds the dynamic and complex nature of students' mathematical thinking. How do students, on a moment-to-moment basis, develop worlds of mathematical significance in which they are not only invited to think mathematically but where it also makes integrated sense for them to do so? How do students, in a setting-to-setting basis, bring forth mathematical worlds?

To guide the study I initially asked:

1. What do students attend to in mathematical tasks?
2. When do shifts in attention to that which is mathematically relevant occur?
3. In what ways does a deeper understanding of what students attend to offer insights into how teachers occasion students' mathematical thinking?

As I navigated the research orientations and methodology and sought to make coherent my observations from preliminary research sessions and conversations, the research question evolved from "what" and "when" do students attend to, to "how" and "in what ways" do students attend. I then asked:

1. How do students attend as they engage in mathematical tasks?
2. Elaborating question 1, in what ways do students, not only as individual beings with mental and physiological structures, but also as learning, organic,

social systems embedded in social collectives, enabled by cultures, extended by language and technology, and as systems embodied with a neuro-motor system and body, attend to mathematical tasks?

3. In what ways do secondary school students await and dwell with mathematical objects?

What might appear as a subtle shift in the research question involved a radical change of focus. At the conception of the study the focus was on the individual child believed to possess particular psychological structures responsible for mirroring given mathematical content. As I observed students working in pairs, in groups and in whole classrooms, broader issues such as collective and distributed sense-making began to call for my attention. Rather than focusing solely on what in the students' minds corresponded to particular mathematical structures, I began to attend to students' actions and interactions, the materials they worked with, and the collectives that sprang from their continued interaction. These aspects began to appear *as if* they were constituents of mathematical thinking, and not just external factors.

This shift also involved a drift toward considering attention and perception as participatory acts. In attending, it appears the attendee also enacts what is attended to. Attention is pregnant contact; it is active and participatory (van Lennep, 1987). It is not, as Merleau-Ponty (1964) critiqued, passive. Probably in learning mathematics, there is more to the mastering of mathematical rules, formulae, definitions, and the like that are usually assumed to exist outside the learner. For Heidegger (1927-1964), the properties and structure that we attend to or think about must also call us to attend and to think. The shift in the research questions also involved a drift toward considering the dynamic nature of mathematical objects, tasks and environments. To Heidegger (1927/1964):

> We are capable of doing only what we are inclined to do. And again, we truly incline toward something only when it in turn inclines toward us, toward our essential being, by appealing to our essential being as what holds us there. (p. 369)

In a Heideggerian, circular manner the research questions could be paraphrased as:

1. How are students inclined to attend in mathematical tasks?
2. In what ways do mathematical tasks *call* secondary school students to attend?
3. In what ways could teachers make mathematics appeal to the essential being of students as what motivates or grips them to attend, as what holds them to attend mathematically?

1.7 Definitions

1.7.1 Mathematics

Bearing in mind that only that which does not have a history can be precisely defined I attempt to offer an observer-laden description of mathematics in the context of this study. Mathematics, at least proto-mathematics, has a history as old as humanity, or perhaps, as some researchers have argued, even older (Butterworth, 1999; Joseph, 1991). Both proto-mathematics—the less rigorous mathematical activities that humans engage in—and modern mathematics appear to be ways of knowing the world; it is difficult to imagine our daily lives without basic mathematical technologies such as counting, shaping, comparing and locating. Historical and contemporary evidence show that every human culture, however small, motivated not only by utility but also by aesthetics, religion or enchantment, is capable of developing some form of mathematics (P. Davis &

Hersh, 1981; Joseph, 1991). Contemporary basic mathematical activities such as enumerating, estimating and calculating are now globally understood languages; they are approaches to engaging with the world.

In this study, mathematics appears not only to be a way of perceiving and understanding the structure of our worlds but also, with advancements in the field of mathematics, it becomes apparent that it is a way of constructing our worlds. Mathematical activity is taken to arise from the engagement of human beings who have particular bodies and brains in a world that has particular forms. Whereas mainstream mathematics incorporated mathematical inventions from diverse traditions, it mainly grew from the classical mathematics of the Greeks, who privileged the formal over the informal and the abstract over the concrete (Joseph, 1991). In my view, it is when educators view school mathematics to be in the service of preparing future mathematicians that school mathematics inevitably continues in the spirit of classical Greek mathematics. In addition to this achievement classical Greek mathematics also had its drawbacks such as elitism, gender bias, religious ties and formalism (Confrey, 1999). For this and other reasons researchers are increasingly contesting that school mathematics has to be distinct from research mathematics (Cobb & Bauersfeld, 1995; Cobb, Wood & Yackel, 1993; Putnam, Lampert & Peterson, 1989).

School and classroom mathematics are related to research and societal mathematics in intimately complex ways. With contemporary societal needs, there is a tendency for mathematics education reform on most continents to shift away from a mathematics curriculum that is true to its origin in the mathematics of the masters, scribes and accountants, toward school mathematics for all contemporary citizens (Bishop, 1997;

D'Ambrosio, 1990). This shift has been supported by historical, cultural, sociological and ecological studies that have called attention to other *strands* of mathematics such as street mathematics (Nunes, Schliemann & Carraher, 1993), ethnomathematics (D'Ambrosio, 1990), children's mathematics (Steffe & Thompson, 2000), pre-classical mathematics (Joseph, 1991) and to the universality of proto-mathematics (Lakoff & Núñez, 2001). In reform studies, mathematical activity has been broadened to include activity such as conjecturing, conversation and inquiry (Putnam et al., 1989; Freudenthal, 1991; Gordon-Calvert, 2001, National Research Council, 1989; NCTM, 2000).

My study appears to be undertaken at a moment in the history of mathematics education when school mathematics is being redefined as a formal domain that emerges from human activities in a particular context and era, rather than being considered as a formalistic domain. School mathematics, as B. Davis (1994/1996) asserts, is a discipline that is centered more on acting and interacting mathematically rather than on acquiring mathematical knowledge. Enactivist researchers maintain that mathematical *actions* rather than mathematical *knowledge* should be the focus of a mathematics curriculum (Gordon-Calvert, 2001; B. Davis, 1994). What counts as adequate school mathematics behavior, activity and knowledge is relational in nature (B. Davis, 1995).

In the reform model of teaching mathematics that is leaning toward more activity- and interaction-based learning, school mathematics—especially in early grades—appears to involve classical mathematics as well as proto-mathematics. Historically, it appears that proto-mathematics, which arises as humans make sense of their immediate activities and environments, leads to formal mathematical systems as people recursively draw generalities from and about these common human activities

(Mac Lane, 1981). The latter is *a recursive elaboration of* the former.[2] Researchers increasingly believe that formal mathematics emerges from basic as well as advanced human behaviors (Lakoff, 1991). According to Mac Lane (1981), such activities range from the mundane, such as counting, measuring, shaping, forming, estimating and moving, to the esoteric, such as calculating, proving, puzzling and grouping. For instance, Mac Lane explains, arithmetic and number theory as tangible formal systems emerge from human engagement in counting, geometry and topology emerge from shaping, symmetry and group theory emerges from forming, and probability, measure theory and statistics emerge from estimating.

In this research, it is emphasized that mathematics is a human activity that is historical, and its historical developments are considered to have implications for the teaching and learning of mathematics in the classroom (Confrey, 1999; Sfard, 1995). As well, mathematical meaning is embedded in contemporary human activities. Mathematics always bears the mark of its spatial and temporal locality.

1.7.2 Mathematical Behavior

In this research, I define mathematical thinking as the sense making that is observed to co-arise with *adequate mathematical behavior.* In enactivism, behavior is considered to be synonymous with induced, planned or spontaneous action of the organism in interaction with its environment. From an ecological perspective, actions are changes in the state of an organism embedded in a medium as seen by an observer (Maturana & Varela, 1987/1992). Behavior is adequate when it fits with the individual's

[2] B. Davis (2002) adopted the phrase *recursive elaboration of* (REO) to refer to cases that the term *beyond,* say, constructivism do not capture the complex relation between two consecutive understandings well. Whereas *beyond* appears to imply to getting over with, REO emphasizes a relation of the latter with the former. The latter owes its existence to the former and the latter does not leave the former unchanged.

world, in his/her interactions or in his/her community.

Dreyfus (1990), like many mathematics researchers, defined mathematical behavior as "behaving like a mathematician." In light of my working definition of mathematics, I prefer to define it in a more contextual manner. For a mathematician, adequate mathematical behavior is behaving in an acceptable manner in a community (or, to use Maturana's (2000) term, in a consensual domain) of mathematicians. Yet for a school mathematics student of a given age and experience, in a given context, who is acting and interacting with a certain others and within a certain environment, his or her behavior may be considered mathematically adequate if it is acceptable and adaptive in that context and community of school mathematics observers at that particular time., Even among mathematicians, mathematical behavior seems to be contextual and highly dependent on the mathematical domains in which particular mathematicians participate (Burton, 1999a, 1999b). A few researchers, contrary to the widely held view, have by studying the behaviors of mathematicians observed that mathematical behavior varies with individuals, institutions and areas of specialization, to mention a few factors (Burton, 1999a, 1999b; P. Davis & Hersh, 1981; Hadamard, 1945/1996). In later chapters I will return to this enactivist understanding of classroom mathematics as a nested *consensual domain*. Next, I explore what I have come to uphold as mathematical thinking for secondary school students.

1.7.3 Mathematical Thinking

In this study, mathematical thinking is inferred by adequate mathematical behavior. This is not simply because thoughts are inaccessible and unobservable, as a radical *behaviorist* might say. Instead, it is because thinking, acting and being are inseparable. In schools, students are not only learning "what to do" (behavior) and "how and why they do it" (knowing). They are also learning "who and how to be" (being, becoming & belonging). Put differently, thinking is partly a description an observer makes of the observable structural changes in a system that is in perpetual interaction. Thinking might involve thinking-to-act, thinking-in-action or thinking-to-reflect on action and more. To use Bateson's (1980) phrase, this *wider thinking* involves structural changes to compensate for recurrent triggers from the *environment* with which and in which a system—be it individual humans, human communities or other adaptive systems—interacts.

In the case of this study, the system will be individual and collective learners whose environment includes the mathematical task, the other students, the physical, cultural and technological objects and more. Therefore, mathematical thinking will be defined locally, in the context of the research participants and settings, but nested within and thus under the constraints of the broader setting of the community of school mathematics. It will also be seen to extend beyond the individual learner. Like Pirie and Kieren (1994) in their model of understanding, I, as an observer, will consider students' and my own potentially fruitful mathematical actions and interactions as adequate thinking-in-action.

1.7.4 Mathematical Tasks

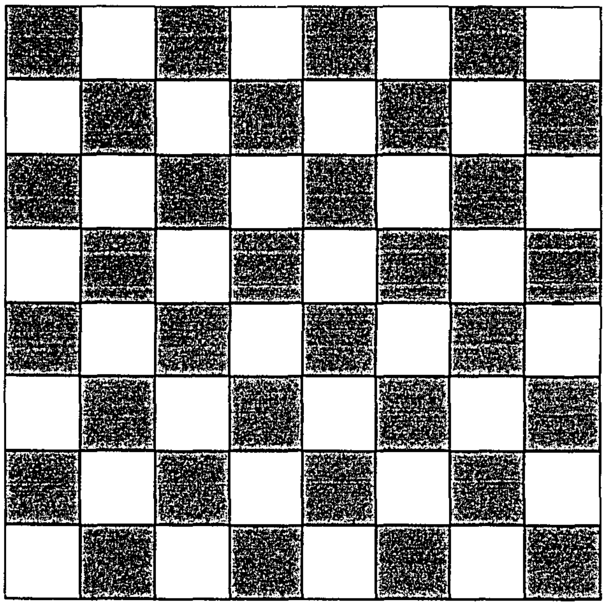
This research considers a mathematical task to include questions, exercises and prompts that may be offered to students in a mathematics classroom. However, in a manner aligned to research that supports current reforms in mathematics education, I specify that the tasks should be *good enough*, *non-routine* and *variable-entry*. Simmt

(2000) adopted the term variable-entry to refer to prompts that are triggers to encourage participation, and at the same time have multiple and varied entry points into mathematical activity. Polya (1945/1973) observes that a problem is *routine* "if it can be solved either by substituting special data into a formerly solved problem or by following step by step, without any trace of originality, some well-worn conspicuous example" (p. 171). Elsewhere in the research literature, to emphasize both the mathematical and situational analyses required in adopting them, tasks that are non-routine and variable-entry are referred to as *insight* (Sfard, 2000a), *non-standard* (Schoenfeld, 1985), *rich learning tasks* (Flewelling & Higginson, 2001), or *structured problems* (Lampert, 1991).

In preparing a task, one can only determine whether the problem is good enough with respect to the particular learners. Yet what happens when learners are engaged in the task, though contingent upon the task is not caused by the task. The outcome, the mathematical activity, is triggered by the task. In this study, I attempt as much as possible to offer students tasks that have the potential to, in Heidegger's words, *incline* students of varied interest and background to behave in ways that are mathematically adequate. An example of such a task that I have used is the Chessboard Squares (CS) task in Figure 1. Other tasks I used in the study are available in Appendix A. I invite the reader who has not encountered this problem before to solve it.

Figure 1. Chessboard Square Task[a]

How many squares are on a chessboard?



[a]Adopted from Mason, Burton & Stacey (1982/1985)

How is this Chessboard Squares task a variable-entry, non-routine and good-enough task? I answer this question in the context of the junior high students, student-teachers and colleagues I have observed engaging in the task. Junior high school students in the study usually responded to the question right away by saying, "There are 64 squares." But even before the teacher interrupted many were quick to correct themselves. "Wait, there is one more square—the big one", "There are a few more than 64", or "There are many more." Such responses marked the beginning of an engagement in the task. The checkerboard appeared to call students to attend. To the extent that all junior high students in the study who attempted the problem appeared to be motivated to engage in the task, at least for the first ten minutes, the task was good enough.

It is also a variable-entry prompt since youth in different contexts as well as student-teachers and adults have had variable ways of engaging in it. To the students who engaged in the study, however, this prompt and many other prompts would after successful engagement on this task no longer be considered non-routine.

I noticed a fourth aspect of good enough, non-routine and variable-entry tasks during the study. The CS task and other tasks on an ongoing basis triggered students to behave in ways that were mathematically fruitful such as recording, organizing work and looking for generalizations. Rather than being a one-time *activation* at the beginning, the task appeared to contribute to guiding, sustaining and occasioning of students' mathematical behavior. For example, in the process of engaging in this task, many students realized that there are many squares of different sizes, so they needed to record what they were counting systematically. A few students, for some reasons, needed to be prompted to record and to organize their work; otherwise they were stuck. With time when no easy answer was up and coming, many students realized there had to be "a shorter way of figuring out the answer". They then began to search for varied patterns, and patterns that led to solving the task and to posing new problems. To students who were inclined towards, say, recording systematically and generalizing, this task appealed to their essential being attracting their attention and holding them to engage in the task. To these students the CS task was good enough and was, for lack of a better term, a dynamical *attractor*, in the sense that it occasioned students to think mathematically. It structured their mathematical behavior. For a majority sessions this and some other tasks I adopted were good enough, non-routine, variable-entry as well as *dynamically attracting* and *mathematically structuring*.

1.8 Contextualizing the study

I have limited my investigation on mathematical thinking to focusing on the dynamics of the ways students attend to as they engage in mathematical tasks. All observations and interpretations in this study are confined to aspects related to understanding mathematical thinking within the enactivists' perspective and drawing from ecological metaphors of complexity science. Other aspects, such as the mental states and maps, cognitive images or information structures that are postulated by *information processing* psychologists, and other frameworks whose relevance to an *ecological-complexity* inquiry cannot be demonstrated, lie outside the scope of the study.

Furthermore, a small-scale and short-term study elucidates the general nature of students' mathematical thinking only in particular ways. I attempt to briefly address the broader issues only when they are evoked by my interpretations of students' activities. Also, there are specific research design limitations that I introduced right from the start for feasibility of the study: to work mainly with junior high school students and to focus mainly on students' engagement in good enough, *non-routine, variable entry* tasks.

The study will make no claim that what is observed or told by the participants is the thinking of the individuals and collectives involved. Observations and interpretations made in the study, as *second-order observations* in Maturana (1988a, 1988b) and von Foerster's (1992, 2003) sense will only reveal aspects of students' mathematical thinking from *my* understanding.

Furthermore, like any human effort, the view of mathematical thinking as effective behavior is likely to leave many aspects of learning mathematics out of the study. Nonetheless, it includes much more than the strictly mental and formally defined reasoning. Lakoff and Johnson (1999), Lakoff and Núñez (2001), and Nørrentranders (1998) assert that most "mental" processes occur at a *non-conscious* level. Even through trained informant introspection, one does not have direct access to these deepest[3] forms of understanding. Therefore this study by way of introspection might be limited to that which bubbles into the consciousness of a learner and of the researcher. Nevertheless, the whole body, the instruments we use, the collectives we form and the works we create are cognitive. Much more could be inferred from close observation of the worlds enacted by the learners.

Like any other study this study is limited by human perception and interpretation. In von Foerster's (2003) terms, the distinctions made in this study have their own blind spots. Wherever possible I will attempt to illuminate these blind spots by interrogating the world I enact in this research, by repeatedly returning to earlier interpretations, by varying scales of observation, and by reflecting on the choices made during the study. Nevertheless, every observation, reflection and scale still operates within a domain of coherences.

1.9 Significance of the Study

Researchers in mathematics education seek "better developed ways of looking [or of understanding students' mathematical behavior], organized into more penetrating theories of mathematical thinking" (Sfard, 2001a, p. 18). This study, even though it has been done mainly with students in extra-curricular settings, appears to have knowledge implications for classroom research and teaching. As with many studies, distinctions

[3] I mean "deepest" in the sense that these forms of understanding are not always available to the actor for interrogation; they represent different ways of knowing that we are not yet able to tap into; and they happen mostly as not-conscious and not personal activity.

made, questions reframed or answered and further questions posed will inevitably contribute to how the mathematics education community views mathematics learning in classrooms. The study will contribute insights to the emerging conversation on embodied, embedded and extended learning. Enactivists hope that these conversations will help researchers, educators, and eventually teachers, textbook authors and policy-makers to think differently about teaching and learning.

The conversations found in my writing may provide a language that will enable the community to observe and to comment closely on how students make sense of their mathematical worlds in novel ways. A study that closely examines mathematical attention as a consequence is bound to elaborate on mathematics subject matter for prospective teachers.

Much research has been done in the area of mathematical thinking, mostly from a traditional point of view of teaching students to think about and to practice given mathematical content. The next part of this writing is a review of the literature on mathematical thinking and students' mathematical behavior.

2. PARADIGMS ON MATHEMATICAL THINKING

The question at the heart of most mathematics education research is how to meaningfully engage students in acting and thinking mathematically. Most researchers emphasize that learning to think mathematically is an important goal of mathematics education (English & Halford, 1995; Núñez, Edwards & Matos, 1999; Schoenfeld, 1985). There are a variety of views on what mathematical thinking is, all depending on theory of learning espoused and views about the aims of school mathematics and nature of mathematics.

Each school of thought focuses on a distinct area of analysis, as evidenced in their overriding metaphors, which they use to understand mathematical thinking. Although the theories of learning and schools of thoughts might appear distinct, B. Davis, Sumara and Luce-Kapler (2000) observe that some of them share basic assumptions about cognition in general and mathematical thinking in particular. Table 1 offers an organizational chart of the schools of thought that I explore in this chapter and their views about mathematical thinking. I have grouped schools of thought that share fundamental assumptions into paradigms. In my view educators in North America, and English-speaking scholars,[4] over the last half century appear to have studied mathematical thinking from four different paradigms: the *child* and *structural* (individual and content) *psychology* paradigm, the *cognitive* paradigm, the *co-emergent* paradigm, the *coherent* and *post-structural* paradigm and, recently, the *ecological* and *systems* paradigm. To situate my study, I examine how mathematical thinking is construed in

[4] In reviewing the literature I am limited by language to research published in English.

these paradigms here. In Chapter 3, I explore insights from the existing studies on mathematical thinking.

Dissertation Landscape Forming

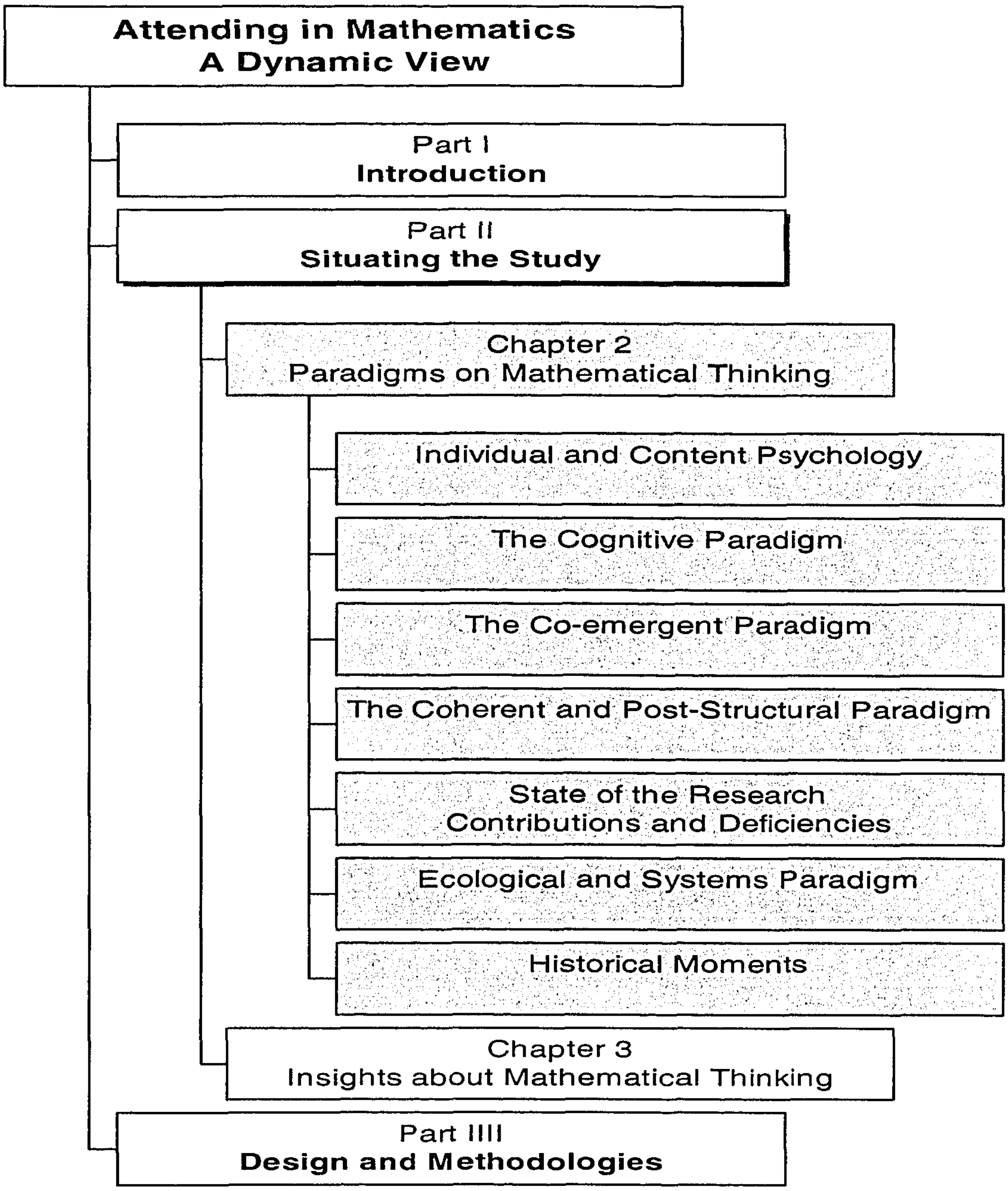


Table 1. *Various Views about Mathematical Thinking*

School of Thought	***Views and Overriding Metaphors***
Individual and Content Psychology	
Content Structuralism	Thinking is mathematically proficient when it has mastered content. *Structures of Content, Acquisition*
Individual Psychology	Thinking is mathematically well developed when a child can conceptualize things the way mathematicians do. *Stages, Growth*
The Cognitive Paradigm	
Computer Psychology[5]	Mathematical thinking is mental processing to solve problems. *Processes*
The Co-emergent Paradigm	
Radical Constructivism	Mathematical thinking involves reflection on one's actions and operations. *Revised or New Conceptualizations*
Social Constructivism	Thinking is internalized social discourse. *Communicating Mathematical Knowledge.*
Socio-Practice Theorists	Mathematical thinking is reflected by proper participation in socially organized practices. *Practice*
Situated Cognitionists	Thinking the way mathematics practitioners do. Mimic Experts, Produce knowledge *Work/Inquiry*
The Coherence and Post-structuralism Paradigm	
Socio-cultural Theorists	Cultural and linguistics mathematical activities are the bases for mathematical thinking. *Enculturation*
Critical and Political Theory	Students as cultural, gendered and socially situated beings should think critically about socio-political settings. *Emancipation*
Symbolic Interactionism & Didactic theory	Thinking is a process of meaning making, reflexively revising your meaning in relation to what others think. *Negotiating meaning*
Connectionism	Thought processes involve re-organizing and networking earlier made connections. *Connections and Recalibration*
Distributed Cognitionists	Thinking is extended to how one uses mathematical tools and materials. *Distributed Intelligence*
Ecological and Systems Paradigm	
See Chapter 6	

[5] First schools in cognitivism were symbolism and representationism and later connectionism and dynamicism.

2.1 Individual and Content Psychology

Brownell (1935/1970) championed a movement toward meaningful arithmetic as a backlash to meaningless drill and practice and incidental learning. This movement emphasized sense-making in what students learnt. It later consisted of psychological studies, most of which were based on *Piagetian theory,* though a few arose out of *structuralists'* studies and *gestalts* and others remained loyal to Thorndike's (1970/1924) *connectionist* studies. The structuralists and conceptionalists of the 1960s, mostly research mathematicians and school teachers, among them Cuisenaire and Gattegno (1957), Dienes and Goldings (1971), Hadamard (1945/1996) and Polya (1945/1973), argued that subject matter was the basis for students' thoughts and emphasized the fundamental structures of mathematics (Bruner, 1960). The dominant metaphor of mathematical thinking for structuralists was having students attain and be able to demonstrate *mathematical structures* that are assumed to be mind independent (Steffe & Kieren, 1994); A metaphor that, while it enabled a rigorous analysis of mathematics, limited the consideration of the factors in which learning is embedded.

From a psychological perspective, Dreyfus (1990) critiqued structuralists for insufficiently taking into account the details of children's thinking. Confrey (1991) remarked that teachers found it difficult to understand the structuralists' analyses of learning in terms of logic and/or sets, for example. In contrast to the structuralists, who turned to mathematics for founding learning in content, other researchers turned to psychology,[6] particularly to experimental and behavioral psychology, for a tradition of scientific inquiry (Schoenfeld, 1994). Schoenfeld (1994) observes that it was in the late

[6] This was at a time when psychology as an objective discipline was advancing in methodology.

1960s that structuralism was replaced by studies in *behavioral* psychology. The behaviorists argued that it was impossible to study thought objectively. They considered the mind to be, for matters of analysis, a black box. They criticized introspective observation. Instead, they focused on the observable behaviors induced by external stimulus. In the early 1970s, Piagetian studies, such as Steffe's (1970), began to draw from Piaget's *developmental* psychology to demonstrate how Piagetian *genetic structures* could, alongside the basic mathematical structures, explain children's thinking in terms of stages, Steffe and Kieren (1994) observe. Together with Piaget's and Bruner's work on play, Dienes and Goldings (1971), Cuisenaire and Gattegno (1957) and Gattegno's (1970) modern mathematics work ushered in studies, such as Kieren (1971), that explored the role of children's hands-on activities, story and play in enhancing concept learning (English & Halford, 1995; Steffe & Kieren, 1994).

During the early years in mathematics education, learning pre- and well-defined mathematical structures and concepts appears to have been the overriding metaphor for understanding mathematical thinking: A child who had mastered a given logical concept and carried out mathematical procedures proficiently was said to be thinking mathematically. Whereas the unit of analysis for the structuralists was the mathematical concepts, for the behavioral psychologist it was the child's performance. Yet for the Piagetian psychologists it was the child's abilities in relation to stages of conceptual growth. In this paradigm, as evidenced in a review of the research published in the early years of the *Journal for Research in Mathematics Education* (see the Journal's subject index for 1970-1981[7]), researchers mainly focused on instruction in an effort to prompt

[7] In 1981, JRME stopped classifying its publications in terms of Title index and Subject index. They now go by Author index under the classifications: Articles, Brief reports and reviews

thinking. The structure of a child did not appear to contribute much beyond their functions of acquiring knowledge.

2.2 The Cognitive Paradigm

Later in the 1970s and 1980s, the cognitive revolution renewed interest in studying mind, thinking and perception. At this time, research came to be dominated by cognitive psychology in which the human mind was construed, among other things, as a thinking machine. Distinctions such as spaces, loads or data were evoked to explain thinking (see Barnard, 1999). In the information processing view, mathematical thinking was and, for many, still is construed as acquiring, processing and mentally representing given concepts. Whereas many researchers in the 1980s only drew tools for thinking and metaphorical language from information-processing to understand mental states, the cognitivists strongly drew from the computer to study mathematical thinking.

In strongly viewing the mind as a *trivial*[8] machine, cognitivists construed mathematical thinking, especially conscious thinking as the symbolic processing to produce expected output (Bereiter, 1997). Mathematical processing is presumed to precede mathematical responses and behavior. Thus cognitivists considered what was presumed to happen in the child's mind during learning as the unit of analysis for studying mathematical thought. The cognitivists held a functionalist view of the brain, mind and body, in which their physical nature and experiences were not considered to be as crucial as the mental processes—the processing unit or software of the mind (Dehaene, 1997). Cognition was understood to be far from a biological and an experiential phenomenon.

[8] I am using the distinction trivial/non-trivial in von Foerster's sense. Non-Trivial in this sense refers to machines whose output might widely vary even when the input remains the same.

Cognitive and structural psychology paradigms maintained a division among mathematics, the environmental and the internal dynamics of the learner. Nevertheless, they raised educators' awareness of the significance of the environment and of the thinker's structure—particularly the age and instruction. More importantly, Kieren and Steffe (1994) maintain that constructivism was implicit in the critiques to some of the perspectives in the structuralists and cognitive paradigm. To most contemporary researchers, these early paradigms were limited in their ability to explain the richness and messiness of mathematical behavior as it occurs in the mathematics classrooms (Cobb & Bauersfeld, 1995). Also, the assumption of the pre-existing fixed mathematical structures that exist independently of the individual learner's activity is considered to be flawed, by the *radical constructivists* (Núñez et al., 1999). Researchers that pay attention to the social and cultural factors find the consideration of mathematical thinking as solely an aspect of an individual's psychological processes to be too narrow.

2.3 The Co-emergent Paradigm

Social theorists contended that mathematical thinking is an aspect of social practices and social discourses (Balacheff, 1990b; Boaler, 2000a; Lerman, 2001). To these didactic theorists, the classroom situation determines what is accepted as thinking mathematically (Balacheff, 1990c). Research work of social theorists such as Balacheff (1986), Bauersfeld (1995), Lave and Wenger (1991), and Walkerdine (1988) have pioneered a movement away from individual psychology and mathematics to other disciplines such as sociology, activity theory, anthropology, cultural studies, linguistic studies, and critical and political discourses.

I refer to those schools of thought that seek explanations from either the material, social, cultural or political contexts or those that look for explanations from connectionism and distributed learning as the *co-emergent* schools of thought. I also include in this paradigm attempts to juxtapose[9] the various *co-emergent* perspectives (e.g. Cobb, 1989, 2000). I use the term *co-emergence* for three reasons. Firstly, most researchers in this paradigm view cognition as a property that occurs within and through—co-emerges with—individual or social activity, community or expert practices, and socio-cultural or political contexts. Secondly, from an ecological point of view, although these perspectives may seem disparate, each of them seems to be focusing on cognition at a different scale. As I will demonstrate in Chapter 6, different theories in this paradigm focus on a different emergent level. Thirdly, from a complexity sensibility, when insights from these varied schools of thought are made to interact in significant ways, novel metaphors for understanding mathematical thinking are born. Historically speaking, research in the co-emergent paradigm was necessary for further paradigm shifts toward the post-structural, ecological and systems theories.

2.3.1 Constructivism

First among the researchers who turned to other disciplines to frame investigations on mathematics learning are the constructivists, including *radical constructivists* (e.g. Confrey, 1987; Steffe & Kieren, 1994; Steffe & Wiegel, 1992; von Glasersfeld, 1995; von Glasersfeld & Steffe, 1991). Radical constructivists focus on the individual child actively constructing his/her own knowledge. For these theorists, thought develops when an individual reflects on their practical-object and material oriented activity. In explicating a theory of learning as active engagement, radical constructivists

[9] Or even to dissociate the perspectives—See Lerman (2000)

draw heavily on Piaget's genetic epistemology and von Glasersfeld's conceptual structure development (Confrey, 1994a; Steffe & Thompson, 2000). They mainly focus on aspects such as mental operations and conceptual structures, and knowledge schemas. However, the *social constructivists* have critiqued the radical constructivists for not paying attention to the influence of social participation. They are also criticized for completely eliminating the structuralist' emphasis from their analyses (Ginsburg & Seo, 1999).

2.3.2 Social Constructivism

Second, the *social constructivists* (Balacheff, 1986, 1990a; Bauersfeld, 1992; Cobb, Yackel & Wood, 1991, 1992; Driver et al. 1994; Ernest, 1991, 1994) focus on the individual in social interaction. Social activity is more than a background to thinking, they contend (Cobb & Bauersfeld, 1998; Cobb, Yackel & Wood, 1992). Participation in social interaction is crucial in enhancing a child's mathematical thinking. Social constructivists focus on the child's interactions, such as communication, arguments and explanations (Cobb & Bauersfeld, 1995). Some researchers specifically draw analogies from social activities, such as speech, conflict, negotiation and dialogue, to understand thinking. For instance, after Vygotsky (1978), mathematical thinking is considered by some to be synonymous with mathematical communication, except that thinking is an internal dialogue with oneself whereas communication is an external dialogue with others. However, most social constructivists, like their radical counterparts, also delimit their focus by ignoring linkages with broader socio-cultural-political settings. They too focus on the individual learner as the only cognizing system in the classroom (Burton, 1999c; Cobb, Yackel & Wood, 1992; Kieren & Simmt, 2002; Sfard, 2001a).

2.3.3 Social Practice Theorists

Third, *social practice theorists,* particularly the *situated cognition* scientists such as Greeno (1991), and Lave and Wenger (1991) draw from the apprenticeship model of learning to consider mathematical thinking as an aspect of *participation in* specialized community practices and work settings. They assert that mathematical thinking is situated within the context and community in which it is invented as a tool to be used (Griffin & Griffin, 1996). When students engage in or even mimic mathematical practices they are thinking mathematically. Cobb et al. (1993) contend that since research mathematics (the practice of experts) is different from school mathematics (the practice of novices) then situated cognitive studies have little to offer to school mathematics. To Bereiter (1997), from a cognitivist framework situated thinking is a small portion of human thinking.

Researchers who searched outside cognitive science study the influence of individual and social activity or the influence of socio-cultural and political contexts on learning. It seems researchers in this paradigm construe mathematical thinking as the sense that emerges through individual activity, inter-individual interactions or community and cultural practices. This is the discursive view of thinking as inner or internalized action and interaction—whether speaking or writing (Lerman, 2001; Sfard, 2000a, 2001; Vygotsky, 1978).

2.4 The Coherent and Post-structuralist Paradigm

Recently the mathematics education community draws from coherence and post-structural human science theories to understanding mathematical thinking and learning. Drawing from post-structural theories, they seek to explore how the varied views about learning cohere.

2.4.1 Socio-Cultural Theorists

The *socio-cultural theorists* draw from Bakhtin and other social theorists to examine the dialectic between thinking and culture and language (e.g. Boaler, 2000a; Lerman, 2001; van Oers, 2001). Another group of researchers that is closely related to socio-cultural theorists are the *interaction theorists* (such as Bauersfeld, 1995; Voigt, 1994, 1995), who draw from Blumer's *symbolic interaction* theory to highlight the social negotiation of *mathematical meaning*. Voigt (1994) in a manner similar to social-practice theorists, asserts that mathematical knowing happens between, not inside or outside, individuals. Students' mathematical thinking develops reflexively as students socialize their own behavior in relation to other participants' understanding. Mathematical thinking is synonymous with adjusting one's interpretations to converge with mathematical conventions (Voigt, 1994).

French scholars such as Balacheff (1990c) and Chevallard integrate Piaget's work with Vergnaud's situational analysis and Brousseau's didactical theory. They consider the socio-cultural in broader terms to include situational and institutional factors that give status to some thoughts as mathematical and others as non-mathematical (Balacheff, 1990c).

To some social-cultural theorists, mathematical meaning–making and shared understanding are better theoretical constructs than mathematical thinking. Social practice theorists and socio-cultural theorists believe that mathematical thinking develops with interactions within socially organized practices and cultural activities (Cobb & Bauersfeld, 1995). (Mental activity involves refining and revising what first appears on the individual plane in response to what appears on the social plane rather than *appropriating* what appears first on the social plane, Wenger (1998) asserts.) These schools of thought delimit their analysis to how the micro-politics of a classroom, school and community institutionalize students' mathematical thinking. Whereas for constructivists mathematical thinking is re-organizing one's conceptions to fit either empirical facts, for socio-culturalists and the interactionists it is to fit with social conventions. To both constructivists and social theorists thinking is considered to be about avoiding social or cognitive conflicts or as solving problems.

2.4.2 Politico-Critical Theorists

The *social critical* theorists (e.g. Apple, 1992; Lerman, 2000; Walkerdine, 1990) focus on macro aspects of mathematics education that economically, anthropologically and politically subjugate and those that emancipate learners. Critical theorists critique *traditional* as well as reform-learning theories such as constructivism for further perpetuating undemocratic societies. Mathematical thinking is mainly construed in terms of critical thinking and political awareness about the *formatting* power of mathematics (Skovsmose, 1990, 1992, 2001). Politico-critical theorists attempt to contest thinking that is governed by institutional expectations, and therefore produce uncritical citizens. Some scholars, being pragmatic, argue that critical mathematical thinking is synonymous with an equity-seeking stance that examines relationships between mathematics and the larger societal setting. Thinking mathematically should not only be about correctness but also about consistency and appropriateness, Apple (2000) and Skovsmose (1992) contend. The overriding metaphor of thinking is reasoning and seeing through political injustices and inequalities of organized society.

Walkerdine (1990) further questions the whole discourse around what mathematical thinking is. To her it was unsoundly constructed around the fear of the marginalized other—be it the girl, the poor or the colonized other—whom this discourse so to speak baptizes as a non-mathematical thinker.

Efforts at inclusive mathematics circular, however, in some places such as South Africa, are seen by the marginalized people themselves as efforts by White male researchers to perpetuate social economic differences (Vithal & Skovsmose, 1997). Also in advancing the notion of social determinism, critical theorists brushed aside the influence of biological and environmental constraints, just as many social and cultural theorists did.

2.4.3 Connectionism

The search outside cognitive science was paralleled by a shift in perspective within cognitive science itself. In the 1990s there was a movement from cognitivism to *connectionism*. Recently there is a shift from connectionism to dynamicism. Connectionism arose as an alternative to the cognitivists' view of learning as a phenomenon based on rules and symbols. Designs of intelligent systems and neural networks led to stressing that understanding involves recognizing similarities and making connections. It involves network dynamics rather than mastering rules (Bereiter, 1991; English & Halford, 1995). The overriding metaphor of mathematical thinking for the connectionists appears to be *recalibrating* in response to recurrent interaction to strengthen the connections. Connectionists nonetheless retain the basic cognitivists' assumption of a mind with architecture that is capable of knowing an independent world.

2.4.4 Distributed Learning Theorists

Another recent perspective on learning that is yet to influence views about mathematics thinking is that of *distributed learning systems* (Wertsch, Tulviste & Hangstrom, 1993). Distributed learning theorists contend that not only is intelligence distributed within in an individual's body, it is also distributed among environmental dimensions. This being a recent school of thought, as is the case with connectionism and dynamicism, not much work in mathematics education draws directly from it.

2.5 State of the Research: Contributions and Deficiencies

Research in the co-emergent, coherent and post-structural paradigm has contributed to the recognition that most learning processes are not as linear and easily defined as was described by earlier psychology-based research such as *bond, incidental learning* or *meaning* theories. It specifically challenges the passive and computational mode of thinking. Socio-cultural and critical-political theories have gradually expanded the perceived limits of cognitive activities away from strictly head-based structures (Confrey, 1994b; Núñez et al., 1999). Practically speaking, they have offered better[10] metaphors of thinking. Today, mathematical thinking is construed in broader terms than simply the mastery of arithmetic facts.

In addition to offering a wider conceptual focus, co-emergent, coherent and post-structural theorists have led to the widening of foci in investigative analyses away from solely measuring the effectiveness of instruction. Instead, they analyze a range of aspects: individual children's actions, student interactions and micro classroom cultures; and social practices, cultures of groups and political structures of organized society. More importantly, from an ecological and systems perspective, when the varied perspectives of

[10] They are better in that they enable us to see through the myths of the old (Hoy, 1991; Rorty, 1982), also to the extent that they allow us to see, to do and to say a lot of things not possible before.

the co-emergent paradigm are considered in light of each other, they generate theoretical expansions that are not only reconciliations[11] between perspectives or departures from them. Rather, they produce dramatic iterations to bring forth hybrid perspectives (B. Davis & Sumara, 2000; Kieren, 2000). B. Davis & Sumara (2000) assert that these schools share a metaphoric commitment to a single body—biological, social, cultural, epistemological or political. As I will explain each of these bodies could be construed as emerging from the other body at yet another body of cognition. More ecological, distributed and situated theories of learning have arisen from the incompatibility of theories in these paradigms. This is another connotation of the term co-emergent theories, one related to the complexity research notion of *emergence* (see Chapter 6). Broader and deeper understandings of mathematical thinking are springing forth from discussions of the limitations of earlier paradigms. As such, researchers such as Confrey (1995a, 1995b), Kieren (2000) and Sfard (2000b) have desired to understand these often-contradictory perspectives. A view of each of these theories as intertwined bodies of mathematical thinking is evolving. Many theorists now believe each of these schools of thought shows a small part of the big picture (Cobb, 1994; Kieren, 2000). Many researchers are exploring what the whole picture could be. They are looking at the oversights of earlier theories.

[11] At times these juxtapositions have been simplistic. For pragmatic reasons, some researchers haven't been bothered by the inconsistencies (Kieren, 2000).

In summary, the aspects of mathematical thinking that needed a recursive elaboration by the late 1990s include:

- an understanding of the centrality of the physical nature of the brain and body in cognition;
- a recognition of other cognizing systems in the classroom;
- a positive construal of thinking as more than problem solving, overcoming obstacles or negotiating conflicts;
- a revision of the linear view that thinking is separate from and always precedes action;
- an inclusion of biological, contextual and historical influences of thinking;
- a closer examination of the dialectic between individual actions, social activity, organized practice, physical environments, broader political and cultural milieu and mathematical thinking;
- an exploration of how the physical environments and tools determine what it means to think mathematically;
- an admission of novelty and divergence in students' mathematical thought.

Most co-emergent, coherent and post-structural perspectives, apart from radical constructivism, have neglected the behaviorists' focus on the biological contexts that shape knowing (Núñez et al., 1999). To all theorists, including those who focus on social practice, cultural and political aspects thought activities are still limited to the inner and conscious activities of the individual student (Burton, 1999c ; Kieren & Simmt, 2002; Namukasa & Simmt, 2003). Some studies are tacitly underlain by Modern assumptions of the mind as machine creating internal representations through symbolic manipulation.

Also these paradigms do not explicitly challenge the Cartesian "either-or" approach or the "reduction add-on" model. Lerman (1999, 2001) insists on the impossibility of theoretical conflation of the individualistic psychological views with the discursive views. Rightly so, the overemphasis on the individual person as the only cognizing system is a problematic view that will not go away by conflating radical constructivism with social constructivism. As well, although researchers in the co-emergent paradigm construe mathematical thinking as the sense that arises, say through individual activity, they have not yet elaborated on how students' thinking turns back to influence the individual activity, inter-individual interactions or community practices within which it happens. Students' mathematical thinking, as I will elaborate in this writing, is far from being an epiphenomenon; it is in reciprocal influence with actions, interactions and practices.

Furthermore, a person reviewing the studies on mathematical thinking is left to wonder how the categorized aspects and influences of mathematical thinking cohere. Many aspects appear to compete with one another. Also problematic is the empiricist unattainable desire to study students' mathematical thinking as a fixed form, "a fixed object that needs nothing except itself in order to exist" (Jardine, 1998, p. 43), and that should therefore be prescribed explicitly. Kieren (2000), in his paper entitled "Binoculars or Dichotomies", calls on researchers to view their theories as partial truths in order to "occasion for new and perhaps different ways of thinking/acting" (p. 231). It is not the 1990s question about which one is the most crucial influence on mathematical thinking. Beyond this debate, researchers have begun to draw from juxtapositions of disciplines such as ecology and feminism, post-structuralism and complexity research theories, neuroscience and discursive psychology to inform the complex discussions on cognition.

2.6 Ecological and Systems Paradigm

Certain researchers have attempted to address the deficiencies by studying cognitions in its complexity without proclaiming that it is primarily psychological, social or cultural. To do this, they have drawn from the work of: (a) *social biologists* such as Bateson (1979), Maturana (1987) and Varela (1992, 1987), (b) adherents to *complexity theory* such as Capra (1996), Johnson (2001) and Waldrop (1992), (c) *post-structural theorists* such as Bahktin, Bourdieu, Brousseau, Foucault, and (d) *ecological mathematicians* and *mathematics educators* such as Dehaene (1997), Jardine (1998), and Lakoff and Núñez (2001). By questioning the assumptions that underlie the desire to reduce complex phenomena to sums of their components, researchers in the ecological and systems paradigm are recursively elaborating of the preceding paradigms. Enactivists and complexivists,[12] as I shall elucidate in Chapter 6, maintain that not everything in the preceding perspectives, such as the *radical behaviorist* and *radical constructivists* schools, need be thrown out.

Some of the perspectives that have been taken on by mathematics educators in this paradigm are eco-feminism (Confrey, 1995b), enactivism (B. Davis et al., 2000; B. Davis et. al., 1996; Gordon, 2002; Kieren, 2000; Simmt, 2000, 1998; Towers 1998), and neuro-biology (Butterworth, 1999).

In particular, Confrey (1995b) draws from eco-feminism to propose an evolutionary biology metaphor of intellectual development that iterates on both Piaget's

[12] Davis, Sumara and Simmt (2003) adopt the term *complexivist* to refer to researchers in human sciences who draw from complexity science theories.

and Vygotsky's theories. Boaler (2000a), drawing from situated cognition and narrative inquiry, explores how human agency is closely intertwined with mathematical cognition. Furthermore, other researchers like Confrey (1999) and Sfard (1995) extend the constructivist belief of individual construction of knowledge to trace the history of mathematics in an evolutionary manner and on a grander scale. Sfard's and Confrey's developmental and historical analyses, Núñez' (2000) mathematical idea analysis and Freudenthal's (1991) mathematical studies anticipate that the phylogeny (evolutionary development over generations) and the ontogeny (development in an individual) of mathematics will reveal more than marginal similarities. They expect that the mathematical thinking of an individual learner re-constructing knowledge may be quite close to that of generations and eras of mathematicians as they constructed new knowledge.

Lakoff and Johnson (1980, 1999) and Lakoff and Núñez (2001) study how the peculiarities of our bodies and brains, and our environments have evolved to create mathematical thought. By inferring that the mind is embodied in the "deep sense that our conceptual system and our capacity for thought are shaped by the nature of our brains, our bodies and our bodily interactions" (Lakoff & Johnson, 1999, p. 265), Lakoff and his colleagues seek to study how mathematical thinking is grounded in everyday experiences. Núñez (2000) analyzes mathematical intuitions by developing ways of studying largely unconscious ordinary everyday mathematical conceptual structures. Sfard (1994; 2000b) draws on Lakoff and Johnson's explorations of the embodiment of conceptual metaphors to elucidate her earlier work on reification of mathematical objects. Radford (2003) also juxtaposes his critique to Lakoff and Núñez' (2001) work with Peirce's semiotics to explicate a theory of social-cultural semiotics. Due to the limitations of space I have only mentioned a few of such studies.

Also included in this paradigm are researchers dubbed *complexivists* who draw metaphors from the complexity science theories to investigate mathematical cognition as a complex, dynamic and adaptive phenomenon (see Davis & Simmt, 2002; Kieren & Simmt, 2002; Towers & Davis, 2002). It is from the enactivist and complexivist perspectives that I would like to explore the nature of students' mathematical thinking. Chapter 6 explores these perspectives. One wonders, however, whether coherent and post-structural schools of thought such as social-cultural semiotics and enactivism do address the deficiencies of prior theories I identified earlier. If the mathematics education community is to broaden its views on mathematical thinking there is need for perspectives that pose novel questions about mathematical thinking in addition to addressing these deficiencies.

2.7 Historic Moments in Mathematical Thinking Research

I have briefly situated my interests on students' mathematics thinking in the mathematics education community. The exploration is a historical sketch. Different paradigms appear to have marked different moments in mathematics education. At a macro level different questions dominated different decades or continents. Yet at a micro level, researchers such as Kieren and Schoenfeld, akin to Bruner, have asked different questions and worked with different epistemological presuppositions at different times. For example, it appears that in 1960 Bruner worked with a structuralist epistemology. In 1986, he shifted to a more constructivist-linguistic approach that by 1990 had evolved into a socio-cultural epistemology. Yet in 1996 Bruner exhibits a more ecological perspective. For another detailed example of a mathematics education researcher who has subscribed to different epistemological presuppositions at different times see Schoenfeld (1986). Also for an analysis of a theoretical model that has evolved from the radical constructivist perspective through a discursive psychology to an ecological complexity perspective see Kieren and Simmt (2002), Kieren, Pirie and Gordon-Calvert (1999), and Pirie & Kieren (1989).

These examples have occasioned me to view each of the five paradigms—psychological, cognitivist, co-emergent, coherent and post-structural, and ecological and systems paradigms—in an ecological complexity manner as a *recursive elaboration of* preceding paradigms, at least in the Western context.[13] For instance, although researchers in the co-emergence paradigm may not agree on presuppositions made by researchers in the cognitive paradigm, studies in cognitivism were necessary for the co-emergence paradigm to evolve. More accurately, perspectives such as social constructivism to a large extent were embedded in the critique of earlier perspectives such as radical constructivism. Hence, even though this work is guided by an ecological-complexity orientation, I acknowledge that my research questions unfolded from questions that were investigated in the co-emergence and post-structural paradigms. Indeed later questions in this research, as we shall see in Chapter 8, evolved as my individualist and cognitivistic assumptions were progressively challenged. It is an ongoing process. Our participation expands the theories of learning.

I have reviewed varied perspectives that continue to inform research on mathematical thinking, but my exploration is bound by space and time. For instance, my

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[13] There are many examples on non-Western contexts in which ecological-complexity sensibilities arose not as recursive elaborations of Cartesian and positivistic sensibilities.

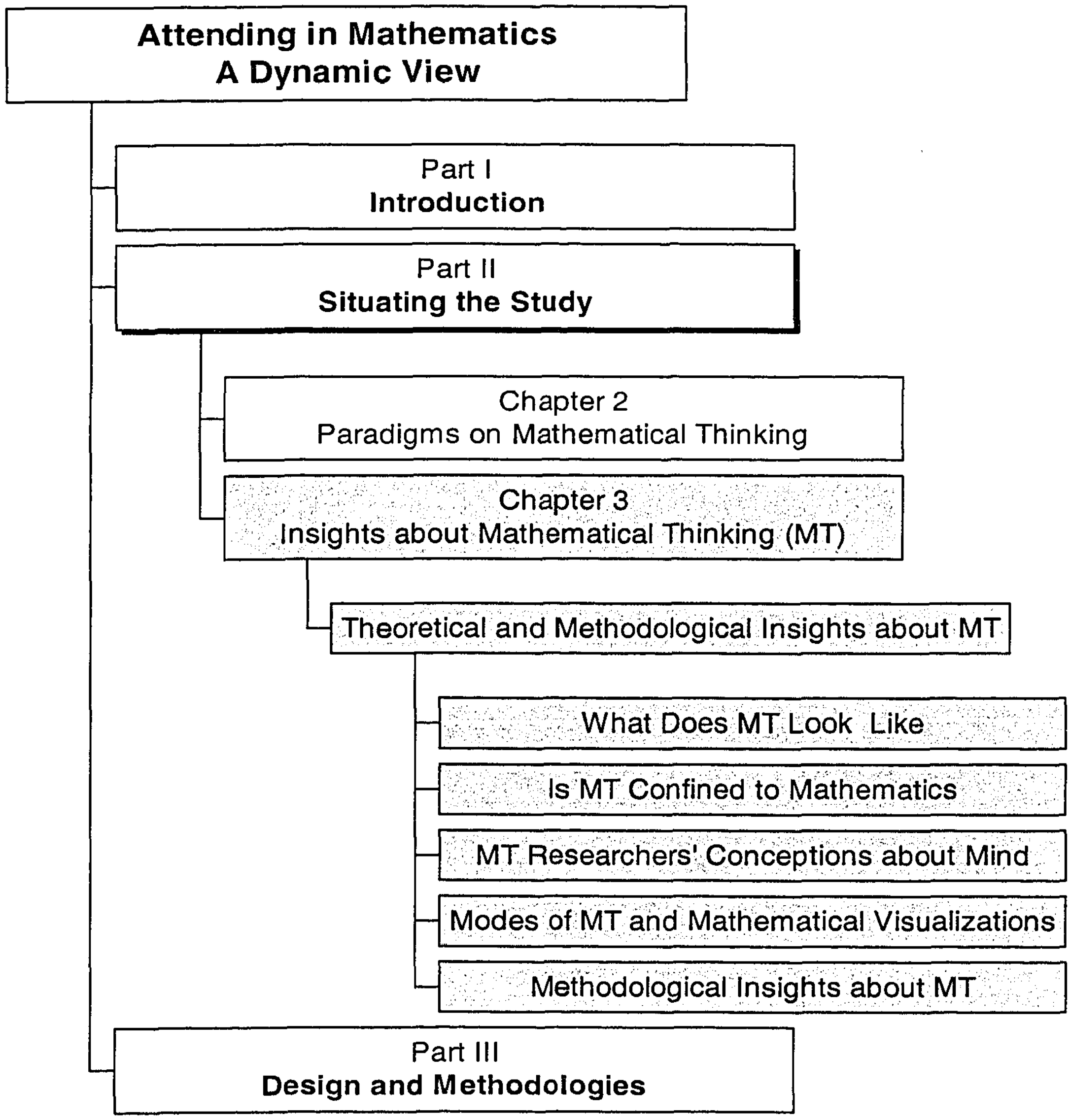
review might be considered largely blind to theories that were central to developments in mathematics education on other continents and in other centuries, especially those theories that have not largely been taken on in North American research. Examples of such theories include *experiential mathematics* by Freudenthal in the Netherlands, the *theory of didactical situations* in *mathematics* (see Herbst & Kilpatrick, 1999 for a preliminary exploration of the theory) and Mary Boole's work in the later 1800s. For purposes of delimiting the review to theories that have implication to mathematical thinking I have selectively left out theories such as *social constructionism* and *historical analyses* (see for instance Gray, 1999) whose influence on studies on secondary school students' mathematical thinking is still minimal. I have nonetheless attempted, as much as the locale of this writing at a Canadian university in the early 21st century allows, an elaborate situating of my research interests. Because the purpose of this review was to situate my research, I have not concerned myself with more specific variations in what researchers consider central to mathematical thinking. In the next chapter, I look at insights from more specific studies on mathematical thinking.

3. INSIGHTS FROM THE REVIEWED STUDIES

Studies on mathematical thinking concern themselves with aspects that enhance mathematics learning. A large group of researchers has for a significant time concerned itself with studying *representation* and *visualization* as an aspect of mathematical thinking and interpretation. Recently, some researchers drawing from semiotics concern themselves with symbolization and signs. A few have studied the nature of students' errors and difficulties in learning mathematics. Researchers have studied the role of these aspects in learning mathematics with the aim to help students to experience mathematics in a more meaningful and accessible way (Presmeg, 1986).

The nature of mathematical representation, symbolization, visualization and difficulties and their relation to the nature of mathematical thinking have been approached from varied perspectives: psychological, mathematical, philosophical, anthropological and, recently, neurobiological and ecological. All of these approaches provide important insights in understanding the nature of students' mathematical thinking. In Chapter 2, I delimited the exploration to broader underlying assumptions and questions investigated about mathematical thinking by the schools of thought. In this chapter I examine more specifically the theoretical, methodological and practical insights gained from the varied studies. Most of the insights are theoretical insights; I categorize them around four questions about mathematical thinking in particular and thinking and perception in general.

Dissertation Landscape Forming



3.1 Theoretical and Methodological Insights

In attempts to investigate the nature of students' mathematical thinking the following questions have been central:

1. What does it look like to think mathematically? Is thinking taken to include only the rational process of reasoning or does it include the non-conscious, emotive and affective domains?
2. Is mathematical thinking confined to mathematics or is it applicable to learning in general? Further, is it considered domain or level-specific with in mathematics?
3. What is the role of body, brain and mind—its structure and functions in mathematical thinking? What, for the thinker, does it mean to think?
4. What is the role of perception in mathematical thinking? How about the role of the environment?

3.1.1 Mathematical Thinking: What Does it Look Like

A researcher's view of the nature of mathematics appears to influence how they define and study mathematical thinking. Researchers who take mathematics to be algorithmic and mainly to encompass abstract structures will tend to emphasize formal mathematical thinking. On the other hand, researchers who take mathematics to be intuitive and experiential will emphasize intuitive and affective aspects like conjecturing, expressing and the role of emotions (Burton, 1999a, 1999b; Drodge & Reid, 2000; Lampert, 1990). There are variations along this spectrum from formal to intuitive aspects. For example, the National Research Council (1989), emphasizing that mathematics is a science of patterns but at the same time foregrounding formal processes, refers to mathematical modes of thought like "modeling, abstraction, optimization, logical analysis, inference from data and use of symbols" to describe mathematical thinking (p. 31). Burton (1984), focusing on mathematics as a problem-solving activity, describes mathematical thinking to consist of operations like iterating, enumerating and ordering; processes like specializing, conjecturing and generalizing; dynamics like manipulating and making sense; and the affective phase of articulating *(entry, attack* and *review).*

There are recent shifts away from focusing on mathematics, hence mathematical thinking, as a solitary attribute that is entirely rational and highly structured, toward a more communal and situated approach in which emotions and social interactions also have a role to play. These shifts have resulted in the expansion of mathematical processes to include social processes such as *adjusting to socio-mathematical norms,* and being able to engage in *mathematical conversations* (e.g. Cobb et al., 1993; Mason et al., 1985; Schoenfeld, 1992; Gordon Calvert, 2001). What mathematical thinking might look like is changing. My view of mathematics as a human activity inevitably influences what I study as mathematical thinking.

As well researchers' perceived aims of school mathematics tacitly affect how they define mathematical thinking. When researchers view mathematics as a school discipline mainly geared toward preparing future mathematicians, they construe mathematical thinking in relation to presumed experiences of mathematicians. Researchers such as Putman, Lampert and Peterson (1989) assert that the discourse of school mathematics should imitate the discourse of mathematicians, so that by engaging in *inquiry mathematics* (a parallel of research mathematics) students might think like mathematicians. In this case, researchers would study mathematicians' practices to define mathematical thinking. Although an understanding of adequate mathematical behavior among mathematicians might lead to a better understanding of the nature of students' mathematical thinking, this view is increasingly contested. To some researchers it is faulty to assume that the goal of school mathematics is solely in service of research mathematics (D'Ambrosio, 2001; B. Davis, 1995; P. Davis & Hersh, 1981). Do not school mathematics students, just like mathematicians, bring forth their own worlds of mathematical significance, worlds that might be somewhat different from mathematicians' worlds but nonetheless mathematically legitimate? Should students' mathematical thinking solely be defined in terms of research mathematicians' ways of being? D'Ambrosio (2001), and P. Davis and Hersh (1981) assert that the tacit view that school mathematics (or schooling in general) is preparation for work does not fit in times of rapid technological change. Researchers are considering learning mathematics in more than utilitarian and futurist ways. For instance, B. Davis (1995, 2001) asserts that school mathematics expands the domain of the possible for humans. In the mathematics education community school mathematics is increasingly seen to serve other aims including professional, cultural, social, civic and aesthetic. This broadened view of the aim of mathematics is redefining what students' mathematical thinking might look like.

The belief that school mathematics should be in the service of research mathematics, when coupled with the belief that mathematical ability is a talent, casts mathematical thinking as a propensity of the few students who demonstrate higher mathematical ability. Social, historical and cultural investigations in mathematics education seem to be settling some of the motivations behind the question of whether mathematical ability is innate or learned. Many researchers do value both the influence of biological and environmental factors. In embodied mathematics, Lakoff and Núñez (2001) argue that the "basic cognitive mechanisms used by mathematical inferences are innate, others develop through childhood, and some develop only with special training" (p. 353). Dehaene (1997) studied behavior of both professional mathematicians and mentally deficient prodigies. He concludes, "Genes and other biological factors do not weigh much when compared to the power of learning, fueled by the passion of numbers" (p.170) and perhaps the passion of other mathematical attributes such as change and patterns. It appears the environment, including educational experience, family, and social and economic factors, intertwines with biological factors such as hormones and genes in the emergence of what is commonly defined as adequate mathematical thinking (Dehaene, 1997; Walkerdine, 1990).

When researchers view mathematics as geared toward offering learners skills, attitudes and knowledge that they need in life, then mathematical thinking is defined in the contexts of the students' experiences, needs and abilities.

Other researchers, such as Lutfiyya (1998), have rephrased the question of the nature of mathematical thinking to "what characterizes the thinking of individuals who demonstrate a high ability in mathematics" (p. 56). Such studies distinguish behaviors of strong mathematics students from behavior of weak students. For instance, Presmeg (1997) and Tall (1999) assert that low-achieving students focus on details whereas high-achieving students focus on abstractions. Comparing what characterizes students' mathematical thinking with what strong mathematics students do is only part of what is needed. Researchers also need to investigate adequate mathematical behavior as it co-emerges when as students (whether fast, average or slow learners) work on mathematical tasks.

In studying mathematical thinkers, methods of observation such as introspection and speak-aloud protocols have been limited to conscious and explicit mathematical thinking. Yet recent advancements in neurological studies have resulted in what Nørrentranders (1998) has dubbed the *composure* of conscious activity. Conscious and formulated activity is a limited portion of human knowing (Núñez, 2000). As well, it might be the case that not questioning objective (observer-free) observations limits what some studies have defined as mathematical thinking. Is whatever is defined as mathematical thinking confined to mathematics? This is another question whose answer tacitly influences how one investigates mathematical thinking.

3.1.2 Mathematical Thinking: Is it Confined to Mathematics

Variations in the description of mathematical thinking at times point to whether or not a researcher considers mathematical thinking to be confined to mathematics, and whether or not within mathematics it is domain and level-specific. For some researchers mathematical thinking is viewed to be inclusive of other learning. Mason et al. (1985) describe mathematical thinking as a "dynamic process which, by enabling us to increase the complexity of ideas we can handle, expands our understanding" (p. 158). Burton (1992, quoted in English & Halford, 1995) preferring the term *thinking mathematically* to *mathematical thinking*, describes mathematical thinking as "the style of processing which supports an enquiry which might ultimately lead to the learning of some mathematics but equally might lead to the learning in other subject areas" (p. 259).

To other researchers, mathematical thinking is specific to mathematics. Love

(1988) identifies the need to distinguish between mathematical and non-mathematical thinking. For instance, Schoenfeld (1992) describes it as having a mathematical point of view—a way of seeing, and of being able to develop competence in mathematical sense-making and to use mathematical tools meaningfully. E. P. Goldenberg (1996) describes mathematical thinking as part of a set of mathematical "habits of mind," and for Lutfiyya (1998) "mathematical thinking involves using mathematically rich thinking skills to understand ideas" (pp. 55-56). Bauersfeld (1995) describes mathematical thinking as a way of seeing the world, of approaching the world in the ways mathematicians do. To Mason (1989), Radford (2003), Sfard (1991b) and others mathematical thinking involves abstracting as well as objectifying imagined, esoteric mathematical objects so as to act on them. It appears these researchers view mathematical thinking in terms of habits, tool use, language, attitude, skills or point of views that may not be applicable to learning in general but nonetheless central to learning mathematics.

In addition to describing mathematical thinking as specific to mathematics learning, certain researchers stress that mathematical thinking is also level- and content-specific. For example, Peterson (1988) contrasts low-level and high-level thinking skills and, like other researchers, likens high-level thinking skills to adequate mathematical thinking. Gray, Pinto, Pitta-Pantazi and Tall (1999) embrace a developmental view of mathematical thinking. They illustrate advanced mathematical thinking as a progression from the level of procedure—doing routine mathematics accurately—through the procedure and process level of performing mathematics flexibly and efficiently, to the procedure, processes and procept[14] level of thinking about mathematics symbolically. Dreyfus (1990), Gray et al. (1999) and Tall (1991), however, observe that although advanced mathematical thinking tends to involve more abstract actions, such as formal proofs, it is not considerably different from elementary mathematical thinking. There is mounting research evidence that elementary and junior high students do act and think in complex ways, though they might lack the formal linguistic tools to represent their actions in symbolic terms (Kieren, Mason & Davis, 1996). Kieren et al. (1999) demonstrate that learning is not linear and developmental from the concrete to the abstract, from procedural and routine to conceptual understanding, from the elementary to the advanced ways of thinking. Rather, many times it involves moving back and forth, folding back to inner levels and to the primitive forms of knowing. Also from the Brousseau's situational analysis, elementary mathematics might not be a mere "elementarization" of advanced or research mathematics (Balacheff, 1990c). One might conceptualize it as a distinct domain altogether.

Also, in addition to studying mathematical thinking at different levels and stages, some researchers have explored its content specificity in such areas as *algebraic thinking, statistical thinking* and *geometrical thinking*. A few researchers even consider content specific thinking such as *algebraic thinking* to be a counterpart of mathematical thinking (Lee, 1997). If one believes mathematical thinking to be specific to mathematics and within mathematics to be specific to particular areas, then one might get caught up by the cognitivist assumption that the brain is compartmentalized by function, with specific regions of the brain dedicated to particular domains of learning. The question of whether

[14] *Procept* is a term coined by Gray and Tall (1994) to refer to the amalgam of a process and a concept produced by that process (quoted in Gray et al. 1999).

mathematical thinking is exclusively specific to particular domains and age groups might need to be revisited in light of recent developments in neurological research. Are there such pre-demarcated and specialized compartments or selves as a *mathematics module or* even more specific modules for statistics or algebra in human bodies and worlds?

The argument that mathematical thinking is specific to mathematics and that distinct concepts utilize specific thinking might be an oversimplification (Butterworth, 1999; Dehaene, 1997). In neuro-physiological studies, scans of brain activity show that whereas distinct regions of the brain seem to be active, say when one is doing arithmetic, also distinct brain regions are active for subtly different mathematical activities, such as when remembering basic facts and performing a multiplication procedure. As well, neuropsychological studies of patients with damage to particular brain regions illustrate that damage usually affects mathematical performance only in particular ways that might not be limited to domains of mathematics but to miniature skills. For example a patient might have lost their number facts yet can perform other mathematical procedures, all within the same domain of arithmetic thinking (Butterworth, 1999). On the other hand, in many cases, the same regions activated during specific mathematical tasks are also activated during non-mathematical tasks such as verbal and motor tasks. These hypotheses from brain studies challenge views about content specific mathematics and what cognitive researchers call mathematical processing. In the next section, I continue to review how conceptions about the mind tacitly influence theoretical insights gained from research on mathematical thinking.

3.1.3 Conceptions about the Mind, its Structure and its Functions

Researchers' conceptions of the mind greatly influence what they consider the role of mind and brain in mathematical thinking to be. Does mind operate like a machine? Do not some systems without complex brains demonstrate qualities of mind? What does it mean to think? How do our views on mathematical thinking change when mind is construed as the collective and emergent capability of brain and body embedded in larger systems? Apart from the individual student what other agents could be observed to demonstrate thinking qualities? How can we construe *thinking about thinking*? Might thinking be a phenomenon that pertains to the domain of descriptions, the meta-domain? Is thinking a commentary by an observer about a system acting and changing within its medium? Or could it be the case that other than being observer-relative thinking and other *logical* forms of mind just like properties of life are grounded in the intrinsic organizational complexity of particular material systems?

Mind, and consequently thinking, have largely been conceived as solely psychological human phenomena. Thought and action are taken to be separable. Many researchers for whom thinking is synonymous with reasoning about something, transitive thinking, have pervasively focused on mathematical thinking as reasoning about mathematical patterns and generalizations and even as thinking about this mathematical reasoning (R. B. Davis, 1983; Ginsburg, Jacobs & Lopez, 1993). Such researchers in an almost Cartesian dualist manner focus on thinking as a rational process or range of processing styles and skills involving non-corporeal, timeless and universal facts that need to be mirrored in the learner. To other researchers mathematical errors, illusions and difficulties have been the focus of analysis (Borasi, 1987; Sierpinska, 1990; Zazkis & Liljedahl, 2004). These researchers equate mathematical thinking to overcoming, eliminating or circumnavigating erroneous intuitions and perceptions. However, studies on epistemological and cognitive obstacles do not consider the extent to which thinking attributes such as errors are a commentary by an observer about a system acting and changing within its medium. Indeed, a majority of the studies have been based upon *representationists'* and mechanists, views of mind. In many studies the information processing analogies to thinking limit researchers to the computational, rule governed and symbolic views of thinking. One wonders how construing mathematical thinking, however abstract or advanced, in broader terms as recursive and organic sense-making of bodily-grounded ideas can challenge the thinker and thought dichotomy. Isn't thinking also intransitive (non-object-oriented)? Isn't much of it deeper than what can be accessed consciously and introspectively or felt subjectively? How about pre-verbal, pre-reflective, self-referential, *primitive* thinking?

With advancements in neuroscience and artificial intelligence, researchers such as Bruner (1996), Butterworth (1999), and Dehaene, Spelke, Pinel, Stanescu, and Tsivkin (1999) have begun to view mathematical thinking as constrained and enabled by evolutionary, biological, experiential and other micro factors, as well as grander ones. Bruner (1996) encourages educators to see that neurobiological studies "offer useful hints about mind for much more clearer hypothesis" (p. 164). Adding neurobiology to the bricolage of disciplines informing investigations on mathematical learning might inevitably redefine questions about what mathematical thinking is.

3.1.4 Views about the Role of Perception in Thinking

Studies of mathematical thinking provide important insights about the role of perception in learning. A large group of researchers has for a significant time concerned itself with studying *representation* and *visualization* as an aspect of mathematical thinking and perception. They emphasize the importance of visualization and representation in mathematics, saying that it is crucial because both ideograms (symbols, notations) and diagrams (drawings, sketches, graphs, etc) frequently accompany mathematical thinking (Presmeg, 1986). Some work on representations emerged in the 1990s as a response to practical questions on multiple settings of representing concepts offered by graphical, analytic and diagrammatic as well as computerized means (Kaput, 2002). Other researchers have studied representations with the aim of helping students to experience mathematics in a more concrete and visual way (Presmeg, 1986).

To many scholars who focus on formal and symbolic mathematical structures, the question of mathematical thinking is synonymous with that of mathematical representations. It is argued that humans only have access to mathematical objects through representations (Otte, 2002). To some researchers, perception of mathematical entities and properties basically involves perceiving attributes that exist independently—transcendently or otherwise—of the observer. Writing, for instance, is seen as a medium for representing mathematics ideas from in there (intuitionism), from out there (Platonism), or from social games (formalism). Most studies on mathematical representations emphasize the structural factors, especially the symbolic and formal factors, in what students perceive in mathematical tasks. Much insight has been gained from the enormous work on images, visualization and representation. Indeed, these studies have provided a way of talking about how students think mathematically. I try to outline these insights below.

3.1.4.1 Modes of thinking and visualizations

Researchers in this approach to studying mathematical thinking have distinguished between verbal-logical (symbolic/abstract) and visual thinking (a geometrical/pictorial representation). Presmeg and Balderas-Cañas (2002) view the visual mode of cognition as a precursor to symbolic cognition. Flores (2002) asserts that particular representations might "provide a way of shifting students' attention, from the purely procedural approach to considering the terms and operations involved in a numerical relationship as entities that are worthwhile to pay attention to" (p. 10). Representations, visual tools (diagrams and imagery) and geometrical forms are observed to smoothen the shifts from the numeric, the concrete, the arithmetic and the particular to the formal, the general, the abstract and the symbolic. However, it is noted that although visual illustrations and settings are helpful in formulating solutions to problems and in understanding unfamiliar concepts, they have limitations. Students at higher levels shy away from them. And many theorists maintain that the verbal-logical is a more central component to mathematical ability.

Work that illuminates the modes of representations that are common among weak students has been directly linked to how students think mathematically (Presmeg, 2002). To Stylianou and Pitta-Pantazi (2002), and Tall (1999), there is a relation between modes of visualizations and students' strength in mathematics. Low-achieving students focus on detail, perceptual items, real things and actions, whereas high achieving students focus on abstractions, general strategies, symbols and objects of actions (Presmeg, 1997; Tall, 1999).

In the literature on mathematical representations, further distinction is made between kinds of visualization: figural, concrete-visual images and relational, verbal images (Stylianou & Pitta-Pantazi, 2002). In Stylianou and Pitta-Pantazi's study of elementary, secondary and advanced level students, low achievers introspectively reported to use figural and concrete images whereas successful visualizers reported to use images that were relational and verbal. Stylianou and Pitta interpreted their results to mean that low achievers use the former while high achievers use the latter images. Many researchers have noted that the one-case concreteness of an image may tie mathematical thought to insignificant detail and even introduce false data (Presmeg, 1986; Watson, 2001). Also Presmeg (1986) observes that *visualizers*, that is, students who preferred to use visual methods when engaging in tasks used more time and had difficulty communicating mathematical concepts. Presmeg & Balderas-Cañas (2002) observe, however, that it might be more useful to maintain that a combination of visual and symbolic methods is useful, especially in the preparatory stage of solving problems. Historically, classical mathematicians saw geometrical proofs as intuitive and hence inadequate (Watson, 2001). They were steadily replaced by analytical proofs. Nonetheless, for didactic purposes, in my view, the visual, the particular and the concrete might offer a preliminary understanding to mathematical concepts. As well exploration of other forms of imagery other than the pictorial might broaden our understanding of mathematical thinking.[15] It appears the mathematics education community needs a conceptual framework that will consider perception to play more than a representing role.

Rotman (2000) warns against the view that diagrams are inferior to ideograms. To him it is platonic rigor that always seeks to replace diagrams with ideograms. Perhaps a closer look at the historical relationship among forms of representation might reveal the necessity of the diagrammatic as a condition of existence for more abstract settings of

[15] Presmeg (1986) identifies five kinds of imagery: the concrete, pictorial imagery; pattern imagery; memory images of formulae; kinesthetic imagery; and dynamic imagery.

representation. Furthermore attempting to categorize which modes of visualization are common among weak students is likely to misguide how we define adequate mathematical behavior, especially since this categorization carries the message that the concrete, kinesthetic, dynamic and visual are insignificant in mathematical knowing. Some mathematical concepts require one mode of imagery more than another. For example, geometric concepts are more figural than other concepts, Fischbein (1999) maintains. They can be represented using figures and might trigger mental images of sensorial representations. They are figural-visual but at the same time conceptual-logical. Could evolutional metaphors adopted from complexity science help educators to avoid the tendency to privilege one mode of complexity at the expense of the other?

For humans with basic human capacities such as touching, listening, visualizing and imagining, one wonders what the repercussions of privileging one mode over the others are. It might be important to examine how physical, social and cultural experiences, as well as imaginative and volitional exertions, influence what students perceive even in abstract mathematical tasks.

3.1.4.2 Kinds of representations

Some literature, in addition to classifying modes of representation common among weak students, classifies representations used in mathematics as internal representations, external representations and representations systems. English and Halford (1995) have classified representations into the symbolic, mathematical, cognitive, computer and explanatory representations. The internal or cognitive representations are taken to be the mental representations, such as concept images, that are not directly observable. External representations are observable representations such as graphs, figures, words and computerized versions of these. Representation systems are those configurations that are specific to a particular science, such as mathematical representations. Hitt (2002) and others further the discussion from mental representations to include semiotic representations. To them semiotic representations, unlike mental representations, are external (but not organic) representations (see Appendix B). Hitt emphasizes the role of both mental representations and semiotic representations in mathematical thinking. This diverse classification of representations is evidence that mathematics is conceptualized as populated by what are commonly known as representations. But whether these representations are representations as in copies or resemblances is a distinction that the majority of researchers have not attended to. Researchers might be drawing the distinctions verbal-logical versus visual-concrete, internal versus external, organic versus inorganic, weak versus strong too sharply. Also the term representation with the connections it evokes in terms of mirror images is problematic. Do not mathematical representations participate in the evolution of the mathematical objects themselves? Drawing from Dienes' construct of *multiple embodiments,* a few researchers such as Noble, Nemirovsky, Wright and Tierney (2002) have looked at mathematical representations as part of the origins of mathematics (Dienes & Goldings, 1971).

