Instantons in Parton Gauge Theories of Condensed Matter

by

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Abstract

This thesis broadly investigates monopole-driven confinement transitions in parton gauge theories of fractionalized phases of matter. Chapter 2 studies the Dirac spin liquid, a $2d$ fractionalized Mott insulator with gapless Dirac fermion excitations coupled to a compact $U(1)$ gauge field. We use semiclassical instanton methods not relying on conformal symmetry to construct all monopole operators as 't Hooft vertices – instanton-induced interactions between fermions that have their origin in zero modes of the Euclidean Dirac operator in an instanton background. These monopole operators serve as order parameters for conventionally ordered states proximate to the Dirac spin liquid, as determined by their quantum numbers under lattice symmetries, which we are able to capture on bipartite lattices. Chapter 3 is a detailed technical description of instanton-induced interactions and their symmetry-breaking effects in CQED_3, motivated by a fermionic parton description of hardcore bosons on a $2d$ lattice with $U(1)$ symmetry. We show how the proliferation of instantons carrying fermion zero modes can lead to spontaneous breakdown of this symmetry, leading to either a conventional superfluid or an exotic 'paired superfluid'. Chapter 4 generalizes this study to Ising spins on a $2d$ lattice with $\mathbb{Z}_2$ symmetry, represented by N Majorana partons. By varying the Chern number of the Majorana bandstructure, we access chiral spin liquid, paramagnetic, and long-range ordered phases of the Ising spins. A certain $SO(N)$ Chern-Simons gauge theory with massless Majorana fermions is argued to be the critical theory that interpolates between these different phases. Finally, $\mathbb{Z}_2$-charged monopoles are shown to drive confinement in such a theory, leading to a magnetic phase with spontaneous breakdown of the Ising $\mathbb{Z}_2$ symmetry.

Preface

This dissertation is built on the following publications:

- [1] G. Shankar and J. Maciejko, “Symmetry-breaking effects of instantons in parton gauge theories”, Phys. Rev. B **104**, 035134 (2021),

- [2] G. Shankar, C.-H. Lin, and J. Maciejko, ”Continuous transition between Ising magnetic order and a chiral spin liquid”, Phys. Rev. B **106**, 245107 (2022),

- [3] G. Shankar and J. Maciejko, ”Monopoles in Dirac spin liquids and their symmetries from instanton calculus”, SciPost Phys. **16**, 118 (2024),

which form, respectively, chapters 3, 4, and 2.

॥ श्रीगुरुभ्यो नमः ॥

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Abbreviations

CFT Conformal field theory.

CQED Compact quantum electrodynamics.

CS Chern-Simons.

DSL Dirac spin liquid.

FP Faddeev-Popov.

FQH Fractional quantum Hall.

IQH Integer quantum Hall.

PSG Projective symmetry group.

QCD Quantum chromodynamics.

QED Quantum electrodynamics.

QSL Quantum spin liquid.

RG Renormalization group.

VBS Valence bond solid.

WZW Wess-Zumino-Witten.

YM Yang-Mills.

ZM Zero mode.

Chapter 1

Introduction

This thesis is broadly on theories of phase transitions between so-called fractionalized phases and conventional symmetry-broken phases. To place such a study in context, we begin with a review of the Landau-Ginzburg-Wilson theory of phase transitions and discuss why this theory is not straightforwardly applicable to the transitions studied in this work.

1.1 Landau-Ginzburg-Wilson Theory

In 1937, L. D. Landau published two extraordinary papers [4] formulating a theory of continuous phase transitions, initiating a research program the results of which now form the basis of our understanding of much of condensed matter and quantum field theory. Landau noted that the majority of then known transitions, for example between solids of different crystal structures or between a ferromagnet and paramagnet, involved the "sudden disappearance or appearance of some elements of symmetry".[1] A quantitative description of this phenomenon is afforded by the notion of a Landau order parameter $\phi(x)$, which is nonzero on average in the symmetry-broken/ordered phase and zero in the symmetric/disordered phase. It is clear that ϕ must transform nontrivially under the symmetry in question. In the example of a uniaxial magnet, $\phi(x)$ is the local magnetic moment along the preferred axis, and a nonzero average $\langle\phi(x)\rangle = m$ results in a magnetized state that spontaneously

[1] *"The problem is not so much to see what nobody has yet seen, but to think what nobody has yet thought concerning that which everybody sees"*, Arthur Schopenhauer reflects in his *Parerga und Paralipomena* (1851).

breaks the $\mathbb{Z}_2$ symmetry $\phi\to-\phi$ present in the disordered phase – characterized by random values of $\phi(x)$ across the system. An investigation of the phase diagram of a certain material thus begins with the identification of an appropriate order parameter. The next task is to obtain the free energy $F(T,m)$ from which thermodynamical quantities of interest can be calculated; for example, the specific heat or susceptibilities of various kinds and associated critical exponents that characterize their singular behavior at the transition.

For concreteness, let us briefly sketch this program for the Ising model of a uniaxial ferromagnet on a hypercubic lattice, described by the partition function

$$Z = \sum_{\{s_x=\pm 1\}} e^{\sum_{x,x'} J_{x,x'} s_x s_{x'}}. \tag{1.1}$$

The discrete spins s_x attain the binary values ± 1 on any lattice site x, and the function $J_{x,x'} = \beta \sum_{\hat{\mu}} \delta_{x,x'+\hat{\mu}}$ is assumed to couple nearest neighbors, with $\beta = T^{-1}$ as the inverse temperature and $\hat{\mu}$ being vectors from a lattice site to all its nearest neighbors. Straightforward expansions of the partition function in the low and high temperature limits show that there are two distinct phases in any dimension $D>1$. Naïvely, at low temperatures $T\to 0$, the energy is minimized by the symmetry-breaking ferromagnetic state $s_i=1$ on all sites, which forms the dominant contribution to the partition function. At high temperatures, the energy becomes unimportant and all configurations contribute more or less equally to the partition function suggesting a disordered phase. A more careful analysis accounts for entropic effects of excitations in the low temperature phase, but supports the conclusion that ordered and disordered phases exist in dimensions $D>1$.

Introducing a continuous field $\phi_x \in \mathbb{R}$, the partition function can be exactly rewritten as

$$\begin{aligned} Z &\propto \int D\phi\, e^{-\frac{1}{4}\sum_{x,x'} \phi_x J^{-1}_{x,x'} \phi_{x'}} \sum_{\{s_x\}} e^{\sum_x \phi_x s_x}, \\ &= \int D\phi\, e^{-\frac{1}{4}\sum_{x,x'} \phi_x J^{-1}_{x,x'} \phi_{x'} + \sum_x \ln 2\cosh\phi_x}. \end{aligned} \tag{1.2}$$

From the first line, we observe that the interaction between spins is encoded in the interaction of a spin with a local field ϕ_x. The interpretation of the latter as the local magnetization at x is enforced by the observation that the variational saddle point of ϕ_x is at $\langle s_x \rangle$. Therefore, ϕ_x is the Landau order parameter for the Ising model. Assuming ϕ to be a uniform constant that minimizes the action is precisely the content of mean-field theory. As the transition temperature is approached from above, we may assume that ϕ_x is infinitesimally small and Taylor expand the potential derived above. Moreover, as the ordered phase is characterized by a uniform magnetization, we may further posit that its variation from site to site on the lattice is small near the transition, allowing us to replace the lattice variable ϕ_x and its lattice derivatives with a smooth field $\phi(x)$. We thus arrive at the *Landau-Ginzburg action*[2] that governs the dynamics of the order parameter

$$S = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \sum_{n=2}^{\infty} \frac{g_{2n}}{(2n)!} \Lambda^{D-n(D-2)} \phi^{2n} \right] \tag{1.3}$$

where the "bare" coupling constants g_{2n} are $O(1)$ numbers that can be explicitly derived from eq. (1.2), and the factors of $\Lambda = a^{-1}$ are obtained on converting sums to integrals. The continuum fields have been defined as $\phi(x) \sim \Lambda^{d/2-1} \phi_x$. It is clear that our continuum theory can only be valid on length scales larger than Λ^{-1}, so the Fourier modes $\phi(k)$ have nonzero support only for $k \in (0, \Lambda)$.

The first task of identifying an order parameter and its dynamics now complete, the goal is to now perform the path integral over ϕ and thus compute the free energy $F = -T \ln Z$, which encapsulates the thermodynamics of the transition. Let us briefly discuss how this is traditionally done in the context of the Ising transition. Unfortunately, in the vast majority of cases, the calculation of the free energy can only be done perturbatively in the interactions g_{2n}, the matter being all the more complicated by the presence of an infinite number of them. This turns out to be a general feature; the Landau-Ginzburg action for any order parameter in the representation of a symmetry group will involve all possible

[2] The lattice propagator has been replaced with the continuum propagator, which is an excellent approximation even for $|x - x'| \sim a$ the lattice spacing. The lattice propagator has the sole effect of regulating the divergence in the coincidence limit $x = x'$, which can equivalently be done with a cutoff $\Lambda \sim a^{-1}$ on momenta. Strictly speaking, one must perform a gradient expansion of the lattice propagator, which leads to derivative interactions of the form $\phi \nabla^{2n} \phi$. Such interactions are almost always irrelevant in a sense to be described below. A general principle in writing the Landau-Ginzburg action is to allow all possible interactions allowed by symmetry with a minimal number of derivatives.

terms allowed by symmetry. Still, we may hope that the leading perturbative corrections near the transition (where ϕ is small) will come from monomials of small degree, for example the ϕ^4 term in the potential.[3] For instance, we may perturbatively calculate the inverse of the susceptibility

$$\begin{aligned}\chi^{-1}(k) &= k^2 + [g_2\Lambda^2 - \Sigma(0)] - [\Sigma(k) - \Sigma(0)], \\ &\equiv k^2 + \mu - [\Sigma(k) - \Sigma(0)], \end{aligned} \tag{1.4}$$

where $\Sigma(k)$ is the self-energy and μ the renormalized mass in the language of field theory. The transition is defined by the divergence of the *exact* static susceptibility $\chi(k=0)$, that is $\mu = 0$. A direct application of diagrammatic perturbation theory reveals that corrections from interactions to the susceptibility are ordered in powers of $g_4(\mu\Lambda^{-2})^{(D-4)/2}$. Evidently for $D > 4$, the corrections are smaller than μ in the critical region $\mu \to 0$, validating perturbation theory. However, for the interesting cases of dimensions $D < 4$, the perturbative corrections overwhelm the mean-field result and it becomes necessary to account for "all orders", quashing any hope of a direct perturbative analysis of the thermodynamics near the transition.[4]

The culprit behind the failure of direct perturbation theory in the critical region turns out to be the low momentum fluctuations of the order parameter ϕ, which lead to infrared divergences in various Feynman diagrams. The renormalization group (RG) was invented in part to address this problem. While there were many contributors to the formalism, the deepest insights inarguably came from K. G. Wilson [5, 6]. In its full generality, the RG is a set of metaphysical ideas that have to be adapted to the problem in question. We shall limit ourselves here to a brief qualitative discussion of its application in the theory of the Ising transition being discussed. Since the invalidity of direct perturbation theory is caused by $k \sim 0$ Fourier modes of the order parameter, the idea is to leave these alone

[3] This hope can be bolstered by an analysis of the superficial degree of divergence of various Feynman diagrams.

[4] In the 'upper critical dimension' $D = 4$, the perturbative corrections are logarithmic of the form $g_4 \ln\left(\Lambda^2/\mu\right) + O(g_4^2)$,

and first account for the effects of large momentum modes. This is done by means of a decomposition

$$\begin{aligned}\phi(x) &= \int_0^{s\Lambda} \frac{d^D k}{(2\pi)^D}\, e^{ikx}\phi(k) + \int_{s\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D}\, e^{ikx}\phi(k), \\ &\equiv \phi_<(x) + \phi_>(x),\end{aligned} \tag{1.5}$$

where $\phi_<$ and $\phi_>$ are called *slow* and *fast modes*, respectively. Inserting this into the action (1.3), the path integral over the fast modes $\phi_>(x)$ can be computed in perturbation theory. For inappropriate but historical reasons, this process of integrating out fast modes is called a renormalization group transformation. All perturbative corrections coming from the fast modes turn out to be finite, as they involve momentum integrals over a shell $(s\Lambda, \Lambda)$ that excludes a neighborhood of $k=0$. The result is a new effective action for the slow modes that is exactly of the same form as the original action, but with renormalized couplings $g_{2n}(s\Lambda)$. By making the momentum shell $(s\Lambda, \Lambda)$ infinitesimally thin, a set of differential "flow equations"

$$\beta_{2n} = \frac{\mathrm{d}g_{2n}}{\mathrm{d}\ln s}, \tag{1.6}$$

can be perturbatively derived to describe the "RG flow" of the couplings as modes in momentum shells $(s\Lambda, \Lambda)$ are iteratively integrated out. This suggests a hope that may salvage our failed perturbative endeavor earlier – that is $\beta_{2n} > 0$ so that the couplings $g_{2n}(s\Lambda)$ decrease as s is decreased, so that the effective interaction that governs the $k \sim 0$ modes is weak enough to validate perturbation theory in the renormalized couplings in the critical region. This turns out to not be true, but the line of thought nevertheless leads to great progress and new ideas that form the foundations of modern field theory. It will also give us the occasion to introduce language that is used throughout this thesis.

For the Ising transition, the first few terms in the first few beta functions are

$$\begin{aligned}\beta_2 &\approx -2g_2 - \frac{K_D g_4}{1+g_2},\\ \beta_4 &\approx (D-4)g_4 + \frac{3K_D g_4^2}{(1+g_2)^2} - \frac{K_D g_6}{1+g_2},\\ \beta_6 &\approx (2D-6)g_6 + \frac{15K_D g_4 g_6}{(1+g_2)^2},\end{aligned} \tag{1.7}$$

where K_D is the surface area of a unit sphere in D dimensions. Trivially, we first note that turning off all couplings $g_{2n} = \mu = 0$ implies that all beta functions are zero. This defines a *fixed point* of the RG flow, called the *Gaussian* or *free fixed point.* In mean-field theory, this is the choice of couplings that corresponds to the critical point $g_2 \propto (T - T_c) = 0$. Invariance of the couplings in a theory under RG flow implies *scale invariance*, for small and large momentum modes all interact with the same numerical value of the couplings.[5] As implied by $\beta_2 < 0$, a nonzero g_2 is amplified under RG flow in any dimension D, and takes us away from the Gaussian fixed point. Depending on the initial sign of g_2, we end up in the paramagnetic or ferromagnetic phases of the Ising model. This means the Gaussian fixed point is *unstable* to the mass term $g_2\phi^2$, which is said to be a *relevant* operator at this fixed point. On the other hand, all the interaction couplings g_{4n} flow to zero in $D \geq 4$, so that the $k \sim 0$ modes of the order parameter roughly do not interact in those dimensions. This means the operators $g_{2n}\phi^{2n}$ are *irrelevant* at the Gaussian fixed point. This causes critical exponents to simply attain their mean-field values as obtained from the free part of the Landau-Ginzburg action which defines the Gaussian fixed point.

Unfortunately, the flow equations in $D < 4$ instead show that small momentum modes do interact strongly due to the relevancy of the coupling g_4 at the Gaussian fixed point. Since $g_4(s\Lambda)$ grows as s is lowered, our earlier hope of formulating a perturbation series in this renormalized coupling seems to have been crushed. A simple but elegant solution to this problem was provided by K. G. Wilson and M. E. Fisher in the charmingly titled paper [7], "Critical Exponents in 3.99 Dimensions", by formulating a perturbative expansion in

[5]For many fixed points, including the Wilson-Fisher and Gaussian, scale symmetry is actually extended to conformal symmetry.

the quantity $\epsilon = 4 - D$ in addition to the interaction g_4. The benefit of treating dimension as a continuous variable is that in a neighborhood of $\epsilon = 0$, the interactions are relevant but still weak. In this limit, the flow equations can be explicitly solved to find that the bare couplings at the transition flow to a new fixed point

$$(g_2^*, g_4^*, g_6^*, ...) \sim (-\epsilon, \epsilon, 0, ...) + O(\epsilon^2). \tag{1.8}$$

This is the celebrated *Wilson-Fisher fixed point* that governs the Ising transition in $4-\epsilon$ dimensions.[6] In the limit $\epsilon \to 0$, it fuses with the Gaussian fixed point. Treating ϵ as small, a perturbation series can be formulated in the couplings g_{2n}^* to redo the previous calculation of the susceptibility [8]. Perhaps surprisingly, the critical exponents calculated as a series in ϵ turn out to be fairly accurate even if we set $\epsilon = 1$ to obtain the integer dimension $D = 3$. Of course, the existence of the fixed point as we move away from $D = 4$ can only be perturbatively established in this scheme. For interacting fixed points such as the Wilson-Fisher that are a finite 'distance' away from the free fixed point, the strategy is always to introduce a parameter into the theory that brings it close to the free fixed point, making it perturbatively accessible. One is the ϵ-expansion discussed above. The other is a $1/N$-expansion. The basic idea of the latter, in the context of the Ising transition, is to consider generally a vector order parameter ϕ_α, where $\alpha = 1, ..., N$. In the limit $N \to \infty$ and *fixed dimension*, one can show that the theory is again weakly coupled and organize a perturbative expansion in powers of $1/N$, hoping that accurate results for the Ising transition will be obtained for $N = 1$. More detail can be found in the beautiful monograph by A. M. Polyakov [9].

This concludes our brief tour of the theory of phase transitions in conventional phases described by a local Landau order parameter. In this thesis, we will be concerned with phase and transitions where this formalism is not at all straightforward to implement. To

[6] By linearizing the flow equations (1.7) at this point, one observes that there is only one relevant operator at the Wilson-Fisher fixed point that is a linear combination of the mass $(g_2 - g_2^*)\phi^2$ and interaction $(g_4 - g_4^*)\phi^4$ operators. This combination tunes us away from criticality into the ferromagnetic or paramagnetic phases.

segue into this next topic, we end with a few additional observations from the discussion above. The Gaussian and Wilson-Fisher fixed points discussed above are both unstable, for there exist relevant operators that amplify under RG flow and take us away from the fixed point. Such unstable fixed points generically describe phase transitions. The relevant operators have the simple interpretation of various parameters (temperature, pressure, external fields etc.) that one can tune to pass into proximate phases. The (ir)relevancy of an operator at a fixed point can be deduced from its *scaling dimension* at that point, which characterizes how operators respond to the scale transformations that form an emergent symmetry at fixed points. Generally given a basis set of scaling operators $\mathcal{O}_i$ that transform homogeneously under scaling,

$$x \to \lambda x, \qquad \mathcal{O}_i(x) \to \mathcal{O}_i(\lambda x) = \lambda^{-\Delta_i}\mathcal{O}_i(x), \tag{1.9}$$

where Δ_i is the scaling dimension of $\mathcal{O}_i$ at the fixed point. This is similar to how rotation invariance can be used to decompose a general operator into angular momentum components. The scaling dimension fixes the correlator of $\mathcal{O}_i$ at the fixed point to be

$$\langle \mathcal{O}_i(x)\mathcal{O}_i(0)\rangle \sim \frac{1}{|x|^{2\Delta_i}}. \tag{1.10}$$

Perturbing a fixed point action S_* by a local operator $\mathcal{O}_i(x)$, and scaling coordinates as above,

$$S_* + \int d^D x\, \mathcal{O}_i(x) \to S_* + \int d^D x\, \lambda^{D-\Delta_i}\mathcal{O}_i(x). \tag{1.11}$$

If $\Delta_i > D$, then evidently $\mathcal{O}_i(x)$ is irrelevant at the fixed point, being driven to zero under RG flow. On the other hand, $\Delta_i < D$ implies the operator is relevant at the fixed point, and forms a perturbation that grows in importance at low momenta. A fixed point with no relevant operators is said to be *stable*. These correspond to *critical phases of matter*.[7] The Dirac spin liquid studied in chapter 2 is purported to be such a phase.

[7] Strictly speaking, fixed points can have either zero or infinite renormalized correlation length ξ, the characteristic length scale of fluctuations of the order parameter. We neglect the $\xi = 0$ case, for they imply only on-site or "ultralocal" correlations; the connected correlation function vanishes. This does not mean such phases are trivial, for they could be described by a nontrivial topological quantum field theory. The chiral spin liquid studied in chapter 4 is one example of such a phase.

1.2 Fractionalization and parton gauge theories

The early 1970s marked the dawn of the era of topology in condensed matter physics, with the discovery of phases that could not be distinguished on the basis of broken symmetry. The historical example is the superfluid transition in thin helium films. The Landau theory, by means of the Mermin-Wagner theorem[8], predicts that the superfluid order parameter cannot have a nonzero average in two dimensional systems. It was realized by V. L. Berezinskii [10, 11], and independently by J. M. Kosterlitz, and D. J. Thouless [12] that there nonetheless exists a transition in such films driven by proliferation of vortices, which arise as topological defects. The two phases on either side of this BKT transition can be distinguished by the behavior of the correlations $\langle\varphi(x)\varphi^*(0)\rangle$ of the superfluid order parameter $\varphi\in\mathbb{C}$, although $\langle\varphi\rangle$ itself is zero in both phases so the epithet 'order parameter' is not to be taken seriously in the Landau sense. In the low temperature phase, the correlation decays as a power law $|x|^{-\eta}$ with a temperature dependent exponent $\eta(T)$. In the high temperature phase, there is exponential decay similar to other disordered phases in the Landau framework. However, the formalism used to study the transition is not different from the Landau-Ginzburg-Wilson theory discussed above. There still exists a Landau-Ginzburg action for the order parameter φ (which never condenses in either phase). Upon identifying vortex defects in the phase of φ as playing an important role, one can rewrite the Landau-Ginzburg action in terms of 'vortex fields' (called a duality in field theory) at which point the theory outlined in the previous section can be straightforwardly applied.

It is with the experimental discovery of the integer and fractional quantum Hall effects in the 1980s [13, 14] that our ability to devise such descriptions began to be truly tested. Of these, the former belong to a now wide class of phases that can mostly be understood in the framework of electronic band theory, with interactions playing little role.[9] The frac-

[8]A no-go theorem on the possibility of spontaneous symmetry breaking in low dimensions. This was first demonstrated for crystal structures by L. D. Landau, building on an argument of R. E. Peierls, already in his seminal paper on phase transitions [4]. By the 'Arnold principle', it is therefore attributed now to Hohenberg, Mermin, Wagner, Coleman, and Berezinskii.

[9]There is an ongoing attempt to bring these into an 'extended' Landau paradigm by describing topo-

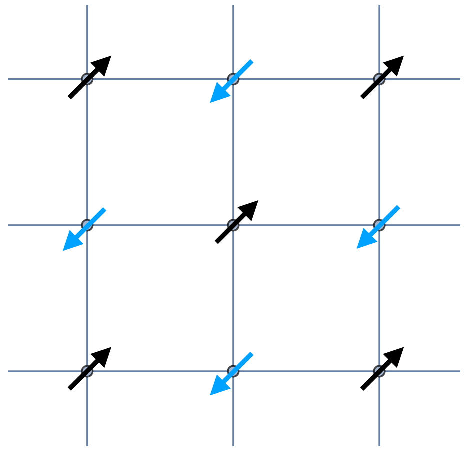

Figure 1.1: Néel state on a square lattice.

tional quantum Hall effect belongs to a class of phases that cannot be understood without interactions, involving the phenomenon of *fractionalization* – the low energy excitations of such phases are described by quantum numbers that are fractions of those of the elementary constituents (lattice bosons or fermions). The first example of such a phase, actually predating the discovery of quantum Hall effects, is the "resonating valence bond" (RVB) state proposed by P. W. Anderson in 1973 as a possible ground state of an antiferromagnetic Mott insulator on the triangular lattice [16].[10] A 'metamodel' for antiferromagnets is the spin-1/2 Heisenberg model

$$H = \sum_{i<j} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j, \qquad [S_i^\alpha, S_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma} S_j^\gamma, \tag{1.12}$$

on a lattice of interest. If the spins are treated classically and $J_{ij} > 0$ couples only nearest neighbors, the obvious ground state for a square lattice is the Néel state depicted in figure 1.1. This is a Landau symmetry-breaking phase with the staggered magnetization $(-1)^{i_x+i_y}\boldsymbol{m}$ in some randomly chosen direction serving as a Landau order parameter. In the quantum case, this is no longer the true ground state as it is not eigenstate of H. How-

logical phases as states with spontaneously broken 'higher-form symmetries' [15].

[10]The original proposal was for the spin-1/2 Heisenberg model with only nearest neighbor antiferromagnetic interactions. This model is now known to exhibit an ordered state in which the moments are angled at 120° relative to each other. However, next nearest neighbor interactions can change this picture considerably.

ever, it turns out to remain an excellent approximation to the true ground state, which simply has the average local magnetization reduced due to quantum fluctuations [17].

However, it is clear that the Néel state ceases to be the ground state even classically if J_{ij} also couples next nearest neighbors with comparable strength, leading to *frustration* of magnetic order. It is in such a context that Anderson proposed the RVB state. Noting that the energy of the exchange interaction $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$ is minimized if the spins form a singlet $(|\uparrow\rangle_i \otimes |\downarrow\rangle_j - |\downarrow\rangle_i \otimes |\uparrow\rangle_j)$, the RVB state is a superposition of various such singlets. More precisely, it is a linear superposition of *dimer coverings* $\mathcal{D}$ of the lattice, with each dimer being a spin singlet:

$$|\mathrm{RVB}\rangle = \sum_{\mathcal{D}} d(\mathcal{D})\, |\mathcal{D}\rangle \, . \tag{1.13}$$

Two example terms in the sum are depicted in figure 1.2. The coefficients $d(\mathcal{D})$ can be further written as a product over dimers (ij) of functions d_{ij}, which serve as variational parameters. If these are functions only of the distance $|x_i - x_j|$, then it is clear that such a state preserves all lattice symmetries in addition to full spin rotation symmetry (no magnetization). However, the state as written also seems quite different to a trivially disordered state such as a paramagnetic state found at infinite temperature. The key difference turns out to be *long-range entanglement* of well-separated spins on the lattice; the RVB state cannot be smoothly transformed, by tuning $d_{ij}(\mathcal{D})$, to a product state like $\prod_i |S_i^z\rangle$. This is the defining feature of a *quantum spin liquid* (QSL). It is clear that no straightforward order parameter exists to describe such phases. We shall see later that the low energy excitations above such states carry fractional quantum numbers. Similar variational states, called Laughlin states, were proposed as descriptions of ground states of FQH systems [18].

Early studies of fractionalized phases relied on such explicit variational constructions. While these allow to build intuition and emphasizes the entanglement structure of such states, it is not obvious what a "parent Hamiltonian" could be or what the low energy excitations (quasiparticles) are. It is also not clear where such states fit within the broader

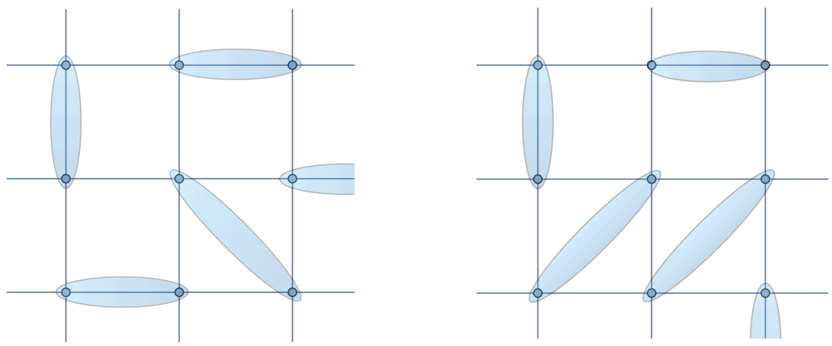

Figure 1.2: Two dimer coverings that form components of the RVB state on a square lattice.

phase diagram of antiferromagnetic materials, which certainly includes Landau-ordered states such as the Néel state in figure 1.1. An appropriate theory of such phases must then allow us to study transitions between a QSL and any proximate phases, similar to the Landau-Ginzburg-Wilson theory. *Parton methods* achieve precisely this feat, rephrasing the description of fractionalized phases in a language amenable to the application of Landau-Ginzburg-Wilson theory. The basic idea behind such methods is to represent a local physical degree of freedom, for example a lattice spin or electron, as a bound state of fictitious particles called partons. In fractionalized phases, it so happens that these partons become unbound at low energies, carrying quantum numbers that are a fraction of that of the physical composite. This is similar to the liberation of quarks in the Standard Model, with one key difference. Quarks are bound into hadrons at energies much smaller than $\Lambda_{\rm QCD} \sim 250$ MeV, and become asymptotically free at high energies. Precisely the reverse occurs in fractionalized phases of condensed matter – partons become unbound and acquire a physical reality only at low energies. This analogy hints that a parton description will be a gauge theory. This is reviewed in detail for the various parton constructions used in this thesis in the main chapters below, and only a brief sketch is provided here to convey the essential ideas.

In the context of spin models, representing the local spin-1/2 basis as $|S_i^z\rangle$, one possible parton construction begins by noting that

$$|S_i^z\rangle \mapsto c_{i\sigma}^\dagger |0\rangle \tag{1.14}$$

is an isomorphism between span$\{|{\uparrow_i}\rangle, |{\downarrow_i}\rangle\}$ and span$\{c_{i\uparrow}^\dagger|0\rangle, c_{i\downarrow}^\dagger|0\rangle\}$, the latter being a subspace of the four dimensional Fock space of the *fermionic partons* $c_{i\sigma}$. This physical subspace is specified by the single-occupancy constraint

$$\sum_\sigma c_{i\sigma}^\dagger c_{i\sigma} = 1. \tag{1.15}$$

The isomorphism (1.14) maps the spin-1/2 operator

$$\boldsymbol{S}_i \mapsto \frac{1}{2} c_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{i\beta}, \tag{1.16}$$

where repeated spin (Greek) indices are always summed over. Any spin-1/2 Hamiltonian $H\{\boldsymbol{S}\}$ can thus be written as a parton Hamiltonian $H\{c\}$ using such a representation of the spin operators; for instance the Heisenberg model is

$$H = \sum_{ij} J_{ij} (c_{i\mu}^\dagger \boldsymbol{\sigma}_{\mu\nu} c_{i\nu}) \cdot (c_{j\lambda}^\dagger \boldsymbol{\sigma}_{\lambda\rho} c_{j\rho}) \tag{1.17}$$

The virtue of this representation is that it allows for a broader range of mean-field theories than the original spin Hamiltonian. The latter generally leads only to magnetic states with order parameter $\langle \boldsymbol{S}_i \rangle$ when subjected to a mean-field decoupling. The crux of the problem is to then interpret the meaning of the parton mean-field states in terms of the original spins; this is one of the main problems addressed in this thesis. For instance, one possible mean-field decoupling of the quartic interaction in the above model leads to the mean-field model

$$H_{\text{mf}} \approx \sum_{ij} [t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.}] + \sum_i a_0(i) [c_{i\sigma}^\dagger c_{i\sigma} - 1], \qquad t_{ij} \equiv \left\langle c_{i\sigma}^\dagger c_{j\sigma} \right\rangle, \tag{1.18}$$

where $a_0(i)$ is a Lagrange multiplier field that imposes the half-filling constraint (1.15). The mean-field t_{ij} above is one choice that preserves spin-rotation invariance, and motivated by a desire to produce states akin to the RVB state. Of course, this choice is not unique by any means. There could also be pairing terms for the partons, which we have neglected to include, or a more general hopping term that mixes the parton spins. As such, there is clearly enormous freedom in the choice of mean-field ansatz in the parton framework, made possible by various decouplings and patterns of t_{ij}. The correct choice for a given spin model is undoubtedly decided by symmetry and energetic considerations, but it is typically impractical to search for the lowest energy state among all possible mean-field solutions. Instead, the utility of parton methods is mostly in the discovery of novel phases that can plausibly occur in a given system of (in this case) spin-1/2 moments. In particular, it allows us to combine fractionalization with band topology (as t_{ij} amounts to a choice of parton bandstructure) to discover an entire host of unconventional phases. For instance, by including pairing terms between the spinons, one can arrive at a $\mathbb{Z}_2$ gauge theory that describes the RVB state of Anderson [19, 20]. Instead, choosing $t_{ij} = t$ real on a honeycomb lattice leads to relativistic spinons and the so-called *Dirac spin liquid,* which forms the subject of chapter 2. Yet another is the *Kitaev spin liquid* discussed in chapter 4 that can be obtained from a Majorana representation of spin-1/2. For *each* of these spin liquids, there is an enormous experimental literature on candidate materials, synthetic designs, and possible means of detection. As the field is still rapidly evolving, we preclude here a discussion of the experimental status of the search for various QSLs and refer the reader to the review [21]. However, we stress that the existence of such phases in theory has been established beyond any shred of doubt, in large part due to exactly solvable models with spin liquid ground states [22–24].

Returning to the development of the parton method, to check if the mean-field ansatz (1.18) really does correspond to a plausible phase of spins, we must investigate its stability under fluctuations of the mean-field t_{ij}, of which there are two kinds – amplitude and phase. Amplitude fluctuations (of t_{ij}) are typically gapped [19, 20, 25] or correspond to higher derivative terms in a continuum limit, thus irrelevant compared to phase fluctuations.

Including the latter by replacing $t_{ij} \to t_{ij} e^{ia_{ij}}$, the parton model can be written as

$$H = \sum_{ij} [t_{ij} e^{ia_{ij}} c^\dagger_{i\sigma} c_{j\sigma} + \text{h.c.}] + \sum_i a_0(i) [c^\dagger_{i\sigma} c_{i\sigma} - 1] + H_g, \tag{1.19}$$

where H_g specifies the dynamics of a_{ij} that we shall shortly discuss. The matter part above describes a gauge theory as foretold earlier, being invariant under the $U(1)$ gauge transformations,

$$\begin{aligned} c_i &\to e^{i\lambda_i} c_i, \\ a_{ij} &\to a_{ij} + (\lambda_i - \lambda_j). \end{aligned} \tag{1.20}$$

Of course, this is a consequence of the ansatz chosen. If pairing terms are included in Eqs. (1.18)-(1.19), then there may only be a $\mathbb{Z}_2$ gauge invariance. This gauge invariance actually *emerges* from the parton decomposition (1.16) itself. Clearly, a $U(1)$ transformation of the partons as just defined leaves the local spin operator invariant, and thus any spin model will also be gauge invariant.[11]

The dynamics for a_{ij} specified by H_g in Eq. (1.19) must preserve gauge invariance, and all the symmetries of the mean-field ansatz specified by the matter part. This includes compactness $a_{ij} \sim a_{ij} + 2\pi$. These constraints uniquely determine the gauge dynamics to be that of c*ompact* $U(1)$ *gauge theory:*

$$H_g = \sum_\ell e_\ell^2 + K \sum_\Box (1 - \cos f_\Box), \qquad f_\Box = \sum_{\ell \in \partial \Box} a_\ell, \tag{1.21}$$

where e_ℓ and $f_\Box$ denote the electric field on edge ℓ and magnetic field on the face $\Box$. The latter is the lattice curl of a_ℓ, i.e. the sum of a_ℓ over the set of boundary edges $\partial\Box$ of the face $\Box$. As usual, the electric field is a time-derivative of a_ℓ and obeys the canonical commutation relation $[a_\ell, e_{\ell'}] = i\delta_{\ell\ell'}$. The presence of the electric field means the gauge constraint is modified from the hard relation (1.15) enforcing half-filling, to the Gauss law

$$(\operatorname{div} e)_i = \sum_\sigma c^\dagger_{i\sigma} c_{i\sigma} - 1. \tag{1.22}$$

[11] In fact, as discussed later in chapter 2, the largest group of transformations on the partons that leaves the spin invariant is actually $SU(2)$. However, this gauge group will be broken down to a $U(1)$ or $\mathbb{Z}_2$ subgroup by a specific choice of mean-field ansatz such as (1.19).

On short distances comparable to the lattice constant, the electric field must be zero and the hard half-filling constraint is recovered, so that only spins are visible. At long distances, the partons $c_{i\sigma}$ can transition from being purely formal degrees of freedom to physical quasiparticles. These renormalized low energy partons are called *spinons* in the context of spin liquids.

1.3 Polyakov confinement

Most of the parton constructions in this thesis involve compact $U(1)$ gauge theory directly or indirectly, coupled to partons with various bandstructures. It is therefore expedient to discuss at this point some general properties of the pure gauge theory, i.e. in the absence of any matter, described by the Hamiltonian and Gauss law

$$H_g = \frac{1}{2}\sum_{\ell} e_\ell^2 + K\sum_{\square}(1-\cos f_\square), \qquad (\mathrm{div}\, e)_i = 0. \tag{1.23}$$

Traditionally, $K \propto g^{-2}$ where g is the gauge coupling. Some aspects of this theory are discussed in chapter 3. In this section, we shall demonstrate that test charges in this theory are subject to a confining force that grows linearly with their separation, a result established in a historic paper by A. M. Polyakov [26].

In the limit $K\to 0$, the 'electric term' (kinetic part) in (1.23) dominates and the eigenstates are then those of e_ℓ, but subject to the Gauss constraint $(\mathrm{div} e)_i = 0$ on every site i. The compactness of a_ℓ and the canonical relation $[a_\ell, e_{\ell'}] = i\delta_{\ell\ell'}$ implies that the spectrum of the operator e_ℓ is integral. Therefore, the operator $(\mathrm{div} e)_i$ also has an integral spectrum implying the quantization of charge in integral units. It is then easy to see that the ground state has $e_\ell = 0$ on all links of the lattice, and that gauge invariant excited states are oriented loops of links carrying an integer valued electric flux. Explicit formulae for excited states and energies can be inductively found starting from the ground state. Let the latter be denoted $|0\rangle$. Excited states are obtained by increasing the electric field e_ℓ on the links of a set of closed, oriented loops $\{C\}$ by $\{n(C)\}$, where $n(C)\in\mathbb{Z}$. The commutation relation

$[a_\ell, e_{\ell'}] = i\delta_{\ell\ell'}$ allows us to construct an operator that does this. The excited states are then

$$|\{C_i, n_i\}\rangle = e^{in_1 \oint_{C_1} a_\ell} ... e^{in_N \oint_{C_N} a_\ell} |0\rangle , \qquad E(C_i, n_i) = \sum_i \frac{1}{2} [n_i]^2 P(C_i), \tag{1.24}$$

where $E(C_i, n_i)$ is the energy of the state and $P(C_i)$ is the perimeter of the loop C_i. Unlike standard $U(1)$ gauge theory with free gapless photons, the compact theory is gapped in the electric limit $K = 0$ due to the integral spectrum of the electric field.

Equation (1.24) provides the energy of excited states in the absence of gauge charges. Let us now place two test charges, ± 1 at sites i and j. The Gauss constraint is modified on those sites to $(\mathrm{div} e)_i = 1$ and $(\mathrm{div} e)_j = -1$. Then every gauge invariant state must contain an electric field line that begins and ends on i and j respectively. In addition to this field line, there can be closed loops at various places. The new ground state is the field line with the shortest length, that is

$$|0_{ij}\rangle = e^{i \int_i^j a_\ell} |0\rangle , \qquad E_{0,ij} = \frac{1}{2}|i - j|. \tag{1.25}$$

There is an extensive cost in energy to separate opposite test charges, which are thus *linearly confined.* The tension in the electric flux line confining the test charges is the energy per unit length, 1/2. This linear confinement of test charges is quite different to the standard Coulomb force in two dimensions, which logarithmically confines.

In the opposite limit $K \to \infty$, the magnetic term in Eq. (1.23) must be minimized. Since the Hamiltonian in this limit is a sum over plaquettes of commuting terms, we can restrict attention to a single plaquette. It is incorrect to conclude that there exist degenerate vacua $f_\square = 2\pi n_\square$, which we shall call *n-vacua.*[12] The true dynamical variable in the compact theory is $\exp(ia_\ell)$, which implies the total flux is defined only modulo 2π. A close analogy is offered by the problem of a quantum particle on a ring, immersed in a gravitational potential $(1 - \cos\theta)$. The position θ of the particle is defined only modulo 2π, so the labels $\theta_n = 2n\pi$ for the classical ground state are all identified as the same physical point. However, there is a finite tunneling amplitude (due to the kinetic term) for

[12]The full ground state is then a product state over plaquettes: $\otimes_\square |n_\square\rangle$, which has total flux $\sum_\square 2\pi n_\square$.

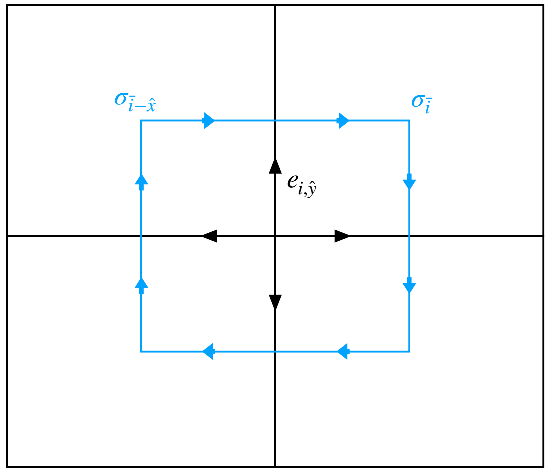

Figure 1.3: Construction of fields $\sigma_{\bar{i}}$ on vertices of the dual lattice (blue), defined by $\sigma_{\bar{i}} - \sigma_{\bar{i}-\hat{x}} = e_{i,\hat{y}}$. The same is done for other links in a consistent way (right-left on the dual link pierced by $e_{i,\hat{\mu}}$), as denoted by the relative orientation of black and blue arrows.

the particle to wind around the ring an integer number of times. As we shall see shortly, such tunneling events also occur in the compact $U(1)$ gauge theory between the different n-vacua.

We shall proceed with the analysis by first solving the Gauss constraint in order to determine the physical states (on the square lattice for simplicity). It is easy to do this in the continuum and then adapt the solution to the lattice. In the continuum, the Gauss law $\text{div}(e) = \partial_\mu e_\mu = 0$ can be solved by setting $e_\mu = \epsilon_{\mu\nu}\partial_\nu \sigma$ for some scalar field σ. To adapt this to the lattice, let us consider the *dual lattice* as shown in figure 1.3 and place a scalar field $\sigma_{\bar{i}}$ on its sites $\bar{i}$. Consistent with the relative orientations of blue and black arrows in figure 1.3, we define

$$\begin{aligned} 2\pi e_{i,\hat{x}} &= -\Delta_y \sigma(\bar{i} - \hat{y}), \\ 2\pi e_{i,\hat{y}} &= \Delta_x \sigma(\bar{i} - \hat{x}), \\ 2\pi e_{i,-\hat{x}} &= \Delta_y \sigma(\bar{i} - \hat{x} - \hat{y}), \\ 2\pi e_{i,-\hat{y}} &= -\Delta_x \sigma(\bar{i} - \hat{x} - \hat{y}), \end{aligned} \tag{1.26}$$

where Δ_μ is the lattice derivative along the direction $\hat{\mu}$. It can be checked that the Gauss constraint is automatically satisfied on expressing the electric field in terms of $\sigma_{\bar{i}}$ as above.

The scalar $\sigma_{\bar{i}}$ is called the *dual photon,* and it is clear that its spectrum consists of integral multiples of 2π. The terminology is because σ describes the single polarization state of the photon in two spatial dimensions; as expected, the momentum $k_\mu \sim \Delta_\mu \sigma$ of σ is transverse to both electric and magnetic (out of plane) fields. It is also clear from the definitions above that electric flux lines correspond to kinks/domain walls of σ. It is known that the energy of a domain wall scales with its length in two dimensions[13], which is the signature of confinement in the dual language, which we shall now demonstrate explicitly.

We have already established that the $K=0$ limit of the theory is in a confined phase. It remains to show that this is also true in the opposite limit $K \to \infty$. For this purpose, we rewrite the Hamiltonian (1.23) in terms of σ. To express the magnetic term in the new variable, note that (1.24) tells us that $\exp\big(in \oint_C a\big)$ increases the electric field on the links forming the oriented loop C by n. If C is chosen to be the anti-clockwise elementary plaquette centered on the dual site $\bar{i}$ (see figure 1.3), then this operator can be written in terms of $f_{\bar{i}}$ with action

$$
\begin{aligned}
e^{-inf_{\bar{i}}} e_{i,\hat{y}} e^{inf_{\bar{i}}} &= e_{i,\hat{y}} - n = \frac{1}{2\pi}(\sigma_{\bar{i}} - \sigma_{\bar{i}-\hat{x}}) - n, \\
e^{-inf_{\bar{i}}} e_{i,\hat{x}} e^{inf_{\bar{i}}} &= e_{i,\hat{x}} + n = \frac{1}{2\pi}(\sigma_{\bar{i}-\hat{y}} - \sigma_{\bar{i}}) + n.
\end{aligned} \tag{1.27}
$$

This result shows that

$$
e^{inf_{\bar{i}}} \sigma_{\bar{i}} e^{-inf_{\bar{i}}} = \sigma_{\bar{i}} + 2\pi n, \tag{1.28}
$$

which identifies the momentum canonically conjugate to σ as

$$
\Pi_{\bar{i}} \equiv \frac{f_{\bar{i}}}{2\pi}, \qquad \big[\sigma_{\bar{i}}, \Pi_{\bar{j}}\big] = i\delta_{\bar{i},\bar{j}}. \tag{1.29}
$$

In terms of these dual variables, the gauge theory Hamiltonian (1.23) is

$$
H = K \sum_{\bar{i}} (1 - \cos 2\pi \Pi_{\bar{i}}) + \frac{1}{2} \sum_{\bar{i},\mu} (\Delta_\mu \sigma_{\bar{i}})^2. \tag{1.30}
$$

[13] An instant way to see this is to consider the energy of a domain wall in the ferromagnetic state of the $2d$ Ising model $J\sum_{\langle ij\rangle} s_i s_j$.

Compactness $a_\ell \sim a_\ell + 2\pi$ translates to compactness $\Pi \sim \Pi+1$ as evident from (1.29), which again means that σ has a spectrum in $2\pi\mathbb{Z}$. The dual variables enable an easy analysis of the magnetic limit $K \to \infty$. We cannot proceed with a Taylor expansion of $\cos \Pi$, for this destroys the compactness $\Pi \sim \Pi+1$, or equivalently the integrality of the spectrum of σ. However, we can attempt to impose the latter energetically by adding a perturbation $\cos\sigma$. The result is the *sine-Gordon theory*

$$H \approx \frac{K}{2}\sum_{\bar{i}} \Pi_{\bar{i}}^2 + \frac{1}{2}\sum_{\bar{i},\mu}(\Delta_\mu \sigma_{\bar{i}})^2 - \lambda \sum_{\bar{i}} \cos\sigma_{\bar{i}}. \tag{1.31}$$

The operator $\exp(i\sigma)$ generates translations in $\Pi = f/2\pi$ by unity, so it inserts 2π flux on a plaquette. The operator

$$\mathcal{M}_n(\bar{i}) \equiv e^{in\sigma_{\bar{i}}}, \qquad \mathcal{M}_n^\dagger(\bar{i}) f_{\bar{i}} \mathcal{M}(\bar{i}) = f_{\bar{i}} + 2\pi n, \tag{1.32}$$

is thus called a *monopole operator* of charge n. Evidently, $\mathcal{M}_n$ connects the different n-vacua discussed earlier in this section. The coupling λ is then interpreted as the transition amplitude between $|n\rangle$ and $|n\pm 1\rangle$.[14] The importance of such tunneling events can be studied in the framework of the RG. If the coupling λ is relevant, then instantons are important and play an important role in determining the ground state by pinning $\sigma \in 2\pi\mathbb{Z}$. This also pins the electric field $e_{i\mu} \sim \epsilon_{\mu\nu}\Delta_\nu \sigma_{\bar{i}}$ to integral values, which leads to confinement just as in the electric limit $(K=0)$ due to an energy gap to create electric flux on a link. On the other hand, if λ is irrelevant, then σ can fluctuate as it pleases, monopoles do not form an important class of low energy excitations, and the electric flux on a link can be as small as it likes. This obviously implies *deconfinement*, for test charges can be far separated with little cost in energy. This RG analysis is easiest to carry out in the continuum theory with a Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2K}(\partial_\mu \sigma)^2 - \lambda \cos\sigma. \tag{1.33}$$

[14] In the framework of field theory, tunneling events between classical vacua correspond to *instantons,* which are finite-action saddle points of the Euclidean action.

This is a monopole perturbation of the Gaussian fixed point at which $[\sigma]=0$ and $[K]=-1$. In the absence of the monopole perturbation, there is a shift symmetry,

$$U(1)_{\text{top}}: \qquad \sigma \to \sigma + c, \tag{1.34}$$

corresponding to the conservation of magnetic flux. Monopole operators can be abstractly defined as operators charged under this symmetry, a fact obvious from our explicit construction of such operators above.

The (ir)relevancy of monopoles is determined by the scaling dimension of the monopole operator $\exp(i\sigma)$ at the Gaussian fixed point, which can be inferred from the long distance behavior of the correlator

$$\left\langle e^{i\sigma(x)} e^{-i\sigma(0)} \right\rangle_0 = e^{\left\langle \sigma(0)^2 \right\rangle_0 - \langle \sigma(x)\sigma(0) \rangle_0}. \tag{1.35}$$

The propagator is

$$\langle \sigma(x)\sigma(0) \rangle_0 = K \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{k^2} = \frac{K}{4\pi|x|}, \tag{1.36}$$

which shows that $\langle \sigma(0)^2 \rangle$ is UV divergent. Regulating with a finite lattice spacing a,

$$\left\langle e^{i\sigma(x)} e^{-i\sigma(0)} \right\rangle_0 = \exp\left(\frac{K}{4\pi a} - \frac{K}{4\pi|x|} \right). \tag{1.37}$$

The UV cutoff a can be absorbed into a definition of renormalized monopole operators

$$\mathcal{M}_R(x) = \sqrt{Z_{\mathcal{M}}}\, \mathcal{M}(x) = e^{K/8\pi a} e^{i\sigma(x)}, \tag{1.38}$$

from which it follows that

$$\langle \mathcal{M}_R(x) \mathcal{M}_R(0) \rangle_0 = e^{-K/4\pi|x|} \xrightarrow[x\to\infty]{} 1. \tag{1.39}$$

The scaling dimension of $\mathcal{M}_R(x)$ at the Gaussian point is therefore $0 < (D=3)$, implying that monopole operators are very relevant and pin σ to $2\pi\mathbb{Z}$ in the infrared, causing confinement.

These arguments indicate that compact $U(1)$ gauge theory always confines in three spacetime dimensions. This might seem sobering in that QSLs require a deconfinement of partons, the gauge charges in the theory. One obvious way to obtain QSLs is Higgs the $U(1)$ down to a $\mathbb{Z}_2$ gauge group, which does not feature monopoles. An alternative is to add gapless matter. This is discussed in chapter 2 in the context of the Dirac spin liquid, which is described by coupling the compact gauge theory above to $N_f = 4$ flavors of relativistic spinons (CQED_3). It has been shown that in the limit of large N_f, monopole operators[15] in CQED_3 have scaling dimensions that scale linearly with N_f, rendering them irrelevant for large N_f [27–29]. Since the theory confines for $N_f = 0$ (pure gauge theory), there must exist a critical N_c that separates confinement and deconfinement regimes. There is as yet no consensus on what this critical N_c is, and estimates range from $N_c \leq 1.5$ all the way to $N_c \sim 10$. A useful table of different N_c obtained using various methods is provided in Ref. [30]. In the following chapters, we shall see that monopole operators function in effect as Landau order parameters for conventional phases proximate to a fractionalized phase.

1.4 Structure of this thesis

This thesis presents three case studies of phase transitions out of fractionalized states. The structure is fully modular with few cross-references between chapters (at the cost of some redundancy), which can therefore be read in any order.

Chapter 2 uses the parton representation of spin-1/2 discussed earlier in the introduction to develop CQED_3 as a theory of the Dirac spin liquid in spin-1/2 systems. Monopole operators analogous to the those discussed for $U(1)$ gauge theory in Sec. 1.3 are explicitly constructed for CQED_3 by means of an instanton gas calculation. By studying the response of monopole operators to lattice symmetries, we argue that these function as Landau order parameters of conventional phases proximate to the Dirac spin liquid in the parton framework. The methods developed in this chapter allow the classification of all such proximate phases on bipartite lattices.

[15] Monopole operators in CQED_3 have a different structure to those of the pure gauge theory discussed in this section. In particular, as shown in chapter 2, the flux operator $\exp(i\sigma)$ gets dressed by instanton-bound fermion zero modes.

Chapter 3, chronologically the first, is in effect a detailed description of instanton-induced interactions between fermions and their symmetry-breaking effects. This is studied in a context where the fermions emerge from a parton gauge theory of a system of $2d$ hardcore bosons with $U(1)$ symmetry, which we show is spontaneously broken by instanton-induced interactions. This chapter fills in many of the fine details and subtleties underlying similar calculations appearing in other chapters, in addition to expounding on some intricate features of the vacuum structure of compact $U(1)$ gauge theory in three spacetime dimensions.

Chapter 4 presents the theory of a continuous transition from a chiral spin liquid to a phase with magnetic long range order in $2d$ Ising spin systems (i.e. with $\mathbb{Z}_2$ symmetry). This is achieved by first representing an Ising spin by N Majorana partons. By varying the total Chern number of the parton bandstructure, paramagnetic, long-range ordered, and chiral spin liquid phases are accessed. The critical theory that governs the transitions between these phases is constructed by utilizing recently discovered dualities between $(2+1)D$ quantum field theories, and turns out to be an $SO(N)$ gauge theory with massless Majorana fermions. $\mathbb{Z}_2$-charged monopoles in this theory are shown, when proliferated, to lead to confinement of partons and long-range ordering of Ising spins.

Chapter 2

Monopoles in Dirac spin liquids and their symmetries

2.1 Introduction

A quantum spin liquid is a quintessential example of a fractionalized phase in strongly correlated systems, whose low-energy description is best afforded by a deconfined gauge theory [24]. The parton construction is a systematic approach to derive such a description [19, 20, 31]. In such an approach, the lattice spins are rewritten as a composite of fermions or bosons (partons) glued together by an emergent gauge field. While these partons remain confined in conventional phases, a quantum spin liquid is characterized by their deconfinement at low energy.

Of the various spin liquids that have been proposed, the Dirac spin liquid (DSL) is of special renown for its candidate role as a “parent state” for several competing orders in two spatial dimensions (2d) on various lattice geometries [32–38]. As known and reviewed below, a low-energy description of the DSL state is afforded by compact quantum electrodynamics in three spacetime dimensions (CQED_3) with $N_f = 4$ flavors of massless Dirac fermions. This theory is strongly coupled in the infrared and is expected to flow, at least for sufficiently large N_f, to an interacting conformal field theory (CFT) with an emergent $SU(N_f)$ flavor symmetry, at which one observes power-law correlations in order parameters for several microscopic competing orders [32, 35, 39].

In this story, the first question to be asked concerns the *stability* of the DSL. Are there relevant operators in this CFT with the same microscopic lattice symmetries as the DSL? Fermion bilinears are of course relevant, but always violate microscopic symmetries [32, 35–37]. Of special concern are monopole operators in CQED_3 [9, 26, 27, 40], which have their origin in the compactness of the emergent gauge field that results from the parton construction on the lattice. At least for sufficiently large N_f, all monopole operators are irrelevant [27, 28, 39, 41] and CQED_3 remains in a deconfined phase, thus guaranteeing stability of the DSL. In contrast, the fate of the DSL for small N_f, including the value of interest $N_f = 4$, is murkier. The issue is the possible renewed relevancy of monopoles, in which case one then has to determine if there are monopoles with the same symmetries as the microscopic realization of the DSL on a given lattice. Correctly determining how monopole operators transform under lattice symmetries (i.e., their "quantum numbers") has been the subject of a longstanding theoretical program [35–37, 42–45]. To be specific, as monopole operators in CQED_3 are dressed by fermion zero modes [1, 27, 46], their transformation under lattice symmetries has two contributions: from the zero modes themselves, and from a $U(1)_{\mathrm{top}}$ phase shift of the bare monopole interpreted as a Berry phase obtained on dragging the monopole through a Dirac sea. (Here $U(1)_{\mathrm{top}}$ denotes the $U(1)$ topological symmetry of planar $U(1)$ gauge theories, whose global charge is the total magnetic flux.) The latter Berry phase has been difficult to compute, and a general framework to do so has only recently emerged in two works by Song *et al.* [36, 37]. Their conclusions indicate that, in realizations of the DSL on bipartite lattices, there always exist monopoles that transform trivially under all lattice symmetries of the state. The relevancy of such monopoles will then destabilize the DSL, and a transition into one of the proximate competing orders is then expected.

The second part of the DSL story is then determining the various competing orders for a given microscopic realization of a DSL [32, 35–37, 45, 47, 48]. The immediately available "order parameters" in the continuum field theory are the gauge-invariant fermion bilinears

$\bar{\psi}t^a\psi$, where ψ is a spinor in the fundamental representation of the $SU(4)$ flavor symmetry group and $t^a \in \mathfrak{su}(4)$. However, the spontaneous generation of an expectation value for such a fermion bilinear is not enough to drive the DSL into the corresponding ordered phase, for the fermions are still deconfined. To obtain phases with conventional long-range order, one further requires a mechanism by which the gauge charges confine. This is assumed to be due to monopole proliferation in the gauge theory, whose consideration we are again led to. The state-operator correspondence allows one to classify all monopole operators by their scaling dimension [27–29, 49–58]. Combined with the methods developed in Refs. [36, 37] to compute the quantum numbers of the monopoles, one can determine the correct monopoles to add to the Lagrangian. As argued in those references, the transition from the DSL into a proximate conventionally ordered phase then consists of a two-step process in which a fermion bilinear is first spontaneously generated, due for instance to a sufficiently strong symmetry-allowed four-fermion interaction [59], followed by the proliferation of the relevant monopoles to drive confinement. In certain cases, the fermion bilinear does not encode all the broken symmetries of a given microscopic order, and monopole proliferation is responsible for breaking the remaining symmetries.

To construct these monopole operators, Ref. [27] utilized the conformal invariance of massless CQED_3 at large N_f and defined monopole operators as states in the large-N_f CFT in a background flux on $S^2 \times \mathbb{R}$. In this chapter, we use the definition of monopole operators as instanton defects in the path integral [9, 26, 40, 60, 61] to explicitly reconstruct these directly on $\mathbb{R}^3$ as terms in an effective Lagrangian, in the specific context of a DSL. Moreover, our construction is not reliant on conformal symmetry. Indeed, we specifically focus on the dynamics of confinement once a fermion mass $\bar{\psi}t^a\psi$ is added to the DSL Lagrangian. We find that such an "adjoint mass" results in the existence of Euclidean zero modes (of the 3D massive Dirac operator) bound to instantons, distinct from the zero-*energy* modes that appear in the massless limit. Resumming the instanton gas results in the generation of an instanton-induced term in the effective Lagrangian

$$,$$

dubbed the 't Hooft vertex [1, 2, 62–65], which in this case turns out to be equivalent to the zero mode-dressed monopole operator found in the CFT approach. For ordered phases with (broken) symmetries fully captured by a fermion mass, we show that requiring the associated 't Hooft vertex to satisfy the same symmetries can be sufficient to compute monopole quantum numbers under microscopic symmetries. As observed in Refs. [36, 37], the DSL on square and honeycomb lattices possesses such proximate orders, in contrast to non-bipartite lattices.

