Topological Recursion and Genus One Quantum Curves: An Accessible Exploration

by

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Abstract

We explore the connection between Eynard-Orantin Topological Recursion (EOTR) and the asymptotic solutions to differential equations constructed with the WKB method (named for its creators Wentzel, Kramers and Brillouin). Using the Airy spectral curve as an initial example, we propose a general connection between topological recursion and WKB solutions to the quantum curve generated via quantization of the defining algebraic curve.

We proceed further by examining the proposed connection in the context of the genus one family of Weierstrass spectral curves. We construct the perturbative wave-function and show that it is annihilated by a differential operator which is not a quantization of the spectral curve. Furthermore, as a consequence of equivalent approaches we also obtain an infinite collection of identities relating cycle integrals of elliptic functions to quasi-modular forms.

Dedicated to my loving parents, Kerri and Kevin.

Thank you for all that you have sacrificed for me.

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Chapter 1

Introduction

A spectral curve can, for the purpose of the introduction, be described as an irreducible algebraic curve (call this Σ) on $\mathbb{C}^2$. By definition we can express this curve as the zero locus of some polynomial expression $P(x, y) = 0$. *Eynard-Orantin topological recursion* (or EOTR) is a recursive algorithm that takes in the data of a spectral curve, and outputs an infinite sequence of objects known as *correlation functions*. Despite their name they are actually symmetric meromorphic differentials on Σ^n with $g \geq 0$ and $n \geq 1$.

These correlation functions can be prescribed a meaning depending on the spectral curve. For example they can be related to a number of enumerative invariants such as Gromov-Witten invariants, Hurwitz numbers, knot invariants, and others (See for example [5, 6, 11, 12, 13, 15, 16, 18, 20, 19, 24, 27, 25, 26, 28, 30, 29, 32, 34, 36]).

EOTR's origins are firmly rooted in the context of matrix models [14, 21, 27, 25] and as it turns out these origins can suggest connections between EOTR and some *a priori* unrelated mathematical constructs.

For example it is not immediately obvious that a method typically used to solve the Schrodinger equation in non-relativistic quantum mechanics is related to a set of objects generated from a quantum field theory. However matrix model theory tells us that EOTR and the WKB method[1] are in fact related to one another, albeit indirectly.

More concretely matrix model theory provides us with a method for extracting a *perturbative wave-function* $\psi(z)$ from topological recursion. It is then expected that for some spectral curves this perturbative wave-function is "killed" by a differential operator $\widehat{P}(\hat{x}, \hat{y})$ such that,

$$\widehat{P}(\hat{x}, \hat{y})\psi(z) = 0. \tag{1.1}$$

This operator is a *quantization* (a mapping of variables to operators) of the defining algebraic curve (i.e. $P(x, y) \mapsto \widehat{P}(\hat{x}, \hat{y})$) known as a *quantum curve*. Furthermore this implies that the perturbative wave-function should be the WKB asymptotic solution for the above differential equation, since they have the same exponential form. This expectation follows from determinantal formulae in matrix models [2, 3].

We can explore the existence of such quantum curves using only topological recursion with no reference to its matrix model roots. In fact there is a straightforward method for constructing $\psi(z)$ directly from the correlation functions generated by topological recursion. However the question remains is this expectation from matrix model theory a fact? That is to say, is this perturbative wave-function in fact the WKB asymptotic solution to the quan-

[1]The reader may refer to [1] for more details about the WKB method in particular.

tization of the spectral curve?

This question was answered for a large class of genus zero spectral curves in [8] using the formulation posited in [10, 9]. More precisely: all spectral curves whose Newton polygons have no interior point and that are smooth as affine curves[2]. One such curve that belongs to this class is the Airy curve, which we will explore explicitly.

We can then pose another question, does this connection hold for higher genus spectral curves? In [7] we set out to answer this question for a family of genus one spectral curves known as the *Weierstrass* spectral curves, given by

$$y^2 - (4x^3 - g_2(\tau)x - g_3(\tau)) = 0. \tag{1.2}$$

This work was split among three approaches, all of which were seeking a quantum curve generated by building a wave-function $\psi(z)$. The first approach adapted the work presented in [8], producing a perturbative wave-function and subsequently a quantum curve.

The second approach is equivalent to the first, and produces the same perturbative wave-function. The quantum curve, although equivalent, had a much different form which led to an infinite number of identities for cycle integrals of elliptic functions. This approach will be the main focus of this document, as it was my contribution to [7].

Both perturbative approaches generated the quantized spectral curve but with an infinite number of corrections that are unobtainable from the quantization procedure. Implying that the perturbative formulation is not the best

[2] For a more detailed explanation of these properties see [8].

approach to generating higher genus quantum curves.

This spurred the exploration with the third approach, which is nonperturbative in nature. This approach also generated a quantum curve with an infinite number of corrections. These corrections however are “nicer” since they certainly can be generated through a quantization of the classical equation.

The primary goal of this document is to elaborate and expand on my contribution to [7], namely the second perturbative approach eluded to above. As a secondary goal, we intend to fill a gap in the literature by providing a more informal exposition of the connection between WKB and topological recursion, with the hopes that an advanced undergraduate or new graduate student may be introduced to the topic via this document.

1.1 Outline

In Chapter 2 we introduce the background material required to define Eynard-Orantin topological recursion (typically referred to as simply *topological recursion*). We then proceed to explicitly define the structure of topological recursion, including both the input data and the objects it produces.

In Chapter 3 we start with the simplest genus zero case as a foothold, the *Airy spectral curve*, satisfying the following equation,

$$y^2 = x. \tag{1.3}$$

We focus on the second perturbative approach from [7] to find the quantum curve directly from topological recursion. The known result is replicated and the connection between topological recursion and the WKB expansion is then stated more formally.

In Chapter 4 we introduce the background information required for topological recursion on a genus one spectral curve. Namely we introduce the torus, and the elliptic functions on said torus.

In Chapter 5 the original results are tabulated and proven. We explore the connection between WKB and topological recursion for the family of genus one spectral curves, described by the *Weierstrass* equation, given by:

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau). \tag{1.4}$$

More precisely we will construct the quantum curve from the correlation functions produced by topological recursion on this family of curves.

In doing so we obtain a quantum curve which, as expected (from matrix model theory), is not a straightforward quantization of the spectral curve. Instead we obtain an infinite number of corrections, which cannot be resolved by the ambiguity in the defining classical equation.

We then compare this result to that obtained from the approach given in [8]. The equivalence of these two approaches leads to an infinite collection of identities for elliptic functions, the first of which we calculate explicitly.

Chapter 2

Topological Recursion

2.1 Spectral Curve

Definition 2.1.1. A *spectral curve* is a triple (Σ, x, y) where Σ is a Torelli marked, genus $\hat{g}$, compact Riemann surface[3] and x and y are meromorphic functions on Σ, such that the zeros of dx do not coincide with the zeros of dy.

In topological recursion we are interested in the branched covering $\pi : \Sigma \mapsto \mathbb{P}^1$ given by the meromorphic function x. Given this branched covering we define the set of *ramification points* as the points that satisfy $\mathrm{d}x = 0$ or the poles of x of order greater than or equal to two. Essentially, a ramification point is wherever the function behaves locally as either z^n or $\frac{1}{z^n}$ for $n \geq 2$. We will denote this set of ramification points as R.

For the remainder of the document we will assume that the meromorphic functions x and y satisfy a polynomial equation of the form:

[3] A Torelli marked compact Riemann surface Σ is a genus $\hat{g}$ Riemann surface Σ with a choice of symplectic basis of cycles $(A_1, \dots, A_{\hat{g}}, B_1, \dots, B_{\hat{g}}) \in H_1(\Sigma, \mathbb{Z})$.

$$P(x, y) = y^2 - f(x) = 0. \tag{2.1}$$

This requirement implies that x generates only a double cover, hence there are only two sheets to this covering. Furthermore this implies that all ramification points are simple (since there are only two sheets that can meet). This simplifies both the formulation of topological recursion as well as the properties of the extracted objects.

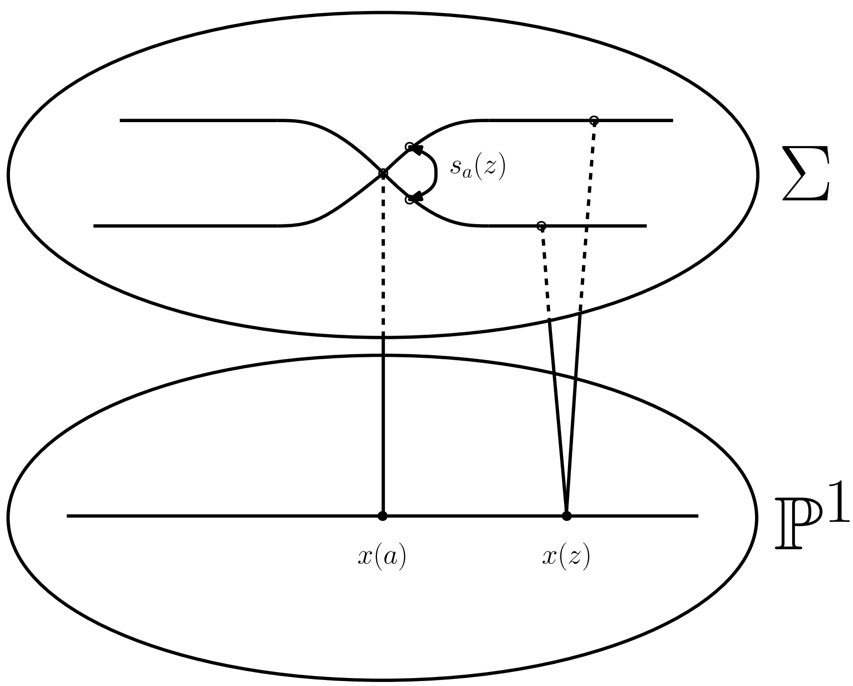

Figure 2.1: Pictorial representation of a double covering with the deck transformation $s_a(z)$. Every point in $\mathbb{P}^1$ is mapped by the inverse projection π^{-1} to two points in Σ. The deck transformation sends one point to other and vice versa. These two points meet at the simple ramification points.

For a given ramification point a in R we can define a local deck transformation map $s_a(z)$ that satisfies the following two properties:

$$x(s_a(z)) = x(z) \text{ and } s_a(a) = a. \tag{2.2}$$

This deck transformation map jumps from one sheet of the covering to the other. We can visualize this double covering alone with the deck transformation map in figure 2.1.

We now define a number of objects essential to formulating topological recursion.

2.2 Fundamental Objects

2.2.1 Billinear Differential

Definition 2.2.1. The *canonical bilinear differential of the second kind* (denoted $B(z_1, z_2)$) is the unique bilinear differential on Σ^2 satisfying the conditions:

- It is symmetric, $B(z_1, z_2) = B(z_2, z_1)$;
- It has only one pole, of order two, along the diagonal $z_1 = z_2$, with leading order term (in any local coordinate z)

$$B(z_1, z_2) \underset{z_1 \to z_2}{\to} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \ldots; \tag{2.3}$$

- It is normalized on A-cycles:

$$\oint_{z_1 \in A_i} B(z_1, z_2) = 0, \qquad \text{for } i = 1, \ldots, \hat{g}. \tag{2.4}$$

Remark 2.2.2. This definition refers to being *normalized over the A cycle*. This is trivially true for genus zero spectral curves and as such we will withhold the discussion of cycles for the time being, introducing it more explicitly in the context of the genus one case.

2.2.2 Recursion Kernels

Given a spectral curve $\mathcal{L} = (\Sigma, x, y)$ we can define a *recursion kernel* for each of the ramification points resulting in a "family" of recursion kernels.

Definition 2.2.3. For each a in the set of ramification points R, we define a meromorphic one form in z_0, known as a recursion kernel:

$$K_a(z_0; z) = \frac{\int_{t=\alpha}^{z} B(t, z_0)}{(y(z) - y(s_a(z)))\,\mathrm{d}x(z)}, \tag{2.5}$$

where $\alpha \in \Sigma$ is an arbitrary base point.

Remark 2.2.4. It is important to note that $\frac{1}{\mathrm{d}x(z)}$ is actually the contraction operator with respect to the vector field $\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^{-1} \frac{\partial}{\partial z}$. It can be assumed that for the remainder of the document "division" by a differential means precisely this contraction.

2.3 Eynard-Orantin Topological Recursion

With the preliminary definitions settled we can now define Eynard-Orantin Topological Recursion, which will be referred to as simply topological recursion for the remainder of the document.

Definition 2.3.1. Given a spectral curve $\mathcal{L} = (\Sigma, x, y)$, where x gives rise to a double covering, a set of simple ramification points denoted R, and corresponding deck transformations $s_a(z)$.

We first define the "initial conditions" by the following:

$$W_{0,1}(z) = y(z)\mathrm{d}x(z) \quad \text{and} \quad W_{0,2}(z_1, z_2) = B(z_1, z_2). \tag{2.6}$$

For all $g, n \in \mathbb{N}$ such that $2g+n-1 \geq 1$ there exists the following relation,

$$W_{g,n+1}(z_0, \mathbf{z}) = \sum_{a \in R} \operatorname*{Res}_{z=a} K_a(z_0; z)\mathcal{R}^{(2)}W_{g,n+1}(z, s_a(z); \mathbf{z}), \tag{2.7}$$

with the "recursive structure" explicitly defined as:

$$\begin{aligned} \mathcal{R}^{(2)}W_{g,n+1}(z, s_a(z); \mathbf{z}) &= W_{g-1,n+2}(z, s_a(z), \mathbf{z}) \\ &\quad + \sum_{g_1+g_2=g} \sum_{I \cup J = \mathbf{z}}' W_{g_1,|I|+1}(z, I)W_{g_2,|J|+1}(s_a(z), J). \end{aligned} \tag{2.8}$$

The primed sum in (2.8) denotes that we are excluding the cases where either $(g_1, |I|)$ or $(g_2, |J|)$ equals $(0, 0)$.

Remark 2.3.2. It is worth emphasizing that we always exclude the cases where g is negative, effectively setting any $W_{g,n} = 0$ with $g < 0$.

We call the $W_{g,n}$ objects *correlation functions*, however they are not functions but instead differential forms. Nevertheless we will stick to this nomenclature as it is standard throughout the literature.

We can segment the correlation functions into levels characterized by the number $2g + n - 1 = k$. We call the forms found via recursion "stable forms" i.e. those on the levels $k \geq 2$, and those not found with recursion ($W_{0,1}$ and $W_{0,2}$) unstable forms.

Using the recursion structure we can write down the general form for the $k = 1, 2$ and 3 levels.

k = 1

$$W_{0,2}(z_1, z_2) = B(z_1, z_2) \tag{2.9}$$

k = 2

$$\begin{aligned} W_{0,3}(z_0, z_1, z_2) = \sum_{a \in R} \operatorname*{Res}_{z=a} K_a(z_0, z) \{ & W_{0,2}(z, z_1) W_{0,2}(s_a(z), z_2) \\ & + W_{0,2}(s_a(z), z_1) W_{0,2}(z, z_2) \} \end{aligned} \tag{2.10}$$

$$W_{1,1}(z_0) = \sum_{a \in R} \operatorname*{Res}_{z=a} K_a(z_0, z) W_{0,2}(s_a(z), z) \tag{2.11}$$

k = 3

$$\begin{aligned} W_{0,4}(z_0, z_1, z_2, z_3) = & \sum_{a \in R} \operatorname*{Res}_{z=a} K_a(z_0, z) \\ & \times \{ W_{0,3}(z, z_1, z_2) W_{0,2}(s_a(z), z_3) + W_{0,3}(s_a(z), z_1, z_2) W_{0,2}(z, z_3) \\ & + W_{0,3}(z, z_2, z_3) W_{0,2}(s_a(z), z_1) + W_{0,3}(s_a(z), z_2, z_3) W_{0,2}(z, z_1) \\ & + W_{0,3}(z, z_1, z_3) W_{0,2}(s_a(z), z_2) + W_{0,3}(s_a(z), z_1, z_3) W_{0,2}(z, z_2) \} \end{aligned} \tag{2.12}$$

$$W_{1,2}(z_0, z_1) = \sum_{a \in R} \operatorname*{Res}_{z=a} K_a(z_0, z) \{W_{0,3}(z, s_a(z), z_1) +$$

$$+ W_{1,1}(z) W_{0,2}(s_a(z), z_1) + W_{1,1}(s_a(z)) W_{0,2}(z, z_1)\} \quad (2.13)$$

We can also list a number of useful properties for the stable forms.