In a mathematics classroom there is drawing, sketching, writing, imagining, constructing and, recently, clicking and dragging. One wonders how evident and primary the division between external (organic vs. non-organic/semiotic) and internal representations actually is in mathematics learning? When and to whom is a representation a representation? Could a representation be appropriately considered solely external without invoking internal interpretations? What constructs, other than representations, could be evoked to understand perception and cognition in more useful ways? Such are the questions about representations and perception that are central in studying mathematical thinking.

Some researchers such as Ball (2002) and Confrey (1999) are instead studying mathematical interpretations. Others such as Marton and Booth (1997) are studying mathematical experiences. Duval (2002), Hitt (2002), Otte (2002), Presmeg (2002), Radford (2003) and Rotman (2000) study mathematics signs and sign practices. Sáenz-Ludlow (2002), critiques the detachment of internal from external representations. She argues that interpretation and representations are intertwined. Like many mathematics educators, Sáenz-Ludlow and other semiotic theorists offer Peircean semiotics as a way forward in studying mathematical interpretations (see Appendix B for a discussion on semiotic representations and signs).

3.1.4.3 Representationism

A study of what students attend to in mathematical tasks begs some understanding about why the traditional frameworks of representation are almost irresistible in understanding mathematical thinking. In representationists' models, perception is taken to be a reaction to an object that is initiated at the receptor or sense level. Perception is considered to be pick-up, detection or re-presentation of some event or scene by an organism (Skarda, 1992). During a perceptual act, the organism centrally forms a more or less adequate internal representation of the event and its features.

In mathematics education most research done has taken this representationists' approach. This is not surprising, given the fact that mathematics is at most times foundationally viewed in a formalist way as a solely symbolic subject, and in a platonic way as a subject about a priori objects. The intuitionist view of mathematics as a subject about observer constructions also fits with the recovering (instead of grasping) metaphor of perception. Any view of mathematical objects and properties that upholds the dichotomy between in here and out there, between the perceiver (machine) and the perceived (raw data), perception and action lends itself to a cognitivistic role of perception.

The representationists' view of perception further supports the stage-by-stage, machine-like understanding of cognition. The machine's body, its context and history are taken not to participate in the processing to the same extent that the software, the thought processes does (Thompson, 1997). Additionally, the perceiver, his or her tools are co-implicated in what he/she perceives only to the extent of the quality of their representation.

If we are to consider cognition as ongoing interpretation and experience that is a matter of action and history and mathematics as a human activity, then what becomes of the role of perception? There have been alternative views of cognition, of foundations (or anti-foundations) of mathematics, and of perception. The mathematics education community has begun to consider that non-symbolic aspects of cognition, its context and its so-called hardware—the body-in-space and materiality—might have more to do with perception than what representationist frameworks allow. What these new views mean for the question of mathematical thinking is unfolding.

The analysis of theoretical insights from the studies has involved examining views about what I thought were central issues to studies on mathematical thinking. For instance, I have demonstrated that views about school mathematics, mind and perception influence not only the questions asked, but also the methods used to research mathematical thinking, as well as the interpretation of research results. In the next sub-section, I return to an overview of the methodological insight from studies on mathematical thinking. This will lead to the next part of this writing: an exploration of the research design and methodological framework of this research.

3.1.5 Methodological Insights

Most of the investigations done in the literature I have reviewed have been empirical studies involving school children. Some, such as Hadamard's (1945/1996) and Burton's (1999a, 1999b) studies, have involved the self-observation and introspection of mathematicians. Others, such as Sfard's (1991a, 1995) have been historical studies. Some studies have been philosophical studies. Yet a few more recent ones such as Butterworth (1999), Lakoff and Núñez (2001), and Dehaene et al. (1999) have been a bricolage, drawing from a juxtaposition of disciplines including neurobiological studies. My study as we will see is also a bricolage.

Whether empirical or theoretical research, the units of analysis for the different studies have varied with respect to the paradigm and general views espoused. For example, research that has led to the development of Cognitively Guided Instruction (Carpenter, Fennema, Franke, Levi, & Empson, 1999) by studying actions of children has categorized ways in which children think about mathematical concepts and operations. For child and structural psychology the unit of investigation is instructional material and isolated children's performance in terms of accuracy and speed. For the radical and social constructivists the unit of investigation is mainly the individual child's conceptions (or misconceptions), while for the socio-practice theorists the practices of a community and social patterns are the unit of investigation; yet for socio-cultural and critical theorists the unit is how individual subjectivities that are produced by discursive discourses subjugate learners. In my study, I ponder the metaphors that will help in examining how these units relate to each other. As well, I explore other units of analysis that could enhance the understanding of students' mathematical thinking.

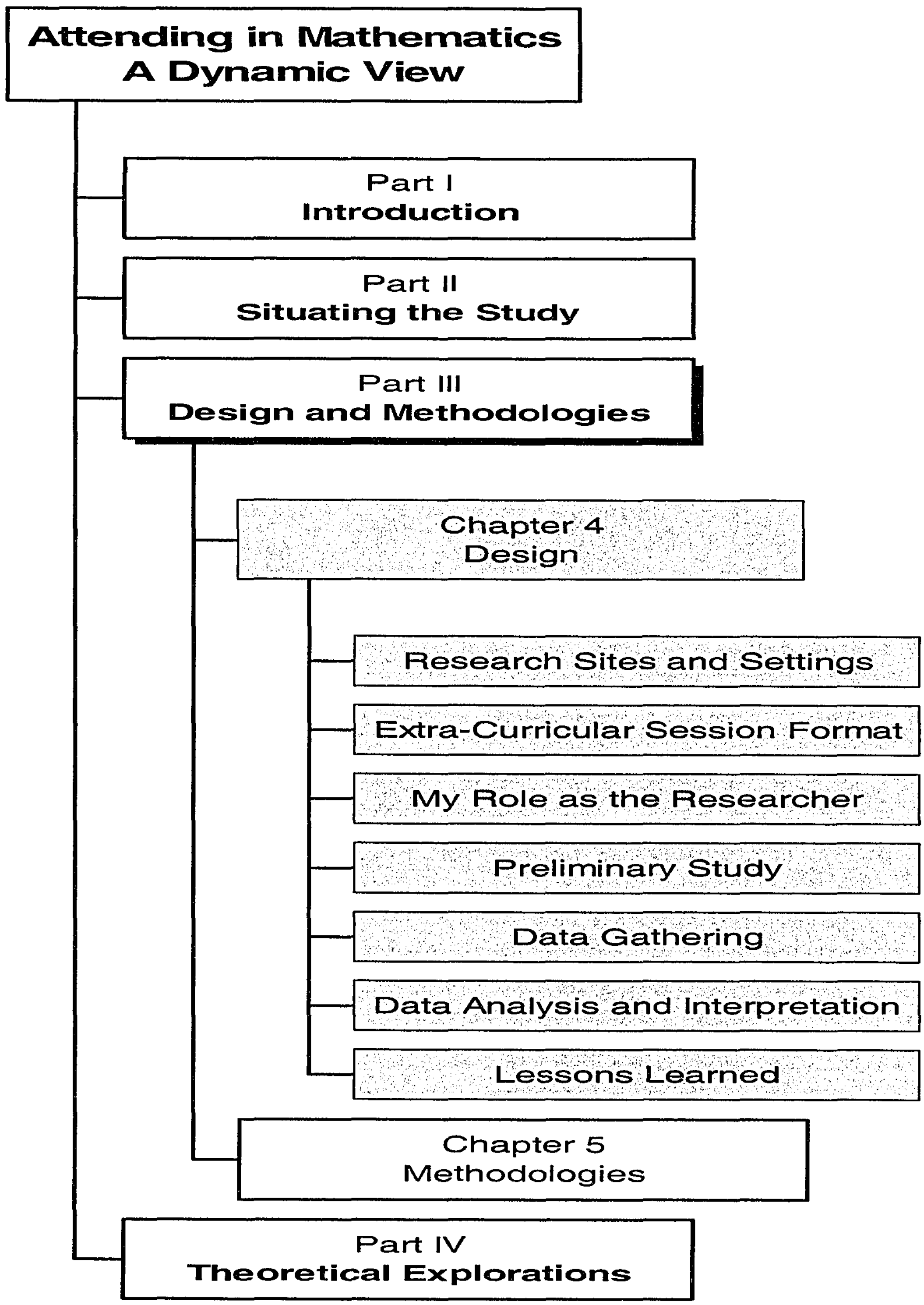
Although the nature of investigations varies, from the 1970s and 1980s studies with controlled experiments to the more recent prevalent naturalistic studies, most of them aim at generating recommendations for practice, usually in the form of learning or teaching *models*. However, the meaning of the term *model* or of any form of recommendations offered appears to vary with whether the study is framed by information processing psychology or by different kinds of constructivism. For example, studies that construe mind by using the machine metaphor consider a model as a mechanism that supports a child's mathematical thinking. On the other hand, radical constructivists claim to use models as observer constructions of the children's mathematical constructions, a view that is consistent with enactivism (Steffe & Kieren, 1994; von Glasersfeld & Steffe, 1991). Pirie and Kieren (1989, 1994) provide an example of a radical constructivist model. The social constructivists, together with the interactionists, claim that their models describe group dynamics of students solving problems rather than offering models of individual mental actions and reflection (Cobb, Wood & Yackel, 1992; Voigt, 1994). In the next part of this dissertation, I explore the research design, elucidating the units of analysis and the observation models which I adopt.

4. DESIGN OF THE STUDY

I have introduced and situated my study, but three tensions persist. How does one manifest the interrelatedness of question, reviewed literature and research orientation in a linear text? In what ways can the openness required by interpretive research be preserved in designing and carrying out a study, and in writing a dissertation? How does one work within research distinctions such as collecting data, and designing and piloting a study while remaining open to possibilities outside of the technical workings of these distinctions? I lessen the first two tensions by describing the evolution of the research design, research focus and questions respectively in Chapters 4, 7 and 8. I lessen the last tension in Chapter 5 by exploring the organic nature of this research. Although I make an effort to address the first tension, it persists until the last chapters where, by returning to the motivations of the research, I endeavor to loop back onto the first chapter so as to make the writing more circular than linear.

Even though I outlined some form of a research design when I proposed the study, much of it has developed with time. This chapter is about the research design that emerged with the study. In it, I discuss the empirical aspects including: choices about research sites and settings, the nature of participants, my participation, and the means of gathering and analyzing data. For purposes of maintaining the openness of interpretive inquiry, I weave aspects of the research design around lessons learned from the preliminary study. I discuss the ongoing choices I made and the happenstances that occurred, reflecting on the contexts that might have prompted me to value some particular designs. In most cases, the actualized designs were those enabled by the parts in the study, of which I was just one. The design that arose was the emergent order that the system as a whole finally settled on given the relations among the context, time, participants and community.

Dissertation Landscape Forming



4.1 Research Sites and Setting

The study involved five interpretive sites:

- working with students in a research project outside the classroom;
- participating as a research assistant in a junior high classroom;
- reflecting on my experiences as a student and as a teacher;
- studying related research, discussing and interacting in the community;
- some writing and re-writing, conversations and informal observations with students, with pre-service and in-service teachers.

These sites can be grouped into two: the planned—the first two—and the spontaneous—the last three. Emergent sites come from my participation in activities such as research assistantships and in-service teacher education study; others arose from interactions with my supervisors, other researchers and graduate students at conferences, in classrooms and seminars.

I view my participation in the varied sites and contexts as participation in a hermeneutical conversation. Data was gathered through participation with secondary school students. Participation in informal sites informed the analyses of data. The students I observed were in two settings: extra-curricular research project and classroom research project, both in urban settings. The multi-site conversation has given me a more global understanding of the problem of study.

4.1.1 Extra Curricular Research Project

One of the planned sites of my research involved secondary school students in a project outside the classroom. Twenty-seven students (6 from Edmonton and 21 from Uganda) participated. Whereas two students in Alberta who joined the study towards the end participated in only one session, the rest took part in an average of three sessions. Table 2 offers information about the participants, including the tasks they engaged in.

Table 2. *Demographic and Participation Information*

	Edmonton	**Uganda**
Boys	3	0
Girls	3	21
Grades at time of sessions	7, 8, 9, 10	Senior 2
Years of birth	1987 (1), 1988 (3), 1989(1), 1990 (1),	1986 (1), 1987(1), 1988(5), 1989(5) [a]
Year of participation	2001, 2002, 2003	2001, 2003
Schools represented	4 Schools [b]	1 School [c]
Number of sessions	1 - 2 Sessions	3 – 5 Sessions
Tasks done	Pirates Aboard (PA) Chessboard Squares (CS) Bee Genealogy (BG) Consecutive Terms (CT) (see Appendix A for description of the tasks)	PA, CS, BG, CT, plus Dominoes, Cubes Cubed, Ladies Luncheon, Fifteen, Matches 1, Circular Disks, Ins and outs, Paper Strips, Triangular Count.

Note. [a]Only 12 out of 21 Ugandan students completed the demographic survey (see Appendix C). [b]Day mixed public schools. [c]All girls boarding Christian secondary school

In Alberta, the children I recruited had parents or parents' friends who were my colleagues who had shown interest in my research. The sessions were conducted at the Faculty of Education at the University of Alberta. In Uganda, I solicited students through the school principal, at the school where I taught before entering graduate school. Sessions were conducted at the school. I also taught a Senior Two class in 2001 for a school term. Senior 2 is an equivalent to grade 8 when you go by age—13 to 15 years old. In terms of syllabus coverage it is more equivalent to grade 7. In addition, I participated in a junior high year-long research project as a participant-observer in a school in Canada.

4.1.2 Classroom-Based Research Site

At a school site another researcher taught 27 grade 7 mathematics students (15 boys and 12 girls) in a public school during the entire 2001-2002 school year. I participated in the classroom as a research assistant helping out during group work and individual seatwork. I video-recorded the lessons and took field notes. The project posed questions such as, what can be done to enlarge the space of the possible when engaging students in mathematical activity? This project was also about theory building. The principal researcher and I were interested in developing ways of making our observations about learning coherent. I found myself creating explanations based on my observations about students' mathematical thinking and attentiveness. Moments when students made diverse sense of activities,, had their understanding shift, or worked in novel or divergent ways, as in the Fraction Kit activity in Chapter 1, were of particular interest to me.

4.2 Extra Curricular Research Session Format

In the extra-curricular project, I invited pairs of students to work on mathematical tasks. In each hour-long session, pairs of students engaged in a mathematical task and later participated in a *conversational interview* based on their activity. I provided the basic materials such as pen and paper, and tools such as geometry sets and calculators as well as furnishing concrete materials such as Fraction Kits and counters. I interacted with, observed and video-recorded students' engagement and collected their written work. Follow-up sessions involved conversation about participation in earlier sessions as well.

In each conversational interview, I encouraged students to talk about, or observe and reflect on their actions. Carson (1986) has distinguished a conversational interview from the commonly undertaken research interview. He said that the latter is meant to gather information about the researched whereas a conversational interview reveals something held in common and so allows meaning to emerge through language. It is a continuation of the hermeneutic conversation between question and answer. In conversational interviews the researcher does not ask for proofs of assertions, but rather for examples and vivid recollections.

The conversations at the sessions were not structured; they were contingent on my observations. During the students' engagement with the task, I noted possible conversational prompts. Sample prompts include: Can you explain how you got your answer? What did you have in mind when approaching the task in this way? By the last three sessions in Edmonton and the 2003 study in Uganda I had developed a more semi-structured list of conversational prompts (see Appendix D). In some conversational interviews, I also I prompted students to talk about their mathematics classroom experiences. In a few sessions I showed the students parts of their video recording. In two sessions I showed the students a transcript of their earlier participation. I did this mainly to *stimulate recall* so as to talk about what interested the students and me. In Alberta, where I had access to more than one camera, I recorded the conversation around the viewing as well.

4.3 Role of the Researcher in the Project

Firstly, my role was to select and present *non-routine*, *good enough* and *variable entry, dynamically attracting* tasks that were likely to prompt students' sustained engagement (see Chapter 1 for definitions). I adopted the mathematical tasks from problem-solving books such as *Thinking Mathematically* by Mason et al. (1985), and from research studies. During the preliminary study, I used a range of tasks; however, in the actual study I had honed down the tasks to include *Chessboard Squares (CS), Consecutive Terms* (CT), *Pirates Aboard* (PA) and *Bee Genealogy* (BG). These tasks in addition to being good enough, non-routine, variable-entry prompts had proved to sustain students' mathematical engagement dynamically and to structure their behavior in mathematical ways. In Edmonton, I decided both pairs who had two sessions would do similar tasks. Tony & Ronald did the CT task in the first session in 2001 and the BG task in the second session in 2003. Tanya & Tammy worked on the BG task during their first session in 2002 and on CT task in the second session in 2003. Deo and Laura, who joined in July 2003, worked on PA task.

Secondly, through all the sessions I was a participant-observer (i.e. observed and participated with), specifically a *close observer*. van Manen (1998/1988) differentiates between a close observer and an *observer* by saying that a close observer maintains "a certain orientation of reflectivity while guarding against the more manipulative and artificial attitude that a reflective attitude tends to insert in a social situation" (p. 69). As a researcher who embraces the complexity of living, I was always first and foremost a participant in the tasks that I set for the students to engage in. I thought observing students engaging with tasks was not enough; during the activities, I offered the students additional prompts to facilitate their engagement (see Appendix D for prompts).

Thirdly, while students were engaged with the task I took note of the students' actions, phrases, gestures, artifacts and voice inflections. In the latter half of the study, my notebook was organized under three headings: (a) things to do differently in or before the next session, (b) areas that I would require comments on from the students, and (c) comments on emerging themes and moments of possible interest.

4.4 Preliminary Study

One of the very difficult decisions was what kind of data was necessary and/or useful to explore my questions. To prepare myself on the specific nature of data to gather I did an *exploratory study* both formally and informally. In traditional terms, the exploratory study would be referred to as the pilot study; however, I prefer to call it the *preliminary study,* since it was more of an exploration. Its analyses are presented in this writing. Informally, I introspectively worked on mathematical tasks and had colleagues or family work on them. Formally, I worked with two boys, Tony (grade 7) and Ronald (grade 8) in Canada and with eight girls—Irene, Lillian, Rose, Norah and four other, all *senior 2s*—in Uganda. None of the eight Ugandan students were able to take part in the 3rd year of the study.[16]

Insights gained from the preliminary study were crucial in gradually reducing my research tension. They offered me an opportunity to maintain the openness required by interpretive research. The preliminary study informed my ongoing literature review along with a reframing of the research questions, as it manifested the interrelatedness of the question, the literature and the research. It informed the design of the *actual study.* Also it was an initial step in gathering data and in the analysis and writing. After I describe the design that emerged for the actual study, I discuss three lessons learned from the "pilot" study.

[16] This was because the study took place during their end of O-level year, when the students were focusing all their energy and time on preparing for the national achievement tests.

4.5 Data Gathering

Simmt (2000) maintains that data collected are, to a large extent, also data created by the researcher. Etymologically, the term *datum* is Latin for that which is given or granted. In my study, the elements that were given were the students who participated, the research questions I initially posed, the research orientations I embraced, the community and the literature I interacted with, as well as my teaching experiences. It is from the interaction of these influences, which in complexity research are referred to as *agents* that the data surfaced.

As I proposed the study I anticipated three sources of data: direct participation with and observation of students, video records of students' activity, and copies of students' writing. As the study progressed, I realized that the notes I scribbled during the sessions, my comments on the copies of students' artifacts, the marks I made on the transcripts, the comments colleagues made on my observations, and the earlier drafts of this writing also became data for further analysis. Simmt (2000) refers to this as the layer of *secondary data*, which, in addition to the first layer of data, occasions interpretation.

4.6 Data Analysis and Interpretation

Morse (1994) observes that analyses do not just emerge from collected data. A stage has to be set, data organized and the researcher must organize herself for the analysis. She further says that

> [D]ata analysis is a process that requires astute questioning, ...active observation...It is a process of...making the invisible obvious, of recognizing the significant from the insignificant, of linking seemingly unrelated facts.... It is a process of conjecture and verification, ...of suggestion and defense. (p. 25)

During the study, the research artifacts—videotapes, transcripts, students' written work and my journal entries—were conceptually organized around moments that interested me, *interpretive moments.* I will elaborate on the concept of interpretive moments in the next chapter.

To observe students' thinking-in-action, I adopted the stance of focusing on the mathematical worlds students enact. I understood students' actions and interactions—what they said, used or wrote and how they said, used or wrote it—not as consequences of thinking, but as acts of cognition in themselves. Therefore it was crucial to pay close attention to the whole learning bodies along with their extensions to the concrete and social world. At first, I sought to analyze the selective nature of students' perception that filters meaning from what is taught. But with time, as I illustrate later, my focus drifted toward the dynamics of how students bring forth what they attend to.

I interpreted and re-interpreted individual data sets in light of preceding and subsequent moments of interest. For each of the interpretive moments I asked:

- What are the other experiences that this moment resonates with?
- What is this moment an example of?
- Are there any micro or macro themes emerging in the data interpreted?
- In what other sessions are these themes evident or contradicted?

As I will demonstrate, interpretive moments served to organize the data, and acted as both the analysis and writing in progress. They were the ongoing, spontaneous and revisable study *results* and micro themes. Micro-themes began to constitute a developing whole. Toward the end of the study, I also carried out a brief retrospective analysis of the video clips and transcripts of interest, together with my research notes, the micro research narratives and student writings. This rigorous data analysis was only possible once I had stopped taking part in the projects. It was a way of illuminating macro themes. The formal analysis was organized around themes that had become apparent in earlier interpretive moments. These included:

- What are mathematically adequate actions and ways of being?
- What are the other agents of mathematical thinking in addition to writing, utterances, and students' actions and interactions?
- In what ways could we think about mathematical concepts usefully as structures that cut though the mind-body-environment chasm?

In the formal analysis, as with the ongoing data analysis, the objects of analysis mainly included (a) the individual's interactions and (b) each pair's or group's collective actions. At times I found myself reflecting on (c) each individual's embodiment and available tools, and (d) on the school social practices and larger cultural and institutional contexts that appeared to constrain the students' interactions. Simmt (1998) observes:

> Features of the in-person embodiment are inferred from observing and listening to a person's mathematical actions, body language, and tone of voice and by attending to the content of his or her utterances and written work. ...Features of this [collective] embodiment are noted by observing one's interactions with others—usually in discourse...[The community embodiment] is observed in the actions and interactions of persons engaged in mathematical activity that sustain and contribute to the body of mathematics. (p. 12)

In the analysis, I maintained that individual's mathematical thinking is nested in collective mathematical thinking, which in turn is nested in the larger body of school mathematics, which is also nested in other larger bodies such as research mathematics.

Although I engaged students in conversational interviews, I did not consider analysis of the interviews to be central to the formal analysis. It is the students' non-verbal and verbal gestures and expressions during engagement with the task that I transcribed. The transcribing process, though laborious, was invaluable. It not only provided ready excerpts to cut and paste into my writing, but in pausing, rewinding and re-playing I got a deeper sense of the sessions. The process of transcribing was part of data observation and analysis. It amazed me to discover how much I either did not attend to, or had observed differently by just participating in the session or watching the video records only once. The transcripts also presented written traces for easy access to events.

When I proposed this study, I planned the analysis in definite phases. However, I did not follow the phases religiously. Below are the stages as stated in my research proposal (July 2002).

Phase I: After transcribing the tapes study the transcripts to select excerpts.

Phase II: Begin to scrutinize students' actions. On the basis of emerging interpretations, have a conversation with my supervisors. Re-watch the tapes, re-view writings of earlier analyses and weave micro narratives around emerging interpretations in preparation for the conversations.

Phase III: In enactivist inquiry, researchers consider interpretation to be a co-emergent phenomenon that happens with others. At this later phase of the analysis, prepare to present a mini-report to other researchers.

Nested within each of the above phases would be the micro-inquiries, research moments of interest that I anticipated to materialize at any phase.[17] With the layering of the phases by nearly complete micro-inquiries, the data analysis would take on a fractal nature, which is crucial in interpretive studies framed by enactivism (Simmt, 2000). It is a feature of the circularity between the spontaneous and the formal analyses.

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[17] I use the term *micro-inquiries* to point to moments during the research when I pursue a particular event of interest.

Phase IV: Watch more sessions in light of the interpretation to select clips that complement existing interpretations. If necessary, repeat the preceding three phases with the newly selected clips. Also begin to study the whole set of data in light of the micro-inquiries to generate counter-examples.

Phase V: Make attempts to vary interpretation, prepare to share interpretations at seminars, participate in classroom teaching and teachers' workshop, write articles for publications and edit the research narrative.

In retrospect, this phased formal analysis with its clear-cut boundaries was more prescriptive than is required for an interpretative study. To my surprise the analysis in the study did not wait for the rigorous viewing of the tapes to begin. By the time I carried out the final sessions, I had engaged in a couple of micro-inquiries and shared emerging themes. It is these themes that I pondered in the final sessions. What the pre-specified data analysis offered was a pool of potentialities that in the event of analyzing the data increased the probability for me to act in suitable ways. In the organic analysis that was occasioned by the rigidly phased plan only some possibilities were pulled into existence, and novel ones sprouted.

Data collection, analysis and writing were inextricably linked. In this organic analysis, I did not work in clear-cut phases, but instead moved back and forth. I also did not transcribe all the tapes. After watching some of the tapes many times, possible counter examples and examples often involuntarily popped up into my consciousness so that all I had to do was to reach out for the tapes or transcripts where I recalled a particular example to be. I have yet to examine the whole set of data in light of the macro-themes.

My research has involved engagement with the community of researchers. The main artifacts of my engagement in this community are papers presented at conferences or published. These include: Redefining school and progress (Namukasa, 2002a); The role of the observer (Namukasa, 2002c); The phenomenology of seeing (Namukasa, forthcoming); The relationship between globalization and school mathematics (Namukasa, 2004); What counts as knowing (Namukasa, 2003b); Collective mathematical thinking (Namukasa, 2002b; Namukasa & Simmt, 2003); and A theoretical rationale for multiplicity of mathematics models (Namukasa, 2003a).

4.7 Lessons Learned from the Preliminary Study

Given that the focus on student thinking could be approached from many frameworks my preliminary explorations engaged clarifying frameworks and tools of observation. To appreciate the choices made for the research design, I saw a need to articulate the major lessons I learned from the preliminary study: (a) the role of the participant observer, (b) observer co-implicitness, (c) the need for a theoretical observational tool, and (d) the ways in which observable features might indicate thinking. I explore the last lesson at length offering vignettes that I return to in Chapter 7.

4.7.1 The Role of the Participant-Observer

During the first sessions, I found myself reflecting on my role as a researcher. Was I to participate as an investigator or as a detached observer? What did it mean to be a *close observer* of Tony and Ronald, for example? While working with the participants aged 13 to 16, I realized that my role was always better construed as teacher. Students always looked up to me as a teacher, and their presence, given my history in teaching, occasioned me to participate as a teacher. Boostrom (1994) observes that the observer is transformed by the act of observing as he/she learns how to observe and what to observe concurrently. The teacher-student relationship co-emerged with my complex role as a researcher interested in mathematical thinking. In addition to offering the tasks and prompts, I explored with the students the mathematics evoked by the tasks, observed as well as listened to their experiences with mathematics. Towers (2001) looks at the participatory role of a teacher. She identifies intervention modes of an enactivist teacher.

> *Shepherding* is an extended stream of interventions directing a student towards understanding through subtle coaxing .... *Inviting* is the suggesting of a new and potentially fruitful avenue of exploration. *Retreating* is a deliberate strategy where by the teacher leaves the student(s) to ponder a problem. *Rug-pulling* is a deliberate shift of the student's attention to something that confuses and forces the student to reassess what she or he is doing. (Towers, 2001, p. 334)

During the sessions I participated in an *interventionist teacher mode* (Towers, 1998). However I soon learned that it is the students' response to any particular mode of a teacher's participation that determines whether the teacher shepherded, invited, and so on (B. Davis, 1994). In most of the sessions, in order to observe what the students attended to in the tasks, I retreated or rug-pulled. However, there were moments when the ethical act was to invite students into fruitful avenues. For a particular pair of students who engaged in less mathematical ways, I shepherded using prompts. And for students who took big leaps that they were not ready for, after fruitless efforts at inviting and shepherding, I acted as an expert and shared the "facts". I elaborate using a vignette from the exploratory study in Appendix E, Vignette E2.

4.7.2 Observer Co-implicitness

My participation with the students extended toward being a factor in their histories. With such co-implicitness in students' mathematical behavior, I had to deal with the positivistic tension that underlies most neo-Piagetian clinical interviews: If you want to discover the students' thought patterns, then let it be their thinking without tainting it by your influence. As Tower (1998) explains, the realization of thorough implication of the researcher in the participation of the researched might

> [p]rove troubling to a researcher convinced of his or her ability to remain removed from the data leaving those data "untainted" from bias. Instead, this realization simply foregrounds for me the growing call that researchers recognize and acknowledge their complicity in shaping the findings of their research. (p. 33)

That being said, in retrospect, some of my participation during the initial sessions seemed regrettable; at least this is how I felt when transcribing some tapes. Specifically, as I was transcribing Tony and Ronald's first, Lillian and Irene's fourth session, and Rose and Norah's fifth session, it occurred to me that some of my participation interrupted their actions. In some interruptions I had given unconscious clues that the students were onto a right or wrong path. Some of my responses drastically changed students' actions. This was an interpretive moment.

4.7.2.1 Tacit Influences at Work in my Observations

To explore ways in which my participation could be more supportive, I did a micro-inquiry using Lillian and Irene's transcript. I marked all the moments where I had overtly participated. I then studied how students' participation had been altered after my participation. I began by classifying participations as supportive or not. I later teased out the particulars of the actions that were supportive or not. Prior to the inquiry, I had expected my participation to be significant whenever I offered prompts, listened actively and carefully, encouraged active participation or reflected on students' participation. I had also considered my participation to be unfortunate whenever I failed to listen, missed opportunities to offer prompts or interrupted students' utterances. The following surprises, however, arose from the inquiry:

- In some instances I barely listened to inaudible and non-verbal conversations and to silent and subtle participations.
- At times the questions I asked triggered the students to participate as if they were in a question-answer interview.
- The appropriateness of each of my interventional prompts highly depended on the responses it occasioned more than on my intention for it to be supportive.

As a lesson learned from the micro-inquiry, I tried to prevent myself from turning the sessions into *structured clinical interviews* by avoiding directed questions during students' engagement in tasks and by reserving inquiry questions either for the conversational interviews or for further observation. Some prompts that were tacitly intended to encourage students to think aloud, so as to determine the mechanism at work in their thinking, were indeed unfortunate for a researcher who considers her input as engagement with the students' unfolding worlds. With the ecological frameworks (see Chapter 6), unlike other dominant theories, it is exactly what happens in the interactions (rather than in the individuals' heads) that a teacher is able to influence and therefore it is what ought to be of interest to a researcher.

Recognizing that some of the less helpful modes of participation may be inevitable, I reframed my concern about the role of the observer: When I define myself as a researcher-observer, what does that mean to the students, to me and to the study? What are the implicit factors influencing observations of my own research participation?

4.7.3 The Need for an Observational Tool

During the study, especially after I reflected on substantive issues such as *where to look for what students attended to,* it occurred to me that I needed to be inclined, consciously and unconsciously, towards what to pay attention to before I saw it. Although I had theoretically decided to draw upon a complexity and ecological orientation, prevailing theories on mathematical thinking continued to influence my observations reflexively. For instance, in the case of the Fraction Kit task, was I going to reflect on the difference between the two approaches in terms of differences in Piagetian stages of mathematical thinking or what? As we will see when I discuss students' mathematical attentiveness, perceiving in particular ways takes orientation at many levels. This is applicable to the investigator as well. Eisner (1997) observes, "Perception is selective and the motives for selection are influenced by the tools [cultural, technical or otherwise] one has or knows how to use: we tend to seek what we know how to find." (p. 7) Interpreting behavior requires a fore-structure, a sense of tradition that both shapes the observations and is modified by the observations (Kieren, 1992). I needed a background and a specific lens to be able to see things in specific and new ways. The lens I eventually brought to the study not only guided technical decisions like where to place the camera, but it acted as a springboard for the evolution of other lenses. This called for an ongoing theoretical study, especially in enactivism and complexity research theories as well as in studies on human thought and perception. To be engaged deeply by what I observed I participated in ongoing conversations with researchers about observations. I also adopted tools of observation from other researchers.

To find a language for and a way of observing both individual and collective activities, I began by drawing from Pirie and Kieren (1989), and Simmt's (2000) models of learning (Figure 2 & 4). Simmt's model is helpful when observing person-person and person-environment interactions. Pirie and Kieren's (1989) model is helpful when we hone down to the conceptual signification space of both individual and collective learning bodies. Both models offer a consideration of knowing as a recursive rather than a linear hierarchical process (Pirie & Kieren, 1994). They regard knowing not as a static state, but as a phenomenon that continuously unfolds as students interact.

The inner level distinctions—primitive knowing, image making, image having and property noticing— of the Pirie and Kieren's model might allow an observer to comment closely on students' activity. For instance, in most of my preliminary study sessions, I saw students begin to engage with a mathematical task by working with their already existing knowledge (*primitive knowing*) before proceeding to form, articulate and revise the images (*image forming* and *image making*), and later to *formalize* the images as they began to *notice properties*, often after *folding back* to primitive knowing or image forming.

Understood in terms of B. Davis et al.'s (2000) *nested learning bodies* metaphor (see Figure 3), these models are observational tools at different orders of observation. I offer more about these in Chapter 6 and in Appendix B.

Figure 2. Pirie and Kieren's model of Growth of Mathematical Understanding[a]

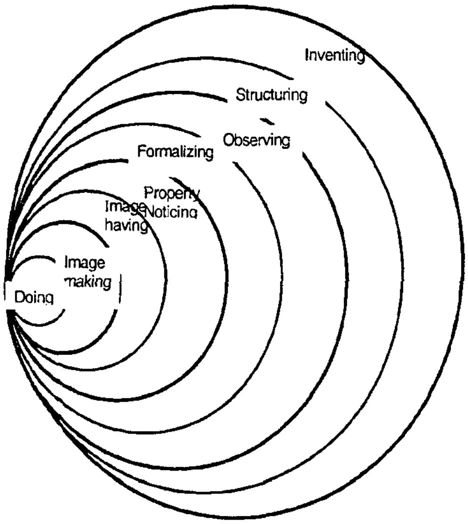


Figure 3. Nested Knowing bodies[b]

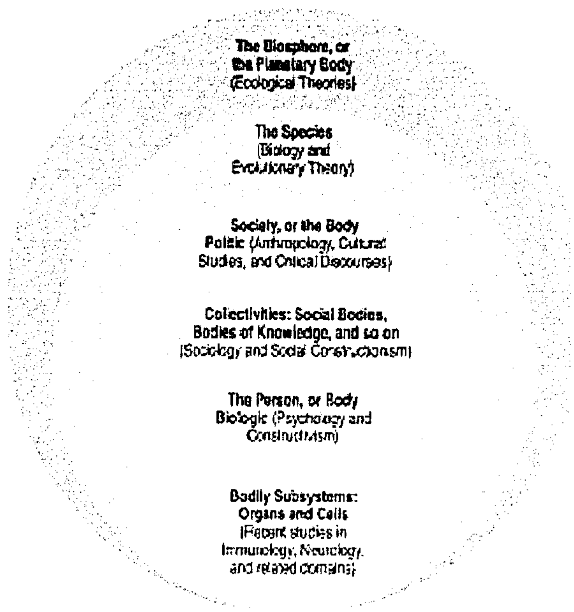
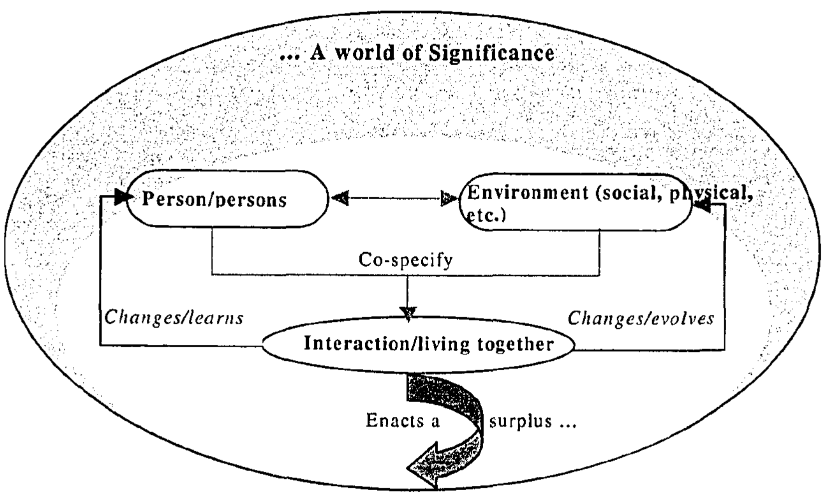


Figure 4. Simmt's (2000) model for interaction and enaction[c]



[a]Image copyright Tom Kieren. Reprinted with permission. Originally published in Pirie and Kieren (1989).
[b]Image copyright Brent Davis. Reprinted with permission. Originally published in B. Davis et al. (2000)
[c]Image copyright Elaine Simmt. Reprinted with permission. Originally published in Simmt (2000)

4.7.4 Written Work, Concrete Manipulations and Utterances

Also the debate between mentalism and behaviorism was evoked at the onset of observing the dynamics of what students attended to. What was I to focus on? What indicated differences and shifts in how students attended? What would the data look like? To answer the questions I had to outgrow any mentalist tendencies that I tacitly held without falling back to behaviorists' trivializing of the learner's structure. Thinking was not "in there" and, therefore, separate from actions, language and culture.

I gradually began to notice things that were not separate from students' mathematical thinking. The things they did, used and said appeared not to simply capture their conceptions. Actions and expressions give birth to and are themselves conceptions. They illuminate worlds which we bring forth. The deep sediments of the worlds students enact—including what they articulate, manipulate and write—became the main forms of data that I needed to collect. In this section I elaborate on how this came to be.

4.7.4.1 Episodic Writing

In their first session Tony and Ronald worked individually yet collaboratively. They, like Rose and Norah, wrote on separate sheets and convened at regular intervals (at times after my prompting) to share the progress of their work. While reviewing their videotape, I noticed that the boys had written in an intervallic form (I present their work in detail in Chapter 7). The shifts in their written work varied from seemingly subtle changes, such as where to place the *equals* sign, to what looked like a different focus of inquiry. On noticing the periodic nature of their written work, I conjectured that each new episode began after Tony and Ronald had convened to talk about their work. To pursue this hunch, I decided to re-watch the videotape wondering about what triggered the changes. To my surprise, most of the breaks happened *before* the two boys reconvened. If the shifts in their written work were not occasioned by their talk, what could have occasioned them? Did the end of each episode mark a new conjecture made or a new world about to be enacted? Or was it about shifts to outer layers of knowing in terms of Pirie and Kieren's (1989) model?

In the study, some students, especially those in Uganda, did not frequently use concrete materials as they engaged with mathematical tasks. This might have been due to the fact that in *traditional* secondary mathematics classes one does not often see concrete materials used. However, when the students did use concrete materials, their actions with these materials offered a rich source for observing mathematical thinking-in-action. To elucidate on this assertion let me share Irene and Lillian's third session. Irene and Lillian took turns to write on shared paper and so, unlike Rose and Norah (see appendix E), and Ronald and Tony, verbalized most of their actions. In analyzing their engagement, I examine ways in which episodes in written work and students' actions with concrete materials may indicate what students attend to.

4.7.4.2 Loud and Bold Utterances

In addition to written work and actions with concrete materials, another observable feature that appeared to indicate shifts in attention is voice inflection and content of student utterances. At some moments during the sessions came the animated utterances, "But", "Wait", "Why don't we?" "Oh Yeah" and "Oh just wait, I know". I will call these animated utterances, *aha utterances*. During Irene and Lillian's third session, most animated utterances appeared to mark a moment of shift in thought which evoked at many times a change in the direction of the students' line of inquiry. I elucidate these claims by drawing three excerpts from the girls' transcript.

Vignette 2. ***Irene and Lillian's Third Session: The Consecutive Terms (CT) Task***

Some numbers can be expressed as the sum of a string of consecutive positive integers.

Exactly which numbers have this property? For example,

$9 = 2 + 3 + 4$ $\qquad$ $11 = 5 + 6$ $\qquad$ $18 = 3 + 4 + 5 + 6$

EXCERPT 1

In the first seven minutes on the *CT* task, Irene and Lillian generated a list of numbers 3, 5, 6, 9, 10, 11...that had the property. They began by working with dominoes and mental computations before they proceeded to write. The excerpt is taken from a time when they were investigating for a pattern in their list.

74[a]	Irene:	Not much of a pattern
75	Lillian:	Because it is 2 1 3 1 1
76	Teacher:	Okay. So the interval?
77	Lillian:	The interval is not helping
78	Irene:	No [*They both look at the list of numbers and Lillian interrupts*]
79	Lillian:	I think we ...|
80	Irene:	**|Why don't we list down the numbers in a pattern?**
81	Lillian:	Yes [*Lillian replies after looking at Irene for a while. Irene turns to write and Lillian watches. When it comes to writing 11, Irene pauses and without saying anything she and Lillian look in quest at 11. Each of them appears to be doing some mental calculations.*]
82	Irene:	**No it can't, so let's just go on with the list"** [*Irene writes down 12 as she says*] **12** [*They loudly count together*] 1 plus 2 plus 3...

Note. [...] Stands for text that I left out. ... Stands for pauses in speech or inaudible utterances. | Marks a point at which a current speaker's talk is interrupted by the talk of another, with the interrupting talk directly beneath. Utterances written in **bold** are animated or considerably louder utterances.
[a] This excerpt begins at the 74th turn in the transcript

At a moment when they had concluded that the interval was not helping (turn 77) Irene suggested, "Why don't we list down the numbers in a pattern?" (turn 80). On the videotape this utterance is unique, for it is loud and bold. In addition to interrupting Lillian's utterance, it interrupts their line of inquiry. Grammatically, the utterance is a question, but it comes across as a suggestion to re-write the records. It is proactive rather than reactive (Sfard, 2001a). It suggests a new line of action rather than being a response to Lillian's previous utterances. To the extent that an utterance evokes a changed focus it would be reasonable to hypothesize that it indicates a moment of insight. Let me use Irene and Lillian's written work to explore the actions that Irene's utterance evoked. Figure 5 is an excerpt from what the girls wrote. The writing in Episode A was done before Irene's *aha utterance*. The writing in Episode B follows immediately after.

Figure 5. Irene and Lillian's written work A and B

Episode A (Lillian writes)	**Episode B (Irene writes)**[a]
1-	3 = 1 + 2
2-- ~~1 + 1~~	5 = 2 + 3
3-- 1 + 2	6 = 1 + 2 + 3
4-- ~~2 + 2~~	9 = 2 + 3 + 4
5-- 2 + 3	10 = 1 + 2 + 3 + 4
6-- 1 + 2 + 3	11 = 5 + 6
~~7-~~	
~~8~~-- ~~2 + 3~~	12 = 3 + 4 + 5
9-- 2 + 3 + 4	~~14~~ 15 = 1 + 2 + 3 + 4 + 5
10- 1 + 2 + 3 + 4	14 = 2 + 3 + 4 + 5
	16
$3^{+3}, 5^{+1}, 6^{+1}, 9^{+1}, 10^{+1}$, 11 ...	

Note. [a]The writing that follows 11 = 5 + 6 comes moments later. I leave the space to mark this break.

In analyzing Episodes A and B in relation to each other, it appears that in Episode A the girls systematically checked the numbers from 1 to 10, striking out numbers that could not be arranged as a string of consecutive numbers. After approaching each number separately they stepped back to check for the interval in the sequence. Irene's assertion came at the moment when they were attending to the intervals in the set {3, 5, 6, 9, 10, 11 ...} at the bottom of Episode A. It appears her utterance suggested that they re-write the numbers in the form shown in Episode B. I infer that Irene's utterance induced a *technology* that shifted their attention toward attending to something else.

In Episode A, the work was organized, recorded and systematic, but the recording in Episode B presents some aspects that Episode A did not. As Irene re-wrote the sums, she hesitated at 11. After doing some silent and ungestured work they concluded, "It [11] can't, let's go on with list" (turn 82). To the extent that a different form of recording allowed the girls to see a different aspect of 11, as I will show in Chapter 7, writing was more than a representation of what they thought.

4.7.4.3 Joint Thinking

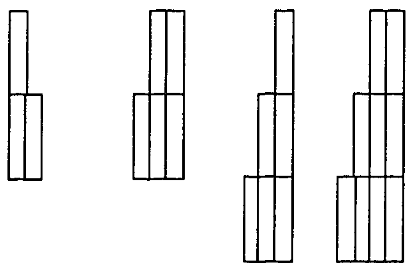
It is interesting that Lillian did not object to or question Irene's idea to record differently. Rather, she followed closely as Irene listed the numbers. One might say the students engaged in a *joint project.* At these moments it was not so clear whether one of them was merely emulating and following the other without sharing in what the other was seeing or whether they were jointly engaging. When Irene hesitated before writing 11, Lillian also saw the need to recheck 11. This appears to be evidence that the girls were engaged in a joint project in which there was a possibility of *joint thinking.* The girls collectively formed an image that 11 violated. It seems it is because they as a collective held this image that neither of them was in position instantly to get an arrangement for 12.

EXCERPT 2

This excerpt follows two minutes after Excerpt 1. The girls had been trying to write 12 as 1 + 2 +...or as 2 + 3 +.... Lillian then asked, "Isn't it possible for us to have 1 + 2 + 3 + 1 + 2 + 3?" (turn 85). Without hesitation, Irene replied, "I don't think so, because it is a string of consecutive numbers...like they just ..." (uses her hand to imitate an on going process). "Keep on being consecutive", Lillian joined in. She then asked for the materials, which they had worked with earlier as they searched for geometrical arrangements of the numbers 5, 9 and 11.

94	Lillian:	**Bring the dominoes** [*Irene passes the dominoes over to Lillian*]
95	Lillian:	Will give us 3. Plus 2…1, 2 [*She speaks loudly as she arranges the dominoes for 3, 5, 6, 9.*]
96	Irene:	5...6 [*Irene now does the arranging.*]
97	Lillian:	For 5 we have 1 plus 2 plus 3…Then 2 plus 3 plus 4 …
98	Irene:	They really…[A*s she slides the dominoes for 9*]
99	Lillian:	They don't have a similar…|
100	Irene:	|Yeah|
101	Lillian:	|Pictorial [*Looks at dominoes arranged as shown in Figure 6*]
102	Irene:	**Wait. This one the one is there, this one the one is gone (not), the next one** the one is there, this one the one has a two [*Irene looks and sounds pleased at the beginning but half way through she changes*]

Figure 6. Arrangement of Dominoes



For a moment Irene and Lillian engaged in *folding back* actions. Lillian checked whether it was possible to arrange 12 as a double of 6 arranged as a string of consecutive terms before they returned to using the dominoes. Pirie and Kieren (1994) observe that students usually *fold back* to inner levels of understanding when faced with difficulty at an outer level. In folding back to the dominoes Irene and Lillian seemed to be, once again, inspecting the *pictorials*—the geometry of the sums. However, it appears that this time they looked for something more than whether the shapes were triangles or not. Irene joined in with another *aha utterance* about how *one* appeared and disappeared— "this one the one is there, this one the one is gone, this next one the one is there, this one the one has a two" (turn 102). It appears Irene was attending to the four arrangements as a collection, possibly with a pattern. Moreover, Irene was now able to articulate the image, which they had been *doing* all along in Episode B—they imagined all numbers could be arranged starting with either one or two. Were 11 and 12 exceptions, or was the image faulty? The girls were getting frustrated for some numbers did not fit their pattern.

4.7.4.4 Concrete Materials

The role of the dominoes as concrete materials appeared critical. What the students did with the dominoes seems inextricably linked with what they thought. They did not use the dominoes merely to illustrate what they had already formulated. Just as Lillian's understanding intertwined with Irene's understanding, that which they both attended to was thoroughly tied up with the materials they used. B. Davis (1997) postulates that manipulatives "serve as a common place for learners to talk about ideas, enabling the process of re-presentation and revision [re-vision]" (p. 365). The act of falling back on the dominoes, with the history of the records in episode B embodied in the sensibilities that they had generated, enabled them to see a different aspect in the materials—the pattern of the beginning digits. The domino arrangement, together with the writing in Episode B, presented the pattern of 1-2-1-2-1-2-1, which broke at 11.

EXCERPT 3

This excerpt comes 15 minutes after Excerpt 2. In those 15 minutes the girls checked out possible ways of arranging 11. When Lillian suggested that 11 was an odd man out, Irene disagreed, saying, "I don't think so because 18 starts with 3". Now Irene was able to stress that 18 began with a 3. All along it appears she attended to 18 as a number that had the property without noticing that it did not necessarily begin with a 1 or a 2. Lillian then suggested, "So let's find some of the numbers in between 11 and 18". As they tried to find a string for 13 (beginning with 1, 2 and then 3) they realized they had "one for 12 actually"—12 = 3 + 4 + 5 (turn 85). But they failed to find one for 13. As they proceeded to find an arrangement for 14, they accidentally got one for 15—1 + 2 + 3 + 4 + 5. Excerpt 3 occurred when, together with the teacher, they were trying to make sense of the pattern "one two one two one", which breaks at 11 and 12 and then "comes back" at 15.

202	Lillian:	Fifteen is.
203	Irene:	Even fourteen is...fourteen is 2 plus 3 plus 4 plus 5
204	Lillian:	**If we can't find one |**
205	Irene:	|After all...|
206	Lillian:	**|We try to find the next, and then we find that we can actually have one...a pattern**
207	Irene:	I guess... [*She shakes her head imitating a balancing scale, as if to weigh the idea*]
208	Teacher:	**You try to find...what do you begin with? ...Do you begin with the number and then find the pattern or you...**
209	Lillian:	Yes
210	Teacher:	[...] What if you try the pattern and then find the numbers? [...]. But this time when you began with picking 2 plus 3 then there you were sure you were going to get a number [that satisfies].
211	Lillian:	By the way...[*she accompanies her speech with a subtle laugh*] Actually when you start with a pattern, **obviously there will be a sum, and that sum will be a number.** [Lillian speaks fast and rhythmically at the end. *Irene shakes her head as if to be slowly coming to agreement.*]
212	Teacher:	Irene, are you with us? Do you get that?
213	Irene:	Y-e-a-h. [*She nods as she picks up a paper to write something.*]
214	Lillian:	Okay.
215	Lillian:	So...
216	Irene:	Are we using the pattern first? Okay. [*She mumbles, hands pen to Lillian who smiles back, asks for a fresh sheet of paper and writes as she initiates a (first-order) reflective conversation with the teacher*]
217	Lillian:	We didn't think we were very right for the first one
[Five turns later]		
222	Irene:	**| I guess what we have to...do...is we start with one number, like we are starting with one and add on the next,** and then we start with 2 and keep on adding the next, and then we start with 3 to get all the|

4.7.4.5 Gradual Shifts

The *aha utterance* in Excerpt 3 is not as seamless as the ones in Excerpts 1 and 2; it is a chain of utterances. Lillian observes, "We try to find the next, and then we find that we can actually have one..." (turn 206). Whereas for an observer it might have been clear that this utterance evoked the shift towards generating the numbers by beginning with sums, in the moment both the teacher and Irene seemed not to catch on to Lillian's suggestion. When they came to their moments of insight later, in both the teacher's (turn 208) and Irene's (turn 222) utterances there is no acknowledgment that they were paraphrasing Lillian's earlier remark. Although the shift took time to spread to Irene and the teacher, like the shifts in the first two excerpts, it appears to be spontaneous. This insight happened more at the individual rather than collective level. One might also say that it was a gradual shift; but are not all shifts in understanding gradual? Figure 7 shows the written work that corresponds to this shift. No doubt this was a shift in attention that the girls experienced. Witness Lillian asking for a fresh sheet of paper (turn 216).

Figure 7. Irene and Lillian's Written Work B & C

Episode B (Irene writes)

2 = 1 + 2
5 = 2 + 3
6 = 1 + 2 + 3
9 = 2 + 3 + 4
10 = 1 + 2 + 3 + 4
11 = 5 + 6

12 = 3 + 4 + 5
~~14~~15 = 1 + 2 + 3 + 4 + 5
14 = 2 + 3 + 4 + 5
16

Episode C (Lillian writes)

1 + 2 = 3
1 + 2 + 3 = 6
1 + 2 + 3 + 4 = 10
1 + 2 + 3 + 4 + 5 = 15
1 + 2 + 3 + 4 + 5 + 6 = 21
1 + 2 + 3 + 4 + 5 + 6 + 7 = 28
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36

3, 6, 10, 15, 21, 28, 36 ...

2 + 3 = 5
2 + 3 + 4 = 9
2 + 3 + 4 + 5 = 14
2 + 3 + 4 + 5 + 6 = 20
2 + 3 + 4 + 5 + 6 + 7 = 27
2 + 3 + 4 + 5 + 6 + 7 + 8 = 35
2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 44

5, 9, 14, 20, 27,35, 44 ...

3, 5, 6, 9, 10, 14, 15, 21, 27, 28, 35, 36, 44, 45, 54,55 ...

In Episode C, as we shall see, the girls' actions were more systematic, easier and elegant. They got strings for numbers as big as 44. Also, their attention had shifted toward randomly generating numbers by summing consecutive natural numbers. What could have caused this shift is a central question in this inquiry that I return to later. For now, I continue illustrating how students' actions and interactions are observable aspects of their mathematical thinking-in-action.

In Episode C, the girls altered the positioning of the equals sign, wrote more and did not cancel work. Why this was the case begs an explanation that I will explore in light of other students' written work on this task. Also in this episode they began with the strings themselves. The form of writing in Episode C is different in ways that seem to have created space for the girls to easily generate as many numbers as possible. This potentially shifted their attention towards noticing that they could solve the task by finding a property of the numbers that did not satisfy the consecutive terms property.

By looking closely at Lillian and Irene's written work, together with their utterances, I have briefly illustrated how I observed what they attended to. I have also noted moments of shifts in their attention, particularly those marked by a shift in written records, ways of manipulating materials and aha utterances. Some shifts are individual. And some are more gradual than others. It appears that what students attend to includes what they write, their actions, what they manipulate, and what their pair mate attends to. My analysis suggests the temporal, contextual and relational nature of what students attend to in mathematical tasks. It also alludes to the possibility of joint attention and the radical role of re-presentations. To escape the mentalists' narrow view about thought, I had to conceive the mind in new terms. Conceiving the mind in broader terms as a complex organization with novel properties, Kieren (1992) postulates that a person or community's thoughts usually manifest themselves in the world that the person brings forth. To Bruner (1996), "Mind is an extension of the hands and tools that you use and of the jobs to which you apply them" (p. 151). Thus, we may seek to observe students' works and expressions hermeneutically as the deep sediments of their (collective as well as individual) lives that emerge from lived experience (D. G. Smith, 1991). In the next chapter, I explore what such a hermeneutic stance for this study might be.

5. RESEARCH METHODOLOGIES

5.1 What is Research?

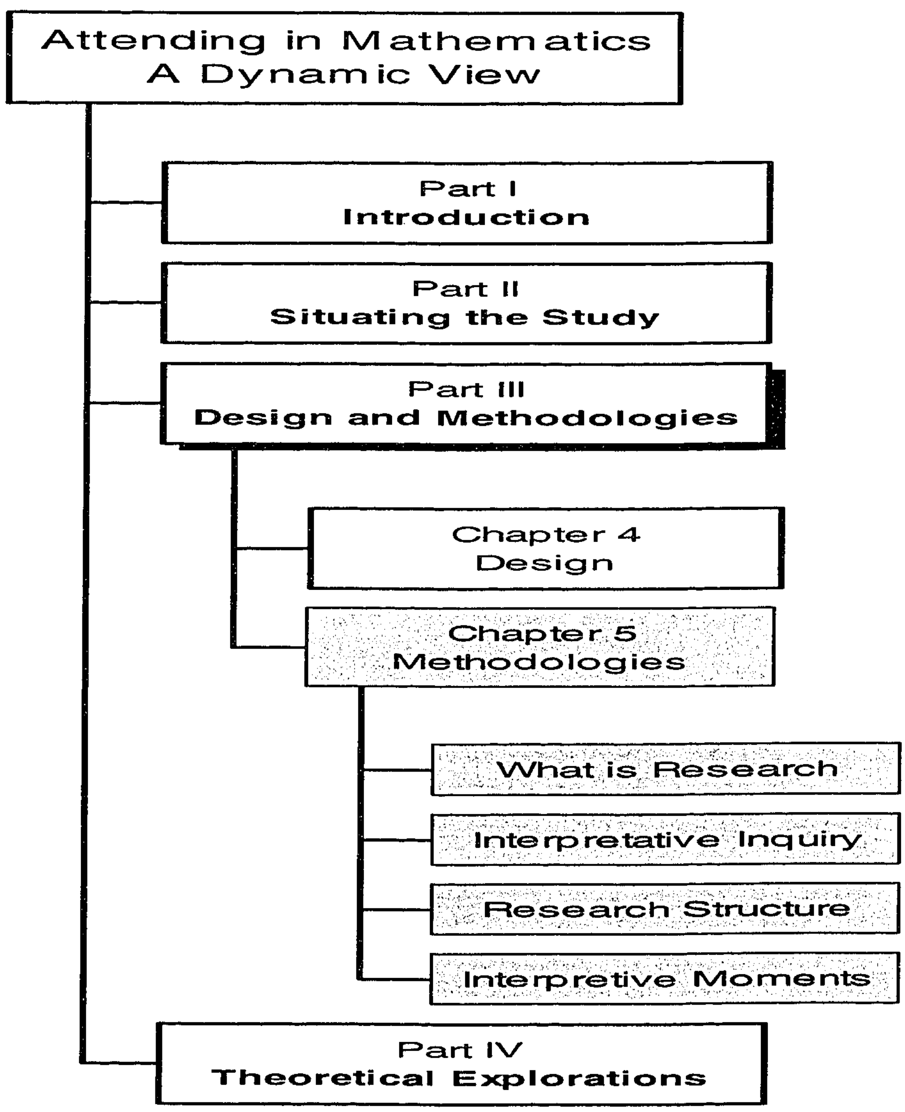
As I prepared the research proposal I found myself wondering about what it means to do research. Was research just a systematic investigation that discovers facts? In proposing to study about mathematical thinking, was I setting out to contribute generalizable knowledge to the field? Although this is how some fields define research, I gradually found myself uncomfortable with the *findings-reporting* metaphor. I take on the approach to research as, literally, to *re-search* and *re-look.*

Re-observing suggests that as a researcher I may not just be looking to establish facts but may be re-searching more closely that which I already wonder about. Re-looking in hermeneutics is aimed at finding alternate ways of interpreting a phenomenon. In *second-order cybernetic* terms, it is to *observe observing* in order to illuminate the conditions for prior observations. It is this generative re-searching, the interpretive orientation, that I explore as the investigative orientation for my study.

The basic aim of my study is to gain insight into the nature of students' mathematical thinking. I seek alternate ways in which this phenomenon—that we as teachers, educators and researchers always find ourselves with—can be re-interpreted. In enactivist theory, there is no place where objects can appear (or be discovered) apart from where there is an observer who observes, describes or explains them. It is in the history of describing and making distinctions that both the object of observation and the *subject* who observes already dwell (Crusius, 1991; Maturana & Varela 1987/1992).

This study requires a stance that will investigate mathematical thinking not as an objective property that can be ascribed to an individual, warranted by method and decided in general, but rather as a contingent and fluid accomplishment, which is enacted in doing (Jardine, 1998). In Maturana's (1998a, 1988b) theory of observation, research requires the researcher to take the stance of objectivity-in-parenthesis. It is particularly important that I acknowledge that my own understandings and prior involvements as a student and teacher are the starting points of observation, the grounds of interpretation of students' learning. Because this study does not seek to establish objective facts or make observerless observations, it is an interpretive study.

Dissertation Landscape Forming



5.2 Interpretive Inquiry

Etymologically, *interpretation* is deliberate construal, intended to pull together different categories of experience (Oxford English Dictionary, 1989). It involves realizing (making real) the meaning of something, often in light of one's own circumstances. Thus an interpretation may say as much about the interpreter, his world and the conditions of interpretation as it does about the interpreted. Unlike analytic inquiry, interpretive inquiry never seeks to discover facts that can be verified to exist "out there" (Crusius, 1991). Even in the natural sciences, where explanation by pure reason has reigned for a long time, aspirations for a single refined and certain interpretation are being given up as undesirable ideals (Kuhn, 1991). Hiley, Bohman and Shusterman (1991) observe that in both the human and natural sciences there is a gradual shift from positivism toward a view in which observations are not considered to be neutral, empirical data are not considered independent of theoretical frameworks, and the ideal of a univocal language and the belief in the rational progress of science are questioned. Attempts are made to investigate complex human phenomena in their complexity without desiring to control them (B. Davis et al., 1996).

To Maturana (2000), even the hard scientific *laws of nature* are observer constructions. "Anything said is said by an observer" to another observer who may be observing him or herself (von Foerster, 2003, p. 283). This has a significant theoretical impact. No longer are laws seen as objective phenomena but they are understood as human observations. As Maturana (1987, 2000) tells us, scientists generate the laws of nature. The choice to see research as interpretation rather than discovery is, for Maturana (1988a), an *emotional choice.* To Rorty (1991), it is a recontextualization of the desire to know essences. To Varela (1999a), it is an *ethical choice*. Yet for von Foerster (2003), it is *taking responsibility* for our actions and interpretations.

5.2.1 The Hermeneutic Orientation

The hermeneutic orientation arose as an interpretation of historically distant texts, but is now increasingly conceived as the theory and practice of interpretation of texts and textures of life (Jardine, 1998). Hermeneutics derives from *Hermes,* the deliverer of messages in the Greek Pantheon (D. G. Smith, 1991; van Manen, 1988/1998). J. K. Smith (1993) describes hermeneutics as a philosophy of messages and meaning for all human expression. The interpretive problems we face in our attempts to understand historically distant texts and cultures also appear in our attempts to understand our current and intimate cultures, practices, relationships and lives.

The hermeneutic stance sees interpretation as the human mode of being, and therefore all human endeavors including theories are interpretive. Ellis (1998a) postulates that the dynamics of interpretation constitute our very mode of being; we are continually interpreting events, people and objects in order to be in ways that make sense. On a moment-to-moment basis as we interpret the world we change it (Osberg & Biesta, 2003). By this reasoning, the ecological complexity stance that guides my study may be viewed as a hermeneutical endeavor that seeks to understand cognition so that the education community can go on living their lives to enhance students' learning.

Ellis (1998a, 1998b) maintains that a hermeneutical study begins with an entry question, which is a practical concern, an openness to behold life in its wholeness. In Maturana's (1991, 2000) terms, inquiry is evoked by the emotion of curiosity. As such, interpretive inquiry focuses on understanding *concrete vicissitudes* of human activity. It is when a researcher brings concerned engagement and genuine wonder to the problem that the creativity required for a fruitful inquiry is generated. We engage in the inquiry to unearth, or should we say, to *enact* "alternate ways in which a phenomenon can be understood" (Misgeld & Jardine, 1989, p. 260).

Further, hermeneutic research is guided by an awareness of the intricacy of the relation between the researched and the researcher. Both parties engage in a conversation that Jardine (1998) calls the interplay of lives. D. G. Smith (1991) observes, "[M]eaning is always arrived at referentially and relationally rather than absolutely" (p. 197). It is at the moment of belonging together of the researchers and the researched that research begins (Misgeld & Jardine, 1989). Research is oriented toward enacting conversations, fusing of each other's horizons and creating mutual understanding. In ecological complexity terms it is about bringing forth a collective, researchers-researched system. Therefore, a hermeneutic imagination involves a care and concern for the researched, without which the inquiry would be different, if not impossible. As the researcher recursively interacts with the researched and with the other researchers they have the potential to enact collective and emergent phenomena as well as a world of research significance.

Interpretive inquiry attempts to be self-conscious as a researcher is open to transformation. To use enactivist language, a question that engages us engages our beings and worlds. As we reflect on the conditions of earlier interpretation we rise to the level of collective self-consciousness. From a hermeneutic stance, understanding cannot be separated from self-understanding; hermeneutics is pedagogical as it leads to a deeper understanding of ourselves, albeit in relation to others.

5.2.2 Central Tenets of Hermeneutics and the Hermeneutic Circle

Educational researchers have noted three central themes in hermeneutics. First, interpretation is a creative activity. Hermeneutics is always more about creating meaning (Ellis, 1998b; D. G. Smith, 1991). Second, hermeneutics insists on the articulation between the parts and the whole, between the general and the particular, the emergent and the agents in the development of understanding. The key role of language and tradition in interpretation is a third central theme of hermeneutics. Language and tradition frame our interpretation. They are considered "double-edged" tools, as they both enable and constrain our interpretations.

The hermeneutic emphasis on language finds resonance in the enactivism work of Maturana and Varela (1992) which stresses that language is not a set of words used to designate things. Rather, as I will demonstrate in Chapter 9, we human beings happen in language and it is in *languaging*—socially acting and interacting—that we bring forth human worlds. D. G. Smith (1991) and Gallagher (1992) contend that it is crucial for a researcher to develop a deep attentiveness to language itself, to notice how it is used. van Manen (1988/1998) observes that it helps to listen to the language that speaks through us. We may search for etymological sources of words, examine idiomatic phrases and reflect on the sources of the common metaphors we use to "reflectively hold onto verbal manifestations that appear to possess interpretive significance" (p. 62).

In *ontological hermeneutics* it is acknowledged that what one already knows constrains what one can understand. Put differently, prior knowledge and experience are conditions of possibility for what the researcher can know. Thus the energy of the hermeneutic is in its acceptance of the interconnectedness between an interpretation and its basis of interpretation. Essentially, the researcher's practical understanding is the starting place. It configures the conditional probabilities for further observations. Packer and Addison (1989) define interpretation as the "working out of possibilities that have become apparent in a preliminary, dim understanding of events" (p. 277).

Interpretation involves reflecting on ways of understanding and searching the assumptions that have been taken for granted. In von Foerster's (2003) terms, it is understanding understanding—a *second-order observation*—of not only the problem but also of the looking itself. Observing observing involves an examination of what the grounds of interpretation and what the observing systems were in prior understandings. While interpreting, we read and re-read the text for the possibilities of understanding that it evokes. Hermeneutists deliberately (as far as it is possible) shift perspectives and call into question earlier readings. Read in a back and forth manner, outward and inward manner, the *given* reading and the *evaluative* reading together frame what is referred to as the hermeneutic cycle. In second order cybernetic terms, the interpretive circle is a recursion between first order and second order observation. It is an attempt to illuminate the blind spots, to see through the myths and assumptions of earlier observations.

The concept of the hermeneutic circle illustrates the circularity of inquiry. Our pre-understandings affect what we know and are in turn re-configured by what we come to know. This circularity is not considered a tautology in hermeneutics. Rather, as is the case with theories of observation, it is essential. It is a creative circle rather than a vicious cycle. Because it does not leave earlier observations unchanged, it is a *recursive elaboration*, for example, of the traditions that we are thrown into (B. Davis, 2002).[18] In

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[18] For Maturana (2000) what constitutes a recursion "is the coupling of the cyclical dynamics from recurrent interactions...with the linear dynamics constituted by the displacement of" understanding and

the *forward arc* or *first-order observing*, a researcher approaches a phenomenon with some understandings that form the fore-structure of the inquiry. In turn, the researcher's understanding is suddenly enriched by the phenomenon. Moreover, as a researcher engages with, or is engaged by, that which comes to perturb him or her, novel understandings are generated.

Equally important in the hermeneutical circle is the *backward arc* of the circle, the loop that is marked by a deliberate and radical re-reading of the text. It is in this backward arc that *uncoverings,* to use a depth metaphor, the different ways of understanding and the unexpected insights are likely to happen. Also called *evaluation,* the backward arc entails an attempt to see what went unseen in the initial interpretation of the forward arc. In the evaluative stance, empirical materials are re-examined for contradictions and inconsistencies, interpretations are checked against counter-examples and probable counter-interpretation, and the conceptual framework is read against alternative frameworks. The backward arc is where interpretive re-search begins. It involves reflecting on the basis and usefulness of observations made.

5.2.3 Recursion in the Observation Circle

A comparison can be made between the recursion in the hermeneutic circle and orders of observation studied in cybernetics. Consider the claim of cybernetics that there are (at least) three orders of observation which create a recursively layered interpretation: zero-order, first-order and second-order.

At the level of the zero-order observing, observers act. Zero-order cybernetics is when activity becomes structured, when purposeful behaviour emerges. But one does not

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relations that occurs in each interaction (p. 463). Recursive elaboration is an elaboration that inevitably changes the original understanding and also sprouts novel understandings.

ask about the *why* and the *how* of this behaviour. In this study, when learners or researchers act mathematically but do not seek to comprehend the nature of their mathematical thinking, they are engaging in zero-order observation. They just live in their actions.

First-order observing involves observing the what, how or why of behaviours. For instance, the education community studies the characteristics of learning behaviours. It is in first-order cybernetics that one reflects on how students think mathematically. First-order observation leads to the development of notions like *stages* and *modes* of mathematical thinking. In first-order observation researchers observe mathematical thinking as a phenomenon with fixed properties.

Just like scientific observations, first-order observation is motivated by the desire to describe and to explain. However, every observation at this level is not aware of itself and the distinctions it makes. It is not even aware that it is making distinctions. Thus, Luhman (2002a, 2002b) and von Foerster (2003) argue that there is a need for observation to ascend above itself and attempt to interrogate its conditions and to reflect on the consequences of its observations. Since, as we shall see later, observations are operations that shape the world, then people who build theories about mathematical thinking ought to let the underlying ways of talking about aspects such as representations, modes, styles and thinking processes become explicit. Second-order observation reflects about these mechanisms that have been presumed to generate mathematical behaviour as well as on the properties of the observing system. The researcher as the observer re-enters the form which he or she distinguishes (Kauffman, 200; Varela, 1974).

Second-order cybernetic observing deals with observing *observing* (von Foerster

(1992, 2003). We observe how prior distinctions and ways of researching were enacted. We illuminate the conditions and blind spots of first-order constructions. For instance, we ask: What is the motivation behind distinctions such as phases and stages of mathematical thinking? Why are the notions of thinking, objects and imagery used in the first place? Why not use notions of patterns in behavior and transformation in worlds, for instance? In what ways do current distinctions such as cognitive representations and obstacles influence our actions and world? In second-order observation, which appears to include the hermeneutic backward arc, the observer stipulates his or her own conditions and properties of observing, or those of another observing system. Second-order observation is about illuminating observing systems and seeking useful ones. Whereas in first-order cybernetics researchers just explain and solve problems in second–order cybernetics the motivations to solving problems and the way the solution might transform the world are reflected upon. In Chapter 3, I reflected upon the motivations behind current research on mathematical thinking.

Von Foerster (2003) explains that if we enter into third-order cybernetics by reflecting upon our reflections we do not create a new order of observation, because by moving into second-order observation we have ascended into a circle that closes upon itself. In *theories of distinction*, as I will show, one is no longer an independent observer who watches the world go by, but a participant in the circularity of human conditions.

5.2.4 Dynamic Landscapes: A Metaphor for Interpretive Research

During interpretive analysis (first- and second-order observing), research questions are re-framed, one's understanding changes and a deeper relationship with the setting and participants evolves. The spiral used in hermeneutic research might be better illustrated by a *non-Euclidean geometry of fractals* or better still by the *geographical metaphor of landscapes.* Such metaphors offer a more natural and continually changing form. Consider the hermeneutic circle. Through hermeneutic questioning a research landscape forms and through the process of layered interpretations that research landscape is gradually transformed.

The metaphor of landscapes offers us some interesting interpretive possibilities. Firstly, it evokes images of valleys and hills of varied contours. Landscapes are always in the process of transformation both through gradual changes and radical alterations. Landscapes have similar detail in close up and on the larger scale. Mathematically speaking, landscapes illustrate fractal form, in contrast to planes, lines, circles and spirals which are understood as Euclidean. Euclidean surfaces and shapes are closed and regular; fractals like landscapes are irregular and open.

There are many factors in place that contribute to the transformation of a landscape: some external and others belonging to the topography and geology of the landscape itself. The hermeneutic research process could be visualized as a landscape forming and being transformed. At whatever stage of the research, there will be widening basins, smoothened valleys, raised ridges and the like. The space of the possible will be in continual fractal change. This landscape metaphor highlights the phenomena of interdependency among the aspects of research, the non-linearity of the process, the influence of community, the inbuilt temporal dimension, and the relative stability as well as the gradual and sometimes sudden changes. More importantly, the metaphor emphasizes that the scale of observing research themes can be local or global and that there is a benefit in its being both. Although we can learn some things about a landscape by observing/studying the pebbles, and the rushing water, we must step back to see the river that has (trans)formed the valley.

Each valley in the research landscape represents a different attempt to get close to what interests a researcher or to what he or she hopes to understand. In complexity research, what one hopes to know changes during the inquiry process, hence a recalibration in the research landscape. It is a dynamic landscape whose basins of varied sizes can be seen as dynamical attractors of research themes and understanding. With time, interaction and action the valleys of understanding deepen and widen. In addition, there is the possibility of novel and imaginary experiences that might arise globally at critical moments, from catastrophic changes in the landscape. It is a multithreaded and dynamic landscape formed in doing and living interpretive research.

5.2.5 The Role of Complexity Theory in Interpretive Research

In interpretive inquiry one may adopt a theory as a functional tool to assist in deepening understanding. One may choose a model to set boundaries or to reduce complexity so that they can understand better (Osberg & Biesta, 2003). As with Freeman's (1999) studies, interpretive inquiry requires that the theories, tools of analysis and observational models as well as the spatial, technical and temporal scale of observation of the theoretical framework be suitable for the problem under study.[19]

In my study, complexity theories function to constitute ecological metaphors, shapes and figures for understanding learning. Those metaphors, however, are not mirror

[19] Studies in neuro-physiology of animals and etymological studies of the behavioral ecology of ants illustrate well that an adequate scale of observation counts. For example, a study of activities of single neurons in the brain is inadequate in comparison with the study of neural assemblies that transiently link multiple brain regions and areas during activity (Lutz & Thompson, 2003; Freeman, 1991). Similarly, observing and thinking about individual ants or even surveying given colonies of ants for days is not adequate for understanding the organization of ant colonies which cycle through infancy, adolescence and maturity in up to fifteen years (Johnson, 2001).

images of reality. Hermeneutically, theory as a critical reflection (a higher order observation) in practice transforms reality. Moreover, the *macro-theories*—the culturally determined ways of seeing—the folk theories and technologies of our times tacitly frame how we perceive. For this reason, D. G. Smith (1991) maintains that macro paradigms too should be reflexively worked through to deepen our sense of what is at work in our interpretations. Smith further suggests that in evaluating our interpretations we should not only rely "on the more conventional perhaps-on-the-verge-of-exhaustion grand narratives...but also, importantly, on the more suffocated narratives of our time" (p. 199). The grand narrative for construing mathematical thinking, as I expressed in Chapters 2 and 3, seems to be the psychological view of thinking as the effortful processing of information. Emerging narratives include the discursive view of thinking as communication (Lerman, 2001; Sfard, 2000a, 2001; Vygotsky, 1978). Also, there are grand and emerging narratives about data-analysis procedures.

5.2.6 Nature of Data, Analysis, Writing and Evaluation

In hermeneutic (interpretive) research, data collection and analysis procedures, adapt and co-evolve, although conceptualized at the beginning. As the study proceeds and the researcher reflects on the process, a detailed methodology is laid down. Data analysis is ongoing; even the writing, as a vital part of the research, does not wait until the study ends.

In analytic studies, writing usually comes at the end of the inquiry as the researcher sets out to report the findings. However, in interpretive inquiry the "research process is itself practically inseparable from the writing process" (van Manen, 1988/1998, p. 167). In complexity terms, writing, data collections, reviewing literature, data analysis, and the like are coherent forms, agents from which novel interpretations surface. We at most times write our way into data analysis rather than writing results from analyzed data. One may make field notes, mark, write and rewrite *micro research* reports around events of interest while paying incredible attention to the emergent themes that present themselves.