The rest of the chapter is structured as follows. After a review of the parton construction of the DSL in Sec. 2.2, we organize the effects of monopoles in the path integral as an instanton-gas sum in Sec. 2.3.1, where it is also shown that such instanton-bound zero modes cause the path integral to vanish. The physical meaning of these Euclidean zero modes, and their relation to zero-energy modes found in previous constructions in the literature, are discussed in Sec. 2.3.2. Section 2.3.3 discusses the technical computation of the 't Hooft vertex by resumming the instanton gas. This 't Hooft vertex is rewritten by introducing "zero-mode operators" in Sec. 2.4, which reveals the relation to monopole operators constructed in the CFT approach. After discussing the continuum symmetries of the instanton-induced monopole operators, we comment in Sec. 2.5 on their quantum numbers under lattice symmetries for bipartite lattices, and finally conclude in Sec. 2.6.

2.2 Review of Dirac spin liquids

For concreteness, we consider the spin-1/2 antiferromagnetic Heisenberg model,

$$H = \sum_{ij} \mathcal{J}_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j, \tag{2.1}$$

on an arbitrary planar lattice, although one really has in mind an equivalence class of lattice models differing by symmetry-allowed terms. To obtain spin-liquid states, one typically begins with a parton representation [20],

$$\boldsymbol{S}_i = \frac{1}{2} \sum_{\alpha,\beta=\uparrow,\downarrow} c_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{i\beta}, \tag{2.2}$$

where $c_{i\alpha}$ are fermions of spin-1/2 and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector. Since the local spin-1/2 Hilbert space is only two-dimensional, the parton representation introduces a gauge redundancy, and one must project out unphysical states using the single-occupancy constraint:

$$\sum_{\alpha} c^{\dagger}_{i\alpha} c_{i\alpha} = 1. \tag{2.3}$$

The gauge group can be seen to be $SU(2)$, for a local $SU(2)$ rotation of the Nambu spinor $(c_{i\uparrow} \quad c^{\dagger}_{i\downarrow})$ leaves the spin operator (2.2) invariant.

The Heisenberg model then becomes a quartic interaction of fermions, which can be exactly decoupled inside a path integral using Hubbard-Stratonovich (HS) fields, as a prelude to mean-field theory. Motivated by a search for translationally and rotationally invariant spin liquids[1], the most general decoupling consistent with these requirements results in a Lagrangian (assuming sums over repeated spin indices):

$$\begin{aligned} L = \sum_i c^{\dagger}_{i\alpha} \partial_\tau c_{i\alpha} - \sum_{ij} \frac{\mathcal{J}_{ij}}{4} (c^{\dagger}_{i\alpha} z_{ij} c_{j\alpha} + \text{h.c.}) - \sum_{ij} \frac{\mathcal{J}_{ij}}{4} (\epsilon_{\alpha\beta} c^{\dagger}_{i\alpha} w_{ij} c^{\dagger}_{j\beta} + \text{h.c.}) \\ + \sum_{ij} \frac{\mathcal{J}_{ij}}{4} (|z_{ij}|^2 + |w_{ij}|^2) - i \sum_i a_0(i,\tau)(c^{\dagger}_{i\alpha} c_{i\alpha} - 1). \end{aligned} \tag{2.4}$$

Here $a_0(i,\tau)$, with τ being Euclidean time, is a Lagrange multiplier field that imposes the half-filling constraint on every site, and z_{ij} and w_{ij} are complex-valued HS link fields. The saddles of z_{ij} and w_{ij} are respectively at $c^{\dagger}_{i\alpha} c_{j\alpha}$ and $\epsilon_{\alpha\beta} c^{\dagger}_{i\alpha} c^{\dagger}_{j\beta}$, so a mean-field ansatz for z_{ij} and w_{ij} is equivalent to condensing those fermion bilinears. Introducing the Nambu variables,

$$\psi_i = \begin{pmatrix} c_{i\uparrow} \\ c^{\dagger}_{i\downarrow} \end{pmatrix}, \qquad T_{ij} = \begin{pmatrix} z_{ij} & w_{ij} \\ w^{\dagger}_{ij} & -z^{\dagger}_{ij} \end{pmatrix}, \tag{2.5}$$

and Pauli matrices τ^l, $l = 1, 2, 3$ that act in this Nambu space, and relabeling $a_0 \to a_0^3$, the Lagrangian can be rewritten as:

$$L = \sum_i \psi^{\dagger}_i (\partial_\tau - i a_0^{\ell} \tau^{\ell}) \psi_i - \sum_{ij} \frac{\mathcal{J}_{ij}}{4} (\psi^{\dagger}_i T_{ij} \psi_j + \text{h.c.}) + \sum_{ij} \frac{\mathcal{J}_{ij}}{8} \operatorname{tr} T^{\dagger}_{ij} T_{ij}, \tag{2.6}$$

[1]While breaking spin-rotation symmetry does not preclude a spin-liquid ground state [66], the Dirac spin liquid is a state that preserves this symmetry.

where the half-filling constraint is redundantly imposed using two more Lagrange multipliers, a_0^1 and a_0^2, to produce the temporal component $a_0 \equiv a_0^\ell \tau^\ell$ of an $\mathfrak{su}(2)$ gauge field. Indeed, the Lagrangian is now invariant under an $SU(2)$ gauge transformation:

$$\begin{aligned} a_0(i) &\to \Omega_i (a_0 + i\partial_\tau)\Omega_i^\dagger, \\ T_{ij} &\to \Omega_i T_{ij} \Omega_j^\dagger, \\ \psi_i &\to \Omega_i \psi_i. \end{aligned} \tag{2.7}$$

The Lagrangian (2.6) is an exact representation of the spin-1/2 Heisenberg model on an arbitrary lattice, and describes a lattice $SU(2)$ gauge theory at infinite gauge coupling (i.e., with no dynamics for the gauge fields), but with the group elements U_{ij} on every link being arbitrary complex matrices instead of $SU(2)$ matrices. However, any complex matrix admits a polar decomposition

$$T = \sqrt{T^\dagger T} U \equiv \rho U, \tag{2.8}$$

where U is unitary, and ρ is positive semi-definite and Hermitian.

At this point, one chooses a mean-field ansatz $\langle T_{jk} \rangle$ that renders the parton Hamiltonian quadratic. As T_{jk} is gauge-covariant, this ansatz generically violates gauge invariance, and the mean-field Hamiltonian H_{mf} will not commute with the constraint operators $\psi_i^\dagger \tau^\ell \psi_i$. However, some measure of gauge invariance is restored by considering fluctuations in T_{jk} about its mean-field value. Of these, there are "amplitude fluctuations" in ρ and "phase fluctuations" in U, as evident from (2.8). Since ρ only modulates the magnitude of the hopping, it is expected that the fluctuations of qualitative importance are those of the "phase matrix" U. Since we are interested in the infrared fate of the system, these gauge fluctuations will have dynamics due to a renormalization of the gauge coupling to finite values under RG flow of (2.6). This means the hard gauge constraint (2.3) will be softened in the infrared to

$$(\partial E^\ell)_i = \psi_i^\dagger \tau^\ell \psi_i, \tag{2.9}$$

where the left-hand side is t he lattice divergence of the electric field. It is understood that the fermions on the right-hand side are now renormalized fermions, and thus need not obey the hard constraint of the ultraviolet partons originally used in the parton construction. The mean-field Hamiltonian is then understood as written in terms of these renormalized partons, dubbed spinons.

Then writing $T_{ij} = \bar{T}_{ij} \exp(ia_{ij})$ to allow for phase fluctuations, it is intuitive from (2.6) that a generic mean-field value $\bar{T}$, which translates to condensing bilinears of type $c_{i\alpha}c^{\dagger}_{j\alpha}$ and $c_{i\uparrow}c_{j\downarrow}$, might Higgs the $\mathfrak{su}(2)$ gauge bosons down to some subgroup. A criterion given by Wen determines the infrared gauge group [19, 20, 67]. Considering all based loops on the lattice, a collinear flux (in some direction in $SU(2)$ space) of the mean-field $\bar{T}$ through all such loops results in a Higgsing of $SU(2) \to U(1)$, and generic non-collinear fluxes will break it down to $\mathbb{Z}_2$, completely gapping out all gauge bosons. In contrast, a trivial $SU(2)$ flux ($\propto \mathbb{I}$) ensures all the $\mathfrak{su}(2)$ gauge bosons remain massless. We shall be specifically interested in mean-field states that Higgs $SU(2) \to U(1)$ on various lattices. Examples include the staggered flux state on the square lattice [32], or the π flux state on the kagome lattice [33–35]. The spinons $(c_\uparrow, c_\downarrow)$ in these states have relativistic dispersions, with generically two Dirac nodes ($\alpha = \pm$) in the bandstructure. A linearized description at these nodes with low-energy fermions $\psi_{\alpha\sigma}$, that also accounts for U(1) gauge fluctuations with an emergent gauge field a_μ, is then given by the continuum (Euclidean) Lagrangian:

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ - i a\!\!\!/)\psi + \frac{1}{4e^2} f^2, \tag{2.10}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength tensor, ψ is a Dirac 2-spinor in the fundamental representation of $SU(4)$[2], the gamma matrices $(\gamma_1, \gamma_2, \gamma_3) = (\gamma_x, \gamma_y, \gamma_z)$ are chosen as the three Pauli matrices, and the Dirac adjoint is $\bar{\psi} = \psi^\dagger \gamma_3$. Since the gauge coupling e^2 has dimensions of inverse length, the Lagrangian is expected to be strongly coupled in the infrared, flowing to an interacting conformal fixed point which we shall call the DSL fixed point. In the $1/N_f$ expansion, one can show that this fixed point becomes nearly

[2] i.e. there are $N_f = 4$ flavors of relativistic 2-spinors forming a vector of $SU(4)$.

free, characterized by $e_*^2 \propto N_f^{-1}$, so that in the limit $N_f \to \infty$, gauge fluctuations are suppressed and spinons are free [39, 68, 69]. While it is unclear if this fixed point persists as N_f is lowered to the physically relevant value $N_f = 4$, conformal invariance at large N_f provides an accessible window to find relevant operators that can destabilize the DSL. Of central importance are monopole operators arising from the compactness of a, which when proliferated act to confine spinons into gauge-neutral spins [9, 26, 40], yielding conventional phases of the parent spin system.

These monopole operators can be defined at the large-N_f DSL fixed point via the state-operator correspondence in radial quantization, by considering free fermions on a sphere containing a monopole (plus fluctuations controlled by the $1/N_f$ expansion) [27]. The monopole with smallest scaling dimension corresponds to the ground state of the fermions. In a 2π flux background created by a minimal monopole, there is one zero-energy mode per flavor of relativistic fermion as required by the Atiyah–Singer index theorem. To obtain a gauge-invariant state respecting the constraint (2.3), half of the four zero-energy modes have to be filled. There are thus $\binom{4}{2} = 6$ monopole operators of minimal charge. If there is a symmetry-allowed relevant monopole, then the DSL is an unstable critical point separating ordered phases. If all monopoles are irrelevant, then there is no confinement and a stable DSL is obtained. However, there could be other interactions that drive symmetry-breaking by generating a fermion mass, allowing a previously disallowed monopole to then condense, causing confinement. We will now proceed to explicitly construct these monopole operators without relying on conformal invariance. As a byproduct of such a construction, we will obtain the exact monopole that proliferates for a given pattern of symmetry breaking described by the "adjoint masses":

$$M^a = m\bar{\psi}t^a\psi, \qquad t^a \in \{\sigma^i, \mu^i, \sigma^i\mu^j\}, \tag{2.11}$$

where σ_i, μ_i are Pauli matrices that act on spin and nodal indices respectively. The 15 mass terms considered above are the most general (Hermitian and Lorentz-invariant) fermion masses that do not radiatively generate a Chern-Simons term for the gauge field a_μ. Indeed, along with the identity matrix, they form a basis for the space of all 4×4 Hermitian matrices. The identity itself, that is the fermion mass $\bar{\psi}\psi$, generates a Chern-Simons term for a_μ and is not expected to lead to a symmetry-broken phase with confined gauge charges. In the CFT picture, an adjoint mass spoils conformal invariance and splits the degeneracy between the four zero-energy modes, causing one particular combination of the six monopole operators to lower its scaling dimension compared to the rest [48]. Our construction will directly yield this monopole, and by varying the adjoint mass yields all linearly independent monopole operators.

2.3 The 't Hooft vertex

The basic idea behind our construction is to (1) formulate the instanton problem in its original Euclidean path-integral language, rather than the canonical-quantization formalism of CFT, and (2) utilize semiclassical instanton calculus [61, 63–65] to resum a monopole-instanton gas in the presence of massive fermions [1]. We show that the existence of instanton-bound fermion zero modes (ZMs) of the *Euclidean* Dirac operator on $\mathbb{R}^3$ cause transition amplitudes to vanish unless fermion insertions can "soak up" these ZMs in the path-integral measure. This is in contrast to the case of massless Dirac fermions, for which no normalizable Euclidean ZMs exist [70]. These insertions will then "dress" the bare monopole operator that simply creates 2π flux in the gauge theory.

2.3.1 Euclidean fermion zero modes

To set up our semiclassical calculation, we decompose the emergent gauge field as:

$$a = \mathcal{A} + \delta a, \tag{2.12}$$

where $\mathcal{A}$ is a monopole-instanton solving the Euclidean equations of motion, and δa describes smooth fluctuations (photons) around the instanton solution. Temporarily neglecting the coupling of fermions to photons[3], the partition function can be written as a sum

[3]This is justified in a large-N_f approximation, but one can improve the calculation by considering fluctuations around the instanton just as in Ref. [28].

over an instanton gas [1]:

$$Z = \int Da\, e^{-\frac{e^2}{2}\int d^3x\,(\partial_\mu\sigma)^2} \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{k=1}^{N} \left(\int d^3 z_k \sum_{q_k\in\mathbb{Z}} e^{-q_k^2/e^2\ell} e^{iq_k\sigma(z_k)} \int D(\bar{\psi},\psi) e^{-S_f[\mathcal{A}(q_k)]} \right), \tag{2.13}$$

where σ is the dual photon [9], N is the number of monopoles in the gas, q_k their charges, z_k their locations (a collective coordinate), and $q_k^2/e^2\ell$ with short-distance cutoff ℓ (on the order of the lattice constant) the action cost for a charge-q_k monopole. Finally, $S_f[\mathcal{A}(q_k)]$ is the fermion action in a *single-instanton background* specified by (q_k, z_k). A dilute-gas approximation has been made in the partition above, which allows one to partition an N-instanton background as $\mathcal{A} = \sum_{k=1}^{N} \mathcal{A}_{(k)}$, describing N well-separated boxes containing a single instanton each. Assuming a dilute gas of monopoles allows one to bring the fermion path integral inside the product in (2.13), and consider fermions moving in a single-instanton background instead of that of a correlated instanton liquid. This is formally accomplished by decomposing ψ into fields localized in large boxes around each instanton [64], with zero overlap between boxes. This is justified in hindsight by the observation that fermion ZMs are exponentially localized on the instantons.

A brief remark on the validity of the dilute gas approximation is in order. In the pure gauge theory (without matter), monopoles interact via a Coulomb potential, which is known not to form bound states in 3D. The diluteness of this Coulomb gas can be argued using a standard Debye screening theory that assumes weak coupling $e^2\ell \ll 1$, as in Polyakov's original work [9, 40]. In the theory with fermion matter considered in this work, we will see that there exist pair exchanges of fermions between monopoles. Such pair exchanges are expected to correct the Coulomb potential between bare monopoles. However, since the fermions considered are massive, we suspect that such a correction is not enough to overwhelm the long-range Coulomb interaction and result in monopole bound states that would invalidate a dilute gas approximation.

We will only be concerned with monopole operators of lowest charge $q = \pm 1$ since these have the smallest scaling dimension, although the computation straightforwardly generalizes to higher charges in an obvious way. The fermion path integral in Eq. (2.13), which we separately write as:

$$Z_f[\mathcal{A}_q] = \int D(\bar{\psi},\psi) e^{-\int \bar{\psi}(\not{\partial} - i\not{\mathcal{A}}_q + mt^a)\psi}, \tag{2.14}$$

evaluates to zero for a gauge-field configuration $\mathcal{A}_q$ with nonzero monopole charge q. This is because the Euclidean Dirac operator:

$$\mathfrak{D}_q = \not{\partial} - i\mathcal{A}_q + mt^a, \tag{2.15}$$

has nontrivial ZMs in an instanton background. Unlike zero-*energy* modes of the Hamiltonian [71] that are typically bound to solitons, these zero modes of $\mathfrak{D}$ are bound to instantons. The relation between energy ZMs and these Euclidean ZMs will be further elucidated in Sec. 2.3.2.

Explicit solutions for these ZMs are obtained in Appendix 2.7. For a fixed mass $mt^a \in \mathfrak{su}(4)$ with $m>0$, the normalizable ZMs of $\mathfrak{D}_\pm$ in $q=\pm 1$ backgrounds are (respectively):

$$u_+^{(i)}(r,\theta,\varphi) = \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^{1/2}_{1,0,0}(\theta,\varphi) \left|i\right\rangle_a, \quad i=2,4; \tag{2.16}$$

$$u_-^{(i)}(r,\theta,\varphi) = \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^{1/2}_{-1,0,0}(\theta,\varphi) \left|i\right\rangle_a, \quad i=1,3, \tag{2.17}$$

and those of $\mathfrak{D}_\pm^\dagger$ are respectively:

$$v_+^{(i)}(r,\theta,\varphi) = \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^{1/2}_{1,0,0}(\theta,\varphi) \left|i\right\rangle_a, \quad i=1,3; \tag{2.18}$$

$$v_-^{(i)}(r,\theta,\varphi) = \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^{1/2}_{-1,0,0}(\theta,\varphi) \left|i\right\rangle_a, \quad i=2,4, \tag{2.19}$$

where the four eigenvectors of the $\mathfrak{su}(4)$ mass are defined by:

$$t^a \left|i\right\rangle_a = (-1)^i \left|i\right\rangle_a, \quad i = 1,2,3,4, \tag{2.20}$$

and $\mathcal{Y}^{j\pm 1/2}_{q,j,M}$ are monopole spinor harmonics as defined in Appendix 2.7. As discussed in the subsequent section, gauge invariance mandates that only two of these ZMs can be filled in any fixed instanton background. It will turn out to be sufficient to consider the ZMs

$u_+^{(i)}$ and $v_-^{(i)}$ of $\mathfrak{D}_+$ and $\mathfrak{D}_-^\dagger$ to obtain nearly all the results in this chapter. As shown in Sec. 2.3.3, these lead to spontaneous fermion pair creation (in an instanton background) and annihilation events (in an anti-instanton background).

The topological guarantee of these Euclidean ZMs is provided by their relation to *energy ZMs* of a *massless* Dirac Hamiltonian in a static $2\pi q$ flux background, which are protected by an Atiyah–Singer index theorem [46]. Indeed, the Euclidean ZMs above in the limit $m \to 0$ (ignoring the normalization) have precisely the form of the energy ZMs observed in radial quantization on $S^2 \times \mathbb{R}$, on recognizing the Weyl rescaling factor $r^{-1} = \exp(-\tau)$ [27]. A nonzero mass gaps out these energy ZMs, which reincarnate as normalizable (exponentially localized) ZMs of the Euclidean Dirac operator $\mathfrak{D}$. As we show below, the physical consequence of these Euclidean ZMs is that instanton events are correlated with fermion-number violating processes.

2.3.2 Euclidean fermion zero modes in a Hamiltonian view

We first present an intuitive argument for the heretofore claimed fermion-number violating processes caused by instantons. Instead of modeling the instanton as a point source of flux spatially localized in 2d, we can distribute the $2\pi q$ flux uniformly across the area A of a finite system. This is physically reasonable as a nonzero gauge coupling will supply monopoles with momentum, effectively delocalizing them. What is important is that the total flux through the system can only jump discretely through instanton events. Massive Dirac fermions under this uniform magnetic field $2\pi q/A$ are then housed in relativistic Landau levels (for each flavor),

$$\begin{aligned} E_{n\pm} &= \pm\sqrt{2\pi n|q|A^{-1} + m^2}, \qquad n \geq 1, \\ E_0 &= m\,\mathrm{sgn}(q), \end{aligned} \tag{2.21}$$

where E_0 is the "zero Landau level" obtained in the massless limit. The degeneracy of the levels is $|q|$, so that for $q = +1$ there is precisely one "zero mode" per flavor of Dirac fermion, in agreement with the Atiyah–Singer index theorem, giving a total of four modes for $N_f = 4$.

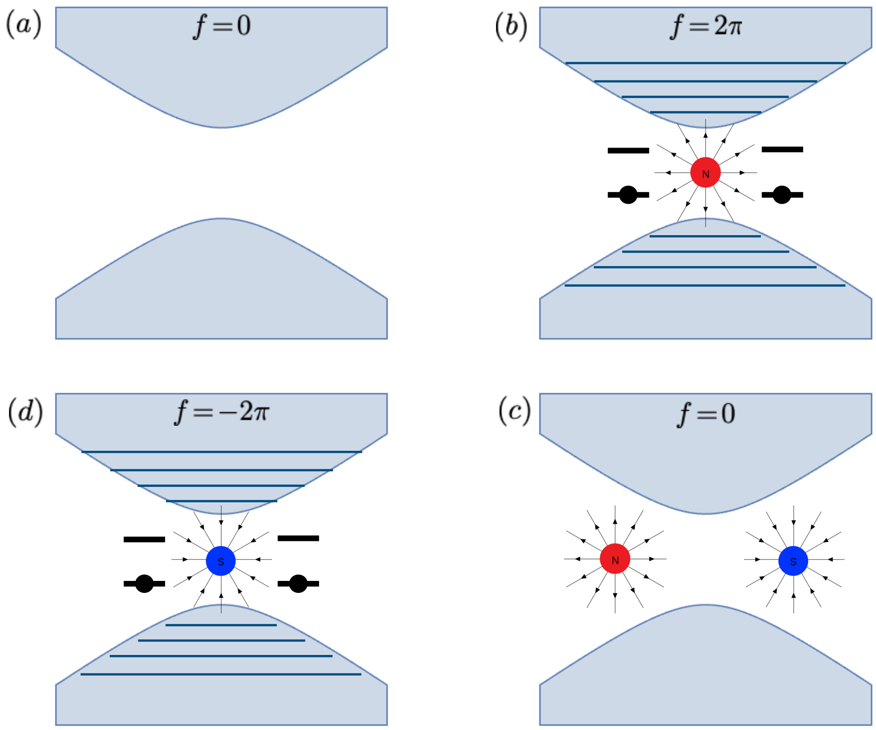

Figure 2.1: Massive Dirac fermions in various flux backgrounds at half-filling. Clockwise from top left: [(a) to (b)] Instanton $f = 0 \to 2\pi$ accompanied by spinon pair creation in two midgap modes. [(b) to (c)] Anti-instanton $f = 2\pi \to 0$ accompanied by pair annihilation of spinons in occupied midgap modes. [(c) to (d)] Anti-instanton $f = 0 \to -2\pi$ with pair creation of spinons in two midgap modes. [(d) to (a)] Instanton $f = -2\pi \to 0$ accompanied by pair annihilation of spinons in the occupied midgap modes.

In preparation for an interpretation of instanton events, let us imagine adiabatically dialing the flux from 0 to 2π. In the zero-flux limit, we simply have two bands formed by gapping a Dirac cone [Fig. 2.1(a)]. The single-occupancy constraint (2.3) ordained by the parton decomposition (2.2), which is equivalent to a Gauss' law constraint, mandates a half-filling of these bands (for each flavor of Dirac fermion). As a 2π flux is adiabatically turned on, the $E_{n\pm}$ levels evolve in perfect tandem out of the upper and lower bands, while the "zero mode" captures the *spectral asymmetry* of the Hamiltonian. Depending on the relative sign of m and q, it either descends from the upper band [$\text{sgn}(mq) > 0$] or ascends from the lower one [$\text{sgn}(mq) < 0$]. Since we are working with $\mathfrak{su}(4)$-valued masses mt^a that preserve time-reversal (TR) invariance, it follows that there are a total of 4 displaced energy

"ZMs", two with energy m and two with energy $-m$ [Fig. 2.1(b)]. Gauge invariance (i.e., the single-occupancy constraint) again requires us to fill two of these modes. The ground state is *uniquely* obtained by filling the two negative-energy modes. This is to be contrasted with the massless limit, in which all four modes are degenerate at zero energy, and there are six possible ways to fill two of them. Selecting a specific $\mathfrak{su}(4)$-mass mt^a gaps the four degenerate ZMs in a TR-invariant manner, selecting precisely two of them to fill.

Therefore, when the flux tunnels from $0 \to 2\pi$ by means of an instanton, two negative-energy modes suddenly appear in the spectrum. If these remain unfilled[4], the instanton would have caused an unphysical transition from a gauge invariant state to a non-invariant state violating the half-filling condition. The resolution is that an instanton event *must* be accompanied by fermion pair creation in the two new unfilled levels. Proceeding then in reverse from $2\pi \to 0$ flux sectors by means of an "anti-instanton", we immediately observe that anti-instantons should cause fermion pair annihilation as the two "ZMs" disappear into the lower bands [Fig. 2.1(c)].

These considerations lead to the conclusion that instantons cause fermion pair creation and annihilation. However, such processes must be reflected in an appropriate effective Lagrangian by means of "dressed" monopole operators of the form:

$$\mathcal{M}\bar{\psi}\Delta_+\bar{\psi}^\intercal + \mathcal{M}^\dagger\psi^\intercal\Delta_+^\dagger\psi, \tag{2.22}$$

where $\mathcal{M}$ is a "bare" monopole operator that creates 2π flux, $^\intercal$ denotes the transpose, and Δ_+ is a vertex factor valued in $\mathfrak{su}(4)$ that will select precisely two flavors from the four $\psi_{\alpha\sigma}$ to fill the two displaced energy ZMs just discussed. Determination of this vertex factor for a specific $\mathfrak{su}(4)$ mass is one of the central goals of this work, a task that shall be taken up in the next section.

Finally, the above considerations can be equally applied to flux tunneling from $0 \to (-2\pi)$, which leads to the conclusion that anti-instantons can also create fermions [Fig. 2.1(d)].

[4]It is assumed that we are at sufficiently low temperature that the leading order contribution is the filling of the two *negative* modes rather than the positive mid-gap modes that are also present. In any case, we shall see in the next section that the selection of two modes automatically falls out of the calculation.

This would yield a vertex contribution

$$\mathcal{M}^\dagger \bar{\psi}\Delta_- \bar{\psi}^\intercal + \mathcal{M}\psi^\intercal \Delta_-^\dagger \psi. \tag{2.23}$$

2.3.3 Resummation of the instanton gas

In this section, the intuitive picture sketched in the previous section will be formally laid out in the path-integral framework, and the monopole operators (2.22-2.23) completely determined by a resummation of the instanton gas in the partition function (2.13). To do so in the presence of ZMs of the Euclidean Dirac operator, we shall use a slight variant of the technique originally devised by 't Hooft in his resolution of the $U(1)$ problem in QCD$_4$ [63–65]. More technical details can be found in Ref. [1]; see also Ref. [62] for a symmetry-based argument. An analogous calculation for $SO(N)$ gauge theory with Majorana matter was done in Ref. [2], which studied confinement transitions out of a chiral spin liquid. Readers uninterested in the technical details of the calculation can safely proceed to the next section with just the final result [Eq. (2.36)] in hand.

As observed in Sec. 2.3.1, the fermion path integral vanishes in a nontrivial instanton background, implying that only the sector with zero instanton charge contributes to the partition function itself in Eq. (2.13). However, sectors with nonzero charge will contribute to correlation functions that can "soak up" the ZMs (to be explained below). From the discussion in the previous section, we expect these to be correlators of the form $\langle\psi\psi\rangle$. This is best seen with mode expansions of the spinons $(\psi, \bar{\psi})$ in eigenfunctions of the self-adjoint operators $(\mathfrak{D}_+^\dagger \mathfrak{D}_+, \mathfrak{D}_+ \mathfrak{D}_+^\dagger)$ for a $q=+1$ background:

$$\begin{aligned}\psi &= u_{2+}(x-z_+)\eta_2 + u_{4+}(x-z_+)\eta_4 + {\sum_i}' w_i(x-z_+)\xi_i,\\ \bar{\psi} &= {\sum_i}' \bar{w}_i(x-z_+)\bar{\xi}_i,\end{aligned} \tag{2.24}$$

where w_i are nonzero modes (indicated by the primed sums) of $\mathfrak{D}_+^\dagger \mathfrak{D}_+$, which occur in pairs with $\bar{w}_i$ of $\mathfrak{D}_+ \mathfrak{D}_+^\dagger$, and $\{\eta, \xi, \bar{\xi}\}$ are Grassmann numbers to ensure the correct Fermi statistics. The functions $u_{i+}(x - z_+)$ are the ZMs of $\mathfrak{D}_+^\dagger \mathfrak{D}_+$, localized on a charge $+1$

instanton at z_+, whose explicit expressions are given in Eq. (2.16). Only two ZMs have been included as mandated by the gauge-invariance arguments in Sec. 2.3.2, and the ZMs $v_+^{(i)}$ of $\mathfrak{D}_+^\dagger$ have not been "filled" by including them in the mode expansion of $\bar{\psi}$. Strictly speaking, we should sum over all possibilities by doing a separate calculation that only includes the two ZMs of $\mathfrak{D}_+^\dagger$ and not those of $\mathfrak{D}_+$. However, it will be easy to write down the result of such a calculation after our considerations below.

The functional measure can now be defined as:

$$D(\bar{\psi}, \psi) = d\eta_2 \, d\eta_4 \prod_i{}' \, d\bar{\xi}_i d\xi_i, \tag{2.25}$$

where the prime again denotes the exclusion of ZMs in the product. Since the ZMs $\{\eta_2, \eta_4\}$ do not appear in the Lagrangian $\bar{\psi}\mathfrak{D}_+\psi$, the Grassmann integrals over these cause the partition function to vanish. However, pair correlators of the form $\langle\psi\psi\rangle$ involve enough insertions to "soak up" the ZMs in the measure and produce a nonzero path integral. An explicit calculation, using the mode expansions (2.24), shows that:

$$\left\langle \psi^a(x)\psi^b(y)^\intercal \right\rangle_+ = -K_+ u_{2+}^{[a}(x - z_+) u_{4+}^{b]}(y - z_+)^\intercal, \tag{2.26}$$

where a, b are $SU(4)$ indices that have been here antisymmetrized (i.e., $v^{[a}w^{b]} \equiv v^a w^b - v^b w^a$), and K_+ is the fermion path integral over the nonzero modes $(\xi, \bar{\xi})$ in the instanton background. While this amplitude looks neither Lorentz nor gauge invariant at present, we reassure the reader that these issues will be addressed towards the end of the calculation.

Since the fermion ZMs are exponentially bound to the instanton with a width m^{-1}, this result shows that anomalous correlations also decay exponentially away from the instanton with a length scale m^{-1}. This also reinforces the conclusion reached intuitively in the previous section; the fermion vacuum in the presence of 2π flux has two additional fermions compared to the one with zero flux. A transition between the two states is possible only if these two extra fermions are annihilated, and this is precisely what the $\psi\psi$ insertion achieves.

We now ask for an effective Lagrangian that reproduces such correlation functions, which will amount to resumming or "integrating out" instantons. In the pure gauge theory, it is well known that the result of such a resummation is a sine-Gordon term $\propto \cos(\sigma)$ that gaps out the dual photon σ in the infrared [9, 26, 40]. With fermionic matter, a 2π flux is associated with two additional fermions, so we expect an effective Lagrangian to contain a term of the form $e^{i\sigma}\bar{\psi}\Delta_+\bar{\psi}^\intercal$. To determine Δ_+, let us perturb with a generic anti-symmetrized source[5] $\psi(x)^\intercal J(x,y)\psi(y)$, with suppressed $\mathfrak{su}(4)$ and Lorentz indices, and perturbatively expand to $\mathcal{O}(J)$:

$$\begin{aligned} Z_f[\mathcal{A}_+, J] &= \int D(\bar{\psi},\psi) e^{-\int \bar{\psi}(\not{D}_+ + mt^a)\psi - \int \psi^\intercal J \psi} \\ &= K_+ \int d^3x\, d^3y\, u_{2+}^\intercal(x - z_+) J(x,y) u_{4+}(y - z_+) + \mathcal{O}(J^2), \end{aligned} \tag{2.27}$$

where the second line is obtained by using the mode expansions (2.24), and K_+ is the fermion integral over non-ZMs as in Eq. (2.26). Our arguments in this and previous sections have indicated that such an amplitude can be reproduced by a path integral of the form [1]:

$$\begin{aligned} I_+[J] = \int D(\bar{\psi},\psi)\, e^{-\int \bar{\psi}(\not{\partial} + mt^a)\psi - \int \psi^\intercal J \psi} \\ \times \int d^3x'\, d^3y'\, C_+\, \bar{\psi}(x')\omega_+(x' - z_+)\zeta_+(y' - z_+)^\intercal \bar{\psi}(y')^\intercal, \end{aligned} \tag{2.28}$$

where the vertex Δ_+ has been written as a dyadic product $\omega_+\zeta_+^\intercal$ of vectors with possible spinor and $\mathfrak{su}(4)$ indices. We will determine C_+, ω_+, ζ_+ by demanding equality with Eq. (2.27) to $\mathcal{O}(J)$. Note that the fermions are no longer in a flux background in $I_+[J]$. Expanding the above integral to $\mathcal{O}(J)$ and Wick contracting gives:

$$I_+[J] = C_+ \int d^3(x, x', y, y') [G_f(x - x')\zeta_+]^\intercal J(x,y) [G_f(y - y')\omega_+], \tag{2.29}$$

where $G_f = \langle \psi\bar{\psi} \rangle_0$ is the free fermion propagator. Comparing with Eq. (2.27) and demand-

[5] One should strictly add $\psi^\intercal J\psi + \text{h.c.}$, but the conjugate term cannot soak up the zero modes in the path-integral measure in the $q = +1$ sector, so we drop it to reduce clutter.

ing equality gives the vertex factors:

$$
\begin{aligned}
C_+ &= K_+ = \det{}'\mathfrak{D}_+, \\
\zeta_+ &= G_f^{-1} u_{2+} \approx -2\sqrt{2}\pi \mathcal{Y}^{1/2}_{1,0,0}(\theta,\varphi)\, |2\rangle_a\,, \\
\omega_+ &= G_f^{-1} u_{4+} \approx -2\sqrt{2}\pi \mathcal{Y}^{1/2}_{1,0,0}(\theta,\varphi)\, |4\rangle_a\,,
\end{aligned}
\tag{2.30}
$$

where we have used explicit expressions for the free propagator (Appendix 2.8) and ZM solutions [Eq. (2.16)], and the approximation holds at distances $r \gg m^{-1}$ (the width of the instanton-bound ZM). One can now replace $Z_f[\mathcal{A}_+, J]$ with $I_+[J]$ in the instanton-gas sum appearing in the partition function (2.13).

To obtain a path integral $I_-[J] = Z_f[\mathcal{A}_-, J]$ in the anti-instanton sector, the calculation above should be repeated with ZMs of $\mathfrak{D}_-\mathfrak{D}_-^\dagger$ in the mode expansion of $\bar{\psi}$. One can write down the result based solely on reflection positivity (not reality) of the Euclidean action [72, 73], but since this is somewhat subtle as we shall see later, it is more prudent to just repeat the above calculation. The result is:

$$
I_-[J] = C_- \int d^3(x,x',y,y') [G_f^\dagger(x-x')\zeta_-]^\intercal J(x,y) [G_f^\dagger(y-y')\omega_-],
\tag{2.31}
$$

with

$$
\begin{aligned}
C_- &= K_- = \det{}'\mathfrak{D}_-^\dagger, \\
\zeta_- &= (-\partial\!\!\!/ + m t^a)^{-1} u_{2-} \approx -2\sqrt{2}\pi \mathcal{Y}^{1/2}_{-1,0,0}(\theta,\varphi)\, |2\rangle_a\,, \\
\omega_- &= (-\partial\!\!\!/ + m t^a)^{-1} u_{4-} \approx -2\sqrt{2}\pi \mathcal{Y}^{1/2}_{-1,0,0}(\theta,\varphi)\, |4\rangle_a\,.
\end{aligned}
\tag{2.32}
$$

Substituting in $I_\pm[J]$ for $Z_f[\mathcal{A}_\pm, J]$ in the partition function (2.13), we obtain:

$$
\begin{aligned}
Z[J] = \int D\sigma\, D(\bar{\psi},\psi)\, e^{-S_0 - \int(\psi^\intercal J \psi + \text{h.c.})} \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{k=1}^{N} \int d^3 z_k \int d^3x\, d^3y \\
\times \Big[-K_+ e^{i\sigma(z_k)} \bar{\psi}(x) \mathcal{Y}^{1/2}_{1,0,0}(x - z_k)\, |2\rangle\langle 4|\, \mathcal{Y}^{1/2}_{1,0,0}(y - z_k)^\intercal \bar{\psi}^\intercal(y) + \text{r.c.} \Big],
\end{aligned}
\tag{2.33}
$$

where "r.c." denotes the reflection conjugate[6], dimensionless constants have been lumped

[6]This is the analog of the hermitian conjugate in Euclidean signature, and is discussed in Sec. 2.4.1.

into K, and the free action S_0 is:

$$S_0 = \int d^3x \left[\frac{e^2}{2}(\partial_\mu\sigma)^2 + \bar{\psi}(\not{\partial} + mt^a)\psi\right]. \tag{2.34}$$

As remarked below the mode expansions in Eq. (2.24), one must also sum over a transition amplitude that involves the two ZMs of $\mathfrak{D}_+^\dagger$ but not those of $\mathfrak{D}_+$. The calculations leading to Eq. (2.33) clearly indicate that resumming instantons with these ZMs would lead to further insertions of the kind:

$$-K_- e^{-i\sigma(z_k)}\bar{\psi}(x)\mathcal{Y}^{1/2}_{-1,0,0}(x-z_k)\,|1\rangle\langle 3|\,\mathcal{Y}^{1/2}_{-1,0,0}(y-z_k)^\intercal\bar{\psi}^\intercal(y) + \text{r.c.}, \tag{2.35}$$

where the ZMs (2.17) and (2.18) have been used. As predicted at the end of Sec. 2.3.2, this vertex corresponds to spinon-pair creation by anti-instantons. Including these terms in Eq. (2.33), re-exponentiating the instanton-gas sum and then setting the source J to zero results in an instanton-induced contribution to the effective action: the *'t Hooft vertex*,

$$\begin{aligned} S^a_{\text{inst}} = K_+ &\int d^3z\, e^{i\sigma(z)}\left[\int d^3x\,\bar{\psi}(x)\mathcal{Y}^{1/2}_{1,0,0}(x-z)\right]|2\rangle\langle 4|\left[\int d^3y\,\mathcal{Y}^{1/2}_{1,0,0}(y-z)\bar{\psi}(y)\right]^\intercal \\ + K_- &\int d^3z\, e^{-i\sigma(z)}\left[\int d^3x\,\mathcal{Y}^{1/2}_{-1,0,0}(x-z)^\dagger\psi(x)\right]^\intercal|4\rangle\langle 2|\left[\int d^3y\,\mathcal{Y}^{1/2}_{-1,0,0}(y-z)^\dagger\psi(y)\right] \\ + K_- &\int d^3z\, e^{-i\sigma(z)}\left[\int d^3x\,\bar{\psi}(x)\mathcal{Y}^{1/2}_{-1,0,0}(x-z)\right]|1\rangle\langle 3|\left[\int d^3y\,\mathcal{Y}^{1/2}_{-1,0,0}(y-z)\bar{\psi}(y)\right]^\intercal \\ + K_+ &\int d^3z\, e^{i\sigma(z)}\left[\int d^3x\,\mathcal{Y}^{1/2}_{1,0,0}(x-z)^\dagger\psi(x)\right]^\intercal|3\rangle\langle 1|\left[\int d^3y\,\mathcal{Y}^{1/2}_{1,0,0}(y-z)^\dagger\psi(y)\right], \end{aligned} \tag{2.36}$$

where the superscript a in S^a_{inst} serves to remind that this effective interaction is associated with a given adjoint mass mt^a, whose eigenvectors $|i\rangle$ feature in the vertex. However, at this point, we note that the role of the fermion mass is solely to regulate $K_\pm = (\det'\mathfrak{D}_\pm^\dagger\mathfrak{D}_\pm)^{1/2}$ (our discussion below of reflection positivity will imply $K_+ = K_- \equiv K$), and the derived instanton-induced vertex is sensible in the massless limit, with the functional determinant being regulated in some other way. The adjoint mass then serves a role similar to a symmetry-breaking source for a specific ordered state in our calculation. When the massless limit, which does not "commute" with the resummation of the instanton gas, is taken at the end, the adjoint mass leaves behind in its wake a monopole which in turn will drive a confining transition into a proximate ordered state.

2.4 Monopole operators and their symmetries

We will now rewrite the 't Hooft vertex (2.36) using "zero-mode operators" in a form that makes explicit its relation to the CFT monopole operators constructed in Ref. [27]. To this end, we define the mode operators:

$$\begin{aligned} \bar{c}_{qjM}(z) &= \int d^3x\, \bar{\psi}(x)\mathcal{Y}^{j+1/2}_{qjM}(x-z), \\ c_{qjM}(z) &= \int d^3x\, \mathcal{Y}^{j+1/2}_{qjM}(x-z)^\dagger \psi(x), \\ c_{\pm 1,0,0} &\equiv d_\pm, \end{aligned} \tag{2.37}$$

where flavor indices have been suppressed. These can be thought of as a spacetime analog of a change of basis with coefficients $\langle jM|x\rangle$. In fact, this follows from a mode expansion of the fermion fields in monopole harmonics (see Eq. (7.3) of Ref. [29]), and thereby identifies c_{qjM} as the $\mathbb{R}^3$ analog of the "zero-mode operators" of Refs. [27, 29], there defined in radial quantization on $S^2\times\mathbb{R}$. The 't Hooft vertex (2.36) can be written in terms of these operators as:

$$\begin{aligned} S_{\text{inst}} = K\int d^3z\, \Big[& e^{i\sigma(z)}\bar{d}_+(z)\,|2\rangle\langle 4|\,\bar{d}_+(z)^\intercal + e^{-i\sigma(z)} d_-(z)^\intercal\,|4\rangle\langle 2|\,d_-(z) \\ & + e^{-i\sigma(z)}\bar{d}_-(z)\,|1\rangle\langle 3|\,\bar{d}_-(z)^\intercal + e^{i\sigma(z)} d_+(z)^\intercal\,|3\rangle\langle 1|\,d_+(z)\Big], \end{aligned} \tag{2.38}$$

which should be understood as the monopole operator spawned by a given adjoint mass mt^a with eigenvectors as in Eq. (2.20). This form makes it manifestly clear that the $\mathfrak{su}(4)$ part of the vertices, $|2\rangle\langle 4|$ and $|1\rangle\langle 3|$, must be antisymmetrized, in accordance with the observation in Refs. [27, 29] that monopole operators of minimal charge transform in the antisymmetric representation of the flavor group with $N_f/2$ indices. Before discussing flavor symmetry in greater detail, we first derive how the ZM operators (2.37) transform under spacetime symmetries, reflection positivity, and gauge transformations.

2.4.1 Spacetime symmetries, reflection positivity, and gauge invariance

Lorentz invariance

Since the ZM operator $\bar{d}_\pm$ defined in Eq. (2.37) creates a fermion in a $j = 0$ state, one might intuitively expect it to be Lorentz invariant. To see that this bears out, consider a Lorentz transformation Λ (rotation in Euclidean signature) with $U(\Lambda)$ the corresponding $SU(2)_{\text{rot}}$ action on spinors. Since the monopole spinor harmonics $\mathcal{Y}^{1/2}_{\pm1,0,0}$ have total angular momentum $j = 0$, they must satisfy the identity:[7]

$$\mathcal{Y}^{1/2}_{\pm1,0,0}(\Lambda x) = U(\Lambda)\mathcal{Y}^{1/2}_{\pm1,0,0}(x), \tag{2.39}$$

using which we see that:

$$\begin{aligned}\Lambda : \bar{d}_\pm(z) \to & \int d^3x\, \bar{\psi}(\Lambda^{-1}x)U^\dagger(\Lambda)\mathcal{Y}^{1/2}_{\pm1,0,0}(x - z),\\ &= \int d^3x\, \bar{\psi}(x)U^\dagger(\Lambda)\mathcal{Y}^{1/2}_{\pm1,0,0}\left(\Lambda(x - \Lambda^{-1}z)\right),\\ &= \int d^3x\, \bar{\psi}(x)\mathcal{Y}^{1/2}_{\pm1,0,0}(x - \Lambda^{-1}z),\\ &= \bar{d}_\pm(\Lambda^{-1}z),\end{aligned} \tag{2.40}$$

as expected of a Lorentz scalar.

$\mathcal{CRT}$

Here we consider how the ZM operators transform under the discrete symmetries of continuum Euclidean QED_3: reflection $\mathcal{R}$, charge conjugation $\mathcal{C}$, and time reversal $\mathcal{T}$. These are to be distinguished from the microscopic symmetries of the projective symmetry group (PSG) [20] for Dirac spin liquids on various lattices, to be discussed later in Sec. 2.5.

We define reflections $\mathcal{R}_\mu$ to be in the μ-coordinate. Let us consider reflections $\mathcal{R}_1$ in the x^1 coordinate for concreteness. On spinors, this acts as $\psi \to \gamma_1\psi$ and $\bar{\psi} \to \bar{\psi}(-\gamma_1)$

[7] To explicitly verify this with the expressions for the harmonics in Appendix 2.7 requires some care, for such expressions are derived by solving the Euclidean Dirac equation in a fixed gauge. The background gauge field $\mathcal{A}_\mu dx^\mu$ also transforms under rotations, and one must make a subsequent gauge transformation to bring back $\mathcal{Y}^L_{qjM}$ to its original form, as discussed in Ref. [74].

so that a flavor-singlet mass $\bar{\psi}\psi$ breaks reflection symmetry. Under $\mathcal{R}_1$, the unit vector $\hat{\varphi}=-(\sin\varphi)\hat{x}+(\cos\varphi)\hat{y}\to-\hat{\varphi}$ so that the monopole background in the Wu-Yang gauge [74] transforms as $\mathcal{A}_\mu=(0,0,\mathcal{A}_\varphi)\to(0,0,-\mathcal{A}_\varphi)$, which amounts to reversing the monopole charge $q\to-q$ as expected of reflections. Explicitly, the monopole spinor harmonics obey:

$$\mathcal{Y}^{1/2}_{\pm1,0,0}(\theta,\pi-\varphi)=(-\gamma_1)\mathcal{Y}^{1/2}_{\mp1,0,0}(\theta,\varphi), \tag{2.41}$$

under reflection $\mathcal{R}_1$ in the x^1-coordinate, so that

$$\begin{aligned}\mathcal{R}_1:\bar{d}_\pm(z)\to&\int d^3x\,\bar{\psi}(\mathcal{R}_1x)(-\gamma_1)\mathcal{Y}^{1/2}_{\pm1,0,0}(x-z),\\ &=\int d^3x\,\bar{\psi}(x)(-\gamma_1)\mathcal{Y}^{1/2}_{\pm1,0,0}\left(\mathcal{R}_1(x-\mathcal{R}_1z)\right),\\ &=\int d^3x\,\bar{\psi}(x)(-\gamma_1)^2\mathcal{Y}^{1/2}_{\mp1,0,0}(x-\mathcal{R}_1z),\\ &=\bar{d}_\mp(\mathcal{R}_1z).\end{aligned} \tag{2.42}$$

Charge conjugation is a unitary symmetry that acts to send:

$$\begin{aligned}\psi&\to-\gamma_1\psi^*=-\gamma_1\gamma_3\bar{\psi}^\intercal=(i\gamma_2)\bar{\psi}^\intercal,\\ \bar{\psi}&\to\psi^\intercal(-\gamma_1\gamma_3)=\psi^\intercal(i\gamma_2),\end{aligned} \tag{2.43}$$

which flips the sign of the Dirac current, but not the mass. Using:

$$(i\gamma_2)\mathcal{Y}^{1/2}_{\pm1,0,0}(\theta,\varphi)=\pm\mathcal{Y}^{1/2}_{\mp1,0,0}(\theta,\varphi)^*, \tag{2.44}$$

it is straightforward to verify that:

$$\mathcal{C}:\,\bar{d}_\pm(z)\to\pm d_\mp(z),\quad d_\pm(z)\to\mp\bar{d}_\mp(z). \tag{2.45}$$

In Euclidean signature, time reversal as a spacetime symmetry behaves identically to reflections [75], and is specifically unitary. It can be defined as:

$$\begin{aligned}\mathcal{T}:\,\psi(x)&\to\gamma_3\psi(\mathcal{R}_3x),\\ \bar{\psi}(x)&\to\bar{\psi}(\mathcal{R}_3x)\gamma_3,\end{aligned} \tag{2.46}$$

where $\mathcal{R}_3$ is a reflection in the Euclidean time (x^3) coordinate. One can alternatively define a modified time-reversal operation $\mathcal{CT}$ that also involves charge conjugation. On the ZM operators,

$$\mathcal{T}\colon \bar{d}_\pm(z) \to \bar{d}_\mp(\mathcal{R}_3 z), \tag{2.47}$$

using the fact that, under $\mathcal{R}_3$ reflections,

$$\mathcal{Y}^{1/2,(N)}_{\pm 1,0,0}(\pi-\theta,\varphi) = \gamma_3 \mathcal{Y}^{1/2,(S)}_{\mp 1,0,0}(\theta,\varphi), \tag{2.48}$$

as one can verify from explicit expressions for the monopole harmonics.

Reflection positivity

Reality of the real-time action (and thus unitarity of the corresponding quantum field theory) is guaranteed by *reflection positivity* $\vartheta(S)=S$ of the Euclidean action [72, 73]. This is a form of complex conjugation accompanied by a reversal of Euclidean time and an involution of the Grassmann algebra. With our choices of coordinates and Dirac matrices [1],

$$\begin{aligned}
\vartheta(\lambda\psi(x)) &\coloneqq \lambda^* \bar{\psi}(\mathcal{R}_3 x)\gamma_3, \\
\vartheta(\lambda\bar{\psi}(x)) &\coloneqq \lambda^* \gamma_3 \psi(\mathcal{R}_3 x), \quad \lambda\in\mathbb{C}, \\
\vartheta(a_\mu(x)dx^\mu) &\coloneqq a_\mu(\mathcal{R}_3 x) d(\mathcal{R}_3 x)^\mu,
\end{aligned} \tag{2.49}$$

and ϑ also reverses the order of Grassmann variables, e.g., $\vartheta(\psi_\alpha\psi_\beta\psi_\gamma) = \vartheta(\psi_\gamma)\vartheta(\psi_\beta)\vartheta(\psi_\alpha)$. For instance, one can check that the usual Berry phase term $\int \bar{\psi}\gamma_3\partial_\tau\psi$ is reflection positive using the definitions above. On the ZM operators $d_\pm$, we observe that

$$\begin{aligned}
\vartheta(\bar{d}_\pm(z)) &= \int d^3x\, \mathcal{Y}^{1/2}_{\mp 1,0,0}(x-\mathcal{R}_3 z)^\dagger \psi(x), \\
&= d_\mp(\mathcal{R}_3 z),
\end{aligned} \tag{2.50}$$

where we have used the fact that reflections invert the monopole charge [see Eq. (2.48)]. Together with the transformation $\vartheta(\sigma(z)) = \sigma(\mathcal{R}_3 z)$ for the dual photon [1], the transformation (2.50) ensures that the 't Hooft vertex (2.38) is reflection positive, thereby implying the reality of the real-time action or hermiticity of the Hamiltonian. It is important to note that reflection conjugation $\vartheta(d_q)$ replaces the notion of hermitian conjugation in Euclidean signature. In particular, we will *define*:

$$d_{\pm}^{\dagger} \coloneqq \vartheta(d_{\pm}) = \bar{d}_{\mp}. \tag{2.51}$$

Local gauge invariance

We will prove invariance of the ZM operators under gauge transformations with nonzero support on a sphere of fixed radius in $\mathbb{R}^3$. By radial quantization, this suffices to prove gauge invariance in general. The integrand of the expressions (2.37) should be viewed as sections of a $U(1)$ bundle over punctured $\mathbb{R}^3$. Charting a fixed sphere surrounding the monopole with "northern" (N) and "southern" (S) gauges à la Wu-Yang [74], it is clear that ψ should gauge transform identically to the spinor harmonics $\mathcal{Y}^{1/2}_{q00}$:

$$\begin{aligned}\psi^{(N)}(x) &= e^{-iq\varphi}\psi^{(S)}(x),\\ \mathcal{Y}^{1/2,(N)}_{q,0,0}(x) &= e^{-iq\varphi}\mathcal{Y}^{1/2,(S)}_{q,0,0}(x),\end{aligned} \tag{2.52}$$

for φ the azimuthal coordinate on $S^2 \subset \mathbb{R}^3$. Since $\int d^3x = \int_{N\cap S} d^3x$, because the N and S poles are a set of measure zero in the integral, the mode operators transform as:

$$\begin{aligned}\bar{d}_q &= \int_{N\cap S} d^3x\, \bar{\psi}^{(N)}(x)\mathcal{Y}^{1/2,(N)}_{q,0,0}(x-z)\\ &= \int_{N\cap S} d^3x\, \bar{\psi}^{(S)}(x)(e^{-iq\varphi})^* e^{-iq\varphi}\mathcal{Y}^{1/2,(S)}_{q,0,0}(x-z)\\ &= \int_{N\cap S} d^3x\, \bar{\psi}^{(S)}(x)\mathcal{Y}^{1/2,(S)}_{q,0,0}(x-z)\\ &= \bar{d}_q.\end{aligned} \tag{2.53}$$

A similar calculation shows invariance of d_q.

2.4.2 Flavor symmetry

The global form of the symmetry group of compact QED$_3$ with N_f flavors has been nicely summarized in Ref. [76]; let us review the necessary aspects here in our framework and notation, for general N_f. The Lagrangian $\bar{\psi} \not{D} \psi$ is invariant under $U(N_f)$ rotations of the fermions, but the center $U(1)$ is a gauge redundancy as it leaves spin operators invariant. Moreover, it acts trivially on gauge-invariant fermion bilinears such as $\bar{\psi} t^a \psi$. One might then conclude that the symmetry group of the DSL is $PU(N_f) \times U(1)_{\mathcal{M}} \cong PSU(N_f) \times U(1)_{\mathcal{M}}$, where $PSU(N_f) \cong SU(N_f)/\mathbb{Z}_{N_f}$ and $U(1)_{\mathcal{M}}$ is the topological "magnetic" symmetry corresponding to conservation of magnetic charge $\frac{1}{2\pi}\int f$ on any 2-cycle. However, monopole operators do not transform well as a representation of this group. The monopoles of minimal charge are precisely the 't Hooft vertices calculated previously, and are of the form (for general N_f):

$$e^{i\sigma} \Delta_{a_1 \cdots a_{N_f/2}} d^\dagger_{a_1} \cdots d^\dagger_{a_{N_f/2}}, \tag{2.54}$$

with Δ totally antisymmetric in its $N_f/2$ indices. Under the center of $SU(N_f)$ generated by $e^{2\pi i/N_f}$, the vertex transforms by an overall phase of $(e^{2\pi i/N_f})^{N_f/2} = -1$. This is identical to a π shift in $U(1)_{\mathcal{M}}$, which implies the symmetry group is really:

$$\frac{SU(N_f) \times U(1)_{\mathcal{M}}}{\mathbb{Z}_{N_f}}, \tag{2.55}$$

where the $\mathbb{Z}_{N_f}$ in the quotient is generated by:

$$(e^{2\pi i/N_f}, -1) \in SU(N_f) \times U(1)_{\mathcal{M}}. \tag{2.56}$$

For $N_f = 4$, the isomorphism $SU(4)/\mathbb{Z}_2 \cong SO(6)$ can be used to equivalently write the symmetry group of the DSL as:

$$\frac{SO(6) \times U(1)_{\mathcal{M}}}{\mathbb{Z}_2}, \tag{2.57}$$

as concluded by Ref. [37].

A basis for the vector space of $q = \pm 1$ monopole operators can then be constructed from the six antisymmetric generators of $\mathfrak{su}(4)$. Doing so, we obtain three spin-singlet, valley-triplet monopoles:

$$
\begin{aligned}
e^{iq\sigma}\bar{d}_q(-i\sigma_2\mu_3)(\bar{d}_q)^\intercal &\equiv \mathcal{V}_{1q},\\
e^{iq\sigma}\bar{d}_q(\sigma_2)(\bar{d}_q)^\intercal &\equiv \mathcal{V}_{2q},\\
e^{iq\sigma}\bar{d}_q(i\sigma_2\mu_1)(\bar{d}_q)^\intercal &\equiv \mathcal{V}_{3q},
\end{aligned}
\tag{2.58}
$$

and three spin-triplet, valley-singlet monopoles:

$$
\begin{aligned}
e^{iq\sigma}\bar{d}_q(-\sigma_3\mu_2)(\bar{d}_q)^\intercal &\equiv \mathcal{S}_{1q},\\
e^{iq\sigma}\bar{d}_q(i\mu_2)(\bar{d}_q)^\intercal &\equiv \mathcal{S}_{2q},\\
e^{iq\sigma}\bar{d}_q(\sigma_1\mu_2)(\bar{d}_q)^\intercal &\equiv \mathcal{S}_{3q}.
\end{aligned}
\tag{2.59}
$$

It is straightforward to verify that these have the same spin/valley structure as the monopole operators defined in Refs. [36, 37], up to some signs chosen so that the six monopoles map to the standard basis of $\mathbb{C}^6$, under the isomorphism from the $\bigwedge^2 \mathbb{C}^4$ irrep of $SU(4)$ to the vector irrep of $SO(6)$. In addition, there are operators reflection conjugate to those defined above:

$$
\mathcal{V}_{iq}^\dagger \equiv \vartheta(\mathcal{V}_{iq}), \qquad \mathcal{S}_{iq}^\dagger \equiv \vartheta(\mathcal{S}_{iq}),
\tag{2.60}
$$

which we can use to construct the six operators

$$
\mathcal{V}_i = \mathcal{V}_{i+} + \mathcal{V}_{i-}^\dagger, \quad \mathcal{S}_i = \mathcal{S}_{i+} + \mathcal{S}_{i-}^\dagger,
\tag{2.61}
$$

For example,

$$
\mathcal{V}_2 = e^{i\sigma}(\bar{d}_+\sigma_2\bar{d}_+^\intercal + d_+^\intercal\sigma_2 d_+),
\tag{2.62}
$$

is a monopole of definite magnetic charge $(+1)$ that can create or annihilate pairs of spinons, as illustrated earlier in Fig. 2.1.