2.3.1 General Properties of Stable Correlation Functions

The correlation functions at levels $k \geq 2$ have a few general properties that are worth elaborating.

1. **Symmetric with respect to exchange of coordinates**

   If we define a map that exchanges the ith and jth entries in the argument vector (denoted $\sigma_{ij}(\mathbf{z})$), it can be proven that the stable correlation functions satisfy the following property:

   $$W_{g,n}(\sigma_{ij}(\mathbf{z})) = W_{g,n}(\mathbf{z}). \quad (2.14)$$

   That is to say that the correlation functions are *symmetric*.

2. **Poles only at Ramification points**

   It can also be proven that if a stable correlation function has poles they must be at ramification points.

3. **Stable Forms are odd**

   It is also possible to show that when x gives rise to a double cover, all of the correlation functions are odd *except for* $W_{0,2}(z_1, z_2)$. In other words they satisfy the property:

$$W_{g,n}(z_1, \cdots, z_n) + W_{g,n}(-z_1, \cdots, z_n) = 0 \qquad \text{for } (g,n) \neq (0,2), \tag{2.15}$$

which by the symmetry property extends to all variables in the argument vector.

$W_{0,2}(z_1, z_2)$ on the other hand satisfies its own special property:

$$W_{0,2}(z_1, z_2) + W_{0,2}(-z_1, z_2) = \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}. \tag{2.16}$$

Chapter 3

Airy Spectral Curve

With the general framework of topological recursion established, we can now demonstrate more explicitly how to work with this structure to calculate the correlation functions. We will then replicate the known quantum curve result (for example see [17]).

3.1 Topological Recursion on the Airy curve

The Airy spectral curve is given by the spectral curve $\mathcal{L} = (\mathbb{P}^1, x, y)$, with x and y taking the following forms:

$$x(z) = z^2, \text{ and } y(z) = z. \tag{3.1}$$

x and y obviously satisfy the following characteristic equation:

$$y^2 - x = 0. \tag{3.2}$$

There are two ramification points, one coming from the single zero of $\mathrm{d}x$ at $z = 0$ and the other from the double pole of x at $z = \infty$. Thus the set of ramification points is $R = \{0, \infty\}$. The deck transformation map is globally defined in this case and is the same for both ramification points, it is given by $s_a(z) = -z$.

We must also nail down the "initial conditions", to do this we need to find an appropriate differential for $B(z_1, z_2)$ that satisfies all the conditions in Definition 2.2.1. The only option we have is

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}, \tag{3.3}$$

since there is no holomorphic differential on $\mathbb{P}^1$, and hence nothing we can add to this.

Finally we can write down our recursion kernel which is identical for both ramification points,

$$K(z; z_0) = -\frac{\mathrm{d}z_0}{4z^2 \mathrm{d}z} \left[\frac{1}{z - z_0} - \frac{1}{\alpha - z_0} \right]. \tag{3.4}$$

Remark 3.1.1. It can be shown via recursion that the pole at $z = \infty$ does not contribute a residue for any correlation function calculation using topological recursion. Hence we will exclude it from the sum over residues.

We can now use topological recursion to calculate the first few correlation functions (those at the $k = 2$ level), beginning with the general form for $W_{0,3}(z_0, z_1, z_2)$ given in (2.10) and making the appropriate substitutions (again noting that the residue at $z = \infty$ does not contribute) results in:

$$
W_{0,3}(z_0,z_1,z_2) = -\mathrm{d}z_0 \operatorname*{Res}_{z=0} \frac{1}{4z^2\mathrm{d}z}\left[\frac{1}{z-z_0} - \frac{1}{\alpha - z_0}\right] \times \left\{ \frac{\mathrm{d}z\mathrm{d}z_1}{(z-z_1)^2}\frac{-\mathrm{d}z\mathrm{d}z_2}{(z+z_2)^2} + \frac{-\mathrm{d}z\mathrm{d}z_1}{(z+z_1)^2}\frac{\mathrm{d}z\mathrm{d}z_2}{(z-z_2)^2} \right\}. \quad (3.5)
$$

It is easy to see that none of the terms in brackets have a pole at $z = 0$. This means that the residue will pick out the linear term in the bracketed part's power series. That is to say that we simply evaluate the partial derivative (with respect to z) of the bracketed parts at $z = 0$. We can calculate this derivative quite quickly by noting that

$$
\frac{\partial}{\partial z}\left[\frac{1}{(z-z_1)^2}\frac{1}{(z+z_2)^2} + \frac{1}{(z+z_1)^2}\frac{1}{(z-z_2)^2}\right]\Bigg|_{z=0} = 0. \quad (3.6)
$$

We obtain the following result:

$$
W_{0,3}(z_0,z_1,z_2) = -\frac{\mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2}{2z_0^2z_1^2z_2^2}. \quad (3.7)
$$

We can proceed in a similar fashion to obtain $W_{1,1}(z_0)$, starting with the general form given in (2.11) and continuing as before

$$
W_{1,1}(z_0) = -\mathrm{d}z_0 \operatorname*{Res}_{z=0} \frac{1}{4z^2\mathrm{d}z}\left[\frac{1}{z-z_0} - \frac{1}{\alpha - z_0}\right]\frac{-\mathrm{d}z^2}{4z^2}. \quad (3.8)
$$

Again the bracketed part has no pole at $z = 0$, meaning the residue will pick out the cubic coefficient in its power series, leading to:

$$
W_{1,1}(z_0) = -\frac{\mathrm{d}z_0}{16z_0^4}. \quad (3.9)
$$

We can continue this process indefinitely to calculate any $W_{g,n}$, however for both brevity and sanity we will move onto something more exciting.

3.1.1 Seemingly Arbitrary Combination

Now we will calculate a few quantities that appear to be pulled from thin air, but as we will see this single calculation in conjunction with the next section, will provide the core for the rest of our investigation.

$$S_0(z) = \int_\infty^z y(z)\mathrm{d}x(z) = \int_\infty^z 2z^2\mathrm{d}z, \tag{3.10}$$

$$S_1(z) = \int_\infty^z \int_\infty^z W_{0,2}(-z_1, z_2), \tag{3.11}$$

$$S_2(z) = \frac{1}{3!}\int_\infty^z \int_\infty^z \int_\infty^z W_{0,3}(z_0, z_1, z_2) + \frac{1}{1!}\int_\infty^z W_{1,1}(z_0). \tag{3.12}$$

Proceeding with these calculations, ignoring the infinite constants[4] that arise in S_0, and S_1 (as physicists love to do), and expressing the results in terms of $x = z^2$ we obtain the following:

$$S_0(x) = \frac{2}{3}x^{3/2}, \tag{3.13}$$

$$S_1(x) = -\frac{1}{4}\log(x), \tag{3.14}$$

$$S_2(x) = \frac{5}{48x^{3/2}}. \tag{3.15}$$

With these quantities lodged firmly in the back of our minds we will shift the focus to a topic that seems unrelated to the preceding discussion. However

[4]More precisely we are not simply "ignoring" the constants, this is a perfectly well defined *regularization*.

as we was previously eluded to, there is a very interesting thread tying it into the web we have woven thus far.

3.2 WKB Method

Typically discussed within the context of non-relativistic quantum mechanics, the WKB method (named after its creators Wentzel, Kramers, and Brillouin) is used to provide approximate solutions to ordinary differential equations, more specifically the Schrodinger equation.

The treatment presented in most undergraduate textbooks focuses only on a very rough version of this approximation, that is to say it only includes the first term in a much larger asymptotic solution. In addition to this they also obtain so called "connection formulae" for stitching a series of these solutions together at special points. Most of this is irrelevant to the current discussion and instead we will focus on the expansion itself.

3.2.1 Calculating the Expansion

Beginning with a differential equation of the following form[5]:

$$\left[\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - V(x)\right]\psi(x) = 0 \qquad \text{where } \hbar \text{ is a small parameter,} \tag{3.16}$$

[5]The WKB method can be used to provide asymptotic solutions to higher order differential equations, but since we deal only with second order equations in this document we restrict its definition here.

and inserting the ansatz given by

$$\psi(x) = \exp\left(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k(x)\right), \tag{3.17}$$

we obtain the following equation:

$$\left[\left(\sum_{k=0}^{\infty}\hbar^{k+1}S_k''(x)\right) + \left(\sum_{k=0}^{\infty}\hbar^k S_k'(x)\right)^2 - V(x)\right]\psi(x) = 0. \tag{3.18}$$

Collecting terms according to powers of $\hbar$ leads to

$$\sum_{k=1}^{\infty}\hbar^k\left(S_{k-1}''(x) + \sum_{m=0}^{k}S_m'(x)S_{k-m}'(x)\right) + S_0'(x)^2 - V(x) = 0. \tag{3.19}$$

Solving this order by order in $\hbar$ we obtain a recursion relation for the constituent functions $S_k(x)$:

$$S_{k-1}''(x) + \sum_{m=0}^{k}S_m'(x)S_{k-m}'(x) = 0. \tag{3.20}$$

Expanding this for the first few orders we see that the initial case $S_0(x)$ is completely determined by the function $V(x)$, and the rest are recursively determined via the general formula:

$$\mathcal{O}(\hbar^0) \qquad (S_0'(x))^2 = V(x), \tag{3.21}$$

$$\mathcal{O}(\hbar^1) \qquad S_1'(x) = -\frac{S_0''(x)}{2S_0'(x)}, \tag{3.22}$$

$$\mathcal{O}(\hbar^2) \qquad S_2'(x) = -\frac{1}{2S_0'(x)}\left[S_1''(x) + (S_1'(x))^2\right]. \tag{3.23}$$

As you may have guessed by now these quantities are related to the ones calculated above from the topological recursion.

3.2.2 Constructing the Operator

Let us outline a way of constructing a differential operator from the spectral curves characteristic equation $P(x, y) = 0$. We will assign operators to each variable in this equation as follows:

$$y \mapsto \hat{y} = \hbar \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{and} \quad x \mapsto \hat{x} = x. \tag{3.24}$$

Remark 3.2.1. There is ambiguity in the ordering of terms involving products of x and y, since the operators do not commute but the classical variables do. This will be discussed when stating the connection between WKB and topological recursion formally.

Labelling the resulting differential operator as $\widehat{P}(\hat{x}, \hat{y})$ and having it act on its homogeneous solution denoted $\psi(x)$ results in the differential equation:

$$\widehat{P}(\hat{x}, \hat{y})\psi(x) = 0. \tag{3.25}$$

We will now construct an operator from the Airy spectral curve with this mapping and apply the WKB method to the resultant differential equation.

3.2.3 Airy Operator

If we apply this mapping to the Airy spectral curve we obtain the following differential equation:

$$\left[\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\right] \psi(x) = 0. \tag{3.26}$$

If we apply the WKB "algorithm" to the above differential equation, we obtain the following results for the first few S_k's:

$$S_0'(x) = \sqrt{x}, \tag{3.27}$$

$$S_1'(x) = -\frac{1}{4x}, \tag{3.28}$$

$$S_2'(x) = -\frac{5}{32x^{5/2}}. \tag{3.29}$$

Integrating each of these expressions and comparing the results to (3.13), (3.14) and (3.15), we see that they are identical! Surely this hints to the deeper connection between topological recursion and the WKB expansion.

We will first more formally define the general form of this connection. After this we will explore it in the context of the Airy case, recovering this differential operator (known as a *quantum curve*) from topological recursion, and in doing so proving these identities for the Airy case at all orders of $\hbar$.

3.3 Definition of TR-WKB Connection

Conjecture 3.3.1. *Given a spectral curve* $\mathcal{L} = (\Sigma, x, y)$ *we first submit its data to topological recursion and construct the following perturbative wave-*

function:

$$\psi(z) = \exp\left\{\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_Q^z\cdots\int_Q^z\left(W_{g,n}(z_1,\cdots,z_n)\right.\right.$$
$$\left.\left. - \delta_{g,0}\delta_{n,2}\frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right\}. \tag{3.30}$$

where Q is a chosen pole of the meromorphic function $x(z)$.

It is then proposed that there exists a differential operator $\widehat{P}(\hat{x},\hat{y})$ such that,

$$\widehat{P}(\hat{x},\hat{y})\psi(z) = 0. \tag{3.31}$$

This operator is constructed by applying the mappings given by

$$y \mapsto \hat{y} = \hbar\frac{d}{dx} \quad \text{and} \quad x \mapsto \hat{x} = x, \tag{3.32}$$

to the spectral curve's characteristic equation $P(x,y) = 0$. We call this process quantizing the spectral curve and the result is known as a quantum curve.

There is ambiguity in the ordering of non-commutative operators, and as a consequence this quantization is not unique, and the choice of $\widehat{P}(\hat{x},\hat{y})$ is related to the choice of pole Q.

To elaborate on the uniqueness issue, suppose we have a spectral curve that includes terms with products of x and y (i.e. a term of the form xy). When quantized the resulting quantum curve has some ambiguity, since $\hat{x}$ and $\hat{y}$ do not commute. What this implies is we have a choice of ordering when dealing with terms that commute in the classical equation but do not after quantization.

It is shown in [8] that the choice of pole Q used in the construction of $\psi(z)$, affects this ordering. That is to say that if we pick a pole Q and the "correct" ordering this conjectured connection is uniquely satisfied. How to choose this correct ordering based solely on the pole Q is not clear however.

This ordering issue does not arise in the following investigation since both the Airy and Weierstrass spectral curves do not have terms involving products of x and y. There is however, another issue pertaining to commutativity.

Using the Airy spectral curve as an example we see that classically the following two polynomial expressions are equivalent:

$$y^2 - x = 0 \quad \text{and} \quad y^2 - x + (xy - yx) = 0. \tag{3.33}$$

However upon quantizing, the latter equation gains an extra additive constant $\hbar$ due to the commutation relation of the operators. What this means is it is possible to have classically "transparent" terms that become opaque upon quantization. This will be relevant when discussing the quantum curve in the Weierstrass case since we end up with an infinite number of corrections in $\hbar$ to the quantum curve.

As previously mentioned, this conjecture was proven for a large class of genus zero curves (see [8]), which includes the Airy spectral curve. In the following section we will provide a proof for the Airy curve through alternative means.

3.4 Airy Quantum Curve

We will seek to show that the quantization of $P(x, y) = 0$, kills the perturbative wave-function $\psi(z)$ by following the steps outlined below.

1. **Residues:** We must first find the correlation functions, instead of appealing to a direct approach (which would be infinitely time consuming) we use the fact that the sum of residues of a meromorphic differential on a Riemann surface is zero. Using this fact we can generate a new recursion relation that does not contain a residue calculation.

2. **Integration:** We then integrate the resulting expressions in all variables along a path with base point at the single pole of $x(z)$ at $z = \infty$.

3. **Specialization:** We then specialize the result of the previous step by setting all variables to the same value, i.e. send $z_i \to z$ for all $i \in \{0, 1, \cdots, n\}$.

4. **Summation:** Finally we sum over these expressions in the same fashion as the exponential argument of the perturbative wave-function, that is we couple each $W_{g,n}$ with an appropriate factor of $\hbar$ and sum them according to their level $2g + n - 1 = k$.