This suggests that a research narrative usually rolls up in a manner related to the question and the research orientation. In any interpretive text, distinct parts—the chapters, structure and themes—and the whole pattern—research tone and transformation triggers—are present. The autobiographical material that predisposed interpretations is as important as the literature related to the question. Researchers holistically trace the multithreaded paths they have laid, the transformations that the researcher and the researched underwent during the study (D. G. Smith, 1991). That being said, Ellis (2001) and van Manen (1988/1998) caution that a hermeneutic text is not a compilation of these presences—parts and the whole, the autobiography, journey and more. It is a well-argued essay with the researcher's interpretations together with illustrative cases offered to the reader. The essay is offered to occasion the collective mind.

Appropriately, to evaluate an interpretive account "one should ask whether the concern that motivated the inquiry has been advanced" (Ellis, 1998b, p. 20). Empiricist legacies of validity and technical rigor give way to issues of viability and convergence. J. K. Smith (1993) observes that criteria for evaluation of an interpretive work are conceptualized in ethical and practical terms rather than in epistemological terms. To evaluate the adequacy of an account, Ellis (1998b) observes, hermeneutists ask,

> " Is it plausible…? Does it fit with other material we know? Does it have power to change practice? Has the researcher's understanding been transformed? Has a

solution been uncovered? Have new sensibilities been opened up?" (p. 30)

In saying that a particular study is "good or "not good" research, we find ourselves once again making a claim on the criteria for evaluating research. In hermeneutics the criteria themselves co-evolve with adequate studies (Gallagher, 1992; J. K. Smith, 1993).

In complexity research terms, in addition to issues of plausibility and fitness, a researcher might ask whether the study is coherent, and whether it is an agent in the emergence of novel and grander understandings. Was it done on the appropriate scale and with adequate tools? Did it expand the space of the possible for research, teaching and learning? Theories and models generated by research are not adequate only when they fit with other existing theories (though that could be the case). Researchers might want to evaluate the models and conclusions for their functionality: Do they help those concerned to (re)negotiate their worlds (Osberg & Biesta, 2003)? Are they cast in pragmatic and temporal terms rather than in terms of truths? In a word, do they embrace complexity?

5.3 The Research Structure

As I write this research narrative I have a concern that labeling the phases in the study as literature review, research design, theoretical framework and so on will evoke technical tendencies of looking at each phase as a separate entity. In Chapter 4, I dealt with two tensions that I face: (a) maintaining the interrelations among research question, literature and orientation in a linear text; (b) maintaining the openness required by interpretive studies of complex phenomena. In this chapter, I deal with the third tension: "How can I work within distinctions such as collecting data, and designing and piloting a study while embracing complexity?" I resolve this tension by locating my study in the traditions of educational research and by introducing an important dimension of my study—*interpretive moments.*

5.3.1 Educational Research Traditions

In spite of my critique of the positivistic research paradigm, I have been acculturated in this paradigm, which continues to dominate educational research. Its legacies continue to shape my knowing. Gallagher (1992) observed that we cannot escape traditions, even those that hinder our understanding. Bruner (1996) and Merleau-Ponty (1964) say we use them instead. It is through hermeneutical reflection and second order observation that we consciously work at transforming and transcending ourselves and the limits of our traditions toward the generative.

My research question is qualitative rather than quantitative. How do students, on a moment-to-moment basis, bring forth worlds of mathematical significance and relevance in which they find themselves thinking mathematically? In order to address the larger concern, I specifically focus on a phenomenon that cuts across knowing bodies. I investigate students' mathematical attentiveness.

Creswell (1998) suggests that for "how" questions *qualitative methods* have a distinct advantage. My questions require both a design and an attitude that allows for an in-depth exploration of students attention in its messiness and interconnectedness. In terms of the anticipated outcomes of the research, the study could be viewed as a second order, *meta-descriptive project,* rather than an *evaluation project* (Peshkin, 1993).

Merrian (1998) classifies my starting place for the inquiry as *basic (generic)* qualitative research. She maintains that basic qualitative studies exemplify the characteristics of naturalistic studies, and those studies

> [d]o not focus on culture or build a grounded theory; nor are they intensive case studies of a single unit or a bounded system. Rather researchers who conduct these studies...seek to...understand a phenomenon. (p.11)

According to Bogdan and Biklen (1992), and Merrian (1998), qualitative studies have the researcher as the key instrument. They usually involve natural settings. Their empirical material and narratives are richly descriptive, and they tend to analyze data inductively. Denzin and Lincoln (1994) posit that qualitative studies may involve a wide repertoire of methods and perspectives, with the aim of *deeply* studying the phenomenon.

My main focus has been re-framed to seek insight into

1. How do students attend as they engage in mathematical tasks?
2. In what ways do students as embodied, embedded and extended agents attend?
3. In what ways do secondary school students await and dwell with mathematical concepts?

Had I found myself in subsequent loops of the inquiry analyzing narratives of students' experiences of how they attend and had I had the narrative inquiry skills, this study would have been a narrative inquiry. A *phenomenological* orientation to writing has in many respects been a more workable approach to this inquiry. Generally, to the extent that I have sought to reflect on students' mathematical thinking a lived phenomenon and have recognized phenomenological sub-questions and sources when they emerged, the study is a *hermeneutical phenomenology* study (van Manen, 1988/1998). Issues of *perception* and *thinking* lend themselves to neuro-physiological and philosophical research on perception; this study draws heavily from these fields. As a result, I have found myself with what Denzin and Lincoln (1994) named a *bricolage* methodology. Quantitative methods were never among my cartography, though some mathematical theories of dynamical and stochastic processes provided metaphorical tools to think about non-deterministic complex phenomena. Statistical and analytic methods were not evoked—not even in efforts to create a *thicker* description.

Furthermore, because I specifically worked with a group of junior high school students, I initially understood my study as a set of case studies of students' mathematical thinking (Merrian, 1998). According to Confrey (1994b), my study might be viewed as a mathematics *teaching experiment*. Confrey, as well as Steffe and Kieren (1994), explain that unlike clinical interviews, teaching interviews extend beyond a single episode and emphasize the student's activity to facilitate an understanding of children's conceptualizations. My study, however, is not a *psychological clinical interview* as described by Ginsburg (1981). Even though I had students engage in mathematical tasks, the study was not *problem-solving research* in the mathematical research gist.

The co-emergent nature of interpretive projects requires flexibility on the side of the researcher. Just like life or adaptation, when proposing a study we do not know how the structure will evolve. It is in doing the study that we happen to choose directions, explore techniques and identify subsequent data sources, thus laying down a detailed design of the study. Educated fore-structuring and entry design are required at the onset. For that reason, instead of thinking in terms of a pre-specified research design, as I illustrated in Chapter 4, I now look back on the proposed design as a research structure. This explains the title of this subsection. I entitled it "research structure" because for me it evokes the organic structure in addition to the pre-planned (modern architectural) structure. A biological structure embraces emergent and spontaneous forms in addition to the pre-determined forms. With time I began to envision general principles encompassing a "myriad of potentialities, one of which, [as we shall see,] will be pulled into existence—but only by living through the event" (B. Davis & Sumara, 2000, p. 842). Additional to the co-emergent research structure are the micro emergent structures that arose as moments of interest. These micro-structures sparked off micro-inquiries—fractal parts. It is the interpretive moments that sustained a non-technical stance to inquiry in this study. Interpretive moments open up possibilities for novel forms that might arise while working within the structures that help organize the study and writing.

5.4 Interpretive Moments: The Focal Sites in the Study

Before I proceed to the next chapter, let me introduce the notion of interpretive moments by reflecting on the example I narrated at the beginning of this writing.

In the study, I specifically worked with moments that happened to be salient for me. It is these moments that I took as *micro-phenomena* under study. The moments at times arose as questions, hunches or moments of surprise or discomfort. They seemed to have been the relevant moments, at particular moment. For example, as I read, a researcher's remarks would strike me as interesting, or as I participated with students I might have noticed that different students were approaching the same exercise differently. Most of these moments were salient only for a short time. The majority of them were subtle, and some of them were only observable in hindsight. I had to learn to listen for them (B. Davis, 1996) and to remark on them (Mason, 1994).

B. Davis (1994) refers to interpretive moments as *watershed moments*. He says that unlike orchestrated aspects of research, such moments are the day-to-day experiences that are part of the research conversation. Jardine (1998) refers to them as *fecund* *individual cases.* He observes that such cases are generative in gathering data, and in occasioning interpretations. In *the Discipline of Noticing,* Mason (1989, 2002) refers to them as *moments of choice.* Elsewhere in the literature they have been referred to as *critical events* (Confrey, 1998).

Mason (1994) observes that noticing is what we do all the time, but to engage in a conversation with and about a *moment of noticing* requires a deliberate choice on the part of the researcher. Mason developed the discipline of noticing as a practice for working with and dwelling with moments of noticing. Although most interpretive moments in the study were unformulated, for those I could mark so as to contemplate and record them, interpretive moments became the data that I gathered (as well as created). They were micro-data that I analyzed and made further inquiries about—*micro-analyses and -inquiries*—during the past four or so years.

The practice that I adopted in working with these moments drew from phenomenological inquiry and was methodically informed by the discipline of noticing. To mark the moments when they happened, I collected artifacts and wrote *brief-but-vivid vignettes* about what engaged me. The anecdotes were a lived description of my experiences of what I both observed and participated in. In Mason's (2003) terms, the vignettes, such as the ones I offered in Chapter 4, were the *accounts-of* the moment. Together with the vignettes, I also recorded my reflections or justifications of what I thought was going on in the moment, the *accounts-for.* It is the accounts-of and the immediate accounts-for that I began from and recursively returned to during the analysis. The practice of working with interpretive moments was both anticipated and actualized in reading and re-reading related literature, in participating in varied research sites, in sharing interpretations, and in writing and re-writing *micro-narratives*. In the next section, by returning to Fraction Kit anecdote, I elaborate how generative interpretive moments can be. I offered other examples from the extra curricular project while looking at the lessons learned from the preliminary study in Chapter 4.

5.4.1 Fraction Kit Activity: An Example of an Interpretive Moment

The story at the beginning of Chapter 1 is a description of an interpretive moment—vivid accounts-of followed by scanty accounts-for. It happened in a lesson in which 27 students were exploring the sizes of Fraction Kit pieces. As a research assistant, I participated in different ways. I observed what the students were doing, and I assisted a student, Peter, who remarked that his kit was missing one pink piece. It was then that I observed that one student had her pieces neatly stacked at the corner of her desk. I was surprised that she would consider stacking the pieces before figuring out their sizes, for that is what I thought she was doing. At first I pre-judged it as a "girl thing" of liking to organize and color work. Little did I know the stacking of pieces was more fundamental. As I walked around the class I noticed that some other students, including "rough" boys, were engaging in the task in an equally orderly way; they finished the task without covering a larger piece. Because this *new* approach of stacking did not involve *covering,* I began to consider it as a distinct approach. But was the *stacking* approach adequate (rather than optimal), and in what way? I continued to wonder.

5.4.1.1 Mathematical Methods or Strategies

A major contribution of radical constructivist research is its support for diversity among students' interpretations (Confrey, 1994a; Steffe & Wiegel, 1992). Constructivist teachers offer opportunities for students to use multiple approaches in solving problems and making meaning (Ball, 1990; Ball & Bass, 2000). For instance, teachers may teach fractions using measuring cups, paper pieces, fraction blocks or other materials. But, are different forms of manipulative materials (or their symbolic records) merely varied versions of a concept?

Tony and Ronald, junior high student participants, as a pair engaged in the Consecutive Terms tasks. Tony used bingo counters whereas Ronald just worked with number symbols. Ronald worked with bigger numbers than Tony. When asked to explain his strategy, Ronald commented, "Which is like the same as the counters, pretty much, …except he [Tony] worked with bigger numbers".

In such an environment it is crucial to ask: In what ways was working with counters and with smaller numbers the same as working with number symbols and bigger numbers? In what ways are any two methods or concrete, pictorial or symbolic representations similar? Thinking in terms of the Fraction Kit task, in what ways was the *stacking method* an alternative method to the *covering method*? The stacking approach was a "less traditional other", but was it less significant? In what ways did the students choose one approach over the other? Did the two methods require as well as generate similar mathematical experiences?

An apparent difference between the two approaches is that when using the stacking approach, a student was likely to say that each pink piece was a $\frac{1}{23}$ instead of $\frac{1}{24}$ if he or she, like Peter, saw only 23 instead of 24 pink pieces. However, there might be essential differences between the two approaches. Indeed, in what ways can we say a student who stacked the fraction pieces into piles and one who covered a larger sheet undertook the same *sensory-motor, perceptual, conceptual, linguistic* and *formal* tasks?

5.4.1.2 Embodied Tasks: Analyzing Tasks

In some sense the differences in the stacking and covering methods are differences in how students attended. For instance, as the students explored their kits, to some students the task-at-hand happened to be stacking. These students *deeply chose* to perceive the pieces as a set of discrete objects not so different from *counting stones*. The area of the pieces was a background attribute against which color was stressed. Hence, by stacking, a student was likely to have attended differently. In attending more to the numerical aspects of the pieces the students who stacked are likely to have overlooked the geometrical aspects—the area of the whole covered by any single piece. Consequently, the students who used the stacking approach brought forth a task different from (yet compatible with) the task intended by the teacher.

To figure out the size of pink pieces students asked, "Out of how many total pink pieces in the kit is a single pink piece?" They did not ask, "What portion (how much) of the whole paper does a single pink piece cover?" The former question seems more likely to evoke a chain of thought from numerical aspects to stacking, to multiplicative ratios, and finally to a multiplicative relationship. By contrast, covering evokes portion of a whole, followed by an additive process and comparing of fraction sizes (Kieren, 1990). As a result, it is likely that different imaginations, visualizations and metaphors were called forth for students who used the stacking approach. In an ongoing cognitive process the students brought forth an embodied task (Gordon-Calvert, 2001).

Kieren, Pirie and Gordon-Calvert (1999), for instance, have observed that when some elementary students read six "eights" *(6 $^1/_8$'s)* instead of six eighths ($^6/_8$), this is not a reading error as it might appear on the surface. They explain that it is a conceptual issue, and eights is a legitimate label in a specific transient fractional understanding. The setting of paper folding stresses plurality of fraction pieces—2 halves, 3 thirds, 4 fourths and so on. Students might shade or tear off small pieces, say two thirds, but still these are two objects of the kind *third.* When eight equal parts are shared among four children, each of them gets *two* "one-eights" pieces (not so much a quarter of the cake)—adjective is *two*, which modifies the idiosyncratic noun "one-eights". It is the number of pieces—2 pieces—rather than the size of the piece, plurality that is the substantial concern in a *folding-based fractional* world. Children at this moment in learning fractions might write four *eights* as *four-one-eights 4* $\frac{1}{8}$. And it might follow that 6 $\frac{1}{8}$—*six-one-eights*—is the same amount as 3 $\frac{1}{4}$—*three-one-fours*. Kieren observed the same children when they operated with the full Fraction Kit. The students then saw fractions differently with the numerators as modifiers of denominators. They read $\frac{6}{8}$ as *six eighths*, for in this fractional setting, the size of a fraction piece was stressed against the background of multiplicity of pieces.

Since students worked independently on this seatwork activity, no verbal expressions are available for *conversational analysis*. Nevertheless, it is discernible that the sensory-motor acts, perceptual images and conceptual structures of the students who saw the pieces as *discrete numerical models* were different from those who saw the pieces as *continuous geometrical models*. For instance, the students who used the covering approach are likely to have developed an image of fractions as combinable, additive quantities, an idea that was required for the task that followed. Yet for the students who stacked, it appears relations across sets were most likely played down (e.g.

yellow pieces (quarters) had no immediate relation with red pieces (halves)).[20]

At one level of description—the level of the task posed by the teacher—all the students may have appeared to be solving the same task (figuring out the sizes of the pieces). They even got the same values for the piece sizes. At this level, as I will expand later, we can examine what students attended to by examining properties of the mathematical task, as if the properties existed without the problem solver. Task analyses omit another level of description, in which particular students dependent on their histories and contexts bring forth relevant issues to be addressed. It appears necessary to analyze fractional worlds that students enact in interactions, as I am doing in this section. Just because the mathematical analysis is inadequate does not mean it is not necessary. On the contrary, it is a preliminary step in adopting tasks that are appropriate. Nonetheless, a task analysis ought to be interpretive. The objects, in this case the fractions, it infers are not objects of a transcendent task, but are dependent on students' activities.

A task analysis might not explain why different students enacted a different Fraction Kit task. Thus, to ask about "what students attend to in mathematical tasks" is not merely to ask about "What there is in a task to be attended to?" In a way it is to ask, "What are the students attuned to?" What choices and possibilities are allowed by their histories and structures, including external structures? As we shall see in Chapter 8, it is to ask about students' experiences, their biological nature, traditions, pre-judgments, purposes and hopes, as well as language and cognitive domains they constitute. Indeed focusing on the ways students attend as a means to understand mathematical thinking evokes layered interpretational activity.

[20] This is especially so because paper, unlike more voluminous material, did not offer opportunities for comparison of stacks by height

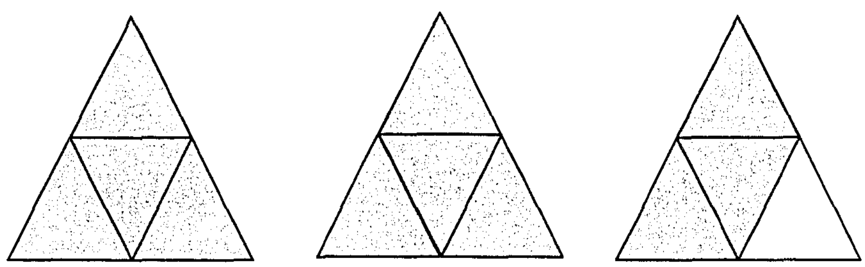
Within the broader limits of figuring out the size of the different fraction pieces, to some students the relevant issue to be addressed, given their enacted worlds, was "out of how many in a set" rather than "what region of a whole was covered". To most students who stacked, $\frac{1}{4}$ was likely to be *one of four*; $\frac{1}{4}$ was an act of selecting a single piece out of a set, a collection of four yellow pieces. This also seems to be the case with a student who covered, except that to a student who covered $\frac{1}{4}$ was also a portion of the unit sheet. In the fraction space, a share unit of *one* must be presumed ($\frac{1}{4}$ piece is $\frac{1}{4}$ of a whole piece not one of "four"). In most cases, quantities are stressed over ratios, and $\frac{11}{4} = 2\frac{3}{4}$ as well as $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ (Confrey, 1998; Kieren, 1990). According to Confrey (1998), a student who stacked was in the ratio space, which complements the fraction space. In Vergnaud's (1988) terms, ratios are tied to the conceptual field of fractional amounts. Kieren (1990) maintains that children have to experience rational numbers both as quantities (covering, *muchness*) and as ratios (stacking, *manyness*). For this reason I wonder about the fractional worlds that were enacted by students who neither stacked nor covered, but assembled wholes of different colors to figure our the size of the pieces.

What is more, what the students attend to in a given task is likely to determine what they attended to in subsequent tasks. This appears to have been the case in the second task, when most students who used the stacking approach asked, "What do you mean by covering? Does it matter if the pieces overlap? Do the pieces have to be of the same color?" For the students who stacked, the distinction of "covering a half piece" did not signify a common action. Covering was not in their embodied history and therefore its conditional probability of being enacted was low. I wondered whether the students who used the stacking approach would experience further problems with other fractional tasks. Let us see how a latter occurrence could have been originated by the stacking approach.

5.4.1.3 Rooted Errors: Dynamic Error Analyses

Two days later, the students approached *conversions* (converting mixed fractions to improper numbers) by working out the number of unit fractions—pieces—in given mixed fractions. Arlene was one of the students who seemed to be in need of one more illustrated example before working on her own. After listening to a student's example, she animatedly turned to her notebook to work out how many quarters ($\frac{1}{4}$) there were in two wholes and three quarters ($2\frac{3}{4}$). She drew three "fancy" triangular wholes, each with four equal portions, and shaded all four portions in the first two triangles and only three portions in the third triangle. In an excited tone she then said, "It [two wholes three-fourths] is eleven-twelfths." In total she had shaded 11 of the 12 small triangles. Indeed in Figure 8, eleven is an amount whose kind is twelfths not fourth! Because she was a relatively *slow learner*, Arlene's performance on this task might not necessarily mean that the stacking approach constrained her actions. Or might it?

Figure 8. Diagram used by Arlene to write 2 as a fraction



Many researchers seek to identify the origin of common errors so as to eliminate the source or to circum-navigate them (Zazkis & Liljedahl, 2004). Sierpinska (1990) and others have explored *epistemological obstacles* posed by particular tasks. Other researchers, in a manner aligned to analyzing worlds enacted, seek to determine the domain of validity of a perceived error (Balacheff, 1990c; Borasi, 1987). For in addition to analyzing tasks and speculating about sources of errors and how to overcome them, I speculate about what students might be attending to given their experiences. I ask, "What world of significance does a student bring forth with others as he/she engages in activity?" This level of description, unlike the task and error analyses, is addressed in the fact or after-the-fact and is typically context-dependent. The view presented in this writing looks at Arlene's response $2\frac{3}{4} = \frac{11}{12}$ from the perspective of the learner as not errors but legitimate sense in the world she enacted. It requires participating *sympathetically*—as a synonym for thoughtfully and helpfully—with the students in their mathematical worlds. In the world brought forth by stacking fraction pieces, it might have made perfect sense for Arlene to conclude that $2\frac{3}{4} = \frac{11}{12}$. Unlike the other students who used the same figure to realize that $2\frac{3}{4} = \frac{11}{4}$, to Arlene it was 11 small (but whole) shaded triangles out of a collection of 12. With the history of the Fraction Kit activity, it is plausible that Arlene attended to the collection, the numerosity of the smaller triangles. She stacked the small triangles, rather than covering the whole big triangle.

The challenge for interpretive research on mathematical thinking is, "In what ways can we invite students to bring forth a space in which the conventional is highlighted and concepts are understood in general terms?" It is important to hermeneutically attend with Arlene so that she will, in grade 8, be able to see $\frac{1}{4} + \frac{1}{4}$ in fractional contexts as $\frac{1}{2}$ rather than as $\frac{2}{8}$, which is also the case in many other contexts.
fractional contexts as $\frac{1}{2}$ rather than as $\frac{2}{8}$, which is also the case in many other contexts.

In Chapter 8 I return to this point to discuss how task, error and world analyses ought to be layered by hermeneutic as well as systems analyses. To conclude this chapter, I reflect on the interpretive function of analyzing events such as the Fraction Kit vignette.

5.4.2 Generativity of Interpretive Moments

In marking and reflecting on an interpretive moment, a person might be led to do further noticing, engage some literature or recall related events. Also, in a latter event of noticing, this particular interpretation may be called forth and iterated upon. No single interpretation is static, done once and for all and in a way that rules out other possible interpretations. To *evaluate* my Fraction Kit interpretations I ask: What are the other possible interpretations? Such a question requires me to review the data I collected in the form of vignettes, video records and photocopies of students' work. More importantly, the question requires that I read related research and share my interpretations. In interacting with colleagues, both present and virtual, I ponder the following questions:

- What do colleagues say about my interpretations?
- Why was this moment salient for me? Why did I interpret it the way I did?
- What traditions are at work in my specific observations?
- What else could be said about the moment?
- What experiences does the moment cause to resonate in other researchers?
- How does it relate to other interpretive moments I have explored?
- What is the observing system that evokes constructs like *levels of analysis*?
- What are the blind spots, the observational properties of such a system?
- What shapes and metaphors am I thinking in terms of? Can I find better ones?

I have used the Fraction Kit event to exemplify the centrality of interpretive moments in my research methodology. Interpretive moments are likely to occasion what one observes down stream and to recursively transform earlier (upstream) observations. They are opportunities to explore new avenues of inquiry and to examine one's own biases. By sharing interpretive moments with colleagues, one can participate in research conversations. Indeed, ongoing data gathering, data analysis and research writing were centered on interpretive moments in this study. Jardine (1998) has a caution though: in interpretive inquiry one ought to know when to stop before one goes too far, gets carried away and stretches the interpretation of an individual case "out of all proportion." (p. 47) He re-affirms, "The implications [of any interpretation] are not meaningless. In spite of the fact that they can easily become too 'wild', they are not altogether 'unfitting'. The 'analogical kinships' of meaning still seem to persist." (p. 47)

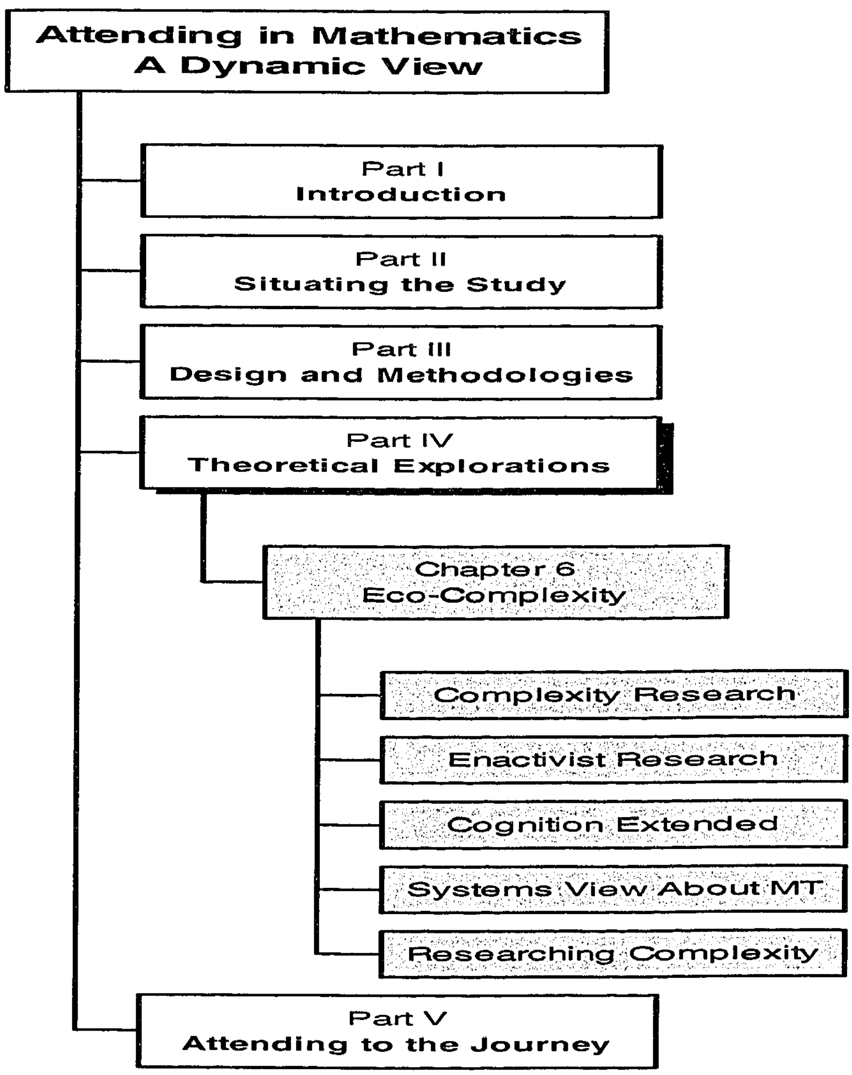
I have elaborated on a hermeneutic stance from which I re-search students' mathematical thinking. Hermeneutics has its counterpart in the hard science—second-order cybernetics. I have attempted to understand these two orientations in light of each other. Also I have illustrated how I juxtapose organic and designed research structures. I have shown how I work within traditional research distinctions and constructs while remaining open to possibilities outside of the technical workings of these distinctions. In the next part of the dissertation I turn to a more detailed exploration of enactivism and of ecological-complexity as the frameworks that guide this study.

6. THEORETICAL EXPLORATIONS: ECOLOGICAL COMPLEXITY

Theoretical frameworks such as semiotics and discursive psychology are being appropriated to study the complexities of mathematical cognition. However, there is need to incorporate dynamic and embodied aspects of cognition if post-structural theories are to go far enough. Studies on mathematical thinking could benefit from a broader framework that accounts for the role of the body, of context, of history, and of materials in cognition. Enactivism is a theoretical framework that attends to these dimensions. It recursively elaborates on frameworks, such as constructivism, which are prevalent in the mathematics education research. Recently enactivist researchers, such as Kieren and Simmt (2002), Simmt et al. (2003), Towers and Davis (2002), have drawn from complexity science to further understand mathematics learning. Complexity science has been adopted as an umbrella term to refer to theories with common themes of complexity, adaptation, recursion and emergence that allow for understanding of dynamic phenomena in ecological and non reductictionist ways. In these theories, at times referred to as *dynamic systems theories*, there is a rigorous attempt to transcend the values inherent in mechanistic world-views (Capra, 1996; Waldrop, 1992). Complexity researchers consider problems as systems problems (Capra, 1996). They question underlying assumptions such as the reductionistic mentality of modernism. Complexity theorists who study human cognition emphasize metaphors of learning which are centered on being and co-evolution. The study of human cognition as a complex phenomenon involves the study of a range of systems including the nervous system's activity, individual cognition, group intelligence, cultural evolution, global, social, economic and political systems and similar temporal and spatial systems.

I take complexity theory as a source of metaphors for investigating mathematical thinking. In a manner closely aligned to the enactivist sub-discourse in cognitive studies, I adopt metaphors that emerge from biological, ecological and geographical disciplines. My study therefore takes an ecological slant on complexity science. It is an ecological-complexity study.

Dissertation Landscape Forming



6.1 Complexity Research

As a meta-narrative, complexity *research* has unfolded from a synthesis of insights in specialized areas such as community and behavioral ecology, neuroscience, bio-mathematics, cybernetics, enactivism, socio-biology, non-Euclidean geometry, non-computationalist artificial intelligence and deep ecology. Its theories range from the mathematical theories of observation and chaos, through physical theories such as *far from equilibrium thermodynamics* to evolutional biological theories. Hence it is far from being a well-defined science; it has been criticized for over emphasizing theory but lacking practice (Rasch & Wolfe, 2000). It is also criticized for the possibility of its becoming a theory of everything. Also its central tenets such as organization and emergence might mean different things in its different sub-discourses. Post-structural, particularly critical, theorists might doubt whether complexity research, given its genesis in varied fields of the cold war era, have anything to offer to human sciences.

Complexity research is an interconnected organic (rather than a linear) meta-narrative. It appears many complexity theorists are aware of the shortcomings of grand narratives. Many metaphors and notions of complexity research are ecological and relational. Within the field of second-order cybernetics, for example, objective observations are problematized as we are reminded that our actions affect the world. This perspective demands reflexivity and ethical responsibility. Complexity theory is increasingly embraced as a promising theoretical orientation even in social science such as in organizational theory. It offers an understanding of the dynamism of highly complex adaptive systems ranging from bodily, family and urbanite to ecosystem dynamics. To consider cognition as an *organized* complex phenomenon affects educational practice and research in significant ways. (This will become obvious as the dissertation progresses.)

What research methodologies might resonate with ecological complexity? Adopting a complexity stance encourages the study of broader social, historical, cultural and political contexts that shape and are shaped by human thinking, without eclipsing the biological and psychological aspects. Johnson (2001) challenges research at this stage in history to seek more than hermeneutical interpretations. In my work, I use complexity to identify and create research settings, frame my observations, and create tools for observing mathematical thinking.

6.1.1 Complex Systems

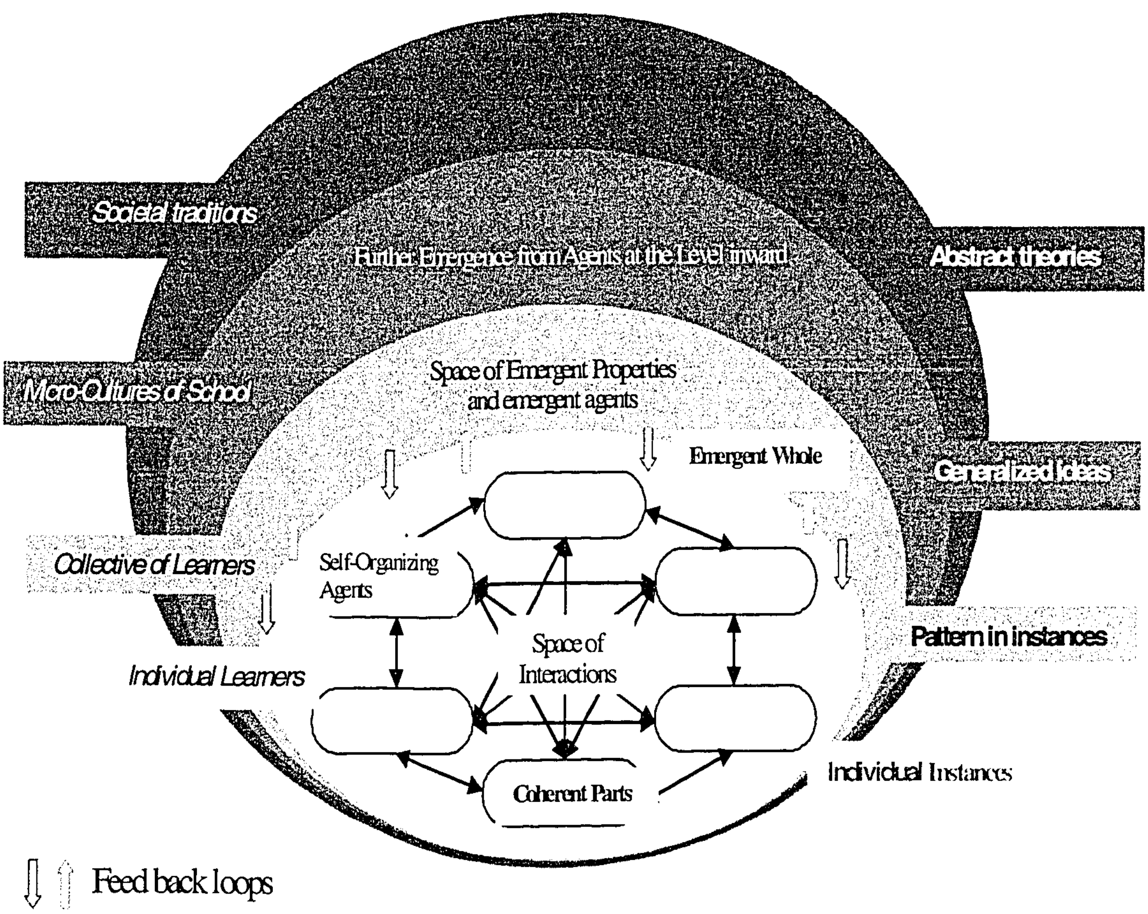
Complexity theorists concern themselves with systems that are: self-organizing (their organization changes as a result of their activity); adaptive (they learn from experience); and, co-evolving with other systems and with their environments.

Systems such as the immune system, bodily organs, and organisms are complex. They are spontaneous and adaptive; their output and behavior largely depend on their internal states (Waldrop, 1992). Individual humans, human communities and cultural varieties are dynamic and adaptive. Johnson (2001) defines *adaptive* as the quality of growing smarter over time. It is keeping fit in a changing environment: Adaptive systems "rarely settle in on a single, frozen shape; they form patterns in time" (p. 20). Adaptive systems are also more organic (B. Davis et al., 2000; Waldrop, 1992). They are composite systems with multiple, independent agents dynamically interacting in numerous ways (Johnson, 2001) and they possess a vitality that resists mechanical analyses. Indeed they are better understood in relation to other systems with which they interact and to their internal dynamics. Complex systems are self-organizing, *autonomous* systems which change their structure but maintain their organization and overall patterns of living. Whatever perturbations or input they experience, they in-form it according to their *internal coherences* rather than in correspondence to some external agent (Varela, 1987). Unique to complex systems is a *logic of emergence.* Multiple coherent parts following local rules result in discernible behavior at the level of a collective system. Observing complex systems one can note properties that lie on a scale above that of agents and parts. These properties arise from relationships among agents that interact in a great many ways (Capra, 1996). The whole or collective unity that emerges has a new environment of interaction (Johnson, 2001). For example, many organs and systems come together and from them emerge the human body; where many humans come together in interaction human collectives form. This three layered system displays a nested organization; nested organizations are common to complex systems.

As soon as a broader system arises local systems become related inner-level parts of an outer-level whole. Put differently, as soon as the emergent whole coheres, it turns back to nest and regulate the behavioral domain of its agents in intricate ways. Thompson and Varela (2001) call this agent-system causation. It is two-way feedback from nesting systems to nested agents and vice-versa. Other researchers have referred to it as local-to-global and global-to-local determination. Juarrero (1999) explains that the dynamics of emergent properties "serve as orderly context that structures the behavioral characteristics and activities of the parts" (p. 130). As soon as parts are embedded in a complex whole, they are unable to access some of the states that might have been available to them as independent unities. The good news is that they can access novel states that might not have been available to them before (Juarrero, 1999). This feature will be further discussed in the last section of this thesis as it is significant to rethinking mathematical thinking.

In Figure 9, I use nested ellipses to illustrate the agents and global wholes that participate in human learning. To the left of the figure, I illustrate the human collectives that emerge from individual learners and to the right the collectives that emerge from classroom experiences and instances. From the interactions with other agents at that level, an emergent system arises at yet another level, the darker nesting ellipse. Moving outwards from the inner white ellipse we may talk about conceptualizations, behaviors, experiences, interpretations, practices, communities, domains and traditions. This figure is similar to the Davis et al. (2000) model of nested bodies (see Figure 3 in Chapter 4).

Figure 9. The logic of emergence and coherence



As in the Simmt (2000) model, in my model I have highlighted interaction between agents. As well, I have illuminated how outer coherent layers spring forth from inner coherent layers. I have also emphasized how outer and inner layers, to use Luhman's (2002a, 2002b) language, *interpenetrate* each other (feedback loops).

Enactivist and complexivist researchers study cognition in its complexity without reducing it to simple processes or aggregates, without dichotomizing its aspects, and without proclaiming that it is primarily psychological. They emphasize the dynamic interrelatedness and self-similarity among the psychology of the individual, the behavior of the classroom collective, the body of mathematical ideas, and the socio-cultural aspects of school and society (shown by the nesting ellipses).

Another useful principle is that of *positive feedback*, in addition to the traditional principle of diminishing returns. Positive feedback might keep systems such as economies far from equilibrium. Positive feedback may at times propel a system onwards. Some complex systems are catastrophic as they may be driven to bifurcations into systems with totally new dynamics. N*on-linear dynamics* and *far from equilibrium theory* would explain why at times small changes in prior conditions might lead to huge differences as well as why some systems when perturbed are likely to settle on particular behaviors—*dynamical attractors*; yet after some threshold point they may break into unexpected behavior or disintegrate. In my view, notions of *emergence*, *feedback loops*, *degrees of autonomy*, *mutualistic* constraints and *nested organizational levels* are promising for studying human learning. For instance, they allow researchers to move beyond the debate over whether learning is psychological, social or institutional.

Using the principle of emergence, many human abilities are viewed as a property arising, at a higher organizational layer, from the interplay of bodily and environmental agents at an inner layer. These emergent properties such as thinking and perception provide boundary conditions for inner-level human dynamics. Because with the emergent whole novel states can be accessed, studying one variable at a time and in controlled settings and drawing from atomism and *laws of additivity* of elementary processes are adversely limited approaches. The understanding of emergent wholes that causally nest coherent agents at an inner level is a radical departure of complexity perspectives from behaviorist, mentalist and cognitivist perspectives, as well as from positivist studies.

6.1.2 Complex Systems in a Classroom

The mathematics education community has moved beyond exploring reflex behavior, computational mental process and isolated psychological attributes (see Chapter 2). Since most learning involves many factors interacting in varied ways, it cannot be understood by a reductionist process.

There is need to go further than acknowledging the multifaceted nature of learning: cognition is dynamic and hard to predict; learners are complex adaptive systems; in classrooms there are many coherent agents acting and interacting in multiple ways to produce emergent orders that in turn regulate and expand what is possible.

The individual psychological system is not the only system with a high degree of autonomy that can be distinguished in a classroom. In classrooms, individual cognition is nested in social cognition in the neighborhood of other complex systems and nested in larger societal and cultural bodies.

6.1.3 Collective Learning Agents

Individual students are autonomous learning agents, but as they continuously act and interact with each other dynamics of the collective (rather than an aggregate) arise. When students work in groups of two or more, complexity research posits that collective learning behaviour buds in ways that would not have been imaginable if students worked on their own. From simple rules and local issues—we are solving the task, what I know is valuable, you are good at drawing—in the interaction of students a complex organism assembles itself. Two, three or even twenty-seven or more students self-organize into a coherent whole—a student collective that has its own domain of interaction (Namukasa & Simmt, 2003). Patterns of collective human engagement such as social norms and culture in turn become feedback for the individual's habits, preferences, knowing patterns and ideas. With collectives and joint projects arising from individual students' actions and interactions, individual students find themselves behaving and thinking in ways that were not possible for them as individuals. Classroom collectives, in a recursive manner, are a prompt to individual students' learning (Kieren & Simmt, 2002; Namukasa & Simmt, 2003). Simmt et al. (2003) observe that it is important for collectives of students to emerge around the subject matter itself.

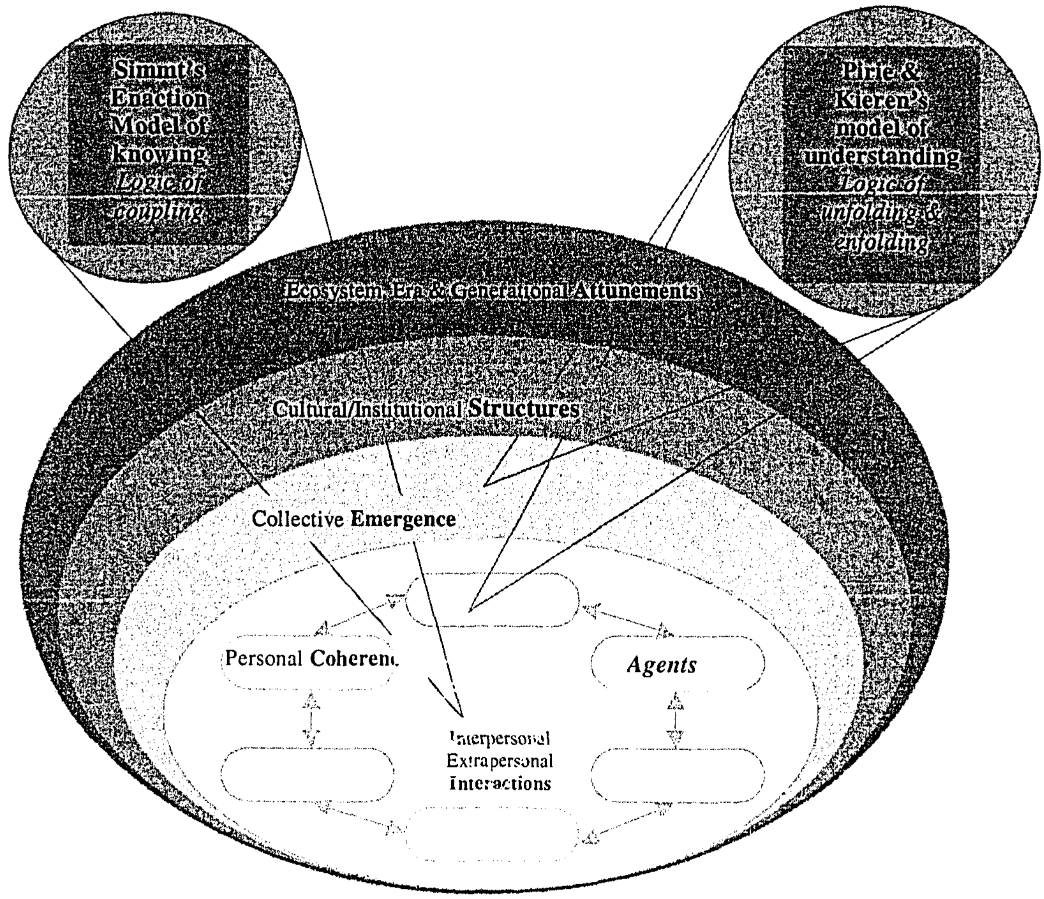
Collectives adapt and evolve, as well. Rather than evacuating individual student experiential and conceptual accounts, as suggested by some socio-cultural theorists, metaphors drawn from complexity research can be adopted to understand the behavior of collectivities in a classroom, in a different domain which embeds an individual's sense-making. Using a complexity frame, mathematics students in a classroom could be observed as a system of agents that are engaging in doing mathematics at both the individual and social level. Recurrent actions and interactions around classroom mathematics would then be valued as they allow students to transcend themselves to acquire new understandings that they might never have possessed individually or in previous environments. Parallel to individual and collective learning is the evolution of ideas and insights as evolving systems (B. Davis & Simmt, 2002; Namukasa & Simmt, 2003; Waldrop, 1992). For example, abstract ideas, ways of doing things and patterns of behaviors evolve from ordinary classroom activities (Davis & Simmt, 2002). Cognizing agents in classrooms are in turn nested in broader linguistic, material, social, cultural, political and ecological influences that they recursively compose. What might cognition that involves emergence of collective learning systems look like? How are we to view individual learners in relation to all these systems?

6.1.4 Individual Students

A student as an individual coherent unity is already a composite system (Maturana, 1988b). Learners are composed of many agents such as the neuro-motor, hormonal and sensorial, affective and emotive, volitional and intentional, and behavioral and semantic systems. Viewing individuals as complex systems that have embedded within themselves other learning systems illuminates how various divergent theories about mathematical thinking might interact as parts of whole. Individual humans are organic systems embodying sub-personal systems and embedded within supra-personal systems. Indeed many cognitive spaces intersect in an individual learner: the neuro-motor, perceptual and pre-conceptual behaviors, conceptual and pre-linguistic experiences, linguistic and social practices, and formal and rigorous language traditions. Learners ought to be seen as systems comprised of coherent agents and as agents who are building blocks for emergent-level behavior.

Complexity metaphors offer me an organizational tool for observing mathematical thinking that is embodied and embedded. In Chapter 4, I related two observational tools taken from Pirie and Kieren (1989) and Simmt (2000) that I adopted in the study. In Figure 10, I develop an encompassing model that juxtaposes these two models with the notion of emergent layers and extended structures of cognition.

Figure 10. Models for Observing Emergent Mathematical Thinking-in-Action



6.2 Enactivism

Notions of identity, interaction and meaning are at the heart of studying mathematical thinking. How autonomous are learners? What is the extent of the influence from their environments? What meanings do learners make of mathematics? Many mathematics educators have rephrased questions about mathematical thinking in terms of relevance and signification. Enactivists pay particular attention to living. They view human behavior as essentially biological and experiential. Varela and Maturana, the originators of enactivism, are considered to be among researchers who laid a foundation for the broader framework of complexity science. Maturana and Varela (1980) drew from neuro-physiology and biology. Varela (1991, 1999a) later drew from phenomenology to understand cognition as a condition for living. Three of their theoretical constructs are central to my study: *Autopoiesis*, *operational closure* and *bringing forth worlds.*

To Maturana and Varela living systems are autopoietic entities in the way that they are bounded as autonomous unities and they self-produce their components. To complement their earlier notion of autopoiesis, Varela (1987) develops the notion of closure as another condition for living. Because it relies essentially on internal dynamics to specify a specific mode of coherence, the organism is operationally and functionally (but not interactively and structurally) closed. Enactivists further emphasize that autopoietic unities, since they are operationally closed, do *bring forth a world that is of significance and relevance* to them.

Mason (1988) asserts that relevance is a property of an appropriate fit between the topic of study and the learner's structure, mathematical background, interest and context. It is relevance that stimulates involvement, he claims. Even though *significance* is often used as a synonym for relevance, there is more to it. Significance has, via *signification,* gained semiotic value as the relation between signifier and signified. It comes from the word *signify*. In enactivism, however, significance has more to do with potentialities arising from the interplay, the co-emergent causality between the organism and its environment. As I will show, significance has to do with foreshadowing and suggestiveness of matters-at-hand for learning. It is about motivation and interest that select what is attended to, based on experiences from the past and conjectures about the future. In a way, signification evokes the centrality of the interpreter, the learning organism as a *webbed* system. According to Varela (1991), in every event of interpretation there is a *surplus of signification,* the emergent regularity added by the whole matrix of the interpreter in relation to the interpreted. In an ecological complexity manner, significance points to a complex adaptive system generating a space of possibilities, a world relevant to its continued living. Enactivism explores signification in relation to living.

6.2.1 A World Brought Forth

Merleau-Ponty (1974) has differentiated between *environment* and *world*, a distinction that for Heidegger was between *universe* and world. Merleau-Ponty (1964) quoted Scheler, saying that "Human perception is directed to the world; animal perception is directed to an environment" (p. 40). Gadamer (1992/1975) pursues Scheler's distinction further by saying that all living beings have environments but "[m]an rises above the environment, ... he arises to the 'world' as he constitutes it linguistically" (Gadamer, 1992, p.443). Varela (1997, 1999a) asserts that the cognitive subject—whether human or non-human—is in both the environment and the world, but not in the same way. Clearly the mind of the collective of philosophers—the *world mind*—is trying its best to study this difference. What is so engaging about the distinction between a world that is relational to being and one that is not? Maturana (1988a, 1988b) and Varela (1991, 1999a, 1999b) ask what counts as an environment *for* any coherent unity?

In *Autopoiesis and Cognition,* Maturana and Varela (1980) make reference to the organism to talk about enacted worlds. Maturana's (1988a, 1988b) later work makes reference to a coherent *unity* while in Varela's (1997, 1999a, 1999b) later work, perhaps after the emergence of complexity discourse, he refers to *a living system.*

Maturana and Varela begin by considering a single-celled organism that interacts with its environment. For this cell, the environment in which it is embedded has a survival value and curious status, for instance, as a physical-chemical milieu (Varela, 1999a). Organisms as coherent autonomous forms are *structurally determined*; nothing external instructs or designs them, Maturana and Varela's autopoietic theory asserts. They are systems to which input is a marginal, triggering factor. The organism as a complex system has a perspective, a standpoint from which the *exterior* is one and from which perturbations are shaped. The organism's orientation to its *exterior* cannot be confused with the *material surroundings* as they appear to human observers (Varela, 1999a). The case is similar, albeit more complex, for multi-cellular organisms, Varela maintains. Whatever perturbations reach an organism from the environment they are in-formed according to the internal coherences of the multi-cellular system—food/not food, obstacle/penetrable etc. Such perturbations cannot be information and signals until they are, according to the organism's operational coherences, interpreted as a difference to which the system can respond by a structural change (Nørretranders, 1998). According to Bateson (1980), these changes that result in more changes, the differences that make a difference, are in essence what *information* is.

6.2.1.1 A System's World

To Maturana (1988b) and Varela (1991), the *environment* is what the observer distinguishes without reference to the unity and the medium that is identified when he/she distinguishes an autonomous entity to exist (Maturana, 1988b). Observation is key. The environment as the system interacts with and operates in is distinct from the medium. Varela (1999a) refers to the environment that the organism *knows* as the *environment for* the organism. In other phrasing, Merleau-Ponty (1964) refers to it as the *world* whereas Maturana (1988b) calls it the *environmental niche*; for Spencer-Brown (1972/1979) it is *a cut universe;* and for Rosch (1999a) it is a *categorized world.*

A system's world only exists in the mutual specification of the system. The niche changes on an ongoing basis. As the interactive domain of the unity changes, the unity undergoes the dynamics of structural change to compensate for the recurrent triggers (Maturana 1988b). In the end, the features of the world are inseparable from the history of the system's coupling. This has the following implications: in a way, the organism obscures its world; the organism-environment is a supra system; the organism in many ways extends through and beyond it (Juarrero, 1999; Osberg & Biesta, 2003). Other theorists have used the term *person-plus* to describe how the learner not only *extends* to the class collective but also to the physical environment. The environment for the system is not just a source of stimuli. It is more. The same breath that embeds an embodied system in its environment extends the system's structure into the world.

Although the behaviorists

> [r]esuscitated the role of the environment, [their] ideal was doomed from the start [because they] never quite embedded the agent in the environment. They just plunked the organism in the environment and assumed that when the appropriate stimulus occurs, boom! The organism would automatically respond. (Juarrero, 1999, p. 74-75)

When we conflate the organism's world, its niche with its environment as we observe it, like the behaviorists we miss completely the signification added by the organism's perspective (Varela, 1999a). Put differently, it is necessary that the students be permanently embedded in their worlds, in their extended structure as we explain their mathematical attentiveness. Viewing the student-learning environment as a larger complex system has wider implications. Had they considered a learner to be embedded, embodied and extended rather than just dropped into an environment, the logical and experimental behaviorists' focus on observable behavior would have been quite appropriate. Ecological-complexity prompts us to consider the environment as an *external structure* of a learner who has also an *internal structure*.

6.2.1.2 Organismic Relevance and Significance

The adaptive system in its perpetual interactions with the environment is constantly valuing or not valuing (or perceiving, interpreting or not) triggers from the environment to bring forth significance that is not pre-existent. This gives rise to constantly emerging regularity, dynamic meaning, and order and structure. This might be what Merleau-Ponty (1964) meant when he said the coupling is double.

> I have tried to re-establish the roots of mind in its body and in its world. The body has a double function—it models the world, but as an active body it turns back on the world to signify it. (p. 12)

In this way the system does not represent properties from the environment; it does not pay attention to objects in an exotic environment. Rather, it perceives as well as transforms the properties of its world from its perspective. Put differently a complex adaptive system's ontogenic pre-dispositions and phylogenetic dispositions are attuned to and they respond to its niche in ways that are intimately significant, be they of thermodynamic, chemical, biological, psychological, social, technological or cultural significance (Juarrero, 1999). The system as a functioning whole furnishes what is perceived. In this case, significance refers to the necessary emergence of a world that is proper *for* that particular organism. An encounter, an instruction or a representation acquires significance in the context of an entire organism-environment system.

A student's mathematical world, for instance, does not come completely structured into events with features and properties, definitions, proofs and procedures whose patterns and meaning he or she construes. Cognition and perception are the bringing forth of a world of action, perception and cognition. Maturana and Varela (1987/1992) assert that the world as seen from the *point of view* of the living system points scientific observers to aspects that would not have been relevant had it not been for the existence of the organism. This view of relevance occasions me to wonder about those aspects of students' mathematical thinking to which their mathematical worlds point us?

The *world for* the organism or system is "the particular way in which it has maintained a continuous history of interacting and coupling ... without disintegrating", Varela (1987, p. 52) explains. Through a history of changing its structure as it *couples* with its environment the organism in a manner contingent to the flow of interactions maintains its coherence within an ever-changing environment. The relentless and permanent engagement with what is lacking for the system's coherence becomes, from the observer side, the system's ongoing cognitive activity. The difference between the environment and the world "haunts the understanding of the living and of cognition" (Varela, 1991, p. 7). The living system, through its maintenance of coherence and its actions, shapes and changes its world, which changed world suddenly changes the space of the possible for the living system. Varela (1991, 1997) defines cognition as the constant bringing forth of signification. Herein lies a wider understanding of cognition as changing the world and as adaptation to that ever-changing world.

6.3 Cognition Extended

Cognition, in enactivism, is viewed as perceptually guided action; it is an act of specifying, of in-forming the relevant features of the environment (Varela, Thompson & Rosch, 1991). For any living system, including humans, cognition involves bringing forth a world that is of significance to living (Merleau-Ponty, 1964; Varela, 2000). The enactive view of cognition, in the spirit of educators such as John Dewey and Mary Boole, questions the stance that bases most learning on the doctrines of *instruction, training* and *telling*.[21] Cognition is wider than consciousness and habitual behavior.

In the ecological-complexity view such phenomena as thought and learning are construed not as solely individual-psychological events, but rather as part of the more inclusive phenomenon that is cognition (von Foerster, 1981). Cognition is described as an emergent property of a level of organization that is above the level of the internal dynamics of a system and its interactive dynamics (Maturana, 2000; Varela, 1979). For instance, in humans, cognitive properties arise from the interplay of brain, body and

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[21] In the late 1890s Mary Boole decried the practice of explaining rules and presenting ideas to the mind and *consciousness* of children who have not laid the basis of subconscious, non-conscious and unconscious knowledge in actual experience.

social and physical environment.

Human cognition is embodied in that it is shaped by the structure of our brains, our bodies and our everyday interactions. It is *co-emergent* in that it arises in activities (Maturana & Varela, 1987/1992; Varela, Thompson & Rosch, 1991, see also Kieren, Calvert, Reid & Simmt, 1995; Lakoff & Núñez, 2001; Núñez et al., 1999). Ecological-complexity theorists view cognition to *extend throughout and beyond the human body to encompass* to social and symbolic-technological levels and other emergent and neighboring structures. That humans *know with* and *through* technologies of *intelligence* and in collectives is more apparent in today's highly technological and globally networked civilization. What does it mean to say that a learner is embedded as well as *extended* in the world?

The exploration of the difference between the environment presumed by the observer and the world that the unity brings forth offers broader conception to perception, action and imagination. Learning is also redefined, from adapting to a static and independent environment, to co-adapting with a world that a dynamic learner structurally and socially enacts on an ongoing basis. Knowledge is also cast in functional, temporal and historical terms (Juarrero, 1999; Osberg & Biesta, 2003). Also, the histories of complex systems matter for they learn, adapt and evolve. Complex systems embody in their very structure the conditions under which they have coupled (evolutionarily and developmentally) with their environments (Juarrero, 1999). Experiences and interpretations emerge with a surplus (never a deficit) of signification.[22] The cognitive agent specifies its own domain of actions and problems to be solved (Varela, 1987;

[22] In a way emergence implies an extra, a bonus at yet an outer level than that where the interactions are taking place.

Varela et al., 1991). For example, in enacting mathematical worlds students pose problems and specify paths that must be trod to obtain solutions.

In this study, to embrace the extended, embodied and embedded nature of cognition evokes the following research orientations:

- Foreground the co-implicitness of the observer and the instruments of measurement. Take responsibility for our observations and actions.
- Observe students as systems interconnected in the web of learning. Study cognitive systems as dynamic wholes, or as networks of activity (Juarrero, 1999).
- Focus on individual students as nested systems within larger systems such as classrooms collectives. Use the notion of nestedness to study the phenomenon in question at the appropriate spatial configuration scale.
- View psychic systems as just one kind of system constrained by social, cultural and political systems while at the same time offering constraints to bodily systems. Use the notion of feedback loops and extended internal structures.
- Focus on patterns, stable structures and relationships rather than on isolated factors observable in students' mathematical thinking. Observe complexity.
- Reflect on the extent to which distinctions made are in the observer domain or the behavioral domain of the system. Admit the active role of the observer.
- Interrogate preconceptions about students' mathematical thinking to generate better metaphors for understanding and viable ways of acting. Observe the

systems engaged in observation.

6.3.1 Wider Knowing and Thinking

Education scholars who challenge the dominant view of factual knowing resonate with the enactivist notion of "knowing is doing". Gray et al. (1999) observe that mathematical thinking is *knowing-to-act* in a situation rather than doing routine mathematics accurately. Ernest (1999), and Mason and Spence (1999) differentiate knowing-to-act in the moment from other kinds of knowing, such as *knowing-about*. Mason and Spence maintain that knowing-to-act, *phronesis* in Aristotle's terminology, is dynamic and situated. Knowing that is inseparable from doing is Michael Polanyi's *tacit knowing*: a state of awareness and a *preparedness* to attend that enables people to act creatively. In enactivism, Varela (1992) refers to knowing-to-act as common sense, "[A] *readiness-to-hand*, a know-how based on lived experience and a vast number of cases, which entails an embodied history" (p. 252, italics mine). In hermeneutics, Gallagher (1992), after Aristotle, refers to it as *practical wisdom* that entails *technical knowing* (*techne*).

In light of discussions about knowing and thinking–in-action, the terms *thinking mathematically* and *mathematical knowing* might offer a better distinction than *mathematical understanding* and *mathematical knowledge*. Thinking mathematically exists only where students themselves through embodied coupling have brought forth mathematical worlds in which there is perpetual novelty. Mathematics educators have alluded to the fact that in learning mathematics students learn to operate in a mathematical world (Schoenfeld, 1992). However researchers are yet to explore how students enact these worlds of mathematical significance. In this study, I wonder about aspects of *wider* knowing that the ecological complexity view evokes. I also ponder instances when humans *act-to-know* rather than know-to-act or even when they *know-with* and *through* others than know-about.

6.4 Systems Views about Students' Mathematical Thinking

Learning and improved behaviors, Maturana (1988a, 1988b) maintains, could be commentaries made by observers who see an organism acting and interacting in ways that they could not before. According to Maturana (2001), the learner does not learn to do this or that. The learner changes in the continuous process of living together with others, with mathematical tools, and other learning circumstances. The learner couples such that he/she now can act in ways which he/she could not before. But the learner is also in a different world and his/her structures have changed (Maturana, 2001). A student does not learn to think mathematically; neither does he or she merely expand their cognitive repertoire and capacities; nor do they progress towards thinking about more mathematical knowledge. Thinking mathematically is all at once about *expanding behavioral* and *cognitive possibilities* as well as enacting unique worlds of significance (Davis, Sumara & Simmt, 2003; Kieren, 2000; Simmt, 2000). This is compatible with Rorty's (1982) pragmatic sensibility about "progress towards new possibilities for humanity" (p. 8). From the ecological complexity perspective, learning involves transforming the conditional probabilities of actions and interactions, shrinking some to zero while increasing some actions to one (Juarrero, 1999).

From this ecological and systems paradigm, a student's mathematical thinking can "be observed as co-emergent: codetermined by and codetermining the personal structural dynamics, the dynamics of the interpersonal and the environment, and the cultural conversations in which he or she exists" (Kieren, 2000, p. 232). The overriding metaphor for mathematical thinking I develop in this study is *bringing forth a mathematical world with a potential of signification. Bringing forth* captures the complexity sensibility better than the modern synonyms such *as develop* and *create.* Bringing forth akin to sprouting, leading out and springing forth points to the dance between novelty and habit, happenstance and expected, readiness-to-hand and technical knowing, as well as between the internal and external. To Kieren et al. (1995), after Maturana &Varela (1987/1992), thinking arises in action as the learner co-adapts with an ever-adapting world. That is to say, thinking or knowing is what is observed as the learner functions, as he/she is acting and being in a way that permits him/her to continue in existence in his/her ongoing unique world or to shape a new one (B. Davis, Sumara & Kieren, 1996; Varela, 1992).

Cognition encompasses the being of the thinker within whom the ideas co-emerge (Lakoff & Núñez, 2001; Núñez et al., 1999). Therefore, mathematical thinking from this point of view is best studied in the moment and in the context that it *co-emerges.* It is inseparable from the functional and embodied history of the student. As any adaptive system interacts, thinking and knowing are seen to emerge from the simple rules and actions of agents such as the neural and sensori-motor agents, and outward-inward from the nuances of community and culture. In the ecological and systems narrative, thinking is not necessarily an antecedent to action and actions are not the consequence of thinking: rather, as Kieren and Gordon-Calvert (1999) put it, thinking is in-action. Bruner (1996) would say that most of the time we implicitly act our way into thinking. Mathematical thinking does not only emerge solely from activity, interaction or participation. It is not simply about mental operations that students build as they reflect on their actions, nor does it just arise as the learners are initiated in mathematical practice. Rather, the cognizing agent, either an individual learner or a group of learners, enacts mathematical knowing from their ongoing interaction within a dynamic environment. It is a conflation of knowing, of actions, of identity, of agency, of interactions and of enactment of worlds, of these varieties that are unique to living cognitions (B. Davis, 1995; Varela, 1999a). Herein is a radical departure of ecological complexity studies from many studies: cognition is action and living. Thinking is not contemplation on a given set of ideas but rather an enactment of these concepts in actions such as speaking, gesturing, writing and interacting. The dynamic grounds of an agent's mathematical thinking is laid by her own structure (Butterworth, 1999; Nørretranders, 1998; Waldrop, 1992). Learning involves a natural drift as a cognizing system self-organizes. What is more, learning systems self-organize at a distributed level (Johnson, 2001) across emergent and embodied levels of signification.

In my writing mathematics thinking is construed as enacting a world that is relevant to the continued existence of the learner. With thinking, as with evolution, many multi-dimensional paths are potentially possible. The multi-threaded path that is laid down in walking, the landscape that shifts in response to an agent's steps is the one that had a higher conditional probability than others. The cognitive landscape laid in living is an expression of the particular kind of embodied history the system has lived in continuous tinkering and emergent constraints.

An *enactivist* approach provides an understanding of mathematical knowing as adequately adapted mathematical behavior (B. Davis et al., 1996; Kieren, 2000; Simmt, 1998). From the enactivist perspective, although thinking might include conscious imagination and reasoning it is more than this. It is webbing of formulated with unformulated thinking, as well as of logical with instinctive sense. From this perspective, mathematical thinking is studied as *all-at-once* biological, psychological, socio-cultural and formal. These domains cannot be collapsed into each other. However, this in itself presents a problem. How does one study everything, all at once? Perhaps we ought to study aspects of cognition that offer an interface of all domains? Does the study of the dynamics of students' attention cut through most layers of signification?

It should be noted that ecological-complexity perspectives have escaped the pitfall of precluding preceding perspectives by evoking notions of embodied, embedded and extended cognition. Like concentric ellipses, different theories of learning, just like Davis et al.'s (2000) knowing bodies, can be seen to recursively nest into each other. Although varied schools of thought study different aspects of mathematical thinking, each school of thought is legitimate; each helps educators to understand the nature of students' mathematical thinking, albeit on a different nesting scale. However ecological-complexity promises to focus on aspects that have largely been outside the focus of co-emergent, coherent and post-structural paradigms: (a) the collectives of learners that can be observed to think (b) other dynamic systems, such as abstract mathematical ideas, which bud as students continue to act and interact. (c) as well the ways by which students' thinking is braided outwards through the physical learning environments as well as inwards through their physiological and emotional states.

In my research mathematical thought is not only interpreted as a resultant, an effect of mathematical activities, but since it suddenly *turns back* to constrain the activities it is inseparable from the learning activities themselves (Dienes, 2003; Kieren, Pirie & Gordon-Calvert, 1999). Adequate mathematical behavior and interaction are aspects of thinking mathematically. It is the aim of complexivist studies to tease out the agency of actions, interactions, history, community and observer descriptions in enacting students' mathematical thinking.

6.4.1 Embodied Mathematical Action

As I explored in Chapter 3, most educational research and practice is tacitly influenced by the *representationists'* views of cognition. For example, the structuring of mathematical instruction into *exposition-example-practice* is based on the view that knowledge is cumulative internalization of facts (B. Davis et al., 2000). On the other hand, based on the view that knowledge is individually and actively constructed, the radical constructivists propose, structuring instruction by providing rich mathematical activities that might occasion re-organization in the students' conceptual structures (Steffe & Thompson, 2000). To social theorists who view knowledge as established practices, teaching involves initiating learners into the practices of a community. The view of knowledge as pre-given, as individually constructed or as communally established is inadequate to address the question of mathematical thinking-in-action. Kieren et al. (1996) observed that:

> At the time when there seem to be conflicting views on mathematical cognition between those which observe it as personally driven and those which observe it as externally driven; between individually based and socially based views; between cognition as fundamentally active or fundamentally receptive; ...it is important and perhaps necessary to seek and apply ways of thinking about cognition which are in the middle (p. 9).

In this middle-way view, mind and body, mathematical structures and learners, the individual and the collective exist in relation to each other (Maturana & Varela, 1992). Furthermore, knowing, doing and being are brought together as embodied action. To quote Kieren (1995), "[m]athematical cognition is seen as an activity fully determined by a person's structure" (p. 7). Mathematical knowing is about the dynamic co-emergence of the knower and known, of a fluid self and others in what Mason (1994) specifies as the *I-You* relation (B. Davis, 1995; Kieren et al., 1996; Kieren, 2000; Simmt, 1998, 2000). The students and teacher, along with their practices and mathematical traditions, all exist in relation to each other. Two quotations from Maturana and Varela (1987/1992) are worth noting. "[A]ll doing is knowing and all knowing is doing" (p. 26). "Cognition is effective action" (p. 244). In this view, school mathematics is neither pre-given (product) nor does it just arise with activity (process), but it is enacted as a world in which effective mathematical action is the main required condition (Varela, 1992; see also Kieren, Calvert, Reid & Simmt, 1995). Here Simmt's (2000) model for knowing in interaction is helpful (see Figure 4 in Chapter 4). This study, in line with further elaborations on Simmt's understanding (B. Davis & Simmt, 2002, Kieren & Simmt, 2002; Namukasa & Simmt, 2003), extends the model to focus on the knowing of a pair or a bigger collective of students. As persons continue living together they adopt both to their physical and social environments. As well the ideas themselves could be interactive agents co-specifying each others' environments and bumping into each other to enact hybrid and more general ideas.

Maturana (1987) says that if we want to know if someone knows about something we look for adequate conduct—patterns of appropriate action. It is more fruitful to point to something mathematical in a moment than to define characteristics of mathematical thinking once and for all. Nevertheless, mathematical thinking is not always identical to observable behaviour so that to enhance it we would seek to reproduce those behaviours. Rather, it resides in that which makes the action mathematically adequate within a mathematical landscape laid down and transformed in living. Rather than being an antecedent of adequate mathematical action, mathematical thinking inheres in students' actions, responses and living.

6.4.2 Enactivist Views about Mind

Thought processes are not exactly "in there". Mind is the collective, emergent capability that arises in the recursive interactions of systems with their environments and in their layered embedment in larger systems (Varela, 2000). From the perspective of complexity research, it is not only the person that is capable of properties of mind. At a micro-level, human organs and the immune system could be said to cognize, and at a macro-level, communities, cities and generations may also be observed to demonstrate emergent top-down control qualities (Johnson, 2001). The extent to which activities of mind pertain to the domain of descriptions and so are to a larger extent a commentary by an observer is an open question in this research.

Activities of mind such as thinking, perception and reflection in this dissertation are considered first and foremost to be a distinction made by the observer. As we will see in Chapter 9, we humans make such distinctions in order to converse about the coherences of our experiences of others, be they human or non-human systems. Our conversations about mathematical thinking, for instance, come to partially constitute what we will ever know as mathematical thinking (Bruner, 1996). Maturana (1988a, 1988b) and von Foerster (1981) maintain that what we describe of a phenomenon or object is a property of our description more than of the phenomenon. As observers we can comment on how students think mathematically when they are seen engaging in mathematical activities. In the Fraction Kit task, for instance, it appeared to me that students thought mathematically by handling the pieces in particular ways—stacking or covering.

With advancements in neuroscience and artificial intelligence, researchers such as Bruner (1996), Butterworth (1999) and Dehaene, et al. (1999) have begun to view the mathematical mind as constrained and enabled by evolutionary, biological, experiential and other micro factors, as well as grander ones. The cartography of disciplines informing research necessarily increases when the community views mathematical learning in ways developed in this chapter. This inevitably redefines the ways in which students' actions and interactions may be investigated.

6.5 Researching Eco-Complexity

Ecological and complexity perspectives—eco-complexity perspectives—have relevance to classroom research. In education research where, without exception, clinical research has prevailed, adopting metaphors from complexity theories faces challenges. Like certain researchers, I have taken complexity research, including its theories such as dynamic systems theory, as a *theory* and *image-constitutive metaphor* for classroom research and practice (Juarrero, 1999). Embracing complexity evokes evolutional, narrative, historical, interpretative and other context sensitive stances. Since complexity theory is a relatively new paradigm, my study is as well an engagement in a conversation regarding possible contributions and challenges of adopting complexity metaphors to investigate learning.

Most learning is an ongoing behavior that is context dependent, fluid, evolving and organic. This implies that, in a way, we may never predict or control learning. We can nevertheless understand it enough to make decisions and to act in ways that respect and occasion its complexity. We can observe learners' behavior, seeking to understand its dynamics. Most complex phenomena depict patterns, patterns of patterns that are not necessarily repeatable, but not radically different either. Since complex behaviors are not totally without regularities, they can be studied to understand local rules of lower agents, broader level qualities and focal level behaviors.

A study of a complex phenomenon may seek to explain the recurrent themes and to observe problems that transcend local cases. In education there are not many examples of empirical studies done with complexity research sensibilities. For purposes of exemplifying, I take lessons from Freeman's (1991) studies on the neuro-physiology of perception. While studying the physiology of perception, Freeman (1990) akin to Johnson (2001) maintains that an adequate understanding of particular phenomenon is only obtainable given the methods and metaphors proper at an appropriate temporal scale (milliseconds, seconds, hours, days etc.), level of description (cell, cell assemblies, brain regions, whole brain, etc.) and discourse. Freeman studies odor recognition and recall, for example, at the level of spatially as well as transiently integrated and extended patterns of activity in the nervous system. He maintains that a large-scale level of observation is more appropriate than the level of individual neurons, the genome or neuronal networks (Freeman, 1990, 1991, Freeman & Skarda, 1990). He draws metaphors from chaos theory that he uses as tools for understanding. As well he seeks to explore what his hypotheses on odor perception might mean for other kinds of perception as well as for cognition as a whole. Freeman too, akin to researchers such as Schoenfeld and Kieren in mathematics education, has on an ongoing basis revised his understanding of perception, all the time seeking newer and more useful functional and temporal tools and perspectives.

Maturana (1988a, 1988b) maintains that in everchanging worlds, there are only opinions. In the spirit of the discussions about objectivity, subjectivity and inter-subjectivity, Maturana refers to research findings as "objectivity-in-parenthesis", functional tools and opinions. A study informed by enactivism gives up the idea of facts or "objectivity-without-parenthesis". Observers are co-implicit in what they say. And as I will illustrate in Chapter 9 the objects of observation arise from continuous participation in particular communities of observers. Researchers can only make conjectures. They can only by induction learn about a different reality or enact a new reality. In an analogous manner, the education community may seek for and open viable possibilities in the hope that they will be able to act in ways that are viable and appropriate.

In my research, ecological and systems interpretations offer more useful tools for understanding students' mathematical thinking than mechanistic ones. I have illustrated this by interweaving in the discussion how systems' views are closely aligned with or different from the prevalent views. In Chapter 9, I return to many of the theorists referenced here as I expand my theoretical framework for understanding the dynamics of students' mathematical attentiveness. In the next two chapters I explore the insights I gained from empirical observations in my study.

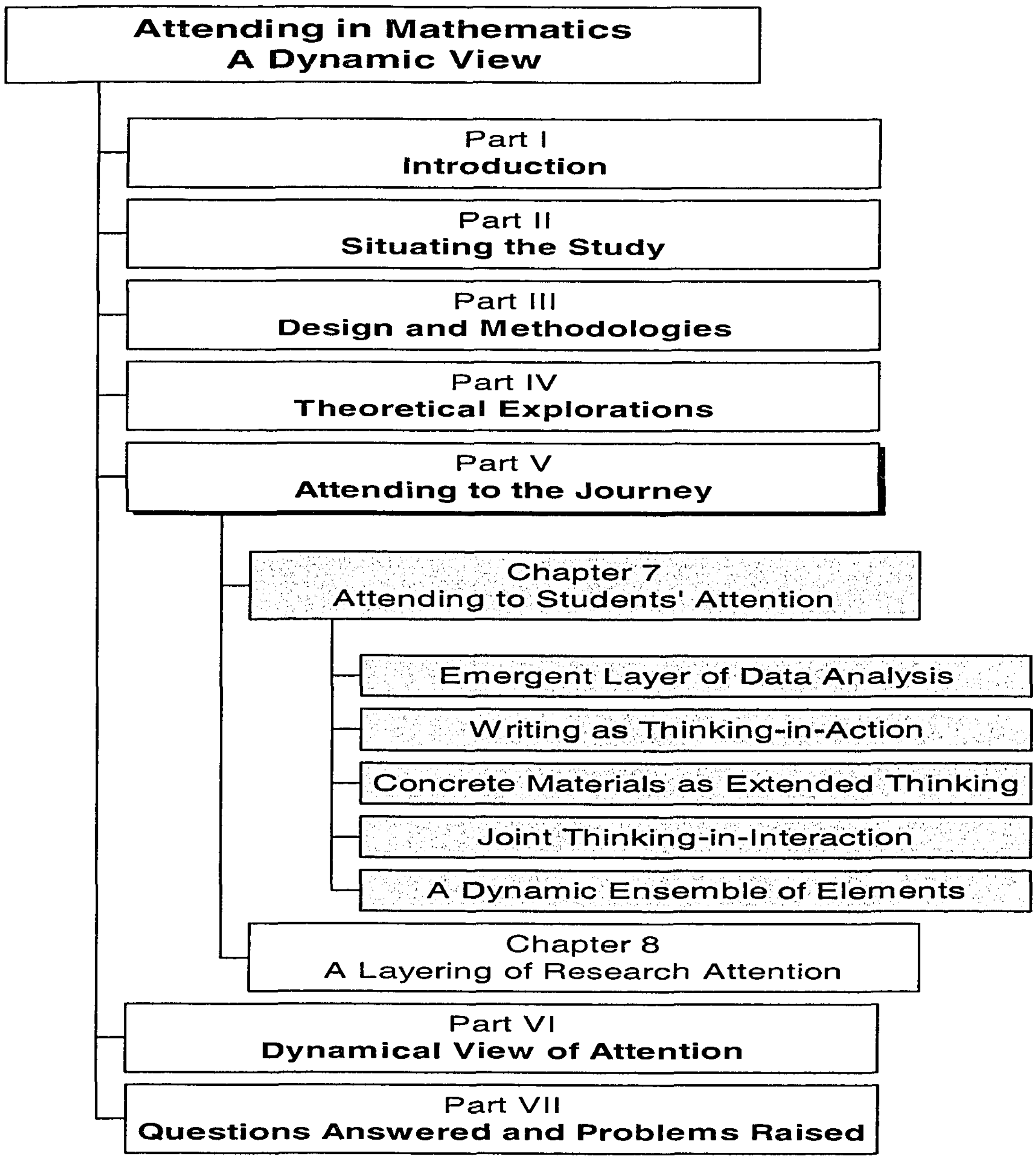
7. ATTENDING TO ATTENTION

At the onset of the study I asked about: what students attend to, when shifts in their attention occur, and how ways in which a deeper understanding of what they attend to offer insights into occasioning their mathematical thinking. With experience the questions were broadened to focus on: how students attend, and the dynamics of what they attend to. In what ways do students—not only as individual persons with conceptual structures, but also as organic, learning systems embodied with a neural system and a body, embedded in social collectives and cultures, and enabled by and extended to technological, symbolic and material environments—attend as they engage in mathematical tasks? In this part of the writing I map the path through which my understanding has co-evolved.