By examining the instanton-induced 't Hooft vertex (2.38), we observe that a choice of $\mathfrak{su}(4)$-adjoint mass proliferates a linear combination of two of the six monopoles $\{\mathcal{V}_i, \mathcal{S}_i\}$. There are 15 such combinations, in correspondence with the 15 generators of $\mathfrak{su}(4)$. As an

Adjoint mass	Monopole proliferated
M_{01}	$\mathcal{V}_3 + i\mathcal{V}_2 + \text{r.c.}$
M_{02}	$\mathcal{V}_3 + i\mathcal{V}_1 + \text{r.c.}$
M_{03}	$-\mathcal{V}_1 + i\mathcal{V}_2 + \text{r.c.}$
M_{i1}	$\mathcal{S}_i - i\mathcal{V}_1 + \text{r.c.}$
M_{i2}	$\mathcal{S}_i + i\mathcal{V}_2 + \text{r.c.}$
M_{i3}	$\mathcal{S}_i - i\mathcal{V}_3 + \text{r.c.}$
M_{i0}	$\mathcal{S}_j + i\mathcal{S}_k + \text{r.c.}$

Table 2.1: Monopoles proliferated by the 15 adjoint masses. "r.c." denotes the reflection conjugate. In the last row, (ijk) is an even permutation of (123).

example, the 't Hooft vertex (2.38) for a spin-Hall mass $M_{30} = \bar{\psi}\sigma_3\psi$ can be written in the above basis as

$$\begin{aligned} \mathcal{L}_{30} &= \mathcal{S}_{1+} + i\mathcal{S}_{2+} + \mathcal{S}_{1-} - i\mathcal{S}_{2-} + \text{r.c.} \\ &= \operatorname{Re}\mathcal{S}_1 + \operatorname{Im}\mathcal{S}_2, \end{aligned} \tag{2.63}$$

defining $\operatorname{Re}\mathcal{S}_i \equiv \mathcal{S}_i + \mathcal{S}_i^\dagger$ and $\operatorname{Im}\mathcal{S}_i \equiv i(\mathcal{S}_i - \mathcal{S}_i^\dagger)$. Again, the adjoint $\S_i^\dagger$ of a monopole operator should really be viewed in Euclidean signature as the "reflection conjugate" $\vartheta(\S_i)$ defined earlier in Sec. 2.4.1. In this way we can find the monopole operators spawned by all 15 adjoint masses, and we tabulate them in Table 2.1.

2.5 Monopole quantum numbers on bipartite lattices

It was observed in Refs. [36, 37] that there exist orders on bipartite lattices whose microscopic symmetries are completely captured by appropriate adjoint masses. Using such orders, we can demand that the 't Hooft vertex induced by the given adjoint mass—i.e., the monopole proliferated by such a mass (Table 2.1)—must not break additional symmetries, in order to fix its quantum numbers under certain lattice symmetries. As we show below for the square lattice (Sec. 2.5.1) and the honeycomb lattice (Sec. 2.5.2), monopole quantum numbers on bipartite lattices are reproduced accurately by this method. We expect that this is true for any microscopic order that can be described in the continuum by condensing a fermion bilinear. Conversely, there exist conventional orders whose broken symmetries are not fully captured by condensing a fermion bilinear. Examples include the $\boldsymbol{q}=0$ noncollinear magnetic states on the kagome lattice [35, 77]. Such orders have a C_6-breaking spin-ordering pattern which is invisible to all 15 adjoint masses, but is captured by the spin-triplet monopoles that serve as the correct order parameter for such states [36, 37]. (Precisely, it turns out that C_6 embeds into a $\mathbb{Z}_3^{\mathcal{M}}$ subgroup of $U(1)_{\mathcal{M}}$, as suspected initially in Ref. [35].) On non-bipartite lattices, monopole proliferation breaks additional symmetries beyond those broken by the adjoint mass [36], thus our method for determining monopole quantum numbers does not apply to those cases.

2.5.1 Square lattice

On a square lattice, a DSL is obtained by coupling a staggered flux mean-field state to $U(1)$ gauge fluctuations [32]. We work with the gauge choice of Refs. [36, 37] (but a different gamma matrix convention) which yields the following PSG action on the continuum Dirac spinor ψ:

$$
\begin{aligned}
T_x\colon \psi &\to (-i\sigma_2\mu_3)(i\gamma_2)\bar{\psi}^\intercal, & \bar{\psi} &\to \psi^\intercal(i\gamma_2)(i\sigma_2\mu_3),\\
T_y\colon \psi &\to (-i\sigma_2\mu_1)(i\gamma_2)\bar{\psi}^\intercal, & \bar{\psi} &\to \psi^\intercal(i\gamma_2)(i\sigma_2\mu_1),\\
r_x\colon \psi &\to (\mu_3\gamma_1)\psi, & \bar{\psi} &\to \bar{\psi}(-\gamma_1\mu_3),\\
C_{4s}\colon \psi &\to \frac{1}{\sqrt{2}}\sigma_2(i\mu_2-1)e^{-i\frac{\pi}{4}\gamma_2}(i\gamma_2)\bar{\psi}^\intercal, & \bar{\psi} &\to \psi^\intercal e^{-i\frac{\pi}{4}\gamma_2}(i\gamma_2)\sigma_2(-i\mu_2-1)\frac{1}{\sqrt{2}},\\
\Theta\colon \psi &\to Ki\mu_2(i\gamma_2)\gamma_3\bar{\psi}^\intercal, & \bar{\psi} &\to \psi^\intercal(-i\mu_2)(i\gamma_2)\gamma_3 K,
\end{aligned}
\tag{2.64}
$$

for x and y translations (T_x, T_y), reflections in the x coordinate (r_x), site-centered four-fold rotations (C_{4s}), and time reversal (Θ), respectively, and K denotes complex conjugation only on spin/valley matrices.

The embedding of the PSG into flavor (Sec. 2.4.2) and spacetime (Sec. 2.4.1) symmetries in the continuum completely fixes how the zero-mode part of the monopole operators

	T_x	T_y	r_x	C_{4s}	Θ
M_{i0}	$-$	$-$	$-$	$-$	$-$
M_{01}	$-$	$+$	$+$	M_{03}	$+$
M_{03}	$+$	$-$	$-$	$-M_{01}$	$+$
M_{02}	$+$	$+$	$+$	$-$	$-$
M_{i1}	$+$	$-$	$+$	$-M_{i3}$	$+$
M_{i3}	$-$	$+$	$-$	M_{i1}	$+$
M_{i2}	$-$	$-$	$+$	$+$	$-$

Table 2.2: Transformation of the adjoint masses $M_{ij} = \bar{\psi}\sigma_i\mu_j\psi$ under the symmetries of the staggered-flux state on the square lattice.

transform. However, the lattice symmetries also embed into $U(1)_{\mathcal{M}}$, which acts on the bare monopole $\exp(i\sigma)$, and this information is not present in the mean-field state from which the above PSG is derived. The most general approach to calculating this action, developed in Ref. [37], is to consider the Wannier limit, and the associated charge centers, of the spinon insulator obtained on gapping the DSL with a given adjoint mass. In this limit, the $U(1)_{\mathcal{M}}$ phase rotations of the monopole under lattice symmetries are interpreted as Aharonov-Bohm phases. For instance, a C_{4s} action on a $q = +1$ monopole in an insulating state with gauge charges Q at lattice sites will yield a phase $\exp(iQ\pi/2)$.

While no substitute for such rigorous microscopic arguments, we simply note here that the existence of orders whose symmetries are fully encapsulated by a fermion bilinear provides a simple means to compute some, if not all, of the monopole quantum numbers. For example, on the square lattice, the symmetries of Néel and valence-bond-solid (VBS) states are completely encapsulated in the adjoint masses M_{i2} and $M_{01/3}$, respectively (see Table 2.2). Let us demand that the monopoles proliferated by those masses (Table 2.1) also remain invariant under the latter's symmetries. As the VBS mass M_{03} is T_x invariant, we require that the monopole $(-\operatorname{Re}\mathcal{V}_1 + \operatorname{Im}\mathcal{V}_2)$ also be T_x invariant. Likewise, C_{4s} is a symmetry of the Néel mass M_{i2} which proliferates the monopole $\operatorname{Re}\mathcal{S}_i + \operatorname{Im}\mathcal{V}_2$. This means we can demand that $\operatorname{Im}\mathcal{V}_2 = i(\mathcal{V}_2 - \mathcal{V}_2^\dagger)$ be invariant under both T_x and C_{4s}. However, from Eq. (2.64) we see the corresponding PSG transformations involve charge conjugation $\psi \to (i\gamma_2)\bar{\psi}^\intercal$. Thus, the ZM operators $d_\pm$ will also undergo charge conjugation [Eq. (2.45)], and from Eq. (2.58), $\mathcal{V}_2$ will be mapped to its reflection conjugate $\mathcal{V}_2^\dagger$. The only way for $\operatorname{Im}\mathcal{V}_2$ to remain invariant is thus to demand:

$$\begin{aligned} T_x(\mathcal{V}_2) &= T_x(e^{i\sigma})(d_-^\intercal\sigma_2 d_- + \bar{d}_-\sigma_2\bar{d}_-^\intercal) \quad \overset{!}{=} -\mathcal{V}_2^\dagger, \\ C_{4s}(\mathcal{V}_2) &= C_{4s}(e^{i\sigma})(-d_-^\intercal\sigma_2 d_- - \bar{d}_-\sigma_2\bar{d}_-^\intercal) \overset{!}{=} -\mathcal{V}_2^\dagger, \end{aligned} \tag{2.65}$$

which determines:

$$T_x(\sigma) = -\sigma + \pi, \qquad C_{4s}(\sigma) = -\sigma. \tag{2.66}$$

The quantum numbers of σ under other lattice symmetries can be similarly calculated, but one can also exploit relational constraints among the generators of the PSG (see Supplemental Material of Ref. [36]). Using that T_xT_y and ΘT_x are symmetries of the Néel order M_{i2} leads to:

$$T_y(\sigma) = \Theta(\sigma) = -\sigma + \pi. \tag{2.67}$$

Finally, we look at reflections r_x on the square lattice. Its embedding into the continuum symmetries involves the continuum reflection $\mathcal{R}_1$, which has an action $\mathcal{R}_1 : \bar{d}_\pm \to \bar{d}_\mp$ on ZM operators [Eq. (2.42)]. On the monopole $\mathcal{V}_2$, we find that:

$$r_x(\mathcal{V}_2) = r_x(e^{i\sigma})(\bar{d}_-\sigma_2\bar{d}_-^\intercal + d_-^\intercal\sigma_2 d_-) = e^{i\theta_r}\mathcal{V}_2^\dagger. \tag{2.68}$$

As reflections are a symmetry of the Néel mass M_{i2}, we can demand invariance under r_x of the $\operatorname{Im}\mathcal{V}_2$ monopole it proliferates. This sets $\theta_r = \pi$ in Eq. (2.68) and therefore

$$r_x(\sigma) = -\sigma + \pi. \tag{2.69}$$

The set of equations (2.66)-(2.69) completely determines the Berry phases of monopoles under the lattice symmetries (2.64). The total action of these symmetries on monopole operators has been summarized in Table 2.3. We note that our results are identical to the

	T_x	T_y	r_x	C_{4s}	Θ
$\mathcal{V}_1$	$\mathcal{V}_1^\dagger$	$-\mathcal{V}_1^\dagger$	$-\mathcal{V}_1^\dagger$	$-\mathcal{V}_3^\dagger$	$\mathcal{V}_1^\dagger$
$\mathcal{V}_2$	$-\mathcal{V}_2^\dagger$	$-\mathcal{V}_2^\dagger$	$-\mathcal{V}_2^\dagger$	$-\mathcal{V}_2^\dagger$	$-\mathcal{V}_2^\dagger$
$\mathcal{V}_3$	$-\mathcal{V}_3^\dagger$	$\mathcal{V}_3^\dagger$	$\mathcal{V}_3^\dagger$	$\mathcal{V}_1^\dagger$	$\mathcal{V}_3^\dagger$
$\mathcal{S}_i$	$-\mathcal{S}_i^\dagger$	$-\mathcal{S}_i^\dagger$	$\mathcal{S}_i^\dagger$	$\mathcal{S}_i^\dagger$	$-\mathcal{S}_i^\dagger$

Table 2.3: Monopole quantum numbers on the square lattice.

first four rows of Table 1 of Ref. [36]. In particular, we also find that the monopole $\operatorname{Im}\mathcal{V}_2$ is trivial under all lattice symmetries.

We caution that one cannot expect the 't Hooft vertex to respect the symmetries of the adjoint mass *in general*, as demonstrated by the results of Refs. [36, 37]. As an example, consider the unconventional order:

$$M_{i3} \sim \sum_{\boldsymbol{r}} (-1)^{r_x} (\boldsymbol{S}_{\boldsymbol{r}} \times \boldsymbol{S}_{\boldsymbol{r}+\hat{y}})_i, \tag{2.70}$$

where the right-hand side is a spin operator on the square lattice with the same microscopic symmetries as the fermion bilinear on the left-hand side [32, 36]. This equation suggests that M_{i3} describes a spin-triplet VBS state invariant under T_y. However, the monopole proliferated by M_{i3} is $\mathcal{S}_i - i\mathcal{V}_3$. By condensing this as $\langle \mathcal{S}_i - i\mathcal{V}_3\rangle = 1 - i$, one observes that there is Néel order along σ_i *in addition to* the order described by M_{i3} (2.70). This follows from the fact that $\operatorname{Re}\mathcal{S}_i$ and $\operatorname{Re}\mathcal{V}_3$ have the symmetries of Néel order M_{i2} and the triplet VBS order (2.70), respectively. The additional broken symmetries of the Néel order are not visible to the adjoint mass M_{i3} but are captured by the associated 't Hooft vertex, which additionally breaks T_y and Θ symmetries. However, the above method offers a quick way to compute quantum numbers when there exist orders with symmetries completely encoded in a fermion bilinear, paradigmatic examples being Néel and VBS orders.

2.5.2 Honeycomb lattice

On a honeycomb lattice, a parton mean-field Hamiltonian describing uniform nearest-neighbor hopping has a relativistic dispersion with gapless Dirac nodes at $\boldsymbol{K}_\pm = \pm\frac{4\pi}{3\sqrt{3}}\hat{y}$. As is well-known, this model has a particle-hole symmetry which acts trivially on the physical spin operators, and when combined with $U(1)$ gauge fluctuations yields an $SU(2)$ gauge theory (QCD_3) at low energies [78]. However, the addition of longer-range hopping breaks particle-hole symmetry and yields a DSL described by CQED_3 in the infrared. Since the particle-hole symmetric state is adiabatically connected to the DSL, we may calculate monopole quantum numbers in the former for simplicity, and to make useful comparison with the results of Refs. [36, 37]. Choosing a two-site (AB) unit cell on armchair graphene with Bravais lattice vectors $\boldsymbol{a}_{1/2}=(1/2,\pm\sqrt{3}/2)$, the PSG for the particle-hole symmetric ansatz is:

$$\begin{aligned}
T_{1/2}\colon \psi &\to e^{-i2\pi\mu_3/3}\psi, & \bar{\psi} &\to \bar{\psi}e^{i2\pi\mu_3/3},\\
C_6\colon \psi &\to -i\mu_1 e^{-i2\pi\mu_3/3}e^{-i\pi\gamma_1/6}\psi, & \bar{\psi} &\to \bar{\psi}(ie^{i\pi\gamma_1/6}e^{i2\pi\mu_3/3}\mu_1),\\
r_x\colon \psi &\to \mu_2\gamma_3\psi, & \bar{\psi} &\to \bar{\psi}(-\mu_2\gamma_3)\\
\Theta\colon \psi &\to K(i\sigma_2\mu_2\gamma_3)\psi, & \bar{\psi} &\to \bar{\psi}(i\sigma_2\mu_2\gamma_3)K,
\end{aligned} \tag{2.71}$$

for (respectively) translations $T_{1/2}$ along $\boldsymbol{a}_{1/2}$, plaquette-centered six-fold rotations (C_6), reflections about the vertical axis through an AB unit cell (r_x), and time reversal (Θ). K acts to complex conjugate only within the spin-valley space (i.e., the matrices σ_i and μ_i).

Similar to the square lattice in Sec. 2.5.1, we first tabulate the transformation of the 15 adjoint masses $M_{ij} = \bar{\psi}\sigma_i\mu_j\psi$ under the above PSG. From Table 2.4, it is clear that M_{i3} encapsulates all the symmetries of Néel order on the honeycomb lattice. We can then expect the associated proliferated monopole $\mathcal{S}_i - i\mathcal{V}_3$ (see Table 2.1) to not break any additional symmetries, and thus demand:

$$T_{1/2}(\mathcal{V}_3) = T_{1/2}(e^{i\sigma})[\bar{d}_+(i\sigma_2\mu_1)\bar{d}_+ - d_+(i\sigma_2\mu_1)d_+] \stackrel{!}{=} \mathcal{V}_3, \tag{2.72}$$

	$T_{1/2}$	C_6	r_x	Θ
M_{i0}	$+$	$+$	$-$	$+$
M_{01}	$\alpha M_{01}+\beta M_{02}$	$\alpha M_{01}+\beta M_{02}$	$+$	$+$
M_{02}	$\alpha M_{02}-\beta M_{01}$	$-\alpha M_{02}+\beta M_{01}$	$-$	$+$
M_{03}	$+$	$-$	$+$	$+$
M_{i1}	$\alpha M_{i1}+\beta M_{i2}$	$\alpha M_{i1}+\beta M_{i2}$	$+$	$-$
M_{i2}	$\alpha M_{i2}-\beta M_{i1}$	$-\alpha M_{i2}+\beta M_{i1}$	$-$	$-$
M_{i3}	$+$	$-$	$+$	$-$

Table 2.4: Transformation of the adjoint masses $M_{ij}=\bar{\psi}\sigma_i\mu_j\psi$ under the PSG (2.71) on the honeycomb lattice, with $\alpha=\cos\left(\frac{2\pi}{3}\right)$ and $\beta=\sin\left(\frac{2\pi}{3}\right)$.

noting that to the difference of Eq. (2.64), the PSG here does not involve charge conjugation. Equation (2.72) implies that lattice translations act trivially on the dual photon. Turning to reflections, we similarly demand that $r_x(\mathcal{V}_3)=e^{i\theta_r}\mathcal{V}_3^\dagger$ be equal to $\mathcal{V}_3$, which leads to the action $r_x(\sigma)=-\sigma$ with no Berry phase.

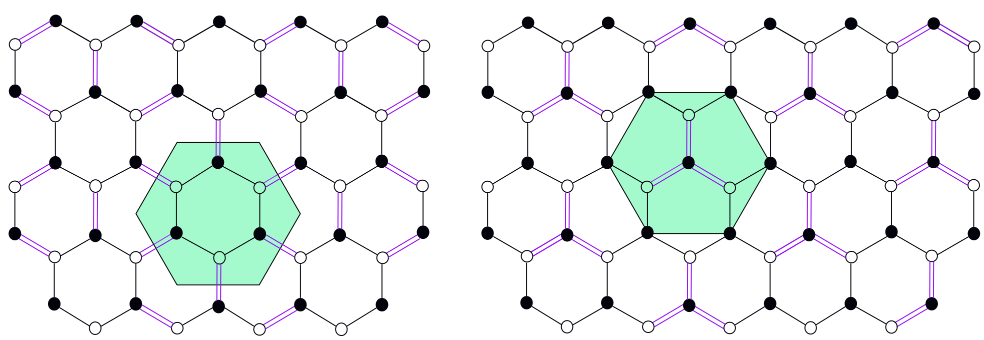

Figure 2.2: Kekulé-O (left) and Kekulé-Y (right) patterns on the honeycomb lattice.

Similarly, M_{01} and M_{02} account for all symmetries of the Kekulé-O and Kekulé-Y VBS states, respectively (Fig. 2.2). We can use the time-reversal invariance of these orders to demand invariance of the monopole $\mathrm{Re}\,\mathcal{V}_3$ that both proliferate:

$$\Theta(\mathcal{V}_3) = \Theta(e^{i\sigma})[\bar{d}_-(i\sigma_2\mu_1)\bar{d}_-^\intercal - d_-^\intercal(i\sigma_2\mu_1)d_-] = e^{i\theta_\Theta}\mathcal{V}_3^\dagger \overset{!}{=} \mathcal{V}_3^\dagger, \tag{2.73}$$

and so $\theta_\Theta = 0$. Finally, to compute quantum numbers under C_6, we may use the $C_6\Theta$

symmetry of the Néel mass M_{i3} to demand invariance of $\mathrm{Im}\,\mathcal{V}_3$:

$$\begin{aligned}(C_6\circ\Theta)(i(\mathcal{V}_3-\mathcal{V}_3^\dagger)) &= -iC_6(\mathcal{V}_3-\mathcal{V}_3^\dagger),\\ &= -ie^{-i\theta_6}(\mathcal{V}_3-\mathcal{V}_3^\dagger),\\ &\overset{!}{=} i(\mathcal{V}_3-\mathcal{V}_3^\dagger),\end{aligned} \tag{2.74}$$

which requires $\theta_6=\pi$. Collecting our results, the lattice symmetries act on the dual photon as follows:

$$\begin{aligned}T_{1/2}(\sigma) &= \sigma,\\ r_x(\sigma) &= -\sigma,\\ \Theta(\sigma) &= -\sigma,\\ C_6(\sigma) &= \sigma+\pi,\end{aligned} \tag{2.75}$$

from which transformations of all six monopole operators can be determined. These results are summarized in Table 2.5, and agree with the results in Table 1 of Ref. [36] for the honeycomb lattice. The monopole $\mathrm{Re}\,\mathcal{V}_3$ is trivial under all lattice symmetries and is thus a symmetry-allowed perturbation to the DSL.

	$T_{1/2}$	r_x	C_6	Θ
$\mathcal{V}_1$	$\alpha\mathcal{V}_1-\beta\mathcal{V}_2$	$\mathcal{V}_1^\dagger$	$-\alpha\mathcal{V}_1+\beta\mathcal{V}_2$	$\mathcal{V}_1^\dagger$
$\mathcal{V}_2$	$\alpha\mathcal{V}_2+\beta\mathcal{V}_1$	$-\mathcal{V}_2^\dagger$	$\alpha\mathcal{V}_2+\beta\mathcal{V}_1$	$\mathcal{V}_2^\dagger$
$\mathcal{V}_3$	$\mathcal{V}_3$	$\mathcal{V}_3^\dagger$	$\mathcal{V}_3$	$\mathcal{V}_3^\dagger$
$\mathcal{S}_i$	$\mathcal{S}_i$	$-\mathcal{S}_i^\dagger$	$-\mathcal{S}_i^\dagger$	$-\mathcal{S}_i^\dagger$

Table 2.5: Monopole quantum numbers on the honeycomb lattice, with $\alpha=\cos\left(\frac{2\pi}{3}\right)$ and $\beta=\sin\left(\frac{2\pi}{3}\right)$.

2.6 Conclusion

In summary, we have constructed monopole operators in the DSL directly on $\mathbb{R}^3$ without assuming conformal invariance, and have computed their quantum numbers under lattice symmetries on the square and honeycomb (bipartite) lattices. The first task was accomplished by first deforming the DSL with a choice of an $\mathfrak{su}(4)$-valued fermion mass. This was shown to lead to ZMs of the Euclidean Dirac operator exponentially bound to monopole-instantons. The interpretation of these ZMs in the Hamiltonian framework and their relation to zero-energy modes was also discussed. We then showed that resumming a semiclassical instanton gas in the presence of such ZMs leads to an instanton-induced effective interaction, designated as the 't Hooft vertex in analogy with a similar effect in QCD_4. By introducing ZM creation/annihilation operators, we then identified this vertex as a linear combination of two of six possible monopole operators in the DSL, previously constructed in radially-quantized conformal CQED_3.

Our next result involved an analysis of the effects of lattice symmetries in specific microscopic realizations of the DSL. By recognizing the existence of orders on bipartite lattices with symmetries fully encapsulated in a specific fermion bilinear, we were able to compute quantum numbers of all monopoles under symmetries of the DSL on square and honeycomb lattices. Specifically, from a symmetry standpoint, Néel and VBS orders on these lattices could be described in the continuum by *either* appropriate fermion bilinears *or* monopole operators (although monopole proliferation is necessary to confine spinons). By knowing the 't Hooft vertex associated to a given bilinear, we could then demand that the former *not* break additional symmetries of the DSL to fix the lattice symmetry action on monopoles. Néel and VBS orders on the square and honeycomb lattices together possess enough unbroken lattice symmetries to fully determine the transformations of all monopole operators. In particular, our results for the "Berry phase" of monopoles, arising from the embedding of the lattice symmetries into the magnetic symmetry $U(1)_{\mathcal{M}}$ of the dual photon, were shown to be consistent with the more general Wannier center calculations of Refs. [36, 37]. On both square and honeycomb lattices, we showed the existence of a monopole transforming trivially under all lattice symmetries, and thus an allowed perturbation to the DSL on these lattices likely to lead to its instability.

2.7 Appendix: Zero modes of Dirac operators

Since an index theorem for Dirac operators with abelian gauge fields on odd-dimensional noncompact manifolds has not been established, we resort to an explicit calculation of zero modes.

A charge-$q\in\mathbb{Z}$ monopole-instanton can be described by a Wu-Yang connection,

$$\mathcal{A}_q = \begin{cases} -\frac{q}{2r\sin\theta}(\cos\theta-1)\hat{\boldsymbol{\varphi}}, & \theta\in(0,\pi/2), \\ -\frac{q}{2r\sin\theta}(\cos\theta+1)\hat{\boldsymbol{\varphi}}, & \theta\in(\pi/2,\pi), \end{cases} \tag{2.76}$$

in spherical coordinates with an orthonormal frame $(\hat{\boldsymbol{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})$. We will explicitly solve for the zero modes of the Euclidean (non-self-adjoint) Dirac operator in an instanton background,

$$\mathfrak{D}_{aq} = \not{\partial} - i\not{\mathcal{A}}_q + mt^a, \tag{2.77}$$

where $t^a\in\mathfrak{su}(4)$. Using the fact that $(\boldsymbol{\gamma}\cdot\hat{\boldsymbol{r}})^2\equiv\gamma_r^2=1$, with $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$ the Pauli vector, the Dirac operator can be rewritten as [1, 27]:

$$\gamma_r^2\,\mathfrak{D}_{aq} = \gamma_r\left(\partial_r - \frac{1}{r}\boldsymbol{\gamma}\cdot\boldsymbol{L} - \frac{q}{2r}\gamma_r\right) + mt^a, \tag{2.78}$$

where:

$$\boldsymbol{L} = \boldsymbol{r}\times(\boldsymbol{p}-\boldsymbol{a}) - \frac{q}{2}\hat{\boldsymbol{r}}, \tag{2.79}$$

is the conserved angular momentum in a monopole field. Defining the total angular momentum:

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\gamma}, \tag{2.80}$$

the Dirac operator takes the form:

$$\mathfrak{D}_{aq} = \gamma_r\left[\partial_r - \frac{1}{r}(\boldsymbol{J}^2-\boldsymbol{L}^2-\frac{3}{4}) - \frac{q}{2r}\gamma_r\right] + mt^a. \tag{2.81}$$

To find the eigenfunctions, note that $\boldsymbol{J}^2$, J_z, t^a and $\mathfrak{D}_{aq}$ commute. This prompts an eigenfunction ansatz:

$$u^{qi}_{jM} = R(r)\mathcal{Y}^{j+1/2}_{qjM}(\theta,\varphi)\,|i\rangle_a + S(r)\mathcal{Y}^{j-1/2}_{qjM}(\theta,\varphi)\,|i\rangle_a\,, \tag{2.82}$$

where $|i\rangle_a$ is one of the four eigenvectors of t^a with eigenvalue $(-1)^i$, and $\mathcal{Y}^L_{qjM}$ are monopole spinor harmonics defined in Appendix A of Ref. [1], and also in [27]. Their necessary properties are summarized as follows:

$$\begin{aligned}
\boldsymbol{J}^2\mathcal{Y}^L_{qjM} &= j(j+1)\mathcal{Y}^L_{qjM},\\
\boldsymbol{L}^2\mathcal{Y}^L_{qjM} &= L(L+1)\mathcal{Y}^L_{qjM},\\
J_z\mathcal{Y}^L_{qjM} &= M\mathcal{Y}^L_{qjM},\\
\gamma_r\mathcal{Y}^{j\pm 1/2}_{qjM} &= a_\pm\mathcal{Y}^{j+1/2}_{qjM} + b_\pm\mathcal{Y}^{j-1/2}_{qjM},
\end{aligned} \tag{2.83}$$

where:

$$\begin{aligned}
&j \in \left\{\frac{|q|}{2}-\frac{1}{2}, \frac{|q|}{2}+\frac{1}{2}, \dots\right\},\ (j>0),\\
&M \in \{-j, -j+1, \dots, j\},\\
&L \in \left\{j-\frac{1}{2}, j+\frac{1}{2}\right\},\ \left(L\geq\frac{|q|}{2}\right),\\
&a_+ = -b_- = \frac{q}{2j+1}, \quad a_- = b_+ = -\frac{\sqrt{(2j+1)^2-q^2}}{2j+1}.
\end{aligned} \tag{2.84}$$

The condition $j>0$ implies $j=(|q|-1)/2$ is excluded when $q=0$, and the condition $L\geq|q|$ requires that $L=j-1/2$ be excluded when $j=(|q|-1)/2$. Therefore, for a fixed q, the lowest angular momentum states with $j=(|q|-1)/2$ have $S(r)=0$ in the ansatz (2.82). As we shall now show, these states are zero modes. The zero mode equation for $\mathfrak{D}_{aq}$ then separates to:

$$\left(\partial_r R + \frac{1}{r}R + \mathrm{sgn}(q)m_i\right)R(r) = 0, \tag{2.85}$$

where $m_i = (-1)^i m$ corresponding to $|i\rangle_a$, the $SU(4)$ part of the zero mode. Solving for the radial function $R(r)$, the zero modes can be written as:

$$\begin{aligned}
u^{qi}_{(q-1)/2,M} &= R^{qi}(r)\mathcal{Y}^q_{q,(q-1)/2,M}(\theta,\varphi)\,|i\rangle_a\,,\\
&= \frac{\sqrt{2m}}{r}e^{-\,\mathrm{sgn}(q)(-1)^i mr}\mathcal{Y}^q_{q,(q-1)/2,M}\,|i\rangle_a\,.
\end{aligned} \tag{2.86}$$

For a fixed monopole charge q and $\mathfrak{su}(4)$ mass mt^a, it is clear that there are $2q\times\frac{N_f}{2} =$

qN_f linearly independent *normalizable* zero modes.[8] We have utilized the fact that for $j = (q-1)/2$, the quantum number M ranges over the $2q$ values $-j, \ldots, j$, and that a given sign of q results in precisely two of the four eigenvectors $|i = 1, 2, 3, 4\rangle_a$ contributing normalizable ZMs.

It is also important to consider zero modes of the adjoint Dirac operator $\mathfrak{D}^\dagger$, for the Dirac action can be rewritten after an integration by parts and throwing away boundary terms as:

$$\begin{aligned} S_f &= \int d^3x \, \bar{\psi}(\partial\!\!\!/ - i a\!\!\!/ + mt^a)\psi, \\ &= \int d^3x \, [(-\partial\!\!\!/ + i a\!\!\!/ + mt^a)\bar{\psi}^\dagger]^\dagger \psi, \end{aligned} \tag{2.87}$$

where it is to be remembered that $\bar{\psi}$ and ψ are independent variables in the Euclidean path integral, unrelated by any notion of complex conjugation. Repeating the calculation above leads to the zero modes:

$$\begin{aligned} v^{qi}_{(q-1)/2,M} &= R^{qi}(r)\mathcal{Y}^q_{q,(q-1)/2,M}(\theta,\varphi)\,|i\rangle_a\,, \\ &= \frac{\sqrt{2m}}{r} e^{\mathrm{sgn}(q)(-1)^i mr}\mathcal{Y}^q_{q,(q-1)/2,M}\,|i\rangle_a\,, \end{aligned} \tag{2.88}$$

where again, for a given q, i must be chosen to ensure normalizability.

For reference, we give expressions for the monopole spinor harmonics, also given in Appendix A of Ref. [1]:

$$\begin{aligned} \mathcal{Y}^{j-1/2}_{q,j,m}(\theta,\varphi) &= \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j}Y_{q,j-\frac{1}{2},m-\frac{1}{2}} \\ \sqrt{j-m}Y_{q,j-\frac{1}{2},m+\frac{1}{2}} \end{pmatrix}, \\ \mathcal{Y}^{j+1/2}_{q,j,m}(\theta,\varphi) &= \frac{1}{\sqrt{2j+2}} \begin{pmatrix} -\sqrt{j-m_j+1}Y_{q,j+\frac{1}{2},m-\frac{1}{2}} \\ \sqrt{j+m+1}Y_{q,j+\frac{1}{2},m+\frac{1}{2}} \end{pmatrix}. \end{aligned} \tag{2.89}$$

[8] It is assumed that N_f is even, so that there is no parity anomaly. In the case of N_f odd, the non-anomalous theory has a half-integral Chern-Simons term that can be regarded as the result of integrating out an extra Dirac fermion.

The monopole harmonics Y_{qLM} are defined in terms of the Wigner D-matrices $D^J_{MM'}(\alpha, \beta, \gamma)$ [74, 79]. In the northern chart on a sphere that surrounds the monopole-instanton,

$$Y_{q,L,M}(\theta_N, \varphi) = \sqrt{\frac{2L+1}{4\pi}} \left[D^L_{M,-q/2}(\varphi, \theta, -\varphi) \right]^*, \tag{2.90}$$

where $\theta_N \in [0, \pi)$. The southern versions (which are valid on the south pole) can be obtained via a gauge transformation on the overlapping region between northern and southern charts:

$$Y_{q,L,M}(\theta_S, \varphi) = e^{-i2q\varphi} Y_{q,L,M}(\theta_N, \varphi). \tag{2.91}$$

From the above formula, the first two $q=1$ harmonics are given by

$$\begin{aligned} Y_{1,\frac{1}{2},\frac{1}{2}}(\theta_N, \varphi) &= -\frac{1}{\sqrt{2\pi}} e^{i\varphi} \sin\frac{\theta}{2}, \\ Y_{1,\frac{1}{2},-\frac{1}{2}}(\theta_N, \varphi) &= \frac{1}{\sqrt{2\pi}} \cos\frac{\theta}{2}, \end{aligned} \tag{2.92}$$

in the northern chart. Their analogs on the southern chart are obtained from the gauge transformation $\exp(-i\varphi)$.

For $q=-1$, the first two harmonics on the northern chart are

$$\begin{aligned} Y_{-1,\frac{1}{2},\frac{1}{2}}(\theta_N, \varphi) &= \frac{1}{\sqrt{2\pi}} \cos\frac{\theta}{2}, \\ Y_{-1,\frac{1}{2},-\frac{1}{2}}(\theta_N, \varphi) &= \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \sin\frac{\theta}{2}, \end{aligned} \tag{2.93}$$

with their versions in the southern chart now obtained from the gauge transformation $\exp(i\varphi)$.

2.8 Appendix: Real-space Dirac propagator

With the Lagrangian $\bar{\psi}(\not{\partial}+m)\psi$ where m is a signed quantity, the free Dirac propagator on $\mathbb{R}^3$ is:

$$\begin{aligned} G_f(x) &= \int \frac{d^3k}{(2\pi)^3} e^{ikx} \frac{i\not{k} - m}{k^2 + m^2}, \\ &= (\not{\partial} - m) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{k^2 + m^2}, \\ &= (\gamma_r \partial_r - m) \frac{e^{-|m|r}}{4\pi r}, \\ &= -\frac{e^{-|m|r}(1 + |m|r)}{4\pi r^2} \gamma_r - \frac{m e^{-|m|r}}{4\pi r}. \end{aligned} \tag{2.94}$$

Chapter 3

Parton theory of superfluidity in $2d$ hardcore bosons

3.1 Introduction

The parton or projective construction is one of the most versatile and conceptually fruitful approaches to a theoretical understanding of strongly correlated systems [20]. This approach is based on rewriting microscopic degrees of freedom in terms of fractionalized ones that are charged under an emergent gauge field, and thus transform projectively under microscopic symmetries. The emergent gauge structure strongly constrains the low-energy physics, which is progressively revealed as high-energy degrees of freedom are integrated out. A lattice gauge theory with dynamical gauge fields first emerges, and is then replaced by a continuum gauge theory once lattice-scale fluctuations have been decimated. The universal low-energy physics of the original quantum many-body system is then dictated by the infrared fate of this continuum parton gauge theory.

Fractionalized phases of matter, such as spin liquids, fractionalized Fermi liquids, or fractional quantum Hall states, correspond to deconfined phases of parton gauge theories. Whether such phases exist at all for $U(1)$ parton gauge theories in 2+1 dimensions—our prime focus—is a nontrivial question, due to the strong infrared relevance of the gauge coupling and the ensuing tendency to confinement. Nonperturbative confinement-inducing effects in such gauge theories, notably monopole-instantons [9, 26, 40], can be suppressed by a variety of mechanisms, including large-flavor screening effects [39, 80], the Higgs mechanism [31], and Chern-Simons topological masses [81, 82]. If the suppression of monopole-instantons does obtain, the appropriate fractionalized phase is adiabatically connected to a weakly coupled phase of the parton gauge theory, despite being highly nonperturbative from the point of view of the microscopic Hamiltonian.

While fractionalized phases are thus perturbatively accessible in the parton framework, conventional broken-symmetry phases are more difficult to describe, as nonperturbative confinement effects must then necessarily play a role. An ability to describe conventional phases within the framework of parton gauge theory is however necessary for overall consistency of the theory, as well as to understand the mechanism underlying confinement transitions between a fractionalized phase and proximate conventional phases. This question was studied carefully in recent work [36, 37] in the context of the Dirac spin liquid, described at low energies by $U(1)$ quantum electrodynamics (QED_3) with four flavors of two-component massless Dirac fermions [32, 68, 69, 83]. Extending earlier work by Alicea and collaborators [42–45], Song *et al.* [36, 37] utilized the state-operator correspondence of conformal field theory [27] to determine the quantum numbers of monopole operators $\mathcal{M}$ for microscopic realizations of the Dirac spin liquid state on various lattices. The insertion of a (single) monopole operator in the Hamiltonian formalism corresponds to an instanton event in (2+1)D spacetime whereby a localized source of 2π magnetic flux is suddenly added to the system [9, 26, 40]. In the Hamiltonian picture, a conventional phase is argued to be accompanied by a monopole condensate $\langle\mathcal{M}\rangle \neq 0$ which confines excitations with nonzero gauge charge, gives a mass to the emergent photon, and breaks physical symmetries if $\mathcal{M}$ transforms nontrivially under the latter.

While these arguments are undoubtedly correct, there exist few explicit computations of the nonperturbative *dynamics* that would substantiate these general symmetry considerations. Song *et al.* assume a two-step scenario in which a gauge-invariant fermion mass bilinear first acquires an expectation value, a process described by an effective theory of the QED_3-Gross-Neveu-Yukawa type [38, 55, 84–93] in which compactness of the gauge field is assumed to not play a role. After the fermionic matter is gapped out, instanton proliferation is further assumed to proceed as in the pure compact gauge theory [9, 26, 40].

In the presence of fermionic matter, however, gauge instantons may be accompanied by fermion zero modes (ZMs) [94], which can qualitatively affect the dynamics of instanton proliferation. Such Euclidean ZMs are traditionally associated with massless fermions, and are responsible for symmetry-breaking effects in the fermion sector. In (3+1)D Yang-Mills theory with massless fermions in the fundamental representation, 't Hooft showed [63–65] that fermion ZMs on the Belavin-Polyakov-Schwartz-Tyupkin instanton [95] are responsible for the explicit breaking of chiral symmetry in the fermion sector, in a manner consistent with the Adler-Bell-Jackiw anomaly equation [96, 97]. Fermion ZMs on gauge instantons in the (2+1)D Georgi-Glashow model [40] with massless fermions in the adjoint representation were shown by Affleck, Harvey, and Witten [62] to possibly lead to spontaneous breaking of the global $U(1)$ fermion number conservation symmetry. In both cases, Euclidean fermion ZMs generate, via resummation of the semiclassical instanton gas, an effective fermionic interaction—the 't Hooft vertex—that manifests the desired broken symmetry. The existence of fermion ZMs in the above theories is guaranteed by the Atiyah-Singer index theorem in (3+1)D [98] and the Callias index theorem in (2+1)D [99, 100]. The latter in particular crucially relies on the non-Abelian nature of the gauge field and the presence of a scalar Higgs field in the adjoint representation which winds nontrivially at infinity in the instanton solution [40].

The examples above involve non-Abelian gauge fields and do not directly apply to our prime focus, but nonetheless suggest that fermion ZMs on gauge instantons may play an important role in the description of conventional phases and their broken symmetries in $U(1)$ parton gauge theories. A natural starting point to investigate this question is compact QED_3 with massless Dirac fermions, relevant for the Dirac spin liquid. At late times and long distances, the Polyakov lattice instanton can be modeled as a Dirac monopole in 3D

Euclidean space. The corresponding Euclidean massless Dirac equation was studied by Marston [70], but shown by explicit calculation to *not* exhibit any normalizable ZM bound to the instanton. Further, there appears to exist no generalization of the Callias index theorem to compact QED_3 [101], despite the similar infrared fate of the (2+1)D Georgi-Glashow model and compact QED_3 without fermions [9, 26, 40]. In the absence of explicit fermion ZM solutions or a general theorem guaranteeing their existence, their relevance to the infrared dynamics of $U(1)$ parton gauge theories is at best speculative.

We emphasize here that we are interested in fermion ZMs bound to instantons in non-compact Euclidean spacetime $\mathbb{R}^3$, as opposed to ZMs of the Dirac Hamiltonian on a 2-sphere S^2 surrounding a monopole insertion in the state-operator correspondence of conformal QED_3 [27]. The existence of the latter ZMs is guaranteed by the Atiyah-Singer index theorem applied to the massless Dirac operator on the compact space S^2. As the Marston calculation [70] indicates, however, the existence of Hamiltonian ZMs in the latter context does not automatically imply the existence of Euclidean ZMs in noncompact spacetime.

In this chapter, we present a study of nonperturbative effects in a $U(1)$ parton gauge theory, in which we show by explicit calculation that Euclidean fermion ZMs bound to gauge instantons exist and lead to symmetry-breaking effects. The gauge theory we consider arises as the effective continuum description of interacting lattice bosons in the vicinity of a multicritical point separating superfluid, Mott insulating, and fractional quantum Hall ground states [102]. While the parton description is introduced as a means to access the fractional quantum Hall state, in which a Chern-Simons term for the emergent $U(1)$ gauge field leads to deconfinement, our focus here is on the nonperturbative gauge dynamics that obtains in the superfluid phase, which must result simultaneously in the confinement of gauge-charged excitations and the spontaneous breakdown of the global $U(1)$ boson number conservation symmetry. Ref. [102] argues from general considerations that the Affleck-Harvey-Witten mechanism [62] should be operative and yield the desired physics, but does not provide an explicit derivation of the underlying instanton dynamics.

Here we show by explicit calculation that, by contrast with massless QED_3, QED_3 with *massive* Dirac fermions admits normalizable Euclidean ZM solutions in a $U(1)$ instanton background. Such solutions are exponentially localized to the center of the instanton with a length scale inversely proportional to the fermion mass. Using semiclassical methods [63–65], we then explicitly compute the 't Hooft vertex induced by those ZMs and show that it naturally leads to *two* possible superfluid phases: a conventional superfluid phase with single-particle condensation [102], but also an exotic paired superfluid phase with a residual $\mathbb{Z}_2$ symmetry.

The rest of the chapter is structured as follows. We briefly review Ref. [102]'s parton description of the interacting boson problem in Sec. 3.2. In Sec. 3.3, we formulate the imaginary-time partition function of the system in a way that makes the contribution of Polyakov instantons manifest, and allows us to introduce θ parameters analogous to those of 4D Yang-Mills theory [103, 104]. In Sec. 3.4, we show that massive fermions support Euclidean zero modes localized on such instantons, and discuss their relationship to Hamiltonian (quasi-)zero modes in canonical quantization. Sec. 3.5 details the calculation of the 't Hooft vertex and Sec. 3.6 explores its symmetry-breaking consequences. We end the chapter with brief concluding remarks in Sec. 3.7; accessory technical results are collated in Appendices 3.8 and 3.9.

3.2 Parton gauge theory

We begin by reviewing the parton gauge theory introduced in Ref. [102]. We consider a system of charge +1 (in appropriate units) hard-core bosons on a 2d lattice described by operators $b(\boldsymbol{x})$ and $b^\dagger(\boldsymbol{x})$. The hard-core condition imposes on these operators the algebra

$$[b(\boldsymbol{x}), b^\dagger(\boldsymbol{x}')] = [1 - 2b^\dagger(\boldsymbol{x})b(\boldsymbol{x})]\delta_{\boldsymbol{x}\boldsymbol{x}'}, \tag{3.1}$$

$$[b(\boldsymbol{x}), b(\boldsymbol{x}')] = [b^\dagger(\boldsymbol{x}), b^\dagger(\boldsymbol{x}')] = 0. \tag{3.2}$$

The hard-core boson then admits a parton decomposition

$$b(\boldsymbol{x}) = f_1(\boldsymbol{x}) f_2(\boldsymbol{x}), \tag{3.3}$$

where $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ are fermion annihilation operators. We associate the physical boson charge with f_1 and couple this to a background gauge field A_μ when it is necessary to keep track of the physical $U(1)$ symmetry associated with conservation of the boson number. The parton decomposition (3.3) also introduces a local $SU(2)$ gauge redundancy $f_i(\boldsymbol{x}) \to W_{ij}(\boldsymbol{x}) f_j(\boldsymbol{x})$, under which the boson operators remain invariant. In the parton approach [20], one first ignores this gauge structure and postulates a mean-field ansatz for the partons. Gauge fluctuations above the mean-field fermion ground state are then reintroduced, which ensures the parton dynamics is projected onto the physical boson Hilbert space. In what follows, we shall assume a mean-field ansatz for the partons that breaks the $SU(2)$ redundancy down to a $U(1)$ subgroup, for example via a lattice analog of the Higgs mechanism [31], which leaves a single gauge boson massless. Under the leftover $U(1)$ gauge redundancy, the partons f_1 and f_2 are assigned gauge charges ± 1 respectively, so that the boson operator remains gauge invariant.

We further consider a mean-field ansatz for the partons in which f_1 and f_2 form independent Chern insulators with Chern numbers ± 1, respectively, described for instance by Haldane models [105] or their analog on the lattice of interest. In the vicinity of Chern-number-changing transitions in the parton bandstructure, this theory is described in the continuum limit by a 3D Euclidean Lagrangian,

$$\mathcal{L} = \sum_{\alpha=\pm} \left[\bar{\psi}_{1\alpha} (\not{\partial} - i\not{A} + m) \psi_{1\alpha} + \bar{\psi}_{2\alpha} (\not{\partial} - m) \psi_{2\alpha} \right], \tag{3.4}$$

where A_μ is the background field that tracks the physical $U(1)$ symmetry, and $\{\psi_{1\pm}, \psi_{2\pm}\}$ are two-component Dirac fermions obtained in a linearization of the partons $\{f_1, f_2\}$ at the two Dirac points $K_\pm$ that generically appear in the parton bandstructure [1]. Importantly,

[1] In our convention, the 3D Euclidean Dirac matrices are just Pauli matrices with $\hat{z}$ being the Euclideanized time direction. Matter of general charge e gauge transforms as $\psi \to \psi e^{ie\lambda(x)}$, and the gauge covariant derivative is $(\partial_\mu - iea_\mu)$.

the fermion masses for $\psi_{1\pm}$ and $\psi_{2\pm}$ are opposite in sign, since the Chern numbers are opposite in sign for f_1 and f_2.

For the above mean-field parton ansatz to correspond in fact to a physical state of bosons, we must reintroduce gauge fluctuations. To study the effect of those fluctuations, the lattice fermions (i.e., the partons) are minimally coupled to an emergent $U(1)$ gauge field a_μ. For example, the parton f_1 of gauge charge $+1$ minimally couples to the gauge field on link $(\boldsymbol{x}, i)$ as $f^\dagger_{1\boldsymbol{x}} t_{\boldsymbol{x},i} \exp(-i a_{\boldsymbol{x},i}) f_{1\boldsymbol{x}+i}$, where $t_{\boldsymbol{x},i}$ is a hopping integral, $\boldsymbol{x}$ is a lattice site, and i is a lattice vector. The invariance of such a term under $2n\pi$ shifts of $a_{\boldsymbol{x},i}$ can be viewed as a gauge redundancy or as a true local symmetry. These two perspectives will be discussed in Sec. 3.3. In either case, at low energies, the renormalization group endows the emergent field a with dynamics that preserves this periodicity, implying an effective gauge field Hamiltonian of the form

$$H_g = \frac{1}{2}\sum_l e_l^2 + K \sum_{\Box}(1 - \cos f_{\Box}), \tag{3.5}$$

where e_l is the electric field on link l satisfying $[a_l, e_{l'}] = i\delta_{ll'}$, and $f_\Box$ is the flux (lattice curl) of a through the plaquette $\Box$ (we shall henceforth assume a square lattice for simplicity). The physical Hilbert space of the gauge theory (with fermions) is the gauge invariant subspace specified by a Gauss constraint. Weak fluctuations of a_μ correspond to the $K \gg 1$ limit, in which H_g is energetically appeased by $f_\Box = 2\pi n_\Box$, where $n_\Box \in \mathbb{Z}$ is a plaquette-dependent integer. Expanding about any one of these minima leads to the usual Maxwell theory with a massless photon. However, it is well known that tunneling events $f_\Box \to f_\Box + 2\pi Q$, where $Q \in \mathbb{Z}$, on a plaquette cannot be ignored, for these give the photon a mass exponentially small in the gauge coupling K. These tunneling events, corresponding to $2\pi Q$ flux insertions on a plaquette, feature as instantons (Dirac monopoles of charge Q) with finite action in the 3D Euclidean theory [9, 26, 40].

In a naïve continuum limit, the effective parton Lagrangian with gauge fluctuations is

$$\mathcal{L} = \sum_{\alpha=\pm} \left[\bar\psi_{1\alpha}(\not\partial - i\not A - i\not a + m)\psi_{1\alpha} + \bar\psi_{2\alpha}(\not\partial + i\not a - m)\psi_{2\alpha}\right] + \frac{1}{4e^2} f^2, \tag{3.6}$$

where e is the renormalized gauge coupling (some function of the lattice coupling K). However, a finite UV regulator (lattice constant) and the fact that the lattice theory is invariant under $a \to a+2n\pi$ imply that the effects of instantons must be accounted for in this continuum limit. This theory is termed compact QED$_3$ (CQED$_3$). We note however that by contrast with the CQED$_3$ theory of the Dirac spin liquid, which also has four flavors of two-component Dirac fermions, the fermions in our case (i) are massive, and (ii) do not all have the same sign of the gauge charge.

3.3 θ parameters and instantons

In this section, we use canonical quantization to derive a path integral representation of the partition function of the pure gauge theory without matter, which makes the contribution of instantons explicit and allows us to introduce θ parameters [106, 107] analogous to those of 4D Yang-Mills theory [71, 104]. This sets the stage for our computation of the 't Hooft vertex using path integral methods in Sec. 3.5, after explicit fermion ZM solutions in the background of a single instanton are obtained in Sec. 3.4.

We begin with the pure gauge theory, described by the Hamiltonian (3.5), which we shall consider in the absence of background charges. This means that the Gauss constraint on every site is $(\text{div} e)_{\boldsymbol{x}} |\Psi\rangle = 0$ on all physical states $|\Psi\rangle$. As stated previously, the invariance of H_g under $2\pi Q$ translations of the gauge flux on a plaquette can be viewed as either a true local symmetry, or as a gauge redundancy due to rotor-valued link variables $a_{\boldsymbol{x},i} \in U(1) \cong \mathbb{R}/2\pi\mathbb{Z}$. The former view will be called *minimal compactness*, and the latter *forced compactness*. In what follows, we shall mostly be concerned with the "magnetic limit" $K \gg 1$, in which gauge fluctuations are weak.

3.3.1 Minimal compactness

In the minimal compactness picture, the gauge field $a_{\boldsymbol{x},i} \in \mathbb{R}$. A general state in the Hilbert space is given by a wavefunctional

$$\Psi[a_{\boldsymbol{x},i}] = \langle \{a_{\boldsymbol{x},i}\} | \Psi \rangle \,, \tag{3.7}$$

where $\{a_{\boldsymbol{x},i}\}$ denotes the collection of a on all links, and $|\{a_{\boldsymbol{x},i}\}\rangle$ forms a basis. The electric fields $e_{\boldsymbol{x},i}$ generate translations of these wavefunctionals. On a single link,

$$e^{-i\alpha e} |a\rangle = |a + \alpha\rangle \,, \tag{3.8}$$

which means

$$\begin{aligned} e^{-i\alpha \frac{\delta}{\delta a}} \Psi[a] &= \langle a| e^{-i\alpha e} |\Psi\rangle \\ &= \langle a - \alpha | \Psi \rangle \\ &= \Psi[a - \alpha]. \end{aligned} \tag{3.9}$$

A gauge transformation $\exp[-i\phi(\mathrm{div} e)_{\boldsymbol{x}}]$ on a site $\boldsymbol{x}$ is a translation that leaves all plaquette fluxes $f_\square$ invariant. Since the magnetic term is a periodic function of $f_\square$, H_g is not only invariant under these gauge transformations, but also under a discrete group of local flux translations $f_\square \to f_\square + 2\pi Q$, $Q \in \mathbb{Z}$ on a plaquette. This group is generated by monopole operators $\mathcal{M}_Q(\bar{\boldsymbol{x}})$, where $\bar{\boldsymbol{x}}$ denotes a plaquette (or equivalently, a site on the dual lattice), and

$$\mathcal{M}_Q^\dagger(\bar{\boldsymbol{x}}) f_{\bar{\boldsymbol{x}}} \mathcal{M}_Q(\bar{\boldsymbol{x}}) = f_{\bar{\boldsymbol{x}}} + 2\pi Q. \tag{3.10}$$

This translation is also generated by electric fields, but one must use an infinite string of fields [108], since only the flux in plaquette $\bar{\boldsymbol{x}}$ must be changed. One possibility is to consider an infinite product of all horizontal links below $\bar{\boldsymbol{x}}$, and non-uniquely define

$$\mathcal{M}_Q(\bar{\boldsymbol{x}}) = e^{i2\pi Q \sigma_{\bar{\boldsymbol{x}}}}, \qquad \sigma_{\bar{\boldsymbol{x}}} \equiv \sum_{p=-\infty}^{0} e_{\boldsymbol{x}+p\hat{x}_2, \hat{x}_1}, \tag{3.11}$$

where $\hat{x}_1, \hat{x}_2$ are unit vectors in the positive horizontal and vertical directions, respectively.

The minimally compact theory has similarities with the Bloch problem of electrons in a crystal lattice, in which a discrete translation by a lattice constant is a physical symmetry as opposed to a gauge redundancy. In the Bloch problem, there occur instantons that tunnel between the minima of the crystal potential, and the true ground state is a superposition of all local minima. In minimally compact CQED$_3$, the analogs are monopole-instantons that tunnel between physically distinct minima $f_{\bar{\boldsymbol{x}}}=2n\pi$ to $f'_{\bar{\boldsymbol{x}}}=2m\pi$ on a given plaquette $\bar{\boldsymbol{x}}$. Since $[H_g, \mathcal{M}_Q(\bar{\boldsymbol{x}})]=0$ on every plaquette, the physical eigenstates of H_g fall in representations of these symmetries. The irreducible representations of this Abelian group are all one-dimensional, and are simply phase factors (Bloch theorem). The eigenstates of H_g must thus obey,

$$\forall \bar{\boldsymbol{x}}, \quad \mathcal{M}_Q(\bar{\boldsymbol{x}}) \left|\Psi_{n,\theta}\right\rangle = e^{iQ\theta_{\bar{\boldsymbol{x}}}} \left|\Psi_{n,\theta}\right\rangle, \quad \theta_{\bar{\boldsymbol{x}}} \in [0, 2\pi), \tag{3.12}$$

where $\theta_{\bar{\boldsymbol{x}}}$ is an analog of crystal momentum, and n is a collective index denoting all the other quantum numbers necessary to specify the state. The corresponding eigenenergies will be continuous functions of $\theta_{\bar{\boldsymbol{x}}}$, as in conventional band theory.

For example, a single square plaquette in the "electric limit" $K \to 0$ (the analog of the "empty lattice approximation" in the Bloch problem) is governed by the Hamiltonian

$$H_g \approx \frac{1}{2} \sum_{\boldsymbol{x},i} e_{\boldsymbol{x},i}^2. \tag{3.13}$$

There is a single site $\bar{\boldsymbol{x}}$ on the dual lattice, and we thus drop the site index. Eigenstates of H_g are eigenstates of all four electric fields bordering the plaquette, but subject to the Bloch condition (3.12) and the Gauss constraint $(\text{div} e)_{\boldsymbol{x}} \left|\Psi\right\rangle = 0$. For this single-plaquette system, Eq. (3.11) implies $\sigma_{\bar{\boldsymbol{x}}} = e_{\boldsymbol{x},\hat{x}_1}$ and thus $\mathcal{M}_Q(\bar{\boldsymbol{x}}) = \exp(i2\pi Q e_{\boldsymbol{x},\hat{x}_1})$. The Bloch condition (3.12) is then

$$e^{i2\pi Q e_{\boldsymbol{x},\hat{x}_1}} \left|\Psi_{n,\theta}\right\rangle = e^{iQ\theta} \left|\Psi_{n,\theta}\right\rangle. \tag{3.14}$$

This implies physical eigenstates of $e_{\boldsymbol{x},\hat{x}_1}$ (and of H_g in the limit $K \to 0$) satisfying the Bloch condition are restricted to those with eigenvalues

$$e_{\boldsymbol{x},\hat{x}_1} = n + \frac{\theta}{2\pi}, \qquad n \in \mathbb{Z}. \tag{3.15}$$

The electric fields on the other links can be found using the Gauss constraint. The physical states are loops of electric flux circling the plaquette, with an integer level spacing. Substitution of these values into the Hamiltonian (3.13) gives the bandstructure

$$E_n(\theta) = 2\left(n + \frac{\theta}{2\pi}\right)^2. \tag{3.16}$$

In the minimal compactness picture, $\{\theta_{\bar{\boldsymbol{x}}}\}$ are a set of quantum numbers specifying states in the same Hilbert space.

