Paraconsistent Logic for Dialethic Arithmetics

by

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Abstract

Inconsistent and collapse models of arithmetic are presented in the language and semantics of the simple paraconsistent logic LP. I present a logic which extends LP by the addition of a sensible conditional connective and quantifiers. This logic, called A_3, is specified as a Hilbert style axiom system and a Gentzen-style sequent calculus, and these systems are shown to be equivalent. I show the sequent calculus to be sound and complete for the A_3 semantics and prove the elimination theorem. Finally, I specify arithmetical axiom systems for the collapse models and show that these axiom systems capture some salient properties of their associated models.

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Table of Contents

List of Figures

Introduction

"... we can do everything you can do, only better, and we can do more."

– Richard Routley [43, 927] –

"The resulting change in viewpoint cannot impoverish insight into the nature of mathematical structures, but rather can only enrich it."

– Robert K. Meyer & Chris Mortensen [25, 929] –

Formalised Arithmetics and Relevant Arithmetic

The study of formalised arithmetics – collections of axioms governing arithmetical operations against a logic governing inferences from them – is very important for the history of logic, as well as for philosophy and mathematics. It is important for two related reasons. First, it played a major role in two of the major early traditions in the foundations of mathematics, namely, logicism and formalism. In the logicist tradition, it was to be by appeal to the analytic claims of logic alone that the rest of the grand structure or artifice of mathematics was to be shown to consist only in extensions of logic; that even the most abstract of set theoretic analyses were to be justified by appeal to very simple and evidently correct axioms and inference forms. The fact that much of this work was carried out not with axiomatic presentations of arithmetic, but rather of a suitably strong set theory from which these claims where to be justified[1] is not essential to the program, so long as the chain of definitional extensions from the axioms suffice to bring all of mathematics into one system, taking

[1]Presentations such as this are given in [15] and [45] where the Peano axioms are given among many theorems of arithmetic following from the respective set theoretical axioms of each. This is what Richard Heck has called "Frege's Theorem" [19].

its form largely or entirely from the underlying logic. The role of formalised arithmetic in the school of formalism is structurally similar - i.e. it was to be by appeal to the consistency of the basic systems of mathematical definitions that the work with other kinds of numbers was to be justified. In either case, there was quite a lot of motivation to develop a consistent, complete formalised arithmetic.

The second reason that formalised arithmetics are so important for logic, philosophy, and mathematics, and their histories, is that these theories failed at their task. The first of Gödel's Incompleteness results [18] states, in short, that there is no consistent, complete axiomatisation of arithmetic. Any axiomatic theory strong enough to prove all truths about arithmetic must prove some contradiction. Since so much of the motivation for both the logicist and formalist programs was to show that other areas of mathematics was free from contradiction, this result was fairly devastating. It was taken as being simply part of the landscape that the facts of arithmetic outrun all finite, consistent axiomatic theories.

There are, of course, a number of possible responses to this fact. By far the most popular has been to accept that true arithmetic just does in fact outrun finite consistent axiomatic systems, though it is, *of course*, consistent. This is in spite of Gödel's second result which indicates that a consistency proof for arithmetic, if it is consistent, cannot be given which does not rely on a stronger theory yet. So, the consistency of other mathematical theories cannot be justified by appeal to the consistency of arithmetic, since this is not provable. Perhaps the best take-away message available given these results is that axiomatic theories just do not stand up to the pressure placed on them by these approaches, and that the ideal of doing mathematics in an entirely finitist (in Hilbert's sense of the word) syntactic structure is not attainable.

However there are other options available, and they have to do with taking different approaches to Gödel's results. In the logic used in Gödel's proof, the properties of inconsistency and triviality are identified. I shall distinguish these properties as follows:

- We shall call any theory *trivial* iff every sentence of its language is a theorem.[2]
- We shall call any theory *inconsistent* iff it is such that, for some sentence A, both A and its negation $\neg A$ are theorems. Call the conjunction of A and $\neg A$ a *contradiction*.

The general form of the inference rule which has the effect of identifying inconsistency with triviality is:

$$A, \neg A \vdash B \qquad \text{(Explosion)}$$

Where $\neg$ is negation and the comma is a kind of conjunction, $C \vdash D$ states that D follows from C or that C implies D. The A, B occurring above can be uniformly replaced by any formulae whatsoever. This rule has been called *ex contradictione quodlibet*, *explosion*, and *Pseudo Scotus' rule*, among others – we stick to *explosion*. It is clear that this rule ensures that inconsistent theories trivialize, since any contradiction will allow the derivation of an arbitrary sentence. With this in mind, we can see the possibility for another approach to Gödel's theorem. We may distinguish varieties of "consistency" available for arithmetic theories, following the work of Robert K. Meyer [23], ranging from full non-triviality to simply omitting particular sentences - such as $0 \neq 0$ - as theorems, and consider how the second incompleteness result may be stated with different versions of consistency. It may be the case that a strong enough theory of arithmetic may be inconsistent, but non-trivial. If arithmetic is consistent, then we would be capturing all the truths but not only the truths, and if arithmetic is not consistent, then we may be able to capture all and only the truths, where some of those truths are contradictory. In any case, what arithmetic certainly is not is *trivial*, and so neither can our arithmetic theories be. However, to make these distinctions at all it is necessary that we employ a logic which enables us to distinguish inconsistent theories and trivial theories, and which therefore does not validate explosion. Such logics are called *paraconsistent* - meaning "partially consis-

[2]Trivial theories are also standardly referred to as *absolutely inconsistent*, since they prove not just some contradiction or other, but every contradiction.

tent" – that is, they admit theories that are negation inconsistent but not absolutely inconsistent.[3]

Paraconsistent logics have been used as the basis for formalised arithmetics only, to my knowledge, in the work of Meyer who attempted to show that arithmetics built onto a logic properly contained within classical logic could be made as strong as arithmetic built onto classical logic and in the work of Chris Mortensen [28] who developed a general approach to mathematics from a paraconsistent framework. The upshot of this is that the reasoning forms encoded in the paradoxes of material implication (where $\rightarrow$ is a conditional connective) - $A, \neg A \vdash B$ and $A \vdash B \rightarrow A$ - are not necessary for elementary mathematical reasoning. This was to add to the motivational program set out for Relevant logics (from which many of the most developed paraconsistent logics originated) which claimed that these reasoning forms, and the material implication, do not actually describe our common notion of entailment. If it were shown that even in mathematics these forms of reasoning needn't be appealed to, it would better justify their philosophical claims, given the relative weakness of the logics produced by the school. Some similar work in substructural logics has been tried, but with a fairly different aim or methodology. See Chapter 11 of [41].

Much of the work I'll appeal to here has been carried out in and in response to a research program set out by Richard Routley to showcase the power and value of paraconsistent logics, in particular as it applies to the case of mathematics. The idea is to indicate that paraconsistent modes of reasoning are sufficient for all standard reasoning and can subsume classical inference. However, as it applies to mathematics, Routley's program is somewhat more specific.

Routley's Program and The Meyer-Mortensen Line

The program of paraconsistent mathematics, as first and best expressed by Richard Routley in the central programmatic paper of the school, "Ultralogic as Universal"

[3]For a time, Graham Priest preferred the term *transconsistent*, as indicating that we were going *beyond* a facile and limiting fear of inconsistency, however, paraconsistent – apparently Newton Da Costa's term, is without a doubt the most popular name.

in [43], has two major aims. The first of these, called *Classical Recapture* is simply that of showing that paraconsistent logics can, by appropriate theoretical extensions, account for classical mathematical reasoning as well as classical logic can, but without the questionable appeal to ill-defended principles of classical logic. The idea, in rough, is that no actual mathematical reasoning relies upon explosion – even when working with infinitesimals, which were inconsistent, no one would have accepted a proof of some theorem simply stating that some infinitesimal was and was not equal to 0.[4] So, it ought to be the case that by reasonable alterations to the extra-logical theories laid on top of the logic, one can do without the principle of explosion, so that paraconsistent logics can do the same work that classical logic does. In broad, all the modes of reasoning invoked in mathematical activity are themselves actually, or can be uniformly substituted for, paraconsistent modes of reasoning. This has the advantage, for those already convinced that paraconsistent modes of reasoning are preferable, of simply giving better justification for their contradiction-tolerant epistemic approach. For those unconvinced by other arguments given in favour of paraconsistent reasoning more generally, justification for paraconsistent mathematics could be that of an argument from simplicity, of a kind. The conclusions one reaches in a paraconsistent mode of inference are justified by a proper subset of the theorems of classical logic - namely, those assumptions which give rise to explosion. However, this variety of simplicity is bought at the expense of another kind of simplicity. Whatever paraconsistent logic one appeals to, its propositional fragment will not be simpler than the boolean algebra which characterises the propositional fragment of classical logic. This is without considering the first order case which, depending on the logic, will likely require more axioms or inference rules than classical first order logic.

However, recapture is only half of Routley's program. Another part, *broadening* (my term) is also called for. The idea is that by moving to a paraconsistent basis for mathematical reasoning we also open the possibility of investigating structures out of the reach of classical logics. This is the domain of inconsistent, but non-trivial and

[4]Some work has been done about inconsistency in science from a paraconsistent background. For instance, consider [40].

potentially rich, interesting theories, trivialized by classical consequence. These kinds of structures are badly mishandled by classical logic in the same vocabulary. Their potentially interesting features bulldozed over by explosion, leaving only useless and trivial theories. Paraconsistent logics are unique in allowing for the study of such theories, some of which, beyond simple interesting curiosities, may well provide for new insights into open questions and problems. A similar motivational approach to paraconsistent mathematics, set out clearly by Meyer & Mortensen [25, 928-929], essentially just appeals to broadening. Their claim is not that relevant or paraconsistent reasoning can or should uproot classical reasoning, but rather that classical reasoning and classical mathematics are special cases of relevant reasoning, which allows one to reason about a larger class of theories. This line develops from the observation that negation consistent theories are a special case of non-trivial theories, and so a logic which allows one to discuss negation inconsistent but non-trivial theories has a broader scope than one which conflates the two. On their line, by allowing paraconsistent modes of reasoning into areas of mathematics which are currently studied only through a classical lens, we can only gain more insight, and that this is reason enough to be interested in the area.[5]

A very nice example of a project which has promise to get at both recapture and broadening is that of naive set theory in depth relevant logic (see Brady [12] and Weber [48] for some recent work in the area). This project has some promise to ensure recapture by a detour through broadening. Naive set theory, the inconsistent set theory originally given by Georg Cantor [13], given the right kind of logic, is a rich, non-trivial, though inconsistent theory (it turns out that the axiom of comprehension is extremely strong, and that the way to get a great deal of richness out of it, without going overboard into triviality, is to weaken the logic in proportion). Interesting results are coming out of this work, placing Naive Set Theory in a Depth Relevant logic

[5]As Meyer's later work [24] indicates, he continued to be interested in this area for some time afterwards, despite an unpleasant limitative result in [16]. In particular, he was interested in re-evaluating Gödel's theorems in light of the broadening afforded by paraconsistent modes of inference. So, in this sense, his claim for broadening is stronger yet – that these central mathematical limitative results may not be applicable to the more general structure afforded by paraconsistent reasoning.

as a strictly stronger theory than ZF, a standard classical set theory, for instance.[6] Should this follow through, we would have an excellent case of a simpler set theory (though in a more complicated logic) providing for more than that regularly appealed to in standard mathematical practice. Other than this, however, most projects in the area are involved in recapture at the apparent expense of broadening. The motivation for this is clear, since if recapture can be completed then a response could be made against some detractors of paraconsistent logics, those that reject these logics on the grounds that they are too weak to account for most inferential practice. However, a healthy and rich research program should not simply be built upon showing that one can do what others have done with different tools, but should rather indicate that these new tools allow one to do new things, and that these new things are interesting in their own right. This is a way to win new adherents to the program – by showing them that exciting things await them if they should come.

An example of Broadening by itself is in the application of paraconsistent model theory to first order Peano arithmetic, hereafter PA. Non-standard models of PA have garnered some attention in the last century, since PA proved to be an incomplete theory for the standard model itself. However, one can also, given a paraconsistent model theory, produce non-trivial models of PA which contain a wide variety of interesting structures, all of which are proper extensions of classical PA models – thus are inhabited by not just numbers but also what Meyer has called "alien intruders" [26], this name coming originally from Dedekind [47]. These are entities which have all the properties ascribed by the Peano axioms, and yet which behave in odd ways which natural numbers do not. These models, the inconsistent models of arithmetic, have been used to some interesting ends in the philosophy of paraconsistent logic (particularly in terms of claims made about finitist paraconsistent philosophy of mathematics, as well as to provide the grounding of Graham Priest's general notion of "number" and his associated claim that arithmetic actually is inconsistent, after a finite point), and have attracted the attention of Jeff Paris, who is generally very

[6]The primary justification for this claim being that it proves the principle of choice, which is independent of ZF. However, it must be noted that these results are, as yet, unstable since they rely on one unproved conjecture which is currently under investigation.

much a classicalist (to the extent that, though he has studied these models, he does so only with an explicitly classical and consistent metatheory). It would seem that this is a very nice field and there are a handful of open or partially open questions answers to which might well indicate that paraconsistent logics do actually provide us with some things we cannot get in classical logic - and things we might well want too. The semantics which have been used as the basis of the model theory used give form to these Inconsistent models of arithmetic is that of the Logic of Paradox – LP. One of the problems posed is whether or not Inconsistent models are axiomatizable. It is claimed that the finite ones are, but no axiom system in the standard sense of PA is given, and the question of the infinite among them is still unanswered.

I'm interested to see whether there is any suitable language and axiom system for these models. The question is one of what kind of paraconsistent logic is up to the task of being adequate to this class of models. Since the model theory used is that of LP it seems reasonable to suspect that some arithmetical extension to LP, in the form of PA itself or some variation on the same, might provide a syntax. However failing that there are plenty of other potential logics in the region, either extensions of LP – RM3, for instance, or A_3, to be developed in this thesis – or similar enough to allow for a straightforward translation of LP model theory. In addition there are plenty of ways to get to formal arithmetical content (either PA or some sub-theory, a set theoretic construction as per Frege's *Grundgesetze*, a category theoretic construction). I shall present the inconsistent and collapse models of arithmetic and develop a logic, A_3, which captures the LP semantics in a pleasing way. With this logic, I shall specify some sets of axioms to capture the salient features of these models, pointing in directions for future research.

Chapter 1

LP Semantics and collapse models

The class of inconsistent models of arithmetic developed by applying Dunn-Meyer extensions or "collapses" to the natural numbers (henceforth referred to as "collapse models of arithmetic" or just "collapse models"), as its description suggests, are models of arithmetic: that is to say that any formula true in the standard model of arithmetic is true in each collapse model, under translation of the standard model into the language of the logic of paradox, hereafter LP. However, they are peculiar in that they have this property while also including some extra, contradictory, claims about the natural numbers. Notwithstanding, they are non-trivial: there are claims in the language which are not true in these models. This is a peculiar admixture of properties for many reasons. First, the standard model, which is classical, and so consistent, is very widely accepted as the correct account, and so, at least in part, because it is classical. Such wide acceptance seems much less likely for an inconsistent model of the same phenomenon. So, this casts some doubt on the value of an inconsistent theory which purports to be arithmetic at all. Second, it is still widely doubted, though perhaps less than it has been in the past, that there can be non-trivial, inconsistent models, or theories of them, let alone involving a mathematical subject matter. Though this hard line appears to be ever weakening, it continues to have force in convincing many in the logical community that paraconsistent logics represent, at best, a futile endeavour. However, what is most interesting about these collapse models is that they provide a potential inroad for the study of paraconsistent

logics with possibly valuable applications in number theory.[1] These are interesting mathematical structures which are simply not amenable to analysis within the semantics of classical logic, which would bulldoze over the peculiar features of these models, leaving only triviality in its wake. It is, largely, because of allowing inconsistency into the model that it is a model of arithmetic - otherwise, it would simply be a collection of cycles of equivalence classes or cycles of equivalence classes with a tail, and would not express arithmetic fully. If we invoke real inconsistencies, we can compress the information of infinite models of arithmetic into a finite space. This would be valuable, and simply couldn't be carried out in a logic which conflates inconsistency and triviality. So, if these models can provide some grist for logical and mathematical investigation, then it must speak in favor of paraconsistent logics that they are capable of sustaining such investigation, while classical logic simply cannot. For, indeed, it is not just the structure of the models which are peculiar – some similar structures exist in classical mathematics in the form of modular arithmetics – but the semantic properties of the model, which are thoroughly dialethic.[2]

In this chapter, I shall set out the semantics of the Logic of Paradox (LP), and describe the kind of congruence relation, taking the closure of the standard model

[1]Perhaps most interesting is that there are collapse models which are finite models of Peano arithmetic. A better working knowledge of the finite models may well provide means to study Peano arithmetic, and perhaps the standard model of arithmetic, in terms of finite structures as opposed to the usually infinite models - either denumerably, as in the case of the standard model, or non-denumerably in some non-standard models. Priest in [34, 339] makes the claim that the collapse result to be stated as Theorem 1.2 is "the ultimate downward Löwenheim-Skolem Theorem: arithmetic has a model of *every* cardinality." Berto [10] generalises this point to say that "one can reduce a model with a denumerably infinite domain into one of any smaller size." While these are both very strong claims, it seems that we are given some tools to wrestle with infinities which classical methods do not obviously allow us.

[2]The word *dialethic* is derived from ancient Greek and means something like "two truths". It is regularly used to refer to researchers, theories, and logics which aim to capture inconsistent facts in a non-trivial way, where *fact* is taken seriously – that there really are true or correct contradictory assertions. It seems that the term initiated with Richard Routley, who named one of the earliest paraconsistent logic DL or "dialethic logic" in the appendix of [43]. This logic was originally called *dialectical* logic as it purported to capture the logic invoked by the Hegelian dialectic, though this conceit appears to have been abandoned early. See [42]. Probably, the most standard use is to refer to a researcher who treats contradictions as real as a dialetheist, and it is certainly Priest [38] who popularized this usage. I use the term only sparingly, favouring more precise terminology, with the exception of the title, which refers to dialethic arithmetics. By this, I mean only arithmetics, theories or models, which respectively contain or validate contradictions without being trivial.

under which produces inconsistent, non-trivial models which retain all the facts of the standard model. These models are variously called "collapses" [35] or "Dunn-Meyer extensions" [29]. I shall first expound LP, then the collapse construction, and finally, the basic kinds of the models themselves, some interesting results about them, and how the basic models are made to produce more complex structures.

1.1 The Logic of Paradox

LP is a very simple three-valued paraconsistent logic most famously expounded and studied by Graham Priest in [32].[3] It operates on the basis of weakening the classical negation symbol from a contradictory forming operator to a subcontrary forming operator. That is, the LP negation $\neg$, when applied to a formula A, produces a formula $\neg A$, which is in some cases true alongside A. A formula is valid on this semantics just in case it is assigned a designated value by any valuation function. The logic is many-valued, which is to say that it contains more than simple truth and falsity as possible truth-values. Of the three possible values some are designated as being "correct" in some way. Informally, sentences which are always assigned designated values are the kind which one would like the system to prove, while those which are possibly non-designated one wishes to avoid proofs of.

So, on an LP valuation it can be, though is not necessarily the case that A and $\neg A$ are both designated, and thus are both correctly assertable. Beside this point, LP is very similar to classical logic, up to having all the same theorems as classical logic has. Its conjunction and disjunction are, algebraically speaking, meet and join, and its conditional (to the extent that it has one) is defined as the material implication. LP has been presented in a number of places, particularly in [32] and [39], however the treatment in [36], where many of Priest's general claims about the collapse models are made, is of first order LP, which is necessary for the axiomatic developments in this project. Thus, I shall follow the presentation given there, though I shall go into more detail than is given there in order to clarify which extensions are required in

[3]LP, though not so called there, was first introduced in the work of Asenjo [4]. Credit goes to JC Beall [9] for pointing out this fact.

order to better highlight what is necessary to produce a sensible proof system for models in this semantics.

1.1.1 LP Semantics

The language consists in the following:

- Variable symbols: $x, y, z \ldots$
- Constant name symbols: $a, b, c, \ldots$
- Function symbols: $f^n, f^m \ldots$
- Predicate symbols: $P^n, Q^n, R^n \ldots$
- Meta-variables: $A, B, C \ldots$
- Sets: $\Phi, \Psi, \Theta \ldots$
- Connectives: $\wedge, \vee, \neg, \supset, \equiv, \rightarrow, \leftrightarrow$
- Quantifiers: $\forall, \exists$
- Arithmetical symbols: $=, ', +, \cdot, 0$

In general, lower case letters from the end of the Latin alphabet with or without prime symbols[4] are variables and lower case letters from the beginning of the alphabet are name constants, also with or without prime symbols. The function letters are restricted to occurrences of f with some number of prime symbols superscripted, where the superscripted n indicates the arity of the function, that is, the number of arguments it takes. The function symbols which are most often used shall be provided by the language of arithmetic. Capital Latin letters from the end of the alphabet are predicates while capital Latin letters from the beginning of the alphabet are meta-variables ranging over formulae. Capital Greek letters are sets or sequences of sentences, as need demands. The ambiguity between sets and sequences will be disambiguated by context – they are almost universally sets. Capital Greek letters which resemble capital Latin letters (eg. the capital Latin p and the capital ρ) will be studiously avoided, in favour of the Latin. So, in the sequence Γ, Δ, A, read A not as the capital α, and thus as a set, but as the capital Latin a, and, thus, as a meta-variable. The arithmetical language I shall deal with more later, but, as standard, 0

[4]For my purposes here, terms almost always denote numbers, 0 and its successors in an ω sequence. A usual convention, which I employ, is that the letter x with n appended prime symbols may be abbreviated x^n.

is a name constant, $'$ is a unary function, $+$ and $\cdot$ are binary functions, and $=$ is the identity predicate.

Definition 1.1. The set of *terms* is defined recursively by the following clauses:

1. If t is a name constant a, then t is a term.
2. If t is a variable x, then t is a term.
3. If $t_1, \ldots t_n$ are terms and f^n is a function symbol, then $f^n(t_1, \ldots t_n)$ is a term.
4. Nothing is a term save being so in virtue of clauses 1-3 of this definition.

Definition 1.2. The set of *formulae* is defined recursively by the following clauses:

1. If P^n is a predicate letter, and $t_1, \ldots t_n$ are terms, then $P^n(t_1, \ldots t_n)$ is a formula.
2. If A is a formula, then $\neg A$ is a formula.
3. If A, B are formulae, then $A * B$ is a formula, where $* \in \{\wedge, \vee, \supset, \equiv, \rightarrow, \leftrightarrow\}$[5]
4. If A is a formula and x is a variable then $\forall x A$ and $\exists x A$ are formulae.
5. Nothing is a formula save by clauses 1-5 of this definition.

Definition 1.3. An LP *interpretation* is a tuple $\langle D, I, v\rangle$ such that:

- D, the domain, is a non-empty set of objects.
- I, the interpretation, is a function from non-logical expressions of the language to domain objects and set-theoretical constructs of them.
  - for every name constant a, $I(a) \in D$ - the domain object called the name's "referent".

[5]I include the $\rightarrow$ and $\leftrightarrow$ here since they will be used in the latter chapters. These connectives are not in LP, and it is their addition to LP which occupies most of the second chapter, and so they have been included here so as to be available as formulae in the following work.

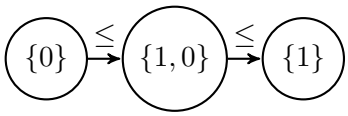
- for every function f^n, $I(f^n) \in D^{D^n}$.
- for every n-place predicate, P^n, $I(P^n)$ is the pair $\langle I^+(P^n), I^-(P^n)\rangle$, of, respectively, the extension and anti-extension of P. These are such that $I^+(P^n) \cap I^-(P^n)$ may be non-empty and $I^+(P^n) \cup I^-(P^n) = \{\langle d_1, \ldots d_n\rangle; d_i \in D\}$. So $I^+(P^n) \subseteq D^n$ and $I^-(P^n) \subseteq D^n$.

- v, the valuation, takes formulae to truth-values, non-empty subsets of $\{1, 0\}$ and variables to domain objects, ie. $v(x) \in D$. We shall often have to specify the particular value assigned to a variable at a particular valuation – $v[d/x](Ax)$. This is the truth-value of Ax where d is assigned to x, that is $v[d/x](x) = d$. However, all other variables which are not displayed are left alone under that valuation – if $x \neq y$ then $v[d/x](y) = v(y)$.

So v does two distinct jobs; if A is a formula, then $v(A)$ is a truth value and if x is a variable, then $v(x)$ is a domain object, and if Ax is a formula with a free variable, then $v(Ax)$ is a truth-value assignment for A with a particular domain object assigned to x in all its instances in A.

I shall use lattice-theoretic notions to describe some facts about LP and related logics. For this purpose, the ordering on the LP values is as follows:

Figure 1.



Definition 1.4. The function i is an extension of I and v, which assigns an element of the domain to every term:

1. Where a is a name constant, $i(a) = I(a)$.
2. Where x is a variable, $i(x) = v(x)$.
3. Where f^n is a function, and $t_1, \ldots t_n$ are terms, then $i(f^n(t_1, \ldots t_n)) = I(f^n)(i(t_1), \ldots i(t_n))$.

It follows that for every term t there is a $d \in D$ such that $i(t) = d$. For atomic formulae, v assigns values according to the following:

- Where A is a meta-variable, $v(A) \subseteq \mathcal{P}(\{1, 0\}) - \{\varnothing\}$.
- Where P^n is a predicate letter, and $t_1, \ldots t_n$ are terms, $1 \in v(P(t_1, \ldots t_n))$ if $\langle i(t_1), \ldots i(t_n)\rangle \in I^+(P)$ and $0 \in v(P(t_1, \ldots t_n))$ if $\langle i(t_1), \ldots i(t_n)\rangle \in I^-(P)$.

The most often used predicate in this work is $=$, the extension of which is defined as follows:

- $I^+(=)$ is $\{\langle d, d\rangle; d \in D\}$.

We shall, for the moment, leave the anti-extension of $=$ open, except that for any d and e which are distinct domain objects, $\langle d, e\rangle \in I^-(=)$. This definition shall be extended to give rise to a basic class of inconsistent models which are not collapse models, to be introduced in this chapter.

To move on to v as it applies to complex formulae, for each kind of formula there are two cases to consider - where 1 is in the value of that formula - read that the formula is "true" or "at least true" - and where 0 is in the value of that formula - read that it is "false" or "at least false". The other cases are similar - $\vee$ can be defined from $\wedge$ in the usual way, and the conditional is just the material implication, defined in terms of $\neg, \vee$.

- $1 \in v(\neg A)$ iff $0 \in v(A)$
- $0 \in v(\neg A)$ iff $1 \in v(A)$
- $1 \in v(A \wedge B)$ iff $1 \in v(A)$ and $1 \in v(B)$
- $0 \in v(A \wedge B)$ iff $0 \in v(A)$ or $0 \in v(B)$
- $v(A \vee B) = v(\neg(\neg A \wedge \neg B))$
- $v(A \supset B) = v(\neg A \vee B)$

- $1 \in v(\forall x Ax)$ iff for all d, $1 \in v[d/x](Ax)$.
- $0 \in v(\forall x Ax)$ iff for some d, $0 \in v[d/x](Ax)$.
- $v(\exists x Ax) = v(\neg \forall x \neg Ax)$

These cases of v indicate that LP is a truly *dialethic* logic[6] - it can be the case that for some predicate P, there is a $d \in D$ such that $d \in I^+(P)$ and $d \in I^-(P)$, and the valuation of Pa, where $I(a) = d$ contains both 1 and 0 and is thus both true and false - ie. $v(Pa) = \{1, 0\}$. That this is still a designated value is of most importance in the LP definitions of validity, valid argument, and model.

Definition 1.5. A formula A is an *LP-validity* iff for any valuation v, $1 \in v(A)$.

A notion which I'll make ample use of is that of *sequents*. A sequent is an expression of the form $\Gamma \vdash \Delta$, where Γ, Δ are sets of sentences, and it expresses that the formulae of Δ follow from those of Γ. The symbol $\vdash$ or "turnstile" is a piece of the meta-language used to indicate the conjunction of what occurs to the left of it, or the "antecedent", implies the disjunction of what occurs to the right of it, the "succedent". More informally: where $G_1, \ldots G_n \in \Gamma$ and $D_1, \ldots D_n \in \Delta$, $\Gamma \vdash \Delta$ may be read as $G_1 \wedge \cdots \wedge G_n \vdash D_1 \vee \cdots \vee D_n$. If $\Gamma = \varnothing$, then the sequent, written $\vdash \Delta$ expresses that at least one $D_i \in \Delta$ is a logical validity and, at least to begin with, we shall not allow an empty succedent Δ. The advantage of using sequents is that they explicitly invoke the notion of consequence, so that in reasoning with sequents one reasons explicitly about acceptable *inferences*, not just acceptable formulae. This is of particular value in non-classical and substructural logics, when these are constructed by rejecting certain classical inference forms, and so a sequent-system allows one to make quite explicit the underlying principles allowing these inferences, and thus one can trace the problem to its source, as it were. This will be more developed in the next chapter, with the introduction of a sequent calculus for reasoning about inconsistent models. To that end, the definition of *valid sequent* is as follows:

[6]There is another sense in which LP is dialethic in the original meaning of the word. There are, in fact, two "truths" in LP in the sense that two values are taken as designated or correct - both $\{1\}$ and $\{1, 0\}$.

Definition 1.6. A sequent $\Gamma \vdash \Delta$ is valid in LP iff there is no valuation v such that $\forall G \in \Gamma(1 \in v(G))$ and $\forall D \in \Delta(1 \notin v(D))$.

This is the multiple conclusion variation of the definition, from which the single conclusion version is easily obtainable – just restate the requirement so that if all premises are distinguished, then so must *the* conclusion be.

Given these definitions, it is clear that the truth-values which contain 1 - namely $\{1\}$ and $\{1, 0\}$ - are designated while $\{0\}$ is not. With this the above criterion for validity can be simply restated - A sequent is valid iff it cannot be the case that all of its premises have designated values and all of its conclusions do not. So, there are cases where a sequent can have all true premises - where 1 occurs in the valuation of each formula - and yet have all false conclusions - where 0 occurs in the valuation of each - so long as it is the case that some conclusions are designated but false - that is, their values are $\{1, 0\}$. In this case, these conclusions are false, but also true, and thus correct.

Now, this logic has some intriguing properties which make it particularly well suited to interpreting classical theories in a paraconsistent framework. While it is dialethic, it is also very strong in relation to Classical Logic - with the readings of the connectives given above the set of validities is the same as that of classical tautologies. The proof is given by Priest in [32, 223] and [32, 230] for the first order case. It is as follows.