How and exactly when the questioning was reframed is not easy to answer. But as I reflect on the landscape formed during this study, a juxtaposition of factors has contributed to this growth: experiences from research sessions, interactions with other researchers and literature, engagement in preliminary analyses and writing, and further exploration of the complexivist and enactivist conceptual frameworks.

These factors can be summarized as embeddedness of the research and researcher in time and space. I elaborate only the last two of the three factors: preliminary inquiry and framework explorations. These two factors appear to be most influential and I can articulate them. To elaborate I offer an analysis of the dynamics of how students in the research sessions attended in this chapter. In Chapter 8, I explore how, through initial *readings* of the data, the questions evolved to more ecological and systems questions. Looking back at the changes in questions, I see that when my questions changed my attention had shifted; a new world had been enacted. For the questions I ask speak to what I am attending and how I attend as a researcher investigating students' mathematical thinking.

Dissertation Landscape Forming



7.1 An Emergent Layer of Data Analysis

It was during the preliminary analysis of some of my data collected early in the study that most of my tacit assumptions, some of which were positivistic, were challenged. For instance, by looking closely at the artefacts of Irene and Lillian's engagement with the Consecutive Terms (CT) prompt, presented in Chapter 4, I attempted to study what they attended to in that particular task, and to trace the shifts that occurred. Analysing the students' written work together with their utterances and gestures, their actions and the concrete materials they worked with I looked at the shifts in attention and the objects that they attended to at different times. Although this was fruitful, like many first-order observations, it was narrow and was likely to generate conclusions prototypical to Irene and Lillian, to particular tasks, as well as unreflected upon assumptions and contexts. To observe patterns and regularities in engagement, I began to analyse Irene and Lillian's engagement with the Consecutive Terms task in relation both to their engagement with other tasks, and to other students' engagement with similar tasks. I related these analyses to my experiences at other research sites and to related research. I also interrogated the observing systems, the observational tools and tacit assumptions at work. This back-and-forth analysis of parts in relation to other parts and to the emerging unity evoked a question about the dynamics of what students attended to and the ways by which they attended.

When I returned to data gathering, analysis and writing with the broader question of the dynamics of students' mathematical attentiveness, I observed the following influences:

- Writing is an agent in mathematical thinking.

- Actions with concrete materials, tools and media are extended thinking.
- Students talk and gesture their way into thinking-in-action.
- Actions and interactions with others are joint thinking-in-interaction.
- Thinking is a dynamic integration of co-related coherent forms.

As I share these insights I primarily discuss the role of writing. I briefly return to the other influences—concrete materials, communication, joint activity and integrated forms. Most of my discussion of the written and concrete materials is also applicable in these other areas. For this reason the reader will find the last three subsections much less detailed than the first two.

7.2 Writing as Thinking-in-action

In Chapter 4, I recounted that while analysing a session in which two boys—Tony and Ronald—engaged with the CT task, I noticed that they seemed to have written at intervals (see Figure 11 and 12). I dubbed each distinct phase in their writing an *episode*. This episodic writing was not unique to Tony and Ronald, nor was it unique to this task. The writing of other students on this and other tasks also appeared episodic (see Appendix E for more examples). For specific tasks there were episodes that were common among different students and some that were unique. I analysed Tony and Ronald's as well as Lillian and Irene's engagement with the CT task to ascertain when and why their writing shifted. This raised the questions: were there conceptual differences between the episodes? Did differences and shifts in writing necessarily correspond with shifts in what the students attended to and how they attended? Did shifts in writing point to something about the dynamics of students' attention?

Figure 11. Tony's written work

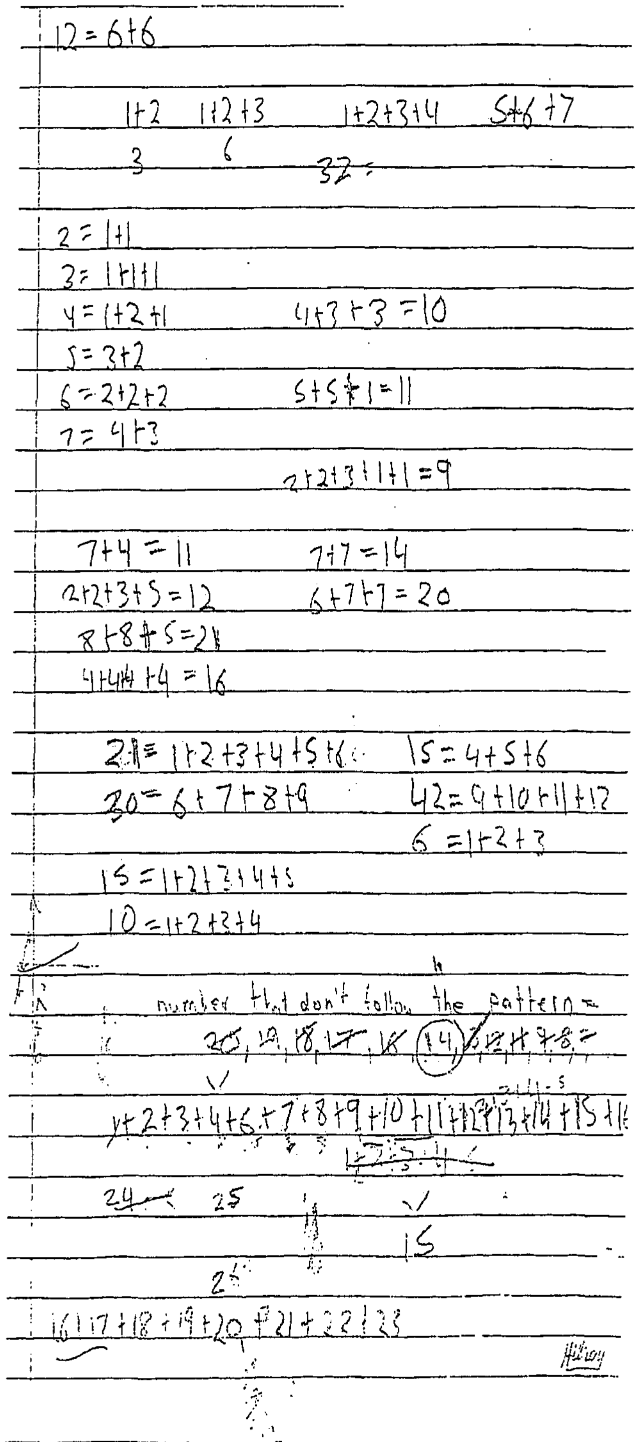
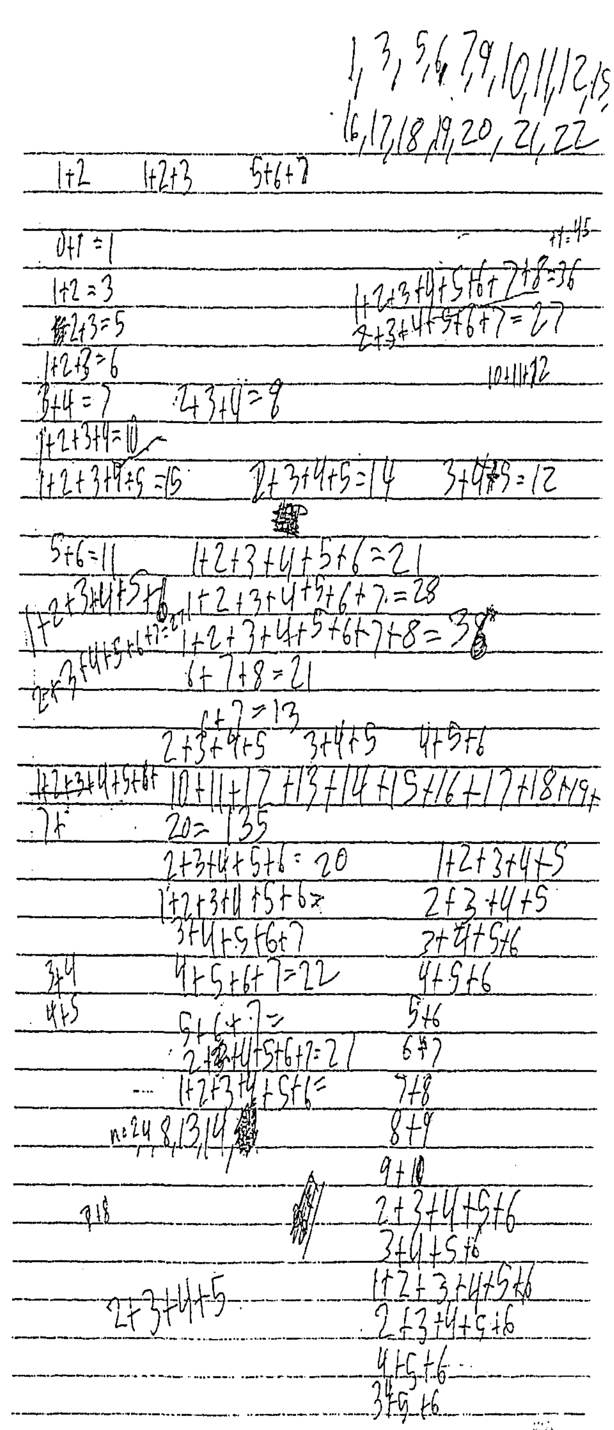


Figure 12. Ronald's written work



As I analysed students' written records over time and in light of other factors—such as use of concrete materials, pictorial representations, and engagement in speech and joint projects—the following questions emerged: (a) What is the role of written records in what students attend to? (b) When students record their work differently, what do they see and/or not see? For example, what is made salient when students horizontally (rather than vertically) list numbers? (c) What distinctions does a particular form of writing enact? And (d) how do writing activities expand the space of the possible?

Given that many students wrote in episodes, I reflected on the ways in which what and how they wrote constrained how they perceived, acted and thought. The analysis revealed that students' mathematical thoughts extend to include what they record and the writing activity itself. I use the word *extend* in the same ways as Juarrero (1999) when she refers to the world we enact as our external structure. She says that we extend, what other theorists call *leak*, into our environmental niches.

In the analysis that follows, I draw from Irene and Lillian's episodes because they are representative of other students' writing. I present other students' unique engagements whenever they are useful as illustrative cases. In Figure 13 below, I summarize Irene and Lillian's written work into 4 episodes. For contrast I have added an Episode from another pair of students, Sonia and Gertrude's work, Episode D_{ii}.

Figure 13. Irene and Lillian's written work ABCD

Episode A (Lillian writes)	**Episode B (Irene writes)** [a]	**Episode C (Lillian writes)**	**Episode D**
1-	3 = 1 + 2	1 + 2 = 3	D_i (Lillian writes)
2-- ~~1 + 1~~	5 = 2 + 3	1 + 2 + 3 = 6	16 = 1 + 2 + 3 + 4 + 5
3-- 1 + 2	6 = 1 + 2 + 3	1 + 2 + 3 + 4 = 10	= 2 + 3 + 4 + 5
4-- ~~2 + 2~~	9 = 2 + 3 + 4	1 + 2 + 3 + 4 + 5 = 15	= 3 + 4 + 5
5-- 2 + 3	10 = 1 + 2 + 3 + 4	1 + 2 + 3 + 4 + 5 + 6 = 21	4 + 5 + 6
6-- 1 + 2 + 3	11 = 5 + 6	1 + 2 + 3 + 4 + 5 + 6 + 7 = 28	5 + 6 + 7
~~7~~-	12	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36	6 + 7 + 8
~~8~~-- ~~2 + 3~~	12 = 3 + 4 + 5		7 + 8
9-- 2 + 3 + 4	~~14~~15 = 1 + 2 + 3 + 4 + 5	3, 6, 10, 15, 21, 28, 36 ...	8 + 9
10- 1 + 2 + 3 + 4	14 = 2 + 3 + 4 + 5		9 + 10
	16	2 + 3 = 5	
$3^{+3}, 5^{+1}, 6^{+1}, 9^{+1}, 10^{+1}$, 11		2 + 3 + 4 = 9	
...		2 + 3 + 4 + 5 = 14	
		2 + 3 + 4 + 5 + 6 = 20	*[D_{ii} (Sonia & Gertrude)*
		2 + 3 + 4 + 5 + 6 + 7 = 27	*2 consecutive number added*
		2 + 3 + 4 + 5 + 6 + 7 + 8 = 35	
		2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 44	*1 + 2 → 3*
			2 + 3 → 5
		5, 9, 14, 20, 27,35, 44 ...	*3 + 4 → 7*
			4 + 5 = 9
		3, 5, 6, 9, 10, 14, 15, 21, 27, 28, 35,	*5 + 6 = 11*
		36, 44, 45, 54,55 ...	*6 + 7 = 13*
			7 + 8 = 15
			8 + 9 = 17
			9 + 10 = 19
			10 + 11 = 21
			11 + 12 = 23
			12 + 13 = 25
			Conjecture: Odd numbers have the property]

Notes. I have organized the episodes that were evident from Irene and Lillian's work in at least four episodes. For more episodes see a copy of their work in Appendix F. Episode D_i is written work for Irene and Lillian that comes after they had noticed that the numbers {1, 2, 4, 8} that did not satisfy the property were unique. In the written work it appears after episode B but on the video recording it comes after episode C. Through deduction they added 16 onto this list. They then sought to verify whether it really did not have the CT property. Episode D_{ii} is written work from Sonia and Gertrude, who solved the task in a somewhat different manner by including a "paragraph" of 2, 3, 4, and 5 consecutive numbers added and recording their conjectures. Episode D_{ii} specifically illustrates two consecutive numbers added. They also included commentaries on their work in their writing.

Studying Episodes A, B and C in relation to each other, and with Episode D in the background, the noticeable differences and similarities among the episodes include the:

- Use of relation signs—dashes in A, equals sign in B and C, equals sign positioned differently in B and C
- Manner in which numbers are written—numbers are listed vertically elsewhere,

but horizontally in a list, at the end of A and C. Intervals are included in the list in A but not in C; some numbers are crossed out in A and B

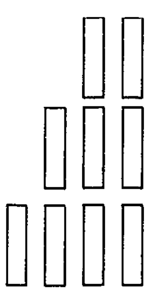
- Size and nature of numbers written—relatively higher numerical value and more numbers in C than in A and B.

Are these differences and similarities fundamental? As more than representations written records, it appears, open up spaces for learners to articulate and formulate the unformulable. Subtle variations in format of recording may call forth different thoughts, percepts and acts. Although I analyse students' *written* work here, I do not look at it in isolation. In a later section I briefly argue that students' actions with concrete materials and with tools are thinking-in-action as well. I also briefly discuss how other students' understanding was co-implicated in how students attended.

7.2.1 Writing Down: Representing Mental and Concrete Work

In the students' written excerpt that I have labelled Episode A as well as in concrete and *mental* [23] work that preceded it we note the following. Irene and Lillian appear to have attended to each number as a set of units, an amount of dominoes to be decomposed into consecutive number of dominoes or to be arranged in stairs as shown figure 13a on the left. For instance, they asked, "What will 9 be?" They answered this question at first by physically arranging 9 dominoes in a staircase of 2, 3 and 4 or by using trial and error computation—9 will be 1 + 2 + 3 + 4 that does not work,

Figure 13a.



Or
9-- 2 + 3 + 4

[23] My usage of the qualification *mental* is aligned to its use in curriculum documents when they talk about mental mathematics. In my research whenever students gave a response that was not immediate and automatic yet it did not involve any overt actions such as writing, talking and acting with material I label their work mental work. The term *mental* however is theoretically loaded, and therefore there is need for a more practical label, especially one that integrates brain and the rest of body, the individual and collective thinking, and internal and external structure.

2 + 3 + 4, that works! That is to say, in Episode A they split numbers as collections of units. Whenever they failed to split a number, as was the case for 2, 4, 7, 8, 13 and 14, they would cross it out of the list or omit it from the list in B. In Episodes A and B Irene and Lillian physically, perceptually and conceptually attended to the numbers more *concretely* than *symbolically*. Their actions and images with numbers—whether as number symbols or as a collection of dominoes were fairly specific and local. They did not yet appear to form any generalizations about numbers that have or do not have the property.

Episode A involved a systematic checking—by arranging concrete materials and/or computing sums—of the numbers from 1 to 10, approaching each *number separately*. At the end of Episode A their approach changed: At turn 53 of the transcript presented in Chapter 4 Lillian said, "So 10 does, I am going to list down [all] the numbers ... that can satisfy." She then listed 3, 5, 6, 9, 10 and 11 at the bottom of Episode A. "Then [what is] the pattern [in list 3, 5, 6, 9, 10, 11, ...]?" Irene asked Lillian (turn 59). At first they wondered about classification of numbers: "Odd and even", Lillian thought aloud. "Doesn't", Irene interrupted (turn, 72). "Doesn't really make up anything," (turn, 73) Lillian concluded as Irene turned to write the intervals +2, +1, +3, +1, +1. Lillian looked on the new list "3^{+3}, 5^{+1}, 6^{+1}, 9^{+1}, 10^{+1}, 11" and said, "Not much of a pattern." (turn 74)

After generating a reasonable amount of numbers that had the property the students began to look at these numbers as a group that might have a common pattern. As they sought to describe the *category* of numbers not crossed out, they examined whether it was a *set* of odd, even, triangular arrangements or prime numbers (turn 68, 69 & 165).

They did not yet examine pairs of the numbers in relation to each other (i.e. in two dimensions) nor did they examine numbers that did not have the property. Examining the group of numbers as a set of special numbers proved fruitless. At first they unconsciously and then later consciously began to look for what the pattern(s) among pairs could be. The list at the end of Episode A evidences a move towards explicitly treating the group of numbers as a special set, *a sequence.*[24] This is evidenced by how they listed the numbers *horizontally.* Perhaps it is the writing action of listing of numbers horizontally that evoked the consideration of the numbers as a sequence and not vice versa.

The girls' attention had drifted to explicitly attending to the set as a sequence—a class of numbers with a recursive rule that could generate further terms, a pattern of patterns. They attended to the additive intervals in the sequence {3, 5, 6, 9, 10, 11...} as a possible *rule*. It was after they failed to find a *common difference* that Lillian responded to the teacher, "The interval is not helping" (turn 77). Irene then suggested, "Why don't we list down the numbers in a *pattern*?" (turn 80) But what did she mean by this? She proceeded to *re-write* the numbers in the form shown in Episode B as Lillian closely followed. It might be argued that Irene had formed a conjecture that she was seeking to articulate or verify. I believe that her recording in Episode B, rather than being a step to test an already formed conjecture, potentially shifted both students' attention beyond attending to the summing or splitting itself. I explain.

What appears to be a neater re-write of Episode A, leaving out the numbers that did not have the property, in B is actually a mathematical *technology* that shifted the girls' attention toward attending to the *pattern in the strings* of consecutive numbers.

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[24] In two dimensions they examined the patterns in pairs such as (3, 5), (5, 6) and (6, 9). The pattern in Fibonacci sequence is in three dimensions, powers of two in one but only after looking at it in 2 dimensions.

When it came to writing 11 in Episode B (see figure insert 13b, to the left), without saying anything Irene paused, she and Lillian looked puzzled (turn 81). Each of them appeared to be doing some unspoken, ungestured and unwritten—*mental*—computations. After the silent but busy pose, they concluded,

Figure 13b.

3 = 1 + 2
5 = 2 + 3
6 = 1 + 2 + 3
9 = 2 + 3 + 4
10 = 1 + 2 + 3 + 4
11 = 5 + 6

12 = 3 + 4 + 5
~~14~~15 = 1 + 2 + 3 + 4 + 5
14 = 2 + 3 + 4 + 5
16

"No it [11] can't, so let's just go on with the list." (turn 82) If what they were checking was 11 indeed, in what ways did it make sense to say 11 could not? The shift from Episode A to Episode B was deeper than just a shift in recording. Because it involved recording only the numbers that satisfy the property, it illuminated different differences from A. The writing in B was a distinction-making act. Episode B thus triggered a focus on unique patterns.

The girls' attention had shifted from reflecting on the set of numbers as a group to viewing it as a sequence and later on to focusing on the strings in the sums. To Bateson (1980), from our random, spontaneous actions and interactions emerge regularities, habits, signs and patterns. Meaningful cognition and perception involves recognizing differences and patterns as well as perceiving unities. But these patterns and rules rather than being latent patterns awaiting recognition, to the cognising agent they arise when experiences are ordered, rhythms in action are sensed, regularities in successive interactions stabilize, and when perceptions become lawful and iconic signs progressively become indices and then symbols.[25] Repeated and recurrent actions and interactions are an opportunity to synthesize patterns and to order experiences. As von Foerster (2003) points out, from the restless dance of human actions, interactions and observations,

[25] *More* and *later* is differ for human cognitive agents.

irreversible, regular, lawful patterns, sign tokens and habits—*eigenvalues*—emerge. These patterns in turn guide cognitive acts.

With these observations one may wonder about the numbers such as 7 and 13 whose strings the girls took so long to come up with in Episodes A and B. It might be the case that the strings of these numbers did not harmonize with or should we say they were not illuminated by the regularities, what Pirie and Kieren (1989) call *image having*, that had surfaced.

7.2.2 Re-writing: Re-presenting Mathematical Work

"Bring the dominoes," Lillian asked Irene (turn 94). They then worked with the dominoes. Was this return to the materials a switch in actions or was it a more dynamic shift? When Irene and Lillian folded back to the dominoes after the writing in Episode B they seemed to be once again inspecting the *pictorials*—the geometry of the sums. But with the history of records in Episode B, the students enacted a different aspect in the arrangements—the pattern in the beginning digits. With a sense of urgency, in an aha utterance Irene said, "Wait. This one the one is there, this one the one is gone (not), the next one the ..." (turn 102) Beyond attending to whether or not the shapes were triangles; beyond attending to the four arrangements (of 3, 5, 6, & 9) as a group, they attended to the initial number in the arrangements. It appears the *sensibilities* and *regularities* cultivated during Episode B enabled them to stress a different aspect of the task. The records in Episode B as indexical, pointing signs directed attention.

In Episode A, the students' work was recorded in a fairly organized and systematic way. It seems that the form of recording in A, although mathematically adequate, had gone as far as it could in evoking what the pupils needed in order to engage further with the task. Indeed, the recording in Episode B *presented* an aspect of the arrangements that Episode A did not. As the girls attended jointly in B they seemed to have *formed an image* that 11 violated. It seems that because of this conjecture they took a long time to get an arrangement for 7 and 12. For 12 they tried, "1 plus 2 plus 3 plus 4 " (turn 84) and "5 plus 6 can't, 2 plus 3 is 5 … 5…" (turn 86) The image they had formed at that time did not appear to recognize 7 as 3 + 4 nor 12 as 3 + 4 + 5, nor could it create a possibility for them to comfortably record 11 as 5 + 6 the way it was stated on the question sheet.

Later in work that preceded transcript Excerpt 3 (see Chapter 4, p. 95), when Lillian suggested that 11 was an odd man out, Irene disagreed, saying, "I don't think so because 18 starts with 3." (turn 136) Irene had been able to see that in addition to having the property, 18 began with a 3. It appears all through Excerpt 1 and Excerpt 2 (p. 91 & 93) they attended to numbers that had the property without paying particular attention to the initial numbers in their string. Alternatively the girls might have noticed that 11 and 18 began with numbers other than 1 and 2. But because they did not fit some tacit expectation—the image they had formed—they delayed recording these *odd* strings. They might have thought that there must be a way of arranging 11 and 18 beginning with a 1 or 2. 7, 11, 12 and 13 did not fit the pattern 1-2-1-2 formed in Episode B.

After explicitly attending to the string of 18 as a string that began with a three, the girls then were able to accept 11 as 5 + 6 and to find a string for 12, 13 and 14 many turns later. "We have one for 12 actually", Lillian discovered as they were searching for one for 14, starting with a 3 (turn 152). The writing in Episode B focused the girls' attention on variation and order in the first digit—start with one, then with two, three and so on as you check a number. It originated this pattern. (Other students noticed this pattern at the onset.) Irene and Lillian then returned to recording numbers in what they, together with the teacher, called "start with one number, like we are starting with one and add on the next" (Lillian, turns 206-212). With the shift from Episode B to Episode C, they altered the *positioning of the equal sign.* Why did they change the sign to write $1 + 2 = 3$ in Episode C instead of writing $3 = 1 + 2$ as in A and B? What distinctions did this shift originate?

To the extent that a different form of recording allowed the students to see a different aspect of their work, writing was a *re-presentation* in the true sense of the word. That which they re-wrote re-presented—presented differently—ideas in a distinct form. Different forms of writing might not merely be varied re-writes of static ideas. Different formats are also a form of dynamic presentation that evokes distinct sensations, perceptions and observations. As I see it, mathematical writing is a source of metaphors, visualizations and ways of articulating. Writing assisted Irene and Lillian and the other students in grasping aspects that could easily have eluded them. Osberg and Biesta (2003) and von Glasersfeld (1992) have distinguished between *representations* and *re-presentations* to explore the co-implicit relation between the knower and the known, the pattern recogniser and the recognised patterns. Drawing from Davis (1994) and Gadamer's (1992) exploration of works of art, I add *presentation* as the third aspect of this distinction.

7.2.3 Writing as an Order of Signification: Presenting

Up until Episode A, it appears that the students' writing activity served mainly to keep records of their findings. In Episode B, however, the *writing activity* itself began to emerge as a problem-solving and problem-posing strategy. It became a source of more general images and began to present novel insights. In writing, the students were posing new problems and at the same time treading down paths to their solutions.

7.2.3.1. Same Signifier Varied Signified: A Case for the Equals Sign

Figure 13c. Episode D_i

```
16 = 1 + 2 + 3 + 4 + 5
   = 2 + 3 + 4 + 5
   = 3 + 4 + 5
     4 + 5 + 6
     5 + 6 + 7
     6 + 7 + 8
     7 + 8
     8 + 9
     9 + 10
```

In Episode A, Irene and Lillian used a dash as a relational symbol between a number and its string (i.e. 2-- ~~1+~~ 1 and 3-- 1 + 2). In Episode B they introduced the equal sign 3 = 1 + 2. In Episode C they altered its position 1 + 2 = 3. In D_i, as shown in Figure 13c, it gradually disappeared.

Other students, such as Sonia and Gertrude used an arrow, $1 + 2 \rightarrow 3$ in D_{ii}, before using the equal sign to relate the two sides. Some students had no sign at all for an episode where they *started with 1* to produce triangular numbers—1, 1+ 2, 1 + 2 + 3, …. Also students who tabulated their records did not use any relational sign. In one of Ronald's episodes, in which he used a "number line" he did not *need* the equals sign; instead he drew lines to mark the strings (see Figure 16). Are these differences in use of a symbol really insignificant?

Indeed, dashes are more appropriate signifiers for Episode A where all numbers, whether they had the property or not, were being recorded; an equal sign on the left when recording strings of consecutive terms; and on the right in C when recording numbers for given strings. An equal sign on the right was appropriate in Episode C, where, as Lillian put it, "When you start with a pattern, obviously there will be a sum, and that sum will be a number" (turn 211). It appears that in Episodes B and C the equals sign was used to indicate equivalence, rather than an operation of splitting or summing. Ginsburg and Seo

(1999) and Sfard (1991b) have distinguished two uses of the equals sign—symbol of *equivalence* and *a sign indicating a computation*. Researchers argue that for students to function in algebra they should be able to see the equals sign as signifying equivalence. At an advanced level equality might represent true statements, introduce new variables or other relations (Artigue, 1999). When it is not necessary to compute 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 to get 44 the strings of consecutive positive integers can appear first; 9 could be added to 35 from the previous string to get 44, that is using the iterative process. It is probable that the next steps in C will not require an equals sign nor will the string of integers, 2 + 3 + 4 +.... This happened in the writing that followed. Irene and Lillian just wrote 54 and 65 without recording their strings. Instead they added 10, then 11 onto 44 and 54 respectively. Tammy and Tanya replaced the *plus sign* with commas in latter episodes. The written records indicate many shifts during students' engagement.

The writing and thinking activity in Episode C involved *continuity* in generating the strings. One wonders about how this continuity focuses attention. What pattern and tones form, little by little, fainter and fainter from this writing activity? In the semiotic theoretic, what new and enlarged signs present themselves as a new signifier to represent the signified? The students' actions were more systematic and computationally elegant in C, even for numbers as high as 55. As the pupils worked on the sums in C Lillian could attend elsewhere. She managed to carry on a conversation with the teacher as she completed the sums. "We didn't think we were very right for the first one" (turn 215). Mason (2002), after Gattegno (in Cuisenaire & Gattegno, 1957), would say that part of the pupils' attention had usefully been released from doing to expressing and reflecting on the doing. This shift in awareness is important when doing mathematics.

7.2.3.2 Problem-solving Strategies

By Episode C, Irene and Lillian's attention had shifted toward generating numbers by summing consecutive natural numbers the way Ivy and Noela approached the task from the very beginning (they considered strings of numbers generated by 1, 2, 3, 4 and so on). The form of writing in Episode C, together with the attunements that Irene and Lillian had cultivated in prior engagement in A and B allowed them to generate strings for many numbers easily in a short time. As they continuously generated strings for bigger and bigger numbers their actions and attention reallocated toward noticing that the ongoing process, the set of numbers that have the property had an infinite size and was explicitly unique. Their attention then drifted to examining the numbers that did *not* have the property. Lillian began the aha utterance "But, one thing I'm getting to realize is that every number, every number is made up by something" (turn 284). Irene qualified, "Almost every … apart from one." (turns 288, 290) Lillian then added, "And two." They then added 4 and 8 to the list of numbers that could not be made up by strings of sums. From my experience with observing students' engagement on the CT task, the shift to focusing on numbers that do not satisfy the property is significant. Solving a problem or generating a proof by working with the opposite case is a general *problem solving strategy*. Many junior high students I observed in the study were not inclined to use this problem solving strategy prior to engaging in the CT task. This raises a question about particular tasks that have the potential to evoke specific mathematical behaviour: should students experience such prompts specifically because they foster particular kinds and means of mathematical thinking and attentiveness? As Tony and Ronald engaged in the CT task they debated whether or not by describing numbers that do not have the property they would have described the numbers that do (see Appendix B). In Chapter 1, while describing non-routine, good enough and variable-entry tasks, I instantiated that the Chessboard Squares (CB) task prompted systematic recording. Because such prompts on an occurring basis structure students' behaviour in ways that are mathematical they are *dynamically attracting* and mathematically *structuring* tasks. I will say more about dynamically structuring tasks in Chapter 11. For all of the students the writing activity was a site for transforming understanding.

7.2.3.3 Abstracting from the Concrete

In Episode A, as I explained, the approach to numbers involved local and non formalized, image based activities with the numbers as amounts. They split each amount into consecutive amounts. This activity could be seen as parallel to the *concrete stage* in the historical evolution of numbers. Historically, before numbers were conceived as abstract entities, people used them in ways tied to the concrete units they counted (Schmandt-Besserat, 1994).[26] While in Episode A the girls used *eight* as an adjective that modified dominoes, in Episode C the girls seemed to be attending to numbers in fairly general and *abstract* terms. ("Eight" was at once a number of dominoes, a noun and a manifold of things). Also in Episode C, the girls seemed to have been attending to number symbols as objects in themselves. "Forty-four" was more than an adjective (44 dominoes); it was the noun "forty-four", distilled from any amount it counts or measures. Mason (1989) and Sfard (2000b) refer to this as reification—a shift in attention toward *imaginary actions*.

The form of writing in Episode C was computationally and notationally economic and mathematically effective. It allowed the girls to realize that both sets of

[26] Remnants of concrete counting exist in our languages—twins for 2 babies or duet for 2 musicians (Schmandt-Besserat, 1999; Swertz, 1994).

numbers, those that had the property and those that did not, were infinite in size. However the latter grew faster and in a relational way, and was therefore easier to examine. They now focused on the set {2, 4, 8, 16, ...} of numbers without the property.

7.2.3.4 Writing as a Coherent Site of Interpretation

For Lillian and Irene's engagement, writing served a third purpose *presentation,* in addition to *representation* and *re-presentation.* What the students wrote down presented the numbers in a different way, a way that focused their attention in mathematically adequate ways. As students worked on a moment-to-moment basis, the product together with the activity of writing itself was an evocative site for mathematical interpretations. Besides being *re-presentations* of thoughts and actions they had already formulated, they, like works of art rather than photocopies, were *presentations* in the true sense of the word. As presentations, the written and writing opened up spaces for the girls to articulate the previously unformulated, and to formulate what was not possible at the beginning of the task. Written records are at once: (a) memory/records—a preservation of actions, and articulations of thoughts, (b) a reformulation of ideas and a re-organization and meta-stabilization of what we attend to, and (c) a space for writing ourselves into more novel and deep intuitions. I have summarized these three roles in Table 3.

It appears what students attend to *extended* to include the writing activity itself. Moreover, writing is a way of writing oneself into thinking mathematically. *Mathematical writing,* like telling phenomenological stories and giving examples, is likely to assist in grasping aspects what would otherwise be nonexistent. Paper and pencil work facilitates the invention of algorithms, and the noticing of patterns plus the enactment of mathematical properties. It presents yet-to-be conceived ideas. Writing technologies define and redefine what is mathematically articulate, elegant, visible, knowable, sensible and possible. This attending to students' written work enables the study of thinking to be extended to what students write and to their other activities.

Table 3. *Representations, Re-presentations and Presentations*

Writing to Record and Externalise	**Writing as a Space for Noticing**	**Writing as an Agent in the Emergence of Insight**
Episode A: Arranging and splitting amounts	Episode B: Noticing patterns in strings of numbers	Episode C: Experiencing recursion and continuity
Concrete numbers	Forming an image	Abstract numbers & objects
Individual numbers	Articulating conjectures	Elegance and economy
Group, set, sequence—interval	Focusing individual attention	Computational effectiveness
Having an image	Attending jointly	Problem solving strategy
		Insight on numbers that do not fit
Writing down	*Re-writing*	*Writing activity*
Representing work	*Re-presenting*	*Presenting insight*
Writing to avail	*Writing to bring into focus*	*Writing to call ourselves to think (mathematically)*
Extending memory	*Affording joint attention*	*Changing focus of attention*
Records and artefacts	*Technology of thinking*	*Agent in growth of ideas*

7.2.3.5 Writing as a Component in the Emergence of Mathematical Consciousness

Writing is one among many other technologies that have encouraged us to mathematize our worlds. The written, however, is likely to be less dynamic when done in isolation from the touched, the felt, the spoken, the made, the shared and the experienced. At the advent of writing, Plato wrote the following about written words, "[They] seem to talk to you as though they were intelligent, but if you ask them anything about what they say, from a desire to be instructed, they go on telling you just the same thing forever" (Phaedrus, 275d; quoted in Abram, 1996, p. 301). Indeed mathematical writing ought to be in interaction with other mathematical activities. In interaction with other agents, *scribbling* (as well as the surfaces and interfaces for writing and drawing, and possibly for clicking, selecting and dragging) calls forth the possible and thus transforms human understanding. Rotman (2000) recognizes writing as an order of signification (see Appendix B). Written signifiers and signified arise together (Sfard, 2001a). To use Bruner's (1996) phrasing, *we implicitly* scribble, sketch and click *ourselves into* ways of perceiving and being. Sumara (2002) would say writing is a site of interpretation and transformation. In complexity terms, writing mathematically is a coherent unity, an inner level agent. It is a *condition of possibility* in the evolution of mathematical concepts and strategies. Writing—whether ideographic, diagrammatic or dynamical—when coupled with other elements in mathematical activity expands the space of what is mathematically thinkable and comprehensible. It is an act of distinction.

Research on the role of writing in mathematics has recommended that students write about their mathematical experiences (see Mowatt, 1992 for a review). What I am discussing here is different. It is about the mathematical writing experience and artefact itself. Simmt (2000) maintains that records are among the *occasions* for thinking. They open up a space for rigorous *collective attention*. Schoenfeld (1992) observes that it makes a difference when students *look back* at their earlier written work as they solve problems. In Heidegger's terminology, organized records and the act of organizing and recording itself are among the many things that *call us to think*. They are *what hold us there* to enact mathematical worlds, to make mathematical distinctions.

Joseph (1991) and Kaput (2002), who study the evolution of mathematical technologies, note that in history different surfaces and forms of inscriptions opened up spaces for the development of insightful mathematical ideas. Butterworth (1999) traces the transition through different surfaces, forms and systems of writing. Kaput and Joseph remind us that permanent writing has moved from marks on cave walls and sand boards, to phonetic systems and inscriptions on clay tablets and papyrus rinds, to alphabetic-phonetic systems and placeholder systems on paper and in printing press, to algebra symbolism as well as visuo-graphic systems of the electronic interfaces. Technologies such as the printing press and dynamical environments allow for the development of conceptions that would have been next to impossible using surfaces such as animal hides and bones. With these distinction making acts of recording, interacting and thinking, the human consciousness can access novel states that were not previously available (Abram, 1996). Many inventions (including early and "primitive" ones) change "the nature of what it means to be human by changing conditions, culture and the societies in which" these technologies ensue (Kaput, 2002, p. 81). While new signifying technologies and acts of distinctions allow human consciousness to access novel states, they may also layer or even decimate states available before (Abram, 1996). Static and dynamic inscribing ought to be recognized as relational parts that change the nature of what it means to make mathematical sense.

Spencer-Brown (1972/1979) asserts, "[I]f a different surface is used, what is written on it, although identical in marking, may not be identical in meaning" (p. 86). Writing and the media of writing are agents, conditions in the evolution of abstract mathematical objects. Mathematical activities such as writing, talking and manipulating are agents of cognition at a level before that of mathematical activity. Moreover, as I argue in Chapter 10, writing systematically, carefully and promptly is a mathematically adequate action.

For now, I return to the other coherent agents in mathematical activity before looking at how the research question evolved with further exploration of the eco-complexity framework. The idea of dynamic representations, re-presentations and presentations is analogously applicable to students' use of concrete materials, as well as their spoken and gestured communication.

7.3 Concrete Materials as Extended Thinking

Vignette 3. ***Tony and Ronald's First Session: The Consecutive Terms Task***

Like Irene and Lillian, Tony and Ronald used manipulative materials on the CT task. But they turned to them after getting stuck with their pencil and paper work. Tony used materials to try out higher numerical value numbers whose strings had not been easily discovered by *mental computation*. He commented that working with the discs he was sure not to miss an arrangement that worked for a number.

Half way into the session, when the boys collected the numbers they had generated independently, 16 appeared on both the *yes* and *no* sides, as shown in their written work in Figure 14. How were the boys and the teacher going to explain this? Excerpt 4 is taken from a moment when Ronald noticed that 16 appeared on both sides.

Figure 14. An excerpt of Tony and Ronald's written work

No [does not have the property]	Yes [has the consecutive terms property]
2, 4, 8, 13, 14,	1, 3, 5, 6, 7, 9, 10, 11, 12, 15,
16, 24, 26, 32	**16**, 17, 18, 19, 20, 21, 11, 23
	14, 23, 31, 40, 50, 25, 13, 24

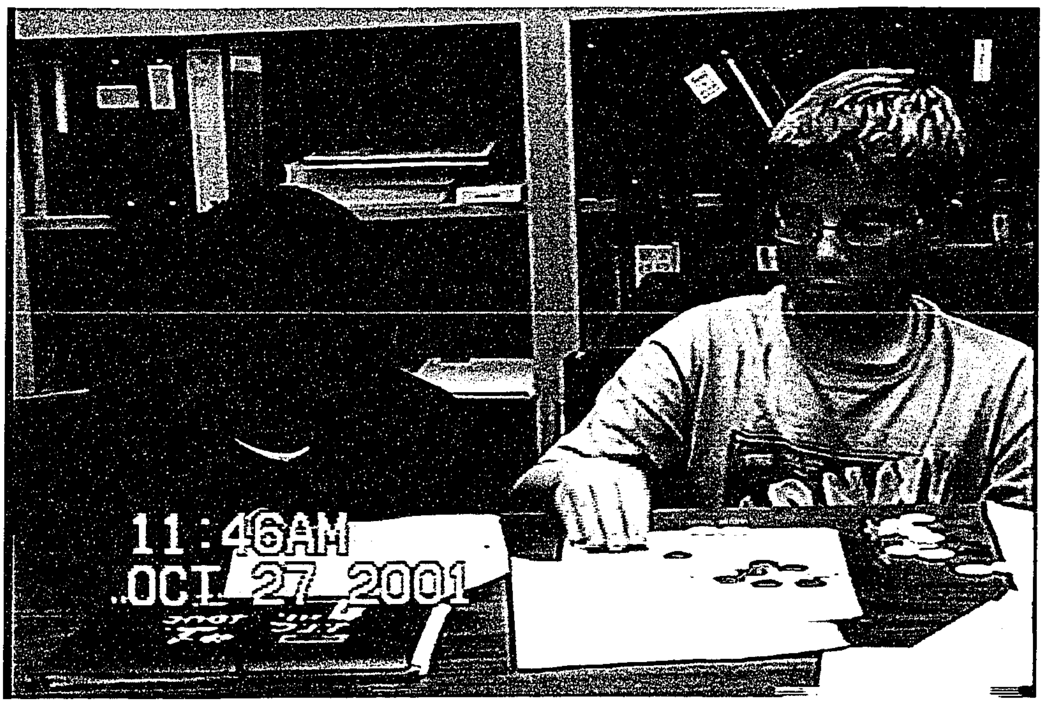
Notes: Ronald wrote first. He drew the chart headers on the shared paper. Tony wrote his numbers, drew a line below them and handed the paper to Ronald, who wrote below the line. Ronald only wrote numbers that were not on Tony's list.

EXCERPT 4

88	Ronald:	|16 appears on both sides!
89	Teacher:	Oh! You have 16 this side, so it is ... like it can?
90	Tony:	So 16 ... Oh yeah. *[Both turn to check on their papers for the string of consecutive terms they had generated for 16. There was none, not even on Tony's paper]*
91	Teacher:	Can we try to have 16 and see whether there are any ways to do 16
92	Tony:	No *[It does not have the property, Tony assures.]*

[Tony reaches for counters to test 16, Ronald looks on as Tony moves counters to find an arrangement for 16. The teacher asks Tony to explain how he uses the counters]

Picture 1. *Tony Moving the Counters to Check 16*

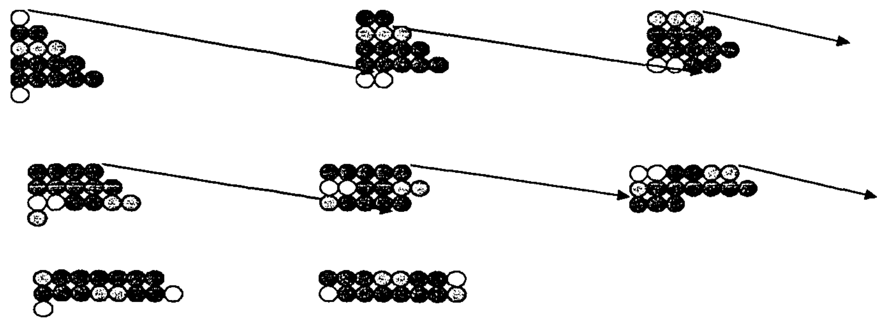


[Seven turns later]

100	Tony:	[...] Just I use them and ... if there are no left over chips then that will mean that there is a way. But if there are *[leftover counters]* ... then you know you did it the wrong way or there is no way.
101	Teacher:	Tony can you explain a little bit about that. I am lost. With the arrangement... First you said that that doesn't work, but ...|
102	Tony:	|There two together here, three, four then five, then these are the left over counters *[He illustrates, taking care to be slower, as the teacher watches]*
103	Ronald:	*[Ronald says something inaudible, as he looks on. Picture 1 shows Tony moving the counters.]*
104	Tony:	Four, five, six, two, ...six, two, ...seven, ...eight *[slowly re-arranging the discs, and is at times inaudible and at times fast]*. Sure I can't find a way. *[He concludes as Ronald reaches for paper, cancels 16 off the yes side of the list]*

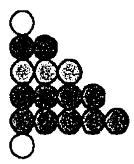
As Tony explained how he used the counters to check 16, in the moment I was not able to understand how he did so. He explained once more. But I was yet to get it. It was after watching the videotape and arranging the discs myself that I could follow his procedure. In Figure 15, I offer my reconstruction of his actions with the counters.

Figure 15. Tony's verification for 16 using manipulative materials[a]



[a] I have colour-coded the rows for a clear illustration. Tony used red and yellow counters. As in Picture 1 he did not pay attention to the colours; as well he did not keep the arrangements in a grid as I have done for clarification.

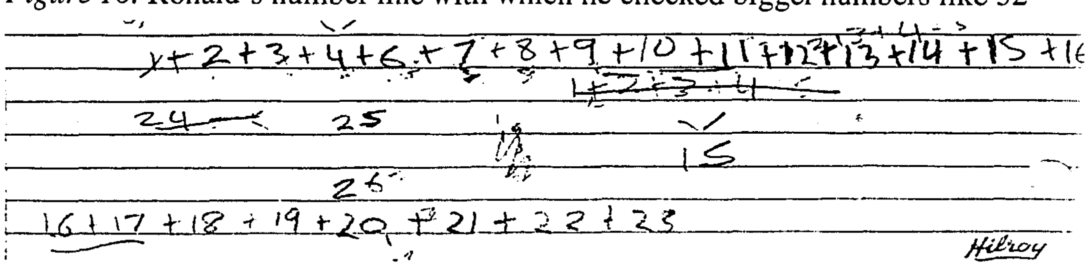
Figure 15a.
Step 1



Tony began by counting off 16 counters. He then began with the *triangular* arrangement (Figure 15a), with 1 as the first number in a string. This gave him a leftover of one disc. He stack it at the bottom to begin the next row. He then slid the top row and the next to the bottom successively to complete the bottom row as shown in Figure 16, step 2. In the Figure 15a on the left he would move the yellow disc to join the left over white disc, then the pink row, the orange and so on, all the time ensuring he had consecutive rows and a left over row. He would do this until there were no left over discs. None leftover would mean he had a string of consecutive terms. If there were left over discs at any n-1 step he continued, but not ad infinitum! (A student could go on moving counters or summing for an unnecessarily longer time.) Tony had a point at which he stopped. At this point he somehow knew that he had exhausted all the possibilities. After a few moves, in less than a minute, he was able to conclude, "Sure I can't find a way [for 16]" (turn 104). There were left overs. "If there are [left overs] then you know you did it the wrong way or there is no way." (turn 100) But how he knew that he had not missed a string for 16 begs further analysis. Were his actions with the counters organized, general and sophisticated enough to guarantee verification of numbers?

Tony and Ronald later checked 32, for which Ronald continued working with number symbols. Ronald wrote down a string of sums from 1 to 23 (see Figure 16). When I asked about how he had used the *number line*, he explained, "I began with, [pointing at what he had written] from one all the way to 23, if you keep on like adding them up. (turn 236). "I… Just like what he [Tony] is doing, except I wrote it. And I kept on adding it, and see if I can get 32, and if I could not get 32 I would cut off a lower number and add the next ones and then keep on doing it" (turn 251). But how were Ronald's actions with the number line similar or distinct from Tony's with the counters?

Figure 16. Ronald's number line with which he checked bigger numbers like 32

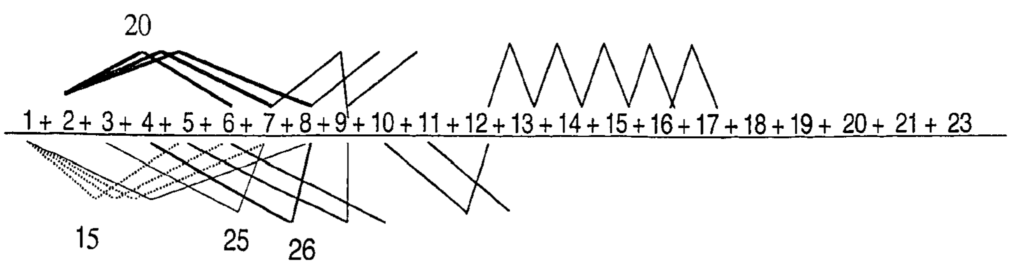


Mathematics education researchers have recently turned to closely examining particular aspects of students' activity. Some have reconceptualized the notion of thinking mathematically in terms of viewing, acting, using tools, attending or even *emotioning* mathematically (Davis, 1994; Mason, 1989; Schoenfeld, 1992; Sfard, 2000b; Simmt, 2000). By looking at students' written work, together with how students used manipulative materials, I attempt to understand how what students attend to in mathematical tasks extends to include their actions with manipulative materials, cultural artefacts, technological media and symbolic tools. Most times—especially if we do not pay close attention to students' activities—variations, details and shifts in students' attention pass unnoticed. At other times they are interpreted as styles, repetitions, obstacles or idiosyncrasies. In unfortunate cases varied ways of working might be assessed as simply wrong, and their wrongness may not be examined further by teachers. Yet, as my analysis demonstrates, variations, details and shifts in students' work could be taken as invitations to closely attend to students' mathematical worlds. According to Steffe and Thompson (2000), a teacher ought to engage with students' enacted mathematics so as to engender and sustain modifications in it.

7.3.1 Bodily Attending with Instruments

It was interesting how Tony, by checking with counters, knew for sure that he had verified that a number could not have the property. I asked, "Tony can you explain a little bit about that." (turn 101) To explain, he re-did the illustration, this time a little slower and counting along, but again I did not see how he knew when to stop for a number that did not have the property. For a number that has the property you stop when you get a string of consecutive numbers. But when do you stop for one that doesn't? In the next section I reflect on how one could kinaesthetically know when to stop checking.

Figure 16a. Ronald's verification of 16 and 32 using number symbols



A closer look at how Ronald did his written computations illustrates how systematic and procedural Tony might have been in moving the discs. Indeed, there are many ways in which Ronald's method, which I have elaborated upon in Figure 16a, is similar to Tony's. In fact it is highly sophisticated.

First, with counters, as is the case with number symbols, it appears important to be systematic when checking any number. As with Tony's use of counters, Ronald began with the triangular string, 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 (the bottom dashed arrows) which does not work for 32. He then would drop the 1 and begin with a 2 (the upper thicker arrows), then begin with a 3 and so on until beginning with 16 + 17 = 33, that does not work, 17 + 18 = 35, that doesn't either. After this he stopped at 17 computations in the least. But does one have to check all these sums. Tony using discs did not have to. Irene and Lillian using systematic records did not have to either.

Second, with the number of computations involved in checking a number as big as 32, it is important to come up with a method that minimises errors. Tony's use of counters was systematic and it ruled out errors. He began with 1 and worked through, each time beginning with the next integer. Note that when using counters one does not have to compute what 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 is. But Ronald using number symbols has to in order to check a sum. When Tony sees a left over he moves onto the next string by sliding the top row to the bottom

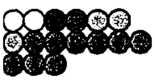
Third, to speculate about the point at which Tony knew he had exhausted the arrangements we look at work from other students. Tony and Ronald's ways of checking numbers were also similar to what Irene and Lillian did when they, in Episode D_i–Figure 13, were verifying that 16 did not have the property. See Insert 13d to the left.

Irene and Lillian stopped checking 16 after they had sums 17, 19. Tony using counters stopped before these last two sums after which he would have had 8 + 8. In a way Irene and Lillian in Figure 13d stopped checking a little too late; one pair of students said they always stopped checking a number when the first term was a little over a half the number tested, for 32 at 17 + 18. Was this what Irene and Lillian did to stop at 9 + 10 when checking 16? Did Tony have an explicit relation about when to stop checking? Probably it was after he had moved the leftover chip—the white one in Figure 15, I have thought. Or could it have been the turn shown in Insert 15b, the 6th step in figure 15? After this step, 6 + 7 + left over 3, a few things happen: The next step is 7 + 8 + left over 1. And the next could not begin with 8 and have 8 left over. For numbers like 4, 8, 16 and 32 you also get 2 equal rows (2 + 2, 4 + 4, 8 + 8, …) with no remainder. By the time Irene and Lillian were summing 8 + 9, and 9 + 10 they had exceeded the steps that could be accommodated using counters. The counters make it senseless to proceed beyond the steps 7 + 8 + 1 for 16. Before sliding the counters you realize that the next step will be 8 + 8. "Sure I can't find a way", Tony declared

Figure 13d. Lillian and Irene's verification of 16, Episode D

16 = 1 + 2 + 3 + 4 + 5
= 2 + 3 + 4 + 5
= 3 + 4 + 5
4 + 5 + 6
5 + 6 + 7
6 + 7 + 8
7 + 8
8 + 9
9 + 10

Figure 15b.
6th step



Watching Tony know when to stop illuminated a conceptual knowing that was embodied in his gestures, phrasings, timing and actions with the counters—you stop at the kind of arrangement that always feels right to stop at! "If there are [left over counters] ... then you know … there is no way" (turn 100). (With the towers of Hanoi problem some participants also demonstrated this bodily knowing and attending. While moving the discs, one said some movements felt right. Another said after a few movements he picked up the rhythm of optimal moves.) But would his bodily and material knowing work for all numbers, including those that satisfied the property? Had there been more time in the session, these would have been good avenues to pursue with the pupils. How was Tony's method distinct from Ronald's? Did their methods work for all numbers? I came up against the broader institutional issue of time and curricular structures that teachers have to work around. Even extra-curricular explorations had to be carried out within a specified length of time. I was left to ponder the questions on my own as a teacher-researcher and during sessions with other students.

7.3.2. Importance of Kinaesthetic Knowing as a Perceptive Element

Another pair of students, Eva and Faith approached the checking of a randomly selected number with actions derived from symbolic materials, as did Irene, Lillian and Ronald. They adopted a new symbol "|", probably for the remainder or perhaps to mark a point at which to stop while checking a particular number. Figure 17 is work with which they checked 19. After 4 steps they concluded that 19 did not have the property.

Figure 17. Eva and Faith checking Whether 19 satisfies the property

19→1+2+3+4+5+|6
2+3+4+5+|6
3+4+5+6
4+5+6

Eva and Faith's intuitive method did not work for many numbers, including 19. They stopped short of 9 + 10, which works for 19. Perhaps this foresight originated from the way they attended: They systematically increased the initial number in the sum, kept no records of leftovers and had an upper bound at 6. In a way not recording (or keeping, in terms of discs) the leftover 4 in the first sum as 19 →*1* + *2* + *3* + *4* + *5* + |*4*, but instead recording the next number 6 in the sum as in *19* →*1* + *2* + *3* + *4* + *5* + |*6*, appears to incline one to ignore the left over 4 as they focus on the upper bound. And this is not without repercussion. Inturn Eva and Faith were not checking 19, but discovering strings in the neighborhood of 19.

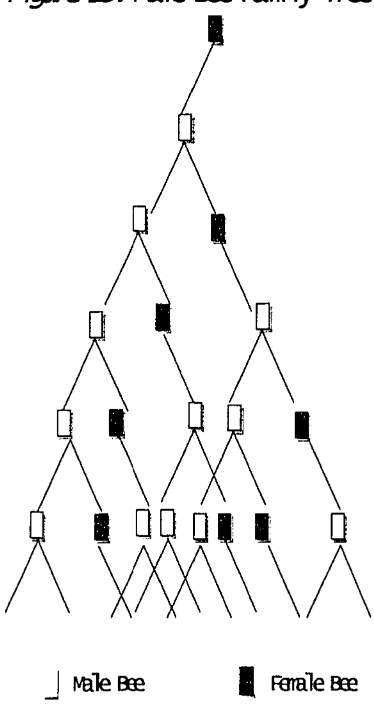
I speculate that had Tony been prompted to symbolically record the actions derived from concrete materials, in a way parallel to what Eva and Faith did, his method would have worked since it did not neglect the left over counters (see Figure 18). Also to the extent that the use of concrete materials ensured that the left over was *indicated* it was a computationally more effective method. Working with discs highlighted the importance of the ever-changing left over strings as shown in Figure 18—when you adjust the first term in the string the left over changes but the amount 19 stays fixed. Actions with concrete materials provide perceptive clues to what is important to mark and notate. It offers ground for significant regularities and patterns to be illuminated. It too is a distinction making act. But is it safe to speculate that Tony's method articulated in a symbolic manner would have allowed him to get an arrangement for 19? Or would he, like Eva and Faith, have stopped checking short of 9+10 for some other reason? Rather, how are these significant questions? Or should the question be about how we can help students to articulate their bodily knowing and attending?

Figure 18. Symbolic records Tony and Ronald might have used

19→1+2+3+4+5+|4
2+3+4+5+|5
3+4+5+6+|1
4+5+6+|4
5+6+7+|1
6+7+|6
7+8+|4
8+9+|2
9+10

For classroom teachers observing students working, it is important to ponder and ask about how students know what they know, Ball (2002), and Simmt, Davis, Gordon and Towers (2003) observe. Ball (2002) observes that among the mathematical issues that teachers have to face in a classroom is the need to verify whether students' *invented* methods work for particular domains of numbers or for all domains. Simmt et al. (2003) observe that such circumstances require that a mathematics teacher be curious about classroom mathematics. Simply using manipulative materials differently calls for a teacher's close listening and interest in the *mathematical enactments* of students. It would have helped if I had encouraged Tony to express and reflect on his bodily knowing. Nonetheless, even pondering on Tony's actions expands the space of the possible for me as a teacher. I can appreciate that Tony was working with discs in a more than informal way. His actions appeared to have gone beyond local manipulations of the discs.

Figure 19. Male Bee Family Tree

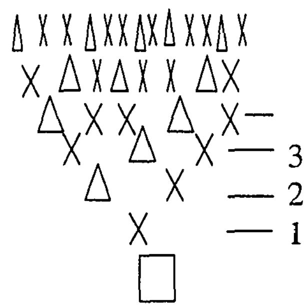


7.3.3 Thinking and Attending With Materials

Tony used the discrete manipulative materials in ways that were distinct from how Irene and Lillian used them when they worked with dominoes. They used them at the beginning of the task, then folded back to them when they were stuck. Tony used them after working with symbols and it is discernible that he was engaged in fairly sophisticated and general mathematical activity with the materials. Witness how systematically he was in checking a number and how he took care of the remainders as well as attending to when to stop.

The students' use of materials—whether in demonstrating, acting out, or verifying an idea—are inextricably tied to the dynamics of their mathematical attentiveness. Different forms of manipulative materials (or their symbolic records) might not be merely varied representations of static ideas. Different actions with concrete materials are a form of re-presentation and presentation; they are aesthetic texts that have potential to present a particular aspect of a concept and problem to a learner. Let us revisit the concrete tree diagram in Appendix E that student teachers used to solve the Bee Genealogy (BG) task. Actions with the tree diagram made of two coloured cubes and toothpicks appeared to have made the Fibonacci sequence more salient. As a dynamic sign, it aided the student teachers who worked with it to solve the task in less time. Had they used a less illuminating representation or not used one at all the situation would have been different. Although not predictable in detail, in my view, every kind of representation as a node in the network of mathematical distinctions evokes specific patterns of oral, gestured, silent and written actions.

Figure 20. Ivy and Neole used checks and triangles to draw male Bee genealogy



Concrete or pictorial representations students used varied for the BG task. Tammy and Tanya used two coloured counters before they drew a genealogy tree. Lillian and Irene used dominoes before they tabulated their results. Tony and Ronald also used a tree diagram for the BG task. And other students used stick people pictures, yet some did not use materials, drawings or tabulate their results at all. Ivy and Neola used a shaded circle and a blank one before they used checks and triangles, as shown in Figure 20, to the left. As they completed drawing the tree, Ivy and Neola concurrently tabulated the results under the columns first generation, females and males. Had they completed the tree and tabulated the results consecutively, I conjecture that this would not have been a trivial variation. Each of these workings might trigger, sustain and structure down stream mathematical behaviours in distinct ways.

Pimm (1995) reviews the role of manipulative materials. Many educators now are open to the use of materials in mathematics learning. However there might not be consensus on how central the materials and artefacts are to learning. What is their role? Like any reform recommendation, there is likelihood that students' actions with concrete materials will be trivialized to making mathematics fun, colourful, tactile oriented and so on. To some researchers concrete materials are representations and illustrations of abstract ideas; to others they aid visualization, yet to others they are helpful for students to learn harder concepts or for weaker learners. Less is said about how different it is to know a concept with particular physical materials. In the BG and Pirates Aboard tasks students did not only use materials to represent, they also *subjectively* put themselves in the place of the materials that respectively represented the bees and sailors. Papert (1986), and Rasmussen and Nemirovsky (2003) have noted that kinaesthetic, emotive, linguistic and semantic relations that students develop as they puppeteer instrument extend students' senses, emotions, motives and knowing. When we focus on the role of touching and moving materials just as mere manipulations, we neglect to highlight that these are not independent of the manipulator's bodily orientation and actions. They are not independent of expressions, spontaneous gestures as well as linguistic expressions perceptions and thoughts. Gordon-Calvert (2001) maintains that we do not manipulate concrete materials; our bodies become unified in action with them. In the eco-complexity understanding where mathematical properties are considered to be emergent properties, bodily actions—both spontaneous and cultural—and use of materials are central to mathematical thinking.

B. Davis (1997) proposes that concrete materials, in addition to being illustrative cases, are sources of metaphors and ways of articulating the *verbally indescribable*. Manipulative, symbolic, cultural and technological interfaces are central in deepening our intuitions. They participate in how we order, pattern and regularize our mathematical interactions. We implicitly manipulate concrete materials into our own mathematical thinking. For distributed cognitivists as with eco-complexity theorists there is nothing surprising about these assertions: thinking extends to include the tools we use, the media we work in, and concrete materials and surfaces we work with (Hutchins, 1995; Wertsch, Tulviste & Hangstrom, 1993). To Juarrero (1999) our knowing is looped through the external structures including the physical and social environments. Materials of *intelligence* and distinction making artefacts change the probabilities of our conceptual possibilities.

Students' actions with manipulative objects as well as semi- and symbolic representations provide grounds for the emergent. They are coherent sites for mathematical interpretation that are correlated with other component parts by the unities that emerge in the form of mathematical insights. Subtle variations in the use of concrete materials, as witnessed in the Fraction Kit activity discussed in Chapter 5, gradually present different thoughts. The surfacing of a distinct way of handling the kit points to the distinctions in worlds enacted by students. Indeed, mathematical worlds are of an emergent order. This outward order *hems in* the order of tools used. Students *know-with* artefacts, interfaces and instruments.

7.3.4 Mathematical Thinking: An Emergent Unity Surrounding Mathematical Activity

Students, when engaged in rich mathematical activity together with their teachers, might suddenly find themselves conjecturing, thinking, reasoning and visualizing new patterns. This emergent mathematical behaviour, in eco-complexity terms, is far more than the sum of concrete materials, students' interactions and written work. As an emergent order it surrounds that of the written and the manipulated. Often the emergent is sudden—a *phase transition,* to borrow a word from catastrophe theory—and at other times it is a gradual progression—an unfolding. At all state and operation transitions, what Piaget would refer to as *accommodation* and *equilibration*, isolated patterns, hunches and noticings coalesce into sudden insights and broadened students' conceptual understanding (Capra, 1996; Johnson, 2001). These insights, many of which arise from regularities, lawful linkages and habits in ongoing actions and interactions, suddenly change the students' attentive landscape.

When I began the study, I observed students' actions with materials and what they wrote down, made or said as *articulations* and representations of their thinking. However, my focus has drifted to the dynamics of students' attention—how they act themselves while attending to mathematical objects. I now analyse activities as more than illustrative cases. They are not mere visualizations or externalisation, nor are they solely triggers for recall of already existent ideas. To consider them as growth of potentially existing ideas is teleological. Mathematical activity is an agent that inclines students to think mathematically. The inner-level agents of mathematical thinking discussed in this section *hold students there*, in an expanded *space* to pose mathematical questions, to lay down mathematical worlds tread to solve mathematical problems, and recalibrate their mathematical attentiveness spaces.

7.4 Speaking and Gesturing our Way into Thinking

At the moment when looking for the interval in the sequence of numbers was not helping, Irene said to Lillian, "Why don't we list down the numbers in the pattern?" What followed this utterance appears to be a radically different and fruitful focus of attention. They recorded only the numbers that satisfied the CT property in Episode B and generated patterns in Episode C. Simmt (2000) asserts that words are all-at-once recording, thinking and conversational tools. We talk and gesture our way into thinking both in a gradual and radical manner. Moreover our articulations are agents in the thinking of the people with whom we interact. Speech, mood, expressions and gestures are also conditions that facilitate mathematical thinking. Radford (2002) refers to them as means of objectification. They are marks that generate meaning

7.5 Joint Thinking-in-Interaction

What Irene attended to appeared to be thoroughly intertwined with Lillian's understanding, and vice versa. When they checked 11 in Episode B, they appeared to be acting in accord, without one of them having to signal what they were silently computing. In some of these instances of joint acting and thinking the girls appeared to be collectively engaging. Even their spontaneous bodily actions such as gestures, expressions, postures and mood seemed to be synchronized. Seamless and swift shifts in attention of a pair of students might be taken as evidence of joint attention. Kieran (2001) notes that these moments when one of the interlocutors enters the world of thought of the other in the moment of action and interaction are important in mathematical thinking.

During such moments interlocutors are caught up in the emergent collective's behaviour. For Bruner (1986), there is a "loan of consciousness" in the course of such moments—moments of thinking-in-interaction, especially when the interaction is between an adult and a child or an expert and a novice. In neurological and second-order cybernetic terms working jointly is likely to be a source of energy-rich matter and enabling constraints for individual students (Newberg et al., 2001; Simmt 2000; von Foerster, 1981). As we saw in Chapter 6, once in place, the dynamics of the collective as a unity influence the behavioural characteristics of individuals. And a collective of students can attend in ways that each of the students could not.

Even though each individual human being is organizationally closed and distinct at the matter level, at the interaction and dynamics level we are contextually, thermodynamically and operationally open systems. Individual student's attending is in mutual causation with the collective mathematical attending of the groups that span them. Self-organizing systems in perpetual interaction live on the available energy, order, structuring processes, context-sensitive constraints, patterns and social habits from the neighbouring and embedding bodies. It takes an agent, nonetheless, with certain structural properties, sensibilities and history—a certain gradient—to take in socio-cultural energy and to generate order from noise, as it does for the perception of patterns.

In some pairs of students, moments of joint attention were limited. Kieran (2001) has referred to such pairs as *non-mutually productive pairs* because of the asymmetries evident in their participation. To Davis and Simmt (2003) the individuals must have much in common yet they must also have enough differences to keep the collective from disbanding. The pedagogical question would then be, "How do teachers enhance mutually productive collectives in the classroom?" Davis and Simmt (2003), and Davis, Sumara and Simmt (2003) have discussed this topic in detail. In this research I find the social psychology distinction between a community and a collective (Moscovici, 2001) quite telling. For Blumer (1969), human groups or societies exist in action either as individuals or as collectives.[27] When more than one individual is jointly participating in an immediate action such as solving a problem, there is potential for a collective project to arise around the collaborative project. Otherwise the group of people is an aggregate of people living and working together, a community not immediately and repeatedly coordinating and ordering actions toward the tasks-at-hand. Joint thinking-in action is not always a good thing, as non-mathematical ways of attending that were once accessible may become inaccessible. Under what circumstances would the emergence of a collective be at the expense of individual student's mathematical thinking? In which kind of collectives does mathematical thinking sprout? These become the questions as we seek to create conditions for students to attend in mathematical ways.

7.6 Thinking-in-action as a Dynamic Ensemble of Elements

Students who completed mathematical tasks for this study attended not to a single item but to a chain of relationships. In discussing the agency of the writing activity, I illustrated how cognition involves a search for patterns, connections and relationships. When analysing students' activity it becomes apparent that students'

[27] In my work, to avoid pairing students only at the community level I had to provide only one set of materials, including writing materials. When I recruited the Ugandan students as a group from the same class, I asked the students to suggest people they thought they would work better with. Usually these groups worked well as collectives, but when they did not a random re-organization of the groups or pairs many times rescued the students who had become unhelpfully dependent or domineering. Most importantly, it helped in the research sessions to provide a task that was likely to, on an on going basis, trigger the involvement of both students in a pair. In the sessions a few pairs of students divided the labour, some worked in parallel, but many worked fully collaboratively.

attention wanders across many relations and drifts on a milliseconds timescale.

Irene and Lillian's, and Tony and Ronald's written work has provided a focal space for articulating and gaining insight into what I consider as the *dynamics* of what students attend to and how they attend. Other students who worked with the same tasks seemed to share in these dynamics. However, there were differences in how fast and keen students wrote, for example. There were also differences in how students were systematic and organized in their writing and in using concrete materials. Some behaviour was prototypical to particular mathematical tasks and others to particular students. Irene and Lillian's engagement with the BG task, a task they engaged with during their second session, seemed to share some dynamics of what they attended to in the CT task. During the BG task, they began by using the manipulative materials to represent the bees, and then moved on to pictorial representation as they drew a generation tree when the dominoes they were using ran out. After finding the male ancestors in a few generations and after realizing that the generation tree was growing so fast and that it was becoming laborious and prone to error, at the fifth generation they set out to look for a pattern. They, however, did not fold back to using manipulative materials again during this task. Other students did not proceed in this order, from materials, to pictures, to writing, and to pattern noticing.

Some pairs of students were not keen to use manipulative materials; a few students, such as Norah and Rose, needed prompting to represent the bees concretely or pictorially. A few students were not keen on recording systematically, yet some students, like Tammy and Tanya and Ivy and Neola, drew the genealogy tree for all the twelve generations, with some errors.

With such varied observations I continue to wonder broadly about the dynamics of what students attended to in mathematical tasks. For instance, what dynamics, if any, were prototypical to particular tasks and which ones are prototypical to secondary school students? At the meta-level of my attending to the dynamics of students' attentiveness, what were the patterns that became salient for me as I observed and analysed more sessions?

A second layer of analysis emerged after the preliminary analysis and writing. It was a layer supported by further exploration of the principles of complex adaptive systems. In a study of dynamic systems it makes more sense to study regularities even among differences, the patterns of patterns formed in time rather than to hold onto particular shapes as if adaptive systems could ever settle in static equilibrium. When observed at the appropriate temporal and spatial scale with appropriate tools, meta-stabilities are discernible. One such meta stability that I began to attend to is the presence of coherent forms that themselves are subagents in the emergences of mathematical thinking. Neither written records and activity, concrete materials, utterances and bodily gestures, and joint projects are mathematical thinking. Yet put them together in the appropriate way, let them successfully interact, and dynamical behaviour, multi-stable regularities emerge. I conjecture that these regularities are what the mathematics educator calls mathematical thinking.

Whatever task, session, or aspect I analysed it became a node in the complex web that included theoretical understandings. In the next chapter I explore how the research question evolved with further exploration of the theoretical framework. It is a chapter about how my attention shifted during the research.

8. A LAYERING OF RESEARCH ATTENTION

As I analyzed the data, interacted with other researchers about it and did further readings in eco-complexity research, my orientation to the research questions drifted. My question about the specifics of what students attend to was reframed. As I demonstrated in Chapter 7, I began to ask questions about the dynamics of students' mathematical attentiveness. Through successive research sessions and analyses the research focus extended beyond looking at: the structural aspects of the mathematical task—what do students attend to? The psychological aspects of children—what mental mechanism do students attend with? My personal experiences—what do I attend to and how?

As I sought to make my observations and analyses coherent and as I gained a deeper understanding of cognition as embodied action, embedded action and extended action[28], it made more sense to ask *how* or *in what ways* rather than *what* and *when* questions. In particular, eco-complexity views about human perception and observation provoked me to ask interpretive questions: In what ways do secondary school students as embodied, embedded and extended systems attend as they engage in mathematical tasks? In what ways do students attend in mathematically adequate ways?

In this chapter, I elucidate how further exploration of the theoretical framework induced a re-framing of the question. I make an effort to relate this multi-threaded path to the historical development of research on mathematics learning. It appears to me that what appeared on the historical plane and time scale is somewhat parallel to how my understanding has drifted during the study. I have organized this section around the six questions I have asked at various points in this study:

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[28] All actions are extended to the learner-environment unity.

I. What is there to be attended to?

II. What psychological structures underlie mathematical thinking?

III. What do I attend to and how do I attend?

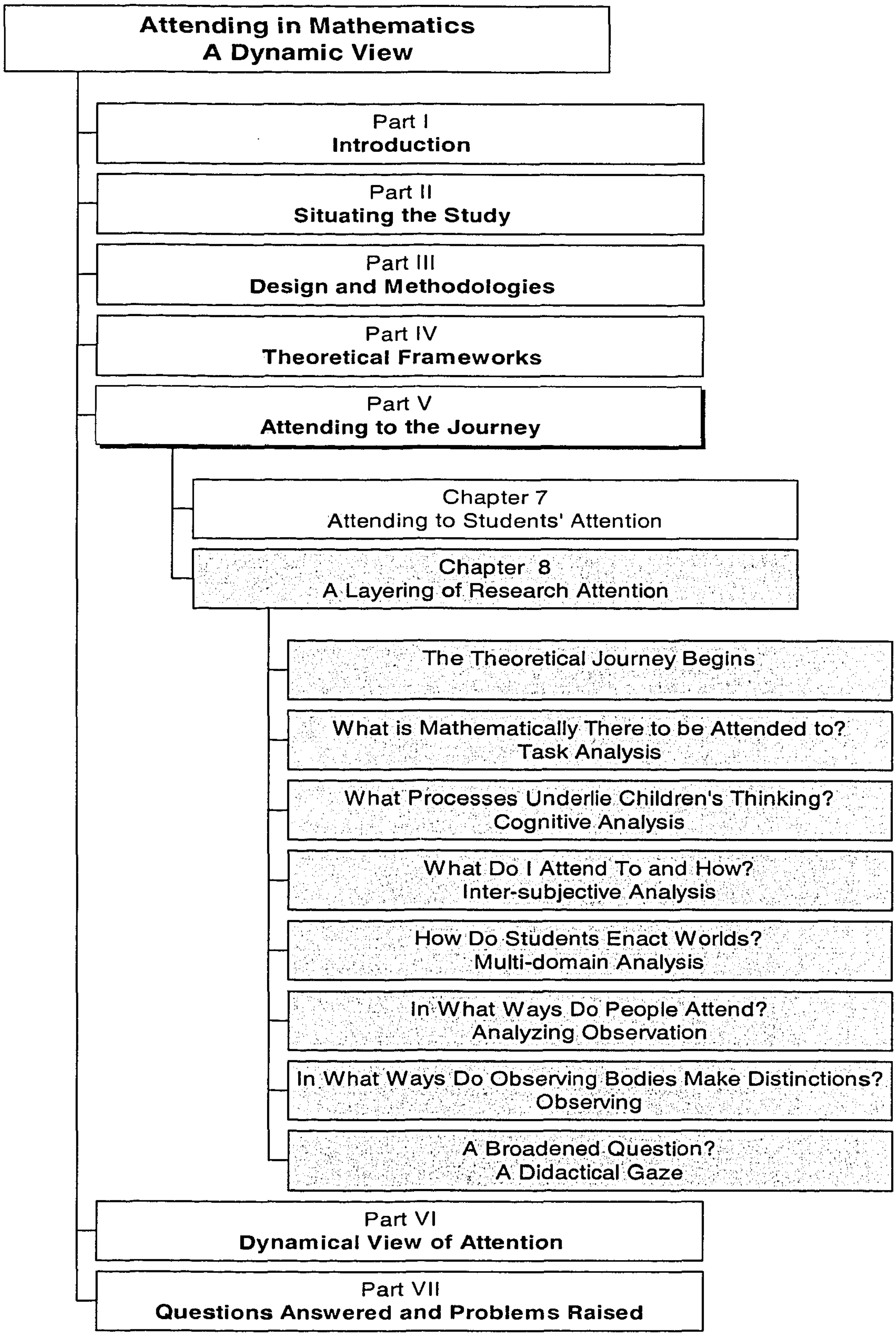
IV. How do students enact what they attend to?

