STOCHASTIC CONTROL FOR OPTIMAL GOVERNMENT DEBT MANAGEMENT

by

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Abstract

We develop stochastic control models for optimal government policies. We study three problems: (1) the optimal debt ceiling, (2) the optimal currency portfolio, and (3) the optimal management of the stabilization funds. The results of this research provide insights that are useful to policy-makers.

For the first problem we present theoretical models for a government that wants to control its debt by imposing a ceiling on its debt-to-GDP ratio. We find explicit solutions for the optimal debt ceiling and we derive a practical recommendation for debt policy based on it.

In the second problem we study the optimal currency portfolio and debt payments in a model that considers debt aversion and jumps in the exchange rates. We find that higher debt aversion and jumps in the exchange rates lead to a lower proportion of optimal debt in foreign currencies. In addition, we show that for a government with extreme debt aversion it is optimal not to issue debt in foreign currencies.

In the last problem we consider a government that wants to control the stabilization fund by depositing money in and withdrawing money from the fund. We obtain explicit solutions for the optimal bands. Furthermore, we derive a practical recommendation for the management of the stabilization fund based on the optimal bands.

Preface

This research has been conducted under the supervision of Professor Abel Cadenillas.

Chapters 2 and 3 were motivated by the debt crises in the world, and the USA debt ceiling problem. My supervisor proposed to study the optimal debt ceiling. The main results of Chapter 2 resulted in "Explicit Formula for the Optimal Government Debt Ceiling" (see Cadenillas and Huamán-Aguilar 2015). This paper is a joint work with Professor Abel Cadenillas, we were both involved in all sections of the document.

Chapter 4 was motivated from an observation made by Associate Professor Christoph Frei during my presentation at my candidacy PhD examination. He pointed out that the interest rate in the model of Chapter 2 could depend on the size of debt.

Chapter 5 emerged while I was working at the Debt Department of the Ministry of Finance, Peru, 2004-2008. A version of this chapter, the paper "Government Debt Control: Optimal Currency Portfolio and Payments", has been accepted for publication in *Operations Research* (see Huamán-Aguilar and Cadenillas 2015). This paper is a joint work with Professor Abel Cadenillas, we were both involved in all sections of the document.

Chapter 6 originated in 2013 after some discussions with Marco Sal y Rosas Muñoz, analyst at the Ministry of Finance, Peru.

In Chapters 3, 4, and 6, I was responsible for all the mathematical proofs, the economic analysis, the conclusions as well as the manuscript composition. Professor Abel Cadenillas provided feedback and suggestions to improve earlier versions of these chapters.

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Chapter 1

Introduction

Public debt is a key macroeconomic variable, for both developed and developing countries. Indeed, some past and recent economic crises have been triggered by debt crises. As a result, today there is an absolute consensus that high debt-to-GDP ratios are undesirable, and controlling them has become one of the most important decisions for every government. On the other hand, to prevent or smooth the extreme consequences of an economic and financial crisis some countries have created stabilization funds as a mechanism of fiscal policy to save money in the good economic times to be used in the bad economic times.

In this thesis we develop models for government debt control as well as a model to study the stabilization fund band. In order to make this research policy-oriented, we focus on obtaining explicit solutions for the problems, along with their corresponding economic analysis, that provides insights that are useful to policy-makers in their decision-making process.

The pioneering work in this thesis is interesting not only to the academic communities of Mathematics, Operations Research, Economics, and Finance, but also to the people in authority at government offices who want to manage their debt and their stabilization funds.

1.1 Objectives

The general objective of this research is to develop mathematical models for optimal government policies, which are solved by applying the theory of classical and singular stochastic control, with regime switching when considered suitable.

We study the following problems:

1. The optimal government debt ceiling:

   How can one find the optimal debt ceiling for a country?

2. The optimal government currency portfolio:

   How can one construct a model to explain the fact that developing countries have reduced their composition of foreign debt in their portfolios?

3. The optimal management of the government stabilization funds:

   How can one find the optimal stabilization fund bands in a model with business cycles?

1.2 Motivation and contributions

In this section we provide the motivations and contributions for each of the problems mentioned above.

On the optimal debt ceiling

Due to recent debt crises, controlling the debt has become one of the most important decisions of every government. That is why some countries follow rules that consider a ceiling for their debt in terms of either percentage of GDP or monetary units. As Wyplosz (2005) describes it, the British Code for Fiscal Stability includes an explicit statement about the net public debt, saying that it must remain at a stable and prudent level, in the understanding that it is 40% of GDP. The Maastricht Treaty in 1992 set the debt-to-GDP ratio at 60%, which was meant to be a debt threshold for countries to be members of the European Economic Community. Moreover, in the USA, the nominal debt (measured in USA dollars, not debt-to-GDP ratio) is subject to a ceiling that must be changed by the Congress. Although, in general, this ceiling has been changed whenever it was required, it was extremely difficult to reach an agreement recently. This was known as the debt ceiling problem in 2011, but it is still a matter of current debate.

However, on the theoretical side, the literature has not studied the optimal debt ceiling yet. Apparently, the rationale for the specific debt ceilings mentioned above come from empirical data. In fact, based only on statistical analysis but without a theoretical model, in pages 43-44 of IMF (2002) it is claimed that a debt-to-GDP ratio of 40% provides a very rough guide for assessing a country's debt ratio. Furthermore, according to Chowdhury and Islam (2010), the 60% was simply the median of the debt ratio of some European countries to prepare for the formation of the Euro zone. There are other theoretical models about public debt (such as Barro (1974, 1999), Bulow and Rogoff (1989), and Stein (2006, 2012)), but they do not study the debt ceiling.

Given the above discussion, we define *debt ceiling* as the maximum level of debt ratio at which fiscal interventions are not required (or, equivalently, the minimum level of debt ratio at which the government should intervene). Consequently, if the debt ratio of a country is above that level, the government should generate fiscal surpluses to reduce the debt ratio; otherwise, the debt is considered to be under control and there is no call for interventions. In particular, this definition is consistent with the meaning given to the 60% in the Maastricht Treaty (Article 104c).

For the sake of further clarification, we provide the definition of other terms related to government debt (but different from "*debt ceiling*"). "*Optimal debt*" is the level of debt that comes as a result of a welfare analysis that considers both the benefits and costs of increasing debt. See, for example, the study of optimal debt by Aiyagari and McGrattan (1998) and Barro (1999). "*Credit ceiling*" (or "*credit limit*") is the maximum level of debt that a country is allowed to borrow. This level is imposed by lenders based on the characteristics of the country, such as its history of default. For a reference of this term, see Eaton and Gersovitz (1981). "*Debt limit*" is the level of debt at which a debt crisis occurs. In particular, at this level of debt, the debt service payments are unsustainable (see, for example, Stein (2006)). Thus, the "*debt limit*" is strictly larger than the "*debt ceiling*". Another difference between these two terms is that a government can select its "*debt ceiling*", but a government cannot select its "*debt limit*". Summarizing, "*optimal debt*", "*credit ceiling*", "*debt limit*", and "*optimal debt ceiling*" are different terms.

Our contribution is to propose a theoretical framework for government debt control to study rigorously the optimal debt ceiling. This research topic is motivated by: (i) the current debt crisis in the world, (ii) the fact that some countries or community of countries (for example, USA and the European Economic Union) impose debt ceilings to control their debts, and (iii) the lack of a theoretical model to study the optimal government debt ceiling.

Specifically, we consider theoretical models for a government that wants to control its debt by imposing an upper bound or ceiling on its debt-to-GDP ratio. The goal of the government is to find the optimal control that minimizes the expected total cost, the cost of having debt plus the cost of the interventions of the government to reduce the debt ratio. In such models we find explicit solutions for the optimal debt ceiling. In one of the models, we even find an explicit formula for the optimal debt ceiling. In all models for the debt ceiling we derive a practical recommendation for debt policy based on the optimal debt ceiling.

On the optimal currency debt portfolio

The government debt portfolio is, in most cases, the largest financial portfolio in a country. In particular, the currency composition of the public debt is an important variable that could exacerbate a currency debt crisis, such as in Mexico in 1994. High foreign currency debt in the portfolio (especially, short term) exposes the country to the fluctuations of the exchange rates, and this becomes dramatic when unexpected and huge devaluations of the relevant exchange rates occur.

As Panizza (2008a) points out, developing countries have been reducing consistently the proportion of foreign debt in their portfolios in favor of local currency debt. This empirical fact is happening in the context in which most developing countries have access to the international capital markets. That is, although the countries can borrow in external currencies, there is a deliberate tendency to borrow in domestic currencies. Some authors (see Borensztein et al. 2008, for example) find that debt crises are the factor that has urged countries to pursue such strength in their domestic markets. In other words, the underlying factor that explains this tendency is the goal of reducing the exchange rate vulnerabilities, and hence the chances of a debt crisis. This can be interpreted as the countries, due to their past experiences, have become more risk averse.

What does the theoretical literature on currency debt management say about the fact above? Surprisingly, although the order of magnitude of the debt of a country could be of thousands of millions of dollars and, as we mentioned above, the government debt portfolio is in general the largest in the country, the theoretical literature has paid almost no attention to currency government debt portfolios. As far as we know, Giavazzi and Missale (2004) and Licandro and Masoller (2000) are the only theoretical references that deal with government currency debt portfolios. These approaches have the following weaknesses: (i) the debt ratio dynamics is not realistic because they consider only one period models, (ii) the jumps in the exchange rates are not considered explicitly, and (iii) the role of the government risk aversion is not included. Thus, important elements of the currency debt analysis have been neglected.

Our contribution is to present and solve a problem for currency government debt portfolio that overcomes the shortcomings mentioned above. We study the optimal currency portfolio and debt payments in a model that considers debt aversion and jumps in the exchange rates. Before solving the classical stochastic control problem, we derive a realistic stochastic differential equation for public debt. We obtain explicit solution for the currency portfolio and payments. We show that higher debt aversion and jumps in the exchange rates lead to a lower proportion of optimal debt in foreign currencies. In addition, we show that for a government with extreme debt aversion it is optimal not to issue debt in foreign currencies.

