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ISBN 0-315-55585-8

THE UNIVERSITY OF ALBERTA

MARTIN'S AXIOM AND TOPOLOGY

BY

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1989

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: DARIUS NAQVI

TITLE OF THESIS: MARTIN'S AXIOM AND TOPOLOGY

DEGREE FOR WHICH THESIS WAS PRESENTED: MASTER OF SCIENCE

YEAR THIS DEGREE GRANTED: 1989

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Date: Sept. 5, 1989

ABSTRACT

Martin's axiom, together with background information and several equivalent forms, is presented. Then follows a survey of results of Martin's axiom in point-set topology, including results concerning the existence of S- and L-spaces.

ACKNOWLEDGEMENT

One cannot accomplish anything without accumulating numerous debts of gratitude. Here I would like to express my gratitude to those who helped me complete this thesis. Firstly, I would like to express my indebtedness to Dr. S.W. Willard for his guidance and excellent teaching, and for helping me believe in the value of my mathematical work. To all others who have instilled in me an interest in and love for mathematics, among whom stands out Dr. W.D. Montgomery, I also extend my thanks. Finally, I must thank three individuals without whom the completion of this paper would have been impossible. I would like to thank my parents for helping and encouraging me at all stages of my life, and I would like to thank my wife Nooshin for her kindness, patience and encouragement during my studies.

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0. Introduction

In recent years, a lot of the interesting work in set theory and topology has been that involving independence proofs. It turns out, in fact, that certain problems in general topology cannot be solved in ZFC, the usual axioms of set theory including the axiom of choice. However, many independence proofs involve forcing, which is a method used to develop models of set theory with certain additional properties, and usually involves elaborate and complicated arguments. Martin's axiom then becomes very useful to topologists, because it enables them to examine and answer many topological problems without using forcing, or at least keeping forcing arguments to a minimum.

The purpose of this thesis is to provide an introduction to Martin's axiom and survey some of the problems in point-set topology that it solves. In Chapter 1, we introduce the necessary notation, present some background information concerning partially ordered sets, Boolean algebras, and filters and ultrafilters defined on them: then we present Martin's axiom and derive some of 9ts equivalent forms. In Chapter 2, we present some results of Martin's axiom, and $MA_{\aleph_1}$, a related statement, concerning compact spaces, c.c.c. spaces, and normality. Finally, in Chapter 3, we provide a brief survey of S- and L-spaces and their existence under $MA_{\aleph_1}$.

Since this thesis is concerned with Martin's axiom, results which involve forcing are not included, although such results may be stated without proof whenever they apply to the discussion at hand.

1. Martin's Axiom

In this chapter, we will present some basic notions about partially ordered sets, Boolean algebras, filters and ultrafilters in order to introduce Martin's axiom, which is usually stated in terms of generic filters on partially ordered sets. Then we shall consider Martin's axiom as the collection of statements MA_κ, for all cardinals $\kappa < 2^{\aleph_0}$ to show that MA_κ is equivalent to MA_κ restricted to partially ordered sets of cardinality κ, which is equivalent to a similar statement concerning complete ultrafilters on Boolean algebras. Finally, we shall reformulate Martin's axiom in terms of point-set topology.

Before starting, we must first review some basic concepts and notation. Throughout, familiarity is assumed with the ordinal numbers; a good development can be found in [6]. In particular, each ordinal is the set of its predecessors, so that $\alpha < \beta \iff \alpha \in \beta$. A cardinal number is an initial ordinal, *i.e.*, λ is a cardinal iff it is the smallest ordinal α such that there is a bijection from α onto λ; for a given set A, $|A|$ denotes the cardinality of A, *i.e.*, the smallest ordinal α for which there is a bijection from α onto A. A cardinal λ is called ***regular*** iff λ is the smallest limit ordinal α such that there exists a sequence $\{\, \gamma_\beta \mid \beta < \alpha \,\}$ of ordinals $\gamma_\beta < \lambda$ with $\sup_{\beta<\alpha} \gamma_\beta = \lambda$; since $\sup_{\beta<\alpha} \gamma_\beta = \sup(\bigcup_{\beta<\alpha} \gamma_\beta)$, this is equivalent to the statement that λ is the smallest limit ordinal α such that there is a collection of sets $\{\, A_\beta \mid \beta < \alpha \,\}$ with each $|A_\beta| < \lambda$ and $|\bigcup_{\beta<\alpha} A_\beta| = \lambda$.

For a given set A and a cardinal λ, let $[A]^{\lambda}$ denote the collection of all cardinality-λ subsets of A, let $[A]^{<\lambda}$ denote the collection of all subsets of A with cardinality less than λ, and let $[A]^{\leq\lambda}$ denote the collection of all subsets of A with cardinality less than or equal to λ.

ZFC denotes the usual formulation of axiomatic set theory, which includes the axiom of choice (cf. [6]). There is no proof of the consistency of ZFC, so whenever S is a statement, the phrase "S is consistent" is taken to mean "if ZFC is consistent, then so is ZFC + S". This is known as *relative consistency*, and is standard in independence proofs.

The definitions for such terms as Lindelöf, separable, regular, completely regular, T_i for $i = 1, \ldots, 4$ and so on will follow the conventions set forth in [13], and will be re-stated as necessary.

Now let us proceed with developing the necessary tools to study Martin's axiom.

1.1 Definition. Let $(P, \leq)$ be a partially ordered set. Then $D \subset P$ is a *dense* subset of P iff whenever $x \in P$, there exists $d \in D$ with $d \leq x$.

Two elements $x, y \in P$ are called *compatible* iff there exists $z \in P$ such that $z \leq x$ and $z \leq y$. ■

1.2 Definition. Let B be a Boolean algebra, with the binary operations $\cdot$ and $+$, the unary operation $-$, and the universal bounds 1 and 0.

For $x, y \in B$ let us write $x - y$ as an abbreviation for $x \cdot (-y)$, and let us define $x \leq y$ iff $x - y = 0$ ■

It is trivial to show that with this definition of $\leq$, $(B, \leq)$ is a partially ordered set, and

$$x \leq y \iff x \cdot y = x \iff x + y = y.$$

(It is also easy to show that $x \cdot y$ is the greatest lower bound, or infimum, of $\{x, y\}$, and $x + y$ is the lowest upper bound, or supremum, of $\{x, y\}$).

Henceforth whenever a Boolean algebra is spoken of as a partially ordered set, 1.2 is intended.

1.3 Definition. Let $(P, \leq)$ be a partially ordered set. A subset $F \subset P$ is called a *filter* on P iff

i) $F \neq \emptyset$

ii) if $x, y \in P$, and $x \leq y$, then $y \in F$

iii) if $x, y \in F$, then there exists $z \in F$ such that $z \leq x$ and $z \leq y$.

Furthermore, if $\mathcal{D}$ is a collection of subsets of P, a filter F on P is called $\mathcal{D}$-*generic* iff for each $D \in \mathcal{D}$, $D \cap F \neq \emptyset$. ■

1.4 Definition. Let B be a Boolean algebra. A set $F \subset B$ is called a *filter* on B iff

i) $1 \in F$, $0 \notin F$

ii) if $x, y \in B$, $x \in F$ and $x \leq y$, then $y \in F$

iii) if $x, y \in F$, then $x \cdot y \in F$. ■

The usual definition of a filter is that a *filter* F over a set S is a collection of subsets of S such that

i) $S \in F$

ii) if $X, Y \in F$, then $X \cap Y \in F$

iii) if $X, Y \subset S$, $X \in F$, and $X \subset Y$, then $Y \in F$.

In addition, a *proper* filter F satisfies $\emptyset \notin F$. This is, in fact, a definition for *filter* on the Boolean algebra of all subsets of a given set S; we can then think of 1.4 as a generalization of the concept of a (proper) filter to all Boolean algebras. Similarly, via 1.2, every Boolean algebra is a partially ordered set, so 1.3 is a generalization of the notion of filter to all partially ordered sets.

Given a filter F on a Boolean algebra B, we say F is an *ultrafilter* iff F is maximal, i.e. F is not strictly contained in another filter. Then using Zorn's lemma, we can easily show that any subset of B having the finite intersection property is contained in an ultrafilter on B. We also have, for a filter F on B,

$$F \text{ is an ultrafilter} \iff \forall x, y \in B,\ x + y \in F \iff x \in F \text{ or } y \in F$$

$$\iff \forall x \in B, \text{ either } x \in F \text{ or } -x \in F$$

The proof of this statement is identical to the proof of the similar statement for a filter as a collection of subsets of a given set.

1.5 Definition. Let B be a Boolean algebra. B is called a *complete* Boolean algebra iff for any subset $X \subset B$, $\sup X$ and $\inf X$ exist. In that case, we refer to $\sup X$ as $\sum X$, and to $\inf X$ as $\prod X$

Let B be a complete Boolean algebra, $\mathcal{X}$ a collection of sets, F an ultrafilter on B. F is called *$\mathcal{X}$-complete* iff $X \subset F$ and $X \in \mathcal{X}$ implies $\prod X \in F$. ∎

It is not difficult to show that the deMorgan laws and the distributive laws hold for the above infinite sum and product:

$$a \cdot \sum\{\, u \mid u \in X \,\} = \sum\{\, a \cdot u \mid u \in X \,\}$$

$$a + \prod\{\, u \mid u \in X \,\} = \prod\{\, a + u \mid u \in X \,\}$$

$$-\sum\{\, u \mid u \in X \,\} = \prod\{\, -u \mid u \in X \,\}$$

$$-\prod\{\, u \mid u \in X \,\} = \sum\{\, -u \mid u \in X \,\}.$$

1.6 Definition. Let B be a Boolean algebra. A subset $W \subset B$ is called a *partition* of B iff it is pairwise disjoint (*i.e.* for $x, y \in W$, $x \neq y$, we have $x \cdot y = 0$) and $\sum W = 1$. B is said to satisfy the *countable chain condition (c.c.c.)* iff every partition of B is at most countable.

A partially ordered set $(P, \leq)$ is said to satisfy the countable chain condition iff every pairwise incompatible subset of P is at most countable. That is, if $W \subset P$ is such that whenever $x, y \in W$ are distinct, there is no $z \in P$ with $z \leq x$ and $z \leq y$, then $|W| \leq \aleph_0$.

A linearly ordered set $(L, \leq)$ is said to satisfy the countable chain condition iff every family of disjoint open intervals is at most countable, where an open interval is a set of the form $(a, b) = \{\, x \in L \mid a < x < b \,\}$.

A topological space (X, τ) is said to satisfy the countable chain condition iff every family of disjoint open sets is at most countable. ∎

Note that the definition of c.c.c. for a linearly ordered set is exactly that for a topological space, where we give $(L, \leq)$ the order topology. This is done because it is not useful to apply the concept of c.c.c. on a partially ordered set to a linearly ordered set, since the latter has no incompatible elements.

Now we are in a position to state Martin's Axiom.

1.7 Martin's Axiom (MA). *If $(P, \leq)$ is a c.c.c. partially ordered set and $\mathcal{D}$ is a collection of less than $2^{\aleph_0}$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.* ∎

We can also think of Martin's axiom as the statement that MA_κ holds for every $\kappa < 2^{\aleph_0}$, where MA_κ is as follows, for every infinite cardinal κ:

1.8 MA_κ. *If $(P, \leq)$ is a c.c.c. partially ordered set and $\mathcal{D}$ is a collection of at most κ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.* ∎

$\mathrm{MA}_{\aleph_0}$ is provable in ZFC and the proof follows.

1.9 Theorem.[6] *Let $(P, \leq)$ be a partially ordered set, $\mathcal{D}$ a countable collection of dense subsets of P. Then there exists a $\mathcal{D}$-generic filter on P.*

Proof. Let $\mathcal{D} = \{D_1, D_2, \ldots\}$. Choose $x_1 \in P$. For each integer n, choose x_n such that $x_n \leq x_{n-1}$ and $x_n \in D_n$. This is possible by the density in P of each D_n. Now consider the set $F = \{x \in P \mid \exists n < \omega \text{ such that } x \geq x_n\}$. Since $x_n \in F \cap D_n$ for each n, it is easy to see that F is a $\mathcal{D}$-generic filter on P. ∎

Thus since $\mathrm{MA}_{\aleph_0}$ holds in ZFC, Martin's axiom follows from the continuum hypothesis. However, Martin's axiom does not imply the continuum hypothesis, as there exists a model of ZFC[6,11,12] in which Martin's axiom holds and $2^{\aleph_0} > \aleph_0$. More recent work[2] has shown that it is possible for Martin's axiom to fail, so Martin's axiom is independent of ZFC.

Note that in 1.9 we did not even assume that $(P, \leq)$ was c.c.c.; 1.10 below shows that we cannot extend 1.9 to uncountable ordinals, that is, if MA'_κ is the statement identical to MA_κ but not requiring $(P, \leq)$ to be c.c.c., then MA'_κ is false in ZFC for $\kappa > \aleph_0$.

1.10 Example. Let P be the set of all finite sequences of countable ordinals (*i.e.* all functions mapping a finite subset of ω of the form $\{0, 1, \ldots, n\}$ into ω_1). For $x, y \in P$, define $x \leq y$ whenever x extends y, *i.e.* $\mathrm{dom}(y) \subset \mathrm{dom}(x)$ and $x | \mathrm{dom}(y) = y$. Easily $(P \leq)$ is a partially ordered set. Now for each $\alpha \leq \omega_1$, let $D_\alpha = \{ x \in P \mid \alpha \in \mathrm{ran}(p) \}$. Each D_α is dense in P, for given $x \in P$, say $x = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, define $y \in P$ by $y = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha)$. Then $y \in D_\alpha$ and $y \leq x$.

Now setting $\mathcal{D} = \{ D_\alpha \mid \alpha < \omega_1 \}$, we have $|\mathcal{D}| = \aleph_1$, and there cannot be a $\mathcal{D}$-generic ultrafilter F on P, for if there were, we would get a mapping $f = \bigcup F$ of a subset of ω onto ω_1 (given $n \in \bigcup_{x \in F} \mathrm{dom}(x)$, pick $y \in F$ with $n \in \mathrm{dom}(y)$, set $f(n) = y(n)$; this is well-defined by pairwise compatibility of F, and onto ω_1, since F meets each D_α, for each $\alpha < \omega_1$). ∎

Now we will show that MA_κ implies $\kappa < 2^{\aleph_0}$; to do this it is sufficient to show that $\mathrm{MA}_{2^{\aleph_0}}$ is false, which the next example does.

1.11 Example. Define P to be the set of all finite sequences of 0's and 1's (whence the set of all functions mapping an initial sequence of ω into $\{0,1\}$) and define "$x \leq y$" to mean "x extends y", as in 1.10. Clearly $|P| = \aleph_0$, so P is c.c.c. For each $g \in \{0,1\}^\omega$, let

$$D_g = \{ x \in P \mid x \text{ is not extended by } g, \text{ i.e. } g|\mathrm{dom}(x) \neq x \}.$$

Clearly each D_g is dense in P, for given $x \in P$, say $x = (a_0, a_1, \ldots, a_{n-1})$, set $y = (a_0, a_1, \ldots, a_n)$ where $a_n \neq g(n)$. Then $y \in D_g$ and $y \leq x$.

Now setting $\mathcal{D} = \{D_g \mid g \in \{0,1\}^\omega\}$, we have $|\mathcal{D}| = 2^{\aleph_0}$. If there exists a $\mathcal{D}$-generic filter F on P, then form $\bigcup F$ as in 1.10. We have $\mathrm{dom}(\bigcup F) \subset \omega$; if $\mathrm{dom}(\bigcup F) \neq \omega$, take any $g \in \{0,1\}^\omega$ extending $\bigcup F$; otherwise, set $g = \bigcup F$. Now pick $x \in D_g \cap F$. Then $x \in F$ implies x is extended by $\bigcup F$, whence by g, and $x \in D_g$ implies x is not extended by g, a contradiction. Thus there is no $\mathcal{D}$-generic filter on $(P, \leq)$, whence $\mathrm{MA}_{2^{\aleph_0}}$ fails. ■

Now we reformulate Martin's axiom in two ways, one of which is in terms of Boolean algebras.

1.12 Theorem.[6] *Let κ be an infinite cardinal. Then the following are equivalent:*

(a) MA_κ

(b) If $(P, \leq)$ is a c.c.c. partially ordered set with $|P| \leq \kappa$ and $\mathcal{D}$ is a collection of at most κ dense subsets of P, then there is a $\mathcal{D}$-generic filter on P.

(c) If B is a complete, c.c.c. Boolean algebra and $\mathcal{X}$ is a family of at most κ subsets of B, then there exists an $\mathcal{X}$-complete ultrafilter on B.

Proof. (a) $\Longrightarrow$ (c): Assume MA_κ. Let B be a complete, c.c.c. Boolean algebra, and let $\mathcal{X}$ be a collection of at most κ subsets of B. Let $P = B \setminus \{0\}$. For each $X \in \mathcal{X}$, define

$$D_X = \{ u \in P \mid (u \leq \prod X) \text{ or } (\text{for some } x \in X,\ u \cdot x = 0) \}.$$

Each D_X is dense in P, since for $p \in P$, if $p \leq \prod X$, then $p \in D_X$ and $p \leq p$, and if $p \not\leq \prod X$, then

$$\begin{aligned} 0 \neq p \cdot \left(-\prod X\right) &= p \cdot \left(\sum \{ -x \mid x \in X \}\right) \\ &= \sum \{ p \cdot (-x) \mid x \in X \}. \end{aligned}$$

Since "$\sum$" means "sup", there exists $x \in X$ such that $p \cdot (-x) \neq 0$. Then $x \in X$ and $(p \cdot (-x)) \cdot x = 0$, so $p \cdot (-x) \in D_X$, and $p \cdot (-x) \leq p$.

Then by MA_κ there exists a filter F on P meeting each D_X. But then F is also a filter on B, and it is contained in some ultrafilter U. Now whenever $X \subset U$ and $X \in \mathcal{X}$, we have $\exists u \in U \cap D_X$. Since $u \in U$ and $X \subset U$, $u \cdot x \neq 0$ for all $x \in X$, so $u \in D_X$ implies $u \leq \prod X$, whence $\prod X \in U$. Therefore, U is an $\mathcal{X}$-complete ultrafilter on B.

(c) $\Longrightarrow$ (b): (For this step, we need to use the following result concerning complete Boolean algebras, the proof of which can be found in [6]: for every partially ordered set $(P, \leq)$ there is a complete Boolean algebra r.o. P (unique up to isomorphism) and a mapping $e : P \longrightarrow \text{r.o.}\,P \setminus \{0\}$ (called the ***canonical embedding*** of P in r.o. P, although it may not be one-to-one) such that (for $x, y \in P$):

i) if $x \leq y$, then $e(x) \leq e(y)$

ii) x and y are compatible iff $e(x) \cdot e(y) \neq 0$

iii) $\{ e(p) \mid p \in P \}$ is dense in r.o. P.

Note: a subset $D \subset B$ of a Boolean algebra B is said to be *dense* in B iff for every *non-zero* $b \in B$, there exists a *non-zero* $d \in D$ such that $d \leq b$, whence, iff $D \setminus \{0\}$ is dense in $B \setminus \{0\}$ in the sense of partially ordered sets.)