V. In what ways do students attend?

VI. Which observing systems are at work when students attend?

Metaphors and shapes adopted from eco-complexity research influenced my thinking. Old questions were answered or reframed and new ones arose. In particular, the theory of distinction and observation informed my later analyses. In the last section of this chapter I introduce the theories of distinction. These theories have the potential to evoke a listening, sympathetic and participatory stance on the part of a teacher and educator.

Dissertation Landscape Forming



8.1 My Theoretical Journey

As a schoolteacher faced with the challenge of teaching mathematics better, especially to some *struggling* students, I used to wonder what needed to be emphasized for each topic. Most of my explorations involved reading collections of textbooks from which I would, to the extent that it was possible, make comprehensive notes. The bulk of the notes and explanations, as well as the examples and exercises I offered to high school students grew considerably. Yet this did not appear to help the struggling learners to think mathematically. It was always challenging and disappointing to see a few students who had excelled at their junior high school mathematics losing interest in mathematics while in my high school classes. In my teaching, the question remained: How can I occasion mathematical thinking in students for all topics and at all times?

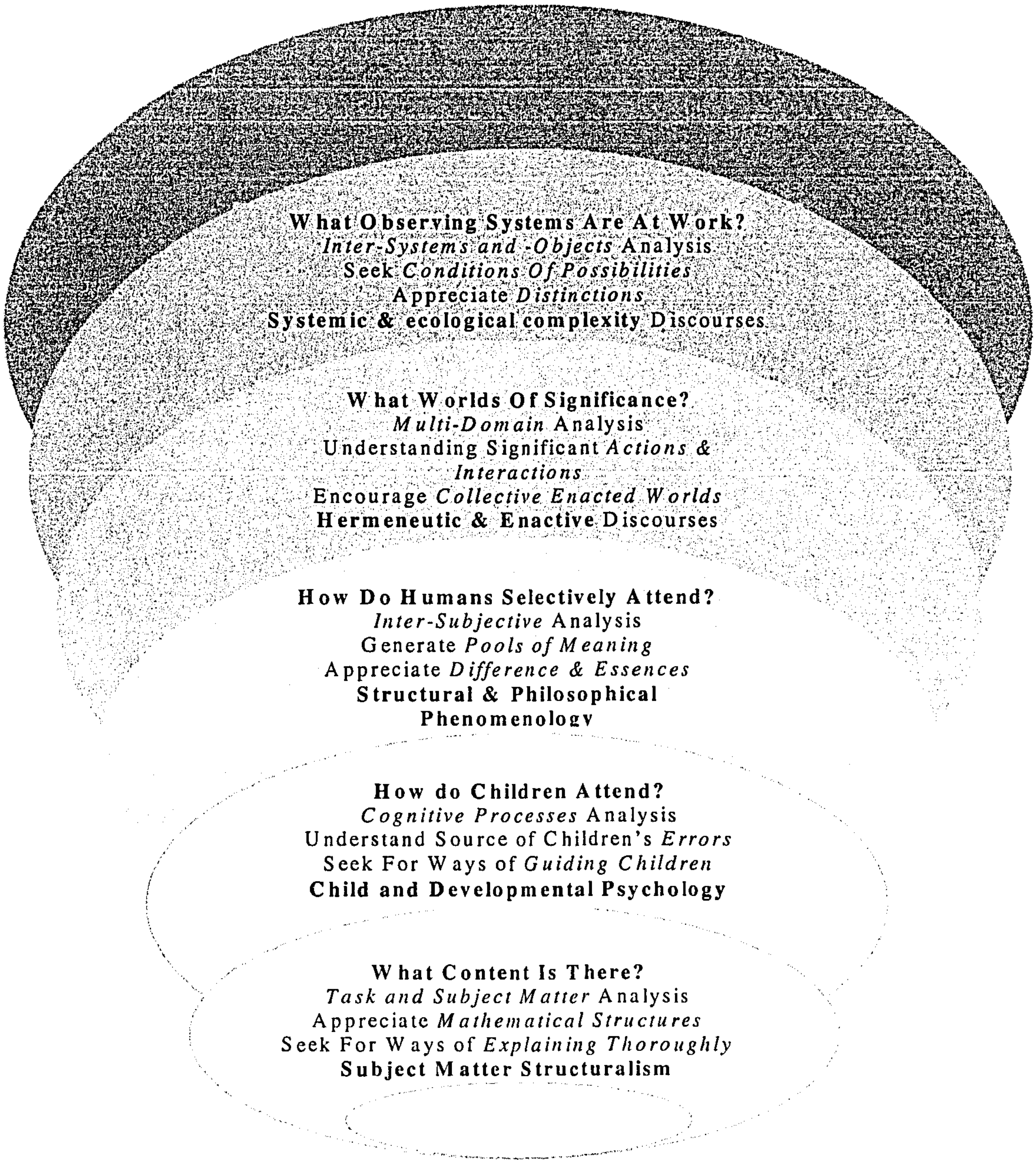
When I returned to graduate school my hopes were raised. I thought that by discovering what psychological structures underlie mathematical thinking I would be able to understand how to occasion mathematical thinking. My starting point was to explore work in the area of cognitive studies. Yet cognitivists' assumptions about mind and its architecture appeared too technical and theoretical for a teacher's pragmatic problems. Nor did these studies address why students found particular levels or topics difficult.

I also explored research work that acknowledged that exposure to well-explained mathematical concepts does not guarantee that students will come to think mathematically. It was early in my graduate studies that I came to a turning point in my understanding of learning. I began to recognize experiential, socio-cultural and institutional influences on students' mathematical thinking. My interest in matters of attention was evoked by the work of Greeno (1991), Mason (1989), Schoenfeld (1992) and Sfard (2001a); these are researchers who suggest mathematical knowing as coming to see what is mathematically significant. This resonated with my experiences. The question of mathematical thinking now appeared to be synonymous with *mathematical ways of perceiving*. John Mason introduced me to the question, what do students attend to? At that point I began to investigate mathematical tasks. I asked colleagues to consider the tasks and I attempted to determine what they saw or did not see. I learned that perception is selective. Different people attend to different aspects of tasks, I thought. I began to wonder whether the variations actually pointed to the selective nature of perception or to the dynamic nature of what is perceived as well. Drawing from hermeneutics, I began to emphasize that perception was selective but also prejudiced—never objective—and that it holistically included other sensory modalities. Enactivism, a theory that I had begun to explore, problematized the assumption of pre-existing objects of attention. It emphasized *mathematical ways of acting* and *being*.

Enactivist theories challenged my earlier assumption of an isomorphic mapping from given mathematical structures to corresponding psychological architecture. Perception, I came to understand, was enactive, hence selective and prejudiced. In attending, as we shall see in Chapter 9, perception enacts a world containing the objects, categories and properties to be attended to. I asked: What do students—conceived as embodied, extended and embedded—attend to? In enactivism, epistemological questions were cast for me as ontological questions. Through enactivism I began exploring eco-complexity theories. I then considered the notion of learners being extended to their environments and their worlds of significance. Perception, cognition and action were recast, for me, as intertwined phenomena that are illuminated by *attached* attention.

In Figure 21 I have included the observing I focused on and the metaphor at work at each layer of questioning. To show that my research question is unfinished I have included ellipses for further layers of questioning beneath and beyond.

Figure 21. A Layering of Questions



8.2 What is Mathematically Present to be Attended to? Task Analysis

My original question was focused on mathematical tasks and the mathematical structures that humans are conscious of when they are doing mathematics. I will call that approach structuralism. Here, the question of what students attend to is synonymous with asking what is structurally present to be attended to. What mathematical concepts make up a given task? A mathematical appreciation of tasks—subject matter analysis—is the main focus. As I illustrated in Chapter 5, the structural question is helpful to the teacher or researcher in selecting rich mathematical tasks for students to engage in. Indeed it is important, when approaching any topic, that he or she consider what is important to learn (or to observe) and how the task will present critical aspects of a particular concept for students. However the question of what mathematical concepts the Fraction Kit illuminates, as we saw, is insufficient in analyzing students' activity. For example, it does not address the primary activity the grade 7 students engaged in which was figuring out the size of the pieces.

With only a task analysis, the teacher or researcher in the context of observing students engaged in the task might be biased to look only at what fits or does not fit with mathematical content assumed in the task. Both historically and in this study, a structural stance has not been found sufficient to study students' attentiveness in mathematical tasks. From my earliest observations, it was evident that the students attend to legitimate mathematical structures, the relative size or amount of fractional pieces, even when they were observed to be attending in ways divergent from what the teacher and I expected.

From a structural stance there is a limited framework for conceiving students' novel and divergent understandings. At best these understandings are explained in terms of epistemological obstacles and misconceptions. The structural stance does not problematize the ontology of mathematical ideas, and thus one is bound to consider any understanding that differs from the explicitly stated as erroneous. Historically, there has been a shift from solely analyzing tasks for structure and concepts and from considering students' inadequate methods as errors toward looking closer at children's work and toward conceiving divergent interpretations as legitimate, though at times unfitting, constructions (Confrey, 1994b). Many researchers see children's inadequate methods as valid mathematics, *children's mathematics* (Balacheff, 1990c). This was a shift from "what is there" to be attended to, towards "what do children attend to and how do they attend?" With the fraction activity, stacking the pieces rather than covering a whole was a legitimate way of attending that was valid in the ratio-multiplicative fractional world. As an observer, I had thought that all students would attend to the covering of the whole piece, but many students attended differently.

8.3 What Processes Underlie Children's Thinking? Cognitive Analysis

In response to structuralism that turned only to mathematics for understanding the foundation of children's mathematical thinking, other researchers turned to psychology to study the nature of children's thinking. This psychological stance investigates the cognitive processes that support students' understanding of a task. It is important to note here that the tasks themselves are still assumed to be conceptually precise. Therefore any difference in what children attend to can only be explained by inadequacies in their cognitive structures which lead to cognitive obstacles. The second question I asked focused on the child as a subject: What structures do learners impose on mathematical objects? Among other things, studies on cognitive structures allowed me to see hypothetical growth phases and concept images that were speculated to be at work as students engaged in a task. It is believed that it is these structures which evoke in children divergent *perspectives* on standard mathematical objects. During my observations in classrooms and research situations, students' errors and novel approaches frequently triggered interpretive moments for me. But how was I going to interpret their actions, as epistemological errors or ontological differences?

At the cognitive psychology level, students' errors are to be understood as children's partial conceptions—primitive knowing. In a way, children's mathematics studies involve recognizing that children experience aspects of the world differently; they do not experience it as adults do. By exploring children's cognitive capabilities, researchers wanted to gain a window into how mathematical worlds (note pre-existing mathematical worlds) appear to children.

Many researchers are interested in the space between concepts as they are precisely reported—the objects—and how students come to understand them—the images. This is a useful shift from referring to students' inadequate methods as misconceptions, to referring to them with such terms as students' unmatching concept images (Tall, 1989), idiosyncratic inventions or problematics (Confrey, 1987, 1994b), cognitive obstacles (Sierpinska, 1990), inappropriately carried-over discursive templates, or pseudo-structural conceptions (Sfard, 2000a; Sfard & Lincheski, 2000). My work comes close to these researchers' work. However, in most of these observer constructs the problem observed in the students' actions is still essentially seen as an undesirable roadblock to be eliminated. For example, in the Fraction Kit activity it may be noted that the students who stacked pieces face a cognitive obstacle by working with as part of a discrete space rather than of a continuous area. From the enactivist/hermeneutic view although students who stacked brought forth a distinct task, stacking was not an obstacle that needed to be avoided. Stacking the pieces was one of many possibilities that students were disposed to lay down.

In my early exploration of enactivism and hermeneutics I began to realize that the space between students' "idiosyncratic" conceptions and the "standard" mathematical concepts could be a space of mathematical thinking-in-action and -interaction. In hermeneutics such a space is positively cast as fore-structures that are necessary conditions for further understanding.

To the extent that cognitive interpretation foregrounds the epistemological without reference to the ontological question about the human world, it maintains the structural stance. Studies by mathematics educators that involve isolating individual students in clinical interviews make sense at the structural and cognitive psychology level of investigation. Clinical interviews produce analyses that are much easier (than mathematical analyses) for teachers to understand (Confrey, 1991). However, this view assumes that the adult, the enlightened person's or expert's understanding is complete. Hermeneutically speaking, all understanding is partial. Some might be more desirable from one perspective or another but all is genuine and equally legitimate. For Maturana (2000) both questions: what is… and how things are for the children seek reference to a mathematical reality, whether empirical, rational or ideal. In this path of questioning, which Maturana refers to as *objectivity-without-parenthesis*, the claim to mathematical knowledge is a demand for obedience. It is a negation of the realities of learners as it entails a claim of privileged access to objective mathematics.

Many researchers now observe that there is no divide between how children and adults attend. It is rather a question of how people experience the same event differently given their experiences and contexts. This is the phenomenological stance. Mason (2003) asks the question, "How am I attending?" He argues that if a teacher is to appreciate how his or her students think, he or she needs to reflect on his or her own engagement in mathematical tasks.

8.4 What Do I Attend To and How? Inter-subjective Analysis

According to Mason (1994), in order to answer the question, "What do students attend to in a mathematical task?" one ought to ask, "What do I attend to as I think mathematically?" This question adds a phenomenology flavor to the earlier two questions. It studies the experience of the trained informant. Indeed, as human beings with the same evolutional endowment and history, and with common socio-cultural backgrounds and environments, we are likely to bring forth compatible tasks. In Rosch's (1999a, 1999b) terms, some aspects of perception are focal aspects; they do not vary much across individual humans and across cultures. Our bodies place us in a relationship with other bodies (present, ancestral or virtual). Given our species and locale, specific needs and capacities, we relate to and interact with triggers from the environment in particular ways. We attend in *shared* and *compatible* ways. Gordon-Calvert (2001) asserts that humans bring forth common and overlapping cognitive realities and perspectives, partly due to the conversations they co-exist in. To her, our worlds are filled with *echoes* of other humans' worlds. This phenomenological level analyses human experiences. Mason's question is therefore crucial as long as it, unlike the structural question, "What objects make up a given mathematical task?" does not assume a mathematical task that is independent of the mathematician, the teacher or the student. Nor does it assume that children attend partially as the cognitive psychology question would.

A phenomenological treatment of questions on how students attend places primacy on the experience of attending mathematically. From this perspective the question of what students attend to is broader than the structural and the psychological which focus on the mathematical and cognitive structures respectively. With the phenomenological inquiry children's mathematical conceptions are not viewed as inferior to adults' conceptions. What both adults and children already know is viewed as pre-understanding at any instant. It is a condition of possibility and grounds for further attending. The phenomenological layer of observing what we ourselves attend to in order to better understand what students attend to is crucial especially when it does not result in defining structural essences of phenomena that are independent of the attendee. It allows us to anticipate what the students may attend to in a particular task. In my research, I ask Mason's question not only to anticipate the students' possible worlds. Through an awareness of what I myself attend to in a particular task, I may be able to tease out other ways in which people think mathematically, because I know that what I attend to in a task is one of the many possible worlds (Bruner, 1986; Varela, 1992). Further I ask Mason's question to guard against the tendency to listen unreflexively for what I attend to and judge students against that; rather than listening for the students' experience of attending within a particular context. I refer to this level of analysis as the *inter-subjective analysis*. Listening to what a student in a particular setting might be attending to without negating their mathematical worlds is at the center of my study.

Some researchers assert that any phenomenon we encounter is experienced in a *limited* number of qualitatively different ways. Marton (1989) and others, drawing from a sub-field of phenomenology called *phenomenography*, seek to ascertain from the experience of others the critical aspects that make an experience an experience of that particular object. In mathematics education, Booth, Wistedt and Halldén (1999), and Marton and Booth (1997) examine children's experiences rigorously. They look across experiences for commonalities about how a phenomenon is seen, handled, related to or known. The discernible commonalities in the descriptions are taken to form a set of categories of descriptions, the collective space of how a phenomenon is experienced, the various ways with which people experience a particular task (Booth et al., 1999; Marton & Booth, 1997). These researchers suggest the thing-in-itself, the mathematical task itself is constituted by the outcome space—the constellation of meaning—of the qualitatively distinct ways of experiencing. For example, constellations of experience and, what Booth et al. (1999) dub *collective understandings* of number concepts correspond to qualitatively different ways of experiencing number concepts. Knowing is recast from acquiring mathematical objects, developing cognitive structures, overcoming obstacles or seeing relevant objects, to being capable of *experiencing things in certain ways*. In the frame of phenomenography, varied ways of figuring out the size of the pieces—whether by covering, stacking or assembling—would be the different ways of experiencing the Fraction Kit activity as well as its meaning. Similarly the different ways students used to describe the set {2, 4, 8, and 16} as doubling, 2 times 2 times, even numbers that are not multiples of odd numbers would be what constitutes the meaning of exponents of two.

Phenomenography and other interpretive studies have enormously enriched descriptions of concepts from the precise, narrow and formal definitive approach common in mathematics textbooks. They are pedagogically helpful since they guide a teacher to listen to children and to welcome various interpretations. To the extent that it recognizes that what we observe is to some extent unique, the phenomenographic theory appears compatible to the enactivist view. But a closer examination reveals that the underlying assumptions and emotional desire guiding their research is different from the enactivists.

Enactivism asks, "In what ways do students attend as they enact mathematical worlds?" A set of invariants, identified by the phenomenographers may fall apart when interrogated with empirical studies across radically different cultures or classroom contexts; there might be no invariance in how *non-basic* events are perceived (Lakoff, 1991; Namukasa, 2003b). In the enactivist view what is attended to is not taken to be fixed, once and for all. The mathematical world is considered to be a perceiver-dependent world.

Confrey (1994b) has developed a method of *close listening* as a way of paying attention to students' "inventions". B. Davis (1997) identifies three forms in which a teacher attends to students as the students attend to the mathematics: the evaluative, the interpretive and the hermeneutic modes. In evaluative listening, a teacher assesses students' sense-making against "*what is there*" to be attended to, looking for matching and un-matching understanding. As I see it, evaluative listening is based on task analyses. In interpretive listening, a teacher tries *to make sense of what students may be attending to*; he or she takes note of the *problematics*. Interpretive listening requires an inter-subjective analysis. Yet in hermeneutic listening, a teacher, in addition to anticipating differences, *participates with the students in what they appear to be attending to*; he or she seeks to engage in their embodied tasks. A teacher lingers in students' inventions (Gordon-Calvert, 2001).

It is in hermeneutic and enactivist attending/listening that we could seek to bring students to dwell in the required sensibilities for an effective and collective attending. However, hermeneutic listening calls for yet a deeper level of analysis, one related to analyzing the worlds enacted as students engage in the task—*multi- domains* or *worlds analysis*. This later level of questioning involves participating with the students in their mathematical realities. From such a perspective we begin to rethink the role of the observer or teacher as we note the possibilities for attending *with* students.

8.5 How Do Students Enact Worlds? Multi-domain Analysis

> We do not see the "space" of the world; we live in our field of vision. We do not see the "colors" of the world; we live our chromatic space .... But when we examine more closely how we get to know this world, we invariably find that we cannot separate our history of actions—biological and social—from how this world appears to us. (Maturana & Varela, 1987/1992, p. 23)

My study looks at the invariants as well as the regularities in what people attend to in mathematical tasks. It attempts to narrow the gap between the properties of the attendee and the attended. From an enactivist perspective to say something about what students attend to in mathematical tasks is to make statements about the broader domains, including:

- Structure of the attendee—biological, historical and contextual;
- Collectives in which the attendee participates. The attendee never attends only as an *I*, but as a *we*.

- Communicative, symbolic, technical and material media available to the attendee;

In a sentence, my focus is on the world brought forth by the attendee.

As Merleau-Ponty (1974) concludes, perception is participation with the world that we are in (see Chapter 9). The desire to describe pools of meaning so as to explain a concept thoroughly to students still lingered in me at both the phenomenological-inter-subjective and enactivist—multi-domain level of observing. Yet a century of research on mathematics teaching informs us that teaching is not all about explicit and thorough expositions of concepts (Bass, 2002). Observing different individuals or pairs of students as they engage in a task might allow a researcher to access qualitatively different foci of attention, but they are not so much aspects of a fixed task as they are a world enacted in doing.

In enactive observing, a teacher or a researcher tries *to make sense of what students are possibly attending to,* the mathematical worlds rolled up in living. Even that which is different from conventional mathematics is a legitimate conception in a particular world. Directing students' awareness to what teachers and educators attend to is thus a complex task. It involves studying how students enact their mathematical worlds. Moreover the differences in what a student might be attending to can be seen as an invitation to dwell in the domain in which the particular object of attention and ways of attending make sense. It is in a particular cognitive domain, for example, that the statement "a minus and a minus make a positive" makes sense.

I need not only seek to understand the mathematical worlds that students bring forth, but also to investigate the conventional mathematical domain as one of many possible worlds. I seek to understand the conditions of possibility for enacting adequate mathematical worlds. This is a desire to appreciate the conditions, patterns of behaviors and emotional urges that have a potential to generate mathematical worlds.

By analyzing mathematical worlds enacted, we, researchers and educators, might be able to offer ways that occasion adequate mathematical actions among students. By gazing at what students attend to and the dynamics of this attention even when students are attending conventionally, we glimpse into the dynamics of thinking and its constraints. We can systemically and ecologically seek to understand the complexity and the dynamics of enacting adequate mathematical worlds. Also by studying the dynamics of the space between and shifts from children's mathematics toward conventional understanding, it becomes possible to analyze the conditions under which such shifts are likely to occur so as to *engineer* them and to be able to trigger significant shifts in other contexts. The eco-complexity question on mathematical attentiveness is driven by a desire to understand mathematical thinking in order to build conditions of possibility or to alter conditional probabilities for it to happen (Juarrero, 1999).

8.5.1 Embracing Eco-Complexity

I began with the *traditional* gaze at the structure of an isolated concept. Then I followed it with the psychological gaze at children as yet-to-be adults; then took on the phenomenological gaze at phenomenal invariants in experiences of children and the enactivist gaze at worlds enacted in living. Enfolding all these four gazes, the eco-complexity orientation provokes a gaze at the nature of mathematical worlds as ever-changing, rooted in history and context dependent.

Eco-complexity researchers gaze at the conditions of learning at many integrated levels: the neurological, the experiential, the collective, the symbolic, the material, the institutional and the ecological. Lakoff and Johnson (1999) assert that such a focus is aimed at generating convergent conclusions. To Bruner (1996), it serves to generate better-informed hypotheses. To me, human beings do not attend at only one level, but at many nested levels and so it appears necessary and useful that we study learners as systems that attend—observing systems. My questioning has evolved to focus on humans as observing systems who, in the operation of observing, enact worlds with blind spots of observation. In the next section, I elucidate the outer eco-complexity layer of questioning—studying students as observing systems. But before I do that let me summarize so far.

I have explored levels of analysis of students' engagement, including task analysis (mathematical and epistemological questions), cognitive analysis (child and developmental psychology questions), inter-subjective analysis (philosophical/interpretive questions) and inter-domains analysis (enactivist/hermeneutic questions). I asked these questions at different moments throughout my research; now they are *layered* by the questions about nested *observing systems*. I use the verb *layered* to invoke nested levels and the metaphor of recursive elaboration that I explored in Chapter 3 while discussing how different theories of learning relate to one another (see also Appendix B where I explore how spaces of signification relate to one another). Steffe and Thompson (2000) explain that a superseded question is not rejected but rather it is re-structured by the emergence of a new, enfolding structure. The complexity metaphor of emergence helps us to understand how emergent layers are dependent on superseded layers, but as soon as latter layers emerge they influence and co-relate the nested layers. In practical terms it would be less useful to compare inner nested questions with an outer surrounding one. Figure 21, at the beginning of this chapter, provides a summary of the landscape formed for me in this research. Different questions interested me, but they were not unrelated.

8.6 In What Ways Do People Attend? Analyzing Observation

Lakoff (1991) asks a question parallel to that of how students attend: What do people attend to in an environment? Recognizing that people seem to cope with complex environments by categorizing, he has specifically asked: How do people categorize objects? For Rosch (1999b), categorized objects could be perceptual, semantic, biological, social, formal, biological or goal-derived objects. Lakoff's question is close to the question of how people synthesize patterns as well as conceptualize abstract events.

Lakoff (1991), drawing heavily from Eleanor Rosch's work, has concluded that what people observe in their environments and how they observe grows fundamentally out of embodiment and imagination. What people experience is limited, at the inner, sub-personal layer of description, by individuals' perception, speech and motor movement. At the higher species and society layers of description it is constrained by genetic organization of the species as well as by the nature of the physical and social environments in which the individuals integrate. What people see, as I will discuss in Chapter 9, are more than pre-existing symbols and signs. In attending, Lakoff would maintain, people categorize according to their experiences. Categories themselves are dynamic rather than static and arbitrary forms. They are humanly motivated. The properties of people's descriptions and schemas, to use the gestalt psychologists' terminology, are not as defined and well-formulated as postulated by the classical theory of categorization and concept formation, Rosch (1999b) explains. Rather, they are a cross-product of nature, human biological capacities and experiences of functioning in dynamic environments. Lakoff (1991) further observes that abstract events and entities, as those involved in mathematics, are attended to more or less as physical objects, for example by part-whole structuring. Following Rosch, he also says that while what people attend to may vary with culture, individuals, domains of experience and age, the principles of attending appear similar. Specific to mathematics, Gattegno (1970), Mason (2000), Peirce (1965/1839-1914) and Watson (2003) claim that people have a general propensity to spot and use pattern, to generalize, to organize and to order. For Mason and Watson these human inclinations could be strengthened for learning mathematics.

8.6.1 Eleanor Rosch on Categorization

Rosch (1999b) claims that *categorization* is "[o]ne of the most basic functions of living creatures." (p. 61) "Every object and event is unique but we act towards them as members of classes" (Rosch, 1999a, p. 4). We live in a world we have regularized and categorized. Rosch developed the thesis that "categories form around ... salient rich or highly imaginable stimuli which become prototypes for the category. Other items are judged in relation to these prototypes; that is the way they form gradients of category membership" (Rosch, 1999a, p.5). Some stimuli provide good examples of a category while others are not. For example, some birds or triangles are better examples of birds or triangles, respectively. Better examples are more universal while others are not easily agreed on. There need not be any defining attributes that all category members have in common; there are no defining attributes, no overall invariants, no logic sets at all, Rosch (1999a, 1999b) explains. She further asserts that the content of concepts or categories is not universal. But the structure of categories and the processes by which people organize what they attend to are universal. How (rather than what) people categorize is universal. People in different environmental settings map out pieces of meaningful lived reality and perceptual experiences. Some categories like *chair*, and, I should add, *take away* that derive from every day basic life categories are basic level prototypes, while others like *furniture* or *office chair* and *subtract* or *minus* are non-basic and general. "Basic categories and concepts are panhuman, species-specific perceptual universals, they are perceptually more salient, can be learned more rapidly, more easily remembered." (Rosch, 1999a, p. 5) They are usually biologically and species determined and are similar among groups of people who share culture or context.

Further initial categories are essential, for as soon as you make a distinction, then a myriad of things are learnt and invented and just as many things are not. Culture and language intricately influence general categories held by groups of people. According to Rosch, the instant you shift ever so slightly away from *primary knowing*, it is recursions of distinctions, concepts and categorizations all the way down. *Thingness* and definite objects is one of the delusions of our constricted mind. My exploration of Rosch's work raises these questions: Where on the spectrum of basic and non-basic does a given mathematical category lie? To what extent is mathematical content basic, thus easily attended?

Following after Rosch's philosophical work, Lakoff (1991) emphasizes that it is important to study distinctions and categories, including distinctions that are frozen as our primary knowing. His work is an example of a study at the layer of *observing systems* among people. What I learn from Rosch and Lakoff is that how people attend can be more hermeneutically studied than what people attend to. Lakoff's experientialist work resonates with recent neuroscience hypothesis that humans are gifted at pattern recognition, organizing, seeing differences and giving meaning to percepts. There are numerous experimental research studies on how humans attend, albeit at the sensory attentiveness layer. Research within the ecological and systems paradigm prompts the question: How do people order and increase the energy so at to regularize their worlds?

The human brain is capable of recognizing even nuanced patterns of patterns (Johnson, 2001). Human beings as individuals or as collectives are observing systems that pattern and abstract the worlds they both attend to and create (Doll, 2003). Questions about humans as observing systems are central in systems and ecological paradigms, where, as I will explore, perception and attention are taken to play a participatory role. I will defer discussion of the eco-complexity theories of observing systems until the next chapter in which I discuss in detail the work of Luhman (2002a, 2002b), Maturana (1988a, 1988b), Spencer-Brown (1972/1979), Varela (1999a) and von Foerster (1981, 2003). I now briefly return to the question that I have asked most recently in my research work.

8.7 In What Ways Do Observing Bodies Make Distinctions?

Observing *Observing* Systems

I, through my recursive and enfolded questioning, intend not only to understand what students attend to, but seek to participate in what they attend to and to make it more probable that they will attend mathematically. How can I, as a researcher, educator or teacher, be a part of a student's mathematical world? How can I invite my students to enact mathematically significant worlds? Arlene after the Fraction Kit activity thought that $2\frac{3}{4} = \frac{11}{12}$ instead of $2\frac{3}{4} = \frac{11}{4}$. Rather than look at this as an error, in Chapter 5 we saw it is an embodied task that makes sense in the ratio and folding fractional world. Arlene as an individual learner had categorized fractions as a collection of numerous smaller parts. At an observing-system layer of analysis, the question becomes: How can we participate with Arlene in ways that will incline her to enact an adequate fractional world?

Maturana (1988a, 1988b, 2000) explains that cognitive beings on an operational basis constitute objects, entities or relations that make up their worlds. "Every cognitive domain is a domain of co-ordinations of actions in the praxis of living of a community of observers." (Maturana, 1988a, p. 29) In the explanatory path of objectivity-in-parenthesis we are aware that there are different domains of reality, all equally valid. I will say more on this in Chapter 9.

If we observe that students are making mistakes or acting inadequately in mathematics, this implies that the students have made a distinction in an operational domain different from the one we expected. Such an explanation of the students' behavior is made from an explanatory path of *objectivity-in-parenthesis*. The world has objectivity but the operational domain in which a student finds herself specifies that objectivity. In the objective path that concerns itself with a one-size-fits-all world, what there is to be attended to identifies mistakes, poor instruction or disobedience. In the subjective path, mistakes point to children's ill structured and illogical concepts. Yet in the inter-subjective paths they point to different ways of experiencing or to aspects of a concept. With objectivity in parenthesis—inter-objects and inter-systems analysis—mistakes point to the existence of a different cognitive domain from which the observer views the actions of a student as mistaken. Conventional mathematical distinctions could also be construed as mistakes when observed from another cognitive domain or observing system.

In mathematics education terms, the question about observing systems invites us to listen to students as if the students had invited us to share in their operational coherences, for it is within different operational coherences that it makes sense, for instance, for a student to conclude that $2\,\frac{3}{4} = 1\frac{1}{12}$. It is in the same world of stacking fraction pieces rather than covering a whole that the multiplicative-ratio embodiment of fractions makes sense (Kieren & Gordon-Calvert, 1999; Kieren & Simmt, 2002). In this ratio world $\frac{1}{4} + \frac{1}{4} = \frac{1}{4}$.[29] The sign = as well as the set {1, 2, 4, 8, …} involves nested and varied signifiers (see Appendix B). It all depends on the world enacted. In taking on invitations from students, a teacher seeks to *participate with the students in what they appear to be attending to so as to alter the space of the possible for the student.* This questioning is interested in collectively constituting possible and adequate mathematical realities.

Having analyzed a task such as the Fractional Kit activity and reflected on what I attend to in it, and having listened to varied and emergent tasks posed by learners, it seems vital to explore the possibilities of inviting students to enact jointly mathematically significant worlds. Through this participation, the teacher also invites students to dwell in cultural and conventional, yet embodied mathematical distinctions. Students are offered environments that are likely to trigger from them and structure mathematically adequate distinction while narrowing the conditional probability of non-mathematical distinctions.

[29] Two classes, each with a ratio of one textbook to four students, when combined in one classroom still gives a ratio of one textbook to four students rather than one textbook to two students!

It seems important to be attentive to the operational coherences necessary to see $2\frac{3}{4} = 1\frac{1}{4}$ and $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ in the part-whole (rational number) world. This appears to be what Davis and Simmt (2002) mean when they say that subject matter analyses must be part of broadened appreciations of mathematical concepts' experiential requisites. As I observe students' thinking and attentiveness, I affirm that $\frac{1}{4} + \frac{1}{4} = \frac{1}{4}$ must make sense, but I do not terminate at this. I wonder in what ways it makes sense. I also ponder the ways by which $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ makes sense. Núñez (2000), drawing from Lakoff and Johnson's (1980) work on unconscious conceptual structures, seeks to understand the full complexity of ideas and intuitions. He studies the ways in which "standard" mathematical ideas *really* make sense. In what ways does $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ make perfect sense? Lakoff (1991), Lakoff and Johnson (1999), and Lakoff and Núñez (2001) explore grounding and linking metaphors, metonym, image schemas, metaphorical projection, conceptual blending and imagination as cognitively implicit ways, the operational coherences by which mathematical intuitions make sense. These illuminate worlds enacted.

8.7.1 Objectivity-in-parenthesis

Whereas Lakoff (1991), Lakoff and Johnson (1999) and Lakoff and Núñez (2001) study the non-conscious linguistic and conceptual ways in which mathematical ideas make sense, Maturana (1988a) and others such as von Foerster (2003) seek to explore how we create the objects of our attention. How do objects, concepts and tools arise? This question calls for acknowledging the contingency of observation. The term observation is taken to refer inclusively to perception, conception and action—the fundamental operations of being. It spans body, mind and world. Luhman (2002a, 2002b), Maturana (1988a, 1988b), and von Foerster (1981, 2003) appear to have elaborated on Spencer-Brown's (1972/1979) theory of making indications. In a manner related to Merleau-Ponty's (1964) work on perception and to recent neuro-physiological and neuro-psychological hypotheses in brain research, these researchers emphasize that senses do not work in isolation from each other, nor do they work in isolation from cognition.

All humans observe. By observing they make distinctions; they stress and ignore, categorize and generalize, pattern and organize, and abstract and reify. But what they abstract does not precede their operations of making distinctions. To Luhman (2002a, 2002b) and Spencer-Brown (1972/1979), observing involves carving our universes into a marked and unmarked state. Each observation generates the unmarked state of a distinction—a blind spot, according to Luhman (2002a). Thus to ask what students attend to is also to ask: What distinction is made? What are the marked and unmarked states of such a distinction? What are the properties and conditions of the operation that makes such a distinction?

With an eco-complexity stance, I am motivated to attend systemically to students as observers whose being and state are precisely the distinctions, the operations they make (Maturana, 1988a, 1988b; Spencer-Brown, 1972/1979). In chapters 9 and 10, I will illustrate that a distinction arises as a coherent form from the integration of students' experiences including linguistic, political and emotional experiences. As well, the system's stance considers humans as observing systems nested within larger observing systems and interacting with other observing systems. By considering students, teachers, mathematicians, collectives of students and the culture of mathematics, each and all as observing systems, second-order observation theorists attempt to ascertain the distinctions those systems make as they attend. A teacher engages in inter-systems analyses.

The inter-systems gaze emphasizes the temporal and contingent validity of observations. Just as there is a variety of biological observing systems—the species—there is a variety of formal communicative observing systems of which the mathematical one is just one. Schoenfeld (1992) describes mathematical thinking as having a mathematical point of view—a way of seeing and using mathematical tools. Theories of distinctions are a radical interpretation of *points of view*. When students view negative 4 times positive 3 as negative 12, this is a point of view—a distinction by an observing system with specific conditions of possibilities. When a grade 7 student explains that the result negative 12 is because the negative number in the product is bigger, second-order observers appreciate the totally different distinctions the student is making. Different distinctions bring forth different objects as integers. Since it is the intention of mathematics teaching to trigger students to enact mathematical objects, it is important that teachers and educators seek to understand the realities and distinctions that students operationally constitute as observers.[30] In Chapter 11, as I conclude this writing, I ask how different points of view relate to mathematical concepts. Are they universals, invariants, generalities, abstractions, habits of mind, or what?

With observing observation, what was previously accepted as self-evident, $2\frac{3}{4} = \frac{11}{4}$, becomes visible as a peculiar way of observing, with a particular web of

[30] B. Davis and Simmt (2003) illustrate the variety of points of view that emerged when they asked students to explain how they knew that 3 X -4 = -12. From the charts that the students drew, the researchers saw a diversity of understanding integers as vectors, as directed displacements on a number line, as black and red counters, as negative and positive charges, or as rises and falls.

meaning, or what semiotic theorists would call, particular series of signifiers. An observation is possible in a given network of settings and operational coherences, which include tools, objects, media and collectives—the properties of the observing system. Any distinction that students may make is a distinction made with particular kind of initial conditions, within current state of possibilities. To paraphrase Heidegger (1927/1964), humans are capable of making distinctions that they are inclined to make, and when the distinctions in turn incline towards their essential being. Thus, as teachers, educators and researchers we may want to attempt to create observing domains in which students are prompted to make distinctions that mathematicians make. To do this it might be helpful to observe mathematics as an observer-constituted reality.

What human observers do is to observe or to make distinctions so as to constitute their ontologies—their very beings (Maturana, 1988b). There are operational preferences or choices, even where a volitional choice to attend to this and not to the other is just a small fraction of the choices we make at deeper levels. Therefore what an observer observing another observer system could appropriately do would be to attempt to understand the distinctions together with their properties that other particular observing systems make. One could attempt to make explicit the formal and the informal conditions of observing.

In our daily lives and practices we, especially mathematics teachers, mostly act as if we observe the same stable and transcendental objects—*objectivity-without-parenthesis*. However, for pedagogical and world citizenship reasons the standpoint of objectivity-in-parenthesis is different. A mathematician reporting his or her findings or explicitly stating his or her definitions and axioms acts and explains with objectivity-without-parenthesis, for he or she is acting within a community of observers who share constituted mathematical ontologies. Yet a teacher who interacts with students whose bodies are intersections of various collectives (of which the teacher belongs to only a few) the case is different. This appears to be what Ball (2002) intends to evoke when she says that traditional college mathematics curriculum for mathematics students is surely not sufficient for mathematics student teachers. Mathematics courses are necessary for teachers of mathematics, but there is more that a teacher needs in order to be with students in sympathetic ways.[31] A teacher can act with objectivity-in-parenthesis, be present with an attitude that seeks to perceive observer-constituted ontologies when they recognize that mathematical objects are inter-objects and students are intersections of many observing systems. This is as well an *inter-objective* stance to matters of students' attentiveness that enfolds the inter-subjective, subjective and objective stances. How do students attend as systems that observe at many levels including subsystems, system and global systems?

The findings and conclusions from this systems layer of questioning and observing could never be considered complete. Nonetheless those finding and conclusions change our worlds.

8.8 A Broadened Question? A Didactical Gaze

I consider the question of *how* students attend in mathematical tasks to be broader than the initial research question of *what* students attend to. This level of questioning leads to the following: In what ways do students as coherent sensory-motor

[31] The value of acquiring advanced and enriched descriptions of mathematical ideas lies more in how it affects the teacher's action in the presence of students than in what it does to the teacher's mathematical knowledge (Simmt et al., 2003).

and somatic systems make distinctions? In what ways do they attend as conscious and emotional beings? In what ways do they attend as systems nested within particular collectives and cultures? In what ways do they attend as systems that extend to the material, technological and symbolic environments? To the extent that it is layered, this level of questioning is not positivistic. It embraces complexity. Conceptual distinctions that a student makes are construed to be a coherent form that spontaneously emerges from observations made by nested and nesting systems.

In asking in what ways students attend, the distinctions, whether cultural or not, can be viewed as different observer systems each operating within an experiential network that has a distinct history, initial conditions, and an internal and external structure. As researchers when we observe observation we are able to pay particular attention to the kinds of distinctions that the observed learning system is able to see, what does and does not yet exist for an observing system such as school mathematics.

Operating in the same observing systems in which the observed system operates we may not be able to observe the condition of a specific observation. If we are to reflect on a cognitive domain, second order cybernetics theorists say, we need to specify another domain in which the domain of the observed distinction is an object of observation. I speculate that one such domain is the mathematics education domain, a domain from which we can observe conditions of possibility for mathematical observations. I wonder what such a domain might be for the school mathematics students themselves.

Take the task I posed to grade 7 students on Consecutive Terms, for example. Students looked at the {1, 2, 4, 8, 16} additively, multiplicatively or exponentially at different moments (see Appendix B where I discuss these differences in terms of

Vergnaud's (1988) construct of conceptual field). For a researcher who had not only done the task but who had also analyzed its content, it was very evident that the set was a set of positive powers of two—an exponential structure. When I asked another researcher to ascertain why it made sense for powers of two not to have the consecutive terms property, he observed that it made perfect sense for him. He offered an explanation based on a binary structure—his tools, settings, media and collectives that enacted his observation were distinct.

To the extent that theories of distinction motivate researchers to await a glimpse of the conditions of possibility for a reality in which the operationally preferred description for the set {1, 2, 4, 8, 16, ...} is $\{2^x; \text{for } x \geq 0\}$, they offer a didactical gaze. But every distinction has a blind spot, yet this blind spot is a necessary condition for sense-making—for understanding (Luhman, 2002a; von Foerster, 2003). Observing observation illuminates the blind spots, the properties, and conditions of possibilities of mathematical observations. As Mason (2003), Sfard (2001a, 2001b) and others have put it, for any seeing there is something in focus while something else is peripherally attended to and another totally ignored. Since individual humans are embedded in socio-political observing systems and they embody biological observing systems the stressing and ignoring (Gattegno, 1970) is mostly done at a level beneath or beyond the one of conscious and formal attending.

Structural, psychological, philosophical and post structural layers of questions are for me conditions of possibilities for the latter question of observing of observation. Both historically, and in my research, systems questioning recursively loops back onto the structural question, as demonstrated by Núñez's (2000) mathematical idea analysis.

When I now ask questions about what mathematics is there to be attended to, I am inclined to approach them in ways that implicate students as complex observing systems.

To use Merleau-Ponty's (1974) terminology, theories of distinction "measure the distance between" understanding learning as bringing forth worlds of significance and understanding learning as acquiring knowledge (p. 29). All the layers of questioning that historically preceded the eco-complexity question mark the distance. Further, the question, "In what ways do students as embodied, embedded and extended observing systems attend?" is more than a one-discipline question. It calls for mathematical expositions after pondering about living, cognition, socio-cultural structures and neurological hypotheses.

In this chapter, I further explored how my research questions evolved. My current questioning is of an emergent order. It arose from the interaction of the literature I reviewed, from earlier interpretive moments, from preliminary analyses, from interactions with other researchers and from collective exploration of both the methodological and theoretical frameworks. In research, it appears, questioning is one level above the other components of research, of which *time* and place are important ones.

9. PERCEPTION AND OBSERVATION: A SYSTEM'S VIEW

When work on the philosophy of perception by Merleau-Ponty (1964, 1974) is read together with recent research into brain dynamics and with complexity research metaphors, they emphasize that what we attend to guides our actions just as our actions guide what we attend to. Attending is all at once a biological, psychological, linguistic, social, cultural and institutional phenomenon. What we observe, however *supra-natural, mystic or esoteric*, is co-implied by the species we belong to, the tools we have, the collective observing systems we participate in, and what we have observed before.

In this chapter, I explore a systems and dynamic view of perception. While in Chapter 6, I explored the theoretical frameworks for this study, in this later chapter I explore a dynamic theory that is specific to perception and observation, a theory that takes attention to span levels of signification. I differentiate between sensation, perception and observation. Non-cognitivists' views emphasize that perception is intertwined with action. Merleau-Ponty's (1964) work offers a philosophical approach, which views perception as a point of view. Recently, certain neuro-physiologists such as Newberg et al. (2001) and Freeman (1991) have illustrated that perception is an internally organized activity, a neurological choice to attend. Maturana and Varela (1987/1992) observe that any organism's actions are perceptually guided. To von Foerster (1981) non-trivial systems, since their output-input relation is determined by their previous operations, are observing systems. Maturana (1988a, 1988b), Luhman (2002a, 2002b) and Spencer-Brown (1972/1979) advance a theory of distinctions, which illustrates that humans observe, both at the personal and collective level.

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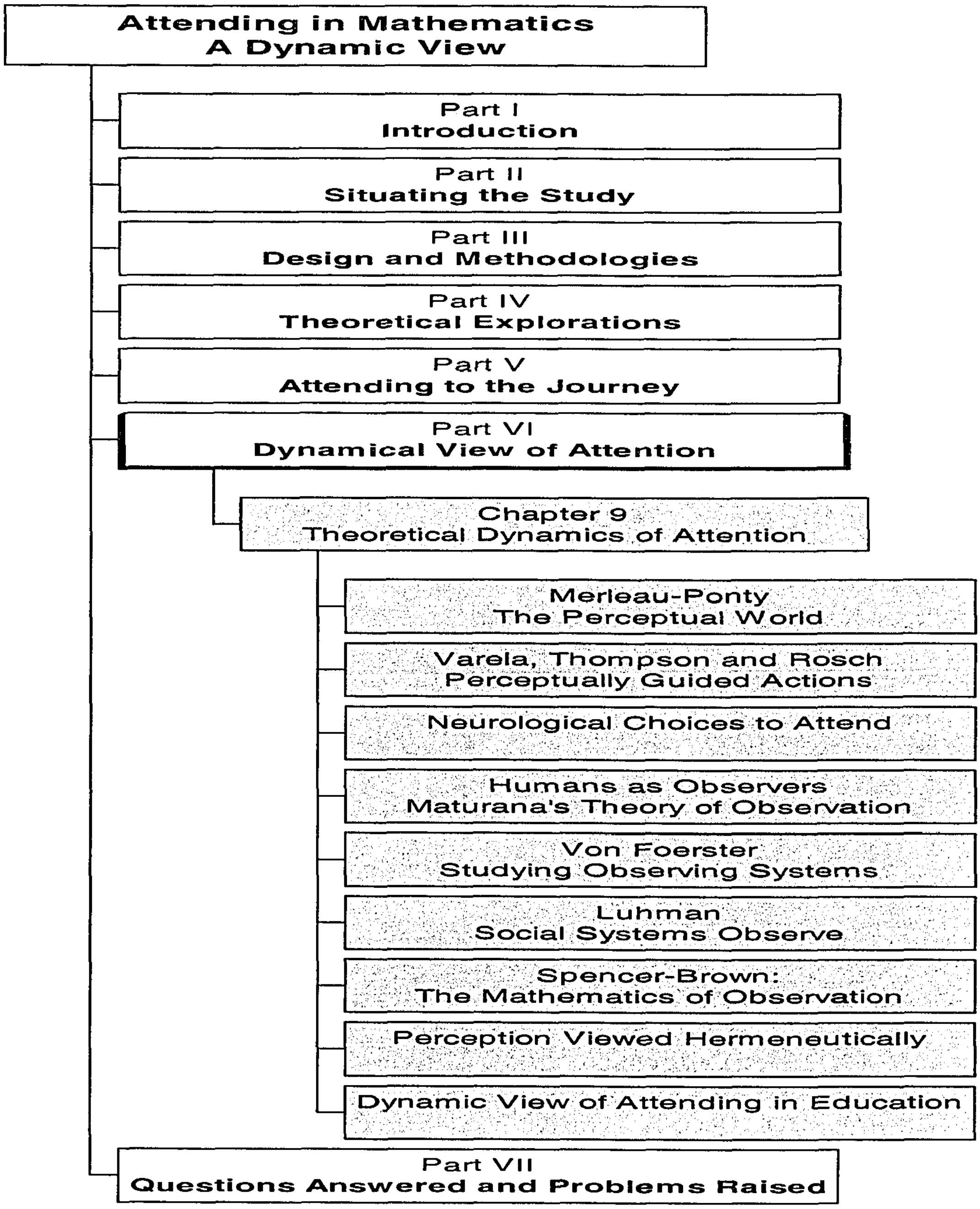


9.1 Merleau-Ponty: The Perceptual World

Although Merleau-Ponty (1964) interpreted the same experimental data as his behaviorist counterparts, he offered an alternative stance to the classical view that perception is a taking up of external stimuli and the subsequent processing of those stimuli into internal representations and external responses. He drew from the work of other philosophers such as Heidegger, Scheler and Husserl to uphold the idea that perception is not an individualistic process and is more than reacting to stimuli (Freeman & Skarda, 1990; O'Regan & Noë, 2001). Merleau-Ponty's philosophical insights guide scholars that challenge perception as recovering reality.

> Perception is not science of the world, it is not even an act, a deliberate taking up of a position; it is the background from which all acts stand out, and is also presupposed by them (p. xi). Matter is pregnant with its forms.... It is necessary that the meaning and signs, the form and matter of perception, be related from the beginning. (Merleau-Ponty, 1964, p. 12)

Merleau-Ponty re-establishes the linkage between perception and action and between matter and form. He maintains that what is perceived is not a result of the perceiver interpreting a sensible matter according to a law. The duality of what is observed—the matter—and the sense made due to observation—the form—is not necessary.

> [T]he alleged signs [representations, signifiers] are not given to me separately from what they signify [so]...there is no deciphering, no immediate inference from sign what is signified. (p. 15)

For Merleau-Ponty, the perceptual world "is always a presupposed foundation of all rationality" (p. 15). Although what one may observe is necessarily unique, Merleau-Ponty observes that in some sense it is not entirely unique, since "[t]he thing [that I observe] imposes itself not as true for every intellect, but as real for everyone standing where I am" (p. 17). Accordingly, to perceive is to "render one self-present through the body" (p. 161). Moreover,

> Associated bodies to mine must be brought forward along with my body—the 'others', not merely as my congeners [sic], as the Zoologist says, but the others who haunt me and whom I haunt; the 'others' along with whom I haunt a single present, and actual beings as no animal haunted the beings of his own species, locale or habitat. (p. 161)

It appears crucial that a study on students' ways of attending in addition to emphasizing the primacy of perception, must also view perception as a process in which the body, the context and the participatory collectives are not extraneous to what we perceive. For Merleau-Ponty, the interpretation and the interpreted, the signifier and signified, form and matter, perception and action can only be separated for purposes of analysis.

When discussing perception Merleau-Ponty (1964) seems to keep in mind the entire sensory range of modalities. He uses examples from seeing and touching, and at most times avoids using the verb *to see* as a synonym for the verbs *to perceive.* In this way, he does not seem to restrict his ideas to visual perception, as is a common tendency. Examples from one sensory modality usually illuminate the general nature of perception, however for visual perception there is a physical distance maintained between the seer and the seen. As classical theories consider visual perception as an ideal metaphor for knowing, it is not surprising that a distance between the knower and the known appears necessary. Gadamer (1975/1992) and B. Davis (2000) suggest the auditory as a metaphor for interrupting this tendency, since with hearing one is literally embedded in what one hears. Seeing, touching, hearing, tasting and smelling involve almost the entire sensory-motor range of modalities together with *memory* and *expectation* in ways that are complex. How can we capture their integration?

Merleau-Ponty (1974) also embraces socio-cultural embodiment. He talks about *associated bodies* to mine that must be brought along with my body during perception. His views about perception as taking up a position, a perspective, are echoed in the idea that mathematical thinking is a mathematical *point of view* or a culture. For him, akin to Peirce, the object of perception "is given as an infinite sum of an indefinite series of perspectival views" (p. 15). Thus we cannot "decompose a perception, to make it into a collection of sensations, because in it the whole is prior to the parts" (p. 15). The perceptual views "blend with one another according to a given style, which determines the object in question." (p. 16) Moreover, the interpretations, the perceptual objects, are real for every one who stands where I stand.

9.2 Varela, Thompson and Rosch: Perceptually-Guided Actions

"Perception does not just happen in the world; it contributes to the enactment of the world," Varela et al. (1991, p. 174) assert. Varela is among the cognitive scientists who have rejected the representationists' view. He adopts Merleau-Ponty's (1964) phenomenological interest in perception and integrates it with a neurological orientation.

> The point of departure for understanding perception is the study of how the perceiver guides his [sic] actions in local situations. Since these local situations constantly change as a result of the perceiver's activity, the reference point for understanding perception is no longer a pre-given, perceiver-independent world, but rather the sensorimotor structure of the cognitive agent, the way in which the nervous system links sensory and motor surfaces .... The perceiver can act and be modulated by environmental events. (Varela, 1999a, p. 13)

For Varela (1999a), perceptually-guided actions are the basis of perception. Recurrent sensori-motor patterns, lawful linkages and contingencies enable action to be perceptually guided in a perceiver dependent world. Varela illustrates that perception and action, sensorium and motorium, are linked together by common principles and order, and successively arise from their mutual feedback. Since what counts as a perceptual world is inseparable from the structure of the perceiver, neuro-physiologists have begun to examine the feeling of a perceiver independent world as a *grand illusion*. Even though we are aware of experiencing a coherent world, when we turn to examine how we get to perceive this world, "[W]e invariably find that we cannot separate our history of actions—biological and social—from how the world appears to us" (Maturana & Varela, 1987/1992, p. 23).

To Varela, after Merleau-Ponty, a "situated observer has a perspective" a position from which he or she stands and acts (Varela, 1999a, p. 54). Said differently, this perspective is not a matter of occupying a position from which to perceive by the sensory extraction of features. It is the sensory guidance of actions.

If as Merleau-Ponty asserts we participate in a not pregiven world then how do we perceive its attributes? In school mathematics, how do students perceive attributes such as measurement? What about the more abstract attributes? Varela (1997) says "the nervous system is such a gifted synthesizer of regularities, that any basic material suffices as an environment to bring forth a compelling world.... That shows up through the enactment of the perceptuo-motor regularities" (p. 84-85).

In a similar manner we may study mathematical attributes as psychophysical, neurological and phenomenological experiences, and each of these expositions assumes a perceiving organism. Attributes are regularities that organisms synthesize. How an attribute appears, is perceived and is experienced might vary with species. It also likely varies with locality and culture. Recall how the same Fraction Kit was viewed and handled differently by students. Also Tony in moving the counters was able to synthesize regularities when testing whether a number had the Consecutive Terms (CT) property.

If movement and other biological abilities affect what and how we perceive, how then do other abilities such as language and consciousness—whether individual or collective—affect our perception? Perception from the perspective I am exploring includes sensual and higher-level recursion on sensations. To summarize, basic-level perceptions are embodied (i.e. the sensori-motor structures participate), experiential (i.e. what has been sensed before and concurrently participates) and consensual (i.e. our cultural history and social interactions structure it). Drawing from the enactivist logic of coherence and the complexity logic of emergence we can see that the percept is the emergent layer above the coherence of embodied, experiential and consensual perception (Varela, 1999a). This coherence offers entire *readiness*-for-action (the percept) in the next moment. Indeed as Merleau-Ponty (1964) observes, what we attend to is the background for actions. Our actions and what we attend to are inseparable (Varela et al., 1991). But does this apply to mathematical cognition? Do students perceive mathematical objects, like the set {1, 2, 4, 8}, the same way they, say, smell a flower?

Some neuro-physiologists such as O'Regan and Noë (2001) radically maintain that perception is a way of acting. It is about mastering the regularities of one's possibilities. To put it in terms of my study, the nature of students' mathematical thinking (conceptions), action (behavior) and what they attend to (perceptions) as they engage in mathematical activities are co-implicative. What a learner attends to is not only a feature of the task; it is also determined by his/her structures—the biological, socio-cultural and historical *readiness* (to use Bruner's (1960) term) to attend. What we attend to is at once a feature of the task, a feature of our structures and a feature of our experiences. Human perception is a way of acting, interacting and being. In this understanding of perception, teaching and education in general might have to do with prompting students to notice, or should I say to regularize and pattern their worlds in particular ways (B. Davis et al. 2000; Mason, 1989, 1994). Tony and Ronald, for example, attended to the CT properties in ways that pointed to their experiences with numbers, number lines and systematically manipulating material. In verifying whether a number had a property or not students in the study perceived numbers in particular ways, as decomposable entities.

How are non-basic level perceptions enacted? On issues of explaining the symbolic dimension that clearly exists in mammals, Varela draws from Rosch's distinction of experiential, embodied categories as the most basic ones. He then goes on to explain that higher cognitive structures emerge from recurrent patterns of perceptually guided actions (Varela, 1999a). They are outer-layer emergent properties. What we call the abstract, as in the mathematical domain, is an aggregate of readiness-for mathematical action.

While the embodied view holds that perception and imagination of abstract entities is emergent from experiential perception, there is little agreement on how this is possible. To me, it appears that to explain the perception of mathematical objects we have to evoke the role of language and socialization. O'Regan and Noë (2001) and other neuro-physiologists speculate that in addition to movement, speech, imagination and thought play important roles in human perception. This is evidenced by effects of lesions in human brains that show two visual systems where impairment in one may exist independently of impairment in another. A patient may be able to make verbal reports about the shape, features, categories and location of the object (using the *ventral visual* system) in the absence of the ability to locate the object with respect to the body and to grasp and manipulate it (using the *dorsal visual* system). These two parallel systems appear to have evolved separately, one with movement and the other with language and thought. But does this mean that the two (or three) systems do not interact in the perception of people without this impairment? If they do not, then this might be viewed as an argument that abstract perception is independent of basic-object perception.

Some researchers consider imaginary and symbolic perception in living systems to be assigned to the system by the observer (Searle, 1997), while others consider self-conscious and intellectual perception as a resultant that has neural correlates and causes, and others consider it to be intrinsic to the biology of organizationally complex organs like the brain (Thompson, 1997). To some the imaginary is metaphysical and so cannot be explained biologically. Each of these views will affect how we view and explain how students attend as they engage in mathematical tasks.

Many approaches to explaining mathematical perception have been top down—metaphysical. Advancements in neurology have shown that the relation between conscious and symbolic abilities and the physiology of the brain exists in terms of emergence and mutual causation, and thus there is need for bottom-up views to perception, including perception of mathematical attributes (Thompson, 1997; Varela, 1999b). While the overlap between bodily movement and perception is increasingly demonstrated, the overlap between imagination, thought, consciousness, technologies of intelligence and perception has yet to be explored by neurologists.

9.3 Neurological Basis for Attending

Recent studies in neurobiology consider perception to be more than reception (Freeman, 1991; Freeman & Skarda, 1990). At the basic neurological level, sensory data enter the neural system in the form of billions of tiny bursts of electrochemical energy gathered by countless sensors of the skin, eyes, ears, mouth and nose—sensation is happening. Later they are channeled along appropriate pathways, be they visual (color, depth, spatial or form), proprieceptive, auditory, and so on (Freeman, 1991). Individual impulses are re-routed to appropriate cortical areas where they are sorted, cross-referenced, amplified or inhibited, and integrated with input from other centers and senses (Newberg et al., 2001). Finally, they are assembled into a percept, according to the emergent global states, that has useful, individual meaning to the owner of the particular brain. Perception has happened. Many contemporary neuro-scientists believe global brain activity like nerve cell assemblies are the basis for memory, adapting to new situations and learning (Freeman & Skarda, 1990; E. H. Goldenberg, 2003). Large and distributed sub-ensembles of neurons are involved in any given percept they argue. Many now take the neurons that fire in synchrony or together to be the signatures of cognitive states. Newberg et al. (2001) postulate various cognitive operators or modules—various brain structures—that function collectively, such as the quantitative-number operator and abstract-categorizing cognitive operator. Even with the anatomical brain regions and lobes and hypothesized functional modules, each of these structures "has a set of highly specialized functions, but each also cooperates with the rest of the brain as a whole in complex and elegant ways, giving it the ability to channel, interpret and respond to the rush" (Newberg et al., 2001, p. 47) and torrent of sensory data flooding the body's neural pathways. The neural circuits and emergent nerve cell assemblies in humans afford highly efficient and harmonious ways of coupling action, thought and perception. In terms of learning, this points to how sensory data is just a small fraction in the emergence of a multi-threaded percept. This might address the variations in what students attend to even when they are given the same stimulus such as a mathematical task or an activity.

Newberg et al. (2001) point to neurological obligations to perceive in particular ways. Choices to sense the way we do are made at the neurological level, they maintain. Species-specific, modality-specific and individual-specific neuro-physiology compels us to attend in particular ways. Humans are different from animals in the nuances created by their cortex. Specifically, the complexity of the human brain adds new and significant wrinkles to any triggering stimulus. Because the cortical structures are so intimately linked to the more *primitive* functions of the *limbic* and *autonomous* systems (hormones and primal emotion centers), humans are able, for example, to trigger a biological fear response by simply thinking of danger, Newberg et al. explain. The human mind is able to think of many attributes in abstract and non-immediate terms.

The choices to attend extend beyond direct stimulus. You may instantaneously perceive and act in particular ways because your global neurological activity allowed you no other choice. Dimensional fullness, past experiences and emotional meaning are added to perception in the *associational areas* to develop a realistic, fully integrated experience of gazing upon something, be it in ordinary life activities or in specialized domains like mathematics. But does this mean each human attends in a unique way?

Lakoff (1991) refers to this extra ability in humans as non-basic *imaginative structures,* in contrast to the basic level bodily structures. In the view I am developing here, it is important to couple this imaginary component, which is immensely strengthened by language, with the embodied dimension (Varela, 1991). Although each individual has an autonomous sensori-motor structure, perceptions and thoughts of archetypes that have been noted to exist as panhuman universals appear to exist in every human mind (Lakoff, 1991). Does this prompt a classification of mathematical thoughts and meanings in elementary topics to be among perceptions that cannot help but be similar because they are shaped by unchanging aspects of the brain irrespective of languages spoken, culture grown into and geography known?

Newberg et al. (2001) go further to explain seemingly non-neurological phenomena, such as mystical experiences, as basically biological phenomena. This would imply that attending even to the most formal mathematical domains could in principle be explained neurologically. From an eco-complexity orientation, I would add that mathematical perception in addition to emerging from biological activity in turn influences neurological varieties.

With technological advancements in brain research, more work has been done to trace the biological signatures of symbolic expressions like thought and consciousness. It is imaginable that soon neurobiologists and educators will be together in laboratories attempting to figure out brain structures that are active during as well as those that change with mathematical attentiveness. Newberg et al. talk about the attention association area, a brain area that appears active when attention is focused on an object, idea or desired goal. Butterworth (1999) talks about regions which when damaged affect specific mathematical performance. Some of these assertions are not widely agreed upon by neuro-physiologists, one of the reasons being that many human studies have been done on anaesthetized patients and on animals. Even the recent ones that are done on healthy, awake humans could be used to support contradicting theories. In my view, neurological assertions, just like any other explanations and observations, are observations enacted by particular tools within particular human domains. The extent to which educators should depend on them is a matter of degree. Nonetheless I find them helpful in dislodging non-useful views about mathematical attentiveness. Let me offer a few more examples.

Of particular interest to the discussions on learning and perception are findings in neuro-physiological research that challenge the traditional view of perception as serial, stage-by-stage inputting (sensation), processing (thought) and outputting (action). For example, the ratio of inter-neurons to sensory and motor neurons highly challenges serial processing. Motor neurons (neurons that serve the motor surfaces) relate closely to sensory neurons (neurons that serve the sensory surfaces). This relation is mediated and modulated by intervening neurons, the *inter-neurons*. The ratio of inter-neurons to sensory-neurons to motor-neurons is 10,000: 10: 1 (actual figures that Newberg et al. and others give are 10^{11}: 10^{7}: 10^{6}). To have almost a negligible number of neurons serving receptors and effectors surfaces in comparison to connections across inner surfaces illuminates the following:

- Perception goes beyond sensations.
- Perception is a highly cooperative, distributed phenomenon with interconnected feedback loops. It begins with self-organized neural activity.
- The brain itself, through both negative (inhibitory) and positive (excitatory) feedback, creates conditions for perceptual-motor coherences.
- Perception is active, not passive; it occurs when the organism interacts—moves,

communicates and so on. It is co-emergent.

- The percept is an emergent global phenomenon that arises from a background of incoherent activity of the network of overlapping assemblies.
- The emergent percept, in turn, has causal powers, it feeds backward on neuronal activity.

The complexity of perception is also demonstrated by neuro-psychological and behavioral experimental findings about *priming, change blindness, the blind spot, inattentional blindness* and *masking* and the like, aspects of which normal observers would have no experience (James, 1981/1842-1910; Langer, 1997; Nørretranders, 1998; O'Regan & Noë, 2001). Neuro-physiologists, like neuro-psychologists, draw from these *paradigms* to inform their quests. Consider the fact that contents of a card flashed so quickly on a screen you are looking at that you do not even take note of its presence might affect your *downstream* actions. This phenomenon is called *priming*. The most recent paradigm is one of change blindness—while you observe a scene, some of its characters might be changed without you ever noticing as long as the moment of transition is disguised. In many cases, the stranger a person was asking directions from was switched after an interruption without a majority of participants ever noticing! Magicians and comedians appear for a long time to draw from these perceptual phenomena. It appears the ways we think, or we have been made to think we perceive in our daily lives are very different from the ways we actually perceive. If this is the case for daily life, then I think we need to pay more attention to how we think students perceive mathematical objects, patterns and changes. How many of them do they notice the way we do?

The multidirectional multiplicity in the neuronal network is reflected in the natural temporal parsing of 200-500 msec in humans before the "now-ness" of a perceptuo-motor unity (Nørretranders, 1998; Varela, 1992). It is interesting that we human beings do not notice this temporal parsing. Is this because it is too close for us to notice it? It happens on a timescale—milliseconds. Could it be that this timescale is behaviorally insignificant? Or is it that perhaps we adapt to it? For me this temporal parsing together with the view that perceptual awareness is a small fraction of what we perceive has implications for how I view the role of pointing students to volitionally see mathematical patterns.[32]

To summarize so far, perception is a coherence of somewhat independent, parallel and distributed but cross-correlated sensations that are intimately coupled with motor, cognitive and affective aspects. There is more to perception than what a perceiving being can be aware of. Perception happens within the mobile, thinking and feeling humans as an internally generated neural activity that lays ground for further reception, thought and action. Perceivable attributes such as color and dimension do not just exist out there but are brought forth by the perceiver through a mostly hereditary and experiential participation that defines what counts as a perceivable world (Varela, 1992). We do not view the world from a point of view per se; the world and its objects are enacted through perceptually guided activity. In the next sections I extend this discussion from sensation and perception to the domain of observation.

[32] As well, studies from humans with bodily and psychological abilities that are different in pronounced ways do illuminate the complexities of human perception.

9.4 Humans as Observers: Maturana's Theory of Observation

Maturana (1988a, 1988b) has a radical take on human perception. He argues that observation is a fundamental human operation. For Maturana (1988b) objects, language and knowing arise with the operations of making distinctions.

> We cannot operate with objects ... as if they existed outside the distinctions of distinction that constitute them .... Without observers nothing exists, and with observers everything that exists [exists] in explanations. (p. 37) The understanding of the ontological primacy of observing is the basis for understanding the phenomenon of cognition. (Maturana, 1988b, p.42)

All distinctions an observer makes take place in the network of operational and structural coherences specified by his or her operation. When I make a *distinction,* I cannot claim that I am distinguishing something that pre-existed the state or operation of distinction that brought it forth. All that I can claim is that as I make a distinction, I specify what I bring forth with it— the entity distinguished, *the observed* together with its domain of existence. In addition, every distinction specifies a domain of possible distinctions as a constituted ontology, a world among many possible worlds. Each distinction an observer makes constitutes a node in the *operational matrix* of the observer. Through *reflective languaging*[33], second-order observation according to von Foerster, an observer can become aware of the *system of coherences* implied by his or her local distinctions, and so make many locales of such a matrix accessible to his or her actions.

An observer does more than specify the observed, with its domain and domains of possible distinctions. In the act of observing he/she as the *agent,* the coherent form that observes arises. All observers live in the domain of inter-objectivity, generated in

[33] For a constructivist, reflective languaging means reflecting on action to create knowledge.

language with other observers. The *act of distinction* that an observer performs could not be used to make a claim about an *external reality*. Distinctions that an observer makes exist more in the observer *domain of descriptions* as commentaries rather than in the *phenomenal domain* of the observed.

On the question of what humans do, Maturana (1988a, 1988b) explains that humans observe and make distinctions, both concrete and conceptual. As beings that happen in language they generate notions and relations (Maturana, 2000). Observing is an ongoing process "in which each distinction appears as the ground for the next," without questioning its own ground. (Maturana, 2000, p. 461) The consideration that observations are done with a succession of coherences of experiences appears to be key in mathematical observations. Mathematicians are mathematical observers that continually exist and live with others in a co-specified mathematical domain.

Human beings as biological entities exist in the *praxis of living* but, unlike organisms that do not reflect or attend to what they do or what happens to them, humans appear in their experience with a *presence* (Maturana, 1988a, 1988b). Maturana develops a theory of observation that might make us empathize with students at moments when they are not able to perceive what is evident for us. It appears crucial that mathematics students become mathematical observers as well. Maturana explains that this is possible by virtue of the domain of their coherences. And again students might need to be mathematical observers that do not just live in the flow of their classroom activities, but that are present in their mathematical worlds.

Human behavior, communication, domains of inter-objectivity and knowing are all aspects of Maturana's evolutionary cognitive theory. Maturana takes the middle way between the observed object and the observer's coherences; however this is necessarily circular. The encouraging point about such an approach is that it includes recursion; Maturana chooses a starting point by defining an observer, but he does not end where he began. With the help of community, time and space, Maturana (2000) gains a displacement, what B. Davis (2002) has referred to as a *recursive elaboration*, as he lays down his theory. He transforms the collective landscape. On returning to the point of departure where he chose to begin his theory, the theory and the world he is theorizing about have been changed with the action of theorizing. I will demonstrate how this is done. (This is a key feature of my research, as well.)

9.4.1 Enacted Objects

For Maturana (1988a, 1988b), it is the observer who points at behaviors or doings as he or she distinguishes an entity that interacts in and with its medium. On the side of the organism it is just changing to adapt to its medium. Actions occur in the domain of operation of the whole and embedded organism, in its *relational, the interactional* domain. This might involve only its medium.

When two or more organisms live together they coordinate their actions, which are in turn transformed. This transformation might be by way of chemical, gestural, postural, vocal or tactile coupling, or by higher communicative behavior. The flow of interactions between systems might appear to an observer as *communication.* When in addition to innate behavior organisms come to develop new behavior through which they coordinate actions, Maturana and Varela (1987/1992) say the organisms are socially coupling in the *linguistic domain* that includes instinctive and learned coordination of action. If learned communicative behavior is observed to span generations, say among social animals, the observer refers to it as *culture*. However, humans are unique because "in their linguistic coordination of behavior, they give rise to a new phenomenal domain, viz., the domain of *language*" (Maturana & Varela, 1987/1992, p. 209, italics mine).

Here Maturana's theory begins its ascent into nested phenomenal domains—the *operational domain* of internal dynamics, the *relational domain* of behaviors, the *linguistic domain* of social behavior including language and culture, and the *semantic domain* of inter-objects—in which humans make distinctions. For me his theory further clarifies how mathematical attentiveness might span more than one level—at one level as sensations, at another as perception and at yet a higher level as observation.

In the behavioral domain there is a flow in the dance of coordination of actions, which allows linguistic distinctions. As organisms recurrently make linguistic distinctions of actions in their behavioral domains, this leads to other *phenomenal domains*. The domains of language and of *objects* arise as linguistic distinctions of linguistic distinctions. Objects of distinction arise as tokens, eigenvalues[34] of the dance of actions and distinctions. These tokens as patterns of interaction, as dynamical equilibrium in turn obscure the relations they coordinate. Numbers are stable tokens for counting and subitizing, and functions as dynamically stable tokens for measuring, moving and calculating (in Appendix B, I offer two basic examples of numbers as tokens from students' work). Also in Chapter 7 we saw that the students began to attend to mathematical patterns through their writing, physical and communicative activities. In languaging we see ourselves as if we distinguish and handle objects that exist independent of what we do as we couple with other human beings and with our

[34] An Eigenvalue is a mathematical value that remains stable amidst change.

environments. By hiding the behaviors they coordinate objects as *operational analogies* enable further coordination of behavior. As well they enable different *domains* of activity to arise. Maturana dubs them different domains of *inter-objects*. They might be non-academic or academic. Mathematical inter-objects differ from other inter-objects such as hunting or biological inter-objects as they arise from a particular manner of languaging. Let us explore more.

9.4.2 Inter-Objectivity

For matters of analysis, Maturana (1998a) attempts to trace the initial interaction between two organisms. I find Bateson's (1980) experience with a dolphin helpful for understanding Maturana. Bateson decided to experiment with different dolphins in a pool. Meanwhile he watched how the flow of their interaction proceeded given his ongoing interaction. With one dolphin he just stayed still as the dolphin swam around him getting closer each time. Finally after Bateson did not move even when the dolphin used its limb to strike him, the dolphin swam away. But on another occasion when he decided to initiate interaction and respond to a dolphin say by stroking it there was more activity between the dolphin and him. The emergence of meaningful behavior between humans and animals and human and humans is a well-known experience. Maturana explains that with the initial coordination of behavior, usually in a form of a gaze or gesture, mammals are still in the relational domain of doings. It is after the response to the initial coordination of actions, in the choreography of coordinated actions that they begin acting in a linguistic domain. If this process recurs often enough, at each stage of the cycle there is *displacement* of the relation and meaning of behavior. In "recurrent social interactions, language appears when the operations in a linguistic domain result in repeated coordination of actions about actions that pertain to the linguistic domain itself" (Maturana & Varela, 1987/1992, p. 209-210). Language occurs in the domain of *regularities* of actions, in which particular communicative behaviors have *consensual* meanings. It is in this new operational domain of descriptions of descriptions that *observers* arise. Through linguistic coordination of actions, which appear to an observer as distinctions, language allows humans new phenomenal domains such as the domain of formal behavior, reflection and self-awareness.

As an object or entity arises in our new relations, meanings—the distinctions—it arises in the domain in which it is distinguished. By explaining any ongoing coordination of human actions as a *domain of inter-objectivity,* Maturana's theory stresses that objects could not exist as independent entities. To apply his theory to mathematics: Mathematics objects such as numbers, lines and curves are dependent on mathematical observers—the mathematicians who live the existence of the mathematical inter-objects as though they were independent of their interactions.

Maturana being a scientist confines his illustrations to the scientific domain of inter-objectivity. I draw analogies from his explanations to understand school mathematics as a domain of inter-objectivity in which particular kinds of objects arise as a network of mathematically significant interactions. Maturana's notion of inter-objectivity underscores the importance of interactions in enacting mathematical domains. In eco-complexity terms, mathematical objects, things and entities emerge, as signs, as patterns and patterns of patterns that, with the help of time, become the ground for further mathematical actions.

Mathematical descriptions and generalizations are abstractions of the coherences of our operation that we distinguish as we explain our mathematical experiences. They arise in our existence in mathematical *inter-objectivity* as a way of understanding some of the regularities of what we do or what happens to us as we engage in mathematical activity. This means that what we mean when we speak of objects such as numbers, lines and functions we them about in our indications.