On the optimal stabilization fund

The recent global and fiscal crises have led to more interest in countercyclical fiscal policies to mitigate the negative consequences of a crisis. That is why many governments have created stabilization funds, which is a mechanism of fiscal policy used to save money in the good economic times to be used in the bad economic times. This balances the budget over the business cycle. According to Joyce (2001) and Vasche and Williams (2001), in many circumstances, it is the best way to handle fiscal disruptions, in comparison to other options. The stabilization funds have been implemented in a number of countries (such as Mexico, Russia, Venezuela, Peru, Chile, and Bolivia), and in most states of USA, in which they are called Budget Stabilization Funds (BSF) (or Rainy Day Funds).

In general, the natural level of the stabilization fund is associated with the price of a commodity (such as oil, cooper and hydrocarbon) and/or with the annual budget surplus (or deficit). As a result, the fluctuations of the natural level of the fund are closely related to the price of such commodities and the government budget. The stabilization funds we want to study have in common that the government makes deposits in and withdrawals from the fund according to some predetermined rules (see Rodriguez-Tejedo 2012 for details). In other words, it intervenes to modify the natural level of the stabilization fund. Typically, there exists a band for the stabilization fund. For instance, in Russia a maximum level in terms of domestic currency units for the stabilization fund itself was established in 2007. The amount that exceeds such maximum is withdrawn in order to pay foreign public debt or cover part of the Pension Funds' deficit. A minimum level of zero is implicitly being considered. As another example, Peru set up explicitly the stabilization fund minimum at zero and the maximum level at 4% of the GDP.

Our contribution is to present the first theoretical model to compute the optimal bands for the government stabilization fund. We model it as a stochastic singular problem with regime switching. We consider a government that wants to control the stabilization fund by depositing money in and withdrawing money from the fund. We obtain explicit solutions for the optimal bands that characterize the optimal control. These bands depend on the regime of the economy. Furthermore, we derive a practical recommendation for the management of the stabilization fund based on the optimal bands.

1.3 Organization of the thesis

Including this chapter, this thesis consists of seven chapters and three appendices.

Chapters 2-4 are devoted to the study of the optimal debt ceiling. In Chapter 2 we study the optimal debt ceiling in which the government intervention to reduce the debt ratio is unbounded. We model it as a stochastic singular control problem. In contrast to Chapter 2, in Chapter 3 we consider that the ability of the government to reduce the debt ratio is bounded. We model it as a classical stochastic control problem. In Chapter 4 we extend Chapter 2 to consider that the interest rate and the rate of economic growth depend on the debt ratio.

In Chapter 5 we study the optimal currency portfolio for a government, in a model that includes debt aversion and jumps in the exchange rates. We model it as a classical stochastic control problem.

Chapter 6 addresses the stabilization fund problem. We model it as a stochastic singular control problem with regime switching.

In the last chapter, Chapter 7, we review the contributions and results of this research.

We finish this thesis with three appendices. Appendices A and B provide the proofs of auxiliary results that are used in the thesis. Appendix C presents the definitions of some special functions.

Chapter 2

The optimal debt ceiling: unbounded intervention

We present, for the first time in the literature, a theoretical model for a government that wants to control its debt by imposing an upper bound or ceiling on its debt-to-GDP ratio. We obtain an explicit formula for the optimal government debt ratio ceiling (or equivalently the optimal ceiling for its debt ratio). Moreover, we derive a recommendation for government debt management based on the optimal government debt ceiling.

The formula we obtain is tailor-made for each country, in the sense that it depends on specific variables such as the rate of economic growth, the interest rate on debt, the marginal cost of debt reduction, the debt volatility, and the aversion to debt. Hence each country has a different optimal debt ceiling depending on these variables. This is consistent with Wyplosz (2005) observation that it does not make sense to apply a unified ceiling to countries with a wide range of debt ratios, essentially because the current levels of debt ratios show that different countries can deal with different levels of debt ratios without necessarily incurring in debt problems.

We consider a government that wants to control optimally the debt ratio of a country. We assume that during the periods in which the government does not intervene, the dynamics of the debt ratio follows a geometric Brownian motion. This model is the stochastic version of the standard debt ratio dynamics presented in macroeconomics textbooks.

We acknowledge that public debt may be beneficial for the economy, but also that it has adverse financial and real consequences. Since we are specifying a model inspired by the current debt crises in Europe, we consider that the disadvantages far outweigh the advantages. Thus, we assume that debt generates a cost for the country, and that this cost is an increasing and convex function of the debt ratio (we call it *the cost of having debt*). The government can reduce the debt ratio, but there is a cost associated with this reduction (we call it *the intervention cost*). This cost is generated by fiscal adjustments, which can take the form of raising taxes or reducing expenses. To keep track of these (cumulative) interventions we define the control variable: a non-negative, non-decreasing process that represents the cumulative reduction of the debt ratio.

We define the total cost as the sum of the cost of having debt and the intervention cost. In this framework, the government faces a trade-off: the greater the control process, the greater the cost of intervention, and the lower the cost of having debt; and the other way around. The government wants to find the optimal control, which is the control that minimizes the expected total cost. The function that gives the minimum expected total cost is called the value function.

We show that the optimal control and the value function can be associated with a debt ratio ceiling, which we call the optimal government debt ratio ceiling. By construction, it is the minimum level of debt ratio at which the government should intervene. Hence, the government should not intervene when the debt ratio is less than the optimal debt ratio ceiling. If the debt ratio is greater than the optimal debt ceiling, it is optimal for the government to generate positive primary surpluses aiming at reducing the debt ratio.

We would like to emphasize that the optimal debt ceiling, optimal debt, debt limit, and credit ceiling (or credit limit) are different concepts. As a first example, if the actual debt of a country is below the *optimal debt ceiling*, then it is optimal for the government not to intervene to control the debt. In contrast, if the actual debt of a country is below the *optimal debt*, then it might be optimal to intervene (provided that the cost of intervention is not too high). As a second example, if the actual debt ratio is greater or equal than the *optimal debt ceiling*, it implies that it is optimal for the government to intervene to reduce its debt ratio; there is no implication whatsoever of crisis. In contrast, if the actual debt ratio is equal to the *debt limit*, then a crisis does occur. Finally, since the *credit ceiling* (or *credit limit*) is determined by the lender's perception of the country, this term is not associated to optimality (from the country's perspective) or debt crisis.

We succeed in obtaining an explicit solution for the government debt control problem. In particular, we obtain an explicit formula for the optimal government debt ratio ceiling as well as for the value function. Thus, we can perform comparative statics to analyze the effects of the underlying parameters. We find, among other results, that an increase in the growth rate of the GDP increases the optimal government debt ceiling. An improvement in the characteristics of debt and an increase of debt volatility have the same effect. On the contrary, when the interest rate on debt increases, the optimal government debt ceiling decreases. Furthermore, as expected, an increment in the rate of economic growth, or an improvement in the characteristics of debt, reduce the expected total cost, and thus generates a better result for the government. The other parameters have the opposite effect on the expected total cost.

2.1 The model

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$, which is the P-augmentation of the filtration generated by a one-dimensional Brownian motion W.

The state variable is the debt ratio $X = \{X_t, t \in [0, \infty)\}$ of a country, where

$$X_t := \frac{\text{gross public debt at time t}}{\text{gross domestic product (GDP) at time t}}. \tag{2.1}$$

The gross public debt is the cumulative total of all government borrowings less repayments; that is, it includes the central and local government debt, and the domestic and external debt. This definition of debt is consistent with the debt ceiling in the USA, and the empirical work on this topic (see, for instance, Uctum and Wickens 2000). Moreover, this definition of debt ratio is standard in the economics literature. We note that both the gross public debt and the GDP are denominated in domestic currency.

We assume that $X = \{X_t, t \in [0, \infty)\}$ is an $\mathbb{F}$-adapted stochastic process that follows the dynamics

$$X_t \ = \ x + \int_0^t \mu X_s ds + \int_0^t \sigma X_s \, dW_s - Z_t \, , \tag{2.2}$$

or, equivalently,

$$dX_t = \mu X_t dt + \sigma X_t dW_t - dZ_t \, ,$$

where $\mu := (r - g) \in \mathbb{R}$ and $\sigma \in (0, \infty)$ are constants, and W is a Brownian motion. Here $r \in [0, \infty)$ represents the real interest rate on debt, $g \in \mathbb{R}$ the rate of economic growth, and σ the debt volatility. We denote the initial debt ratio of the country by $x \in (0, \infty)$. Sometimes, the government intervenes to reduce the debt ratio of the country. The process Z represents the cumulative primary balance that is deliberately generated by the government through fiscal interventions with the specific purpose of controlling the debt ratio. We note that debt volatility comes from either the primary deficits/surpluses (see Remark 2.1 below) or the structure of the debt itself. For the former, we recall that negative surpluses end up being more government debt (and positive surpluses become less debt). For the latter, for instance, if part of the debt is issued in foreign currencies, then σ accounts for the volatility of the exchange rates. We note that all the above parameters (r, g, σ, x) are specific to each country. In particular, if a debt of a country is represented by a smooth time series, then their corresponding volatility σ would be small.

2.1 Remark. *The rationale for Eq. (2.2) comes from the well-known macroeconomic identity in discrete time (see, for example, Blanchard 2009)*

$$X_{n+1} \,=\, X_n \,+\, (r-g)X_n \,-\, s_{n+1}\,,$$

or, in continuous time (see, for instance, Blanchard and Fischer 1989),

$$dX_t = (r-g)X_t dt \,-\, dZ_t \quad t \in (0, \infty);$$

where X is the debt ratio, r is the interest rate on debt, g is the growth rate of GDP, s is the primary balance, and Z is the cumulative primary balance. Since in reality we have uncertainty, we extend the previous dynamics to this stochastic setting:

$$dX_t = (r-g)X_t dt + \sigma X_t dW_t - dZ_t, \tag{2.3}$$

where σ represents volatility and W is a standard Brownian motion. The stochastic integral component $\sigma X_t\, dW_t$ in the debt dynamics represents the fact that the debt ratio can increase (respectively, decrease) due to government deficits (respectively, surpluses) that are beyond the control of the government. On the other hand, the process Z is the control variable that represents the cumulative surpluses generated by the government with the explicit goal of reducing the debt ratio.

Since our work is motivated by the current debt crisis in the world, we consider a government that needs to reduce its debt and thereby $Z = \{Z_t, t \in [0, \infty)\}$ is a non-negative and non-decreasing stochastic control process. Hence, in our continuous-time model, Z is formally an $\mathbb{F}$-adapted, non-negative, and non-decreasing stochastic control process $Z : [0, \infty) \times \Omega \to [0, \infty)$.