Finally, we observe that $[H, \mathcal{M}_Q(\bar{\boldsymbol{x}})] = 0$ for the full lattice Hamiltonian H with gauge fields and fermions, since the gauged fermion hopping term discussed in Sec. 3.2 is invariant under local shifts of the link field $a_{\boldsymbol{x},i}$ by arbitrary integer multiples of 2π, including those produced by conjugation with the monopole operator (3.11). Thus the Bloch condition (3.12) applies to eigenstates of H as well. As with the Bloch theorem in solid-state physics, an eigenfunctional of H satisfying this Bloch condition can be written as the product of a "plane wave" and a periodic function,

$$\Psi_{n,\theta}[a_{\boldsymbol{x},i}] = e^{\frac{i}{2\pi}\sum_{\bar{\boldsymbol{x}}}\theta_{\bar{\boldsymbol{x}}} f_{\bar{\boldsymbol{x}}}}\Phi_{n,\theta}[a_{\boldsymbol{x},i}], \tag{3.17}$$

where $\Phi_{n,\theta}$ is invariant under $2\pi Q$ flux translations on a plaquette, i.e., $\mathcal{M}_Q(\bar{\boldsymbol{x}})\Phi = \Phi$, and we have suppressed the dependence of Φ on fermionic coordinates, which does not play a role in this analysis. As in band theory, we can reduce the solution of the Schrödinger equation over $a_{\boldsymbol{x},i} \in \mathbb{R}$ to that over a single "unit cell" $a_{\boldsymbol{x},i} \in [0, 2\pi)$ by either solving the original equation over that domain with the twisted periodic boundary conditions (3.12), or by deriving an equation for the periodic part Φ. Defining the unitary

$$U_\theta = e^{\frac{i}{2\pi}\sum_{\bar{\boldsymbol{x}}}\theta_{\bar{\boldsymbol{x}}} f_{\bar{\boldsymbol{x}}}}, \tag{3.18}$$

we see that $\Phi_{n,\theta}$ obeys the modified Schrödinger equation $H_\theta \Phi_{n,\theta} = E_n(\theta)\Phi_{n,\theta}$, where

$$H_\theta \equiv U_\theta^\dagger H U_\theta = \frac{1}{2}\sum_{\boldsymbol{x},i}\left[e_{\boldsymbol{x},i} + \frac{1}{2\pi}\epsilon_{ij}\Delta_j \theta_{\bar{\boldsymbol{x}}-\hat{x}_j}\right]^2 + K\sum_{\bar{\boldsymbol{x}}}(1 - \cos f_{\bar{\boldsymbol{x}}}). \tag{3.19}$$

Since $\theta_{\bar{\boldsymbol{x}}}$ only enters H_θ through its spatial lattice derivative, a uniform parameter $\theta_{\bar{\boldsymbol{x}}} = \theta$ has no effect in the bulk [106, 107], but will affect energetics in a system with boundary as the single square plaquette considered here.

The partition function for a fixed set $\{\theta_{\bar{\boldsymbol{x}}}\}$ is [109]

$$Z_\theta = \operatorname{tr} e^{-\beta H_\theta} = \sum_{\{Q_{\bar{\boldsymbol{x}}}\} \in \mathbb{Z}} e^{i \sum_{\bar{\boldsymbol{x}}} Q_{\bar{\boldsymbol{x}}} \theta_{\bar{\boldsymbol{x}}}} Z_Q, \tag{3.20}$$

where the second equality is a Fourier decomposition, since Z_θ is periodic in all the $\theta_{\bar{\boldsymbol{x}}}$. The Fourier coefficients are given by

$$Z_Q = \int_0^{2\pi} D\{\theta_{\bar{\boldsymbol{x}}}\}\, e^{-i \sum_{\bar{\boldsymbol{x}}} Q_{\bar{\boldsymbol{x}}} \theta_{\bar{\boldsymbol{x}}}} Z_\theta, \tag{3.21}$$

where we define $D\{\theta_{\bar{\boldsymbol{x}}}\} \equiv \prod_{\bar{\boldsymbol{x}}} \frac{\mathrm{d}\theta_{\bar{\boldsymbol{x}}}}{2\pi}$. These Fourier coefficients can be interpreted as a partition function of the original Hamiltonian H with monopole insertions as follows. Let $\mathcal{M}(Q_{\bar{\boldsymbol{x}}})$ be a product of monopole operators that inserts flux across the system in a manner determined uniquely by the configuration function $Q_{\bar{\boldsymbol{x}}}$. Using the completeness of flux $(\hat{f})$ eigenstates in the gauge-invariant subspace, we obtain:

$$\begin{aligned} \operatorname{tr} e^{-\beta H} \mathcal{M}(Q_{\bar{\boldsymbol{x}}}) &= \int_{\mathbb{R}} D\{f_{\bar{\boldsymbol{y}}}\} \left\langle \{f_{\bar{\boldsymbol{y}}}\} \right| e^{-\beta H} \mathcal{M}(Q_{\bar{\boldsymbol{x}}}) \left| \{f_{\bar{\boldsymbol{y}}}\} \right\rangle \\ &= \int_{\mathbb{R}} D\{f_{\bar{\boldsymbol{y}}}\} \left\langle \{f_{\bar{\boldsymbol{y}}}\} \right| e^{-\beta H} \left| \{f_{\bar{\boldsymbol{y}}} + 2\pi Q_{\bar{\boldsymbol{y}}}\} \right\rangle, \end{aligned} \tag{3.22}$$

where $D\{f_{\bar{\boldsymbol{y}}}\} \equiv \prod_{\bar{\boldsymbol{y}}} \frac{\mathrm{d}f_{\bar{\boldsymbol{y}}}}{2\pi}$. Inserting a complete set of eigenstates of H using

$$1 = \int_0^{2\pi} D\{\theta_{\bar{\boldsymbol{x}}}\} \sum_{\{n_{\bar{\boldsymbol{x}}}\}} \left| \Psi_{n,\theta} \right\rangle \left\langle \Psi_{n,\theta} \right|, \tag{3.23}$$

we find that

$$\begin{aligned} \operatorname{tr} e^{-\beta H} \mathcal{M}(Q_{\bar{\boldsymbol{x}}}) &= \int_0^{2\pi} D\{\theta_{\bar{\boldsymbol{x}}}\} \int_{\mathbb{R}} D\{f_{\bar{\boldsymbol{y}}}\} \sum_{\{n_{\bar{\boldsymbol{x}}}\}} \left\langle \{f_{\bar{\boldsymbol{y}}}\} \right| e^{-\beta H} \left| \Psi_{n,\theta} \right\rangle \left\langle \Psi_{n,\theta} \middle| \{f_{\bar{\boldsymbol{y}}} + 2\pi Q_{\bar{\boldsymbol{y}}}\} \right\rangle \\ &= \int_0^{2\pi} D\{\theta_{\bar{\boldsymbol{x}}}\} \sum_{\{n_{\bar{\boldsymbol{x}}}\}} e^{-\beta E_n(\theta)} e^{-i \sum_{\bar{\boldsymbol{x}}} Q_{\bar{\boldsymbol{x}}} \theta_{\bar{\boldsymbol{x}}}} \int_{\mathbb{R}} D\{f_{\bar{\boldsymbol{y}}}\} \Psi^*_{n,\theta}[f] \Psi_{n,\theta}[f] \\ &= \int_0^{2\pi} D\{\theta_{\bar{\boldsymbol{x}}}\}\, e^{-i \sum_{\bar{\boldsymbol{x}}} Q_{\bar{\boldsymbol{x}}} \theta_{\bar{\boldsymbol{x}}}} \operatorname{tr} e^{-\beta H_\theta} \\ &= Z_Q. \end{aligned} \tag{3.24}$$

In the second line, we have used the Bloch condition satisfied by the gauge-invariant wave-functional $\Psi_{n,\theta}[f]$ as seen in Eq. (3.17), and the third line follows from its normalization to unity. Therefore, the partition function for a fixed set $\{\theta_{\bar{\boldsymbol{x}}}\}$ can be written as

$$Z_\theta = \sum_{\{Q_{\bar{\boldsymbol{x}}}\}\in\mathbb{Z}} e^{i\sum_{\bar{\boldsymbol{x}}} Q_{\bar{\boldsymbol{x}}}\theta_{\bar{\boldsymbol{x}}}} \operatorname{tr} e^{-\beta H}\mathcal{M}(Q_{\bar{\boldsymbol{x}}}). \tag{3.25}$$

Each term of this series can be written as a path integral in a fixed monopole configuration background. The Q-dependent exponential prefactor can be absorbed into the trace by explicitly including a θ-term $i\int \theta(x)\star \mathrm{d}f(x)$ in the action, where $f(x)$ is now the Euclidean electromagnetic 2-form and $\star$ denotes the Hodge star.

3.3.2 Forced compactness

In the forced compactness picture, the gauge field $a_{\boldsymbol{x},i}\in U(1)\cong\mathbb{R}/2\pi\mathbb{Z}$ is a rotor-valued variable. The canonically conjugate electric fields then have a spectrum valued in $\mathbb{Z}$. In this perspective, the various minima $f_{\bar{\boldsymbol{x}}}=2n_{\boldsymbol{x}}\pi$ for $n_{\boldsymbol{x}}\in\mathbb{Z}$ are identified as the *same* state, and a flux translation $f_{\bar{\boldsymbol{x}}}\to f_{\bar{\boldsymbol{x}}}+2\pi Q$ becomes a gauge redundancy.

In this perspective, the problem is akin to that of a quantum particle on a ring, where a translation by a length equal to the circumference is a gauge transformation. If the ring is suspended in a gravitational potential, then the unique classical ground state that minimizes the potential is at the bottom of the ring. However, in the quantum problem, there are tunneling events (instantons) that correspond to the particle winding around the ring an integer number of times, which involves overcoming a potential barrier.

The exact analog in CQED_3 in the forced compactness picture are monopole-instantons that cause $f_{\bar{\boldsymbol{x}}}\to f_{\bar{\boldsymbol{x}}}+2\pi Q$ on a plaquette. The monopole operator defined by Eqs. (3.10)-(3.11) is thus a gauge transformation (a "do-nothing" operator) that connects different labels for the same physical state. These are called *large gauge transformations,* terminology inspired by analogous concepts in 4D Yang-Mills theory [71, 104]. Large gauge transformations are distinguished from the usual *small* ones in that the former crucially utilize the multi-valuedness of the gauge function.[2] A small gauge transformation is one of the form $\prod_{\boldsymbol{x}} \exp[i\phi_{\boldsymbol{x}}(\mathrm{div} e)_{\boldsymbol{x}}]$, where $\{\phi_{\boldsymbol{x}}\}$ are single-valued gauge functions, i.e., all of them lie in a single branch of $\mathbb{R}/2\pi\mathbb{Z}$, for example $[0, 2\pi)$. One example of a large gauge transformation is $\prod_{p=0}^{\infty} \exp[i0(\mathrm{div} e)_{\boldsymbol{x}+p\hat{x}_1}]$ which, since $0 \sim 2\pi$ in $\mathbb{R}/2\pi\mathbb{Z}$, can be written as $\prod_{p=0}^{\infty} \exp(i2\pi e_{x+p\hat{x}_1,\hat{x}_2})$, which is the same operator as (3.11), but with a displaced Dirac string.

Physical states are required to be invariant under all gauge transformations, small and large. This imposes $\theta_{\bar{\boldsymbol{x}}} = 0$ for all plaquettes $\bar{\boldsymbol{x}}$ in the Bloch condition (3.12). For the particle on a ring, a background flux can be threaded through the ring, which *changes* the Hamiltonian and the Hilbert space of the problem, as we are dealing with a physically different system. The background flux can be unitarily removed from the Hamiltonian, at the expense of twisting the boundary conditions on wavefunctions, which in winding around the ring, will then gain an Aharonov-Bohm phase. Similarly, in CQED$_3$, one can introduce a *theta term*, which changes the Hamiltonian from Eq. (3.5) to Eq. (3.19). There is a macroscopic number of such possible theta terms, corresponding to a choice $\{\theta_{\bar{\boldsymbol{x}}}\}$. Again, one can remove the theta terms from the Hamiltonian, but at the expense of introducing twisted boundary conditions under large gauge transformations, as in Eq. (3.12). The key difference with the minimal compactness picture is that a given set $\{\theta_{\bar{\boldsymbol{x}}}\}$ labels the entire Hilbert space of the theory, sometimes called a given *theta universe*. States with different $\{\theta_{\bar{\boldsymbol{x}}}\}$ belong in different Hilbert spaces; conversely, states in the same Hilbert space have the same $\{\theta_{\bar{\boldsymbol{x}}}\}$.

The expression (3.25) for a partition function with fixed $\{\theta_{\bar{\boldsymbol{x}}}\}$ remains valid in the forced compactness perspective. In fact, it is the full (i.e., unrestricted) partition function here since the entire Hilbert space is characterised by the fixed set of parameters $\{\theta_{\bar{\boldsymbol{x}}}\}$.

[2] Precisely, gauge functions defining small gauge transformations vanish at infinity.

3.4 Zero modes of massive fermions in instanton backgrounds

Having discussed instantons in the pure gauge theory, we now include fermions. As mentioned previously, the presence of fermionic ZMs in instanton backgrounds is typically associated with massless fermions in non-Abelian gauge fields [62–64, 99, 100]. Marston [70] considered massless Dirac fermions in an Abelian instanton background in (2+1)D and found no fermion ZMs bound to the instanton. Motivated by the $U(1)$ parton gauge theory (3.6), we consider here *massive* Dirac fermions in the same background and show by explicit construction that fermion ZMs now exist. This result is in accordance with the existence of zero-energy bound states for relativistic fermions in a (soliton) monopole background in (3+1)D (see Ref. [110] and references therein). In the soliton version of the problem, the fermion ZM is found by a self-adjoint extension of the Dirac Hamiltonian. Such a technique is inapplicable for the instanton version of the problem as the Euclidean Dirac operator $\mathcal{D}$ appearing in the action [see Eq. (3.26)] is not Hermitian, nor is there any requirement for it to be. Rather, $\mathcal{D}$ must obey reflection positivity (see Appendix 3.9). Therefore, the calculation of the ZM solution must be done anew in the context of the instanton problem.

As Marston himself notes, the Callias index theorem for odd-dimensional noncompact manifolds provides the number of fermion ZMs in the case of massless fermions in the background of non-Abelian instantons [99, 100]. This index theorem crucially relies on (i) the existence of a Higgs field, and (ii) on relating the index of the Dirac operator $\mathcal{D}$ to $(\dim\ker\mathcal{D}^\dagger\mathcal{D}-\dim\ker\mathcal{D}\mathcal{D}^\dagger)$, both of which fail to hold in the current setting of massive fermions in Abelian instanton backgrounds. The reason for the failure of (ii) might seem surprising, and is discussed in Appendix 3.9. Despite the absence of a rigorous index theorem for the current problem, and as we discuss below, the ZMs we find by explicit solution can be given a topological interpretation by analogy with Hamiltonian (quasi-)zero modes associated with the Atiyah-Singer index theorem.

3.4.1 Setting up in spherical coordinates

A 3D Dirac fermion ψ of charge e (with sign) and mass m, in an instanton background $a_\mu^{(g)}$ of topological charge g, is specified by the Euclidean Lagrangian

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ - ie a\!\!\!/^{(g)} + m)\psi \equiv \bar{\psi}\mathcal{D}\psi. \tag{3.26}$$

The instanton is assumed to sit at the origin, with its Dirac-monopole vector potential defined à la Wu and Yang [74]. Working in spherical coordinates (r, θ, φ), we have $a_r^{(g)} = a_\theta^{(g)} = 0$ and

$$a_\varphi^{(g)} = \begin{cases} -\frac{g}{r\sin\theta}(\cos\theta - 1), & \mathbf{r} \in R_N, \\ -\frac{g}{r\sin\theta}(\cos\theta + 1), & \mathbf{r} \in R_S, \end{cases} \tag{3.27}$$

where charts in the northern and southern hemispheres, R_N and R_S, are defined by $R_N : \theta \in [0, \pi/2+\delta)$ and $R_S : \theta \in (\pi/2-\delta, \pi]$, and a choice of $\delta \in [0, \pi/2)$ defines the chart overlap region $R_{NS} : \theta \in (\pi/2-\delta, \pi/2+\delta)$. The Dirac matrices are Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$, with $\hat{z}$ being the Euclideanized time direction. The ZM ψ_0 of the Dirac operator $\mathcal{D}$ solves

$$-i\mathcal{D}\psi_0 \equiv (-i\partial\!\!\!/ - e a\!\!\!/^{(g)} - im)\psi_0 = 0. \tag{3.28}$$

We will treat this formally as a quantum mechanics problem in three spatial dimensions, regarding $-i\partial_j$ as a canonical momentum operator p_j. The ZM equation is then

$$(\sigma_j p_j - e\sigma_j a_j^{(g)} - im)\psi_0 \equiv (\sigma_j \pi_j - im)\psi_0 = 0, \tag{3.29}$$

where the mechanical momentum $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{a}^{(g)}$. Defining $\pi\!\!\!/ = \boldsymbol{\sigma} \cdot \boldsymbol{\pi}$, we use the fact that $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^2 = \sigma_r^2 = 1$ to write

$$\begin{aligned} \pi\!\!\!/ &= (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) \\ &= (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})[\hat{\mathbf{r}} \cdot \boldsymbol{\pi} + i\boldsymbol{\sigma} \cdot (\hat{\mathbf{r}} \times \boldsymbol{\pi})]. \end{aligned} \tag{3.30}$$

The canonical angular momentum that is conserved in a monopole field is (see Appendix 3.8)

$$\boldsymbol{L} = \mathbf{r} \times \boldsymbol{\pi} - eg\hat{\mathbf{r}}, \tag{3.31}$$

where $eg \in \mathbb{Z}/2$ by the Dirac quantization condition. This can be used to rewrite Eq. (3.30) as

$$\not{\pi} = (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(\hat{\mathbf{r}} \cdot \boldsymbol{\pi} + \frac{i}{r} \boldsymbol{\sigma} \cdot \boldsymbol{L} + \frac{ieg}{r} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \right). \tag{3.32}$$

Since $\boldsymbol{L}$ generates spatial rotations, which leave $r = |\mathbf{r}|$ invariant, $[r, \boldsymbol{L}] = 0$ and the placement of r does not matter in the above equation. Using $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{a}^{(g)}$ and since $\hat{\mathbf{r}} \cdot \boldsymbol{a}^{(g)} = 0$ for the Wu-Yang potential,

$$\begin{aligned} \not{\pi} &= (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(\hat{\mathbf{r}} \cdot \boldsymbol{p} + \frac{i}{r} \boldsymbol{\sigma} \cdot \boldsymbol{L} + \frac{ieg}{r} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \right) \\ &= -i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(\partial_r - \frac{1}{r} \boldsymbol{\sigma} \cdot \boldsymbol{L} - \frac{eg}{r} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \right). \end{aligned} \tag{3.33}$$

The Dirac operator is thus

$$\begin{aligned} \mathcal{D} &= i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(\hat{\mathbf{r}} \cdot \boldsymbol{p} + \frac{i}{r} \boldsymbol{\sigma} \cdot \boldsymbol{L} + \frac{ieg}{r} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \right) + m \\ &= (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(\partial_r - \frac{1}{r} \boldsymbol{\sigma} \cdot \boldsymbol{L} - \frac{eg}{r} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \right) + m. \end{aligned} \tag{3.34}$$

3.4.2 Zero modes of the Dirac operator

Firstly, we note that $[\boldsymbol{J}, \mathcal{D}] = [\boldsymbol{J}, i\not{\pi}] = 0$, since $\not{\pi} = \boldsymbol{\sigma} \cdot \boldsymbol{\pi}$ is a dot product that remains invariant under simultaneous rotations of $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ generated by the *total* angular momentum $\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\sigma}$. As a set of commuting observables, we take

$$[\boldsymbol{J}^2, \mathcal{D}] = [J_z, \mathcal{D}] = 0. \tag{3.35}$$

This means the angular part of eigenspinors of $\mathcal{D}$ are the monopole spinor harmonics $\mathcal{Y}^L_{eg,j,m_j}(\theta, \varphi)$ (see Appendix 3.8), which informs the eigenspinor ansatz:

$$\psi_{j,m_j} = A_{j,m_j}(r) \mathcal{Y}^{j+1/2}_{eg,j,m_j}(\theta, \varphi) + B_{j,m_j}(r) \mathcal{Y}^{j-1/2}_{eg,j,m_j}(\theta, \varphi), \tag{3.36}$$

for a given (eg, j, m_j). It is necessary to superpose both values of the orbital angular momentum, $L_\pm = j \pm 1/2$, that give rise to a given total j as L is not a good quantum number.

For a monopole of the lowest magnetic charge $g = \pm 1/2e$, the total angular momentum can assume $j = 0, 1$. We will focus on the lowest spherical wave $j = 0$, for which the orbital angular momentum $L_- = -1/2$ is excluded. To reduce notational clutter, we suppress (eg, j, m_j) labels everywhere except on the monopole spinor harmonics, and write the ZM ansatz as:

$$\psi_0(r,\theta,\varphi) = A_\pm(r)\mathcal{Y}^{1/2}_{\pm 1/2,0,0}(\theta,\varphi). \tag{3.37}$$

The zero index on ψ_0 indicates that it is a ZM. The $\pm$ indices on the radial part $A_\pm(r)$ denote the value of $eg = \pm 1/2$. Finally, $\mathcal{Y}^{1/2}_{\pm 1/2,0,0}$ stands for the $(L = 1/2, eg = \pm 1/2, j = 0, m_j = 0)$ monopole spinor harmonic. Since this ansatz is coincidentally also an eigenstate of $\boldsymbol{L}^2$, we rewrite the Dirac operator in Eq. (3.34) as:

$$\mathcal{D} = (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left[\partial_r - \frac{1}{r}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \frac{3}{4}\right) - \frac{eg}{r}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right] + m. \tag{3.38}$$

The action of $\mathcal{D}$ on the ZM ansatz (3.37) is then (for $eg = \pm 1/2$),

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\Bigg[\mathcal{Y}^{1/2}_{\pm 1/2,0,0}\partial_r A_\pm + \frac{3}{2r}A_\pm \mathcal{Y}^{1/2}_{\pm 1/2,0,0} \\ \mp\frac{1}{2r}A_\pm(\boldsymbol{\sigma}\cdot\hat{r})\mathcal{Y}^{1/2}_{\pm 1/2,0,0}\Bigg] + mA_\pm \mathcal{Y}^{1/2}_{\pm 1/2,0,0} = 0.\end{aligned} \tag{3.39}$$

Using Eq. (3.119) of Appendix 3.8 with $eg = \pm 1/2$ yields:

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\mathcal{Y}^{1/2}_{\pm 1/2,0,0} = \pm\mathcal{Y}^{1/2}_{\pm 1/2,0,0}, \tag{3.40}$$

and therefore:

$$\partial_r A_\pm(r) + \left(\frac{1}{r} \pm m\right)A_\pm = 0, \tag{3.41}$$

which have the obvious exponential solutions. Therefore, the ZMs for $eg = \pm 1/2$ are

$$\begin{aligned}\psi_0^+(r,\theta,\varphi) &= \frac{\sqrt{2m}}{r}e^{-mr}\mathcal{Y}^{1/2}_{1/2,0,0}(\theta,\varphi), \\ \psi_0^-(r,\theta,\varphi) &= \frac{\sqrt{-2m}}{r}e^{mr}\mathcal{Y}^{1/2}_{-1/2,0,0}(\theta,\varphi),\end{aligned} \tag{3.42}$$

which are normalizable for $m > 0$ and $m < 0$, respectively (recall that there sits an r^2 in the Jacobian for spherical integrations). The divergence at $r = 0$ is superficial. The origin is excluded in the problem due to the instanton there, mathematically implemented by working in spherical coordinates. Alternatively, we know that the instanton has a finite core due to the underlying lattice, and so the derived form of the ZM is valid only at large distances compared to the lattice constant. This has been discussed further in the soliton version of the problem by Yamagishi [110].

3.4.3 Zero modes of the adjoint Dirac operator

Under an integration by parts, the Lagrangian changes from $\bar{\psi}\mathcal{D}\psi$ to $(\mathcal{D}^\dagger\bar{\psi}^\dagger)^\dagger\psi$, so the ZMs of the adjoint Dirac operator are also important. The adjoint Dirac operator is

$$\mathcal{D}^\dagger = -\not{\partial} + ie\not{A}^{(g)} + m, \tag{3.43}$$

where the adjoint is defined with respect to the standard inner product on $L^2(\mathbb{R}^3)$. In fact, the correct domains of $\mathcal{D}$ and $\mathcal{D}^\dagger$ are subsets of $L^2(\mathbb{R}^3)$, as discussed in Appendix 3.9. The ZM equation for $\mathcal{D}^\dagger$ is

$$i\mathcal{D}^\dagger\tilde{\psi}_0 = (-i\not{\partial} - e\not{A}^{(g)} + im)\tilde{\psi}_0 = 0, \tag{3.44}$$

which is the same as that of $\mathcal{D}$, but with the sign of m reversed. This implies the ZMs of $\mathcal{D}^\dagger$ for $eg = \pm 1/2$ are

$$\begin{aligned}\tilde{\psi}_0^+(r,\theta,\varphi) &= \frac{\sqrt{-2m}}{r}e^{mr}\mathcal{Y}^{1/2}_{1/2,0,0}(\theta,\varphi),\\ \tilde{\psi}_0^-(r,\theta,\varphi) &= \frac{\sqrt{2m}}{r}e^{-mr}\mathcal{Y}^{1/2}_{-1/2,0,0}(\theta,\varphi),\end{aligned} \tag{3.45}$$

which are normalizable for $m < 0$ and $m > 0$, respectively. We note that $\mathcal{D}$ and $\mathcal{D}^\dagger$ cannot both possess normalizable ZMs simultaneously.

3.4.4 Hamiltonian picture and quasi-zero modes

Although the Callias index theorem does not directly apply to the problem considered here, the topological nature of the fermion ZMs explicitly found in the previous subsections can be understood by considering an approximate treatment of the same problem but in the Hamiltonian picture, following the approach of Refs. [42–45]. In this approach, a single instanton is modeled as a static, infinitely thin 2π solenoidal flux to which the fermions respond: $\nabla \times \boldsymbol{a} = 2\pi\delta(\boldsymbol{r})$, where $\boldsymbol{r} = (x, y)$ now denotes the spatial coordinate. Consider first fermions of gauge charge $+1$ and zero mass. As is well known, for each fermion flavor, the corresponding 2d massless Dirac Hamiltonian possesses a single quasi-normalizable "chiral" zero-energy eigenstate $\psi_0 = (u, 0)^T$ with $u \sim f(x+iy)/r$, which can be understood as a manifestation of the Atiyah-Singer index theorem [46].

In the presence of a nonzero mass $m > 0$, a "zero-mode" solution persists, again of the form $\psi_0 = (u, 0)^T$ with $u \sim f(x+iy)/r$, but its energy E is shifted from zero to $E = m$. For fermions of gauge charge -1 and negative mass $-m$, there is also a single such quasi-zero mode per fermion flavor, now with wavefunction $\psi_0 = (0, v)^T$ and $v \sim g(x-iy)/r$, but again energy $E = m$. For a -2π flux background corresponding to an anti-instanton, the situation is reversed: fermions of gauge charge $+1$ and mass m possess a quasi-zero mode $\psi_0 = (0, v)^T$ with $v \sim g(x-iy)/r$, and fermions of gauge charge -1 and mass $-m$ possess a quasi-zero mode $\psi_0 = (u, 0)^T$ with $u \sim f(x+iy)/r$, both with energy $E = -m$.

The Hamiltonian quasi-zero modes discussed above are similar to those appearing in the "zero" mode dressing of monopole operators at the quantum critical point between a Dirac spin liquid and an antiferromagnet [55, 56, 111]. In the latter context, a spin-Hall mass $m\sigma_z$ appears spontaneously in the saddle-point free energy of the associated conformal field theory quantized on a sphere surrounding a monopole insertion, following the approach of Ref. [27] to calculate the scaling dimension of monopole operators at conformal fixed points. This spin-Hall mass gives a mass of opposite sign to fermion flavors $\psi_\uparrow$ and $\psi_\downarrow$ of opposite spin, but the gauge charge is the same for both flavors. In our case, the two parton flavors ψ_1 and ψ_2 in Sec. 3.2 play the role of $\psi_\uparrow$ and $\psi_\downarrow$, and the "spin-Hall" mass comes from the parton Chern numbers appropriate to a superfluid state [102]. Furthermore, the gauge charge is opposite for both flavors on account of the parton decomposition (3.3). Nonetheless, in both cases a single normalizable quasi-zero mode with energy $\pm m$ appears for each Dirac fermion flavor, as expected from the Atiyah-Singer index theorem.

Finally, the counting of instanton zero modes in Sec. 3.4.2 and 3.4.3 is consistent with that of the Hamiltonian quasi-zero modes just discussed, if both $\mathcal{D}$ and $\mathcal{D}^\dagger$ are considered. For an instanton ($g > 0$), and for fermions of positive gauge charge ($eg = 1/2$) and mass $m > 0$, the Euclidean Dirac operator $\mathcal{D}$ has a single normalizable zero mode $\psi_0^+ \propto e^{-mr}/r$, where r now denotes Euclidean spacetime distance from the center of the instanton [see Eq. (3.42)]. For fermions with negative gauge charge ($eg = -1/2$) and mass $m < 0$, $\mathcal{D}$ likewise possesses a single normalizable zero mode, $\psi_0^- \propto e^{mr}/r$. The adjoint Dirac operator $\mathcal{D}^\dagger$ has no zero modes in this case. For an anti-instanton ($g < 0$), the situation is reversed, as in the Hamiltonian picture: $\mathcal{D}^\dagger$ now has normalizable zero modes, but $\mathcal{D}$ has none. For $e > 0$ and $m > 0$, $\mathcal{D}^\dagger$ has a single zero mode $\tilde{\psi}_0^- \propto e^{-mr}/r$; for $e < 0$ and $m < 0$, the $\mathcal{D}^\dagger$ zero mode is $\tilde{\psi}_0^+ \propto e^{mr}/r$ [Eq. (3.45)]. The fact that instantons (anti-instantons) are associated with zero modes of $\mathcal{D}$ ($\mathcal{D}^\dagger$) is further discussed in Sec. 3.5.2 and 3.5.3 and has important consequences for instanton-induced symmetry breaking.

3.5 The 't Hooft vertex

Using the explicit ZM solutions for massive 3D Dirac fermions found in Sec. 3.4, we now show that Abelian instantons in CQED$_3$ induce symmetry-breaking interactions for such massive fermions. The calculation here is analogous to 't Hooft's groundbreaking solution of the $U(1)$ problem in 4D quantum chromodynamics [63, 64], which is well summarized in a review article [65] by the same author. In short, fermion ZMs cause transition amplitudes with nonzero instanton charge $Q \neq 0$ to vanish when evaluated between states of equal fermion number. Instead, nonvanishing amplitudes occur between states of different fermion number: fermionic field insertions appearing in such amplitudes "soak up" the fermion ZMs appearing in Grassmann integration. Resumming these insertions in the Coulomb gas formalism produces a fermionic interaction term known as the 't Hooft vertex, whose symmetry is lower than that of the classical Lagrangian.

3.5.1 Partitioning the partition function into instanton sectors

To make the calculations less tedious, we consider only two species of 3D Dirac fermions ψ_1, ψ_2 with opposite gauge charges $e_1 = -e_2 = e$ and zero net Chern number, so that their masses satisfy $m_1 = -m_2 = m$. This corresponds to effectively ignoring the valley ($\pm$) subindex in the original Lagrangian (3.6), which can be easily restored at the end of the calculation (see Sec. 3.6). The Lagrangian of interest is then

$$\mathcal{L} = \bar{\psi}_1(\not{\partial} - i\not{a} + m)\psi_1 + \bar{\psi}_2(\not{\partial} + i\not{a} - m)\psi_2 + \frac{1}{4e^2} f_{\mu\nu}^2 + i\theta(x)\epsilon_{\mu\nu\lambda}\partial_\mu f_{\nu\lambda}, \tag{3.46}$$

where an explicit θ term has been added to keep the discussion general (see Sec. 3.3). The fermion part of the action will be denoted as $S_F[a_\mu]$ when there is need to refer to it separately. The presence of a lattice regulator permits monopole-instantons in this theory to have finite action. The qualitative effects of those instantons on fermions are what we wish to understand. To formulate this problem in terms of a path integral, we use Eq. (3.25) to write the partition function in a fixed θ universe as

$$Z = \sum_{Q \in \mathbb{Z}} \int D(\bar{\psi}_\alpha, \psi_\alpha) \, [Da_\mu]_Q \, e^{-S}, \tag{3.47}$$

where $[Da_\mu]_Q$ indicates a restriction of the integration over a_μ to configurations with total instanton charge $Q/2e$. The sum over instanton configurations in Eq. (3.25) has been replaced by a sum solely over total charge Q, with integrations over instanton collective coordinates subsumed in the measure $[Da_\mu]_Q$. We will "integrate out" the instantons in the $Q \neq 0$ sectors and write a theory of fermions. A direct coupling between photons—quantum fluctuations of the gauge field about the classical instanton background—and fermions remains in this final theory, but can be neglected in the computation of the 't Hooft vertex, which is a semiclassical effect [63–65].

Insofar as the pure gauge theory is concerned, the effect of an instanton of charge $Q/2e$ is captured by an insertion

$$e^{-\frac{\pi^2}{2e^2}Q^2} \int \mathrm{d}^3x \, e^{i2\pi Q\sigma(x)}, \tag{3.48}$$

in the path integral [9, 26, 40]. Here, $\pi^2 Q^2/2e^2$ is the action of a charge-$Q/2e$ instanton, $\sigma(x)$ is a compact scalar called the dual photon—the continuum limit of the lattice variable $\sigma_{\tilde{x}}$ in Eq. (3.11)—and the integration is over the instanton position x. Likewise, the factor $\exp(i2\pi Q\sigma)$ is just the spacetime representation of the monopole operator $\mathcal{M}_Q$ [27], whose lattice form was given in Eq. (3.11). An attempt to dualize this theory in the same vein as the classic Polyakov duality between the compact $U(1)$ gauge theory and the sine-Gordon theory [9, 26, 40] leads to the path integral

$$Z\int D\sigma\, e^{-\frac{e^2}{2}\int \mathrm{d}^3x(\partial_\mu\sigma)^2}\sum_{N=0}^{\infty}\frac{1}{N!}\prod_{k=1}^{N}\int \mathrm{d}^3 z_k \\ \times \sum_{Q_k=-\infty}^{\infty} e^{-\frac{\pi^2}{2e^2}Q_k^2} e^{iQ_k[2\pi\sigma(z_k)+\theta(z_k)]}\int D(\bar{\psi}_\alpha,\psi_\alpha)e^{-S_F[a_\mu^{Q_k}]}. \quad (3.49)$$

There are a number of results used in writing Eq. (3.49), especially with fermions present. The gauge field in each Q-sector in Eq. (3.47) has been formally decomposed as

$$a_\mu = a_\mu^{SW} + a_\mu^{Q}, \quad (3.50)$$

where a_μ^{SW} is the photon (analog of "spin wave" in the 2d XY/sine-Gordon duality [112]) part whose coupling to fermions is neglected at the semiclassical level, and a_μ^{Q} describes an instanton configuration of total charge $Q/2e$. The photon part has been dualized to the Gaussian action for the compact scalar σ. In the above decomposition, the photon part can be thought of as finite-action fluctuations (of arbitrary size) around the fixed instanton solution. In addition, the sum over charges Q, and integration over positions implicit in the measure $[Da_\mu]_Q$ in Eq. (3.47) have been rewritten as sums over the number N of instantons, their charges Q_k, and their locations z_k.

An important assumption used here is the dilute gas approximation, which gains new significance in the presence of fermions for the following reason. Consider, for instance, $N = 2$ with charges $Q_1/2e$ and $Q_2/2e$. It has been assumed [63–65] that the gauge field for such a configuration can be written as

$$a_\mu^{Q_1+Q_2}(x; z_1, z_2) \approx a_\mu^{Q_1}(x; z_1) + a_\mu^{Q_2}(x; z_2), \quad (3.51)$$

with $|z_1 - z_2| \gg 1$. This might seem plausible given the assumption of a dilute instanton gas, but the consequence is severe, for it implies that the Dirac action also separates,

$$S_F[a_\mu^{Q_1+Q_2}] \approx S_F[a_\mu^{Q_1}] + S_F[a_\mu^{Q_2}]. \tag{3.52}$$

This allows the fermion path integral to be written inside the N and Q sums, as in Eq. (3.49). If $Q_1 = -Q_2$ so that the *total* instanton charge is zero, one might expect that there exist no normalizable fermion ZMs. However, the decomposition of the action in the above form (which filters into a decomposition of the Dirac operator) clearly allows ZMs. We shall nevertheless assume such a decomposition; the error in the resulting partition function turns out be of $\mathcal{O}(\lambda^2)$, where

$$\lambda = e^{-\pi^2/2e^2}, \tag{3.53}$$

is the action for an instanton of lowest charge, and we consider the semiclassical limit $\lambda \ll 1$. Moreover, even if there are no strict (topologically protected) ZMs in this case, there will likely be eigenmodes of the Dirac operator lying arbitrarily close to zero, with splitting proportional to $\exp(-m|z_1 - z_2|)$, since the ZM wave functions (3.42) for an isolated instanton decay exponentially away from the center of the instanton.

In what follows, we shall only consider the effects of a dilute gas of instantons of elementary charge, thereby restricting the sum over Q_k in Eq. (3.49) to ± 1. The contributions of instantons of higher topological charge are suppressed by increasing powers of λ.

3.5.2 $Q = 1$ instanton sector

In the $Q = 1$ sector, the fermion part of the partition function is

$$Z_F[a_\mu^+] \equiv \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F[a_\mu^+]}, \tag{3.54}$$

where a_μ^+ describes a single charge $1/2e$ instanton located at z_+. We will show that this path integral is precisely zero. Before the zero is revealed, the fermion functional measure needs definition. For technical reasons discussed in Appendix 3.9, the standard means of definition using the eigenfunctions of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ does not work, since these only span subspaces of $L^2(\mathbb{R}^3)$ over which $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are self-adjoint (not merely Hermitian), and it so happens that the ZM lies outside these subspaces. Instead, we shall proceed along physical lines; the effects of ZMs are what we are interested in. The results of Sec. 3.4 indicate that the Euclidean Dirac operators,

$$\begin{aligned}\mathcal{D}_1 &\equiv \gamma^\mu\partial_\mu - i\gamma^\mu a^+_\mu + m,\\ \mathcal{D}_2 &\equiv \gamma^\mu\partial_\mu + i\gamma^\mu a^+_\mu - m,\end{aligned} \tag{3.55}$$

each possess a normalizable ZM. We shall thus use a mode expansion of the Fermi fields as

$$\begin{aligned}\psi_1(x) &= u_0(x-z_+)\eta_0 + \sum_i{}' u_i(x-z_+)\eta_i,\\ \psi_2(x) &= v_0(x-z_+)\chi_0 + \sum_i{}' v_i(x-z_+)\chi_i,\end{aligned} \tag{3.56}$$

where η_i, χ_i are single-component Grassmann variables. u_0 and v_0 are the respective ZMs of $\mathcal{D}_1$ and $\mathcal{D}_2$, calculated in Sec. 3.4.2, and the primed sum denotes non-ZM contributions. As discussed in Appendix 3.9, we can either assume that there exists some self-adjoint operator whose domain includes the ZM, or we can use the eigenfunctions of a self-adjoint $\mathcal{D}^\dagger_\alpha\mathcal{D}_\alpha$ (which excludes the ZM) to account for non-ZM contributions and add the ZM by hand. Either way, the ZM contribution has to be accounted for on physical grounds.

Since $\mathcal{D}^\dagger_1$ and $\mathcal{D}^\dagger_2$ are deprived of normalizable ZMs in the $Q=+1$ sector, we write mode expansions of $\bar\psi_1 \in \mathrm{Dom}(\mathcal{D}^\dagger_1)$ and $\bar\psi_2 \in \mathrm{Dom}(\mathcal{D}^\dagger_2)$ as

$$\begin{aligned}\bar\psi_1(x) &= \sum_i{}' \bar u_i(z_+-x)\bar\eta_i, \quad \bar u_i(z_+-x) \equiv \tilde u_i^{\intercal}(x-z_+),\\ \bar\psi_2(x) &= \sum_i{}' \bar v_i(z_+-x)\bar\chi_i, \quad \bar v_i(z_+-x) \equiv \tilde v_i^{\intercal}(x-z_+),\end{aligned} \tag{3.57}$$

where the *transpose acts on spin and spatial indices*, no matter where the $^\intercal$ is placed. The functional measure for fermions can now be defined as

$$D(\bar{\psi}_{1,}\psi_1,\bar{\psi}_2,\psi_2) = \mathrm{d}\eta_0\,\mathrm{d}\chi_0\,\prod_i{}' \mathrm{d}\bar{\eta}_i\,\mathrm{d}\eta_i\,\mathrm{d}\bar{\chi}_i\,\mathrm{d}\chi_i$$
$$= \mathrm{d}\eta_0\,\mathrm{d}\chi_0\,D'(\bar{\eta},\eta)D'(\bar{\chi},\chi). \tag{3.58}$$

Since the ZMs η_0, χ_0 do not appear in the action, the path integral for $Z_F[a_\mu^+]$ is killed by the measure. Therefore, only the sector with zero instanton charge contributes to the full partition function (Z) of the theory.

However, sectors with nonzero instanton charge will contribute to correlation functions that can "soak up" the ZMs. For instance, (only) the $Q = 1$ sector contributes to the correlation function

$$\langle \psi_1^\intercal(x_1)\psi_2(x_2)\rangle \propto \int \mathrm{d}\eta_0\,\mathrm{d}\chi_0\,D'(\bar{\eta},\eta,\bar{\chi},\chi)e^{-S_F[a_\mu^+]}\sum_i u_i^\intercal(x_1-z_+)\eta_i\sum_j v_j(x_2-z_+)\chi_j$$
$$= -u_0^\intercal(x_1-z_+)v_0(x_2-z_+)\int D'(\bar{\eta},\eta,\bar{\chi},\chi)e^{-S_F[a_\mu^+]}, \tag{3.59}$$

where only the fermion part of the path integral has been written. This correlation function is non-zero provided

$$|x_1 - z_+| \sim |x_2 - z_+| \lesssim 1/m, \tag{3.60}$$

where m^{-1} is the width of the ZM bound to the instanton at z_+.

A nonzero value for such an anomalous (Gor'kov) Green's function is suggestive of symmetry breaking. It is indeed a gauge-invariant object, since ψ_1 and ψ_2 couple to the dynamical $U(1)$ gauge field a_μ with opposite charge, but transforms nontrivially under the global $U(1)$ symmetry associated with boson number conservation in the original microscopic model (see Sec. 3.2). To investigate whether symmetry breaking indeed occurs, we add a weak symmetry-breaking source J and consider the fermion part of the path integral in Eq. (3.54),

$$Z_F[a_\mu^+,J] = \int D(\bar{\psi}_\alpha,\psi_\alpha)e^{-S_F[a_\mu^+]-\int \mathrm{d}^3x\mathrm{d}^3y\,\psi_1^\intercal J\psi_2}, \tag{3.61}$$

where the source $J(x,y)$ is generically nonlocal with some spinor structure. This is not a valid term by itself, since it renders the Hamiltonian non-Hermitian (or the action non reflection positive). However, the $Q=-1$ sector will provide the needed conjugate term (Sec. 3.5.3). In nonlocal expressions like the source term in (3.61), Wilson lines should be inserted to maintain local gauge invariance. In accordance with our neglect of fermion-photon interactions at this stage, and because the final form of the 't Hooft vertex will be a local interaction, we do not write down these Wilson lines explicitly. Working to linear order in J, the action can be simplified using the mode expansions in Eqs. (3.56)-(3.57) as

$$\begin{aligned} S_F[a_\mu^+, J] &= S_F[a_\mu^+] + \int \mathrm{d}^3(x,y) \sum_{i,j} u_i^\intercal(x-z_+)J(x,y)v_j(y-z_+)\eta_i\chi_j \\ &= S'_F[a_\mu^+, J] + \int \mathrm{d}^3(x,y)\Big[(u_0^\intercal J v_0)\eta_0\chi_0 + {\sum_j}'(u_0^\intercal J v_j)\eta_0\chi_j + {\sum_i}'(u_i^\intercal J v_0)\eta_i\chi_0\Big], \end{aligned} \tag{3.62}$$

where $\mathrm{d}^3(x,y) \equiv \mathrm{d}^3x\,\mathrm{d}^3y$, and all the non-ZM contributions have been subsumed into $S'_F[a_\mu^+, J]$. Using the functional measure (3.58),

$$\begin{aligned} Z_F[a_\mu^+, J] = \int \mathrm{d}\eta_0\,\mathrm{d}\chi_0\, D'(\bar\eta,\eta,\bar\chi,\chi)e^{-S'_F[a_\mu^+,J]} &\Big[1 - \int \mathrm{d}^3(x,y)(u_0^\intercal J v_0)\eta_0\chi_0\Big] \\ &\times \Big[1 - {\sum_i}' \int \mathrm{d}^3(x,y)(u_0^\intercal J v_i)\eta_0\chi_i\Big] \\ &\times \Big[1 - {\sum_i}' \int \mathrm{d}^3(x,y)(u_i^\intercal J v_0)\eta_i\chi_0\Big], \end{aligned} \tag{3.63}$$

the square brackets coalesce into the expression

$$\begin{aligned} 1 - \int \mathrm{d}^3(x,y)&\Big[(u_0^\intercal J v_0)\eta_0\chi_0 + {\sum_i}'(u_0^\intercal J v_i)\eta_0\chi_i + {\sum_i}'(u_i^\intercal J v_0)\eta_i\chi_0\Big] \\ &+ {\sum_{i,j}}' \int \mathrm{d}^3(x,y)\mathrm{d}^3(x',y')(u_0^\intercal J v_j)(u_i^\intercal J v_0)\eta_0\chi_j\eta_i\chi_0. \end{aligned} \tag{3.64}$$

All the terms except the second vanish because of the functional measure (3.58); $\mathrm{d}\chi_0$ kills the third, $\mathrm{d}\eta_0$ the fourth, both kill the first, and $\mathrm{d}\bar\eta_{i\neq 0}\,\mathrm{d}\bar\chi_{i\neq 0}$ kills the last. Therefore, we obtain a nonvanishing result:

$$Z_F[a_\mu^+, J] = \int \mathrm{d}^3x\mathrm{d}^3y\, u_0^\intercal(x-z_+)J(x,y)v_0(y-z_+)K, \tag{3.65}$$

where K denotes the contribution of nonzero modes. The order of integration, $\mathrm{d}\eta_0\,\mathrm{d}\chi_0$, has yielded a minus sign. The factor K is independent of J when working to first order in the weak source J. Recalling that the ZM spinors u_0, v_0 have radial parts of the form $\exp(-m|x-z_+|)$, similar to the free-fermion propagator, we note that Eq. (3.65) resembles the structure of the Feynman diagram in Fig. 3.1.

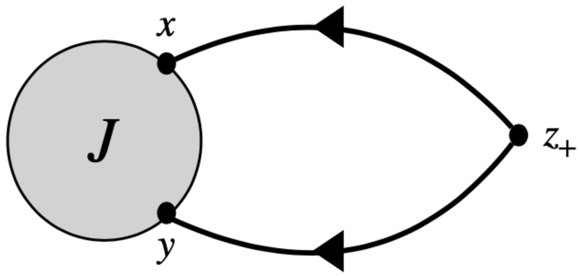

Figure 3.1: Fermion pair annihilation due to the source $J(x, y)$ and instanton at z_+.

As an ansatz for the result of integrating out instantons, we are thus motivated to consider the path integral,

$$I[J] = \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F - \int \mathrm{d}^3x \mathrm{d}^3y\, \psi_1^\intercal(x) J(x,y) \psi_2(y)} \int \mathrm{d}^3x_1 \mathrm{d}^3x_2\, A\, \bar{\psi}_2(x_2) \omega_2 \omega_1^\intercal \bar{\psi}_1^\intercal(x_1), \quad (3.66)$$

where S_F (written without a source argument) is the free Dirac action in the *absence* of instantons, A is some constant to be determined, and $\omega_{1,2}$ are spinors (possibly spacetime dependent) to be determined. A $\bar{\psi}$ in the insertion can pair up with a ψ in the source term to give the free propagator that we desire. A, ω are malleable quantities that must be fixed to obtain the exact result in Eq. (3.65). The specific form of the insertion is motivated in hindsight by the calculation that follows. Taylor expanding the source exponential, we obtain:

$$\begin{aligned} I[J] &= \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F} \left[1 - \int \mathrm{d}^3(x,y) \psi_1^\intercal(x) J(x,y) \psi_2(y) \right] \int \mathrm{d}^3(x_1, x_2) A \bar{\psi}_2(x_2) \omega_2 \omega_1^\intercal \bar{\psi}_1^\intercal(x_1) \\ &= -A I[0] \int \mathrm{d}^3(x, y, x_1, x_2) \left\langle \psi_1^\alpha(x) J^{\alpha\beta}(x,y) \psi_2^\beta(y) \bar{\psi}_2^\gamma(x_2) \omega_2^\gamma \omega_1^\lambda \bar{\psi}_1^\lambda(x_1) \right\rangle_0, \end{aligned} \quad (3.67)$$

defining

$$\langle O(\bar{\psi}_\alpha, \psi_\alpha) \rangle_0 \equiv \frac{1}{I[0]} \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F} O(\bar{\psi}_\alpha, \psi_\alpha). \quad (3.68)$$

The normalization $I[0]$ serves to restrict Wick contractions to connected diagrams, and therefore

$$I[J] = A \int \mathrm{d}^3(x, y, x_1, x_2)[G_1(x - x_1)\omega_1]^{\intercal} J(x, y)[G_2(y - x_2)\omega_2], \tag{3.69}$$

where

$$G(x - y) = -\left\langle \psi(x)\bar{\psi}(y)\right\rangle_0, \tag{3.70}$$

is the free Dirac propagator in Euclidean signature.

Comparison of Eq. (3.69) with Eq. (3.65) tells us $I[J] = Z_F[a_\mu^+, J]$ if we make the identifications

$$\begin{aligned} A &= K, \\ \int \mathrm{d}^3x_1 G_1(x - x_1)\omega_1 &= -u_0(x - z_+), \\ \int \mathrm{d}^3x_2 G_2(y - x_2)\omega_2 &= -v_0(y - z_+). \end{aligned} \tag{3.71}$$

The minus signs on the right-hand side are conventional, and $I[J] = Z_F[a_\mu^+, J]$ even without them. Clearly, the second and third equalities demand $\omega_i = \omega_i(x_i - z_+)$. A shift of integration variables $x_i \to x_i + z_+$ reveals that these are Fredholm integral equations of the first kind, with solutions

$$\begin{aligned} \omega_1 &= -G_1^{-1}u_0 = (\partial\!\!\!/ + m)u_0, \\ \omega_2 &= -G_2^{-1}v_0 = (\partial\!\!\!/ - m)v_0. \end{aligned} \tag{3.72}$$

Substituting the results for A and ω into Eq. (3.66) gives

$$Z_F[a_\mu^+, J] = \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F - \int_{x,y} \psi_1^{\intercal} J \psi_2} \int_{x_1, x_2} \bar{\psi}_2(x_2) \left[K\omega_2(x_2 - z_+)\omega_1^{\intercal}(x_1 - z_+)\right] \bar{\psi}_1^{\intercal}(x_1). \tag{3.73}$$

The single instanton of the $Q = 1$ sector has been integrated out. This path integral still evaluates to zero if the source J is switched off, since the free Dirac action S_F conserves fermion number. So far, all we have done is rewrite zero in different garb.

3.5.3 $Q = -1$ anti-instanton sector

A similar analysis can be carried out for the $Q = -1$ sector, assuming a single charge $-1/2e$ instanton (anti-instanton) localized at z_- and described by a background gauge field a_μ^-. The final result can actually be written down by the requirement of reflection positivity alone, but for the sake of completeness we explicitly rederive the result here.

The anti-instanton sectors gift the adjoint Dirac operators $\mathcal{D}_\alpha^\dagger$ with ZMs, and thus contribute to anomalous Green's functions of the form $\langle \bar{\psi}_2 \bar{\psi}_1 \rangle$. This means one has to add a source term of the form $\bar{\psi}_2 \tilde{J} \bar{\psi}_1^\intercal$ and consider

$$Z_F[a_\mu^-, J] = \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F[a_\mu^-] - \int \mathrm{d}^3(x,y) \bar{\psi}_2 \tilde{J} \bar{\psi}_1^\intercal}, \tag{3.74}$$

where $\tilde{J} = \sigma_z J^\dagger \sigma_z$ with J being the source added in the discussion of the $Q = 1$ instanton sector. This is determined by the requirement of reflection positivity of the action with the full source term $(\psi_1^\intercal J \psi_2 + \bar{\psi}_2 \tilde{J} \psi_1^\intercal)$.

The mode expansions are now

$$\begin{aligned}
\psi_1(x) &= \sum_i{}' u_i(x - z_-)\eta_i, \\
\psi_2(x) &= \sum_i{}' v_i(x - z_-)\chi_i, \\
\bar{\psi}_1(x) &= \bar{u}_0(z_- - x)\bar{\eta}_0 + \sum_i{}' \bar{u}_i(z_- - x)\bar{\eta}_i, \\
\bar{\psi}_2(x) &= \bar{v}_0(z_- - x)\bar{\chi}_0 + \sum_i{}' \bar{v}_i(z_- - x)\bar{\chi}_i,
\end{aligned} \tag{3.75}$$

where $\bar{u}_i = \tilde{u}_i^\intercal$ and $\tilde{u}_0$ is the ZM of $\mathcal{D}^\dagger$ calculated in Sec. 3.4.3. Inserting this into $Z_F[a_\mu^-, J]$ gives the analog of Eq. (3.65) for an anti-instanton as

$$\begin{aligned}
Z_F[a_\mu^-, J] &= -K \int \mathrm{d}^3x \mathrm{d}^3y \, \bar{v}_0(z_- - x) \tilde{J}(x,y) \bar{u}_0^\intercal(z_- - y) \\
&= -K \int \mathrm{d}^3x \mathrm{d}^3y \, \tilde{v}_0^\intercal(x - z_-) \tilde{J}(x,y) \tilde{u}_0(y - z_-).
\end{aligned} \tag{3.76}$$

The Feynman-diagram interpretation of Fig. 3.1 holds again but for fermion pair creation due to an anti-instanton at spacetime coordinate z_-. There is an extra minus sign here compared to Eq. (3.65), due to the order of the measure $\mathrm{d}\eta_0\,\mathrm{d}\chi_0$. This is important to obtain a reflection positive action at the end. The contribution from nonzero modes, subsumed into K, is the same as the $Q=+1$ sector if the eigenfunctions of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are used to account for the non-ZM contributions in mode expansions of fermion fields. This is because the nonzero eigenmodes of both operators are paired with the same eigenvalues. In any case, the precise numerical factor is unimportant here.