Theorem 1.1. *A is a classical tautology iff A is an LP-validity.*

Proof. The left-to-right direction is immediate since every two-valued valuation is a three-valued valuation, since the values assigned to complex formulae the components of which are assigned classical valuations are always classical, as per the definitions of the connectives and quantifiers. For the right-to-left direction suppose that A is a classical tautology and, for any LP valuation v, let v_1 be a two-valued valuation formed by identifying $\{1, 0\}$ with $\{1\}$ - so $v_1(A) = 1$ iff $1 \in v(A)$ and $v_1(A) = 0$ iff $v(A) = \{0\}$. This can be checked by appeal to the LP truth tables - which are Kleene's strong three-valued tables [21] except that the intermediate value is taken to

be designated. Below are these truth tables, where $\{1, 0\}$ is abbreviated as a Fraktur $\mathfrak{i}$.

$\neg$	
1	0
$\mathfrak{i}$	$\mathfrak{i}$
0	1

$\supset$	1	$\mathfrak{i}$	0
1	1	$\mathfrak{i}$	0
$\mathfrak{i}$	1	$\mathfrak{i}$	$\mathfrak{i}$
0	1	1	1

$\wedge$	1	$\mathfrak{i}$	0
1	1	$\mathfrak{i}$	0
$\mathfrak{i}$	$\mathfrak{i}$	$\mathfrak{i}$	0
0	0	0	0

$\vee$	1	$\mathfrak{i}$	0
1	1	1	1
$\mathfrak{i}$	1	$\mathfrak{i}$	$\mathfrak{i}$
0	1	i	0

By an obvious structural induction on formulae, it is clear that $v(A)_1 = 1$ if $1 \in v(A)$, and thus that A is a classical tautology if it is an LP-validity. □

So, counter-intuitively $(A \wedge \neg A) \supset B$ is an LP-validity as is $A \supset (\neg A \supset B)$. However, what fails are the sequent versions of these; $A \wedge \neg A \nvdash B$. This is a result of the fact that when $v(C) = \{1, 0\}$ and $v(D) = \{0\}$, $v(C \supset D) = \{1, 0\}$, and yet $C \nvdash D$. It is a result of this that LP is ill-suited to axiomatic developments of the usual sort. This is due to the fact that basic conditional principles fail in LP - like Modus Ponens. As a result, certain standard proof systems which rely on the conditional cannot be given for LP. The standard proof theory given for LP is a tableaux system, given in [39], which does not generally fit our project of axiomatising theories of arithmetic. This is a point to be revisited in the next chapter, where I shall present a logic which, I'll argue, retains the spirit of LP while, at the same time opens the door for syntactical presentations which better match our aims. However, before that, there are two metatheoretical results about LP which are necessary for the development of the collapse models themselves.

1.1.2 Some LP metatheory

Both of the extension lemma and the collapse theorem, taken from Priest [35], are prime movers in the model-theoretic construction.

Definition 1.7. Given two interpretations, I and J, J is an *extension* of I iff for every predicate P, $I^+(P) \subseteq J^+(P)$ and $I^-(P) \subseteq J^-(P)$.

The collapse models are extensions, given a type-lifting of arithmetic, so that the numbers are taken as singleton sets. Then, they are interpretations which subtly alter the meanings of certain parts of the arithmetical language.

Lemma 1.1 (Extension)**.** - *If I, J are interpretations and J is an extension of I then for any formula A of the language, $v_I(A) \subseteq v_J(A)$, where v_I and v_J are valuations based on the interpretations I and J, respectively.*

Proof. By a structural induction on formulae. The base case is straightforward, and in general we shall omit cases where $v_I = v_J$, since these clearly follow. Suppose that A is $P^n(t_1, \ldots t_n)$. Since $I^+(P^n) \subseteq J^+(P^n)$ and $I^-(P^n) \subseteq I^-(P^n)$, then if $1 \in v_I(P^n(t_1, \ldots t_n))$ then $1 \in v_J(P^n(t_1, \ldots t_n))$ and if $0 \in v_I(P^n(t_1, \ldots t_n))$ then $0 \in v_J(P^n(t_1, \ldots t_n))$. The most drastic effect that J can have on the atomic formulae is to change their values from $\{1\}$ or $\{0\}$ to $\{1, 0\}$, otherwise it leaves their values as they were under I.

Case ($\neg$): Suppose that $v_I(A) = \{1\}$, then $v_I(\neg A) = \{0\}$. If $v_J(A) = \{1\}$, then $v_J(\neg A) = \{0\}$. If $v_J(A) = \{1, 0\}$, then $v(\neg A) = \{1, 0\}$, and $v_I(\neg A) \subseteq v_J(\neg A)$ as desired. Suppose that $v_I(A) = \{0\}$, then $v_I(\neg A) = \{1\}$. If $v_J(A) = v_I(A)$ then $v_J(\neg A) = v_I(\neg A)$. If $v_J(A) = \{1, 0\}$, then $v_J(\neg A) = \{1, 0\}$ and, in either case $v_I(\neg A) \subseteq v_J(\neg A)$.

Case ($\wedge$): Suppose that $v_I(A)$ or $v_I(B)$ is $\{0\}$, then $v_I(A \wedge B)$ is also $\{0\}$. If $v_J(A) = \{0\}$ or $v_J(B) = \{0\}$, then $v_J(A \wedge B) = \{0\}$, as desired. If $v_J(A) = v_J(B) = \{1, 0\}$ then $v_J(A \wedge B) = \{1, 0\}$ and still $v_I(A \wedge B) \subseteq v_J(A \wedge B)$. If either $v_J(A) = \{1, 0\}$ or $v_J(B) = \{1, 0\}$ then $v_J(A \wedge B) = \{1, 0\}$ and $v_I(A \wedge B) \subseteq v_J(A \wedge B)$. If $v_I(A) = \{1\}$ or $v_I(B) = \{1\}$, then the case is completely straightforward, given that 1 must, then, be in the value of whichever of A, B v_I assigned to $\{1\}$, and thus its v_J-assignment will also be designated.

Case ($\vee$): Suppose that $v_I(A) = v_I(B) = \{0\}$, then $v_I(A \vee B) = \{0\}$. If $v_J(A) = \{1, 0\}$ or $v_J(B) = \{1, 0\}$ then $v_J(A \vee B) = \{1, 0\}$ and $v_I(A \vee B) \subseteq v_J(A \vee B)$ as desired. Suppose that $v_I(A) = \{1\}$ or $v_I(B) = \{1\}$, then $v_I(A \vee B) = \{1\}$. If one of $v_J(A)$ or $v_J(B)$ is $\{1, 0\}$ while the other is $\{1\}$, then $v_J(A \vee B) = \{1\}$ and $v_I(A \vee B) \subseteq v_J(A \vee B)$. If $v_J(A) = v_J(B) = \{1, 0\}$ then $v_J(A \vee B) = \{1, 0\}$ and $v_I(A \vee B) \subseteq v_J(A \vee B)$.

Case ($\supset$): Since $v(A \supset B) = v(\neg A \vee B)$ for all v, then this case follows from that for $\neg$ and that for $\vee$.

Case ($\forall$): Suppose that for all d, $v_I[d/x](Ax) = \{1\}$, then $v_I(\forall xAx) = \{1\}$. J may result in some d being such that $v_J[d/x](Ax) = \{1, 0\}$ and thus $v_J(\forall xAx) = \{1, 0\}$ and $v_I(\forall xAx) \subseteq v_J(\forall xAx)$. Suppose that for some d, $v_I[d/x](Ax) = \{0\}$. Then $v_I(\forall xAx)$ is either $\{0\}$ or $\{1, 0\}$. If the former, then for all d, $v_I[d/x](Ax) = \{0\}$. J may either leave all values or result in some d being such that $v_J[d/x](Ax) = \{1, 0\}$. If $v_J[d/x](Ax)$ for some d, then $v_J(\forall xAx) = \{1, 0\}$ and $v_I(\forall xAx) \subseteq v_J(\forall xAx)$. If for all d $v_J[d/x](Ax) = \{0\}$, then $v_J(\forall xAx) = \{0\}$ and $v_I(\forall xAx) \subseteq v_J(\forall xAx)$. Suppose that $v_I(\forall xAx) = \{1, 0\}$. Then for all d, $1 \in v_I[d/x](Ax)$ and for some d, $v_I[d/x](Ax) = \{1, 0\}$. Then, $v_J[d/x](Ax) = v_J[d/x](Ax)$, and thus $v_I(\forall xAx) \subseteq v_J(\forall xAx)$, since the existence of one d such that $v_J[d/x](Ax) = \{1, 0\}$ guarantees that $v_J(\forall xAx) = \{1, 0\}$ if for all other d, $1 \in v_J[d/x](Ax)$, which is true by the hypothesis of the induction.

□

The effect of this lemma is most interesting given another result - Theorem 1.2 or the collapse theorem - which gives way to the class of collapse models of arithmetic. For this a definition of a *collapsed* interpretation is necessary. This is the result of the closure of the interpretation under an equivalence relation on the domain which is also a congruence relation on the interpretations of all functions - a quotient algebra of the initial interpretation.

Definition 1.8. Given an interpretation $A = \langle D, I, v\rangle$ and an equivalence relation $\sim$ on D, which is a congruence for all functions[7] of the language, define $A^{\sim} = \langle D^{\sim}, I^{\sim}, v^{\sim}\rangle$, the collapsed interpretation, according to the following cases, where $[d]$ is the equivalence class of $d \in D$ under an equivalence relation $\sim$, that is, $D^{\sim} = \{[d]; d \in D\}$:

- for every constant a, $I^{\sim}(a) = [I(a)]$
- for every n-place function f, $I^{\sim}(f)([d_1], \ldots, [d_n]) = [I(f)(d_1, \ldots, d_n)]$

[7]Where f is an n-place function letter and where $d_i, e_i \in D$, if $d_i \sim e_i$ for all $1 \leq i \leq n$ then $I(f)(d_1, \ldots, d_n) \sim I(f)(e_1, \ldots, e_n)$. [35, 225]

- for every predicate P, $\langle [d_1], \ldots, [d_n] \rangle \in I^{\sim}_{+}(P)$ iff for all $e_1 \sim d_1, \ldots, e_n \sim d_n$ such that $\langle e_1, \ldots, e_n \rangle \in I^{+}(P)$ and the same for the anti-extension – $\langle [d_1], \ldots, [d_n] \rangle \in I^{\sim}_{-}(P)$ iff for all $e_1 \sim d_1, \ldots, e_n \sim d_n$ such that $\langle e_1, \ldots, e_n \rangle \in I^{-}(P)$.

The equivalence classes, made up of domain objects identified by $\sim$, may well have inconsistent properties, as in the case of some collapses of the standard model of arithmetic. This kind of maneuver was first presented by J. Michael Dunn in [14] - the collapse theorem below is given there as his basic theorem, just as lemma 1.1 is a restatement of his preservation theorem. However, it was Robert K. Meyer and Chris Mortensen who initially put this to use in research into relevant arithmetic - particularly in [25], which was a precursor of Priest's work in the area. As a result, another name for a congruence relation inducing collapse which has been used is "Dunn-Meyer extension", introduced by Mortensen [29]. From this point on, I shall use "collapse" to refer to the theorem and class of models developed by Priest in [35] and [36].

It follows from the definition of a collapsed interpretation, and the type-difference, that $I^{+} \subseteq I^{\sim}_{+}$ and $I^{-} \subseteq I^{\sim}_{-}$ since in each point in the above definition, $I^{\sim}$ is stipulated to include I - if $\langle d_1 \ldots d_n \rangle \in I^{+(-)}(P)$ [8] then it must also be in $I^{\sim}_{+(-)}$, since $\sim$ is an equivalence relation and thus for all d_i, $d_i \sim d_i$ where $1 \leq i \leq n$. So, collapses can be viewed as extensions as per Definition 1.7, and thus Lemma 1.1 applies.

Theorem 1.2 (Collapse)**.** *For any formula, A,* $v(A) \subseteq v^{\sim}(A)$.[9]

Proof. By structural induction on A. All terms are interpreted under $i^{\sim}$ by equivalence classes, and all predicates are type-lifted to take equivalence classes as arguments under $I^{\sim}_{+(-)}$. The proof is quite short and obvious in view of the proof given for the previous lemma.

For the basis, suppose that $1 \in v(P(t_1, \ldots, t_n))$ for a predicate P.
Then $\langle i(t_1), \ldots i(t_n) \rangle \in I^{+}(P)$. Then for each of $t_1, \ldots, t_n$ there is something in each

[8] Read $I^{+(-)}$ as "either I^{+} or I^{-}".
[9] This is the theorem which Priest [34] has called the ultimate downward Löwenheim-Skolem theorem – see footnote 2 of this chapter.

of $i(t_1), \ldots, i(t_n)$ which is in the extension of P, namely d_i $(1 \leq i \leq n)$. Thus $\langle d_1, \ldots, d_n \rangle \in I^{\sim}_{+}(P)$ and $\langle i^{\sim}(t_1), \ldots, i^{\sim}(t_n) \rangle \in I^{\sim}_{+}(P)$. Thus, $1 \in v^{\sim}(Pt_1, \ldots, t_n)$.

Case 1 ($\wedge$): Suppose that $1 \in v(A)$ and $1 \in v(B)$ and thus $1 \in v(A \wedge B)$. Then, by the Extension lemma, $1 \in v^{\sim}(A)$ and $1 \in v^{\sim}(B)$ and thus $1 \in v^{\sim}(A \wedge B)$.

Case 2 ($\neg$): Suppose that $1 \in v(\neg A)$. Then $0 \in v(A)$ and, by Extension, $0 \in v^{\sim}(A)$ and thus $1 \in v^{\sim}(\neg A)$.

Case 3 ($\forall$): Suppose that $1 \in v(\forall x Ax)$. Then $1 \in v[d/x](Ax)$ for all d and by the Extension lemma, $1 \in v^{\sim}[d/x](Ax)$. Thus $1 \in v^{\sim}(\forall x Ax)$.

The negative cases involving the anti-extension can be obtained from the above by uniformly substituting "I^{-}" for I^{+} and 0 for 1 (and conversely). All connectives and the universal quantifier can be shown similarly, and the theorem easily follows. [10]

□

The result of this theorem is that 1s and 0s, as they occur in the valuations of propositions, are inherited across a collapse. The value of this is, as Priest puts it, where $\vDash$ is semantic entailment in analogy to $\vdash$, "in particular, if $A \vDash \Sigma$ then $A^{\sim} \vDash \Sigma$: if we collapse a model of a theory, we therefore produce another model." [35, 226] The import of this last claim is in the fact that a Dunn-Meyer Extension of a model of arithmetic is still a model of arithmetic - even though the properties of the equivalence classes formed by such an extension may be inconsistent. We can take a quotient algebra of $\langle \mathbb{N}, ', +, \cdot \rangle$ using a Dunn-Meyer extension, producing an inconsistent model of the same. This will, essentially, have the result of identifying distinct elements of $\mathbb{N}$ into equivalence classes, while still retaining the truth of their non-identity. All of the inconsistent models with which we'll be concerned here are collapse models, though there is a class of inconsistent models not arising from a collapse. It will be made clear that these models are, in general, non-trivial. That is to say, not all $n \in \mathbb{N}$ fall into the same equivalence class, and thus not every formula over the language of arithmetic is valid. So the fact that these models capture arithmetic

[10] The statement of this proof is adapted from [33] and [14].

is not simply a trivial result of their validating every sentence in the language.

1.2 Taxonomy of extended and collapse models

Let $\mathcal{L}$ be the language of arithmetic - including the name constant 0, the singulary function symbol $'$ (successor), and the binary function symbols $+, \cdot$ (addition and multiplication). Since it is useful to have a simple shorthand to indicate the predecessor of a number, I shall follow Priest in using $'n$ to indicate the predecessor of n, where convenient. In this language the numeral n is 0 followed by that many (that is n) occurrences of $'$. The standard model is just the usual linear well-ordering with unique and successors and predecessors bottoming out at 0, but there are non-standard models of arithmetic to which we shall appeal as well - these include natural numbers and copies of the natural numbers in ordering patterns which can produce peculiar results when collapsed.[11]

A Dunn-Meyer extension of a model in this language will have the following properties, by the definition. This very clear statement of the interpretations of the elements of the language of arithmetic is found in [31, 214], which makes it explicit that the elements of the language are type-lifted to take equivalence classes of numbers rather than simply numbers. An interesting case, of course, is that which involves negation applied to our only predicate $=$. The equivalence class of a, produced by a collapse, shall simply be written as a, while the members of the equivalence class, standard numbers, shall be written with an overline, $\overline{a}$.

$$I^{\sim}(\overline{0}) = 0 \qquad I^{\sim}(')(\overline{x}) = x'$$
$$I^{\sim}(+)(\overline{x}, \overline{y}) = x + y \qquad I^{\sim}(\cdot)(\overline{x}, \overline{y}) = x \cdot y$$
$$I^{\sim}_{+}(=) = \{\langle x, y\rangle; \overline{x} \sim \overline{y}\} \qquad I^{\sim}_{-}(=) = \{\langle x, y\rangle : \overline{x} \neq \overline{y}\}$$

Since these are the only elements of the language, barring definitional extension, these are all the salient facts about the collapse models generally - the other facts about each model being determined by the specific closure operation which produces

[11] My reference material for the non-standard models of arithmetic are Richard Kaye's *Models of Peano Arithmetic* [20] and the seventeenth chapter of *Computability and Logic*, third edition [11].

it. There is some variety in the kinds of closure operations which produce inconsistent models - and for the remains of this chapter I shall set out the standard categories into which they fall. This way, it becomes fairly natural to produce axioms which capture the particular features of each.

Since the language only has one predicate, along with apparatus for term-formation, every formula added will contain as subformulae either equations or the negation of equations. Most any proper extension, that is, one where $\sim$ is not just the standard $=$, will result in a contradiction of some equation with the negation of that equation, since either the equation in question or its negation will be added to $I^{\sim}_{+}$ or $I^{\sim}_{-}$, respectively. Those cases of formulae independent of arithmetic aside, most extensions of arithmetic will produce an inconsistency and will be the result, at bottom - that is, after breaking the formula into its components, either of adding some ordered pair $\langle x, x\rangle$ to $I^{\sim}_{-}(=)$ - saying that $x \neq x$ - or of adding some ordered pair $\langle x, y\rangle$, where $\overline{x} \neq \overline{y}$, to $I^{\sim}_{+}(=)$ - that $\overline{x} \sim \overline{y}$ and thus $x = y$. The latter case is clearly involved in collapse models produced by Dunn-Meyer extensions.

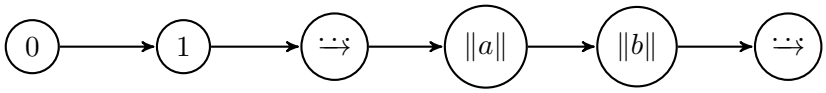
In [36], Priest sets out a general taxonomy of collapse models of arithmetic which later on has been pruned by Paris and Pathmanathan in [30]. First I shall set out each of a few categories of models and then state the general features that they have in common.

1.2.1 Simple extended models

These are the former of the two kinds of extensions I sketched earlier - those resulting from adding some ordered pair of the same number to $I^{-}(=)$. The nominal simple extensions are the addition of $\langle a, a\rangle$ to $I(=)$, for some a, so that $a \neq a$ thus becomes true in the model. These are, as the title I've given them suggests, extremely simple and, while they are not collapse models they provide something of a nice counterpoint to collapse models. Since $a = a$ is valid on the Standard Model, this means that $a = a$ becomes a point of contradiction, or a glut - $a = a \wedge a \neq a$ is made correct. Given no congruence, we are not guaranteed that any standard principles for the functions of arithmetic hold, however, we are, in this instance, bound by the kind of model which

Priest outlines, so we shall try to stick quite close to that presentation. Since $x \neq y$ must imply $x' \neq y'$ if we are to capture the kinds of models Priest [35] describes, the result is that for every $b > a$, $b = b \wedge b \neq b$. This is hard to depict but quite simple - after the number for which identity is denied in the ordering, the model becomes simply an ordering of gluts, depicted by writing the number in double vertical bars – $\| \|$ – which do not interact with the segment of the model occurring before a in the ordering.

Figure 2.



However, there is a decision point as to whether the numbers lower than a ought to be similarly inflicted - since, as I'll argue, the principles which would imply this fail for some of the collapse models, we may have some reason to force them to fail for these models as well. As such, it seems to be something like a decision point in the construction of these models whether that is to be true - it is not decided simply by the LP model theory. However, in either case these are hardly the rich models we might be interested in studying. In the case where gluttiness is only inherited up, the models are quite dull - one has a consistent tail isomorphic to an initial segment of the standard model and after that a denumerably long glutty tail. However, if the gluttiness is inherited in both directions, then no matter where one introduces the glut, the entire model becomes straightforwardly glutty - simply a copy of the standard model with the extra information that, where x ranges over numbers, $\forall x(x = x \wedge x \neq x)$. This might be an undesirable result, and the principle which would produce the behaviour in the latter case – that $x' \neq y'$ implies $x \neq y$ – is invalid on all collapsed models (which are not just equivalent to the standard model), under the definition of $I(=)$. So, I lean toward its rejection for the purposes of these models, even though they are not the result of collapse. However, even if one does reject the principle, one can produce basically the same effect in at least one model. It is the limiting case - that where $a = 0$, so that the result is that $0 = 0 \wedge 0 \neq 0$ and

thus, whether or not the gluttiness is inherited down the ordering, $\forall x(x = x \wedge x \neq x)$ - this is the only simple extended model which Priest discusses. [35, 226] In this model, every equation and non-equation where the same term appears on both sides of $=$ will be true, as will be the standard stock of non-equations of distinct numerals - those which are formed without appeal to binary functions $+, \cdot$. However, it is not the case that the model verifies every equation between distinct numerals – ie. 0 with some number of applications of $'$. The remaining discriminateness of the model remains true for terms formed with the binary functions. A classical consequence relation would validate every formula, trivially, however, the LP consequence relation does not.

Consider some equation $b + c = a$, which is true of the standard model, where $d \neq d$ is introduced, where $d < a$. Then $a \neq a$ will be true, and, thus, so will be $b + c \neq a$, by Leibniz's law. So, if the formula is a theorem of the standard model and the term occurring on one side of the equation is glutty, then, the negation of the formula will be provable. However the converse does not need to be the case – if $b+c \neq a$ classically, the model does not, necessarily, verify $b+c = a$ as a result of the forced gluttiness of a. Consider some $a = b$. If this is true on the standard model, and a, b are glutty, then one can show $a \neq b$ - but if $a \neq b$ is true, it can still be the case that $a = b$ is simply false. These examples indicate that the gluttiness associated with single elements of the number line does not, without a classical consequence relation enforcing explosion, necessarily produce triviality.

These models are valuable really only to indicate that one can produce inconsistent models of arithmetic without appeal to collapse, but they do not have much in the way of inherent interest beyond that. They are included mainly as a limiting case - though to provide an axiomatization of these models would be, perhaps, illuminating for the rest of the project. As such, I include them here, even though they are unlike the other models I shall be considering. Since they are introduced to be, at least similar to the collapse models, I shall, from here on, stipulate that '$a' = b'$ implies $a = b$' is not true of these models, so that they can be treated uniformly under the same banner as the others. The logic to be developed to axiomatise these models will

generally reject this principle, so the decision is made to retain some cohesion in the subject matter I'll consider here.

1.2.2 Cyclic models

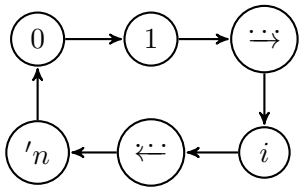
This and all the remaining classes of models considered in this chapter are collapse models, unlike the simple extended models. As such we must deal with two distinct entities - numbers in a model and equivalence classes of those numbers. In general, we shall most often be dealing with equivalence classes, usually using standard numbers only to specify $\sim$ in order to define the classes on the particular model.

The cyclic models result of applications of Dunn-Meyer extensions to produce models which are very similar to basic modular arithmetics, with a twist resulting from the LP semantics. Their forms are those of closed loops. They are produced by the closure of the standard model under the following equivalence relation $\sim$:

$\overline{x} \sim \overline{y}$ iff $\overline{x} \equiv \overline{y} \pmod{\overline{n}}$.

The results are quite simple - it is still the case that if $\overline{a}$ is distinct from $\overline{b}$, but $\overline{a} \sim \overline{b}$ then, by Theorem 1.2, $\overline{a} \neq \overline{b}$, but $\overline{a} \sim \overline{b}$ and so $\langle \overline{a}, \overline{b} \rangle \in I_+^{\sim}(=)$ and thus $a = b$. So $a = b \wedge a \neq b$, as expected, and the model, quite familiar, looks like this, for $0 \leq i \leq n$:

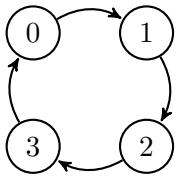
Figure 3.



Consider the following simple class and model as a concrete example:

$\overline{x} \sim \overline{y}$ iff $\overline{x} \equiv \overline{y} \pmod{\overline{4}}$:

Figure 4.



where $[1] = \{\overline{x}; \overline{x} \equiv \overline{1} \pmod{\overline{4}}\} = \{\overline{1}, \overline{5}, \overline{9}, \dots\}$, $[2] = \{\overline{2}, \overline{6}, \overline{10}, \dots\}$, $[3] = \{\overline{3}, \overline{7}, \overline{11}, \dots\}$, and $[0] = \{\overline{0}, \overline{4}, \overline{8}, \dots\}$. So, as a result of the construction, $3 = 3$ is true, as is $3 = 7$, and yet $3 \neq 7$ is also true, given the definition of $\neq$-clauses and since $\overline{3} \neq \overline{7}$. So $3 = 7$ is a glut, as is $3 = 11$, however claims like $3 = 2$ are still straightforwardly false. The result being that these models, though inconsistent, are so in a way that is regular and restricted. There are claims which are, on this model, straightforwardly false, despite the fact that the model admits of some glutty claims. These glutty claims, though inconsistent, fit into a regular structure which admits of a concise and sensible presentation.

These are very similar to modular arithmetics, even in their method of construction. The only difference is the presence of gluts in the equivalence classes - but even these are reasonable and are clearly results of the structure of the model. These are at least sensible structures, despite having inherent contradictions. Here there is a natural motivation to distinguish inconsistency and triviality – just because these models are inconsistent, it clearly does not follow that they are somehow ill-defined, and, on top of this, they capture some reasonable intuitions about their subjects. Consider the inconsistent cyclic models as a gloss on the usual modular arithmetics. The difference being that we actively treat the equivalences invoked as glutty, so that there is a sense in which 1=13 on a standard 12 hour clock, just as it remains the case that this is not actually true that 1=13 - both facts are encompassed in this simple model, and we can reason about these facts in an interesting way.

Beside the general structure, a feature particularly of note is that it remains the case here that each domain element has a unique successor (class) and a unique prede-

cessor (class) - that is to say, for each equivalence class a there is only one equivalence class b such that $b = a'$ and only one c such that $c' = a$. For instance, in the above, 1 has a unique predecessor, 0 and a unique successor, 2. It is straightforwardly false that $3' = 1$ and that $3 = 1'$. The fully injective successor function of these models is a feature which none of the other basic collapse models share. A result of these facts is that the full complement of standard successor principles still apply.

$$x = y \vdash x' = y' \qquad x \neq y \vdash x' \neq y'$$
$$x' = y' \vdash x = y \qquad x' \neq y' \vdash x \neq y$$

These four principles, usually run together, are worth distinguishing in the case of collapse models for reasons to be enumerated in the next chapter. Namely, that some of these principles are independent of the others is shown by the fact that some inconsistent models validate some but not all of them. However, for the purposes of cyclic models, they are all valid, and successor is basically similar to what it is classically.

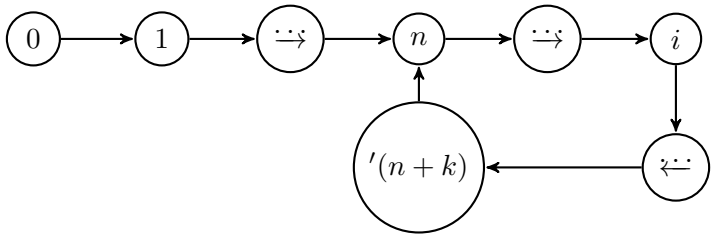
1.2.3 Heap models

Heap models are essentially cyclic models but beginning with finite tails of standard numbers, and then collapsing into a cycle with a first member which is the successor of a standard number. They are formed by closures under equivalence classes of the following structure, where $\overline{n} > \overline{0}$:

$\overline{x} \sim \overline{y}$ iff $((\overline{x}, \overline{y} < \overline{n}$ and $\overline{x} = \overline{y})$ or $(\overline{x}, \overline{y} \geq \overline{n}$ and $\overline{x} \equiv \overline{y} \pmod{\overline{k}}))$

So, every number occurring before $\overline{n}$ in the standard sequence occupies precisely the same place and has most of the same properties, but from $\overline{n}$ onwards forms a cycle of period $\overline{k}$. The resulting model looks like this:

Figure 5.

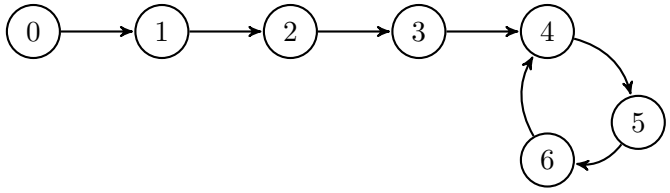


For example, consider the following equivalence:

$\overline{x} \sim \overline{y}$ iff $((\overline{x}, \overline{y} < \overline{4}$ and $\overline{x} = \overline{y})$ or $(\overline{x}, \overline{y} \geq \overline{4}$ and $\overline{x} \equiv \overline{y} \pmod{\overline{3}}))$

which produces the following:

Figure 6.



$4 = \{\overline{4}, \overline{7}, \overline{10} \dots\}$ — $5 = \{\overline{5}, \overline{8}, \overline{11} \dots\}$ — $6 = \{\overline{6}, \overline{9}, \overline{12} \dots\}$

So, for instance, $6 = 12$ is true, as is $6 \neq 12$, since $\overline{6} \neq \overline{12}$, and yet $6 = 5$ is straightforwardly false, as is $3 = 4$ and the like, for $0, 1, 2, 3$, which operate similarly to the classical numbers $\overline{0}, \overline{1}, \overline{2}, \overline{3}$ do. So, the cycle operates just as a cyclical model, while the tail operates as the standard ordering.

An interesting features of these models is that n, the least member of the cycle, has two predecessors, $n - 1$ and $n + \overline{k} - 1$ - so in the example $3' = 6'$ does not imply that $3 = 6$, since the succedent is simply false while the antecedent is a glut. Notice, that $(n - 1)' = ((n + \overline{k}) - 1)'$ has a designated value - namely $\{1, 0\}$ since $(\overline{n} - \overline{1})$ and $((\overline{n} + \overline{k}) - \overline{1})$ must be distinct in $\mathbb{N}$, so long as $k > 0$ - we have no negative integers, and yet $(n - 1)' = n$ and $((n + \overline{k}) - 1)' = n$, by the construction. However, $(n - 1) = ((n + \overline{k}) - 1)$ is assigned $\{0\}$ - that it is just false can be read off of the model.