The government wants to select the control Z that minimizes the total cost functional J defined by

$$J(x;Z) \ := \ E_x\left[\int_0^\infty e^{-\lambda t}h(X_t)dt + \int_0^\infty e^{-\lambda t}kdZ_t\right].$$

Here $k \in (0,\infty)$ is the proportional (marginal) cost for reducing the debt, $\lambda \in (0,\infty)$ is the government's discount rate, and h is a cost function, which we assume nonnegative and convex, with $h(0) \geq 0$.

In the above functional J, the integral $\int_0^\infty e^{-\lambda t}k\,dZ_t$ represents the cumulative discounted cost associated with the specific and deliberate goal of reducing debt. This cost is generated by fiscal adjustments, which can take the form of raising taxes or reducing expenses. Furthermore, k represents the marginal cost of debt reduction, i.e., for each unit of debt ratio reduction the government has to pay the cost k, which is a positive constant.

2.2 Remark. *We preclude debt repudiation (default) as a means of reducing the debt ratio. In other words, we assume that the government uses market mechanisms and its sovereignty to control its debt, and precisely wants to avoid such a potential situation. We provide two complementary arguments to justify this assumption. Firstly, Bulow and Rogoff (1989) provide a theoretical model in which they conclude that loans can take place only in presence of direct sanctions available to creditors, which can include the ability to impede a country's trade, or to seize its financial assets abroad. Secondly, Blanchard (2009) and Eaton and Gersovitz (1981) point out that not paying the debt is a last resort, and not a sound solution, because the international markets are closed for those countries that default. (For references on models that consider default, but do not study the debt ceiling, see Eaton and Gersovitz 1981, Aguiar and Giponath 2006, Arellano 2008, and Mendoza and Yue 2012.)*

The function h represents the economic cost of having debt. According to the Ricardian equivalence (see Ricardo 1951a and Ricardo 1951b), government debt has no effect on the economy and thereby the size of the debt does not matter. Hence, according to that theory, h should be a function that does not depend on the debt ratio. However, as shown by Barro (1974), that theory relies on unrealistic assumptions. Following Barro (1974), we assume that the Ricardian equivalence does not hold, i.e., the public debt does matter.

Having public debt may be beneficial for the economy. For instance, Holmstrong and Tirole (1998) claim that sovereign debt in circulation enhances market liquidity, which provides households with a means of smoothing consumption. On the other hand, high public debt has negative effects on the economy. Blanchard (2009) points out that high public debt means less growth of the capital stock and more tax distortions (for empirical works regarding the negative effects of public debt on the economy, see, for instance, Kumar and Woo 2010). Moreover, it could cause vicious circles and make the fiscal policy extremely difficult. Besides, as stated above, according to Das et al. (2010), it could lead to a debt crisis which in turn may cause an economic crisis.

Since we are specifying a model inspired by the current debt crisis, we consider that the disadvantages (negative effects) of government debt on the economy are significant, whereas the advantages (positive effects) are negligible. Accordingly, we will assume that the cost (or loss) function $h : (0, \infty) \to [0, \infty)$ is given by

$$h(y) = \alpha y^{2n} + \beta, \tag{2.4}$$

where α is a strictly positive constant, β is a nonnegative constant, and $n \geq 1$ is an integer number. Here, n is a subjective parameter that captures the aversion of the policy-makers towards the debt ratio, and β is a scale parameter. For instance, countries that have never had a default or have never suffered a severe debt crisis (such as Canada and USA) have a lower n than countries that have experienced serious debt problems (such as Argentina and Greece). Since Reinhart et al. (1993) call more debt intolerant countries to those with higher associated default probabilities, we can think of n as a measure of debt intolerance as well. On the other hand, the parameter α represents the importance of public debt to the government. The stronger the importance, the larger the parameter α.

We note that our above specification for loss function h generalizes the quadratic function, which is the loss function most widely used in the economics literature (see, for example, Cadenillas et al. 2013, Cadenillas and Zapatero 1999, Kydland and Prescott 1977, and Taylor 1979).

To illustrate our framework, let us consider the example in which the government never intervenes.

2.3 Example. *If the government would never intervene, then* $Z \equiv 0$*. Hence*

$$X_t = x \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}.$$

Then, the total cost function would be

$$\begin{aligned} J(x;0) &= E_x\left[\int_0^\infty e^{-\lambda t} h(X_t)dt + 0\right] \\ &= E_x\left[\int_0^\infty e^{-\lambda t}\big(\alpha X_t^{2n} + \beta\big)dt\right] \\ &= \int_0^\infty e^{-\lambda t}\alpha E_x\left[X_t^{2n}\right]dt + \frac{\beta}{\lambda} \\ &= \alpha\int_0^\infty e^{-\lambda t}x^{2n}\exp\Big\{\Big(\sigma^2 n(2n-1) + 2\mu n\Big)t\Big\}dt + \frac{\beta}{\lambda} \\ &= \alpha x^{2n}\int_0^\infty \exp\Big\{-\Big(\lambda - \sigma^2 n(2n-1) - 2\mu n\Big)t\Big\}dt + \frac{\beta}{\lambda} \\ &= \begin{cases} \frac{\alpha x^{2n}}{\lambda - \sigma^2 n(2n-1) - 2\mu n} + \frac{\beta}{\lambda} & \text{if } \lambda - \sigma^2 n(2n-1) - 2\mu n > 0 \\ \infty & \text{if } \lambda - \sigma^2 n(2n-1) - 2\mu n \le 0\,. \end{cases} \end{aligned}$$

Thus, the total cost of the policy of no-intervention is finite if and only if the follow-

ing condition is satisfied:

$$\lambda > \sigma^2 n(2n-1) + 2\mu n. \tag{2.5}$$

It would be interesting to compare the no-intervention policy with the optimal control policy. This motivates us to consider the policy of no intervention $Z \equiv 0$ as admissible. Certainly, we expect that policy not to be optimal (we confirm this result in Section 2.5). Accordingly, we will assume condition (2.5) from now on. Furthermore, condition (2.5) is consistent with the real world, where governments are by far more concerned about the present than the future. Indeed, the larger the λ, the more concerned the government is about the present costs than about the future costs. To some extent, we are assuming some level of government myopia (see Collard et al. 2013). We also remark that imposing a condition on the discount rate similar to (2.5) is a common practice in macroeconomic models with infinite horizon (see, for example, Romer 2002).

We want to provide a mathematically rigorous definition of the control process.

We will allow the process Z to have jumps. Specifically, we assume the process to be left-continuous with right-limits. We define $\Delta := \{t \geq 0 : Z_t \neq Z_{t+}\}$, the set of times when Z has a discontinuity. The set Δ is countable because Z is increasing and, hence, can jump only a countable number of times during $[0, \infty)$. We will denote the discontinuous part of Z by Z^d, that is $Z_t^d := \sum_{0 \leq s < t,\, s \in \Delta} (Z_{s+} - Z_s)$. The continuous part of Z will be denoted by Z^c, that is $Z_t^c = Z_t - Z_t^d$.

2.4 Definition. *Let $x \in (0, \infty)$. An $\mathbb{F}$-adapted, non-negative, and non-decreasing stochastic control process $Z : [0, \infty) \times \Omega \longmapsto [0, \infty)$, with sample paths that are left-continuous with right-limits, is called an admissible stochastic singular control if $J(x; Z) < \infty$. The set of all admissible controls is denoted by $\mathcal{A}(x) = \mathcal{A}$. By convention, we set $Z_0 = 0$.*

2.5 Remark. *We observe that for every $Z \in \mathcal{A}(x)$:*

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T} X_T^{2n}\right] \;=\; 0. \tag{2.6}$$

Proof. See Appendix A.1. □

We will use condition (2.6) in the proof of Theorem 2.11 below.

To complete this section, we state formally the stochastic debt control problem.

2.6 Problem. *The government wants to select the control $Z \in \mathcal{A}(x)$ that minimizes the functional J defined by*

$$J(x; Z) \;:=\; E_x\left[\int_0^\infty e^{-\lambda t} h(X_t)dt + \int_0^\infty e^{-\lambda t} k dZ_t\right].$$

From a mathematical point of view, Problem 2.6 is a stochastic singular control problem. (Fleming and Stein 2004, Stein 2006, and Stein 2012 present a classical stochastic control model with the goal of providing a technique to predict debt crisis.) The theory of stochastic singular control has been studied, for instance, in Cadenillas and Haussmann (1994), Fleming and Soner (2006), Karatzas (1983), and Karatzas (1985).

2.2 A verification theorem

We define the value function $V : (0, \infty) \to \mathbb{R}$ by

$$V(x) \;:=\; \inf_{Z\in\mathcal{A}(x)} J(x; Z).$$

This represents the smallest cost that can be achieved when the initial debt ratio is x and we consider all the admissible controls. As such, it is a measure of the government well-being.

2.7 Proposition. *The value function is non-negative, increasing and convex. Furthermore,* $V(0+) = \frac{\beta}{\lambda}$.

Proof. See Appendix A.2. □

Let $\psi : (0, \infty) \to \mathbb{R}$ be a function in $C^2(0, \infty)$. We define the operator $\mathcal{L}$ by

$$\mathcal{L}\psi(x) \ := \ \frac{1}{2}\sigma^2 x^2 \psi''(x) + \mu x \psi'(x) - \lambda \psi(x).$$

For a function $v : (0, \infty) \to \mathbb{R}$ in $C^2(0, \infty)$, consider the Hamilton-Jacobi-Bellman (HJB) equation

$$\forall x > 0 : \quad \min \left\{ \mathcal{L}v(x) + h(x), k - v'(x) \right\} = 0. \tag{2.7}$$

This equation is equivalent to the variational inequalities (see Bensoussan and Lions (1982) for a classical reference on variational inequalities)

$$\begin{aligned} \mathcal{L}v(x) + h(x) \ &\geq \ 0, \\ k - v'(x) \ &\geq \ 0, \\ \big(\mathcal{L}v(x) + h(x)\big)\big(k - v'(x)\big) \ &= \ 0. \end{aligned}$$

We observe that a solution v of the HJB equation defines the regions $\mathcal{C} = \mathcal{C}^v$ and $\Sigma = \Sigma^v$ by

$$\mathcal{C} = \mathcal{C}^v := \Big\{ x \in (0, \infty) : \mathcal{L}v(x) + h(x) = 0 \quad \text{and} \quad k - v'(x) > 0 \Big\}, \tag{2.8}$$

$$\Sigma = \Sigma^v := \Big\{ x \in (0, \infty) : \mathcal{L}v(x) + h(x) \geq 0 \quad \text{and} \quad k - v'(x) = 0 \Big\}. \tag{2.9}$$

We note that the regions $\mathcal{C}$ and Σ form a partition of $(0, \infty)$. That is, if v solves the HJB equation, then $\mathcal{C} \cup \Sigma = (0, \infty)$ and $\mathcal{C} \cap \Sigma = \emptyset$.