Similar to the analysis in the previous section, we consider with an insertion a path integral

$$\tilde{I}[J] = \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F - \int \mathrm{d}^3(x,y)\bar{\psi}_2 \tilde{J}\bar{\psi}_1^\intercal} \int \mathrm{d}^3(x_1, x_2) A \psi_1^\intercal(x_1)\xi_1\xi_2^\intercal\psi_2(x_2), \tag{3.77}$$

which can simplified as before to

$$\tilde{I}[J] = -A \int \mathrm{d}^3(x, y, x_1, x_2)[\xi_2^\intercal G_2(x_2 - x)]\tilde{J}(x, y)[G_1^\intercal(x_1 - y)\xi_1]. \tag{3.78}$$

Demanding equality with Eq. (3.76) sets $A=K$ and provides equations to solve for $\xi_{1,2}$:

$$\begin{aligned}
&\int \mathrm{d}^3x_1 G_1^\intercal(x_1 - y)\xi_1(x_1 - z_-) = \tilde{u}_0(y - z_-),\\
&\int \mathrm{d}^3x_1 G_2^\intercal(x_2 - x)\xi_2(x_2 - z_-) = \tilde{v}_0(x - z_-).
\end{aligned} \tag{3.79}$$

Again, these are Fredholm integral equations of the first kind, with solutions

$$\begin{aligned}
\xi_1 &= -(-\not{\partial} + m)\tilde{u}_0,\\
\xi_2 &= -(-\not{\partial} - m)\tilde{v}_0.
\end{aligned} \tag{3.80}$$

The explicit forms of the ZMs, quoted here from Sec. 3.4, are

$$\begin{aligned}
\tilde{u}_0 &= \frac{\sqrt{2m}}{r} e^{-mr}\mathcal{Y}^{1/2}_{-1/2,0,0}(\theta,\varphi) = v_0,\\
\tilde{v}_0 &= \frac{\sqrt{2m}}{r} e^{-mr}\mathcal{Y}^{1/2}_{1/2,0,0}(\theta,\varphi) = u_0,
\end{aligned} \tag{3.81}$$

where u_0, v_0 are the respective ZMs of $\mathcal{D}_1, \mathcal{D}_2$. The equations (3.80) for ξ_i thus become

$$\begin{aligned}
\xi_1 &= (\not{\partial} - m)v_0 = \omega_2,\\
\xi_2 &= (\not{\partial} + m)u_0 = \omega_1,
\end{aligned} \tag{3.82}$$

where ω_i were defined in Eq. (3.72). Therefore, the final result for the fermion path integral in the $Q=-1$ sector is

$$Z_F[a^-_\mu, J] = \int D(\bar{\psi}_\alpha, \psi_\alpha) e^{-S_F - \int_{x,y} \bar{\psi}_2 \tilde{J} \bar{\psi}_1^\intercal} \int_{x_1,x_2} \psi_1^\intercal(x_1) \left[K\omega_2(x_1 - z_-)\omega_1^\intercal(x_2 - z_-)\right] \psi_2(x_2). \tag{3.83}$$

3.5.4 Resummation and a local Lagrangian

Inserting the results of Sec. 3.5.2 and 3.5.3 into the partition function (3.49) of the full theory, where only $Q = \pm 1$ instantons are kept, we obtain:

$$\begin{aligned} Z[J] = &\int D(\bar{\psi}_\alpha, \psi_\alpha) D\sigma\, e^{-\int \mathrm{d}^3x\, \mathcal{L}_0 - \int \mathrm{d}^3(x,y)(\psi_1^\intercal J \psi_2 + \bar{\psi}_2 \tilde{J} \bar{\psi}_1^\intercal)} \\ &\times \sum_{N=0}^{\infty} \frac{(\lambda K)^N}{N!} \prod_{k=1}^{N} \int \mathrm{d}^3 z_k \int \mathrm{d}^3x \mathrm{d}^3y \\ &\times \left[e^{-i[2\pi\sigma(z_k) + \theta(z_k)]} \psi_1^\intercal(x)\omega_2(x - z_k)\omega_1^\intercal(y - z_k)\psi_2(y) \right. \\ &\left. + e^{i[2\pi\sigma(z_k) + \theta(z_k)]} \bar{\psi}_2(x)\omega_2(x - z_k)\omega_1^\intercal(y - z_k)\bar{\psi}_1^\intercal(y)\right], \end{aligned} \tag{3.84}$$

where

$$\mathcal{L}_0 = \bar{\psi}_1(\partial\!\!\!/ - i a\!\!\!/ + m)\psi_1 + \bar{\psi}_2(\partial\!\!\!/ + i a\!\!\!/ - m)\psi_2 + \frac{e^2}{2}(\partial\sigma)^2, \tag{3.85}$$

is the Lagrangian of Eq. (3.46) but absent instanton effects, with the Maxwell term dualized. We have also reinstated fermion-photon interactions to maintain explicit gauge invariance. The k-product in Eq. (3.84) just gives the N^{th} power of the insertion and, summing over N, an exponential is born. Exponentiating and *then* setting the source $J=0$ results in a nonlocal effective action

$$\begin{aligned} S_{\text{eff}} = &\int \mathrm{d}^3x\, \mathcal{L}_0 - \lambda K \int \mathrm{d}^3z \mathrm{d}^3x \mathrm{d}^3y \\ &\times \left[e^{-i[2\pi\sigma(z) + \theta(z)]} \psi_1^\intercal(x)\omega_2(x - z)\omega_1^\intercal(y - z)\psi_2(y) \right. \\ &\left. + e^{i[2\pi\sigma(z) + \theta(z)]} \bar{\psi}_2(x)\omega_2(x - z)\omega_1^\intercal(y - z)\bar{\psi}_1^\intercal(y)\right]. \end{aligned} \tag{3.86}$$

Can this action be approximated by a local one? Because the ZM wavefunctions decay exponentially in spacetime, it is reasonable to expect so (recall Fig. 3.1 and the discussion surrounding it). A change of integration variables, $x \to x+z$ and $y \to y+z$, allows the rewriting of one of the terms in the 't Hooft vertex (i.e., the instanton-induced action) as:

$$\Delta S_{\text{eff}} \equiv -\lambda K \int \mathrm{d}^3 z \mathrm{d}^3 x \mathrm{d}^3 y \, e^{-i[2\pi\sigma(z)+i\theta(z)]} \\ \times \psi_1^{\intercal}(x+z)\omega_2(x)\omega_1^{\intercal}(y)\psi_2(y+z). \tag{3.87}$$

Since ω_1 and ω_2 are proportional to the radial part (e^{-mr}/r) of the ZM, the dominant contributions to the x and y integrals are from small neighborhoods of $x=0$ and $y=0$. Taylor expanding the Fermi fields in powers of x and y to leading (zeroth) order gives

$$\Delta S_{\text{eff}} \approx -\lambda K \int \mathrm{d}^3 z \, e^{-i[2\pi\sigma(z)+\theta(z)]} \\ \times \psi_1^{\intercal}(z) \left(\int \mathrm{d}^3 x \, \omega_2(x) \int \mathrm{d}^3 y \, \omega_1^{\intercal}(y) \right) \psi_2(z). \tag{3.88}$$

Using Eq. (3.72), and denoting $v_0(p) = \int \mathrm{d}^3 x \, e^{-ipx} v_0(x)$,

$$\begin{aligned} \int \mathrm{d}^3 x \, \omega_2(x) &= \int \mathrm{d}^3 x (\not{\partial} - m) v_0(x) \\ &= \lim_{p \to 0} (i\not{p} - m) v_0(p) \\ &= -m \int \mathrm{d}^3 x \, v_0(x) \\ &= -\sqrt{2} m^{3/2} \int_0^\infty \mathrm{d}r \, r e^{-mr} \int \mathrm{d}\Omega \, \mathcal{Y}^{1/2}_{-1/2,0,0} \\ &= \sqrt{\frac{2\pi}{m}} \cdot \frac{4-\sqrt{2}}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned} \tag{3.89}$$

and similarly,

$$\begin{aligned} \int \mathrm{d}^3 x \, \omega_1^{\intercal}(x) &= \int \mathrm{d}^3 x (\not{\partial} + m) u_0(x) \\ &= \sqrt{2} m^{3/2} \int_0^\infty \mathrm{d}r \, r e^{-mr} \int \mathrm{d}\Omega \left(\mathcal{Y}^{1/2}_{1/2,0,0} \right)^{\intercal} \\ &= \sqrt{\frac{2\pi}{m}} \cdot \frac{4-\sqrt{2}}{3} \begin{pmatrix} -1 & -1 \end{pmatrix}, \end{aligned} \tag{3.90}$$

we find that the quantity appearing in brackets between $\psi_1^{\intercal}(z)$ and $\psi_2(z)$ in Eq. (3.88) is

$$\int \mathrm{d}^3 x \, \omega_2(x) \int \mathrm{d}^3 y \, \omega_1^{\intercal}(y) \propto -\frac{1}{m}(\sigma_z + i\sigma_y), \tag{3.91}$$

where the proportionality constant is some number which shall be subsumed into K. This implies the effective action is specified by the local Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \frac{Ke^{-\pi^2/4e^2}}{m}\Big[e^{-2\pi i(\sigma+\theta/2\pi)}\psi_1^{\intercal}(\sigma_z+i\sigma_y)\psi_2 + e^{2\pi i(\sigma+\theta/2\pi)}\bar{\psi}_2(\sigma_z+i\sigma_y)\bar{\psi}_1^{\intercal}\Big]. \quad (3.92)$$

Using the transformation Θ in Appendix 3.9, with additionally $\Theta(\sigma(x)) = \sigma(\theta x)$ for a scalar field, one can check explicitly that the local 't Hooft vertex thus derived is reflection positive, and thus corresponds to an interaction that preserves unitarity of the underlying real-time quantum field theory.

3.6 Partons and symmetry breaking

The original parton gauge theory had $2N_f = 4$ fermion flavors, described by the Lagrangian (3.6). In the preceding Sec. 3.5, to simplify the calculation of the 't Hooft vertex, we retained only two fermion flavors while preserving the $U(1)$ global and gauge symmetries. It can be seen from the calculations in that section that instantons in CQED$_3$ with $2N_f$ flavors of fermions with the given mass and charge assignments will induce a 't Hooft vertex with $2N_f$ fermion operators. For example, in the case of the original Lagrangian (3.6) with four fermion flavors $\{\psi_{1\pm}, \psi_{2\pm}\}$, since the mass and charge assignments are independent of the valley ($\pm$) index, one considers the Euclidean Dirac operators:

$$\begin{aligned}\mathcal{D}_1 &\equiv \gamma^\mu\partial_\mu - i\gamma^\mu a_\mu + m,\\ \mathcal{D}_2 &\equiv \gamma^\mu\partial_\mu + i\gamma^\mu a_\mu - m.\end{aligned} \quad (3.93)$$

The presence of a valley index for the fermion fields simply doubles the number of fermion ZMs in each instanton charge sector. For instance, in the $Q=1$ sector, there are four ZMs, and the mode expansions of the four fermion fields now become:

$$\begin{aligned}\psi_{1\pm}(x) &= u_0(x-z_+)\eta_{0\pm} + \sum_i{}' u_i(x-z_+)\eta_{i\pm},\\ \psi_{2\pm}(x) &= v_0(x-z_+)\chi_{0\pm} + \sum_i{}' v_i(x-z_+)\chi_{i\pm}.\end{aligned} \quad (3.94)$$

This results in a path integral measure:

$$D(\bar{\psi}_{1\pm,}\psi_{1\pm},\bar{\psi}_{2\pm,}\psi_{2\pm}) = \mathrm{d}\eta_0^+\,\mathrm{d}\eta_0^-\,\mathrm{d}\chi_0^+\,\mathrm{d}\chi_0^-\,D'(\bar{\eta}_\pm,\eta_\pm)D'(\bar{\chi}_\pm,\chi_\pm), \tag{3.95}$$

where the primed measure includes contributions from nonzero modes in the expansion (3.94). Therefore, to obtain a nonzero path integral, a four-fermion insertion of the form $\psi_{1+}^\intercal\psi_{2+}\psi_{1-}^\intercal\psi_{2-}$ is required. Repeating the calculation of the 't Hooft vertex in Sec. 3.5 with a four-fermion source and insertion yields an instanton-induced term (the exponentiated insertion) in the Lagrangian of the form:

$$e^{-2\pi i\sigma}e^{-i\theta}\psi_{1+}^\intercal\gamma\psi_{2+}\psi_{1-}^\intercal\gamma\psi_{2-} + \text{H.c.}, \tag{3.96}$$

where $\gamma\equiv\sigma_z+i\sigma_y$, and we have used $\bar{\psi}=\psi^\dagger\sigma_z$ to rewrite the second term as the Hermitian conjugate of the first. Because ψ_+ and ψ_- create excitations with lattice momenta near the Dirac points K_+ and K_-, respectively, the presence of an equal number of ψ_+ and ψ_- fields in (3.96) guarantees the 't Hooft vertex respects the microscopic translation symmetry (since $K_+ + K_-=0$ modulo a reciprocal lattice vector).

In the absence of instantons, (noncompact) QED$_3$ has a global topological $U(1)_{\text{top}}$ symmetry associated with the conservation of the topological current $j_\mu^{\text{top}}=\frac{i}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$. In the dual formulation, this symmetry is a shift symmetry of the dual photon $\sigma\to\sigma+\alpha$, manifest in the Lagrangian (3.85). The parton theory with noncompact gauge fluctuations thus has the global symmetry $U(1)\times U(1)_{\text{top}}$, where the first $U(1)$ is the boson number conservation symmetry under which $\psi_{1\pm}\to e^{i\beta}\psi_{1\pm}$ and $\psi_{2\pm}\to\psi_{2\pm}$ [recall the choice of global charge assignments in Eq. (3.6)]. The 't Hooft vertex (3.96) shows that instantons have the effect of explicitly breaking this $U(1)\times U(1)_{\text{top}}$ symmetry to a diagonal $U(1)$ subgroup under which

$$\psi_{1\pm}\to e^{i\beta}\psi_{1\pm},\quad \psi_{2\pm}\to\psi_{2\pm},\quad \sigma\to\sigma+\frac{\beta}{\pi}\,(\text{mod }1). \tag{3.97}$$

The latter transformation makes clear the fact that σ is a compact scalar field of compactification radius 1. The diagonal $U(1)$ symmetry (3.97) is to be understood as the correct incarnation of the unique microscopic $U(1)$ boson number conservation symmetry in the low-energy parton theory with *compact* gauge fluctuations, i.e., where Polyakov instantons are accounted for.

Although instantons have been explicitly taken into account in the derivation of the four-fermion 't Hooft vertex (3.96), the resulting effective theory is still an interacting gauge theory, and its infrared fate not altogether obvious. A natural route to confinement—our primary focus—is the instanton proliferation scenario, whereby the coefficient of the 't Hooft vertex (3.96) is assumed to run to strong coupling under renormalization group flow. One then expects spontaneous breaking of the global $U(1)$ symmetry (3.97), with σ acquiring an expectation value [62, 101, 113]. The σ field itself is the Goldstone mode of the broken continuous symmetry, and the microscopic hard-core boson system becomes superfluid, as discussed in Ref. [102]. Additionally, (3.96) shows that a constant θ parameter can be given a natural interpretation as a global shift in the phase of the condensate.

However, $\langle\sigma\rangle\neq 0$ only implies that the $U(1)$ symmetry is broken to a $\mathbb{Z}_2$ subgroup under which $\psi_{1\pm}\to-\psi_{1\pm}$ ($\beta=\pi$), since the 't Hooft vertex contains two ψ_1 fields. In terms of the original constituent bosons, this corresponds to a boson pair condensate $\langle b(\boldsymbol{x})b(\boldsymbol{x}')\rangle\neq 0$ without single-particle condensation, $\langle b(\boldsymbol{x})\rangle=0$, which preserves an Ising symmetry $b(\boldsymbol{x})\to -b(\boldsymbol{x})$ (see, e.g., Refs. [114, 115]). In terms of the fermionic partons, the order parameter $\langle\psi_{1+}^\intercal\gamma\psi_{2+}\psi_{1-}^\intercal\gamma\psi_{2-}\rangle\neq 0$ is analogous to that for charge-$4e$ superconductivity [116], but without concomitant Higgsing of the $U(1)$ gauge symmetry since (3.96) is manifestly gauge invariant (recall that ψ_1 and ψ_2 carry opposite gauge charge under the dynamical gauge field).

The residual global $\mathbb{Z}_2$ symmetry in such a paired superfluid can be further broken [114, 115], yielding a conventional superfluid phase with single-particle condensate $\langle b(\boldsymbol{x})\rangle\neq 0$. In the current context, this occurs if a gauge-invariant fermion bilinear condenses, $\langle\psi_1\psi_2\rangle\neq 0$. The various possible spinor/valley index structures of such a bilinear (suppressed here) allow in principle for both translationally invariant condensates and spatially modulated ones, i.e., supersolid phases.

3.7 Conclusion

In summary, we have presented a nonperturbative study of monopole-instanton effects in a (2+1)D parton gauge theory featuring Dirac fermions coupled to a compact $U(1)$ gauge field—CQED$_3$. This parton gauge theory is meant to encapsulate the universal low-energy physics of hard-core lattice bosons in the vicinity of a multicritical point separating fractionalized phases, such as boson fractional quantum Hall states, and conventional ones. While the compactness of the gauge field becomes irrelevant in fractionalized phases, which support deconfined excitations, we focused on developing an explicit understanding of the instanton dynamics that leads to confinement in conventional phases. As our first main result, we showed that in contrast to CQED$_3$ with massless fermions—an effective gauge theory describing the Dirac spin liquid—CQED$_3$ with *massive* fermions supports Euclidean fermion zero modes exponentially localized on instantons. The localization length of the zero mode "wavefunction" is found to be inversely proportional to the fermion mass, which in hindsight elucidates the absence, first observed by Marston, of normalizable zero modes in massless CQED$_3$. While we did not prove a rigorous index theorem guaranteeing the topological stability of such Euclidean zero modes, they were found to be in one-to-one correspondence with *Hamiltonian* quasi-zero modes occurring in the context of monopole operator dressing in conformal field theories associated with spin ordering transitions out of the Dirac spin liquid. In such theories, a nonzero fermion mass arises when the theory is canonically quantized on the sphere, and the resulting Hamiltonian quasi-zero modes can be understood as "massive deformations" of true zero modes protected by the Atiyah-Singer index theorem.

As our second main result, we combined semiclassical methods with our zero mode solutions to show by explicit derivation that instantons mediate an effective four-fermion interaction in the gauge theory, known as the 't Hooft vertex. This effective interaction explicitly breaks a spurious $U(1) \times U(1)_{\text{top}}$ symmetry of the classical parton Lagrangian to a diagonal $U(1)$ subgroup, corresponding to the physical boson number conservation symmetry of the microscopic model. Under the further assumption of confinement via instanton proliferation, we found that the 't Hooft vertex could naturally lead to two distinct superfluid phases: an ordinary single-particle condensate, but also a boson pair condensate without single-particle condensation, in which the global $U(1)$ symmetry is only broken to $\mathbb{Z}_2$.

Looking ahead, our approach based on semiclassical instanton techniques could be used to complement the Hamiltonian monopole-operator dressing approach to confinement transitions out of the Dirac spin liquid [36, 37]. Song *et al.* rely solely on microscopic symmetries and write down deformations of the conformal QED_3 Lagrangian consisting of (dressed) monopole operator/fermion composites allowed by those symmetries. Alternatively, 't Hooft vertices containing similar physics could be explicitly derived as follows. In the two-step route to confinement advocated by Song *et al.* and mentioned earlier, a fermion mass bilinear acquires an expectation value before instanton proliferation proceeds. Applying the semiclassical methods employed here after the first step, Euclidean zero modes for the resulting massive fermions could be searched for and used to derive a 't Hooft vertex that would encapsulate the range of symmetry-breaking phases made possible by instanton proliferation. Finally, the proof of an index theorem for massive Dirac fermions in 3D Abelian instanton backgrounds would be a desirable extension of the results presented here.

3.8 Appendix: Monopole miscellanea

3.8.1 Monopole harmonics

This appendix collates some well-known results on the theory of Dirac monopoles and is mostly self-contained. We first consider a spinless charge e in the field of a static point monopole at the origin,

$$\mathbf{B} = \frac{g}{r^2}\hat{\mathbf{r}}, \tag{3.98}$$

described by a Wu-Yang vector potential $\boldsymbol{A}$ [see Eq. (3.27)]. Classically, the spherical symmetry of the problem suggests conservation of angular momentum. The natural guess $\boldsymbol{l}=\mathbf{r}\times(\boldsymbol{p}-e\boldsymbol{A}) = \mathbf{r}\times m\boldsymbol{v}$, by minimal coupling, does not work because

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times m\boldsymbol{v}), \\
&= \mathbf{r}\times m\ddot{\mathbf{r}}, \\
&= \mathbf{r}\times e(\boldsymbol{v}\times\mathbf{B}), \\
&= \mathbf{r}\times\frac{eg}{r^3}(\boldsymbol{v}\times\mathbf{r}), \qquad (3.99)\\
&= \frac{eg}{mr^3}\boldsymbol{l}\times\mathbf{r}. \qquad (3.100)
\end{aligned}$$

This is generically non-zero, suggesting $\boldsymbol{l}$ is not conserved. Using a formula for the vector triple product, Eq. (3.99) can be rewritten as [117]

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} &= \mathbf{r}\times\frac{eg}{r^3}(\dot{\mathbf{r}}\times\mathbf{r}), \\
&= \frac{eg}{r^3}\left[r^2\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}\right], \\
&= \frac{eg}{r^3}\left[r^2\dot{\mathbf{r}} - \frac{1}{2}\frac{\mathrm{d}r^2}{\mathrm{d}t}\mathbf{r}\right], \\
&= eg\left[\frac{1}{r}\mathbf{r} - \frac{1}{r^2}|\dot{\mathbf{r}}|\mathbf{r}\right], \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(eg\frac{\mathbf{r}}{r}\right). \qquad (3.101)
\end{aligned}$$

This implies a conserved angular momentum

$$\boldsymbol{L} = \mathbf{r}\times(\boldsymbol{p}-e\boldsymbol{A}) - q\hat{\mathbf{r}}, \qquad q\equiv eg\in\mathbb{Z}/2. \tag{3.102}$$

One can explicitly prove (post-quantization) that

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{3.103}$$

Since $[r^2, \boldsymbol{L}]=0$, these two operators can be simultaneously diagonalized and $\boldsymbol{L}$ can be studied for fixed r. Also, since $[L_z, \boldsymbol{L}^2]=0$ and $[L_i, L_j]=i\epsilon_{ijk}L_k$, we have the familiar

$$\begin{aligned} \boldsymbol{L}^2 Y_{q,L,M}(\theta,\varphi) &= L(L+1)Y_{q,L,M}(\theta,\varphi), \\ L_z Y_{q,L,M}(\theta,\varphi) &= M Y_{q,L,M}(\theta,\varphi). \end{aligned} \tag{3.104}$$

The sections $Y_{q,L,M}$ are called *monopole harmonics,* and their exact form is gauge dependent, which means northern and southern versions differ by a gauge transformation in the Wu-Yang formulation. Only L, M are quantum numbers, while q is a parameter that determines one complete set of harmonics. Just based on the $\mathfrak{su}(2)$ algebra, allowed values of L must be a subset of $\{0, 1/2, 1, ...\}$, while $M \in \{-L, -L+1, ..., L\}$. However,

$$\begin{aligned} \boldsymbol{L}^2 &= (\boldsymbol{l} - q\hat{\mathbf{r}})^2 \\ &= \boldsymbol{l}^2 + q^2 - q(\boldsymbol{l}\cdot\hat{\mathbf{r}} + \hat{\mathbf{r}}\cdot\boldsymbol{l}) \\ &= \boldsymbol{l}^2 + q^2. \end{aligned} \tag{3.105}$$

For fixed $2q \in \mathbb{Z}$, this gives a bound on the eigenvalues

$$L(L+1) \geq q^2. \tag{3.106}$$

The solution of the inequality above is $L \geq |q|$. To prove this, we may take $q \geq 0$ without loss of generality as the inequality is independent of $\operatorname{sgn} q$. Factorization and substitution of $q = n/2$, where $n \in \mathbb{N}$, gives

$$\left(L - \frac{-1+\sqrt{1+n^2}}{2}\right)\left(L - \frac{-1-\sqrt{1+n^2}}{2}\right) \geq 0. \tag{3.107}$$

Both brackets must be of the same sign. Positivity of L implies

$$L \geq \frac{-1+\sqrt{1+n^2}}{2}.$$

$L \geq n/2 = q$ satisfies this inequality, since for $n \geq 0$,

$$\frac{-1+\sqrt{1+n^2}}{2} \leq \frac{-1+\sqrt{1+n^2+2n}}{2} = \frac{n}{2},$$

To see that this is the smallest satisfactory half-integral L, note that the next smallest value of L does not satisfy:

$$\frac{n-1}{2} = \frac{-1+\sqrt{n^2}}{2} \leq \frac{-1+\sqrt{1+n^2}}{2}.$$

We thus have the result that $L \geq |q|$.

Written in spherical coordinates, Eq. (3.102) reads

$$\boldsymbol{L}^{N/S} = -q\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}[i\csc\theta\partial_\varphi - q(\cot\theta \mp \csc\theta)] - \hat{\boldsymbol{\varphi}}i\partial_\theta. \tag{3.108}$$

This implies

$$L_z = -i\partial_\varphi \mp q, \tag{3.109}$$

for the $\hat{z}$ component, which has northern and southern eigenfunctions of the form $\exp[i(M \pm q)\varphi]$. The requirement of a single-valued wavefunction then mandates $(M \pm q) \in \mathbb{Z}$, which is satisfied if M is (half-)integral whenever q is (half-)integral. Together with $L \geq |q|$, this determines the allowed values of (L, M) as

$$L \in \{|q|, |q|+1, ...\}, \quad M \in \{-L, -L+1, ..., L\}. \tag{3.110}$$

For completeness, we provide a general formula for the monopole harmonics $Y_{q,L,M}$ in terms of the Wigner D-matrix. An elegant derivation can be found in Ref. [79]. In the *northern hemisphere,*

$$Y_{q,L,M}(\theta_N, \varphi) = \sqrt{\frac{2L+1}{4\pi}} \left[D^L_{M,-q}(\varphi, \theta, -\varphi)\right]^*, \tag{3.111}$$

where $\theta_N \in [0, \pi)$. The southern versions (valid on the south pole) are obtained by a gauge transformation,

$$Y_{q,L,M}(\theta_S, \varphi) = e^{-i2q\varphi} Y_{q,L,M}(\theta_N, \varphi). \tag{3.112}$$

The Wigner D-matrix is defined in terms of Euler angles (α, β, γ) as

$$\begin{aligned} D^j_{m',m}(\alpha, \beta, \gamma) &= \langle jm'| \, e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \, |jm\rangle \,, \\ &= e^{-im'\alpha} d^j_{m'm}(\beta) e^{-im\gamma}. \end{aligned} \tag{3.113}$$

Using the formula above, the first two $q=1/2$ harmonics are given by

$$\begin{aligned} Y_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}(\theta_N,\varphi) &= -\frac{1}{\sqrt{2\pi}}e^{i\varphi}\sin\frac{\theta}{2}, \\ Y_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}(\theta_N,\varphi) &= \frac{1}{\sqrt{2\pi}}\cos\frac{\theta}{2}, \end{aligned} \tag{3.114}$$

in the north. Their southern versions are given by a gauge transformation $\exp(-i\varphi)$.

For $q=-1/2$, the first two northern harmonics are

$$\begin{aligned} Y_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(\theta_N,\varphi) &= \frac{1}{\sqrt{2\pi}}\cos\frac{\theta}{2}, \\ Y_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}}(\theta_N,\varphi) &= \frac{1}{\sqrt{2\pi}}e^{-i\varphi}\sin\frac{\theta}{2}, \end{aligned} \tag{3.115}$$

with their southern versions now obtained by a gauge transformation $\exp(i\varphi)$.

3.8.2 Monopole spinor harmonics

We now consider a spin-1/2 particle of charge e in a monopole background. The total angular momentum for a spin-1/2 is

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\sigma}. \tag{3.116}$$

The allowed eigenvalues of $\boldsymbol{J}^2$ and J_z, respectively denoted $j(j+1)$ and m_j, follow from rules for addition of angular momenta:

$$\begin{aligned} j &\in \{L-1/2, L+1/2\} = \left\{|q|-\frac{1}{2}, |q|+\frac{1}{2}, ...\right\}, \\ m_j &\in \{-j,-j+1,...,j\}. \end{aligned} \tag{3.117}$$

The same rules also provide the (angular) eigensections $\mathcal{Y}^L_{q,j,m_j}$, called *monopole spinor harmonics*,

$$\begin{aligned} \mathcal{Y}^{j-1/2}_{q,j,m_j}(\theta,\varphi) &= \frac{1}{\sqrt{2j}}\begin{pmatrix} \sqrt{j+m_j}Y_{q,j-\frac{1}{2},m_j-\frac{1}{2}} \\ \sqrt{j-m_j}Y_{q,j-\frac{1}{2},m_j+\frac{1}{2}} \end{pmatrix}, \\ \mathcal{Y}^{j+1/2}_{q,j,m_j}(\theta,\varphi) &= \frac{1}{\sqrt{2j+2}}\begin{pmatrix} -\sqrt{j-m_j+1}Y_{q,j+\frac{1}{2},m_j-\frac{1}{2}} \\ \sqrt{j+m_j+1}Y_{q,j+\frac{1}{2},m_j+\frac{1}{2}} \end{pmatrix}, \end{aligned} \tag{3.118}$$

where $Y_{q,L,M}$ are the monopole harmonics defined in Sec. 3.8.2, and their coefficients are Clebsch-Gordan. For a given q, these spinor harmonics are a complete, orthonormal set of 2-spinor eigensections of $\boldsymbol{L}^2, \boldsymbol{\sigma}^2, \boldsymbol{J}^2,\ J_z$.

For use in the main text, we also record here the action of $\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$ on $\mathcal{Y}^L_{q,j,m_j}$, which can be explicitly evaluated using Eq. (3.118). Alternatively, since $\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$ commutes with $\boldsymbol{J}$, the most it can do is mix the $L{=}j{\pm}1/2$ states. A general formula is

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\mathcal{Y}^{j\pm1/2}_{q,j,m_j} = a_\pm \mathcal{Y}^{j+1/2}_{q,j,m_j} + b_\pm \mathcal{Y}^{j-1/2}_{q,j,m_j}. \tag{3.119}$$

Substituting in Eq. (3.118) and using the known forms of the monopole harmonics provides linear equations for the coefficients, which turn out to be [118]

$$\begin{aligned} a_+ = -b_- &= \frac{2q}{2j+1}, \\ a_- = b_+ &= -\frac{\sqrt{(2j+1)^2-4q^2}}{2j+1}. \end{aligned} \tag{3.120}$$

3.9 Appendix: Self-adjoint operators

This appendix elaborates on some technical aspects of the path integrals studied in Sec. 3.5.2-3.5.3, particularly the difficulties involved in suitably defining the functional measure and connections with index theorems. We shall first proceed along a standard route used in the physics literature to define path integral measures. The path integral of interest is

$$Z_F[a^+_\mu] \equiv \int D(\bar{\psi},\psi) e^{-\int \mathrm{d}^3x \bar{\psi}\mathcal{D}\psi}, \tag{3.121}$$

where a^+_μ describes a single charge $1/2e$ instanton located at z_+. Although an even number of fermion flavors are required for this theory to make physical sense, a single flavor is sufficient to highlight some of the general mathematical difficulties that arise in this problem. The (massive) Euclidean Dirac operator $\mathcal{D}$ is defined as

$$\begin{aligned} \mathcal{D} &\equiv \gamma^\mu\partial_\mu - i\gamma^\mu a^+_\mu + m, \\ &= (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left[\partial_r - \frac{1}{r}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \frac{3}{4}\right) - \frac{q}{r}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right] + m. \end{aligned} \tag{3.122}$$

The (naïvely taken) adjoint of the Dirac operator is

$$\begin{aligned}\mathcal{D}^\dagger &= -\gamma^\mu\partial_\mu + i\gamma^\mu a_\mu^+ + m \\ &= -(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left[\partial_r - \frac{1}{r}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \frac{3}{4}\right) - \frac{q}{r}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right] + m.\end{aligned} \tag{3.123}$$

The second lines of Eqs. (3.122)-(3.123) are the results of Sec. 3.4.2-3.4.3. These operators are not Hermitian, but one can consider the Hermitian combinations $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ with eigenvalue equations

$$\begin{aligned}\mathcal{D}^\dagger\mathcal{D}u_i(x-z_+) &= a_i u_i(x-z_+), \\ \mathcal{D}\mathcal{D}^\dagger\tilde{u}_i(x-z_+) &= \tilde{a}_i\tilde{u}_i(x-z_+),\end{aligned} \tag{3.124}$$

for an instanton located at z_+.

We have used the term "Hermitian operator" to refer to what is called a "symmetric operator" in the mathematical literature on unbounded operators [119, 120]. Acting on a Hilbert space, a densely defined Hermitian (or symmetric) operator Λ satisfies $(u, \Lambda v) = (\Lambda u, v)$, for any $u, v \in \mathrm{Dom}(\Lambda)$. To be self-adjoint, that is for $\Lambda = \Lambda^\dagger$ to hold as an operator equation, one also requires $\mathrm{Dom}(\Lambda) = \mathrm{Dom}(\Lambda^\dagger)$, which does not follow from the Hermiticity condition for unbounded operators (such as the Dirac operator under consideration), and usually $\mathrm{Dom}(\Lambda) \subset \mathrm{Dom}(\Lambda^\dagger)$. Only self-adjoint operators have the desirable properties of possessing a complete set of eigenfunctions and real eigenvalues. However, this fact is typically ignored, and one proceeds to use the eigenfunctions of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ as a basis to facilitate a mode expansion of the Fermi fields in the path integral, assuming such operators are indeed self-adjoint. This is usually harmless, but not so in current circumstances as we shall momentarily show. In any case, a loose mathematical justification of this ignorance can be made by assuming that the Hermitian operators above possess a unique, or a family, of self-adjoint extensions. An arbitrary choice in this family will have the required property of possessing a complete basis of eigenfunctions to facilitate mode expansions.

However, a self-adjoint extension of a Hermitian operator Λ involves imposing boundary conditions on its eigenfunctions which effectively shrink or enlarge $\mathrm{Dom}(\Lambda)$ and $\mathrm{Dom}(\Lambda^\dagger)$

until they are equal. The eigenfunctions of the self-adjoint extension will form a complete basis only for the final domain $\mathrm{Dom}'(\Lambda)$. Assuming $\mathcal{D}^\dagger\mathcal{D}$ or $\mathcal{D}\mathcal{D}^\dagger$ have been made self-adjoint, mode expansions of Fermi fields in terms of the eigenfunctions of these operators effectively assume that the space of fields being integrated over in the path integral is the same as $\mathrm{Dom}(\mathcal{D}^\dagger\mathcal{D})$ or $\mathrm{Dom}(\mathcal{D}\mathcal{D}^\dagger)$. This typically does not warrant close analysis since one hopes (usually correctly) that all important physical effects are accounted for in this procedure. In the present case, as we show below, the ZM solution lies outside these domains and is thus missed if the eigenfunctions of $\mathcal{D}^\dagger\mathcal{D}$ or $\mathcal{D}\mathcal{D}^\dagger$ are used for a mode expansion or in the definition of the functional measure.

We start with the following paradox. The ZMs of $\mathcal{D}$ and $\mathcal{D}^\dagger$ were calculated in Sec. 3.4.2 and 3.4.3 respectively. For $g=-1/2e$, the operator $\mathcal{D}^\dagger$ has a normalizable ZM

$$\tilde{\psi}_0^-(r,\theta,\varphi) = \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^{1/2}_{-1/2,0,0}(\theta,\varphi), \tag{3.125}$$

which implies

$$(\tilde{\psi}_0^-, \mathcal{D}^\dagger \tilde{\psi}_0^-) = 0. \tag{3.126}$$

However, using Eq. (3.122), one finds explicitly that $\mathcal{D}\tilde{\psi}_0^- = 2m\tilde{\psi}_0^-$, which seems to imply

$$(\mathcal{D}\tilde{\psi}_0^-, \tilde{\psi}_0^-) = 2m\left\|\tilde{\psi}_0^-\right\|^2 \neq 0. \tag{3.127}$$

The implied consequence is that $(\tilde{\psi}_0^-, \mathcal{D}^\dagger\tilde{\psi}_0^-) \neq (\mathcal{D}\tilde{\psi}_0^-, \tilde{\psi}_0^-)$. Before a resolution of this is pointed out, we note that $\mathcal{D}\tilde{\psi}_0^- = 2m\tilde{\psi}_0^-$ implies then that $\mathcal{D}^\dagger\mathcal{D}\tilde{\psi}_0^- = 0$ so that $\dim\ker\mathcal{D}^\dagger\mathcal{D} - \dim\ker\mathcal{D}\mathcal{D}^\dagger = 0$. It is the latter that is typically calculated as $\mathrm{index}(\mathcal{D})$ in typical proofs of index theorems used in physics. This highlights the difficulty in producing an index theorem for the current scenario, and also in defining the functional measure of the path integral using eigenfunctions of $\mathcal{D}^\dagger\mathcal{D}$ or $\mathcal{D}\mathcal{D}^\dagger$.

The resolution of the paradox lies in a careful examination of the domains of the operators $\mathcal{D}^\dagger$ and $\mathcal{D}$, which turn out to be subspaces of square integrable functions. In the subspace of spinors ψ with fixed angular part $\mathcal{Y}^{1/2}_{1/2,0,0}(\theta,\varphi)$, and for $eg = 1/2$ (i.e., in the

$Q = 1$ instanton sector discussed in Sec. 3.5.2), the action becomes:

$$S = \int \mathrm{d}^3x \, \bar{\psi}(ip_r + m)\psi, \tag{3.128}$$

where one defines a "radial momentum operator" [121],

$$p_r = -\frac{i}{2}(\hat{\mathbf{r}} \cdot \nabla + \nabla \cdot \hat{\mathbf{r}}) = -i\left(\partial_r + \frac{1}{r}\right). \tag{3.129}$$

The adjoint Dirac operator $\mathcal{D}^\dagger$ in Eq. (3.123) has been naïvely derived from the form of $\mathcal{D}$ by essentially assuming p_r is Hermitian on $\mathrm{Dom}(\mathcal{D})$. However, this Hermiticity condition is violated on the ZM,

$$\psi_0(r) \propto \frac{1}{r} e^{-mr}, \tag{3.130}$$

of the operator $(ip_r + m)$ appearing in the action (3.128), for $(p_r\psi_0, \psi_0) \neq (\psi_0, p_r\psi_0)$. This is the reason for the paradoxical equations (3.126)-(3.127). However, to restrict the path integral over ψ to a function space on which p_r *is* Hermitian is to exclude ZMs and their associated physics. To determine the space of fields (ψ and $\bar{\psi}$) that one should integrate over, we use the necessary condition that the Minkowski action must be real-valued.

The reality of the Minkowski action in a unitary quantum field theory translates to Osterwalder-Schrader or reflection positivity of the corresponding Euclidean action, which is invariance $\Theta(S) = S$ under a form of complex conjugation followed by Euclidean time reversal [72]. This transformation acts on fermions as an involution of the Grassmann algebra [73], which for our particular choice of Dirac matrices can be chosen as

$$\Theta(\psi_\alpha(x)) = \sigma_z^{\alpha\beta}\bar{\psi}_\beta(\theta x), \tag{3.131}$$

$$\Theta(\bar{\psi}_\alpha(x)) = \sigma_z^{\alpha\beta}\psi_\beta(\theta x), \tag{3.132}$$

where θ flips the sign of the time (z) coordinate. Additionally, Θ complex conjugates c-numbers and reverses the order of Grassmann variables, e.g., $\Theta(\psi_\alpha\psi_\beta\psi_\gamma) = \Theta(\psi_\gamma)\Theta(\psi_\beta)\Theta(\psi_\alpha)$. Gauge fields transform as $\Theta(a_0(x)) = -a_0(\theta x)$ and $\Theta(a_i(x)) = a_i(\theta x)$ [122]. One can show that under Θ, Eq. (3.128) transforms as:

$$\Theta(S) = S + \int \mathrm{d}\Omega \, (-r^2\bar{\psi}\psi)\big|_{r=0}^{r=\infty}, \tag{3.133}$$

where the boundary term follows from an integration by parts. However, for reflection positivity to hold, the boundary term is required to vanish.

The upper limit of the boundary term vanishes if we require all fields to be square integrable. Say $\psi \sim r^{-\beta}\chi$ as $r \to 0$, where χ is a Grassmann number without r dependence. Then square integrability requires $\lim_{r\to 0} r^{3-2\beta}$ to exist, which means $\beta < 3/2$. Since square integrable functions are therefore at most as singular as $r^{-3/2+\epsilon}$, where $\epsilon > 0$, the lower limit of the boundary term is at most as singular as

$$\lim_{r\to 0} \frac{r^2}{r^{3/2-\epsilon}\, r^{3/2-\delta}}, \tag{3.134}$$

where $\delta > 0$. The existence of this limit requires $\delta + \epsilon > 1$. The choice $\epsilon = \delta > 1/2$ restricts both field integrations (over ψ and $\bar{\psi}$) to the subset of square integrable functions that are less singular than $1/r$ at the origin. As discussed earlier, this excludes the ZM from both path integrals, over ψ and $\bar{\psi}$. To remedy this, we may set $\epsilon = 0$ and $\delta > 1$, so that functions that behave as $1/r$ as $r \to 0$ (such as the ZM ψ_0) are included in the path integration over ψ, but not in that over $\bar{\psi}$ to maintain reflection positivity of the Euclidean action. The problem now reduces to finding self-adjoint operators with domains as these new subspaces of $L^2(\mathbb{R}^3)$, so that the path integral measure can be adequately defined. We will simply assume such operators, with eigenfunctions $\{\psi_i(r,\theta,\varphi)\}$ and $\{\tilde{\psi}_i(r,\theta,\varphi)\}$, exist and will expand the Fermi fields as

$$\begin{aligned} \psi &= \psi_0(r,\theta,\varphi)\eta_0 + \sum_i{}' \psi_i(r,\theta,\varphi)\eta_i, \\ \bar{\psi} &= \sum_i{}' \tilde{\psi}_i(r,\theta,\varphi)\bar{\eta}_i, \end{aligned} \tag{3.135}$$

where the primed sum includes non-ZM contributions and $\bar{\eta}_i$, η_i are independent Grassmann variables.

For $eg = -1/2$, i.e., in the anti-instanton sector $Q = -1$, the situation is reversed. In a subspace of spinors with fixed angular part $\mathcal{Y}^{1/2}_{-1/2,0,0}(\theta,\varphi)$, the action is

$$S = \int \mathrm{d}^3x\, \bar{\psi}(-ip_r + m)\psi. \tag{3.136}$$

The operator $(-ip_r+m)$ does not have normalizable ZMs. However, integrating by parts, we obtain

$$S = \int \mathrm{d}^3x \, [(ip_r+m)\bar{\psi}]\psi + \int \mathrm{d}\Omega \, (-r^2\bar{\psi}\psi)\Big|_{r=0}^{r=\infty} , \tag{3.137}$$

and the operator (ip_r+m) does have the ZM (3.130). Contrary to the $Q=1$ sector, we now include functions with the limiting behavior of the ZM in the path integral over $\bar{\psi}$, but exclude them from that over ψ so that the boundary term in Eq. (3.137) vanishes and reflection positivity is maintained. This implies mode expansions of the form

$$\begin{aligned} \psi &= \sum_i{}' \, \tilde{\psi}_i(r,\theta,\varphi)\eta_i, \\ \bar{\psi} &= \tilde{\psi}_0(r,\theta,\varphi)\bar{\eta}_0 + \sum_i{}' \, \tilde{\psi}_i(r,\theta,\varphi)\bar{\eta}_i. \end{aligned} \tag{3.138}$$

Chapter 4

Magnet to chiral spin liquid in $2d$ Ising spins

4.1 Introduction

Quantum phase transitions out of fractionalized spin-liquid states in two spatial dimensions (2d) are an active area of research in the study of quantum matter [24, 123, 124]. From a theoretical standpoint, the universal properties of spin liquids are captured by slave-particle gauge theories with bosonic or fermionic spinons [20]. In describing a transition from a spin liquid to a conventional ordered phase, two effects must be accounted for: spontaneous symmetry breaking, and confinement of excitations with nonzero gauge charge. In theories with bosonic spinons, such as Schwinger boson theories of $\mathbb{Z}_2$ spin liquids [125], confinement concomitant with symmetry breaking results from the condensation of bosonic visons and/or spinons which carry nontrivial quantum numbers under global symmetries [126]. In descriptions of spin liquids with fermionic gauge theories, such as the $U(1)$ gauge theory of the Dirac spin liquid [32], condensation of a fermion bilinear leads to symmetry breaking and opening of a fermion mass gap [36, 55, 77, 87, 90, 91, 127, 128]. The opening of this gap is followed by the proliferation of monopole-instantons, which induces confinement [9, 26, 40]. In addition to those of fermion bilinears, the symmetry quantum numbers of monopole operators are important to determine the precise patterns of symmetry breaking [35–37, 45, 56].

An emerging platform for the observation of spin liquids and their competition with various ordered phases is spin-orbit coupled Mott insulators; such systems have been a focus of quantum materials research in recent years [129, 130]. In those systems, strong correlations promote the formation of local magnetic moments, while spin-orbit coupling entangles spin and orbital degrees of freedom and introduces anisotropy in the magnetic exchange interactions. The paradigmatic class of materials in this context is Kitaev materials [66, 131], described at low energies by effective spin-1/2 moments on the honeycomb lattice and governed by a Kitaev-like Hamiltonian [23] in which $SU(2)$ spin rotation symmetry is broken to a discrete subgroup. Recently, the Kitaev material α-$RuCl_3$ has attracted much attention due to the observation of a magnetic-field-induced transition from a zigzag-ordered state at low fields to a paramagnetic state at higher fields [132–138]. Remarkably, the latter state appears to exhibit a quantized thermal Hall conductance $\kappa_{xy}/T = 1/2$ in units of $\pi k_B^2/6\hbar$, suggestive of a gapped chiral spin liquid phase with intrinsic topological order [137, 138]. In Ref. [139], a theory of the transition between the zigzag-ordered state and the chiral spin liquid was developed, based on dualities of (2+1)D gauge theories with $U(N)$ gauge groups.

Motivated by these recent developments, we ask the general question whether, from an effective field theory point of view, the phase diagram of spin systems with Ising spin-flip symmetry can exhibit a continuous transition from Ising magnetic order to a gapped chiral spin liquid. Given that the chiral spin liquid is a topological phase without a local order parameter, while the Ising-ordered phase exhibits conventional symmetry breaking, such a transition is necessarily an exotic non-Landau transition involving the fractionalized degrees of freedom of the spin liquid, and possibly monopole-instanton configurations in the associated emergent gauge field.

In this chapter, we use effective field theory methods to show that such an exotic transition is in general possible. Our approach is based on a parton decomposition of the Ising spin operator, involving fractionalized Majorana fermion degrees of freedom coupled to an emergent non-Abelian $SO(N)$ gauge field. Our study can be viewed as a generalization of Ref. [102]—which studies transitions between Mott insulating, fractional quantum Hall, and superfluid states of bosons with continuous $U(1)$ symmetry—to systems of Ising spins with a discrete $\mathbb{Z}_2$ symmetry. Chern-number changing transitions between different topologically superconducting states of the Majorana partons correspond to different phases of the Ising spin system. While various spin-liquid states can be accessed in this way, including spin liquids with non-Abelian topological order, we focus on Abelian chiral spin liquids with the topological order of the bosonic fractional quantum Hall (Laughlin) state. In our construction, such a chiral spin liquid is naturally proximate to a trivial paramagnet and to a magnetically ordered phase with broken $\mathbb{Z}_2$ symmetry. Using recently conjectured dualities of $SO(N)$ gauge theories in (2+1)D, we find that the critical theory for the ordering transition from the trivial paramagnet is, as expected, dual to the standard 3D Ising Wilson-Fisher theory, while transitions involving the chiral spin liquid are described by theories of massless Majorana fields coupled to an $SO(N)$ gauge field with a Chern-Simons term. In particular, we show that a direct transition from the chiral spin liquid to the Ising-ordered phase is possible and can be protected by inversion symmetry on the honeycomb lattice.

Finally, in analogy with our previous work on bosons with $U(1)$ symmetry [1], we show that the breaking of $\mathbb{Z}_2$ symmetry in the confined, ordered phase can be understood as a nontrivial consequence of Euclidean Majorana zero modes (ZMs) bound to monopole-instantons. By contrast with monopole-instantons in $U(1)$ theories, the latter carry here a $\mathbb{Z}_2$ topological charge under the $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry of $SO(N)$ gauge theory in (2+1)D. Under the assumption that the infrared effects of such instantons is adequately captured by a semiclassical instanton-gas treatment, the Euclidean ZMs lead to an effective interaction among Majorana fermions that is analogous to the 't Hooft vertex in quantum chromodynamics [63–65]. This interaction intertwines the Ising $\mathbb{Z}_2$ symmetry with the $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry. As a consequence of this intertwinement, the spontaneous breakdown of $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry expected in a confined phase [140, 141] automatically results in long-range Ising order for the underlying spin system.

The rest of the chapter is structured as follows. In Sec. 4.2, we review the parton description of bosons with $U(1)$ symmetry [1, 102], as a means to introduce the basic ideas and methods that we will generalize to Ising spins with $\mathbb{Z}_2$ spin-flip symmetry. In Sec. 4.3, we introduce our parton decomposition of Ising spins and discuss the various phases that can be accessed within the parton mean-field framework: chiral spin liquids, a trivial paramagnet, and an ordered phase with broken $\mathbb{Z}_2$ symmetry. Using $SO(N)$ dualities in (2+1)D, we discuss transitions between these phases. In Sec. 4.4, we turn our focus to the broken phase. Although conventional from the microscopic standpoint, its description within the parton framework necessitates accounting for nonperturbative confinement effects. We discuss $\mathbb{Z}_2$ monopole-instantons in $SO(N)$ gauge theory, show that Euclidean fermion ZMs are bound to them, and resum the instanton gas to exhibit the 't Hooft vertex that properly accounts for the broken $\mathbb{Z}_2$ symmetry. We conclude in Sec. 4.5 with a summary of our main results and suggestions for future research.

4.2 Warm-up: bosons with $U(1)$ symmetry

We begin by briefly reviewing the problem of continuous quantum phase transitions in systems of hardcore bosons with the global $U(1)$ symmetry associated with particle-number conservation. To aid the passage from $U(1)$ bosons to $\mathbb{Z}_2$ spins, we re-interpret the results of Ref. [102] in the context of dualities of (2+1)D quantum field theories with unitary gauge groups [142–151]. We also point out the key role of monopole-instantons and the Euclidean fermion ZMs bound to them in accounting for the physics of broken-symmetry phases [1].

4.2.1 Parton construction

We begin by considering a system of charge-1 hardcore bosons on a 2d lattice described by operators $b(\boldsymbol{r})$ ($b^\dagger(\boldsymbol{r})$) that annihilate (create) a boson on lattice site $\boldsymbol{r}$. We then write the boson operator as

$$b(\boldsymbol{r}) = f_1(\boldsymbol{r}) f_2(\boldsymbol{r}), \tag{4.1}$$

where $f_1(\boldsymbol{r})$ and $f_2(\boldsymbol{r})$ are fermionic annihilation operators. This parton decomposition [152, 153] introduces a local gauge redundancy. We consider parton mean-field ansätze such that f_1 forms a Chern insulator with Chern number 1 and f_2 forms a Chern insulator with Chern number C. In general, such ansätze have a $U(1)$ gauge structure with emergent gauge field a_μ; we assume f_1 (f_2) carries gauge charge -1 ($+1$), and f_2 carries the unit global $U(1)$ charge of the boson system.

Upon integrating out the massive partons f_1 and f_2, we obtain the low-energy effective Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi} ada + \frac{C}{4\pi}(a + A)d(a + A), \tag{4.2}$$

where $adb \equiv \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda$ for any two gauge fields a_μ and b_μ, and we have added a background gauge field A_μ which couples to the global $U(1)$ symmetry. Performing the shift $a_\mu \to a_\mu - \frac{C}{C+1} A_\mu$ to eliminate the cross terms, we obtain:

$$\mathcal{L} = \frac{C+1}{4\pi} ada + \frac{1}{4\pi}\frac{C}{C+1} AdA. \tag{4.3}$$

For values of C other than $C = -2, -1, 0$, this describes an Abelian fractional quantum Hall state with ground-state degeneracy $|C+1|^g$ on a genus-g surface and quantized Hall conductance $\sigma_{xy} = C/(C+1)$. For $C = 0$, the ground state is unique and the Hall conductance vanishes: this is the Bose Mott insulator. For $C = -2$, the ground state is again unique, but the Hall conductance is nonzero, $\sigma_{xy} = 2$: this is a bosonic integer quantum Hall state [154–157]. For $C = -1$, the Chern-Simons term for a_μ cancels and we must keep a Maxwell term. Integrating out a_μ, we obtain in the low-energy limit,

$$S_{\text{eff}}[A_\mu] = \frac{1}{8\pi^2} \int \frac{d^3q}{(2\pi)^3} A_\mu(-q) \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) A_\nu(q), \tag{4.4}$$

where $\eta^{\mu\nu}$ is the (2+1)D Minkowski metric. Equation (4.4) describes the (transverse) Meissner response of a charged superfluid, thus the $C = -1$ phase is a superfluid of the b bosons.

4.2.2 Phase transitions and $U(1)$ dualities

At the parton mean-field level, transitions between the different bosonic phases mentioned above are Chern-number-changing (topological) transitions in the f_2 parton band structure. For simplicity, we focus on transitions between the Mott insulator ($C = 0$), superfluid ($C = -1$), and $\nu = 1/2$ bosonic fractional quantum Hall state ($C = 1$) [158]. To derive a critical theory for the transition, we can integrate out f_1, which remains gapped across the transition. This generates a $U(1)_1$ Chern-Simons term in the effective theory. By contrast, f_2 becomes gapless at the transition and must be kept in the critical theory. In the low-energy limit and near the transition, the f_2 band structure will generically consist of two Dirac points $\boldsymbol{K}_+$ and $\boldsymbol{K}_-$, such that $f_2(\boldsymbol{r})$ can be expanded near the Dirac points: $f_2(\boldsymbol{r}) \approx \sum_{k=\pm} e^{i\boldsymbol{K}_k\cdot\boldsymbol{r}}\psi_{2k}(\boldsymbol{r})$, where ψ_{2+}, ψ_{2-} are slow two-component Dirac fields with mass m_+, m_- respectively. The low-energy effective theory interpolating between all three phases is thus:

$$\mathcal{L} = \frac{1}{4\pi} ada + \sum_{k=\pm} \bar{\psi}_{2k}(i\not{D} - m_k)\psi_{2k}, \tag{4.5}$$

where $\not{D} = \gamma^\mu D_\mu$ with $D_\mu = \partial_\mu - i(a_\mu + A_\mu)$ the gauge-covariant derivative, and γ^μ are (2+1)D Dirac matrices.

The phases described in Sec. 4.2.1 are recovered when both Dirac fermions are massive and can be integrated out (Fig. 4.1). Since a single massive two-component Dirac fermion with mass m carries a partial Chern number of $\frac{1}{2}\operatorname{sgn} m$ [159–161], when both $m_\pm > 0$, we recover Eq. (4.2) with $C = 1$, i.e., the bosonic $\nu = 1/2$ Laughlin state. When both

$m_\pm < 0$, we find the superfluid with $C = -1$. When the masses are of opposite sign, the partial Chern numbers from the two Dirac fermions cancel out and we obtain the trivial Mott insulator with $C = 0$.

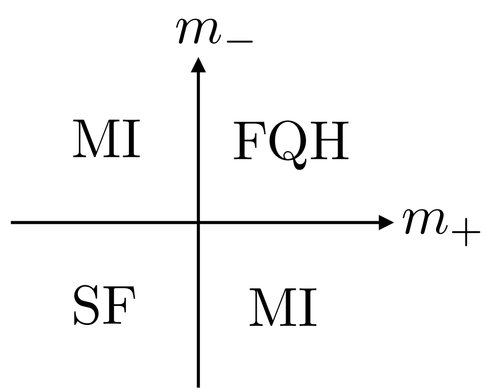

Figure 4.1: Phase diagram for bosons with $U(1)$ symmetry as a function of the two tuning parameters m_+, m_-. MI: trivial Mott insulator; SF: superfluid; FQH: $\nu = 1/2$ bosonic Laughlin state.

Transitions between the Mott insulator and superfluid and between the Mott insulator and the bosonic Laughlin state can be accessed by tuning m_- through zero at $m_+ < 0$ and $m_+ > 0$, respectively. When $m_+ < 0$, integrating out ψ_{2+} and setting $m_- = 0$ gives

$$\mathcal{L} = \frac{1}{8\pi} ada + \bar{\psi}_{2-} i \not{D} \psi_{2-} - \frac{1}{4\pi} Ada - \frac{1}{8\pi} AdA, \tag{4.6}$$

a single two-component Dirac fermion coupled to a $U(1)$ Chern-Simons gauge field at level 1/2, which is conjectured [142, 146–148, 150, 151] to be dual in the infrared to the (2+1)D Wilson-Fisher fixed point of a single complex scalar ϕ,

$$\mathcal{L}_{\text{dual}} = |(\partial_\mu - iA_\mu)\phi|^2 - \lambda|\phi|^4. \tag{4.7}$$

We thus recover the known fact that the boson superfluid-Mott insulator transition is in the 3D XY universality class (in the presence of particle-hole symmetry, which is assumed here due to the relativistic Dirac dispersions). Furthermore, a fermion mass term for ψ_{2-} maps to a mass term for the scalar of the same sign[1], such that $m_- > 0$ corresponds to

[1] The reason the fermion and scalar masses are of the same sign is that we are in fact using a time-reversed version of the duality in Ref. [146].

the disordered (Mott insulating) phase of the scalar and $m_- < 0$ to its broken symmetry (superfluid) phase. As is clear from Eq. (4.7), the dual scalar field ϕ carries charge 1 under the background gauge field A_μ and can thus be directly interpreted as the continuum limit of the boson operator b near the superfluid-insulator transition.

When $m_+ > 0$, integrating out ψ_{2+} and setting $m_- = 0$ yields

$$\mathcal{L} = \frac{3}{8\pi} ada + \bar{\psi}_{2-} i \not{D} \psi_{2-} + \frac{1}{4\pi} Ada + \frac{1}{8\pi} AdA, \tag{4.8}$$

a Dirac fermion coupled to $U(1)_{3/2}$ Chern-Simons theory, which is dual to a single complex scalar coupled to $U(1)_{-2}$ Chern-Simons theory,

$$\mathcal{L}_{\text{dual}} = |(\partial_\mu - i\tilde{a}_\mu)\phi|^2 - \lambda|\phi|^4 - \frac{2}{4\pi}\tilde{a}d\tilde{a} + \frac{1}{2\pi}Ad\tilde{a}. \tag{4.9}$$

The derivation of this duality is reviewed in Appendix 4.6. If we add a positive mass term for the scalar (corresponding to $m_- > 0$ for the fermion), the scalar can be integrated out and upon shifting $\tilde{a} \to \tilde{a} + \frac{1}{2}A$ we obtain $U(1)$ Chern-Simons terms of level -2 and $1/2$ for the $\tilde{a}$ and A gauge fields, respectively, in accordance with the $\nu = 1/2$ bosonic Laughlin state expected for $m_- > 0$. Setting $A = 0$, approximate critical exponents for the Mott insulator-bosonic Laughlin state transition can be obtained by studying Eq. (4.8) in the $1/N_f$ expansion where N_f denotes the number of Dirac fermion flavors [142, 162], or by studying the dual theory (4.9) in a bosonic $1/N_b$ expansion [163].

Finally, exact microscopic symmetries may force $m_+ = m_-$ and protect a topological transition in which C changes by 2 (e.g., inversion symmetry in the Haldane model [105]). In this case a direct transition from the superfluid to the bosonic Laughlin state is generically allowed, and the critical theory is (4.5) with $m_+ = m_- = 0$. Critical exponents can be estimated using the large-N_f expansion mentioned above [142, 162].