So, this model invalidates the injectivity of the successor function, as every heap model does. In each heap model there will be one number which has two predecessors

- namely, the first number of the cycle, which is preceded by one standard number and one number in the cycle - though even it has a unique successor. However, every other number has a unique predecessor and a unique successor. The structure of the model makes this fairly clear. Since every number $< \overline{n}$ is isomorphic to itself in the standard model, it has a unique predecessor and a unique successor. For some equivalence class $i > n$, its successor i' will be in the cycle, as will all of its successors. For some a, $i^a \in n$, but the successor of no number in the cycle is properly outside of the cycle - applications of the successor function never take one out of the cycle.

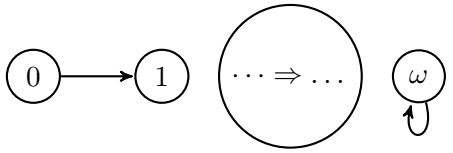
The fact that one of the standard successor principles fails will constrain the available options regarding axioms in the development of syntaxes to capture these models - however, in a way which proves fairly interesting in its own right. Beside this, the heap models are quite simple and quite well behaved - something Priest refers to in his defense of them as the potential model for the natural numbers as they are generally used (with the first inconsistent number being very large - larger than has been practically used yet).

A particular special case of these models is infinite. It involves a closure over a classical non-standard model of arithmetic. This is that given by the following congruence:

$\overline{x} \sim \overline{y}$ iff $(\overline{x}, \overline{y} < \omega$ and $\overline{x} = \overline{y})$ or $\overline{x}, \overline{y} \geq \omega$.

So, in this model all of $\mathbb{N}$ is as usual, and all non-standard numbers are identified in one equivalence class, with the successor function essentially just looping back on itself – $\omega' = \omega$. In the following depiction, the occurrence of $\cdots \Rightarrow \ldots$ is not meant to represent a successor-sequence, but rather that there is a split in the model, and that by moving along the order one arrives at a different chunk of the model with a different successor principle.

Figure 7.



This model may well have some philosophical interest associated with it, but it is not particularly interesting here except as just another example of a heap model. This sort of thing can be done with any finite number as well. Let the congruence be:

$\overline{x} \sim \overline{y}$ iff $(\overline{x}, \overline{y} < \overline{n}$ and $\overline{x} = \overline{y})$ or $\overline{x}, \overline{y} \geq \overline{n}$

This produces a model where all numbers $< \overline{n}$ are standard and then the equivalence class of $\overline{n}$ contains all numbers in the sequence from $\overline{n}$ on. The only notable difference between this model and the ω-model given above is that this is finite – simply consider the diagram given above but with $\overline{n}$ in the place of ω and a finite successor sequence in place of the $\cdots \Rightarrow \ldots$.

1.2.4 Infinite collapse models and the general case

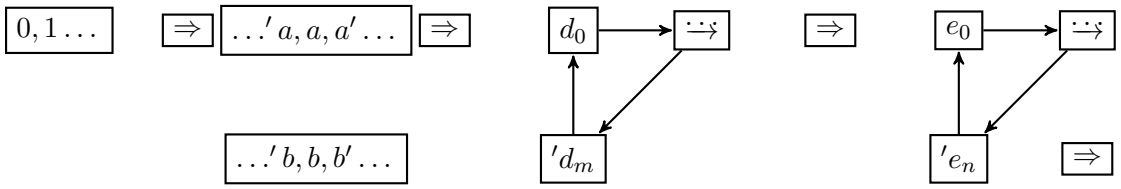
Unlike the basic cyclic and heap models displayed above, which are only finite, one can have infinite collapse models produced by a variety of different collapses. Probably the most interesting are those including $\mathbb{Z}$-like sequences. These are sequences obtained by taking a non-standard model of arithmetic, in the sense of [11] and others, and taking their closure under equivalence relations which leave alone the part of the model isomorphic to $\mathbb{N}$ and collapse parts of the densely-ordered sequence of $\mathbb{Z}$-isomorphic structures occurring afterwards. There are a number of possible results of the collapse of a non-standard model of arithmetic, or perhaps even the collapse of a collapse model, however anything more than a cursory glance at them is beyond the scope of this work, and, as such, I shall do nothing more than to indicate that these models exist. They may include a densely ordered sequence of structures isomorphic to $\mathbb{Z}$ which result from the collapse of a segment of a non-standard model into a something like a psuedo-cycle, the period of which is a non-standard number.

The classical picture of the non-standard models of arithmetic includes the sequence of natural numbers $\mathbb{N}$, followed by a densely ordered sequence of $\mathbb{Z}$-isomorphic structures. This results of a few basic assumptions involving ordering principles and the effect they have on numbers which are not the successors of anything in $\mathbb{N}$. Essentially, infinite collapse models can result of including some of these $\mathbb{Z}$-like sequences,

and collapsing others, or from collapsing the densely ordered sequence of $\mathbb{Z}$-like sequences into some kind of cycle or pseudo-cycle. Even as an initial consideration indicates, the options available for collapses of this kind are many and multi-faceted.So, the infinite collapse models are still under investigation, for instance, by Paris and Sirokofskich in [31], and while they are interesting, it is still very unclear whether a decidability or axiomatisability result can be proven for them. However, it may be hoped that insights gotten by the attempt to axiomatise the finite collapse models may shed light upon the question beyond what has been shown already. In any case, we shall not treat of these any further than this, which is set out to allow us to specify the general case of the collapse models.

This survey covers the basic kinds of Dunn-Meyer extension available[12] and as such any collapse model will be either one of the above basic models or a composite model constructed in some way out of the basic models. The general form is of a possibly non-empty subsequence of $\mathbb{N}$ - some standard numbers - followed by a possibly non-empty collection of disjoint $\mathbb{Z}$-like sequences, followed by a possibly non-empty collection of disjoint cycles, with some restrictions on the periods of the cycles.[13]

Figure 8.



In the above diagram, the occurrences of $\Rightarrow$ distinguish different non-standard parts of the model, either cycles or $\mathbb{Z}$-like sequences, each of which may have many occurrences - there is a dense ordering of $\mathbb{Z}$-like clumps among the a's and b's, and there may be similarly many cycles of d's and e's, where the periods of the cycles

[12]Priest, in [35, 233] he considers another kind of model - the *clique* models, inspired by graph theory. However, Paris and Pathmanathan show that an argument of Priest's is flawed and that such models cannot be produced, since their production would require an equivalence relation which is not properly a congruence on successors. [30, 531-532]

[13]In the following diagram squares are used instead of circles for purely aesthetic reasons.

occurring later are factors of the periods of cycles occurring earlier in the $\Rightarrow$-ordering. The number of non-standard structures depends on the nature of the non-standard model which is collapsed - ie, how many $\mathbb{Z}$-sequences occur after $\mathbb{N}$.

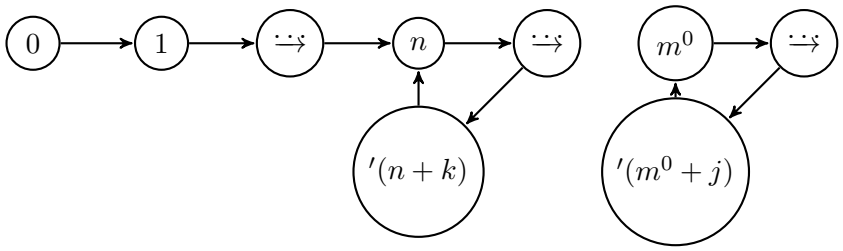
As a simple example of a collapsed model produced from a non-standard model with only the standard numbers and one finite cyclic sequence occurring after it, consider a congruence as follows:

$\overline{x} \sim \overline{y}$ iff $(\overline{x}, \overline{y} \leq \overline{n}$ and $\overline{x} = \overline{y})$ or $(\overline{x}, \overline{y} > \overline{n}$ and $\overline{x} \equiv \overline{y} \pmod{\overline{k}})$

or $(\overline{x}, \overline{y} > \omega$ and $\overline{x} \equiv \overline{y} \pmod{\overline{j}})$

which results in the numbers in the finite sequence in a non-standard model of arithmetic greater than $\overline{n}$ collected in a cycle of period $\overline{k}$ and all non-standard numbers in a cycle of period $\overline{j}$. So, the first sequence is a heap, followed by a finite cycle:

Figure 9.



Priest introduces terminology with a background in cellular biology to describe general collapse models. I shall not go into a lot of detail about the general case of infinite collapse models, however, for ease of statement I shall introduce these terms and present his general account of the structure of collapse models in these terms.[14]

- If $i \in$ some model M, $N(i)$, the *nucleus* of i, is defined as all those numbers which both precede and succeed i – $\{x \in M; i \leq x \leq i\}$. The period of a nucleus, p, is such that $i + p = i$, as usual. Call any nucleus with a non-zero period *proper* - so, any nucleus which forms a cycle is proper, be it a finite cycle or a sequence isomorphic to $\mathbb{Z}$ - else it is *improper*. The result of this is that every classical model is such that each number is in an improper nucleus - each

[14] I think that keeping these terms is probably best, at least for continuity's sake, even if they are potentially misleading given the biological background.

improper nucleus is a singleton, whereas, any inconsistent model produced by collapse must have at least one member with a proper nucleus.

- Let N be any proper nucleus, and $i \in N$. Then the *chromosome* of i is the sequence $\ldots'' i,' i, i, i', i'' \ldots$ (where not all members of the sequence need be distinct). Again, these can either be finite cycles or infinite sequences isomorphic to $\mathbb{Z}$. The basic models containing these types of chromosome I have called cyclic and $\mathbb{Z}$-like, respectively, and these are the terms I shall continue to use for the most part.

For the purposes of dealing with models involving multiple chromosomes, Priest defines a simple ordering on nuclei, where N_1, N_2 are nuclei: $N_1 \preceq N_2$ iff $\forall i \in N_1 \forall j \in N_2 (i \leq j)$, which is an order such that $N(0) \preceq N(1) \preceq, \ldots$. This ordering enforces an interesting property with regard to the periods of the nuclei involved. If $i \leq j$ and $N(i)$ has period p then so must $N(j)$. Since, if $i \leq j$, then for some x, $j = i + x$, and so $p + j = p + i + x$. Since $p + i = i$ then $p + j = i + x$ and thus $p + j = j$. This holds not just when $j \in N(i)$ and thus $N(i) = N(j)$, but also when i, j have separate nuclei. The result of which is that if p is the period of $N(i)$ then it is also the period of $N(j)$. So, nuclei occurring later in the sequence must have periods which are factors of those preceding, where that period is 0 in the case of a sequence of improper nuclei. I shall consider in more detail later on the effect of alterations of $\leq$ on the properties of $\preceq$.

With this information, Priest claims that:

> The general structure of a collapse model of arithmetic is a linear sequence of nuclei with periods inherited up the ordering. There are three segments (any of which may be empty). The first contains only improper nuclei. The second contains proper nuclei with linear chromosomes. The final segment contains proper nuclei with cyclical chromosomes. [36, 1522–1523]

So, after a finite tail similar to the natural numbers, each of which is an improper nucleus, is a collection of distinct linear orderings, each with a non-standard proper

period, followed by sets of cycles of numbers with decreasing standard proper periods, each of which is a factor of the previous. Each of these segments may be empty - producing simple models like those described in the previous sections, or something slightly more complicated like that in this section where the second segment was empty, and the third contained two cycles. But the result of this is that all collapse models can be described as either simple models like those previously presented or as a composition with parts like those, such as that given in the example presented earlier in this section.

1.3 Open problems of the collapse models

Priest has developed and disseminated the collapse models with a handful of goals. These include to promote interest in paraconsistent logic more generally, to develop his favoured dialethic view in the philosophy of mathematics, and to forward the *broadening* element of Routley's program – though he doesn't use this term. To the first and last of these ends he has set out open problems and intellectual puzzles in a variety of publications designed to attract the interest of researchers in different areas. Many of the purely number-theoretic questions, posed in [35] and [36] have been answered by Paris and his coauthors in [30] and [31] - while his claims about dialethic philosophy of mathematics, developed in [38] alongside a suite of papers, have been responded to - for instance in [46]. However, it is more a purely logical investigation which I here attempt to make progress on. We know that, while classical models of first order arithmetic are never finite, some inconsistent models are finite, and that if an inconsistent model is finite, then its set of theorems is decidable. [38, 234] This is because the functions and predicates employed are computable given a finite domain, so, since the truth tables are finite structures and the truth values of quantified statements are equivalent to finite conjunctions (or disjunctions), each conjunct of which is an application of the quantified predicate to one of the (finitely many) domain objects.

Now, decidability implies axiomatisability, ie, that all the truths of the model can

be stated by a finite set of formula schemata and finitary rules on the formulae.[15] This set of formulae is an *axiomatisation* of the model, and axiomatisability is a property which Priest claims to hold of the finite collapse models in [37, 267], where states as a problem, still open to my knowledge, whether any of the infinite models are axiomatisable. However, nowhere, again, to my knowledge, does he give an axiomatisation of any of the models. The result he states simply relies on the fact that one can simply enumerate the truths of the model and, in doing so, also enumerate axioms for it. I am more interested to see whether a more concise list of axioms can be given for the finite models – an axiom system comprising a proper subset of the true formulae of the model. I would also like to give an axiom system which is clearly similar to the standard classical arithmetical axiom system of Peano Arithmetic, since this is so familiar a system, and since such an axiomatization would give us a natural ground to compare the proof-theoretic features of the collapse models with those of the classical axiom system. Such a ground for comparison may well give us a valuable framework for translating classical arithmetic theorems into theorems of paraconsistent arithmetic, supposing a one-to-one translation of the connectives and the elements of the arithmetical language. This might provide us means to adjudicate some debates regarding paraconsistent and dialethic approaches to mathematics more generally, such as that between Shapiro [46] and Priest [38] on the Gödel sentence, which turns on the details of dialethic number theory. To this end, I shall give sensible axioms for the finite collapse models similar to those of Peano Arithmetic (PA) in an extension of LP, Arnon Avron's 3-valued paraconsistent logic PAC or A_3, to be developed in the next chapter.

The finite collapse models, at least, should be axiomatisable, and perhaps the development of a syntax for them may provide insight into whether it is possible to do the same for the infinite models. However, even in the case of the finite models, there are considerations to suggest that to give an axiom system for them is difficult. In particular, it is a roadblock that each collapse model models PA, since each is

[15]This not to say that the truths of *only* that model are captured by the axiom system - ie. that the model is categorical. This is not even true of Peano arithmetic, the fact of which is what gives rise to non-standard models (and collapse models) in the first place.

a collapse of the standard model, assuming again a one-to-one translation of the components of the language of PA to similar components in LP. If the addition of the Peano axioms to a maximally paraconsistent logic does produce a system in which a proper subset of the theorems of classical PA are provable, then no non-trivial axiom system can be given for the inconsistent models, since they would be trivialised under classical consequence. This is a formidable concern, and I shall attempt to respond to it later in this thesis. First, however, we need a better syntax to capture the essential parts of LP as well as some basic arithmetical axioms to provide a groundwork to talk about the special axioms needed for extensional and finite collapse models.

Chapter 2

A_3 and Peano Arithmetic

As shown in the previous chapter, collapse models of arithmetic exhibit a handful of peculiar properties, and these delimit the choice of proof systems capable of expressing them. The results are two distinct sets of criteria. First, loosely grouped, are those properties which limit the background logic in which an appropriate theory of arithmetic may be couched. These have mostly to do with best capturing the basic LP semantics. The models, though they are expressed in a more of less standard language of arithmetic, are not profitably studied in a non-paraconsistent logic, since they're generated by appeal to inconsistency in a fundamental and natural way, as considered in the previous chapter. While the form of any particular model may be reproducible in classical terms, the semantic features of these models which make them more interesting than standard modular arithmetics cannot be non-trivially treated of in a logic admitting explosion. Namely, the clearly non-trivialising inconsistencies resulting from the Dunn-Meyer extension cannot be dealt with in classical PA. Thus, it is a paraconsistent logic needed to capture the interesting elements of these models, and since the models are developed in the LP model theory, it is a natural starting point to consider conservative extensions of LP (since any non-conservative extension would yield classical logic, as is shown in [3]).

However, what is necessary is an *extension* of LP. As it stands, LP (with the conditional defined as a material implication) does not lend itself to the kind of syntax regularly used for axiomatic mathematical reasoning. This is because the material implication in LP does not obey detachment, and as a result it does not

validate most of the standard conditional inferences. Thus, the logic does not admit of a usual Hilbert-style proof system. So, what is needed is a conservative extension of LP resulting from the addition of a detachable conditional suited to it. That is, a conditional which fully reflects the consequence relation. As such, the basis of a proof system for these models should be a paraconsistent logic conservatively extending LP with respect to treating conjunctions, disjunctions, negations, and atomic sentences similarly, but with a conditional better suited to axiomatic reasoning. In particular, this conditional must, as closely as possible, express the LP consequence relation in its object language. I shall more fully detail these criteria and present a logic developed by Arnon Avron, in semantic, axiomatic, and sequent calculus formulations, which does this job nicely. His original name for this is PAC, but I shall, following tradition in logical nomenclature and as a nod to Avron, call this logic A_3.

The second set of criteria for expressing the LP models is that of how the Peano axioms must be altered to account for the somewhat unusual properties exhibited by the function symbols in such models. For instance, in some LP-models of arithmetic the successor function is not injective. I shall detail these changes in the specifically arithmetical part of the construction and indicate what kinds of additions can be made to mitigate these alterations - in order to keep the theory as close to Peano Arithmetic as possible. These alterations will, as I'll show, nicely fit in with the peculiar elements of A_3's consequence relation and conditional, particularly the failure of contraposition, and will largely come down to the rejection of one variation on a Peano axiom for successors, and its consequences.

With this completed, the axiomatic basis for these models will be presented, allowing for the specifications necessary to account for particular classes of finite inconsistent models in the following chapter.

2.1 Proof System for collapse models

Proof systems have been presented for LP, most prominently a tableaux system produced by Priest and presented in [39]. However, it is difficult to specify a Hilbert-style

axiom system for LP since its conditional is so degenerate. Recall that the LP conditional connective is just the material conditional, $A \supset B =_{Def} \neg A \vee B$, resulting in the failure of detachment. Hilbert systems are generally specified in terms of axioms with the conditional as the main connective and detachment, so since this rule is invalid in LP it is unclear to me how one would give a Hilbert style syntax for it. These same issues, though perhaps without the same decisiveness, make it unclear what kind of natural-seeming treatment of axiomatic systems of reasoning one could give in LP. As if to match this intuition of mine, LP is most often employed purely model theoretically, as a semantics which allows one to make distinctions which are impossible in classical logic, as in the case of the papers on LP-models of arithmetic - those of Priest and those of Paris and Pathmanathan and Paris and Sirokofskich. This lack of a strong grounding in multiple proof systems – what Anderson and Belnap referred to as the naturalness or substantiality of a system [2, 50] – has not hindered LP, since it gives rise to so strong a paraconsistent model theory as to be generally applicable in cases where classical logic is. The many-valued semantics of LP differs from the bivalent classical semantics only in cases related to explosion, and so it is great as the first paraconsistent logic to bring on the scene to study some paradoxical or inconsistent phenomenon. For these purposes, a wide selection of proof systems is not necessary, and so not missed.

However, it seems unsatisfying to have an axiomatisability result, such as that stated for the collapse models in [36] and elsewhere, for purely model theoretic reasons - i.e. to do with the size of the models. Axioms are proof theoretic objects, and the claim that one can axiomatise a model or some system of information is a proof theoretic claim. It is the claim that the information in the system can be represented by strings of symbols, perhaps governed by some rules of inference, such that the list of basic string-schemata and the list of rules are finite. The sense in which the finite LP models are axiomatisable is that one can indeed provide a finite list of the true claims of the models, each of which is an axiom. So they are clearly axiomatisable, but it seems that we could want more in the way of explanatory clarity and parsimony. Why is it the case that these statements come out true? In addition to this, exactly

which basic assumptions operative in standard arithmetic fail to hold of these models? What we'd like is an axiom system which lists some proper subset of the theorems, preferably one which is small and elegant, with independent axioms and rules. We'd also like this system to be as similar to Peano arithmetic as possible, allowing us to make easy comparisons between them, allowing us to determine exactly what alterations are necessary, and thus, which we can avoid making. As a general rule, we'd like to keep as much of the classical system as possible, excising only those parts which we must. So, we want an axiomatisability result for a system bearing some resemblance to Peano arithmetic.

So, we need a logic that retains the elements of the LP semantics, but adds a more effective conditional connective. Note that in falsifying detachment the LP material conditional also fails to match the LP consequence relation. That is, $\vdash A \supset B$ may hold while $A \vdash B$ does not. The conditions for the former allow more cases than do the latter. Now, the standard deduction theorem holds for the LP material conditional: $A \vdash B$ only if $\vdash A \supset B$, but the other direction - which is generally assumed, being a necessary condition for detachment, fails. The upshot of a deduction theorem seems to be to give us some information about the relationship between the metalinguistic construction of the consequence relation and an object language connective - the conditional.[1] It indicates that both constructs behave similarly, though they belong to different languages and perform importantly different roles. Having the tight match between these means that a consequence is valid just in case the premise set, conjunctively aggregated, entails the conclusion set, disjunctively aggregated. As such, the material conditional, though it has a truth-functional definition in the LP semantics, does not capture the LP consequence relation, alongside with its being nearly useless for the purposes of derivation. So, another conditional which can be added to LP, and that does capture the consequence relation would be valuable. If this conditional connective were to properly capture the consequence relation, it would obey detachment, as well as a number of other, standard, conditional proper-

[1]The claim which goes in both directions: $\vdash A \supset B$ iff $A \vdash B$ has been called the *naïve validity scheme* by Zach Weber [49], and it is this which gives the strong metalinguistic connection which I discuss here.

ties, making a Hilbert style system feasible. Thus, it would be a conditional which could be usefully identified with that in which arithmetical axioms are stated.

Now, every theorem of LP is a theorem of classical logic, and conversely (Theorem 1.1). So, in the usual sense of "extension" having to do with increasing the set of a logic's *theorems*, if the material implication of LP were to be uniformly replaced with a detachable conditional connective, the result would be a reduction of LP in terms of its conditional theorems.[2] This is because the LP consequence relation is *less* permissive than the material implication, a fact which will feature in the properties of the new connective. Note, however, that the extension of LP by the proposed conditional should be conservative, ie. the set of theorems and sequents stated in LP's language would remain unaltered, in the expanded language. This is a result of the constraints we place on candidate logics - namely, that they feature the same negation, conjunction, and disjunction as LP. The logic one aims to produce by adding a conditional to LP should not include as theorems statements equivalent to $\vdash (A \wedge \neg A) \rightarrow B$, where $\rightarrow$ is the detachable conditional to be introduced. These are not valid across the LP consequence relation, since $A \wedge \neg A \nvdash B$ is not a valid sequent. This is in distinction to $(A \wedge \neg A) \supset B$, which, of course, is a theorem.

So, $A \vdash B$, if and only if it is not the case that at some valuation the value of A is designated (either $\{1\}$ or $\{1, 0\}$) and that of B non-designated ($\{0\}$). LP's consequence relation is nicely described by Arnon Avron in [5] and, in the same chapter, he provides a conditional which captures it and begins to develop the resulting logic. The consequence relation, to start with, fits the standard requirements of being a relation between sets of sentences which is reflexive and transitive. In addition to this, it also obeys thinning, contraction and permutation, trivially since it is defined in terms of sets of formulae. In addition to these, it validates the law of excluded middle. These are listed, with a short combinator name for each, to which I'll refer throughout the chapter:

Reflexivity (I): $A \vdash A$ for every formula A

[2]It is a question which connective plays the role of the conditional in a logic - what is clear is that the material conditional in unacceptable for this role in LP.

Transitivity (B): If $\Gamma_1 \vdash \Delta_1, A$ and $A, \Gamma_2 \vdash \Delta_2$ then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$

Thinning (K): If $\Gamma_1 \vdash \Gamma_2$ then $A, \Gamma_1 \vdash \Gamma_2$ and $\Gamma_1 \vdash \Gamma_2, A$

Contraction (W): If $A, A, \Gamma_1 \vdash \Gamma_2$ then $A, \Gamma_1 \vdash \Gamma_2$

and $\Gamma_1 \vdash \Gamma_1, A, A$ only if $\Gamma_1 \vdash \Gamma_2, A$

Permutation (C): If $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$ then $\Gamma_1, B, A, \Gamma_2 \vdash \Delta$

and if $\Gamma \vdash \Delta_1, A, B, \Delta_2$ then $\Gamma \vdash \Delta_1, B, A, \Delta_2$

Law of Excluded Middle (LEM): $\vdash A, \neg A$

What the consequence relation does not allow is any form of explosion. In particular, while LEM is valid, the dual to LEM is not. This occurs directly below and, at least for the basic sequent calculus to be developed in this chapter, is not even a well-formed sequent.

$$A, \neg A \vdash$$

This sequent, in the presence of unrestricted K in the succedent allows for the derivation of $A, \neg A \vdash B$ - a most blatant statement of explosion, and one which, with the standard rules, allows for the proof of $\vdash A \to (\neg A \to B)$ and $\vdash (A \wedge \neg A) \to B$. Since the rejection of these is characteristic of paraconsistent logic, any treatment of such a logic in terms of sequents must either reject $A, \neg A \vdash$ or must place some restrictions on K in the succedent. I shall, however, leave open the possibility that we may stipulate a different formula as producing this behaviour. Perhaps something like f where $f \vdash$ holds. I shall refer to f as an "absolutely inconsistent" formula, since it produces absolute inconsistency.[3]

[3]It is a question which I'll consider somewhat in the third chapter whether any other formula behaves as an absolute inconsistency. That is, whether any other formula trivialises to the right, since arbitrary contradictions do not. Some have proposed that we may, on a case by case basis introduce specific principles, depending on what we take to be seriously bad things to derive. Now, we may want to distinguish between a formula's being bad to be derived and its being inconsistent, indeed, that is just another way to state the driving motivation behind paraconsistency. As it stands, in the logic to be developed there are no absolutely false formulae, which may occur in the antecedent of a sequent with an empty succedent. So, there is some flexibility as to whether we may simply add some absolutely false formulae, depending on what we desire to use the logic to study. We shall return to this point in the third chapter.

This is a shortened form of the statement of the LP consequence relation given by Avron in [5] and [6], though he does not refer to it as such. In [5, 17], he argues as follows towards the most natural conditional for the LP semantics - I replace the occurrences of $\supset$ (the horseshoe) as the material conditional by occurrences of $\rightarrow$ (the arrow) as the detachable conditional to be introduced, since I would like to use the former to refer specifically to the material conditional in LP and the latter to refer to the new conditional, and I alter the symbols to match those used so far in this thesis. Note that statements like $A \rightarrow 0$ are shorthand for "that conditional with the antecedent A and a consequent which has the truth value $\{0\}$". The language of LP, as I have defined it, does not contain constants for the truth values, nor can they be defined. However, for these purposes, it is sufficient to treat these terms as shorthand for formulae taking those values.

> The condition $A, A \rightarrow B \vdash B$ implies that $A \rightarrow 0 = 0$ if $A \in \{\{1\}, \{1, 0\}\}$ (ie, A is designated). The conditions $A \vdash B \rightarrow A$, (related to K), and $\vdash A \rightarrow A$, (related to I), imply that $A \rightarrow B$ is designated in all other cases. The conditions $\neg(A \rightarrow B) \vdash A$ and $\neg(A \rightarrow B) \vdash \neg B$ imply respectively that $0 \rightarrow A = 1$ and $A \rightarrow 1 = 1$. The condition $A, \neg B \vdash \neg(A \rightarrow B)$ implies that if A is designated and $B = \mathfrak{i}$ then $A \rightarrow B$ cannot be 1. Since it cannot be 0, it must be $\mathfrak{i}$. [5, 17]

This reasoning narrows the possibilities to a single conditional:

$$v(A \rightarrow B) = \begin{cases} 1 & \text{if } v(A) = 0 \\ v(B) & \text{else} \end{cases}$$

or

$\rightarrow$	1	$\mathfrak{i}$	0
1	1	$\mathfrak{i}$	0
$\mathfrak{i}$	1	$\mathfrak{i}$	0
0	1	1	1

The logic which results from adding this conditional to LP is given by Avron in [5], I call it A_3 and it differs from LP semantically only in that its conditional is identified to be the above connective instead of the material implication. I shall leave off giving A_3 a distinct semantic definition, and instead simply point out that it results by taking the semantics of LP (as in the last chapter) and adding the above conditional

to it. That is, A_3's conjunction and disjunction are the same as the LP conjunction and disjunction - namely, they are, respectively, meet (least upper bound) and join (greatest lower bound) - and its negation is also the same - a subcontrary forming operator where $v(\neg A) = v(A)$ if $v(A) = \{1, 0\}$. As I'll go on to show, the extension to the first order can be carried out in A_3 as in LP. So, A_3 meets the basic condition required, and is conservative over LP.

However, we do not simply wind up with a conditional obeying all the standard conditional principles of classical logic. For instance, in the LP semantics it is not generally the case that if $A \vdash B$ then $\neg B \vdash \neg A$. Suppose that $v(A) = \{1\}$ and $v(B) = \{1, 0\}$, validating the former. Then $v(\neg A) = \{0\}$ and $v(\neg B) = \{1, 0\}$, invalidating the latter. As such, a conditional which captures the LP consequence relation should not validate contraposition which, as we'll see, the A_3 conditional does not. In fact, this is probably the most notable feature of A_3, as nearly all other properties of the classical conditional hold for its conditional. The strength of A_3 lends an intuitive clarity to the claim that it is strongly maximal, as shown in [3]. That is, any extension to A_3 would have to be the result of the addition of some formula which would allow the proof of explosion. This provides some grounds for hope that a formal theory of arithmetic in A_3 may well be strong enough to closely resemble axiomatic arithmetics as presented in classical logic.[4]

It is interesting to note that one can define a conditional connective in A_3 which does contrapose, let us $\Rightarrow$ for this connective. This definition is given by $A \Rightarrow B$ iff $(A \rightarrow B) \wedge (\neg B \rightarrow \neg A)$, and this corresponds with the conditional of the logic RM_3, which can also be produced by the addition of a conditional to the LP semantics. Its truth table is:

[4]In fact, A_3 is far too strong for the purposes of naïve set theory, another mathematical theory which paraconsistent logics are uniquely suited to - for which non-triviality demands either rejecting detachment or contraction in order to avoid Curry's paradox. However, for our purposes here no such things are required, none of the Peano axioms, even induction, being nearly so strong as comprehension, and so not giving rise to any obvious paradoxes. They do not, to my knowledge, give rise to trivialising paradoxes. In fact, one property of A_3 which Avron [6] claims is that there is no right-trivialising formula native to A_3. A formula such as this could, of course, be added as a primitive, perhaps, as a means to prove expressive completeness.