Now we assume that (c) holds, and we let $(P, \leq)$ be a partially ordered set with $|P| \leq \kappa$. Let $B = \text{r.o.}\,P$, and let $e : P \rightarrow B$ be the canonical embedding. Suppose that $\mathcal{D}$ is a collection of at most κ dense subsets of B; then we claim there exists an ultrafilter U on B that meets all $D \in \mathcal{D}$. Let $\mathcal{X} = \{ X_D \mid D \in \mathcal{D} \}$, where $X_D = \{ -x \mid x \in D \}$. Then by (c), there exists an $\mathcal{X}$-complete ultrafilter U on B. Now suppose that for some $D \in \mathcal{D}$, $D \cap U = \emptyset$. Then since U is an ultrafilter, $X_D = \{ -x \mid x \in D \} \subset U$, and since U is $\mathcal{X}$-complete, $\prod X_D \in U$. But

$$\begin{aligned} \prod X_D &= \prod \{ -x \mid x \in D \} \\ &= -\sum \{ x \mid x \in D \} \\ &= -\sum D. \end{aligned}$$

So $-\sum D \in U$, whence $-\sum D \neq 0$, so there exists a non-zero $d \in D$ with $d \leq -\sum D$. Then we have

$$0 = d \cdot \sum D = \sum \{ d \cdot x \mid x \in D \}$$

whence $d \cdot x = 0$ for all $x \in D$. Letting $x = d$, we have $d = 0$, a contradiction, so our claim is proved.

Suppose now that $\mathcal{D}$ is a collection of dense sets in P, with $|\mathcal{D}| \leq \kappa$. Consider the sets $e[D]$ for $D \in \mathcal{D}$, and $e[E_{x,y}]$ for $x, y \in P$, where

$$E_{x,y} = \{ r \in P \mid (r \leq x \text{ and } r \leq y) \text{ or } (r \text{ is incompatible with } x) \text{ or } (r \text{ is incompatible with } y) \}.$$

It is clear that $e[D]$ is dense in B for each $D \in \mathcal{D}$, and we show that each $e[E_{x,y}]$ is dense in B. It suffices to show that $E_{x,y}$ is dense in P. Let $p \in P$. If p is incompatible with x, then $p \in E_{x,y}$. If p is compatible with x, then there exists $r \in P$ with $r \leq x$, $r \leq p$. Now if r is incompatible with y, then $r \in E_{x,y}$ and $r \leq p$. If r is compatible with y, then there exists $r' \in P$ with $r' \leq y$ and $r' \leq r \leq x$, whence $r' \in E_{x,y}$ and $r' \leq p$. Now

$$\{ e[D] \mid D \in \mathcal{D} \} \cup \{ e[E_{x,y}] \mid x, y \in P \}$$

is a collection of dense subsets of P, with cardinality less than or equal to

$$|\mathcal{D}| + |P \times P| \leq \kappa + \kappa^2 = \kappa$$

whence by our claim above, there exists an ultrafilter F on B meeting each $e[D]$ and each $e[E_{x,y}]$. Then setting $F' = \{ x \in P \mid e(x) \in F \}$, we have that F' meets each $D \in \mathcal{D}$ and each $E_{x,y}$, for $x, y \in P$. Then F' is a filter, for

i) $F' \neq \emptyset$ since F' meets $e[D]$.

ii) $x \in F'$ and $x \leq y \Longrightarrow e(x) \leq e(y)$ and $e(x) \in F$

$$\Longrightarrow e(y) \in F$$

$$\Longrightarrow y \in F'.$$

iii) $x, y \in F' \Longrightarrow \exists z \in F' \cap E_{x,y}$

$$\Longrightarrow z \leq x,\ z \leq y \text{ and } z \in F'.$$

So F' is a $\mathcal{D}$-generic filter on P.

(b) $\Longrightarrow$ (a): Let $(P, \leq)$ be a c.c.c. partially ordered set, and assume that (b) holds. Let $\mathcal{D}$ be a collection of dense subsets of P, with $|\mathcal{D}| \leq \kappa$. Now we use $\mathcal{D}$ to construct a set $Q \subset P$ with $|Q| \leq \kappa$.

For each $D \in \mathcal{D}$, (by Zorn's lemma) let W_D be a maximal pairwise incompatible subset of D. Note that by c.c.c., $|W_D| \leq \aleph_0$ for each $D \in \mathcal{D}$. Now let $Q_0 = \bigcup\{ W_D \mid D \in \mathcal{D} \}$. Given Q_n, for each $x, y \in Q_n$ such that x, y are compatible, choose an element $r_{x,y} \in P$ with $r_{x,y} \leq x$, $r_{x,y} \leq y$. Now let

$$Q_{n+1} = Q_n \cup \{ r_{x,y} \mid x, y \in Q_n \text{ and } x, y \text{ are compatible } \}$$

$$\text{and} \qquad Q = \bigcup_{n=1}^{\infty} Q_n.$$

We note Q has the properties

(a) $|Q| \leq \kappa$

(b) $W_D \subset Q$ for each $D \in \mathcal{D}$

(c) if $x, y \in Q$ are compatible, then there exists $z \in Q$ with $z \leq x$, $z \leq y$ (whence $x, y \in Q$ are compatible in P iff they are compatible in Q).

We show (a) by induction; (b) and (c) are clear.

$$|Q_0| = |\bigcup_{D \in \mathcal{D}} W_D| \leq |\mathcal{D}| \cdot \sup_{D \in \mathcal{D}} |W_D|$$
$$\leq |\mathcal{D}| \cdot \aleph_0$$
$$\leq \kappa \cdot \aleph_0 \leq \kappa$$

If $|Q_n| \leq \kappa$, then

$$|Q_{n+1}| \leq |Q_n| + |\{\, r_{x,y} \mid x, y \in Q_n \text{ are compatible } \}|$$
$$\leq |Q_n| + |Q_n|^2$$
$$\leq \kappa + \kappa^2 = \kappa.$$

Then $|Q| = |\bigcup_{n<\omega} Q_n| \leq \aleph_0 \cdot \kappa \leq \kappa$.

Now each W_D is a maximal incompatible subset of Q: if $q \in Q$ is incompatible with each $w \in W_D$, then choose $d \in D$ with $d \leq q$, and d is also incompatible with each $w \in W_D$, whence we can extend W_D (in D) to $W_D \cup \{\, d\,\}$, a contradiction. Since each element $q \in Q$ is compatible with some $w \in W_D$, each set

$$E_D \stackrel{\text{def}}{=} \{\, q \in Q \mid q \leq w \text{ for some } w \in W_D \,\}$$

is dense in Q. Then $(Q, \leq)$ and the collection $\{\, E_D \mid D \in \mathcal{D} \,\}$ satisfy the hypothesis of (b), whence there exists a filter F on Q meeting each E_D, $D \in \mathcal{D}$.

Letting

$$F' = \{x \in P \mid x \geq y \text{ for some } y \in F\},$$

we easily have that F' is a filter on P, and furthermore F' is $\mathcal{D}$-generic since

$$x \in F \cap E_D \implies x \leq w, \text{ some } w \in W_D \subset D$$
$$\implies w \in F' \cap D. \blacksquare$$

Now we conclude this chapter with a reformulation of Martin's axiom in terms of point-set topology.

1.13 Theorem. *Martin's axiom holds iff in every compact Hausdorff c.c.c. space, the intersection of fewer than $2^{\aleph_0}$ dense open sets is dense.*

Proof. In fact we will prove that if κ is a cardinal with $\aleph_0 \leq \kappa < 2^{\aleph_0}$, then MA_κ holds iff in every compact Hausdorff c.c.c. space, the intersection of κ dense open sets is dense.

[$\implies$]: Let (X, τ) be a compact Hausdorff c.c.c. space, and $\mathcal{D}$ a collection of dense open subsets of X, with $|\mathcal{D}| = \kappa$. Let $V \subset X$ be a nonempty open set; we need to show that $V \cap (\bigcap \mathcal{D}) \neq \emptyset$.

Consider the set $Y = \{U \in \tau \setminus \emptyset \mid U \cap V \neq \emptyset\}$, i.e., the collection of all nonempty open sets in X that meet V. Y is partially ordered by set inclusion, and it is c.c.c., since any pairwise incompatible subset of Y is clearly a pairwise disjoint collection of open sets in X, whence at most countable. Now for each $D \in \mathcal{D}$, let $Y_D = \{U \in Y \mid \text{Cl}_X U \subset D \cap V\}$. Then each Y_D is dense in $(Y, \leq)$:

if $U \in Y$, then $U \cap V$ is open and nonempty, whence $U \cap V \cap D$ is as well, by density and openness of D in X. Then since X is compact Hausdorff, it is regular, so there exists an open set $W \subset X$ with $\emptyset \neq W \subset \overline{W} \subset U \cap V \cap D$. Then we have $W \in Y_D$ and $W \subset U$, as desired.

Now since MA_κ holds, there exists a filter F on $(Y, \leq)$ meeting each Y_D. Now $\emptyset \notin F$ and F satisfies the finite intersection property, so if we choose $U_D \in F \cap Y_D$ for each $D \in \mathcal{D}$, then $\{ \mathrm{Cl}_X(U_D) \mid D \in \mathcal{D} \}$ is a collection of closed sets with the finite intersection property, so since (X, τ) is compact, we have $\bigcap \overline{U_D} \neq \emptyset$.

But $\bigcap_{D \in \mathcal{D}} \overline{U_D} \subset \bigcap_{D \in \mathcal{D}} (D \cap V) = V \cap (\bigcap \mathcal{D})$, so $V \cap (\bigcap \mathcal{D}) \neq \emptyset$, as desired.

[$\Longleftarrow$]: Given an infinite cardinal κ with $\kappa < 2^{\aleph_0}$, let $(P, \leq)$ be a c.c.c. partially ordered set with $|P| < \kappa$, and let $\mathcal{X}$ be a family of dense subsets of $(P, \leq)$ with $|\mathcal{X}| < \kappa$. Now we form $\mathrm{r.o.}\, P$, and let $e : P \longrightarrow \mathrm{r.o.}\, P$ be the canonical embedding. Since $\mathrm{r.o.}\, P$ is a Boolean algebra, we can form the *Stone space* $\mathcal{S}(\mathrm{r.o.}\, P)$ of $\mathrm{r.o.}\, P$, *i.e.*, the space of all ultrafilters on $\mathrm{r.o.}\, P$ topologized by taking the sets

$$\Psi(a) = \{ u \in \mathcal{S}(\mathrm{r.o.}\, P) \mid a \in u \} \qquad \text{for each } a \in \mathrm{r.o.}\, P$$

as a base. It is well-known, and not difficult to prove, that $\mathcal{S}(\mathrm{r.o.}\, P)$ is a compact Hausdorff space. We show now that it is c.c.c. First $\mathrm{r.o.}\, P$ is c.c.c., for if W is a partition of $\mathrm{r.o.}\, P$, then for each $w \in W$, choose $x_w \in P$ with

$e(x_w) \leq w$; then $\{ x_w \mid w \in W \}$ is pairwise disjoint in P, whence $|\{ x_w \mid w \in W \}| = |W| < \aleph_0$. Now given a collection of pairwise disjoint basic open sets in $\mathcal{S}(\text{r.o.}\, P)$, say $\{ \Psi(a) \mid a \in I \}$ for some $I \subset \text{r.o.}\, P$, then I is pairwise disjoint in r.o. P (otherwise for $a, b \in I$ with $a \cdot b \neq 0$, there exists $u \in \text{r.o.}\, P$ with $a \in u$ and $b \in u$, so that $u \in \Psi(a) \cap \Psi(b)$). Using Zorn's lemma, we can extend I to a partition W of r.o. P, so $|I| \leq |W| < \aleph_0$.

Now for each $X \in \mathcal{X}$, let

$$D_X = \{ u \in \mathcal{S}(\text{r.o.}\, P) \mid e(x) \in u \text{ for some } x \in X \}.$$

Quite clearly each D_X is open in $\mathcal{S}(\text{r.o.}\, P)$. Given a basic open set $\Psi(a)$ for some $a \in \text{r.o.}\, P$, we can choose $x \in X$ with $e(x) \leq a$. Then any ultrafilter on r.o. P containing $e(x)$ must contain a, and any such ultrafilter is in D_X, whence $\emptyset \neq \Psi(e(x)) \subset \Psi(a) \cap D_X$, so each D_X is dense in $\mathcal{S}(\text{r.o.}\, P)$.

Now consider the sets

$$E_{x,y} = \{ z \in P \mid (z \leq x \text{ and } z \leq y) \text{ or } (z \text{ is incompatible with } x) \text{ or } (z \text{ is incompatible with } y) \} \qquad \text{for } x, y \in P.$$

We showed in the proof of 1.10 that each $E_{x,y}$ is dense in r.o. P. Then if we let

$$D_{x,y} = \{ u \in \mathcal{S}(\text{r.o.}\, P) \mid e(z) = u \text{ for some } z \in E_{x,y} \}.$$

using the same argument as for D_X, each $D_{x,y}$ is open and dense in $\mathcal{S}(\text{r.o.}\, P)$. Then $\{ D_{x,y} \mid x, y \in P \} \cup \{ D_X \mid X \in \mathcal{X} \}$ is a collection of open dense sets in

$\mathcal{S}(\text{r.o.}\,P)$ with cardinality at most

$$|P \times P| + |\mathcal{X}| = \kappa^2 + \kappa = \kappa,$$

so, since the topological version of MA_κ holds, $(\bigcap_{x,y\in P} D_{x,y}) \cap (\bigcap_{X\in\mathcal{X}} D_X)$ is dense in $\mathcal{S}(\text{r.o.}\,P)$, whence nonempty. Then there exists an ultrafilter u on r.o. P meeting each D_X and each $D_{x,y}$. Setting

$$F = \{\, x \in P \mid e(x) \in U \,\},$$

we have that F meets each $X \in \mathcal{X}$ and each $E_{x,y}$. Clearly if $x \in F$ and $y \geq x$, then $y \in F$. Now if $x, y \in F$, then there exists $z \in F \cap E_{x,y}$, whence $z \leq x$ and $z \leq y$ (we can't have z incompatible with x or y, since $e(z)$, $e(y)$ and $e(x)$ are all in some ultrafilter and x and z are incompatible iff $e(x) \cdot e(z) = 0$). So F is an $\mathcal{X}$-generic ultrafilter on P, and we are done.∎

Note that in proving that the topological version of MA_κ implies the set-theoretic version, we only used the fact that the intersection of fewer than $2^{\aleph_0}$ open dense sets is *nonempty*, not that it is dense; thus we have that for $\aleph_0 \leq \kappa < 2^{\aleph_0}$, MA_κ holds iff in every compact Hausdorff c.c.c. space, the intersection of κ open dense subsets is nonempty. Since the complement of an open dense set is nowhere dense, and the complement of the closure of a nowhere dense set is open and dense, via deMorgan's laws we have that for $\aleph_0 \leq \kappa < 2^{\aleph_0}$, MA_κ holds iff no compact Hausdorff c.c.c. space is the union of κ nowhere dense sets.

2. Martin's Axiom and Topology

In this chapter, we present some results of Martin's axiom in point-set topology; we will also consider Martin's axiom together with the negation of the continuum hypothesis, abbreviated MA+¬CH. Actually, these particular results will hold if we assume $MA_{\aleph_1}$, which is strictly weaker than MA+¬CH, since $\aleph_1 < 2^{\aleph_0}$ and MA imply $MA_{\aleph_1}$, and it has been shown[12] that there is a model of set theory in which $2^{\aleph_0} > \aleph_2$, $MA_{\aleph_1}$ holds, but $MA_{\aleph_2}$ fails. We shall show that $MA_{\aleph_1}$ implies the following: any compact perfectly normal space is hereditarily separable, any first-countable c.c.c. compact space is separable, any product of c.c.c spaces is c.c.c. As well, we shall show that whenever κ is an uncountable regular cardinal, MA_κ implies that every κ-Sorgenfrey line has a normal square, but is contained in a κ-Sorgenfrey line whose square is not collectionwise normal. (Then in particular, $MA_{\aleph_1}$ implies that every $\aleph_1$-Sorgenfrey line has a normal square and is contained in an $\aleph_1$-Sorgenfrey line whose square is not collectionwise normal).

For the next result, recall that a *normal space* is one in which pairs of disjoint closed sets can be enclosed in disjoint open sets; a T_4 space is a T_1 normal space. By Urysohn's lemma, X is normal iff whenever A, B are disjoint closed sets in X, there exists a continuous function $f : X \longrightarrow \mathbf{I} = [0,1]$ with $f(A) = 0$ and $f(B) = 1$. A *perfectly normal* space X is a T_1 space in which whenever A and B are disjoint closed sets, there exists a continuous function $f : X \longrightarrow \mathbf{I}$ with $A = f^{-1}(0)$, $B = f^{-1}(1)$. Equivalently, X is perfectly normal iff X is T_4 and each closed set in X is a G_δ set (*i.e.* the intersection of a countable collection of open sets). Also, note that a closed subset Y of a perfectly normal space X is perfectly normal (closed disjoint $A, B \subset Y$ are closed and disjoint in X; if $f : X \longrightarrow \mathbf{I}$ is a continuous function with $A = f^{-1}(0)$, $b = f^{-1}(1)$, then consider the function $f|Y$). Now we are ready for the next theorem.

2.1 Theorem.[10]

(a) *If X is a perfectly normal compact space, then X is first countable and c.c.c.*

(b) *If X is a compact Hausdorff c.c.c. space, and λ a regular cardinal with $\aleph_0 < \lambda < 2^{\aleph_0}$, then MA_λ implies that any family $\mathcal{G}$ of open sets in X with $|\mathcal{G}| = \lambda$ has a cardinality-λ subfamily with nonempty intersection.*

(c) *$MA_{\aleph_1}$ implies that every compact perfectly normal space X is hereditarily separable.*

Proof. (a) First, suppose X is not c.c.c.; in particular, suppose $\{ U_\alpha \mid \alpha < \omega_1 \}$ is a collection of pairwise disjoint, nonempty open sets (with $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$). For each $\alpha < \omega_1$, pick $p_\alpha \in U_\alpha$. Let $H = \overline{\bigcup_{\alpha<\omega_1} U_\alpha} \setminus \bigcup_{\alpha<\omega_1} U_\alpha$; H is closed, so (by perfect normality) it is a G_δ set, so $H = \bigcap_{n<\omega} W_n$, where each W_n is open in X.

Clearly $\{ p_\alpha \}_{\alpha<\omega_1} \subset X \setminus H = \bigcup_{n<\omega}(X \setminus W_n)$, so by a cardinality argument, for some $n_0 < \omega$, $X \setminus W_{n_0}$ contains an uncountable number of the p_α's; *i.e.* there exists $n_0 < \omega$ and an uncountable set $M \subset \omega_1$ with $\{ p_\alpha \}_{\alpha \in M} \subset X \setminus W_{n_0}$. Now clearly $\{p_\alpha\}_{\alpha\in M}$ is a net in the compact space X; it therefore has a cluster point $x \in X$. But

$$x \text{ is a cluster point of } \{p_\alpha\}_{\alpha\in M} \implies x \in \mathrm{Cl}_X(\{p_\alpha\}_{\alpha\in M}) \subset \overline{\bigcup_{\alpha<\omega_1} U_\alpha}.$$

Also $x \notin \bigcup_{\alpha<\omega_1} U_\alpha$ since if $x \in U_\alpha$ for some $\alpha < \omega_1$, then (since $M \subset \omega_1$ is uncountable) there exists $\beta \in M$, $\beta > \alpha$; now there is *no* $\beta' \geq \beta$ with $p_{\beta'} \in U_\alpha$ (since $U_\alpha \cap U_{\beta'} = \emptyset$).

So we have $x \in \overline{\bigcup_{\alpha<\omega_1} U_\alpha} \setminus \bigcup_{\alpha<\omega_1} U_\alpha = H = \bigcap_{n<\omega} W_n$, which is a contradiction, for now each W_n is a neighbourhood of x disjoint from $\{p_\alpha\}_{\alpha\in M}$.