It is possible to construct a control process associated with v in the following manner.

2.8 Definition. *Let v satisfy the HJB equation (2.7). An $\{\mathcal{F}_t\}$-adapted, non-negative, and non-decreasing control process Z^v, with $Z_0^v = 0$ and sample paths that are left-continuous with right-limits, is said to be associated with the function v if the following three conditions are satisfied:*

$$(i) \qquad X_t^v \ = \ x + \int_0^t \mu X_s^v ds + \int_0^t \sigma X_s^v dW_s - Z_t^v, \quad \forall t \in [0,\infty), P-a.s.,$$

$$(ii) \qquad X_t^v \in \bar{\mathcal{C}}, \quad \forall t \in (0,\infty),\ P-a.s.,$$

$$(iii) \qquad \int_0^\infty I_{\{X_t^v \in \mathcal{C}\}} dZ_t^v = 0, \ \ P-a.s..$$

Here I_A denotes the indicator function of the event $A \subset [0,\infty)$.

2.9 Remark. *According to Definitions 2.4 and 2.8, if an associate control process Z^v satisfies $J(x;Z^v) < \infty$, then it is admissible.*

Now we state a lemma that will be used in the proof of the Verification Theorem 2.11 below.

2.10 Lemma. *Suppose that v is increasing and satisfies that HJB equation (2.7). Let Z^v be the control associated to v, and X^v the process generated by Z^v. Then*

$$\int_0^T e^{-\lambda t} v'(X_t^v) d(Z^v)_t^c = \int_0^T e^{-\lambda t} k\, d(Z^v)_t^c \tag{2.10}$$

and

$$v(X_t^v) - v(X_{t+}^v) = k(Z_{t+}^v - Z_t^v), \quad \forall t \in \Delta. \tag{2.11}$$

Proof. The left-hand side of (2.10) can be expressed as

$$\int_0^T e^{-\lambda t} v'(X_t^v) I_{\{X_t^v \in \mathcal{C}\}} d(Z^v)_t^c \ + \ \int_0^T e^{-\lambda t} v'(X_t^v) I_{\{X_t^v \in \Sigma\}} d(Z^v)_t^c. \tag{2.12}$$

We will show that the first term of (2.12) equals zero, while the second equals the right hand side of (2.10). Indeed,

$$\left|\int_0^T e^{-\lambda t} v'(X_t^v) I_{\{X_t^v \in \mathcal{C}\}} d(Z^v)_t^c\right| \le \int_0^T \left|e^{-\lambda t} v'(X_t^v) I_{\{X_t^v \in \mathcal{C}\}}\right| d(Z^v)_t^c$$
$$\le \int_0^T e^{-\lambda t} k I_{\{X_t^v \in \mathcal{C}\}} d(Z^v)_t^c \le k \int_0^\infty I_{\{X_t^v \in \mathcal{C}\}} dZ_t^v = 0,$$

where the last equality follows from condition (iii) of Definition 2.8. Note that, as a byproduct, we have also proved that

$$\int_0^T e^{-\lambda t} k I_{\{X_t^v \in \mathcal{C}\}} d(Z^v)_t^c = 0. \tag{2.13}$$

On the other hand, since $v'(x) = k$ for every x in Σ, we have

$$\int_0^T e^{-\lambda t} v'(X_t^v) I_{\{X_t^v \in \Sigma\}} d(Z^v)_t^c = \int_0^T e^{-\lambda t} k I_{\{X_t^v \in \Sigma\}} d(Z^v)_t^c. \tag{2.14}$$

Adding (2.13) to (2.14),

$$\int_0^T e^{-\lambda t} v'(X_t^v) I_{\{X_t^v \in \Sigma\}} d(Z^v)_t^c = \int_0^T e^{-\lambda t} k \, d(Z^v)_t^c. \tag{2.15}$$

Next we show (2.11). By condition (iii) in Definition 2.8, if $t \in \Delta$, then X_t^v and X_{t+}^v are in Σ. Since $v'(x) = k$ for every x in Σ, we have

$$v(X_t^v) - v(X_{t+}^v) = k(X_t^v - X_{t+}^v) = k(Z_{t+}^v - Z_t^v).$$

□

We next state a sufficient condition for a policy to be optimal.

2.11 Theorem. *Let $v \in C^2(0,\infty)$ be an increasing and convex function on $(0,\infty)$, with $v(0+) = \frac{\beta}{\lambda}$. Suppose that v satisfies the Hamilton-Jacobi-Bellman equation (2.7) for every $x \in (0,\infty)$, and there exists $d \in (0,\infty)$ such that the region $\mathcal{C}$ associ-*

ated with v is $\mathcal{C}^v = (0, d)$. Then, for every $Z \in \mathcal{A}(\S)(x)$:

$$v(x) \leq J(x; Z).$$

Furthermore, the stochastic control Z^v associated with v satisfies

$$v(x) = J(x; Z^v).$$

In other words, $\hat{Z} = Z^v$ is optimal control and $V = v$ is the value function for Problem 2.6.

Proof. Since v is twice continuously differentiable, and v' and v'' are bounded functions, we may apply an appropriate version of Ito's formula. Thus, according to Meyer (1976) or Chapter 4 of Harrison (1985),

$$\begin{aligned} v(x) &= E_x\left[e^{-\lambda T} v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t} v'(X_t) dZ_t^c\right] - E_x\left[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t\right] \\ &\quad - E_x\left[\int_0^T e^{-\lambda t}\left\{\frac{1}{2}\sigma^2 X_t^2 v''(X_t) + \mu X_t v'(X_t) - \lambda v(X_t)\right\} dt\right] \\ &\quad - E_x\left[\sum_{\substack{t \in \Delta \\ 0 \leq t < T}} e^{-\lambda t}\left\{v(X_{t+}) - v(X_t)\right\}\right]. \end{aligned} \tag{2.16}$$

Since v satisfies the HJB equation (2.7), we have $\mathcal{L}v(x) + h(x) \geq 0$ and $v'(x) \leq k$ for all $x \in (0, \infty)$. Thus

$$-\int_0^T e^{-\lambda t}\left\{\frac{1}{2}\sigma^2 X_t^2 v''(X_t) + \mu X_t v'(X_t) - \lambda v(X_t)\right\} dt \leq \int_0^T e^{-\lambda t} h(X_t) dt. \tag{2.17}$$

$$\int_0^T e^{-\lambda t} v'(X_t) dZ_t^c \leq \int_0^T e^{-\lambda t} k dZ_t^c \tag{2.18}$$

and

$$v(X_t) - v(X_{t+}) \leq k(Z_{t+} - Z_t), \quad \forall t \in \Delta. \tag{2.19}$$

Hence

$$\begin{aligned} v(x) &\leq E_x\left[e^{-\lambda T}v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t}h(X_t)dt\right] + E_x\left[\int_0^T e^{-\lambda t}kdZ_t^c\right] \\ &\quad + E_x\left[\sum_{\substack{t\in\Delta \\ 0\leq t<T}} e^{-\lambda t}k\left\{Z_{t+}-Z_t\right\}\right] - E_x\left[\int_0^T e^{-\lambda t}X_t\sigma v'(X_t)dW_t\right] \\ &= E_x\left[e^{-\lambda T}v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t}h(X_t)dt + \int_0^T e^{-\lambda t}kdZ_t\right] \\ &\quad - E_x\left[\int_0^T e^{-\lambda t}X_t\sigma v'(X_t)dW_t\right]. \end{aligned} \tag{2.20}$$

Suppose $Z \in \mathcal{A}(x)$. From (2.6), v' bounded, and the linear growth of v on the interval $\Sigma^v = [d, \infty)$, we have

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T}v(X_T)\right] = 0,$$

and

$$E_x\left[\int_0^T e^{-\lambda t}X_t\sigma v'(X_t)dW_t\right] = 0.$$

In addition, letting $t \to \infty$, by the Monotone Convergence Theorem,

$$\lim_{T\to\infty} E_x\left[\int_0^T e^{-\lambda t}h(X_t)dt\right] = E_x\left[\int_0^\infty e^{-\lambda t}h(X_t)dt\right]$$

and

$$\lim_{T\to\infty} E_x\left[\int_0^T e^{-\lambda t}kdZ_t\right] = E_x\left[\int_0^\infty e^{-\lambda t}kdZ_t\right].$$

This proves the first part of this theorem.