3.4.1 Residues

Proposition 3.4.1. *For $2g + n - 1 \geq 1$,*

$$\frac{W_{g,n+1}(z_0, \mathbf{z})}{dz_0} = \frac{1}{4z_0^2\, dz_0^2}\left(W_{g-1,n+2}(z_0, -z_0, \mathbf{z}) + \sum_{\substack{g_1+g_2=g \\ I\cup J=\mathbf{z}}}' W_{g_1,|I|+1}(z_0, I) W_{g_2,|J|+1}(-z_0, J) \right) + \sum_{i=1}^{n} d_{z_i}\left(\frac{1}{4z_i^2\, dz_i}\left[\frac{1}{z_i - z_0} + \frac{1}{z_i + z_0} \right] W_{g,n}(-z_i, \mathbf{z} \setminus \{z_i\}) \right). \quad (3.34)$$

Proof. Starting with the Airy topological recursion (yet again noting that the residue at $z = \infty$ does not contribute),

$$W_{g,n+1}(z_0, \mathbf{z}) = -\operatorname*{Res}_{z=0} \frac{\mathrm{d}z_0}{4z^2 \mathrm{d}z}\left[\frac{1}{z - z_0} - \frac{1}{\alpha - z_0} \right] \mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z}). \quad (3.35)$$

Now using the fact that the sum of residues of a meromorphic form on a compact Riemann surface is zero, we can replace the residue calculation above with an equivalent expression,

$$W_{g,n+1}(z_0, \mathbf{z}) = \sum_{\substack{\text{Poles} \\ Q \notin R}} \operatorname*{Res}_{z=Q} \frac{\mathrm{d}z_0}{4z^2 \mathrm{d}z}\left[\frac{1}{z - z_0} - \frac{1}{\alpha - z_0} \right] \mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z}). \quad (3.36)$$

Clearly there is a pole at $z = z_0$ but there is also a collection of second order poles at each of the marked points $z = \pm z_i$. To see where these extra poles come from we must examine the recursive structure in (2.8) more closely. More specifically we need to dissect the following summation

$$
\begin{aligned}
\sum_{g_1+g_2=g} \sum_{I\cup J=\mathbf{z}}' W_{g_1,|I|+1}(z,I) W_{g_2,|J|+1}(-z,J) \\
= \sum_{\text{stable}} W_{g_1,|I|+1}(z,I) W_{g_2,|J|+1}(-z,J) \\
+ \sum_{i=1}^{n} W_{0,2}(z,z_i) W_{g,n}(-z,\mathbf{z}\setminus\{z_i\}) + W_{0,2}(-z,z_i) W_{g,n}(z,\mathbf{z}\setminus\{z_i\}),
\end{aligned} \tag{3.37}
$$

where the stable[6] sum indicates that we exclude the terms with $(g_1, |I|), (g_2, |J|) = (0,0)$ or $(0,1)$.

Keeping in mind the fact that all the correlation functions only have poles at $z = 0$ except for $W_{0,2}$ means we only need to seek out such terms containing $W_{0,2}$. If we focus on the last line in (3.37) we see that there are two instances of $W_{0,2}$ for each marked point z_i, these are the source of the additional poles because of their limiting behavior

$$
W_{0,2}(z,z_i) \to \frac{\mathrm{d}z\mathrm{d}z_i}{(z-z_i)^2} \text{ as } z \to z_i, \qquad W_{0,2}(-z,z_i) \to -\frac{\mathrm{d}z\mathrm{d}z_i}{(z+z_i)^2} \text{ as } z \to -z_i. \tag{3.38}
$$

All of this together means we only need to calculate the residue around z_0 and each of the marked points $z = z_i$ for $i \in \{1, \cdots, n\}$.

For the pole at $z = z_0$ we obtain

$$
\frac{1}{4z_0^2 \mathrm{d}z_0} \left(W_{g-1,n+2}(z_0,-z_0,\mathbf{z}) + \sum_{\substack{g_1+g_2=g \\ I\cup J=\mathbf{z}}}' W_{g_1,|I|+1}(z_0,I) W_{g_2,|J|+1}(-z_0,J) \right). \tag{3.39}
$$

[6] We define the *stable* correlation functions as those calculated via recursion. In other words all correlation functions except $W_{0,2}$ and $W_{0,1}$.

The contributions from the poles at $z = \pm z_i$ is given by

$$\mathrm{d}_{z_i}\left(\frac{1}{4z_i^2\mathrm{d}z_i}\left[\frac{1}{z_i - z_0} + \frac{1}{z_i + z_0}\right] W_{g,n}(-z_i, \mathbf{z}\setminus\{z_i\})\right)\mathrm{d}z_0. \tag{3.40}$$

Summing up all the contributions and dividing everything by $\mathrm{d}z_0$ results in the proposed expression. □

Corollary 3.4.2. *For* $2g + n - 1 \geq 0$,

$$\begin{aligned}
&-\frac{W_{g-1,n+2}(-z_0, z_0, \mathbf{z})}{dx(z_0)^2} + \sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}\left(\frac{W_{g_1,|I|+1}(-z_0, I)}{dx(z_0)}\right)\left(\frac{W_{g_2,|J|+1}(-z_0, J)}{dx(z_0)}\right)\\
&+\sum_{i=1}^{n}\left(\left(\frac{dx(z_i)}{(x(z_0) - x(z_i))^2}\right)\frac{W_{g,n}(-z_0, \mathbf{z}\setminus\{z_i\})}{dx(z_0)}\right.\\
&\qquad\left. - d_{z_i}\left(\frac{1}{(x(z_0) - x(z_i))}\frac{W_{g,n}(-z_i, \mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right)\right) = 0,
\end{aligned} \tag{3.41}$$

while for $(g, n) = (0, 0)$,

$$\frac{W_{0,1}(-z_0)}{dx(z_0)}\frac{W_{0,1}(-z_0)}{dx(z_0)} - x(z_0) = 0. \tag{3.42}$$

Proof. Starting with the previous proposition, we must massage this into a form more suitable for integration and specialization. To do this we must deal with any $W_{0,2}(z_1, z_2)$ terms in this expression since

$$W_{0,2}(z_1, z_2) \to \infty \text{ as } z_1 \to z_2. \tag{3.43}$$

However if we negate an argument, this issue disappears:

$$W_{0,2}(-z, z) = -\frac{\mathrm{d}z^2}{4z^2}. \tag{3.44}$$

Therefore we must seek to replace any instance of $W_{0,2}(z_1, z_2)$ with $W_{0,2}(-z_1, z_2)$ and deal with the singularity later. To begin we note that

$$\frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} - \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 + z_2)^2} = \frac{(2z_1 \mathrm{d}z_1)(2z_2 \mathrm{d}z_2)}{(z_1^2 - z_2^2)^2}, \tag{3.45}$$

rewriting this in a more convenient form we see

$$W_{0,2}(z_1, z_2) + W_{0,2}(-z_1, z_2) = \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}. \tag{3.46}$$

This along with the fact that all other correlation functions are odd in all arguments implies that we can simply switch the argument's sign in the relevant correlation functions.

$$\begin{aligned}
\frac{W_{g,n+1}(z_0, \mathbf{z})}{\mathrm{d}z_0} &= \sum_{i=1}^{n} \frac{1}{4z_0^2 \mathrm{d}z_0} \left(\frac{\mathrm{d}x(z_0)\mathrm{d}x(z_i)}{(x(z_0) - x(z_i))^2} W_{g,n}(-z_0, \mathbf{z} \setminus \{z_i\}) \right) \\
&+ \frac{1}{4z_0^2 \mathrm{d}z_0^2} \left(W_{g-1,n+2}(z_0, -z_0, \mathbf{z}) - \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{z}}}' W_{g_1,|I|+1}(-z_0, I) W_{g_2,|J|+1}(-z_0, J) \right) \\
&+ \sum_{i=1}^{n} \mathrm{d}_{z_i} \left(\frac{1}{4z_i^2 \mathrm{d}z_i} \left[\frac{1}{z_i - z_0} + \frac{1}{z_i + z_0} \right] W_{g,n}(-z_i, \mathbf{z} \setminus \{z_i\}) \right)
\end{aligned} \tag{3.47}$$

We can also clean up the final line by noting that

$$\frac{1}{4z_i^2 \mathrm{d}z_i} \left[\frac{1}{z_i - z_0} + \frac{1}{z_i + z_0} \right] = -\frac{1}{\mathrm{d}x(z_i)} \frac{1}{x(z_0) - x(z_i)}. \tag{3.48}$$

Finally we make the following substitution [7],

$$\frac{W_{g,n+1}(z_0, \mathbf{z})}{\mathrm{d}z_0} = -\frac{W_{g,n+1}(-z_0, \mathbf{z})}{\mathrm{d}z_0} = 2\frac{W_{0,1}(-z_0)}{\mathrm{d}x(z_0)}\frac{W_{g,n+1}(-z_0, \mathbf{z})}{\mathrm{d}x(z_0)}, \qquad (3.49)$$

which results in the given corollary. □

3.4.2 Integration

The next step is to integrate the expressions from Corollary 3.4.2 in each of the coordinates except z_0. These integrals are along a path with the base point at the single pole of $x(z)$ to the marked point itself. As was mentioned previously, the correlation functions only have poles at $z = 0$, hence these integrals do converge.

Definition 3.4.3. We define the following quantity:

$$G_{g,n+1}(z_0; \mathbf{z}) := \int_{\infty}^{z_1} \cdots \int_{\infty}^{z_n} W_{g,n+1}(-z_0, z_1, \cdots, z_n). \qquad (3.50)$$

It is important to note that the first variable is negated and we do not integrate over it.

[7] This is true since $W_{0,1}(-z_0) = y(-z_0)\mathrm{d}x(-z_0)$.

Lemma 3.4.4. *For $2g + n - 1 \geq 0$,*

$$
\begin{aligned}
&- \left(\frac{\partial}{\partial x(z_{n+1})} \frac{G_{g-1,n+2}(z_0; \mathbf{z}, z_{n+1})}{dx(z_0)} \right)_{z_{n+1}=z_0} \\
&\qquad\qquad + \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{z}}} \left(\frac{G_{g_1,|I|+1}(z_0, I)}{dx(z_0)} \right) \left(\frac{G_{g_2,|J|+1}(z_0, J)}{dx(z_0)} \right) \\
&- \sum_{i=1}^{n} \frac{1}{x(z_0) - x(z_i)} \left(\frac{G_{g,n}(z_0; \mathbf{z} \setminus \{z_i\})}{dx(z_0)} - \frac{G_{g,n}(z_i; \mathbf{z} \setminus \{z_i\})}{dx(z_i)} \right) = 0, \quad (3.51)
\end{aligned}
$$

while for $(g, n) = (0, 0)$,

$$
\frac{G_{0,1}(z_0)}{dx(z_0)} \frac{G_{0,1}(z_0)}{dx(z_0)} - x(z_0) = 0. \tag{3.52}
$$

Proof. For $2g + n - 1 \geq 0$, all of the terms follow directly from the definition of $G_{g,n}$ except for the last term, where we must be careful of the base point evaluation:

$$
\lim_{z_i \to \infty} \frac{1}{x(z_0) - x(z_i)} \left(\frac{G_{g,n}(z_0; \mathbf{z} \setminus \{z_i\})}{\mathrm{d}x(z_0)} - \frac{G_{g,n}(z_i; \mathbf{z} \setminus \{z_i\})}{\mathrm{d}x(z_i)} \right) = 0. \tag{3.53}
$$

This is true since we know that $x(z)$ has a pole at $z = \infty$ and the $W_{g,n}$'s only have poles at the ramification point $z = 0$.

For $(g, n) = (0, 0)$ we do not need to integrate the expression so it follows directly with the identification $W_{0,1}(-z_0) = G_{0,1}(z_0)$. □

3.4.3 Principal Specialization

Now we will send all of the variables to the same value, in other words we will send $z_i \to z$ for all the coordinates, including z_0.

Definition 3.4.5. We define the *partial specialization* of $G_{g,n+1}(z_0;\mathbf{z})$ as:

$$\widehat{G}_{g,n+1}(z_0;z) = G_{g,n+1}(z_0;z,\cdots,z), \tag{3.54}$$

and the *full specialization* is then given by

$$\widehat{G}_{g,n+1}(z;z). \tag{3.55}$$

Before we can specialize the entire expression in Lemma 3.4.4 we must first focus on the derivative term, which requires a few general results related to specialization.

Proposition 3.4.6. *For* $2g+n-1 \geq 0$,

$$\left(\frac{\partial}{\partial x(z_{n+1})}\frac{G_{g-1,n+2}(z_0;\mathbf{z},z_{n+1})}{dx(z_0)}\right)_{\substack{z_0=z\\ \vdots \\ z_{n+1}=z}} = \frac{1}{n+1}\left(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g-1,n+2}(z_0;z)}{dx(z_0)}\right)_{z_0=z}. \tag{3.56}$$

Proof. Letting each variable be a function of z except for z_0 which we will treat as independent of z:

$$\frac{\partial}{\partial x(z)}\frac{G_{g-1,n+2}(z_0;\mathbf{z},z_{n+1})}{\mathrm{d}x(z_0)} = \sum_{i=1}^{n+1}\left(\frac{\partial}{\partial x(z_i)}\frac{G_{g-1,n+2}(z_0;\mathbf{z},z_{n+1})}{\mathrm{d}x(z_0)}\right)\left(\frac{\partial x(z_i)}{\partial x(z)}\right). \tag{3.57}$$

Letting $z_i(z) = z$ for each $i \in \{1,\cdots,n+1\}$ in the preceding expression leads to:

$$\frac{\partial}{\partial x(z)}\frac{G_{g-1,n+2}(z_0;z,\cdots,z)}{\mathrm{d}x(z_0)} = \sum_{i=1}^{n+1}\left(\frac{\partial}{\partial x(z_i)}\frac{G_{g-1,n+2}(z_0;\mathbf{z},z_{n+1})}{\mathrm{d}x(z_0)}\right)_{\substack{z_1=z\\ \vdots \\ z_{n+1}=z}}. \tag{3.58}$$

Finally using the following fact that the derivatives are symmetric when specialized:

$$\left(\frac{\partial}{\partial x(z_i)} \frac{G_{g-1,n+2}(z_0; \mathbf{z}, z_{n+1})}{\mathrm{d}x(z_0)} \right) \Bigg|_{\substack{z_1=z \\ \vdots \\ z_{n+1}=z}} = \left(\frac{\partial}{\partial x(z_{n+1})} \frac{G_{g-1,n+2}(z_0; \mathbf{z}, z_{n+1})}{\mathrm{d}x(z_0)} \right) \Bigg|_{\substack{z_1=z \\ \vdots \\ z_{n+1}=z}} \tag{3.59}$$

for all $i \neq 0$. Setting $z_0 = z$ after this substitution leads to the proposed result. □

Lemma 3.4.7. *For $2g + n - 1 \geq 0$,*

$$-\frac{1}{n+1} \left(\frac{\partial}{\partial x(z)} \frac{\widehat{G}_{g-1,n+2}(z_0; z)}{dx(z_0)} \right)_{z_0=z} + \sum_{g_1+g_2=g} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \frac{\widehat{G}_{g_1,m+1}(z;z)}{dx(z)} \frac{\widehat{G}_{g_2,n-m+1}(z;z)}{dx(z)} - n \left(\frac{\partial}{\partial x(z_0)} \frac{\widehat{G}_{g,n}(z_0; z)}{dx(z_0)} \right)_{z_0=z} = 0, \tag{3.60}$$

while for $(g, n) = (0, 0)$,

$$\frac{G_{0,1}(z)}{dx(z)} \frac{G_{0,1}(z)}{dx(z)} - x(z) = 0. \tag{3.61}$$

Proof. We proceed to specialize the expressions in Lemma 3.4.4.

For $2g + n - 1 \geq 0$ the first term follows directly from the preceding proposition.

The second term specializes trivially, however once specialized the choice of the sets I and J are no longer relevant, only their sizes. Hence the coefficient is simply the number of ways to partition the variables into these sets, which happens to be $\binom{n}{|I|}$.