Now we show X is first-countable. Let $x \in X$; $\{x\}$ is closed, whence G_σ, in X, so $\{x\} = \bigcap_{n<\omega} G_n$ where each G_n is open. We can assume $G_1 \supset G_2 \supset \ldots$ (if not, let $\tilde{G}_n = G_1 \cap G_2 \cap \ldots G_n$). Let $U_1 = G_1$. By regularity of X, given U_n, choose U_{n+1} with $x \in U_{n+1} \subset \overline{U_{n+1}} \subset U_n \cap G_n$. Thus we have

$$U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \ldots \supset U_n \supset \overline{U_{n+1}} \supset U_{n+1} \ldots$$

and $\{x\} \subset \bigcap_{n<\omega} U_n \subset \bigcap_{n<\omega} \overline{U_n} \subset \bigcap_{n<\omega} G_n = \{x\}$, whence $\{x\} = \bigcap_{n<\omega} \overline{U_n}$. Now if we show that $\{U_n\}_{n<\omega}$ is a local base at x, we are finished.

Let V be open with $x \in V$. Now

$$X \setminus V \subset X \setminus \{x\} = X \setminus \bigcap_{n<\omega} \overline{U_n} = \bigcup_{n<\omega} (X \setminus \overline{U_n}).$$

By compactness of X, there exists $n_1, \ldots n_k < \omega$ such that

$$X \setminus V \subset \bigcup_{i=1}^{k} (X \setminus \overline{U_{n_i}})$$

$$\text{whence } V \supset \bigcap_{i=1}^{k} \overline{U_{n_i}} = \overline{U_n} \supset U_n$$

where $n = \max\{n_1, \ldots, n_k\}$.

(b) By Zorn's lemma, let $\mathcal{F}$ be a maximal family of pairwise disjoint sets each meeting less than λ of the elements of $\mathcal{G}$. By c.c.c., $|\mathcal{F}| \leq \aleph_0$. For each $F \in \mathcal{F}$, let $\mathcal{G}_F = \{G \in \mathcal{G} \mid G \cap F \neq \emptyset\}$; now since $|\mathcal{F}| \leq \aleph_0 < \lambda$ and $|\mathcal{G}_F| < \lambda$ for each $F \in \mathcal{F}$ and λ is regular, we have

$$\begin{aligned} |\{G \in \mathcal{G} \mid G \cap F \neq \emptyset \text{ for some } F \in \mathcal{F}\}| &\leq |\mathcal{F}| \cdot \sup_{F \in \mathcal{F}} |\mathcal{G}_F| \\ &< |\mathcal{F}| \cdot \lambda \\ &\leq \aleph_0 \cdot \lambda = \lambda. \end{aligned}$$

Since $|\mathcal{G}| = \lambda$, there exists a nonempty $G \in \mathcal{G}$ with $G \cap F = \emptyset$ for each $F \in \mathcal{F}$. Then $G \notin \mathcal{F}$, but G is open and disjoint from all $F \in \mathcal{F}$, so by the maximality of $\mathcal{F}$, G meets λ elements of $\mathcal{G}$. Also, $\overline{G} \cap F = \emptyset$ for each $F \in \mathcal{F}$ (otherwise $x \in \overline{G} \cap F$ implies F is a neighbourhood of a point of $\overline{G}$, which implies $F \cap G \neq \emptyset$) so by the same argument as that used for G, any nonempty open subset of $\overline{G}$ meets λ elements of $\mathcal{G}$.

Clearly $\overline{G}$ is compact. Since G is open in X, any set open in the relative topology on G is open in X; therefore G is c.c.c. Now G is dense in $\overline{G}$, both in the topology of X and in the relative topology on $\overline{G}$. Given any collection of nonempty disjoint open sets in $\overline{G}$, form the intersection of each with G; the result is a pairwise disjoint collection of nonempty open sets in G, so the collection is at most countable. Thus $\overline{G}$ is compact, Hausdorff and c.c.c.

Since $|\mathcal{G}| = \lambda$, write $\mathcal{G} = \{ G_\alpha \mid \alpha < \lambda \}$ where the map $\alpha \longmapsto G_\alpha$ is a one-to-one map of λ onto $\mathcal{G}$. For each $\beta < \lambda$ let $\widetilde{H}_\beta = \bigcup_{\beta<\alpha<\lambda} G_\alpha$, and let $H_\beta = \widetilde{H}_\beta \cap \overline{G}$. Clearly each H_β is open in $\overline{G}$; we claim it is dense as well. Let $W \subset \overline{G}$ be nonempty and open in $\overline{G}$; then $W = \widetilde{W} \cap \overline{G}$ where $\widetilde{W}$ is open in X. Consider $\widetilde{W} \cap G$; this is open in X and $\widetilde{W} \cap G \subset \overline{G}$, whence $W \supset \widetilde{W} \cap G$ meets λ elements of G. Therefore it must be the case that $W \cap (\bigcup_{\beta<\alpha<\lambda} G_\alpha) \neq \emptyset$, or else W would meet at most each G_α for $\alpha \leq \beta$, which (since $\beta + 1 < \lambda$) is less than λ elements of $\mathcal{G}$.

Now by MA_λ, $\bigcap_{\beta<\lambda} H_\beta \neq \emptyset$. Let $x \in \bigcap_{\beta<\lambda} H_\beta \subset \bigcap_{\beta<\lambda} \widetilde{H}_\beta$. Then $x \in \bigcup_{\beta<\alpha<\lambda} G_\alpha$ for each $\beta < \lambda$. By transfinite induction, we will find a strictly increasing sequence $\{ \alpha_\gamma \mid \gamma < \lambda \}$ with $\alpha_\gamma < \lambda$ and $x \in G_{\alpha_\gamma}$ for each $\gamma < \lambda$; having done so, we will have finished, since $\{ G_{\alpha_\gamma} \mid \gamma < \lambda \}$ is a cardinality-λ subset of $\mathcal{G}$ with nonempty intersection.

i) Given α_γ with $\alpha_\gamma < \lambda$ and $x \in G_{\alpha_\gamma}$, we have $x \in \bigcup_{\alpha_\gamma<\alpha<\lambda} G_\alpha$, so let $\alpha_{\gamma+1}$ be the smallest ordinal α with $\alpha_\gamma < \alpha$ and $x \in G_\alpha$.

ii) Let $\gamma < \lambda$ be a limit ordinal, and suppose we have α_δ with $x \in G_{\alpha_\delta}$ and $\alpha_\delta < \lambda$ for each $\delta < \gamma$. Then λ is regular and $\gamma < \lambda$, so letting $\beta = \sup\{ \alpha_\delta \mid \delta < \lambda \}$, we have $\beta < \lambda$. If $x \in G_\beta$, let $\alpha_\gamma = \beta$; otherwise $x \in \bigcup_{\beta<\alpha<\lambda} G_\alpha$, so let α_γ be the smallest ordinal $\alpha > \beta$ with $x \in G_\alpha$.

(c) Let X be compact and perfectly normal. Suppose there exists a non-separable subset $Y \subset X$; then for any sequence of distinct points $\{x_n \mid n < \omega\}$, we have $\overline{\{x_n\}_{n<\omega}} \neq Y$. (Also, clearly $x_n \notin \overline{\{x_i\}_{i<n}}$ for each $n < \omega$). Given $\beta < \omega_1$, the sequence $\{x_\alpha\}_{\alpha<\beta}$ is countable, so choose x_β with $x_\beta \in Y \setminus \overline{\{x_\alpha\}_{\alpha<\beta}}$; by transfinite induction we have a sequence $\{x_\alpha\}_{\alpha<\omega_1}$ with $x_\beta \notin \overline{\{x_\alpha\}_{\alpha<\beta}}$ for each $\beta < \omega_1$. Let $\tilde{X} = \overline{\{x_\alpha\}_{\alpha<\omega_1}}$; $\tilde{X}$ is closed in X, so it is compact and perfectly normal, whence by (a), $\tilde{X}$ is c.c.c. and first-countable (and Hausdorff as well). For each $\beta < \omega_1$, let $G_\beta = \tilde{X} \setminus \overline{\{x_\alpha\}_{\alpha<\beta}}$; now $\{G_\beta \mid \beta < \omega_1\}$ is a strictly decreasing cardinality-$\aleph_1$ collection of open subsets of $\tilde{X}$. Since we are assuming $\mathrm{MA}_{\aleph_1}$, by (b) there is a cardinality-$\aleph_1$ subfamily with nonempty intersection, *i.e.* there exists a strictly increasing sequence $\{\beta_\gamma\}_{\gamma<\omega_1}$ and $x \in \tilde{X}$ with $x \in \bigcap_{\alpha<\omega_1} G_{\beta_\gamma}$. But then since $\{G_\beta\}_{\beta<\omega_1}$ is strictly decreasing, we also have $x \in \bigcap_{\beta<\omega_1} G_\beta$. We will show that this implies x has no countable local base (which contradicts (a), so we will have finished). Given an open neighbourhood U of x in $\tilde{X}$, since $\{x_\beta\}_{\beta<\omega_1}$ is dense in $\tilde{X}$, there exists $x_\alpha \in U \cap \{x_\beta\}_{\beta<\omega_1}$, so $U \not\subset G_{\alpha+1}$, so U is a subset of at most each G_β for $\beta < \alpha + 1$, *i.e.* a countable number of elements of $\mathcal{G}$. Then via a cardinality argument, there is no countable collection $\mathcal{U}$ of neighbourhoods of x such that each G_β for $\beta < \omega_1$ is a superset of an element of $\mathcal{U}$. ■

Note that 2.1(a) is a theorem of ZFC.

The argument used in 2.1(c) can be used to prove another result about separability. For this, we need the concept of *density* of a topological space; if (X, τ) is a topological space, we define the *density* of X to be $d(X) = \aleph_0 \cdot \min\{|U| \mid U \subset X \text{ and } \overline{U} = X\}$.

2.2 Theorem.[7] *If $MA_{\aleph_1}$ holds, then every compact Hausdorff* c.c.c. *first-countable space is separable.*

Proof. Let X be a compact Hausdorff c.c.c. space; we must show $d(X) = \aleph_0$, or, equivalently, we must show that $d(X) = \aleph_1$ and $d(X) > \aleph_1$ are impossible.

Suppose $d(X) = \aleph_1$. Then we have a dense subset $\{x_\alpha \mid \alpha < \omega_1\}$; without loss of generality, we can assume $x_\beta \notin \overline{\{x_\alpha\}_{\alpha<\beta}}$ for each $\beta < \omega_1$. (If this doesn't hold, then for each $\beta < \omega_1$, $|\{x_\alpha\}_{\alpha<\beta}| \leq \aleph_0 < d(X)$, so $\overline{\{x_\alpha\}_{\alpha<\beta}} \neq X$, whence $\{x_\alpha\}_{\alpha<\omega_1} \not\subset \overline{\{x_\alpha\}_{\alpha<\beta}}$; let $\alpha_\beta = \min\{\gamma < \omega_1 \mid x_\alpha \notin \overline{\{x_\alpha\}_{\alpha<\beta}}\}$. Setting $y_\beta = x_{\alpha_\beta}$ for each $\beta < \omega_1$, we have a dense subset $\{y_\beta\}_{\beta<\omega_1}$ of X with $y_\gamma \notin \overline{\{y_\beta\}_{\beta<\gamma}}$). Now the situation is exactly the same as it was in the proof of 2.1(c); our sequence $\{x_\alpha\}_{\alpha<\omega_1}$ with $x_\beta \notin \overline{\{x_\alpha\}_{\alpha<\beta}}$, which is dense in a compact Hausdorff c.c.c. first-countable space, gives us a contradiction, via $MA_{\aleph_1}$, as in 2.1(c).

Now suppose $d(X) > \aleph_1$ (whence $d(X) > \aleph_0$); we will find $S \subset X$ such that S is c.c.c. and $d(S) = |S| = \aleph_1$. (In fact, this is a special case of the theorem[7] $\chi(X) = \alpha$ and $d(X) > \alpha \implies \exists S \subset X$ with $c(S) \leq c(X)$ and

$|S| = d(S) = \alpha^+$ where for any topological space X, $\chi(X)$ is the *character*

$$\chi(X) = \sup_{x \in X}\Big(\min\{\,|\mathcal{U}| \mid \mathcal{U} \text{ a neighbourhood base at } x\}\Big)$$

of X and $c(X)$ is the *cellularity*

$$c(X) = \aleph_0 \cdot \sup\{\,|\mathcal{U}| \mid \mathcal{U} \text{ is a pairwise disjoint open collection in } X\}$$

of X. Replacing, respectively, all occurences of $\aleph_0$ (sometimes ω) and ω_1 with α and α^+ in the following argument provides a proof of the theorem.)

For each $x \in X$, take a neighbourhood base $\mathcal{U}_x$ at x, where $\mathcal{U}_x = \{\, U_{x,n} \mid n < \omega\}$. For every $x, y \in X$ and $n, m < \omega$ such that $U_{x,n} \cap U_{y,n} \neq \emptyset$, pick $z_{x,n;y,m} \in U_{x,n} \cap U_{y,m}$. Given arbitrary $A \subset X$, define

$$A' = \{\, z_{x,n;y,m} \mid x, y \in A,\ \ n, m \in \omega \text{ and } U_{x,n} \cap U_{y,n} \neq \emptyset\,\}.$$

By finite induction, define $A^0 = A$ and $A^n = (A^{n-1})'$. Now define $\mathrm{Cl}(A) \stackrel{\text{def}}{=} \bigcup_{n<\omega} A^n$; whenever $|A| \leq \aleph_0$ we have $|\mathrm{Cl}(A)| \leq \aleph_0$. ($|A^0| \leq \aleph_0$; if $|A^n| \leq \aleph_0$, then

$$|A^{n+1}| \leq |A^n \times A^n| + |\omega \times \omega|$$

$$\leq \aleph_0^2 + \aleph_0^2 = \aleph_0.$$

Then $|\mathrm{Cl}(A)| = |\bigcup_{n<\omega} |A^n| \leq |\omega| \cdot \sup_{n<\omega} A^n| \leq \aleph_0 \cdot \aleph_0 = \aleph_0$.)

Using transfinite induction, for each $\alpha < \omega_1$ define $A_\alpha \subset X$ with $|A_\alpha| \leq \aleph_0$ as follows: choose any nonempty set $A_0 \subset X$ with $|A_0| \leq \aleph_0$; given $\beta < \omega_1$, if we have A_α for each $\alpha < \beta$ with $A_\alpha \subset X$ and $|A_\alpha| \leq \aleph_0$, then set

$B_\beta = \mathrm{Cl}(\bigcup_{\alpha<\beta} A_\alpha)$. Now $|B_\beta| \leq \aleph_0$ and $d(X) > \aleph_1$, so pick $x_\beta \in X \setminus \overline{B_\beta}$; set $A_\beta = \{ x_\beta \} \cup B_\beta$. Clearly $|A_\beta| \leq \aleph_0$, so we have A_α for each $\alpha < \omega_1$.

Set $S = \bigcup_{\alpha<\omega_1} A_\alpha$; now $|S| \leq |\omega_1| \cdot \sup_{\alpha<\omega_1} |A_\alpha| \leq \aleph_1 \cdot \aleph_0 = \aleph_1$, so clearly $d(S) \leq \aleph_1$; if we show $d(S) \neq \aleph_0$ then we have $d(S) = |S| = \aleph_1$. If $R \subset S$ and $|R| = \aleph_0$, then for each $r \in R$, let $\alpha_r = \min\{ \alpha < \omega_1 \mid r \in A_\alpha \}$; setting $\beta = \sup_{r\in R} \alpha_r$, we have $\beta < \omega_1$, so $R \subset A_\beta \subset B_{\beta+1}$. Then since $x_{\beta+1} \in X \setminus \overline{B_\beta + 1} \subset X \setminus \overline{R}$, R is not dense in S; thus $d(S) \neq \aleph_0$. Also, from our construction, nonempty basic open sets in S are disjoint iff they can be extended to disjoint basic open sets in X (otherwise for $x, y \in S$, $U_{x,n} \cap U_{y,n} \neq \emptyset \implies z_{x,n;y,m} \in (U_{x,n} \cap S) \cap (U_{y,m} \cap S)$) so S is c.c.c. Thus S has the desired properties.

Now we shall use S to derive a contradiction. Since S is c.c.c. and dense in $\overline{S}$, $\overline{S}$ is c.c.c. Also any dense subset of S is dense in $\overline{S}$, so $d(\overline{S}) \leq d(S) = \aleph_1$. We showed above that whenever X is a compact Hausdorff c.c.c. space, $d(X) \neq \aleph_1$; since $\overline{S}$ is a compact Hausdorff c.c.c. space, $d(\overline{S}) \neq \aleph_1$, whence $d(\overline{S}) = \aleph_0$. Let $R \subset \overline{S}$ be dense in $\overline{S}$, with $|R| \leq \aleph_0$. Now for each $r \in R$, $U_{r,n} \cap S \neq \emptyset$, so pick $x_{r,n} \in U_{r,n} \cap S$. Then setting $V_r = \{ x_{r,n} \mid n < \omega \}$, we have $|V_r| \leq \aleph_0$. Setting $V = \bigcup_{r\in R} V_r$, we have $|V| \leq |R| \cdot \sup_{r\in R} |V_r| \leq \aleph_0 \cdot \aleph_0 = \aleph_0$; also V is dense in S (since $x \in S$ implies each basic neighbourhood $U_{x,n} \cap S$ meets V at $x_{r,n}$) contradicting $d(S) = \aleph_1$.

Thus $d(X) = \aleph_1$ and $d(X) > \aleph_1$ are both impossible. ∎

The next result concerning products of c.c.c. spaces is not provable in ZFC, as it has been shown[10] that the existence of a Souslin line implies the existence of a c.c.c space whose square is not c.c.c.

2.3 Theorem.[6,10] *If* $MA_{\aleph_1}$ *holds, then every product of c.c.c. spaces is c.c.c.*

Proof. We will prove this theorem in two steps:

(a) If $X = \prod_{\alpha \in A} X_\alpha$ is not c.c.c., then there exists a finite $B \subset A$ such that $\prod_{\alpha \in B} X_\alpha$ is not c.c.c. (This holds in ZFC).

(b) $MA_{\aleph_1}$ implies that if X, Y are c.c.c., then so is $X \times Y$.

(The theorem then follows from (a) plus finite induction on (b)).

Proof of (a): Suppose $\mathcal{G}$ is a pairwise disjoint collection of nonempty basic open sets in the space $X = \prod_{\alpha \in A} X_\alpha$, with $|\mathcal{G}| = \aleph_1$. Then let $\mathcal{G} = \{ G_\alpha \mid \alpha < \omega_1 \}$ where $\alpha \longmapsto G_\alpha$ is a 1-1 map of ω_1 onto $\mathcal{G}$. Now for each $\alpha < \omega_1$, there exists a finite set $F_\alpha \subset A$ with $G_\alpha = \bigcap_{\beta \in F_\alpha} \pi_\beta^{-1}(U_{\alpha\beta})$, where $U_{\alpha\beta}$ is nonempty and open in X_β and π_β is the projection map of X onto X_β. Now we claim that there exists an uncountable set $D \subset \omega_1$ and a finite set $B \subset A$ such that $F_\alpha \cap F_\beta = B$ for all $\alpha, \beta \in D$. Since the G_α's are pairwise disjoint, we have $F_\alpha \cap F_\beta \neq \emptyset$ for each $\alpha, \beta < \omega_1$ (and each $F_\alpha \cap F_\beta$ is finite). (If $F_\alpha \cap F_\beta = \emptyset$, then for $\alpha \in A$, choose x_γ such that $x_\gamma \in U_{\alpha\gamma}$ for $\gamma \in F_\alpha$, $x_\gamma \in U_{\beta\gamma}$ for $\gamma \in F_\beta$, and $x_\gamma \in X_\gamma$ for $\gamma \in A \setminus (F_\alpha \cup F_\beta)$. Then $x \overset{\text{def}}{=} (x_\gamma \mid \gamma \in A) \in G_\alpha \cap G_\beta$.) So, for each nonempty $J \in F_0$, let (via Zorn's lemma) $\mathcal{F}_J$ be a maximal subset of ω_1 such that

i) $0 \in \mathcal{F}_J$

ii) $\forall \alpha, \beta \in \mathcal{F},\ F_\alpha \cap F_\beta = J$.