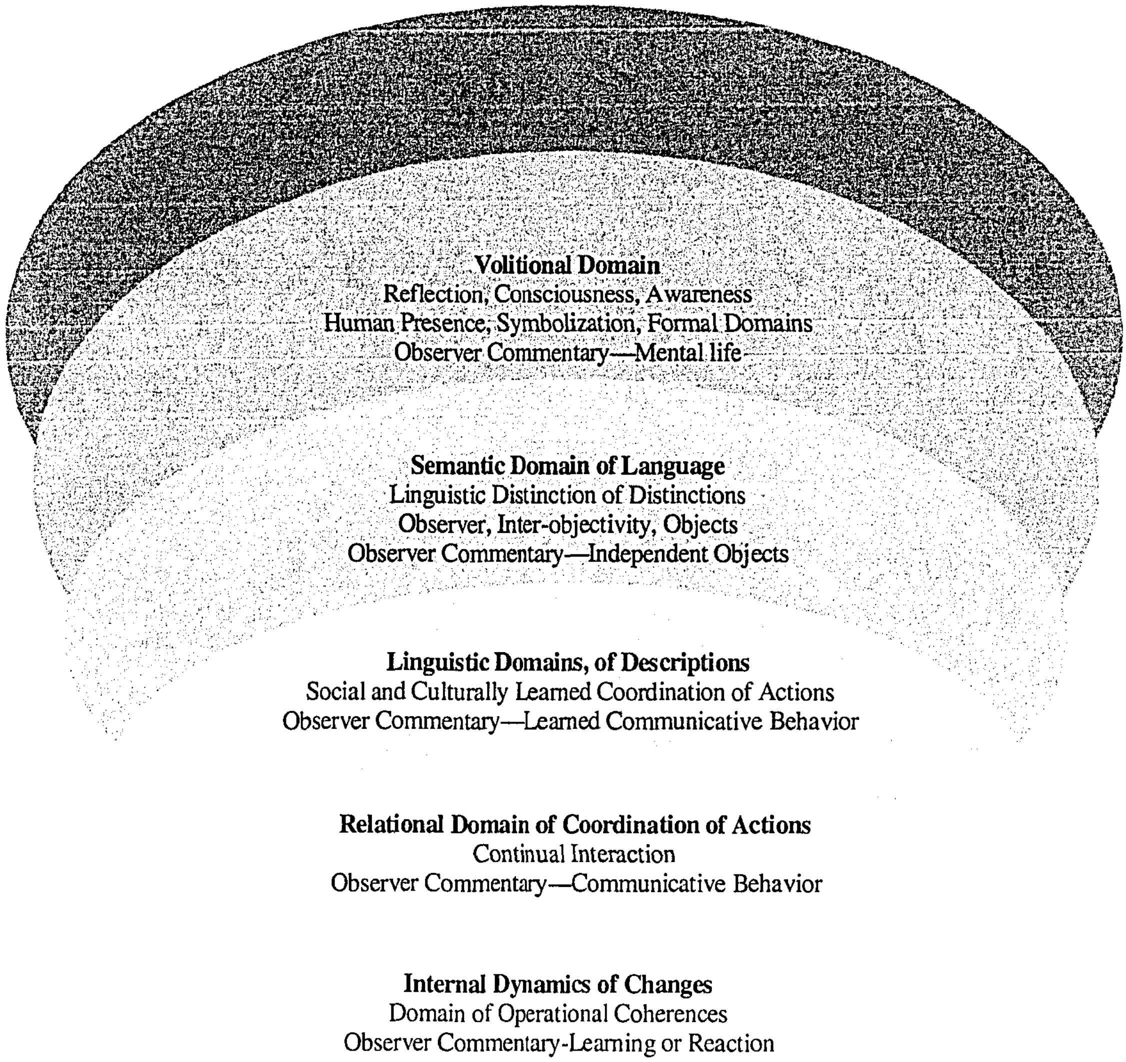
Symbolization is secondary to language; it takes place in the explicit acceptance that one particular operation of distinction will participate in the flow of continued coordination of behaviors in lieu of some other operation. In the language of semiotics it appears Maturana (2000) is making a distinction that symbolization and formal signs, what most semiotic theorists focus on, happen in a domain higher than that of language, linguistic communication and coordinations of doing, the formal-communicative domain. In this domain observers formally, institutionally or politically deliberate on how they wish to continue coordinating mathematical actions. That formal mathematics involves a multitude of symbolization, for me, is an indicator that phenomenal domains—internal dynamics, relational behaviors, linguistic social behavior and the semantic domain of inter-objects layer it.

For teachers and educators, Maturana's theory on observing appears pedagogical. Identifying the objects of the domains we teach as inter-objects offers an *ontological stance* to matters of attending. Some theorists lament that for a long time the epistemological stance has dominated issues of learning (Osberg & Biesta, 2003; Rorty, 1982). Wouldn't it be ethical for the education community to realize that the domain of objects—whether material or ideas—is always a *new operational domain* that arises from ongoing interaction? Perhaps physical objects belong to domains of interaction that most humans participate in. They are basic distinctions, whereas cultural and more academic ones are restricted to formally organized domains of interactions. Maturana's theory of observation implies that the foundational and epistemological background that we need to support mathematical thinking and activity cannot be spoken. All that exists are the domains of interaction, the perceptual world that humans bring about as they exist in matrices of inter-objectivity.

> To *know* is not a manner of reference to entities that are assumed to exist with independence from what we do. To know is to do, and all human knowing occurs as doings in the realization of our living in the domain of inter-objectivity that arises in our living in language. (Maturana, 2000, p. 466)

Maturana explains that we claim that we *know* when we do something adequate in a domain of inter-objectivity. In Figure 22 I summarize Maturana's observation theory. The theory explains the centrality of observation, actions and interactions, and reflection in attending and being. It is a theory about how we come to attend in increasingly abstract and shared domains. In theories of distinction, observation is an operation that accounts for the ontology of human attentiveness and cognition. But is it only individual humans that observe?

Figure 22 Layers of Distinctions and Centrality of Observation



9.5 von Foerster: Studying Observing Systems

In second-order cybernetics[35], von Foerster (1981, 2003) emphasizes that observation is the most basic operation for non-trivial systems. For von Foerster (1981) trivial systems like thermostats are not autonomous; their output remains the same when the input is the same. Their designers govern them externally. They are predictable and

[35] Cybernetics is the mathematical study of control, recursiveness and information, of purposive systems.

mostly independent of their past. Humans are non-trivial observing systems (von Foerster, 2003). Their feedback systems lead to changes in their operation. Over time non-trivial systems approach strange attractors and eigen behaviors as they exist in dynamic equilibrium. Whereas Maturana focuses his theory on individual human beings—perceiving-within-language, Von Foerster extends his theory to other systems. This is important since as we saw in Chapter 6 humans act as individuals as well as collectives. An exploration of observing systems allows us to extend the study of the dynamics of attending to collectivities.

Out of an infinite abundance of possibilities, humans, as non-trivial systems, after recurrent actions and interactions have the potential to behave in particular ways because they make distinctions. They mark. They designate and indicate what is observed. Also as we saw in Chapter 5, it is of paramount importance to study observing itself. Systems and ecological scientists, of whom von Foerster is one, shift to explore perceptual properties *as if* they resided within the operation of observing (von Foerster, 2003). Observing observing systems is a shift that transcends the simplistic shift from looking at things "out there" to looking at things "in here". For this study on how perception guides mathematical actions, the study of observing—perceiving-within-dynamical stabilities in the temporal course of actions and interactions—is central, especially since mathematics is one of those domains that appears to be uniquely human, socially restricted and ever evolving. Von Foerster's work leaves the list of observing systems open.

9.6 Luhman: Social Systems Observe and Attend

Luhman (2002a, 2002b) extends Maturana's notions of observing toward human social systems. He draws from Spencer-Brown (1972/1979) and from von Foerster (1981) to conclude that all autopoietic—what von Foerster refers to as non-trivial—systems make distinctions. For Luhman, observing means indicating one side (and not the other side) of the distinction. Even socio-systems invoke differences and distinctions; they discriminate in a manner similar to individual human beings. Luhman defines observation abstractly to apply to all operations that discriminate: to perception in biological autopoietic organs, to thinking in humans, and to communication in social groups. Luhman would, for example, maintain that cognitive domains like mathematics and organized religion as well as social and political institutions observe at the socio-communicative level. Part of this appears to be what Peirce was attempting to articulate by noting:

> Men [sic] who pursue a given branch [of science] herd together. They understand one another; they live in the same world. ... Sciences must be classified according to a peculiar means of observations that they employ ... great landmarks of history of science are placed at the points where new instruments, or other means of observation, are introduced. (CP 1.99-102)

Observing systems whether scientific or not are determined by the distinctions they make, which are determined by their conditions of observing.

Observing systems cannot choose the distinctions—food/not food, antigen/not antigen, exciting/not exciting, logical/illogical, mathematical/nonmathematical and moral/immoral—with which they dissect the world, Luhman (2002a) maintains. Conscious choice is an untenable notion; for all observing systems, only conditioned (or should we say coherent and emergent?) designations are possible (Luhman, 2002a). To a degree humans choose only those frameworks of observation that have been made accessible to them in material, biological and social ways. Distinctions function blindly at the moment of observation, Luhman emphasizes. Merleau-Ponty (1964) says perception is blind to itself. Thus we might need observation of observation, a second order observation, to pay particular attention to what kinds of regularities and distinctions the observed observer uses. Second-order observations have the potential to illuminate what the observing system is able and unable to perceive with its distinctions. Second-order observations also observe the conditions by which the first-order observer discriminates. In Maturana's (2000) language, through reflection observers can become aware of the *networks* of coherences implied by their local distinctions. Nonetheless, as we saw in Chapter 5, the second-order observer is blind to the conditions of his or her distinctions at that level of observation. Von Foerster (2003) observes that it is hard to find out what generates these stabilities and distinctions. Perhaps, we can hypothetically retrace their genesis as Maturana does with his theory.

To review so far, the theories of distinctions by Luhman (2002a, 200b), Maturana (2000, 1988a, 1988b) and von Foerster (1981, 2003) imply that complex adaptive systems whether they are individual learners, collectives of learners, or domains of human activities make distinctions. Bateson (1980) would say, such systems operate on differences. The work of these scholars parallels the mathematical logical work of Spencer-Brown (1972/1979) on bringing into being a universe by marking out a space.

9.7 Spencer-Brown: The Mathematics of Observation

Spencer-Brown (1972/1979) asserts,

> A universe comes into being when a space is severed or taken apart.... By tracing the way we represent such a severance we begin to reconstruct, with an accuracy and coverage that appear almost uncanny, the basic forms underlying ... mathematical, biological and physical science, and can begin to see how the familiar laws of our own experience follow inexorably from the original act of severance. The act itself is already remembered, even if unconsciously, as our first attempt to distinguish different things in a world where, in the first place, the boundaries can be drawn anywhere we please. At this stage the universe cannot be distinguished from how we act upon it, and the world may seem like shifting sand beneath our feet... All forms, thus all universes are possible. (p. xxix)

Spencer-Brown's interest in observation is of the Boolean logical kind. His explanations of how a universe is severed include vocabulary such as *distinctions, marking, states, form* and *the observer.* He writes, "We see now that the first distinction, the mark and the observer are not only interchangeable, but in the form, identical" (p. 76). From Spencer-Brown's assertion we might say that mathematicians are interchangeable with the distinctions they are making, for example: they too are marks that distinguish the mathematical domain. And so are mathematics students.

In Peirce's terms, the sign and the explanation of what it signifies make up another sign. This larger sign may require an additional explanation producing an enlarged sign that will still make up a larger sign. And proceeding in this regression, we ultimately reach a sign of itself. It appears an observing system is this reflexive sign. An observer is a sign of distinction and the distinction is a sign of the observer (Peirce, CP; 2. 231; See also Kauffman, 2001). Drawing from Spence-Brown, Varela (1974) dubs the observer the third state in the form. The first is the marked and the second is the unmarked state. The autonomous third state arises by self-indication. It is the distinction.

Spencer-Brown (1972/1979) uses the *calculus of circles*. By drawing a circle, a distinction is made that distinguishes points in the circle from the points outside the circle. New distinctions can nest preceding ones as in nested circles, and there is a possibility of self-reference and re-entry when marked states might appear to equate with unmarked ones (Varela, 1979; Kauffman, 2001). Kauffman (2001) has observed that the notation Spencer-Brown uses with his calculus of indication is in line with C. S. Peirce's notation on existential graphs, fifty years earlier. As a circumference of a circle cuts off the inside and outside in a plane, so does the skin of an organism, Spencer-Brown explains; and I would add, so do the boundaries of collectives and manners of languaging and domains of inter-objectivity. Spencer-Brown views distinction-making as a major operation that causes a universe to come into being. Recall Merleau-Ponty and Varela's notions of a perceptual and perceiver dependent world, respectively.

In my view, theories of observation are aimed at facilitating reconstruction of the basic distinctions underlying individual and collective understandings such as mathematics. Interrogating acts of severance that underlie regularities in human actions and interactions could lead to seeing how the invariants and regularities in experience follow inexorably from the earlier acts of distinctions. Spencer-Brown describes the act of severance as early attempts to distinguish different things. In the first case, the borders of the forms could be marked anywhere. To Varela (1974), "We, observers, distinguish ourselves precisely by distinguishing what we apparently are not, the world" (p. 22). Indeed mathematics observers have marked and cut mathematical worlds.

In Chapter 8, by asking the question how do observing systems observe, I wondered about what the original act of severance could be for students' mathematical thinking. At such an original stage of observation, the mathematical universe would not be distinguishable from how students acted upon it. The mathematical universe would seem like shifting sand beneath their feet. To pursue Spencer-Brown's (1972/1979) insight further we may ask: How do familiar laws of our mathematical experiences, such as number operations, follow inexorably from earlier acts of severance? Where else could we seek these original acts of severance than in the historical evolution of particular mathematical ideas? What happens among children learning to count? Should we say the original act of numerical mathematics is a distinction other than reciting and practicing writing numbers? Is it not the case that at the stage of classifying units, grouping, unitizing and seeing plurality and symmetry, all forms, all universes, and many number worlds are possible? Anthropologically speaking, at the original stage of severance, many number systems were possible. Pragmatically speaking, the Hindu-Arabic is just one system among those that were easy to learn and use. In a similar manner, at the stage of shaping and seeing form, many geometrical worlds were possible, of which Euclidean geometry was just one.

In my research study as I observed students making mathematical distinctions, I acknowledge that at every stage many other worlds—some non-mathematical, others mathematically unique—might be possible. Indeed the latest layer of my research questioning: "In what ways do observing bodies make distinctions?" is central in studying students' mathematical thinking. For as students learn mathematics they sense, perceive and observe, all-at-once. In the dynamic view of attending I am developing, the operation of observing is taken to be primary to cognition. Peirce noted this: "All reasoning whatever has observation as its most essential part" (Collected Papers, 2.605). Individuals as well as collectives are observing, cognitive systems. From the philosophical and scientific views explored above, perception is divorced from an external world, to be placed by re-entry into the form, between the external and the internal. The learner who attends or perceives, whether it is individually or in a social collective or any other non-trivial system, is taken to specify a reality rather than to grasp one. That this is so is evidenced by students' invention of signifiers that are meaningful to them. Leo, a student in the grade 7 class I observed, explained to the class how he generated rectangular arrangements for numbers. He said he would begin with one times the number (1 X n) and continue increasing one and reducing the number "until the sides met." It appears Leo had specified a reality with it distinctions such as sides. A second-order observer attempts to understand what reality the student had enacted. After the sides had met, he would know he had exhaustively generated factors of a number.

This recent development in the theories of distinction is an idea that has a long tradition in hermeneutics. Varela (1999a) admits that the philosophical source for his attitude has its origin in philosophical hermeneutics. In Chapter 6 I elucidated how enactivist and complexivist stances inevitably draw from and elaborate on hermeneutics. Next I briefly relate Gadamer's notion of prejudices to theories of distinction.

9.8 Perception Viewed Hermeneutically

Gadamer (1992) observes that our prejudices constrain and enable our perception. Our tradition, historicity and language are conditions for understanding. "While it is true that prejudices limit what we are able to see, it is also true that, were it not for our prejudices (our pre-judgments) we would not be able to see at all." (Davis, 1994, p. 160) In Merleau-Ponty's (1964) language, "We do not suppress our ties to time and space; *in fact* we utilized them" (p. 40) "[O]ur prejudices constitute our being" (B.

Davis, 1994, p. 160). Gadamer (like Davis) casts prejudices in enabling terms, as theories of distinction would cast the blind spot as an enabling constraint. Re-reading hermeneutics alongside ecological complexity prompts me to look beyond and within formal-mathematical factors—mathematical traditions, inherited histories and formal languages—as I study the dynamics of how students attend. Prejudices could be taken as *readiness*-to-attend in particular ways. My analyses illustrate that more than formal factors such as symbolization are at work when students are attending. For the most part they are *inclined* to perceive and observe the way they do in ways that structurally embody their neuro-physiological and emotional make-up as well as their lived, shared and inherited histories. Students as individual unities, composite unities, and socially embedded and technologically enabled attending systems attend in complex ways. But the radically embodied, embedded and extended view of perception urged in this chapter is far from being a folk theory or an educational theory. Is it just a philosophical view that has nothing to do with education practice? To answer this question involves studying senses of the verb to attend that are common in education practice. It may also involve a pondering on why educators as is the case with psychologists focus on conscious, outward, sensorial, focal and effortful attention at the expense of non-conscious, peripheral, derived and intellectual (James 1992/1878-1899) attention.

9.9 Views of Attending in Education: William James

How can the education community study attention in ways that are not delimited to overt attention? To James (1981/1842-1910), attending is "the taking possession by mind, in clear and vivid form, of one out of what seem several simultaneously possible objects or trains of thought" (p. 381). In theories of distinction it could be paraphrased as the making of a distinction, be it at the conscious level or a deeper one, as enacting a world of significance. Given the dynamic views of perception explored in this chapter, it might help to search for the suffocated senses of the verb to attend such as stretching, being present, waiting and awaiting. Attending involves being present in a broadened sense. To James (1981/1842-1910, 1992/ 1878-1899), an *associationist*, attending involves habits, interest, past experiences, emotions and social participations. When we attend, we are present in ways that span the biological, psychological, social, technological and cultural, this chapter adds. Perhaps instead of linking attention to *interest* the way James does, following Varela (1999a), we can use the terms *preparedness-to-act* that would apply even to other cognitive agents. It might be this readiness to act that arouses, orients, divides, selects and fixes as well as sustains attention. As Varela explains, when it falls apart then effortful attention and interest is needed among humans. But effortful attention cannot be sustained for many seconds without the support of passive, derived and global attention (James, 1992/1878-1899). James explains that continued (strong) attentiveness in a particular domain inevitably makes one a genius in that domain. In manners supportive of the eco-complexity view I have presented, James asserts that by attending we choose a universe to experience and inhabit. For him education should be geared toward adaptation of attentiveness to ideas and sensorial objects, say mathematical attentiveness.

In education, and in humanities in general, other scholars such as Langer (a sociologist) have studied the centrality of perception in terms of awareness. Langer (1997) develops a theory about mindful awareness.[36] Some researchers, such as Rosch

[36] She contrasts her mindfulness learning theory to the traditional intelligence theories. Mindfulness, and at times awareness, is in the Tibetan sense taken to be open-ended reflection on the experiences of mind

(1999a) and Varela et al. (1992) are increasingly drawing from Eastern meditating practices to extend the discussion to the phenomenological aspects of perception. Rosch (1999a) maintains that if the senses do not actually perceive the world, if they are instead *participating* parts of the mind-world whole, then a radical re-understanding of perception is necessary. These independent explorations all elucidate that teaching and learning need a dynamic view of perception if they are to understand students' difficulty in *getting* what is taught.

The work I have explored in this chapter broadens attention from sense impression and volitional effort toward sensation, perception and observation. Attention orients. It sustains engagement as it deeply chooses a universe to inhabit, it enacts an *attentional* world. Attention participates in making distinctions and creating its objects. Education could usefully draw from this dynamic view of attention. Given the nature of school subject domains as subjects that involve sensorial activity as well as spatial imagination, mental imagery and abstraction, the participatory and dynamic view of perception explored here is useful. It is a philosophical and physiological basis for understanding perception in dynamic ways. In the next chapter I explore how mathematics educators could *sharpen* students' mathematical attentiveness in ways that await mathematical objects and enact mathematical worlds.

(Varela et al., 1992). Langer critiques traditional intelligence theories' emphasis on perception as representation of a world. She says this view has upheld the notion of one right answer and one right perspective.

10. ORIENTING MATHEMATICAL ATTENTIVENESS

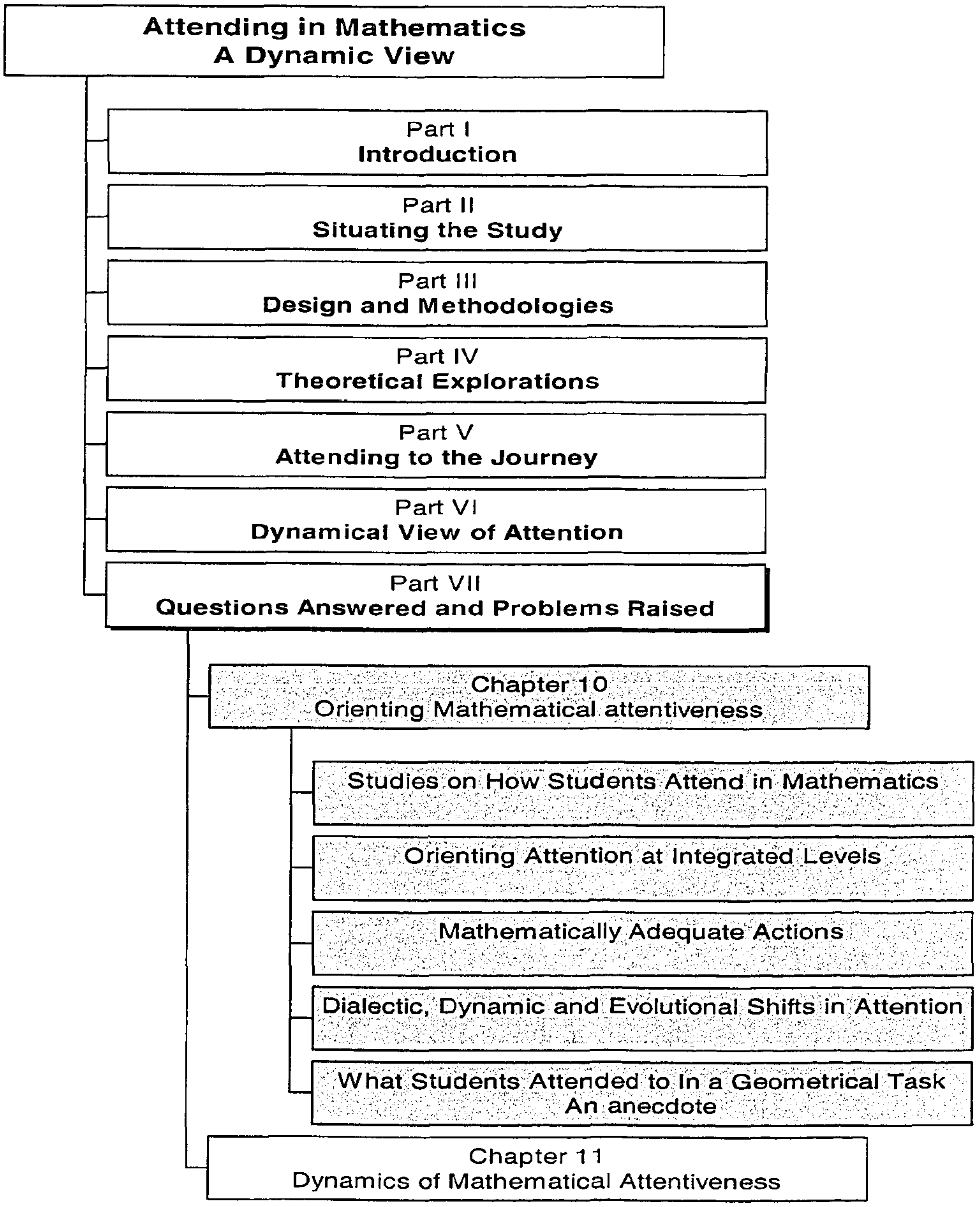
The adoption of computer models to understand cognitive states revived studies on the learner's internal dynamics, particularly the presumed mental processes. These studies elaborate on gestalt psychologists and mentalists' interest in *mental* aspects such as memory, perception and object recognition. Results of studies on perception and attention in experimental psychology that were interpreted in support of behaviorist theory—stimulus-response patterns—are being re-interpreted in ways that do not bypass the internal coherences of the learner. With cognitive scientists' interest in the role of internal dynamics, specifically brain dynamics, research on consciousness, perceptual awareness and similar activities of mind is, so to speak, out of the closet. Constructivist theorists with their strong tradition of research in mathematics education have paid particular attention to students' conceptual structures. In my view, research on mathematical thinking has much to benefit from drawing from recent non-cognitivists' interpretations.

The emergent view developed in this writing underscores that studying the dynamics of students' perception and attention in a particular human domain is central to investigating the nature of students' cognition. How do students attend? In the last chapter I developed a view of attention, which does not delimit itself to visual, conscious and formal ways of attending, much less to individual and sensory attention. It is now time to return to the question that motivated my interest in students' attentiveness: when do shifts in attention happen and how do teachers occasion students' mathematical attentiveness?

There are multiple and mutual interactions in the dynamics of attending.

Therefore a layered and interconnected way of orienting students to attend in mathematically adequate ways is needed. Further, since perception and attending are inseparable from action, an exploration of mathematically adequate activities is also in order. Already identified constructs in mathematics education offer a starting point for exploring the dynamics of students' attention. In this chapter I examine the ways in which shifts in attention in addition to being switches in modes could be irreversible changes in ways of attending as well as transformations in worlds enacted. I explore the dialectics that involve radical or irreversible changes. I draw from a classroom anecdote to illuminate these complexities of attending. But before exploring all these, I look at isolated cases of research on how students attend.

Dissertation Landscape Forming



10.1 Studies on How Students Attend in Mathematics

Students' mathematical attentiveness has not been a common focus in education. Apart from research done on representations, signs and visualizations, a few mathematics educators such as Mason (1989), Hewitt (2001a, 2001b), Watson (2003), Sfard (2000b, 2001a), Marton and Booth (1997), and Booth and others (1999) study the significance of understanding what students pay attention to. In some other studies, such as Simmt (2000), Kieran (2001) and Davis (2000), the topic of students' attention has been explored on the periphery. In earlier chapters, I briefly reviewed the insights gained from studies on, epistemological errors, visualization and representations. For the reason that Mason and associates, Marton and Booth, and Sfard and Kieran's research on mathematical attentiveness appeared to be a special case that closely focuses on what students see, experience and communicate I chose not to explore it until now.

Mason (1982) asserts, "[L]earning consists of shifts in the structure of attention" (p. 9). Mason (1989, 2003) focuses on the *form* of attention—what is attended—and the structure—how it is attended. Hewitt (2001a, 2001b) and Mason maintain that it is important that students' *awareness* be developed. Mason appears to use *awareness* and *attention* interchangeably. Hewitt (2001b) uses *awareness* in its everyday usage to refer to observation, be it sensory, idea or automatic. Both Mason and Hewitt offer ways in which students' awareness can be studied, worked with and developed. Watson and Mason (1998), Mason (2000, 2003), Watson (2003), and Hewitt (2001b) offer some methodical ways to provoke and *direct* students to focus on mathematically significant forms. Among other things, they emphasize that the style of tasks influence students' mathematical attentiveness (Mason, 2000). Watson and Mason (1998), and Mason (2000) offer exposition formats, questioning styles and task designs such as student-generated examples that provoke core mathematical ways of attending. Similarly, Hewitt (2001b) works with students' common errors and structure of questions to *direct* or even *force* students' awareness to be focused on the attributes mathematicians attend to. As we saw Mason, Hewitt and Watson infer how students should attend mainly by interrogating their own experiences. Sfard does so by closely analyzing students' utterances.

To Sfard (2000b, 2001a), what students attend to as evidenced in their communication is significant, since communicating mathematically is *thinking mathematically.* Sfard (2000b) asserts that in any student utterance there is the *pronounced focus*, the *attended focus* and the *intended focus.* For example, when Lillian said, "The interval is not helping" as she worked with Irene on the Consecutive Terms (CT) task, her pronounced focus was "the interval". Her attended focus was the *set of numbers* {3, 5, 6, 9 and 10}, and she and Irene were working at finding the *common difference*, their intended focus. With the same pronounced focus, students might be attending to varied things ranging, in the interval case, from *a table*, an *arrangement of concrete materials*, a *set such as* $\{3^{+2}, 5^{+1}, 6^{+1}, 9^{+1}, 10^{+1}\}$, or even an *equation*, all focus that involve relations among pairs of numbers. Sfard acknowledges that the attended focus—what students perceive—is usually instinctive, habitual and below conscious and thus not explicit, Vergnaud's theorems-in-action (see Appendix B). Nonetheless, the attended focus, as a constituted ontology, has aspects that can be perceptually shown, linguistically explained, formally compared and externally regulated. These are what we observe to be mathematical structures. To *focally* analyze students' discourse Sfard (2000b) closely maps their gestures and verbalizations. She traces the mathematical objects that students might be attending to—the attended focus. She, in a manner similar to my work, emphasizes that students by attending engender mathematical objects.[37] Although Sfard restricts her investigation to individual students' utterances, paying less attention to their written work, joint attention, use of materials and tools, her analyses focus on the variations in what can possibly be attended with one pronounced focus.

Like Sfard, Kieran (2001) focuses on analyzing students' talk as a way of understanding the individual student's mathematical thinking that emerges through discourse. Sfard (2000b, 2001a, 2001b) and Kieran (2001) observe that students attend not only to the mathematical objects—object level attention—but also to the discursive patterns and meta-discursive rules—non-object level attention. In my research work, students have at times attended to clues, to each other's attitudes and to non-mathematical contexts in the tasks. For example, Rose and Norah interpreted the phrase "how many ancestors does a male bee have in the twelfth generation back" as a clue to a fractional solution for the Bee Genealogy (BG) task, (refer to Vignette E2, Appendix E for detail). At many moments non-object level attention is not distinct from object level attention. Thus sometimes the duality between object and non-object levels, like any other observer-constructed duality, has been limiting. Kieran and Sfard nonetheless raise awareness of the fact that what students attend to, the form of attention, includes more-than-object-level aspects.

Marton and Booth (1997), as I showed in Chapter 8, study what students attend to from a phenomenology framework. They argue that studying what students attend to contributes to exploring how students experience mathematical concepts, or indeed, what

[37] In mathematics, the scarcity of sensorial mediation of what is attended might be one of the reasons why students find learning it difficult, Sfard emphasizes.

the concepts are in themselves (Booth et al., 1999).

Simmt (1998) asserts that emotions have a role to play. She observes that shifts in attention might sometimes be caused by mathematical constraints, and at other times by emotions such as excitement or frustration. Davis (2000), following Gadamer, notes that we seem to attend to our habits and prejudices, and our expectations *cloud* our perception. For instance, Rose and Norah considered themselves done with the BG task in less than 7 minutes. They appeared to engage with the task in ways that were aligned better with the well-defined, quick-fix and simple pattern questions found in textbooks. It is not surprising that they did so, since when we enact new perceptual worlds we do not totally transcend our familiar perceptual worlds.

Whereas Sfard (200b, 2001a) views thinking mathematically in a linguistic-cultural manner as *communicating mathematically*, Mason (2003) uses philosophical-phenomenology to view thinking mathematically as *coming to sense mathematically*. To Mason, Hewitt and Watson mathematical attentiveness is a skill that can be encouraged (Watson, 2003). In addition to teasing out the sense made of mathematics by students, Sfard attempts to tease out the mathematical objects students engender. Whereas Marton and Booth (1997), in a manner closely aligned to phenomenology, elaborate on concept descriptions.

Thinking mathematically, as I pointed out earlier on, in addition to being synonymous with *speaking and sensing mathematically*, is synonymous with perceiving and observing and acting, including touching, *emotioning*, gesturing, listening and writing mathematically. Herein lies the point of departure from some frameworks that view the role of perception in learning to be peripheral and limited to already formulated and observable mathematical features. My own view is akin to the theorist noted above, except that my approach draws from a radically embodied view to examine not only the matter (what is attended) and structure (how it is attended), but also the coherences and dynamics of students' mathematical attentiveness as well as the objects enacted by attending in particular ways. Attentiveness is participatory in enacting mathematical objects.

Further, as an embodied experience and a way of enacting worlds it can be oriented in complex ways. When we view students as embodied, embedded and extended systems we ask deeper questions about students' mathematical attention or inattention. In my study, what counts as mathematical attentiveness expands to integrate many modalities of sensing, perceiving and observing, as well it spans more than one order of signification. Mason (2000), Hewitt (2001b) and Watson (2003) all emphasize that to encourage and orient students to attend mathematically takes degrees of explicitness and variation and happens over time. The dynamics of multidimensional attentiveness, I explore below, support their assertion.

10.2 Orienting Attention at Integrated Levels

The question at the heart of my research is how to occasion students to attend in mathematically adequate ways. But to answer this question an understanding of the dynamics of students' mathematical attentiveness has to be sought. If mathematical structures do not *cause* perceptions, but there is instead a *dynamic context* that orients what students attend to and how they attend, then we ought to look at mathematical structure in relation to learning ecologies including students' ways of attending. As we saw in Chapter 9 humans make biological, perceptual, semantic, social, political, formal and goal-directed distinctions—some of these are within (or constitute the) mathematical objects level, others are not. What makes mathematical objects salient for students derives from an assembly and synchrony of all these and more aspects. As an observer I note that the principles that children and learners use are the same as those used by adults and experts. However learners are unique in that they may not be aware of culturally and formally significant attributes (Lakoff, 1991; Rosch, 1999a), they may add "false" attributes or give salience to different attributes from those that experts would highlight. Their worlds might be distinct. Here are principles framing my speculation on how students may be oriented to attend mathematically:

- Learners are self-organizing. Teachers are limited when it comes to directly instructing students, but can create conditions that trigger certain attributes to spring forth.
- Learners are embodied systems. They have nested autonomous sub-personal systems and signification spaces, such as nervous and motor systems. Teachers could be attentive and responsive to how learners attend by targeting more than the personal, conscious space of signification. The somatic and experiential spaces all participate.
- Individual learners are embedded systems whose behavioral states exist in mutual *causation* with the behavioral states of neighboring personal systems as well as of embedding supra-personal systems such as social collectives, cultural practices and formal structures. The collective and cultural, with its technological, linguistic and symbolization possibilities, provide another level at which mathematical attentiveness can be awakened.

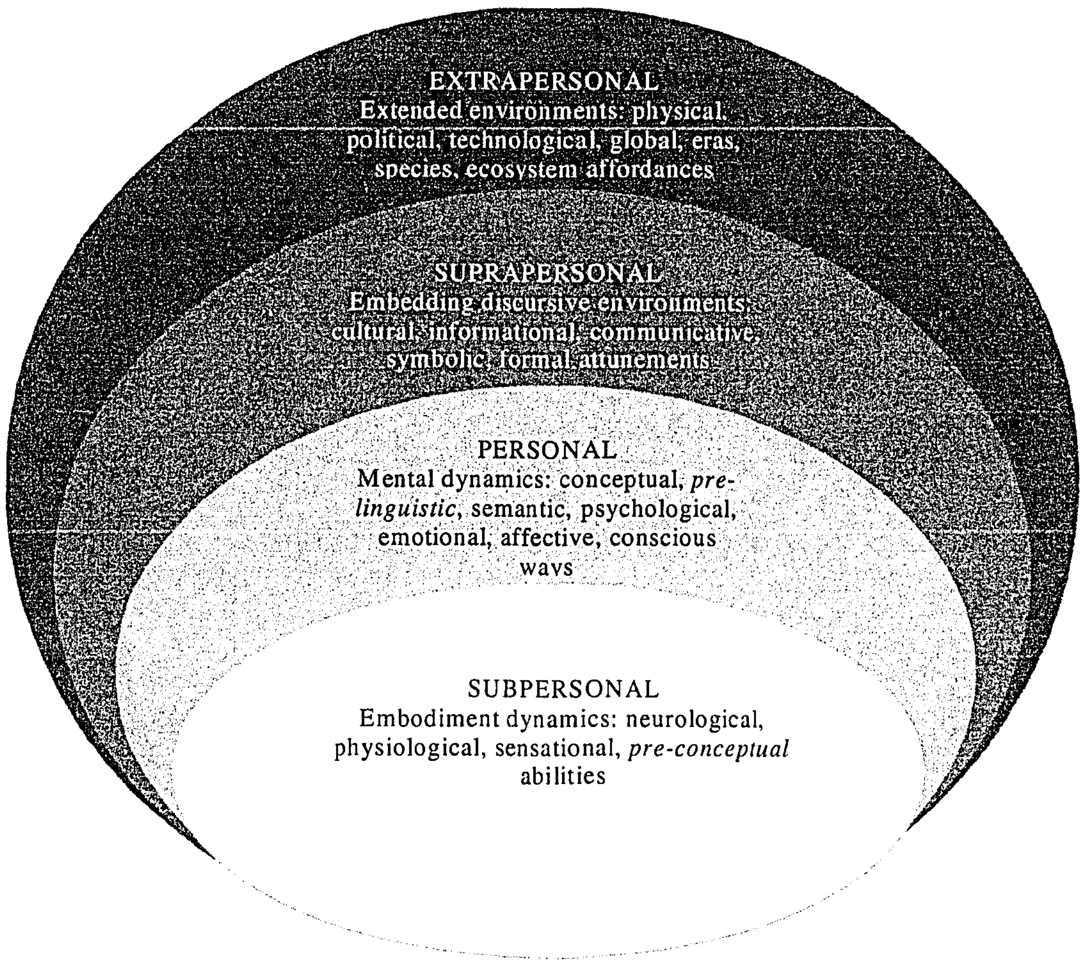
- Learners, be they individual or collective, are extended systems with an external structure that includes symbolic and cultural tools, technological media, and material instruments. Mathematical attention can be occasioned by mindfully braiding it through these extra-personal contexts.
- Learners are personal and conscious individuals who have intentions and goals. At this self-conscious level they can be motivated to attend in significant ways, but this space is just one in the patchwork of intertwined levels of signification.

Guided by the above principles it is helpful to talk about *attuning, triggering, orienting, calling, disposing* and *occasioning* learners' attention, as well as enhancing mathematical *propensities.* When we dynamically view students' attentiveness then the space from which a teacher can orient students expands to include the physiological, emotional and neurological, as well as social, technical and cultural experiences. A number of these levels have been addressed in the existing work in mathematics education, particularly the conscious and formal symbolic levels. My contribution here is to provide a broader framework that synthesizes and orients existing suggestions. Understanding that the levels are integrated is key. Complexity research's notions of emergence, nestedness and inter-level causality are most helpful. For purposes of analysis, I explore each level separately. Revisiting layered diagrams would be helpful at imagining how the levels are interconnected. Figure 23 organizes the learner's contextual dependencies and internal dynamics into a unity.

10.2.1 Sub-Personal layer: Sensory-Motor and Pre-Conceptual Salience

In mathematics education, scholars such as Dienes and Goldings (1971), Cuisenaire and Gattegno (1957), and Bruner (1960) emphasize targeting the perceptuo-motor sphere by including concrete or enactive activities. A teacher can perturb changes by providing materials that students touch, feel, move, play with and orient their bodies to. To the extent that radical constructivists such as Steffe and Wiegel (1992) encourage offering students physical activity, they too attune students to attend mathematically at the perceptuo-motor, the sub-personal layer in Figure 23. In Chapter 7, I explored the role of manipulative materials as physical and sensory embodiments and linguistic enactions of mathematical concepts. But there is more to sensory and perceptuo-motor mediation than the use of manipulative materials.

Figure 23. Multi-dimension Approach to Orienting Attention



There is emerging work on perceptuo-motor activity and *knowing with* instruments. At a research forum at the 27th International Psychology of Mathematics Education meeting (PME) Nemirovsky (2003b) asked:

> - What are the roles of perceptuo-motor activity, by which we mean bodily actions, gesture, manipulation of materials, acts of drawing, etc., in the learning of mathematics?
> - How do classroom experiences, as constituted by the body in interaction with others, tools, technologies and materials, open up spaces for mathematics learning?
> - How does bodily activity become part of imagining the motion and shape of mathematical entities?
> - How does language reflect and shape kinesthetic experiences? (p. 104)

In my view, these are promising questions that put the body and bodily activity back into what it means to attend mathematically. Nemirovsky and other researchers emphasize gesture, rhythm, utterances, vocalization, and kinesthetic, tactile and proprieceptive resources in cognition. To a larger extent, mathematical thinking evolves from and with bodily attentiveness. Hence the structures noted as outer spheres in the diagram have feedback loops with the bodily action and ability.

10.2.2 Personal Mind: The Sentient and Rational Attendee

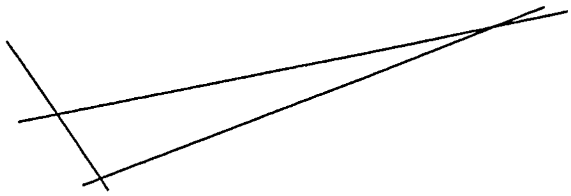
More work has been done in understanding what students attend to and how they attend at the layer of the personal mind than at the other layers in my diagram. When radical constructivists focus on bodily activity they are interested in how this activity triggers the re-organization of a child's conceptual structures. Most work on representations, signs and visualization focuses on students conceptual, semantic and psychological attributes. As well work on meta cognition and reflection focuses on the sentient, rational and logical personal level. It should be noted that there is more to this level than forming mental images, deciphering formal labels and learning concepts definitively. The personal, conceptual layer is interconnected with bodily activity at the sub-personal layer by reciprocal causation: it emerges from coupling of bodily activity with the environment in which it occurs. Semantic and conceptual distinctions at the personal layer in turn provide contextual dependencies for bodily perceptions and pre-conceptual experiences at the sub-personal layer. Language and linguistic forms at the supra-personal layer above the conceptual engagement nuance the dynamics of students' conceptualization and pre-linguistic experiences of the personal layer. Let me elaborate by exploring work on attending to logical definitions and on meta cognition.

10.2.2.1 Rational and Formal linguistic Ways

Pimm (1988) and Mason (1998) note that names and labels, especially those agreed upon in the wider mathematics community (the supra-personal ellipse in my diagram), evoke connections in the minds of students (the personal ellipse). Even arbitrary notations may assist recall (Hewitt, 2001a; Simmt, 2000). Mason identifies frameworks generated in a classroom as either *metonym* or *metaphorical triggers* of different associations for the personal mind. A label like *powers of two* could trigger shifts in the attention of a student to that which is (or is not) mathematically significant (Mason, 1998), all depending on the student's internal (sub-personal) dynamics and social and cultural (supra-personal) embodiment. For students from different cultures, social economic statuses and religious backgrounds, labels such as *powers* and *mass* may trigger political and religious connections (Zevenbergen, 1996). For example, Rose and Norah had a more colloquial understanding of generation than what was intended in the BG task.

Semantic structures and linguistic forms of the supra-personal layer influence what students attend to in ways that do not exclude non-linguistic experiences at the other three layers. That this is the case is evident in the observation that it is not sufficient to teach mathematics definitively and by using only prototypical examples. For example, even when we formally define a triangle as a three-sided polygon, the diagrammatic examples offered in textbooks and in teaching as well as the technological environments used (the extra-personal aspects) further constrain students' imagery of a triangle. Teachers and textbook authors unconsciously use geometrical prototypes—usually triangles with non-obtuse angles or triangles with one side horizontal or vertical—to illustrate the triangle geometrical concept. Yet Monoghan (2000) observes that a generalized diagram or illustrative example for a given concept does not really exist. Given all these observation how can we teach students in ways that help recognize even special cases of triangles? Sfard (2001b), and Ball and Bass (2003) point to examples of students who know the definition of a triangle but cannot recognize non-prototypical triangles, such as very thin ones. Sfard (2001b), favoring the verbal-definitive (the supra-personal and personal layer) way of orienting students' mathematical attention, speculates that when students attend to the verbal definition of a triangle as a three-sided polygon rather than to the overall visual similarity to the prototype examples, the triangle shown in Figure 24 definitely appears to them as a triangle. Is favoring logical definitions against, say, sensory-motor experiences at the sub personal level a viable solution?

Figure 24. A peripheral example of a triangle



Ball and Bass (2003), and Rotman (2000) offer a different view. Rotman (2000) says that using mere definitions to teach geometry is absurd.

> The triangle-as-geometrical-object ... is not only what makes it possible to think that there could be purely abstract or formal or mental triangles but also an always available point of return for geometrical abstractions that ensures it never being abstracted out of frame of mathematical discourse. (p. 57)

For Ball, verbal definitions are no better than prototypical examples, especially since a formal definition does not exist in isolation of other related definitions. Some standard definitions and categories are too narrow to include all cases that could be considered (Ball 2002; Ball & Bass, 2003). Ball offers, however, no ways of generating rich and interconnected mathematical definitions. Logical definitions are only one aspect of the supra-personal layer—a formal linguistic aspect. As I will show later, definitions surface meaningfully from the interaction between aspects within the supra-personal layer including the cultural and informational as well as with aspects from the other three layers. Even in attending to concepts definitively we repeatedly return to perceptual and conceptual and draw from material experiences, and sensorial and bodily images. Affect is a factor of the sub-personal layer that nuances rational discourse. I consider cultivating desires and emotions that are central in the mathematical domain of inter-objectivity as one of the ways of occasioning mathematical thinking. Also I speculate that concrete, rich situations do not only engender but they sustain relational mathematical objects. The question of how the general, the formal and the patterned arise and in turn govern the concrete and particular is crucial. Eco-complexity theorists embrace the possibility of novelties and idiosyncrasies in emergent ideas that might result from different personal, sub-personal, extra-personal experiences.

In this study, mathematical thinking is more than formal linguistic mathematical discourse that concerns itself only with activities of the supra-personal space. It is a way of being, acting and interacting within a community of mathematical observers who have bodies and minds that are intimately and on an ongoing basis coupled with the environment. It is a particular drift.

10.2.2.2 Heuristic and Metacognitive Ways of Orienting Attention

At the personal level of signification, students can be invited to reflect on and discuss the dynamics of their attention. They can be compelled to set goals to attend in mathematical ways. One way to do that is by making their mathematical attentiveness the object of attention by way of heuristics, problem-solving strategies and reflecting (Mason, 2000). Mason explores the difference between *awareness-in-action* and *awareness of awareness-in-action*. He claims that the former involves a high degree of being immersed in what you are doing, whereas the latter involves freeing part of your attention to focus on how you act, perceive and think. Experts usually are aware of what structure of attending is required by their discipline, Mason claims. Humans, as well-integrated psychic systems that engage in the personal behavioral domain with possibilities such as free will, control, planning actions and experiencing their observations, could eventually learn to guide themselves to pay attention to the themes, heuristics and processes of a discipline. Maturana's theory of observation points to the significance of becoming a second-order observer of one's own mathematical activity.

In the educational research there is a distinction made between cognitive and meta-cognitive shifts in attention. Mathematical heuristics could provide learners with readiness-to-attend (Mason, 1989; 2000), in other words with a meta-cognitive heuristic. Maturana (2000) alerts us that when humans as observing beings do not attend or reflect about what they do or what happens to or in them, they do not experience. Without reflection or awareness of awareness the "human being" will not arise in the doing of what they do. For example, the students will not appear in their mathematical experiences, and they will not exist in the mathematical domain of inter-objectivity. Said differently, their being and identity will not include a mathematical dimension. Schoenfeld (1985), among others, has examined how teacher awareness of what is mathematically relevant can be taught to students' strategies. Such researchers argue that learners can, through monitoring and reflection, develop their mathematical attentiveness (Albert, 2000; Mason et al., 1985; Powell, 1997; Watson & Mason, 1998). The contribution of such studies cannot be overestimated. For instance, the problem-solving strategies developed by Polya (1945/1973), Schoenfeld (1985) and other researchers may help students learn from the structure of mathematicians' attentions (Romberg & Carpenter, 1990). Nonetheless, practices such as self-monitoring, developing an inner teacher and meta-discursive rules, when reduced to content or turned to routines, are not of much use (P. J. Davis & Hersh, 1981; Love, 1988; Mason, 1988).

Meta-cognitive generalizations need to be viewed in the light of their biological constraints, for most thoughts and actions (including those that we would consider intelligible or those that were once formulated) are unformulated—they are just lived habits or *deep ways* of being. Indeed, as part of those processes enabled by tradition and by the environment, much thinking is ineffable even to the reflecting and thinking agent (Blumer, 1969; Hadamard 1945/1996; Lakoff & Johnson 1999). We seem more able to act our way into thinking than we are able to think our way into acting and interacting (Bruner, 1996). Higher levels of self-consciousness are not involved at all times—many times they have subsided into habitual behaviors and at other times it is patterns in behavior or established observation practices, at yet other times it is emergent surprises and sometimes it is attending in accord with evolving collectives and emerging cultural revolutions.

In the language of theories of distinction, by paying attention to the distinctions they make, together with their blind spots, people occasion more reflexive attending of their mathematical attentiveness. Nonetheless conscious awareness, if we are to act in real time, needs to be assisted by habits, bodily knowing, technologies of knowing, external memories and mathematically rich cultures and contexts. The attentive fabric is multi-dimensional. It has parallel yet correlated surfaces, external and deep structures. The contexts of mathematical attentiveness are dynamic. As in other complex global systems, strengthening unified and interwoven threads rather than isolating them is central.

10.2.3 Supra-Personal Layer: Social and Human Collectives Attunements

The social collective is a learning agent that constrains what is perceptually and conceptually possible for an individual learner. In the classroom, through interaction with other students and with the teacher, students might find themselves making mathematically adequate distinctions. As a significant part of a student's external structure, by focusing on the drift of the collective mind in the classroom, a teacher, can structure what and how students attend. Since the other students and the classroom intellectual environment constrain what students might be mathematically attentive to, the teacher might also focus on orienting the attention of the classroom as a collective. In Merleau-Ponty's (1964) phrasing, the other bodies could *haunt* a student's body into attending mathematically.[38] Davis et al. (2003) maintain that it is easier to observe and occasion the cognitive activity of the classroom collective than to do so for each individual student. Participating in varied and rich mathematical collectives enhances individual mathematical attentiveness. Work by social constructivists and socio-cultural theorists appears to be targeted at attuning habits and abilities of collectives, cultures and institutions so as to incline students to attend mathematically.

It is increasingly emphasized that how we attend in mathematical tasks is not always a conscious and rational activity (Kieren & Simmt, 2002). We are aware of the world through the collectives that intersect in us and through our histories (Gui Gnon, 1991; Nørretranders, 1998). Interpretively speaking, the traditional practices we are born into, the languages we speak, our experiences, and the collectives we constitute condition what we attend to. To put it differently, we attend in conscious and personal as well as unconscious, non-conscious and collective ways. This appears to be what Booth et al. (1999) point to when they describe learning as a joint constitution of insights in two senses, both as a broadening of an individual's cognitive repertoire and as an experience that happens when individuals jointly make sense. Collective attending in the latter sense arises bottom-up from what individuals in a class initially and successively attend to. The dispositions that emerge from this collective attention in turn constrain individual

[38] *Haunting* is not used here in the behaviorist sense of reinforcing and punishing. For a complexity science theorist who consents to the idea that learning is being is living, haunting could be taken as a constraint to the states that are accessible to individual learning.

student's attention. Within the cognitive systems that emerge in the classroom, broader and invariant ideas emerge from a pool of situations. As a shift in what has been enacted, these in turn, suddenly and radically shift what can possibly be attended to.

In this system's view of attentiveness, we can note the dynamics of attention for individual students as well as those of a collective of students. Cobb et al. (2000) and Cobb (1999) identify reflective discourse as one of the shifts in attention that can be performed jointly in mathematical classrooms. Kieren and Simmt (2002) have elaborated on Pirie and Kieren's (1989) model for observing the growth of mathematical understanding to trace the dynamics of students working in pairs. Distinctions made collectively have the potential to become embedded in the social dynamics creating patterns of behavior that trigger and are nuance by cultural evolutions. We *attend* and *think within* collectives and cultural consciousness as well as institutional boundaries.

Newberg et al. (2001), in their neurological work on ritual, caution that all routine work should not be eliminated in an attempt to embrace meaningful learning. Collective rituals, as is the case with individual routines, in themselves by virtue of their effect on sensory, motor and pre-conceptual experiences have the potential to contribute to mathematical attentiveness and thinking. Rituals seem to induce affection and collegial conditions that are in themselves conducive to joint attention. They also affect bodily and psychological states and abilities.

10.2.4 Extra-Personal layer: Technological and Material Affordances

The teacher can influence the dynamics of attention in the case of both the individual student and the collective of students by ordering cultural artifacts, symbolic media and technological means of observing. Although distributed cognitionists have done work in this area, in mathematics education the area needs more research. Rather than considering tools such as computer dynamical geometry environments as media for representing existing mathematical ideas, eco-complexity metaphors prompt us to study them as fundamentally cognitive factors that change what it means to know geometry. As I showed in Chapter 7, mathematical insights and ideas emerge from the interactions at an inner layer of activity. Bartolini's (2000) work on semiotic mediation focuses on this tool and artifacts level of orienting students' mathematical thinking. Nemirovsky's (2003a) research forum and Radford (2003) also focus on classroom experiences as constituted by the body in interaction with tools, technologies, artifacts and materials.

Robutti and Arzarello (2003), and Borba and Scheffer (2003) explore the role of instruments in students' thinking. They explore the idea of *knowing-with* technologies such as orality, writing, reading, computer technologies and measuring instruments. They study how these media of intelligence, acts of distinction combine with bodily activity, with everyday and mathematical language, and with imagination to enhance or redefine what is known. In the case of my study, students came to know which elements do not have the CT property through writing, drawing, using manipulative materials and joint actions. Through physical environments the teacher may offer, students might be triggered to observe in mathematically adequate ways. We do not only think about. We think with tools, instruments, media and materials such as the Fraction Kits, counters, and number line to enact varied worlds of mathematical significance.

What it means to learn a disciplined and practiced way of living or working extends to include fluency and imagination with available tools. "The one who knows is not a 'lonely knower' nor a collective formed only of humans. The basic unit of knowing always involves non-human actors such as media" (Borba & Scheffer, 2003, p. 126). It is evident in daily life that during human perception, intelligence and memory the human skin and senses become connecting circuits to material and technological tools and other learning environments. At this level of the extra-personal, it shall be recalled that cognitive systems have extended structures. (Unfortunately, in mathematics education there has been less sensitivity to the relation of media to the concepts to which they claim to offer access.)

10.2.5 Integrating the Layers into a Coherent Unity

What students attend to and the dynamics of their attentiveness could not depend on states and abilities at one layer. It depends on the learner's overall dynamic state. My exploration of mathematical attentiveness might explain why a person may attend differently over time. It explains that original internal dynamics, personal history, interactions, current dynamics, physical, social, cultural or temporal contexts or unique circumstances do matter. When a student does not notice a mathematical pattern that another observer notes, it may be that the student's attention is directed differently or elsewhere.

In my observations within the research sites and in my interpretations of the work of others I have come to appreciate how the background from which specific objects of attention stand out involves a flexible dance of the overall learning dynamics. I presume that to attend away from the mathematical entities, as some students do in their "erroneous" visualizations, is in some sense to attend—given the current structures, history and environment—to some other co-constituted ontologies.

To claim that students' attention can be occasioned does not imply that it must be explicitly focused and directed. It is more a question of creating a space for students to attend mathematically, which could be done by concurrently and over time making it conducive for them to carve out mathematical worlds. In enactivism and complexity research, we theorize that conscious, personal and communicative structures are a portion of the structures that participate in learning (Nørretranders 1998; Núñez, 2000).

More-than-conscious and rational directing of personal attention guides our anticipative, concentrated and sustained attention. Linguistic attunements, bodily histories and activity, cultural dispositions, subliminal and external memory, collective orientations and extended structuring structures are among the many fundamental *persons-plus* (Perkins, 1993). These factors orient the ways in which we attend.

My work has been an unpacking of what had been thrown in the closet as *un- or not-conscious* influences, in metaphysical terms, the talents, abilities and energies. They needed to be taken out of the closet and brought back as foci of mathematics education researchers' attention. Mathematical attention binds body, mind and world together; both inward and outward aspects participate. In this way, it is a more useful aspect of mathematical cognition to investigate than representations.

When students interact with mathematical tasks, a myriad of mathematical worlds are possible, as has been shown in my and other research on children's mathematics. Yet whatever sense is made, whatever object is attended or enacted it is not unique. The same distinction imposes itself as real for *every one standing* where that particular student as a complex system is "standing". If it were a non-mathematical distinction, the student would view his way of attending as non-mathematical only if he or she were invited, consciously and in many other ways, to stand where mathematicians stand in relation to the task at hand. An attitude of inviting students to stand where mathematical distinctions would impose themselves as *real* is what my development of a complexivist view of occasioning mathematical attentiveness is about (see Chapter 9). With such an inviting attitude teachers are likely to notice and legitimize the proto-, novel and creative mathematical forms that students enact.

In being attentive to how students attend we may work toward building the conditions that enable attention to mathematical ideas. We may also create spaces for novelties and hybrid ideas to emerge more regularly. Since mathematical attentiveness is an ongoing mathematical *preparedness-to-hand*, we might want to think about orienting attention by encouraging students to put themselves in situations that incline them to think mathematically. In the next section, I elaborate on mathematically adequate actions as one way of ensuring that mathematics students are mathematical sensors, perceivers and observers.

10.3 Mathematically Adequate Actions

During my study, I noticed that the ways students attend braided physical and social environments, as well as lived experiences, into what they attended to. In the analysis it became apparent that students' mathematical thinking is circularly originated by what I have referred to as *adequate mathematical ways of being*. These may include actions such as the act of recording systematically, carefully and creatively, desires such as the desire to make and test conjectures, and the urge for more elegant and general conclusions. In this section I elaborate on mathematically adequate actions as one way of looking at domains of distinctions made by mathematical observers. These are dynamic regularities that emerged as I observed students' actions.

If mathematics is a human activity, it must draw from species-specific endowments such as what Mason (2000) calls powers to categorize, pattern, reify and generalize. It must flourish from recurrent actions, such as repeatedly grouping together, and from interactions common in the human environment. Such interactions are prompted by specific language games, supported by patterns of behavior and practiced within social criteria of acceptance. Mathematics is a domain of inter-objectivity in which peculiar ways of attending (Peirce, C.P., 1.99) are practiced and hence particular kinds of objects and entities enacted. Observers in the school mathematics domain of distinctions acts have a particular manner of languaging and being.

During my research sessions, mathematically adequate ways and desires appeared to co-emerge with adequate mathematical actions. By the final sessions I had begun to notice mathematically adequate writing, speaking, reading, sensing, handling of manipulative materials and interacting in ways that were almost predictive. When students acted in certain ways, they enacted mathematical entities and attended to mathematically significant aspects; they, on an ongoing basis, acted themselves into thinking mathematically. The initial and ongoing dynamics that channeled and triggered mathematical ways of attending included:

- Organizing work
- Recording results and working systematically
- Searching for generalizations—looking for invariants
- Beginning with a particular case or simpler example—specializing
- Seeking a shorter and less laborious method—seeking elegance.
- Using notations that reduce the bulk of written work—

compressing

- Seeking error-free approaches
- Looking for patterns and regularities while working

This list could also be seen as part of the criteria for acceptance of actions as mathematically adequate. In my view, when students', actions fit these criteria they were working under the mathematical desire for systematic, general, organized, elegant or *certain* observations. Simmt (2000) maintains that the desire for simpler or elegant ways of doing what one is already capable of doing is an emotion implicated with mathematical thinking. Rather than reducing the numerical size of the task, making random guesses, looking for clues to appropriate operations or shying away from the complexity and magnitude of a mathematical problem, many students sought structures, relationships and generalized patterns. They recursively threw away the *nonessential* for the sake of economy, elegance and computational effectiveness. When they made conjectures, they sought to verify and justify them. Consequently, their behaviors converged on mathematical insights and rich conceptual understanding. My list focuses educators' attention on students' ways of doing things even at basic levels such as reading the question, listening and writing. Freudenthal (1991) observes that in the activity of mathematizing, humans progressively organize, structure and schematize their experiences to generate mathematical ideas and concepts whose proofs are as important as their descriptions. Mason (2000) lists the mathematical powers of stressing and ignoring, specializing and generalizing, imagining and expressing, conjecturing and convincing, ordering and classifying, and abstracting and instantiating. Like Freudenthal's, Mason's list appears to be a level above, and so emergent from and thus providing context to the adequate mathematical ways of being I have outlined above.

In sum: If you are to behave mathematically there are some things (although there are many more things including novel ones that you can do) you simply do not do. At this point a caution is in order. Outlining mathematically adequate actions is not the same as saying these actions linearly cause mathematical attentiveness; there is reciprocal causation, co-emergence. Each of these aspects is a cause and an effect. Stating that adequate mathematical actions are inseparable from mathematical attentiveness emphasizes this double bind of originating and constraining one another to a range of potentials. Acting mathematically is attending and thinking mathematically. In the last chapter after developing a new beginning and returning to the classroom to observe mathematical attentiveness, I elaborate on mathematically adequate actions as dynamical attractors of mathematical thinking.

10.4 Dialectic, Dynamic and Evolutional Shifts in Attention

One of the initial questions of this study was: When do students shift their attention to mathematically significant observation as they engage in mathematical tasks? In Chapter 3, I reviewed various categorizations, some related to modes of thinking and others to ability in mathematics. Images are classified as relational or verbal, and modes of thinking could be algorithmic or visual and concrete. Concepts are classified as conceptual/logical or figural/visual, and representations are external or internal, semiotic or non-semiotic, and not-formal or formal. We could examine shifts in attention along these distinctions, say from concrete images to relational ones. That is to say we might observe students' attention shifting from one pole to another—observe the switches. For dyads where researchers have specified one mode as more mathematical than another, desirable shifts would then be from non-mathematical to mathematical foci. But given the dynamic view of mathematical attentiveness presented here we may want to pause and ask about the nature of relations between the poles. Must they be oppositional relations? Or might one pole be a node in a complex network of other poles? Rather than always viewing shifts as a switch from one mode of thinking to another, in Chapter 11 I emphasize their developmental and evolutional nature.

Other poles describing what students attend to in mathematical tasks have included the trio of process, products and *procepts*, and the duality of object-level and non-object level responses. The process-product-procept triad has been helpful when related to the means of objectification in mathematics; especially in tracing how mathematical objects historically or developmentally emerge from successive empirical and imagined actions. Sfard and Linchevski (1994) offer the example of algebra that they trace from generalized arithmetic—the operational and functional phase—to abstract algebra—formal operations and abstract structures. For example, students as we saw with the CT task, may be observed to be working with algebra at the abstract level or arithmetic at a more concrete level. It should be emphasized, however, that sometimes these shifts involve folding back, and they always seem to involve leaky boundaries.

Distinctions such as those between abstract and concrete, images and concepts, and cognitive and meta-cognitive levels have limitations, as shifts between them not always appear to be linear and are, at times, seamless parts of a unity. In some cases it is a dynamic interaction between two poles that is more important, as what was once abstract becomes concrete in the activity of mathematization (Mason, 1998; Sfard, 1991a, 1991b). Pirie and Kieren's (1989) model is structured in terms of *vertical* levels along which one could observe growth instead of horizontal switches in students' understanding. Given an eco-complexity sensibility we could think about different foci of attention within mathematics as interrelated levels through which students' understanding is observed to drift recursively and in an ongoing process. Further studies that will seek ways in which a multi-layered view of extended and deep mathematical thinking affects mathematics teaching are needed.

The fact that mathematics is not a unified domain adds additional shifts among sub domains (Cuisenaire & Gattegno, 1957; Freudenthal, 1991; Mac Lane 1981). Artigue (1999) identifies relations among incompatible mathematical worlds. She observes that adequate modes of thinking in calculus might not be adequate in algebra. In Chapter 7, I showed that the equals sign might mean something different depending on the sub-domain of mathematics being considered. There is also forgetting, including positive forgetting, and habituation. Shifts that are transformational may involve broadening or traversing existing mathematical thinking. Radical shifts that result in re-organization of earlier understandings as well as re-activation of parallel ways of knowing appear to be central in learning. Many shifts involve enacting new perceptual and mathematical worlds. Some involve integration, yet others call for exclusion. How is this possible in real time? Complexity theory's logic of emergence may help conceptualize radical and irreversible evolutions in addition to switches, shifts and growths. But is it able to explain all transformations and differences in objects and ways of attending mathematically? We need better metaphors to help visualize shifts in mathematical attention as a distributed yet integrated process. One such metaphor is the landscape metaphor that guides this writing. Offering the landscape metaphor, in Chapter 11 I illustrate evolutional leaps in students' mathematical attentiveness. Let me juxtapose the ongoing discussion about orienting students' mathematical attentiveness with an anecdote.

10.5 An Anecdote on What Students Attended to in a Geometrical Task

This anecdote is drawn from two consecutive lessons on transformational geometry. It is from the classroom research project. Appendix G contains the entire transcript of the introductory part of the first lesson. I present a brief summary of the first lesson here, with excerpts from the transcript. I will present the second lesson in the concluding chapter.

*Vignette 4: **Symmetry Review Lesson***

The teacher started a class discussion by asking about objects that have three lines of symmetry. One student offered, "Triangle". The teacher requested clarification, "What kind of triangle?" After a number of contributions, there was soon some agreement that an equilateral triangle was the only triangle that had three lines of symmetry. The teacher then asked, "Okay, how many lines of symmetry does a square have? Joseph." Joseph paused, "Ummm, eight". "Not a cube but a square," the teacher responded as she drew a square on the overhead. A number of students began to call out, "Four". Another student agreed with Joseph's first response. "No. Eight." "Let's see …" The teacher began drawing a vertical bisector on the square. " There is a line here …" "Horizontally and two diagonally," Joseph said, guiding the teacher. In a soft voice another student said, "Eight". "Eight?" The teacher acted confused. "Four," another student re-asserted. "Can you think of an object that would have eight?" the teacher asked. In a chorus most of the students shouted, "Octagon". John's hand shot up. The teacher called on him to offer an answer. "I think it has more than eight". As the teacher drew the octagon, Tim also, speaking to himself in an excited tone, said, "A circle, oh!" Janelle sitting close to Tim said, "A circle has 180." In the meantime, the teacher was still drawing bisecting lines in the octagon. It was obvious that she was unaware of Tim and Janelle's conversation. The teacher completed her drawing and counted together with the students "So if we have 1, 2, 3, 4; 1, 2, 3, 4. I think there are 8. Not 16. Where would the 16 be?" The excerpt below follows immediately.

EXCERPT 5

38	James:	I know one that has infi—nite!
39	Teacher:	You know one that has an infinite? *[Playfully]* Don't say it.
40	Kayla:	There is a shape with lots.
41	Teacher:	You know one that has ... lots. *[Teacher points at individual students as they raise their hands one by one]*
42	Ken:	*[Nearly inaudible]* Me too.
43	Teacher:	You know one that has what?
44	Ken:	Lots.

Note: In the discussion that follows, the turn numbers refer to the transcript in Appendix G.

10.5.1 How Can a Teacher Attend to and with Students?

In the class discussions, most of the students were attending to what the teacher drew their attention to—a kind of object with three lines of symmetry, nature of symmetry in a square then in an octagon. However a few seemed to be behind—checking whether a square really had 4 lines of symmetry—and a few others seemed to be ahead, attending to the circle as an object with lots of symmetry. Put differently, students attended as a whole class to symmetry in polygons. They also attended in sub-collectives to the symmetry of a circle, for instance (turn 31). As individuals, some students attended to distinct aspects or idiosyncrasies of symmetry (Joseph-turn 10, Stella-turn 15, another student in turn18—a square has 8 lines of symmetry). The path of the lesson appears to have been laid in ways that had not been directly anticipated by the teacher. The collective classroom mind drifted toward thinking about an object with the most lines of symmetry. This was not unfortunate, though.

After checking for an object with three lines of symmetry, followed by checking for the kind of symmetry in a square, some students' locus of attention might inevitably have wandered to lines of symmetry in a pentagon, a hexagon and so on, or to objects with five, six, then seven or more lines of symmetry. As in most lessons in the class, the moment-to-moment actions of individuals unfolded into what could be observed as the foci and loci of attention of the collective as well as of individual students. It was not the teacher's explicit intention to explore with seventh graders the symmetrical properties of a circle, much less of a sphere. While the teacher was drawing a square to assist students in determining whether it had four or eight lines (turns 1-20), in the cognitive domain of the collective it appears that the teacher and the students, given that some thought a square had 8 lines of symmetry, were drifting into naming objects with 4, 8, 16, 32, ... lines of symmetry. As the class was exploring objects with eight lines of symmetry, James interrupted the discussion (turn 38) saying that he knew an object with infinite lines. Prior to that in an independent conversation among Tim, Edwin and Janelle they had considered whether a circle has 180 lines of symmetry. In the collective, a focus of attention had gradually but not slowly drifted to finding an object with most lines of symmetry. The individual and initial events, as well as ongoing dynamics and classroom context, determined the drift of students' attention, from square to objects with 8 to 16 and more lines of symmetry. The students agreed that the circle was the object with lots of symmetry. The next excerpt comes at a time when the teacher is concluding this review part of the lesson.

EXCERPT 6

51	Teacher	Okay, at the count of 3, *[the teacher instructed.]* "An object with an infinite number of lines of symmetry. 1, 2, 3."
52	Students	Circle *[the students called out.]*
53	Edwin	Nothing *[Edwin was a lone voice that called out a different answer]*

[The teacher did not take up his suggestion, (It is not clear whether she heard it, on the video record Janelle, John and Tim can be observed discussing the question, of whether it would be possible to draw lines of symmetry for nothing. It was in that conversation that Tim turned to his colleagues saying,]

55	Tim	I was thinking a sphere with the same diameter as a circle; a sphere will have more lines of symmetry than the circle.

[At the end of the class the researcher-observer asked Tim about his conjecture. He responded]

56 Tim A sphere might have 360 times more lines than the circle.
The teacher followed up Tim's conjecture in the lesson next day, on Monday. In the lesson as we will see in the concluding chapter students discussed how they knew that a circle had infinite symmetry.

10.5.2 Mathematical Attentiveness is more than Paying Attention

If we were to look at attention as paying focused attention to what the teacher was doing, then it would be easy to say that some students, like Tim, Janelle and Esther, were not paying attention. But does this description really fit Tim and Janelle, who were examining the properties of symmetry in a circle and, later in turns 55 and 56, a sphere? How about Esther and Janelle who were pondering 16 lines of symmetry for an octagon (turn 50)? The non-dynamic way of looking at attentiveness would reduce this complexity. The dynamic view suggests that how students attend is much more complex than keeping quiet and sitting in rows; it is a dimension of cognition that explicitly implicates the history, initial conditions, and internal and external dynamics of the students. Students like Tim and James, as well as Joseph, Stella and Esther were more than present in the class. They not only directed their ears and energies to what the teacher was reviewing, but they looked out for—the other etymological sense of *attend*—mathematical insights. Drawing from their past experiences and current interactions, they stretched their understanding. From their ways of engaging and their interaction emerged the whole classroom's examination of the symmetrical properties of a circle, a sphere and of nothing as well as students' understanding of the infinite in the lesson that followed. But how did each individual student attend to symmetry?

10.5.3 Attending to Symmetry in Polygons

Of the above anecdote I ask: In what ways did the students attend in the task of finding the symmetrical properties of a square, for example? In what ways were the students attending when they said a square has eight lines of symmetry? The teacher in this class appears to have asked this question. Witness her response, "Not a cube but a square." As well the action of drawing a square on the overhead projector could be interpreted as an attempt to construe what Joseph was attending to when he thought that a square has eight lines of symmetry. What is more interesting is that after the teacher drew in the vertical bisector of the square, Joseph is the one who guided her to draw the remaining three, "Horizontally and two diagonally." How can we reconcile the fact that Joseph's drawing of a square would not have been different from the teacher's—a horizontal, a vertical and two diagonal lines with the fact that he thought it had eight lines of symmetry?

We might argue that Joseph's perception changed as he guided the teacher to draw in the lines of symmetry, but then what of those other students such as Stella who even after the drawing activity insisted, "An octagon doesn't have ... [8 lines of symmetry]" (turn 27). How was Stella attending? Could it have been that she was making errors in counting the lines of symmetry? At least she did not disagree on the visualization of the polygons and on the drawn lines of symmetry.

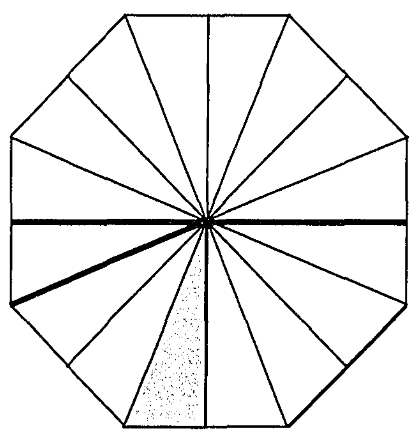
By offering to draw on the OHP, the teacher seemed to implicitly rule out many possibilities of attending differently. In addition, the teacher loudly and systematically counted the lines of symmetry with the students. "Where would the 16 be?" (turn 360) she asked after counting the lines in an octagon. In the flow of events the teacher had attempted to attend to students who attended to symmetry differently. It is then that James interrupted that he knew an object with lots.

With this interpretive moment I have continued to speculate about how students attend to mathematical concepts like symmetry. How did the few students enact 8 and 16 lines of symmetry for a square and octagon, respectively? How about those students who were in agreement with the teacher? What could be the distinctions, acts of severance or observations that were the links of symmetry for these students? How students attended in this activity points to broader questions—what was the distinction "symmetry" for these grade 7 students? What ways of attending engender symmetry? What does the complexivist view of cognition have to say about the nature of the concept of symmetry? Is it a basic or non-basic category? How do people enact it?

Recall that in this analysis I am not solely examining the individual child's understanding. Rather, embracing complexity and the assumption it makes about the emergence of collectivities, I recognize that the individual student, as Bruner (1986) would put it, is not totally free from his or her culture, the language that speaks through him or her, the environment that he or she interacts with, as well as the broader social conditions, and human history at large. In addition, the individual child's conceptual understandings are finely tuned by his or her neuronal activities, physiological dynamics and lived history. Mathematical attentiveness does require as many aspects as possible to override the aspects that might be driving the students' state and operations of awareness in non-mathematical ways.

Asserting that all sense and all worlds are possible, how does it make sense, for a square not a cube to have eight symmetry lines? With the dynamic view of attending that goes beyond sensation to perception and observation it is possible to see that even some of the students who gave correct answers had enacted structures other than those conventionally attended to. For example, some students might have been attending to the vertices—"the corners," as one student was heard saying in the lesson. In fact an equilateral triangle has three corners and three lines of symmetry. An octagon, as shown in Figure 25, has eight corners and eight lines of symmetry. All regular polygons have as many lines of symmetry as they have vertices. For a student who learned symmetry by examining only regular (prototypical, best examples of) polygons, it appears legitimate to identify symmetry with the number of vertices. Thus even among the students who attended correctly, there might have been some students who attended in ways that are mathematically inadequate, especially if they had learnt about and experienced symmetry in impoverished environments that did not explore rich and complex situations. As well even adequate ways of attending to symmetry in polygons are multiply realizable.

Figure 25. Symmetry in an Octagon



On the other hand, what could a regular octagon have 16 of? Taking on Stella's invitation to see what an octagon has 16 of, in Figure 25 we see that an octagon, with all lines of symmetry drawn, has 16 radii, 16 possible images and objects, and 16 sectors. A student need only mark a state that is not conventionally marked to enact more or fewer lines of symmetry. Stella could have given salience to these background attributes as symmetry. Recall the Fraction Kit activity in which some students stressed the quantity rather than the size of the pieces. Students attend to their own perceptual worlds. Stella probably also added false attributes in addition to giving salience to attributes that are distinct from culturally significant attributes. But why this was the case is another focus of this analysis.

Reflecting on what students attended to as symmetry in polygons raises broader questions about how the concept of symmetry was historically enacted. What is the nature of its grounding like in ordinary activities? How much of it can children enact before formal instruction? What distinctions do mathematicians make when they talk about symmetry? What are the conditions for its emergence as an autonomous idea or concept that later becomes the grounds in reference to advanced concepts such as symmetrical matrices in linear algebra? What experiences and sensibilities should be provided to students in order that they may stand where they can see symmetry the way mathematics requires at a given time?

The view of distinctions and concepts as emergent properties, rather than as classical categories, helps in addressing these questions. From the non-cognitivists' theories of concepts, general ideas or categories appear to emerge from many cases of a graded structure, perhaps as the abstractions, invariances or emergent structures. They are not classical universals, commonalities or abstractions, nor are they logical sets per se (Juarrero, 1999). Concepts are bottom-up properties, never conceived as mere definitions or general categories but rather as global properties that arise from interaction at a level below that of general concepts.

Reading this chapter may tie up some threads that have emerged through this work and also prompt some new questions. The chapter you have just read is the beginning of the end. To conclude my dissertation I will not engage in the traditional practice of summarizing the theory, themes and issues discussed. Rather I use the last chapter to end where I began with an interpretation of students enacting mathematics. However, I now do that interpretation with new tools, new concepts, and new interpretation of mathematical thinking. After all, the real test of a theory is in its generative and interpretative power.