To calculate the final term we will treat each term in the sum separately and combine them at the end. To begin we first send $z_i \to z_0$, leading to

$$\begin{aligned}\lim_{z_i\to z_0} \frac{1}{x(z_0)-x(z_i)}\left(\frac{G_{g,n}(z_0;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_0)}-\frac{G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_i)}\right)\\ =\lim_{z_i\to z_0}\frac{1}{x'(z_0)(z_i-z_0)}\left(\frac{G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_i)}-\frac{G_{g,n}(z_0;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_0)}\right)\\ =\frac{\partial}{\partial x(z_0)}\left(\frac{G_{g,n}(z_0;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_0)}\right). \quad (3.62)\end{aligned}$$

Then setting all variables to z and summing up all the contributions results in

$$n\left(\frac{\partial}{\partial x(z_0)}\frac{G_{g,n}(z_0;z,\cdots,z)}{\mathrm{d}x(z_0)}\right)_{z_0=z}. \quad (3.63)$$

Finally the $(g,n)=(0,0)$ case is again trivial, we simply make the substitution $z_0=z$. □

3.4.4 Summation

Now we must sum up these $G_{g,n}$'s and pair them with the small parameter $\hbar$ accordingly.

Definition 3.4.8. We define the following quantity:

$$\xi_1(z';z) = -\sum_{g,n=0}^{\infty}\frac{\hbar^{2g+n}}{n!}\frac{\widehat{G}_{g,n+1}(z';z)}{\mathrm{d}x(z')}. \quad (3.64)$$

Proposition 3.4.9.

$$\hbar\frac{d}{dx(z)}\xi_1(z;z) = -\sum_{2g+n-1\geq 0}\left(\frac{\hbar^{2g+n}}{(n+1)!}\left(\frac{\partial}{\partial x(z)}\ \frac{\widehat{G}_{g-1,n+2}(z_0;z)}{dx(z_0)}\right)_{z_0=z} + \frac{\hbar^{2g+n}}{(n-1)!}\left(\frac{\partial}{\partial x(z_0)}\frac{G_{g,n}(z_0;z)}{dx(z_0)}\right)_{z_0=z}\right). \tag{3.65}$$

Proof. Rewriting the sum in the proposition as:

$$-\hbar\sum_{2g+n-1\geq 0}\left(\frac{\hbar^{2(g-1)+(n+1)}}{(n+1)!}\left(\frac{\partial}{\partial x(z)}\ \frac{\widehat{G}_{g-1,n+2}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z} + \frac{\hbar^{2g+n-1}}{(n-1)!}\left(\frac{\partial}{\partial x(z_0)}\frac{G_{g,n}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z}\right), \tag{3.66}$$

and reindexing the first term by sending $g-1\to g$ and $n+2\to n$, we see that the sum becomes:

$$-\hbar\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{(n-1)!}\left(\left(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g,n}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z} + \left(\frac{\partial}{\partial x(z_0)}\frac{G_{g,n}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z}\right). \tag{3.67}$$

Making a substitution in the preceding expression according to

$$\frac{\mathrm{d}}{\mathrm{d}x(z)}\frac{\widehat{G}_{g,n}(z;z)}{\mathrm{d}x(z)} = \left(\frac{\partial}{\partial x(z_0)}\frac{\widehat{G}_{g,n}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z} + \left(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g,n}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z}, \tag{3.68}$$

results in the proposed expression. □

Lemma 3.4.10.

$$\hbar\frac{d}{dx(z)}\xi_1(z;z) + \xi_1(z;z)^2 - x(z) = 0. \tag{3.69}$$

Proof. Multiplying the expressions in Lemma 3.4.7 by $\frac{\hbar^{2g+n}}{n!}$ and summing over g, n, in conjunction with Proposition 3.4.9, leads directly to the result. □

Now we define the perturbative wave-function which we will extract a quantum curve from.

Definition 3.4.11. The *perturbative wave-function* is given by:

$$\psi(z) = \exp\left\{\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_{\infty}^{z}\cdots\int_{\infty}^{z}\Bigg(W_{g,n}(z_1,\cdots,z_n) - \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\Bigg)\right\}. \tag{3.70}$$

In addition to this we also define the following quantity:

$$\psi_1(z';z) := \psi(z)\xi_1(z';z). \tag{3.71}$$

We can relate the two quantities in the definition via the following proposition.

Proposition 3.4.12.

$$\hbar\frac{d}{dx(z)}\psi(z) = \psi_1(z;z). \tag{3.72}$$

Proof. Taking the derivative of $\psi(z)$ with respect to $x(z)$ results in:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x(z)}\psi(z) \\
&= \frac{\psi(z)}{x'(z)}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_{\infty}^{z}\cdots\int_{\infty}^{z}\left(W_{g,n}(z_1,\cdots,z_n)\right.\right. \\
&\qquad\qquad\left.\left. -\,\delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right)\right\}.
\end{aligned} \tag{3.73}$$

If we focus on the term in curly brackets and assume that each z_i is a function of z, then application of the partial derivative chain rule leads to:

$$\begin{aligned}
&\frac{1}{x'(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_{\infty}^{z_1(z)}\cdots\int_{\infty}^{z_n(z)}W_{g,n}(z_1,\cdots,z_n)-\delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right) \\
&\qquad = -\frac{1}{x'(z)}\sum_{i=1}^{n}\left(\frac{\widehat{G}_{g,n}(z_i;z)}{\mathrm{d}z_i}\right)_{z_i=z} \\
&\qquad = -n\left(\frac{\widehat{G}_{g,n}(z;z)}{\mathrm{d}x(z)}\right).
\end{aligned} \tag{3.74}$$

Substituting the preceding expression into the derivative of $\psi(z)$ results in the proposed result. □

Corollary 3.4.13.

$$\hbar^2\frac{d^2}{dx^2}\psi(z) = \psi_1(z;z)\xi_1(z;z) + \hbar\psi(z)\frac{d}{dx}\xi_1(z;z). \tag{3.75}$$

Proof. Starting with the definition of $\psi_1(z;z)$ and taking its derivative with respect to $x(z)$ results in:

$$\hbar\frac{\mathrm{d}}{\mathrm{d}x}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(z)\right) = \hbar\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi(z)\xi_1(z;z)\right), \tag{3.76}$$

applying the product rule on the right side of this expression leads directly to the result. □

Now we can construct the operator that acts on this perturbative wave-function

Theorem 3.4.14.

$$\left(\hbar^2 \frac{d^2}{dx(z)^2} - x(z)\right)\psi(z) = 0. \tag{3.77}$$

Proof. Starting with 3.69, multiplying it by $\psi(z)$ and utilizing the definition of $\psi_1(z;z)$ results in

$$\hbar\psi(z)\frac{\mathrm{d}}{\mathrm{d}x(z)}\xi_1(z;z) + \xi_1(z;z)\psi_1(z;z) - x(z)\psi(z) = 0, \tag{3.78}$$

recognizing the first two terms as precisely the right hand side of corollary 3.4.13, leads to the result.

□

Theorem 3.4.14 proves that the Airy spectral curve does satisfy Conjecture 3.3.1. That is to say that we recover the straightforward quantization of the spectral curve from topological recursion. In fact this is true for a large class of genus zero spectral curves[8] as proven in [8].

This begs the question: Is this true for higher genus spectral curves? From matrix model theory we expect that this is not the case, and as we will see for the Weierstrass spectral curve, Conjecture 3.3.1 is not satisfied. Instead the constructed quantum curve has an infinite number of corrections in $\hbar$.

[8]As mentioned previously, this is true for all spectral curves whose Newton polygons have no interior point and that are smooth as affine curves.

Chapter 4

Elliptic Functions

We wish to apply an analogous analysis to a spectral curve with a higher genus Riemann surface, namely a torus. To do so we must first get a brief overview of both the structure of this Riemann surface as well as the functions that live on it, known as *elliptic functions*. A good reference for the following is [33].

4.1 Complex Lattice

Given a pair of complex numbers τ and σ such that $\text{Arg}(\tau) \neq \pm\text{Arg}(\sigma)$, we can construct a collection of evenly spaced points in the complex plane called a lattice. This lattice, denoted by Λ, is the set of integer combinations of the two parameters τ and σ:

$$\Lambda = \{m\tau + n\sigma \big| (m, n) \in \mathbb{Z}^2\}. \tag{4.1}$$

We can simplify this definition by requiring that one of these parameters be unity and the other be contained in the upper half complex plane:

Definition 4.1.1. We define *complex lattice* as the set of points given by

$$\Lambda = \{m\tau + n \big| (m, n) \in \mathbb{Z}^2\}, \tag{4.2}$$

where τ (known as the *structure parameter*) is a complex number with $\mathrm{Im}(\tau) > 0$.

This definition automatically segments the complex plane into a collection of parallelograms, with the vertices lying on adjacent lattice points.

We can pick any one of these tiles to be the *fundamental parallelogram* (or *fundamental domain*). The most obvious choice [9] is the tile whose bottom left vertex lies at the origin. This can be visualized in Figure 4.1.

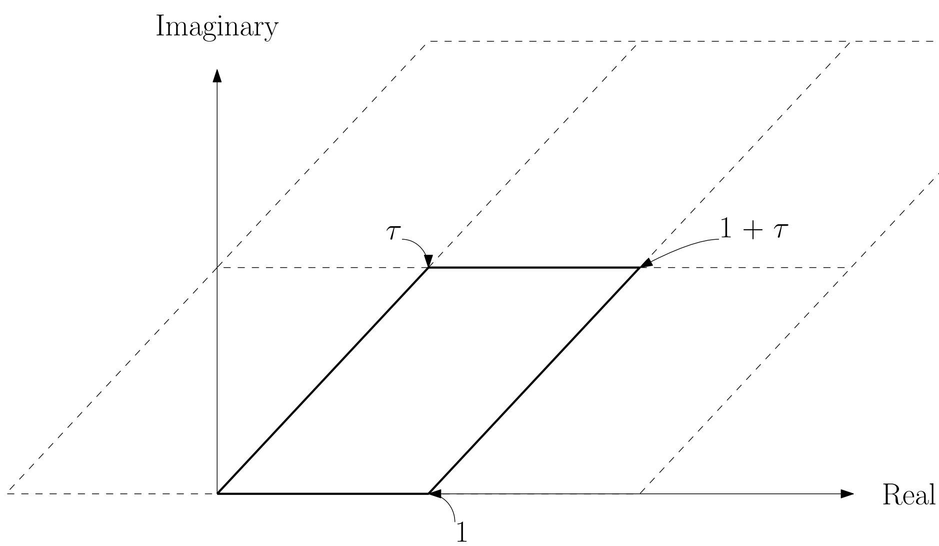

Figure 4.1: Lattice points define a "tiling" of the complex plane with copies of the fundamental parallelogram (the tile with a heavy outline).

As it turns out if we identify opposite sides of this fundamental domain

[9]This tiling of the complex plane can also be shifted by a universal translation, that is to say that the vertices don't need to coincide with the lattice points, it is merely convenient to define it this way initially. Furthermore, it becomes necessary to introduce a translation when dealing with certain integration paths.

with one another, we end up with a space that is topologically a torus. That is to say that the quotient space $\mathbb{C}/\Lambda$ is the genus one Riemann surface known as a torus. Again we can visualize this in Figure 4.2.

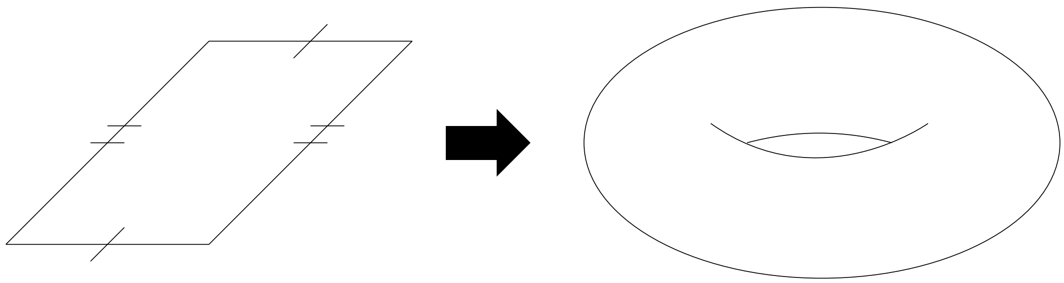

Figure 4.2: Identifying opposite sides of the fundamental domain results in a space that is isomorphic to the torus (donut shape pictured on the right).

The functions that live on this surface are known as *elliptic functions* and before we venture to describe them we should first discuss a vital consequence of having a hole in our surface.

4.2 The Torus and its Cycles

On the Riemann sphere it is true that any two paths with the same endpoints are *homotopic*. That is to say they can be continuously deformed into one another. What this means is that integrals of a meromorphic function over (non-closed) homotopic paths are equivalent. As a consequence of this we only need the endpoints to fully describe a path integral on the Riemann sphere, i.e.

$$\int_a^b f(z)\mathrm{d}z. \tag{4.3}$$

On the torus however, it is possible for two paths with the same end points to give different results. This is because we can "wind" around the torus an arbitrary number of times, effectively multiplying the result.

More precisely we say that paths with the same endpoints are not necessarily homotopic. Instead they are grouped according to how they "wind" around the torus. This is called a *homology class* and we say that any two paths with the same endpoints and the same homology class are homotopic. These homology classes are distinguished based on the paths decomposition in the *basis of cycles* (more informally these are the directions we can wind around the torus). We define this basis as the edges of the fundamental domain, as in Figure 4.3.

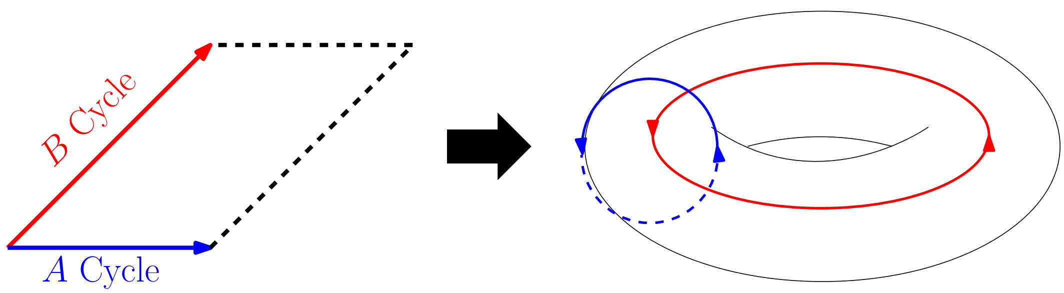

Figure 4.3: The representation of the canonical cycles on both the fundamental domain, and the torus itself.

What this means is in order to fully describe a definite integral using only the endpoints, we must also restrict the homology class of the path. The easiest way to do this is to require the path does not wind around the torus at all. Practically speaking it means that for an integral, we must enforce that the argument of a function remains inside the fundamental domain on the entire path. In other words the path does not intersect the basis of cycles.

Remark 4.2.1. For the remainder of the discussion, we will choose these cycles such that: the A cycle is the linear path from zero to one, and the B cycle is the linear path from zero to τ.

We adopt the following notation to denote cycle integrals (remembering that the path must remain in the fundamental domain.):

$$\oint_A f(z)\mathrm{d}z = \int_0^1 f(z)\mathrm{d}z \quad \text{and} \quad \oint_B f(z)\mathrm{d}z = \int_0^\tau f(z)\mathrm{d}z \tag{4.4}$$

Now that this has been specified, we can now move on to define the family of elliptic functions that is essential to the forthcoming discussion.