4.2.3 Instantons, fermion zero modes, and superfluidity

So far, we have ignored the compactness of the emergent $U(1)$ gauge field a_μ. In the superfluid phase, the low-energy effective action does not feature a Chern-Simons term, thus confinement effects due to the proliferation of monopole-instantons [9, 26, 40] are expected to play a key role. In the presence of fermions, such instantons can additionally lead to symmetry-breaking effects [62]. In an instanton background $a_\mu^{(g)}$ of topological charge g, a (2+1)D Dirac fermion ψ of charge e and mass m possesses Euclidean ZMs given by [1]:

$$\psi_0^+(r,\theta,\phi) = \frac{\sqrt{2m}}{r}e^{-mr}\mathcal{Y}^{1/2}_{1/2,0,0}(\theta,\phi), \tag{4.10}$$

$$\psi_0^-(r,\theta,\phi) = \frac{\sqrt{-2m}}{r}e^{mr}\mathcal{Y}^{1/2}_{-1/2,0,0}(\theta,\phi), \tag{4.11}$$

for $eg = +1/2$ and $eg = -1/2$, respectively, according to the Dirac quantization condition. Here $\mathcal{Y}^{j\pm1/2}_{eg,j,m_j}(\theta,\phi)$ are monopole spinor harmonics with total angular momentum j and its projection m_j [74, 118], and (r,θ,ϕ) are spherical coordinates in 3D Euclidean spacetime. In the semiclassical instanton gas approximation, those fermion ZMs induce an effective four-fermion interaction, known as the 't Hooft vertex [63–65]:

$$e^{-2\pi i\sigma}e^{-i\vartheta}\psi^\intercal_{1+}\gamma\psi_{2+}\psi^\intercal_{1-}\gamma\psi_{2-} + \text{H.c.}, \tag{4.12}$$

where $\psi_{1\pm}$ are the slow fields in the low-energy expansion $f_1(\boldsymbol{r}) \approx \sum_{k=\pm} e^{i\boldsymbol{K}_k\cdot\boldsymbol{r}}\psi_{1k}(\boldsymbol{r})$, σ is the dual photon, ϑ is a topological angle analogous to the theta angle of 4D Yang-Mills theory, and γ is a certain 2×2 matrix in spinor space [1]. The operator $\mathcal{M}(x) = e^{2\pi i\sigma(x)}$ is a monopole operator that inserts 2π flux at a given point x in spacetime.

The importance of this term is that it properly accounts for the unique $U(1)$ global symmetry of the microscopic boson system. Absent instanton effects, the Lagrangian (4.5) has a spurious $U(1)\times U(1)_{\text{top}}$ symmetry, where the first factor corresponds to the conservation of ψ_{2+} particle number, and the second factor represents the conservation of the topological current $j^\mu_{\text{top}} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = \frac{1}{(2\pi)^2}\partial^\mu\sigma$. The 't Hooft vertex (4.12) intertwines these two symmetries, reducing them to the diagonal $U(1)$ subgroup under which a phase rotation of $\psi_{2\pm}$ is compensated by a shift of σ. Assuming instanton proliferation in the confined phase, σ acquires an expectation value, which breaks this $U(1)$ symmetry spontaneously and results in a superfluid phase.

4.3 Ising spins

We now turn to our main focus, a system of quantum Ising spins $\tau^z(\boldsymbol{r}) = \pm 1$ living on the sites $\boldsymbol{r}$ of a 2d lattice. We assume the Hamiltonian of the system is invariant under the global $\mathbb{Z}_2$ symmetry $\tau^z(\boldsymbol{r}) \to -\tau^z(\boldsymbol{r})$, for all $\boldsymbol{r}$. We wish to describe ordered and disordered phases of this quantum spin system, as well as transitions between them, using a parton construction.

4.3.1 Parton construction

In analogy with (4.1), we introduce the parton decomposition

$$\tau^z(\boldsymbol{r}) = i^{N/2}\chi^1(\boldsymbol{r})\chi^2(\boldsymbol{r})\cdots\chi^N(\boldsymbol{r}), \tag{4.13}$$

where N is even and $\chi^\alpha(\boldsymbol{r})$, $\alpha = 1, \ldots, N$ are Hermitian (Majorana) fermion operators obeying the $SO(N)$ Clifford algebra $\{\chi^\alpha(\boldsymbol{r}), \chi^\beta(\boldsymbol{r}')\} = 2\delta^{\alpha\beta}\delta_{\boldsymbol{r}\boldsymbol{r}'}$. One easily checks that (4.13) implies the expected properties $\tau^z(\boldsymbol{r}) = \tau^z(\boldsymbol{r})^\dagger$, $\tau^z(\boldsymbol{r})^2 = 1$, and that the $\tau^z(\boldsymbol{r})$ commute on different sites. This parton decomposition introduces a local $SO(N)$ gauge redundancy $\chi(\boldsymbol{r}) \to R(\boldsymbol{r})\chi(\boldsymbol{r})$ under which $\tau^z(\boldsymbol{r})$ remains invariant, where $R(\boldsymbol{r}) \in SO(N)$ and we group the Majorana operators into an $SO(N)$ vector $\chi = (\chi^1, \ldots, \chi^N)^\intercal$. The charge under the global Ising symmetry can be assigned to any odd number of the N Majorana modes; we choose to assign the global $\mathbb{Z}_2$ charge to χ^1, i.e., χ transforms as $\chi(\boldsymbol{r}) \to W\chi(\boldsymbol{r})$ where $W = \mathrm{diag}(-1, 1, \ldots, 1)$. This action of the global symmetry does not commute with $SO(N)$ gauge transformations in the parton Hilbert space. Therefore, unlike for the $U(1)$ boson problem (Sec. 4.2), we cannot couple the parton system to a background $\mathbb{Z}_2$ gauge field while maintaining $SO(N)$ invariance at the Lagrangian level. However, the global symmetry action is well defined on gauge-invariant operators and gauge-invariant states. To the difference of other Majorana-based parton decompositions of spin operators [164], explicit expressions for the operators $\tau^x(\boldsymbol{r})$ and $\tau^y(\boldsymbol{r})$ in terms of the Majorana fermions $\chi^\alpha(\boldsymbol{r})$ are expected to be nonlocal and cannot be easily written down.

Nonetheless, in Appendix 4.7, we show that the strong-coupling limit of an $SO(N)$ lattice gauge theory with Majorana matter naturally describes a quantum Ising spin system, thus lending support to (4.13) as a valid parton representation.

We consider an $SO(N)$-invariant parton mean-field ansatz in which all N partons of the multiplet form a class-D topological superconductor [165] with Chern number C. Considering fluctuations above the mean-field ground state, the partons couple to an emergent $SO(N)$ gauge field denoted by a_μ. Integrating out the partons, we obtain the effective Lagrangian

$$\mathcal{L} = \mathrm{CS}_{SO(N)_C}[a] + \dots, \tag{4.14}$$

where $\mathrm{CS}_{SO(N)_k}[a]$ denotes a level-k non-Abelian $SO(N)$ Chern-Simons term for the gauge field a [166–168],

$$\mathrm{CS}_{SO(N)_k}[a] = \frac{k}{2 \cdot 4\pi} \operatorname{tr}\left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right), \tag{4.15}$$

with the trace in the vector representation of $SO(N)$. The dots in (4.14) denote non-topological gauge invariant terms such as the Yang-Mills action $\propto \operatorname{tr}(f \wedge *f)$ where $f = da + a \wedge a$ is the non-Abelian field-strength 2-form.

We now investigate the different phases of the original spin system that can be reached by varying C. When $C = 0$, the Chern-Simons term is absent and the Yang-Mills term dominates the action. At least when regularized on a lattice, as is the case here, pure $SO(N)$ gauge theory in $2+1$ dimensions and without a Chern-Simons term is believed to be confining at zero temperature [169, 170]. The confining theory is massive, thus $C = 0$ corresponds to a gapped phase of the original Ising spin system. When $C = 1$, the effective Lagrangian contains an $SO(N)_1$ Chern-Simons term,

$$\mathcal{L} = \mathrm{CS}_{SO(N)_1}[a] + \dots \tag{4.16}$$

This theory is also massive, but the Chern-Simons term leads to deconfinement at zero temperature. Below the mass gap and in the long-wavelength limit, the system is described by a pure topological $SO(N)_1$ theory. Similarly to $U(1)_1$ Chern-Simons theory, this is an invertible topological quantum field theory with a unique ground state on all closed manifolds [146]. Note that $SO(N)_k$ Chern-Simons theory is a consistent theory of microscopic bosons, even if k is odd, provided that only fermionic matter fields couple to the $SO(N)$ gauge field [167]; both conditions are satisfied here. Thus for $C = 1$ the original Ising spin system forms a gapped paramagnet without topological order. For $|C| > 1$, one obtains a deconfined phase described by $SO(N)_C$ Chern-Simons theory. Such theories are not invertible, and thus the Ising spin system is in a phase with intrinsic topological order, i.e., a gapped spin liquid.

Another way to see that the $C = 1$ phase corresponds to a gapped paramagnet without topological order is by looking at the edge degrees of freedom. At the mean-field level, the $C = 1$ phase features N free chiral Majorana modes on the boundary, which is a noninteracting conformal field theory (CFT) with chiral $\mathfrak{so}(N)_1$ current algebra and chiral central charge $c_- = N/2$ [171]. However, when projecting to the physical Hilbert space the $SO(N)$ symmetry is gauged, which gives a trivial coset CFT on the edge with vanishing chiral central charge.

More generally, for $C > 1$ one obtains NC free chiral Majorana modes on the boundary at the parton mean-field level, corresponding to a chiral $\mathfrak{so}(NC)_1$ current algebra (we assume C is positive without loss of generality, as a sign reversal of C simply corresponds to a reversal of chirality). In Appendix 4.8, we show that this current algebra obeys the following conformal embedding:

$$\mathfrak{so}(N)_C \otimes \mathfrak{so}(C)_N \subseteq \mathfrak{so}(NC)_1. \tag{4.17}$$

Gauging the $SO(N)$ symmetry leaves a chiral $\mathfrak{so}(C)_N$ current algebra [172], with chiral central charge

$$c_- = \frac{NC(C-1)}{2(N+C-2)}. \tag{4.18}$$

Thus the $C > 1$ phases are chiral topological phases with protected edge modes described by the chiral $\mathfrak{so}(C)_N$ Wess-Zumino-Witten (WZW) CFT. Since the microscopic system consists of interacting Ising spins, these are chiral spin-liquid phases with broken time-reversal symmetry but unbroken $\mathbb{Z}_2$ spin-flip symmetry. For $C = 2$, the edge theory is a $\mathfrak{so}(2)_N = u(1)_N$ CFT with $c_- = 1$. This is consistent with the fact that the bulk $SO(N)_2$ Chern-Simons theory is equivalent to $SO(2)_N = U(1)_N$ by level-rank duality [173]. Thus the $C = 2$ phase is an Abelian topological phase, with the topological order of a $\nu = 1/N$ bosonic fractional quantum Hall state [158, 174, 175]. Such a chiral spin liquid has anyonic spinon excitations with statistical angle $\theta = \pi/N$. Likewise, for $C > 2$ but $N = 2$, the edge theory is a $\mathfrak{so}(C)_2$ CFT which is equivalent to $\mathfrak{su}(C)_1$ [168], with chiral central charge $c_- = C - 1$. This can be understood intuitively since $N = 2$ corresponds to two Majorana fermions, which is equivalent to a single Dirac fermion. The mean-field state is a Chern insulator of this Dirac fermion with Chern number C, and gauging the internal $SO(2)$ symmetry corresponds to gauging the $U(1)$ symmetry of the Dirac fermion. For $N > 2$ and $C > 2$, the bulk $SO(N)_C$ Chern-Simons theory is level-rank dual to $SO(C)_N$, consistent with the $\mathfrak{so}(C)_N$ edge CFT. In the following, we will be interested exclusively in the case $N > 2$, for reasons to be clarified shortly.

4.3.2 Phase transitions and $SO(N)$ dualities

For simplicity, and to make an analogy with Sec. 4.2.2, we focus on transitions between the three phases with $C = 0, 1, 2$. At the mean-field level, those are topological transitions that proceed by linear (Majorana) crossings of the Bogoliubov-de Gennes bands of the topological superconductor. We consider a parton bandstructure such that a low-lying band with Chern number one remains filled across the transition, and there is a linear crossing at zero energy of two other bands (for examples of multiband Majorana models with topological transitions, see Refs. [176–178]). Since the total Chern number must be integer, and a single massive two-component Majorana fermion carries a partial Chern number of $\frac{1}{2}\,\mathrm{sgn}\, m$, the low-energy bandstructure in the vicinity of the transitions can be described by two slow, two-component Majorana fields ψ_+, ψ_- in the vector representation of $SO(N)$ with masses m_+, m_- respectively. In Appendix 4.9, we give an example of

Majorana hopping model that produces such a low-energy bandstructure with tunable masses $m_\pm$. Considering gauge fluctuations, the theory interpolating between all three phases is

$$\mathcal{L} = \mathrm{CS}_{SO(N)_1}[a] + \frac{1}{4}\sum_{k=\pm} \psi_k^\intercal \mathcal{C}(i\not{D} - m_k)\psi_k, \tag{4.19}$$

where $\not{D} = \gamma^\mu(\partial_\mu - ia_\mu)$ is a gauge-covariant derivative involving the internal $SO(N)$ gauge field a_μ, $\mathcal{C}$ is a charge-conjugation matrix, and the level-1 Chern-Simons term comes from the response of the low-lying band. The phases described in Sec. 4.3.1 for $C = 0, 1, 2$ are obtained when m_+ and m_- are nonzero. When $m_\pm > 0$, the two Majorana fermions $\psi_\pm$ can be integrated out, yielding an $SO(N)_2$ Chern-Simons term. As seen above, this is a chiral spin liquid with the topological order of the $\nu = 1/N$ bosonic fractional quantum Hall state. When both $m_\pm < 0$, the Chern-Simons level vanishes and one obtains a pure Yang-Mills theory which confines. As we discuss below, this is a phase with magnetic long-range order. Finally, when the masses are of opposite sign, one has an $SO(N)_1$ Chern-Simons term which corresponds to a topologically trivial, gapped paramagnet. A schematic phase diagram is given in Fig. 4.2.

The critical theories for the transitions in Fig. 4.2 can all be written in the following general form:

$$\mathcal{L}_{N_f,\nu} = \mathrm{CS}_{SO(N)_\nu}[a] + \frac{1}{4}\sum_{j=1}^{N_f} \psi_j^\intercal \mathcal{C} i\not{D}\psi_j, \tag{4.20}$$

where ν is the Chern-Simons level and N_f is the number of Majorana fields that become massless at the transition. We first consider transitions involving the chiral spin liquid. The transition between the chiral spin liquid and the paramagnet is tuned by m_+ (m_-) crossing zero at constant $m_- > 0$ ($m_+ > 0$); integrating out the massive fermion, we find Eq. (4.20) with $N_f = 1$ and $\nu = 3/2$. A direct transition from the chiral spin liquid to the ordered phase is obtained by tuning the mass of both fermions through zero, and is described by Eq. (4.20) with $N_f = 2$ and $\nu = 1$. Such a transition can be protected by microscopic symmetries enforcing $m_+ = m_-$; for example, this is achieved by requiring inversion symmetry in the honeycomb lattice model presented in Appendix 4.9. Assuming both theories flow to a *bona fide* critical fixed point in the infrared, they correspond to novel universality classes.

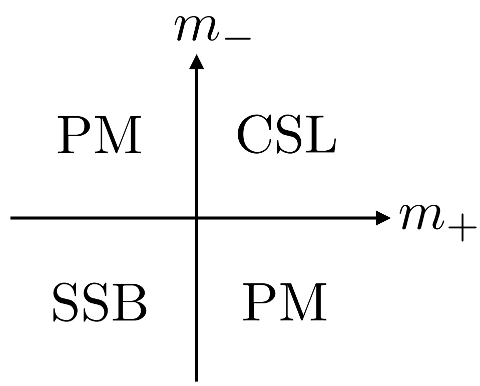

Figure 4.2: Phase diagram for Ising spins as a function of the two tuning parameters m_+, m_-. PM: trivial paramagnet; SSB: ordered phase with $\mathbb{Z}_2$ spontaneous symmetry breaking; CSL: chiral spin liquid.

We next argue that the confining phase at $m_\pm < 0$ (i.e., $C = 0$) has a spontaneously broken $\mathbb{Z}_2$ symmetry: that is, it possesses Ising-type magnetic long-range order. The transition between the $C = 1$ (paramagnetic) and $C = 0$ phases is obtained by tuning m_+ (m_-) through zero at constant $m_- < 0$ ($m_+ < 0$). It is thus described by $\mathcal{L}_{1,1/2}$ in Eq. (4.20), i.e., a single flavor of Majorana fermions in the vector representation coupled to an $SO(N)_{1/2}$ theory. It was conjectured in Ref. [168] that an $SO(k)_{-M+\frac{N_f}{2}}$ theory coupled to N_f flavors of vector Majorana fermions is dual to an $SO(M)_k$ theory coupled to N_f flavors of real scalars $\boldsymbol{\phi}$ in the vector representation with $(\boldsymbol{\phi}^2)^2$ interactions, i.e., an $SO(M)_k$ theory coupled to the bosonic $O(M)$ vector model at its Wilson-Fisher fixed point. For $N_f = M = 1$, this stipulates that an $SO(k)_{-1/2}$ theory coupled to a single Majorana fermion is dual to a single real scalar ϕ at its Wilson-Fisher fixed point, since $SO(1)$ on the scalar side is trivial:

$$\text{Majorana} + SO(k)_{-1/2} \longleftrightarrow \text{real scalar}, \tag{4.21}$$

which can be viewed as a fermionization of the 3D Ising transition. Note that this duality is conjectured to hold only for $k > 2$, which corresponds in our case to $N > 2$, i.e., a minimum of four partons in the decomposition (4.13). Equation (4.21) can also be viewed as the Majorana counterpart of the $U(1)$ boson-fermion duality [142, 146–148, 150, 151]

$$\text{Dirac} + U(1)_{-1/2} \longleftrightarrow \text{complex scalar}, \tag{4.22}$$

whose time-reversed version we have used in Eqs. (4.6-4.7), or as the "inverse" of the Majorana bosonization duality [167, 168]

$$\text{Majorana} \longleftrightarrow O(M)\ \text{scalar} + SO(M)_1, \tag{4.23}$$

obtained by considering $N_f = k = 1$ (and $M \geq 3$). Setting $k = N$ and performing time reversal to reverse the sign of the Chern-Simons level, Eq. (4.21) is precisely the critical theory $\mathcal{L}_{1,1/2}$. Thus one obtains the dual critical theory

$$\mathcal{L}_{\text{dual}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4. \tag{4.24}$$

The dual Lagrangian (4.24) has a global $\mathbb{Z}_2$ symmetry $\phi \to -\phi$ and two massive phases, the unbroken phase $\langle\phi\rangle = 0$ and the phase $\langle\phi\rangle \neq 0$ with spontaneously broken $\mathbb{Z}_2$ symmetry. This transition can be tuned by adding a scalar mass term $\propto \phi^2$. Since $m_- > 0$ in the original theory corresponds to a trivial paramagnet with no broken symmetries, this must correspond to the unbroken phase in the dual theory, i.e., a positive scalar mass term. Therefore, $m_- < 0$ must correspond to a negative scalar mass term in the dual theory, i.e., the phase with spontaneously broken $\mathbb{Z}_2$ symmetry.

It remains to be shown that the $\mathbb{Z}_2$ symmetry under $\phi \to -\phi$ of the dual theory (4.24) is nothing but the original global $\mathbb{Z}_2$ symmetry of the spin system. As in Sec. 4.2.3, this is properly achieved by the consideration of nonperturbative instanton effects, to which we now turn.

4.4 Instantons, Majorana zero modes, and Ising symmetry

In this section, we wish to understand how the low-energy $SO(N)$ gauge theory accounts for the broken Ising symmetry in the phase with parton Chern number $C = 0$. In our description of the $C = 1 \to C = 0$ transition in Sec. 4.3.2, the $C = 0$ Chern number of the occupied bands resulted from the cancellation between a spectator $C = 1$ band and a nearly-critical $C = -1$ band (fermions $\psi_\pm$ with masses $m_\pm < 0$). To understand the physics deep in the $C = 0$ phase, it is simpler to work with a topologically equivalent theory, that is, a single $C = 0$ band producing two continuum Majorana fields Ψ_+ and Ψ_- with opposite masses $\pm m$ (see again Appendix 4.9 for a lattice representative), specified by the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{4}\Psi_+^\intercal \mathcal{C}(\not{\partial} - i\not{a} + m)\Psi_+ + \frac{1}{4}\Psi_-^\intercal \mathcal{C}(\not{\partial} - i\not{a} - m)\Psi_- + \frac{1}{2g^2}\,\mathrm{tr}\, f^2, \tag{4.25}$$

where we have explicitly included a Yang-Mills term, in the absence of a net Chern-Simons level. As we discuss below, the breaking of Ising symmetry ultimately results from Euclidean ZMs supported by those massive Majorana fields in the presence of instantons in the $SO(N)$ gauge field, which have heretofore been ignored.

The rest of this section is structured as follows. We first discuss the monopole operator of $SO(N)$ gauge theory in (2+1)D, which is charged under a topological (magnetic) $\mathbb{Z}_2^{\mathcal{M}}$ symmetry and becomes a $\mathbb{Z}_2$ instanton in Euclidean spacetime (Sec. 4.4.1). Together with the global $\mathbb{Z}_2$ symmetry action W on the Majorana partons χ (recall Sec. 4.3.1), the Lagrangian (4.25) has a spurious global $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathcal{M}}$ symmetry absent instanton effects. We show that in the presence of massive fermions coupled to the $SO(N)$ gauge field, these instantons are dressed by Euclidean Majorana ZMs bound to the instanton (Sec. 4.4.2). We then show that semiclassical resummation of the $\mathbb{Z}_2$ instanton gas produces an interaction term among the fermionic partons, the 't Hooft vertex, that explicitly breaks this spurious $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathcal{M}}$ symmetry down to its diagonal $\mathbb{Z}_2$ subgroup (Sec. 4.4.3). This intertwinement ensures that if the $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry is spontaneously broken, as is typical in a confined phase [140, 141], the microscopic Ising symmetry $\tau^z \to -\tau^z$ is broken also. The $C = 0$ confined phase is thus naturally identified as a broken-symmetry phase, in agreement with the duality arguments of Sec. 4.3.2.

4.4.1 $\mathbb{Z}_2$ instantons in $SO(N)$ gauge theory

In the absence of instantons, $SO(N)$ gauge theory in (2+1)D with $N > 2$ possesses a magnetic $\mathbb{Z}_2^{\mathcal{M}}$ symmetry [140, 141, 168, 179–181], analogous to the topological $U(1)_{\text{top}}$ symmetry of $U(1)$ gauge theory in (2+1)D. Unlike the latter, $\mathbb{Z}_2^{\mathcal{M}}$ lacks a conserved current, being a discrete symmetry. The similarity between the two is that both result in the existence of disorder operators that create topological excitations, in this case monopole-instantons.

The operator charged under the $\mathbb{Z}_2^{\mathcal{M}}$ symmetry is a local monopole operator $\mathcal{M}(x)$, whose charge is defined by a nontrivial second Stiefel-Whitney class $w_2 \in H^2(\Sigma, \mathbb{Z}_2)$ on a closed surface Σ surrounding the operator insertion [2]:

$$\int_\Sigma w_2 \in \mathbb{Z}_2. \tag{4.26}$$

A nontrivial Stiefel-Whitney class is an obstruction to lifting an $SO(N)$ bundle to its double cover, a $\text{Spin}(N)$ bundle. For Σ a sphere, the $\mathbb{Z}_2$ monopole charge corresponds to the nontrivial homotopy group $\pi_1(SO(N)) \cong \mathbb{Z}_2$, $N > 2$. It measures the winding number of an $SO(N)$ gauge transformation that relates the gauge fields a_μ^{I} and a_μ^{II} on the overlap of two coordinate charts I & II on Σ. As in the $U(1)$ theory [9, 26, 40], the $\mathbb{Z}_2$ monopoles are regarded here as instantons in 3D Euclidean spacetime.

An explicit semiclassical representative [185] is obtained by placing a Dirac monopole in a specific $SO(2)$ subgroup of $SO(N)$, so that one may use the Wu-Yang connection 1-form [74]:

$$\mathcal{A}_n = \frac{n}{2}(1 - \cos\theta) t_c \, \mathrm{d}\phi\,, \quad n \in \mathbb{Z}, \tag{4.27}$$

where n is the Dirac monopole charge, ϕ is the azimuthal coordinate on a sphere Σ surrounding the monopole, and $t_c \in \mathfrak{so}(N)$ generates a subgroup $SO(2) \subset SO(N)$. By means of gauge rotations, any $t_c \in \mathfrak{so}(N)$ can be rotated to a Cartan generator, which shall be taken as $t_{(12)}$, the generator of rotations in the (χ^1, χ^2) plane of the Majorana vector $(\chi^1, ..., \chi^N)$.

[2] For an introduction to Stiefel-Whitney classes in a condensed matter context, see, e.g., Refs. [182–184]

Such a semiclassical characterization explicitly breaks the $SO(N)$ gauge symmetry down to an $[O(2)\times O(N-2)]/\mathbb{Z}_2$ subgroup, where quotienting by $\mathbb{Z}_2$ restricts the determinant of the overall transformation to be positive. Note that a suitable $SO(N)$ gauge transformation can invert the $SO(2)$ monopole charge n; thus an $n=2=1+1$ monopole is topologically equivalent to an $n=0=1+(-1)$ monopole, and the unique $\mathbb{Z}_2$-nontrivial monopole is given by $n=1$ (or $n=-1$). Although it does not preserve full $SO(N)$ gauge invariance, this semiclassical description will allow us to perform an explicit instanton-gas calculation analogous to that in Ref. [1] for $U(1)$ gauge theory.

Incarnating the $SO(N)$ monopole as a Dirac monopole in a specific $SO(2)$ subgroup can be regarded as a partial gauge choice. Formally notating this gauge condition as $G(a)=0$, the Euclidean path integral for the theory can be expressed using the Faddeev-Popov (FP) method as

$$Z=\int_{SO(N)}\mathscr{D}R\int\mathscr{D}a\mathscr{D}\Psi_\pm\,\Delta_G[a]\delta[G(a^{(R)})]e^{-S[a,\Psi_\pm]}, \tag{4.28}$$

where $a^{(R)}$ is related to a by a gauge transformation $R(x)$, and we denote $\mathscr{D}\Psi_\pm\equiv\mathscr{D}\Psi_+\mathscr{D}\Psi_-$ for simplicity. The FP determinant $\Delta_G[a]$ and the delta functional $\delta[G(a^{(R)})]$ can be written respectively as ghost and gauge-fixing terms in the Lagrangian. The essential idea expressed by Eq. (4.28) is that one can perform a path integral calculation in a fixed gauge (gauge slice), and then integrate the result over its gauge orbit ($\int\mathscr{D}R$) to recover gauge invariance. This will allow us to use a $U(1)$ monopole operator that creates 2π flux in the $SO(2)$ subgroup, for which an explicit expression is known. In the following, we will omit explicit integration over the gauge orbit but invoke heuristic arguments to (partially) restore $SO(N)$ gauge invariance at the end of the calculation, focusing on its physical consequences.

As stated earlier, incarnating the $SO(N)$ monopole as a Dirac monopole in an $SO(2)$ subgroup only partially fixes the gauge. The group $S[O(2)\times O(N-2)]\equiv[O(2)\times O(N-2)]/\mathbb{Z}_2$ of global gauge rotations is a stabilizer for such a monopole configuration. Naïvely, the existence of a nontrivial stabilizer leads to ghost ZMs, which are ZMs in the FP determinant $\Delta_G[a] = |\det \delta G/\delta\omega|$, where $\omega \in \mathfrak{so}(N)$ generates the rotation $R = \exp(-\omega)$, and $G(a)$ is the aforementioned gauge function partially determining the gauge. By employing the background-field gauge method [64], we show in Appendix 4.10 that such ZMs can be removed from the FP determinant at the cost of introducing an overall factor of $\mathrm{vol}(SO(N)/S[O(2)\times O(N-2)])^{\mathcal{N}}$ in the $\mathcal{N}$-instanton contribution to the partition function. This is interpreted as the volume of the moduli space of "gauge collective coordinates"—global gauge rotations that act to move the Dirac monopole to distinct $SO(2)$ subgroups of $SO(N)$, thus yielding other viable instanton solutions [186]. Besides this, the ghost and gauge-fixing terms will simply spectate in the instanton gas calculation to follow, and will thus henceforth be suppressed to reduce clutter.

The coupling of fermions to finite-action fluctuations ("gluons") around the instanton solution $\mathcal{A}_n$ is ignored in the semiclassical approximation [63–65]. In our choice of gauge, the contribution to the path integral from $\mathbb{Z}_2$ instantons can be separated and written as [1]

$$Z = \int \mathscr{D}a\, e^{-\frac{1}{2g^2}\int d^3x\, \mathrm{tr}\, f^2} \sum_{\mathcal{N}=0}^{\infty} \frac{\lambda^{\mathcal{N}}}{\mathcal{N}!} \prod_{k=1}^{\mathcal{N}} \int \mathrm{d}^3 z_k \sum_{n_k=\pm 1} \mathcal{M}^{(12)}_{n_k}(z_k) \int \mathscr{D}\Psi_\pm e^{-S_F[\mathcal{A}_{n_k},\Psi_\pm]}, \quad (4.29)$$

where $\mathcal{M}^{(12)}_n$ is a monopole operator that creates a Dirac monopole of charge n in the $SO(2)$ subgroup generated by $t_{(12)} \in \mathfrak{so}(N)$, and λ is the fugacity of an $n = \pm 1$ instanton. In this fixed gauge, the monopole operator has an explicit representation $\exp\big(in\gamma_{(12)}\big)$ in terms of the dual photon $\gamma_{(12)}$ [9, 26, 27, 40]. Unlike in $U(1)$ gauge theory, $\gamma_{(12)}$ is no longer gauge invariant, as evident from the presence of the gauge-dependent susbcript that selects an $SO(2)$ subgroup in $SO(N)$. The fermion action in the instanton background is

$$S_F[\mathcal{A}_n] = \frac{1}{4} \int \mathrm{d}^3x\, [\Psi_+^\intercal \mathcal{C}(\partial\!\!\!/ - i\mathcal{A}\!\!\!/_n + m)\Psi_+ + \Psi_-^\intercal \mathcal{C}(\partial\!\!\!/ - i\mathcal{A}\!\!\!/_n - m)\Psi_-]. \quad (4.30)$$

The inclusion of charge $n = \pm 1$ monopoles in Eq. (4.29) deserves further explanation in light of the $\mathbb{Z}_2$ nature of the topological charge of $SO(N)$ monopoles. A simple explanation is that in our fixed choice of gauge, these two charges are distinct configurations and must both be accounted for. Alternatively, one can resort to a stability argument. In one higher spacetime dimension (4D), $\mathbb{Z}_2$ monopoles feature as solitons in the gauge theory. A monopole of topological charge 0 or 1 can be "dynamically" represented as Dirac monopoles of charges $\{0, \pm 2, \pm 4, ...\}$ or $\{\pm 1, \pm 3, ...\}$ respectively, in some $SO(2) \subset SO(N)$. A stability analysis [187, 188] indicates that the uniquely stable dynamical configurations in the two topological classes are the charge 0 and ± 1 Dirac monopoles. This result can be used to determine the stable dynamical configuration of multimonopole solutions. At distances large compared to their separation, two monopoles with Dirac charges $+1$ look like a single monopole of Dirac charge 2, which is unstable to the charge 0 configuration. This implies the instability of the $1+1$ to the $1-1$ configuration, which proceeds by the emission of gluon radiation. While such a stability analysis has been applied to monopoles as soliton excitations in 4D, we expect that a similar result holds for monopole-instantons, with the charge 0 and ± 1 configurations being the most probable instanton events. Since the instanton gas calculation is performed with the Dirac charge instead of the topological $\mathbb{Z}_2$ charge, one must account for both ± 1 charges in the instanton sum (4.29), as both are expected to be equally probable. Finally, we find that inclusion of both ± 1 charges is required to maintain reflection positivity of the instanton-induced 't Hooft vertex, as discussed in Sec. 4.4.3.

4.4.2 Euclidean Majorana zero modes

A natural question to ask now is if there are (Euclidean) Majorana ZMs, associated with zero-eigenvalue modes of the Euclidean Dirac operators

$$\mathcal{D}_{\pm} \equiv \partial\!\!\!/ - i\mathcal{A}\!\!\!/_n \pm m, \tag{4.31}$$

appearing in the fermion action S_F, for Dirac instantons of charges $n = \pm 1$. In the absence of a Callias index theorem for Dirac instantons in Abelian $SO(2) \cong U(1)$ gauge theory [70, 101], we resort to an explicit solution of the Dirac equation.

As stated previously, we assume a gauge in which the instanton incarnates as a Dirac monopole in the $SO(2)$ subgroup generated by

$$t_{(12)} = \left(\begin{array}{cc|c} 0 & -i & \mathbf{0} \\ i & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right) \in \mathfrak{so}(N), \tag{4.32}$$

where the upper-left block corresponds to the (12) subspace, and $\mathbf{0}$ denotes a zero matrix of the appropriate size, involving the remaining $N-2$ directions in color space. Writing $\mathcal{A}_n = a_n t_{(12)}$, and working in the Cartan (diagonal) basis of $\mathfrak{so}(N)$, the Dirac operators can be written as

$$\mathcal{D}_\pm = U \left(\begin{array}{cc|c} \not{\partial} - i\not{a}_n \pm m & 0 & \mathbf{0} \\ 0 & \not{\partial} + i\not{a}_n \pm m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & (\not{\partial} \pm m)\mathbf{1} \end{array}\right) U^\dagger, \tag{4.33}$$

where U is the unitary matrix that diagonalizes $t_{(12)}$. Borrowing results from Ref. [1], in an $n=1$ instanton background, $(\not{\partial} - i\not{a}_+ + m)$ and $(\not{\partial} + i\not{a}_+ - m)$ have the respective normalizable ZMs:

$$\begin{aligned} \psi_0^+ &= \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^0_{1/2,0,0}(\theta,\phi), \\ \psi_0^- &= \frac{\sqrt{2m}}{r} e^{-mr} \mathcal{Y}^0_{-1/2,0,0}(\theta,\phi), \end{aligned} \tag{4.34}$$

where $\mathcal{Y}^{j\pm 1/2}_{n/2,j,m_j}(\theta,\phi)$ are monopole spinor harmonics. The rest of the operators on the diagonal of Eq. (4.33) do not have any normalizable ZMs. Therefore, the normalizable ZMs of $\mathcal{D}_\pm$ in an $n=1$ instanton background are, respectively,

$$\begin{aligned} u_0 &= U(\psi_0^+, 0, ..., 0)^\intercal = \frac{\psi_0^+}{\sqrt{2}}(-i, 1, 0, ..., 0)^\intercal, \\ v_0 &= e^{i\alpha} U(0, \psi_0^-, 0, ..., 0)^\intercal = \frac{e^{i\alpha}\psi_0^-}{\sqrt{2}}(i, 1, 0, ..., 0)^\intercal. \end{aligned} \tag{4.35}$$

Any phase multiplying a ZM still produces a normalized ZM, and this apparent freedom has been encoded in an arbitrary relative phase $e^{i\alpha}$.

Similarly, in an $n\!=\!-1$ instanton background, the operators $\mathcal{D}_\pm$ have the ZMs

$$\tilde{u}_0 = U(0, \psi_0^+, 0, ..., 0)^\intercal = \frac{\psi_0^+}{\sqrt{2}}(i, 1, 0, ..., 0)^\intercal,$$
$$\tilde{v}_0 = e^{-i\beta}U(\psi_0^-, 0, ..., 0)^\intercal = \frac{e^{-i\beta}\psi_0^-}{\sqrt{2}}(-i, 1, 0, ..., 0)^\intercal. \tag{4.36}$$

The $\mathbb{Z}_2$ topological equivalence of the $n = \pm 1$ field configurations under the full $SO(N)$ gauge structure will be discussed later, as well as constraints on the relative phases α and β.

4.4.3 The 't Hooft vertex and Ising symmetry

In this subsection, we show that the Euclidean Majorana ZMs found in the previous subsection induce symmetry-breaking interactions in the $SO(N)$ gauge theory. (As mentioned previously, Appendix 4.10 shows that FP ghosts do not give rise to physical ZMs bound to instantons.) Specifically, these ZMs imply that instanton events are correlated with creation (or annihilation) of Majorana fermions. Resumming the instanton gas results in a new fermion interaction, called the 't Hooft vertex, which reduces the symmetry of the initial Lagrangian (4.25).

We now sketch a derivation of this 't Hooft vertex; more details regarding the structure of such a calculation can be found in Ref. [1]. In the background of an $n\!=\!1$ instanton fixed at location z_+, the measure of the fermion part of the path integral (4.29) can be defined by means of the mode expansions

$$\Psi_+(x) = u_0(x - z_+)\eta_0 + \sum_i{}' u_i(x - z_+)\eta_i,$$
$$\Psi_-(x) = v_0(x - z_+)\chi_0 + \sum_i{}' v_i(x - z_+)\chi_i, \tag{4.37}$$

where η_i, χ_i are single-component Grassmann variables, u_0 and v_0 are the respective ZMs of $\mathcal{D}_+$ and $\mathcal{D}_-$ in an $n\!=\!1$ instanton background, and the primed sums denote non-ZM contributions. The functions that form the non-ZM contributions can be taken to be eigenfunctions of a self-adjoint extension of the Hermitian operator $\mathcal{D}_\pm^\dagger\mathcal{D}_\pm$, whose non-ZM eigenfunctions occur in pairs that share the same eigenvalue [62, 189].

Defining the fermion functional measure as

$$\mathscr{D}\Psi_{\pm} = \mathscr{D}\Psi_{+}\mathscr{D}\Psi_{-} = \mathrm{d}\eta_0\,\mathrm{d}\chi_0 \prod_i{}' \mathrm{d}\eta_i\,\mathrm{d}\chi_i\,, \tag{4.38}$$

we observe that the mode expansions (4.37) diagonalize the fermion action S_F, but the ZMs do not appear in the diagonalized action, by virtue of being annihilated by the Dirac operators $\mathcal{D}_{\pm}$. This causes the integral over the ZMs (η_0, χ_0) to vanish, killing the path integral. As in Ref. [1], instantons do not contribute to the partition function itself, but to correlation functions that can "soak up" the ZMs, such as $\left\langle \Psi_+^{\alpha}\Psi_-^{\beta}\right\rangle$. Such correlation functions generically violate the apparent $\mathbb{Z}_2\times\mathbb{Z}_2^{\mathcal{M}}$ symmetry of the naive continuum Lagrangian (4.25). To find the true effective theory, we add a weak symmetry-breaking source to the action and re-evaluate the fermion part of the path integral to linear order in the source J. Explicitly, using the mode expansions (4.37),

$$\begin{aligned} Z_F[A_+, J] &= \int \mathscr{D}\Psi_{\pm} e^{-S_F[A_+]-\int \mathrm{d}^3(x,y)\Psi_+^{\intercal}(x)J(x,y)\Psi_-(y)}, \\ &= \int \mathrm{d}^3(x,y) u_0^{\intercal}(x-z_+)J(x,y)v_0(y-z_+)K, \end{aligned} \tag{4.39}$$

where K denotes the path integral over non-ZMs, and $\mathrm{d}^3(x,y)=\mathrm{d}^3x\mathrm{d}^3y$. Strictly, nonlocal expressions like the source term require an insertion of Wilson lines to maintain gauge invariance, but we do not write these explicitly, as the final form of the 't Hooft vertex will turn out to be local. This is also consistent with our neglect of fermion-gluon interactions at this stage.

Demanding an effective theory that reproduces this path integral amounts to "integrating out" the instantons in the full partition function (4.29). As an ansatz for the resulting partition function, consider

$$I_+[J] = \int \mathscr{D}\Psi_{\pm} e^{-S_F-\int \mathrm{d}^3(x,y)\Psi_+^{\intercal}(x)J(x,y)\Psi_-(y)} \int \mathrm{d}^3(x_1,x_2)\rho\Psi_-^{\intercal}(x_2)\omega_2\omega_1^{\intercal}\Psi_+(x_1), \tag{4.40}$$

where ρ and $\omega_{1,2}$ are fixed by requiring equality with $Z_F[A_+, J]$ in Eq. (4.39). Note that the action S_F written without source arguments is the free Majorana action. This leads to

$$\begin{aligned}\rho &= K,\\ \omega_1 &= \frac{1}{2}\mathcal{C}(\not{\partial}+m)u_0,\\ \omega_2 &= \frac{1}{2}\mathcal{C}(\not{\partial}-m)v_0.\end{aligned} \tag{4.41}$$

The above calculations can be repeated for an $n=-1$ instanton background using the mode expansions

$$\begin{aligned}\Psi_+(x) &= \tilde{u}_0(x-z_-)\eta_0 + \sum_i{}' \tilde{u}_i(x-z_-)\eta_i,\\ \Psi_-(x) &= \tilde{v}_0(x-z_-)\chi_0 + \sum_i{}' \tilde{v}_i(x-z_-)\chi_i,\end{aligned} \tag{4.42}$$

where $\tilde{u}_0$ and $\tilde{v}_0$ are the respective ZMs of the Dirac operators $\mathcal{D}_\pm$ in an $n=-1$ background, discussed in Sec. 4.4.2. The fermion path integral $Z_F[A_-,J]$ can be shown to be equal to

$$I_-[J] = \int \mathscr{D}\Psi_\pm e^{-S_F-\int \mathrm{d}^3(x,y)\Psi_+^\intercal(x)J(x,y)\Psi_-(y)} \int \mathrm{d}^3(x_1,x_2) K \Psi_-^\intercal(x_2)\tilde{\omega}_2\tilde{\omega}_1^\intercal\Psi_+(x_1), \tag{4.43}$$

provided

$$\tilde{\omega}_1 = \frac{1}{2}\mathcal{C}(\not{\partial}+m)\tilde{u}_0, \qquad \tilde{\omega}_2 = \frac{1}{2}\mathcal{C}(\not{\partial}-m)\tilde{v}_0. \tag{4.44}$$

Substituting $I_\pm[J]$ instead of $Z_F[A_\mu^\pm, J]$ in the full partition function (4.29) and resumming the instanton gas leads to an instanton-induced action of the form

$$S_{\text{inst}} = -\lambda K \int_{x,y,z} \Psi_-^\intercal(x)\left[e^{i\gamma_{(12)}(z)}\omega_2(x-z)\omega_1^\intercal(y-z) + e^{-i\gamma_{(12)}(z)}\tilde{\omega}_2(x-z)\tilde{\omega}_1^\intercal(y-z)\right]\Psi_+(y). \tag{4.45}$$

As $\omega_{1,2}$ and $\tilde{\omega}_{1,2}$ are proportional to the radial part (e^{-mr}/r) of the ZMs, the contribution to the x and y integrals are mainly from small neighborhoods of $x=z$ and $y=z$. A change of integration variables $x\to x+z$ and $y\to y+z$, and subsequent Taylor expansions of the fermion fields $\Psi_-(x+z)$ and $\Psi_+(y+z)$ to leading (zeroth) order in x and y, yield a local action. Substituting the explicit forms of $\omega_{1,2}$ and $\tilde{\omega}_{1,2}$, this local action is

$$S_{\text{inst}} = \frac{\lambda K}{m}\int \mathrm{d}^3 z\, \Psi_-^\intercal(z)\Delta(z)\Psi_+(z), \tag{4.46}$$

where the instanton-induced 't Hooft vertex is defined as

$$\Delta(z) = \frac{1}{2}(-\sigma_z + i\sigma_y)\left[e^{i\gamma_{(12)}} e^{i\alpha} \left(\begin{array}{cc|c} 1 & -i & \mathbf{0} \\ i & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) + e^{-i\gamma_{(12)}} e^{-i\beta} \left(\begin{array}{cc|c} 1 & i & \mathbf{0} \\ -i & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) \right], \tag{4.47}$$

where the z dependence comes through the dual photon $\gamma_{(12)} = \gamma_{(12)}(z)$.

We next address the issue of $SO(N)$ gauge invariance, which the derived 't Hooft vertex currently lacks. Indeed, its matrix structure is invariant only under the $[O(2)\times O(N-2)]/\mathbb{Z}_2$ subgroup, and involves gauge-dependent variables α, β, and $\gamma_{(12)}$. The key physical feature that full $SO(N)$ invariance brings, for $N>2$, is the gauge equivalence between instantons and anti-instantons in any $SO(2)$ subgroup, given the $\mathbb{Z}_2$ topological charge discussed in Sec. 4.4.1. It is expected this feature will be restored upon performing the $SO(N)$ Haar integral in Eq. (4.28), but this is analytically intractable. A more physically transparent way is to impose by hand the gauge equivalence between the $\pm$ monopole operators

$$e^{i\gamma_{(12)}} \sim e^{-i\gamma_{(12)}}, \tag{4.48}$$

where the $\sim$ implies the two operators can be made equal by an $SO(N)$ gauge rotation. Writing the $SO(N)$-invariant monopole operator as

$$\mathcal{M} = e^{i\gamma}, \tag{4.49}$$

the constraint (4.48) requires that $\gamma \in \{0, \pi\} \mod 2\pi$. Accordingly, the continuous $U(1)$ shift symmetry of the dual photon reduces to a discrete $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry, under which the monopole operator is charged:

$$\begin{aligned} \mathbb{Z}_2^{\mathcal{M}} : \gamma &\mapsto \gamma + \pi \\ \mathcal{M} &\mapsto -\mathcal{M}. \end{aligned} \tag{4.50}$$

This is the behavior expected of monopole operators in 3D $SO(N>2)$ Yang-Mills theories, which are charged under $\mathbb{Z}_2^{\mathcal{M}}$ [140, 141, 168, 179–181].

The 't Hooft vertex can be further simplified by imposing reflection positivity of the Euclidean action as well as an anti-unitary time-reversal symmetry. First, reflection positivity[3] sets $\alpha=\beta$ in Eq. (4.47), partially constraining the phases of the ZM functions. This simplifies the vertex to

$$\Delta(z) = \mathcal{M}(z)(-\sigma_z+i\sigma_y)\left(\begin{array}{cc|c} \cos\alpha & \sin\alpha & \mathbf{0} \\ -\sin\alpha & \cos\alpha & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right). \tag{4.51}$$

From this form of the vertex, it is clear that α is analogous to the theta angle in compact $U(1)$ gauge theory in 3D [1], bar complications arising here from the lack of $SO(N)$ gauge invariance. To fix the value of α, we demand that the 't Hooft vertex (4.51) satisfies the same discrete spacetime symmetries as the rest of the action obtained from the Lagrangian (4.25). As can be checked explicitly, the corresponding Hamiltonian possesses an anti-unitary time-reversal symmetry $\mathcal{T}$, which is defined by

$$\mathcal{T}\Psi_\pm\mathcal{T}^{-1} = i\sigma_y\Psi_\mp, \qquad \mathcal{T}a_i\mathcal{T}^{-1} = a_i. \tag{4.52}$$

The nonstandard transformation of the vector potential a_i comes from the fact that the generators of $\mathfrak{so}(N)$ are pure imaginary antisymmetric matrices [e.g., Eq. (4.32)] which pick up an additional minus sign under complex conjugation. To determine the action of $\mathcal{T}$ on $\mathcal{M}$, we use a physical argument. $\mathcal{T}$ can at most reverse the direction of $SO(2)$ flux created by the monopole operator; but monopoles and anti-monopoles are gauge equivalent. Thus we conclude that $\mathcal{M}$ transforms trivially under $\mathcal{T}$.

To study the effect of $\mathcal{T}$ on the 't Hooft vertex, we first rewrite it using the Majorana condition (4.148) as

$$\begin{aligned} \mathcal{L}_{\text{inst}} &= \frac{\lambda K}{2m}(\Psi_-^\dagger\sigma_x\Delta\Psi_+ + \Psi_+^\dagger\Delta^\dagger\sigma_x\Psi_-) \\ &= \frac{\lambda K}{2m}(\Psi_-^\dagger\Delta\Psi_+ + \Psi_+^\dagger\Delta^\dagger\Psi_-). \end{aligned} \tag{4.53}$$

[3] In Euclidean signature, reality of the Minkowski action requires reflection-positivity of the Euclidean one. However, as the instanton-induced term is free of time derivatives, it is also a term in the effective Hamiltonian, which is required to be Hermitian, so it suffices to check Hermiticity.

It is then readily observed that

$$\mathcal{T}\mathcal{L}_{\rm inst}\mathcal{T}^{-1} = -\frac{\lambda K}{2m}(\Psi_+^\dagger\sigma_y\Delta^*\sigma_y\Psi_- + \Psi_-^\dagger\sigma_y\Delta^\intercal\sigma_y\Psi_+). \tag{4.54}$$

Demanding $\mathcal{T}$ invariance then yields the condition

$$\sigma_y\Delta^\intercal\sigma_y = \Delta, \tag{4.55}$$

which requires that the $SO(N)$ matrix in (4.51) be antisymmetric. Thus we obtain

$$\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \mod 2\pi, \tag{4.56}$$

which can be interpreted as a $\mathbb{Z}_2$ theta angle. The two resulting 't Hooft vertices only differ by an overall sign that can be absorbed in the coupling constant. Choosing $\alpha = \pi/2$, the effective Lagrangian that accounts for instanton effects is

$$\mathcal{L}_{\rm eff} = \frac{1}{4}\Psi_+^\intercal\mathcal{C}(\not{\partial} - i\not{a} + m)\Psi_+ + \frac{1}{4}\Psi_-^\intercal\mathcal{C}(\not{\partial} - i\not{a} - m)\Psi_- \\ + \frac{i\lambda K}{m}\mathcal{M}\Psi_-^\intercal(-\sigma_z + i\sigma_y)t_{(12)}\Psi_+ + \frac{1}{2g^2}\,\mathrm{tr}\, f^2, \tag{4.57}$$

where the coupling of fermions to a gluon field a has been restored.

Clearly, $\mathcal{L}_{\rm eff}$ is still only gauge invariant under $[O(2)\times O(N-2)]/\mathbb{Z}_2$, the presence of $t_{(12)}$ indicating memory of the specific $SO(2)$ subgroup the instanton was placed in. Yet, $\mathcal{L}_{\rm eff}$ encapsulates all the correct physical symmetries expected of the gauge-invariant Lagrangian. Without instanton corrections, the parton theory has the spurious global symmetry $\mathbb{Z}_2\times\mathbb{Z}_2^{\mathcal{M}}$, where $\mathbb{Z}_2$ is the microscopic parton representation of the Ising symmetry, under which

$$\Psi_\pm \to W\Psi_\pm, \qquad W = \mathrm{diag}(-1, 1, \dots, 1)_{N\times N}, \tag{4.58}$$

as per our choice of global charge assignment in Sec. 4.3.1, and $\mathbb{Z}_2^{\mathcal{M}}$ is the magnetic symmetry (4.50). This enlarged symmetry is absent in the physical spin model. The low-energy effective theory (4.57) indicates that instantons have the effect of explicitly breaking this spurious $\mathbb{Z}_2\times\mathbb{Z}_2^{\mathcal{M}}$ symmetry to the diagonal subgroup, under which

$$\Psi_\pm \to W\Psi_\pm, \qquad \mathcal{M} \to -\mathcal{M}. \tag{4.59}$$

Indeed, since $W^\intercal t_{(12)} W = -t_{(12)}$, the fermion bilinear in the 't Hooft vertex acquires a minus sign under the action of the first $\mathbb{Z}_2$ factor, which can be compensated by another minus sign coming from the $\mathbb{Z}_2^{\mathcal{M}}$ symmetry action on the monopole operator $\mathcal{M}$. This diagonal symmetry is finally understood as the correct incarnation, in the low-energy parton theory, of the microscopic Ising symmetry $\tau^z \to -\tau^z$. Although it is not presently clear whether nor how this may be derived analytically, we speculate that full averaging over the $SO(N)$ gauge orbit ($\int \mathscr{D}R$) in the partition function (4.28,4.29) produces a fully $SO(N)$-invariant 't Hooft vertex of the form

$$\mathcal{L}_{\text{eff}} \overset{?}{\sim} \mathcal{M}\epsilon_{\alpha_1\cdots\alpha_N}\Psi_-^{\alpha_1}(-\sigma_z+i\sigma_y)\Psi_+^{\alpha_2}\cdots\cdots\Psi_-^{\alpha_{N-1}}(-\sigma_z+i\sigma_y)\Psi_+^{\alpha_N}. \tag{4.60}$$

Under the $\mathbb{Z}_2$ symmetry W, the fermionic "baryon" operator is multiplied by a factor $\det W = -1$ which is compensated by the transformation of the monopole operator $\mathcal{M}$ under the $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry. Note that those transformation properties are now properly independent of the choice of global charge assignment to the fermionic partons, since $\det W$ is invariant under gauge-equivalent redefinitions $W \to RWR^\intercal$ with $R \in SO(N)$.

In either its $[O(2)\times O(N-2)]/\mathbb{Z}_2$ or $SO(N)$ invariant incarnations, the 't Hooft vertex implies that a breakdown of magnetic symmetry, which is typically associated with confinement [140, 141], is concomitant with a breakdown of the Ising symmetry implemented by W in the parton theory. We thus conclude that the $C = 0$ phase, which is described by a confining pure Yang-Mills theory at low energies, is indeed a phase in which the microscopic Ising symmetry is spontaneously broken.

4.5 Conclusion

In summary, we have employed slave-particle methods to discuss universal aspects of quantum phase transitions between magnetically ordered, trivially paramagnetic, and gapped topological phases of Ising spin systems. Our theory can be viewed as a generalization of the work of Ref. [102] from hardcore bosons with $U(1)$ symmetry to Ising spins with $\mathbb{Z}_2$ symmetry. Using a slave-particle decomposition of Ising spins in terms of fermionic Majorana partons with $SO(N)$ gauge structure, we argued that placing the partons in topologically superconducting mean-field states with Chern number $C = 0, 1, 2$ corresponds respectively to magnetically ordered, trivially paramagnetic, and chiral spin liquid phases of the constituent spins. Accounting for gauge fluctuations beyond mean-field, the corresponding Chern-number changing transitions were described by theories of Majorana fields coupled to $SO(N)$ gauge fields with a Chern-Simons term. Using recently conjectured $SO(N)$ dualities with Majorana fermions, the critical theory for the ordering transition from the trivial paramagnet was found to be dual to the usual Wilson-Fisher theory with a single scalar field, as expected for a standard Ising transition. We found that a direct ordering transition from the chiral spin liquid was also possible, and could be protected by lattice symmetries such as inversion symmetry on the honeycomb lattice.

Finally, we turned our attention to the ordered phase itself, in order to identify the symmetry-breaking mechanism from the point of view of the parton gauge theory. The latter was characterized by a spurious apparent $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathcal{M}}$ symmetry, with the first $\mathbb{Z}_2$ factor a global symmetry action on the Majorana partons, and $\mathbb{Z}_2^{\mathcal{M}}$ the magnetic symmetry associated with $SO(N)$ monopole operators. We then showed that the resolution of this problem is to account for nonperturbative instanton effects. First, the massive Majorana fields of the $C = 0$ phase support Euclidean ZMs bound to instantons. Second, resumming the instanton gas using semiclassical methods produces an interaction vertex ('t Hooft vertex) involving Majorana fields and monopole operators, that is only invariant under the diagonal $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathcal{M}}$. Under the plausible assumption of spontaneously broken $\mathbb{Z}_2^{\mathcal{M}}$ magnetic symmetry in the (confined) $C = 0$ phase, the 't Hooft vertex naturally led to simultaneous breaking of the global Ising symmetry in the parton sector. Thus, as in our earlier work on $U(1)$ bosons [1], we found that nonperturbative instanton effects are instrumental in accounting for spontaneous symmetry breaking in the relevant parton gauge theory. The precise pattern of symmetry breaking (e.g., ferromagnetism vs antiferromagnetism) in the physical spin system depends on the microscopic interpretation of the continuum Majorana spinors $\Psi_\pm$ in a specific lattice model.

We finally outline a few avenues for future research. First, it would be interesting to perform tests of the fermionization duality (4.21) using large-N methods, as done in Ref. [142] for the fermionization (4.6) of the 3D XY transition, or in Refs. [190, 191] for non-Abelian dualities with unitary gauge groups. In particular, the duality predicts that the scaling dimension of the Majorana mass operator $[\psi^\intercal\mathcal{C}\psi] = 3-\nu^{-1}$, which is dual to the ϕ^2 operator on the scalar side and related to the correlation length exponent ν, should be independent of the rank of the $SO(N)$ gauge group. It would be interesting to test this prediction by performing computations in the 't Hooft limit with $N \to \infty$ [192]. Second, while the transition between magnetic order and trivial paramagnet is ultimately a standard Ising transition, a direct transition between magnetic order and the $\nu = 1/N$ chiral spin liquid is described by a theory of $N_f = 2$ massless Majorana fermions coupled to an $SO(N)_1$ Chern-Simons term. This presumably defines a new universality class of Ising transitions in 2+1 dimensions, and it would be interesting to compute critical exponents using either large-N_f or large-N expansions. Third, to complement the semiclassical instanton gas calculation we have presented here, it would be interesting to study the scaling dimensions of $\mathbb{Z}_2$ monopole operators in critical $SO(N)$ gauge theories with Majorana matter, using the state-operator correspondence of conformal field theory [27]. The latter has been successfully used in $U(1)$ gauge theories with massless Dirac matter [27, 28, 55–57]. Finally, from a more microscopic standpoint, it would be desirable to construct variational many-body wave functions based on the parton ansätze discussed here (i.e., N-flavor wave functions of Majorana fermions projected to the $SO(N)$ gauge-invariant sector) and use them to study frustrated lattice models of interacting spins with Ising symmetry. Such models could include antiferromagnetic quantum Ising models defined on geometrically frustrated lattices like the kagome lattice, or on non-frustrated lattices but with competing anisotropic interactions, as in the Kitaev model on the honeycomb lattice.

4.6 Appendix: Dualities for bosons with $U(1)$ symmetry

For the reader's convenience, we provide here a derivation of the dualities between the fermionic critical theories (4.6), (4.8) and their respective bosonic duals (4.7), (4.9), respectively, based on Ref. [147].

The starting point is the duality of relativistic flux attachment, whereby coupling a level-1 Chern-Simons gauge field to a relativistic complex scalar attaches one flux quantum to the latter and turns it into a Dirac fermion [193]. This can be expressed by the following equivalence between the partition functions

$$Z_\psi[A]e^{\frac{i}{2}S_{\text{CS}}[A]} = \int \mathscr{D}a\, Z_\phi[a]e^{-iS_{\text{CS}}[a]+iS_{\text{BF}}[a,A]}, \tag{4.61}$$

where we define the fermionic and bosonic partition functions

$$Z_\psi[A] = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\, e^{i\int d^3x\, \bar{\psi}i(\partial\!\!\!/-iA\!\!\!/)\psi}, \tag{4.62}$$

$$Z_\phi[A] = \int \mathscr{D}\phi^*\mathscr{D}\phi\, e^{i\int d^3x\left(|(\partial_\mu-iA_\mu)\phi|^2-\lambda|\phi|^4\right)}, \tag{4.63}$$

and the bosonic action is understood as being tuned to criticality. We define the Chern-Simons and BF actions as

$$S_{\text{CS}}[a] = \frac{1}{4\pi}\int ada, \tag{4.64}$$

$$S_{\text{BF}}[a,b] = S_{\text{BF}}[b,a] = \frac{1}{2\pi}\int adb. \tag{4.65}$$

Equation (4.61) for $A=0$ is simply the relativistic version of the statement that attaching a flux quantum to a boson turns it into a fermion. The BF term for nonzero A expresses the fact that the conserved $U(1)$ fermion current $\bar{\psi}\gamma^\mu\psi$ corresponds to $\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$ in the bosonic theory [194]. To understand the level 1/2 Chern-Simons term for A, consider a massive deformation of the theory. If we add a mass term $-r|\phi|^2$ for the scalar with $r>0$, the scalar is gapped and can be integrated out. At low energies the factor $Z_\phi[a]$ in Eq. (4.61) only contains irrelevant terms and reduces to a constant; integrating out a

then produces a Chern-Simons term of level 1 for A. The fermionic theory should also be gapped. Assuming a fermionic mass term $\propto -r\bar{\psi}\psi$, if $r > 0$ integrating out the fermion produces a Chern-Simons term at level 1/2 for A; an additional level-1/2 Chern-Simons term must be added to the fermionic action for the two sides to match. For this assignment to be consistent, the two sides should match also when $r < 0$. In this case, on the fermionic side integrating out the fermion cancels out the Chern-Simons term and the Hall response vanishes. On the bosonic side, the scalar condenses and a is Higgsed; the Chern-Simons term for a becomes irrelevant and the Hall response also vanishes upon integration over a.