Now each F_α meets F_0 in some nonempty set $J \subset F_0$, so $\omega_1 = \bigcup_{\emptyset \neq J \subset F_0} \mathcal{F}_J$; thus there must be some $J \subset F_0$ such that $\mathcal{F}_J$ is uncountable (otherwise

$$\begin{aligned} \aleph_1 = \Big|\bigcup_{\emptyset \neq J \subset F_0} \mathcal{F}_J\Big| &\leq |\{\, J \mid \emptyset \neq J \subset F_0 \,\}| \cdot \max_{J \subset F_0} |\mathcal{F}_J| \\ &= (2^{|F_0|} - 1) \cdot \aleph_0 = \aleph_0) \end{aligned}$$

so setting $D = \mathcal{F}_J$ and $B = J$, we see that the claim holds.

Now for each $\alpha \in D$,

$$G_\alpha = \Big[\bigcap_{\beta \in B} \pi_\beta^{-1}(U_{\alpha\beta})\Big] \cap \Big[\bigcap_{\beta \in F_\alpha \setminus B} \pi_\beta^{-1}(U_{\alpha\beta})\Big]$$

so set

$$\widetilde{G}_\alpha = \bigcap_{\beta \in B} \pi_\beta^{-1}(U_{\alpha\beta}).$$

Now for $\alpha, \alpha' \in D$,

$$\begin{aligned} G_\alpha \cap G_{\alpha'} &= \widetilde{G}_\alpha \cap \widetilde{G}_{\alpha'} \cap \Big[\bigcap_{\gamma \in F_\alpha \setminus B} \pi_\gamma^{-1}(U_{\alpha\gamma})\Big] \cap \Big[\bigcap_{\gamma \in F_{\alpha'} \setminus B} \pi_\gamma^{-1}(U_{\alpha\gamma})\Big] \\ &= \Big[\bigcap_{\gamma \in B} \pi_\gamma^{-1}(U_{\alpha\gamma} \cap U_{\alpha'\gamma})\Big] \cap \Big[\bigcap_{\gamma \in F_\alpha \setminus B} \pi_\gamma^{-1}(U_{\alpha\gamma})\Big] \cap \Big[\bigcap_{\gamma \in F_{\alpha'} \setminus B} \pi_\gamma^{-1}(U_{\alpha'\gamma})\Big]. \end{aligned}$$

Since B, $F_\alpha \setminus B$ and $F_{\alpha'} \setminus B$ are disjoint sets (because $F_\alpha \cap F_{\alpha'} = B$), if $G_\alpha \cap G_{\alpha'} = \emptyset$ then $U_{\alpha\gamma} \cap U_{\alpha'\gamma} = \emptyset$ for some $\gamma \in B$ (otherwise for each $\gamma \in A$, choose x_γ such that $x_\gamma \in U_{\alpha\gamma} \cap U_{\alpha'\gamma}$ for $\gamma \in B$, $x_\gamma \in U_{\alpha\gamma}$ for $\gamma \in (F_\alpha \cup F_\beta) \setminus B$, $x_\gamma \in X_\gamma$ for each $\gamma \in A \setminus (F_\alpha \cup F_\beta)$, and we have $x \stackrel{\text{def}}{=} (x_\gamma)_{\gamma \in A} \in G_\alpha \cap G_{\alpha'}$).

Thus $\{\widetilde{G}_\alpha \mid \alpha \in D\}$ is an uncountable pairwise disjoint collection of nonempty basic open sets in $\prod_{\alpha\in B} X_\alpha$, whence $\prod_{\alpha\in B} X_\alpha$ is not c.c.c.

Proof of (b): Let $\mathcal{G}$ be a pairwise disjoint collection of nonempty basic open sets in $X \times Y$. Suppose $\mathcal{G}$ is uncountable; we shall then derive a contradiction. Write $\mathcal{G} = \{U_{X\alpha} \times U_{Y\alpha} \mid \alpha < \omega_1\}$, where $U_{X\alpha}$ and $U_{Y\alpha}$ are nonempty and open in X and Y, respectively, and $\alpha \longmapsto U_{X\alpha} \times U_{Y\alpha}$ is a 1-1 map of ω_1 onto $\mathcal{G}$ (if $|\mathcal{G}| > \aleph_1$ take $\widetilde{\mathcal{G}} \subset \mathcal{G}$ with $|\widetilde{\mathcal{G}}| = \aleph_1$).

If $\mathcal{F} = \{U_{X\alpha} \mid \alpha < \omega_1\}$ is countable, then for some $\alpha < \omega_1$, $\mathcal{F}_{U_{X\alpha}} \stackrel{\text{def}}{=} \{U_{Y\beta} \mid U_{X\alpha} \times U_{Y\beta} \in \mathcal{G}\}$ is uncountable, otherwise $\mathcal{G} = \bigcup_{U_{X\alpha}\in\mathcal{F}} \{U_{X\alpha} \times U_{Y\beta} \mid U_{Y\beta} \in \mathcal{F}_{U_{X\alpha}}\}$ implies $\aleph_1 = |\mathcal{G}| \le |\mathcal{F}| \cdot \sup_{\alpha<\omega_1} |\mathcal{F}_{U_{X\alpha}}| \le \aleph_0 \cdot \aleph_0 = \aleph_0$. Then since Y is c.c.c., there exists distinct $U_{Y\beta}, U_{Y\gamma} \in \mathcal{F}_{U_{X\alpha}}$ with $U_{Y\beta} \cap U_{Y\gamma} \neq \emptyset$; choosing $x_X \in U_{X\alpha}$, $x_Y \in U_{X\beta} \cap U_{X\gamma}$, we have

$$x = (x_X, x_Y) \in (U_{X\alpha} \times U_{Y\beta}) \cap (U_{X\alpha} \times U_{Y\gamma}),$$

a contradiction. Thus $\{U_{X\alpha} \mid \alpha < \omega_1\}$ is uncountable.

Now recall that a set U is called *regular-open* iff $U = \operatorname{Int}(\operatorname{Cl}(U))$; equivalently, U is regular-open iff $U = \operatorname{Int}(\operatorname{Cl}(V))$ for some set V. Given a topological space X, let $\mathcal{R}(X)$ be the set of all regular-open sets in X. It is known that if we define

$$U \cdot V = U \cap V$$

$$U + V = \mathrm{Int}_X(\mathrm{Cl}_X(U \cup V))$$

$$-U = \mathrm{Int}_X(\mathrm{Cl}_X(X \setminus U))$$

then $\mathcal{R}(X)$ is a Boolean algebra (in fact it is complete with

$$\sum \mathcal{A} = \mathrm{Int}_X(\mathrm{Cl}_X(\bigcup \mathcal{A}))$$

for each $\mathcal{A} \subset \mathcal{R}(X)$). Now for our space X, let $S_X = \mathcal{S}(\mathcal{R}(X))$, the Stone space of $\mathcal{R}(X)$ (recall that $\Psi(U) \stackrel{\text{def}}{=} \{ p \in S_X \mid U \in p \}$ is open in $S_X = \mathcal{S}(\mathcal{R}(X))$ for each $U \in \mathcal{R}(X)$). For each $U_{X\alpha}$, let $U^{\bullet}_{X\alpha} = \mathrm{Int}_X \mathrm{Cl}_X(U_{X\alpha})$; consider $\Psi[\mathcal{F}^{\bullet}] \stackrel{\text{def}}{=} \{ \Psi(U^{\bullet}_{X\alpha}) \mid \alpha < \omega_1 \}$. We claim $\Psi[\mathcal{F}^{\bullet}]$ is uncountable; in fact, the mapping $U^{\bullet}_{X\alpha} \longmapsto \Psi(U^{\bullet}_{X\alpha})$ is 1-1, since $\Psi(U^{\bullet}_{X\alpha}) = \Psi(U^{\bullet}_{X\beta})$ implies that for $u \in S_X$, $U^{\bullet}_{X\alpha} \in u \iff U^{\bullet}_{X\beta} \in u$, so by considering the principal ultrafilters

$$\widetilde{U^{\bullet}_{X\alpha}} \stackrel{\text{def}}{=} \{ U \in \mathcal{R}(X) \mid U^{\bullet}_{X\alpha} \leq U \}$$

$$\widetilde{U^{\bullet}_{X\beta}} \stackrel{\text{def}}{=} \{ U \in \mathcal{R}(X) \mid U^{\bullet}_{X\beta} \leq U \}$$

we see that $U^{\bullet}_{X\alpha} \leq U^{\bullet}_{X\beta} \leq U^{\bullet}_{X\alpha}$, whence $U^{\bullet}_{X\alpha} = U^{\bullet}_{X\beta}$. So if $\{ U^{\bullet}_{X\alpha} \mid \alpha < \omega_1 \}$ is uncountable, then $\Psi[\mathcal{F}^{\bullet}]$ is as well; now since S_X is compact, Hausdorff and c.c.c, by 2.1(b) there exists an uncountable subset of $\Psi[\mathcal{F}^{\bullet}]$ with nonempty intersection, so there exists an uncountable set $D \subset \omega_1$ with $\bigcap_{\alpha \in D} \Psi(U^{\bullet}_{X\alpha}) \neq \emptyset$. Choose $u \in \bigcap_{\alpha \in D} \Psi(U^{\bullet}_{X\alpha})$. Now $U^{\bullet}_{X\alpha} \in u \ \forall \alpha \in D$, so $U^{\bullet}_{X\alpha} \cap U^{\bullet}_{X\beta} \neq \emptyset \implies U_{X\alpha} \cap U_{X\beta} \neq \emptyset \ \forall \alpha, \beta \in D$.

Now if $U_{Y\alpha} \neq U_{Y\beta}$ for all distinct $\alpha, \beta \in D$, then $\{ U_{Y\alpha} \mid \alpha \in D \}$ is uncountable, whence (by c.c.c.) not pairwise disjoint, so there exists distinct $\alpha, \beta \in D$ with $U_{Y\alpha} \cap U_{Y\beta} \neq \emptyset$. On the other hand, if $U_{Y\alpha} = U_{Y\beta}$ for some distinct $\alpha, \beta \in D$, obviously $U_{Y\alpha} \cap U_{Y\beta} \neq \emptyset$. Now by choosing $x_X \in U_{X\alpha} \cap U_{X\beta}$ and $x_Y \in U_{Y\alpha} \cap U_{Y\beta}$, we have

$$x = (x_X, x_Y) \in (U_{X\alpha} \times U_{Y\alpha}) \cap (U_{X\beta} \times U_{Y\beta}),$$

a contradiction.

The above argument assumes $\{ U^*_{X\alpha} \mid \alpha < \omega_1 \}$ is uncountable; if this is not the case, then for some $\alpha < \omega_1$, the set $D_\alpha \overset{\text{def}}{=} \{ \beta < \omega_1 \mid U^*_{X\beta} = U^*_{X\alpha} \}$ is uncountable (since $\{ U_{X\alpha} \mid \alpha < \omega_1 \}$ is). Now set $D = D_\alpha$, and use the same argument as above. ■

Note that whenever a product is c.c.c., each factor is c.c.c., because it is the continuous image of a c.c.c. space (using the projection maps). Also note that by 2.3 and the comment preceeding it, the statement "whenever X and Y are c.c.c. topological spaces, so is $X \times Y$" is independent of ZFC.

Martin's axiom also has some consequences in terms of normality. Recall that the Sorgenfrey line **E** is the real line, topologized by taking the intervals $[x, y)$, for $x < y$, as a local base at x. The Sorgenfrey line was originally constructed to give an example of a paracompact (whence normal) space whose square is not normal. It is very interesting, then, that Martin's axiom implies that any cardinality κ (for $\aleph_0 < \kappa < 2^{\aleph_0}$) subspace S of $\mathbf{E}$ (called a *κ-Sorgenfrey line*) has normal square.

2.4 Theorem.[9,10] *$\kappa > \aleph_0$ and MA_κ imply that the square of a κ-Sorgenfrey line S is normal.*

Proof. Let H, K be disjoint closed subsets of $S^2 = S \times S$. For $x = (x_1, x_2) \in S^2$ and $n < \omega$ define $U_n(x) = [x_1, x_1 + \frac{1}{n}) \times [x_2, x_2 + \frac{1}{n})$; then $\{ U_n(x) \cap S^2 \mid n < \omega \}$ is a local base at x. For the rest of this proof, let $\overline{U}$ denote $\mathrm{Cl}_{\mathbf{R}^2}(U)$ and let "open (respectively closed, compact) in $\mathbf{R}^2$" mean "open (respectively closed, compact) in the usual tolopogy on $\mathbf{R}^2$." Now clearly $\overline{U_n(x)} = [x_1, x_1 + \frac{1}{n}] \times [x_2, x_2 + \frac{1}{n}]$; thus $\overline{U_{n+1}(x)} \subset U_n(x)$ for each $n < \omega$, $x \in S^2$. Also note that if $U \subset S^2$ is open (respectively closed) in $\mathbf{R}^2$, it is open (respectively closed) in the Sorgenfrey topology on S^2.

For each $x \in H \cup K$, let

$$J_x = \begin{cases} \{ n < \omega \mid \overline{U_n(x)} \cap K = \emptyset \} & \text{for } x \in H; \\ \{ n < \omega \mid \overline{U_n(x)} \cap H = \emptyset \} & \text{for } x \in K. \end{cases}$$

If $x \in H$, then $x \notin K$ with K closed, so for some $n < \omega$, $(U_n(x) \cap S^2) \cap K = U_n(x) \cap (S^2 \cap K) = U_n(x) \cap K = \emptyset$, so $\overline{U_{n+1}(x)} \cap K = \emptyset$. Thus $J_x \neq \emptyset$ for each $x \in H \cup K$ (the same argument holds for $x \in K$). In fact, each $J_x = \omega \setminus \{1, \ldots, n\}$ for some $n < \omega$.

Let P be the set of all functions f from a finite subset of $H \cup K$ into ω such that $x \in (\mathrm{dom}\, f) \cap H$ and $y \in (\mathrm{dom}\, f) \cap K$ implies $f(x) \in J_x$, $f(y) \in J_y$, and $\overline{U_{f(x)}(x)} \cap \overline{U_{f(y)}(y)} = \emptyset$; by Hausdorffness of $\mathbf{R}^2$ and the fact that $\overline{U_{n+1}(x)} \subset$

$U_n(x)$ and the usual topology on $\mathbf{R}^2$ is weaker than the Sorgenfrey topology on $\mathbf{R}^2$, we can construct examples of such functions, so $P \neq \emptyset$. Define $f \leq g$ whenever f extends g, *i.e.*, whenever $(\operatorname{dom} f) \supset (\operatorname{dom} g)$ and $f|(\operatorname{dom} g) = g$. Then $(P, \leq)$ is partially ordered.

For each $x \in H \cup K$, set $X_x = \{ f \in P \mid x \in \operatorname{dom} f \}$. Now we claim each X_x is dense in $(P, \leq)$. Without loss of generality, assume $x \in H$; then there exists $n_0 \in \omega$ with $\overline{U_{n_0}(x)} \cap K = \emptyset$. Let $(\operatorname{dom} f) \cap K = \{ y_1, \ldots, y_m \}$. Then

$$\exists n_1 \text{ such that } \overline{U_{n_1}(x)} \cap \overline{U_{f(y_1)}(y_1)} = \emptyset$$

$$\exists n_2 \text{ such that } \overline{U_{n_2}(x)} \cap \overline{U_{f(y_2)}(y_2)} = \emptyset$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\exists n_m \text{ such that } \overline{U_{n_m}(x)} \cap \overline{U_{f(y_m)}(y_m)} = \emptyset.$$

Set $n = \max\{ n_0, \ldots, n_m \}$. Define $g \in P$ with $\operatorname{dom} g = (\operatorname{dom} f) \cup \{ x \}$ by $g(x) \stackrel{\text{def}}{=} n$, $g|(\operatorname{dom} f) \stackrel{\text{def}}{=} f$. Then $g \leq f$ and $g \in X_x$, so X_x is dense as claimed. Also, $|\{ X_x \mid x \in H \cup K \}| = |H \cup K| \leq |S^2| = \kappa^2 = \kappa$. Thus if $(P, \leq)$ is c.c.c., then by MA_κ there exists a filter F on P meeting each X_x. For each $X \in H \cup K$, let $f_x \in F \cap X_x$. Then the collection $\{ f_x \mid x \in H \cup K \}$ is pairwise compatible, so setting $f(x) = f_x(x)$ for each $x \in H \cup K$, we have that f is a function on all of $H \cup K$ such that $x \in H$, $y \in K$ implies $U_{f(x)} \cap U_{f(y)} = \emptyset$. Then $U \stackrel{\text{def}}{=} S^2 \cap \bigcup_{x \in H} U_{f(x)}(x)$ and $V \stackrel{\text{def}}{=} S^2 \cap \bigcup_{y \in K} U_{f(y)}(y)$ are disjoint open sets with $H \subset U$ and $K \subset V$; thus S^2 is normal.

It remains to show that $(P, \leq)$ is c.c.c. Let $Q \subset P$ be uncountable; we shall show that Q has an uncountable subset pairwise compatible in P. By a series of counting arguments, there exists an uncountable subset $Q' \subset Q$ and integers k and l such that the domain of each $f \in Q'$ has exactly k elements $f_1, \ldots, f_k$ in H and l elements $f_{k+1}, \ldots, f_{k+l}$ in K, and furthermore $f(f_i) = g(g_i)$ for each $i \in \{1, \ldots, k+l\}$ and $f, g \in Q'$. Set $n_i = f(f_i)$ for $i = 1, \ldots, k+l$ and any $f \in Q'$.

Now we claim that for each $f \in Q'$, there exists open sets U, V in $\mathbf{R}^2$ with $(\operatorname{dom} f) \cap H \subset U$, $(\operatorname{dom} f) \cap K \subset V$ and whenever $x \in U$ and $y \in V$, $\bigcup_{i=1}^{k} \overline{U_{n_i}(x)} \cap \bigcup_{i=k+1}^{k+l} \overline{U_{n_i}(y)} = \emptyset$. Then let $\mathcal{U}$ be a countable base for the usual topology of $\mathbf{R}^2$; for each $U, V \in \mathcal{U}$, set

$$R_{U,V} = \{ f \in Q' \mid (\operatorname{dom} f) \cap H \subset U, \ (\operatorname{dom} f) \cap K \subset V \}$$

whenever for $x \in U$ and $y \in V$, $\bigcup_{i=1}^{k} \overline{U_{n_i}(x)} \cap \bigcup_{i=k+1}^{k+l} \overline{U_{n_i}(y)} = \emptyset$, and set $R_{U,V} = \emptyset$ otherwise. Now (assuming that the claim holds) each $f \in Q'$ is contained in some $R_{U,V}$; as well, each $R_{U,V}$ is pairwise compatible in $(P, \leq)$. If each $R_{U,V}$ is at most countable, then $\aleph_1 = |Q'| = |\bigcup_{U,V \in \mathcal{U} \times \mathcal{U}} R_{U,V}| \leq \aleph_0$, a contradiction; then some $R_{U,V}$ is uncountable, so it is an uncountable subset of Q pairwise compatible in $(P, \leq)$, as desired.