4.3 Weierstrass Elliptic Functions

It can be shown that the only possible holomorphic elliptic function is the constant function. The next best thing is a meromorphic function, with isolated distinct poles of finite order. Suppose we also want this function to have zero residue around said poles. We can construct an elliptic function with these properties using a very simple idea:

Given a lattice Λ. We begin with $\frac{1}{z^2}$ and add to it the translation of itself by every possible lattice point $\omega \in \Lambda$ resulting in

$$f(z) = \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^2}. \tag{4.5}$$

Ignoring convergence for the time being, this function is guaranteed to be elliptic by construction, since any translation by a lattice point will be absorbed into this sum:

$$f(z+\omega_0) = \sum_{\omega\in\Lambda} \frac{1}{(z-(\omega-\omega_0))^2} = \sum_{\omega'\in\Lambda} \frac{1}{(z-\omega')^2} = f(z). \tag{4.6}$$

As it turns out this function (with some appropriate modifications to guarantee convergence) is the Weierstrass $\wp(z;\tau)$ function.

4.3.1 Function Definitions

Following the immediately preceding discussion we can define the Weierstrass $\wp(z;\tau)$ function (spoken Weierstrass p function) with the following expression:

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right), \quad \text{where } \Lambda^* = \Lambda\backslash\{0\}. \tag{4.7}$$

This function is elliptic by construction and can be proven to be so with the argument in the previous discussion. The extra $\frac{1}{\omega^2}$ term is included to ensure convergence. It is also worth noting that this function is even.

The Weierstrass $\wp'(z;\tau)$ is the derivative with respect to z of $\wp(z;\tau)$, its definition follows directly from above:

$$\wp'(z;\tau) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}. \tag{4.8}$$

Fortunately this is convergent and again elliptic by the translation argument posed before.

Finally we also need the Weierstrass $\zeta(z;\tau)$ (not to be confused with the Riemann zeta function), its definition is as follows:

$$\zeta(z;\tau) = \frac{1}{z} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{z-\omega} + \frac{z}{\omega^2} + \frac{1}{\omega}\right). \tag{4.9}$$

$\zeta(z;\tau)$ is not elliptic but instead *quasi-elliptic* since:

$$\zeta(z+1;\tau) = \zeta(z;\tau) + 2\zeta(1/2;\tau), \tag{4.10}$$

$$\zeta(z+\tau;\tau) = \zeta(z;\tau) + 2\zeta(\tau/2;\tau). \tag{4.11}$$

The Weierstrass zeta function is related to the Weierstrass $\wp(z;\tau)$ via a derivative with respect to z:

$$-\frac{\mathrm{d}}{\mathrm{d}z}\zeta(z;\tau) = \wp(z;\tau). \tag{4.12}$$

As it turns out these functions satisfy a rather neat differential equation.

4.3.2 Differential Equation

Taking expansions around $z = 0$ of both the Weierstrass $\wp(z;\tau)$ function and its derivative, then comparing terms we can construct a differential equation satisfied by the $\wp(z)$ function:

$$\wp'(z;\tau)^2 = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau). \tag{4.13}$$

Here the two coefficients $g_2(\tau)$ and $g_3(\tau)$ are called *elliptic (or modular) invariants* and denote the following convergent infinite sums:

$$g_2(\tau) = 60 \sum_{\omega\in\Lambda^*} \frac{1}{\omega^4}. \tag{4.14}$$

$$g_3(\tau) = 140 \sum_{\omega\in\Lambda^*} \frac{1}{\omega^6}. \tag{4.15}$$

We also define the *modular discriminant* as,

$$\Delta(\tau) = g_2(\tau)^2 - 27g_3(\tau)^3. \tag{4.16}$$

4.3.3 Eisenstein Series

The sums used to define the modular invariants are called *Eisenstein series* (we refer the reader to [38] for more detail) and they are given by the following uniformly convergent sums:

$$G_{2n}(\tau) = \sum_{\omega\in\Lambda^*} \frac{1}{\omega^{2n}} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2n}} \qquad \text{for } n \geq 2. \tag{4.17}$$

As noted this definition is only valid for $n \geq 2$, since for $n = 1$ the sum is not absolutely convergent. That being said, if we define an order in the summation we can define $G_2(\tau)$:

$$G_2(\tau) = \sum_{m\neq 0} \frac{1}{m^2} + \sum_{n\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^2}. \tag{4.18}$$

The absolutely convergent Eisenstein series are called *modular forms of weight* $2n$ because they possess the following transformation properties:

$$G_{2n}(\gamma\tau) = (c\tau+d)^{2n}G_{2n}(\tau), \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{4.19}$$

with

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}. \tag{4.20}$$

$G_2(\tau)$ however, is a *quasi-modular* form of weight 2, since it satisfies the following property.

$$G_2(\gamma\tau) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d). \tag{4.21}$$

4.3.4 Zeroes of the Weierstrass Functions

The zeroes of both the $\wp(z;\tau)$ and $\zeta(z;\tau)$ functions are highly non-trivial to find and express. In addition these are not relevant to the forthcoming discussion. However the zeroes of $\wp'(z;\tau)$ are certainly relevant and very easy to find.

Starting with the fact that the Weierstrass $\wp(z;\tau)$ function is elliptic:

$$\wp(z+1;\tau) = \wp(z;\tau), \tag{4.22}$$

then taking the derivative with respect to z of both sides

$$\wp'(z+1;\tau) = \wp'(z;\tau). \tag{4.23}$$

Substituting $z = -\frac{1}{2}$ in the preceding expression and using the oddness of $\wp'(z;\tau)$ we see that

$$\wp'(1/2;\tau) = -\wp'(1/2;\tau), \tag{4.24}$$

which implies that

$$\wp'(1/2;\tau) = 0. \tag{4.25}$$

Following a similar procedure we obtain two more zeroes:

$$\wp'(\tau/2;\tau) = 0 \quad \text{and} \quad \wp'((1+\tau)/2;\tau) = 0. \tag{4.26}$$

We denote these *half periods* as follows:

$$w_1 = 1/2, \qquad w_2 = \tau/2, \qquad w_3 = (1+\tau)/2. \tag{4.27}$$

For convenience we also denote the value of $\wp(z;\tau)$ at these half periods as follows:

$$e_1 := \wp(w_1;\tau), \quad e_2 := \wp(w_2;\tau), \quad e_3 := \wp(w_3;\tau). \tag{4.28}$$

It is also possible to define the modular discriminant in terms of e_1, e_2 and e_3:

$$\Delta(\tau) := g_2(\tau)^2 - 27g_3(\tau)^3 = 16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2. \tag{4.29}$$

4.3.5 Series Expansion

For completeness we will state the series expansion of $\wp(z;\tau)$ about $z=0$

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{k=0} G_{2k+2}(\tau) z^{2k}. \tag{4.30}$$

4.3.6 Normalization

As in the Airy case we will need to define $W_{0,2}(z_1,z_2) = B(z_1,z_2)$ such that it satisfies the conditions given in Definition 2.2.1. In light of the fact that $\wp(z_1-$

$z_2;\tau) \to \frac{1}{(z_1-z_2)^2}$ as $z_1 \to z_2$, it is clear that the $\wp(z;\tau)$ function satisfies the first few conditions. That is to say that it is symmetric upon coordinate exchange, with a double pole on the diagonal at $z_1 = z_2$ and no other poles. However upon examination of the A-cycle integral, we see that is in fact nonzero,

$$\oint_A \wp(z_1 - z_2;\tau)\mathrm{d}z_1 = -\left(\zeta(1-z_2;\tau) - \zeta(z_2;\tau)\right) \neq 0. \tag{4.31}$$

This means that we need to normalize the function $\wp(z_1-z_2;\tau)$ with respect to the A-cycle. Before we can normalize the function, we must first understand why the function isn't normalized.

Examining the behaviour of the Weierstrass $\zeta(z;\tau)$ function under translations by 1 in its definition (4.9), we see that the offending is the term of the form

$$\sum_{\omega\in\Lambda^*} \frac{1}{\omega^2}. \tag{4.32}$$

Assuming we sum appropriately, we can recognize this as the previously defined Eisenstein series $G_2(\tau)$. Given this fact we can define a new function, denoted as $P_2(z;\tau)$, with this offending term removed,

$$P_2(z;\tau) := \wp(z;\tau) + G_2(\tau). \tag{4.33}$$

Now in order to prove that this indeed does satisfy the final normalization condition we must first define its indefinite integral which we will denote as $P_1(z;\tau)$ given by

$$P_1(z;\tau) := \int P_2(z;\tau)\mathrm{d}z = -\zeta(z;\tau) + G_2(\tau)z + A. \tag{4.34}$$

If we require that this function be odd, then the constant A is forced to be zero.

Using the properties of the zeta function we see,

$$P_1(z+1;\tau) = -\zeta(z+1;\tau) + G_2(\tau)(z+1) \tag{4.35}$$

$$= -(\zeta(z;\tau) + 2\zeta(1/2;\tau)) + G_2(\tau)(z+1) \tag{4.36}$$

$$= P_1(z;\tau) + (G_2(\tau) - 2\zeta(1/2;\tau)). \tag{4.37}$$

It can then be shown by a simple translation argument[10] that $P_1(z;\tau) = P_1(z+1;\tau)$ hence we can conclude that

$$\frac{1}{2}G_2(\tau) = \zeta(1/2;\tau). \tag{4.38}$$

Crafting a similar expression for a translation by τ leads to

$$P_1(z+\tau;\tau) = -\zeta(z+\tau;\tau) + G_2(\tau)(z+\tau) \tag{4.39}$$

$$= -(\zeta(z;\tau) - 2\zeta(\tau/2;\tau)) + G_2(\tau)(z+\tau) \tag{4.40}$$

$$= P_1(z;\tau) + (\tau G_2(\tau) - 2\zeta(\tau/2;\tau)). \tag{4.41}$$

Using the following identity from [37]

$$\tau\zeta(1/2;\tau) - \zeta(\tau/2;\tau) = \pi i. \tag{4.42}$$

[10]There is an alternative argument for this translation invariance involving the q-expansion for $P_1(z;\tau)$. For more details we refer the reader to [35].

In conjunction with (4.38), implies that

$$\tau G_2(\tau) - 2\zeta(\tau/2;\tau) = 2\pi i, \tag{4.43}$$

resulting in the *quasi-elliptic relations* for $P_1(z;\tau)$,

$$P_1(z+1;\tau) = P_1(z;\tau), \qquad P_1(z+\tau;\tau) = P_1(z;\tau) + 2\pi i. \tag{4.44}$$

This finally proves that the following differential can serve as the *canonical billinear differential* for topological recursion on a genus one spectral curve,

$$B(z_1, z_2) = P_2(z_1 - z_2)\mathrm{d}z_1\mathrm{d}z_2. \tag{4.45}$$

Chapter 5

Weierstrass Spectral Curve

We will begin by calculating the first few correlation functions for the Weierstrass spectral curve. Then proceed to construct the perturbative wave-function $\psi(z)$ to be compared with the quantization of the Weierstrass spectral curve.

As was mentioned in the introduction this chapter is an expanded exposition of my contribution to [7].

5.1 Topological Recursion on the Weierstrass Curve

The Weierstrass spectral curve is given by $\mathcal{L} = (\mathbb{C}/\Lambda, x, y)$ where the defining equation is

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau). \tag{5.1}$$

In light of the previous section we can see that this can be parametrized by the Weierstrass $\wp(z;\tau)$ function and its derivative, i.e

$$x(z) = \wp(z;\tau) \quad \text{and} \quad y(z) = \wp'(z;\tau). \tag{5.2}$$

As in the Airy case this is a double cover, with the ramification points[11] given by the zeroes of $\mathrm{d}x(z) = \wp'(z;\tau)$ along with the poles of $x(z) = \wp(z;\tau)$ with order ≥ 2. The zeroes are precisely the half periods as written in (4.27) and the double pole of $x(z)$ is at $z = 0$. All together these form the set of ramification points, denoted by $R = \{w_1, w_2, w_3, 0\}$.

The corresponding deck transformation maps for the half periods are technically given by $s_i(z) = 2w_i - z$, but since it can be shown that the correlation functions are elliptic in each coordinate, we can ignore the $2w_i$ shift and simply write the map as $s_i(z) = -z$ for $i = \{1, 2, 3\}$. In addition the deck transformation around $z = 0$ is given by $s_0(z) = -z$, meaning the deck transformation map is the same for all four ramification points. This fact implies that the kernel is the same for all four ramification points.

We will now proceed to calculate $W_{0,3}(z_0, z_1, z_2)$ and $W_{1,1}(z_0)$. In order to do this, we first need to calculate our kernel. As was mentioned above, the deck transformation map is globally defined and the same for all ramification points and hence we only have a single kernel given by

[11] More precisely the ramification points are the zeroes of $\mathrm{d}x$ and the poles of x *that lie within the fundamental domain*

$$K_\tau(z,z_0) = \frac{\mathrm{d}z_0}{2\wp'(z;\tau)^2}\left(\int_\alpha^z P_2(z'-z_0;\tau)\mathrm{d}z'\right) \tag{5.3}$$
$$= \frac{\mathrm{d}z_0}{2\wp'(z;\tau)^2}\left[P_1(z-z_0;\tau) - P_1(\alpha - z_0;\tau)\right]. \tag{5.4}$$

Remark 5.1.1. It can be shown via recursion that the ramification point $z = 0$ does not contribute to the total residue. Hence we can, and will, exclude its contribution from all of the following calculations.

Unlike the Airy case, the pole structure of the kernel is not immediately obvious. We know that $\wp'(z;\tau)$ has zeros at the half periods and hence $\frac{1}{\wp'(z;\tau)^2}$ has a pole at each of the half periods. We also know that $P_1(z-z_0;\tau)$ only has a pole at $z = z_0$ hence it will not have any principal part to its power series about $z = w_i$.

What this means is we need a power series expansion for both $\frac{1}{\wp'(z;\tau)^2}$ and the bracketed term. Fortunately both are relatively easy to find. We begin by re-expressing $\frac{1}{\wp'(z;\tau)^2}$ in a more power series friendly form:

$$\frac{1}{\wp'(z;\tau)^2} = \frac{12}{\Delta(\tau)}\left(\sum_{i=1}^{3}(20G_4(\tau) - e_i^2)(\wp(z-w_i;\tau) - e_i)\right). \tag{5.5}$$

Which we can expand around $z = w_i$ to obtain

$$\frac{1}{\wp'(z;\tau)^2} = \frac{12}{\Delta(\tau)}\Bigg((20G_4(\tau) - e_i^2)\left(\frac{1}{(z-w_i)^2} - 4e_i\right) + \sum_{k=1}^{\infty} A_{2k}(w_i)(z-w_i)^{2k}\Bigg). \tag{5.6}$$

Where:

$$
\begin{aligned}
A_{2k}(w_i) = (20G_4(\tau) - e_i^2)&(2k+1)G_{2k+2}(\tau) \\
&+ \frac{(20G_4(\tau) - e_j^2)}{(2k)!}\wp^{(2k)}(w_k;\tau) \\
&+ \frac{(20G_4(\tau) - e_k^2)}{(2k)!}\wp^{(2k)}(w_j;\tau) \\
&\qquad \text{for } \{i,j,k\} = \{1,2,3\} \text{ with } i \neq j \neq k.
\end{aligned} \tag{5.7}
$$

Remark 5.1.2. This identity and its power series are proven in appendix B.

We also need the power series around $z = w_i$ of

$$
P_1(z - z_0;\tau) - P_1(\alpha - z_0;\tau). \tag{5.8}
$$

Since this has no poles at any of the half periods, the power series is straightforward:

$$
\begin{aligned}
P_1(z - z_0;\tau) - P_1(\alpha - z_0;\tau) &= P_1(w_i - z_0;\tau) - P_1(\alpha - z_0;\tau) \\
&\quad + (z - w_i)\sum_{k=0}^{n} \frac{P_2^{(k)}(w_i - z_0;\tau)}{(k+1)!}(z - w_i)^k.
\end{aligned} \tag{5.9}
$$

Combining these two power series results in the desired power series about $z = w_i$ for the kernel,

$$
K_\tau(z, z_0) = \frac{\mathrm{d}z_0}{\mathrm{d}z}\frac{6}{\Delta(\tau)}\sum_{k=-2}^{\infty} B_k(w_i)(z - w_i)^k. \tag{5.10}
$$

With the first few coefficients given by:

$$B_{-2}(w_i) = 20G_4(\tau) - e_i^2, \tag{5.11}$$

$$B_{-1}(w_i) = (20G_4(\tau) - e_i^2)(P_2(w_i - z_0;\tau)), \tag{5.12}$$

$$B_0(w_i) = (20G_4(\tau) - e_i^2)\left(\frac{1}{2}P_2'(w_i - z_0;\tau) - 4e_i(P_1(w_i - z_0) - P_1(\alpha - z_0))\right), \tag{5.13}$$

$$B_1(w_i) = (20G_4(\tau) - e_i^2)\left(-4e_iP_2(w_i - z_0;\tau) + \frac{1}{6}P_2^{(2)}(w_i - z_0;\tau)\right). \tag{5.14}$$

The general expression for $B_k(w_i)$ is not required for the current discussion hence we will leave it unspecified.