$\rightarrow$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

Which admits of the proof of contraposition, but not of thinning, that is $\nvdash A \Rightarrow (B \Rightarrow A)$.[5] The definability of this arrow in A_3 puts into our hands three conditional-like connectives. There is the conditional $\rightarrow$ defined by Avron, to be used throughout this thesis, the RM_3 conditional, as well as the material implication $\supset$ (though this admittedly rather less conditional-like than the others, especially in LP).

2.1.1 A_3 - A Hilbert style axiom system

I shall use the standard abbreviation $\leftrightarrow$; $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$. Given (A4), $A \leftrightarrow B$ easily yields $A \rightarrow B$, $B \rightarrow A$, and in the cases of the axioms stated with $\leftrightarrow$, I shall usually just help myself to these easy results when presenting axiomatic proofs.

A1 $A \rightarrow (B \rightarrow A)$

A2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

A3 $((A \rightarrow B) \rightarrow A) \rightarrow A$

A4 $(A \wedge B) \rightarrow A \qquad (A \wedge B) \rightarrow B$

A5 $A \rightarrow (B \rightarrow (A \wedge B))$

A6 $A \rightarrow (A \vee B) \qquad B \rightarrow (A \vee B)$

A7 $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)))$

A8 $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

A9 $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$

A10 $\neg\neg A \leftrightarrow A$

[5] It is also interesting to note that one can define the A_3 $\rightarrow$ in RM_3 by $(A \Rightarrow B) \vee B$.

A11 $\neg(A \to B) \leftrightarrow (A \wedge \neg B)$

A12 $A \vee \neg A$

Rule of Inference

MP A and $A \to B$ imply B

This clearly matches the motivational work done up to this point. Present are axioms for a standard conjunction and disjunction (A4)-(A7). These and the A_3 negation obeys the de Morgan equivalences, double negation, counterexample, and the law of excluded middle (A8)-(A12). In addition to this (A1-3) account for standard properties of the conditional and (MP) is the rule. However, there is no principle from which contraposition can be derived, and even though counterexample (A11) in conjunction with the de Morgan laws does allow the proof of $(A \to B) \to (\neg A \vee B)$, the converse cannot be shown. So this conditional is not equivalent to the material implication.[6]

This axiom system is shown to be adequate to the propositional A_3 semantics in [6]. However, A_3 is not given a first order treatment by Avron. A first order version of the logic is necessary for arithmetical axioms resembling those of Peano

[6]It is interesting to note that $A \to \neg\neg A$ has to be included as an axiom. Usually this is avoided by including a version of contraposition, $(A \to B) \to (\neg B \to \neg A)$, which allows the proof of double negation introduction. However, along with the rest of A_3, the addition of any version of contraposition would simply produce classical logic. To see that this holds at least of the displayed version of contraposition, consider the following proof:

(1) $\vdash (A \vee \neg A) \to (\neg B \to (A \vee \neg A))$

(2) $\vdash A \vee \neg A$

(3) $\vdash \neg B \to (A \vee \neg A)$

(4) $\vdash (\neg B \to (A \vee \neg A)) \to (\neg(A \vee \neg A) \to \neg\neg B)$ (Contraposition)

(5) $\vdash \neg(A \vee \neg A) \to \neg\neg B$

Where (5) is equivalent, by some simple transformations involving two applications of de Morgan laws, double negation elimination, and exportation to $A \to (\neg A \to B)$. Of course, a more direct proof of the conditional form, $A \to (\neg A \to B)$, can be given with $(\neg B \to A) \to (\neg A \to B)$, prefixing, and an instance of K. This proof takes just five lines, and similar proofs involving other versions of contraposition can be given with the aid of double negation rules, as above. Thanks to Hassan Masoud for discussion on this point.

arithmetic. So, the logic shall need to be extended, and the proofs given by Avron for the propositional A_3 also extended to include the quantifiers and the rules governing them. But first, the reasons adduced in this section so far justify a theorem, which it is at least worth stating. The proof of this theorem will be omitted here as proofs of other theorems in the remainder of this chapter provide obvious grounds for the theorem.

Theorem 2.1 (Deduction)**.** $\vdash A \to B$ *iff* $A \vdash B$ *in* A_3.

To extend A_3 to the first order is quite simple. The only potential concern being to ensure that the addition of the quantifiers is conservative. This just amounts to ensuring that no version of explosion can result from the quantifier axioms and their inference rules. However, the positive fragment of classical logic is equivalent to A_3^+, the positive fragment of A_3, when the material conditional $\supset$ is identified with A_3's $\to$. So it is only the negative parts of A_3 which need be considered. The negative part of the logic can be specified by adding utterly standard negated-quantifier equivalences, and these do not offer any obvious way to collapse the logic into classical logic. So, the following axioms and rule should suffice without going overboard:

A13 $\forall x Ax \to Ay$ - where y is a variable and y is free for x in A.

A14 $\forall x(A \to Bx) \to (A \to \forall x Bx)$ where x is not free in A.

A15 $\forall x(A \vee Bx) \to (A \vee \forall x Bx)$ where x is not free in A.

A16 $Ay \to \exists x Ax$

A17 $\forall x(Ax \to B) \to (\exists x Ax \to B)$ where x is not free in B.

A18 $(\forall x \neg Ax) \leftrightarrow (\neg \exists x Ax)$

A19 $(\exists x \neg Ax) \leftrightarrow (\neg \forall x Ax)$

Rule

Gen Ay implies $\forall x Ax$

(A13) is standard, as are (A14) and (A15), which simply state that the universal quantifier interacts with the conditional and disjunction in a natural way. (A16) and (A17) put back into our hands some usual formulae which rely on contraposition involving the existential quantifier, and (A18) and (A19) give the usual interaction between the universal and existential quantifiers.[7] The definition of *proof* and *theorem* in A_3 are just as in standard Hilbert style systems.

While a Hilbert system makes it quite obvious what the inferential properties of the logic are, it is not the most user-friendly method for producing derivations in the logic itself. Since our interest in this topic involves a bit of proving of particular arithmetical theorems in the object language, having a proof system which is more easy to work with, such as a Gentzen system, or sequent calculus, is valuable.

2.1.2 $\mathbf{LA}_3$ – A Gentzen system for $\mathbf{A}_3$

There are two axioms, one, I, enforcing the reflexivity of the consequence relation, and the other a form of the law of excluded middle:

$$A \vdash A \quad (\mathsf{I}) \qquad\qquad \vdash A, \neg A \quad (\text{LEM})$$

In addition to these are the full-strength complement of structural rules. These are simply Cut and K since the antecedent and succedent of a sequent are sets, hence Gentzen's rules C and W are excluded, since they could not be formulated.

Structural Rules

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Theta}{\Pi, \Gamma \vdash \Delta, \Theta}\ \text{Cut}$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \mathsf{K}\vdash \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \vdash \mathsf{K}$$

We may also place a restriction on the K rules to the effect that one may not introduce a formula by K which already occurs in the premise. This is to avoid potential for proofs with the same end-sequent as premise which, though harmless, are a possible irritation in proofs about the system, or in designing a proof-search method.

[7] I am following Brady [12] in this format for the quantifier axioms, and the equivalence theorem given below bears out these choices.

In addition to these there are logical rules. First are the rules for $\rightarrow, \wedge$ and $\vee$ which are the standard sequent rules for those connectives. Next, the negative rules, of which there are one set per connective. Unlike the classical sequent rules for negation, what are here included are rules which allow one to enforce standard equivalences between some formulae and a negated formula which has as its main connective one of the other operations (for instance, there is a separate set of rules to produce formulae like $\neg(A \rightarrow B)$ and $\neg(A \wedge B)$). Note that the negation rules do not allow one to simply produce the negation of an atomic formula by moving it across the turnstile - this would lead to explosion trivially in the presence of unrestricted K. Instead, the only way to introduce the negation of an atomic formula is by one of the axioms - one can either introduce $\neg A \vdash \neg A$ or $\vdash A, \neg A$ and then manipulate these to produce more complicated negated formulae. Finally, the quantificational rules simply follow the same pattern. The negation-free form of each is standard, while the negated form of each enforces the interdefinability conditions explicitly set out in the Hilbert presentation by (A18) and (A19).

Logical Rules

$$(\rightarrow\vdash)\ \frac{\Gamma \vdash \Delta, A \qquad B, \Pi \vdash \Theta}{A \rightarrow B, \Gamma, \Pi \vdash \Delta, \Theta} \qquad (\vdash\rightarrow)\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash \Delta, A \rightarrow B}$$

$$(\wedge\vdash)\ \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad (\vdash\wedge)\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$(\vee\vdash)\ \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad (\vdash\vee)\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$(\neg\neg\vdash)\ \frac{A, \Gamma \vdash \Delta}{\neg\neg A, \Gamma \vdash \Delta} \qquad (\vdash\neg\neg)\ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \neg\neg A}$$

$$(\neg\rightarrow\vdash)\ \frac{A, \neg B, \Gamma \vdash \Delta}{\neg(A \rightarrow B), \Gamma \vdash \Delta} \qquad (\vdash\neg\rightarrow)\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \rightarrow B)}$$

$$(\neg\vee\vdash)\ \frac{\neg A, \neg B, \Gamma \vdash \Delta}{\neg(A \vee B), \Gamma \vdash \Delta} \qquad (\vdash\neg\vee)\ \frac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \vee B)}$$

$$(\neg\wedge\vdash)\ \frac{\neg A, \Gamma \vdash \Delta \qquad \neg B, \Gamma \vdash \Delta}{\neg(A \wedge B), \Gamma \vdash \Delta} \qquad (\vdash\neg\wedge)\ \frac{\Gamma \vdash \Delta, \neg A, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)}$$

$$(\forall\vdash)\ \frac{Ay, \Gamma \vdash \Delta}{\forall x Ax, \Gamma \vdash \Delta} \qquad (\vdash\forall)\ \frac{\Gamma \vdash \Delta, Ay}{\Gamma \vdash \Delta, \forall x Ax}\ (*)$$

$$(\exists\vdash)\ \frac{Ay, \Gamma \vdash \Delta}{\exists x Ax, \Gamma \vdash \Delta}\ (*) \qquad (\vdash\exists)\ \frac{\Gamma \vdash \Delta, Ay}{\Gamma \vdash \Delta, \exists x Ax}$$

$$(\neg\forall\vdash)\ \frac{\neg Ay, \Gamma \vdash \Delta}{\neg\forall xAx, \Gamma \vdash \Delta}\ (*) \qquad (\vdash \neg\forall)\ \frac{\Gamma \vdash \Delta, \neg Ay}{\Gamma \vdash \Delta, \neg\forall xAx}$$

$$(\neg\exists\vdash)\ \frac{\neg Ay, \Gamma \vdash \Delta}{\neg\exists xAx, \Gamma \vdash \Delta} \qquad (\vdash \neg\exists)\ \frac{\Gamma \vdash \Delta, \neg Ay}{\Gamma \vdash \Delta, \neg\exists xAx}\ (*)$$

Now, there are the usual limitations placed on the use of the $\vdash \forall$ and $\exists \vdash$ and this is extended to $\neg\forall \vdash$ and $\vdash \neg\exists$. These rules are marked by (*). In order to introduce $\forall xAx$ in the succedent the variable y must not occur free in the lower sequent of the application of the rule. The same goes with the introduction of $\exists xAx$ in the antecedent, of $\neg\forall xAx$ in the antecedent, and of $\neg\exists xAx$ in the succedent. Again, *theorem* and *proof* are defined as usual for Gentzen style systems.

I shall generally refer to all the rules which introduce a $\neg$ as the negative rules, and all the others as the positive rules. Notice that the positive rules are all similar to rules in some sequent calculi for classical logic and it is a natural result that the positive fragment of the logic, LA_3^+ coincides with some such calculus LK^+.[8] I shall often make use of this fairly obvious claim in the rest of this thesis.

Theorem 2.2. *Axioms can be restricted to literals – atomic formulae or the negations of atomic formulae – without a change in the set of theorems of LA_3.*

Proof. The proof is by cases on the main connective of the formula and the axiom instance, either of reflexivity (I) or law of excluded middle (L). So, (I -$\wedge$) is the case involving the proof of $A \wedge B \vdash A \wedge B$ and (L -$\wedge$) is the case involving the proof of $\vdash A \wedge B, \neg A \wedge B$.

$$(\mathsf{I}\text{-}\wedge)\ \frac{\dfrac{A \vdash A}{A, B \vdash A} \quad \dfrac{B \vdash B}{A, B \vdash B}}{\dfrac{A, B \vdash A \wedge B}{A \wedge B \vdash A \wedge B}} \qquad (\mathsf{I}\text{-}\vee)\ \frac{\dfrac{A \vdash A}{A \vdash A, B} \quad \dfrac{B \vdash B}{B \vdash A, B}}{\dfrac{A \vee B \vdash A, B}{A \vee B \vdash A \vee B}}$$

$$(\mathsf{I}\text{-}\rightarrow)\ \frac{\dfrac{A \vdash A \qquad B \vdash B}{A, A \rightarrow B \vdash B}}{A \rightarrow B \vdash A \rightarrow B} \qquad (\mathsf{I}\text{-}\neg\wedge)\ \frac{\dfrac{\neg A \vdash \neg A}{\neg A \vdash \neg A, \neg B} \quad \dfrac{\neg B \vdash \neg B}{\neg B \vdash \neg A, \neg B}}{\dfrac{\neg(A \wedge B) \vdash \neg A, \neg B}{\neg(A \wedge B) \vdash \neg(A \wedge B)}}$$

[8]Note, LK^+ may or may not be as Gentzen's original system, which included C and W, since it operated with sequences not sets. For the purposes of this work, suppose that LK is a system like Gentzen's but taking sets as the data type, and thus including analogues to all of the positive rules of LA_3.

$$(\mathsf{I}\text{-}\neg\vee)\ \dfrac{\dfrac{\dfrac{\neg A\vdash\neg A}{\neg A,\neg B\vdash\neg A}\quad\dfrac{\neg B\vdash\neg B}{\neg A,\neg B\vdash\neg B}}{\neg A,\neg B\vdash\neg(A\vee B)}}{\neg(A\vee B)\vdash\neg(A\vee B)}$$

$$(\mathsf{I}\text{-}\neg\to)\ \dfrac{\dfrac{\dfrac{A\vdash A}{A,\neg B\vdash A}\quad\dfrac{\neg B\vdash\neg B}{A,\neg B\vdash\neg B}}{A,\neg B\vdash\neg(A\to B)}}{\neg(A\to B)\vdash\neg(A\to B)}$$

$$(\mathsf{I}\text{-}\forall)\ \dfrac{\dfrac{Ay\vdash Ay}{\forall xAx\vdash Ay}}{\forall xAx\vdash\forall xAx}\qquad(\mathsf{I}\text{-}\exists)\ \dfrac{\dfrac{Ay\vdash Ay}{Ay\vdash\exists xAx}}{\exists xAx\vdash\exists xAx}\qquad(\mathsf{I}\text{-}\neg\neg)\ \dfrac{\dfrac{A\vdash A}{\neg\neg A\vdash A}}{\neg\neg A\vdash\neg\neg A}$$

$$(\mathsf{I}\text{-}\neg\forall)\ \dfrac{\dfrac{\neg Ay\vdash\neg Ay}{\neg Ay\vdash\neg\forall xAx}}{\neg\forall xAx\vdash\neg\forall xAx}\qquad(\mathsf{I}\text{-}\neg\exists)\ \dfrac{\dfrac{\neg Ay\vdash\neg Ay}{\neg\exists xAx\vdash Ay}}{\neg\exists xAx\vdash\neg\exists xAx}$$

$$(\text{L-}\vee)\ \dfrac{\dfrac{\vdash A,\neg A}{\vdash A,\neg A,B}\quad\dfrac{\vdash B,\neg B}{\vdash B,\neg B,A}}{\vdash A,B,\neg(A\vee B)}{\vdash A\vee B,\neg(A\vee B)}\qquad(\text{L-}\wedge)\ \dfrac{\dfrac{\vdash A,\neg A}{\vdash A,\neg A,\neg B}\quad\dfrac{\vdash B,\neg B}{\vdash B,\neg B,\neg A}}{\vdash A\wedge B,\neg A,\neg B}{\vdash A\wedge B,\neg(A\wedge B)}$$

$$(\text{L-}\to)\ \dfrac{\dfrac{\dfrac{A\vdash A}{A\vdash B,A}\quad\dfrac{\vdash B,\neg B}{A\vdash B,\neg B}}{A\vdash B,\neg(A\to B)}}{\vdash A\to B,\neg(A\to B)}\qquad(L-\neg\neg)\ \dfrac{\dfrac{\vdash A,\neg A}{\vdash\neg\neg A,\neg A}}{\vdash\neg\neg A,\neg\neg\neg A}$$

$$(\text{L-}\forall)\ \dfrac{\dfrac{\vdash Ay,\neg Ay}{\vdash Ay,\neg\forall xAx}}{\vdash\forall xAx,\neg\forall xAx}\qquad(\text{L-}\exists)\ \dfrac{\dfrac{\vdash Ay,\neg Ay}{\vdash\exists xAx,\neg Ay}}{\vdash\exists xAx,\neg\exists xAx}$$

□

We shall show that A_3 is equivalent to LA_3 before going on. This requires first a definition of a translation τ from sequents to formulae.

Definition 2.1. The translation $\tau(\Gamma\vdash\Delta)$ of a sequent $\Gamma\vdash\Delta$ is $\bigwedge(\Gamma)\to\bigvee(\Delta)$ where $\bigvee$ and $\bigwedge$ are such that:

$\bigwedge() = (A\to A)$

$\bigwedge(A) = A$

$\bigwedge(A_n, A_{n-1},\dots A_1) = (A_n\wedge\bigwedge(A_{n-1},\dots A_1))$

$\bigvee(A) = A$

$\bigvee(A_1,\dots A_{m-1}, A_m) = (\bigvee(A_1,\dots,A_{m-1})\vee A_m)$

The statement of this definition reflects that there can be no empty succedents as $\bigvee()$ is undefined. This will need to be altered in the case that we desire to introduce an absolute inconsistency. However, this shall be dealt with in the final chapter.

Theorem 2.3. *A_3 and LA_3 are equivalent – for any formula A, A is provable in LA_3 iff it is provable in A_3.*

Lemma 2.1. *If A is provable in A_3 then it is provable in LA_3.*

Proof. For the left to right direction, I show that each A_3 axiom has a proof in LA_3, and that MP and UG are admissible in LA_3. The proof is by cases on the axioms and rules of A_3:

Case: A1

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \vdash B \to A}}{\vdash A \to (B \to A)}$$

Case: A2

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{B \vdash B \qquad C \vdash C}{B, B \to C \vdash C}}{A, A \to B, B \to C \vdash C}}{A \to (B \to C), A \to B, A \vdash C}}{A \to (B \to C), A \to B \vdash A \to C}}{A \to (B \to C) \vdash (A \to B) \to (A \to C)}}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}$$

Case: A3

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}}{\vdash A, A \to B} \qquad A \vdash A}{(A \to B) \to A \vdash A}}{\vdash ((A \to B) \to A) \to A}$$

Case: A4

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A}}{\vdash (A \wedge B) \to A} \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{A, B \vdash B}}{A \wedge B \vdash B}}{\vdash (A \wedge B) \to B}$$

Case: A5

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A, B \vdash A, B}}{A, B \vdash A \vee B}}{A \vdash B \to (A \vee B)}}{\vdash A \to (B \to (A \vee B))}$$

Case: A6

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}}{A \vdash A \vee B}}{\vdash A \to (A \vee B)} \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{B \vdash A, B}}{B \vdash A \vee B}}{\vdash B \to (A \vee B)}$$

Case: A7

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad C \vdash C}{A, A \to C \vdash C}}{A, A \to C, B \to C \vdash C} \quad \dfrac{\dfrac{B \vdash B \quad C \vdash C}{B, B \to C \vdash C}}{B, A \to C, B \to C \vdash C}}{A \vee B, A \to C, B \to C \vdash C}}{A \to C, B \to C \vdash (A \vee B) \to C}}{A \to C \vdash (B \to C) \to ((A \vee B) \to C)}}{\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C))}$$

Case: A8

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A \vdash \neg A}{\neg A, \neg B \vdash \neg A}}{\neg(A \vee B) \vdash \neg A} \quad \dfrac{\dfrac{\neg B \vdash \neg B}{\neg A, \neg B \vdash \neg B}}{\neg(A \vee B) \vdash \neg B}}{\neg(A \vee B) \vdash \neg A \wedge \neg B}}{\vdash \neg(A \vee B) \to (\neg A \wedge \neg B)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\neg A \vdash \neg A}{\neg A, \neg B \vdash \neg A} \quad \dfrac{\neg B \vdash \neg B}{\neg A, \neg B \vdash \neg B}}{\neg A, \neg B \vdash \neg(A \vee B)}}{\neg A \wedge \neg B \vdash \neg(A \vee B)}}{\vdash (\neg A \wedge \neg B) \to \neg(A \vee B)}}{\vdash (\neg(A \vee B) \to (\neg A \wedge \neg B)) \wedge ((\neg A \wedge \neg B) \to \neg(A \vee B))}$$

Case: A9

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A \vdash \neg A}{\neg A \vdash \neg A, \neg B} \quad \dfrac{\neg B \vdash \neg B}{\neg B \vdash \neg A, \neg B}}{\neg(A \wedge B) \vdash \neg A, \neg B}}{\neg(A \wedge B) \vdash \neg A \vee \neg B}}{\vdash \neg(A \wedge B) \to (\neg A \vee \neg B)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\neg A \vdash \neg A}{\neg A \vdash \neg A, \neg B} \quad \dfrac{\neg B \vdash \neg B}{\neg B \vdash \neg A, \neg B}}{\neg A \vee \neg B \vdash \neg A, \neg B}}{\neg A \vee \neg B \vdash \neg(A \wedge B)}}{\vdash (\neg A \vee \neg B) \to \neg(A \wedge B)}}{\vdash (\neg(A \wedge B) \to (\neg A \vee \neg B)) \wedge ((\neg A \vee \neg B) \to \neg(A \wedge B))}$$

Case: A10

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\neg\neg A \vdash A}}{\vdash \neg\neg A \to A} \quad \dfrac{\dfrac{A \vdash A}{A \vdash \neg\neg A}}{\vdash A \to \neg\neg A}}{\vdash (A \to \neg\neg A) \wedge (\neg\neg A \to A)}$$

Case: A12

$$\dfrac{\vdash A, \neg A}{\vdash A \vee \neg A}$$

Case: A11

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg B \vdash A} \quad \dfrac{\neg B \vdash \neg B}{A, \neg B \vdash \neg B}}{A, \neg B \vdash A \wedge \neg B}}{\neg(A \to B) \vdash A \wedge \neg B}}{\vdash \neg(A \to B) \to (A \wedge \neg B)} \quad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg B \vdash A} \quad \dfrac{\neg B \vdash \neg B}{A, \neg B \vdash \neg B}}{A \wedge \neg B \vdash A} \quad \dfrac{\neg B \vdash \neg B}{A \wedge \neg B \vdash \neg B}}{A \wedge \neg B \vdash \neg(A \to \neg B)}}{\vdash (A \wedge \neg B) \to \neg(A \to B)}}{\vdash (\neg(A \to B) \to (A \wedge \neg B)) \wedge ((A \wedge \neg B) \to \neg(A \to B))}$$

Case: A13
$$\dfrac{\dfrac{Ay \vdash Ay}{\forall x Ax \vdash Ay}}{\vdash \forall x Ax \to Ay}$$

Case: A14
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad Bx \vdash Bx}{A, A \to Bx \vdash Bx}}{A, \forall x(A \to Bx) \vdash Bx}}{A, \forall x(A \to Bx) \vdash \forall x Bx}}{\forall x(A \to Bx) \vdash A \to \forall x Bx}}{\vdash \forall x(A \to Bx) \to (A \to \forall x Bx)}$$

Case: A15
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, Bx} \quad \dfrac{Bx \vdash Bx}{Bx \vdash A, Bx}}{A \vee Bx \vdash A, Bx}}{\forall x(A \vee Bx) \vdash A, Bx}}{\forall x(A \vee Bx) \vdash A, \forall x Bx}}{\forall x(A \vee Bx) \vdash A \vee \forall x Bx}}{\vdash \forall x(A \vee Bx) \to (A \vee \forall x Bx)}$$

Case: A16
$$\dfrac{\dfrac{Ay \vdash Ay}{Ay \vdash \exists x Ax}}{\vdash Ay \to \exists x Ax}$$

Case: A17
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Ax \vdash Ax \qquad B \vdash B}{Ax, Ax \to B \vdash B}}{Ax, \forall x(Ax \to B) \vdash B}}{\exists x Ax, \forall x(Ax \to B) \vdash B}}{\forall x(Ax \to B) \vdash \exists x Ax \to B}}{\vdash \forall x(Ax \to B) \to (\exists x Ax \to B)}$$

Case: A18
$$\dfrac{\dfrac{\dfrac{\dfrac{\neg Ax \vdash \neg Ax}{\forall x \neg Ax \vdash \neg Ax}}{\forall x \neg Ax \vdash \neg \exists x Ax}}{\vdash \forall x \neg Ax \to \neg \exists x Ax} \quad \dfrac{\dfrac{\dfrac{\neg Ax \vdash \neg Ax}{\neg \exists x Ax \vdash \neg Ax}}{\neg \exists x Ax \vdash \forall x \neg Ax}}{\vdash \neg \exists x Ax \to \forall x \neg Ax}}{\vdash (\forall x \neg Ax \to \neg \exists x Ax) \wedge (\neg \exists x Ax \to \forall x \neg Ax)}$$

Case: A19
$$\dfrac{\dfrac{\dfrac{\dfrac{\neg Ax \vdash \neg Ax}{\neg Ax \vdash \neg \forall x Ax}}{\exists x \neg Ax \vdash \neg \forall x Ax}}{\vdash \exists x \neg Ax \to \neg \forall x Ax} \quad \dfrac{\dfrac{\dfrac{\neg Ax \vdash \neg Ax}{\neg Ax \vdash \exists x \neg Ax}}{\neg \forall x Ax \vdash \exists x \neg Ax}}{\vdash \neg \forall x Ax \to \exists x \neg Ax}}{\vdash (\exists x \neg Ax \to \neg \forall x Ax) \wedge (\neg \forall x Ax \to \exists x \neg Ax)}$$

Applications of MP can be reproduced in the following form:

$$\dfrac{\vdash A \qquad \vdash A \to B}{\vdash B}$$

This can be altered to something like the following, relying on Cut:

$$\dfrac{\vdash A \qquad \dfrac{\vdash A \to B \qquad \dfrac{A \vdash A \qquad B \vdash B}{A, A \to B \vdash B}}{A \vdash B}}{\vdash B}$$

UG is obvious given the rule ($\vdash \forall$), as below.

$$\frac{\vdash Ax}{\vdash \forall x Ax}$$ □

Lemma 2.2. *If A is provable in LA_3 then it is provable in A_3.*

Proof. By induction on the complexity of an LA_3 derivation.

All positive theorems of LK are positive theorems of LA_3, so I shall only provide proofs for those formulae involving negations, as the proofs of positive theorems to be given in LA_3 are just those to be given between LK and a similar axiomatization of classical logic.

For the purposes of these proofs, I shall introduce a couple of obvious derived rules, where HS abbreviates "Hypothetical Syllogism"[9] and C is the name of the combinator the principle type scheme of which the rule resembles:

$$\frac{A \to B \qquad B \to C}{A \to C}\ \text{HS} \qquad\qquad \frac{A \to (B \to C)}{B \to (A \to C)}\ \mathsf{C}$$

These are valid in the positive fragment of classical logic, and is thus valid in A_3.

I shall also appeal to a number of other theorems of the positive fragment of classical logic, which I'll call K^+. This is axiomatised by (A1)-(A7) and (MP). They are listed here, and their proofs in A_3 are all fairly obvious.

$\vdash ((A \wedge B) \to C) \to ((B \wedge A) \to C)$ A-$\wedge$-comm.[10]

$\vdash ((A \wedge B) \to C) \to (A \to (B \to C))$ Exportation

$\vdash (A \to (B \to C)) \to ((A \wedge B) \to C)$ Importation

$\vdash (A \to B) \to ((B \to C) \to (A \to C))$ Suffixing

$\vdash (A \to B) \to ((C \to A) \to (C \to B))$ Prefixing

In a similar vein, I shall attempt to make the proofs shorter by, in some cases, appealing to obvious consequences of the positive implicational fragment of A_3 as simply instances of prefixing or suffixing, where appropriate, instead of proving the specific formulae. This makes for much shorter and more surveyable proofs.

Base: Axioms. I shall omit the case of $A \vdash A$ since its translation, $A \to A$, is obviously provable. This leaves just $\vdash A, \neg A$ and $\tau(\vdash A, \neg A)$ is $\bigvee(A, \neg A)$. This is $A \vee \neg A$, which is an instance of (A12).

[9] This rule has also been called "rule syllogism", as in [44].

[10] This is an abbreviation of "Antecedent $\wedge$-commutativity".

Induction: This falls into several subcases, one for each LA_3 rule. I shall present both of the $\neg\wedge$ cases in close to full detail, and do the same for one half of each of the other negative rules. I shall omit the classical cases and the other half of the negative rules. The former are the same as to be given for LK and an axiom system for the positive fragment of classical logic comprising (A1)-(A7) and (A13), (A14). The cases for the other half of the negative rules are all similar enough to cases given as to be easily reproducible.

Case ($\neg\neg\vdash$): This rule permits an inference from $\tau(\Gamma, A \vdash \Delta)$ to $\tau(\Gamma, \neg\neg A \vdash \Delta)$. In order to show that if $\tau(\Gamma, A \vdash \Delta)$, or $(G \wedge A) \rightarrow D$, is provable then so is $\tau(\Gamma, \neg\neg A \vdash \Delta)$, or $(G \wedge \neg\neg A) \rightarrow D$, it suffices to show that $((G \wedge A) \rightarrow D) \rightarrow ((G \wedge \neg\neg A) \rightarrow D)$ is provable, since an application of (MP) would allow us to make the necessary inference.