Now we consider the second part of the theorem. Let X^v be the process generated by Z^v. We recall that we are assuming $\mathcal{C}^v = (0, d)$. We also note that, since $\mathcal{L}v(x) + h(x)$ is a continuous function for every $x \in (0, \infty)$, using Eq. (2.8), we have $\mathcal{L}v(x) + h(x) = 0$ for every $x \leq d$. These remarks, and condition (ii) of Definition

2.8, yield

$$\int_0^T e^{-\lambda t}\Big\{\frac{1}{2}\sigma^2(X_t^v)^2 v''(X_t^v)+\mu X_t^v v'(X_t^v)-\lambda v(X_t^v)\Big\}dt=-\int_0^T e^{-\lambda t}h(X_t^v)dt.$$

That is, (2.17) turns into an equality. Moreover, by Lemma 2.10, the inequalities (2.18) and (2.19) become equalities as well. As a result, (2.20) is an equality for Z^v, namely

$$\begin{aligned} v(x) &= E_x\left[e^{-\lambda T}v(X_T^v)\right]+E_x\left[\int_0^T e^{-\lambda t}h(X_t^v)dt+\int_0^T e^{-\lambda t}kdZ_t^v\right] \\ &\quad - E_x\left[\int_0^T e^{-\lambda t}X_t\sigma v'(X_t)dW_t\right]. \end{aligned} \tag{2.21}$$

Since the process X_t^v is bounded above by $\max\{x,d\}$, $v(0+)$ is bounded and v is continuous on $(0,d]$, we conclude that the process $v(X_t^v)$ is bounded. Hence

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T}v(X_T^v)\right]=0,$$

and

$$E_x\left[\int_0^T e^{-\lambda t}X_t\sigma v'(X_t)dW_t\right]=0.$$

Letting $T\to\infty$, by the Monotone Convergence Theorem, we conclude

$$v(x)=E_x\left[\int_0^\infty e^{-\lambda t}h(X_t^v)dt+\int_0^\infty e^{-\lambda t}kdZ_t^v\right]=J(x;Z^v).$$

We observe that, in particular, this shows that Z^v is admissible. This completes the proof of this theorem. □

2.3 The explicit solution

At the beginning of this section we are going to make conjectures to obtain a candidate for optimal control and a candidate for value function. At the end of this section, we are going to apply Theorem 2.11 to prove rigorously that these conjectures are valid.

We want to find a function v and a control Z^v that satisfy the conditions of Theorem 2.11. By condition (ii) of Definition 2.8, we note that Z^v makes the corresponding controlled process X^v stay inside the closure of the region $\mathcal{C}$ all the time (except perhaps at time 0). Moreover, by condition (iii), the control process Z^v remains constant on any subset of $(0, \infty)$ in which the controlled process X^v is strictly inside the region $\mathcal{C}$.

Based on these two observations, we conjecture that there exists a debt ceiling $b \in (0, \infty)$ such that the government should intervene when the debt ratio $X \geq b$, and should not intervene when the debt ratio $X < b$. Accordingly, if v satisfies the Hamilton-Jacobi-Bellman equation, we will call $\mathcal{C} = (0, b)$ the continuation region and $\Sigma = [b, \infty)$ the intervention region. Thus, it is natural to define the debt ratio ceiling as follows.

2.12 Definition. *Let v be a function that satisfies the HJB equation (2.7), and $\mathcal{C}$ the corresponding continuation region. If $\mathcal{C} \neq \emptyset$, the debt ratio ceiling b is*

$$b \ := \ \sup\{x \in (0\,,\infty)\,|\, x \in \mathcal{C}\}.$$

Moreover, if v is equal to the value function, then b is said to be the optimal debt ceiling.

This definition is consistent with the 60% established in the Maastricht Treaty, which represents the upper bound for the debt ratio of the countries that wanted to belong to the European Economic Community. However, there is a crucial difference: that 60% was selected by taking the median of the debt ratio of those countries, while here we follow an optimality criterion. Furthermore, Definition 2.12 is also consistent with the term "debt ceiling" used by the USA administration. However, the difference is that the "debt ceiling" used by them is expressed in monetary units (dollars), whereas (following the economic literature) we express the debt ceiling as the ratio of total public debt divided by the GDP.

Thus, to obtain the optimal debt ceiling, we need to find the value function. The Hamilton-Jacobi-Bellman equation (2.7) in the continuation region $\mathcal{C} = (0, b)$ implies

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta,$$

and the Hamilton-Jacobi-Bellman equation (2.7) in the intervention region $\Sigma = [b, \infty)$ implies

$$v(x) = v(b) + k(x - b).$$

We get the differential equation

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta \qquad \text{if } x < b \tag{2.22}$$

and

$$v'(x) = k \qquad \text{if } x \geq b. \tag{2.23}$$

The general solution of (2.22)-(2.23) has the form

$$v(x) = \begin{cases} Ax^{\gamma_1} + Bx^{\gamma_2} + \alpha\zeta x^{2n} + \frac{\beta}{\lambda} & \text{if } x < b \\ kx + D & \text{if } x \geq b. \end{cases}$$

Here

$$\tilde{\mu} := \mu - \frac{1}{2}\sigma^2, \tag{2.24}$$

$$\gamma_1 := \frac{-\tilde{\mu} - \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} < 0, \tag{2.25}$$

$$\gamma_2 := \frac{-\tilde{\mu} + \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} > 0, \tag{2.26}$$

$$\zeta := \frac{1}{\lambda - \sigma^2 n(2n-1) - 2\mu n}. \tag{2.27}$$

We recall that Proposition 2.7 states $V(0+) = \frac{\beta}{\lambda}$. Furthermore, we conjecture that v is twice continuously differentiable. Then, the four constants A, B, b, and D can be found (see below for the explicit solution) from the following system of four equations:

$$v(0^+) = \frac{\beta}{\lambda}, \tag{2.28}$$

$$v(b^+) = v(b^-), \tag{2.29}$$

$$v'(b^+) = v'(b^-), \tag{2.30}$$

$$v''(b^+) = v''(b^-). \tag{2.31}$$

For future reference, we state conditions that the parameters satisfy.

2.13 Lemma. *The following results are valid:*

(i) $\zeta > 0$,

(ii) $\lambda > \mu$,

(iii) $\gamma_2 > 2n$,

(iv) $\lambda - \sigma^2 n(\gamma_2 - 1) - \gamma_2\mu > 0$,

(v) $2\lambda - \tilde{\mu} - \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2} > 0$.

Proof. See Appendix A.3. □

Now we proceed with the solution. Since $\gamma_1 < 0$, condition (2.28) implies $A = 0$ in the equation for v. Thus, the candidate for value function is given by

$$v(x) = \begin{cases} Bx^{\gamma_2} + \alpha\zeta x^{2n} + \frac{\beta}{\lambda} & \text{if } x < b \\ kx + D & \text{if } x \geq b. \end{cases} \tag{2.32}$$

Taking the first and second derivatives, it follows that

$$v'(x) = \begin{cases} B\gamma_2 x^{\gamma_2 - 1} + \alpha\zeta\, 2n x^{2n-1} & \text{if } x < b \\ k & \text{if } x \geq b \end{cases}$$

and

$$v''(x) = \begin{cases} B\gamma_2(\gamma_2 - 1)x^{\gamma_2 - 2} + \alpha\zeta\, 2n(2n-1)x^{2n-2} & \text{if } x < b \\ 0 & \text{if } x \geq b. \end{cases}$$

From Eqs. (2.30)-(2.31), we obtain

$$b = \left(\frac{1}{\alpha\zeta} \frac{k(\gamma_2 - 1)}{(\gamma_2 - 2n)2n} \right)^{\frac{1}{2n-1}} > 0, \tag{2.33}$$

where the strictly positive sign follows from (i) and (iii) of Lemma 2.13. From (2.31), we find B in terms of b. That is,

$$B = -\frac{\alpha\zeta 2n(2n-1)}{\gamma_2(\gamma_2 - 1)}\, b^{2n-\gamma_2} < 0, \tag{2.34}$$

where the strictly negative sign follows again from (i) and (iii) of Lemma 2.13. Using the previous two constants, and Eq. (2.29), we obtain

$$D = B\, b^{\gamma_2} + \alpha\zeta b^{2n} + \frac{\beta}{\lambda} - kb. \tag{2.35}$$

Hence, from Eqs. (2.29)-(2.31), we obtain the constants b, B and D explicitly, as a function of the parameters $(k, n, \lambda, \mu, \sigma, \alpha, \beta)$.

Summarizing, the candidate for value function is given by (2.32), and the candidate for optimal control is then determined by Definition 2.8. Moreover, the candidate for optimal debt ceiling is provided in (2.33).

We make the following observation about the candidates.

2.14 Lemma. *Let v be the candidate for value function given by Eq. (2.32), and let b be given by Eq. (2.33). Let us define $L(x) := \mathcal{L}v(x) + h(x)$ for every $x \in (0, \infty)$. Then $L'(b+) := \lim_{x \downarrow b} L'(x) > 0$.*

Proof. See Appendix A.4. □

To complete this section, we are going to prove rigorously that the above candidate for optimal control is indeed the optimal control, and the above candidate for value function is indeed the value function.

2.15 Theorem. *The value function is given by*

$$V(x) = v(x) = \begin{cases} Bx^{\gamma_2} + \alpha\zeta x^{2n} + \frac{\beta}{\lambda} & \text{if } \; x \; < \; b \\ kx + D & \text{if } \; x \; \geq \; b, \end{cases} \tag{2.36}$$

where b is the optimal debt ceiling given by

$$b = \left(\frac{1}{\alpha\zeta} \frac{k(\gamma_2 - 1)}{(\gamma_2 - 2n)2n} \right)^{\frac{1}{2n-1}} > 0, \tag{2.37}$$

with γ_2, ζ, B, and D, given by (2.26), (2.27), (2.34), and (2.35), respectively.

Furthermore, the optimal debt control is the process $\hat{Z}$ given by

$$(i) \qquad \hat{X}_t \; = \; x + \int_0^t \mu \hat{X}_s ds + \int_0^t \sigma \hat{X}_s dW_s - \hat{Z}_t, \quad \forall t \in [0, \infty), \; P - a.s.,$$

$$(ii) \qquad \hat{X}_t \in [0, b], \quad \forall t \in (0, \infty), \quad P - a.s.,$$

$$(iii) \qquad \int_0^\infty I_{\{\hat{X}_t \in (0,b)\}} d\hat{Z}_t = 0, \quad P - a.s..$$

Here $\hat{X}$ denotes the debt ratio process generated by the optimal debt control $\hat{Z}$.