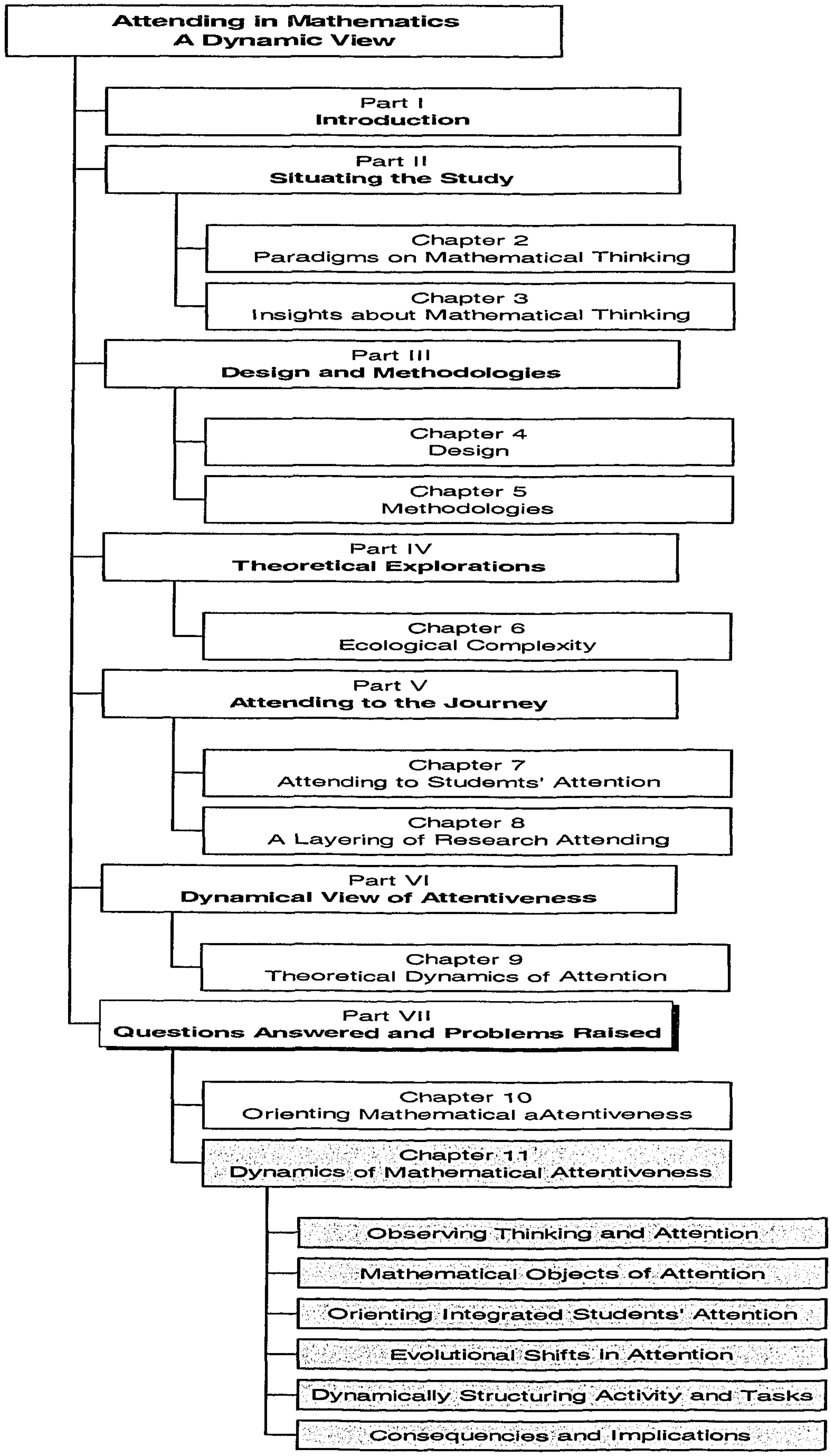
11. DYNAMICS OF MATHEMATICAL ATTENTIVENESS

My study was a theoretical effort to approach teaching as creating necessary conditions for mathematical sense-making. I addressed three questions that remain in the literature on mathematical thinking: What is the nature of mathematical thinking? To what extent or in what ways is it an individual, a social or a contextual phenomenon? And in what ways do teachers or researchers observe and occasion students to think mathematically? This I did by particularly exploring the dynamics of what students attend to in mathematical tasks. I addressed my investigation by specifically asking: How do secondary school students attend as they engage in mathematical tasks? In what ways do students attend in mathematically adequate ways? In what ways do they await and dwell with mathematical objects? When do shifts in attention to that which is mathematically relevant occur? In this writing a number of themes emerged. As well questions surfaced and I find myself in a layer above the one at which my initial questions were posed when beginning my research. As a way of returning to the initial questions that motivated this study I share these emergent themes and questions. Because I believe interpretive research ought to be functional, this chapter also explores the possible consequences of the ideas developed throughout this dissertation.

To offer final thoughts about the study I re-viewed the preceding chapters. This re-looking is an act of research itself. New ideas, images and thoughts continue to present themselves to me. Ideas that had not been well formulated become clearer. Yet others that I thought were well articulated appear to require more attention in order to understand them. Thus I find it impossible to make this conclusion a mere representation or summary of the ideas presented throughout my work. I recognize the impossibility of simply summarizing; every re-reviewing, every thought is a new thought; in each moment, my identity, my very being is transformed.

This chapter is a re-presentation and presentation. It is a recursive elaboration of the ideas of earlier chapters. Indeed as a chapter written after the writing of all the other chapters it is a chapter of an emergent order. It is about exploring the landscape that I have formed and transformed in doing the research as well as envisioning the space of the possible implications that my work evokes.

I began Chapter 1 with a classroom anecdote. I begin this concluding chapter with a classroom anecdote as well. Drawing from this anecdote I explore themes and questions. I conclude by sharing the significances. But first here is a map showing where we have been and where we are.


Attending in Mathematics
A Dynamic View
Part I
Introduction
Part II
Situating the Study
Chapter 2
Paradigms on Mathematical Thinking
Chapter 3
Insights about Mathematical Thinking
Part III
Design and Methodologies
Chapter 4
Design
Chapter 5
Methodologies
Part IV
Theoretical Explorations
Chapter 6
Ecological Complexity
Part V
Attending to the Journey
Chapter 7
Attending to Studemts' Attention
Chapter 8
A Layering of Research Attending
Part VI
Dynamical View of Attentiveness
Chapter 9
Theoretical Dynamics of Attention
Part VII
Questions Answered and Problems Raised
Chapter 10
Orienting Mathematical aAtentiveness
Chapter 11
Dynamics of Mathematical Attentiveness
Observing Thinking and Attention
Mathematical Objects of Attention
Orienting Integrated Students' Attention
Evolutional Shifts in Attention
Dynamically Structuring Activity and Tasks
Consequencies and Implications

*Vignette 5: **Transformational Geometry Lesson 2***

In this second lesson on transformational geometry, there was a twist as students discussed objects with most lines of symmetry. Recall that at the end of the first lesson, described in Chapter 10, while the majority of the students agreed that the circle had many or infinite lines of symmetry, Edwin offered "nothing" and Tim offered a sphere as objects with the most symmetry. The teacher, therefore, intended to follow up with other objects displaying reflectional or rotational symmetry in this next lesson. Instead, the class ended up discussing how it was possible that a circle could have infinite lines of symmetry. In the excerpt that follows we see how part of the class did not agree anymore with their conclusion from the previous lesson. I have bolded **verbs and clauses** of particular interest. Verbs that are interpreted to REFER TO SYMMETRY are capitalized. I present the portions of the conversation that are of interest; the turn numbers on the left show that I am leaving out some of what was said. Excerpt 7 begins when the teacher together with the class examine a line of symmetry drawn by a student in an isosceles triangle.

EXCERPT 7

29 Teacher: I guess our question is: does this actually **REFLECT**.

34 Teacher: You need to imagine an object with three lines of symmetry, something ....which you can **FOLD** in three places and have perfect symmetry. What object might present that? *[Students offer, "Equilateral triangle". One student comes over to the OHP to draw the triangle showing its symmetry lines.]*

42 Teacher: I had asked last day... I would like you to think about or I asked you to imagine an object that has [...] the greatest number of lines of symmetry that you could imagine?

[Student raises a question whether teacher requires a shape or object. In the next thirty turns the class discusses the difference.]

74 Student[a]: Circle. Circle (has most lines of symmetry)*[quietly speaks out but seems not to be heard by teacher]*

75	Teacher:	Do you have one in your head Saul? *[Students begin signaling to the teacher by show of hands when they can imagine an object.]*
78	Teacher:	Julius has one. So does Janelle. Greg, do you have an object in mind?

[After ensuring that everybody has a shape in mind the teacher asks Kimie]

86	Teacher:	Kimie, what if you told me, what shape is in your head?
87	Kimie:	A circle
88	Teacher:	A circle. So let's see, how many lines of symmetry does a circle have?
90	Edwin:	**1800**
91	Students:	**Infinite** *[Other students do not wait to be chosen. In a burst of activity, many say their answers out and many break into small group discussions.]*
93	Allen:	Infinite
94	Teacher:	**What do you mean by infinite?** *[Students have hands up, including Julius and Tim]*
95	Saul:	***[Uses his pen to quickly mark many points on paper,*** *he does not stop until turn 119 when teacher asks him to stop]*
99	Allen:	I was just thinking like basically when they **GO ON FOREVER**|
100	Tim:	|It is **INCALCULABLE**
104	Julius:	**I don't think it is infinite though, because eventually it would not GO ON FOREVER.**
105	Student:	It would **stop.**
107	Joseph:	Yeah, you will|
108	Students:	No No, it will … *[In a burst of energy, most students try to say something; they spontaneously break into small group discussions.]*
109	Teacher:	Okay let's listen to Julius, and everybody gets a turn […]
110	Julius:	**No for like when she said a circle has infinite lines, I don't think it is infinite, because eventually it would WEAR OUT** *[Many students already have their hands up, including Janelle, Stella and Steno. Some want to support Julius' assertion]*
113	Student:	They will **keep going**
114	Student:	Yeah
115	Student:	No
116	Student:	It would **not stop**
118	Janelle:	**It isn't infinite but, but it is INCALCULABLE. You can't, you don't know HOW MANY lines there are.**
123	Teacher:	You may be able to calculate it, but can you **COUNT**?
124	Steno:	**When you CUT a circle in half, only, only you have about one side and you see about how many lines are they. Once I get that...**|
125	Teacher:	|When you **CUT** a circle in half, one side will be, how many

lines will it be?

131 Stella: **The lines would eventually cross like each other. So there is no space in between the line,** [*laughter*]

133 Tim: Yeah. But even if **there are no spaces** in between the lines, the line is um... The **line** is just to tell you in (red/brain?) where the, the um (appropriate?) is. You know. When you go to the molecular level it is more than a line. A line is actually very thin at the molecular level.

143 Teacher: Let us think about this one: Right now let us imagine this circle. *[The teacher is holding up a* ***protractor****. This appears to be an idea that the teacher picked up from Steno in turn 124 when Steno talked about cutting a circle in halves.]*
If we imagine this circle: **how many lines of symmetry are noted at this point on this circle**?|

144 Student: **|360|** *[A student whispers to answer the teachers' question about degree lines marked on a protractor]*

148 Edwin: A--lot

150 Tim: **|Plenty|** *[Stella has her hand up]*

154 Stella: **36**

155 Teacher: 36. Did you figure 36? Why 36?

156 Stella: Because, like um there is a line for every degree.

163 Tim: **If you MAKE them closer together**

168 Tim: **|There is only 18|** *[Tim corrects Stella]*

170 Janelle: **|180|** [Janelle adds]

173 Teacher: 36 in half. There are 18 lines because the top and bottom go together. Good point. *[Teacher re-affirms Tim's correction]*

182 Allen: **Okay you go HALFWAY between each line, then they would technically never touch,**

183 Allen: Because you **are going only half way, and you have to go all the** way, for them to ...touch.

187 Teacher: What if we go every 10th of a degree, not touching! This has something to do with the question of **thickness of line**, which we **DRAW**, which Tim raised|

188 Tim: The question is how **thin** can you go?

190 Tim: Because once you get down to **a single molecule**

193 Tim: |Because ... *[unclear on videotape, explains how we can keep getting thinner to molecules]* you can even **SPLIT** the molecules...

194 Tim: Because you can still **split one thin line and still.** Yeah—**DRAW** another one.

202 Kay: I think it is infinite until like we **don't have material** to actually **see** it... *[Inaudible at end]*

204 John: I... I think it is infinite, **like it keeps on going**. Because if, let's say if even though we don't **have**, we can't **see** how small it is we can still **know** it is possible so we can **IMAGINE** something may be small.

205 John: So let's see, if the smallest would be 1 we **IMAGINE** 0.5. So we just keep on getting smaller no matter how small it is.

207 Kathy: I think it is just like those numbers that go on for like forever. They can keep on getting smaller even though we **necessarily can't see it.**

Note: [a] I use the name Student when the student speaking is not identifiable on the video record

In this Grade 7 discussion on what it might mean for a circle to have infinite symmetry, a number of questions and issues about students' mathematical thinking and attentiveness are apparent. Using the themes developed through the thesis I note them here and follow up with brief discussions of each of the issues I identify.

- What are the observable features of students' thinking and attentiveness in this discussion on symmetry? Observing Thinking and Attention
- What are students attending to as symmetry in a circle? Mathematical Objects of Attention
- How can we occasion students such as Julius and Stella to attend to symmetry and spatial concepts in general in ways that enact continuity so that the lines do **not eventually cross each other** nor does the circle space **wear out**? Orienting Multi Layered and Integrated Attentiveness
- How could a teacher evoke shifts in students' attention to that which is mathematically significant in the understanding of symmetry? Evolutional Shifts in Attention
- In what ways does this task of discussing symmetry of a circle, particularly the use of a protractor, structure students' mathematical thinking and attentiveness? Dynamically Structuring Tasks

In general, in what ways are students attending to symmetry in this lesson? How do they as embodied, extended and embedded systems observe symmetry? Let us explore each of the above as they relate to the themes suggested by the research.

11.1 Observing Thinking and Attention: Nested Level Aspects

What are the observable features of students' thinking and attentiveness in this discussion on symmetry? During the preliminary stages of the research I wondered about the observable features of mathematical thinking. I soon learnt that what students said, used or wrote and how they said, used or wrote it not were always consequences of thinking, but were in themselves acts of thinking. When students explain infinite lines in a circle with phrases and actions such as, "they go on forever," "uses his pen to mark many points on paper," "would not wear out," "keeps going," "it is incalculable," or "would never cross" this is their thinking-in-action. In chapters 4 and 7, I explored students' episodic writing, loud and bold utterances, joint thinking, and concrete materials as observable features of students' thinking. Tony thought with counters during the Consecutive Terms task (CT). Irene and Lillian's writing and re-writing was their observable mathematical thinking.

During mathematical activity, students talk, act and gesture their way into attending mathematically both in a gradual and radical manner. Particular kinds of utterances, written work and use of manipulative materials, as was the case with the Fraction Kit activity and the CT task, may point to different symmetry worlds that students enact. Some students like Julius and Stella had enacted a symmetry world that was still tied to the materials and activities they used to explore symmetry. They were convinced that one could draw or cut plenty or lots of lines but there was a limit. Drawing from Lesson 1 we are not sure about what symmetry meant for the students. In the cutting and drawing actions where was the symmetry for them? Was it in the different ways a shape could be cut into congruent halves, was it in the pair of halves or was it the number of possible cut and fold lines? Were some students attending to symmetry in ways that were not mathematically adequate?

In chapter 6, I explored that the observable features could be understood as inner level aspects of students' mathematical attentiveness. Students come to mathematics classrooms with unique ways of attending and unique mathematical worlds. The continuous interaction of the structures of learners with their external structures, including mathematical tools and each other's actions, as we saw in Chapter 7, provides feedback to the students' ongoing mathematical thinking. This establishes higher order global properties in the form of hunches, insights, mathematical objects, abstract ideas and categories about the content that they are learning. The emergent properties in turn act as top-down constraints; they engender perceptual objects. My research illustrates how thinking can be observed as a dynamic integration of co-related coherent forms including bodily and material forms.

11.1.1 Attending with and Extended Structures that Aid Attentiveness

Learners *know-with* cultural, symbolic and technological interfaces and instruments. Varied learning environments such as dynamical environments appear to change what it means to know a concept. Having been accustomed to geometrical figures on paper, when I used Geometer Sketchpad to construct and deconstruct dynamical shapes I found that this new dynamical environment seemed to stress distinct geometrical sensibilities, including the temporal dimension. It is amazing how commands as simple as "undo" and "iterate" that are available within dynamical environments may radically alter what it means to know geometry. Subtle alterations change the environmental triggers. In this work I found that tools and media influence the mathematics we come to know.

The students when discussing symmetry appeared to know, with the help of the protractor, that a circle could have 18, 36, 360, 1800, 36000, incalculable, uncountable and many more lines of symmetry. Recall that in Lesson 1 in Chapter 10 Tim also talked about a sphere having 360 times more lines of symmetry than a circle. In turn 90, Edwin offered 1800 lines of symmetry. And in turn 143, the teacher explicitly used the protractor. The numerical figures students used were not random. They point to a relevant tool of thought in their symmetry worlds. Steno's reference to cutting a circle into two halves in turn 124 might also have had much to do with the protractor. In Namukasa and Simmt (2003) we explored how what these grade 7 students attended to as symmetry was braided with their familiarity with a protractor and degree measures. The significance of a protractor also exemplifies how signifiers and (inter-) objects can be chained, layered and nested. In the dynamical systems language we may say that the protractor as a cultural measuring instrument had also become a dynamical attractor for what students attended to as symmetry. Students' familiarity with the protractor structured their attention and thoughts. But more about this in later sections. Let us explore the role of sensation, perception and observation in attending to mathematical concepts.

11.1.2 The role of Perception and Visualization in Mathematical Thinking

In mathematics education, many researchers relate inner level aspects of thinking and attentiveness in ways that delegate to these aspects a secondary role. In this study, I have interrogated the metaphors of representation and signs. Otte (2002), and Marton and Booth (1997) say that concrete materials, bodily activity and the like sensorially mediate or offer access to mathematical ideas. For me, writing and drawing are more than representational and sensorial mediators. In Chapter 7 we saw that they have *re*-presentational and presentational roles as well. General and abstract ideas, as emergent properties, come from and in many ways remain dependent on artifactual, sensorial and bodily activity. In the flow of mathematical activity some mathematical aspects are attended to in a bodily and pre-conceptual manner, without the break to linguistically and formally formulate and articulate them. (Recall Tony attending with the counters when verifying numbers that had the CT property.)

During the study I revised my linear views including: thinking is separate from and must precede individual actions; that perception is pick-up of sensorial information; and, that social, material and institutional milieu only provide context. My study shows that the physical environments and tools influence the mathematical objects students attend to. Perception is participatory and co-emergent. As perception guides actions mathematical objects come into existence and continue to evolve. As Merleau-Ponty (1974) put it, perceivable things are not a collection of sensations, nor are observable distinctions a totality of perceptions. Perceived things as Capra (1996) observes, are networks of relationships, embedded in larger networks of perceptual worlds. In sensing, perceiving and observing, in distinction making actions, the attendee also brings forth what is attended to. Moreover as we saw in Chapter 9 this we do as individuals and as collectives nested within institutional and cultural attending systems.

11.1.3 Mathematical Worlds of Mathematical Significance

What worlds of symmetry had the grade 7 students enacted? In my view, the psychology and phenomenology of learning aside, mathematics educators have yet to radically problematize the ontology of mathematical concepts and mathematical observers. While thinking mathematically could be viewed as perceiving, interpreting and experiencing mathematically, in the dynamic view of attending, thinking mathematically is acting and being in ways that expand what is mathematically thinkable and "attendable". It is about enacting mathematically "manipulable", perceivable and conceivable objects. In doing—we at once—as individuals as well as collectives, arise as mathematically attentive observers. When we, with others, attend to our mathematical observations and other distinction making acts we ourselves belong to the mathematical community of observers.

In the Fraction Kit activity I showed that stacking the pieces, covering a larger piece or assembling a whole have far-reaching effects on how students might know fractions since they involve enacting unique fraction worlds. As well I explored how the set {1, 2, 4, 8, 16 ...} could have different signifiers all depending on the world of exponentiation enacted by students. (In Appendix B I explore at length how it could be the case for the signifier {1, 2, 4, 8, 16 ...} to have varied signifieds.) I wondered about the relationship between multiple-signifiers (-experiences, -representations and –interpretations) and the general concepts and categories they illuminate. How do the actions of cutting, splitting, drawing and folding relate to the general concept of symmetry that may apply to symmetrical matrices, for example? Contemplating this question became central to drawing threads from my research. Because the thread on emergent concepts and objects holds all the other themes, I explore it at length. But first I shall finish looking at the inner level aspects of mathematical attentiveness.

11.1.4 Examples, Models, Instances as Inner Level Agents

Considering mathematical concepts as emergent properties and mathematical attentiveness as participatory may help us to approach the components of mathematical thinking at inner levels of organization. It also has theoretical implications as to how we view individual models, meaning, and situations which are used in teaching mathematics. Drawing from the logic of emergence explored in chapters 6 to 10, individual events over time are necessary conditions for the advent of learners' global mathematical concepts. Different embodied meanings of mathematical concepts, such as fractions and multiplication, are seen to emerge from the different models and experiences in enactment of new mathematical worlds. During the symmetry lesson, as students sought to convince each other that the circle has infinite lines of symmetry, the attractors (the agents and components) that enabled the students to participate in the discussion included: cutting the circle into similar halves, folding a shape into two congruent parts, drawing mirror lines to detect similarity, and using protractor degree lines to imagine symmetry in a circle.

11.2 Mathematical Objects of Attention

What are students attending to as symmetry in a circle? By considering this discussion on symmetrical properties of a circle in light of the first lesson and even other activities such as the Fraction Kit Activity we are able to comment about what students attend to as symmetry. Many students such as Kay attended to symmetry in concrete and activity based terms. These students did not use verbs such as reflect, mirror and imagine.

Only Allen, Tim, John and Kathy used IMAGINE and SPLIT. No students used MIRROR or REFLECT in the discussion. Only the teacher used REFLECT. Among those who used more *activity-based* verbs such as cut, draw and fold some could explain how, say, by cutting along lines of symmetry in a circle one could, in their words "go on forever" or "technically never stop". I consider these students to have enacted more general symmetry worlds. A few students such as Julius whose actions of cutting or drawing were less generalized, hardly could imagine how you would DRAW yet another and another, ad infinitum lines of symmetry in a circle. Yet for some such as Kathy and Tim "even though they could not necessarily see it" they could IMAGINE the many more lines in a circle. In Chapter 7 I explored how students even when they use simple concrete materials (the way Tony did with counters) can attend to abstract ideas in ways that are sophisticated and verifiable. Allen explained that even with a protractor, if "you go half way between each line, they would technically never touch" (turn 182). For John, from one degree apart to point five degrees apart then continuously halving, "we just keep on getting smaller no matter how much smaller it is" (turn 205). When students attend to their physical actions and materials in more general ways, when they attend to the beat of the patterns and rhythms in their actions, they attend in more abstract ways. It is then that they can "imagine even though they can't see it," as Kathy explained (turn 207).

To students who had not yet enacted mathematical worlds in which symmetry was the pattern—the imaginary congruence—in folding, cutting, drawing and splitting actions, the lines of symmetry in a circle were many but finite. Recall the student in lesson one who said, "I think eventually ...it will run around" to explain that there are 8 but not 16 symmetry lines in an octagon (turn 49, lesson 1). For some students symmetry lines in a circle would in a similar manner eventually "touch". One student fervently explained that if you were drawing the lines, eventually you would have nowhere to draw them; it would all be thick pencil line covering the circle. Another student explained that if you were successively cutting the circle into halves, you would go on and on until you would have a very tiny piece of paper left. But for those students who had enacted more sophisticated symmetry worlds, the leap from the numerable and foldable lines of symmetry to imaginable infinite lines of symmetry was swift. Such a shift in attention would have to happen for the other students as well in order for them to imagine lines of symmetry.

11.2.1. Mathematical Objects as Inter-objects and Meta-stabilities of Recursive Actions

That students in the class used varied action verbs shows that they had multiple experiences with activities related to symmetry. What did these action verbs have to do with the abstract concept of symmetry? Many of these experiences might have been outside the classroom. These are what I am referring to as the inner nested parts, the actions from which the abstract concept of symmetry emerges as a pattern of patterns. It appears from the recursive coordination of human actions, interactions and observations, irreversible, regular and lawful sign tokens and habits come into light as the meta-stabilities among such actions and interactions. In von Foerster (2003) and Bateson's (1979) language, symmetry is that which stabilizes as people draw, fold, cut, see, detect and mirror congruent halves. The emergent inter-objects and quasi actions enable further acts of distinctions. While observing students noticing number patterns I learnt that what might appear to a mathematician as a fixed pattern, a representation or a sign, for a mathematics student it is dynamic patterns that come out of local actions and situations (see Chapter 6, see also Appendix B). For me, studying patterns and relational properties that emerge from our interactions help us understand how students enact worlds. It also helps us focus on how students' activity engenders abstract concepts?

11.2.2 Concepts as Emergent Properties

The emergent properties such as the development of insights, in combination with the students' history and internal dynamics, may lead to the reshaping of a particular mathematical world. The overall world enacted by a learner proscribes the space of what can possibly be attended. For example, if a student understands the corners in a shape as signaling symmetry then a circle would have no (or have undefined or infinite) symmetry for them. In chapters 8 to 10, I discussed the emergent potential of mathematical activity. I claimed that in attending students enact mathematical concepts and general categories. General concepts arise from the specific instances, happenstances, surfaces of inscription, exemplars, diagrams and settings which students' work with. This could be the reason why it was possible that the students and teachers could use all the varied verbs but still, as a classroom, meaningfully and jointly attend to symmetry. There is a form of resemblance among the actions and imaginations evoked by each of the verbs, albeit this resemblance is by presentation, at an outer level.

Cutting and splitting may involve objects such as paper and scissors, and actions such as folding and cutting along a half line. Yet reflecting involves using a mirror to draw images. Folding and drawing appear to be inner layers of cutting or mirroring: You may fold before you cut. School mathematics activities on symmetry might also involve other activities such as measuring distances and tracing to complete the mirror image of an object. Other researchers have dubbed such resemblance among actions *invariants, generalities, totalities, collections, combinations, classification, universals* or *categories.* For me the metaphor of emergent properties assists in understanding the ways by which these varied actions and verbs come to refer to more abstract verbs and nouns such as symmetry or reflection. The metaphor as well helps with understanding the nature of the resemblances among actions. Indeed calculating, imagining, splitting, counting, halving and making might have so much to do with symmetry. Just like harmony and melody are relational, outer-layer properties at a level above that of the music notes themselves (Varela, 1979), abstract and shared signs, non-basic level verbs and imaginary objects of attention sprout from exemplars and non-exemplars, prototypical and illustrative cases. Symmetry and transformational geometry concepts are at the global level of organization

11.2.3 Ever Shifting Objects of Mathematical Attention

Enacted concepts and objects as global wholes appear to have existed prior to the local actions. This is not the case. Concepts and objects emerge with them discernible boundaries, preciseness and logical structures. Waldrop (1992) claims that the most crucial thing we have got to get at in understanding complex systems such as insights and concepts is how they emerge, how they evolve from many significant parts. They are shifting and ever dynamic. And this is derived from their emergent nature. They are constantly recombining and changing shape in relation to other interacting factors. Concepts are patterns in time that come to light in considering many individual instances. From the enactivist perspective, mathematical ideas and abstract categories are nothing in themselves but (meta-) stable patterns among recurrent local mathematical activity. As I showed in chapters 9 and 10, these relational, inter-objects come about to coordinate further mathematical behavior. To reiterate, mathematical concepts have neither objective existence nor personal sense–making, but they arise through intersection with contexts, artifacts and the community of learners (Kieren, 1995). From this perspective, redundancy in representations, diversity in interpretations, depth in individual instances, extension to symbolic and technological media and multiplicity of experiences are necessary for the emergence and development of concepts, ideas and intuitions (Davis & Simmt, 2002). What might attending to symmetrical matrices be without the student having been offered the space and experience of folding, cutting and drawing symmetry?

In the literature on fractions part-whole, quotient, ratio and multiplicative approaches to fractions are flagged as multiple meanings of the fraction concept. In Zoltan Dienes' terminology, they are the *multi-embodiments* of fractions. Lakoff, Johnson and associates have explored these in terms of metaphors. In mathematics education they are commonly referred to as representations and recently as interpretations. Drawing from the mathematical theory of dynamic systems I talk about these diverse experiences as dynamical attractors of concepts. Dynamical attractors at concrete, physical, sensorial and narrative ellipses enable the sensing, perceiving and observing of, say, multiplication concepts in complex ways that make sense even for imaginary objects such as scalar and vector products.

Unfortunately many teachers appear to view concrete models, physical materials, and illustrative media as merely tangible bases or visual illustrations for abstract ideas that exist "out there" (Towers & Davis, 2002). In methods classes I have taught, pre-service teachers who have recently done educational psychology courses quickly embrace varied approaches as attempts to reach out to learners who are tactile or visual, for example. They also admit that while these methods are helpful for the visuo-tactile learners, they also facilitate recall for weak students. Diversity in interpretations reinforces conceptual understanding of harder topics such as integers, some claim. Also, the use of multiple embodiments fits with the desire to make mathematics more interesting and accessible to all students. To some they are literal metaphors and analogies that offer many ways of expressing the same thing. Towers and Davis (2002) maintain that while reform recommendations have fit well with populist notions such as child-centered learning, individuated learning styles, and multiple intelligence theory, they have not affected how teachers view mathematical concepts in their teaching practice. Populist notions trivialize research discourses. The insights about mathematical attentiveness I have gained during the study, specifically the metaphors about the emergent and dynamical nature of mathematical concepts, offer a theoretical framework that challenges these populist views.

Abstract concepts transcend the inner level agents of experiences, signifiers, and interpretations, at the same time remaining in an evolutionary manner linked to the lower level agents in their life span. There is also circularity: exemplars and specific instances as inner-level coherent forms are related, but only after the emergence of a general idea. Once they emerge they reset the meaning and significance of the individual cases so that in the end it is not clear whether it is the global concepts that are primary or their local components. Concepts illuminate the invariants among the instances. Understood this way, mathematical concepts focus attention on mathematically significant forms. These assertions have resonance in the hermeneutic and interpretive semiotic views.

11.2.4 Hermeneutic Views about the Fluidity of Objects of Attention

Gadamer (1992) asserts that concepts are constantly in the process of being formed, or, more generally, that the thing-in-itself (in this case the mathematical objects and concepts) is nothing but the continuity with which the various perceptual perspectives shade into one another. He further asserts, "[E]very 'shading' of the object of perception is exclusively distinct from every other, and each helps co-constitute the thing in itself as the continuum of these nuances" (p. 447). Gadamer's view that concepts are constantly in the process of being formed is not very far from the post-structural semiotic idea that in an infinite regression signifiers point to yet other signifiers. His use of the color paradigm is close to the idea of signification spaces and to the nested and color graded spheres, which I adopted in the diagrams in chapters 5 to 11. (I also develop the idea of signification from a semiotic point of view in Appendix B.) Phenomenologically, van Manen (1988/1998) explains that *variants* and *invariants* in particular instances illuminate the meaning of an event. Structural phenomenology would refer to these invariants as the *essences of things*, the diverse ways of experiencing or, for Marton and Booth (1997), the pool of meaning. To Rosch (1999a), they are the focal meanings, which once they crop up examples become graded.

As well, anthropological and historical research supports this dynamical and organic view of mathematical concepts (see Joseph, 1991; & Nunes et al., 1993). Knowledge bears deep marks of the practice communities, physical localities and political state of affairs in which it is developed. For more ecological stances, varied explanations and presentations are considered to be the concepts themselves. Jardine (1998), after Gadamer and Wittgenstein, explains that the meaning of the concept is as diverse as the instances in which it arises. Symmetry, for example, is an unnamed feature of the earth and of the young child's experiences and being, rather than a sacred, external structure to be acquired by the child. Akin to Jardine, I have explored the organic and earthly nature of observable objects. My views of the nature of concepts offer a theory from which to reflect on the classical view of concepts and categories as static, logical and infallible universals, and the computationalists' view as a string of symbols as grand illusions of enlightenment. Concepts and objects only appear logical and computational once they have arisen.

11.3 Orienting Integrated Students' Attention

How can we occasion students such as Julius and Stella to attend to symmetry and spatial concepts in general in ways that enact continuity so that the lines do not eventually cross each other nor does the circle space wear out? In Chapter 5, I introduced the landscape metaphor to explain how interpretive research progresses. Spaces enlarge as old landscapes are transformed and new ones are formed with time. In dynamical systems theory, basins of a landscape are seen as basins of attraction; they are dynamical attractors. On the other hand, hills and ridges are perceived as the "separator" or repellers of a system's behavioral states. As students cut, fold, reflect and make congruent halves their understanding of symmetry develops along these activity based lines. We may visualize these different activities as small neighboring basins in a landscape. A general understanding of symmetry arises as a result of gradual re-organization of students' symmetry worlds. This can be seen as a widening and nesting of these neighboring basins. As we saw in Chapter 4, with aha moments, some shifts in attention might involve sudden amalgamation or radical re-organization of attentiveness. The emergence of an abstract and new understanding—be it for an individual, a collective or for a culture—can be visualized as a transformation in the *attentional* landscape or as an expansion in the space of the possible. It involves deepening, enlargement and/or a re-calibration in the basin of attraction for a given concept.

In this way, to make sense of the dynamics of worlds enacted by students I view that any abstract concept and general competences that surfaces for the learner carve out new contours that dynamically channel further mathematical thinking. Coming to know a certain concept in more refined ways is a transition. It involves a transformation of the overall dynamical organization of one's attentional landscape. Such a transformation could be what Julius and Stella needed to understand how a circle had infinite lines of symmetry. In Namukasa (forthcoming) I have explored how at critical occasions people begin to see things as if for the first time.

11.3.1 Dynamics for the Emergence of New Objects of Attention

When we consider mathematical concepts as autonomous abstract objects or insight that emerge from, and in many ways remains tied to, actions and interactions, then the next step is to examine the conditions for emergence for specific concepts. How does a specific idea or percept emerge, say for junior high students? I reflected on this question in earlier chapters and now I tie some of these ideas together.

The idea of *symmetry*, for example, appears to emerge from many cases, some of which, according to Rosch (1999a, 1999b), are better examples, while others are peripheral or even non-examples. In the analysis of Lesson 1 we saw that exploring with students only regular polygons might lead to an understanding of symmetry as the vertices. Not only do cases of the category *symmetrical* bear family resemblance, but also my dynamic understanding of *symmetry* and yours, as Jardine (1998) following Wittgenstein would put it, only bear familial resemblances. Which resemblances are focal and which ones are not?

It would be inappropriate to conceive abstract symmetry, a bottom-up property, as a mere frozen definition. Like other complex properties, concepts involve many-to-many relations and are multiply realizable. Different students might come to know a mathematical concept from varied perspectives and even attend to it, think about it, and explain it in somewhat different ways. My study has shown that these nuances are not arbitrary nor are they insignificant. They point to different worlds enacted, to situations, tools, times, contexts and communities in which particular sense is possible. Historically, this is exemplified by how ancient mathematical traditions of Aztec, China, Egypt, Arabia, Babylonia and others came to make parallel distinction about mathematical concepts such as numbers and geometry (Joseph, 1991). Analogously students' solutions and solution paths may only bear family resemblances.

In this research I have observed individuals, pairs and collectives of students engaging in similar mathematical tasks. Different learners usually start out with different dynamics. Some begin by interpreting questions such as the Bee Genealogy (BG) task in less conventional ways, getting stuck here and there. Others read the question once, and some ask me about what a phrase in the question really means. Yet other learners take their time, reading and re-reading the question. One student underlined statements she thought were central in the question. Another pair discussed at length why the phrasing was the way it was. Even in the first few seconds of engagement in a task, different learners face different problems calling for different solutions. The spaces of what they attend to are different. Although all students solve the problem during the session (a few with intensive shepherding), to the extent that the engagement on the question on an ongoing basis re-configures their states of the possible and attractor dynamics, each pair generates a unique solution. Two pairs of students might find the same solution to the same task, but still their worlds transformed or enacted during their engagement would be distinct. The territories of engagement would have converged at the attractors and states that bear family resemblances, but they would depict different dynamics and regimes before and after the task. They would have attended differently. Julius and Stella had a basis to argue that the circle did not have infinite lines of symmetry.

11.3.2 Observer Constituted Ontologies

Students enact worlds in which it makes sense to attend to some particular objects and to attend in specific ways. At times, the observer may not view these ways or objects as mathematically or culturally significant. However, to refer to Julius' idea of a circle with limited space as an epistemological error, as shown in Chapter 8 with Arlene's conversion of the fraction $2\frac{3}{4}=1\frac{1}{12}$, is to deny the realities students bring forth. Mathematics is an observer constituted ontology itself. Drawing from theories of distinction, I found referring to students' divergent understanding as points of views or perspectives as trivializing as referring to them as errors. Stella and Esther's idea that an octagon has 16 lines of symmetry was deeper than a vantage point from which they could switch back and forth. In their worlds of symmetry it made perfect sense (see Chapter 10). Occasioning students to, on a moment-to-moment basis, collectively co-constitute ontologies which are mathematically significant was the thesis of Chapter 10. In mathematically adequate ontologies students are not only invited to think mathematically but it makes embodied, embedded and extended sense for them to do so.

I have learnt that enacting mathematical worlds involves layered orientation of attention. It involves the structure of the concepts, the way learners attend, the possibilities humans have, the worlds which students enact, the observing systems at work and the nature of the mathematical observing system itself (see Chapter 8). For me, it is the layered stance that supports a positive construal of thinking as more than problem solving, overcoming obstacles or negotiating conflicts but as expanding the space of what is mathematically perceivable as well as transforming the landscapes of what has been mathematically thought about and adopting mathematically adequate ways of attending.

11.3.3 Orienting Multi-dimensional Attention

In what ways does the task of discussing symmetry in a circle structure students' mathematical attentiveness? Complex systems have particular states that have higher-than-average probabilities of occurring—the basins of attraction. A system's behavior is likely to converge on these basins of attractions. "The deeper the valley, the greater the propensity of its being visited and the stronger the entrainment its attractor represents." (Juarrero, 1999, p. 156). Tim and Janelle in many lessons were more inclined to act and interact themselves into attending to concepts in more general terms. Also for this grade 7 class as a whole it was a more common phenomenon for "it" to spontaneously break into small groups for intense discussion. When a learning system finds itself in a basin of attraction, this means its immediate and future behavior will be dependent (but not determined) by the possibilities of that attractor—its depth, width, underlying structures and neighboring attractors and separators—the ridges. Why it was the case that this class in general and students at many occasions were inclined to attend mathematically is a subject of another study.

When talking about attentiveness in Chapter 10, we saw that mathematical attentiveness is intricately threaded with sub-personal, personal, supra-personal and extra-personal dynamics. For example at the supra-personal layer the activities of the classroom collective and sub collectives as well as the broader school, family and societal happenings entrain what an individual student attends to and how they attend at any given moment. As students attended to symmetry the identity of their mathematical attentiveness was the intersections of many factors including: (a) their embodied internal dynamics; (b) embedding social, cultural and institutional systems; (c) extended temporal and material environment; (d) interactions within the learning environment, which now involved a protractor as a dynamically attracting thinking tool; (e) understanding of other related topics; and (f) willingness to offer their understanding to the classroom and to listen to other students. There were ways of attending and objects of attention that were highly possible for the class as a collective and for individual students. This might explain why, during my teaching experience in many cases it appeared to take more than explicit pointing to focus many of the senior high students on mathematically significant structures. But just like natural landscapes, one's mathematical attentiveness continuously drifts. At certain critical points, when unified contours of the attentional landscape drift in synchrony it might shift dramatically. The reverse is an unfortunate situation that happens to some students during impoverished school mathematics experiences—narrowing and contracting of the basins for mathematical attentiveness.

The few students with a more abstract understanding of the infinite offered explanations that you could use a thinner and thinner pencil to be able to draw more and more lines of symmetry. Tim, a student who had compared lines of symmetry in a circle and a sphere during the first lesson, and a few other students explained: "Because you can still split one thin line and still. Yeah—draw another one." (Tim-turn 194) "I think it is infinite until like we don't have material to actually see it…." (John-turn 202) "I think it is infinite, like it keeps on going." (Kay-turn 204) "Because if, let's say if even though we don't have [material], [even if] we can't see how small it is we can still know it is possible so we can imagine something may be small" "even though we necessarily can't see it"(Kathy-turn 207). It appears that the difference between attentiveness to symmetry of learners who understood it in abstract and practiced terms and those who did not is that for the former and for the teacher the evolutional shift in understanding symmetry entirely in terms of imaginary actions of say similarity had already happened.

The mathematical attentiveness of these latter students had yet to be re-organized or radically shifted towards a more complete understanding of symmetry. The question then becomes: How can a teacher occasion a re-configuration of the symmetry worlds of students such as Esther, Joseph, Julius and Stella in the Symmetry activities? Would it take an enaction of new worlds with new emergent objects and novel ways of attending altogether, an upheaval? Or would it require just a subtle prodding in their existing domain?

11.4 Evolutional Shifts in Attention

How could a teacher evoke shifts in students' attention to that which is mathematically significant in the understanding of symmetry? In chapters 4 and 7 while *tracking* how Irene and Lillian's attention shifted to that which is mathematically significant we saw that shifts in attention are gradual. In Chapter 10 I discussed relatively stable and reversible switches and drifts in what one attends to. In this section I discuss discontinuous, dynamic shifts that punctuate reversible shifts. The adoption of novel ways of attending is one such irreversible shift. The emergence of an understanding of an abstract idea from individual situations is another. These shifts might mark evolutional transitions with momentous effects on the dynamics of a student's attention. Such evolutional shifts might be marked by the *aha moments* in students' understanding (see Chapter 4).[39] In a hermeneutic manner, shifts in attention could be considered as moving beyond one's horizon, beyond one's familiar world to a transformed world. This is not abandoning the fore-structures of understanding but challenging and involuntarily revising them (Gallagher, 1992). Shifts might also involve concepts combining and recombining to give rise to hybrid concepts (Davis & Simmt, 2002). They might involve a re-organization—gradual or sudden—in understanding (Sumara, 2002) akin to a geographical land formation.

This study has suggested that what students bring forth during aha moments is

[39] To Hadamard (1945), for an individual, during such moments "ideas pop up with brevity, suddenness and immediate certainty, after an incubation period, from the unconscious [and not conscious] to the conscious." (p. 14) To Davis and Hersh (1981) such moments indicate "that something has been brought forth which is genuinely new... a new understanding for the individual; a new concept placed before the community." (p. 283[0])

not only a mathematical object as well as a novel way of attending but an object in an entire observer constituted ontology. It is a mathematical world in which it makes sense to think about particular objects and attend to them in certain ways.[40] I have learnt that recursive elaborations in what and how students attend mathematically involve concepts recombining to give rise to hybrid concepts and quasi actions.

In topological and dynamical terms, learning, gaining insight, forming abstract concepts and observing general categories and other such radical shifts are equivalent to a cataclysmic adjustments of a space and landscape—an earthquake or an eruption. Such shifts are not simply a question of deepening or smoothening existing valleys or hills: they are about reconfiguration of the overall territory. The dynamics that might have been resetting, widening or deepening part-by-part, basin by basin at *aha* moments of insight, intellectual evolutions, cultural revolutions or paradigm shifts do reset all-at-a go. When Lillian and Irene, in Episode A, understood that when you begin with a sum this actually generates a number and later in a jump of insight, in Episode C when they noticed that by describing the numbers that did not have the property they would have exactly described the numbers that had the property, the entire neighborhood of their attentional regime had undergone a dramatic shift. A similar shift is what Julius and Stella needed. With the help of the teacher, other students and other learning structures, Julius and Stella had to act and interact themselves into attending differently. Then with the understanding that a

[40] Using the analogy of paradigm shifts, Waldrop (1992) explains that paradigm changes are biological, intellectual and cultural evolutions. During paradigm shifts concepts combine and recombine, they leap from mind to mind over miles and generations. To Johnson (2001), while paradigm shifts and the development of novel understanding might appear as a great man's theory (or as a one eureka moment), they are actually complex, multithreaded tales with many agents interacting in a great many ways over time. They are a result of new tools and distinctions appearing on the horizon. They are distributed and communal efforts, which involve new layers of understanding. To Johnson, as it is for Nørretranders (1998), there must be an adequate number of coherent parts interacting in the system before "isolated hunches and obsessions coalesce into a new way of looking" (Johnson, 2001, p. 64).

circle had many and in fact infinite lines of symmetry, their understanding of infinite and infinitesimals, of scales of measurement, angle measurements, of symmetry and the like would also grow. New attractor basins would have appeared.

During phase transitions students gain a more elaborate, imaginary and global understanding of mathematical concepts, their earlier understanding are broadened and related memories re-activated, they learn a new procedure, or they form a unique mathematical habit. Further gradual and radical transitions result in transformed dynamics of attending such that mathematically inadequate ways of attending and objects of attention that were once habitual become impossible. It is a makeover of the overall dynamical identity. Novel constraints, dispositions and propensities to attend to a particular mathematical object in mathematically adequate ways do present themselves. A new perceptual world of mathematical significance and relevance is enacted. And this happens in real time.

11.5 Dynamical Structuring Mathematical Activity and Tasks

In what ways does the task of discussing symmetry of a circle structure the way students' attend and think? My work with students also implies that mathematical tasks and activities that students engage in structure their mathematical attentiveness. Many activities during the study did so in more mathematically adequate ways than others. In Chapter 1, while I offered the Chessboard Squares (CS) as an example of a good enough, variable entry and non-routine mathematical task I mentioned that such a task has the potential to structure students' mathematical attentiveness. The CT task just like the CS task prompted systematic recording among many students. In addition the CT task prompted junior high students to define a set by looking at its non-members. Some activities I modified during the preliminary study and some I dropped after observing one pair engaging in them. Activities might be variable-entry tasks and non-routine tasks but the ways they structured learners' attention might in be many ways mathematically inadequate. I referred to the range of tasks and actions that students engage in which were likely to trigger them to think and attend mathematically as *dynamically structuring tasks* and *acts*. Tasks such as Pirates Aboard, Bee Genealogy (BG), CS and CT I offered to a majority pairs in the final study. In addition to offering a variable and motivating entry, these tasks on an occurring basis appeared to favor mathematical possibilities. I could also see them as a source of preparedness and readiness to attend mathematically in preceding sessions. Unlike the quick fix, textbook exercises they had a higher potential to expand common mathematical ways of attending as well as to evoke novel ones. That was what was routine about them. By engaging in such tasks students were practicing ways of attending mathematically and ways of enacting mathematical worlds. In Chapter 10, I also explored actions and interactions, what I have dubbed mathematically adequate actions and interactions. In a similar manner these actions appeared to originate as well as sustain mathematical attentiveness among students. They dynamically attracted and structured students' attentiveness.

Students' possibilities of mathematical attentiveness during their engagement on the CT might differ according to whether they have solved the CS task before or whether all they have done are routine and clue giving problems. As well it might differ depending on with whom they are solving the task and what other extended structure they have available. I learnt this when analyzing Rose and Norah's engagement on the BG task in light of Irene and Lillian's engagement (see chapters 4 & 7; see also Appendix E, Vignette E2). Learning, thinking and attentiveness appear to be a "process of dynamic self-organization that takes place as a result of ongoing interaction and relationship between" the learner and the learning environment (Juarrero, 1999, p. 159). The worlds of mathematical objects and relations are not one that learners walk into; the learners lay them down and transform them recursively. A Mathematical world rolls up in living and being mathematically.

11.6 Consequences and Implications

This study has been an inquiry into the nature of students' mathematical attentiveness and thinking. Specifically I have explored the embodied, embedded and extended nature of students' mathematical attentiveness. By observing students solving mathematical problems, discussing mathematical concepts and participating in other mathematical activity I have explored ways in which students enact mathematical worlds in which they are not only invited to think mathematically but find it makes deep sense for them to do so. I have drawn upon enactivism and eco-complexity theories to understand aspects of students' engagement that are more than representations and visualizations. Through close observations of the diversity and subtleties among students' interactions my research has suggested many things. To understand the themes I have explored ecological, dynamical systems and geographical metaphors of complexity research such as emergence, mutual feedback and dynamical landscapes. I have illustrated how subtle differences in activity may actually originate larger differences in objects attended. The approach to the inquiry has been hermeneutic I have juxtaposed the interpretive orientation with its counterpart in the hard sciences, second-order observation theories. The ideas presented in this work reflect my ways of attending to students' mathematical attentiveness and thinking. These ways implicate my history, the tools I have had to do this research, the research community in which I participate and the time and locale of my work. This is an observer constituted ontology, a world experienced, transformed and brought forth in doing and living the research.

As a final remark to the study I outline the significance of my work. This last section is about reflecting on the adequacy of the study. I ask whether the study is coherent: Is it an agent in the emergence of novel and grander understandings? Does it expand the space of the possible for research, teaching and learning? Have new sensibilities been opened up? Has the concern that motivated the inquiry—to understand the nature of students' mathematical thinking—been advanced? Do the tools and models offered, and claims made embrace complexity?

11.6.1 Consequences and Implications of the Study

My exploration of students' mathematical thinking in terms of mathematical attentiveness has subtle practical implications for the teacher, curriculum developer and textbook writers. These practical implications need to be gradually brought forth in the experiences of those educators. However, its implications and consequences for the research community are apparent in terms of theorizing and conceptualizing mathematics education.

Researchers who closely observe students' mathematical actions maintain that an interested look at students' thinking elaborates on school mathematics content (Vergnaud, 1988). My study has been a revisiting of issues of mathematical structure from a more hermeneutic, evolutional and historical perspective. Within it one can find *task analyses* for topics such as fractions discussed in the context of students who participated in the research. However, it is clear that elaborating on school mathematics content takes more than one study, researcher and thesis. Hence my study is intended to contribute to the existing body of knowledge. A recursive elaboration of various subject matter and error analyses studies from the ecological and dynamic view in terms of inner level aspects is a subject for further studies. As a consequence of closely observing what students attend to in mathematical tasks, mathematics content that engages prospective teachers "in ways connected to practice" (Ball, 2002, p.11) may arise to enrich the prevalent university mathematics content that primarily engages mathematicians and mathematics students. For instance instructors can examine with student-teachers how subtle variations in students' writing, utterances and use of materials might point to significant differences in attending and objects of attention. In my teaching of pre-service teachers and participation in in-service teachers workshops, that focus on the nuances in what learners attend to has shown potential for engaging teachers in ways that foreground the complex nature of students mathematical thinking.

Educators who conceptualize in-service and pre-service mathematics education may find guidance from claims made in this writing as they prepare mathematics teachers to be responsive to the dynamics of students' mathematical attentiveness and thinking. They may be prompted to work with teachers in an eco-complexivist and enactivist manner that upholds listening and attending to students' interpretations so as to enhance mathematical learning. Teachers with an attitude of attentiveness and responsiveness are likely to occasion students' to learn mathematics meaningfully.

As well, the study contributes to generating novel sensibilities for construing mathematical thinking. Theoretical constructs like signification spaces have been broadened, others like representations have been problematized and new metaphors and constructs such as presentations and re-presentations, structuring tasks have been introduced.

This study also offers the mathematics education research community another "unit of analysis" for classroom research: abstract concepts and mathematical worlds that students enact during classroom activity. It also demonstrates that there are aspects of learning such as mathematical attentiveness that cut through body-mind-environment whole that can be promisingly investigated. My study has been an effort to engage in a broader conversation that seeks to understand the centrality of body and brain in cognition.

Theoretical constructs and new metaphors offered by the study suggest ways of observing closely and commenting students' thinking. They offer observational tools for attending to students' mathematical thinking as well as participation tools for attending with the students. This study was motivated by the need for observational and theoretical tools to understand and occasion students' mathematical thinking in ways that embrace the complexity of cognition.

The conclusions drawn from this study illuminate the relation between mathematical activity and mathematical thinking. For instance, it emphasizes that writing and other mathematical activities are central to problem solving, mathematical understanding and to concept formation. The claims made in this dissertation about the nature of mathematical concepts offer a theoretical rationale for use of multiple mathematical experiences in teaching. In encouraging mathematical thinking, many researchers advocate the use of multiple representations (Confrey, 1999; Ball, 1990). This recent view is penetrating practice from the top down as contemporary teacher resource books encourage teaching with multiple models. However some central questions have not been asked. For instance, what are the relationships between the representations and the mathematical concepts associated with them (Radford, 2002)? To me, there are parts that must be furnished, questions that have to be asked and metaphors that have to be envisioned before systems and ecological views can inform practice in fundamental ways. Otherwise deeper and broader reform views will be swallowed by shallow populist theories (Towers & Davis, 2002). One could use this study, among other contributions, to offer a rationale for why mathematics teachers should use multiple representations in teaching school mathematics.

Beyond the theoretical implications, this research might have gradual contributions to make to the practice community. How might we build classrooms in which students' actions and interactions occasion personal and collective mathematical attentiveness in ways that enlarge the space of the possible? Based on my explorations of students' mathematical attentiveness, it is expected that ecological and systemic conversations on mathematical thinking might eventually might trigger among teachers novel ways of organizing their teaching so as to enhance active, interactive, and *situated* mathematical knowing. Working out direct implications of my study for classroom teaching and learning is a topic for future research.

In investigating the dynamics of student attention and mathematical thinking, I have engaged in a conversation about thinking as complex human phenomenon. The conversation I have participated in triggers new ways of thinking, talking and acting about students' learning among the collective minds of the mathematics education community.

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APPENDICES

APPENDIX A Samples of Mathematical Tasks Adopted

Cubes Cubed

I have eight cubes. Two of them are painted red, two white, two blue, and two yellow but otherwise they are indistinguishable. I wish to assemble them into one large cube with each color appearing on each face. In how many different ways can I assemble the cube?

Chessboard Squares

It was claimed that there are 204 squares on an ordinary chessboard. Can you justify this claim?

Bee Genealogy

Male bees hatch from unfertilized eggs and so have a mother but no father. Female bees hatch from fertilized eggs. How many ancestors does a male bee have in the twelfth generation back? How many of these are male?

Consecutive Terms

Some numbers can be expressed as the sum of a string of consecutive positive integers. Exactly which numbers have this property? For example,

$9 = 2 + 3 + 4$ $11 = 5 + 6$ $18 = 3 + 4 + 5 + 6$

Patterns in Color

Color a pattern on squared paper and ask student to continue it?
Do it for a few more patters and then ask them to make their own patterns which their pair mate would solve. Let them vary easy problems and hard ones.

Pirates Aboard!

Pirates ordered all the sailors to be lined up around the edge of the ship. To the first sailor to the left they said, "You will die!" To the second they said, "You will live!" To the third sailor, you will die; the fourth you will live. They continued in this judgment, repeatedly going around the lined up sailors until only one sailor was left standing. "You will live," they bellowed, "Join our pirate crew!" If you were one of the sailors on the captured ship, where would you stand in order to survive?

Ins and Outs

Take a strip of paper and fold it in half several times in the same fashion as paper strip. Unfold it and observe some of the creases are In and some are OUT. For example
in in out in in out out
What sequence would arise from 10 folds (if that many were possible)?

Ladies Luncheon

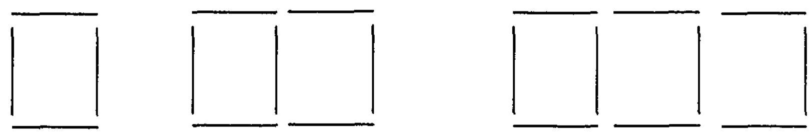
Five women have lunch together seated around a circular table. Ms. Osborne is sitting between Ms Lewis and Ms. Norris. Ms. Lewis is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Ms. Parkes on her left and Ms. Martin on her right. Match the first names to the surnames.

Fifteen

Nine counters marked with the digits 1 to 9 are placed on the table. Two players alternately take one counter from the table. The winner is the first player to obtain, amongst his counters, three with the sums of exactly 12.

Matches 1

How many matchsticks are required to make 14 squares in a row, the side of each being the length of a match, as in the following sequence?

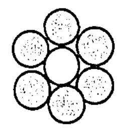


Paper Strip

Imagine a long thin strip of paper stretched out in front of your, left to right. Imagine taking the ends in your hands and placing the right hand end on top of the left. Now press the strip flat so that it is folded in half and has a crease. Repeat the whole operation on a new strip two more times. How many creases are there? How many creases will there be if the operation is repeated 10 times in total?

Circular disks

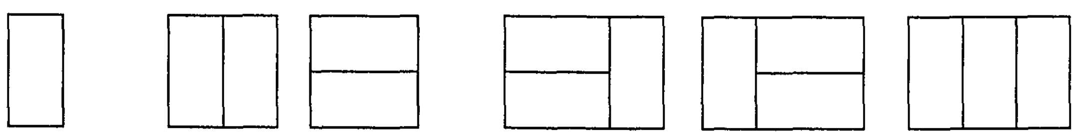
Explore the growth pattern that occurs when rings of disks (pennies or bingo counters) are placed around a central disk.



Think about each ring as a stage. Find the number of disks it will take to make the first, second and third ring. How many disks will it take to make the 10th ring? The 100th ring? Write a rule explaining how to predict the number of disks in any stage.

Tiling Paths/dominoes

How many paths can you tile with a given number of dominoes (2x1 tiles) if the path must be two units wide. There is one path for one tile, two paths for two tiles and three paths for three tiles.



Towers of Hanoi

The object of this puzzle, published in 1883 by a French mathematician Edouard Lucas, is to move all the disks piled on one pole to a different pole. There are three poles altogether. You can only move one disk at a time and you must follow size order -- a bigger disk can't go on a smaller disk.

APPENDIX B Semiotic Interpretations of my Data

SEMIOTIC REPRESENTATIONS AND SIGNS

Mathematics educators have recently appropriated the semiotics of both Peirce and Saussure to examine the concepts of sign and representations. For instance Duval (2002) has elaborated on semiotics to study representations systems. A few post-structural scholars have recently attempted broader conceptions of sign away from its usual delimitation to the formal-linguistic domain. Radford (2003) attempts to develop more contextual semiotics. His elaboration appears to be a step toward embodied mathematical signing when he talks about semiotic *means of objectification*. However, as I discuss in a later section, he does not go far enough. He denies the central influence of the body on the signs. Rotman (2000) develops a more evolutionary approach. Unlike Radford, he emphasizes the role of the body and of materiality. Similarly, Brier (2001), drawing from cybernetics, develops the construct of layered *signification* to illustrate how the concept of signification could accommodate the organic complexities of cognition. How do constructs such as *orders of signifying activity* fit with and contribute to the conceptual underpinnings of this research?

This appendix is more about reading my framework against an alternative post-structural framework, semiotics. I use research data to bring two frameworks, ecological complexity and semiotics, into interplay. I illustrate how semiotics benefits this study. In addition, I illustrate how semiotics theories could benefit from the complexity metaphors.

B.1 Appropriating Semiotics in Mathematics Education

Semiotics is commonly understood to be the study of signs and sign use. It has its origin in structuralism (Saussurean semiotics) as well as in pragmatism and formal logicism (Peircean semiotics). Structural semiotics has been elaborated by its post-structural critique—Lacanian inversion of the Saussurean model. To some scholars, such as Rotman (2000), semiotics is not a kind of philosophical stance but rather a distinct umbrella domain like philosophy, linguistics or science. In the Anglophone mathematics education literature, there is recent interest in semiotics. For instance, in the PME-NA book edited by the working group on representations and visualizations (Hitt, 2002), 8 out of the 22 papers were explicitly framed by semiotics. One of the central questions of the book is, how are meaningful signs in cognition created? Presmeg (2002) and Radford (2002) and others attempt to answer this question by drawing from Peirce. Let me offer a background to their discussion.

Semiotic activity, according to the Peircean model, is triadic. Three elements constitute the Sign: the *object*, *representamen/sign vehicle* and *interpretant*. There are three kinds of signs: the *diagrammatic sign or icon* which exhibits sensuous resemblance, relational likeness, and analogy; the *index,* which like a pronoun demonstratively or by connection directs attention to the object; and the *symbol/token* which signifies either by association, law or habit of mind (Peirce, *Collected Papers*, volume 2, pp. 274-299 [CP, 2. 274-299], 3.362). A few mathematics education researchers, such as Walkerdine, draw from Saussure's and Lacan's semiotics. To Saussure a sign is comprised of *signifier* and *signified.* In this dyadic model the interpreter seems to be implicit or even absent. Saussure focused mainly on linguistic signs, especially the spoken word (Chandler, 2002). He is criticized for ignoring the written and other sign systems, which—as I will illustrate—are historically and developmentally related to linguistic signs. For Saussure, the signified, which to him is purely psychological, takes precedence over the signifier.

B.1.1 Towards an Interpretive Semiotics

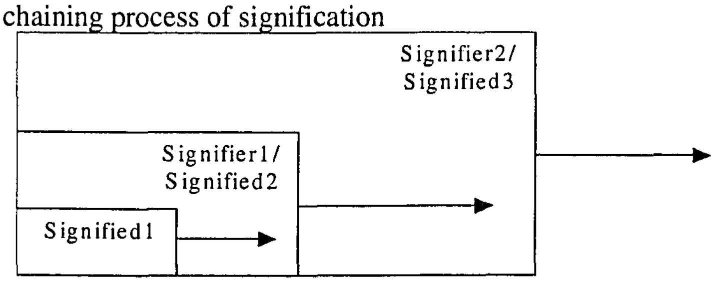
Lacan inverted Saussure's model to stress the priority of the signifier over the signified. The signified inevitably *slips* beneath the signifier, resisting our attempts to delimit it (Chandler, 2002). The precedence of signifier over signified in post-structural semiotics allows for a redefinition of semiotics as a continuous process of *signification* (of what is signified), as the signifier in the previous sign might—via the interpretant for Peirce, via *connotation* for Barthes, via *free play* for Derrida, or via *sliding* for Lacan—become the signified in a new sign combination and so on (Chandler, 2002). According to Peirce this is *limitless* semiosis (Peirce, CP, 2. 303).

> The object of representation can be nothing but a representation of which the first representation is the interpretant. But an endless series of representations, each representing the one behind it, may be conceived to have an absolute object at its limit. The meaning of a representation can be nothing but a representation. ...So there is an infinite regression here. Finally, the interpretant is nothing but another representation to which the torch of truth is handed along; and as representation, it has its interpretant again. Lo, another infinite series. (Peirce, CP, 1. 339)

Sáenz-Ludlow (2003) defines a *chain of signification* as "the embedding of signs within new and more sophisticated signs that for their functioning depend upon prior simpler signs that are kept as a trace" (p. 186). Semiotically, a sign is more than its particular three (object, representamen, interpretant) or two elements (signifier, signified). It is also a relationship between them, what Pierce refers to as *thirdness* and Chandler (2002) as *signification.* Presmeg (2002) also develops the chaining of signs to produce the concept of *nested sign systems.* In her model, the entire first sign with its triad constitutes the next sign in the chain and so on. Figure B1 illustrates the layering of signs. In my view, as soon as what is to be signified, *signified n* is represented by a signifier, *signifier n* there is an excess of signification introduced leading out to a new signified, *signified n + 1* that nests or slips under *signifier n.*

This chaining process has parallels in the mathematical processes that have come to be referred to as *distillation, mathematization, abstraction* or *reification* (Freudenthal, 1991; Kauffman, 2001; Mason, 1989; Sfard, 1991a). These notions attempt to explain how abstract ideas are directly or otherwise generated by more concrete, bodily, elementary and basic experiences and ideas.

Figure B1. The chaining process of signification



Although, not much has been said on the relation of a sign with other signs that the sign user creates, most scholars of semiotics now acknowledge the co-implicitness of the sign user or creator (the interpretant) in the sign. This seems to be a point of departure from the structural approach. "Of course, nothing is a sign unless it is interpreted as a sign" (Peirce, CP, 2. 308). "A sign, or *representamen* is something which stands to *somebody* [italics added] for something" (Peirce, CP 2. 228). A sign must represent "something else, called its Object" (Peirce, CP 2. 230). Where there is interpretation and experience there are signs, information and messages (Bateson, 1980; von Glasersfeld, 1999). Students at many times manipulate symbols only at a syntactic, game level. In fact Peirce uses the term representation in reference to only those signs that have an interpretant (CP, 2.274). In this respect and to the extent that it studies signs systems in relation to human activities semiotics is an alternative to traditional views of human cognition that pay less attention to the representamen.

Presmeg (forthcoming) explores Peirce's ideas about a community's mind—*commens.* Peirce acknowledges that specific disciplines enhance peculiar kinds and means of observations (Peirce, CP, 1.98). Many of Peirce's ideas such as that of a self-reflexive sign (a sign of itself), diagrammatic thoughts, visual language and existential graphs foresee the systems ideas of the: (a) interlocking relation between sign and object, (b) observer as a sign of distinction as well as the distinction as a sign of the observer—a Sign of itself (c) a sign with significant parts, circularity and self-reference, and (d) nested interpretants and collective thought. Sáenz-Ludlow

(2003) draws from Peirce to explore *collective semiosis*. In my view, there is need for educators who draw from semiotics to explore Peirce's semiotic and logical work in light of other authors such as Spencer-Brown (1979), who independently took up similar explorations on patterns that relate logic, speech, writing and drawing to mathematics. Drawing from complexity science might elaborate how collective semiosis and kinds of observation are possible, for instance.

Peirce's work has been interpreted in many different ways. The statement that "a sign is a vehicle conveying into the mind something from without" (Peirce, CP 1. 339) is closely aligned to the presumption that thinking is a matter of processing an external reality (a cognitivists' assertion) via an inevitable bridge, the sign vehicle. This realist interpretation would imply that mathematical signifiers—the notations, numerals, diagrams and the like—have been developed to describe the universe of mathematics. Ernest (1997), Radford (2002) and others have, on the other hand, interpreted the statement in ways that avoid this naïve realism:

> Signifiers of mathematics do not correspond to unique signifieds. The relation is one-many, not a mapping, let alone a one-to-one correspondence.... The signifieds themselves of mathematics are unreachable, except through other signifiers. (Ernest, 1997, p. 26)

With elaborations of the endless series of signification, the elements of the sign are considered continuous (Presmeg, 2002; Sáenz-Ludlow, 2002). However, signs would be considered more dynamic if objectivist views about body, mind and environment are challenged. Putting the cognizing agent together with the cognized world as a global system, as I did in Chapter 6, would allow semiotic research to be radical.

B.2 Semiotic Representations

Is the object of a sign, the object of representation the referent at the particular occasion the sign is uttered, or is it the *totality* of things that might be referred to by uttering the sign? Are signifieds unique to signifiers (Chandler 2002)? Put complexity theoretic, are the objects of representation multiply realizable? Researchers draw from semiotics to explain the role of mathematical representations. Otte (2002) notes, "A mathematical concept does not exist independently of the totality of its possible representations, but must not be confused with any such representation, either" (p. 366). The construction of mathematical objects is based on the use of several *semiotic registers of representations* such as graphics and pictures, Otte (2002) explains. He further differentiates between semiotic and physical or organic and non-semiotic representations. He asserts that although both representations are cognitive representations, organic representations are not semiotic representations. He says it is because material representations are causally and automatically (not intentionally) produced either by an organic system (e.g. dream or memory visual images, footprints in snow) or by a physical device (e.g. reflections, photographs). It is only semiotic, intentional settings such as sentences, graphs, diagrams or drawings that produce semiotic representations. Otte insists that automatically produced registers do not bear an intentional character and therefore are non-semiotic. I wonder, however, whether he considers spontaneous gestures and facial expressions, sensory events and motor movements, kinaesthetic activity and material experiences, bodily postures and impressions, intonations and rhythm as non-semiotic representations? In mathematics, what is the role of non-semiotic registers?

In my work, I do not find it helpful to use Duval's highly contested notions of intention and cause to make what seems to be a critical distinction. In his use of intention, he does not tell us whether he delimits this distinction to deliberate human actions. Would he include the non-conscious, circular and collective, as well as other kinds of non-linear causes and intentions?

Duval (2002) has contributed to the discussion on representations the construct of *mathematical registers* that refers to systems of symbols such as the graphical, analytical, algebraic and arithmetic signs. He says these can be either mental or external processes. Recalling English and Halford (1995), Chapter 3, classification of mathematics representations into the symbolic, mathematical, cognitive, computer and explanatory representations we may add semiotic registers and non-semiotic representations to get:

- Informal representations (*real-life*, natural, intuitive models and language)
- Formal representations (algebraic, geometric and other symbolic settings)
- Semiotic representations or sign systems (graphic and analytic language)
- "Non-semiotic" representations (organic signs, devices produced signs)

Although Otte's distinction raises more questions, his mere observation about the existence of *non-semiotic* signs, if elaborated upon in ways that pay particular attention to full-bodied, rich device produced, material, imagistic and sensory-motor informal signs is informative. Organic, analogical and motivated signs, such as diagrams and spontaneous bodily knowing, appear to "call attention to the materiality of all signs and to the corporeality of those who manipulate them" (Rotman, 2000, p. 57). Besides all signs including mathematical signs have an iconic quality to the sign user. Even, symbols and indices are icons of a peculiar kind. For example, indices are icons whose resemblance is by association (Peirce, CP, 2. 247-249). In complexity terms, symbols are nested into indexical signs that are in turn nested in iconic signs.

B.2.1. Patterns, Regularities and Order in Enacted Worlds

There is need to interrogate the concept of representation itself. It appears the computational view of representations that correspond to some reality is evoked by an assumed chasm between mind, body and world. This raises the question about non-cognitivistic aspects of cognition that fall on the blind spot of such a view? What might appear to a mathematician as static representations and signs, for mathematics students might be dynamic regularities and contingencies, tokens as Peirce referred to them, that emerge out of local actions, signals, anatomy, situations and sequences. And more importantly these regularities have feedback loops that amplify the signs and patterns.

Whatever sense humans make might be more (re-) presentational rather than a representation of the world (Von Glasersfeld, 1999). For it seems, signs and visualizations are "pieces of experience that we have combined in order to form more or less complex structures, in our attempt to order and systematize the world in which we find ourselves living" (Von Glasersfeld, 1999, p. 4). If we fore-grounded aspects such as regularities and patterns (instead of representations), temporal tools that help us negotiate our worlds, in what ways would that change our views about students' thinking?

Considering sign systems and mathematical representations as functional, ontology and pragmatic (instead of solely epistemological) tools that arise from what we do or can do is more useful. For instance, the Fraction Kit that students worked with, the tables in which they tabulated results as well as the number symbols that they worked with during the sessions could be looked at in functional ways. "They are tools which enable us to interact with the world in more complex ways ... [T]hrough experimenting with our world, we are [continuously] led to certain realizations about it which enable us to interact with it differently" (Osberg & Biesta, 2003, p. 9). Drawing from complexity sensibilities, we may begin to see representations as interdependent and open systems that admit novelty and foreground their participatory function in evervarying contexts (Rosch, 1999a).

Mathematical objects like fractions might be distinct in how they are attended. But to the extent that they are enacted objects, they are similar to objects in other domains. In neurological terms there is hardly any event that humans know without participation and interpretation. In a section on inter-objects and inter-objectivity, I show how all objects with which humans interact are necessarily human objects.[41]

If it is to be helpful in researching students' mathematical thinking, semiotics needs to be nested in an ecological view of cognition. In my writing, the concept of a sign or, better still, signification, is taken to arise out of the operations and being of the cognitive organism to which

[41] Objects that draw more from lower recursions of sensation, perception and imagination are easily dubbed natural (not cultural) objects since they appear to be humanly universal.

signs have some value. It arises with interpretive activity that affords peculiar discriminations. Questions on signification call forth ontological questions on how signs relate to the mathematical concepts. Let me explore an example from my research.

Vignette B1. ***The Pirate's Task***

In my work with students I posed the following question:

Pirates ordered all the sailors to be lined up around the edge of the ship. To the first sailor to the left they said, "You will die!" To the second they said, "You will live!" To the third sailor, you will die; the fourth you will live. They continued in this judgment, repeatedly going around the lined up sailors until only one sailor was left standing. "You will live," they bellowed, "Join our pirate crew!" If you were one of the sailors on the captured ship, where would you stand in order to survive?

NARRATION I

After scattered trials including some brainstorming, acting out the problem and written work with simple numerical values, students in the study began to make informed guesses about where in the line would they want to stand in order to survive. They eventually began to work more systematically seeking to ascertain which positions were safe for any numerical values.

B.2.2 Signing to Communicate as well as to Cognize

Students use a variety of signs ranging from informal to formal signs, from non-semiotic to semiotic to as well as iconic, indexical and symbolic signs. At one time all students used numerals—symbolic signs—to stand for sailors. They also used other linguistic signs: They spoke natural language sentences, such as *lives* and *dies,* and mathematical sentences such as 2 times 2. As they worked in pairs they talked as well as gestured spontaneously and used facial expressions. They also adopted notations; say for the sailors and for the action you live. Many sketched drawings of sailors lined up around the edge of the ship. Some used concrete materials like dominoes, knocking down sailors who would die. When they used concrete materials or their bodies—be it just body parts like fingers—to act out the problem many of their signs involved sensory, motor movements, affective, and material experiences. Bodily postures, orientation and images, intonations and rhythm were part of the signs systems they drew from. For whatever signs students used they, as the sign user, their body-in-space, material and environments were are central part of the signing activity. During the activity some of their spontaneous signs and representations grew into more formalized signs. They signed to communicate as well as to cognize. In my study, I am interested as well in how the sign systems interact with students' mathematical thinking and attentiveness.

B.2.3 Mathematical Signs in Relation to Worlds Enacted

With time, as they engaged in the Pirates Aboard task, students noted that for 2 to 3 sailors you were safe if you were in the second position; for 4 to 7, if you were in the fourth; for 8 to 15 if you were in the eighth, for 16 to 31 if you were in the sixteenth and so on. But for bigger numbers solutions diverged. Was this divergence related to the kinds of signs and representations students used? In what ways could it have been related to the worlds that the students notated as well as enacted?

NARRATION II-V

For 50 and 100 sailors:

II. Some students reasoned that it would be the 16 + 16 = 32nd position that would be safe if there were 50 sailors. For over a hundred some of these students kept on repeatedly adding 32: 32 + 32 = 64^{th}, and so on.

III. Other students said it would be the 2 * 16th position for 50 to 63 sailors. For a hundred sailors many of these students got 64 by multiplying 32 by 2.

When asked about 500 sailors this became tricky, especially for students who wanted to make a leap to a non-inductive rule:

IV. Students who continued the sequence—either by repeatedly adding or by iteratively multiplying by two—did not have much trouble, other than the computational problems involved. They gradually generated the sequences 64, 128, 256, 512, 1024. Using this sequence they concluded that for 500 sailors a sailor in the 256th position is safe and for 1000, 512^{th} position.

V. Some junior high students who attempted a leap to using a general rule thought that for 500 sailors it would be 5 * 64. Students reasoned that since for 100 it had been 2 times the safe position for 50 sailors then for 500 it would be 5 * the safe position for 100. They had attempted to articulate the structure which they had been acting out.

By looking at students pursue the pattern to determine where a sailor would want to stand it is discernible that many students had noticed a regularity that involved *doubling*. To double, some students repeatedly added, others multiplied by a factor, say 2. Although for me to repeatedly double also signifies powers of two, none of the junior high students spontaneously used exponents of two to continue the sequence. Yet viewed from the perspective of a mathematician it would be less elegant to find the safe position for bigger number of sailors by thinking in terms of repeated addition and factors (Narration V). While analyzing this anecdote I wonder about how 1, 2, 4, 8, 16, 32, 64, 128, 256, 512 and so on could signify repeated addition to one student, iterative multiplying by two to another and 2^n to yet another student. Why did junior high students not spontaneously think about exponents? One response is that junior high students have not yet looked at exponents, {1, 2, 4, ...} will not signify 2 * 2 * 2 * ... for them. But this was not the case with students I worked with. Many professed to even knowing, "Any number to the power zero is one". If students could not, by resemblance, association or by convention, recognize the objects represented by 1, 2, 4, 8, 16 as *2 times 2 times 2 times,* powers or exponents of two, I wonder how signs and representations relate to mathematical concepts. It appears {1, 2, 4, 8, 16} does not represent the same object to learners.

B.3 Conceptual Fields and Invariant Structures

In my view, the ways in which signs relate to mathematical concepts is a central question to studies on mathematical thinking. Vergnaud (1988) attempts to link the model of representations to subject-matter analysis and students' conceptual understanding. He develops the constructs of *conceptual fields*. For him we need not only pay attention to formal symbols but also to situations in which concepts are rooted plus the invariants that are recognized by students. Akin to Otte (2002) when he talks about the relation between representations and concepts, Vergnaud asserts, "a single concept does not only refer to one type of situation, and a single situation cannot be analyzed with only one concept. Therefore we must study conceptual fields." (p. 141) How is a mathematical idea such as *doublings* distinct from the settings, instances and representations that illuminate it?

Vergnaud views a concept as a triplet of sets: of situations, of invariants and of symbolic representations that can be used to point to the invariants. To Vergnaud (1988) symbols, the signifiers refer to cognitive components. He dubs the learners' and knowers' cognitive aspects conceptual fields. Conceptual fields, according to Vergnaud (1988), are a set of situations whose mastery requires mastery of concepts of varied nature. Examples of conceptual fields, he offers, include exponential, multiplicative and additive structures. Several mathematical concepts, such as multiplication, division, rational number, rate, ratio, fraction and vector spaces, are tied to the multiplicative conceptual field. Multiplicative structures are also related to other conceptual fields such as additive and exponential structures.

Vergnaud by inventing the construct of *theorems-in-action* to refer to students' intuitive knowledge relates intuitive aspects to mathematical symbolism and analytic definitions. Students may not always represent their understanding symbolically and formally. Concepts exist in a larger context and their development extends over time (Rosch, 1999b). Viewed from this perspective concepts and conceptual fields are aspects of mind-in-world. They can be formal or intuitive.