We now turn to deriving the duality between (4.6) and (4.7). The partition function $Z[A]$ for the fermionic theory (4.6) is given by

$$Z[A] = \int \mathscr{D}a\, Z_\psi[a + A] e^{\frac{i}{2}S_{\rm CS}[a] - \frac{i}{2}S_{\rm BF}[a,A] - \frac{i}{2}S_{\rm CS}[A]}. \tag{4.66}$$

Shifting $a \to a - A$ and using

$$S_{\rm CS}[a - A] = S_{\rm CS}[a] - S_{\rm BF}[a, A] + S_{\rm CS}[A], \tag{4.67}$$

$$S_{\rm BF}[a - A, A] = S_{\rm BF}[a, A] - 2S_{\rm CS}[A], \tag{4.68}$$

we obtain

$$Z[A] = \int \mathscr{D}a\, Z_\psi[a] e^{\frac{i}{2}S_{\rm CS}[a] - iS_{\rm BF}[a,A] + iS_{\rm CS}[A]}. \tag{4.69}$$

Apart from an additional Chern-Simons term, this can be interpreted as applying the S operation of Witten's $SL(2,\mathbb{Z})$ action on (2+1)D CFTs with a global $U(1)$ symmetry [195] to the left-hand side of the flux-attachment duality (4.61). Using (4.61), $Z[A]$ becomes

$$\begin{aligned} Z[A] &= \int \mathscr{D}a \mathscr{D}\tilde{a}\, Z_\phi[\tilde{a}] e^{-iS_{\rm CS}[\tilde{a}] + iS_{\rm BF}[\tilde{a},a] - iS_{\rm BF}[a,A] + iS_{\rm CS}[A]} \\ &= \int \mathscr{D}\tilde{a}\, Z_\phi[\tilde{a}] e^{-iS_{\rm CS}[\tilde{a}] + iS_{\rm CS}[A]} \int \mathscr{D}a\, e^{\frac{i}{2\pi}\int ad(\tilde{a} - A)}. \end{aligned} \tag{4.70}$$

Integrating over a enforces $\tilde{a} = A + d\chi$ where χ is an arbitrary function. Exploiting the gauge invariance of the bosonic partition function (4.63) and the Chern-Simons action

(4.64), we find simply $Z[A] = Z_\phi[A]$, thus the gauged Wilson-Fisher theory (4.7) is dual to the fermionic theory (4.6).

We now derive the duality between (4.8) and (4.9). The partition function corresponding to (4.8) is

$$Z[A] = \int \mathscr{D}a\, Z_\psi[a+A] e^{\frac{3i}{2}S_{\mathrm{CS}}[a]+\frac{i}{2}S_{\mathrm{BF}}[a,A]+\frac{i}{2}S_{\mathrm{CS}}[A]}. \tag{4.71}$$

Performing the shift $a \to a - A$ as before, we obtain

$$Z[A] = \int \mathscr{D}a\, Z_\psi[a] e^{\frac{3i}{2}S_{\mathrm{CS}}[a]-iS_{\mathrm{BF}}[a,A]+iS_{\mathrm{CS}}[A]}. \tag{4.72}$$

This can be interpreted as applying the combined ST operation of Witten's $SL(2,\mathbb{Z})$ action to (4.61), whereby one first shifts the Chern-Simons level of the background gauge field by one before making it dynamical [195]. Using (4.61) once again, we have

$$\begin{aligned} Z[A] &= \int \mathscr{D}a\mathscr{D}\tilde{a}\, Z_\phi[\tilde{a}] e^{-iS_{\mathrm{CS}}[\tilde{a}]+iS_{\mathrm{CS}}[a]+iS_{\mathrm{BF}}[a,\tilde{a}-A]+iS_{\mathrm{CS}}[A]} \\ &= \int \mathscr{D}\tilde{a}\, Z_\phi[\tilde{a}] e^{-2iS_{\mathrm{CS}}[\tilde{a}]+iS_{\mathrm{BF}}[\tilde{a},A]}, \end{aligned} \tag{4.73}$$

performing the path integral over a. Thus a single Dirac fermion coupled to $U(1)_{3/2}$ Chern-Simons theory [Eq. (4.8)] is dual to the gauged Wilson-Fisher fixed point coupled to $U(1)_{-2}$ Chern-Simons theory [Eq. (4.9)].

4.7 Appendix: Majorana $SO(N)$ lattice gauge theory in the strong-coupling limit

In this Appendix, we show that in the limit of strong gauge coupling, a theory of N colors of Majorana fermions (N even) coupled to an $SO(N)$ lattice gauge field naturally reduces to a theory of Ising spins corresponding to the gauge-invariant Majorana baryons (4.13).

4.7.1 Euclidean vs Hamiltonian approach

First, we relate the Euclidean and Hamiltonian descriptions of $SO(N)$ lattice gauge theory with Majorana fermions in the vector representation, following the approach of Refs. [196–

198]. We begin with a Euclidean action in discrete 3D spacetime,

$$S = S_\chi + S_U, \tag{4.74}$$

where

$$S_U = -\frac{\beta}{2}\sum_{\Box} \mathrm{tr}\, UUUU - \frac{\beta_\tau}{2}\sum_{\Box_\tau} \mathrm{tr}\, UUUU + \text{c.c.}, \tag{4.75}$$

is the gauge-field action, and

$$S_\chi = \frac{it}{4}\sum_{i,\mu} \chi_i^T h_{i,i+\hat{\mu}} U_{i,i+\hat{\mu}} \chi_{i+\hat{\mu}} + \frac{t_\tau}{4}\sum_i \chi_i^T U_{i,i+\hat{\tau}} \chi_{i+\hat{\tau}}, \tag{4.76}$$

is the gauged Majorana action. Here i, j denote spacetime lattice sites, $\hat{\mu} = (\hat{x}, \hat{y})$ denotes lattice vectors in the two space directions, and $\hat{\tau}$ denotes the lattice vector in the imaginary-time direction. We write $\chi_i = (\chi_i^1, \ldots, \chi_i^N)$ for the N-component vector of Majorana fields on site i, $U_{ij} \in SO(N)$ for the link variable on nearest-neighbor spacetime link ij, with $U_{ji} = U_{ij}^{-1}$, and $\Box$ and $\Box_\tau$ for spacelike and timelike plaquettes, respectively. The real antisymmetric matrix h describes Majorana hopping in the absence of gauge fields [23], but we have factored out the hopping strength t. We consider spacetime-anisotropic couplings in anticipation of taking the τ-continuum limit to relate the discrete-time action formulation to the Hamiltonian formulation [199]. For the same reason, we take the lattice constant in the spatial direction to be unity, and the lattice constant in the temporal direction to be $\epsilon \ll 1$. The action is invariant under local $SO(N)$ gauge transformations,

$$\chi_i \to R_i \chi_i, \quad U_{ij} \to R_i U_{ij} R_j^{-1}, \quad R_i \in SO(N). \tag{4.77}$$

First, we use this gauge freedom to work in the temporal gauge: $U_{i,i+\hat{\tau}} = 1$ on all temporal links. The Majorana action becomes,

$$\begin{aligned} S_\chi &= \sum_\tau \left(\frac{t}{4}\sum_{r,\mu} \chi_r^T(\tau) U_{r,r+\hat{\mu}}(\tau) \chi_{r+\hat{\mu}}(\tau) + \frac{t_\tau}{4}\sum_r \chi_r^T(\tau)\chi_r(\tau+\epsilon) + \text{c.c.} \right) \\ &\approx \epsilon \sum_\tau \left(\frac{t}{4\epsilon}\sum_{r,\mu} \chi_r^T(\tau) U_{r,r+\hat{\mu}}(\tau) \chi_{r+\hat{\mu}}(\tau) + \frac{t_\tau}{4}\sum_r \chi_r^T(\tau)\partial_\tau \chi_r(\tau) + \text{c.c.} \right), \end{aligned} \tag{4.78}$$

to leading order in ϵ, ignoring additive constants. Here we use $i = (r, \tau)$ to denote the dependence on space r and time τ coordinates separately.

The gauge-field action is more subtle. The contribution from spatial plaquettes is obvious; we now focus on temporal plaquettes. In the temporal gauge, we have:

$$\begin{aligned} \operatorname{tr} UUUU\big|_{\square_\tau} &= \operatorname{tr} U_{r,r+\hat{\mu}}(\tau) U_{r+\hat{\mu},r}(\tau+\epsilon) \\ &= \operatorname{tr} U^{-1}_{r,r+\hat{\mu}}(\tau+\epsilon) U_{r,r+\hat{\mu}}(\tau), \end{aligned} \tag{4.79}$$

using the cyclic property of the trace. To work towards the Hamiltonian formulation, we seek an operator $\hat{O}$ such that

$$\langle U_{rr'}(\tau+\epsilon)| e^{-\epsilon \hat{O}} |U_{rr'}(\tau)\rangle = e^{\beta_\tau \operatorname{Re} \operatorname{tr} g}, \tag{4.80}$$

where $g = U^{-1}_{rr'}(\tau+\epsilon) U_{rr'}(\tau) \in SO(N)$, and the equality holds in the limit $\epsilon \ll 1$. We focus on a given spatial link rr'. The state $|U_{rr'}\rangle$ is an eigenstate of the matrix-valued link operator $\hat{U}_{rr'}$,

$$\hat{U}_{rr'} |U_{rr'}\rangle = U_{rr'} |U_{rr'}\rangle. \tag{4.81}$$

We define an electric-field operator $\hat{E}^a_{rr'}$ that is (almost) a canonical conjugate to $\hat{U}_{rr'}$,

$$[\hat{E}^a_{rr'}, \hat{U}_{rr'}] = -T^a \hat{U}_{rr'}, \tag{4.82}$$

where $a = 1, \ldots, N(N-1)/2$ ranges over the generators T^a of $SO(N)$. (Note that on the right-hand side of Eq. (4.82), there is matrix multiplication between the c-number matrix T^a and the matrix-valued operator $\hat{U}_{rr'}$, while for a given a, the operator $\hat{E}^a_{rr'}$ is a scalar.) The electric-field operators satisfy the $\mathfrak{so}(N)$ Lie algebra,

$$[\hat{E}^a_{rr'}, \hat{E}^b_{rr'}] = i f^{abc} \hat{E}^c_{rr'}, \tag{4.83}$$

where f^{abc} are the $\mathfrak{so}(N)$ structure constants. Now consider the operator

$$\hat{R}_{rr'}(g) = e^{-i\omega^a \hat{E}^a_{rr'}}, \tag{4.84}$$

where the $SO(N)$ matrix is parametrized as $g = e^{-i\omega^a T^a}$. We have the property

$$\hat{R}_{rr'}(g)|U_{rr'}\rangle = |gU_{rr'}\rangle. \tag{4.85}$$

Indeed, using Eq. (4.82), we can show that $\hat{R}_{rr'}(g)|U_{rr'}\rangle$ is an eigenstate of $\hat{U}_{rr'}$ with eigenvalue $gU_{rr'}$:

$$\begin{aligned}
\hat{U}_{rr'}\left(\hat{R}_{rr'}(g)|U_{rr'}\rangle\right) &= \left(\hat{U}_{rr'}e^{-i\omega^a\hat{E}^a_{rr'}}\hat{U}^{-1}_{rr'}\right)\hat{U}_{rr'}|U_{rr'}\rangle \\
&= e^{-i\omega^a\hat{U}_{rr'}\hat{E}^a_{rr'}\hat{U}^{-1}_{rr'}}U_{rr'}|U_{rr'}\rangle \\
&= e^{-i\omega^a(\hat{E}^a_{rr'}+T^a)}U_{rr'}|U_{rr'}\rangle \\
&= e^{-i\omega^a\hat{E}^a_{rr'}}e^{-i\omega^a T^a}U_{rr'}|U_{rr'}\rangle \\
&= gU_{rr'}\left(\hat{R}_{rr'}(g)|U_{rr'}\rangle\right).
\end{aligned} \tag{4.86}$$

In the fourth line, we use the fact that $[\hat{E}^a_{rr'}, T^a] = 0$ because T^a is a c-number matrix while $\hat{E}^a_{rr'}$ is a scalar operator.

Using property (4.85), we claim that Eq. (4.80) is satisfied if

$$e^{-\epsilon\hat{O}} = \int dg\, e^{\beta_\tau \operatorname{Re}\operatorname{tr} g}\hat{R}_{rr'}(g), \tag{4.87}$$

where dg denotes the Haar measure on $SO(N)$. Indeed, we then have

$$\begin{aligned}
&\langle U_{rr'}(\tau+\epsilon)|e^{-\epsilon\hat{O}}|U_{rr'}(\tau)\rangle \\
&\qquad = \int dg\, e^{\beta_\tau \operatorname{Re}\operatorname{tr} g}\langle U_{rr'}(\tau+\epsilon)|\hat{R}_{rr'}(g)|U_{rr'}(\tau)\rangle \\
&\qquad = \int dg\, e^{\beta_\tau \operatorname{Re}\operatorname{tr} g}\langle U_{rr'}(\tau+\epsilon)|gU_{rr'}(\tau)\rangle \\
&\qquad = \int dg\, e^{\beta_\tau \operatorname{Re}\operatorname{tr} g}\delta_{U_{rr'}(\tau+\epsilon),gU_{rr'}(\tau)} \\
&\qquad = e^{\beta_\tau \operatorname{Re}\operatorname{tr} U_{rr'}(\tau+\epsilon)U^{-1}_{rr'}(\tau)} \\
&\qquad = e^{\beta_\tau \operatorname{Re}\operatorname{tr} U^{-1}_{rr'}(\tau+\epsilon)U_{rr'}(\tau)},
\end{aligned} \tag{4.88}$$

where in the last line, we have used the fact that $\operatorname{Re}\operatorname{tr} g = \operatorname{Re}\operatorname{tr} g^\dagger$, and $g^{-1} = g^\dagger$ for $g \in SO(N)$. Finally, we consider the $SO(N)$ Haar integral in (4.87). Since $g = e^{-i\omega^a T^a}$, we have $\mathrm{Re}\,\mathrm{tr}\, g = \frac{1}{2}\mathrm{tr}\big(g + g^\dagger\big) = \mathrm{tr}\cos\omega^a T^a$. The integral over g can be converted to an integral over ω:

$$e^{-\epsilon\hat{O}} = \left(\prod_a \int d\omega_a\right) \mathcal{J}(\omega) e^{\beta_\tau \,\mathrm{tr}\cos\omega^a T^a} e^{-i\omega^a \hat{E}^a_{rr'}}, \tag{4.89}$$

where $\mathcal{J}$ is the Jacobian of the transformation. We further write $\beta_\tau = 1/(\epsilon J)$ with fixed J, and consider the limit $\epsilon \ll 1$. In that limit, we can use a saddle-point approximation: the integral is dominated by Gaussian fluctuations around the maximum of $\mathrm{tr}\cos\omega^a T^a$, which is at $\omega^a = 0$. Using

$$\begin{aligned}\mathrm{tr}\cos\omega^a T^a &= \mathrm{tr}\left(1 - \frac{1}{2}\omega^a\omega^b T^a T^b + \dots\right)\\ &= N - \frac{1}{4}\omega^a\omega^a + \mathcal{O}(\omega^4),\end{aligned} \tag{4.90}$$

assuming the $SO(N)$ generators are normalized as $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$. We thus obtain

$$\begin{aligned}e^{-\epsilon\hat{O}} &\propto \mathcal{J}(0)\int d\omega\, e^{-\frac{\beta_\tau}{4}\omega^a\omega^a} e^{-i\omega^a\hat{E}^a_{rr'}}\\ &\propto e^{-\epsilon J \hat{E}^a_{rr'}\hat{E}^a_{rr'}}.\end{aligned} \tag{4.91}$$

Taking the logarithm on both sides and ignoring an irrelevant additive constant, we thus conclude that the desired operator $\hat{O}$ is

$$\hat{O} = J\sum_{r,\mu} \hat{E}^a_{r,r+\hat{\mu}}\hat{E}^a_{r,r+\hat{\mu}}, \tag{4.92}$$

where we have generalized Eq. (4.87) to include a product over all spatial links, since all spatial links decouple in the sum over temporal plaquettes. Finally, writing $\beta = \epsilon K$ with fixed K, $t = \epsilon\kappa$ with fixed κ, and normalizing the action such that $t_\tau = 1$, we obtain

$$\begin{aligned}S &\approx \epsilon\sum_\tau\left(\frac{1}{4}\sum_r \chi_r^T\partial_\tau\chi_r + H\right)\\ &\approx \int d\tau\left(\frac{1}{4}\sum_r \chi_r^T\partial_\tau\chi_r + H\right),\end{aligned} \tag{4.93}$$

where the Hamiltonian is, now dropping hats on operators,

$$\begin{aligned}H = \frac{i\kappa}{4}&\sum_{r,\mu}\chi_r^T h_{r,r+\hat{\mu}}U_{r,r+\hat{\mu}}\chi_{r+\hat{\mu}} + J\sum_{r,\mu}\mathrm{tr}\, E^2_{r,r+\hat{\mu}}\\ &+ K\sum_{\square}\mathrm{Re}\,\mathrm{tr}\, UUUU,\end{aligned} \tag{4.94}$$

where we have defined the matrix-valued electric-field operator $E_{r,r+\hat{\mu}} \equiv E^a_{r,r+\hat{\mu}} T^a$ to arrive at a basis-independent expression (and have absorbed a factor of $\frac{1}{2}$ into J). Note that the hopping matrix $h_{rr'} = -h_{r'r} = h^*_{rr'}$ has no dependence on color indices. To be more precise, we have $\chi^T_r h_{rr'} U_{rr'} \chi_{r'} \equiv \chi^\alpha_r h_{rr'} U^{\alpha\beta}_{rr'} \chi^\beta_{r'}$, where $\alpha, \beta = 1, \ldots, N$ are the color indices. We can check that the constraints of Fermi statistics and Hermiticity of the Hamiltonian both separately imply that $U^{\beta\alpha}_{r'r} = U^{\alpha\beta}_{rr'}$, i.e., that $U^T_{r'r} = U_{rr'}$, which is satisfied for $SO(N)$ gauge fields since $U^T_{r'r} = U^{-1}_{r'r} = U_{rr'}$. Thus lattice Majorana fermions can be consistently coupled to lattice $SO(N)$ gauge fields.

4.7.2 Strong-coupling limit

In the τ-continuum limit, we saw that the relationship between the couplings in the space-time lattice action β, β_τ and those in the Hamiltonian J, K is $\beta = \epsilon K$ and $\beta_\tau = 1/(\epsilon J)$. We now consider the "electric" limit in the Hamiltonian problem: $J \to \infty$ and $K \to 0$. We see that in this limit, $\beta, \beta_\tau \to 0$. Going back to the Euclidean lattice action, the plaquette term S_U disappears in this limit, and the physics is purely governed by the gauged Majorana action: $S(J \to \infty, K \to 0) \approx S_\chi$, where

$$S_\chi = \frac{it}{4} \sum_{i,\mu} \chi^T_i h_{i,i+\hat{\mu}} U_{i,i+\hat{\mu}} \chi_{i+\hat{\mu}} + \frac{1}{4} \sum_i \chi^T_i U_{i,i+\hat{\tau}} \chi_{i+\hat{\tau}}. \tag{4.95}$$

In this limit, all links decouple, and the functional integral over the gauge field reduces to a product of one-link Haar integrals over $SO(N)$ [200, 201]:

$$\begin{aligned} Z &= \int \mathscr{D}\chi \mathscr{D}U \, e^{-S} \\ &= \int \mathscr{D}\chi \left(\prod_{i,\mu} \int dU \, e^{-\frac{it}{4}\chi^T_i h_{i,i+\hat{\mu}} U \chi_{i+\hat{\mu}}} \right) \left(\prod_i \int dU \, e^{-\frac{1}{4}\chi^T_i U \chi_{i+\hat{\tau}}} \right). \end{aligned} \tag{4.96}$$

Consider first the spatial-link term. We perform a formal expansion in the hopping parameter:

$$\int dU \, e^{-\frac{it}{4}\chi^T_i h_{i,i+\hat{\mu}} U \chi_{i+\hat{\mu}}} = \sum_{n=0}^\infty \frac{1}{n!} \left(-\frac{it}{4} h_{i,i+\hat{\mu}} \right)^n \chi^{\alpha_1}_i \chi^{\beta_1}_{i+\hat{\mu}} \cdots \chi^{\alpha_n}_i \chi^{\beta_n}_{i+\hat{\mu}} \int dU \, U^{\alpha_1\beta_1} \cdots U^{\alpha_n\beta_n}. \tag{4.97}$$

Polynomial integrals over compact Lie groups can in principle be computed exactly in the framework of Weingarten calculus [202]. Here we will not attempt to do this, but only use general properties of those integrals to illustrate the physics [203]. The necessary results are given in Ref. [202] for the orthogonal group $O(N)$. To compute integrals over $SO(N)$, we insert the factor $(1 + \det U)/2$ in the integrand:

$$\begin{aligned}\int_{SO(N)} dU\, U^{\alpha_1\beta_1} \dots U^{\alpha_n\beta_n} &= \int_{O(N)} dU \left(\frac{1+\det U}{2}\right) U^{\alpha_1\beta_1} \dots U^{\alpha_n\beta_n} \\ &= \frac{1}{2}\int_{O(N)} dU\, U^{\alpha_1\beta_1} \dots U^{\alpha_n\beta_n} \\ &\quad + \frac{1}{2}\epsilon^{\gamma_1\cdots\gamma_N} \int_{O(N)} dU\, U^{1,\gamma_1} \dots U^{N,\gamma_N} U^{\alpha_1\beta_1} \dots U^{\alpha_n\beta_n}. \end{aligned} \tag{4.98}$$

Consider the first term. For it to be nonzero, n must be even: $n = 2k$, and the sets $\{\alpha_1, \dots, \alpha_{2k}\}$ and $\{\beta_1, \dots, \beta_{2k}\}$ must each contain k pairs of identical entries [202]. Consider such pairs $\alpha_i = \alpha_j = \alpha$ and $\beta_i = \beta_j = \beta$; the corresponding Majorana term is $(\chi_i^\alpha)^2(\chi_{i+\hat{\mu}}^\beta)^2 = \text{const}$. Thus the first term in Eq. (4.98) can be ignored (the gauge-invariant Majorana "mesons" are trivial). Turning to the second term, the pairing rule first requires that the set $\{1, \dots, N, \alpha_1, \dots, \alpha_n\}$ can be grouped into pairs. The smallest n for which this occurs is $n = N$, which implies that $\{\alpha_1, \dots, \alpha_N\} = \{1, \dots, N\}$ in some order. Likewise, the set $\{\gamma_1, \dots, \gamma_N, \beta_1, \dots, \beta_n\}$ must obey the same pair constraint, which also implies that $\{\beta_1, \dots, \beta_N\} = \{1, \dots, N\}$ in some order since $\{\gamma_1, \dots, \gamma_N\} = \{1, \dots, N\}$ by virtue of the epsilon tensor. But since both $\{\alpha_1, \dots, \alpha_N\}$ and $\{\beta_1, \dots, \beta_N\}$ must equal $\{1, \dots, N\}$ in some order, then

$$\chi_i^{\alpha_1}\chi_{i+\hat{\mu}}^{\beta_1} \dots \chi_i^{\alpha_n}\chi_{i+\hat{\mu}}^{\beta_n} \propto (\chi_i^1 \dots \chi_i^N)(\chi_{i+\hat{\mu}}^1 \dots \chi_{i+\hat{\mu}}^N)\epsilon^{\alpha_1\cdots\alpha_N}\epsilon^{\beta_1\cdots\beta_N}, \tag{4.99}$$

since Majorana fields anticommute. Absorbing into a constant B the following integral,

$$B \propto \epsilon^{\gamma_1\cdots\gamma_N}\epsilon^{\alpha_1\cdots\alpha_N}\epsilon^{\beta_1\cdots\beta_N}\int_{O(N)} dU\, U^{1,\gamma_1} \cdots U^{N,\gamma_N} U^{\alpha_1\beta_1} \cdots U^{\alpha_N\beta_N}, \tag{4.100}$$

we obtain:

$$\begin{aligned}\int dU\, e^{-\frac{it}{4}\chi_i^T h_{i,i+\hat{\mu}} U \chi_{i+\hat{\mu}}} &= 1 + \frac{B(-t)^N h_{i,i+\hat{\mu}}^N}{4^N N!}\tau_i^z\tau_{i+\hat{\mu}}^z + \dots \\ &\approx e^{J_{i,i+\hat{\mu}}\tau_i^z\tau_{i+\hat{\mu}}^z + \cdots}, \end{aligned} \tag{4.101}$$

where we have introduced the Ising baryon

$$\tau_i^z = i^{N/2}\chi_i^1 \dots \chi_i^N, \tag{4.102}$$

with N even, and an effective nearest-neighbor exchange $J_{i,i+\hat{\mu}} = B(-t)^N h_{i,i+\hat{\mu}}^N/(4^N N!)$. Likewise for the temporal link integral in Eq. (4.96), the formal expansion gives:

$$\begin{aligned}\int dU\, e^{-\frac{1}{4}\chi_i^T U \chi_{i+\hat{\tau}}} &= 1 + \frac{Bi^N}{4^N N!}\tau_i^z \tau_{i+\hat{\tau}}^z + \dots \\ &\approx e^{K\tau_i^z\tau_{i+\hat{\tau}}^z + \dots},\end{aligned} \tag{4.103}$$

where $K = B(-1)^{N/2}/(4^N N!)$ is the nearest-neighbor coupling in the temporal direction. One thus obtains an effective spacetime lattice Ising action,

$$S_{\text{eff}}[\tau^z] = -\sum_{i,\mu} J_{i,i+\hat{\mu}}\tau_i^z\tau_{i+\hat{\mu}}^z - K\sum_i \tau_i^z\tau_{i+\hat{\tau}}^z + \dots, \tag{4.104}$$

which corresponds to an effective quantum Ising Hamiltonian in the τ-continuum limit [199],

$$H_{\text{eff}}[\hat{\tau}^z, \hat{\tau}^x] = -\sum_{r,\mu} J'_{r,r+\hat{\mu}}\hat{\tau}_r^z\hat{\tau}_{r+\hat{\mu}}^z - K'\sum_r \hat{\tau}_r^x + \dots, \tag{4.105}$$

with a suitably defined exchange coupling $J'_{r,r+\hat{\mu}}$ and transverse field K', neglecting higher-order multi-spin interactions that correspond to neglected higher-order baryon processes in the strong-coupling (hopping) expansion. Thus it is clear that, at least from a strong-coupling perspective, the $SO(N)$ Majorana gauge theory that results from the parton decomposition (4.13) is a theory of interacting Ising spins.

4.8 Appendix: Conformal embeddings in $\mathfrak{so}(n)$ WZW models

In this Appendix, we explain the meaning of the conformal embedding [173]:

$$\mathfrak{so}(N)_k \otimes \mathfrak{so}(k)_N \subseteq \mathfrak{so}(Nk)_1, \tag{4.106}$$

which is a generalization of the embedding $\mathfrak{so}(k)_k \otimes \mathfrak{so}(k)_k \subseteq \mathfrak{so}(k^2)_1$ used in Refs. [204, 205].

4.8.1 Free chiral Majorana fields

The starting point is the 2D CFT of Nk free chiral Majorana fermions $\chi_\alpha(z)$, where $\alpha=1,\ldots,Nk$. The (holomorphic) energy-momentum tensor for this free theory is [206]:

$$T(z) = -\frac{1}{2}\sum_\alpha \chi_\alpha \partial \chi_\alpha, \tag{4.107}$$

where $\partial \equiv \partial_z$. The chiral central charge c_- for this theory is 1/2 per flavor of Majorana fermion, i.e., $c_- = Nk/2$. This theory is equivalent to the critical $\mathfrak{so}(n)$ WZW model at level 1, with $n = Nk$. To establish this, we define the $\mathfrak{so}(n)$ currents:

$$j^a(z) \equiv \frac{i}{2}\chi^T(z) T^a \chi(z), \tag{4.108}$$

where $a = 1, \ldots, n(n-1)/2$ ranges over the real antisymmetric generators T^a of the $\mathfrak{so}(n)$ Lie algebra. These currents satisfy a nontrivial algebra (current algebra) in the sense of the operator product expansion (OPE). To compute the OPE for free fields, we simply need to use Wick's theorem. For now we are only interested in the singular part of the OPE, which is given by the sum of all Wick contractions:

$$\begin{aligned} j^a(z) j^b(w) &\sim -\frac{1}{4}\sum_{\text{Wick}} \chi_\alpha(z) T^a_{\alpha\beta}\chi_\beta(z)\chi_\gamma(w) T^b_{\gamma\delta}\chi_\delta(w) \\ &\sim -\frac{1}{4} T^a_{\alpha\beta} T^b_{\gamma\delta}\Big[-\langle \chi_\alpha(z)\chi_\gamma(w)\rangle \chi_\beta(z)\chi_\delta(w) \\ &\qquad - \langle \chi_\beta(z)\chi_\delta(w)\rangle \chi_\alpha(z)\chi_\gamma(w) \\ &\qquad + \langle \chi_\beta(z)\chi_\gamma(w)\rangle \chi_\alpha(z)\chi_\delta(w) \\ &\qquad + \langle \chi_\alpha(z)\chi_\delta(w)\rangle \chi_\beta(z)\chi_\gamma(w) \\ &\qquad - \langle \chi_\alpha(z)\chi_\gamma(w)\rangle \langle \chi_\beta(z)\chi_\delta(w)\rangle \\ &\qquad + \langle \chi_\alpha(z)\chi_\delta(w)\rangle \langle \chi_\beta(z)\chi_\gamma(w)\rangle \Big]. \end{aligned} \tag{4.109}$$

Next, we use the free Majorana Green's function:

$$\langle \chi_\alpha(z)\chi_\beta(w)\rangle = \frac{\delta_{\alpha\beta}}{z-w}, \tag{4.110}$$

and, in the operator-valued terms, expand $\chi_\alpha(z) = \chi_\alpha(w) + (z-w)\partial\chi_\alpha(w) + \ldots$. Keeping only terms singular as $z \to w$, we obtain:

$$j^a(z)j^b(w) \sim -\frac{1}{4}\left(\frac{2}{z-w}\chi^T[T^a, T^b]\chi + \frac{2}{(z-w)^2}\,\mathrm{tr}\,T^aT^b\right), \tag{4.111}$$

using $\chi^T T^a T^b \chi = -\chi^T T^b T^a \chi$, from Grassmann anticommutation and the antisymmetry of the $\mathfrak{so}(n)$ generators. We assume the (anti-Hermitian) generators obey the following properties:

$$[T^a, T^b] = f^{abc}T^c, \quad \mathrm{tr}\,T^aT^b = -2\delta^{ab}, \tag{4.112}$$

where f^{abc} are the structure constants of $\mathfrak{so}(n)$. We then obtain:

$$j^a(z)j^b(w) \sim \frac{\delta^{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w}j^c(w), \tag{4.113}$$

which is the $\mathfrak{so}(n)_1$ current algebra (Kac-Moody algebra) [206]. The energy-momentum tensor can be expressed in terms of these currents using the Sugawara construction:

$$T_{\mathfrak{so}(n)_1}(z) = \frac{1}{2(n-1)}\sum_a :j^a(z)j^a(z): \,, \tag{4.114}$$

where the colons denote normal ordering, i.e., the product $j^a(z)j^b(w)$ in the limit $z \to w$ (that is, the OPE) but with all singular terms subtracted. To do this computation, we use the identity in Eq. (15.204) of Ref. [206]:

$$\sum_{\alpha\beta}\left(:(\chi_\alpha\chi_\beta)(\chi_\alpha\chi_\beta): - :(\chi_\alpha\chi_\beta)(\chi_\beta\chi_\alpha):\right) = 4(n-1)\sum_\alpha \chi_\alpha\partial\chi_\alpha. \tag{4.115}$$

We also choose a particular basis for the $\mathfrak{so}(n)$ generators [206],

$$T^{(r,s)}_{\alpha\beta} = \delta^r_\alpha\delta^s_\beta - \delta^r_\beta\delta^s_\alpha, \tag{4.116}$$

which is properly normalized according to Eq. (4.112). Here the generators are labeled by the $n(n-1)/2$ pairs (r,s) with $1 \le r < s \le n$. Using the identity

$$\sum_{(r,s)} T^{(r,s)}_{\alpha\beta}T^{(r,s)}_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\beta\gamma}\delta_{\alpha\delta}, \tag{4.117}$$

and Eq. (4.115), we easily find that the Sugawara energy-momentum tensor (4.114) reproduces Eq. (4.107). The chiral central charge can also be checked. The Sugawara energy-momentum tensor of the $\mathfrak{g}_k$ WZW CFT has the general form [206]

$$T_{\mathfrak{g}_k}(z) = \frac{1}{2(k+g)} \sum_a : J^a J^a : , \tag{4.118}$$

where k is the Kac-Moody level and g is the dual Coxeter number. The chiral central charge is then

$$c_-[\mathfrak{g}_k] = \frac{k \dim \mathfrak{g}}{k+g}, \tag{4.119}$$

where $\dim \mathfrak{g} = \delta_{aa}$, i.e., the number of generators of the Lie algebra $\mathfrak{g}$. Here we have $\dim \mathfrak{so}(n) = n(n-1)/2$, and $k = 1$. By comparing (4.118) and (4.114), we find $g = n - 2$, and thus

$$c_-[\mathfrak{so}(n)_1] = \frac{n(n-1)/2}{1+n-2} = \frac{n}{2}, \tag{4.120}$$

as expected for n free Majorana fermions.

4.8.2 Conformal embedding

The conformal embedding (4.106) arises from a natural embedding of the Lie algebras $\mathfrak{so}(N)$ and $\mathfrak{so}(k)$ into $\mathfrak{so}(Nk)$. We first represent the indices for $\mathfrak{so}(Nk)$ matrices as a pair $\boldsymbol{\alpha} = (\alpha, \tilde{\alpha})$ where $\alpha = 1, \ldots, N$ and $\tilde{\alpha} = 1, \ldots, k$. We construct an embedding $\mathfrak{so}(N) \to \mathfrak{so}(Nk)$ as

$$\Sigma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} = (T^a)_{\alpha\beta}\delta_{\tilde{\alpha}\tilde{\beta}}, \tag{4.121}$$

i.e., $\Sigma^a = T^a \otimes \mathbb{1}_k$, where T^a are $\mathfrak{so}(N)$ generators and $\mathbb{1}_k$ the $k \times k$ identity matrix. Likewise, we construct an embedding $\mathfrak{so}(k) \to \mathfrak{so}(Nk)$ as

$$\tilde{\Sigma}^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \delta_{\alpha\beta}(\tilde{T}^a)_{\tilde{\alpha}\tilde{\beta}}, \tag{4.122}$$

i.e., $\tilde{\Sigma}^a = \mathbb{1}_N \otimes \tilde{T}^a$, where $\tilde{T}^a$ are $\mathfrak{so}(k)$ generators and $\mathbb{1}_N$ the $N \times N$ identity matrix. We then define $\mathfrak{so}(N)$ and $\mathfrak{so}(k)$ currents, respectively, as

$$J^a(z) \equiv \frac{i}{2}\chi^T(z)\Sigma^a\chi(z), \qquad \tilde{J}^a(z) \equiv \frac{i}{2}\chi^T(z)\tilde{\Sigma}^a\chi(z), \tag{4.123}$$

analogously to Eq. (4.108). By computing the OPE, we now show these satisfy the $\mathfrak{so}(N)_k$ and $\mathfrak{so}(k)_N$ current algebras, respectively. Using the explicit forms

$$J^a(z) = \frac{i}{2}\chi_{\alpha\tilde{\alpha}}T^a_{\alpha\beta}\chi_{\beta\tilde{\alpha}}, \qquad \tilde{J}^a(z) = \frac{i}{2}\chi_{\alpha\tilde{\alpha}}\tilde{T}^a_{\tilde{\alpha}\tilde{\beta}}\chi_{\alpha\tilde{\beta}}, \tag{4.124}$$

and following the same steps as in Eqs. (4.109-4.113), we find:

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w}J^c(w), \tag{4.125}$$

$$\tilde{J}^a(z)\tilde{J}^b(w) \sim \frac{N\delta^{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w}\tilde{J}^c(w), \tag{4.126}$$

which are indeed the $\mathfrak{so}(N)_k$ and $\mathfrak{so}(k)_N$ current algebras, respectively. By following similar steps and using the fact that $\operatorname{tr} T^a = \operatorname{tr} \tilde{T}^a = 0$, we can show that the mixed $J^a\tilde{J}^b$ OPE has no singular terms. Thus the two current algebras decouple.

Finally, we show that the energy-momentum tensor (4.114) of the $\mathfrak{so}(Nk)_1$ theory decomposes into the sum of the energy-momentum tensors of the $\mathfrak{so}(N)_k$ and $\mathfrak{so}(k)_N$ theories:

$$T_{\mathfrak{so}(Nk)_1}(z) = T_{\mathfrak{so}(N)_k}(z) + T_{\mathfrak{so}(k)_N}(z). \tag{4.127}$$

To do this, we need the following formula [204, 205]:

$$:(\chi_\alpha\chi_\beta)(\chi_\alpha\chi_\beta): \; = \chi_\alpha\partial\chi_\alpha + \chi_\beta\partial\chi_\beta, \qquad \alpha \neq \beta, \tag{4.128}$$

without summation over α, β. We also use Eqs. (4.116) for $T^a, \tilde{T}^a$ and (4.124) to write:

$$J^{(r,s)} = i\sum_{\tilde{\alpha}=1}^{k}\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}, \qquad \tilde{J}^{(\tilde{r},\tilde{s})} = i\sum_{\alpha=1}^{N}\chi_{\alpha\tilde{r}}\chi_{\alpha\tilde{s}}, \tag{4.129}$$

with $1 \le r < s \le N$ and $1 \le \tilde{r} < \tilde{s} \le k$. We have:

$$\begin{aligned}
\sum_{(r,s)} :J^{(r,s)}J^{(r,s)}: &= -\sum_{(r,s)}\sum_{\tilde{\alpha}\tilde{\beta}} :(\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}})(\chi_{r\tilde{\beta}}\chi_{s\tilde{\beta}}): \\
&= -\sum_{r<s}\left(\sum_{\tilde{\alpha}} :(\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}})(\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}): + \sum_{\tilde{\alpha}\neq\tilde{\beta}} :\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}\chi_{r\tilde{\beta}}\chi_{s\tilde{\beta}}:\right) \\
&= -\sum_{r<s}\left[\sum_{\tilde{\alpha}}(\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}} + \chi_{s\tilde{\alpha}}\partial\chi_{s\tilde{\alpha}}) + 2\sum_{\tilde{\alpha}<\tilde{\beta}}\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}\chi_{r\tilde{\beta}}\chi_{s\tilde{\beta}}\right] \\
&= -\frac{1}{2}\sum_{\tilde{\alpha}}\left[\sum_{rs}(\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}} + \chi_{s\tilde{\alpha}}\partial\chi_{s\tilde{\alpha}}) - 2\sum_{r}\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}}\right] - 2\sum_{r<s}\sum_{\tilde{\alpha}<\tilde{\beta}}\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}\chi_{r\tilde{\beta}}\chi_{s\tilde{\beta}} \\
&= -(N-1)\sum_{r\tilde{\alpha}}\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}} - 2O_{\chi\chi\chi\chi},
\end{aligned} \tag{4.130}$$

where we define the four-fermion operator

$$O_{\chi\chi\chi\chi} \equiv \sum_{r<s}\sum_{\tilde{\alpha}<\tilde{\beta}}\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}\chi_{r\tilde{\beta}}\chi_{s\tilde{\beta}}. \tag{4.131}$$

Note that this operator does not need further normal ordering since all fields in the product are different. Similarly, we find:

$$\sum_{(\tilde{r},\tilde{s})} :J^{(\tilde{r},\tilde{s})}J^{(\tilde{r},\tilde{s})}: = -(k-1)\sum_{\alpha\tilde{r}}\chi_{\alpha\tilde{r}}\partial\chi_{\alpha\tilde{r}} - 2\sum_{\alpha<\beta}\sum_{\tilde{r}<\tilde{s}}\chi_{\alpha\tilde{r}}\chi_{\alpha\tilde{s}}\chi_{\beta\tilde{r}}\chi_{\beta\tilde{s}}. \tag{4.132}$$

By performing the changes of dummy summation variables $\alpha, \beta \to r, s$ and $\tilde{r}, \tilde{s} \to \tilde{\alpha}, \tilde{\beta}$, we find:

$$\begin{aligned}
\sum_{(\tilde{r},\tilde{s})} :J^{(\tilde{r},\tilde{s})}J^{(\tilde{r},\tilde{s})}: &= -(k-1)\sum_{r\tilde{\alpha}}\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}} - 2\sum_{r<s}\sum_{\tilde{\alpha}<\tilde{\beta}}\chi_{r\tilde{\alpha}}\chi_{r\tilde{\beta}}\chi_{s\tilde{\alpha}}\chi_{s\tilde{\beta}} \\
&= -(k-1)\sum_{r\tilde{\alpha}}\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}} + 2\sum_{r<s}\sum_{\tilde{\alpha}<\tilde{\beta}}\chi_{r\tilde{\alpha}}\chi_{s\tilde{\alpha}}\chi_{r\tilde{\beta}}\chi_{s\tilde{\beta}} \\
&= -(k-1)\sum_{r\tilde{\alpha}}\chi_{r\tilde{\alpha}}\partial\chi_{r\tilde{\alpha}} + 2O_{\chi\chi\chi\chi}.
\end{aligned} \tag{4.133}$$

Based on Eq. (4.118) with $g = n-2$ for $\mathfrak{g} = \mathfrak{so}(n)$, we expect the following Sugawara forms:

$$T_{\mathfrak{so}(N)_k}(z) = \frac{1}{2(k+N-2)}\sum_{(r,s)} :J^{(r,s)}J^{(r,s)}: , \tag{4.134}$$

$$T_{\mathfrak{so}(k)_N}(z) = \frac{1}{2(N+k-2)}\sum_{(\tilde{r},\tilde{s})} :J^{(\tilde{r},\tilde{s})}J^{(\tilde{r},\tilde{s})}: . \tag{4.135}$$

Using Eqs. (4.130) and (4.133), we thus find:

$$\begin{aligned} T_{\mathfrak{so}(N)_k}(z) + T_{\mathfrak{so}(k)_N}(z) &= -\frac{1}{2}\sum_{r\tilde{\alpha}} \chi_{r\tilde{\alpha}} \partial \chi_{r\tilde{\alpha}} \\ &= T_{\mathfrak{so}(Nk)_1}(z), \end{aligned} \tag{4.136}$$

where we see that the four-fermion contributions $\propto O_{\chi\chi\chi\chi}$ cancel. Note that the $\mathfrak{so}(N)_k$ and $\mathfrak{so}(k)_N$ theories are interacting theories, since their energy-momentum tensors contain four-fermion terms, but their sum is a free theory.

Using Eq. (4.119), we can also check that the chiral central charges add:

$$\begin{aligned} c_-[\mathfrak{so}(N)_k] &= \frac{kN(N-1)}{2(k+N-2)}, \\ c_-[\mathfrak{so}(k)_N] &= \frac{Nk(k-1)}{2(N+k-2)}, \\ c_-[\mathfrak{so}(N)_k] + c_-[\mathfrak{so}(k)_N] &= \frac{1}{2}Nk = c_-[\mathfrak{so}(Nk)_1]. \end{aligned} \tag{4.137}$$

Finally, Ref. [172] shows that a theory of N flavors of Majorana fermions ψ_a^i with an internally gauged $SO(k)$ symmetry ($a = 1, \dots, N$, $i = 1, \dots, k$, thus Nk Majorana fermions in total) is equivalent to the $\mathfrak{so}(N)_k$ WZW model. This is consistent with projecting out the $\mathfrak{so}(k)_N$ sector in the conformal embedding (4.106). The $\mathfrak{su}(n)$ analog [173, 207] of this embedding was used previously in a similar manner to understand the edge physics of fractional quantum Hall states obtained from a parton construction [152].

4.9 Appendix: Kitaev-Kekulé model

In this Appendix, we give an example of noninteracting Majorana hopping model whose low-energy bandstructure consists of two continuum Majorana fields Ψ_+, Ψ_- with tunable masses m_+, m_- [176–178]. We begin with nearest-neighbor Majorana hopping on the honeycomb lattice, which produces two massless Majorana fields at low energies [23]. We then add two perturbations: a second-neighbor hopping term of strength κ, which gives a Haldane-type mass [23, 105] of the same sign for both Majorana fields, and a Kekulé distortion term [208, 209] of strength λ, which gives masses of opposite sign for the Majorana fields. By tuning both κ and λ, the low-energy Majorana masses $m_\pm$ resulting from the combined effect of both perturbations can be tuned independently.

The hopping model consists of two terms:

$$H = H_{\text{Kit}} + H_{\text{Kek}}, \tag{4.138}$$

where

$$H_{\text{Kit}} = \frac{i}{4}\sum_{j,k} A_{jk} c_j c_k, \tag{4.139}$$

is the model specified by Eq. (48) of Ref. [23], with nearest-neighbor hopping amplitude J and second-neighbor hopping amplitude κ for Majorana fermions c_j on the honeycomb lattice. This model gives a topological superconductor with Chern number equal to $\operatorname{sgn}\kappa$. The second term is a spatially non-uniform modulation of the nearest-neighbor hopping amplitude:

$$H_{\text{Kek}} = \frac{i}{4}\sum_{j,k} t_{jk} c_j c_k, \tag{4.140}$$

where t_{jk} specifies the Kekulé distortion pattern:

$$t_{jk} = \begin{cases} -\frac{t}{\sqrt{3}} e^{i\boldsymbol{K}_+\cdot\boldsymbol{\delta}_n} e^{i\boldsymbol{G}\cdot\boldsymbol{r}_j} + \text{c.c.}, & \text{if} \quad \boldsymbol{r}_k = \boldsymbol{r}_j + \boldsymbol{\delta}_n, \\ 0, & \text{otherwise}. \end{cases} \tag{4.141}$$

Here, $\boldsymbol{\delta}_n$ are the 3 nearest-neighbor vectors on the honeycomb lattice, $\boldsymbol{K}_\pm = (\pm 4\pi/3, 0)$ are the two gapless Dirac points obtained in the limit $\kappa = t = 0$, and $\boldsymbol{G} = \boldsymbol{K}_+ - \boldsymbol{K}_-$ is the momentum connecting the two Dirac points. The complex parameter t is such that $|t|$ controls the strength of the distortion. This distortion triples the size of the unit cell of the honeycomb lattice, and thus folds the Brillouin zone three times. The Dirac points are mapped to the Γ point of the reduced Brillouin zone. Since there are six inequivalent sites in the Kekulé-distorted lattice, there are six bands in the bandstructure. When κ and $|t|$ are small compared to J, the low-energy physics is dominated by two bands with avoided crossings near the Γ point. The low-energy degrees of freedom are the spinors

$$\eta_\pm(\boldsymbol{k}) \equiv (c^A_{\boldsymbol{K}_\pm+\boldsymbol{k}} \quad c^B_{\boldsymbol{K}_\pm+\boldsymbol{k}})^\intercal, \qquad \eta^\dagger_\pm(\boldsymbol{k}) = \eta^\intercal_\mp(-\boldsymbol{k}), \tag{4.142}$$

where A and B superscripts indicate the two sublattices of the honeycomb lattice, and the second equality above is a Majorana condition. Linearizing the low-energy bandstructure near the Dirac (Γ) point, we obtain [176]:

$$H \approx \frac{J\sqrt{3}}{4}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\Bigg[\eta_+^\dagger(\boldsymbol{k})\begin{pmatrix}6\kappa/J & -k_y-ik_x\\ -k_y+ik_x & -6\kappa/J\end{pmatrix}\eta_+(\boldsymbol{k})$$
$$+\,\eta_-^\dagger(\boldsymbol{k})\begin{pmatrix}6\kappa/J & -k_y-ik_x\\ -k_y+ik_x & -6\kappa/J\end{pmatrix}\eta_-(\boldsymbol{k})$$
$$+\,\eta_+^\dagger(\boldsymbol{k})\begin{pmatrix}0 & 2it/J\\ -2it/J & 0\end{pmatrix}\eta_-(\boldsymbol{k})$$
$$+\,\eta_-^\dagger(\boldsymbol{k})\begin{pmatrix}0 & 2it^*/J\\ -2it^*/J & 0\end{pmatrix}\eta_+(\boldsymbol{k})\Bigg]. \tag{4.143}$$

The Kekulé distortion couples the gapless excitations from the $\boldsymbol{K}_\pm$ valleys. To diagonalize the Hamiltonian, we define the new spinors

$$\begin{aligned}\Psi_+(\boldsymbol{k}) &\equiv \frac{1}{\sqrt{2}}\big(-i\sigma_z\eta_+(\boldsymbol{k})+\sigma_y\eta_-(\boldsymbol{k})\big),\\ \Psi_-(\boldsymbol{k}) &\equiv \frac{1}{\sqrt{2}}\big(\sigma_z\eta_+(\boldsymbol{k})-i\sigma_y\eta_-(\boldsymbol{k})\big),\\ \Psi_\pm^\dagger(\boldsymbol{k}) &= \Psi_\pm^\intercal(-\boldsymbol{k})\sigma_x.\end{aligned} \tag{4.144}$$

In this model, the $\pm$ indices in $\Psi_\pm$ are no longer valley indices. Indeed, it is obvious from their definition that the $\Psi_\pm$ fermions mix the fermions $\eta_\pm$ from the valleys $\boldsymbol{K}_\pm$. Furthermore, we set the Kekulé coupling $t = i\lambda$, where $\lambda \in \mathbb{R}$, thus removing the phase degree of freedom in the distortion. This diagonalizes in flavor space the linearized Hamiltonian (4.143), which is now written as

$$H=\frac{J\sqrt{3}}{4}\sum_{i=\pm}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\Psi_i^\dagger(\boldsymbol{k})(k_y\sigma_x-k_x\sigma_y+m_i\sigma_z)\Psi_i(\boldsymbol{k}), \tag{4.145}$$

where the low-energy Majorana masses

$$m_\pm = \frac{6\kappa\pm\lambda}{J}, \tag{4.146}$$

can be tuned independently by the lattice couplings κ and λ. When m_+ and m_- are of the same sign (i.e., when κ dominates), the system is a topological superconductor with Chern number ± 1; when m_+ and m_- are of opposite sign (i.e., when λ dominates), the system is a trivial superconductor. Using the Euclidean gamma matrix representation $(\gamma_0, \gamma_1, \gamma_2) = (\sigma_z, \sigma_x, \sigma_y)$, and rescaling the couplings to set the Majorana velocity to unity, we obtain the Euclidean Lagrangian in position space,

$$\mathcal{L} = \frac{1}{4} \sum_{i=\pm} \Psi_i^\intercal \mathcal{C}(\partial\!\!\!/ + m_i)\Psi_i, \tag{4.147}$$

whose gauged version appears in Eq. (4.19). Here, $\mathcal{C} = -i\gamma_2$ is a charge-conjugation matrix, and the fermionic fields $\Psi_\pm$ obey the Majorana condition

$$\bar{\Psi}_\pm \equiv \Psi_\pm^\dagger \gamma_0 = \Psi_\pm^\intercal \mathcal{C}. \tag{4.148}$$

For general nonzero κ and λ, the masses (4.146) break the microscopic time-reversal ($\mathcal{T}$) and inversion ($\mathcal{I}$) symmetries. These can be represented on the Majorana fields $\eta_\pm$ in Eq. (4.142) as

$$\begin{aligned} \mathcal{T}\eta_\pm(\boldsymbol{k})\mathcal{T}^{-1} &= -i\sigma_z K \eta_\mp(-\boldsymbol{k}), \\ \mathcal{I}\eta_\pm(\boldsymbol{k})\mathcal{I}^{-1} &= i\sigma_y \eta_\mp(-\boldsymbol{k}), \end{aligned} \tag{4.149}$$

where we denote complex conjugation by K. These nonstandard transformations deserve further explanation. We recall that the Kitaev honeycomb model (4.139) is in fact a $\mathbb{Z}_2$ gauge theory with static gauge fields u_{jk} that modulate the nearest-neighbor hopping amplitude J. The model (4.139) is obtained as the effective Hamiltonian in the ground-state (zero-flux) sector in standard gauge $u_{jk} = 1$, for all $j \in A$, $k \in B$. The standard definitions of time-reversal ($c_j^{A/B} \to K c_j^{A/B}$) and inversion ($c_j^{A/B} \to c_{-j}^{B/A}$) also flip the sign of u_{jk}, and thus do not preserve the standard gauge. However, the sign change of the latter can be compensated by a $\mathbb{Z}_2$ gauge transformation on either the j or k sites. The definitions (4.149) denote such composite transformations, and are thus projective representations of time reversal and inversion on the Majorana partons. These filter down to the modified spinors $\Psi_{\pm}(\boldsymbol{k})$ in Eq. (4.144) as

$$\mathcal{T}\Psi_{\pm}(\boldsymbol{k})\mathcal{T}^{-1} = i\sigma_y K \Psi_{\mp}(-\boldsymbol{k}),$$
$$\mathcal{I}\Psi_{\pm}(\boldsymbol{k})\mathcal{I}^{-1} = i\sigma_z \Psi_{\mp}(-\boldsymbol{k}). \tag{4.150}$$

Using these transformations on the Lagrangian (4.147), one observes that the Kekulé distortion λ provides a $\mathcal{T}$-invariant mass but breaks $\mathcal{I}$, whereas the Haldane mass λ breaks $\mathcal{T}$, but preserves $\mathcal{I}$. Imposing $\mathcal{I}$, we obtain $m_+ = m_- = 6\kappa/J$, and tuning κ through zero induces a direct continuous transition between the chiral spin liquid and the Ising-ordered phase in Fig. 4.2.

4.10 Appendix: Instanton calculus in the background field gauge

To perform the instanton gas calculation in this chapter, we use a representation of $\mathbb{Z}_2$ monopoles in $SO(N)$ gauge theory as Dirac monopoles in an $SO(2)$ subgroup. This representation breaks the $SO(N)$ invariance down to a $S[O(2)\times O(N-2)] \equiv [O(2)\times O(N-2)]/\mathbb{Z}_2$ subgroup. This is interpreted as a partial choice of gauge, and naïvely leads to ZMs in the Faddeev-Popov (FP) determinant. In this Appendix, we employ the background field gauge to show that such ZMs can be removed [64, 210–212], at the cost of introducing "gauge collective coordinates", which rotate the Dirac monopole between distinct $SO(2)$ subgroups of $SO(N)$.

We shall begin by formulating and gauge-fixing the $\mathcal{N}$-instanton contribution to the partition function. Formally decomposing the gauge field A into an instanton background $\bar{A}$ and a fluctuation part a,

$$A = \bar{A} + a, \tag{4.151}$$

the associated field strength decomposes to[4]

$$F = \bar{F} + d_{\bar{A}}a + a \wedge a. \tag{4.152}$$

Defining a gauge-invariant inner product on the space of $\mathfrak{so}(N)$-valued forms as

$$\langle \alpha, \beta \rangle := \frac{1}{2g^2} \int \mathrm{tr}(\alpha \wedge \star\beta), \tag{4.153}$$

the Yang-Mills action can be decomposed as $S_{\mathrm{YM}} = \bar{S}_{\mathrm{YM}} + S_a[\bar{A}]$ where $\bar{S}_{\mathrm{YM}} = \langle \bar{F}, \bar{F} \rangle$ and

$$S_a[\bar{A}] \equiv \langle d_{\bar{A}}a, d_{\bar{A}}a \rangle + 2 \langle \bar{F}, a \wedge a \rangle + \mathcal{O}(a^3). \tag{4.154}$$

Terms linear in a vanish as $\bar{F}$ satisfies the equations of motion. The fermion action can be similarly decomposed,

$$S_F = \frac{1}{4}\Psi^{\mathsf{T}} C \not{a} \Psi + \frac{1}{4}\Psi^{\mathsf{T}} C(\not{\partial} + \not{\bar{A}} + m)\Psi. \tag{4.155}$$

A single fermion Ψ is considered here, but the derivation is straightforwardly generalized to the case of multiple fermion flavors relevant for the main text.