Proof. To prove this theorem, it suffices to show that all the conditions of Theorem 2.11 are satisfied. By construction, we get immediately that $v \in C^2\big(0,\infty\big)$ and $v(0+) = \frac{\beta}{\lambda}$. Regarding the value function, it remains to verify that v is increasing, convex and satisfies the HJB equation (2.7). Let us show that v is increasing. Taking the first derivative, we have $v'(x) = k > 0$ for every $x > b$. On the other hand, for every $x < b$:

$$\begin{aligned} v'(x) &= 2\alpha\zeta n x^{2n-1} + B\gamma_2 x^{\gamma_2-1} \\ &= 2\alpha\zeta n x^{2n-1} - \Big[2\alpha\zeta n \frac{(2n-1)}{\gamma_2(\gamma_2-1)} b^{2n-\gamma_2}\Big]\gamma_2\, x^{\gamma_2-1} \\ &= 2\alpha\zeta n b^{2n-\gamma_2} x^{\gamma_2-1}\Big\{\Big(\frac{b}{x}\Big)^{\gamma_2-2n} - \Big(\frac{2n-1}{\gamma_2-1}\Big)\Big\}. \end{aligned}$$

We recall that $\zeta > 0$ and $\gamma_2 > 2n$, by Lemma 2.13. Since

$$\Big(\frac{b}{x}\Big)^{\gamma_2-2n} > 1 > \frac{2n-1}{\gamma_2-1},$$

we conclude that $v'(x) > 0$ for every $x < b$. The continuity of v' implies $v'(b) = k$. Hence v is strictly increasing on $(0,\infty)$. Next we prove that v is convex. We observe immediately that $v''(x) = 0$ for every $x > b$. On the other hand, for every $x < b$:

$$\begin{aligned} v''(x) &= 2\alpha\zeta n(2n-1)x^{2n-2} + B\gamma_2(\gamma_2-1)x^{\gamma_2-2} \\ &= 2\alpha\zeta n(2n-1)x^{2n-2} - \Big[2\alpha\zeta n \frac{(2n-1)}{\gamma_2(\gamma_2-1)} b^{2n-\gamma_2}\Big]\gamma_2(\gamma_2-1)\, x^{\gamma_2-2} \\ &= 2\alpha\zeta n(2n-1) b^{2n-\gamma_2} x^{\gamma_2-2}\Big\{\Big(\frac{b}{x}\Big)^{\gamma_2-2n} - 1\Big\} > 0, \end{aligned}$$

due to $\zeta > 0$ and $\gamma_2 > 2n$, again by Lemma 2.13. The continuity of v'' yields $v''(b) = 0$. This proves that v is convex on $(0,\infty)$ and strictly convex on $(0,b)$.

Now we establish the HJB equation (2.7). Let us recall that $L(x) := \mathcal{L}v(x) + h(x)$ for every $x \in (0,\infty)$. We note that, since v is $C^2(0,\infty)$ and h is polynomial, L is continuous on $(0,\infty)$. We proceed by considering cases. First, we consider the case $x < b$. Then, by construction of the candidate for value function, $L(x) = 0$ for

$x < b$. On the other hand, since v is strictly convex on $(0, b)$ and $v'(b) = k$, we obtain $v'(x) < k$ for every $x \in (0, b)$. This shows that $L(x) = 0$ and $v'(x) < k$ for every $x < b$. In other words, the HJB equation (2.7) is satisfied for every $x < b$. We note that $L(b) = L(b-) = 0$. Now we consider the case $x \geq b$. By construction, we have $v'(x) = k$ for every $x \geq b$. It only remains to show $L(x) \geq 0$ for every $x \geq b$. We observe that for every $x \geq b$:

$$L(x) = \mu\, x\, k - \lambda(k\, x + D) + \alpha x^{2n} + \beta.$$

Hence $L \in C^2(b, \infty)$. Computing the first and second derivatives on (b, ∞), we have

$$\begin{aligned} L'(x) &= (\mu - \lambda)k + 2n\alpha x^{2n-1}\,, \\ L''(x) &= 2n(2n-1)\alpha x^{2n-2} \;>\; 0. \end{aligned}$$

Since $L(b) = 0$, it is enough to show that $L(x) \geq L(b)$ for every $x > b$. Hence, we just need to prove that $L'(x) > 0$ on (b, ∞). To this end, since $L''(x)$ is strictly positive for $x > b$, it is sufficient to prove that $L'(b+) > 0$. This inequality is satisfied by Lemma 2.14. This establishes that $v'(x) = k$ and $L(x) \geq 0$ for every $x \geq b$. Hence v satisfies the HJB equation (2.7) for every $x \in (0, \infty)$. We observe that, from (2.8), the corresponding continuation region is indeed $\mathcal{C} = (0, b)$.

From Theorem 2.11, we conclude that $\hat{Z}$ is optimal debt control. Moreover, by Definition 2.12, b is the optimal debt ceiling. This completes the proof of this theorem. □

We observe that conditions (i)-(iii) in the above theorem define a (reflecting) stochastic differential equation for the process $\hat{X}$ and its associated control process $\hat{Z}$. The existence and uniqueness of the solution to that kind of stochastic differential equation can be derived from Theorem 4.1 of Tanaka (1979). (This is known as the "Skorokhod problem". See Skorokhod 1961 in the stochastic analysis literature.)

Now we provide an explanation of how the optimal debt control process $\hat{Z}$ works. When the actual debt ratio $\hat{X}_t$ is below b, condition (iii) in the previous theorem implies that $\hat{Z}$ remains constant. Hence there is no need to generate fiscal surpluses in order to reduce the debt ratio. When the debt ratio reaches b and tries to cross it, condition (ii) implies that the government has to intervene in order to prevent the debt ratio from crossing b. If the initial debt ratio x is strictly greater than the debt ceiling b, conditions (i) and (ii) imply that the control process jumps from $\hat{Z}_0 = 0$ to $\hat{Z}_{0+} = x - b$, and correspondingly, the debt ratio jumps from $\hat{X}_0 = x$ to $\hat{X}_{0+} = b$. Thus, if $x > b$, the government should make fiscal adjustments to increase the primary surplus by $(x - b)$. After this jump, the debt control behaves as described at the beginning of this paragraph, and thereby the controlled ratio process $\hat{X}_t \in [0, b]$ for every $t > 0$. Thus, if necessary, a jump might occur only at time zero.

Therefore, if at any point in time the debt ratio of a country is below the optimal debt ceiling b of Theorem 2.15, then no fiscal intervention is required; if the debt ratio is equal to b, then control should be exerted to prevent the debt ratio from exceeding b; if the initial debt ratio is above the ceiling b, then the government should intervene to bring immediately the debt ratio to the level b (and then continue as described above).

We have obtained the optimal debt ceiling by following a partial equilibrium approach. Merton (1969), Baumol (1952), and Tobin (1956) are some examples of seminal papers in the economics literature that follow a partial equilibrium approach. (Romer 1986 develops a general equilibrium version of Baumol 1952 and Tobin 1956.)

2.4 Time to reach the optimal debt ceiling

In this section, we study the time to reach the optimal debt ceiling. Specifically, we assume that $X(0) = x < b$ and consider the stopping time

$$\begin{aligned}\tau &:= \inf\{t \in (0,\infty) : X(t) \geq b\}\\ &= \inf\left\{t \in (0,\infty) : x\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\} \geq b\right\}\\ &= \inf\left\{t \in (0,\infty) : \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t) \geq \log\left(\frac{b}{x}\right)\right\}.\end{aligned}$$

Hereafter $\log$ stands for the natural logarithm.

We recall that $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2$. We need to consider two cases: $\tilde{\mu} < 0$ and $\tilde{\mu} \geq 0$.

2.16 Proposition. *(a) If $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2 < 0$, then the distribution function of τ is defective. This means that $P\{\tau < \infty\} < 1$ or, equivalently, $P\{\tau = \infty\} > 0$. Hence $E[\tau] = \infty$. Furthermore, the probability that the debt ceiling will eventually be reached is*

$$P\{\tau < \infty\} = \exp\left\{\frac{2\tilde{\mu}}{\sigma^2}\log\left(\frac{b}{x}\right)\right\} = \left(\frac{b}{x}\right)^{\frac{2\mu}{\sigma^2}-1}.$$

(b) If $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2 \geq 0$, then $P\{\tau < \infty\} = 1$ and the probability density function of τ is given by

$$f_\tau(t) = \frac{\log\left(\frac{b}{x}\right)}{\sigma\sqrt{2\pi t^3}}\exp\left\{-\frac{\left(\log\left(\frac{b}{x}\right) - \tilde{\mu}t\right)^2}{2\sigma^2 t}\right\}, \qquad t > 0.$$

Furthermore, for $\tilde{\mu} > 0$ we obtain

$$\begin{aligned} E[\tau] &= \left(\tfrac{\sigma}{\tilde{\mu}}\right)\log\left(\tfrac{b}{x}\right),\\ \mathit{VAR}[\tau] &= \left(\tfrac{\sigma^2}{\tilde{\mu}^3}\right)\log\left(\tfrac{b}{x}\right),\end{aligned}$$

and for every $\theta > 0$:

$$E\left[\exp\{-\theta\tau\}\right] \;=\; \left(\frac{b}{x}\right)^{-\left(\sqrt{\frac{\tilde{\mu}^2}{\sigma^4}+\frac{2\theta}{\sigma^2}}-\frac{\tilde{\mu}}{\sigma^2}\right)}.$$

In addition, for $\tilde{\mu} = 0$ *we have* $E[\tau] = \infty$.

Proof. See Section 8.4 of Ross (1996) and Section 7.5 of Karlin and Taylor (1975). □

Using Proposition 2.16 and the explicit formula (2.37), we can analyze the effects of the parameters on the time τ to reach the optimal debt ceiling. For instance, in Section 2.5 we compute the effects of the economic growth on the probability of reaching the optimal debt ceiling, as well as on the expected time to reach it.

2.5 Economic analysis

In this section we analyze the effects of the parameters on both the optimal debt ceiling and the value function.