Vergnaud's conceptual fields appear to be a subset of what I refer to when I talk about *worlds of mathematical significance.* It appears Vergnaud would refer to my classifications such as the *fractional folding-ratio world* and *the covering-rational number world* as concepts that make up the fraction conceptual field. To him, conceptual fields are not closed to each other in a manner that would apply to worlds enacted. But would Vegnaud's theory of conceptual fields help me understand why students used different actions to double and why their responses diverged for higher numerical values? Or better still, what did students attend to in the pattern 1, 2, 4, 8, and 16? What was signified to learners by these numbers?

B.3.1 Worlds Enacted Vs. Conceptual Structures

I will begin by mapping students' work on three related structures: Additive, multiplicative and exponential as I show in Table B1. The students who repeatedly added the previous number onto itself engaged in the additive structure and those who multiplied by a factor in the multiplicative rather than in the exponential structure.

Table B1. *Additive, Multiplicative and Exponential Structures of the Concept of Doubling*

Number of sailors on the ship	**Additive structure**	**Multiplicative structure**	**Exponential structure**
1	1	1	2^0
2-3	1 + 1 = 2	2 * 1	2^1
4-7	2 + 2 = 4	2 * 2	2^2
8-15	4 + 4	2 * 4	2^3
15-31	8 + 8	2 * 8	2^4
…	…	…	…
Many sailors			2^n

Unlike the Fraction Kit activity in which some students by stacking strips were in the multiplicative ratio world and others by covering the whole were in the additive fractional world, in solving the Pirates Aboard task the additive and multiplicative structures were closely related. Some students who began by adding repeatedly later multiplied. Three more tasks in my study involved powers of two: the Consecutive Terms Rice Board and Chessboard squares. Many junior high students, during their engagement in these tasks, began to talk in terms of multiplying by two when adding to double became laborious. Some students kept doubling by using the operation of multiplication but that too became inefficient computationally and in notation. (Imagine having to keep adding repeatedly to the 63^{rd} term as is required with the Rice Board task!) In ascertaining in which position a student would want to stand if there were 50000 sailors even iteratively multiplying by two was evidently tedious and less elegant. Even then, only a few junior high students in the study would notice that the whole sequence {2, 4, 8, 16, 32, …} was a sequence of *doublings*, of 2 times 2 times 2 …, of *powers of two*—2^0, 2^1, 2^2, 2^3, …. Yet when prompted, by being asked what the sequence had in common with a sequence such as {1, 3, 9, 27, 81…} or to express the sequence in the most simplified form, many students recalled, "O yeah, they are *square numbers*, I mean they are those numbers that you…remember, they are, … I do not mean 2 times 2, 3 times 3, 4 times 4, 5 times 5, I mean those numbers that are … *of* two…" Why did these students not easily use the signifier *exponents of 2*?

Let me offer one more example from another vignette. It is taken from the activity on Consecutive Terms (CT) task. In it Ronald and Tony describe the sequence {2, 4, 8, 16, 32} of the numbers that did not have the CT property.

Vignette B2. ***Ronald and Tony's description of the sequence {2, 4, 8, 16, 32}***

Tony began commenting that the numbers 2, 4, 8, 16 were all even. The excerpt begins when Ronald is responding to Tony's utterance. I underline the signifieds that the students used for the set of numbers. Aha utterances are in **bold.**

EXCERPT B1

199	Ronald:	All even [...]
200	Tony:	They can all be either divided or multiplied by eight. *[Some movements and noises, Ronald looks on as Tony writes]*
201	Tony:	Um
202	Teacher:	They are all even. They are all multiplied or divided by 8. [...]
203	Ronald:	Multiples of 2
204	Tony:	Yeah
205	Teacher:	Multiples of 2. Multiples of 2, what are *[Tony writes down]*
206		*[Inaudible speaker]*
207	Tony:	So far.... *[Inaudible]* Oh yeah.
208	Ronald:	2, 4, 8, 16
209	Teacher:	But we also have ...|
210	Tony:	**O just wait I know. They are doublings. So 2, 4, then 8, 16. So|**
211	Ronald:	|2 plus 2 is 4, 4 plus 4, 8 plus 8 is 16 *[Ronald nodes as he counts loudly. Tony writes down, doublings]*
212	Ronald:	We don't have 32 here [saying that 32 would be the next term]

The boys went ahead to verify whether 32 satisfied the CT property. They worked separately. Tony used concrete materials counters and he also tried using dice and Ronald wrote down his calculations. The next excerpt comes after they have verified that 32 does not have the property. Tony had added it onto the list 2, 4, 8,16, 32, written vertically. They then conjectured that the next term in the sequence 64 would also not have the CT property. They were then reflecting on whether they needed to verify this guess and whether describing numbers that did not have the property would be sufficient to answer the question, "Exactly which numbers have the property?"

EXCERPT B2

271	Ronald:	I think we can make an estimate
277	Tony:	|**So** like numbers that don't are just the...It is just like 2 being doubled so...
278	Teacher:	I will write that down. 2 being doubled ...| *[teacher had taken on role of a scribe to encourage the boys to write on a shared sheet]*
279	Tony:	**And we** are starting with 2 being doubled and the next kind of number doubles. But we didn't really find the property... I guess we didn't really find a property for the yes's [numbers that have the property] *[referring to the fact that they were meant to look for numbers that had the property yet now they were working with numbers that did not]*
282	Teacher:	2 being doubled. Do they have any other ... can we call it anything a name or can we use any another term? *[Teacher probing]*
283	Tony:	I don't know. ... |**doublings.** *[laughter]*
284	Ronald:	*[Inaudible]* ...double...?
285	Tony:	Um...
286	Teacher:	[...] 2 and then 4... I want to write them as ... In terms of 2. 4 in terms of 2 will be ...? *[Sounds like she is writing aloud]*
287	Tony:	Um
288	Ronald:	4
289	Tony:	Um uh Shhh... To the power of two I guess. So 2 will be... 1 to

the power of 2, I think *[laughter].* 4 will be 2 to the power of 2, then 3 to the power of 2, 4 to the power of 2, 5 to the power of 2 *[On his paper he also wrote each of 1^2, 2^2, 3^2, 4^2, 5^2, directly under 2, 4, 8, 16, 32 respectively on the shared paper]*

With the teachers' prompting, Tony had articulated that the sequence was "powers of two." He added, "I guess." Nonetheless he went on to elaborate what he meant, "So 2 will be... 1 to the power of 2". Was this a slip in speech? Excerpt 3 comes when the teacher was checking to see whether Ronald was familiar with exponents.

EXCERPT B3

290	Teacher:	Do you agree Ronald ... [...]
291	Teacher:	Can we check that ... that's...
292	Ronald:	3 to the power of 2 is 9.
293	Tony:	Wait. **O yeah**
294	Teacher:	Tony you are thinking about something else that ...|
295	Tony:	Yeah
296	Teacher:	And then it didn't work here, or you are|
297	Teacher:	**|No would it be uh... 2|**
298	Teacher:	Thinking of powers
299	Tony:	It will be 2 to the power of 2. So I just did these the wrong way, I think.
300	Teacher:	So can we write them the right way ...
301	Teacher:	What you are thinking is the right way.
302:	Tony:	So 2 to the power of 1,
303	Ronald:	4
304	Tony:	Yeah. 2 to the of 2, 2 to the power of 3, 2 to the power 4 and 2 to the power of 5. *[His voice became louder and louder]*
305	Teacher:	Then we could... get a name for the numbers, another name. At first you said being doubled. And in terms of what you are writing we can call them
306	Ronald:	To ... **Powers of 2**
307	Tony:	Powers of 2 (Under lapping) Ronald's

In the ongoing vignette I contemplate about the nature of the students' difficulty in articulating and understanding exponents, much less writing them as x^n. Six students in a Ugandan context had the same difficulty. Irene and Lillian first identified the set {2, 4, 8, and 16} as a special group of even numbers, since 10 and 12 were excluded from the set. They then contemplated that it was a set of "multiples of two", but that was not exact either. Then Lillian attempted, "2 times...Because 2 is 2 times ... Because 2 times 2 is 4 then 2 times 2 times 2 is 8."

Semiotically speaking, the difficulty with describing the set is about the conventional notation, the spoken, written and symbolized signifiers—exponents. But semiotically speaking is also about more. Do the students recognize the numbers 2, 4, 8, 16 to be unique? Are they a set?

2
4
8
16
32

Sequence? Does the list call their attention to some signified? What signified then? Tony wrote the numbers vertically as in the figure to the left and the teacher horizontally as 2, 4, 8, 16. Does this difference in written signs point to differences in mathematical worlds enacted? Was what is signified by *2, 4, 8, 16* the same thing for both Tony and Ronald, for example? To reiterate: the questions I asked about the Fraction Kit activity: What did the students see? What distinctions did the students need to make in order to perceive the set as a sequence of exponents? What regularities were possible for them?

B.3.2 Relation of Signs to Regularities and Worlds in Enacted

Initially to Tony the set was a set of, "all even" numbers (turn 297) and to Ronald of, "multiples of 2" (turn 203). Tony added, they "can all be either divided or multiplied by eight." (turn 200). Then came his *aha utterance*, "O just wait I know. They are doubling. So 2, 4, then 8, 16" (turn 210) Ronald shared in Tony's moment of insight, "2 plus 2 is 4, 4 plus 4, 8 plus 8 is 16 (turn 211). It is just like 2 being doubled ..." (turn, 277). Clearly Tony and Ronald were not yet attending in the exponential-doubling world.

The signified *some even numbers* implies a class not a sequence of numbers. Additive or multiplicative actions relate the numbers by "double the previous to get the next number." This enabled the students to see the next number in the list 32 and the next. Later, Tony revised his description of the sequence as "2 being doubled" (turn 277). This was a more precise sign for it excluded sequences such as {3, 6, 12, 24, ...}. But to the teacher there was a more formal sign, exponents of 2. She checked to see if the boys were familiar with it. Could the students have identified the signifier, powers of two without knowing the signified, what powers of two meant? Or is it that they had just forgotten the signifier but understood the set to be exponents of two?

In turn 282 the teacher prompted for a precise name. To this Tony said, "I don't know |" (turn 283). But he interrupted himself "|doublings" Ronald said, "Double ...?" It appears, the boys were not conversant with the spoken and written description "powers or exponents" Or were they?

When the teacher hinted about expressing the numbers in terms of two "I want to write them ... in terms of 2." After 2 turns, one from each of the boys, Tony said, "Um uh Shhh... to the power of two. But he talked about and wrote 1^2, 2^2, 3^2, 4^2, 5^2. To which Ronald disagreed saying, "3 to the power of 2 is 9." In another aha utterance, Tony then recognized his error, "Wait. O yeah"(turn 293). With some prompting from the teacher the boys corrected, "It will be 2 to the power of 2 [instead of 3 to the power of 2]. So I just did these the wrong way" (Tony, turn 299). At turn 306 Ronald articulated the formal category, "Powers of 2" A lot is happening in this vignette. This includes shifts in attention, familiarity with concepts, joint attention and aha utterances. I will focus on the relation of signs, representations with conceptual structures and worlds brought forth.

In my view, semiotics helps us recognize that students use formal labels—powers of two and 2^2—when they can recall them. However, what the set {2, 4, 8, 16} signifies to students varies in a manner similar to how what the fraction strips signified varied. At one extreme, it is not a unique set. In between, it could be a category. Operating in the additive structure {2, 4, 8, 16} signified special even numbers. After some interactions, it was seen as a sequence that grows by doubling. In the additive, arithmetic progression world, a pair of students looked at, what in high school is described as, the common difference between the terms, which unhelpfully is also 1, 2, 4, and 8. Later they saw it as 2 being added onto itself repeatedly. In the multiplicative world, another pair tried the expression n = 2X. To Tony and Ronald in the multiplicative structure it signified multiples and factors of 8. It was then seen as a geometric progression with first term as one and common ratio 2. In both doubling senses and worlds—additive and multiplicative—students could by doubling generate many numbers. But there was a more precise description of the set, one that enacts and is within an exponential world.

So it seems, before it signifies exponents of 2 the sequence signifies doubles, in the additive then in the multiplicative world. This is a layering of semantic and lived significations. Hence, use of conventional spoken and written signs—2 plus 2, 2 times 2, 2 to the power of 2 and 2^2 are inextricably linked to actions (e.g. repeated adding 2 or iteratively multiplying by 2) and conceptual understandings (i.e. of addition and multiplication) as well as to spontaneous signs—double 2. Signs as tokens that point to regularities, categories and patterns enacted in particular worlds. To me without the enactment of these worlds {2, 4, 8} signify none of these to learners. The aspect that a sign might represent something different to somebody else points to divergences in worlds enacted.

B.3.3 The Exponential Conceptual Structure

For many junior high students the exponential structure is not a comfortable conceptual structure to work. Its objects {1, 2, 4, 8, 16}, 2^n, X^n and the like are not illuminated as such. But would this be the case for senior high participants and pre-service teachers? Given the exponential sequence 2^n, is it easy to imagine where one would stand for a million sailors in the PA task? Even, some pre-service teachers, including those who recognized the sequence as powers of two and the rule as {2^n, where n is the exponent position in the lower neighborhood}, *folded back* to the multiplicative structures—double 1024 to get 2048 as the safe position for 3000 sailors and then double that continuously to approach a million sailors. This left me wondering about how the exponential structure is related to and the ways in which it is distinct from the additive and multiplicative structures?

A multiplicative structure appears to spring forth from repeated addition in developing the sequence {1, 2, 4, 8, 16, 32 ...}. In one interpretation, when it is multiplication by the same multiplicand for a good number of cases, this in turn calls for an exponential structure. Put differently, the additive conceptual structure in this case constitutes the multiplicative structures, and in turn the additive and multiplicative structures are some constituents of the exponential structure as organized in Figure B2. In a way the inner structures of addition and multiplication are always there as available *points of return* and *grounding*. To imagine a power of that is in the neighborhood of a million is not a voluntary action for a majority of people. It is very different, in fact more difficult even for mathematicians, from imagining the power of two near $a2^{bx} + d$. Is it the case that operating across mathematical fields is what makes it hard to imagine an exponent of two that is in the neighborhood of a million sailors?

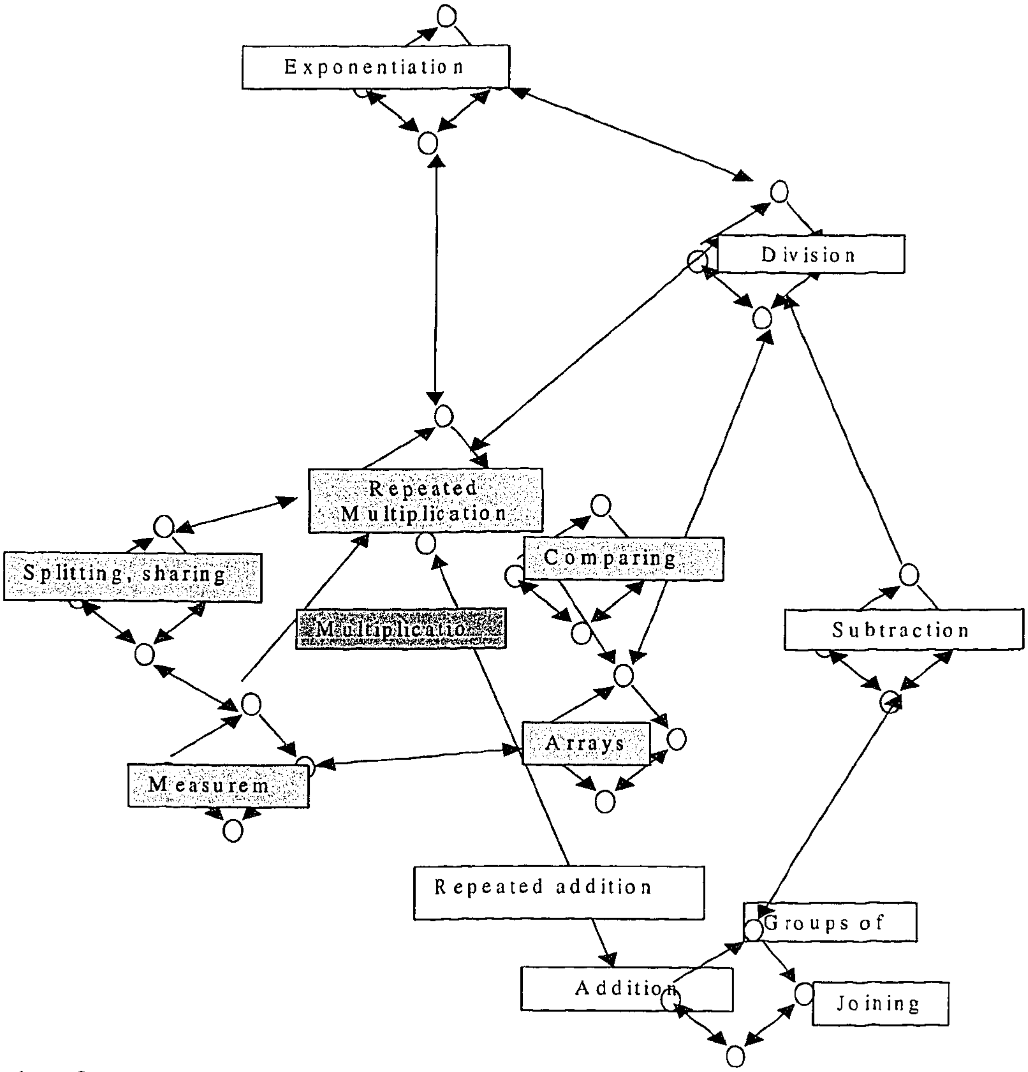
In my research looking at signs and representations is useful; however, it needs to be supported by looking at the worlds student enact, what Vergnaud has explored as conceptual fields. In looking at mathematical structures as *conceptual fields, and, better still, as mathematical worlds enacted,* four points appear to be discernible:

1) Mathematical structures—what researchers recently have dubbed, micro-cultures, views or universes—cut across mind-body-environment and thought-action-perception division. The situations that say the exponentiation field may consist of acts such as splitting, scales and growth are not merely structures of the environment. The symbols that point to the regularities that make up the exponential function are thought-action-perception creations.
2) Conceptual fields are not closed to each other. They have permeable boundaries. Witness the folding back. Some might be genetically related. More abstract and hybrid conceptual fields may spring forth from a juxtaposition of distinct mathematical settings (Artigue, 1999).
3) Many worlds of relevance are radically distinct. At many times there are qualitative incoherences among fields (e.g. multiplying by 5, instead of doubling five times, was not a relevant action in the Pirates Aboard task that grew exponentially. Times 3 is not the same as tripling three times).
4) *Abstract* and *advanced* fields might be grounded in *basic* and *common* actions, interactions and distinctions. Lakoff and Johnson (1980) would say exponential growths are grounded in basic human conceptual fields such as unitizing, identifying plurality, equal partitioning, similarity and repeated splitting or folding.

In Figure B2 I diagrammatically relate the additive, multiplicative and exponential structures. The figure illuminates that mathematical signifiers have many related but partly overlapping signifieds, as has been articulated in many studies. However, the constitution by many, for lack of a better word, meanings, multi-embodiments, micro-worlds or regularities is of a kind that is not carefully depicted by such charts or by modern use of the terms *structure* or *fields*. One wonders how we can illustrate human generalities, categorization and abstractions to illuminate the four qualities: mind-body-environment qualities, permeable boundaries, incoherences across structures and genetic grounding of one structure in another. The noun world seems to capture many of these nuances.

It appears more illuminating to describe the dynamics of students' mathematical thinking in terms of the regularities that arise with actions and interactions and universes enacted in doing. This illuminates the emergent nature of mathematical ideas and knowledge. I develop this further in the main text chapters 10-11. For now, I examine further semiotic constructs.

Figure B2. Networked mathematical conceptual worlds: A zoom in at multiplication



B.4 Systems Elaborations on Semiotics

The recent turn to semiotics to understand mathematical cognition seems to be motivated by the view of mathematics as a domain populated by written notations and conventional representations. Perhaps semiotics researchers ought to study the ontology of mathematical reality that signs stand for as well as calls forth. When talking about the signifier and the signified it is tempting, to gloss over the existence of the interpreter, his or her multi-chaining and nested interpretational activity. There is more to mathematical cognition than formalized linguistic-institutional aspects. Sensuous, motivated (not arbitrary), analogical, metaphorical and other spontaneous signs are easily obscured by symbolic sign systems. Wouldn't it be more useful to acknowledge that the cognitive agent, the sign user has a body with its experiences and related orders of signifying including pre-conceptual signals? In the spirit of Lakoff (1991) one wonders whether by human beings are independent of their "animal" activity being rational and linguistic. In my work I am also interested in the distance, the layers between the zoo-semiotics and formal symbolic signing. A few researchers extend semiotics toward the informal, potentially symbolic or "non-semiotic" signs. It is the connection and trace between human semiotics and bio-, material- and social semiotics, the embodied as well as embedded and extended semiotics that would greatly benefit an ecological study. Certain researchers including Brier (2001), Radford (2003) and Rotman (2000) have attempted such elaborations in ways that I explore below.

B.4.1 Radford on Semiotic Means of Objectification

Radford (2003) both critiques and appropriates Lakoff and Johnson (1999), and Lakoff & Núñez's (2001) discussions on the embodiment of mathematical concepts. He observes that learners have recourse to a broad set of means—not just written symbolic signs. These include manipulating objects, drawing, gesturing, and I will add writing, using tools, making presentations, metaphorically projecting and making commentaries on their own activity. Radford refers to these as (semiotic) means of objectification. Means of objectification are crucial since they "throw mathematical objects" to us. In Peirce (CP. 2. 98) these are the peculiar means and kinds of observation that allow people who herd together to understand each other. From the vignette on powers of two, we see that means of objectification make things perceptually, semantically, socially and linguistically *accessible*. They objectify. To Radford, these means are semiotic even when they are non-intentional, not formal and organic, just like the theorem-in-action.

My earlier list of signifiers involved in learning unfolds to accommodate non-symbolic aspects such as movement and gesture that objectify mathematics. Whereas Vergnaud's conceptual fields are formal, means of objectification may be informal and "not informal". They can be causes and effects, all at once

- Informal means (Real-life situations, mental configurations, etc.)
- Formal means of representation (Mathematical branches)
- Semiotic registers (Graphic, analytic, symbolic, etc.)
- Conceptual structures or fields (Additive, multiplicative, etc.)
- Semiotic means of objectification (actions with concrete things, gesture-based actions, natural language oral speech, written sentences, theorems-in-actions)

Radford distinguishes that occurrence of mathematical objects is related to culturally embodied experiences but not to biological embodiment. The body for him, quoting Foucault, is only a surface of inscription of social, cultural, linguistic and historical experiences. To think mathematically is to transform mathematical-cultural objects into objects of one's understanding (Radford, 2003). The body is thus affected by the socio-cultural, but, to use a complexity term, it has no significant feedback loops with these experiences. He prefers to use the term *empracticed* experiences instead of *embodied* experience. (Peirce, CP, 2.245, uses embodied.)

By appreciating the body and spontaneous actions as sources of biological, material and pre-linguistic constraints for formal mathematics Radford's work adds a new wrinkle to mathematical didactic semiotics. He comes so close to the enactivist stance of "putting body back into mind", but it appears he does not go all the way. He put minds back into body. And he takes on the material embodiment of signs, but he is reluctant to take on their biological embodiment in more fundamental ways. Why linguistic-cultural theorists find it hard to recognize that the body and its informal actions inscribe something on mathematical practices begs attention. The complexity notion of mutual causality would explain that both the body and culture constrain each other.[42] Radford might also benefit from the principle of nested levels of emergence. This metaphor could place the body in its appropriate level of description without

[42] As demonstrated in Chapter 5, whereas the body is a surface of socio-cultural inscriptions, which in many ways is a result of culture and language, it also simultaneously inscribes a lot on mathematical culture and language. "The body is shaped by the world that it participates in shaping" (B. Davis, 1994, p. 56). From an eco-complexity perspective, to deny the biological ties of mathematical experiences does not sit well with any study of human actions, however virtual and esoteric the mathematical actions might be. It is functionalistic to consider the body, the signifying devices, irrelevant in signification (Rotman, 2000).

threatening the importance of culture and history.

B.4.2 Mathematical Activity: Orders of Signification

Rotman (2000) develops a semiotic model of mathematical activity in which he illustrates that a mathematician engages in the signifying activity at three levels: the lingual, sublingual and meta-lingual levels. Mathematical objects for Rotman are not ideal, platonic forms, pure formalistic thoughts or mere intuitionistic constructions of the mind, nor are they located in tangible written products without relation to reality. "Mathematical objects are mentally apprehensible and they owe something to human culture; they are with material, empirical, embodied, or sensory dimension. That is their existence, realness and objectivity." (p. 47) In a manner similar to enactivism, he emphasizes the role of the body, tools and technology, and of materiality in meaning.

To Rotman mathematical signifiers have a creative role rather than a merely descriptive one. It is neither the signifiers (as with Saussure) nor the signified (as with Lacan) that are secondary. The whole signifying activity is important in the generation, sustenance and perpetuation of mathematics. For example, numbers do not precede the numerals (the signifiers that bear them), nor can signifiers (the numerals) occur in advance of signifieds (the numbers). As Sfard (2001b) also put it, they are *co-significant.* Neither 2^n nor the pattern doubling or 2 times 2 times 2 ... is without the other.

For Rotman, mathematics rests on written signs and scribbling. He says written signs are marks with meaning. To explain how writing (in its broadest terms) is inseparable from thinking he states that writing and thinking are the *mathematical orders of signification.* They are, "[O]utside the purpose of analysis and the like, impossible to separate" (Rotman, 2000, p. 58). As orders of signification neither one of them is secondary to nor separate from the other. Rotman's assertion adds yet another column to Mac Lane's (1981) table that maps tangible formal mathematical systems, in a one-to-many relation, onto ordinary human activities. The column Rotman's work suggests is of mathematical objects that arise from and in turn suddenly advance ordinary and mathematical activity. Rotman would add writing and thinking to the first column in each row of Table B2. I would add another column, to would include notating, signifying, puzzling and other ways of being in ways that are mathematically adequate; I said more about this in Chapter 10. The rows of the table are ever incomplete. They grow.

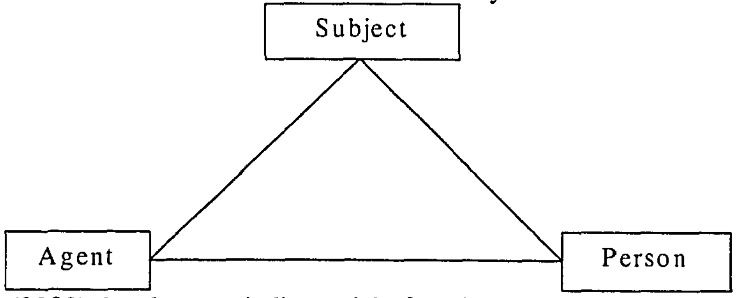
Table B2: *Basic Activities, Formal Domains and Mathematical objects*

Basic Human Activities	**Mathematics Domains**	**Mathematical objects**
Counting, writing, thinking	Arithmetic, Number Theory	Numbers, patterns, sequences
Measuring, writing & T	Calculus and Analysis	Units of measurement, Real Numbers, points, functions, lines
Shaping, W & T	Geometry and Topology	Geometrical shapes, spaces, dimensions,
Forming/Designing, W & T	Symmetry and Group Theory	Curves, groups or sets
Estimating, W & T	Probability, Measure Theory and Statistics	Operations, relations,
Moving, W & T	Mechanics, Calculus and Dynamics	Limits,
Calculating, W & T	Algebra and Numerical Analysis	Equations, roots
Proving, W & T	Logic	operators
Puzzling, W & T	Combinatorics and Number Theory	

Grouping, W & T	Set Theory and Combinatorics	Sets, elements

On the question of whether the body or materiality are primarily and purely cultural constructions, Rotman (2000) argues that to exclude the biological, the experiential, the corporeal and the enactive from what is meant by embodiment "is servitude to theory and high-mindedness" (Rotman, 2000, p.109). All signs are the work of the body. Actions, thoughts and perception makes them human objects. Latour (1996) would say that mathematical objects are quasi-sensorial and quasi-bodily—they are co-determined by sensory-motor, perceptual and imaginary sources (Cohen & Varela, 2000) They, like other emergent wholes, are rooted in the body yet they have an identity different from bodily aspects. Rotman (2000) identifies the belief that "signs are always signs of or about some pre-existing domain of objects" as the "poverty or platonicity of some semiotics." (p. 52). Mathematics educators have to draw from Rotman's semiotics.

Figure B3. Rotman's Triadic model of mathematical activity



Rotman (2000) develops a triadic model of mathematical activity consisting of the *subject*, the *person*, and the *agent* (see Figure B3). For example, a mathematician as a subject engages in counting, measuring, proving, puzzling and, Rotman highlights, writing activities that have potential for formal mathematics. The subject engages in scribbling as well as thinking. At this level mathematical activity is lingual coded and usually involves material and sensory manipulations. Rotman adds, "Counting is not a progression through existing infinite sequence of timeless, spaceless, and originless objects" (Rotman, 2000, p. 78). Although it might appear that way to an agent for whom the universe of numbers has already been brought into being, to the child the subject who is yet to count and make marks about his/her counting, numbers do not yet exist.

From the activities of the subject arises the agent, who engages in the mathematical code. The agent's activity includes sublingual imagined actions in which he/she can, for example, add elements of a divergent sequence, operate with exponential functions, thinks about infinite, work with infinitesimals, or talk about the average of x, y, and z unknown values. This is a level of imaginary action that is made possible by conventional signs and rigorously formulated sign practices. It is a level only carried out in principle: an imaginary, virtual universe.

The objects of the mathematical culture—the agents' nouns such as the points, numbers or dimensions (column 3 in table 5)—are underlain by the activity of the subject, by the subject's verbs such as draw, count, double and measure (column 1). In a sentence, "[M]athematical signifiers are themselves dependent on some prior signifying activity" (Rotman, 2000, p. 23).

Mathematical notations such as $\forall$, $\prod$, ¾, $\div$, $\neq$ and 2^n arise informally or formally, during the activities of the subject, to notate as well as to create the mathematical objects, the signified properties. In my view, mathematical symbols or signifiers such as numerals, symbolized notations and linguistic phrases (e.g. "multiples of" and "exponent") are signs by virtue of the interpretative role of the mathematical agent who has at some time acted, perceived and thought as a subject. To the agent, signs appear as signs about some existing domain, but one enacted by the subject's activities. In enactivist terms objects and signs are, as I explored in chapters 7-11, inter-objects. Mathematical sign practices occur, in the first place, as informal and unrigorous elements in a merely descriptive, motivational or intuitive guise within the meta-code of the mathematician as a *person*, Rotman (2000) adds.

A mathematician as a person arises when he/she thinks and talks about the mathematical universe. This meta-code involves the meta-lingual activities of the person (column 2). All codes considered the junior high students I worked with had not yet engaged fluently in the exponential universe at the meta-code level of the person. They could repeatedly double to raise two to an exponent, but could not notate this conventionally. It might also have been that they had earlier on engaged in the agent and person's code. Generally, mathematical activity might seem formalistic within a person's activities, just as it might seem platonic within the activities of the agent, or as intuitionist within the activities of the subject. There is a layering of mathematical activity and experiences. Mathematical signs include the signs of the subject and agent.[43] More layers can be added on Rotman's three orders of signification. With technological and thought tools that can visualize more than three dimensions theorists may not want to limit themselves to triads much less to dyads. Not any more.

Although Rotman (2000) challenges the roots of the semiotic school of thought in structuralism, realism and rationalism, all his three layers—lingual, sublingual and meta-lingual—are of the linguistic order. A question arises: Is mathematics in particular, and human semiotics in general, basically a linguistic activity?

Brier (2001) elaborates on the concept of signification in ways that are commensurate with bodily and material signification. I take on Brier's (2000, 2001) work for a more ecological semiotics that relates to multi-dimensional signifying systems, most of which are nested in, and therefore intricately related to, the formal-symbolic sign system. Indeed when Peirce's semiotics is read with complexity metaphors it grows.

B.4.3 Nested layers of signification: Embodied Sign Concept

Brier (2001) maintains that Peirce's sign concept is less general than what is needed for a study of signs in its broadest terms. We need a concept of meaning and signification that integrates bodyhood and biological heritage plus post-linguistic sign systems. Perhaps this would be too broad a concept, but there are metaphors and tools that handle the complexity that comes with the consideration of interactions between many layers of signification. In addition to embracing the endless serial nature of sign systems, Brier (2000) has extended human semiotics to illuminate the biological, psychological, socio-linguistic and socio-cultural aspects of meaning making. In my view, even the more restricted understanding of sign as "something which presents itself to the senses as something other than itself to the mind" (Augustine (n.d.) quoted in Radford, 2003, p.42) could be usefully extended to span more than one level. Patterns of the body and regularities in actions, for instance, could be studied as signals, messages and signs at their scale of activity. The broadened concepts of the sign, sense or mind are not only relevant at the human communicative scale of explanation, but they are also relevant for meaning even in simpler organisms and at collective human cognition. I again orient my exploration of Brier's cyber-semiotics with Davis et al.'s (2000) drawing of nested bodies that we saw earlier on. In a way Brier imagines that each of these six knowing bodies or organisms—bodily, personal, collective, societal, species and planetary—engages in signification. Redefining the scale of complexity will allow for the consideration of organic, material and technological signals as signs.

In Figure B4, I juxtapose Rotman's (2000) levels of mathematical activity with Brier's

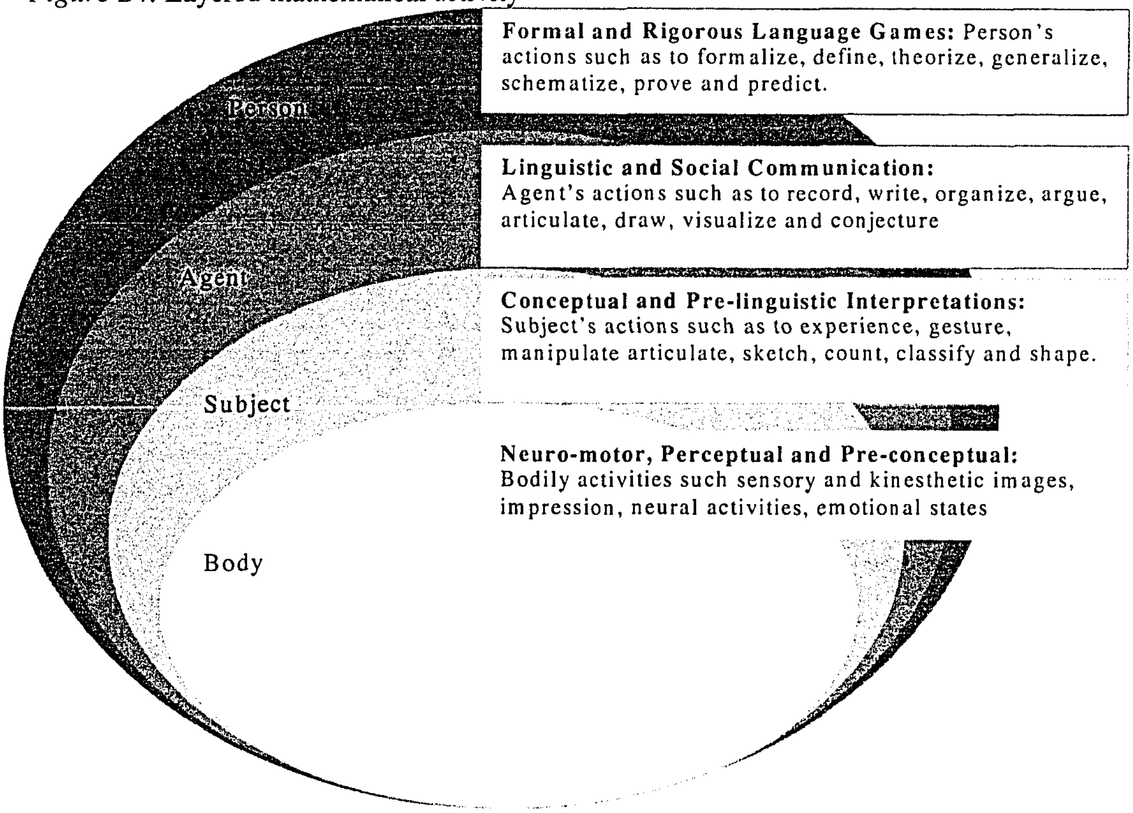
[43] Bateson's implication that anything that triggers a response is a sign, even if it occurs in a layer of complexity other than that of interpersonal communication is a far-reaching elaboration that potentially includes the study of signals and semio-chemical signs as well as *enlarged* signs of signs, patterns and order. It ushers in the study of how humans, both in the daily perception and in the observation of mathematical properties, order their worlds.

(2001) spheres of semiosis. I include the body as an inner level of activity that layers the subject, agent and person's mathematical activities. It is layered signification that includes:

B.4.3.1 Endosemiosis: Sub-personal Semiosis

The nervous, hormonal and immune systems' web of signals is continuously going on inside our bodies. The nervous system itself is a web of signals: synapses, firings and currents. These are organic signs. The semio-chemical, dynamical and physical level of signification might be called the *somatic level of semiosis*. At this level of constituents and bodily activity, neural and emotive signals, gesturing, idiosyncratic coordination of actions and perceptual motor activity by virtue of the fact that they trigger responses in the states of the body, could be considered as pre-conceptual signification.[44] These layer the pre-linguistic sphere of signification, which includes non-verbal thoughts, emotions and hunches.

Figure B4. Layered mathematical activity



B.4.3.2 Psychological: Personal Semiosis

It is from the coherences and interactivity at the bodily level of signification, as we saw in Chapter 5, that global, conceptual and psychological signification springs forth. Each individual human has an emergent mind with imagination, goal-directed behavior and volitional capacities at a level different from the bio-chemical one. Brier (2001) refers to the mind's activity as the *individual psychological* signification sphere.

B.4.3.3 Social-Linguistic Semiosis

The psychological signification space is again perturbed by social interactions, including competition, cooperation, reproduction and socialization. The social, or more specifically in humans, the linguistic-cultural space of signification arises as they live together, as they recurrently interact and coordinate actions with others. When this is done successively

[44] This claim should not be interpreted as saying brain impulses are sentient or that they are signs at the socio-communicative level. Quite the contrary; the different bodily states that spring from collectives of biochemical and physiological activities are causes and effects of conceptual and linguistic sign systems.

within a specific group of humans that share fundamental motives, another level springs forth, one that is more organized and formal.

B.4.3.4 Symbolic-Communicative Signs

Another order of signification is made possible through human imaginative and symbolic capacities such as the capacity for language and symbolization. Brier (2001) explains that formal linguistic semiotics springs from symbiosis. Humans use their social and psychic minds to order and comment on their linguistic behavior with fellow members of a specific culture. This appears to be the case with formal mathematics and other formal disciplines.[45]

B.4.3.5 Integrated and Multi-threaded Signification Spaces

Levels of signification relate to each other in complex ways. Outer symbolic signs layer and are layered by inner instinctual signals. In a way the para-lingual signals such as mental images, sensory images and bodily orientations are *internal* and *embedded*, but not *internalized* social signs. From the emergence principle, in addition to inner layers unfolding into outer ones, inner and outer layers constrain the space of the possible. Thus, in a way all spaces of signification nuance each other. To describe them distinctively in the above hierarchy is only possible in reference to their *evolutional* time scale, from single-celled to multi-celled, then animals, individual humans and human collectives. A more elaborate view of this inter-level interaction's offered in Chapter 5.

B.4.4 Layered Mathematical Signification

Which levels are primary levels of signification in mathematics: The material-bodily, the conceptual and pre-linguistic, the socio-linguistic, or the formal language games level? This question, as explored in Chapter 2 with theories of learning, has been pertinent for studies on mathematical thinking. Rotman (2000) answers this question by saying the outer three spaces—the lingual coded—are primary in mathematical activity. Non-linear dynamics would suggest that all levels are significant. I pursued this further in chapters 9, 10 and 11 as I explore how to orient students' mathematical attentiveness.[46]

Human semiotics, as interpreted after Peirce, Lacan and Saussure, is helpful, but is currently overly restricted to specific levels of description that are sufficient for researchers who consider mathematics to be largely, or even strictly, based on formally written, symbolic meaning. Modern mathematics flourished with and depends on diagrammatic, alphabetic and numeric writing systems and on formalizing. Historically (as well as developmentally), however, we do not have to look so far to remember non-classical, proto- and day-to-day mathematical activities. Classical mathematics has its origin in such activities as Pythagorean mathematics, abacus mathematics and syncopated algebra, which probably involved as much of the proto-sign systems as they did symbolic ones.

Less formal mathematics, from which formal mathematics arises and subsides into, is fundamental from the didactic point of view, particularly if we are to transcend the problems

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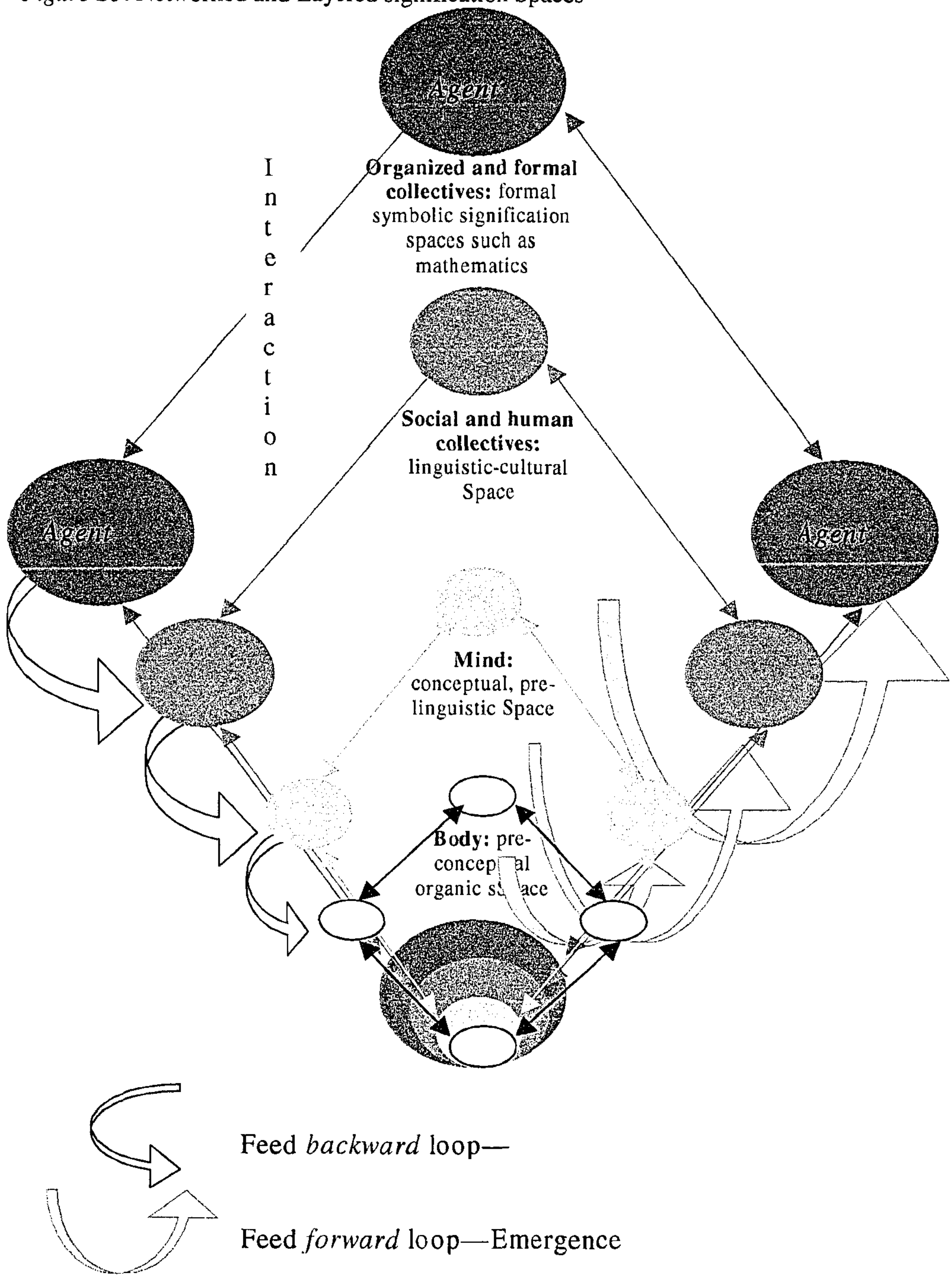
[45] This socio-communicative level of signification does not emerge only from conceptual and pre-linguistic signification. It is a distinct level of signification with a high degree of autonomy, its own sign system, its domain of objects and with novel and at times fantastic processes that have distinct identities—of formal language, politics and culture. Because it is an emergent whole, it turns back to constrain the psychological and then the inner pre-conceptual and paralinguistic spheres. This effect of looping back applies to the other layers as well.

[46] In my writing I infer that the emergent psychological, linguistic-cultural and formal-communicative sign systems remain constitutively and historically tied to their origins in the bodily signal system. Their origins and experiences reconfigure their horizons. In a study about the dynamics of students' mathematical attentiveness, to make no mention about lower layers of signification and about the recursion of levels of signification is to limit the level of complexity of the discussion. As well to limit the discussion to the physiological, bio-chemical and artefactual traces of cognition would be the other side of the coin. There could also be other, perhaps recent on the evolutional scale, outer signification spaces that influence students' mathematical thinking. My study is open to those post-linguistics spaces as well.

encountered in understanding how students think mathematically. When thinking is taken as perception-in-action rather than perception-before-action non-linguistic semiotics has to be called to mind.

Figure B5. Networked and Layered signification Spaces



In many ways semiotics has been recursively elaborated upon, from Saussure's emphasis on the *spoken word* and purely psychological signs to the written and creative role of signifiers, and to include non-symbolic sign systems. Brier's (2001) interest, and hence my interest in his work, does not mean that semiotics at the level of language games is not important.

Rather, the point is that it is important to understand the *depth* and dynamics of formal signs, to understand that there is *connected causation* from the body and materiality to symbolization. In Radford's (2002) language, the body and its artifacts also inscribe so much on culture, language and history.

This discussion on eco-semiotics demonstrates how semiotic studies in mathematics education could benefit from broadening the conception of signs and representations. Figure B5 is an illustration of layered and networked mathematical signification spaces from a systems and ecological orientation that frames this study. It illuminates the nested yet fractal and multi-threaded nature of sign systems.

B.5 Is This Study a Semiotic Study? A Conclusion

I recognize that a study on perception and observation could benefit from post structural frameworks on signs and visualizations. Some semiotic vocabulary such as orders of signification and means of objectification could be usefully adopted in a study on the dynamics of students' mathematical thinking and attentiveness. I view my exploration of semiotics as a recursive elaboration by way of reading and constructively critiquing semiotic theories. By considering sign systems in a broader framework to extend beyond the formal communicative level, semiotic theorists would develop systems understandings that study pre-linguistic and pre-formal signs as well. In light of my exploration, I consider my study to be about how regularities arise and how humans order their worlds; it is more about the development of patterns and of meaningful wholes that happen over time and with experience than about static signs and structures; it is about signification, concepts, representations and points of view understood in broad terms as tokens of actions and interaction. Whichever level of mathematical signification I study, in one way or another it ought to reflect interactions with other levels of signification. The complexities of nested, and multi-threaded signification need to be accommodated if the recent discussions on mathematical thinking that draw from semiotics are to avoid separating thought, action and perception. Eco-complexity frameworks have much to offer the embattled and broadly adopted field of semiotics and other post-structural frameworks.

APPENDIX C Demographic Survey Form

Demographic Information of Participants in the Research On Mathematics Learning by I. Namukasa, University of Alberta

Name: ____

Grade: ____

School: ____

Year of Birth: __________________

Favorite school subjects: ____

____

Rank the following statements ***(Circle whichever best describes you)***

1. I enjoy mathematics.
   - Strongly agree
   - Agree
   - Neutral
   - Disagree
   - Strongly Disagree

2. Mathematics is an easy subject.
   - Strongly agree
   - Agree
   - Neutral
   - Disagree
   - Strongly disagree

3. I have always liked mathematics.
   - Strongly agree
   - Agree
   - Neutral
   - Disagree
   - Strongly Disagree

4. I will do more math at college or University.
   - Strongly agree
   - Agree
   - Neutral
   - Disagree
   - Strongly Disagree

Please answer the following questions as best as you can

Which mathematics topics do you enjoy most?

____

____

Which mathematics topics do you find most difficult?

____

____

What mathematics topics have you done recently at school?

____

____

Do you ever do mathematics outside the mathematics classroom? ___________
If YES, in what situations do you do mathematics outside the classroom?

____

____

If NO, do you have a reason why you do NOT do mathematics outside the classroom?

____

____

APPENDIX D Interview and Observation Prompts

Interview Questions

Interview questions were contingent on observations of the researcher on preceding activities. Some were specific to a particular pair of students and others were generic.

Generic Interview Question and Activities

- Explain how you got your answer.
- What did you have in mind when approaching the task in this way?
- Can you convince me that your answer or strategy works?
- Why did you stop using counters and concentrated on writing instead?
- How can we be sure that the solution you have is right?
- Have you done a similar task in class before?
- What else would you like to find our about this task?

Pair Specific Interview Question

- Do you remember anything about the last task?
- Have you done something at school that relates to the last task?
- Would you like me to show you how I analyze and work with your data?
- How would you describe how you work as a pair or collective?
- What and when do you underline?

Observing Questions

Progressively I developed a list of questions to keep in mind as I observed students' engagement during sessions, watched the video records, transcribed the sessions and studied the transcripts. The list of observational questions included, when the pair is solving a problem:

- What metaphors do they use?
- Are they solving it mathematically? And in what way?
- What actions are the bases of the mathematical concepts they use?
- What knowledge or mathematical objects are enacted by their actions?
- In what ways do the above models help me to understand students' actions and interactions better?
- In what ways do the Simmt (2000), Pirie and Kieren (1989) and B. Davis et al. (2000) above models help me to understand students' actions and interactions better?
- Are they solving the problem mathematically?
- Which concepts do they draw upon?
- Do they use natural language, formal mathematical language or symbolic language?
- What mistakes do they make? What difficulties do they encounter?
- What models, interpretational, images or intuitive tendencies do they have for a particular concept?
- What procedures and skills do they use?
- What level do they work at and how do they progress through Pirie & Kieren's (1989) model of understanding: Image making, image having, ...

APPENDIX E Extra-curricula Anecdotes and Vignettes with Preliminary Analyses

An example of students' engagement in a variable-entry and good enough problem

Chessboard Squares

It was claimed that there are 204 squares on an ordinary chessboard. Can you justify this claim?

Vignette E1. ***Chessboard Squares (CS) Task***

Students worked on the CS problem using varied ways. For example to work out the number of squares in a chessboard, students proceeded in varied manners, some novel, others not:

(a) To a few students what was possible was to find in a systematic way how many 1 X 1 squares, 2 X 2 squares, 3 X 3 squares there are and so on. Finding out how many of each there were was done in a couple of ways. One pair of students systematically considered, for example, the number of 2 X 2 squares in a 2 X 8 row to get 7 squares, which they multiplied by 7, the number of 2 X 2 rows in the chessboard. They used the same method for the 3 X 3, 4 X 4 etc. To me an observer this was a novel approach and an efficient way to approach the problem.

(b) To others what was possible was to randomly recognize that there was an 8 X 8 square, another square of different dimensions, and many more squares of varied dimensions. Usually this was a path full of frustrations.

(c) Other students carefully marked and counted the squares of varied sizes one by one until they got what they thought were all the 85, 100 or 204, or until they realized it was a laborious process. They were then, emotionally, prompted to look for an easier way. "There must be an easier way", one student assured her partner as they contemplated how best to solve the problem.

(d) A few students tried to divide the chessboard, for example into quarters, for example. They worked with the quarter to get, say, nine 2 X 2 squares from which they inferred that the whole chessboard had thirty-six 2 X 2 squares—an error-laden approach that only a few students were able to correct.

(e) A few pairs of students went looking for patterns as soon as they had carefully worked out, for instance, how many squares there were of dimensions 8 X 8, 7 X 7, 5 X 5, 6 X 6 and so on. These students took a relatively short time and were able to detect errors more quickly.

None of the students that I have worked with has approached the task by considering game boards of size 1 X 1, then 2 X 2, then 3 X 3 to look for a pattern. This is an approach that is not only spontaneously possible for me now but one which now I view as optimal, having attempted the task myself, seen many students attempt it in varied ways, and read about other solutions.

There is a variation of this task that I usually offer to students who are not yet used to engaging less definite tasks or as a prompt for students who seem to despair at finding the number of squares in the chessboard.: For such students the task as it stated in Mason et al (1985) is good enough:

> It has been claimed that there are 204 squares on an ordinary chessboard. Can you justify this claim?

As the students work on the problems, I not only closely observe, I participate with them. Let me say more about my role as a participant observer.

Illustration of the Role of Participant Observer

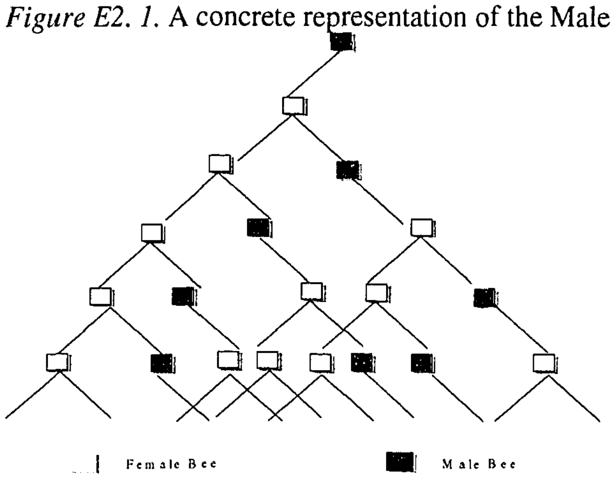
*Vignette E2.**The Bee Genealogy Task: Rose and Norah's Fifth Session***

Male bees hatch from unfertilized eggs and so have a mother but no father. Female bees hatch from fertilized eggs. How many ancestors does a male bee have in the twelfth generation back? How many of these are males?

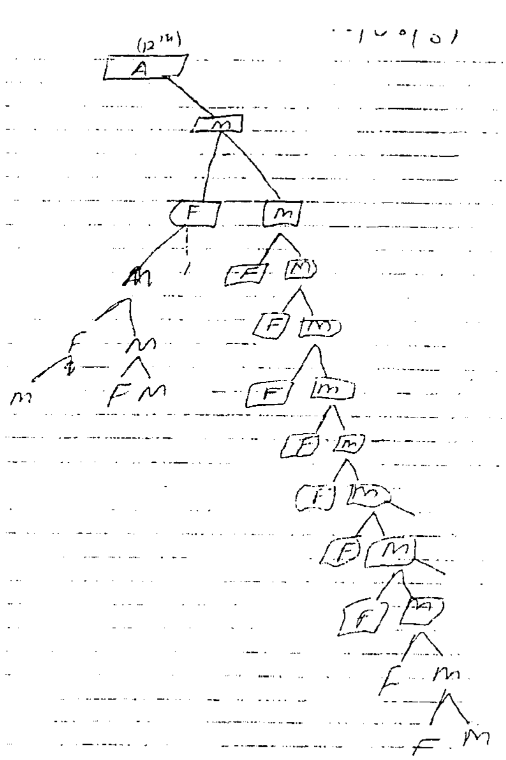
Rose and Norah participated in five sessions: the first on *Dominoes*, the second on *Matches 1*, the third on *Ladies Luncheon,* the fourth on *Fifteen* and the fifth on *Bee Genealogy*. Before narrating Rose and Norah's engagement with the task let me offer a solution to the task to orient the reader to the task. I adopt it from a solution by pre-service teachers. A group of four pre-service teachers they chose to use toothpicks and two colors *cube-a-links*, one color for male bees and the other for female bees. They modeled the generation on the classroom floor as depicted in Figure E1. One student, squatting down by the first bee, saw that there was a pattern in the model. "You add the number of bees in the two preceding rows to get the number of bees in the next row" he blurted out. His group mates listened in. As he pointed at the model he further explained, "1 + 1 = 2 bees in row three, 1 + 2 = 3 bees in row four, 2 + 3 = 5 bees in the fourth generation back, 3 + 5 = 8 ancestors in the fifth generation, 5 + 8 = 13 in the sixth generation and so on. Using this *Fibonacci* sequence in ten minutes the students were then able to find the number of ancestors the male bee had in the 12^{th} generation back and to conjecture the value for the n^{th} generation. Now we return to Rose and Norah's engagement. The girls got their answer in less than 7 minutes! In the vignette my interventions are in **bold**.

After reading the question, Rose asserted, "Unfertilized eggs, it means it [the male bee] only has females [ancestors]" She then turned to me and asked, "A generation, does it mean that it should only be one bee?" I asked, **"What do you think, Norah?"** "I think we should count one generation as one bee," Norah replied. The girls then turned to their papers to write. After 2 $\frac{1}{2}$ minutes, Norah asked Rose, "Did you get the answer?" Rose replied, "Eleven". Then Norah said, "Eleven-twelfths". **I then interrupted the girls saying that I did not understand how Norah had gotten** $\frac{11}{12}$. Rose explained, "If I am the 12^{th} then I have 12 –1 or $\frac{12}{12} - \frac{1}{12}$.... If it were fertilized it would have 24 ancestors."

Figure E2. 1. A concrete representation of the Male Bee Family Tree



When the students advanced to answer the second part of the question, "How many of these are males?" Rose with assurance said, "I think there are no males, they can't be asking for males." To this I retorted, **"A male bee has a mother, how about the mother?"** Norah replied, "I would have only male ancestors if I was the bee." "I even wonder how the generation would continue," Rose interrupted. Noticing that they had done little to represent their work, I prompted, **"Could we have a male bee...could we have something to represent a male bee?"** They then



turned to their papers to write. Each of them drew a tree diagram in which they chose to follow, in Rose's words, only the "exact lineage". At each node they finished the left-most branch, which produced a skewed tree.

In their diagram to the left A is the male bee, M is for Mother and F is for the Father bee at every node. After a few minutes, Norah turned to Rose's tree and counted: 10 male (Fathers) and 11 female (Mothers) ancestors. She asked Rose why they never had to draw the other branches. Rose responded, "To reduce the space it would take, we leave those others". Noticing that they were *making an assumption* that served mainly to reduce the workload, I joined in by drawing attention to the fact that they were answering a question in which only **"first ancestors"** counted as ancestors, yet there was a possibility that all ancestors would count. **When I asked which of the two possibilities they wished to consider**, Norah replied, "The second one". They then proceeded to complete the other lineages.

E2.1. Ongoing Prompts to trigger mathematical behavior

It appears at first the girls read *twelfth* as a clue to a fractional solution. In a couple of minutes they had the answer, a male bee has "Eleven-twelfths" ancestors in the twelfth generation back. They later chose to consider only one branch. This was "to reduce on the space it would take," Rose explained. I then perceived that their ways fell short of mathematical activity. It was unlikely that they would behave in a mathematically adequate manner after choosing to neglect facts that made the task space and time-consuming. Instead of devising ways to make the task less laborious, they changed the question to a simpler one; Instead of searching for patterns, specializing or generalizing situations they reduced the task's magnitude. When mathematicians and students when they behave mathematically do not they repeatedly throw away the nonessentials, not for simplicity but for the sake of elegance and computational effectiveness (Kauffman, 2001)? When they make conjectures, don't they seek to verify them? They organize and systematize, or formalize, and progressively refine their fundamental intuitions and common sense (Dehaene, 1997; Ernest, 1991; Freudenthal, 1991; Joseph, 1991). For Freudenthal (1991) this process of *mathematizing* involves progressive *organizing, formalizing, algorithmzing and symbolizing* to distill, abstract and transform situations.

When I noticed that Rose and Norah had taken 7 minutes to get the answer "10 male and 11 female ancestors", I contemplated: Should I offer them another task? Or should I explore with them the possibility that a boy bee might indeed have male ancestors? I had earlier on tried to *shepherd* Rose and Norah by subtle prompting: "What do you think, Norah?" "I don't understand how you got $11/12$." When Rose concluded that there are no male ancestors I rug pulled, "A male bee has a mother, how about the mother [what parents does she have]?" The girls proceeded to draw the family tree. Seeing that the girls were engaging in the task without any concrete or symbolic records, I *invited* the girls to have something on paper, a potentially fruitful action. "Could we have something to represent a male bee?", I suggested. Even then the girls drew a skewed family tree. I then decided to *invite* them to explore a situation in which ancestors for a bee could mean more than immediate parents of *female ancestors*. I asked, thinking that by considering all ancestors the girls would encounter a constraint—computational size—that would prompt rigorous ordering, as well as noticing structures. The girls then proceeded to complete the other branches of the tree and to look for patterns. After reflecting on Rose and Norah's fifth session I began to watch for basic actions that create the potential for adequate mathematical behavior. A new theme had emerged for me. I explore this theme later.

E2. 2. Relative Suitability of tasks

In retrospect, this incident challenged me to reflect further on the suitability of tasks with respect to a particular pair of students. What did I mean by an appropriate task in the context of Rose and Norah? Did the task dynamically motivate them? The girls appeared to understand generation in colloquial terms. Did they know what is biologically implied by *fertilized egg*? Whereas Lillian and Irene, students from the same classroom, seemed to have engaged on the same task a week before without much dependence on my interventions, Rose and Norah needed more ongoing prompting. Lillian and Irene had used dominoes (labeled the marked face as female and the other side male) with which they modeled an ancestral tree for the first five generations after which they tabulated their results to look for a rule. By contrast, Rose and Norah reduced the task magnitude. They made unfitting assumptions and read clues, a mathematically ineffective practice that Artigue (1999), Boaler (2000b) and Schoenfeld (1988) observed students in traditionally taught schools resort to.

Although Rose and Norah appeared to know that a fertilized egg had both a mother and father, the phrasing of the question might have posed a problem. I realized this during the actual study when a pair of students remarked on the incompleteness of the second sentence in the question. They said the sentence should have read, "Female bees hatch from fertilized eggs *and so have a mother and father*", in parallel to the first one. They thought there had to be a reason why the "examiner" (to use their examination oriented language) had not added the last clause. But were these students reading clues as well, or were they reading more charily? Students stumbled on the phrasing in the Pirates Aboard and Ladies Luncheon tasks as well. Ugandan students particularly could not do the LL task until I switched the names in the task to those common in their context.

E2. 3. Broader Questions about Participation

Like many other students but unlike Rose and Norah, the fact that Lillian and Irene were not overwhelmed by a genealogy tree that grew *exponentially*, may be explained by the types of tasks each group had already been exposed to. Irene and Lillian had done some pattern noticing based on the Chessboard Squares and an Iterated Triangle task, but so had Rose and Norah on tasks like Dominoes and Matches 1. At a closer look, these pairs of tasks vary in computational intricacy—the former involve a faster growth rate and more than one sequence. Thus a task that had been appropriate for Lillian and Irene, given their embodiments that included tasks done in earlier sessions, may have been less appropriate for Rose and Norah. Boaler (2000b) and Schoenfeld (1988) remark that certain "expectations" occasion what students perceive in exercises: The manner of choosing to consider only the *exact maternal lineage* seems to be aligned with an attunement to more defined, quick-fix and easy pattern questions. I say more on attuning attention later.

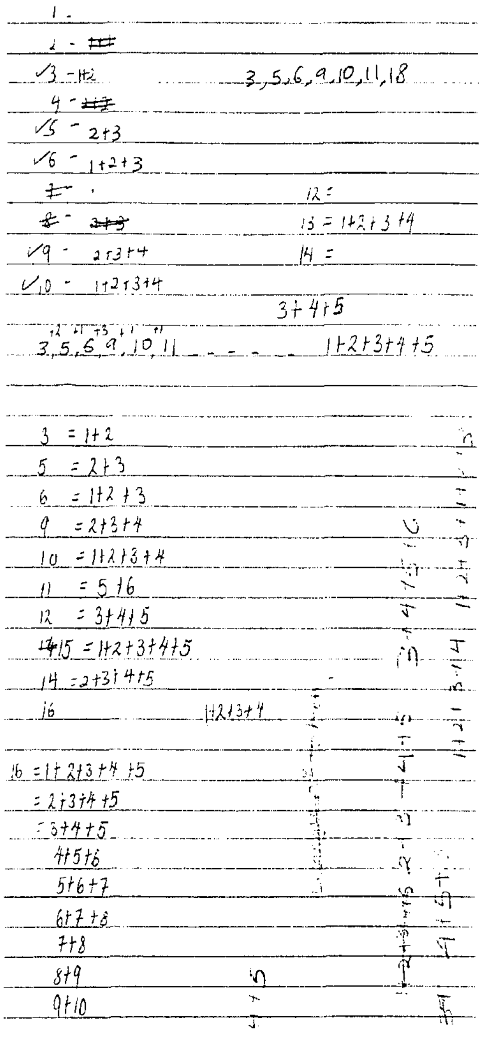
E2. 4. Researchers' Complicitness in Students' Mathematical Behavior

Rose and Norah's session also raises broader questions: Were they motivated to solve the problem? What was going on in their lives that evening? How well had they worked together as a pair? What were their classroom mathematical experiences like? These questions point to personal, interpersonal and institutional structures that structure students' thinking. The BG task had the potential to structure students' behavior in mathematical ways, but the girls did not appear to be inclined toward behaving mathematically. Why was this the case? In the preliminary study, before engaging the students in the tasks in their preliminary study, I had not looked at their demographic information. After analyzing the data from the preliminary study, I designed a two-page survey form (see Appendix C). I also planned to consult school records that documented participants' performance in mathematics whenever possible.

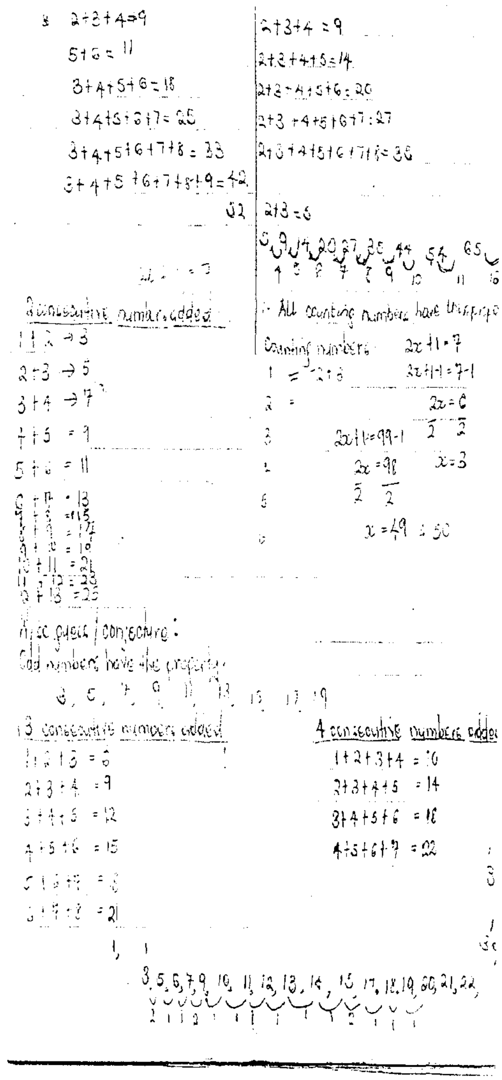
Reflecting on the connection between previous tasks done and engagement in any mathematical task, I began to debate whether or not I was going to expose students to tasks that were unlike most classroom tasks during the first session. In a manner antithetic to mathematical thought many text book tasks, tend to be clue-giving, single-strategy and quick-fix. I wondered whether by offering richer tasks I would jeopardize the specific purpose of the study to explore the dynamics of students' mathematical attention. However, framing my role first and foremost as a teacher helped me to resolve this tension. I decided that I would engage the students in Uganda, who appeared not to have wide experience with non-routine mathematical tasks, in pre-sessions on problem solving in which I prompted them to reflect on what they thought would help in solving problems.

APPENDIX F Sample Students' Written Work

Lillian & Irene's work on the CT task



Sonia & Gertrude's Work on the CT task



Notes. I have only offered the first page of students' work. Irene and Lillian used two pages and so did Susan and Grace. I guided students in the final study to use different ink whenever there appeared to be a shift in line of thought, hence the green, blue and black in Susan and Grace's writing. Tony and Ronald, whose work appears in the main text, used one page each and two pages for joint writing.

APPENDIX G Classroom Anecdotes and Transcripts

*Anecdote G1. **Lesson 1 on Geometry***

1. "Can you tell me the name of an object that would have 3 lines of symmetry? Some objects" A number of students raised their hands immediately but Edwin in a quiet voice blurted out.
2. "Triangle."
3. Unaware that Edwin had made a response the teacher continued, "Imagine in your heads an object that has 3 lines of symmetry. Agnes."
4. "Triangle."
5. "What kind of triangle?" the teacher prompted.
6. "Equilateral."
7. "Yeah. Do all triangles have 3 lines of symmetry?" the teacher asked.
8. In chorus the students responded. "No."
9. After a number of contributions, there was soon some agreement that an equilateral triangle was the only triangle that had three lines of symmetry. "Okay, how many lines of symmetry does a square have? Joseph."
10. "Ummm, eight."
11. "Not a cube but a square," the teacher responded as she drew a square on the overhead. A number of students began to call out,
12. "Four."
13. But Joseph was not sure, "Four?"
14. "Four. Ah-ha, that is what people are saying," the teacher nodded.
15. But another student agreed with Joseph's first response. "No. Eight," Stella added loudly.
16. "Let us see ..." The teacher began drawing in a vertical bisector. "There is a line here ..."
17. "Horizontally and two diagonally," Joseph said, guiding the teacher.
18. In a soft voice another student said, "Eight."
19. "Eight?" the teacher acted confused.
20. "Four," another student asserted. "Can you think of an object that would have eight?" the teacher asked.
21. Again in a chorus most of the students shouted, "Octagon."
22. "This is an important question," the teacher began. "Why did I pose it?"
23. John's hand shot up.
24. The teacher called on him to offer an answer. "John."
25. "I think it has more than eight."
26. "Okay. Save it, save it for a second. Somebody said octagon. Let's take a look." The teacher drew an octagon.
27. Esther made an observation seemingly to herself but out loud. "An octagon doesn't have ..."
28. Tim also speaking to himself in an excited tone, "A circle, oh!"
29. "Has more than 8," Edwin said, possibly responding to Tim.
30. "Y-e-a-h," Tim continued.
31. Janelle sitting close to Tim and Edwin said, "A circle has 180."
32. "Really?" Tim asked Janelle.
33. In the meantime, the teacher was still drawing lines in the octagon. It was obvious that she was unaware of Tim, Janelle and Edwin's conversation.
34. As the teacher was drawing in the diagonals she and the students counted, "two three four ..."
35. A few students audibly interjected her counting with a discussion of whether the octagon has 8 or 16 lines of symmetry: "That is 16." "Eight."
36. The teacher concluded her drawing by counting together with the students "So if we have 1, 2, 3, 4; 1, 2, 3, 4. I think there are 8. Not 16. Where would the 16 be?"
37. James interrupted as the teacher was waiting to hear from the students who thought it had 16.

38. "I know one that has infi—nite!"
39. "You know one that has an infinite?" the teacher asked. "Don't say it," she said playfully.
40. "There is a shape with lots," another student added.
41. By now a number of hands were up. "You know one that has ... lots," the teacher pointed at students one by one as they shot their hands up to show that they knew.
42. "Me too," a student uttered almost inaudibly.
43. "You know one that has what?"
44. "Lots," he replied.
45. "Me too."
46. "Infinite," another student said.
47. "Infinite," the teacher repeated. "Infinite," another student said.
48. "There are 16," another student said to another in the midst of the new "game." She was likely referring back to the octagon that was still being projected on the screen.
49. "I think eventually ... It will run around," another student commented.
50. Esther and Janelle had a side conversation, "There is 16...." "Why did she say [there is 8]?" Esther asked.
51. "Okay, at the count of 3," the teacher instructed. "An object with an infinite number of lines of symmetry. 1, 2, 3."
52. "Circle," the students called out.
53. Edwin was a lone voice, "Nothing," he said.
54. Although the teacher did not take up his suggestion. (It is not clear whether or not she heard it,) on the video record Janelle, John and Tim can be observed discussing the question, of whether it would be possible to draw lines of symmetry for nothing.
55. It was in that conversation that Tim turned to his colleagues saying, "I was thinking a sphere with the same diameter as a circle; a sphere will have more lines of symmetry than the circle."
56. At the end of the class the researcher-observer asked Tim about his conjecture. He responded, "A sphere might have 360 times more lines than the circle.

Transcript G1. ***Irene and Lillian's Engagement on the CT task: Later turns***

296 Irene They are 2, 4, 8
297 Teacher What could follow that sequence?
298 Irene 10 does
299 Lillian & Irene 12 does
300 Lillian 14 does
301 Teacher So they can't be even numbers.
302 Lillian & Irene No
303 Teacher What other property does 2, 8; 2, 4 and 8 have |?
305 Irene |Multiples of two *(Almost inaudible)*
306 Teacher Pardon *(Also almost inaudible)*
307 Teacher Multiples of 2. Multiples of two are even numbers.
308 Irene Yeah
309 Lillian 2 times
311 Teacher Or you meant something else
312 Teacher Did you mean 2 times 2|?
313 Irene |Yeah|
314 Teacher Or 2 times 3?|
315 Lillian 2 times...| *(almost inaudible, gains no audience, and is interrupted)*
316 Irene Yeah, but that
318 Teacher Or you
319 Irene Yeah, but that doesn't ...
230 Lillian 2 times ...
321 Teacher You meant multiples of 2.
322 Lillian Because 2 is 2 times ... *(Still almost inaudible & interrupted)*
333 Irene But that doesn't really ...
334 Lillian Because 2 times 2 is 4 then 2 times 2 times 2 is 8 (Last bit said so fast)
335 Teacher So you are thinking that she thought about even numbers and now you are thinking about ... what numbers are those?
336 Lillian Numbers which..ch..ch *(laughter)*

Transcript G2. ***Tony and Ronald's Engagement on the CT task: Later turns***

197 Tony: |The ones that don't are all even now. (Inaudible)
198 Teacher: You can write that down
199 Ronald: All even [...]
200 Tony: They can all be either divided or multiplied by eight. (some movements noises) (Ronald looks on as Tony writes)
201 Tony: Um
202 Teacher: They are all even. They are all multipliers or dividers by 8. [...]
203 Ronald: Multiples of 2
204 Tony: Yeah
205 Teacher: Multiples of 2. Multiples of 2, what are (Tony writes down)
206 Inaudible speaker
207 Tony: So far (Inaudible) Oh yeah.
208 Ronald: 2, 4, 8, 16
209 Teacher: But we also have ...|
210 Tony: **O just wait I know. They are doubling. So 2, 4, then 8, 16. So|**
211 Ronald: 2 plus 2 is 4, 4 plus 4, 8 plus 8 is 16 (Ronald nodes as he counts loudly.

Tony writes down)

212 Ronald: We don't have 32 here

[Many turns later after they have checked 32 and guessed that 64 will not satisfy the property]

271 Ronald: I think we can make an estimate

277 Tony: |**So** like numbers that don't are just the...It is just like 2 being doubled so...

278 Teacher: I will write that down. 2 being doubled ...|

279 Tony: **And we** are starting with 2 being doubled and the next kind of number doubles. But we didn't really find the property... I guess we didn't really find a property for the yes's

282 Teacher: 2 being doubled do they have any other ... can we call it anything else or can we use any other term?

283 Tony: I don't know |**doublings? (laughter)**

284 Ronald: (Inaudible) double

285 Tony: Um...

286 Teacher: (Sounds like she is writing aloud) Property um, I think (Inaudible). 2 and then 4... I want to write them as ... In terms of 2. 4 in terms of 2 will be

287 Tony: Um

288 Ronald: 4

289 Tony: Um uh Shhh... To the power of two I guess. So 2 will be... 1 to the power of 2, I think (laughter). 4 will be 2 to the power of 2, then 3 to the power of 2, 4 to the power of 2, 5 to the power of 2

290 Teacher: Do you agree Ronald... [...]

291 Teacher: Can we check that ... that's... It looks close

292 Ronald: 3 to the power of 2 is 9.

293 Tony: Wait. **O yeah**

294 Teacher: Tony you are thinking about something else that ...|

295 Tony: Yeah

296 Teacher: And then it didn't work here, or you are|

297 Teacher: **|No would it be uh... 2|**

298 Teacher: Thinking of powers

299 Tony: It will be 2 to the power of 2. So I just did these the wrong way, I think.

300 Teacher: So can we write them the right way...

301 Teacher: What do you think is the right way.

302: Tony: SO 2 to the power of 1,

303 Ronald: 4

304 Tony: Yeah. 2 to the of 2, 2 to the power of 3, 2 to the power 4 and 2 to the power of 5.

305 Teacher: Then we could... get a name for the numbers, another name. At first you said being doubled. And in terms of what you are writing we can call them

306 Ronald: To **Powers of 2**

307 Tony: Powers of 2 (Underlapping) Ronald's