We now have the ability to calculate the $k = 2$ level correlation functions. We will begin with $W_{0,3}(z_0, z_1, z_2)$, using the general expression given in (2.10) and inserting the relevant data we obtain

$$\begin{aligned} W_{0,3}(z_0, z_1, z_2) = \sum_{a\in R} \operatorname*{Res}_{z=a} K_\tau(z, z_0)(P_2(z - z_1;\tau)P_2(z + z_2;\tau) \\ + P_2(z - z_2;\tau)P_2(z + z_1;\tau))(-\mathrm{d}z^2\mathrm{d}z_1\mathrm{d}z_2). \end{aligned} \tag{5.15}$$

It is clear that the expression in the brackets has no pole at any of the half periods, hence the residue at $z = w_i$ will be $B_{-1}(w_i)$ multiplied by the bracketed term evaluated at $z = w_i$, resulting in

$$\begin{aligned} &W_{0,3}(z_0, z_1, z_2) = \\ &\frac{-12}{\Delta(\tau)}\mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2 \sum_{i=1}^{3}(20G_4(\tau) - e_i^2)P_2(z_0 - w_i;\tau)P_2(z_1 - w_i;\tau)P_2(z_2 - w_i;\tau). \end{aligned} \tag{5.16}$$

Finally we will calculate $W_{1,1}(z_0)$ beginning with the general form in (2.11) and again inserting the relevant terms leads to

$$W_{1,1}(z_0) = \sum_{a \in R} \operatorname*{Res}_{z=a} K_\tau(z, z_0) P_2(2z;\tau)(-\mathrm{d}z^2). \tag{5.17}$$

From this expression it doesn't seem obvious that $P_2(2z;\tau)$ has a pole at the ramification points, however using the elliptic property of P_2 we see,

$$P_2(2z;\tau) = P_2(2z - 2w_i;\tau) = P_2(2(z - w_i);\tau), \tag{5.18}$$

which implies that P_2 has a second order pole around $z = w_i$, meaning that the two terms from the kernels power series that will contribute are $(z - w_i)^1$ and $(z - w_i)^{-1}$ leading to:

$$\begin{aligned} W_{1,1}(z_0) = \frac{-6}{\Delta(\tau)} \mathrm{d}z_0 \sum_{i=1}^{3} (20G_4(\tau) - e_i^2) \bigg(&(G_2(\tau) - e_i) P_2(z_0 - w_i;\tau) \\ &+ \frac{1}{4!} P_2^{(2)}(z_0 - w_i;\tau) \bigg). \end{aligned} \tag{5.19}$$

We will now proceed to obtain the quantum curve from topological recursion and compare it to the quantization of (5.1).

5.2 Weierstrass Quantum Curve

We will follow the four steps set out in the Airy chapter to construct the Weierstrass perturbative wave-function and the corresponding quantum curve.

Remark 5.2.1. For readability, we will suppress the τ dependence in the elliptic functions and modular forms for the entirety of this section.

5.2.1 Residues

Beginning with the Weierstrass topological recursion:

$$W_{g,n+1}(z_0,\mathbf{z}) = \mathrm{d}z_0 \sum_{a\in R} \operatorname*{Res}_{z=a} \left(\int_{\alpha}^{z} P_2(z'-z_0)\mathrm{d}z'\right) \frac{\mathcal{R}^{(2)}W_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z)^2\mathrm{d}z} \tag{5.20}$$

The function given by $u(z) = \int_\alpha^z P_2(z'-z_0)\mathrm{d}z'$, is well defined on the complex plane, However it is not elliptic (and hence multiply defined on the torus) since,

$$\begin{aligned} u(z+\tau) &= \int_{\alpha}^{z+\tau} P_2(z'-z_0)\mathrm{d}z' \\ &= u(z) + \oint_B P_2(z'-z_0)\mathrm{d}z' \\ &= u(z) + 2\pi i. \end{aligned} \tag{5.21}$$

A major consequence of this is that the sum of residues at all poles of the right hand side of (5.20) is no longer zero. In fact this total residue is finite and to calculate it we appeal to the *Riemann billinear identity* (For further information we refer the reader to [4]).

Definition 5.2.2. The integral form of the *Riemann bilinear identity* can be stated as follows,

$$\sum_{\text{all poles } b \text{ of } u\eta} \operatorname*{Res}_{z=b} u\eta = \frac{1}{2\pi i}\sum_{j=1}^{g}\left(\oint_{B_j}\omega\oint_{A_j}\eta - \oint_{B_j}\eta\oint_{A_j}\omega\right),$$

where η is a meromorphic differential on the compact Riemann surface Σ of genus g, and (A_j, B_j), $j = 1, \ldots, g$ is a symplectic basis of cycles. Moreover,

$$u(z) = \int_\alpha^z \omega, \tag{5.22}$$

where ω is a residueless meromorphic differential, α is an arbitrary base point, and the line integral is taken in the fundamental domain.

As it turns out this works perfectly for calculating the total residue of the right hand side of the Weierstrass recursion (5.20). By noting that the integral satisfies the properties of $u(z)$ and the remainder meets the criteria for η, meaning we can apply the bilinear identity directly, leading to

$$\begin{aligned}
\sum_{\text{all poles } b} \operatorname*{Res}_{z=b} \left(\int_\alpha^z P_2(z' - z_0) \mathrm{d}z' \right) \frac{\mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z)^2 \mathrm{d}z} \\
= \frac{1}{2\pi i} \left(\oint_B P_2(z - z_0) \mathrm{d}z \oint_A \frac{\mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z)^2 \mathrm{d}z} \right. \\
\left. - \oint_B \frac{\mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z)^2 \mathrm{d}z} \oint_A P_2(z - z_0) \mathrm{d}z \right).
\end{aligned} \tag{5.23}$$

We do not know the cycle integrals of the recursive part just yet, however we do know explicitly the cycle integrals of $P_2(z - z_0)$ (as established in the quasi-elliptic relations of $P_1(z)$).

$$\oint_A P_2(z - z_0) \mathrm{d}z = 0, \qquad \oint_B P_2(z - z_0) \mathrm{d}z = 2\pi i. \tag{5.24}$$

Substituting these into (5.23) leads to the expression for the total residue

$$\sum_{\text{all poles } b} \operatorname*{Res}_{z=b} \left(\int_{\alpha}^{z} P_2(z'-z_0)\mathrm{d}z' \right) \frac{\mathcal{R}^{(2)} W_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z)^2 \mathrm{d}z} = \oint_A \frac{\mathcal{R}^{(2)} W_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z)^2 \mathrm{d}z}. \quad (5.25)$$

We will denote the quantity on the right hand side as follows:

$$B_{g,n+1}(\mathbf{z}) := \oint_A \frac{\mathcal{R}^{(2)} W_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z)^2 \mathrm{d}z}. \quad (5.26)$$

What this means is we can now do a similar replacement of the residue calculation in (5.20) using $B_{g,n+1}(\mathbf{z})$ as the total residue, leading to

$$\frac{W_{g,n+1}(z_0,\mathbf{z})}{\mathrm{d}z_0} = B_{g,n+1}(\mathbf{z}) - \sum_{Q \notin R} \operatorname*{Res}_{z=Q} \left(\int_{\alpha}^{z} P_2(z'-z_0)\mathrm{d}z' \right) \frac{\mathcal{R}^{(2)} W_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z)^2 \mathrm{d}z}. \quad (5.27)$$

With that replacement we can now proceed to construct the quantum curve.

Proposition 5.2.3. *For* $2g+n-1 \geq 1$,

$$
\begin{aligned}
&-\frac{W_{g-1,n+2}(-z_0,z_0,\mathbf{z})}{dx(z_0)^2} + \sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}} \left(\frac{W_{g_1,|I|+1}(-z_0,I)}{dx(z_0)}\right)\left(\frac{W_{g_2,|J|+1}(-z_0,J)}{dx(z_0)}\right)\\
&\qquad + \sum_{i=1}^{n}\left(\left(\frac{dx(z_i)}{(x(z_0)-x(z_i))^2}\right)\frac{W_{g,n}(-z_0,\mathbf{z}\setminus\{z_i\})}{dx(z_0)} - \right.\\
&\qquad\qquad \left. d_{z_i}\left(\frac{1}{(x(z_0)-x(z_i))}\frac{W_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right)\right)\\
&\qquad - 2B_{g,n+1}(\mathbf{z}) + \sum_{i=1}^{n} d_{z_i}\left(\frac{2P_1(z_i)}{\wp'(z_i)}\frac{W_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right) = 0. \quad (5.28)
\end{aligned}
$$

For $(g,n)=(0,1)$,

$$
\begin{aligned}
&2\frac{W_{0,2}(-z_0,z_1)}{dx(z_0)}\frac{W_{0,1}(-z_0)}{dx(z_0)} - \frac{dx(z_1)}{(x(z_0)-x(z_1))^2}\frac{W_{0,1}(-z_0)}{dx(z_0)}\\
&\qquad + d_{z_1}\left(\frac{1}{x(z_0)-x(z_1)}\frac{W_{0,1}(-z_0)}{dx(z_0)}\right) - 2P_2(z_1)dz_1 = 0, \quad (5.29)
\end{aligned}
$$

while for $(g,n)=(0,0)$,

$$
\frac{W_{0,1}(-z_0)}{dx(z_0)}\frac{W_{0,1}(-z_0)}{dx(z_0)} - \left(4x(z_0)^3 - g_2x(z_0) - g_3\right) = 0. \quad (5.30)
$$

Proof. Starting with (5.27), we first note that the only poles not in R, are z_0 and the marked points $z_1,\cdots,z_n$.

The residue at $z=z_0$ is given by

$$
\frac{-1}{2\wp'(z_0)^2\mathrm{d}z_0^2}\left(W_{g-1,n+2}(-z_0,z_0,\mathbf{z}) + \sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}' W_{g_1,|I|+1}(z_0,I)W_{g_2,|J|+1}(-z_0,J)\right), \quad (5.31)
$$

while the total residue at the poles $z = \pm z_i$ is

$$\mathrm{d}_{z_i}\left((P_1(z_i - z_0) + P_1(z_i + z_0))\,\frac{W_{g,n}(-z_i,\mathbf{z})}{2\wp'(z_i)^2\mathrm{d}z_i}\right). \tag{5.32}$$

Summing up all of the contributions results in

$$\begin{aligned}&\frac{W_{g,n+1}(z_0,\mathbf{z})}{\mathrm{d}z_0}\\ &= \frac{1}{2\wp'(z_0)^2}\left(\frac{W_{g-1,n+2}(-z_0,z_0,\mathbf{z})}{\mathrm{d}z_0^2} + \sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}^{\prime}\frac{W_{g_1,|I|+1}(z_0,I)}{\mathrm{d}z_0}\frac{W_{g_2,|J|+1}(-z_0,J)}{\mathrm{d}z_0}\right)\\ &\quad - \sum_{i=1}^{n}\mathrm{d}_{z_i}\left((P_1(z_i - z_0) + P_1(z_i + z_0))\,\frac{W_{g,n}(-z_i,\mathbf{z})}{2\wp'(z_i)^2\mathrm{d}z_i}\right) + B_{g,n+1}(\mathbf{z}). \end{aligned} \tag{5.33}$$

We must prepare this expression to be integrated in a similar fashion to the Airy case, in other words we must make sure that all of the correlation functions have their first argument negated. As we know from experience all of the $W_{g,n}$'s are odd in all of the arguments except for $W_{0,2}(z_1, z_2)$ which has the special property[12]:

$$W_{0,2}(z_1,z_2) + W_{0,2}(-z_1,z_2) = \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}. \tag{5.34}$$

Making the appropriate substitutions in the cross summation term (the term with sums over I and J) in order to negate the first argument of $W_{g_1,|I|+1}$,

[12] This can be proven directly using (A.2) listed in appendix A

we end up with:

$$
\begin{aligned}
&\frac{W_{g,n+1}(z_0,\mathbf{z})}{\mathrm{d}z_0} \\
&= \frac{1}{2\wp'(z_0)^2}\left(\frac{W_{g-1,n+2}(-z_0,z_0,\mathbf{z})}{\mathrm{d}z_0^2} - \sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}' \frac{W_{g_1,|I|+1}(z_0,I)}{\mathrm{d}z_0}\frac{W_{g_2,|J|+1}(-z_0,J)}{\mathrm{d}z_0}\right) \\
&\qquad + \sum_{i=1}^{n}\frac{1}{4z_0^2\mathrm{d}z_0^2}\left(\frac{\mathrm{d}x(z_0)\mathrm{d}(z_i)}{(x(z_0)-x(z_i))^2}W_{g,n}(-z_0,\mathbf{z}\setminus\{z_i\})\right) \\
&\quad - \sum_{i=1}^{n}\mathrm{d}_{z_i}\left((P_1(z_i-z_0)+P_1(z_i+z_0))\frac{W_{g,n}(-z_i,\mathbf{z})}{2\wp'(z_i)^2\mathrm{d}z_i}\right) + B_{g,n+1}(\mathbf{z}). \quad (5.35)
\end{aligned}
$$

Finally making the following two substitutions

$$
P_1(z_i+z_0)+P_1(z_i-z_0) = 2P_1(z_i) + \frac{\wp'(z_i)}{\wp(z_0)-\wp(z_i)}, \tag{5.36}
$$

and

$$
\frac{W_{g,n+1}(z_0,\mathbf{z})}{\mathrm{d}z_0} = \frac{W_{0,1}(-z_0)}{\mathrm{d}x(z_0)}\frac{W_{g,n+1}(-z_0,\mathbf{z})}{\mathrm{d}x(z_0)}, \tag{5.37}
$$

results in the final expression of the proposition for $2g+n-1\geq 1$. For the two remaining cases we do an explicit calculation involving terms that fit with the overall structure. □

5.2.2 Integration

Now we must augment Definition 3.4.3 to reflect the poles of $x(z)$. In the Airy case we integrated from $z=\infty$, however now the pole of $x(z)$ we are interested in is $z=0$. Hence we now need to integrate with the base point $z=0$:

$$G_{g,n+1}(z_0;\mathbf{z}) = \int_0^{z_1}\cdots\int_0^{z_n} W_{g,n+1}(-z_0,z_1,\cdots,z_n) \tag{5.38}$$

Lemma 5.2.4. *For $2g+n-1\geq 1$,*

$$\begin{aligned}
&-\left(\frac{\partial}{\partial x(z_{n+1})}\frac{G_{g-1,n+2}(z_0;\mathbf{z},z_{n+1})}{dx(z_0)}\right)_{z_{n+1}=z_0} \\
&\qquad + \sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}} \left(\frac{G_{g_1,|I|+1}(z_0,I)}{dx(z_0)}\right)\left(\frac{G_{g_2,|J|+1}(z_0,J)}{dx(z_0)}\right) \\
&\qquad - \sum_{i=1}^{n}\frac{1}{x(z_i)-x(z_0)}\left(\frac{G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})}{dx(z_i)} - \frac{G_{g,n}(z_0;\mathbf{z}\setminus\{z_i\})}{dx(z_0)}\right) \\
&\qquad - 2\int_0^{z_1}\cdots\int_0^{z_n} B_{g,n+1}(\mathbf{z}) + \sum_{i=1}^{n}\frac{2P_1(z_i)}{\wp'(z_i)}\frac{G_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)} = 0.
\end{aligned} \tag{5.39}$$

For $(g,n)=(0,1)$,

$$2\frac{G_{0,2}(z_0;z_1)}{dx(z_0)}\frac{G_{0,1}(z_0)}{dx(z_0)} - \frac{1}{x(z_1)-x(z_0)}\left(\frac{G_{0,1}(z_1)}{dx(z_1)} - \frac{G_{0,1}(z_0)}{dx(z_0)}\right) - 2P_1(z_1) = 0, \tag{5.40}$$

while for $(g,n)=(0,0)$,

$$\frac{G_{0,1}(z_0)}{dx(z_0)}\frac{G_{0,1}(z_0)}{dx(z_0)} - (4x(z_0)^3 - g_2x(z_0) - g_3) = 0. \tag{5.41}$$

Proof. We are integrating the expressions found in proposition 5.2.3.