Now we must prove our claim. Let $f \in Q'$ and choose any $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, k+l\}$. Now since $\overline{U_{n_i}(f_i)}$ and $\overline{U_{n_j}(f_j)}$ are disjoint compact sets in $\mathbf{R}^2$, the *distance* between them is positive, *i.e.* $d \stackrel{\text{def}}{=} d(\overline{U_{n_i}(f_i)}, \overline{U_{n_j}(f_j)}) \stackrel{\text{def}}{=} \sup\{\|x - y\| \mid x \in \overline{U_{n_i}(f_i)},\ y \in \overline{U_{n_j}(f_j)}\} > 0$ where $\|\cdot\|$ is the usual norm in $\mathbf{R}^2$. (This must hold, for if $d = 0$, then for each $n < \omega$, choose $x_n \in \overline{U_{n_i}(f_i)}$ and

$y_n \in \overline{U_{n_j}(f_j)}$ with $\|x_n - y_n\| < 1/(2^n)$. Now by compactness, choose convergent subsequences $(x_{n_m})_{m<\omega}$ and $(y_{n_m})_{m<\omega}$ of $(x_n)_{n<\omega}$ and $(y_n)_{n<\omega}$ respectively, say $x_{n_m} \to x \in \overline{U_{n_i}(f_i)}$ and $y_{n_m} \to y \in \overline{U_{n_j}(f_j)}$. Now $(x_{n_m} - y_{n_m}) \to 0 \implies x = y \implies \overline{U_{n_i}(f_i)} \cap \overline{U_{n_j}(f_j)} \neq \emptyset$, a contradiction.) Now set $U = \{x \in \mathbf{R}^2 \mid \|x - f_i\| < d/2\}$ and $V = \{x \in \mathbf{R}^2 \mid \|x - f_j\| < d/2\}$. Now for $x \in U$ and $y \in V$, $\overline{U_{n_i}(x)} \cap \overline{U_{n_j}(y)} = \emptyset$, otherwise $z \in \overline{U_{n_i}(x)} \cap \overline{U_{n_j}(y)}$ implies $z - x + f_i \in \overline{U_{n_i}(f_i)}$, $z - x + f_j \in \overline{U_{n_j}(f_j)}$, and

$$\begin{aligned} \|(z - x + f_i) - (z - x + f_j)\| &= \|f_i - x + y - f_j\| \\ &\le \|f_i - x\| + \|y - f_j\| \\ &< \frac{d}{2} + \frac{d}{2} = d, \end{aligned}$$

contradicting the definition of d. Thus for each $i \in \{1, \ldots, k\}$ and $j \in \{k + 1, \ldots, k + l\}$, there exists open neighbourhoods V_i and V_j in $\mathbf{R}^2$ of f_i and f_j, respectively, such that $x \in V_i$, $y \in V_j \implies \overline{U_{n_i}(x)} \cap \overline{U_{n_j}(y)} = \emptyset$.

We will prove the claim by strengthening the above result. Let $i \in \{1, \ldots, k\}$ be fixed; for each $j \in \{k + 1, \ldots, k + l\}$, let V_{ij} and V_j be open neighbourhoods in $\mathbf{R}^2$ of f_i and f_j, respectively, such that for $x \in V_{ij}$ and $y \in V_j$, $\overline{U_{n_i}(x)} \cap \overline{U_{n_j}(y)} = \emptyset$. Now set $\widetilde{U_i} = \bigcap_{j=k+1}^{k+l} V_{ij}$ and $\widetilde{V_i} = \bigcup_{j=k+1}^{k+l} V_j$; then $\widetilde{U_i}$ and $\widetilde{V_i}$ are open sets in $\mathbf{R}^2$ with $f_i \in \widetilde{U_i}$, $(\operatorname{dom} f) \cap K \subset \widetilde{V_i}$ and for $x \in \widetilde{U_i}$ and $y \in \widetilde{V_i}$, $\overline{U_{n_i}(x)} \cap \overline{U_{n_j}(y)} = \emptyset$ for each $j \in \{k + 1, \ldots, k + l\}$. Now construct $\widetilde{U_i}$ and $\widetilde{V_i}$ for each $i \in \{1, \ldots, k\}$. Let $U = \bigcup_{i=1}^{k} \widetilde{U_i}$ and $V = \bigcap_{i=1}^{k} \widetilde{V_i}$; then U and V are open in $\mathbf{R}^2$ with $(\operatorname{dom} f) \cap H \subset U$, $(\operatorname{dom} f) \cap K \subset V$, and

$\overline{U_{n_i}(x)} \cap \overline{U_{n_j}(y)} = \emptyset$ for all $x \in U$, $y \in V$, $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, k+l\}$, so the claim is proved.■

We can show that under $MA_{\aleph_1}$, there is a κ-Sorgenfrey line whose square is not collectionwise normal. First we show that $\mathbf{E}$ is hereditarily Lindelöf; the proof is identical to the proof that $\mathbf{E}$ is Lindelöf. Let $X \subset \mathbf{E}$, and let $\mathcal{U}$ be an open cover of X in the relative topology on X inherited from $\mathbf{E}$. For each $x \in X$, choose $U_x \in \mathcal{U}$ with $x \in U_x$, and choose $b_x > x$ such that $[x, b_x) \cap X \subset U$. Let $A = \bigcup_{x \in X}(x, b_x)$; now this a cover of A by sets open in $\mathbf{R}$, and $\mathbf{R}$ is second-countable, whence hereditarily Lindelöf. Therefore there exists a countable subset $\{ x_i \mid i \in \omega \}$ of X such that $A = \bigcup_{i \in \omega}(x_i, b_{x_i})$. Now if $x, y \in X \setminus A$ are distinct, then $(x, b_x) \cap (y, b_y) = \emptyset$, otherwise $x \in (y, b_y) \implies x \in A$, or $y \in (x, b_x) \implies y \in A$. Each interval (x, b_x), where $x \in X \setminus A$, contains a rational; therefore, $X \setminus A$ is countable. Then $\{ U_x \mid x \in X \setminus A \}$ is a countable subcover of $\mathcal{U}$ for $X \setminus A$, and $\{ U_{x_i} \mid i \in \omega \}$ is a countable subcover of $\mathcal{U}$ for $A \cap X$. The union of these two collections is then a countable subcover of $\mathcal{U}$ for $(X \setminus A) \cap (A \cap X) = X$, so $\mathbf{E}$ is hereditarily Lindelöf as claimed. Now if $\mathbf{E}$ is not hereditarily separable, then it is a first-countable T_3 hereditarily Lindelöf non-hereditarily separable space; in chapter 3, we shall see that $MA_{\aleph_1}$ implies that there is no such space. Thus (assuming $MA_{\aleph_1}$) if $S, T \subset \mathbf{E}$ then $S \times T$ is separable; but $\mathbf{E}^2$ is not hereditarily separable since $\{ (x, -x) \mid x \in \mathbf{E} \}$ is an uncountable discrete subspace.

Now recall that a topological space X is called ***collectionwise normal*** iff X is T_1 and for every discrete family $\{ F_i \mid i \in I \}$ of closed sets there exists a discrete family $\{ G_i \mid i \in I \}$ of open sets with $F_i \in G_i$ for each $i \in I$. A family of sets is called ***discrete*** iff every point $x \in X$ has a neighbourhood meeting at most one member of the family.

Now if S is a κ-Sorgenfrey line, then so is $S' \stackrel{\text{def}}{=} S \cup \{ -x \mid x \in S \}$. The set $\{ \{ (x, -x) \} \mid x \in S' \}$ is easily an uncountable discrete collection of closed sets in S'; there cannot be a discrete family of open sets $\mathcal{G}$ with each $(x, -x)$ contained in some $G \in \mathcal{G}$, because such a family would be an uncountable pairwise disjoint collection of open sets, contradicting separability of $(S')^2$. Then $(S')^2$ is not collectionwise normal.

To summarize, then, if $\kappa > \aleph_0$, then MA_κ implies that the square of any κ-Sorgenfrey line is normal, and $\mathrm{MA}_{\aleph_1}$ (which is weaker than MA_κ) implies that any κ-Sorgenfrey line is contained in one whose square is not collectionwise normal.

3. Martin's Axiom and S- and L-spaces.

In this chapter, we will present some results concerning the existence of S- and L-spaces under MA+¬CH (actually under $MA_{\aleph_1}$, as in chapter 2). Specifically, we shall show that under $MA_{\aleph_1}$, there are no strong S- or L-spaces, no compact S- or L-spaces, and no first-countable L-spaces. We shall also show that it is consistent with $MA_{\aleph_1}$ that no S-spaces exist. It is, however, consistent with $MA_{\aleph_1}$ that first-countable S-spaces exist[1], and that L-spaces exist[1,8]; it is not known whether it is consistent with ZFC that no L-spaces exist, or whether there exists an S-space iff there exists an L-space.

Recall that a space is called *separable* iff it has a countable dense subset, and *Lindelöf* iff every open cover of the space has a countable subcover; a space is *hereditarily* separable (respectively hereditarily Lindelöf) iff every subspace is separable (respectively Lindelöf). Historically, the question has arisen, "Are *hereditarily separable* and *hereditarily Lindelöf* different from each other?" There are well-known examples[13] in ZFC of spaces that are separable but not Lindelöf and vice-versa, and in fact there is an example[4] of a Hausdorff space that is hereditarily separable and not hereditarily Lindelöf, and a Hausdorff space that is hereditarily Lindelöf and not hereditarily separable[4]. Thus the question is interesting only for T_3 spaces, and with this thought in mind, we form the following definition.

3.1 Definition. An *S-space* is a topological space which is T_3, hereditarily separable and not hereditarily Lindelöf.

An *L-space* is a topological space which is T_3, hereditarily Lindelöf and not hereditarily separable. ■

The obvious question is, of course, do S- or L-spaces exist? In fact, there have been constructions of such spaces using the negation of Souslin's hypothesis (*i.e.*, the existence of a Souslin line), the continuum hypothesis, the combinatorial principle $\diamondsuit$ (which is stronger than CH) and forcing methods, which, it turns out, are ruled out by Martin's axiom and/or $MA_{\aleph_1}$ (as are, obviously, the methods using CH, the existence of a Souslin line, and $\diamondsuit$). The question arises, does Martin's axiom or $MA_{\aleph_1}$ destroy S- and/or L-spaces? In this chapter, we shall show that $MA_{\aleph_1}$ does destroy S- and L-spaces that have certain additional properties, as mentioned above.

It should also be mentioned that an S-space is often defined as T_3, hereditarily separable and *not Lindelöf* (similarly for L-spaces). It is clear that this type of S- (respectively L-) space exists iff an S- (respectively L-) space of the type in 3.1 exists; however, 3.1 is more useful in some ways, *e.g.*, with 3.1 it is possible to discuss compact S-spaces.

In order to discuss the existence of S- and L-spaces, we will need to derive the canonical form for these spaces; in order to do this, we need the following definition and theorem.

3.2 Definition. A topological space X is called *right separated* (in type κ) iff it can be well-ordered (in type κ) so that every initial segment is open, *i.e.*, $X = \{ x_\alpha \mid \alpha < \kappa \}$ and $\{ x_\alpha \mid \alpha < \beta \}$ is open for each $\beta < \kappa$. Similarly, X is called ***left separated*** (in type κ) iff X can be well-ordered (in type κ) so that every initial segment is closed (equivalently, every final segment is open, *i.e.*, $X = \{ x_\alpha \mid \alpha < \kappa \}$ and each $\{ x_\alpha \mid \alpha < \beta \}$ is closed, for $\beta < \kappa$ (equivalently, each $\{ x_\alpha \mid \alpha \geq \beta \text{ and } \alpha \in \kappa \} = \{ x_\alpha \mid \alpha \in \kappa \setminus \beta \}$ is open, for $\beta < \kappa$.■

3.3 Theorem.[8] *A space is hereditarily separable (respectively hereditarily Lindelöf) iff it has no uncountable left separated (respectively right separated) subspace.*

Proof. [$\Longrightarrow$] If $X = \{ x_\alpha \mid \alpha < \omega_1 \}$ is an uncountable right separated space (with the given well-ordering of X), it is clear that the open cover $\{ \{ x_\alpha \mid \alpha < \beta \} \mid \beta < \omega_1 \}$ has no countable subcover, since ω_1 is not equal to the union of a countable collection of countable ordinals; thus X is not Lindelöf. If $X = \{ x_\alpha \mid \alpha < \omega_1 \}$ is left separated (with the given well-ordering of X), then any dense subset of X must meet each $\{ x_\alpha \mid \alpha \in \omega_1 \setminus \beta \}$ for $\beta < \omega_1$, making it uncountable; thus X is not separable.

[$\Longleftarrow$] If X is not hereditarily Lindelöf, then there is a subspace Y of X for which some open cover $\mathcal{U}$ has no countable subcover. By transfinite induction, construct a sequence $\{ U_\alpha \mid \alpha < \omega_1 \}$ of elements of $\mathcal{U}$ such that for each $\alpha < \omega_1$, $U_\alpha \cap (Y \setminus \bigcup_{\beta<\alpha} U_\beta) \neq \emptyset$: given $\alpha < \omega_1$, $\{ U_\beta \mid \beta < \alpha \}$ does not cover Y, so let U_α be an element of $\mathcal{U}$ meeting $Y \setminus \bigcup_{\beta<\alpha} U_\beta$. Now setting $V_\alpha = \bigcup_{\beta<\alpha} U_\beta$ for each $\alpha < \omega_1$, we have that $\{ V_\alpha \mid \alpha < \omega_1 \}$ is a strictly increasing sequence of open sets, so picking $x_\alpha \in V_{\alpha+1} \setminus V_\alpha$ for each $\alpha < \omega_1$, we have that $\{ x_\alpha \mid \alpha < \omega_1 \}$ is an uncountable right separated subspace of X.

If X is not hereditarily separable, then there is a subspace Y of X that has no countable dense subset; as in the proof of 2.2, by transfinite induction construct a sequence $\{ x_\alpha \mid \alpha < \omega_1 \}$ of points in Y such that $x_\alpha \notin \mathrm{Cl}_Y\{ x_\beta \mid \beta < \alpha \}$. Then $\{ x_\alpha \mid \alpha < \omega_1 \}$ is an uncountable left separated subspace of X. ■

Now, from the definition of S- and L-spaces, and from 3.3, we have that every S-space contains a right separated subspace of type ω_1, and every L-space contains a left separated subspace of type ω_1. Furthermore, it is now clear that a T_3 right separated space of type ω_1 is an S-space iff it is hereditarily separable iff it has no uncountable left separated subspace iff it has no uncountable discrete subspace. The last "iff" holds by the following argument: if $X = \{ x_\alpha \mid \alpha < \omega_1 \}$ is right separated, then a discrete subspace of X is left separated in the induced order. On the other hand, if $Y = \{ y_\beta \mid \beta < \omega_1 \}$ is a left separated subspace of X, then each $y_\beta \in \{ x_\alpha \mid \alpha < \omega_1 \}$, so for each $\beta < \omega_1$, there exists $\alpha_\beta < \omega_1$ with $y_\beta = x_{\alpha_\beta}$. Now let $z_0 = y_0 = x_{\alpha_0}$. Given $\{ z_\gamma \mid \gamma < \delta \}$ for some $\delta < \omega_1$, with each $z_\gamma = y_{\beta_\gamma} = x_{\alpha_{\beta_\gamma}}$, let $z_\delta = y_{\beta_\delta} = x_{\alpha_{\beta_\gamma}}$, where β_γ is the smallest ordinal such that $z_\gamma < y_{\beta_\gamma} = x_{\alpha_{\beta_\gamma}}$ for each $\gamma < \delta$. Then $\{ z_\delta \mid \delta < \omega_1 \}$ is a subspace of X left separated (in type ω_1) in the well-ordering induced by the well-ordering of X; therefore it is an uncountable discrete subspace of X. Easily, a similar argument yields a dual statement for L-spaces, so we have just proved the following:

3.4 Theorem.[8] *A T_3 right separated (respectively left separated) space of type ω_1 is an S-space (respectively L-space) iff it has no uncountable discrete subspaces.* ■

Now we are ready to derive the canonical form of S- and L-spaces.

3.5 Theorem.[8] (Canonical Form)

(a) *Every S-space has a zero-dimensional subspace which is right separated in type ω_1, and hence also an S-space.*

(b) *Assume $\neg$CH. Then every L-space has a zero-dimensional subspace which is left separated in type ω_1, and hence also an L-space.*

Proof. Since an L-space is regular and Lindelöf, it is normal, whence completely regular. Also an L-space has a left separated subspace $\{ x_\alpha \mid \alpha < \omega_1 \}$, so each point x_β in this subspace has a neighbourhood (in the subspace) $\{ x_\alpha \mid \alpha \in \omega_1 \setminus \beta \}$ of cardinality $\leq \aleph_1$; assuming $\neg$CH we have that every point has a neighbourhood of cardinality less than $2^{\aleph_0}$.

Now any S-space has a right separated subspace $\{ x_\alpha \mid \alpha < \omega_1 \}$; this subspace is then *locally countable, i.e.* each point has a countable neighbourhood. Now any regular locally countable space X is regular: a regular countable space is Lindelöf, whence normal, whence completely regular. If F is a closed set in X not containing x, then let U be a countable neighbourhood of x missing F. Then by regularity, there exists an open neighbourhood V of x with $x \in V \subset \overline{V} \subset U$, and since U is a countable regular space, it is completely regular, so there exists a continuous function $f : U \longrightarrow [0,1]$ with $f(x) = 0$,

$f(U \setminus V) = 1$. Then extend f to X by defining $f(X \setminus U) = 1$; then $f(X \setminus \overline{V}) = 1$ as well. Now for $a \in (0,1)$, $f^{-1}(a,1] = (X \setminus \overline{V}) \cup (f^{-1}(a,1] \cap U)$ is open in X, and $f^{-1}[0,a)$ is open in U, whence in X; thus $f : X \longrightarrow [0,1]$ is continuous and $f(x) = 0$, $f(F) = 1$, so X is completely regular as claimed. Then a right separated subspace of an S-space is a completely regular space in which each point has a neighbourhood of cardinality less than $2^{\aleph_0}$.

Now we can prove (a) and (b) simultaneously by proving the following: a completely regular space in which each point has a neighbourhood of cardinality less than $2^{\aleph_0}$ is zero-dimensional. Let X be such a space, $x \in X$, and U an open neighbourhood of x with $|U| < 2^{\aleph_0}$. It suffices to show there exists a clopen set containing x and contained in U (since obviously any neighbourhood of x contains such a neighbourhood U of x). Since X is completely regular, there exists a continuous function $f : X \longrightarrow [0,1]$ with $f(x) = 0$, $f(X \setminus U) = 1$. Now $|U| < 2^{\aleph_0} = |(0,1)|$, so there exists $r \in (0,1) \setminus f(X)$. Then $f^{-1}[0,r) = f^{-1}[0,r]$ is a clopen set containing x, and it is contained in U since $f(X \setminus U) = 1 \implies U \supset X \setminus f^{-1}(1) \supset f^{-1}[0,r]$. ■

We will always be able to apply (b), since we will work under the assumption $MA_{\aleph_1}$, which implies $\neg$CH; however, Hajnal and Juhász[5] proved that CH implies the existence of a zero-dimensional L-space. This fact together with 3.5 shows that we can always use the following result, independent of CH:

3.6 Theorem.[8] **(Canonical Form)**

There is an S-space (respectively L-space) iff there is a zero-dimensional one right (respectively left) separated in type ω_1. ■

But note, however, that there is not *exactly* a duality here. Unless we explicitly assume $\neg$CH, all we can say with certainty concerning L-spaces is that each one contains one which is left separated in type ω_1, and there exists a zero-dimensional one left separated in type ω_1; we do not know that every L-space has a *zero-dimensional* subspace left separated in type ω_1.