(1) $((G \wedge A) \rightarrow D) \rightarrow ((A \wedge G) \rightarrow D)$	A-$\wedge$-comm.
(2) $((A \wedge G) \rightarrow D) \rightarrow (A \rightarrow (G \rightarrow D))$	Exportation
(3) $\neg\neg A \rightarrow A$	A10
(4) $(\neg\neg A \rightarrow A) \rightarrow \big[(A \rightarrow (G \rightarrow D)) \rightarrow (\neg\neg A \rightarrow (G \rightarrow D))\big]$	Suffixing
(5) $(A \rightarrow (G \rightarrow D)) \rightarrow (\neg\neg A \rightarrow (G \rightarrow D))$	MP 3,4
(6) $((A \wedge G) \rightarrow D) \rightarrow (\neg\neg A \rightarrow (G \rightarrow D))$	HS 2,5
(7) $(\neg\neg A \rightarrow (G \rightarrow D)) \rightarrow ((\neg\neg A \wedge G) \rightarrow D)$	Importation
(8) $((A \wedge G) \rightarrow D) \rightarrow ((\neg\neg A \wedge G) \rightarrow D)$	HS 6,7
(9) $((\neg\neg A \wedge G) \rightarrow D) \rightarrow ((G \wedge \neg\neg A) \rightarrow D)$	A-$\wedge$-comm.
(10) $((A \wedge G) \rightarrow D) \rightarrow ((G \wedge \neg\neg A) \rightarrow D)$	HS 8,9
(11) $((G \wedge A) \rightarrow D) \rightarrow ((G \wedge \neg\neg A) \rightarrow D)$	HS 1,10

Case ($\vdash \neg\wedge$): From the provability $\tau(\neg A, \Gamma \vdash \Delta)$ and $\tau(\neg B, \Gamma \vdash \Delta)$, the provability of $\tau(\neg(A \wedge B), \Gamma \vdash \Delta)$ follows. As in the previous case, it suffices to show that $(((\neg A \wedge G) \to D) \wedge ((\neg B \wedge G) \to D)) \to ((\neg(A \wedge B) \wedge G) \to D)$ is provable.

First, $\vdash ((A \wedge C) \to D) \wedge (B \wedge C) \to D)) \to (((A \vee B) \wedge C) \to D)$ is a classical tautology, and thus a theorem of classical logic, and of the positive fragment of classical logic, thus it is a theorem of A_3. Thus (1) is provable.

(1) $\big[(\neg A \wedge G) \to D) \wedge (\neg B \wedge G) \to D)\big] \to (((\neg A \vee \neg B) \wedge G) \to D)$	K^+ theorem
(2) $(((\neg A \vee \neg B) \wedge G) \to D) \to ((\neg A \vee \neg B) \to (G \to D))$	Exportation
(3) $\big[(\neg A \wedge G) \to D) \wedge ((\neg B \wedge G) \to D)\big] \to ((\neg A \vee \neg B) \to (G \to D))$	HS 1,2
(4) $((\neg A \vee \neg B) \to (G \to D)) \to \big[(\neg(A \wedge B) \to (\neg A \vee \neg B)) \to (\neg(A \wedge B) \to (G \to D))\big]$	Prefixing
(5) $(((\neg A \wedge G) \to D) \wedge ((\neg B \wedge G) \to D))) \to \big[(\neg(A \wedge B) \to (\neg A \vee \neg B)) \to (\neg(A \wedge B) \to (G \to D))\big]$	HS 3,4
(6) $(\neg(A \wedge B) \to (\neg A \vee \neg B)) \to \big[(((\neg A \wedge G) \to D) \wedge ((\neg B \wedge G) \to D))) \to (\neg(A \wedge B) \to (G \to D))\big]$	C 5
(7) $\neg(A \wedge B) \to (\neg A \vee \neg B)$	A9
(8) $\big[((\neg A \wedge G) \to D) \wedge ((\neg B \wedge G) \to D)\big] \to (\neg(A \wedge B) \to (G \to D))$	MP 6,7
(9) $(\neg(A \wedge B) \to (G \to D)) \to ((\neg(A \wedge B) \wedge G) \to D)$	Importation
(10) $\big[((\neg A \wedge G) \to D) \wedge ((\neg B \wedge G \to D)\big] \to ((\neg(A \wedge B) \wedge G) \to D))$	HS 8,9

Case $(\neg\wedge\vdash)$: From $\tau(\Gamma\vdash\Delta,\neg A,\neg B)$ to $\tau(\Gamma\vdash\Delta,\neg(A\wedge B))$ or
$(G\to(D\vee\neg A\vee\neg B))$ implies $(G\to(D\vee\neg(A\wedge B)))$.
For this, note that $(A\to(B\vee C))\leftrightarrow((A\to C)\vee(A\to B))$ and $(A\vee B)\to((B\to C)\to(A\vee C))$ are both theorems of the positive fragment of classical propositional logic, and are thus also theorems of A_3. I shall appeal to both of these principles, to the former at lines (2) and (10), and the to latter at line (7).

(1) $G\to(D\vee(\neg A\vee\neg B))$	Assumption
(2) $(G\to(D\vee(\neg A\vee\neg B)))\to((G\to D)\vee(G\to(\neg A\vee\neg B)))$	K^+ theorem
(3) $(G\to D)\vee(G\to(\neg A\vee\neg B))$	MP 1,2
(4) $(\neg A\vee\neg B)\to\neg(A\wedge B)$	A9
(5) $((\neg A\vee\neg B)\to\neg(A\wedge B))\to\big[(G\to(\neg A\vee\neg B))\to(G\to\neg(A\wedge B))\big]$	Prefixing
(6) $(G\to(\neg A\vee\neg B))\to(G\to\neg(A\wedge B))$	MP 4,5
(7) $((G\to D)\vee(G\to(\neg A\vee\neg B)))\to\big[((G\to(\neg A\vee\neg B))\to(G\to\neg(A\wedge B)))\to((G\to D)\vee(G\to\neg(A\wedge B)))\big]$	K^+ theorem
(8) $((G\to(\neg A\vee\neg B))\to(G\to\neg(A\wedge B)))\to\big[(G\to D)\vee(G\to\neg(A\wedge B))\big]$	MP 3,7
(9) $(G\to D)\vee(G\to\neg(A\wedge B))$	MP 6,8
(10) $(((G\to D)\vee(G\to\neg(A\wedge B)))\to(G\to(D\vee\neg(A\wedge B)))$	K^+ theorem
(11) $G\to(D\vee\neg(A\wedge B))$	MP 9,10

Case $(\neg \rightarrow \vdash)$: from $\tau(A, \neg B, \Gamma \vdash \Delta)$ to $\tau(\neg(A \rightarrow B), \Gamma \vdash \Delta)$ or
$(((A \wedge \neg B) \wedge G) \rightarrow D) \rightarrow (\neg(A \rightarrow B) \wedge G) \rightarrow D)$

(1) $(((A \wedge \neg B) \wedge G) \rightarrow D) \rightarrow ((A \wedge \neg B) \rightarrow (G \rightarrow D))$	Exportation
(2) $\neg(A \rightarrow B) \rightarrow \neg(A \rightarrow B)$	A11
(3) $(\neg(A \rightarrow B) \rightarrow (A \wedge \neg B)) \rightarrow (((A \wedge \neg B) \rightarrow (G \rightarrow D)) \rightarrow (\neg(A \rightarrow B) \rightarrow (G \rightarrow D)))$	Suffixing
(4) $((A \wedge \neg B) \rightarrow (G \rightarrow D)) \rightarrow (\neg(A \rightarrow B) \rightarrow (G \rightarrow D))$	MP 2,3
(5) $(((A \wedge \neg B) \wedge G) \rightarrow D) \rightarrow (\neg(A \rightarrow B) \rightarrow (G \rightarrow D))$	HS 1,4
(6) $(\neg(A \rightarrow B) \rightarrow (G \rightarrow D)) \rightarrow ((\neg(A \rightarrow B) \wedge G) \rightarrow D)$	Importation
(7) $(((A \wedge \neg B) \wedge G) \rightarrow D) \rightarrow ((\neg(A \rightarrow B) \wedge G) \rightarrow D)$	

Case $(\neg\forall \vdash)$: If $\tau(G, \neg Ay \vdash \Delta)$ is provable then $\tau(G, \neg\forall x Ax \vdash \Delta)$ is also provable. In this I shall introduce the rule that $\forall x Ax \rightarrow \forall y Ay$ is provable, namely, that one can simply relabel the bound variable of a quantified statement, so long as the new variable is also free to be substituted into the formula. This is appealed to at line (7) as "relabeling".

(1) $(G \wedge \neg Ay) \rightarrow D$	Assumption
(2) $((G \wedge \neg Ay) \rightarrow D) \rightarrow ((\neg Ay \wedge G) \rightarrow D)$	A-$\wedge$-comm.
(3) $((\neg Ay \wedge G) \rightarrow D) \rightarrow (\neg Ay \rightarrow (G \rightarrow D))$	Exportation
(4) $((G \wedge \neg Ay) \rightarrow D) \rightarrow (\neg Ay \rightarrow (G \rightarrow D))$	HS 2,3
(5) $\neg Ay \rightarrow (G \rightarrow D)$	MP 1,4
(6) $\forall y(\neg Ay \rightarrow (G \rightarrow D))$	UG 5
(7) $\forall x(\neg Ax \rightarrow (G \rightarrow D))$	Relabeling
(8) $\forall x(\neg Ax \rightarrow (G \rightarrow D)) \rightarrow (\exists x \neg Ax \rightarrow (G \rightarrow D))$	A17
(9) $\exists x \neg Ax \rightarrow (G \rightarrow D)$	MP 6,7
(10) $\neg\forall x Ax \rightarrow \exists x \neg Ax$	A19
(11) $\neg\forall x Ax \rightarrow (G \rightarrow D)$	HS 8,9
(12) $(G \wedge \neg\forall x Ax) \rightarrow D$	Importation and A-$\wedge$-comm.

Case ($\neg\exists \vdash$): If $\tau(G, \neg Ay \vdash \Delta)$ then $\tau(G, \neg\exists xAx \vdash \Delta)$. Suppose that $(G \wedge \neg Ay) \rightarrow D$ is provable in A_3, and we shall show that $(G \wedge \neg\exists xAx) \rightarrow D$ is also provable. Again, relabeling is appealed to.

(1) $(G \wedge \neg Ay) \rightarrow D$	Assumption
(2) $((G \wedge \neg Ay) \rightarrow D) \rightarrow ((\neg Ay \wedge G) \rightarrow D)$	A-$\wedge$-comm.
(3) $((\neg Ay \wedge G) \rightarrow D) \rightarrow (\neg Ay \rightarrow (G \rightarrow D))$	Exportation
(4) $((G \wedge \neg Ay) \rightarrow D) \rightarrow (\neg Ay \rightarrow (G \rightarrow D))$	HS 2,3
(5) $\neg Ay \rightarrow (G \rightarrow D)$	MP 1,4
(6) $\forall y \neg Ay \rightarrow \neg Ay$	A13
(7) $\forall y \neg Ay \rightarrow (G \rightarrow D)$	HS 5,6
(8) $\forall x \neg Ax \rightarrow (G \rightarrow D)$	Relabeling
(8) $\neg\exists xAx \rightarrow \forall x \neg Ax$	A18
(9) $\neg\exists xAx \rightarrow (G \rightarrow D)$	HS 7,8
(10) $(G \wedge \neg\exists xAx) \rightarrow D$	Importation & A-$\wedge$-comm.

The other negated quantifier cases can be easily worked out appealing to the non-negated quantifier cases, either the classical theorem used above or $(A \leftrightarrow B) \rightarrow ((C \rightarrow (A \vee D)) \rightarrow (C \rightarrow (B \vee D)))$ for the succedent cases. The remaining cases are similar, which completes the proof.

□

Now, the proofs for soundness, completeness and cut elimination in [6] only apply to the propositional part of A_3, so I shall have to provide an extension of those proofs into the first order. In the case of soundness, one need only indicate that the extra rules all preserve designation, within the first order semantic structure. The completeness proof, adapted directly from Avron, is somewhat more involved so I'll present it all. I shall give a proof of the cut elimination theorem in the style of Gentzen, in close to full detail.

2.1.3 First-order A_3 Soundness

Theorem 2.4 (Soundness)**.** *Any sequent provable by LA_3 is a valid entailment of the A_3 semantics.*

Proof. The propositional part of this proof is given in [6], and involves checking the rules against the definitions of the valuations in A_3. I shall omit it here with the exception of two rules, to show the general structure of the cases. The first order part is classical except for the cases of the $\neg\exists$ and $\neg\forall$ rules. This is because the class of positive theorems for the propositional A_3 is the same as the set of positive classical theorems, and the positive quantifier rules are the same as in the classical case. Since the positive quantifier rules are sound for the classical case, they are also sound for A_3. On this note, I shall also omit the details for the proofs involving only classical values since they are standard for proofs of this kind.

Case 1 ($\vdash\ \neg\ \rightarrow$): Suppose that both of the two upper sequents, $\Gamma \vdash \Delta, A$ and $\Gamma \vdash \Delta, \neg B$, of the rule are valid. There are a couple of subcases. First, either all $G \in \Gamma$ are designated or one of them isn't.

Case 1.0 Some sentence $G \in \Gamma$ is non-designated. Then both upper sequents are valid, and so is the end-sequent. This is simply a special case of validity for a sequent.

Case 1.1 Every sentence $G \in \Gamma$ is assigned a designated value. Then either A or some $D \in \Delta$ must be designated, since $\Gamma \vdash \Delta, A$, and one of some $D \in \Delta$ or $\neg B$ must be designated, since $\Gamma \vdash \Delta, \neg B$. If some $D \in \Delta$ is designated then the sequent is obviously valid. So, I shall only consider the case when $A, \neg B$ are designated, since if both are non-designated then there must be some element of Δ that is designated, and since the rule requires a shared context, if some $D \in \Delta$ is designated in one premise then it is designated in both premises and the end-sequent.

Case 1.1.0 Suppose $v(\neg B) = \{1\}$. Then $v(B) = \{0\}$. If A is designated, then $v(A \rightarrow B) = \{0\}$ and thus $v(\neg(A \rightarrow B)) = \{1\}$ and the end-sequent is valid for having a designated formula in the succedent. Suppose that A is not designated, then some $D \in \Delta$ must be, assuming the right premise is valid, and thus the end-sequent is valid.

Case 1.1.1 If $v(\neg B) = \{1, 0\}$, then $v(B) = \{1, 0\}$. A must be designated, and so $v(A) = \{1\}$ or $\{1, 0\}$. Suppose that $v(A) = \{1\}$. Then $v(A \rightarrow B) = \{1, 0\}$ and $v(\neg(A \rightarrow B)) = \{1, 0\}$. Suppose that $v(A) = \{1, 0\}$. Then $v(A \rightarrow B) = \{1, 0\}$ and $v(\neg A \rightarrow B) = \{1, 0\}$. So, $v(\neg(A \rightarrow B)) = \{1, 0\}$ and the end-sequent is valid.

This case exhibits the general structure of the cases, and for the remaining cases I shall omit labeling the various subcases. Reinserting them is not difficult.

Case 2 ($\neg \rightarrow \vdash$): Suppose that the upper sequent is valid. Then either the values of all of $A, \neg B$, and all $G \in \Gamma$ are designated, or one of them is not. Suppose that all of $A, \neg B$ and all $G \in \Gamma$ are designated. Then so must be some $D \in \Delta$. Suppose that $v(\neg B) = \{1\}$; then $v(B) = \{0\}$ and $v(\neg(A \rightarrow B)) = \{1\}$. Suppose that $v(\neg B) = \{1, 0\} = v(B)$, then $v(\neg(A \rightarrow B)) = \{1, 0\}$, and the end-sequent is valid. Suppose that one of $A, \neg B$, or some $G \in \Gamma$ is non-designated. Suppose that some $G \in \Gamma$ is non-designated; then both the upper sequent and the end-sequent are valid. Suppose that $v(A) = \{0\}$; then $v(\neg(A \rightarrow B)) = \{0\}$, and the end-sequent is valid. Suppose that $v(\neg B) = \{0\}$; then $v(B) = \{1\}$ and $v(\neg(A \rightarrow B)) = \{0\}$, and the end-sequent is valid.

Case 3 ($\neg\exists \vdash$): Suppose that $v(\neg Ax) = \{1\}$ or $\{0\}$. In the former case, $v(Ax) = \{0\}$, and $v[d/x](Ax) = \{0\}$, and since the valuation $v[d/x]$ was an arbitrary choice, it is true for all d that $v[d/x](Ax) = \{0\}$ and thus $v(\exists x Ax) = \{0\}$ and $v(\neg\exists x Ax) = \{1\}$. The latter case, where $v(\neg Ax) = \{0\}$ is obvious, since it deals also with a classical truth value. Suppose that $v(\neg Ax) = \{1, 0\}$; then $v(Ax) = \{1, 0\}$. Thus $1 \in v(\exists x Ax)$, since, for some d, $1 \in v[d/x](Ax)$. However, since d was chosen arbitrarily, for any d, $v[d/x](Ax) = \{1, 0\}$, so $0 \in v[d/x](Ax)$ and thus $0 \in v(\exists x Ax)$ and so $v(\exists x Ax) = \{1, 0\} = v(\neg\exists x Ax)$. Thus, in any case, if the upper sequent is valid, then the lower sequent will be since Ax and $\exists x Ax$ always retain the same truth value. So if some $G \in \Gamma$ is non-designated then both sequents are valid, and if some $D \in \Delta$ is designated then both sequents are valid. Finally, if $\neg Ax$ is a non-designated formula in the antecedent while all $G \in \Gamma$ are designated. Thus the end-sequent is valid because $\neg\exists x Ax$ is non-designated.

Case 4 ($\vdash \neg\exists$): Suppose that the values of all $G \in \Gamma$ are designated and no value of $D \in \Delta$ is designated. Then, for the upper sequent to be valid $v(\neg Ax)$ must be designated. The other cases are straightforward. Suppose that $v(\neg Ax) = \{1, 0\}$; since there is a d such that $1 \in v[d/x](\neg Ax)$ then $1 \in v(\exists x Ax)$. Similarly, since $0 \in v[d/x](\neg Ax)$, therefore, $0 \in v(\forall x \neg Ax)$, and thus $0 \in v(\neg\exists x Ax)$. The case in which $v(\neg Ax) = \{1\}$ is equivalent to a similar case in classical logic appealing to only two values.

Case 5 ($\neg\forall \vdash$): Once again I shall omit the cases where the value of Ax is classical, since these are obvious. Suppose that $v(\neg Ax) = \{1, 0\}$; then $v(Ax) = \{1, 0\}$ and, since there is an d such that $v[d/x](Ax) = \{0\}$, $0 \in v(\forall x Ax)$. Thus $1 \in v(\neg\forall x Ax)$, $\neg\forall x Ax$ is designated, and the end-sequent is valid.

Case 6 ($\vdash \neg\forall$): Suppose that $v(\neg Ax) = \{1, 0\}$; then for some d, $0 \in v[d/x](Ax)$ so $0 \in v(\forall x Ax)$ and thus $1 \in v(\neg\forall x Ax)$. Since d is arbitrarily chosen, for any d, $1 \in v[d/x](Ax)$ and so $1 \in v(\forall x Ax)$ and $0 \in v(\neg\forall x Ax)$. So $v(\forall x Ax) = \{1, 0\}$ as desired. Suppose that $v(\neg Ax) = \{1\}$; then $v(Ax) = \{0\}$ and $v(\forall x Ax) = \{0\}$, so $v(\neg\forall x Ax) = \{1\}$. If $v(\neg Ax) = \{0\}$ then, for the upper sequent to be valid it must be the case that some $D \in \Delta$ is designated or some $G \in \Gamma$ is non-designated, and this valuation will also validate the end-sequent.

This completes the proof of soundness.

□

With this, the first order apparatus is shown to be sound for first order A_3. These arguments essentially support the claim that the standard equivalences between the universal and existential quantifier hold in this logic. This fact is part of the definition of the first order A_3 semantics, and, as seen, the rules do enforce it.

2.1.4 First-order A_3 Completeness

Theorem 2.5 (Completeness)**.** *If $\Gamma \nvdash \Delta$ in LA_3 then there is an A_3 model which assigns a designated value to each $G \in \Gamma$ and assigns a non-designated value to each $D \in \Delta$.*

Proof. This is following Avron [6], though omitting those elements of his proof which have to do with eliminating cut. He gives a Schütte style proof of cut elimination of which the completeness is only a part. Since I am interested to provide a constructive proof of cut elimination, I shall not appeal to it here. The proof given there is for the propositional fragment of LA_3, and I extend this to the first order version given above.

First, there are two definitions. In these, and below, S is a finite set of sequents, $\{\Gamma_0 \vdash \Delta_0, \Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n\}$ where $n \in \mathbb{N}$. So, $\bigcup_{i=1}^{n} \Gamma_i$ is the union of antecedents of sequents in S, where $(0 \leq i \leq n)$.

Definition 2.2. An S-proof of a sequent $\Gamma \vdash \Delta$ is a proof in which the members of S may be used as extra axioms of the sequent calculus.

Definition 2.3. A sequent $\Gamma^* \vdash \Delta^*$ is saturated iff it has the following properties:

1. There is no S-proof of $\Gamma^* \vdash \Delta^*$.
2. If $A \in \bigcup_{i=1}^{n} \Gamma_i \cup \bigcup_{i=1}^{n} \Delta_i$ then $A \in \Gamma^* \cup \Delta^*$.
3. If $A \to B \in \Gamma^*$ then $A \in \Delta^*$ or $B \in \Gamma^*$.
4. If $A \to B \in \Delta^*$ then $A \in \Gamma^*$ and $B \in \Delta^*$.
5. If $\neg(A \to B) \in \Gamma^*$ then $A \in \Gamma^*$ and $\neg B \in \Gamma^*$.
6. If $\neg(A \to B) \in \Delta^*$ then $A \in \Gamma^*$ or $\neg B \in \Delta^*$.
7. If $A \wedge B \in \Gamma^*$ then $A \in \Gamma^*$ and $B \in \Gamma^*$.
8. If $A \wedge B \in \Delta^*$ then $A \in \Delta^*$ or $B \in \Delta^*$.
9. If $\neg(A \wedge B) \in \Gamma^*$ then $\neg A \in \Gamma^*$ or $\neg B \in \Gamma^*$.
10. If $\neg(A \wedge B) \in \Delta^*$ then $\neg A \in \Delta^*$ and $\neg B \in \Delta^*$.
11. If $A \vee B \in \Gamma^*$ then $A \in \Gamma^*$ or $B \in \Gamma^*$.
12. If $A \vee B \in \Delta^*$ then $A \in \Delta^*$ and $B \in \Delta^*$.
13. If $\neg(A \vee B) \in \Gamma^*$ then $\neg A \in \Gamma^*$ and $\neg B \in \Gamma^*$.
14. If $\neg(A \vee B) \in \Delta^*$ then $\neg A \in \Delta^*$ or $\neg B \in \Delta^*$.
15. If $\neg\neg A \in \Gamma^*$ then $A \in \Gamma^*$.
16. If $\neg\neg A \in \Delta^*$ then $A \in \Delta^*$.

17. If $\forall x Ax \in \Gamma^*$ then for every y occurring in $\Gamma^* \vdash \Delta^*$, $Ay \in \Gamma^*$.
18. If $\forall x Ax \in \Delta^*$ then for some y, $Ay \in \Delta^*$.
19. If $\neg\forall x Ax \in \Gamma^*$ then for some y, $\neg Ay \in \Gamma^*$.
20. If $\neg\forall x Ax \in \Delta^*$ then for every y occurring in $\Gamma^* \vdash \Delta^*$, $\neg Ay \in \Delta^*$.
21. If $\exists x Ax \in \Gamma^*$ then for some y, $Ay \in \Gamma^*$.
22. If $\exists x Ax \in \Delta^*$ then for every y occurring in $\Gamma^* \vdash \Delta^*$, $Ay \in \Delta^*$.
23. If $\neg\exists x Ax \in \Gamma^*$ then for every y occurring in $\Gamma^* \vdash \Delta^*$, $\neg Ay \in \Gamma^*$.
24. If $\neg\exists x Ax \in \Delta^*$ then for some y, $\neg Ay \in \Delta^*$.

For the following, $\Gamma \vdash \Delta$ is a *sub-sequent* of $\Pi \vdash \Theta$ iff $\Gamma \subseteq \Pi$ and $\Delta \subseteq \Theta$. A saturated sequent includes every subformula of each complex formula occurring in it, so that one can derive from it every complex formula occurring in it. In addition to this, no sub-sequent of a saturated sequent is provable, else there would be a trivial S-proof for it. Namely, to use repeated applications of K to derive all the extra formulae. So, it cannot be the case that a sub-sequent of a saturated sequent is an instance of either axiom.

We show that if $\Gamma \vdash \Delta$ does not have an S-proof, then there is a model of S which does not model $\Gamma \vdash \Delta$. This inference is valid supposing contraposition in the metatheory. This rule is invalid in LA_3, however, there are some researchers in non-classical logic willing to appeal to classical inference patterns in the metatheory of non-classical logics, Avron being among them, and I shall also commit this somewhat hypocritical move, recognizing unhappily that it is such. I could not find a suitable completeness proof method for A_3 or LA_3 which did not appeal to contraposition, though I remain hopeful that there is such a method.

Lemma 2.3. *If $\Gamma \vdash \Delta$ has no S-proof then it can be extended to a saturated sequent $\Gamma^* \vdash \Delta^*$.*

Proof. Suppose that $\Gamma \vdash \Delta$ has no S-proof. It is a simple matter to check that adding extra formulae to Γ and Δ in accordance with the properties of a saturated sequent listed above will not produce a provable sequent. For instance, suppose Δ is $A \to B$,

and that $\vdash A \to B$ is not provable. The sequent resulting from the addition of A to the antecedent and B to the succedent, $A \vdash A \to B, B$, will not be provable. For this addition to render the sequent provable, it would have to be the case that A is absolutely false, and there is no such formula, or that B itself were provable, however if this were the case then $\vdash A \to B$ would be provable by appeal to K. There is one tricky point in those rules with multiple upper-sequents. It cannot be the case that for any formula A, it occurs in both Γ and Δ, since $A \vdash A$ is an instance of the axiom, and thus $\Gamma \vdash \Delta$ could be derived by some applications K, and would thus have a trivial S-proof. As such, in those cases where there are two possibilities, choose the one that does not produce a sequent of that form. Similarly, in the case of sequents where A and $\neg A$ both occur in the succedent. □

Lemma 2.4. *If $\Gamma^* \vdash \Delta^*$ is saturated then there is a model of S which does not validate $\Gamma^* \vdash \Delta^*$ – that is, all $G \in \Gamma^*$ are assigned designated values, and no $D \in \Delta^*$ is designated.*

Proof. First, we choose a valuation v which assigns values to literals occurring in $\Gamma^* \vdash \Delta^*$ in the following way:

$$v(P) = \begin{cases} \{1\} & \text{if } \neg P \in \Delta^* \\ \{1,0\} & \text{if } P \notin \Delta^*, \neg P \notin \Delta^* \\ \{0\} & \text{if } P \in \Delta^* \end{cases}$$

I shall show that if a formula $C \in \Gamma^*$ then $1 \in v(C)$ and that if $C \in \Delta^*$ then $v(C) = \{0\}$. This way, the construction of v entails that $\Gamma^* \nvdash \Delta^*$, as desired. This is shown by induction on the structure of C.

Base: C is a literal, either P or $\neg P$. If $P \in \Gamma^*$ then $P \notin \Delta^*$, since otherwise a subsequent of $\Gamma^* \vdash \Delta^*$ would be $P \vdash P$, which is an axiom. If $P \in \Delta^*$ then $\neg P \notin \Delta^*$ and if $\neg P \in \Delta^*$ then $P \notin \Delta^*$ for similar reasons regarding $\vdash P, \neg P$. So, if $P \in \Gamma^*$ then $1 \in v(P)$ since either $\neg P \in \Delta^*$ and thus $v(P) = \{1\}$ or $\neg P \notin \Delta^*$ in which case $v(P) = \{1,0\}$, since $P \notin \Delta^*$ by hypothesis. If $P \in \Delta^*$ then $v(P) = \{0\}$ and if $\neg P \in \Delta^*$ then $v(\neg P) = \{0\}$.

Case 1: Suppose that C is $A \to B$. If $C \in \Gamma^*$, then $A \in \Delta^*$ or $B \in \Gamma^*$. So either $v(A) = \{0\}$ or $1 \in v(B)$. If the former then $1 \in v(A \to B)$, as with the latter.

In either case $\Gamma^* \models \Delta^*$ as desired. Suppose that $A \to B \in \Delta^*$. Then $A \in \Gamma^*$ and $B \in \Delta^*$. Then $1 \in v(A)$ and $v(B) = \{0\}$, and thus $v(A \to B) = \{0\}$, as desired.

Case 2: Suppose that C is $A \wedge B$, and $A \wedge B \in \Gamma^*$. Then $A \in \Gamma^*$ and $B \in \Gamma^*$, and thus $1 \in v(A) \cap v(B)$ and so $1 \in v(A \wedge B)$, as desired. Suppose that $A \wedge B \in \Delta^*$, then $A \in \Delta^*$ or $B \in \Delta^*$, and so $v(A) = \{0\}$ and so $v(A \wedge B) = \{0\}$, as desired.

Case 3: Suppose that C is $A \vee B$. This is dual to the previous case.

Case 4: Suppose that C is $\neg(A \to B)$ and is in Γ^*. Then $A \in \Gamma^*$ and $\neg B \in \Gamma^*$ and so $1 \in v(A)$ and so in $v(\neg B)$, and thus $1 \in v(\neg(A \to B))$, as desired. Suppose that $\neg(A \to B) \in \Delta^*$. Then either $A \in \Delta^*$ and $v(A) = \{0\}$ or $\neg B \in \Gamma^*$ and $1 \in v(\neg B)$. In either case $v(\neg(A \to B)) = \{0\}$, as desired.

Case 5: Suppose that C is $\neg(A \wedge B)$ and $\neg(A \wedge B) \in \Gamma^*$, then $\neg A \in \Gamma^*$ or $\neg B \in \Gamma^*$. In the former case, $1 \in v(\neg A)$, and in the latter $1 \in v(\neg B)$, and in either case $1 \in v(\neg(A \wedge B))$. Suppose that $\neg(A \wedge B) \in \Delta^*$, then $\neg A \in \Delta^*$ and $\neg B \in \Delta^*$, so $v(\neg B) = \{0\} = v(\neg A)$, so $v(A) = \{1\} = v(B)$, and thus $v(\neg(A \wedge B)) = \{0\}$.

Case 6: Suppose that C is $\neg(A \vee B)$. This is dual to the previous case.

Case 7: Suppose that C is $\neg\neg A$ and $\neg\neg A \in \Gamma^*$, then $A \in \Gamma^*$ and thus $1 \in v(A)$, as desired. Suppose that $\neg\neg A \in \Delta^*$; then $A \in \Delta^*$ and $v(A) = \{0\}$ as desired.

For the quantificational cases, we shall restrict the domain D to that part of the domain occurring in $\Gamma^* \vdash \Delta^*$, ie. $D^* = \{i(x); x \text{ occurs free in } \Gamma^* \cup \Delta^*\}$. The objects $d^* \in D^*$ are then the suitably restricted domain elements. In the following cases, the domain is always D^*.