For $2g+n-1\geq 1$, the first, second and fourth term follow directly from integration. The third term has a base point evaluation of the form:

$$\sum_{i=1}^{n}\lim_{z_i\to 0}\frac{1}{x(z_i)-x(z_0)}\left(\frac{G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_i)} - \frac{G_{g,n}(z_0;\mathbf{z}\setminus\{z_i\})}{\mathrm{d}x(z_0)}\right) = 0, \tag{5.42}$$

which vanishes since zero is a pole of $x(z)$ and the $G_{g,n}$'s have poles only at the half periods.

The fifth and final term results in a limit of the form:

$$\sum_{i=1}^{n} \lim_{z_i \to 0} \frac{2P_1(z_i)}{\wp'(z_i)} \frac{G_{g,n}(-z_i, \mathbf{z} \setminus \{z_i\})}{\mathrm{d}x(z_i)}, \tag{5.43}$$

which vanishes since $\wp'(z)$ has a pole of order 3 at $z = 0$ and $P_1(z)$ has only a simple pole at $z = 0$.

For $(g, n) = (0, 1)$, the integration results in a limit of the form:

$$\begin{aligned}\lim_{z_1 \to 0} \frac{1}{x(z_1) - x(z_0)} \left(\frac{G_{0,1}(z_1)}{\mathrm{d}x(z_1)} - \frac{G_{0,1}(z_0)}{\mathrm{d}x(z_0)} \right) - 2P_1(z_1) \\ = \lim_{z_1 \to 0} \frac{1}{\wp(z_1) - \wp(z_0)} \left(-\wp'(z_1) + \wp'(z_0)\right) + 2P_1(z_1),\end{aligned} \tag{5.44}$$

But since as $z_1 \to 0$,

$$\wp'(z_1) \to -\frac{2}{z_1^3} \qquad \wp(z_1) \to \frac{1}{z_1^2} \qquad P_1(z_1) \to -\frac{1}{z_1},$$

we see this limit vanishes.

Finally, for $(g, n) = (0, 0)$ we do not even integrate so the statement is trivially true. □

5.2.3 Principal Specialization

Using Definition 3.4.5 with no augmentations we proceed to principal specialize the expressions from the previous lemma.

Lemma 5.2.5. *For $2g + n - 1 \geq 1$,*

$$- \frac{1}{n+1}\left(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g-1,n+2}(z_0;z)}{dx(z_0)}\right)_{z_0=z}$$

$$+ \sum_{g_1+g_2=g}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{\widehat{G}_{g_1,m+1}(z;z)}{dx(z)}\frac{\widehat{G}_{g_2,n-m+1}(z;z)}{dx(z)}$$

$$- n\left(\frac{\partial}{\partial x(z_0)}\frac{\widehat{G}_{g,n}(z_0;z)}{dx(z_0)}\right)_{z_0=z}$$

$$- 2\int_0^z\cdots\int_0^z B_{g,n+1}(\mathbf{z}) + 2n\frac{P_1(z)}{\wp'(z)}\frac{\widehat{G}_{g,n}(z;z)}{dx(z)} = 0. \quad (5.45)$$

For $(g,n) = (0,1)$,

$$2\frac{\widehat{G}_{0,2}(z;z)}{dx(z)}\frac{\widehat{G}_{0,1}(z)}{dx(z)} - \frac{d}{dx(z)}\frac{\widehat{G}_{0,1}(z)}{dx(z)} - 2P_1(z) = 0, \quad (5.46)$$

while for $(g,n) = (0,0)$,

$$\frac{\widehat{G}_{0,1}(z)}{dx(z)}\frac{\widehat{G}_{0,1}(z)}{dx(z)} - (4x(z)^3 - g_2x(z) - g_3) = 0. \quad (5.47)$$

Proof. For $2g+n-1 \geq 1$, the first term follows from Proposition 3.4.6 since this is a general result. The second and third terms follow from the same arguments as in Lemma 3.4.7 which again are general arguments. The final two terms follow trivially from specialization.

For $(g,n) = (0,1)$, the first and third terms follow trivially. The second results in a limit of the form:

$$
\begin{aligned}
\lim_{z_1\to z_0} \frac{1}{x(z_1)-x(z_0)} &\left(\frac{G_{0,1}(z_1)}{\mathrm{d}x(z_1)} - \frac{G_{0,1}(z_0)}{\mathrm{d}x(z_0)}\right) \\
&= \lim_{z_1\to z_0} \frac{1}{x'(z_0)(z_1-z_0)} \left(\frac{G_{0,1}(z_1)}{\mathrm{d}x(z_1)} - \frac{G_{0,1}(z_0)}{\mathrm{d}x(z_0)}\right) \\
&= \frac{\partial}{\partial x(z_0)}\left(\frac{G_{0,1}(z_0)}{\mathrm{d}x(z_0)}\right), \quad (5.48)
\end{aligned}
$$

setting $z_0 = z$ in the final line completes the proof for this level.

As for $(g, n) = (0, 0)$ we simply set $z_0 = z$.

□

5.2.4 Summation

We now sum over the expressions in the previous lemma with appropriate factors of $\hbar$.

Lemma 5.2.6.

$$
\begin{aligned}
&\hbar \frac{d}{dx(z)}\xi_1(z;z) + \xi_1(z;z)^2 - (4x(z)^3 - g_2(\tau)x(z) - g_3) \\
&\quad - 2\hbar\frac{P_1(z)}{\wp'(z)}\xi_1(z;z) - 2\sum_{2g+n-1\geq 1}\frac{\hbar^{2g+n}}{n!}\int_0^z \cdots \int_0^z B_{g,n+1}(\mathbf{z}) = 0. \quad (5.49)
\end{aligned}
$$

Proof. Multiplying the equations in Lemma 5.2.5 by $\frac{\hbar^{2g+n}}{n!}$ results in the following expressions:

For $2g+n-1 \geq 1$,

$$
\begin{aligned}
&-\frac{\hbar^{2g+n}}{(n+1)!}\left(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g-1,n+2}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z}+\frac{\hbar^{2g+n}}{(n-1)!}\left(\frac{\partial}{\partial x(z_0)}\frac{G_{g,n}(z_0;z)}{\mathrm{d}x(z_0)}\right)_{z_0=z}\\
&\quad+\sum_{g_1+g_2=g}\sum_{m=0}^{n}\left(\frac{\hbar^{2g_1+m}}{m!}\frac{\widehat{G}_{g_1,m+1}(z;z)}{\mathrm{d}x(z)}\right)\left(\frac{\hbar^{2g_2+n-m}}{(n-m)!}\frac{\widehat{G}_{g_2,n-m+1}(z;z)}{\mathrm{d}x(z)}\right)\\
&-2\frac{\hbar^{2g+n}}{n!}\int_0^z\cdots\int_0^z B_{g,n+1}(\mathbf{z})+2\frac{P_1(z)}{\wp'(z)}\left(\frac{\hbar^{2g+n}}{(n-1)!}\frac{\widehat{G}_{g,n}(z;z)}{\mathrm{d}x(z)}\right)=0. \qquad (5.50)
\end{aligned}
$$

For $(g,n)=(0,1)$,

$$
2\left(\hbar\frac{\widehat{G}_{0,2}(z;z)}{\mathrm{d}x(z)}\right)\left(\frac{\widehat{G}_{0,1}(z)}{\mathrm{d}x(z)}\right)-\hbar\frac{\mathrm{d}}{\mathrm{d}x(z)}\frac{\widehat{G}_{0,1}(z)}{\mathrm{d}x(z)}-2\hbar P_1(z)=0, \qquad (5.51)
$$

while for $(g,n)=(0,0)$,

$$
\frac{\widehat{G}_{0,1}(z)}{\mathrm{d}x(z)}\frac{\widehat{G}_{0,1}(z)}{\mathrm{d}x(z)}-(4x(z)^3-g_2x(z)-g_3)=0. \qquad (5.52)
$$

Noting the following fact

$$
P_1(z)=\frac{P_1(z)}{\wp'(z)}\left(\frac{\widehat{G}_{0,1}(z)}{\mathrm{d}x(z)}\right), \qquad (5.53)
$$

we see that the $2\hbar P_1(z)$ term in the $(g,n)=(0,1)$ equation fits nicely into the final term of the $2g+n-1\geq 1$ set of equations.

Finally summing over g and n we obtain the result in the lemma. □

We define the perturbative wave-function in the same fashion as the Airy case except now we integrate from zero instead of infinity (since this is now the pole of $x(z)$):

$$\psi(z) = \exp\left\{\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_0^z\cdots\int_0^z\left(W_{g,n}(z_1,\cdots,z_n) - \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right)\right\}. \tag{5.54}$$

with $\psi_1(z';z) := \psi(z)\xi_1(z';z)$ as before.

Theorem 5.2.7.

$$\left[\hbar^2\frac{d^2}{dx^2} - (4x(z)^3 - g_2(\tau)x(z) - g_3(\tau)) - 2\hbar^2\frac{P_1(z)}{\wp'(z)}\frac{d}{dx} - 2\sum_{2g+n-1\geq 1}\frac{\hbar^{2g+n}}{n!}\int_0^z\cdots\int_0^z B_{g,n+1}(\mathbf{z})\right]\psi(z) = 0. \tag{5.55}$$

Proof. Multiplying the result from Lemma 5.2.6 and using the general results of both Proposition 3.4.12 and its corollary 3.4.13 leads directly to the result.

□

What this means is we no longer recover the differential operator expected from the spectral curve as we did in the Airy case. Instead we get additional corrections in $\hbar$ including an additional first derivative term. This proves that the Weierstrass spectral curve does not satisfy the proposed connection. Furthermore the fact that the corrections are not combinations of $\hat{x}$ and $\hat{y}$ means that we could not construct this operator by taking advantage of the commutativity in the classical equation. More precisely, this differential operator is not a quantization of the spectral curve, since there is no way we could have obtained it from a mapping of the characteristic polynomial.

All of this points to the fact that clearly the perturbative formulation is not what we should be using in this case but instead we need to appeal to the non-perturbative formulation presented in [7], which was derived from the general formulation in [5] (see also [22, 23]). However this non-perturbative formulation has an extra "quantization condition" that must be satisfied. The simplest way to satisfy this condition is to restrict the spectral curves to those with $g_2(\tau) = 0$, i.e. curves of the form:

$$y^2 = 4x^3 - g_3(\tau). \tag{5.56}$$

In [7] it was found that this non-perturbative formulation gives rise to a quantum curve with corrections as well. However these corrections are functions of $\hat{x}$ and $\hat{y}$ meaning it could be constructed by harnessing the ambiguity in the classical quantum curve. This was verified up to order 5 in $\hbar$ with symbolic software.

5.3 Additional Results

5.3.1 Identities for Elliptic Functions

Proposition 5.3.1. *For* $2g + n - 1 \geq 1$

$$B_{g,n+1}(\mathbf{z}) = -\left(\frac{W_{g,n+1}(z_0, \mathbf{z})}{dz_0}\right)_{z_0=0} + \sum_{i=1}^{n} d_{z_i}\left(\frac{P_1(z_i)}{\wp'(z_i)}\frac{W_{g,n}(-z_i, \mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right). \tag{5.57}$$

Proof. Starting with a rearranged topological recursion

$$B_{g,n+1}(\mathbf{z}) = -\frac{W_{g,n+1}(-z_0, \mathbf{z})}{\mathrm{d}z_0} + \sum_{Q \notin R} \operatorname*{Res}_{z=Q} \left(P_1(z - z_0) - P_1(\alpha - z_0)\right) \frac{\mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z)^2 \mathrm{d}z}, \quad (5.58)$$

and taking the limit as z_0 approaches zero on both sides results in,

$$B_{g,n+1}(\mathbf{z}) = -\left(\frac{W_{g,n+1}(-z_0, \mathbf{z})}{\mathrm{d}z_0}\right)_{z_0=0} + \sum_{Q \notin R} \operatorname*{Res}_{z=Q} \left(P_1(z) - P_1(\alpha)\right) \frac{\mathcal{R}^{(2)} W_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z)^2 \mathrm{d}z}. \quad (5.59)$$

The only poles Q not in R are at the marked points $z = \pm z_i$, we can calculate the contribution at each of these points. Beginning with the residue at $z = z_i$ we obtain

$$\mathrm{d}_{z_i} \left(\frac{P_1(z_i) - P_1(\alpha)}{2\wp'(z_i)} \frac{W_{g,n}(-z_i, \mathbf{z} \setminus \{z_i\})}{\mathrm{d}x(z_i)} \right). \quad (5.60)$$

While the matching contribution at $z = -z_i$ results in

$$\mathrm{d}_{z_i} \left(\frac{P_1(z) + P_1(\alpha)}{2\wp'(z)} \frac{W_{g,n}(-z_i, \mathbf{z} \setminus \{z_i\})}{\mathrm{d}x(z_i)} \right). \quad (5.61)$$

Summing up all of the contributions leads directly to the proposed result. □

If we write down the expression in the proposition for $(g, n) = (1, 0)$ we obtain the following,

$$B_{1,1} = -\left(\frac{W_{1,1}(-z_0, \mathbf{z})}{\mathrm{d}x(z_0)}\right)_{z_0=0}. \quad (5.62)$$

Which when explicitly written states:

$$\oint_A \frac{P_2(2z;\tau)}{2\wp'(z;\tau)^2}\mathrm{d}z = \frac{G_4(\tau)(5G_4(\tau) - G_2(\tau)^2)}{30(20G_4(\tau)^3 - 49G_6(\tau)^2)}. \tag{5.63}$$

We can provide an independent proof of this identity by calculating this cycle integral directly. As it turns out this is non-trivial and is provided in appendix C.

These identities could be interesting to study on their own and could potentially be useful for evaluation. It is also possible that this result is related to the work presented in [31]. In addition they also give rise to an equivalence between two very different looking differential operators.

5.3.2 Alternate Quantum Curve

A further result was obtained in [7], where the perturbative wave-function was constructed in a method adapted from [8]. This "alternate" quantum curve has a form *a priori* much different from that in Theorem 5.2.7, and is given by:

$$\Bigg[\hbar^2 \frac{d^2}{dx(z)^2} - (4x(z)^3 - g_2(\tau)x(z) - g_3(\tau)) - 2\hbar P_1(z;\tau) \\ +2\sum_{2g-1+n\geq 1} \frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0}\Bigg]\psi(z) = 0. \tag{5.64}$$

Although they were constructed via two different methods we will see that as a consequence of aforementioned elliptic identities, this equation and that in Theorem 5.2.7 are indeed equivalent.