In addition to the above tools, we will need to use Martin's axiom in a form which slightly easier to apply in order to discuss strong S- and L-spaces and first-countable L-spaces. Thus the following definition and theorem.

3.7 Definition. Let $(P, \leq)$ be a partially ordered set, Q a subset of P. Then Q is said to be *centred* in P (or in $(P, \leq)$) iff for any finite subset $\{q_1, \ldots, q_n\}$ of Q there exists $p \in P$ with $p \leq q_i$ for each $i = 1, \ldots, n$; p is called a *lower bound* for $\{q_1, \ldots, q_n\}$. Q is called *maximally centred* in P iff it is centred in P and not strictly contained in any other centred subset of P. ■

The idea is that we want to prove an analog of 2.1(b) for partially ordered sets; the way to do this is to construct an analog of a Stone space for partially ordered sets. The construction $S(\text{r.o.} P)$ does not work, because although $p \leq q$ implies $\Psi(e(p)) \subset \Psi(e(q))$, the converse (which, it turns out, we need to use) is true iff $(P, \leq)$ is *separative* (cf. [6]) where $(P, \leq)$ is called *separative* iff whenever $p \not\leq q$, there exists an element $r \leq p$ incompatible with q.

On the other hand, trying to work with the collection of all the ultrafilters on P doesn't work either. If we define an ultrafilter on P to be a filter on P not strictly contained in any other filter on P, then it is not difficult to show that $U \subset P$ is an ultrafilter iff U is maximally centred in itself, where we define a set $Q \subset P$ to be *centred in itself* iff each finite subset of Q has lower bound *in* Q, and define Q to be *maximally* centred in itself iff it is centred in itself and not strictly contained in any other subset of Q centred in itself. It turns out, however, that in developing the Stone topology on $\mathcal{S}(B)$, the set of all ultrafilters on the Boolean algebra B, it is necessary to use the fact that any subset of B with the finite intersection property is contained in an ultrafilter. This property does not hold for partially ordered sets, since a set maximally centred in itself need not be maximally centred.

However, it is obvious that (via Zorn's lemma) any centred set is contained in a maximally centred set, so we can attempt to construct $\mathcal{S}(P)$ by considering maximally centred subsets of P. Also note that if U is such a set and $p \geq q$ for some $q \in U$, then $\{p\} \cup U$ is also centred, so $p \in U$.

3.8 Lemma. *Let λ be an uncountable regular cardinal, $(P, \leq)$ a c.c.c. partially ordered set. Then MA_λ implies that any cardinality-λ subset $Q \subset P$ has a cardinality-λ subset Q' which is centred in P.*

Proof. Let $\mathcal{S}(P)$ be the set of all maximally centred subsets of P. For each $p \in P$, let

$$\Psi(p) = \{\, U \in \mathcal{S}(P) \mid p \in U \,\}.$$

Now topologize $\mathcal{S}(P)$ by giving it the topology having the collection $\{\Psi(p) \mid p \in P\}$ as a subbase. Each $\Psi(p)$ is clopen, since if $U \notin \Psi(p)$, then $p \notin U$, so $\{p\} \cup U$ is not centred. Then for some finite subset $\{p_1, \ldots, p_n\} \subset U$, $\{p, p_1, \ldots, p_n\}$ has no lower bound, so $(\Psi(p_1) \cap \ldots \Psi(p_n)) \cap \Psi(p) = \emptyset$ and $U \in \Psi(p_1) \cap \ldots \cap \Psi(p_n)$. Then $\mathcal{S}(P)$ is completely regular, and it is Hausdorff since $U \neq V$ implies that by maximality of U, there exists $x \in U \setminus V$; then $\Psi(x)$ and $\mathcal{S}(P) \setminus \Psi(x)$ are disjoint open sets containing U and V, respectively. Thus $\mathcal{S}(P)$is a *Tychonoff* space, *i.e.*, a completely regular T_1 space; now let $\tilde{\mathcal{S}}(P)$ be the Stone-Čech compactification of $\mathcal{S}(P)$. Recall that $\mathcal{S}(P)$ is dense in $\tilde{\mathcal{S}}(P)$ and $\tilde{\mathcal{S}}(P)$ is compact and Haudsdorff. Now $\{\Psi(p) \mid p \in P\}$ is a *π-basis* for the topology on $\mathcal{S}(P)$, *i.e.*, any nonempty open set contains some $\Psi(p)$. This is true because $U \in \Psi(p_1) \cap \ldots \cap \Psi(p_n) \implies \{p_1, \ldots, p_n\} \subset U \implies$ there exists $p \in P$ with $p \leq p_i$, for $i = 1, \ldots, n \implies \Psi(p) \subset \Psi(p_1) \cap \ldots \cap \Psi(p_n)$. Then since P is c.c.c., so is $\mathcal{S}(P)$: if $\{\Psi(p) \mid p \in A\}$ is pairwise disjoint for some $A \subset P$, then A is a pairwise incompatible subset of P, so $|A| \leq \aleph_0$. Then since $\mathcal{S}(P)$ is c.c.c. and dense in $\tilde{\mathcal{S}}(P)$, the latter is c.c.c.: if $\mathcal{U}$ is a pairwise disjoint collection of nonempty open sets in $\tilde{\mathcal{S}}(P)$, then $\{U \cap \mathcal{S}(P) \mid U \in \mathcal{U}\}$ is a pairwise disjoint collection of nonempty open sets in $\mathcal{S}(P)$, so $|\{U \cap \mathcal{S}(P) \mid U \in \mathcal{U}\}| \leq \aleph_0$. As well, if $U, V \in \mathcal{U}$ are distinct, so are $U \cap \mathcal{S}(P)$ and $V \cap \mathcal{S}(P)$; thus $|\mathcal{U}| \leq |\{U \cap \mathcal{S}(P) \mid U \in \mathcal{U}\}| \leq \aleph_0$.

Now for each $p \in P$, let $\widetilde{\Psi}(p) = \mathrm{Int}_{\widetilde{\mathcal{S}}(P)} \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)}(\Psi(p))$. Since $\Psi(p)$ is closed in $\mathcal{S}(P)$, $\Psi(p) = \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)}(\Psi(p)) \cap \mathcal{S}(P)$. Then

$$\begin{aligned}\widetilde{\Psi}(p) \cap \mathcal{S}(P) &= \mathrm{Int}_{\widetilde{\mathcal{S}}(P)} \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)}(\Psi(p)) \cap \mathcal{S}(P) \\ &\subset \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)}(\Psi(p)) \cap \mathcal{S}(P) \\ &= \Psi(p).\end{aligned}$$

Also each $\widetilde{\Psi}(p)$ is nonempty; if not, then $\widetilde{\mathcal{S}}(P) \setminus \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)} \Psi(p)$ is open and dense in $\widetilde{\mathcal{S}}(P)$, so $\mathcal{S}(P) \cap (\widetilde{\mathcal{S}}(P) \setminus \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)} \Psi(p))$ is dense in $\widetilde{\mathcal{S}}(P)$, so it is dense in $\mathcal{S}(P)$ in its topology; this is a contradiction, since $\Psi(p)$ is a nonempty open set in $\mathcal{S}(P)$ not meeting $\mathcal{S}(P) \cap (\widetilde{\mathcal{S}}(P) \setminus \mathrm{Cl}_{\widetilde{\mathcal{S}}(P)} \Psi(p))$.

Now we have all the facts we need about $\widetilde{\mathcal{S}}(P)$ to prove the lemma. Let $Q \in [P]^\lambda$; then $|\{\widetilde{\Psi}(q) \mid q \in Q\}| \leq \lambda$. Since $\widetilde{\Psi}$ might not be 1-1, we cannot with any certainty replace "$\leq$" with "$=$", but it is possible to get around this obstacle. Define a relation $\sim$ on Q by $p \sim q \overset{\text{def}}{\iff} \widetilde{\Psi}(p) = \widetilde{\Psi}(q)$. It is easy to see that this is an equivalence relation, so it partitions Q into cosets.

Now if $|Q/\sim| = \lambda$, then easily $|\{\widetilde{\Psi}(q) \mid q \in Q\}| = \lambda$; thus $\{\widetilde{\Psi}(q) \mid q \in Q\}$ is a cardinality-λ collection of open sets in a compact, Hausdorff c.c.c. space, so by 2.1(b), there exists a cardinality-λ subset $Q' \subset Q$ with $\bigcap\{\widetilde{\Psi}(q) \mid q \in Q'\} \neq \emptyset$. Now Q' is centred: if $\{q_1, \ldots, q_n\} \subset Q'$, then there exists $U \in \mathcal{S}(P) \cap (\bigcap_{i=1}^n \widetilde{\Psi}(q_i)) \subset \bigcap_{i=1}^n \Psi(q_i)$; thus each $q_i \in U$, so $\{q_1, \ldots, q_n\}$ has a lower bound. Thus Q' is a cardinality-λ subset of Q which is centred in $(P, \leq)$.

On the other hand, Q is equal to the union of its cosets, so if $|Q/\sim| < \lambda$, then there are less than λ cosets, so by regularity of λ, some coset Q' has cardinality λ. Now $\widetilde{\Psi}$ is constant on Q', so let $U \in \widetilde{\Psi}(q) \cap \mathcal{S}(P) \subset \Psi(q)$ for any $q \in Q'$. Then $Q' \subset U$ so Q' is a cardinality-λ subset of Q which is centred in $(P, \leq)$. ■

The construction $\widetilde{\mathcal{S}}(P)$ can also be used to show[12] that the topological version of MA_κ implies the partial order version, as $\mathcal{S}(\text{r.o.}\,P)$ was used in the proof of 1.13. Note, however, that the construction $\mathcal{S}(\text{r.o.}\,P)$ does not appear to be adequate to prove 3.8. If we had used the same argument with the construction $\mathcal{S}(\text{r.o.}\,P)$, we would have gotten to the point that Q has a cardinality-λ subset Q' with $\bigcap\{\,\Psi(e(p)) \mid q \in Q'\,\} \neq \emptyset$. Then there is an ultrafilter U containing $e[Q'] \stackrel{\text{def}}{=} \{\,e(q) \mid q \in Q'\,\}$, so $e[Q']$ is pairwise compatible, whence Q' is as well. Since U contains $e[Q']$, any finite subset $\{\,q_1, \ldots, q_n\,\} \subset Q'$ satisfies $\prod_{i=1}^{n} e(q_i) \neq 0$; by the density in r.o. P of $e[P]$, there exists $p \in P$ with $e(p) \leq \prod_{i=1}^{n} e(q_i)$; then $e(p) \leq e(q_i)$ for $i = 1, \ldots, n$, but we cannot conclude that $p \leq q_i$ for $i = 1, \ldots, n$ unless we know that $e(p) \leq e(q)$ implies $p \leq q$, which holds only for ***separative*** partial orders, as noted earlier. Thus the use of $\mathcal{S}(\text{r.o.}\,P)$ will only prove a weaker version of 3.8, namely, 3.8 with "centred in $(P, \leq)$" replaced by "pairwise compatible in P". However, in the ensuing discussions it will be sufficient to use the following statement: if $(P, \leq)$ is an uncountable c.c.c. partially ordered set, then there exists an uncountable subset $G \subset P$ which is pairwise compatible in $(P, \leq)$.

It is no coincidence that 3.8 resembles 2.1(b); in fact, it is not difficult to see that for any regular uncountable cardinal λ, the following two statements are equivalent:

(a) if $(P, \leq)$ is a c.c.c. partially ordered set, then any cardinality-λ subset Q of P has a cardinality-λ subset Q' which is centred in P,

(b) if X is a compact Hausdorff c.c.c. space, then any cardinality-λ family of open sets in X has a cardinality-λ subfamily with nonempty intersection.

The proof of (b) $\Longrightarrow$ (a) is the argument used in 3.8. For (a) $\Longrightarrow$ (b), let $\mathcal{G}$ be a cardinality-λ family of open sets in a compact, Hausdorff c.c.c. space (X, τ); then $(\tau \setminus \emptyset, \subset)$ is a c.c.c. partially ordered set. By regularity of X, for each $G \in \mathcal{G}$ let U_G be an open set with $\emptyset \neq U_G \subset \overline{U_G} \subset G$; let $\widetilde{\mathcal{G}} \stackrel{\text{def}}{=} \{\, U_G \mid G \in \mathcal{G}\,\}$. If $|\widetilde{\mathcal{G}}| = \lambda$, then by (a) there exists $\mathcal{G}' \in [\mathcal{G}]^{\lambda}$ such that $\{\, U_G \mid G \in \mathcal{G}'\,\}$ has the finite intersection property. Note that $\mathcal{G} = \bigcup_{U_G \in \widetilde{\mathcal{G}}} \mathcal{G}_{U_G}$ where $\mathcal{G}_{U_G} \stackrel{\text{def}}{=} \{\, G' \in \mathcal{G} \mid \overline{U_G} \subset G'\,\}$; thus if $|\widetilde{\mathcal{G}}| < \lambda$, then by regularity of λ there exists some $\mathcal{G}_{U_G}$ with cardinality λ. In this case, take $\mathcal{G}' \stackrel{\text{def}}{=} \mathcal{G}_{U_G}$; now in either case, $\{\, U_G \mid G \in \mathcal{G}'\,\}$ has the finite intersection property, so (by compactness of X) $\emptyset \neq \bigcap \{\, \overline{U_G} \mid G \in \mathcal{G}'\,\} \subset \bigcap \mathcal{G}'$.

Now we wish to discuss *strong* S- and L-spaces.

3.9 Definition. A topological space (X, τ) is a *strong* S- (respectively L-) space iff for each nonzero integer n, the product X^n is an S- (respectively L-) space. ∎

3.10 Theorem.[8] *$MA_{\aleph_1}$ implies that there are no strong L-spaces.*

Proof. Suppose there exists a strong L-space L; then since $MA_{\aleph_1}$ implies $\neg$CH, it has a subspace $X = \{ x_\alpha \mid \alpha < \omega_1 \}$ which is zero-dimensional and left separated. Also X^n is a subspace of L^n for each $n \in \omega \setminus \{0\}$, so X^n is hereditarily Lindelöf. Then for each $\alpha < \omega_1$, let $U_\alpha \subset \{ x_\beta \mid \beta \in \omega \setminus \alpha \}$ be a clopen neighbourhood of x_α. Let $P = \{ p \in [\omega_1]^{<\omega} \mid \alpha, \beta \in p,\ \alpha < \beta \implies x_\beta \notin U_\alpha \}$, and partially order P by defining $p \leq q \iff p \supset q$. Clearly $|P| \geq \aleph_1$, so if P is c.c.c, then $MA_{\aleph_1}$ implies there exists a cardinality-$\aleph_1$ subset $G \subset P$ which is centred in P. Now we have $|\bigcup G| > \aleph_0$, since if $|\bigcup G| \leq \aleph_0$ then there exists $\alpha < \omega_1$ such that $p \in G \implies \max p < \alpha$; thus G is a collection of finite subsets of the countable set α, but $|[\alpha]^{<\omega}| = \aleph_0$, contradicting the uncountability of G. Thus $Y \overset{\text{def}}{=} \{ x_\alpha \mid \alpha \in p \text{ for some } p \in G \}$ is an uncountable subset of X. Now if $\alpha, \beta, \gamma \in \bigcup G$ and $\alpha < \beta < \gamma$, clearly $x_\alpha \notin U_\beta$, and by the fact that G is pairwise compatible, $x_\gamma \notin U_\beta$. Thus for each $x_\beta \in Y$, $U_\beta \cap Y = \{ x_\beta \}$, so Y is an uncountable discrete subspace of X, contradicting hereditary Lindelöfness of X.

Now all we need show is that $(P, \leq)$ is c.c.c. Let $\{ p_\alpha \mid \alpha < \omega_1 \}$ be uncountable and pairwise incompatible. In 3.11, it will be shown that it can be assumed without loss of generality that each p_α has the same cardinality n, and $\max p_\alpha < \min p_\beta$ for $\alpha < \beta$.

Let each $p_\alpha = \{ p_\alpha(1), \ldots, p_\alpha(n) \}$, with $p_\alpha(1) < \cdots < p_\alpha(n)$. Let $V_\alpha = \bigcup_{i=1}^n U_{p_\alpha(i)}$. Then if $\alpha < \beta$, p_α and p_β are incompatible, so there exists $i \in$

$\{1,\ldots,n\}$ with $x_{p_\beta(i)} \in V_\alpha$. Let $\mathbf{x}_\alpha = (x_{p_\alpha(1)},\ldots,x_{p_\alpha(n)}) \in X^n$ and

$$W_\beta = \{\mathbf{x} \in X^n \mid \exists i \in \{1,\ldots,n\} \text{ with } x_i \in V_\beta\} = \bigcup_{i=1}^{n} \pi_i^{-1}(V_\beta),$$

where π_i is the ith projection function on X^n, and x_i is the ith component of $\mathbf{x}$. Thus each W_β is clopen in X^n.

If $\alpha \leq \beta$, then by the previous paragraph, $\mathbf{x}_\beta \in W_\alpha$; if $\beta < \alpha$, then clearly $\mathbf{x}_\beta \notin W_\alpha$. Thus $W_\alpha \cap \{\mathbf{x}_\beta \mid \beta < \omega_1\} = \{\mathbf{x}_\beta \mid \beta \in \omega \setminus \alpha\}$, so each final segment of $\{\mathbf{x}_\beta \mid \beta < \omega_1\}$ is clopen, whence each initial segment is as well; thus $\{\mathbf{x}_\beta \mid \beta < \omega_1\}$ is an uncountable right separated subspace of X^n, contradicting hereditary Lindelöfness of X^n and of L^n.■

Note that by using the dual of the above argument, *i.e.* obtaining a zero-dimensional right separated S-space $X = \{x_\alpha \mid \alpha < \omega_1\}$ contained in a strong S-space, choosing a clopen neighbourhood $U_\alpha \subset \{x_\beta \mid \beta < \alpha+1\}$ of x_α, setting $P = \{p \in [\omega_1]^{<\omega} \mid \alpha,\beta \in p,\ \alpha < \beta \implies x_\alpha \notin U_\beta\}$ with $p \leq q \overset{\text{def}}{\iff} p \supset q$, we can show that $\mathrm{MA}_{\aleph_1}$ also destroys strong S-spaces. (The only difference between the two proofs would be that to destroy strong S-spaces, we do not explicitly use $\neg$CH to construct X). However, this dual argument is not really necessary, since it is known that there exists a strong S-space iff there exists a strong L-space[8]. Thus, in any case, *$MA_{\aleph_1}$ implies that there no strong S- or L-spaces.*

Now here is the lemma as promised in the proof of 3.10.