Case 8: Suppose that C is $\forall x Ax$, and $\forall x Ax \in \Gamma^*$, then for any y, $Ay \in \Gamma^*$, and $1 \in v[d^*/y](Ay)$, and since d^* was arbitrarily chosen, thus $1 \in v(\forall x Ax)$, as desired. Suppose that $\forall x Ax \in \Delta^*$. Then, for some y occurring in $\Gamma^* \vdash \Delta^*$, $Ay \in \Delta^*$. So for some d^*, $v[d^*/y](Ay) = \{0\}$ and so $v(\forall x Ax) = \{0\}$, as desired.

Case 9: Suppose that C is $\neg\forall x Ax$ and $\neg\forall x Ax \in \Gamma^*$, then, for some y occurring in $\Gamma^* \vdash \Delta^*$, $\neg Ay \in \Gamma^*$ and for some d^* $1 \in v[d^*/y](\neg Ay)$, and so $1 \in v(\exists x \neg Ax)$ and thus $1 \in v(\neg\forall x Ax)$, as desired. Suppose that $\neg\forall x Ax \in \Delta^*$. Then, for any y, $\neg Ay \in \Delta^*$ and so $v[d^*/y](\neg Ay) = \{0\}$ so, $1 \in v[d^*/y](Ay)$ and so $v(\neg\forall x Ax) = \{0\}$ since d^* was

arbitrarily chosen.

Case 10: Suppose that C is $\exists xAx$ and $\exists xAx \in \Gamma^*$, then for some y occurring in $\Gamma^* \vdash \Delta^*$, $Ay \in \Gamma^*$ and thus for some d^* $1 \in v[d^*/y](Ay)$ and thus $1 \in v(\exists xAx)$, as desired. Suppose that $\exists xAx \in \Delta^*$, then, for any y, $Ay \in \Delta^*$ so $v[d^*/y](Ay) = \{0\}$ and since d^* was arbitrarily chosen $v(\exists xAx) = \{0\}$, as desired.

Case 11: Suppose that C is $\neg\exists xAx$ and $\neg\exists xAx \in \Gamma^*$. Then for any y, $\neg Ay \in \Gamma^*$ and $1 \in v[d^*/y](\neg Ay)$, so $1 \in v(\forall x\neg Ax)$ and thus $1 \in v(\neg\exists xAx)$. Suppose that $\neg\exists xAx \in \Delta$, then for some y occurring in $\Gamma^* \vdash \Delta^*$, $\neg Ay \in \Delta^*$ and thus for some d^*, $v[d^*/y](\neg Ay) = \{0\}$ and $v(\forall x\neg Ax) = \{0\}$ and thus $v(\exists xAx) = \{0\}$, as desired.

All that is left of the lemma is to show that v models S.

Suppose, $\Gamma_i \vdash \Delta_i \in S$. $\Gamma_i \subseteq \Gamma^*$ and $\Delta_i \subseteq \Delta^*$ cannot both hold since, in that case $\Gamma^* \vdash \Delta^*$ would have a trivial S-proof consisting of some number of applications of thinning. Thus either for some $A \in \Gamma_i$, $A \in \Delta^*$ or for some $A \in \Delta_i$, $A \in \Gamma^*$. If the former, then $v(A) = \{0\}$ and so v models $\Gamma_i \vdash \Delta_i$, and if the latter then $1 \in v(A)$ and thus v models $\Gamma_i \vdash \Delta_i$, as desired. Since $\Gamma_i \vdash \Delta_i$ was chosen arbitrarily, v models S, and yet does not model $\Gamma^* \vdash \Delta^*$. □

So, by lemma 2.3 any sequent without an S-proof can be saturated, and by lemma 2.4, all saturated sequents are invalidated by at least one model. So, if $\Gamma \vdash \Delta$ is valid in the A_3 semantics, then there must be an S-proof of it from S in LA_3.

□

2.1.5 LA_3 Cut Elimination

Cut is a very useful structural rule, but that it can be eliminated indicates that the proof system is well defined, and that the subformula property holds. If LA_3 is to do much of the work which classical logic does, then it is a question of its suitability whether cut can be eliminated from it. This is why I provide a proof of this theorem here. Avron [6] has proven the elimination theorem for the propositional part of LA_3 using a semantic argument in the vein of Schütte. I shall extend this theorem

to the first-order case, however using a syntactical proof in line with that originally presented by Gentzen in [17]. This proof provides an effective procedure for removing cuts from proofs, and is constructive.[11]

Theorem 2.6 (Elimination)**.** *Every LA_3 proof can be transformed into an LA_3 proof of the same end-sequent involving no applications of cut. The only logical rules applied in the new proof also occur in the given proof.*

Proof. I present a demonstration of cut elimination using Gentzen's technique as presented in [17].

The proof is by induction on the number n of applications of the cut rule in a given proof. If $n > 0$ there must occur in it an earliest cut, which has no other cut occurring between it and the leaves of the proof. Consider the part of the given proof which terminates with some sequent $\Gamma, \Pi^* \vdash \Delta^*, \Theta$, where Π^* is Π and Δ^* is Δ but the cut formula excised - this is the "given subtree of the proof", or "given part". Suppose we can alter this given part to obtain another proof in LA_3 of the same sequent without the last use of cut. Call this the "resulting part". The replacement of the given part of the proof by the resulting part produces a new proof the same sequent with $n - 1$ cuts.

Lemma 2.5. *Given a proof in LA_3 of the end-sequent $\Gamma, \Pi^* \vdash \Delta^*, \Theta$ with cut as the final step and no other application of cut, another proof can be found which does not feature an application of cut.*

In the stated lemma, Γ^* is Γ but with the cut formula removed as a result of a cut. Due to Lemma 2.2 we may assume that all formulae instantiated in the axioms are literals.

Definition 2.4. The *left rank* ρ_l of an application of the cut rule is the greatest number of sequents located consecutively one above another at the bottom of any branch terminating with the left premise of the cut, namely $\Gamma \vdash \Delta$, containing the

[11] Part of the project of paraconsistent mathematics, as expressed to me by Zach Weber in personal correspondence, which I am interested to take on, is a general move to constructive and direct proof methods. As such, I would prefer a proof of this kind over the semantic kind given by Avron.

cut formula A in the succedent. The *right rank* ρ_r is similarly defined, but with the right premise $\Pi \vdash \Theta$. The *rank* $\rho = \rho_l + \rho_r$ which is always ≥ 2.

Definition 2.5. The *grade* γ of the cut is the number (≥ 0) of occurrences of logical symbols ($\rightarrow, \wedge, \vee, \neg, \forall, \exists$) in A, the cut formula.

The proof is by induction on the grade γ of the cut. Nested in this is another induction on the rank ρ. If one can directly eliminate cuts on atomic formulae, as per cases 1-8, and one can always replace cuts on complex formulae with cuts on atomic formulae when the rank is 2 as per cases 9-19, and one can always transform cuts of rank > 2 for cuts of rank 2 as per cases 20-22, then one can eliminate all cuts. In each case, I shall present the general form as the following, in keeping with what I have specified as the left and right premises:

$$\frac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Base: Where $\rho = 2$, and the cut is to be completely eliminated.

Case 1: Suppose that the left premise of the cut is an instance of the axiom $A \vdash A$ and the cut formula is A. Then the proof is of the following form, where Π is the same as Π^*, since $\rho = 2$.

$$\frac{A \vdash A \qquad A, \Pi \vdash \Theta}{A, \Pi^* \vdash \Theta}$$

If this is the case then the right premise of the cut is sufficient to establish the result, and the application of the cut is eliminable.

Case 2: The right premise of the cut is an instance of $A \vdash A$ and the cut formula is A. This is symmetric to case 1.

Case 3: The left premise is an instance of $\vdash A, \neg A$ and the cut formula is A, and the right premise is $A \vdash A$. The proof is of the following form, where, again, the cut can clearly be eliminated:

$$\frac{\vdash A, \neg A \qquad A \vdash A}{\vdash A, \neg A}$$

Case 4: The left premise is an instance of $\vdash A, \neg A$ and the cut formula is $\neg A$ and the right premise is $\neg A \vdash \neg A$. The proof is of the following form, where cut is clearly eliminable:

$$\dfrac{\vdash A, \neg A \qquad \neg A \vdash \neg A}{\vdash A, \neg A}$$

Case 5: The left premise is an instance of $\vdash A, \neg A$, the cut formula is A. Since $\rho = 2$, the inference culminating in $A, \Pi \vdash \Theta$ must be a result of either appeal to the reflexivity axiom or to thinning. The former possibility is that of case 3. Suppose that $A, \Pi \vdash \Theta$ follows from $\Pi \vdash \Theta$ by thinning as such:

$$\dfrac{\vdash A, \neg A \qquad \dfrac{\Pi \vdash \Theta}{A, \Pi \vdash \Theta}}{\Pi^* \vdash \Theta, \neg A}$$

So, the right premise must be the result of applying thinning to $\Pi \vdash \Theta$, and as such, one can produce the same end-sequent, without appeal to cut, by thinning on the right premise as follows, where Π must be the same as Π^* since $\rho = 2$:

$$\dfrac{\Pi \vdash \Theta}{\Pi \vdash \Theta, \neg A}$$

Case 6: The left premise is an instance of $\vdash A, \neg A$, the cut formula is $\neg A$, and is the result of thinning applied to $\Gamma \vdash \Delta$. This case is similar to case 5.

Case 7: Suppose that the left premise of the cut is a result of thinning. The proof given is of the form of that on the left:

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad A, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad\qquad \dfrac{\Gamma \vdash \Delta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

The proof on the right can be given as a result of some applications of thinning. As before Δ^* is Δ and Π^* is Π.

Case 8: The right premise of the cut is the result of thinning. This is similar to case 7.

Induction Step (γ): Where $\gamma > 0$, $\rho = 2$, and γ is to be reduced.

Case 9: The cut formula is $A \wedge B$, and thus occurs in both the succedent of the left and the antecedent of the right premise, and is a result of an application of $\vdash \wedge$ in the left, and an application of $\wedge \vdash$ in the right premise. The proof is of the form:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad \dfrac{A, B, \Pi \vdash \Theta}{A \wedge B, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

This proof can be rewritten to include two applications of cut, each on immediate subformulae of $A \wedge B$, and thus each of a lower grade than that given:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, A \qquad A, B, \Pi \vdash \Theta}{\Gamma, \Pi^*, B \vdash \Delta^*, \Theta} \qquad \Gamma \vdash \Delta, B}{\Gamma^*, \Pi^* \vdash \Delta^*, \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

The move here is just a shift in where cut is applied, removing any need to apply the conjunction rules as in the original proof. In the given proof the logical rules are applied first so that one cuts out the complex formula, whereas in the resulting proof we perform two cuts, one on each subformula of the given cut formula. In the above, note that the *s occurring above actual refer to different cut formulae. In the given proof, Π^*, Δ^* are Π, Δ with $A \wedge B$ erased from each. However, in the resulting proof, the Π^* occurring in the second line is Π with A cut out, whereas that in the third line is with B cut out as well. This is a standard convention, and most often the cut formula of the cut occurring directly above the sequent wherein a starred set occurs is the formula which has been cut out. In general, it is fairly obvious which formulae have been cut out and when.

Case 10: The cut formula is $A \vee B$ and is thus a result of an application of $\vdash \vee$ in the left premise, and of $\vee \vdash$ in the right premise. The proof is of the form:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \dfrac{A, \Pi \vdash \Theta \qquad B, \Pi \vdash \Theta}{A \vee B, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

and it can be altered to the following, with lower grade cuts on proper subformulae of $A \vee B$:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, A, B \qquad A, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, B} \qquad B, \Pi \vdash \Theta}{\dfrac{\Gamma, \Pi^* \vdash \Delta^*, \Theta^*}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}}}$$

Case 11: The cut formula is $A \to B$ and is thus a result of the application of $\vdash\to$ in the left, and of $\to\vdash$ in the right premise. The proof is of the form:

$$\dfrac{\dfrac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B} \qquad \dfrac{\Pi \vdash \Theta, A \qquad B, \Pi \vdash \Theta}{A \to B, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

which can be altered to:

$$\dfrac{\dfrac{\dfrac{\Pi \vdash \Theta, A \qquad A, \Gamma \vdash \Delta, B}{\Gamma^*, \Pi \vdash \Delta, \Theta^*, B} \qquad B, \Pi \vdash \Theta}{\dfrac{\Gamma^*, \Pi^* \vdash \Delta^*, \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}}}$$

Case 12: The cut formula is $\neg\neg A$ and is thus a result of the application of $\vdash \neg\neg$ in the left, and of $\neg\neg \vdash$ in the right premise. The proof is of the form:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \neg\neg A} \qquad \dfrac{A, \Pi \vdash \Theta}{\neg\neg A, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

to be altered to:

$$\dfrac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Case 13: The cut formula is $\neg(A \wedge B)$ and is thus the result of the application of $\vdash \neg\wedge$ in the left, and of $\neg\wedge \vdash$ in the right premise.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, \neg A, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)} \qquad \dfrac{\neg A, \Pi \vdash \Theta \qquad \neg B, \Pi \vdash \Theta}{\neg(A \wedge B), \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

again, for a grade reduction:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, \neg A, \neg B \qquad \neg B, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg A} \qquad \neg A, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Case 14: The cut formula is $\neg(A \vee B)$ and is thus the result of the application of $\vdash \neg\vee$ in the left, and of $\neg\vee \vdash$ in the right premise.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \vee B)} \qquad \dfrac{\neg A, \neg B, \Pi \vdash \Theta}{\neg(A \vee B), \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

for a grade reduction alter the above proof to:

$$\dfrac{\Gamma \vdash \Delta, \neg B \qquad \dfrac{\Gamma \vdash \Delta, \neg A \qquad \neg A, \neg B, \Pi \vdash \Theta}{\neg B, \Gamma, \Pi^* \vdash \Delta^*, \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Case 15: The cut formula is $\neg(A \to B)$ and is thus the result of the application of $\vdash \neg \to$ in the left, and of $\neg \to \vdash$ on the right.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \to B)} \qquad \dfrac{A, \neg B, \Pi \vdash \Theta}{\neg(A \to B), \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

For grade reduction, alter the above proof to the following:

$$\dfrac{\Gamma \vdash \Delta, \neg B \qquad \dfrac{\Gamma \vdash \Delta, A \qquad A, \neg B, \Pi \vdash \Theta}{\neg B, \Gamma, \Pi^* \vdash \Delta^*, \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Again, in this case, Π^* and Π are identical, so the altered proof does have the same endsequent as the given proof.

Case 16: The cut formula is $\forall x Ax$ and is thus the result of the application of $\vdash \forall$ on the left, and of $\forall \vdash$ on the right. The proof below on the left is the given proof, the proof on the right results.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, Ay}{\Gamma \vdash \Delta, \forall x Ax} \qquad \dfrac{Az, \Pi \vdash \Theta}{\forall x Ax, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad\qquad \dfrac{\Gamma \vdash \Delta, Ay \qquad Ay, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

That this does case does not fail due to a problem involving the restriction placed on variable replacement is assumed here. Gentzen gives a separate lemma to establish this result, which I omit here, as it is fairly intuitive, though non-trivial.

Lemma 2.6. *An LA_3 sequent or application of a logical rule remains the same sequent or an application of same rule if we uniformly replace a free variable which is not the eigenvariable of the logical rule by another free variable, provided that the new variable is not the eigenvariable of the logical rule.*

The proof of this lemma, given by Gentzen [17, 300], can be easily reproduced here. The rules $\exists \vdash$ and $\vdash \forall$, which were those rules sensitive to variables in Gentzen's system of LK are just the same here, and the restrictions on the new rules closely match the restrictions on the original rules. The classical means of deriving the end-sequents of the new rules, say $\neg\forall \vdash$ simply involves applying $\vdash \forall$, then applying $\neg \vdash$. This does the same job as the rule $\neg\forall \vdash$, which essentially just invokes that $\exists x \neg Ax \rightarrow \neg\forall x Ax$. The other rules are similar, and those rules which are not sensitive to the particular eigenvariable of the premise are no problem in any case.

Case 17: The cut formula is $\exists x Ax$ and is thus the result of the application of $\vdash \exists$ on the left, and of $\exists \vdash$ on the right.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, Az}{\Gamma \vdash \Delta, \exists x Ax} \quad \dfrac{Ay, \Pi \vdash \Theta}{\exists x Ax, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad \dfrac{\Gamma \vdash \Delta, Ay \quad Ay, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Case 18: The cut formula is $\neg\forall x Ax$ and is thus the result of the application of $\vdash \exists$ on the left, and of $\exists \vdash$ on the right.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, \neg Az}{\Gamma \vdash \Delta, \neg\forall x Ax} \quad \dfrac{\neg Ay, \Pi \vdash \Theta}{\neg\forall x Ax, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad \dfrac{\Gamma \vdash \Delta, \neg Ay \quad \neg Ay, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Case 19: The cut formula is $\neg\exists x Ax$ and is thus the result of the application of $\vdash \neg\exists$ on the left, and of $\neg\exists \vdash$ on the right.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, \neg Ay}{\Gamma \vdash \Delta, \neg\exists x Ax} \quad \dfrac{\neg Az, \Pi \vdash \Theta}{\neg\exists x Ax, \Pi \vdash \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad \dfrac{\Gamma \vdash \Delta, \neg Ay \quad \neg Az, \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Thus, any proof featuring a cut on a complex formula can be altered to a proof with cuts on less complex formulae when $\rho = 2$ and in both premises the cut formula is the principle formula of the logical rule. It remains to be shown that any higher rank cut can be altered to one with a cut of lower rank than that given, with a minimum of 2. The cuts which occur on atomic formulae at $\rho = 2$ can be eliminated, by the same kind of procedures applied in the base cases.

Induction Step (ρ): $\rho > 2$, and is to be reduced (or outright eliminated, case 21).

There are two categories into which these cases may fall. First is that where the $\rho_r > 1$, and second is that where $\rho_r = 1$, necessitating that $\rho_l > 1$. These cases are, essentially, symmetrical. As such, I shall present only the first of these, since the other cases are easily reproducible.

Case 20: Suppose the inference resulting in the right premise is an application of K. The principal formula A of K must not be the cut formula C, since $\rho_r > 2$, and A can occur in either the antecedent or the succedent. An example of this kind of proof, with $\mathsf{K} \vdash$ applied, on the left below is that given, that on the right resulting, where the applications of the cut and K are permuted.

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{\Pi \vdash \Theta}{A, \Pi \vdash \Theta}}{\Gamma, A, \Pi^* \vdash \Delta^*, \Theta} \qquad\qquad \dfrac{\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}}{\Gamma, A, \Pi^* \vdash \Delta^*, \Theta}$$

The resulting proof features a cut of ρ_r of 1 fewer than that in the given proof, and so the cut can be eliminated according to the inductive hypothesis. The case involving an application of $\vdash \mathsf{K}$ can be solved in just the same way, but with A occurring in the succedent.

Case 21: Suppose that the cut formula C occurs in either the antecedent of the left premise or the succedent of the right premise. In this case, we can just use some applications of K to derive the desired end-sequent from one of the premises. If C occurs in the antecedent of the left premise, Γ, then, as below, we may use K on the left premise to arrive at the end-sequent. This is allowed since the occurrence of C in Π can be thought of as moved from Π to Γ, where there is already an occurrence, since we do not distinguish location in $\Gamma \cup \Pi$. In the case where C occurs in Θ, we may use K to get the same end-sequent from the right premise.

$$\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad\qquad \dfrac{\Pi \vdash \Theta}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

Given this case, I shall assume for all the remaining cases that the cut formula does not occur in the antecedent of the left premise or in the succedent of the right premise.

Case 22: Suppose the inference resulting in the right premise is an application of a logical rule with one premise. The principal formula of the application of such a rule will occur either in the succedent of the resulting sequent, as in the case of $\vdash\rightarrow$, or in the antecedent, as in the case of $\neg\neg\vdash, \wedge\vdash, \neg\vee\vdash$ and the antecendent quantifier and negated-quantifier rules.

Case 22.0: Suppose that the logical rule resulting in the right premise of the cut is $\vdash\rightarrow$. Then the given proof is as follows:

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{A, \Pi \vdash \Theta, B}{\Pi \vdash \Theta, A \rightarrow B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A \rightarrow B}$$

Given Case 21, we may assume that neither $A \rightarrow B$ nor B are the cut formula, as they appear in the succedent of the right premise, but A may be the cut formula.

Case 22.0.0 Suppose that A is the cut formula, we may need to reintroduce A by K after the application of the cut before applying $\vdash\rightarrow$ again. However, we may well do this, as below:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta \qquad A, \Pi \vdash \Theta, B}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, B}}{A, \Gamma, \Pi^* \vdash \Delta^*, \Theta, B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A \rightarrow B}$$

The cut in the resulting proof has a ρ_r of one fewer than that given, and so it can be eliminated.

Case 22.0.1 Suppose that A is not the cut formula, then we may simply permute the application of cut and $\vdash\rightarrow$:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad A, \Pi \vdash \Theta, B}{\Gamma, A, \Pi^* \vdash \Delta^*, \Theta, B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A \rightarrow B}$$

This cut is also clearly of ρ_r of one fewer than that given.

Case 22.1 Suppose that the principal formula of the logical rule resulting in the right premise occurs in the antecedent of the right premise. These cases are all very

similar, so we shall only present one case ($\neg\neg\vdash$) since the proofs of the others follow the same pattern.

Case 22.1.0 Suppose that the rule resulting in the right premise is $\neg\neg\vdash$. Then the principal formula is $\neg\neg A$, and it may either be the cut formula or not, and the A occurring the antecedent of the right premise of the cut may be the cut formula or not (the case in which A is not the cut formula is covered by the other cases).

Case 22.1.0.0 Suppose that $\neg\neg A$ is the cut formula. Then the proof is as follows:

$$\dfrac{\Gamma\vdash\Delta \qquad \dfrac{A,\Pi\vdash\Theta}{\neg\neg A,\Pi\vdash\Theta}}{\Gamma,\Pi^*\vdash\Delta^*,\Theta}$$

We may perform two cuts, each of lower rank than that given. The first is clearly of ρ_r of one fewer than that given, while for the second cut, the occurrence of the cut formula in Π has been removed, so the ρ_r at the second cut is 1, and so is lowered from whatever it was in the given proof.

$$\dfrac{\Gamma\vdash\Delta \qquad \dfrac{\dfrac{\Gamma\vdash\Delta \qquad A,\Pi\vdash\Theta}{\Gamma,A,\Pi^*\vdash\Delta^*,\Theta}}{\Gamma,\neg\neg A,\Pi^*\vdash\Delta^*,\Theta}}{\Gamma,\Pi^*\vdash\Delta^*,\Theta}$$

Case 22.1.0.1 Suppose that $\neg\neg A$ is not the cut formula. Then the given proof is as on the left below. We may simply permute the application of cut with the application of $\neg\neg\vdash$, as on the right below. Clearly the cut of the resulting proof is of one lower ρ_r than the given.

$$\dfrac{\Gamma\vdash\Delta \qquad \dfrac{A,\Pi\vdash\Theta}{\neg\neg A,\Pi\vdash\Theta}}{\Gamma,\neg\neg A,\Pi^*\vdash\Delta^*,\Theta} \qquad\qquad \dfrac{\dfrac{\Gamma\vdash\Delta \qquad A,\Pi\vdash\Theta}{\Gamma,A,\Pi^*\vdash\Delta^*,\Theta}}{\Gamma,\neg\neg A,\Pi^*\vdash\Delta^*,\Theta}$$

Case 22.1.0.2 Suppose that A is the cut formula. This case is covered under Case 22.1.0.1, with one minor alteration. In the resulting proof, $\neg\neg A$ must be reintroduced by K after the cut.

Case 23: Suppose that the inference figure resulting in the right premise is a two premise rule, namely, one of $\rightarrow\vdash$, $\vee\vdash$, $\vdash\wedge$, $\vdash\neg\rightarrow$, $\vdash\neg\vee$, $\neg\wedge\vdash$.

Case 23.0: Suppose the rule is $\rightarrow\vdash$:

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{\Pi \vdash \Theta, A \qquad B, \Phi \vdash \Omega}{A \rightarrow B, \Pi, \Phi \vdash \Theta, \Omega}}{\Gamma, (A \rightarrow B)^*, \Pi^*, \Phi^* \vdash \Delta^*, \Theta, \Omega}$$

where $(A \rightarrow B)^*$ is either $A \rightarrow B$ or nothing, depending whether it is not or is the cut formula, respectively. There are then two sub-cases:

Case 23.0.0: Suppose that the cut formula occurs in both Π and Φ:

Case 23.0.0.0: Suppose that the cut formula is not $A \rightarrow B$. In this case, we may prove the desired endsequent by the following:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, A}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A} \qquad \dfrac{\Gamma \vdash \Delta \qquad B, \Phi \vdash \Omega}{\Gamma, B^*, \Phi^* \vdash \Delta^*, \Omega}}{A \rightarrow B, \Pi^*, \Gamma, \Phi^* \vdash \Delta^*, \Theta, \Omega}$$

with two instances of cut, each of which can be eliminated, since each has a lower rank than that initially given.

Case 23.0.0.1: Suppose that the cut formula is $A \rightarrow B$. Then another cut can be applied to the endsequent of the proof given in case 23.0.0.0 to remove $A \rightarrow B$ from the antecedent as follows:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad A \rightarrow B, \Pi^*, \Gamma, \Phi^* \vdash \Delta^*, \Theta, \Omega}{\Pi^*, \Gamma^*, \Phi^* \vdash \Delta^*, \Theta, \Omega}}{\Pi^*, \Gamma, \Phi^* \vdash \Delta^*, \Theta, \Omega}$$

again, this cut can be eliminated, since it must be of lower ρ than the original, since ρ_r is one fewer than before, while ρ_l has not changed.

Case 23.0.1: Suppose that the cut formula does not occur in both Π and Φ. It must occur in one or the other, else the right rank of the cut is equal to 1.

Case 23.0.1.0: Suppose that the cut formula occurs in Π but not Φ.

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, A}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A} \qquad B, \Phi \vdash \Omega}{A \rightarrow B, \Gamma, \Pi^*, \Phi \vdash \Delta^*, \Theta, \Omega}$$

where the cut may be eliminated.

Case 23.0.1.0.0: Suppose that the cut formula is $A \to B$. Then we introduce another cut with $\Gamma \vdash \Delta$ to remove $A \to B$, to produce the desired sequent. The procedure is similar to that of case 23.0.0.1.

Case 23.0.1.0.1: Suppose that the cut formula is not $A \to B$. Then, the procedure is similar to that of case 23.0.0.0.

Case 23.0.1.1: Suppose that the cut formula occurs in Φ but not Π. Then the proof can be altered to the following:

$$\dfrac{\dfrac{\Pi \vdash \Theta, A \qquad \dfrac{\Gamma \vdash \Delta \qquad B, \Phi \vdash \Omega}{\Gamma, B, \Phi^* \vdash \Delta^*, \Omega}}{A \to B, \Pi, \Gamma, \Phi^* \vdash \Delta^*, \Omega, \Theta}}$$

where the cut may be eliminated. There are, as above in case 23.0.1.0, two sub-cases, each of which is treated in a way essentially similar to the way in which they are treated above.

Case 23.1: Suppose that the rule is $\vee \vdash$:

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{A, \Pi \vdash \Theta \qquad B, \Pi \vdash \Theta}{A \vee B, \Pi \vdash \Theta}}{\Gamma, (A \vee B)^*, \Pi^* \vdash \Delta^*, \Theta}$$

which can be altered to produce:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad A, \Pi \vdash \Theta}{A, \Gamma, \Pi^* \vdash \Delta^*, \Theta} \qquad \dfrac{\Gamma \vdash \Delta \qquad B, \Pi \vdash \Theta}{B, \Gamma, \Pi^* \vdash \Delta^*, \Theta}}{A \vee B, \Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

where both applications of the cut can be eliminated, by the induction hypothesis.

Case 23.1.0: Suppose that the cut formula is $A \vee B$. Then a cut involving the end-sequent of the second proof presented in 23.1 can be altered to produce the following:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad A \vee B, \Gamma, \Pi^* \vdash \Delta^*, \Theta}{\Gamma^*, \Pi^* \vdash \Delta^*, \Theta}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta}$$

where the cut can be eliminated.

Case 23.1.1: Suppose that the cut formula is not $A \vee B$ and must, therefore, occur in Π. Then, in a procedure similar to 23.0.0.1, the end-sequent of the second proof given in 23.1 is the desired sequent.

Case 23.2: Suppose that the rule is $\vdash \wedge$.

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{\Pi \vdash \Theta, A \qquad \Pi \vdash \Theta, B}{\Pi \vdash \Theta, A \wedge B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A \wedge B}$$

which can be altered to the following, in which both cuts are of lower rank than that given, and thus can both be eliminated:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, A}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A} \qquad \dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, B}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A \wedge B}$$

Case 23.3: Suppose that the rule is $\vdash \neg \rightarrow$.

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{\Pi \vdash \Theta, A \qquad \Pi \vdash \Theta, \neg B}{\Pi \vdash \Theta, \neg(A \rightarrow B)}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg(A \rightarrow B)}$$

This proof can be altered to the following:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, A}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, A} \qquad \dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, \neg B}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg(A \rightarrow B)}$$

in which the cuts may be eliminated.

Case 23.4: Suppose that the rule is $\vdash \neg\vee$.

$$\dfrac{\Gamma \vdash \Delta \qquad \dfrac{\Pi \vdash \Theta, \neg A \qquad \Pi \vdash \Theta, \neg B}{\Pi \vdash \Theta, \neg(A \vee B)}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg(A \vee B)}$$

which can be altered to the following, in which each cut is of a lower rank than that given, and thus each can be eliminated:

$$\dfrac{\dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, \neg A}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg A} \qquad \dfrac{\Gamma \vdash \Delta \qquad \Pi \vdash \Theta, \neg B}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg B}}{\Gamma, \Pi^* \vdash \Delta^*, \Theta, \neg(A \vee B)}$$

Case 23.5: Suppose that the rule is $\neg\wedge\vdash$

$$\dfrac{\Gamma\vdash\Delta \qquad \dfrac{\neg A,\Pi\vdash\Theta \qquad \neg B,\Pi\vdash\Theta}{\neg(A\wedge B),\Pi\vdash\Theta}}{\Gamma,\neg(A\wedge B)^*,\Pi^*\vdash\Delta^*,\Theta}$$

which may be altered to the following, where both cuts may be eliminated, but where, as in case 23.1, there remain two sub-cases:

$$\dfrac{\dfrac{\Gamma\vdash\Delta \qquad \neg A,\Pi\vdash\Theta}{\Gamma,\neg A^*,\Pi^*\vdash\Delta^*,\Theta} \qquad \dfrac{\Gamma\vdash\Delta \qquad \neg B,\Pi\vdash\Theta}{\Gamma,\neg B^*,\Pi^*\vdash\Delta^*,\Theta}}{\neg(A\wedge B),\Gamma,\Pi^*\vdash\Delta^*,\Theta}$$

Case 23.5.0: Suppose that the cut formula is $\neg(A\wedge B)$. The proof is similar to that given in 23.1.0.