First, let us compare the optimal intervention policy with the no-intervention policy. We recall that the value function V represents the minimum total cost, and is a function of the initial debt ratio. Hence the cost $J(x;0)$ of no intervention should be greater than or equal to $V(x)$. Consider first every $x < b$. From Example 2.3 and Theorem 2.15, we can easily verify that

$$G(x) := J(x;0) - V(x) = -Bx^{\gamma_2} \;>\; 0\,,$$

because $B < 0$. For every $x \geq b$,

$$\begin{aligned} G(x) \;&:=\; J(x;0) - V(x) \\ &=\; \alpha\zeta x^{2n} + \frac{\beta}{\lambda} - kx - D \end{aligned}$$

$$
\begin{aligned}
&= \alpha\zeta x^{2n} + \frac{\beta}{\lambda} - kx - \Big(Bb^{\gamma_2} + \alpha\zeta b^{2n} - kb + \frac{\beta}{\lambda}\Big) \\
&= \alpha\zeta(x^{2n} - b^{2n}) + k(b - x) - Bb^{\gamma_2}.
\end{aligned}
$$

We notice that $G(b) = -Bb^{\gamma_2} > 0$. To show that $G(x) > 0$ for $x \geq b$, it is enough to verify that $G'(x) > 0$ for $x > b$. First, we note that Eq. (2.37) implies

$$
k = 2\alpha\zeta n \frac{(\gamma_2 - 2n)}{(\gamma_2 - 1)} b^{2n-1}.
$$

Thus, taking the first derivative,

$$
\begin{aligned}
G'(x) &= \alpha\zeta 2n x^{2n-1} - k \\
&= 2\alpha\zeta n x^{2n-1} - 2\alpha\zeta n \frac{(\gamma_2 - 2n)}{(\gamma_2 - 1)} b^{2n-1} \\
&= 2\alpha\zeta n b^{2n-1} \Big\{ \Big(\frac{x}{b}\Big)^{2n-1} - \frac{(\gamma_2 - 2n)}{(\gamma_2 - 1)} \Big\} > 0\,,
\end{aligned}
$$

because $x > b$ and Lemma 2.13 (iii) implies $0 < (\gamma_2 - 2n) < (\gamma_2 - 1)$.

In other words, regardless of the value of the initial debt ratio, it is always worthy to intervene optimally. Here $G(x) > 0$ represents the advantage of the optimal intervention policy compared to never intervening.

After confirming that the government gets a better result from intervening, we proceed with our comparative analysis. First of all, we recall here the meaning of the parameters: interest rate on debt r, economic growth rate g, debt ratio volatility σ, marginal cost of intervention k, aversion towards debt ratio n, important of debt to government α, scale parameter β, and discount rate λ.

Table 2.1: Basic parameter values

k	r	g	σ	n	α	λ	β
1.0	0.07	0.06	0.05	2	1.0	0.07	0.0

In Table 2.1 we set the basic parameter values we are going to consider in our analysis. Thus, whenever we do not specify the value of a parameter, it means that we are using the ones in this table. As in Cadenillas and Zapatero (1999), and Sotomayor and Cadenillas (2009), we set the discount rate $\lambda = 0.07$. Regarding the parameter α, associated with the importance of debt, we take a neutral position and fix it at 1. We normalize the marginal cost of debt intervention by selecting $k = 1$. In addition, we set the scale parameter β equal to zero, the volatility σ at 0.05, the interest rate r at 7%, the rate of economic growth g at 6% and the debt aversion n at 2.

2.17 Example. *Using the parameter values of Table 2.1, the optimal government debt ceiling is $b = 26.46037561\%$, and the corresponding value function is*

$$V(x) = \begin{cases} -122.9206427616\, x^{4.7613558209} + \frac{200}{3}\, x^4 & \text{if } x < 0.2646037561 \\ x - 0.1567729207 & \text{if } x \geq 0.2646037561. \end{cases}$$

To facilitate the exposition, we focus on the effects of the parameters (α, g, σ). However, as we will remark below, the qualitative impact of g and r are exactly opposite. As a result, we are actually analyzing the effects of (α, g, σ, r).

Effects of the importance of debt (α)

We recall that the parameter α represents the importance of debt to the government (see Section 2.1). From Eq. (2.37), we get

$$\frac{\partial b}{\partial \alpha} = -\frac{1}{\alpha(2n-1)} b < 0\,.$$

Consequently, the greater the importance of debt for the government, the lower the optimal debt ceiling. In other words, the more concerned is the government about its debt, the more control is required. We notice that this result does not depend on the value of the other parameters, in particular, the level n of debt aversion.

In Table 2.2 and Figure 2.1, we show the impact of the parameter α on the

Table 2.2: Effect of α on the value function.

	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$	$\alpha = 1.2$	$\alpha = 1.4$
$x = 0.20$	0.033561	0.041674	0.048913	0.055413	0.06127
$x = 0.50$	0.314125	0.331121	0.343227	0.352471	0.35986
$x = 0.80$	0.614125	0.631121	0.643227	0.652471	0.65986

See Table 2.1 for the values of the other parameters used in these computations.

value function V. We observe that the larger the value of α, the higher the value function, for any initial debt ratio.

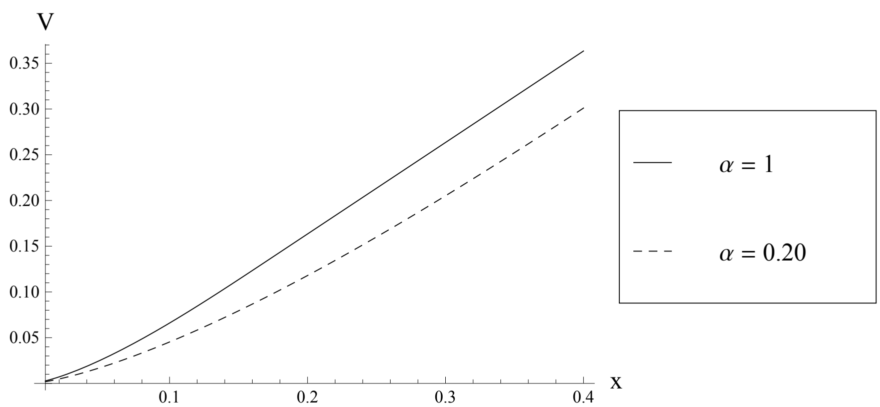

Figure 2.1: Effect of α on the value function V

Effects of the rate of economic growth (g)

From Eq. (2.37) again, we obtain

$$\begin{aligned}\frac{\partial b}{\partial g} &= \frac{k}{4\alpha n(2n-1)}\left((2n+1) - (2n-1)\frac{\tilde{\mu}}{\sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}\right) b^{2-2n} \\ &> \frac{k}{4\alpha n(2n-1)}\Big((2n+1) - (2n-1)\Big) b^{2-2n} \\ &= \frac{k}{2\alpha n(2n-1)}\, b^{2-2n} \; > \; 0,\end{aligned}$$

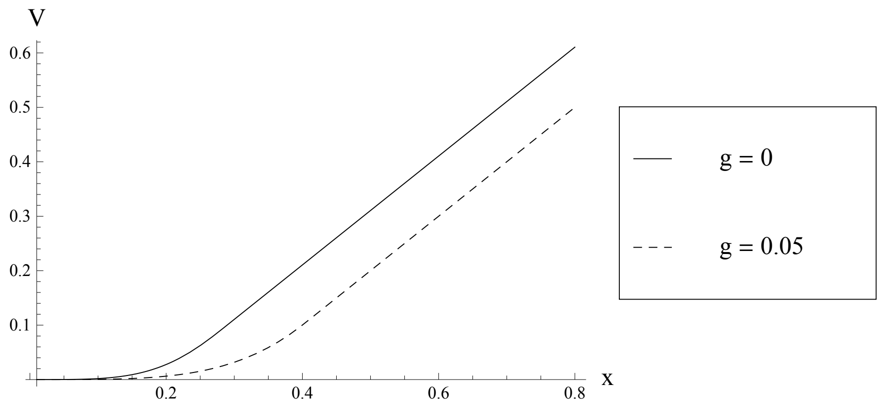

Figure 2.2: Effect of g on the value function V

where we used $\sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2} > \tilde{\mu}$. Therefore, the greater the economic growth the greater the optimal debt ceiling. We note that this result holds regardless of the value of debt aversion n.

2.18 Remark. *The interest rate on debt r has the opposite effect. Specifically, the partial derivative of b with respect to r is the negative of the the partial derivative of b with respect to g. Thus, countries which borrow money at high interest rates should have a low debt ceiling. This comes from the fact that our solution for both the optimal debt ceiling and the value function depends on μ and, by definition, $\mu := r - g$.*

From Table 2.3 and Figure 2.2, it follows that the greater the rate of economic growth g the lower the value function. This has an intuitive explanation: when it comes to control the debt ratio, high rates of economic growth are desirable.

We notice that the difference between the value functions corresponding to two different rates of economic growth is constant for large values of the initial debt ratio. This result follows immediately from Eq. (2.36), in which for values of the initial debt ratio greater than the maximum level of the two optimal debt ceilings, the slope of the two value functions is the same (equal to k, actually).

Proposition 2.16 illustrates that economic growth, in some sense, helps to control the debt ratio. Consider $\tilde{\mu} < 0$ and an initial debt ratio below the debt ceiling,

Table 2.3: Effect of g on the value function

	$g = 0.01$	$g = 0.02$	$g = 0.03$	$g = 0.04$	$g = 0.06$
$x = 0.20$	0.18547	0.163399	0.137189	0.108074	0.048913
$x = 0.50$	0.48547	0.463399	0.437189	0.407923	0.343227
$x = 0.80$	0.78547	0.763399	0.737189	0.707923	0.643227

See Table 2.1 for the values of the other parameters used in these computations.

$x < b$. Then the greater the rate of economic growth g, the lower the probability of reaching the optimal debt ceiling. In fact,

$$\frac{\partial P\{\tau < \infty\}}{\partial g} = -2\frac{P\{\tau < \infty\}}{\sigma^2}\left\{\log\left(\frac{b}{x}\right) - \frac{\tilde{\mu}}{b}\left(\frac{\partial b}{\partial g}\right)\right\} < 0.$$

In the case $\tilde{\mu} > 0$ and again with an initial debt ratio below the debt ceiling ($x < b$), the expected time to reach the debt ceiling increases when the economic growth rate increases. Indeed,

$$\frac{\partial E[\tau]}{\partial g} = \frac{\sigma}{\tilde{\mu}^2}\log\left(\frac{b}{x}\right) + \frac{\sigma}{\tilde{\mu}}\frac{1}{b}\left(\frac{\partial b}{\partial g}\right) > 0.$$

We note that in the case $\tilde{\mu} \leq 0$, $\frac{\partial E[\tau]}{\partial g}$ is not defined because $E[\tau] = \infty$. However, we have noticed above that $\frac{\partial P\{\tau<\infty\}}{\partial g} < 0$ when $\tilde{\mu} < 0$.

Consequently, our model predicts that countries with high economic growth rates should not have low debt ceilings, because basically the economic growth is taking care of the debt ratio.