3.11 Lemma. *Let A be an uncountable collection of finite subsets of ω_1. Then there exists an uncountable subcollection $A' \subset A$, an integer n, and a finite set $b \subset \omega_1$ such that A' can be written in the form $A' = \{ a_\alpha \mid \alpha < \omega_1 \}$ where*

i) $\forall \alpha < \omega_1,\ |a_\alpha| = n,$

ii) $\alpha < \beta \implies \max(a_\alpha \setminus b) < \min(a_\beta \setminus b)$ *and* $a_\alpha \cap a_\beta = b.$

Proof. By a counting argument, for some integer $n < \omega$ there must be an infinite subcollection $A_1 \subset A$ with $|a| = n$ for each $a \in A_1$. Now we proceed by induction. If $n = 1$, then set $A' = A_1$ and $b = \emptyset$. If we let $a \leq c$ iff $\alpha < \beta$ where $a = \{ \alpha \}$ and $c = \{ \beta \}$, then clearly we can well-order A' such that the ordering "$\leq$" is preserved. Thus $A' = \{ a_\alpha \mid \alpha < \omega_1 \}$ where $\alpha < \beta$ implies $\max(a_\alpha \setminus b) < \min(a_\beta \setminus b)$ and $a_\alpha \cap a_\beta = \emptyset = b$.

Now suppose that whenever A_1 is an uncountable subcollection of $[\omega_1]^{n-1}$, there exists an uncountable $A' \subset A$ and finite $b \subset \omega_1$ such that for $\alpha < \beta$, $a_\alpha \cap a_\beta = b$ and $\max(a_\alpha \setminus b) < \min(a_\beta \setminus b)$. Let A_1 be an uncountable subset of $[\omega_1]^n$. If, for some $\alpha < \omega_1$, the set $A_\alpha \stackrel{\text{def}}{=} \{ a \in A_1 \mid \alpha \in a \}$ is uncountable, then by the inductive hypothesis above, there exists an uncountable set $A' \subset \{ a \setminus \{ \alpha \} \mid a \in A_\alpha \}$ and a finite set $b \subset \omega_1$ with $A' = \{ a_\beta \mid \beta < \omega_1 \}$ where $\beta < \gamma$ implies $a_\beta \cap a_\gamma = b$ and $\max(a_\beta \setminus b) < \min(a_\gamma \setminus b)$. Then set $b' = b \cup \{ \alpha \}$, $A'' = \{ a \cup \{ \alpha \} \mid a \in A' \}$, and give A'' the well-ordering that A' has. Now $\beta < \gamma$ implies $a_\beta \cap a_\gamma = b'$ and $\max(a_\beta \setminus b') < \min(a_\gamma \setminus b')$. On the other hand, if each A_α is countable, then we shall construct (by transfinite induction) a sequence $\{ a_\alpha \mid \alpha < \omega_1 \} \subset A_1$ of pairwise disjoint sets where $\alpha < \beta$ implies

$\max a_\alpha < \min a_\beta$; having done so, we shall have completed the proof, since this sequence satisfies the properties of A' in the conclusion of the lemma, where $b \stackrel{\text{def}}{=} \emptyset$.

Suppose, for some $\beta < \omega_1$, we have a sequence $\{ a_\alpha \mid \alpha < \beta \}$ of pairwise disjoint elements of A_1 with $\max a_\gamma < \min a_\sigma$ for $\gamma < \sigma < \beta$. Now if we define $\alpha^* = \sup\{ \xi \mid \xi \in a \}$, clearly $\alpha^* < \omega_1$. Since each $A_\xi = \{ a \in A_1 \mid \xi \in a \}$ is countable, it is the case that $\bigcup_{\xi \in \alpha^*} A_\xi$ is countable; thus there exists $a_\beta \in A_1 \setminus \bigcup_{\xi \in \alpha^*} A_\xi$; now the sequence $\{ a_\xi \mid \xi \leq \beta \}$ is pairwise disjoint, and $\max a_\gamma < \min a_\sigma$ for $\gamma < \sigma \leq \beta$. Thus by transfinite induction, we have our sequence $\{ a_\alpha \mid \alpha < \omega_1 \}$, as desired. ∎

In the proof of 3.10, we assumed that we had an uncountable pairwise incompatible subset $A \subset P$; by 3.11 there exists $A' = \{ a'_\alpha \mid \alpha < \omega_1 \} \subset A$ and a finite set $b \subset \omega_1$ such that $a'_\alpha \cap a'_\beta = b$ and $\max(a'_\alpha \setminus b) < \min(a'_\beta \setminus b)$ for $\alpha < \beta < \omega_1$. Now if for some $\alpha < \omega_1$ the sets $a'_\alpha \setminus b$ and $a'_\beta \setminus b$ are compatible, then for any $\gamma, \sigma \in (a'_\alpha \setminus b) \cup (a'_\beta \setminus b)$, $\gamma < \sigma$ implies $x_\sigma \notin U_\gamma$; however, since $b = a'_\alpha \cap a'_\beta$, the same statement holds for $\gamma, \sigma \in a'_\alpha \cap a'_\beta$, which is a contradiction, since $a'_\alpha, a'_\beta \in A' \subset A$ and A is pairwise incompatible. Thus, if $a_\alpha \stackrel{\text{def}}{=} a'_\alpha \setminus b$ for each $\alpha < \omega_1$, then $\{ a_\alpha \mid \alpha < \omega_1 \}$ is pairwise incompatible and each a_α has the same (finite) cardinality. Thus we have that *if $(P, \leq)$ is as defined in 3.10, then there exists an uncountable pairwise incompatible subset of P iff there exists one of the form $\{ a_\alpha \mid \alpha < \omega_1 \}$, where each a_α has the same cardinality and*

$\max a_\alpha < \min a_\beta$ *for* $\alpha < \beta < \omega_1$. (The same argument holds for the partially ordered set $(P, \leq)$ mentioned in the remark after 3.10 as well).

The partially ordered set of 3.10 can also be used to show that $MA_{\aleph_1}$ destroys first-countable L-spaces. Before we embark on the proof, let us first recall that a point a is called a ***condensation point*** of a set A iff for every neighbourhood U of a, $U \cap A$ is uncountable.

3.12 Theorem.[1] *$MA_{\aleph_1}$ implies that there are no first countable L-spaces.*

Proof. Assume there exists a first-countable L-space; then there exists one which is zero-dimensional and left separated in type ω_1. By identifying x_α with α, we can assume there exists a topology τ on $\omega_1 = \{ \alpha \mid \alpha < \omega_1 \}$ making it a zero-dimensional first-countable L-space with each set $\{ \beta \mid \beta \in \omega \setminus \alpha \}$ open.

As in the proof of 3.10, for each $\alpha < \omega_1$, let $U_\alpha \subset \{ \beta \mid \beta \in \omega \setminus \alpha \}$ be a clopen neighbourhood of α. Let

$$P = \{ p \in [\omega_1]^{<\omega} \mid \text{ if } \alpha, \beta \in p,\ \alpha < \beta, \text{ then } \beta \notin U_\alpha \},$$

and partially order P by defining $p \leq q$ whenever $p \supset q$.

We will show that $(P, \leq)$ is c.c.c. Then $MA_{\aleph_1}$ implies that there exists a cardinality-$\aleph_1$ subset G of P which is centred in P, and as in the proof of 3.10, we have that $|\bigcup G| > \aleph_0$, so $\bigcup G = \{ \alpha \mid \alpha \in p \text{ for some } p \in G \}$ is an uncountable discrete subspace of (ω_1, τ), a contradiction.

Now we show that $(P, \leq)$ is c.c.c. Suppose $\{ p_\alpha \mid \alpha < \omega_1 \}$ is an uncountable subset of P; we shall show that it is not pairwise compatible. We can assume, by use of 3.11, that each p_α has the same finite cardinality n, and $\max p_\alpha < \min p_\beta$ for $\alpha < \beta$. Let $p_\alpha = \{ p_\alpha(1), \ldots, p_\alpha(n) \}$ where $p_\alpha(1) < \cdots < p_\alpha(n)$. Set $V_\alpha = \bigcup_{i=1}^n U_{p_\alpha(i)}$; then each V_α is clopen. Also, notice that with this notation, if $\alpha < \beta$, then p_α and p_β are compatible iff for each $i \in \{1, \ldots, n\}$, $p_\alpha(i) \notin V_\beta$, *i.e.* iff $p_\alpha \cap V_\beta = \emptyset$.

Now we claim that for any uncountable subset A of (ω_1, τ), there exists an open set W such that whenever $A' \subset \omega_1$ is uncountable, the sets $\{ \alpha \in A' \mid W \cap V_\alpha = \emptyset \}$ and $W \cap A$ are uncountable. If for each $\alpha \in A$ there exists an open neighbourhood A_α of α such that $A_\alpha \cap A$ is countable, then $\{ A_\alpha \mid \alpha \in A \}$ is an open cover of A with no countable subcover, contradicting hereditary Lindelöfness of (ω_1, τ). So there exists $\alpha^* \in A$ such that for any open neighbourhood U of α^*, $|U \cap A| = \aleph_1$, *i.e.* α^* is a condensation point of A. Now $A' \subset \omega_1$ is uncountable, so $A' \cap \{ \alpha \mid \alpha^* < \alpha < \omega_1 \} = A' \setminus (\alpha^* + 1)$ is uncountable, since it is A' minus an at most countable set. Thus $\{ V_\alpha \mid \alpha^* < \alpha < \omega_1$ and $\alpha \in A' \}$ is a cardinality-$\aleph_1$ family of closed sets which do not contain α^*; since (ω_1, τ) is first-countable, there exists a countable local base $\{ W_n \mid n < \omega \}$ at α^*, and for each $\alpha \in A' \cap \{ \alpha \mid \alpha^* < \alpha < \omega_1 \}$, there exists $n < \omega$ with $\alpha^* \in W_n \subset \omega_1 \setminus V_\alpha$. Thus for some fixed $n < \omega$, $W_n \subset \omega_1 \setminus V_\alpha$ for an uncountable collection of α's in $A' \cap \{ \alpha \mid \alpha^* < \alpha < \omega_1 \}$. Thus if $W \stackrel{\text{def}}{=} W_n$, then the set $\{ \alpha \in A' \mid W \cap V_\alpha = \emptyset \}$ is uncountable, and $W \cap A$ is as well, since α^* is a condensation point of A; thus the claim is proved.

Now since $\{ p_\alpha(1) \mid \alpha < \omega_1 \}$ is uncountable, there exists an open set W_1 such that the sets

$$A_1 \stackrel{\text{def}}{=} \{ \alpha < \omega_1 \mid W_1 \cap V_\alpha = \emptyset \}$$

$$B_1' \stackrel{\text{def}}{=} W_1 \cap \{ p_\alpha(1) \mid \alpha < \omega_1 \}$$

$$B_1 \stackrel{\text{def}}{=} \{ \alpha \mid p_\alpha(1) \in B_1' \}$$

$$= \{ \alpha \mid p_\alpha(1) \in W_1 \}$$

are uncountable. Since $\{ p_\alpha(2) \mid \alpha \in B_1 \}$ and A_1 are uncountable, there exists an open set W_2 such that the sets

$$A_2 \stackrel{\text{def}}{=} \{ \alpha \in A_1 \mid W_2 \cap V_\alpha = \emptyset \}$$

$$B_2' \stackrel{\text{def}}{=} W_2 \cap \{ p_\alpha(2) \mid \alpha \in B_1 \}$$

$$B_2 \stackrel{\text{def}}{=} \{ \alpha \mid p_\alpha(2) \in B_2' \}$$

$$= B_1 \cap \{ \alpha \mid p_\alpha(2) \in W_2 \}$$

are uncountable. Continuing, we obtain uncountable sets $A_1 \supset \cdots \supset A_n$ and $B_1 \supset \cdots \supset B_n$ such that whenever $l \in \{ 1, \ldots, n \}$, $\alpha \in A_l$ and $\beta \in B_l$, we have $p_\beta(i) \notin V_\alpha$ for $i = 1, \ldots, l$. Then for $\alpha \in A_n$, $\beta \in B_n$ with $\alpha < \beta$ (such ordinals α, β exist since B_n is uncountable) $p_\alpha \cap V_\beta = \emptyset$, whence p_α and p_β are compatible. ■

In this proof, we used certain properties of L-spaces which make it difficult to dualize the proof to show that $\text{MA}_{\aleph_1}$ destroys first-countable S-spaces; in fact, it is known that MA+¬CH+"there exists a first-countable S-space" is relatively consistent with ZFC[1]; the proof involves forcing, and so it is not included here.

The next result is interesting in that it uses $MA_{\aleph_1}$ in its topological form rather than the version involving c.c.c partially ordered sets.

3.13 Theorem.[8] *$MA_{\aleph_1}$ implies there are no compact L-spaces.*

Proof. Suppose that there exists a compact L-space L. Then L has a left separated subspace $\{x_\alpha \mid \alpha < \omega_1\}$. Let $X \stackrel{\text{def}}{=} \mathrm{Cl}_L\{x_\alpha \mid \alpha < \omega_1\}$; then X is also a compact L-space. For each $\alpha < \omega_1$, let $X_\alpha = \mathrm{Cl}_X\{x_\beta \mid \beta < \alpha\}$; let $U_\alpha = \mathrm{Int}_X(X_\alpha)$. X cannot have an uncountable strictly increasing sequence of open sets (equivalently, X cannot have an uncountable strictly decreasing sequence of closed sets), or else there would exist an uncountable right separated subspace of X, contradicting hereditary Lindelöfness. Thus there exists $\alpha^* < \omega_1$ such that $\alpha \geq \alpha^* \implies U_\alpha = U_{\alpha^*}$. Now $X \neq U_{\alpha^*}$, otherwise $\{x_\alpha \mid \alpha < \alpha^*\}$ would be dense in X, whence in $\{x_\alpha \mid \alpha < \omega_1\}$, which is impossible since $\{x_\alpha \mid \alpha \in \omega_1 \setminus \alpha^*\}$ is a nonempty open set missing $\{x_\alpha \mid \alpha < \alpha^*\}$. Setting $Y = X \setminus U_{\alpha^*}$, we have $U_\alpha \cap Y = \emptyset$ for each $\alpha < \omega_1$; we shall show that for each $\alpha \in \omega_1 \setminus \alpha^*$, $X_\alpha \cap Y$ is nowhere dense in Y. Since each X_α is closed in X, all we need show is that for $\alpha \in \omega_1 \setminus \alpha^*$, $Y \setminus X_\alpha$ is dense in Y, *i.e.* any nonempty open (in the relative topology on Y) set $U \subset Y$ meets $Y \setminus X_\alpha$. Let U be such a set; then $U = \tilde{U} \cap Y$ for some set $\tilde{U}$ open in X. Now $\alpha \geq \alpha^* \implies U_{\alpha^*} = U_\alpha \subset X_\alpha \implies X \setminus X_\alpha \subset X \setminus U_{\alpha^*} = Y$. Thus

$$U = \tilde{U} \cap Y = \tilde{U} \cap (X \setminus U_{\alpha^*}) \supset \tilde{U} \cap (X \setminus X_\alpha) \neq \emptyset.$$

(If $\tilde{U}\cap(X\setminus X_\alpha)=\emptyset$, then $\tilde{U}\subset \mathrm{Int}_X(X_\alpha)=U_\alpha$, so $U=\tilde{U}\cap Y=\tilde{U}\cap(X\setminus U_{\alpha^*})=\emptyset$). Then $X_\alpha\cap Y$ is nowhere dense in Y for $\alpha\in\omega_1\setminus\alpha^*$, as claimed.

Let $x\in X=\mathrm{Cl}_X\{x_\alpha\mid\alpha<\omega_1\}$; then for any open neighbourhood U of x, let $\alpha_U=\min\{\alpha<\omega_1\mid x_\alpha\in U\}$. If there exists $\alpha'<\omega_1$ such that for any open neighbourhood V of x, $\alpha_V\le\alpha'$, then $x\in\mathrm{Cl}_X\{x_\beta\mid\beta<\alpha'+1\}=X_{\alpha'+1}$. If not, *i.e.* for each $\alpha<\omega_1$ there exists an open neighbourhood U of x with $\alpha_U>\alpha$, then by transfinite induction we can construct a sequence $\{V_\alpha\mid\alpha<\omega_1\}$ of open neighbourhoods of x such that $\{(\bigcup_{\alpha<\beta}V_\alpha)\cap\{x_\alpha\mid\alpha<\omega_1\}\mid\beta<\omega_1\}$ is an uncountable strictly increasing sequence of open (in $\{x_\alpha\mid\alpha<\omega_1\}$) sets, contradicting hereditary Lindelöfness. Thus $X=\bigcup_{\alpha<\omega_1}X_\alpha$; since $\{X_\alpha\mid\alpha<\omega_1\}$ is an increasing sequence of sets, it is also true that $X=\bigcup_{\alpha\in\omega_1\setminus\alpha^*}X_\alpha$, whence $Y=\bigcup_{\alpha<\omega_1\setminus\alpha^*}(X_\alpha\cap Y)$.

Now Y is a closed subset of X, so it is compact and Hausdorff. It is also c.c.c., since if $\{V_\alpha\mid\alpha<\omega_1\}$ is a pairwise disjoint collection of nonempty open sets in Y, then $\{\bigcup_{\alpha<\beta}V_\alpha\mid\beta<\omega_1\}$ is an uncountable strictly increasing sequence of open sets in Y. Thus Y is a compact T_2 c.c.c. space that is the union of $\aleph_1$ nowhere dense sets, contradicting $MA_{\aleph_1}$. ■

Attempts to use a similar argument to destroy compact S-spaces do not appear to work. For example, if X is the compact closure of a right separated space $\{x_\alpha\mid\alpha<\omega_1\}$, we can define X_α and U_α as in 3.13; since there cannot be an uncountable strictly increasing sequence of closed sets, there exists $\alpha^*<\omega_1$ such that $X_\alpha=X_{\alpha^*}$, whence $U_\alpha=U_{\alpha^*}$, for $\alpha\ge\alpha^*$. The problem, however, is that since X is hereditarily separable, one of the sets $\{x_\beta \mid \beta < \alpha\}$ is dense in $\{x_\alpha \mid \alpha < \omega_1\}$, which is dense in X; thus, in fact, $X_{\alpha^*} = X$, whence $U_{\alpha^*} = X$, so the subspace we *want* to consider, $X \setminus U_{\alpha^*}$, is empty, and so of no use to us. On the other hand, suppose we set $X_\alpha = \mathrm{Cl}_X\{x_\beta \mid \beta \in \omega_1 \setminus \alpha\}$ and $U_\alpha = \mathrm{Int}_X(X_\alpha)$. Now there can be no uncountable strictly decreasing sequence of open sets, so there exists $\alpha^* < \omega_1$ such that for $\alpha \geq \alpha^*$, $U_\alpha = U_{\alpha^*}$. If we define $Y = X \setminus U_{\alpha^*}$, then we can show, using the same argument as in 3.13, that $X_\alpha \cap Y$ is nowhere dense in Y for $\alpha \in \omega_1 \setminus \alpha^*$; the problem is that $Y \neq \bigcup_{\alpha<\omega_1\setminus\alpha^*}(X_\alpha \cap Y)$: $\alpha^* \neq 0$ (otherwise $X = X_{\alpha^*}$ is nowhere dense in itself) so $x_0 \in Y = X \setminus U_{\alpha^*}$ but $x_0 \notin \bigcup_{\alpha<\omega_1\setminus\alpha^*}(X_\alpha \cap Y)$. In fact, this last argument shows that we cannot "cut off" the space, *i.e.* we cannot consider $\tilde{X} \overset{\text{def}}{=} \mathrm{Cl}_X\{x_\alpha \mid \alpha \in \omega_1 \setminus \alpha^*\}$ and try to show that $\tilde{Y} = \bigcup_{\alpha<\omega_1\setminus\alpha^{**}}(X_\alpha \cap Y)$ for some $\alpha^{**} \in \omega_1 \setminus \alpha^*$ where each X_α is nowhere dense in $\tilde{Y}$, for $\alpha \in \omega_1 \setminus \alpha^{**}$.