Case 23.5.1: Suppose that the cut formula is not $\neg(A\wedge B)$. The proof is similar to that given in 23.0.0.1.

The remaining cases are those where $\rho > 2$, the right rank is 1, and therefore the left rank > 1. These transformation of these cases are, generally, little more than the duals of cases 20-23, the biggest change coming about because the rules governing the truth functional connectives (except the rules for $\neg\neg$) tend to have one upper sequent when introducing the formula on one side of the turnstile and two upper sequents when introducing it on the other side. It is perhaps worth flagging that there is some extra work to be done for the arrow rules when $\rho_l > 1$. However, those proofs are still just the same as given for LK in [17], so they are omitted. Thus, cuts can always be traded for cuts of lower ranks until one reaches cuts on $\rho = 2$, at which point the cut formulae can be reduced in grade until they are literals, at which point they can be directly eliminated. This process of separately reducing rank and grade may need to be repeated in some sequence for a given proof, however the process will eventually produce a cut free proof of the same end-sequent as the given proof.

□

Note the following rule, $\vdash\neg$, is one of the two standard sequent negation rules.

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \vdash \neg$$

Corollary 2.1. $\vdash \neg$ *is an admissible rule.*

That is, whenever there is a proof in LA_3 of $A, \Gamma \vdash \Delta$, there is a proof of $\Gamma \vdash \Delta, \neg A$. So, $\neg \vdash$ is admissible.

The result of this is that LA_3 could just as well be stated without $\vdash A, \neg A$ and with $\vdash \neg$ as an additional rule, as the instances of LEM are evidently provable by $\vdash \neg$ from the appropriate instances of the reflexivity axiom.

This completes my treatment of LA_3 - at least all the facts about LA_3 which I'll be appealing to and relying upon for justification in what follows. The soundness and completeness theorems I have presented in order to justify the kind of reasoning which leaps back and forth from the syntax to the semantics and conversely. This will make certain arguments considerably simpler and seems a valuable asset in justifying this logic as natural. The cut elimination proof I have presented because of the value in seeing that LA_3 is syntactically well-formed - also because, as stated, the first order proof, like the first order extension of the logic, is novel. However, the next part of this work is to consider the alterations necessary to the standard arithmetical axioms to account for the unusual facts about the collapse models.

2.2 Arithmetical Axioms for collapse models

The collapse models, though a result of a congruence on the standard model, are fairly unusual in some crucial respects. The major one being the behaviour of the operations, and most particularly that of successor ($'$). All of the alterations necessary here are made in order to match the axioms to the facts about operations congruent to this under various collapses. To start, a fairly standard presentation of the Peano axioms with the LA_3 conditional, where, as standard, $x \neq y =_{Def} \neg x = y$. Given that UG is an inference rule of A_3, we may leave universal quantifiers tacit in the axioms and in the theorems, since we can always introduce those quantifiers by repeated

appeals to UG. As such, I shall omit quantifiers except in cases where it makes an important difference to have them inside a formula (as in the case of inferences by mathematical induction).

Peano Axioms

$\vdash x = y \to (z = y \to x = z)$ $\qquad$ $\vdash x + 0 = x$

$\vdash x' = y' \leftrightarrow x = y$ $\qquad$ $\vdash x + y' = (x + y)'$

$\vdash 0 \neq y'$ $\qquad$ $\vdash x \cdot 0 = 0$

$\vdash x \neq 0 \to \exists y\, x = y'$ $\qquad$ $\vdash x \cdot y' = (x \cdot y) + x$

and finally the schema of mathematical induction,

$\vdash (A0 \wedge \forall x(Ax \to Ax')) \to \forall x Ax$ (MI_1)

To allay any concerns about the usefulness of the weak induction principle in LA_3, below a proof that it is equivalent to a sequent form of induction, namely $A0, \forall x(Ax \to Ax') \vdash \forall x Ax$ (MI_2), is given. This is an easy consequence of the deduction theorem and its converse, the fact of which is clearly carried out in the proof system (see (*) below).

The extent to which this shortens proofs is astounding and makes the presentation much easier.

So, we have a system of first order LA_3 plus the standard suite of Peano Axioms. This system I shall call $\mathrm{LA}_3^{\#}$, analogous to Meyer's system $\mathrm{R}^{\#}$, and it proves many of the same basic theorems which classical Peano Arithmetic (PA). In fact, since LA_3's positive fragment is equivalent to the positive fragment of classical logic and since they both have the same Peano Axioms, PA^{+} and $\mathrm{LA}_3^{\#+}$ are equivalent. The difference between them is only in those theorems containing negations. However, as I'll show in the next chapter, many of these can be proven in $\mathrm{LA}_3^{\#}$, giving it a solid set of arithmetical theorems. Among others, there may be proofs available in $\mathrm{A}_3^{\#}$ of all the theorems necessary to produce the glutty Gödel sentence $\mathfrak{G}$, however I have not proved each of these theorems as of writing. To show this would be to show that many proofs in elementary formal number theory do not rely on inconsistency-trivialising

Theorem 2.7. *MI_2 is equivalent to MI_1 in LA_3. (*)*

$$\dfrac{\dfrac{A0, \forall x(Ax \to Ax') \vdash \forall x Ax}{A0 \wedge \forall x(Ax \to Ax') \vdash \forall x Ax}}{\vdash (A0 \wedge \forall x(Ax \to Ax')) \to \forall x Ax}$$

$$\dfrac{\dfrac{\vdash (A0 \wedge \forall x(Ax \to Ax')) \to \forall x Ax}{A0 \wedge \forall x(Ax \to Ax') \vdash \forall x Ax} \quad \dfrac{\dfrac{A0 \vdash A0}{A0, \forall x(Ax \to Ax') \vdash A0} \quad \dfrac{\forall x(Ax \to Ax') \vdash \forall x(Ax \to Ax')}{A0, \forall x(Ax \to Ax') \vdash \forall x(Ax \to Ax')}}{A0, \forall x(Ax \to Ax') \vdash A0 \wedge \forall x(Ax \to Ax')}}{A0, \forall x(Ax \to Ax') \vdash \forall x Ax}$$

MI_2 is equivalent to the principle MI_3 which I'll use in favour of MI_1 or MI_2 for the sake of surveyability of the proofs. MI_3 is as follows:

$$\mathrm{MI}_3 \ \dfrac{\Gamma \vdash \Delta, A0 \qquad \Gamma \vdash \Delta, \forall x(Ax \to Ax')}{\Gamma \vdash \Delta, \forall x Ax}$$

That this is equivalent to the above formulations is obvious considering each instance of MI_3 could be trivially reproduced with MI_2 by conjoining the two premises of MI_2 by $\vdash \wedge$ and adding a second upper sequent to the end-sequent with the appropriate statement of MI_2, where the end-sequent is justified by cut, and not by MI_3:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A0 \qquad \Gamma \vdash \Delta, \forall x(Ax \to Ax')}{\Gamma \vdash \Delta, A0 \wedge \forall x(Ax \to Ax')} \qquad A0 \wedge \forall x(Ax \to Ax') \vdash \forall x Ax}{\Gamma \vdash \Delta, \forall x Ax}$$

principles, and may be worth it in its own right. The upshots of considering the proofs necessary to express $\mathfrak{G}$ are many – first, $\mathfrak{G}$ would be the first explicit glut to occur "naturally" in the language of $\mathrm{LA}_3^{\#}$, using only predicates definable in the language of arithmetic, and, second, it would better couch Priest's discussion of $\mathfrak{G}$ in paraconsistent number theory [38, 48–50] which takes for granted that the syntax of LP can be arithmetized. Though this is somewhat tangential to the goal of providing adequate syntaxes for the inconsistent models, it is a subject which the development of this theory may well provide some insight into.

Back to the axioms – as discussed in the last chapter, the standard principle $x' = y' \rightarrow x = y$ is false in some of the collapse models. Thus there must be a different theory to deal with these models specifically. First, $x' = y' \leftrightarrow x = y$ will need to be split into its component parts:

$$\vdash x = y \rightarrow x' = y' \qquad\qquad \vdash x' = y' \rightarrow x = y$$

and, since contraposition fails for this conditional, it is also necessary to explicitly state the contraposed forms of these:

$$\vdash x \neq y \rightarrow x' \neq y' \qquad\qquad \vdash x' \neq y' \rightarrow x \neq y.$$

Out of this range of successor axioms we simply reject $\vdash x' = y' \rightarrow x = y$, however all the others are valid on all the collapse models. So, all of the others must be included, in order to ensure that the information we can get about successors is as robust as possible. Since they are not derivable from the other axioms it is necessary to explicitly include each of these axioms individually - this claim seems obvious, though I present no proof it here. It is, perhaps, interesting to note that one of the potential uses for the inconsistent models which is suggested by Paris, [30, 536], is to produce new independence proofs for the Peano axioms in classical logic. As it stands, however, I'll have to rely on the fact that it is intuitively clear that none of the above can be inferred from the other axioms.

The addition and multiplication axioms do not present any obvious problem, however it may be an interesting exercise to consider a version of Presburger's arithmetic to determine whether the classical axiomatisability result, perhaps as presented in

[11, 219], can be adapted to the same theory based on LA_3. However, this aside, I don't see a reason to alter any of these axioms, since the behaviours they produce which might cause problems are avoided by the alteration to the successor axiom presented above.

Transitivity of $=$ will clearly produce no problem, and $x \neq 0 \rightarrow (\exists x)(x = y')$ remains true in all models since even in models in which a, distinct from 0, is identified with 0 under collapse, the consequent remains true since in such a model 0 and a must occur in a cycle such that $a-1$ is the predecessor of 0. The same considerations hold in the case of $x' \neq 0$. In models with a classical tail which includes, minimally, 0 this will be true, and in those cyclic models where 0 and a are identified $(a-1)' = 0$ but also $(a-1)' \neq 0$, so that the formula is still assigned a designated value, $\{1, 0\}$. In fact, it will be by introducing extra axioms which explicitly exploit this feature that I'll attempt to capture the simple cyclic models, as well as those cycles occurring in larger, more complicated models.

So, the basic axiom system I'll be using throughout will look the same as that above with $\vdash x' = y' \leftrightarrow x = y$ replaced with:

$\vdash x = y \rightarrow x' = y'$
$\vdash x \neq y \rightarrow x' \neq y'$
$\vdash x' \neq y' \rightarrow x \neq y$

It is to this basis that I shall add extra axioms designed to govern the specific differences which occur in the particular collapse models. Call this variant system on $\mathrm{LA}_3^{\#}$, $\mathrm{LA}_{3*}^{\#}$. In the next chapter, I shall present some proofs in $\mathrm{LA}_3^{\#}$ to indicate its general suitability as an arithmetical theory and to consider what theorems follow from the undesirable axiom above which are still, however, true in the collapse models. Beyond this, I shall present natural extensions to $\mathrm{LA}_{3*}^{\#}$ for each of the basic classes of finite inconsistent models, and attempt to show that they capture at least the salient properties of their respective models.

Chapter 3

Formal Number Theory in $\mathrm{LA}_3^\#$ and $\mathrm{LA}_{3*}^\#$

In the course of the second chapter, we introduced the systems $\mathrm{A}_3^\#$ and $\mathrm{A}_{3*}^\#$, and in this chapter, I shall present some proofs of these systems to indicate that they are a good match for formal reasoning about the inconsistent models of arithmetic as well as for formal arithmetics more generally. The core of this chapter is to provide some grounds that this formal system is a good match for arithmetical reasoning and, in particular, the finite collapse models, and alongside this we shall consider some more general points about how these theories match up to PA. One of the first things to note regarding the more general point is that $\mathrm{A}_3^\#$ shares a positive fragment with PA - any valid negation-free sequent of PA is a valid sequent of $\mathrm{A}_3^\#$.[1] This is fairly obvious, considering that we already know that the positive fragment of LA_3 is equivalent to the positive fragment of LK, and that all the arithmetical axioms of PA are simply reproduced in $\mathrm{A}_3^\#$. So, I shall be appealing freely to positive theorems and sequents of PA, confident that their proofs in $\mathrm{A}_3^\#$ require little more than straightforward translations of the PA proofs into this formalism. This provides a good starting point to arguing that $\mathrm{A}_3^\#$ is a plausible candidate for studying PA and the standard model of arithmetic from a paraconsistent basis. Namely, we can rest assured that, at very

[1] Of course, this is not the case with $\mathrm{A}_{3*}^\#$, which does not include the injectivity of $'$. In the case of this system, a bit more groundwork is necessary to indicate which theorems of PA are also theorems of $\mathrm{A}_{3*}^\#$

least, we only stand to lose theorems with negations, and it is not clear which of those we do actually lose. As well as this, with one alteration, it indicates that the kind of construction necessary for an arithmetization of the syntax of $\mathrm{A}_3^{\#}$ may possibly be carried out. This would allow us to better couch discussion about the Gödel sentence as a glut of sufficiently strong mathematical systems, such as carried out in the third chapter of [38]. This kind of approach to the incompleteness theorems has, among others, been discussed by Berto in [10, 203–213] and Meyer in [24]. To assess these claims about the incompleteness theorems from a paraconsistent viewpoint, it must be the case that we can *actually* carry out the arithmetisation of the syntax of a paraconsistent logic.[2] A theory like $\mathrm{A}_3^{\#}$ is, at least, a plausible candidate for a system for which this kind of construction can be given. Otherwise, until one has shown that such a construction can be carried out in a formal paraconsistent arithmetic, it must remain in doubt whether or not one can, in fact, produce something like a Gödel sentence at all, which would in turn shed doubt on the philosophical arguments presented by Priest [38] and others on the topic of the Gödel sentence in paraconsistent logic. Of course, one can always make stipulations in order to get the right kind of expressibility, as we may have to, but the question is whether such stipulations do not simply result in classical arithmetic. I shall come back to this point as a final note to this chapter.

The first question, then, is of the recapture of the negative sequents of PA in $\mathrm{A}_3^{\#}$. Because these theorems are of general importance for the collapse models as well as for the standard model, we shall consider this in some detail before moving on to considering the adequacy of various extensions of $\mathrm{A}_{3*}^{\#}$ to the task of axiomatising the collapse models. With this information, we shall be closer to determining the adequacy of $\mathrm{A}_3^{\#}$ for PA – how substantial differences between $\mathrm{A}_3^{\#}$ and PA actually are – though a thorough consideration of this question is another major research project in its own right.

Since A_3 does not validate the rule form of disjunctive syllogism[3] ($\vdash A$ and $\vdash$

[2]Perhaps in something like the construction given by Mendelson in the textbook [22]

[3]What has been called, in the tradition of Relevant Logic, the rule γ, following Ackermann [1].

$\neg A \vee B$ does not imply $\vdash B$) $A_3^\#$ is likely susceptible to the same argument as is given for $R^\#$[4] to indicate that its set of theorems is properly contained in that of PA. Given that this is the case, it is quite likely that $A_3^\#$ will also be strictly weaker, in the sense of having a smaller set of theorems, than PA, and as such will fail to fully capture classical formal arithmetic. However, $A_3^\#$ still quite closely approaches to PA, and even more so if we take on an addition, in the form to be discussed below, of some limited trivialising formulae. Given that all theorems of PA provable without appeal to formule involving negations are theorems of $A_3^\#$ the question then is which of the negative sequents do the systems have in common. That is, how much of the negative part of PA is recaptured in $A_3^\#$? This is of some general importance for the inconsistent models of arithmetic as well as for the general value of $A_3^\#$ as an arithmetical theory.

3.1 Negative sequents of PA

So, it is the negative formulae of $A_3^\#$ which differ from PA - and the standard format for reductio ad absurdum (RAA) style reasoning in natural deduction-style systems, which can be presented axiomatically as $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B)) \rightarrow \neg A$ or $(A \rightarrow (B \wedge \neg B)) \rightarrow \neg A$, is obviously invalid in A_3. It is not obvious how the inferences allowed by this principle are to be recovered, and it seems that there are a number of possibilities, depending on the formulae in question. First, there is a standard dodge of altering RAA inferences from the irrelevant (and potentially explosive) format as those given above to the principle of consequentia mirabilis (CM) [5] – $(A \rightarrow \neg A) \rightarrow \neg A$, which is valid in A_3:

$$\dfrac{\dfrac{\vdash A, \neg A \qquad \neg A \vdash \neg A}{A \rightarrow \neg A \vdash \neg A}}{\vdash (A \rightarrow \neg A) \rightarrow \neg A}$$

[4]This is Meyer's system of relevant arithmetic given in [23] and elsewhere.

[5]This principle is what Whitehead and Russell call reductio ad absurdum or Abs ($*2{\cdot}01$) [45, 100], and in classical logic it is equivalent to $(A \rightarrow (B \wedge \neg B)) \rightarrow A$, which is the explosive version of RAA - however, note that these are not equivalent in A_3, and so I shall distinguish them, as here, by using "RAA" and "CM" or "consequentia mirabilis."

This is a straightforward theorem, which can be applied in many in the same ways as in relevant logics which involve this rule but would invalidate the explosive RAA. In fact, this theorem is also useful in showing that, for instance, the Russell set in the presence of unrestricted comprehension produces a contradiction of the usual conjunctive form.[6] However, in other cases, there are more straightforward solutions just invoking the LA_3 rules and axioms, allowing for more constructive proofs of classical theorems than are given in terms of classical RAA. Often, all that is required is an instance of LEM and some negated-connective rules, indicating that LEM really is very strong, as is the negation in LA_3, despite the lack of the usual negation rules. Often even these are not required. Consider the following proof of $\forall x.x \neq x'$:

$$\dfrac{\dfrac{\dfrac{0 \neq 0' \vdash 0 \neq 0'}{\forall x.0 \neq x' \vdash 0 \neq 0'} \quad \vdash \forall x.0 \neq x'}{\vdash 0 \neq 0'} \quad \dfrac{\vdash x \neq x' \to x' \neq x''}{\vdash \forall x(x \neq x' \to x' \neq x'')}}{\vdash \forall x.x \neq x'}\ \mathrm{SsMI}$$

This proof illustrates a standard dodge which I'll employ to shorten the proofs in the following sections. I shall treat the arithmetical axioms and some derivable theorems as sequents when useful - so, for example, I may use $x = y, x = z \vdash y = z$ or $x = y \vdash y = z \to x = z$ instead of introducing $\vdash x = y \to (y = z \to x = z)$ and then analyzing the formula with cuts, as this kind of procedure tends to quickly expand proof trees beyond clarity or surveyability. Given that $\vdash A \to B$ iff $A \vdash B$, the conjunction and disjunction rules, and cut, these transformations are trivial, and do not introduce anything other than succinctness and clarity of the presentation.

So, the above is one case where a negative theorem of PA can be given in $\mathrm{LA3}^{\#}$ without the use of RAA.[7] Of course, this same proof could be reproduced in PA without RAA. I should be interested to see whether any of the negative formulae provable in PA by RAA are provable without appeal to it. Some evidence that this holds is that many of the basic of these formulae are provable in $\mathrm{R}^{\#}$, which also

[6] Otherwise, one is left with a theorem of the form $R \in R \leftrightarrow R \notin R$, which is, perhaps, not as explicit or disturbing as $R \in R \wedge R \notin R$. At very least [27] distinguishes these two formulae.

[7] The same holds for $\mathrm{LA3}^{\#}_{*}$, since no appeal is made to the axiom distinguishing these.

invalidates the standard form of RAA. However, since this system is also inadequate for PA, this conjecture is not as strong a starting point as we might like. However, what this conjecture indicates is that at least all the arithmetical formulae proved in PA Kleene [21] are provable in $\mathrm{R}^{\#}$, and so long as these proofs do not require contraposition, they should also be provable in $\mathrm{LA}_3^{\#}$. It is worth going through a few more of these proofs to see the standard steps that are made, since many more complex results simply follow from the basic ones to be outlined here.

Now, probably one of the most standard uses of RAA is to justify negative existentials - one supposes that an object matching the proposed description exists, shows that this leads to inconsistency, and thus infers that nothing of the supposed type can exist, after all. There are some proofs of this kind which can also be reproduced in $\mathrm{LA}_3^{\#}$ without appeal to CM. Consider the following proof of $\forall z \neg z < 0$, where $x < y$ shall be defined $\exists z(z \neq 0 \wedge x + z = y)$. Given this definition, the desired theorem is $\vdash \forall z \neg \exists x(x \neq 0 \wedge z + x = 0)$. The proof features an interesting inference involving $\rightarrow$, where counterexample is used involving another theorem of the system, namely the left premise of the fourth line:

$$\dfrac{\dfrac{\vdash z + x = 0 \rightarrow x = 0 \qquad \dfrac{\dfrac{\dfrac{\vdash z + x = 0, \neg z + x = 0 \qquad x = 0 \vdash x = 0}{z + x = 0 \rightarrow x = 0 \vdash \neg z + x = 0, x = 0}}{z + x = 0 \rightarrow x = 0 \vdash \neg z + x = 0, \neg\neg x = 0}}{z + x = 0 \rightarrow x = 0 \vdash \neg(z + x = 0 \wedge \neg x = 0)}}{\dfrac{\dfrac{\vdash \neg(z + x = 0 \wedge \neg x = 0)}{\vdash \neg\exists x(z + x = 0 \wedge \neg x = 0)}}{\vdash \forall z \neg\exists x(z + x = 0 \wedge \neg x = 0)}}$$

This theorem gives us grounds to believe that the subject matter of $\mathrm{A}_3^{\#}$ is $\mathbb{N}$, which is a nice thing to know, and the proof is straightforward, showcasing how the negation rules of A_3 do recapture some of the work which the negation symbol does in PA. A similar proof can be constructed showing that $\forall y \neg y < y$

$$\dfrac{\dfrac{\vdash x+y=y, \neg x+y=y \qquad 0=x \vdash 0=x}{x+y=y \to 0=x \vdash \neg x+y=y, 0=x} \quad \vdash x+y=y \to 0=x}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \neg x+y=y, 0=x}{\vdash \neg x+y=y, \neg\neg 0=x}}{\vdash \neg(x+y=y \wedge \neg 0=x)}}{\vdash \neg\exists x(x+y=y \wedge \neg 0=x)}}{\vdash \forall y \neg\exists x(x+y=y \wedge \neg 0=x)}}$$

This is a principle provable in $\mathrm{LA}_3^{\#}$ which may prove problematic on collapse models involving cycles, and thus should be avoided in $\mathrm{LA}_{3*}^{\#}$. For instance, it is not the case that the left premise of the second application of cut - $\vdash x+y=y \to 0=x$ is true. Any improper nucleus will have some period p such that $p+y=y \wedge p \neq 0$ where, in the case of heap models, the right-most conjunct, $p \neq 0$, is not glutty but simply true. This formula can, however, be proved in $\mathrm{A}_3^{\#}$ as follows:

$$\dfrac{\dfrac{\vdash 0+y=y \qquad 0+y=y, x+y=y \vdash 0+y=x+y}{x+y=y \vdash 0+y=x+y} \qquad 0+y=x+y \vdash 0=x}{\dfrac{x+y=y \vdash 0=x}{\vdash x+y=y \to 0=x}}$$

Of course, this is a proof that could be given in PA using LK as the background logic. However, this proof should not be valid in $\mathrm{A}_{3*}^{\#}$ and so it is necessary that the right premise of the second application of cut in this proof be rejected in that system. The proof of $0+y=x+y \vdash 0=x$ is given on the next page

These are some straightforward examples of theorems, many of which are traditionally given by means of proof by contradiction which do not need to be proved this way. This is something of a trick resulting of the fact that, in $\mathrm{LA}_3^{\#}$, CM is proven by appeal to LEM and the $\to$ rules, and that the same kind of move can be done with the negated connective rules. The general form is of a cut, such that the instance of the cut theorem occurring in the succedent of a premise of the application of cut is a theorem, the main connective of which is a conditional. Then, with this formula occurring in the antecedent of a premise of the cut allows one to prove some negated conjunctive formula by means of counterexample. This seems to be a generally useful trick available to us. This gives us a bag of tricks with which to avoid explosive RAA style reasoning, while still getting us some desirable negative theorems. The only

$$\dfrac{\dfrac{\dfrac{x+0=x \qquad x+0=x, y=x+0 \vdash x=y}{y=x+0 \vdash x=y} \qquad y+0=y, x+0=y+0 \vdash y=x+0}{y+0=y, y+0=x+0 \vdash x=y} \qquad \vdash y+0=y}{x+0=y+0 \vdash x=y}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(x+z)'=(y+z)' \vdash x+z=y+z}{\dagger\ x+z'=y+z' \vdash x+z=y+z}}{x+z'=y+z' \vdash x+z=y+z, x=y}}{\vdash x+z=y+z, x+z'=y+z' \to x=y} \qquad \dfrac{x=y \vdash x=y}{x=y, x+z'=y+z' \vdash x=y}}{\vdash x+z'=y+z' \to x=y, x+z=y+z} \qquad \dfrac{x=y \vdash x=y}{x=y, x+z'=y+z' \vdash x=y}}{x+z=y+z \to x=y \vdash x+z'=y+z' \to x=y, x+z'=y+z' \to x=y}}{x+z=y+z \to x=y \vdash x+z'=y+z \to x=y}}{\vdash (x+z=y+z \to x=y) \to (x+z'=y+z' \to x=y)}}{\vdash \forall z((x+z=y+z) \to x=y) \to (x+z'=y+z' \to x=y))}$$

Thus $\vdash \forall z(x+z=y+z \to x=y)$ follows by induction, though the particular induction inference is left out since the proof would not fit on the page (let alone the margin). Notice, that the culprit does finally come out as the axiom distinguishing $\mathrm{A}_3^{\#}$ from $\mathrm{A}_{3*}^{\#}$, namely $\vdash x'=y' \to x=y$ in the first axiom in the left premise of the proof of the induction step (the step is marked by †). So, there does not need to be any alterations to any of the other axioms of $\mathrm{A}_3^{\#}$ to avoid this result. This is not to say that this same theorem cannot be produced by some other means, though I have not found any proof which does not rely on this axiom. Of interest here is that the failure of injectivity of $'$ results in the failure of right cancellation for $+$ and thus for $\cdot$ as well. This is to be expected, and provides something of a picture of how the arithmetical functions behave in $\mathrm{A}_{3*}^{\#}$.

kind of arithmetical proof which these cannot get us around would be such that, for some formula $\neg A$, say the negation of a theorem A, $A \vdash$ holds in classical PA and K allows for an immediate proof of some desired formula. I have come across examples of theorems such that a move like this would make the proof immediately obvious - however, in none of these cases is it also clear that no other kind of proof is available. One of these is a theorem necessary to get the correct kind of multiplication properties off the ground, namely $\neg x = 0 \rightarrow (y \cdot x = 0 \rightarrow y = 0)$. This theorem is part of the construction of the arithmetisation of classical logic given in Mendelson [22, 156], and it seems to be necessary for results which are necessary for the construction. As such, it is a desirable theorem for one of our stated purposes.

3.1.1 Absolutely false formulae

One potential method for proving $\neg x = 0 \rightarrow (y \cdot x = 0 \rightarrow y = 0)$ in an easy way follows a move used by Meyer and Mortensen [26] which is to introduce some right-trivializing formulae, perhaps just one. This is a novelty for LA_3, which does not admit of empty succedent sequences otherwise - there is no definable absolutely false formula in A_3. This formula ought to be something which there are independent reasons to dislike, beyond simply disliking arbitrary contradictions. That is, it is a formula which we are to take as evidence that we've done something seriously wrong, even under the assumption that we can correctly prove some contradictions, and that this bad formula ought to trivialise, as evidence of our doing something seriously wrong in proving it. For an example, suppose we introduce as another axiom in the arithmetic language either $0' = 0 \vdash$, which holds in the cyclic models, or $\neg 0 = 0 \vdash$, which holds in the heap models with some improper nuclei. This would clearly invoke something like a restricted version of explosion, given unrestricted K. This special case would have the benefit of getting some short and simple proofs of desirable theorems back into our hands, and, perhaps, allow us to finish up the construction to the proof predicate and the Gödel sentence. It is an option left to us by the adoption of paraconsistent modes of reasoning is that we may still have the choice of taking some formulae as seriously bad, which we would desire to trivialise. This is acceptable, if we

can show that, despite this addition, we can still have non-trivial gluts, like $\mathfrak{G}$, then we may still have some motivational ground to stand on. This would have us accept some formulae as seriously bad, while not requiring us to also admit that an arbitrary contradiction is seriously bad. This would give us a way to consider claims made by some paraconsistent or dialethic philosophers of mathematics, while still retaining the spirit of the program. Those presented above are obvious candidates for these kind of sentences. Meyer [23] discusses claims like these as possible *measures* of the consistency of an arithmetical system, of which negation consistency and triviality are distinct *degrees* of inconsistency, and that things like $0 \neq 0 \vdash$ are somewhere in between.

However, for our purposes, it is most interesting to note that these make certain proofs in $\mathrm{A}_3^{\#}$ either possible or much easier. For instance, $0 \neq 0 \vdash$ can be made use of to easily prove the following base cases of a case-based inductive derivation (in the style exhibited in [22]) showing that $\vdash \neg x = 0 \to (y \cdot x = 0 \to y = 0)$. After these are proven, the induction step is straightforward, and so shall be omitted.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash 0 = 0}{0 \cdot x = 0 \vdash 0 = 0}}{\neg x = 0, 0 \cdot x = 0 \vdash 0 = 0}}{\neg x = 0 \vdash 0 \cdot x = 0 \to 0 = 0}}{\vdash \neg x = 0 \to (0 \cdot x = 0 \to 0 = 0)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg 0 = 0 \vdash}{\neg 0 = 0 \vdash y = 0}}{y \cdot 0 = 0, \neg 0 = 0 \vdash y = 0}}{\neg 0 = 0 \vdash y \cdot 0 = 0 \to y = 0}}{\vdash \neg 0 = 0 \to (y \cdot 0 = 0 \to y = 0)}}$$

The addition of $0 \neq 0 \vdash$, as stated, would seem to serve to get us derivations, at least simpler derivations, which are not clearly available in $\mathrm{A}_{3*}^{\#}$. The problem is whether by adding, say $0 \neq 0 \vdash$, we do not undo the paraconsistency of the arithmetic part of the theory - whether it's the case that any old contradiction in the language of arithmetic implies $0 \neq 0$ and thus triviality. One interesting possibility here is that introducing $0 \neq 0 \vdash$ can be done without reducing the system to classical arithmetic by rejecting $\vdash x' \neq y' \to x \neq y$ - since this is pretty clearly a principle necessary to reduce any arithmetical contradiction to $0 \neq 0$, as it allows us to strip successor-symbols off of numbers standing in the $\neq$ relation. This principle is, by dint of nothing more than basic visual similarity, related to $\vdash x' = y' \to x = y$, which is clearly unacceptable for collapse models, so its rejection may provide some reason

to reject $\vdash x' \neq y' \rightarrow x \neq y$, in addition to the reassurance that extra work done by $0 \neq 0 \vdash$ does not simply buy us back classical formal arithmetic.