The net action is invariant under the infinitesimal gauge transformation

$$\begin{aligned} \bar{A} + a &\to \bar{A} + a + d_{\bar{A}}\omega + [a, \omega], \\ \Psi &\to e^{-\omega}\Psi = \Psi - \omega\Psi. \end{aligned} \tag{4.156}$$

The fermions will just spectate in the following discussion, and so will not be discussed further. Since $\bar{A}$ is a classical background field (not integrated over in the path integral), a true gauge transformation must act only on the fluctuation a, so that

$$\begin{aligned} \delta_\omega a &= d_{\bar{A}}\omega + [a, \omega], \\ \delta_\omega \bar{A} &= 0. \end{aligned} \tag{4.157}$$

However, it is useful to define a "pseudo" gauge transformation

$$\begin{aligned} \delta_{\mathrm{pseudo}} \bar{A} &= d_{\bar{A}}\omega, \\ \delta_{\mathrm{pseudo}} a &= [a, \omega], \end{aligned} \tag{4.158}$$

[4]Given a gauge group G and a linear representation $\rho\colon G \to \mathrm{Aut}(V)$, the exterior derivative with respect to a $\mathfrak{g}$-valued connection A is $d_A = d + d\rho(A)\wedge$, where $d\rho$ is the induced representation of $\mathfrak{g}$ on V.

under which the action remains invariant. As far as the parent theory with $A=\bar{A}+a$ is concerned, the pseudo and true gauge transformations are identical. We shall shortly see that the background field method is a clever choice of gauge that retains invariance under the pseudo gauge transformations (4.158) while gauge-fixing the fluctuation part of the path integral

$$Z = e^{-\bar{S}_{\rm YM}} \int \mathscr{D}\Psi \mathscr{D}a \, e^{-S_a[\bar{A}]-S_F[\bar{A},a]}. \tag{4.159}$$

To gauge-fix this path integral, we select an $\mathfrak{so}(N)$-valued gauge function $G(a)$ and employ the FP method by inserting into Z the identity in the form $\Delta_{\rm FP}^{-1}\Delta_{\rm FP}$, where the gauge-invariant FP determinant is defined as the inverse of

$$\begin{aligned}
\Delta_{\rm FP}^{-1} &= \int \mathscr{D}\omega \, \delta[G(a+\delta_\omega a)], \\
&= \int \mathscr{D}G \, \delta[G(a+\delta_\omega a)] \left|\det \frac{\delta G}{\delta \omega}\right|^{-1}, \\
&= \left|\det \frac{\delta G}{\delta \omega}\right|^{-1}_{G=0}.
\end{aligned} \tag{4.160}$$

We will choose the background field gauge,

$$G(a) = \bar{D}_\mu a_\mu = \partial_\mu a_\mu + [\bar{A}_\mu, a_\mu] = 0, \tag{4.161}$$

where $\bar{D}_\mu$ is the gauge covariant derivative with respect to the instanton field $\bar{A}$. The reason for this choice is that the gauge function will eventually feature in a gauge-fixing term $\operatorname{tr} G^2$ in the Lagrangian, and it is easy to show that such a term is invariant under the pseudo gauge transformation (4.158), but *not* under a true gauge transformation (4.157) of the fluctuation field a. In this manner, the gauge invariance of the parent theory with $A=\bar{A}+a$ is retained.

The FP determinant is easily evaluated to be

$$\det \frac{\delta G}{\delta \omega} = \det{}_0 \bar{D}_\mu D_\mu, \tag{4.162}$$

where the subscript 0 indicates that the determinant is to be evaluated in the space of $\mathfrak{so}(N)$-valued 0-forms $\omega(x)$, and the (bar-less) covariant derivative D_μ is with respect to the total field $A = \bar{A} + a$, so that $D_\mu \omega = \partial_\mu \omega + [\bar{A}_\mu + a_\mu, \omega]$. If $\bar{A} = 0$, then this reduces to the familiar result for Lorenz gauge. A good gauge function must satisfy $G(a + \delta_\omega a) \neq G(a)$ for any $\omega \neq 0$, so that the gauge slice $G(a) = 0$ contains only inequivalent configurations of a. If there exists an ω that violates this requirement, then this would result in a ZM contribution to (4.162); these can be interpreted as would-be FP ghost ZMs [see Eq. (4.172)]. To see this explicitly, note that the FP operator is

$$\begin{aligned} \bar{D}_\mu D_\mu \omega &= \bar{D}_\mu^2 \omega + [\bar{D}_\mu a_\mu, \omega] + [a_\mu, \bar{D}_\mu \omega], \\ &= \bar{D}_\mu^2 \omega + [G, \omega] + [a_\mu, \bar{D}_\mu \omega]. \end{aligned} \tag{4.163}$$

If $\bar{D}_\mu \omega = 0$, then ω is a ZM of the FP operator evaluated on the gauge slice $G(a) = 0$ [see Eq. (4.160)]. Noting that $\bar{D}_\mu \omega = 0$ infinitesimally means $e^{-\omega}(\bar{A} + d)e^{\omega} = \bar{A}$, we find that ZMs exist if there is a nontrivial stabilizer (denoted $\mathcal{H}$) of $\bar{A}$ in the group of gauge transformations $\mathcal{G}$.[5] For instance, if $\bar{A} = 0$ then the stabilizer consists of all global gauge transformations so that $\mathcal{H} = SO(N)$, the gauge group. Here, $\bar{A}$ is the instanton embedded in an $SO(2)$ subgroup, which has a stabilizer subgroup of global rotations in $S[O(2) \times O(N-2)]$.

Due to the presence of ZMs in the FP determinant, one must split the domain of the ω integral in Eq. (4.160) into a ZM space consisting of all $\omega \in \ker_0 \bar{D}_\mu$, and its orthogonal complement $\ker_0 \bar{D}_\mu^\perp$, where the subscript 0 indicates that the domain of $\bar{D}_\mu$ is restricted to 0-forms. Such a grading can be achieved by means of the inner product (4.153) defined on this space. Then any gauge transformation can be decomposed as

$$\omega = \phi + \lambda, \quad \phi \in \ker_0 \bar{D}_\mu, \quad \lambda \in \ker_0 \bar{D}_\mu^\perp. \tag{4.164}$$

This also means that

$$\begin{aligned} \Delta_{\mathrm{FP}}^{-1} &= \int \mathscr{D}\omega \, \delta[G(a + \delta_\omega a)], \\ &= \int_{\ker_0 \bar{D}_\mu} \mathscr{D}\phi \int_{\ker_0 \bar{D}_\mu^\perp} \mathscr{D}\lambda \delta[G(a + \delta_\lambda a)], \\ &= \mathrm{vol}(\mathcal{H}) \left| \mathrm{det}_0' \, \bar{D}_\mu D_\mu \right|^{-1}, \end{aligned} \tag{4.165}$$

[5]If M is spacetime and $G = SO(N)$ the gauge group, the group $\mathcal{G} : M \to G$ of gauge transformations acts as $\mathcal{G} : x \mapsto \exp(\omega(x))$.

where the prime indicates that the determinant of $\bar{D}_\mu D_\mu$ is evaluated in the space $\ker_0 \bar{D}_\mu^\perp$, with the ZMs removed. The gauge-fixed path integral is then,

$$\begin{aligned} Z &= \frac{e^{-\bar{S}_{\rm YM}}}{\mathrm{vol}(\mathcal{H})}\int \mathscr{D}a\, \mathscr{D}\Psi \int \mathscr{D}\omega\ \delta[G(a+\delta_\omega a)]\big|\mathrm{det}_0' \,\bar{D}_\mu D_\mu\big| e^{-S_a[\bar{A}]-S_F[\bar{A},a]}, \\ &= \left(\int \mathscr{D}\omega\right) \frac{e^{-\bar{S}_{\rm YM}}}{\mathrm{vol}(\mathcal{H})}\int \mathscr{D}a\mathscr{D}\Psi\delta[G(a)]\big|\mathrm{det}_0' \,\bar{D}_\mu D_\mu\big|\, e^{-S_a[\bar{A}]-S_F[\bar{A},a]}, \end{aligned} \tag{4.166}$$

where the second line is obtained on a gauge transformation by $-\omega$, keeping all gauge invariant quantities fixed. The integral over ω is an infinite constant that can be dropped by defining a suitable normalization.

The fluctuation integral is subject to the gauge condition $G(a)=\bar{D}_\mu a_\mu=0$. Once again, the inner product (4.153) can be used to split the space of $\mathfrak{so}(N)$-valued 1-forms into $\ker_1 \bar{D}_\mu$ and its orthogonal complement $\ker_1 \bar{D}_\mu^\perp$, where the subscript 1 now indicates that the domain of $\bar{D}_\mu$ is the space of 1-forms. Using the inner product (4.153), it is readily seen (using integration by parts) that any nontrivial element of $\ker_1 \bar{D}_\mu^\perp$ is of the form $\bar{D}_\mu\varphi$ for some 0-form $\varphi\in\ker_0 \bar{D}_\mu^\perp$. Then,

$$a_\mu = \alpha_\mu + \bar{D}_\mu\varphi, \quad \alpha_\mu\in\ker_1 \bar{D}_\mu, \quad \varphi\in\ker_0 \bar{D}_\mu^\perp. \tag{4.167}$$

A change of variables from a to α and φ in the path integral now has a nontrivial Jacobian, found by examining the metric in this functional space,

$$\|a_\mu\|^2 = \big\langle \alpha_\mu + \bar{D}_\mu\varphi, \alpha_\mu + \bar{D}_\mu\varphi\big\rangle = \langle \begin{pmatrix}\alpha_\mu & \varphi\end{pmatrix} \begin{pmatrix}1 & 0\\ 0 & -\bar{D}_\mu^2\end{pmatrix} \begin{pmatrix}\alpha_\mu \\ \varphi\end{pmatrix}\rangle. \tag{4.168}$$

The Jacobian is thus $[\mathrm{det}_0'(-\bar{D}_\mu^2)]^{1/2}$, where the operator acts on $\varphi \in \ker_0 \bar{D}_\mu^\perp$, so there are no ZMs in this determinant. Since $-\bar{D}_\mu^2$ is a positive-definite operator on $\ker_0 \bar{D}_\mu^\perp$, its square root is well defined, and an absolute value sign is redundant. The path integral then simplifies to

$$\begin{aligned} Z &= \frac{e^{-\bar{S}_{\rm YM}}}{\mathrm{vol}(\mathcal{H})}\int_{\ker_1 \bar{D}_\mu} \mathscr{D}\alpha \int_{\ker_0 \bar{D}_\mu^\perp} \mathscr{D}\varphi\, \delta[\bar{D}_\mu^2\varphi]\sqrt{\mathrm{det}_0'(-\bar{D}_\mu^2)}\big|\mathrm{det}_0'\bar{D}_\mu D_\mu\big| \int \mathscr{D}\Psi e^{-S_{\alpha+\bar{D}_\mu\varphi}[\bar{A}]-S_F[\bar{A},\alpha+\bar{D}_\mu\varphi]}, \\ &= \frac{e^{-\bar{S}_{\rm YM}}}{\mathrm{vol}(\mathcal{H})}\int_{\ker_1 \bar{D}_\mu} \mathscr{D}\alpha\, [\mathrm{det}_0'(-\bar{D}_\mu^2)]^{-1/2}\big|\mathrm{det}_0'\bar{D}_\mu D_\mu\big| \int \mathscr{D}\Psi e^{-S_\alpha[\bar{A}]-S_F[\bar{A},\alpha]}, \end{aligned} \tag{4.169}$$

where the second line is obtained by using $\delta[\bar{D}_\mu^2\varphi] = [\det_0(-\bar{D}_\mu^2)]^{-1}\delta[\varphi]$, and performing the φ integral.

The path integral (4.169) has actually been derived for a general background $\bar{A}$. We will now specialize to the case of instantons in $SO(N)$ gauge theory, and evaluate an $\mathcal{N}$-instanton contribution to the path integral, such as appears in Eq. (4.29). Each instanton incarnates as a Dirac monopole in some $SO(2)$ subgroup, and the background $\bar{A}$ is a simple sum of the single instanton 1-form (4.27) in the dilute gas approximation. Within such an approximation, the stabilizer for such an instanton configuration on a spacetime M is simply $\mathcal{H}^{\mathcal{N}}$, where the single instanton stabilizer is $\mathcal{H} : M \to (H = S[O(2)\times O(N-2)])$. Writing a general element of $\mathcal{H}$ as $\exp[-\phi^a(x)h^a]$, where $a \in \{1, ..., \dim H\}$,

$$\mathrm{vol}(\mathcal{H}) = \int_{\ker_0 \bar{D}_\mu} \mathscr{D}\phi(x) = \int_H \prod_{a=1}^{\dim H} d\phi^a \sqrt{\frac{\mathrm{vol}M}{2g^2}} = \left(\frac{\mathrm{vol}M}{2g^2}\right)^{\dim H/2} \mathrm{vol}(H). \tag{4.170}$$

The determinant $[\det_0'(-\bar{D}_\mu^2)]^{-1/2}$ appearing in the general path integral (4.169), deviates from $[\det_0'(-\partial_\mu^2)]^{-1/2}$ pertinent to a trivial background $\bar{A} = 0$, only in small (disjoint) neighborhoods of the $\mathcal{N}$ localized instantons. The $\mathcal{N}$-instanton correction to the trivial determinant can be defined via $[\det_0'(-\bar{D}_\mu^2)]^{-1/2} \equiv [\det_0'(-\partial_\mu^2)]^{-1/2} K^{\mathcal{N}}$ [213]. Normalizing the path integral against the trivial background $\bar{A} = 0$, the $\mathcal{N}$-instanton contribution to the partition function is

$$\begin{aligned}\frac{Z_{\mathcal{N}}}{Z_0} = &\mathrm{vol}(G/H)^{\mathcal{N}} \left(\frac{\mathrm{vol}M}{2g^2}\right)^{\frac{\mathcal{N}\dim(G/H)}{2}} e^{-\bar{S}_{\mathrm{YM}}} \\ &\times \int_{\ker_1 \bar{D}_\mu} \mathscr{D}\alpha \int_{\ker_0 \bar{D}_\mu^\perp} \mathscr{D}(\bar{\eta}, \eta) \int \mathscr{D}\Psi \frac{K^{\mathcal{N}}}{\mathcal{K}} e^{-S_\alpha[\bar{A}]-S_F[\bar{A},\alpha]-S_{\mathrm{gh}}[\bar{A},\alpha]},\end{aligned} \tag{4.171}$$

where the action for the ghosts $\eta, \bar{\eta}$ is

$$S_{\mathrm{gh}} = \int \mathrm{d}^D x \,\mathrm{tr}\, \bar{\eta} \bar{D}_\mu D_\mu \eta, \tag{4.172}$$

and the normalization $\mathcal{K}$ is the transverse mode (α) and ghost path integrals evaluated in the trivial background $\bar{A} = 0$.

As stated in the introduction to this Appendix, the coset space $G/H = SO(N)/S[O(2)\times O(N-2)]$ has a collective coordinate interpretation. It is the space of global rotations that move an instanton between distinct $SO(2)$ subgroups of $SO(N)$. Furthermore, the fact that the α integral is restricted to $\ker_1 \bar{D}_\mu$ means that such global rotations that change $\bar{A}$ are excluded from that path integral. However, there are still ZMs corresponding to other non-gauge collective coordinates, such as a translation of an instanton in spacetime. For the monopole-instanton considered here, it is clear that there are no other collective coordinates besides these. Explicitly separating out the collective coordinates $\{z_i\}$ corresponding to the locations of the instantons (for which the Jacobian is a trivial constant that can be absorbed into $\mathcal{K}$), the final result is

$$\begin{aligned}\frac{Z_{\mathcal{N}}}{Z_0} &= \mathrm{vol}(G/H)^{\mathcal{N}} \left(\frac{\mathrm{vol} M}{2g^2}\right)^{\frac{\mathcal{N}\dim(G/H)}{2}} e^{-\bar{S}_{\mathrm{YM}}} \\ &\times \int \left(\prod_{i=1}^{\mathcal{N}} \mathrm{d}z_i\right) \int_{\ker_1 \bar{D}_\mu} \mathscr{D}'\alpha \int_{\ker_0 \bar{D}_\mu^\perp} \mathscr{D}(\bar{\eta},\eta) \int \mathscr{D}\Psi \frac{K^{\mathcal{N}}}{\mathcal{K}} e^{-S_\alpha[\bar{A}] - S_F[\bar{A},\alpha] - S_{\mathrm{gh}}[\bar{A},\alpha]}, \end{aligned} \tag{4.173}$$

where the primed measure $\mathscr{D}'\alpha$ means that ZM solutions of α are excluded from the domain of integration. More precisely, if the Gaussian part of the fluctuation action is $S_\alpha = \langle \alpha, \Omega\alpha \rangle$, then ZMs of the operator Ω are to be discarded in a mode expansion of α. Therefore, the only ZMs still present in the path integral are those of fermions bound to the instantons, which have physical consequences for symmetry breaking.

Chapter 5

Conclusion

The perspective of this thesis has been that parton gauge theories provide unified frameworks potentially capable of describing both Landau-ordered and fractionalized phases. The three chapters above have been case studies in support of this view. In chapter 2, conformal CQED_3 with $N_f = 4$ flavors of relativistic spinons was established as an example of such a theory centered on the Dirac spin liquid, a critical phase of matter in $2d$ spin-$1/2$ systems that exhibits power law correlations for a wide array of fluctuating Landau orders. To investigate the stability of such a critical state, and find proximate ordered states in which spinons must be confined, we were led to the construction of monopole operators in CQED_3. In contrast to previous works on the subject that defined these operators by conformal field theory methods, we used semiclassical instanton methods to directly construct these operators on $\mathbb{R}^3$ without assuming conformal invariance. This was achieved by first deforming the critical theory with a choice of spinon mass, which was shown to lead to instanton-bound zero modes of the Euclidean Dirac operator. A resummation of the instanton gas in the presence of these zero modes resulted in a 't Hooft vertex (an instanton-induced interaction) in the effective action, which we recognized as a monopole operator previously constructed using conformal methods in the literature. By choosing various spinon masses, we were able to obtain all independent monopole operators. Moreover, using the fact that the symmetries of Néel and VBS states on bipartite lattices could be fully encoded in a fermion bilinear, we were able to infer the quantum numbers of all monopole operators under the symmetries of some lattice realization of the Dirac spin liquid. On both square and honeycomb lattices, we confirmed the existence of a monopole that transforms trivially under all lattice symmetries, thus likely to lead to an instability of the Dirac spin liquid on those lattices.

In chapter 3, we emphasized how instantons could account for the spontaneous breakdown of internal symmetries in parton gauge theories. Our arguments were phrased in the context of a parton gauge theory (CQED_3) describing a multicritical point separating fractional quantum Hall, superfluid, and Mott insulating phases in a system of $2d$ hardcore bosons with $U(1)_{\mathrm{b}}$ symmetry. We showed how instantons break a spurious $U(1)_{\mathrm{top}} \times U(1)$ symmetry of CQED_3 to the diagonal subgroup, which we identified as the $U(1)_{\mathrm{b}}$ number conservation of physical bosons. On proliferating instantons, this symmetry was then shown to be spontaneously broken, leading in principle to two possible superfluid phases: a 'paired superfluid' with condensation of only charge-2 bosonic operators (SSB of $U(1)_{\mathrm{b}} \to \mathbb{Z}_2$), or a conventional superfluid with completely broken $U(1)_{\mathrm{b}}$. Some finer points of the vacuum structure of compact $U(1)$ gauge theories and their effects on instantons were also discussed.

Chapter 4 demonstrated the general perspective of the previous chapter in a more sophisticated example of a non-Abelian parton gauge theory – $SO(N)$ Chern-Simons gauge theory with massless Majorana fermions. Such a gauge theory was shown to describe a critical point separating magnetically ordered, trivially paramagnetic, and gapped chiral spin liquid phases in an Ising spin system with $\mathbb{Z}_2$ symmetry. By representing an Ising spin in terms of N Majorana partons on the lattice, we showed how these phases could be accessed by varying the total Chern number of the parton bandstructure. By utilizing recently conjectured dualities for $SO(N)$ Chern-Simons-matter theories, the critical theory for the Ising transition between the paramagnet and long-range ordered phase was found to be dual to the usual Wilson-Fisher theory reviewed in the introductory chapter. A direct continuous transition from the chiral spin liquid to the magnetic state was also found to be possible, protected by lattice symmetries such as inversion symmetry on the honeycomb lattice. Finally, we investigated the dynamics of the spontaneous breakdown of the Ising $\mathbb{Z}_2$ symmetry in the parton description. This was shown to be due to the proliferation of $\mathbb{Z}_2$-charged monopoles which feature as instantons of the $SO(N)$ gauge theory, which also supply the confinement mechanism required to produce a conventionally ordered phase in the parton framework.

In closing, we outline a few open problems that naturally follow from our work. As remarked previously, existing constructions of monopole operators in CQED_3 have focused on exploiting the conformal invariance of the latter, which allows operators to be identified with states in radial quantization. However, this approach is very specific to the Dirac spin liquid featuring relativistic spinons. Part of the motivation for the instanton gas construction in chapter 2 comes from the existence of spin liquids unprivileged with such a powerful symmetry, for instance those defined by parton bandstructures that feature a quadratic band-crossing point or a Fermi surface. Our work can be applied to such systems to see if there is a story paralleling that of the Dirac spin liquid. At least in $(3+1)D$, it is known that QED featuring nonrelativistic fermions can flow to a stable interacting fixed point, the Luttinger-Abrikosov-Beneslavskii point [214–217]. It is not clear to the author if such a fixed point exists in the $(2+1)D$ version of the theory. If so, then could it serve as a 'parent state' encoding several competing orders like the Dirac spin liquid? Are these ordered states classified by monopole operators? These are questions that lie outside the purview of the CFT methods used to study monopoles, but can be studied using the instanton calculus presented in this work.

In our study of the Dirac spin liquid, the proximate conventional phases were identified by the quantum numbers of monopole operators under lattice symmetries. To complete this program, one must also ensure that the correct number of Goldstone modes appropriate for a given symmetry breaking is reproduced correctly in the parton framework. This is also left for future work.

Bibliography

[1] G. Shankar and J. Maciejko, *Symmetry-Breaking Effects of Instantons in Parton Gauge Theories*, Phys. Rev. B **104**, 035134 (2021).

[2] G. Shankar, C.-H. Lin, and J. Maciejko, *Continuous Transition between Ising Magnetic Order and a Chiral Spin Liquid*, Phys. Rev. B **106**, 245107 (2022).

[3] G. Shankar and J. Maciejko, *Monopoles in Dirac spin liquids and their symmetries from instanton calculus*, SciPost Phys. **16**, 118 (2024).

[4] L. D. Landau, *29 - On the Theory of Phase Transitions*, in *Collected Papers of L.D. Landau*, edited by D. ter Haar (Pergamon, 1965) pp. 193–216.

[5] K. G. Wilson, *Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture*, Phys. Rev. B **4**, 3174–3183 (1971).

[6] K. G. Wilson, *Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior*, Phys. Rev. B **4**, 3184–3205 (1971).

[7] K. G. Wilson and M. E. Fisher, *Critical Exponents in 3.99 Dimensions*, Phys. Rev. Lett. **28**, 240–243 (1972).

[8] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, 2011).

[9] A. M. Polyakov, *Gauge Fields and Strings* (Routledge, New York, 2021).

[10] V. L. Berezinskii, *Destruction of Long-range Order in One-dimensional and Two-dimensional Systems having a Continuous Symmetry Group I. Classical Systems*, Sov. Phys. JETP **32**, 493–500 (1971).

[11] V. L. Berezinskii, *Destruction of Long-range Order in One-dimensional and Two-dimensional Systems Possessing a Continuous Symmetry Group. II. Quantum Systems*, Sov. Phys. JETP **34**, 610–616 (1972).

[12] J. M. Kosterlitz and D. J. Thouless, *Ordering, metastability and phase transitions in two-dimensional systems*, J. Phys. C: Solid State Phys. **6**, 1181 (1973).

[13] K. v. Klitzing, G. Dorda, and M. Pepper, *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*, Phys. Rev. Lett. **45**, 494–497 (1980).

[14] D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Two-Dimensional Magnetotransport in the Extreme Quantum Limit*, Phys. Rev. Lett. **48**, 1559–1562 (1982).

[15] J. McGreevy, *Generalized Symmetries in Condensed Matter*, Annual Review of Condensed Matter Physics **14**, 57–82 (2023).

[16] P. Anderson, *Resonating valence bonds: A new kind of insulator?* Mat. Res. Bull. **8**, 153–160 (1973).

[17] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer-Verlag, New York, 1994).

[18] R. B. Laughlin, *Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations*, Phys. Rev. Lett. **50**, 1395–1398 (1983).

[19] X.-G. Wen, *Quantum orders and symmetric spin liquids*, Phys. Rev. B **65**, 165113 (2002).

[20] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2007).

[21] C. Broholm, R. J. Cava, S. A. Kivelson, D. G. Nocera, M. R. Norman, and T. Senthil, *Quantum spin liquids*, Science **367**, eaay0668 (2020).

[22] A. Y. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303**, 2–30 (2003).

[23] A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Phys. **321**, 2–111 (2006).

[24] L. Savary and L. Balents, *Quantum Spin Liquids: A Review*, Rep. Prog. Phys. **80**, 016502 (2016).

[25] S.-S. Lee and P. A. Lee, *U(1) Gauge Theory of the Hubbard Model: Spin Liquid States and Possible Application to* $\kappa-(\mathrm{BEDT-TTF})_2\mathrm{Cu}_2(\mathrm{CN})_3$, Phys. Rev. Lett. **95**, 036403 (2005).

[26] A. M. Polyakov, *Compact gauge fields and the infrared catastrophe*, Phys. Lett. B **59**, 82 – 84 (1975).

[27] V. Borokhov, A. Kapustin, and X. Wu, *Topological Disorder Operators in Three-Dimensional Conformal Field Theory*, JHEP **11**, 049–049 (2002).

[28] S. S. Pufu, *Anomalous Dimensions of Monopole Operators in Three-Dimensional Quantum Electrodynamics*, Phys. Rev. D **89**, 065016 (2014).

[29] E. Dyer, M. Mezei, and S. S. Pufu, *Monopole taxonomy in three-dimensional conformal field theories*, arXiv:1309.1160 (2013).

[30] S. Gukov, *RG flows and bifurcations*, Nucl. Phys. B **919**, 583–638 (2017).

[31] X. G. Wen, *Mean-Field Theory of Spin-Liquid States with Finite Energy Gap and Topological Orders*, Phys. Rev. B **44**, 2664–2672 (1991).

[32] M. Hermele, T. Senthil, and M. P. A. Fisher, *Algebraic spin liquid as the mother of many competing orders*, Phys. Rev. B **72**, 104404 (2005).

[33] M. B. Hastings, *Dirac structure, RVB, and Goldstone modes in the Kagomé antiferromagnet*, Phys. Rev. B **63**, 014413 (2000).

[34] Y. Ran, M. Hermele, P. A. Lee, and X.-G. Wen, *Projected-wave-function study of the spin-1/2 Heisenberg model on the Kagomé lattice*, Phys. Rev. Lett. **98**, 117205 (2007).

[35] M. Hermele, Y. Ran, P. A. Lee, and X.-G. Wen, *Properties of an Algebraic Spin Liquid on the Kagome Lattice*, Phys. Rev. B **77**, 224413 (2008).

[36] X.-Y. Song, C. Wang, A. Vishwanath, and Y.-C. He, *Unifying Description of Competing Orders in Two-Dimensional Quantum Magnets*, Nat. Commun. **10**, 4254 (2019).

[37] X.-Y. Song, Y.-C. He, A. Vishwanath, and C. Wang, *From Spinon Band Topology to the Symmetry Quantum Numbers of Monopoles in Dirac Spin Liquids*, Phys. Rev. X **10**, 011033 (2020).

[38] X. Y. Xu, Y. Qi, L. Zhang, F. F. Assaad, C. Xu, and Z. Y. Meng, *Monte Carlo Study of Lattice Compact Quantum Electrodynamics with Fermionic Matter: The Parent State of Quantum Phases*, Phys. Rev. X **9**, 021022 (2019).

[39] M. Hermele, T. Senthil, M. P. A. Fisher, P. A. Lee, N. Nagaosa, and X.-G. Wen, *Stability of U(1) Spin Liquids in Two Dimensions*, Phys. Rev. B **70**, 214437 (2004).

[40] A. M. Polyakov, *Quark Confinement and Topology of Gauge Theories*, Nucl. Phys. B **120**, 429–458 (1977).

[41] T. Grover, *Entanglement monotonicity and the stability of gauge theories in three spacetime dimensions*, Phys. Rev. Lett. **112**, 151601 (2014).

[42] J. Alicea, O. I. Motrunich, M. Hermele, and M. P. A. Fisher, *Criticality in quantum triangular antiferromagnets via fermionized vortices*, Phys. Rev. B **72**, 064407 (2005).

[43] J. Alicea, O. I. Motrunich, and M. P. A. Fisher, *Algebraic vortex liquid in spin-1/2 triangular antiferromagnets: scenario for* Cs_2CuCl_4, Phys. Rev. Lett. **95**, 247203 (2005).

[44] J. Alicea, O. I. Motrunich, and M. P. A. Fisher, *Theory of the algebraic vortex liquid in an anisotropic spin-1/2 triangular antiferromagnet*, Phys. Rev. B **73**, 174430 (2006).

[45] J. Alicea, *Monopole Quantum Numbers in the Staggered Flux Spin Liquid*, Phys. Rev. B **78**, 035126 (2008).

[46] R. Jackiw, *Fractional charge and zero modes for planar systems in a magnetic field*, Phys. Rev. D **29**, 2375–2377 (1984).

[47] U. F. P. Seifert, J. Willsher, M. Drescher, F. Pollmann, and J. Knolle, *Spin-Peierls instability of the U(1) Dirac spin liquid*, arXiv:2307.12295 (2023).

[48] G. Nambiar, D. Bulmash, and V. Galitski, *Monopole Josephson Effects in a Dirac Spin Liquid*, Phys. Rev. Research **5**, 013169 (2023).

[49] M. A. Metlitski, M. Hermele, T. Senthil, and M. P. A. Fisher, *Monopoles in* CP^{N-1} *model via the state-operator correspondence*, Phys. Rev. B **78**, 214418 (2008).

[50] S. S. Pufu and S. Sachdev, *Monopoles in 2+1-dimensional conformal field theories with global U(1) symmetry*, JHEP **09**, 127 (2013).

[51] E. Dyer, M. Mezei, S. S. Pufu, and S. Sachdev, *Scaling dimensions of monopole operators in the* CP^{N-1} *theory in* $2+1$ *dimensions*, JHEP **06**, 037 (2015).

[52] S. M. Chester, M. Mezei, S. S. Pufu, and I. Yaakov, *Monopole operators from the* $4-\epsilon$ *expansion*, JHEP **12**, 015 (2016).

[53] S. M. Chester, L. V. Iliesiu, M. Mezei, and S. S. Pufu, *Monopole operators in U(1) Chern-Simons-matter theories*, JHEP **05**, 157 (2018).

[54] R. Boyack, C.-H. Lin, N. Zerf, A. Rayyan, and J. Maciejko, *Transition between algebraic and* $\mathbb{Z}_2$ *quantum spin liquids at large* N, Phys. Rev. B **98**, 035137 (2018).

[55] É. Dupuis, M. B. Paranjape, and W. Witczak-Krempa, *Transition from a Dirac spin liquid to an antiferromagnet: Monopoles in a* QED_3*-Gross-Neveu theory*, Phys. Rev. B **100**, 094443 (2019).

[56] É. Dupuis and W. Witczak-Krempa, *Monopole hierarchy in transitions out of a Dirac spin liquid*, Ann. Phys. **435**, 168496 (2021).

[57] É. Dupuis, R. Boyack, and W. Witczak-Krempa, *Anomalous Dimensions of Monopole Operators at the Transitions between Dirac and Topological Spin Liquids*, Phys. Rev. X **12**, 031012 (2022).

[58] S. M. Chester, É. Dupuis, and W. Witczak-Krempa, *Evidence for 3d bosonization from monopole operators*, arXiv:2210.12370 (2022).

[59] C. Xu, *Renormalization group studies on four-fermion interaction instabilities on algebraic spin liquids*, Phys. Rev. B **78**, 054432 (2008).

[60] G. Murthy and S. Sachdev, *Action of hedgehog instantons in the disordered phase of the 2+1-dimensional* CP^{N-1} *model*, Nucl. Phys. B **344**, 557–595 (1990).

[61] M. Ünsal, *Magnetic Bion Condensation: A New Mechanism of Confinement and Mass Gap in Four Dimensions*, Phys. Rev. D **80**, 065001 (2009).

[62] I. Affleck, J. Harvey, and E. Witten, *Instantons and (Super-) Symmetry Breaking in (2+1) Dimensions*, Nucl. Phys. B **206**, 413–439 (1982).

[63] G. 't Hooft, *Symmetry breaking through Bell-Jackiw anomalies*, Phys. Rev. Lett. **37**, 8–11 (1976).

[64] G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, Phys. Rev. D **14**, 3432–3450 (1976).

[65] G. 't Hooft, *How instantons solve the U(1) problem*, Phys. Rep. **142**, 357–387 (1986).

[66] H. Takagi, T. Takayama, G. Jackeli, G. Khaliullin, and S. E. Nagler, *Concept and realization of Kitaev quantum spin liquids*, Nat. Rev. Phys. **1**, 264–280 (2019).

[67] C. Mudry and E. Fradkin, *Separation of Spin and Charge Quantum Numbers in Strongly Correlated Systems*, Phys. Rev. B **49**, 5200–5219 (1994).

[68] D. H. Kim and P. A. Lee, *Theory of spin excitations in undoped and underdoped cuprates*, Ann. Phys. **272**, 130–164 (1999).

[69] W. Rantner and X.-G. Wen, *Spin correlations in the algebraic spin liquid: implications for high-*T_c *superconductors*, Phys. Rev. B **66**, 144501 (2002).

[70] J. B. Marston, *Instantons and Massless Fermions in (2+1)-Dimensional Lattice QED and Antiferromagnets*, Phys. Rev. Lett. **64**, 1166–1169 (1990).

[71] R. Jackiw and C. Rebbi, *Solitons with fermion number* $\frac{1}{2}$, Phys. Rev. D **13**, 3398–3409 (1976).

[72] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions*, Commun. Math. Phys. **31**, 83–112 (1973).

[73] C. Wetterich, *Spinors in Euclidean Field Theory, Complex Structures and Discrete Symmetries*, Nucl. Phys. B **852**, 174–234 (2011).

[74] T. T. Wu and C. N. Yang, *Dirac Monopole without Strings: Monopole Harmonics*, Nucl. Phys. B **107**, 365–380 (1976).

[75] E. Witten, *Fermion Path Integrals and Topological Phases*, Rev. Mod. Phys. **88**, 035001 (2016).

[76] C. Cordova, P.-S. Hsin, N. Seiberg, and P.-S. Hsin, *Time-Reversal Symmetry, Anomalies, and Dualities in 2+1d*, SciPost Phys. **5**, 006 (2018).

[77] Y.-M. Lu, G. Y. Cho, and A. Vishwanath, *Unification of bosonic and fermionic theories of spin liquids on the kagomé lattice*, Phys. Rev. B **96**, 205150 (2017).

[78] M. Hermele, *SU(2) gauge theory of the Hubbard model and application to the honeycomb lattice*, Phys. Rev. B **76**, 035125 (2007).

[79] M. Stone, *Supersymmetry and the Quantum Mechanics of Spin*, Nucl. Phys. B **314**, 557–586 (1989).

[80] T. W. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, *Spontaneous chiral-symmetry breaking in three-dimensional QED*, Phys. Rev. D **33**, 3704–3713 (1986).

[81] R. D. Pisarski, *Monopoles in Topologically Massive Gauge Theories*, Phys. Rev. D **34**, 3851–3857 (1986).

[82] I. Affleck, J. Harvey, L. Palla, and G. Semenoff, *The Chern-Simons Term versus the Monopole*, Nucl. Phys. B **328**, 575–584 (1989).

[83] I. Affleck and J. B. Marston, *Large-n Limit of the Heisenberg-Hubbard Model: Implications for High-Tc Superconductors*, Phys. Rev. B **37**, 3774–3777 (1988).

[84] L. Janssen and Y.-C. He, *Critical behavior of the QED$_3$-Gross-Neveu model: duality and deconfined criticality*, Phys. Rev. B **96**, 205113 (2017).

[85] B. Ihrig, L. Janssen, L. N. Mihaila, and M. M. Scherer, *Deconfined criticality from the QED$_3$-Gross-Neveu model at three loops*, Phys. Rev. B **98**, 115163 (2018).

[86] N. Zerf, P. Marquard, R. Boyack, and J. Maciejko, *Critical behavior of the QED$_3$-Gross-Neveu model at four loops*, Phys. Rev. B **98**, 165125 (2018).

[87] N. Zerf, R. Boyack, P. Marquard, J. A. Gracey, and J. Maciejko, *Critical properties of the Néel–algebraic-spin-liquid transition*, Phys. Rev. B **100**, 235130 (2019).

[88] R. Boyack, A. Rayyan, and J. Maciejko, *Deconfined criticality in the QED$_3$-Gross-Neveu–Yukawa model: the $1/N$ expansion revisited*, Phys. Rev. B **99**, 195135 (2019).

[89] R. Boyack and J. Maciejko, *Critical Exponents for the Valence-Bond-Solid Transition in Lattice Quantum Electrodynamics*, in *Quantum Theory and Symmetries*, edited by M. B. Paranjape, R. MacKenzie, Z. Thomova, P. Winternitz, and W. Witczak-Krempa (Springer International Publishing, Cham, 2021) pp. 337–345.

[90] N. Zerf, R. Boyack, P. Marquard, J. A. Gracey, and J. Maciejko, *Critical properties of the valence-bond-solid transition in lattice quantum electrodynamics*, Phys. Rev. D **101**, 094505 (2020).

[91] L. Janssen, W. Wang, M. M. Scherer, Z. Y. Meng, and X. Y. Xu, *Confinement transition in the QED$_3$-Gross-Neveu-XY universality class*, Phys. Rev. B **101**, 235118 (2020).

[92] R. Boyack, H. Yerzhakov, and J. Maciejko, *Quantum phase transitions in Dirac fermion systems*, Eur. Phys. J. Spec. Top. **230**, 979–992 (2021).

[93] W. Wang, D.-C. Lu, X. Y. Xu, Y.-Z. You, and Z. Y. Meng, *Dynamics of compact quantum electrodynamics at large fermion flavor*, Phys. Rev. B **100**, 085123 (2019).

[94] V. Rubakov, *Classical Theory of Gauge Fields* (Princeton University Press, Princeton NJ, 2002).

[95] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, Phys. Lett. B **59**, 85–87 (1975).

[96] S. L. Adler, *Axial-vector vertex in spinor electrodynamics*, Phys. Rev. **177**, 2426–2438 (1969).

[97] J. S. Bell and R. Jackiw, *A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ-model*, Nuovo Cimento A **60**, 47–61 (1969).

[98] M. F. Atiyah and I. M. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. **69**, 422–433 (1963).

[99] C. Callias, *Axial Anomalies and Index Theorems on Open Spaces*, Commun. Math. Phys. **62**, 213–234 (1978).

[100] R. Bott and R. Seeley, *Some Remarks on the Paper of Callias: "Axial Anomalies and Index Theorems on Open Spaces"*, Commun. Math. Phys. **62**, 235–245 (1978).

[101] M. Ünsal, *Topological symmetry, spin liquids and CFT duals of Polyakov model with massless fermions*, arXiv:0804.4664 (2008).

[102] M. Barkeshli and J. McGreevy, *Continuous transition between fractional quantum Hall and superfluid states*, Phys. Rev. B **89**, 235116 (2014).

[103] R. Jackiw and C. Rebbi, *Vacuum Periodicity in a Yang-Mills Quantum Theory*, Phys. Rev. Lett. **37**, 172 (1976).

[104] C. G. Callan, R. F. Dashen, and D. J. Gross, *The Structure of the Gauge Theory Vacuum*, Phys. Lett. B **63**, 334–340 (1976).

[105] F. D. M. Haldane, *Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"*, Phys. Rev. Lett. **61**, 2015–2018 (1988).

[106] S. N. Vergeles, *On the problem of θ-vacua in compact gauge theories*, Nucl. Phys. B **152**, 330–344 (1979).

[107] W. E. Brown and I. I. Kogan, *Compact three-dimensional QED with a θ term and axionic confining strings*, Phys. Rev. D **56**, 3718–3731 (1997).

[108] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, *Quantum electrodynamics on a lattice: A Hamiltonian variational approach to the physics of the weak-coupling region*, Phys. Rev. D **19**, 619–638 (1979).

[109] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Oxford University Press, New York, 2002).

[110] H. Yamagishi, *Fermion-Monopole System Reexamined*, Phys. Rev. D **27**, 2383–2396 (1983).

[111] É. Dupuis, M. B. Paranjape, and W. Witczak-Krempa, *Monopole Operators and Their Symmetries in QED_3-Gross–Neveu Models*, in *Quantum Theory and Symmetries*, edited by M. B. Paranjape, R. MacKenzie, Z. Thomova, P. Winternitz, and W. Witczak-Krempa (Springer International Publishing, Cham, 2021) pp. 327–336.

[112] J. M. Kosterlitz, *The critical properties of the two-dimensional XY model*, J. Phys. C **7**, 1046–1060 (1974).

[113] M. Ünsal, *Abelian duality, confinement, and chiral-Symmetry Breaking in a SU(2) QCD-like theory*, Phys. Rev. Lett. **100**, 032005 (2008).

[114] R. Bendjama, B. Kumar, and F. Mila, *Absence of single-particle Bose-Einstein condensation at low densities for bosons with correlated hopping*, Phys. Rev. Lett. **95**, 110406 (2005).

[115] K. P. Schmidt, J. Dorier, A. Läuchli, and F. Mila, *Single-particle versus pair condensation of hard-core bosons with correlated hopping*, Phys. Rev. B **74**, 174508 (2006).

[116] E. Berg, E. Fradkin, and S. A. Kivelson, *Charge-4e superconductivity from pair-density-wave order in certain high-temperature superconductors*, Nat. Phys. **5**, 830–833 (2009).

[117] Y. M. Shnir, *Magnetic Monopoles* (Springer-Verlag, Berlin, 2005).

[118] Y. Kazama, C. N. Yang, and A. S. Goldhaber, *Scattering of a Dirac Particle with Charge Ze by a Fixed Magnetic Monopole*, Phys. Rev. D **15**, 2287–2299 (1977).

[119] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis* (Academic Press, New York, 1972).

[120] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert space* (Dover Publications, New York, 1993).

[121] G. Paz, *The non-self-adjointness of the radial momentum operator in n dimensions*, J. Phys. A: Math. Gen. **35**, 3727–3731 (2002).

[122] D. Simmons-Duffin, *TASI Lectures on the Conformal Bootstrap*, arXiv:1602.07982 (2016).

[123] Y. Zhou, K. Kanoda, and T.-K. Ng, *Quantum spin liquid states*, Rev. Mod. Phys. **89**, 025003 (2017).

[124] M. Vojta, *Frustration and quantum criticality*, Rep. Prog. Phys. **81**, 064501 (2018).

[125] N. Read and S. Sachdev, *Large-N expansion for frustrated quantum antiferromagnets*, Phys. Rev. Lett. **66**, 1773–1776 (1991).

[126] C. Xu and S. Sachdev, *Global phase diagrams of frustrated quantum antiferromagnets in two dimensions: Doubled Chern-Simons theory*, Phys. Rev. B **79**, 064405 (2009).

[127] P. Ghaemi and T. Senthil, *Néel order, quantum spin liquids, and quantum criticality in two dimensions*, Phys. Rev. B **73**, 054415 (2006).

[128] W. G. F. Krüger and L. Janssen, *Nesting instability of gapless U(1) spin liquids with spinon Fermi pockets in two dimensions*, Phys. Rev. B **104**, 165133 (2021).

[129] W. Witczak-Krempa, G. Chen, Y. B. Kim, and L. Balents, *Correlated Quantum Phenomena in the Strong Spin-Orbit Regime*, Annu. Rev. Condens. Matter Phys. **5**, 57–82 (2014).

[130] J. G. Rau, E. K.-H. Lee, and H.-Y. Kee, *Spin-Orbit Physics Giving Rise to Novel Phases in Correlated Systems: Iridates and Related Materials*, Annu. Rev. Condens. Matter Phys. **7**, 195–221 (2016).

[131] G. Jackeli and G. Khaliullin, *Mott Insulators in the Strong Spin-Orbit Coupling Limit: From Heisenberg to a Quantum Compass and Kitaev Models*, Phys. Rev. Lett. **102**, 017205 (2009).

[132] A. Banerjee, C. A. Bridges, J.-Q. Yan, A. A. Aczel, L. Li, M. B. Stone, G. E. Granroth, M. D. Lumsden, Y. Yiu, J. Knolle, S. Bhattacharjee, D. L. Kovrizhin, R. Moessner, D. A. Tennant, D. G. Mandrus, and S. E. Nagler, *Proximate Kitaev quantum spin liquid behaviour in a honeycomb magnet*, Nat. Mater. **15**, 733–740 (2016).

[133] A. Banerjee, J. Yan, J. Knolle, C. A. Bridges, M. B. Stone, M. D. Lumsden, D. G. Mandrus, D. A. Tennant, R. Moessner, and S. E. Nagler, *Neutron scattering in the proximate quantum spin liquid α-$RuCl_3$*, Science **356**, 1055–1059 (2017).

[134] N. Janša, A. Zorko, M. Gomilšek, M. Pregelj, K. W. Krämer, D. Biner, A. Biffin, C. Rüegg, and M. Klanjšek, *Observation of two types of fractional excitation in the Kitaev honeycomb magnet*, Nat. Phys. **14** (2018), 10.1038/s41567-018-0129-5.

[135] A. Banerjee, P. Lampen-Kelley, J. Knolle, C. Balz, A. A. Aczel, B. Winn, Y. Liu, D. Pajerowski, J. Yan, C. A. Bridges, A. T. Savici, B. C. Chakoumakos, M. D. Lumsden, D. A. Tennant, R. Moessner, D. G. Mandrus, and S. E. Nagler, *Excitations in the field-induced quantum spin liquid state of α-$RuCl_3$*, npj Quantum Mater. **3**, 1–7 (2018).

[136] C. Balz, L. Janssen, P. Lampen-Kelley, A. Banerjee, Y. H. Liu, J.-Q. Yan, D. G. Mandrus, M. Vojta, and S. E. Nagler, *Field-induced intermediate ordered phase and anisotropic interlayer interactions in α-$RuCl_3$*, Phys. Rev. B **103**, 174417 (2021).

[137] Y. Kasahara, T. Ohnishi, Y. Mizukami, O. Tanaka, S. Ma, K. Sugii, N. Kurita, H. Tanaka, J. Nasu, Y. Motome, T. Shibauchi, and Y. Matsuda, *Majorana quantization and half-integer thermal quantum Hall effect in a Kitaev spin liquid*, Nature **559**, 227–231 (2018).

[138] T. Yokoi, S. Ma, Y. Kasahara, S. Kasahara, T. Shibauchi, N. Kurita, H. Tanaka, J. Nasu, Y. Motome, C. Hickey, S. Trebst, and Y. Matsuda, *Half-integer quantized anomalous thermal Hall effect in the Kitaev material candidate $\alpha - RuCl_3$*, Science **373**, 568–572 (2021).

[139] L. Zou and Y.-C. He, *Field-induced QCD_3-Chern-Simons quantum criticalities in Kitaev materials*, Phys. Rev. Research **2**, 013072 (2020).

[140] Z. Komargodski and N. Seiberg, *A symmetry breaking scenario for QCD_3*, JHEP **01**, 109 (2018).

[141] F. Benini, *Three-dimensional dualities with bosons and fermions*, JHEP **02**, 068 (2018).

[142] W. Chen, M. P. A. Fisher, and Y.-S. Wu, *Mott transition in an anyon gas*, Phys. Rev. B **48**, 13749–13761 (1993).

[143] D. T. Son, *Is the composite fermion a Dirac particle?* Phys. Rev. X **5**, 031027 (2015).

[144] M. A. Metlitski and A. Vishwanath, *Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators*, Phys. Rev. B **93**, 245151 (2016).

[145] C. Wang and T. Senthil, *Dual Dirac Liquid on the Surface of the Electron Topological Insulator*, Phys. Rev. X **5**, 041031 (2015).

[146] N. Seiberg and E. Witten, *Gapped Boundary Phases of Topological Insulators via Weak Coupling*, Prog. Theor. Exp. Phys. **2016**, 12C101 (2016).

[147] A. Karch and D. Tong, *Particle-Vortex Duality from 3D Bosonization*, Phys. Rev. X **6**, 031043 (2016).

[148] J. Murugan and H. Nastase, *Particle-vortex duality in topological insulators and superconductors*, JHEP **05**, 159 (2017).

[149] D. F. Mross, J. Alicea, and O. I. Motrunich, *Explicit derivation of duality between a free Dirac cone and quantum electrodynamics in 2+1 dimensions*, Phys. Rev. Lett. **117**, 016802 (2016).

[150] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, *Nonsupersymmetric dualities from mirror symmetry*, Phys. Rev. Lett. **118**, 011602 (2017).

[151] A. Karch, B. Robinson, and D. Tong, *More Abelian dualities in 2 + 1 dimensions*, JHEP **2017**, 17 (2017).

[152] X. G. Wen, *Non-Abelian statistics in the fractional quantum Hall states*, Phys. Rev. Lett. **66**, 802–805 (1991).

[153] X.-G. Wen, *Projective construction of non-Abelian quantum Hall liquids*, Phys. Rev. B **60**, 8827–8838 (1999).

[154] Y.-M. Lu and A. Vishwanath, *Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach*, Phys. Rev. B **86**, 125119 (2012).

[155] T. Senthil and M. Levin, *Integer quantum Hall effect for bosons*, Phys. Rev. Lett. **110**, 046801 (2013).

[156] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B **87**, 155114 (2013).

[157] M. Barkeshli, *Transitions between chiral spin liquids and $\mathbb{Z}_2$ spin liquids*, arXiv:1307.8194 (2013).

[158] V. Kalmeyer and R. B. Laughlin, *Equivalence of the resonating-valence-bond and fractional quantum Hall states*, Phys. Rev. Lett. **59**, 2095–2098 (1987).

[159] A. J. Niemi and G. W. Semenoff, *Axial-anomaly-induced fermion fractionization and effective gauge theory actions in odd-dimensional spacetimes*, Phys. Rev. Lett. **51**, 2077 (1983).

[160] A. N. Redlich, *Gauge noninvariance and parity nonconservation of three-dimensional fermions*, Phys. Rev. Lett. **52**, 18 (1984).

[161] A. N. Redlich, *Parity violation and gauge noninvariance of the effective gauge field action in three dimensions*, Phys. Rev. D **29**, 2366 (1984).

[162] J. Ye and S. Sachdev, *Coulomb interactions at quantum Hall critical points of systems in a periodic potential*, Phys. Rev. Lett. **80**, 5409–5412 (1998).

[163] X.-G. Wen and Y.-S. Wu, *Transitions between the quantum Hall states and insulators induced by periodic potentials*, Phys. Rev. Lett. **70**, 1501–1504 (1993).

[164] J. Fu, J. Knolle, and N. B. Perkins, *Three types of representation of spin in terms of Majorana fermions and an alternative solution of the Kitaev honeycomb model*, Phys. Rev. B **97**, 115142 (2018).

[165] N. Read and D. Green, *Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect*, Phys. Rev. B **61**, 10267 (2000).

[166] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys. **129**, 393–429 (1990).

[167] M. A. Metlitski, A. Vishwanath, and C. Xu, *Duality and bosonization of 2+1-dimensional Majorana fermions*, Phys. Rev. B **95**, 205137 (2017).

[168] O. Aharony, F. Benini, P.-S. Hsin, and N. Seiberg, *Chern-Simons-matter dualities with SO and USp gauge groups*, JHEP **02**, 072 (2017).

[169] F. Bursa, R. Lau, and M. Teper, *SO(2N) and SU(N) gauge theories in 2+1 dimensions*, JHEP **05**, 025 (2013).

[170] R. Lau and M. Teper, *SO(N) gauge theories in 2+1 dimensions: glueball spectra and confinement*, JHEP **10**, 022 (2017).

[171] E. Witten, *Nonabelian bosonization in two dimensions*, Commun. Math. Phys. **92**, 455–472 (1984).

[172] I. Antoniadis and C. Bachas, *Conformal invariance and parastatistics in two dimensions*, Nucl. Phys. B **278**, 343–352 (1986).

[173] S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, *Group-level duality in WZW models and Chern-Simons theory*, Phys. Lett. B **246**, 417–422 (1990).

[174] X. G. Wen, *Vacuum degeneracy of chiral spin states in compactified space*, Phys. Rev. B **40**, 7387 (1989).

[175] X. G. Wen, F. Wilczek, and A. Zee, *Chiral spin states and superconductivity*, Phys. Rev. B **39**, 11413–11423 (1989).

[176] Z.-C. Yang, T. Iadecola, C. Chamon, and C. Mudry, *Hierarchical Majoranas in a programmable nanowire network*, Phys. Rev. B **99**, 155138 (2019).

[177] F. Mirmojarabian, M. Kargarian, and A. Langari, *Phase diagram and thermal Hall conductivity of the spin-liquid Kekulé-Kitaev model*, Phys. Rev. B **101**, 115116 (2020).

[178] A. Farjami, M. D. Horner, C. N. Self, Z. Papić, and J. K. Pachos, *Geometric description of the Kitaev honeycomb lattice model*, Phys. Rev. B **101**, 245116 (2020).

[179] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, *3d dualities from 4d dualities for orthogonal groups*, JHEP **08**, 099 (2013).

[180] F. Benini, P.-S. Hsin, and N. Seiberg, *Comments on global symmetries, anomalies, and duality in (2+1)d*, JHEP **04**, 135 (2017).

[181] C. Córdova, P.-S. Hsin, and N. Seiberg, *Global Symmetries, Counterterms, and Duality in Chern-Simons Matter Theories with Orthogonal Gauge Groups*, SciPost Phys. **4**, 021 (2018).

[182] T. Senthil, D. T. Son, C. Wang, and C. Xu, *Duality between (2+1)d quantum critical points*, Phys. Rep. **827**, 1–48 (2019).

[183] J. Ahn, D. Kim, Y. Kim, and B.-J. Yang, *Band topology and linking structure of nodal line semimetals with $\mathbb{Z}_2$ monopole charges*, Phys. Rev. Lett. **121**, 106403 (2018).

[184] J. Ahn, S. Park, D. Kim, Y. Kim, and B.-J. Yang, *Stiefel–Whitney classes and topological phases in band theory*, Chin. Phys. B **28**, 117101 (2019).

[185] P. Goddard, J. Nuyts, and D. Olive, *Gauge theories and magnetic charge*, Nucl. Phys. B **125**, 1–28 (1977).

[186] D. Tong, *TASI Lectures on Solitons*, arXiv:hep-th/0509216 (2005).

[187] R. A. Brandt and F. Neri, *Stability analysis for singular non-Abelian magnetic monopoles*, Nucl. Phys. B **161**, 253–282 (1979).

[188] S. Coleman, *The Magnetic Monopole Fifty Years Later*, in *The Unity of the Fundamental Interactions*, edited by A. Zichichi (Plenum Press, 1983).

[189] M. Stone, *Gamma matrices, Majorana fermions, and discrete symmetries in Minkowski and Euclidean signature*, J. Phys. A: Math. Theor. **55**, 205401 (2022).

[190] A. Hui, M. Mulligan, and E.-A. Kim, *Non-Abelian fermionization and fractional quantum Hall transitions*, Phys. Rev. B **97**, 085112 (2018).

[191] A. Hui, E.-A. Kim, and M. Mulligan, *Non-Abelian bosonization and modular transformation approach to superuniversality*, Phys. Rev. B **99**, 125135 (2019).

[192] G. 't Hooft, *A planar diagram theory for strong interactions*, Nucl. Phys. B **72**, 461–473 (1974).

[193] A. M. Polyakov, *Fermi-bose transmutations induced by gauge fields*, Mod. Phys. Lett. A **03**, 325–328 (1988).

[194] E. Fradkin and F. A. Schaposnik, *The fermion-boson mapping in three-dimensional quantum field theory*, Phys. Lett. B **338**, 253–258 (1994).

[195] E. Witten, *$SL(2,Z)$ Action On Three-Dimensional Conformal Field Theories With Abelian Symmetry*, in *From Fields to Strings: Circumnavigating Theoretical Physics (Ian Kogan Memorial Collection)*, Vol. 2, edited by M. Shifman, A. Vainshtein, and J. Wheater (World Scientific, 2005) pp. 1173–1200.

[196] J. Kogut and L. Susskind, *Hamiltonian formulation of Wilson's lattice gauge theories*, Phys. Rev. D **11**, 395–408 (1975).

[197] J. B. Kogut, *An introduction to lattice gauge theory and spin systems*, Rev. Mod. Phys. **51**, 659–713 (1979).

[198] M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, 1983).

[199] E. Fradkin and L. Susskind, *Order and disorder in gauge systems and magnets*, Phys. Rev. D **17**, 2637–2658 (1978).

[200] P. Rossi and U. Wolff, *Lattice QCD with fermions at strong coupling: A dimer system*, Nucl. Phys. B **248**, 105–122 (1984).

[201] U. Wolff, *Baryons in lattice QCD at strong coupling*, Phys. Lett. B **153**, 92–96 (1985).

[202] B. Collins and S. Matsumoto, *On some properties of orthogonal Weingarten functions*, J. Math. Phys. **50**, 113516 (2009).

[203] J.-Y. Chen and M. Zimet, *Strong-weak Chern-Simons-matter dualities from a lattice construction*, JHEP **08**, 015 (2018).

[204] S. Sahoo, Z. Zhang, and J. C. Y. Teo, *Coupled wire model of symmetric Majorana surfaces of topological superconductors*, Phys. Rev. B **94**, 165142 (2016).

[205] M. Cheng, *Microscopic Theory of Surface Topological Order for Topological Crystalline Superconductors*, Phys. Rev. Lett. **120**, 036801 (2018).

[206] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).

[207] I. Affleck, *Exact critical exponents for quantum spin chains, non-linear σ-models at $\theta = \pi$ and the quantum Hall effect*, Nucl. Phys. B **265**, 409–447 (1986).

[208] C. Chamon, *Solitons in carbon nanotubes*, Phys. Rev. B **62**, 2806–2812 (2000).

[209] C.-Y. Hou, C. Chamon, and C. Mudry, *Electron fractionalization in Two-dimensional graphenelike structures*, Phys. Rev. Lett. **98**, 186809 (2007).

[210] C. Bernard, *Gauge zero modes, instanton determinants, and quantum-chromodynamic calculations*, Phys. Rev. D **19**, 3013–3019 (1979).

[211] H. Osborn, *Semiclassical functional integrals for self-dual gauge fields*, Ann. Phys. **135**, 373–415 (1981).

[212] M. Mariño, *Instantons and Large N* (Cambridge University Press, Cambridge, 2015).

[213] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, 1985).

[214] A. A. Abrikosov and S. D. Beneslavskii, *Possible Existence of Substances Intermediate Between Metals and Dielectrics*, Sov. Phys. JETP **32**, 699–708 (1971).

[215] A. A. Abrikosov, *Calculation of critical indices for zero-gap semiconductors*, Sov. Phys. JETP **39**, 709–716 (1974).

[216] E.-G. Moon, C. Xu, Y. B. Kim, and L. Balents, *Non-Fermi-Liquid and Topological States with Strong Spin-Orbit Coupling*, Phys. Rev. Lett. **111**, 206401 (2013).

[217] S. Dey and J. Maciejko, *Quantum-critical electrodynamics of Luttinger fermions*, Phys. Rev. B **106**, 035140 (2022).