However, despite the apparent lack of symmetry, a similar statement to 3.13 does in fact hold for S-spaces. In order to prove this, we will need to use the concept of a *cofinally centred* sequence, and we need a lemma about the existence of such a sequence.

3.14 Definition. A collection of sets $\{B_\alpha \mid \alpha < \omega_1\}$ is called *cofinally centred* on a set A iff for each uncountable subset $C \subset A$, there is an $\alpha < \omega_1$ such that the collection $\{B_\alpha \cap C \mid \beta \in \omega_1 \setminus \alpha\}$ is *centred* with respect to $\in$, *i.e.* for any finite set $b_\alpha \subset \omega_1 \setminus \alpha$, $\bigcap_{\beta\in b_\alpha}(B_\beta \cap C) \neq \emptyset$.■

The following gives a condition under which we can assume the existence of a cofinally centred sequence.

3.15 Lemma.[8] *For each $\alpha < \omega_1$, let U_α be an at most countable set, let $\mathcal{U} \supset \{ U_\alpha \mid \alpha < \omega_1 \}$ be a collection closed under finite unions, with each element of $\mathcal{U}$ being a countable set, and let $P = \{ p \in [\omega_1]^{<\omega} \mid \alpha, \beta \in p, \alpha < \beta \implies \alpha \notin U_\beta \}$, ordered by $p \leq q \overset{\text{def}}{\iff} p \supset q$. Then if $(P, \leq)$ is not c.c.c., there exists an uncountable set $A \subset \omega_1$ and a collection $\{ B_\alpha \mid \alpha < \omega_1 \} \subset \mathcal{U}$ cofinally centred on A.*

Proof. Let $\{ a_\alpha \mid \alpha < \omega_1 \}$ be an uncountable pairwise incompatible subset of P; by 3.11 and the remarks after it, we may assume without loss of generality that each a_α has the same (finite) cardinality n and $\max a_\alpha < \min a_\beta$ for $\alpha < \beta$. For each $\alpha < \omega_1$, let $B_\alpha = \bigcup_{\beta \in a_\alpha} U_\beta$; by pairwise incompatibility, $a_\alpha \cap B_\beta \neq \emptyset$ whenever $\alpha < \beta$. Thus $\{ (a_\alpha, B_\alpha) \mid \alpha < \omega_1 \}$ is a sequence satisfying:

(1) $a_\alpha \in [\omega_1]^n$ for each $\alpha < \omega_1$;

(2) $B_\alpha \in [\omega_1]^{\leq \omega_0} \cap \mathcal{U}$ for each $\alpha < \omega_1$;

(3) $\alpha < \beta < \omega_1 \implies \max a_\alpha < \min a_\beta$ and $a_\alpha \cap B_\beta \neq \emptyset$.

Now let k be the smallest integer such that there exists a sequence $\{ (a_\alpha, B_\alpha) \mid \alpha < \omega_1 \}$ satisfying (1), (2) and (3), with $n = k$ in (1). Let $A = \{ \min a_\alpha \mid \alpha < \omega_1 \}$. We shall show that $\{ B_\alpha \mid \alpha < \omega_1 \}$ is cofinally centred on A; assume it is not. Then there exists a set $C \subset [A]^{\omega_1}$ such that for each $\alpha < \omega_1$, $\{ B_\beta \cap C \mid \beta \in \omega_1 \setminus \alpha \}$ is *not* centred, *i.e.* for each $\alpha < \omega_1$ there exists a finite set $b_\alpha \subset \omega_1 \setminus \alpha$ with $C \cap (\bigcap_{\beta \in b_\alpha} B_\beta) = \emptyset$.

Let $C = \{ \delta_\alpha \mid \alpha < \omega_1 \}$ be the increasing enumeration of C, i.e. $\alpha < \beta \implies \delta_\alpha < \delta_\beta$. Each δ_α is the minimum of a unique a_β; let $a^*_\alpha = a_\beta$ iff $\delta_\alpha = \min a_\beta$. Now $\alpha < \beta \implies \delta_\alpha < \delta_\beta = \min a^*_\beta \implies \max a^*_\alpha < \min a^*_\beta$. Also if $a^*_\alpha = a_\zeta$ and $a^*_\beta = a_\xi$, then $\alpha < \beta \implies \max a_\zeta < \min a_\xi \implies \zeta < \xi$.

Now since $\{ b_\alpha \mid \alpha < \omega_1 \}$ is uncountable, for each $\alpha < \omega_1$ there exists $\beta_\alpha < \omega_1$ with $\zeta < \beta_\alpha \leq \min b_{\beta_\alpha}$ where $a_\zeta = a^*_\alpha$; by setting $\widetilde{b_\alpha} = b_{\beta_\alpha}$ we thus have $\zeta < \min \widetilde{b_\alpha}$ (where $a_\zeta = a^*_\alpha$) for each $\alpha < \omega_1$. For each $\alpha < \omega_1$, let $a'_\alpha = a^*_\alpha \setminus \{ \delta_\alpha \}$, $B'_\alpha = \bigcup_{\beta \in \widetilde{b_\alpha}} B_\beta$. (Now $B'_\alpha \in [\omega_1]^{\leq \omega_0} \cap \mathcal{U}$ for each $\alpha < \omega_1$). If $\alpha < \beta$, then since $C \cap (\bigcap_{\beta \in \widetilde{b_\beta}} B_\beta) = \emptyset$, there exists $\gamma \in \widetilde{b_\beta}$ with $\delta_\alpha = \min a^*_\alpha \notin B_\gamma$; however if $a^*_\alpha = a_\zeta$ and $a^*_\beta = a_\xi$ then $\zeta < \xi < \min \widetilde{b_\beta} \leq \gamma$, so $a_\xi \cap B_\gamma = a^*_\alpha \cap B_\gamma \neq \emptyset$ and $\delta_\alpha = \min a^*_\alpha \notin a^*_\beta \cap B_g$ so

$$\begin{aligned} \emptyset \neq (a^*_\alpha \setminus \{ \delta_\alpha \}) \cap B_\gamma &= a'_\alpha \cap B_\gamma \\ &\subset a'_\alpha \cap \Big(\bigcup_{\gamma \in \widetilde{b_\beta}} B_\gamma \Big) \\ &= a'_\alpha \cap B'_\beta \end{aligned}$$

whence $\{ (a'_\alpha, B'_\alpha) \mid \alpha < \omega_1 \}$ is a sequence satisfying (1), (2) and (3) with $n = k - 1$ in (1), contradicting the minimality of k. ■

It is no coincidence that the partially ordered set mentioned in 3.15 is similar to the one used in 3.12; in fact we use an argument similar to the one in 3.12 to show that the statement "$(P, \leq)$ is c.c.c." results in a contradiction, and we use 3.15 to show that the statement "$(P, \leq)$ is not c.c.c" also results in a contradiction.

3.16 Theorem.[8] *$MA_{\aleph_1}$ implies that there are no compact S-spaces.*

Proof. Assume that there exists an S-space. Then (as in 3.13) there exists one of the form $X = \mathrm{Cl}_X\{\, x_\alpha \mid \alpha < \omega_1 \,\}$, where $\{\, x_\alpha \mid \alpha < \omega_1 \,\}$ is a zero-dimensionsal right separated S-space. Now we can identify x_α with α, and define $A \subset \omega_1$ to be open iff $\{\, x_\alpha \mid \alpha \in A \,\}$ is open in $\{\, x_\alpha \mid \alpha < \omega_1 \,\}$; then if we call this topology τ, (ω_1, τ) is a zero-dimensional S-space with each $\alpha < \omega_1$ open, and since (ω_1, τ) is in fact homeomorphic to $\{\, x_\alpha \mid \alpha < \omega_1 \,\}$, we can write $X = \mathrm{Cl}_X(\omega_1, \tau)$.

Now we want to define the collection $\{\, U_\alpha \mid \alpha < \omega_1 \,\}$ dually to the definition in 3.10 and 3.12, but with stronger conditions on each U_α. For each $\alpha < \omega_1$, let $Y_\alpha = \mathrm{Cl}_X(\omega_1 \setminus \alpha)$; let $Y = \bigcap_{\alpha<\omega_1} Y_\alpha$. Now if $F \subset X \setminus Y$ is closed in X, then $F \cap X_\alpha = \emptyset$ for some $\alpha < \omega_1$; if F is closed in X, it is compact, so there is some finite subcover of $\{\, X \setminus Y_\alpha \mid \alpha < \omega_1 \,\}$ for F, say $\{\, X \setminus Y_{\alpha_1}, \ldots, X \setminus Y_{\alpha_n} \,\}$. But for $\alpha < \beta$, $Y_\alpha \supset Y_\beta$, so $X \setminus Y_\alpha \subset X \setminus Y_\beta$; thus $F \subset X \setminus Y_\alpha$ whence $F \cap Y_\alpha = \emptyset$, where $\alpha \overset{\mathrm{def}}{=} \max\{\, \alpha_1, \ldots, \alpha_n \,\}$. What we would like to do is choose, for each $\alpha < \omega_1$, a clopen neighbourhood $U_\alpha \subset \alpha + 1$ in (ω_1, τ) of α such that $(\mathrm{Cl}_X U_\alpha) \cap Y = \emptyset$; then we can apply the previous condition to each set $\mathrm{Cl}_X U_\alpha$. With this thought in mind, for each $\alpha < \omega_1$, let V_α be an open set in X with $V_\alpha \cap \omega_1 = \alpha + 1$. Now α is a point in X not in the closed set Y, so by regularity, there exists an open neighbourhood W_α of α in X with $(\mathrm{Cl}_X W_\alpha) \cap Y = \emptyset$. Then $W_\alpha \cap V_\alpha$ is an open neighbourhood of α in (ω_1, τ); by zero dimensionality, let $U_\alpha \subset W_\alpha \cap V_\alpha$ be a clopen neighbourhood of α in (ω_1, τ). Then $(\mathrm{Cl}_X U_\alpha) \cap Y = \emptyset$, as desired.

Now as before, we let $P = \{ p \in [\omega_1]^{<\omega} \mid \alpha, \beta \in p,\ \alpha < \beta \implies \alpha \notin U_\beta \}$, partially ordered by defining $p \leq q$ iff $p \supset q$. If $(P, \leq)$ is c.c.c., then by $MA_{\aleph_1}$ there exists an uncountable set $G \subset P$ centred in P, whence (as in the proof of 3.10 and 3.12) $\bigcup G$ is an uncountable discrete subspace of (ω_1, τ), whence of X, contradicting hereditary separability.

On the other hand, if $(P, \leq)$ is not c.c.c., we will derive a contradiction, thus completing the proof. Suppose $(P, \leq)$ is not c.c.c. Let $\mathcal{U}$ be the collection of all finite unions of U_α's; then by 3.15 there exists a sequence $\{ B_\alpha \mid \alpha < \omega_1 \} \subset \mathcal{U}$ cofinal on some uncountable set $A \subset \omega_1$. Now there exists $\alpha_1 < \omega_1$ such that $\{ B_\alpha \cap A \mid \alpha \in \omega_1 \setminus \alpha_1 \}$ has the finite intersection property; then so does $\{ C_\alpha \mid \alpha \in \omega_1 \setminus \alpha_1 \}$, where $C_\alpha \overset{\text{def}}{=} \mathrm{Cl}_X B_\alpha$. Also each B_α is clopen in X, $C_\alpha \cap B_\alpha = \omega_1$, and $C_\alpha \subset X \setminus Y$.

Now let $K_\alpha = \bigcap_{\beta \in \omega_1 \setminus \alpha} C_\alpha$. Since $K_\alpha \subset K_\beta$ for $\alpha < \beta$, and there cannot be an uncountable strictly increasing sequence of closed sets in a hereditarily separable space, there exists $\alpha_2 < \omega_1$ such that for $\alpha \geq \alpha_2$, $K_\alpha = K_{\alpha_2}$. Set $K = K_{\alpha_2}$; by the properties of $\{ K_\alpha \mid \alpha < \omega_1 \}$, $K_\alpha \subset K$ for each $\alpha < \omega_1$. Now each C_α misses Y, so each K_α does as well, so $K \cap Y = \emptyset$; by the property of closed sets in X that miss Y, $K \cap Y_{\alpha_3} = \emptyset$ for some $\alpha_3 < \omega_1$. Now $Y_{\alpha_3} \cap \omega_1 = \omega_1 \setminus \alpha_3$, so $Y_{\alpha_3} \cap A = A \setminus \alpha_3$ whence $Y_{\alpha_3} \cap A$ is an uncountable subset of A, so for some $\alpha_4 < \omega_1$, $\{ B_\alpha \cap Y_{\alpha_3} \cap A \mid \alpha \in \omega_1 \setminus \alpha_4 \}$ has the finite intersection property; since $B_\alpha \subset C_\alpha$ and $Y_{\alpha_3} \cap A \subset Y_{\alpha_3}$, $\{ C_\alpha \cap Y_{\alpha_3} \mid \alpha \in \omega_1 \setminus \alpha_4 \}$

is a collection of closed subsets of X with the finite intersection property, so by compactness of X,

$$\emptyset \neq \bigcap_{\alpha \in \omega_1 \setminus \alpha_4} (C_\alpha \cap Y_{\alpha_3}) = Y_{\alpha_3} \cap \Big(\bigcap_{\alpha \in \omega_1 \setminus \alpha_4} C_\alpha \Big)$$
$$= Y_{\alpha_3} \cap K_{\alpha_4} \subset Y_{\alpha_4} \cap K = \emptyset,$$

an obvious contradiction. ■

It should be mentioned here that 3.12 and 3.16 also imply that $MA_{\aleph_1}$ destroys locally compact S- and L-spaces. Given a locally compact space X, recall that the ***one-point compactification*** X^* of X is formed by adjoining a point $p \notin X$; then we take all sets of the form $\{p\} \cup (X \setminus K)$ where K is compact in X, to be a local base for neighbourhoods of x in X^*. Then we have that X^* is compact Hausdorff, and X is dense in X^*. If X is an L-space, then since X is not hereditarily separable, X^* is also not hereditarily separable. If Y is a subspace of X^*, then if $p \notin Y$, Y is a subspace of X, whence Lindelöf. If $p \in Y$, then $Y \setminus \{p\}$ is Lindelöf, so given an open cover $\mathcal{U}$ of Y, select one element $U \in \mathcal{U}$ with $p \in U$, and select a countable subcollection $\mathcal{U}' \subset \mathcal{U}$ covering $Y \setminus \{p\}$; then $\mathcal{U}' \cup \{U\}$ is a countable open subcover of $\mathcal{U}$ for Y. Thus X^* is hereditarily Lindelöf. Since a similar argument holds for S-spaces, the existence of a locally compact S- (respectively L-) space implies the existence of a compact S- (respectively L-) space, contradicting $MA_{\aleph_1}$.

Note that 3.12 and 3.16 show that under $MA_{\aleph_1}$, any compact space is hereditarily separable iff it is hereditarily Lindelöf. Thus, by 2.1(a), $MA_{\aleph_1}$ im-

plies that any perfectly normal space is hereditarily separable and hereditarily Lindelöf, and so is not an S- or L-space, and in fact has no S- or L-subspaces.

The argument concerning cofinally centred sequences which was used in 3.16 can also be used to show that $MA_{\aleph_1}$ destroys *all* S-spaces, *if* we also assume the validity of another set-theoretic hypothesis known as TOP. This result will not hold without TOP, because forcing arguments have shown[8] that the existence of S-spaces is consistent with $MA_{\aleph_1}$; in fact, a forcing argument has been used to show[1] the consistency of the existence of first-countable S-spaces with $MA_{\aleph_1}$, as mentioned in the remarks after 3.12.

Let us now familiarize ourselves with the principle TOP.

3.17 The Thinning-out Principle (TOP). If Z and B are uncountable subsets of ω_1 and $\{ S_\alpha \mid \alpha \in B \}$ is a collection cofinally centred on Z, with $S_\alpha \subset \alpha$ for each $\alpha < \omega_1$, then there exists an uncountable set $Y \subset Z$ such that $(Y \cap \alpha) \setminus S_\alpha$ is finite for each $\alpha \in B$.■

It has been shown, via a forcing construction[3,8], that MA+¬CH+TOP is relatively consistent with ZFC.

The key to using TOP to destroy S-spaces is that if (ω_1, τ) is an S-space and if each S_α is closed in (ω_1, τ), then each $Y \cap \alpha$ is closed in the relative topology on Y, since TOP will imply that $Y \cap \alpha = F \cup S_\alpha$, where F is finite, whence closed; thus (since $Y \subset \omega_1$ is uncountable) $\{ Y \cap \alpha \mid \alpha < \omega_1 \}$ is (or at least contains) an uncountable strictly increasing sequence of closed sets in Y, contradicting hereditary separability.

3.18 Theorem.[8] *$MA_{\aleph_1}$+TOP implies that there are no S-spaces.*

Proof. Suppose there exists an S-space; then we can assume that there exists some topology τ on ω_1 such that each $\alpha < \omega_1$ is open, and (ω_1, τ) is a zero-dimensional S-space. For each $\alpha < \omega_1$, let $U_\alpha \subset \alpha+1$ be a clopen neighbourhood of α. Let $P = \{ p \in [\omega_1]^{<\omega} \mid \alpha, \beta \in p,\ \alpha < \beta \implies \alpha \notin U_\beta \}$, partially ordered by defining $p \leq q$ whenever $p \supset q$.

As in previous proofs, if $(P, \leq)$ is c.c.c., then $MA_{\aleph_1}$ implies that there is an uncountable subset $G \subset P$ centred in P, so that $\bigcup G$ is an uncountable discrete subspace of (ω_1, τ), contradicting hereditary separability of (ω_1, τ).

Suppose then that $(P, \leq)$ is not c.c.c. Let

$$\mathcal{U} = \{ \bigcup_{\alpha \in b} U_\alpha \mid b \in [\omega_1]^{<\omega} \}.$$

Then by 3.15, there exists an uncountable set $A \subset \omega_1$ and a collection $\{ B_\alpha \mid \alpha < \omega_1 \} \subset \mathcal{U}$ cofinally centred on A. Now each B_α is the union of a finite number of U_α's, and each U_α is countable, so $\sup B_\alpha < \omega_1$. Also each B_α is clopen.

Let $\alpha_0 = \min\{ \sup(B_\alpha) + 1 \mid \alpha < \omega_1 \}$.

Given $\{ \alpha_\gamma \mid \gamma < \beta \}$ for some $\beta < \omega_1$, $\bigcup_{\gamma<\beta} \alpha_\gamma$ is countable, so for some $\alpha < \omega_1$, $\{ B_\delta \cap (A \setminus \bigcup_{\gamma<\beta} \alpha_\gamma) \mid \delta \in \omega_1 \setminus \alpha \}$ has the finite intersection property; in particular, for $\delta > \alpha$, $\sup B_\delta \in A \setminus \bigcup_{\gamma<\beta} \alpha_\gamma$. Let $\alpha_\beta = \min(\{ \sup(B_\alpha) + 1 \mid \alpha \in \omega_1 \} \setminus \bigcup_{\gamma<\beta} \alpha_\gamma)$. By transfinite induction, we have a sequence $\{ \alpha_\gamma \mid \gamma < \omega_1 \}$ such that if $S_\beta \stackrel{\text{def}}{=} B_{\alpha_\beta}$ for each $\beta < \omega_1$, then $\{ S_\beta \mid \beta < \omega_1 \}$ is cofinally centred on A, with each $S_\beta \subset \beta$. Then since each S_β is closed, by TOP and the remark after 3.17, (ω_1, τ) is not hereditarily separable, a contradiction. ■

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