Proposition 5.3.2.

$$\left[\hbar^2 \frac{P_1(z)}{\wp'(z)} \frac{d}{dx} + \sum_{2g+n-1\geq 1} \frac{\hbar^{2g+n}}{n!} \int_0^z \cdots \int_0^z B_{g,n+1}(\mathbf{z})\right] \psi(z)$$
$$= \left[\hbar P_1(z) - \sum_{2g-1+n\geq 1} \frac{\hbar^{2g+n}}{n!} \left(\frac{\hat{G}_{g,n+1}(z_0;z)}{dz_0}\right)_{z'=0}\right] \psi(z). \quad (5.65)$$

Proof. Integrating, Specializing, and summing the result in Proposition 5.3.1, then multiplying the result by $\psi(z)$:

$$\left[\sum_{2g+n-1\geq 1} \frac{\hbar^{2g+n}}{n!} \int_0^z \cdots \int_0^z B_{g,n+1}(\mathbf{z})\right] \psi(z)$$
$$= \left[- \sum_{2g+n-1\geq 1} \frac{\hbar^{2g+n}}{n!} \left(\frac{\widehat{G}_{g,n+1}(z_0;z)}{\mathrm{d}z_0}\right)_{z_0=0} + \frac{P_1(z)}{\wp'(z)} \sum_{2g+n-1\geq 1} \frac{\hbar^{2g+n}}{(n-1)!} \frac{\widehat{G}_{g,n}(z;z)}{\mathrm{d}x(z)}\right] \psi(z). \quad (5.66)$$

Explicitly calculating the derivative of $\psi(z)$

$$\hbar^2 \frac{\mathrm{d}}{\mathrm{d}x} \psi(z) = \left[\hbar \wp'(z) - \sum_{2g+n-1\geq 1} \frac{\hbar^{2g+n}}{(n-1)!} \frac{\widehat{G}_{g,n}(z;z)}{\mathrm{d}x(z)}\right] \psi(z), \quad (5.67)$$

which when mulitplied by $\frac{P_1(z)}{\wp'(z)}$ results in

$$\hbar^2 \frac{P_1(z)}{\wp'(z)} \frac{\mathrm{d}}{\mathrm{d}x} \psi(z) = \left[\hbar P_1(z) - \frac{P_1(z)}{\wp'(z)} \sum_{2g+n-1\geq 1} \frac{\hbar^{2g+n}}{(n-1)!} \frac{\widehat{G}_{g,n}(z;z)}{\mathrm{d}x(z)}\right] \psi(z). \quad (5.68)$$

Substituting this into (5.66) results in the proposed equality. □

Chapter 6

Conclusion

We have constructed the perturbative wave-function for both the Airy and Weierstrass spectral curves.

The Airy case served to exemplify the connection between topological recursion and the WKB expansion. This prompted the general conjecture of the connection, which was proven for a large class of genus zero curves in [8]. The question remained, does this connection (as conjectured) extend to spectral curves of higher genus? This question motivated an analogous treatment of the Weierstrass spectral curve (a spectral curve of genus one).

The Weierstrass case was presented in a similar fashion and as was expected, for reasons from matrix model theory, the quantum curve was not a straightforward quantization of the spectral curve. Instead there were an infinite number of corrections in $\hbar$. These corrections were not functions of $\hat{x}$, which implied that the operator could not be constructed by taking advantage of the commutativity/ambiguity of the classical equation.

The fact that the Weierstrass spectral curve does not satisfy the conjectured connection (as stated) implies that the perturbative wave-function is not the right quantity to look at. Instead we need to look to the non-perturbative formulation which is explored explicitly in [7].

Another consequence of the work presented in [7], is an alternate differential operator that kills the perturbative wave-function. This alternate quantum curve was constructed by adapting the methods in [8]. We showed that the two quantum curves are indeed equivalent as a consequence of a nice side result involving an infinite collection of identities relating elliptic functions and quasi-modular forms.

There remain a number of interesting open questions:

1. As was mentioned in the introduction some spectral curves, when submitted to topological recursion, give rise to correlation functions that correspond to enumerative invariants. So naturally we can ask, what meaning can we attribute to the correlation functions for the Weierstrass spectral curve?

2. Can we expand the non-perturbative treatment beyond the restricted family in (5.56)? And if so...

3. ... is it possible via the non-perturbative formulation to construct a wave-function that will in fact be killed by the straightforward quantization? i.e. is it possible to construct $\psi(z)$ from topological recursion such that:

$$\left[\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (4x^3 - g_2(\tau)x - g_3(\tau))\right] \psi(z) = 0. \tag{6.1}$$

4. What would extension to higher genus spectral curves result in? It would

be interesting to see treatment of higher genus spectral curves, with both the pertubative and non-perturbative constructions.

5. Are the infinite collection of identities for cycle integrals of elliptic functions (from Proposition 5.3.1) interesting in the context of elliptic modular forms? As was mentioned this could be related to the expressions in [31], it would be interesting to explore this possible connection.

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Appendix A

Identities involving Weierstrass Elliptic Functions

For reference we include some well known identities involving the Weierstrass elliptic functions.

A.1 Addition Forms

$$\wp(z_1 + z_2; \tau) = \frac{1}{4}\left(\frac{\wp'(z_1; \tau) - \wp'(z_2; \tau)}{\wp(z_1; \tau) - \wp(z_2; \tau)}\right)^2 - \wp(z_1; \tau) - \wp(z_2; \tau) \tag{A.1}$$

$$\wp(z_1 - z_2; \tau) - \wp(z_1 + z_2; \tau) = \frac{\wp'(z_1; \tau)\wp'(z_2; \tau)}{(\wp(z_1; \tau) - \wp(z_2; \tau))^2} \tag{A.2}$$

$$\zeta(z_1 + z_2; \tau) = \zeta(z_1; \tau) + \zeta(z_2; \tau) + \frac{1}{2}\left(\frac{\wp'(z_1; \tau) - \wp'(z_2; \tau)}{\wp(z_1; \tau) - \wp(z_2; \tau)}\right) \tag{A.3}$$

$$\zeta(z_1 - z_2; \tau) + \zeta(z_1 + z_2; \tau) = 2\zeta(z_1; \tau) + \frac{\wp'(z_1; \tau)}{\wp(z_1; \tau) - \wp(z_2; \tau)} \tag{A.4}$$

A.2 “Double Angle” Forms

$$\wp(2z; \tau) = -2\wp(z; \tau) + \frac{1}{4}\left(\frac{\wp''(z; \tau)}{\wp'(z; \tau)}\right)^2 \tag{A.5}$$

$$\zeta(2z; \tau) = 2\zeta(z; \tau) + \frac{1}{2}\frac{\wp''(z; \tau)}{\wp'(z; \tau)} \tag{A.6}$$

A.3 Differential Equations

$$\wp'(z;\tau)^2 = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau) \tag{A.7}$$

Taking derivatives of both sides of this equation with respect to z, leads to a number of additional differential equations satisfied by the Weierstrass elliptic functions:

$$\wp''(z;\tau) = 6\wp(z;\tau)^2 - \frac{1}{2}g_2(\tau) \tag{A.8}$$

$$\wp'''(z;\tau) = 12\wp(z;\tau)\wp'(z;\tau) \tag{A.9}$$

Appendix B

Power Series Expansion for Part of Weierstrass Kernel

Proposition B.0.1.

$$\frac{1}{\wp'(z;\tau)^2} = \frac{12}{\Delta(\tau)}\left(\sum_{i=1}^{3}(20G_4(\tau) - e_i^2)(\wp(z-w_i) - e_i)\right). \tag{B.1}$$

Proof. Suppressing the τ dependence for the time being, we begin with the Weierstrass differential equation in its factored form

$$\frac{1}{\wp'(z)^2} = \frac{1}{4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}. \tag{B.2}$$

Performing partial fractions decomposition on the right hand side of this expression yields

$$\begin{aligned}\frac{1}{\wp'(z)^2} = \frac{1}{4}\Bigg(&\frac{1}{(e_1-e_2)(e_1-e_3)(\wp(z)-e_1)} \\ &+ \frac{1}{(e_2-e_1)(e_2-e_3)(\wp(z)-e_2)} \\ &+ \frac{1}{(e_3-e_1)(e_3-e_2)(\wp(z)-e_3)}\Bigg).\end{aligned} \tag{B.3}$$

Utilizing the well known identity

$$\wp(z-w_j) - e_j = \frac{(e_j-e_i)(e_j-e_k)}{\wp(z)-e_j} \tag{B.4}$$

for $\{i,j,k\} = \{1,2,3\}$ with $i \neq j \neq k$, and dividing both sides by $(e_j - e_i)^2(e_j - e_k)^2$, results in

$$\frac{\wp(z - w_j) - e_j}{(e_j - e_i)^2(e_j - e_k)^2} = \frac{1}{(e_j - e_i)(e_j - e_k)(\wp(z) - e_j)} \quad \text{for } \{i, j, k\} = \{1, 2, 3\} \text{ with } i \neq j \neq k. \tag{B.5}$$

Using the definition of $\Delta = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_1 - e_3)^2$ we can rewrite the previous expression as:

$$\frac{16(e_i - e_k)^2(\wp(z - w_j) - e_j)}{\Delta} = \frac{1}{(e_j - e_i)(e_j - e_k)(\wp(z) - e_j)} \quad \text{for}\{i, j, k\} = \{1, 2, 3\} \text{ with } i \neq j \neq k. \tag{B.6}$$

We can simplify the previous expression further by making substitutions according to the following identity:

$$(e_i - e_k)^2 = 3(20G_4 - e_j^2) \quad \text{for } \{i, j, k\} = \{1, 2, 3\} \text{ with } i \neq j \neq k. \tag{B.7}$$

Making said substitutions results in

$$\frac{48(20G_4 - e_j^2)(\wp(z - w_j) - e_j)}{\Delta} = \frac{1}{(e_j - e_i)(e_j - e_k)(\wp(z) - e_j)} \quad \text{for } \{i, j, k\} = \{1, 2, 3\} \text{ with } i \neq j \neq k. \tag{B.8}$$

Finally substituting this expression into (B.3) leads to the proposition. □

Corollary B.0.2. *The power series of* $\frac{1}{\wp'(z)^2}$ *around* $z = w_i$ *is given by:*

$$\frac{1}{\wp'(z)^2} = \frac{12}{\Delta(\tau)}\left((20G_4(\tau) - e_i^2)\left(\frac{1}{(z - w_i)^2} - 4e_i\right) + \sum_{k=1}^{\infty} A_{2k}(w_i)(z - w_i)^{2k}\right), \tag{B.9}$$

where

$$A_{2k}(w_i) = (20G_4(\tau) - e_i^2)(2k + 1)G_{2k+2}(\tau) + \frac{(20G_4(\tau) - e_j^2)}{(2k)!}\wp^{(2k)}(w_k) + \frac{(20G_4(\tau) - e_k^2)}{(2k)!}\wp^{(2k)}(w_j). \tag{B.10}$$

Proof. All the terms follow directly from the various power series of $\wp(z)$, except for the constant term which is simple to calculate:

$$
\begin{aligned}
(20G_4 - e_j^2)(\wp(w_i - w_j) - e_j) + (20G_4 - e_k^2)(\wp(w_i - w_k) - e_k) - e_i(20G_4 - e_i^2) \\
&= (e_j - e_k)(e_j^2 - e_k^2) - e_i(20G_4 - e_i^2) \\
&= (e_j - e_k)^2(e_j + e_k) - e_i(20G_4 - e_i^2) \\
&= 3(20G_4 - e_i^2)(e_j + e_k) - e_i(20G_4 - e_i^2) \\
&= -3e_i(20G_4 - e_i^2) - e_i(20G_4 - e_i^2) \\
&= -4e_i(20G_4 - e_i^2)
\end{aligned}
$$

□

Appendix C

Direct Proof of First Elliptic Identity

In this Appendix we provide an independent proof of the following identity,

$$\oint_A \frac{P_2(2z;\tau)}{\wp'(z;\tau)^2}dz = \frac{G_4(\tau)(5G_4(\tau) - G_2(\tau)^2)}{30(20G_4(\tau)^3 - 49G_6(\tau)^2)}. \tag{C.1}$$

Let us evaluate the period integral on the LHS explicitly and show that it is indeed equal to the quasi-modular form on the RHS. In this Appendix we will suppress the τ-dependence for brevity.

First we expand the integrand with a "double angle" identity:

$$P_2(2z) = G_2 - 2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2, \tag{C.2}$$

hence our original integral splits into the following three integrals:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}dz = G_2 \oint_A \frac{\mathrm{d}z}{\wp'(z)^2} - 2\oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z + \frac{1}{4}\oint_A \frac{\wp''(z)^2}{\wp'(z)^4}\mathrm{d}z. \tag{C.3}$$

Let us focus on the third constituent integral. Using integration by parts and the fact that $\wp'''(z) = 12\wp(z)\wp'(z)$ we see that it simplifies into a more familiar form:

$$\frac{1}{4}\oint_A \frac{\wp''(z)^2}{\wp'(z)^4}\mathrm{d}z = \frac{1}{4}\left\{ -\frac{\wp''(z)}{3\wp'(z)^3}\bigg|_0^1 + 4\oint_A \frac{\wp(z)}{\wp'(z)^2}\right\} = \oint_A \frac{\wp(z)}{\wp'(z)^2}. \tag{C.4}$$

This means our original problem reduces to solving the following two integrals on the right hand side:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}dz = G_2 \oint_A \frac{\mathrm{d}z}{\wp'(z)^2} - \oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z. \tag{C.5}$$

To solve both we take advantage a very useful identity, which follows directly from the differential equation for the Weierstrass $\wp$-function (4.13) and the fact that $\frac{2}{3}(\wp''(z) - g_2) = 4\wp(z)^2 - g_2$:

$$\frac{1}{\wp'(z)^2} = \frac{1}{g_3}\left[\frac{2}{3}\frac{\wp(z)\,(\wp''(z) - g_2)}{\wp'(z)^2} - 1\right]. \tag{C.6}$$

As it turns out, using integration by parts we can express these two integrals in terms of one another:

$$\begin{aligned}\oint_A \frac{\mathrm{d}z}{\wp'(z)^2} &= \frac{1}{g_3}\left[-1 + \frac{2}{3}\oint_A \frac{\wp(z)\,(\wp''(z) - g_2)}{\wp'(z)^2}\right] \\ &= -\frac{1}{3g_3}\left[1 + 2g_2 \oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z\right]\end{aligned} \tag{C.7}$$

$$\begin{aligned}\oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z &= \frac{1}{g_3}\left[\frac{2}{3}\oint_A \frac{\wp(z)^2\,(\wp''(z) - g_2)}{\wp'(z)^2} - \oint_A \wp(z)\mathrm{d}z\right] \\ &= -\frac{1}{3g_3}\left[G_2 + \frac{g_2^2}{6}\oint_A \frac{\mathrm{d}z}{\wp'(z)^2}\right]\end{aligned} \tag{C.8}$$

For the last equation, we used the fact that

$$\oint_A \wp(z)\mathrm{d}z = -G_2, \tag{C.9}$$

since

$$0 = \oint_A P_2(z)\mathrm{d}z = \oint_A (\wp(z) + G_2)\mathrm{d}z. \tag{C.10}$$

Solving the system of equations (C.7) and (C.8) results in the following explicit expressions (with $\Delta = g_2^3 - 27g_3^2$):

$$\oint_A \frac{\mathrm{d}z}{\wp'(z)^2} = \frac{18g_3 - 12G_2g_2}{2\Delta}, \tag{C.11}$$

$$\oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z = \frac{18G_2g_3 - g_2^2}{2\Delta}. \tag{C.12}$$

As a result we see that the original integral (C.5) is given by:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}\mathrm{d}z = \frac{18G_2g_3 - 12G_2^2g_2}{2\Delta} - \frac{18G_2g_3 - g_2^2}{2\Delta} = \frac{g_2(g_2 - 12G_2^2)}{2\Delta}. \quad \text{(C.13)}$$

Making the substitutions $g_2 = 60G_4$ and $g_3 = 140G_6$ into this expression we arrive at the final expected result:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}\mathrm{d}z = \frac{G_4(5G_4 - G_2^2)}{30(20G_4^3 - 49G_6^2)}. \quad \text{(C.14)}$$