This kind of move would, in general, be in keeping with the program of paraconsistent mathematics. It need not be the case that *no* contradictions trivialise, but only that *not all* contradictions trivialise, and that none of the contradictions which do trivialise are theorems. Instead, what we are free to do is to choose which contradictions to accept and which to reject, within the confines of the model.[8]

In the case of proper heap models, where the classical tail contains at least one member, it is *straightforwardly false* that $0 \neq 0$, and to prove this would be to fail to capture an important fact about the model. In this case it would be sensible to take $0 \neq 0$ as a sign of failure, and thus allow it to trivialise, like all contradictions do in a classical context. However, in the case of a cyclic model $0 \neq 0 \vdash$ would be totally unacceptable. It is a feature of cyclic models that $0 \neq 0$ is true, since 0 is collapsed into an equivalence class with classically distinct numbers. So, even in the case of collapse models, this axiom is motivated in some models, but not in others. The job of trivialisation, on this account, is to mark out what is unacceptable on a model, and it is coherent to claim that not every contradiction trivialises, but that some may, given some extra-logical facts at our disposal. So, depending on the task at hand (ie. the model under consideration), we are free to specify the stopping points - the formulae which will indicate that the system has gone awry, and it is not necessarily the case that any claim of the form $A \wedge \neg A$ is such a formula.

In any case, even if we cannot recapture all of PA with $\mathrm{A}_3^{\#}$, we can recapture quite a lot of it, and even more if we take the dodge discussed above. $\mathrm{A}_3^{\#}$ is a fairly robust theory which captures some essential parts of formal arithmetic. This gives us some ground to consider $\mathrm{A}_{3*}^{\#}$ and the simple extended and collapse models of arithmetic.

[8]This is related to a point made by Beall in a number of places that it is not a matter of *logic* whether a contradiction is bad or not, but that it is rather a matter of other principles of reasoning. His general account is presented in [7]. On his picture, we must rely on extra-logical claims to tell us what contradictions are acceptable or not, but that logic does not decide the point for us. My point here is along these lines.

3.2 Axioms for inconsistent models of arithmetic

The inconsistent models of arithmetic come in a variety, and each kind will require some distinct extra axioms in order to capture their specific properties. I shall propose to build these model-specific axioms on the background of the basic system of $A_3^{\#}$, in the case of simple cyclic models, and $A_{3*}^{\#}$, in the case of any model including something like a heap, where improper nuclei directly interact with proper nuclei by a finite number of applications of the successor function. So, the following are additional principles for axiom systems for the individual kinds of finite model. In each case, these are given as schemata, where specific terms are input in order to capture the particular equivalence relations which give structure to the models. I shall set out the axioms, give some justification for why the axioms I propose are sensible, and finally state some interesting features of the proof systems produced by the addition of these axioms.

One general point relevant to all of the following developments is what kind of order principles are derivable in $A_{3*}^{\#}$, as these principles will hold for all the proof systems developed here. This will give us at least some background information about the ordering relation of the models, with the possibility that the extra axioms introduced in the cases of the specific models allowing us to derive more order properties. To this end, we shall take $x < y$ to be defined as before $(\exists z(\neg z = 0 \wedge z + x = y))$, and define $x \leq y$ as $x = y \vee \exists z(\neg z = 0 \wedge z + x = y)$. This is at least a fairly standard definition, matching that used in [22], and very similar to that used in [21]. According to the following two results, $\leq$ in $A_{3*}^{\#}$ is at least a pre-order. This matches the models, and in particular satisfies Priest's claims regarding the order-types of the parts of collapse models.

$$\dfrac{\dfrac{\vdash x = x}{\vdash x = x, \exists z(\neg z = 0 \wedge z + x = x)}}{\vdash x = x \vee \exists z(\neg z = 0 \wedge z + x = x)}$$

This proof establishes that $x \leq x$. The proof of transitivity is presented by cases on the next two pages, due to space.

For the proof of transitivity, there are a number of cases which are put together in obvious ways to achieve the desired result.

$$\dfrac{\dfrac{x = y, y = z \vdash x = z}{x = y, y = z \vdash x = z, \exists v(\neg v = 0 \wedge v + x = z)}}{x = y, y = z \vdash x = z \vee \exists v(\neg v = 0 \wedge v + x = z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{1.5pt}{}{\neg w = 0 \vdash \neg w = 0}{x = y, \neg w = 0, w + y = z \vdash \neg w = 0} \qquad \dfrac{\dfrac{x = y \vdash w + x = w + yx \qquad w + x = w + y, w + y = z \vdash w + x = z}{x = y, w + y = z \vdash w + x = z}}{x = y, \neg w = 0, w + y = z \vdash w + x = z}}{x = y, \neg w = 0, w + y = z \vdash \neg w = 0 \wedge w + x = z}}{x = y, \neg w = 0 \wedge w + y = z \vdash \neg w = 0 \wedge w + x = z}}{x = y, \neg w = 0 \wedge w + y = z \vdash \exists v(\neg v = 0 \wedge v + x = z)}}{\dfrac{x = y, \exists w(\neg w = 0 \wedge w + y = z) \vdash \exists v(\neg v = 0 \wedge v + x = z)}{x = y, \exists w(\neg w = 0 \wedge w + y = z) \vdash x = z \vee \exists v(\neg v = 0 \wedge v + x = z)}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{1.5pt}{}{\neg u = 0 \vdash \neg u = 0}{\neg u = 0, u + x = y, y = z \vdash \neg u = 0} \qquad u + x = y, y = z \vdash u + x = z}{\neg u = 0, u + x = y, y = z \vdash \neg u = 0 \wedge u + x = z}}{\neg u = 0 \wedge u + x = y, y = z \vdash \neg u = 0 \wedge u + x = z}}{\dfrac{\neg u = 0 \wedge u + x = y, y = z \vdash \exists v(\neg v = 0 \wedge v + x = z)}{\exists u(\neg u = 0 \wedge u + x = y), y = z \vdash \exists v(\neg v = 0 \wedge v + x = z)}}}{\exists u(\neg u = 0 \wedge u + x = y), y = z \vdash x = z \vee \exists v(\neg v = 0 \wedge v + x = z)}$$

This case is the trickiest, and I shall have to do a bit of cheating to get it to a reasonable length. First, note that $\neg u^w = 0$ if $\neg w = 0$, since u^w shall be something of the form $u^{'\cdots'}$, and $\vdash \neg x' = 0$. Second, note that $u^w = u + w$, and thus that $\neg u + w = 0$. These straightforward things give rise to the first line, which holds for all collapse models given that even in the case where one can add some u to w to get to 0, $u + w$ will also not be equal to 0 by the construction.

$$\dfrac{\dfrac{\neg w = 0 \vdash \neg u + w = 0}{\neg u = 0, \neg w = 0 \vdash \neg u + w = 0}}{\neg u = 0, u + x = y, \neg w = 0, w + y = z \vdash \neg u + w = 0}$$

$$\dfrac{\dfrac{u + x = y \vdash (u + x) + w = y + w \qquad (u + x) + w = y + w, w + y = z \vdash (u + x) + w = z}{u + x = y, w + y = z \vdash (u + x) + w = z}}{\dfrac{u + x = y, w + y = z \vdash (u + w) + x = z}{\neg u = 0, u + x = y, \neg w = 0, w + y = z \vdash (u + w) + x = z}}$$

The above two give rise to the following by $\vdash \wedge$

$$\dfrac{\dfrac{\neg u = 0, u + x = y, \neg w = 0, w + y = z \vdash \neg(u + w) = 0 \wedge (u + w) + x = z}{\neg u = 0, u + x = y, \neg w = 0, w + y = z \vdash \exists v(\neg v = 0 \wedge v + x = z)}}{\exists u(\neg u = 0 \wedge u + x = y), \exists w(\neg w = 0 \wedge w + y = z) \vdash \exists v(\neg v = 0 \wedge v + x = z)}$$

Combining the proofs given above by applications of $\vee \vdash$, gives rise to:

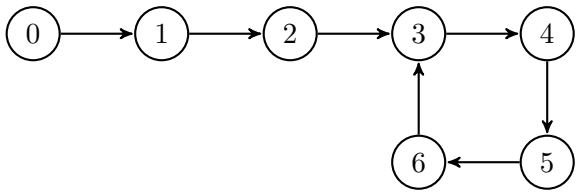
$x = y \vee \exists u(\neg u = 0 \wedge u + x = y), y = z \vee \exists w(\neg w = 0 \wedge w + y = z) \vdash x = z \vee \exists v(\neg v = 0 \wedge v + x = z)$

which is just $x \leq y, y \leq z \vdash x \leq z$, as desired.

These proofs establish that $\leq$ is a pre-order, however, different collapse models have different features as regard potential symmetry conditions which would flesh out these two conditions. The structural features of the proper nuclei of the classes of models dictate the options. First, $\leq$ cannot be anti-symmetric in any of the models involving non-trivial cycles. This would have the effect of collapsing the cycle to a point.

Consider the following heap model:[9]

Figure 10.



$3 \leq 5$ and $5 \leq 3$ are both clearly true, and yet if $3 = 5$ were true it would simply result in all the members of the loop consisting of $3 \ldots 5$ to be equivalent to the point 3. So anti-symmetry of $\leq$ cannot hold here. As we'll see, however, symmetry - $x \leq y \rightarrow y \leq x$ - does generally hold for the cyclic models, meaning that the ordering is an equivalence relation. However, the heap models provide less obvious means to proving other ordering principles. This involves a principle indicating that if some numbers occur in the cycle, then they are symmetrically ordered with respect to each other. We shall come back to this when discussing heap models.

In the following sections, we shall consider the basic classes of finite inconsistent models of arithmetic as described in the first chapter, set out the additions to $\mathrm{A}_3^{\#}$ or $\mathrm{A}_{3*}^{\#}$ necessary to capture their salient features, and provide some proofs that at least some of those features considered by Priest [35] are provable in the appropriate class of axiom systems.

[9]The argument is obviously applicable to cyclic models as well.

3.2.1 Simple extended models – SE_n

These simple models are obtained by adding, for some n, $\langle n, n\rangle$ to $I^-(=)$. The axiomatization should be simple, though this model is difficult to depict. We should only need to add $\vdash n \neq n$ to $\mathrm{A}^{\#}_{3*}$. Axioms of this form shall be called SE_n, where n is the non-identity in question. So, SE_0 is $\vdash 0 \neq 0$ and the associated model is that where $\vdash 0 = 0 \wedge 0 \neq 0$, and, due to the fact that $\vdash x = y \rightarrow x' = y'$ and $\vdash x \neq y \rightarrow x' \neq y'$, it follows that $\vdash \forall x(x = x \wedge x \neq x)$. In general, every ancestral successor of n shall admit of a proof that it is equal to itself and a proof that it is not equal to itself. This matches what is said in Priest's paper about these models (note that there seems to be a decision as to whether the ancestral predecessors of n are affected by this alteration to the model). The result is that the model obtained by adding $\langle 0, 0\rangle$ to $I^-(=)$ is that given above.

As described, these models are essentially just the same as the standard model but with a single glutty sentence (namely $n = n \wedge n \neq n$) leading into a chain of glutty sentences up the ordering. A possible issue is whether to use $\mathrm{A}^{\#}_3$ or $\mathrm{A}^{\#}_{3*}$ for these models. This decision point is just the issue considered in the first chapter, stated in slightly different terms. If it is to be the case that the predecessors are trivialized as well then the full-strength successor axioms will be correct - so $\mathrm{A}^{\#}_3$ is the correct choice. If not, then we must remove some of these principles, and something like $\mathrm{A}^{\#}_{3*}$ is desirable, depending on just which successor principles we may wish to rid ourselves of.

These models do not have many other particularly interesting features. They are essentially just the standard model with some least glutty element such that all elements later than that element are also glutty. The result is that they are non-trivial, but not adding anything remotely interesting to the standard model. As such, we shall not treat of the SE_n models any further than to say what we have. That the axiom schema, and the functionality of $', +, \cdot$ captures the salient properties of these models is pretty clear, as the only other salient properties of the models are those which should be captured by the rest of the Peano-style axioms included in $\mathrm{A}^{\#}_{3*}$. So,

we shall move on to the Collapse models.

3.2.2 Cyclic models – C_n

These are very similar to modular arithmetics induced by equivalences such as $\overline{x} \sim \overline{y}$ iff $\overline{x} \equiv \overline{y} \pmod{\overline{n}}$, resulting in simple cycles. The difference between cyclic models and modular arithmetics is just that the structure is describable in terms of inconsistent facts, as a result of the meanings of $=$ and $\neq$ on the LP semantics. This model is a quotient algebra of the standard model of natural numbers and each number on the above graph is an equivalence class such that $t = \{\overline{x}; \overline{x} = \overline{t} \pmod{\overline{n}}\}$. $\mathrm{A}_3^{\#}$ is appropriate here since every number in the cycle, that is every number, has a unique successor and a unique predecessor, so we may have the full suite of successor axioms displayed in the previous chapter. The salient features of these models seem to be captured by adding to $\mathrm{A}_3^{\#}$ only:

$\vdash n = 0 \qquad (\mathrm{C}_n)$

where into n is substituted the period of the cycle. This axiom allows the following straightforward proof, which comes in handy, where the final line is just a cut on C_n:

$$\dfrac{\dfrac{n = 0 \vdash x + 0 = x + n \qquad \dfrac{x + 0 = x \qquad x + 0 = x, x + 0 = x + n \vdash x + n = x}{x + 0 = x + n \vdash x + n = x}}{n = 0 \vdash x + n = x}}{\vdash x + n = x}$$

C^n, naturally, provides the proof of an explicit contradiction: $\vdash \neg\forall y \neg y' = 0 \wedge \forall y \neg y' = 0$, where n' is the period:

$$\dfrac{\dfrac{\dfrac{\vdash n' = 0}{\vdash \exists y y' = 0}}{\vdash \neg\forall y \neg y' = 0} \qquad \vdash \forall y \neg y' = 0}{\vdash \neg\forall y \neg y' = 0 \wedge \forall y \neg y' = 0}$$

The functionality of the successor function – $\vdash x = y \to x' = y'$ – will ensure that the rest of the simple true equations of the cycle involving only $'$ will be provable from this axiom. It is nothing more than a proof involving a single cut on C_n and an instance of $x = y \to x' = y'$ to get that $\vdash n'' = 0'$, and the other instances of basic equations involving $'$ are just as straightforward. Theorems involving $+, \cdot$ do not

obviously cause any other problems. So, it seems at least intuitively plausible that $\mathrm{LA}^{\#}_{3*} + \mathrm{C}_n$ ought to provide all the right atomic theorems about its associated model in the restricted language involving only that the operations $'$, $+$, and $\cdot$ behave as should be expected, since the axioms involving $+, \cdot$ are unchanged from the usual. As a result, the classical principles generally lost are those related to the loss of injectivity of $'$ - like right cancellection of $+$ as discussed in the previous section.

It is interesting to note that it is a fairly intuitive result of the structure of cyclic models that the ordering relation on their members satisfies symmetry - $x \leq y \rightarrow y \leq x$. Consider the above example of C_n. To say that $x \leq y$ is essentially to say that x, y are in the cycle, i.e. that they are in the model. It is the case that for any object in the cycle it is strictly less than everything else in the cycle (including itself), since one only produces another member of the cycle by taking the sum of a member of the cycle with any other member of the cycle (including the period of the cycle, which is an identity for addition, alongside 0). The proof is quite straightforward, relying on functionality of addition and on C_n. The proof is on the next page.

So, on the C_n family of models, $\leq$ is an equivalence relation. That is, every element of a C_n is structurally equivalent to every other member. So $'$ (something like the image of $\leq$) behaves in a somewhat strange way, and this is likely to be matched in the behaviour of $\leq$. What is of potential interest here, and after we have considered axioms for heap models, is whether the weakness of the ordering within chromosomes, such as a cycle in a cyclic model affects any changes in the ordering over nuclei ($\preceq$) should we move on to consider the general framework and infinite models. Priest [36] claims that $\preceq$ is a partial order, but this is something which almost certainly will not fall out of this axiom system in any obvious way – of course $\preceq$ only comes in with the infinite models, as in the finite models it is indistinguishable from $\leq$. This is a result of the fact that the only nuclei occurring in a finite model are either cycles or improper nuclei (the initial tail of standard natural numbers) - so $\preceq$ could hold either between two improper nuclei, where it would be indistinguishable from $\leq$, or it could hold between an improper nucleus and a cycle, that is, between the final member of the initial tail and the cycle as occurring in a heap model.

The two right-most axioms in the following are simple consequences of $\vdash x + n = x$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg n = 0 \vdash \neg n = 0}}{\neg n = 0, n + x = y \vdash \neg n = 0}\quad \dfrac{\dfrac{\dfrac{}{n + x = y \vdash x = y}\quad \dfrac{}{x = y \vdash n + y = x}}{n + x = y \vdash n + y = x}}{\neg n = 0, n + x = y \vdash n + y = x}}{\neg n = 0, n + x = y \vdash \neg n = 0 \wedge n + y = x}}{\neg n = 0, n + x = y \vdash \exists w(\neg w = 0 \wedge w + y = x)}}{\neg n = 0 \wedge n + x = y \vdash \exists w(\neg w = 0 \wedge w + y = x)}}{\exists u(\neg u = 0 \wedge u + x = y) \vdash \exists w(\neg w = 0 \wedge w + y = x)}}{\exists u(\neg u = 0 \wedge u + x = y) \vdash x = y \vee \exists w(\neg w = 0 \wedge w + y = x)}\quad \dfrac{\dfrac{}{x = y \vdash x = y}}{x = y \vdash x = y \vee \exists w(\neg w = 0 \wedge w + y = x)}}{x = y \vee \exists u(\neg u = 0 \wedge u + x = y) \vdash y = x \vee \exists w(\neg w = 0 \wedge w + y = x)}}{\vdash (x = y \vee \exists u(\neg u = 0 \wedge u + x = y)) \to y = x \vee \exists w(\neg w = 0 \wedge w + y = x)}$$

Which establishes that $\vdash x \leq y \to y \leq x$, as desired.

In any case, there is no clear way to extend these facts about $\leq$ in cyclic models to get information about $\preceq$ in the general framework for infinite models. However, the information given about $\leq$ is at least correct, and we have some justification to think that the axiom systems including C_n capture the salient features of the cyclic models. What is wanted, however, is a decidability result which I do not have and thus cannot present here. A potential way to prove completeness of an axiom system of this kind for its associated cyclic model would seem to rely on a proof that $\vdash \forall x(x = 0 \vee x = 1 \vee \cdots \vee x = n)$. I don't know whether formulae of this kind are generally provable, but it is under investigation.

3.2.3 Heap models – $\mathbf{H}_k^n$

This class of models is characterized by a finitely long 'tail' – in the diagram equivalent to $\langle 0, \ldots, p{-}1\rangle$ – followed by a single cycle of some finite period $\overline{n}$, induced by quotient algebras with defining equivalence relations like $\overline{x} \sim \overline{y}$ iff $(\overline{x}, \overline{y} < \overline{n} \wedge \overline{x} = \overline{y}) \vee (\overline{x}, \overline{y} \geq \overline{n} \wedge \overline{x} \equiv \overline{y} \pmod{\overline{k}})$. This is also the class of models which necessitate the introduction of $\mathrm{A}_{3*}^{\#}$ in that they feature some element with two distinct predecessors - namely, the first element of the cycle which is preceded by the last improper nucleus and the penultimate member of the cycle (where the ultimate member of the cycle is just the first member over again).

Given this consideration, much the same as in the previous case we can specify one obvious additional axiom. This extra axiom states an identity between the first number in the cycle n and itself plus the period of the cycle – $n + \overline{k}$. Note that $\overline{k}$ is a standard number, since it is a member of the collapsed equivalence class k. This is worth noting only to indicate that it is irrelevant and that one can just as easily write $n + k$, as I shall be from now on. There are two possibilities in a finite heap model: k can occur either in the tail of improper nuclei or in the proper nuclei, ie. a cycle. If k is an improper nucleus then its behaviour is no different from $\overline{k}$, as the only numeral occurring in k is $\overline{k}$. On the other hand, if k occurs in the cycle, then $n + \overline{k}$ will have the same result as $n + k$ (without the troublesome mixing of type levels), since $\overline{n} + \overline{k}$ and $\overline{n} + \overline{l}$ where $\overline{k} \sim \overline{l}$ will have the same result. As such the schemata we seek to

add to $A^{\#}_{3*}$, where k is written instead of $\overline{k}$, and where, by stipulation, $n > 0$:

$\vdash n = n + k \qquad (H^n_k)$

The name H^n_k indicates the first member of the cycle, n, while k denotes the cycle of the cycle. In each such model, this particular statement will contradict whatever statement specifies the next number in the sequence - suppose that $k = n = 5$, and so we add H^5_5 to $A^{\#}_{3*}$. We shall have a theorem $\vdash 10 \neq 5$, given the definition of $\neq$, and the axiom shall produce $\vdash 10 = 5$. As in the above example $\vdash x = y \rightarrow x' = y'$ should ensure that the rest of the equalities of the cycle will fall into place, as well as allow the proof of the right equalities up the improper nuclei. What must be done is to ensure that the cycle does not trivialize, or interact with the improper nuclei in any problematic ways. Part of this is to ensure that the ordering is of the appropriate kind. It should be a linear order for the improper nuclei in the tail, but should act as does the ordering on a cycle when applied to members of the proper nucleus. In essence there are two ordering relations which meet in the last improper nucleus in the tail and its successor in the cycle.

So, the kind of principle we would like, where $'n$ is the last improper nucleus, is $\forall x \forall y(x, y \geq n \rightarrow (x \leq y \rightarrow y \leq x))$, as well as something for the improper nuclei which specifies a linear ordering. Perhaps a simple way of stating that x is in the cycle, as would be useful in the above antecedent, is that $x + k = x$, where k is the period of the cycle as introduced in H^n_k. This would enforce the claim that the cycle occurring after the finite tail does behave like a cycle in the cyclic models, and give us some ground for the claim that we've captured the heap models. A similar theorem which expresses something very similar, but with a much shorter proof is $\vdash n \leq x \rightarrow x < x$. Since $x \leq x$ is a theorem as a trivial result of $\vdash 0 + x = x$, if $x < x$ is true, it must be because some $y \neq 0$ is such that $y + x = x$. In the heap models, this is clearly true for any x in the cycle, since the period k should be such that $\vdash k + x = x$. This is just what $\vdash n \leq x \rightarrow x < x$ expresses, however we must add another condition to H^n_k in order to prove one of the cases of this theorem. Namely, it must be stated:

$\vdash n = n + k \wedge k \neq 0$

With this in tow, the theorem is easily provable, and this addition to the axiom is hardly a major addition. It essentially stipulates that the cycle is in fact a cycle and not just a point. The salient parts of the proof of $\vdash n \leq x \rightarrow x < x$ are stated on the following page. This gives us some indication that the cycle in a heap model does behave like a cycle, especially since it is essential to the proof that the x in question be in the cycle, we could not simply derive $\vdash x < x$.

There is much left undone for an appropriately thorough treatment of these models, however, the beginning is in place, and the axiom system is simple enough to give us an easy means to continue work on these models. As we have started to do here, we must ensure that the binary functions behave as expected in the tail and cycle. If the successor function works as it should, then it is plausible that the binary functions which operate in terms of recursive definitions in terms of successor will behave as expected. What remains to be seen is that this is the case, and that there are no strange and undesirable side effects. However, there is very much left in order to do this in a satisfactory way.

All of the finite inconsistent models are just like these models or are composed of models similar to these. The axiom systems for the other models should simply involve adding to $\mathrm{LA}^{\#}_{3*}$ some instances of SE_n, C_n, or H^n_k. There are a handful of possibilities for such composition, but they have been put aside for the purposes of this chapter. This task may well be complicated by having to deal with the interactions of various proper nuclei, but at least in the finite case this seems feasible. Along the lines of these necessary alterations, axiomatisablity results for the former kinds of models should provide an obvious generalisation for any finite collapse model composed of other finite collapse models. This may not be the case if we consider infinite models which can have multiple proper nuclei. In this case, we must show that, under the right kind of axiom system, $\preceq$ can be defined to have the right properties, and that the elements of different lots don't interact in ways we do not want. This would seem to require typing the language to reflect the different lots a number may fall into and restating parts of the arithmetical axiom system with these types in mind to be sure that one cannot get from one lot to another by means of successor. Perhaps the way

$$\dfrac{\dfrac{(k+n)=x,(k+n)=(k+x)\vdash(k+x)=x}{(k+n)=x\wedge(k+n)=(k+x)\vdash(k+x)=x}\,(*) \qquad \dfrac{\dfrac{(k+n)=n,n=x\vdash(k+n)=x}{n=x\vdash(k+n)=x} \qquad n=x\vdash(k+n)=(k+x)}{n=x\vdash(k+n)=x\wedge(k+n)=(k+x)}}{n=x\vdash(k+x)=x}$$

In the above proof, at the line marked $(*)$, a cut with $\vdash(k+n)=n$, the axiom H^n_k. This is done for space reasons.

$$\dfrac{\vdash k\neq 0}{n=x\vdash k\neq 0}$$

These give rise to $n=x\vdash k\neq 0\wedge k+x=x$, which implies

$$(\dagger)\ n=x\vdash\exists z(z\neq 0\wedge z+x=x)$$

$$\dfrac{\dfrac{k+n=n \vdash y+(k+n)=y+n}{\vdash k+(y+n)=y+n} \qquad y+n=x, k+y+n)=y+n \vdash k(y+n)=x}{y+n=x \vdash k+(y+n)=x}$$

$$\dfrac{y+n=x \vdash k+(y+n)=k+x \qquad k+(y+n)=k+x, k+(y+n)=x \vdash k+x=x}{y+n=x, k+(y+n)=x \vdash k+x=x}$$

From the above two by an application of cut we arrive at:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y+n=x \vdash k+x=x}{y\neq 0, y+n=x \vdash k+x=x} \quad \dfrac{\vdash k\neq 0}{y\neq 0, y+n=x \vdash k\neq 0}}{y\neq 0, y+n=x \vdash k\neq 0 \wedge k+x=x}}{y\neq 0, y+n=x \vdash \exists z(z\neq 0 \wedge z+x=x)}}{y\neq 0 \wedge y+n=x \vdash \exists z(z\neq 0 \wedge z+x=x)}}{\exists y(y\neq 0 \wedge y+n=x) \vdash \exists z(z\neq 0 \wedge z+x=x)}$$

By one application of $\vee \vdash$, on (†) and the sequent just derived we arrive at $n=x \vee \exists y(y\neq 0 \wedge y+n=x) \vdash \exists z(z\neq 0 \wedge z+x=x)$, from which we can easily arrive at $\vdash (n=x \vee \exists y(y\neq 0 \wedge y+n=x)) \rightarrow \exists z(z\neq 0 \wedge z+x=x)$, which is just $\vdash n\leq x \rightarrow x<x$, as desired.

forward on this front is to define $\preceq$ as allowing us to move through different types, each type answering either to an improper nucleus or to a sequence of proper nuclei (the classical tail of a heap-like model).

In any case, these are little more than speculation, as we have not considered the infinite models in anything more than a cursory glance in the first chapter, designed to provide a complete picture of the collapse models. Clearly, there is very much left to do in the way of completing the work on the finite models, let alone the infinite models. However, it is also clear that just these problems provide the way forward for the program, hopefully giving rise, eventually, to adequacy results for the axiom systems we have presented. In the meantime, however, if one is convinced that this topic is worth considering, then one ought also to be convinced that the open problems are *interesting* and that answers to them would be *valuable* and worth the effort required to find them.

Conclusions

I have, in this thesis, attempted to provide a sensible axiom system for the finite collapse models of arithmetic and, in so doing, to set out a logic which is generally well suited to doing mathematics in a paraconsistent framework. A_3 is a powerful and versatile system, differing from classical logic in a minimal way while still rejecting explosion. In general, I would like to claim that most any logical work which can be done with LP can also be done with A_3, except those topics where the failure of modus ponens in LP is particularly appealed to - such as in some philosophical work of JC Beall [8]. These projects aside, A_3 is a logic which captures the best elements of LP while avoiding what is, generally, considered most unpleasant about it - invalidating modus ponens and other standard conditional principles. In particular, $A_3^{\#}$ seems the basis of a solid toolset for studying mathematical subject matter from a paraconsistent perspective, whether that subject matter be dialethic or not.

A_3's strong points come in the form of, first, intuitive correctness. Admitting the deduction theorem and modus ponens are pretty fundamental notions for a conditional connective, and having a connective which behaves in the ways specified by these principles opens the way for a logic otherwise very much like LP to be used to investigate conditional claims in a natural way. Second, A_3 is simple: it can be specified in a many-valued semantics and some natural proof systems are available for it, at least two of which are presented in this work.[10] A very simple signed tableaux system can easily be presented, just following the truth table definitions of A_3's connectives. This is in distinction to some other proposals for a conditional connective

[10]There is a third if one considers the sequent calculus which replaces the axiom $\vdash A, \neg A$ with the rule $(\vdash \neg)$.

to be added to LP, which require a presentation in terms of a ternary accessibility relation in a relational semantic structure. While these proposals are very interesting, and allow for a lot of flexibility in the specification of similar connectives, their semantics is much less simple than that which can be given for A_3, and this simplicity is a virtue, at least in terms of trying to disseminate the logic. Third, A_3, at least in the propositional part, has shown to be maximally paraconsistent, that is, no non-conservative extension of A_3 would reject explosion. This ought to give one confidence that A_3 captures paraconsistent reasoning in as 'lossless' a way as possible. On this grounds, if someone who is only familiar with classical or intuitionist modes of reasoning should find themselves interested in paraconsistent reasoning, A_3 is a great logic for such a person. In the same note, it is a good choice for an initial application of paraconsistent logic to some area where it has not been used.

These benefits I claim for this logic. I am much less sure about exactly what can be done with $A_3^{\#}$ or $A_{3*}^{\#}$, however, these initial results are at least intriguing, if not promising, and there is a great deal of ground left to cover. Some of this ground, at least, may well provide insight into the paraconsistent approach to mathematics beyond simple arithmetical theories. In any case, there are a number of potential directions for future research which are opened up by this thesis, the first of which, perhaps, is to provide proofs that the appropriate extensions of $A_{3*}^{\#}$ actually do axiomatise their associated collapse models. Beyond this, there are natural extensions into related areas of classical mathematics which may be amenable to a paraconsistent treatment on the basis of A_3. However, this initial presentation serves to set out the background of the logic, and give some initial indication of what can be done in it.

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