Effects of debt volatility (σ)

From Eq. (2.37) it follows that

$$\frac{\partial b}{\partial \sigma} = \frac{k\sigma}{4\alpha n\sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}\left(2\lambda - \tilde{\mu} - \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}\right) b^{2-2n} > 0.$$

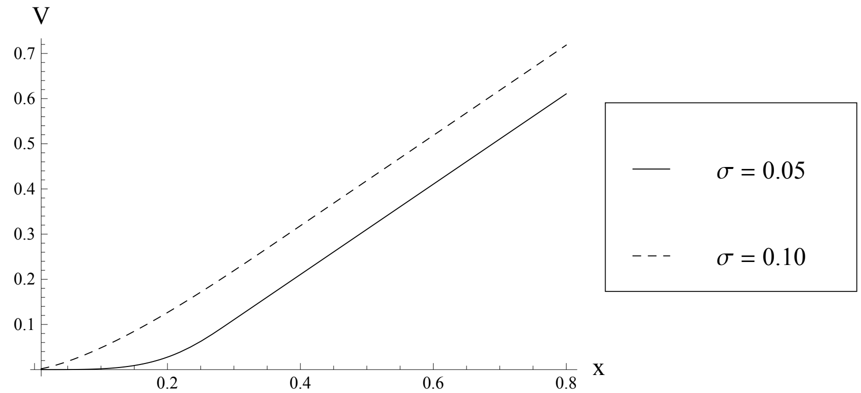

Figure 2.3: Effect of σ on the value function V

The positive sign follows from inequality (v) in Lemma 2.13. Hence, if debt volatility increases, then the optimal debt ceiling increases as well.

Table 2.4: Effect of σ on the value function

	$\sigma = 0.01$	$\sigma = 0.03$	$\sigma = 0.05$	$\sigma = 0.07$	$\sigma = 0.09$
$x = 0.20$	0.045348	0.046544	0.048913	0.052177	0.056059
$x = 0.50$	0.341465	0.341845	0.343227	0.345638	0.348861
$x = 0.80$	0.641465	0.641845	0.643227	0.645638	0.648861

See Table 2.1 for the values of the other parameters used in these computations.

Consequently, countries in which their primary balances are linked to exogenous factors, such as the price of raw materials or the evolution of the financial markets should have a higher debt ceiling. Since debt is denominated in domestic currency, this is specially important for developing countries that have a large proportion of their debts in foreign currencies. (See Panizza 2008b for a study regarding domestic and external public debt in developing countries.)

Likewise, an increment of debt volatility increases the value function, as shown in Table 2.4 and Figure 2.3. Thus, high volatility is clearly not desirable. Besides, we observe a similar pattern to the one we noticed on the effect of the rate of economic growth on the value function. Indeed, in Figure 2.3 we see that the difference between the value functions corresponding to different volatilities remains the same for large values of the initial debt ratio.

Summary of economic results

We have found that the increments in the rate of economic growth g, or the debt volatility σ, increase the optimal debt ceiling b. In contrast, as the interest rate on debt r or the importance of debt α increases, the optimal debt ceiling b decreases.

Regarding the value function, we have found that the higher the economic growth, the lower the expected total cost. All the other parameters (interest rate on debt, the importance of debt, and volatility) have the opposite effect.

2.6 Concluding remarks

We have considered a government that wants to control its debt ratio taking into account the cost of having debt and the cost of fiscal interventions to reduce the debt ratio. We have obtained an explicit solution for the government debt problem and, for the first time in the literature, an explicit formula for the optimal government debt ceiling (2.37). This formula is tailor-made for each country, in the sense that it depends on specific variables such as the rate of economic growth, the interest rate on debt, the importance of debt to the government, the cost of debt reduction, the debt volatility, and the aversion to debt.

Furthermore, we have found that the optimal debt control implies the following recommendation for debt policy: if at any point in time the actual debt ratio of a country is below the optimal government debt ceiling b given by (2.37), then fiscal intervention is not required; if the debt ratio is equal to b, then control should be exerted to prevent the debt ratio from being greater than b; if the initial debt ratio is above the government debt ceiling b, then the government should intervene to bring the debt ratio to the level b, and then continue as described above. Overall, the controlled government debt ratio process remains in the interval $[0, b]$ all the time (except perhaps at time $t = 0$).

This pioneering work sheds light on the connection between the optimal government debt ceiling and key macro-financial variables. We have found, among other results, that increments in the rate of economic growth or debt volatility, increase the optimal government debt ceiling. In contrast, as the interest rate on debt or the importance to debt increases, the optimal government debt ceiling decreases.

Chapter 3

The optimal debt ceiling: bounded intervention

In Chapter 2 we have solved the debt control problem, and obtained an explicit formula for the optimal debt ceiling, under the assumption that the government intervention to reduce the debt ratio is unbounded. As a result, the optimal debt policy states that if the actual debt ratio is above the optimal debt ceiling, then the government should generate primary surpluses to reduce the initial debt ratio to the optimal debt ceiling immediately.

However, in reality, it is hard for a country to generate primary surpluses. Indeed, the ceiling 60% set in the Maastricht Treaty was part of the "convergence criteria", in the understanding that countries whose debt ratio was above that threshold should reach the 60% over a number of years. This implies the existence of constraints that governments face to reduce their debt ratio. Due to this motivation, in this chapter we study the optimal debt ceiling assuming that the ability of the government to generate primary surpluses to reduce the debt ratio is bounded above.

3.1 The model

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$, which is the P-augmentation of the filtration generated by a one-dimensional Brownian motion W.

The state variable is the debt ratio $X = \{X_t, t \in [0, \infty)\}$ of a country, defined in (2.1). We assume that it is an $\mathbb{F}$-adapted stochastic process. Sometimes the government intervenes to control the debt ratio via the $\mathbb{F}$-adapted stochastic process $u = \{u_t, t \in [0, \infty)\}$. This process represents the rate of intervention of the government, and it is associated to the generation of primary surpluses with the specific goal of reducing the debt ratio. For $\overline{U} > 0$, we assume that $u(t) \in [0, \overline{U}]$ for every $t \geq 0$, which means that the ability of the government to produce primary surpluses is limited. Clearly, each country has a different bound $\overline{U}$, which depends on its structural economic (and political) characteristics.

We assume that the debt dynamics is given by

$$X_t \; = \; x + \int_0^t \mu X_s ds + \int_0^t \sigma X_s \, dW_s - \int_0^t u_s \, ds \,, \tag{3.1}$$

where $\mu := (r - g) \in \mathbb{R}$ and $\sigma \in (0, \infty)$ are constants. Here $r \in [0, \infty)$ represents the interest rate on debt, $g \in \mathbb{R}$ the rate of economic growth, and σ the volatility. We denote the initial debt ratio of the country by $x \in (0, \infty)$.

We note that, without governmente intervention, the debt dynamics in Chapter 2 and the one in (3.1) above are the same.

The intervention of the government generates some costs. This cost is generated by fiscal adjustments, which can take the form of raising taxes or reducing expenses. Furthermore, k represents the marginal cost of debt reduction, i.e., for each unit of debt ratio reduction the government has to pay the cost k, which is a positive constant. Let λ be the government's discount rate. Thus $\int_0^\infty e^{-\lambda t} k \, u_t \, dt$ represents the cumulative discounted cost associated with the specific and deliberate goal of reducing debt.

There are also costs for having debt. High public debt has negative effects on the economy. Indeed, Blanchard (2009) points out that high public debt means less growth of the capital stock and more tax distortions (for empirical works regarding the negative effects of public debt on the economy, see, for instance, Kumar and Woo 2010). Moreover, it can cause vicious circles and make the fiscal policy extremely difficult. Accordingly, we assume

$$h(y) \;=\; \alpha y^{2n} + \beta,$$

where α is a strictly positive constant, β is a nonnegative constant, and $n \geq 1$ is an integer number. Here n is a subjective parameter that captures the aversion of the policy-makers towards the debt ratio, and β is a scale parameter. On the other hand, the parameter α represents the importance of debt for the government. The stronger the importance, the larger the parameter α.

The expected total cost is given by

$$J(x;u) \;:=\; E_x\left[\int_0^\infty e^{-\lambda t} h(X_t)dt + \int_0^\infty e^{-\lambda t} k\, u_t\, dt\right],$$

where $k \in (0,\infty)$ is the proportional cost of reducing debt and $\lambda \in (0,\infty)$ is the discount rate of the government.

3.1 Definition. *An $\mathbb{F}$-adapted process $u : [0,\infty) \times \Omega \to [0,\infty)$ which satisfies $u(t,\omega) \in [0,\overline{U}]$ is called an admissible stochastic control if $J(x;u) < \infty$. The set of all admissible controls is denoted by $\mathcal{A}(x) = \mathcal{A}$.*

3.2 Problem. *The government wants to select the control $u \in \mathcal{A}$ that minimizes the functional J defined by*

$$J(x;u) \;:=\; E_x\left[\int_0^\infty e^{-\lambda t} h(X_t)dt + \int_0^\infty e^{-\lambda t} k\, u_t\, dt\right].$$

Thus we can think of J as a loss function, a function that the governments wants to minimize.

3.3 Remark. *We observe that for every* $u \in \mathcal{A}$*:*

$$\int_0^\infty e^{-\lambda t} E_x \left[\alpha X_t^{2n} + \beta\right] dt = \int_0^\infty e^{-\lambda t} E_x \left[h(X_t)\right] dt$$
$$= E_x \left[\int_0^\infty e^{-\lambda t} h(X_t) dt\right] < \infty.$$

Thus, for every $u \in \mathcal{A}$*,*

$$\lim_{T\to\infty} E_x \left[e^{-\lambda T} X_T^{2n}\right] = 0. \tag{3.2}$$

We will use condition (3.2) in the proof of Theorem 3.5 below.

As in the case of the unbounded model in Chapter 2, we assume the following condition on the discount rate:

$$\lambda > \sigma^2 n(2n-1) + 2\mu n \tag{3.3}$$

3.2 A verification theorem

We define the value function $V : (0, \infty) \to \mathbb{R}$ by

$$V(x) := \inf_{u\in\mathcal{A}} J(x; u).$$

This represents the smallest cost that can be achieved when the initial debt ratio is x and we consider all the admissible controls.

3.4 Proposition. *The value function is non-negative, increasing and convex. Furthermore,* $V(0+) = \frac{\beta}{\lambda}$*.*

Proof. Similar to the proof of Proposition 2.7. □

Let $\psi : (0, \infty) \to \mathbb{R}$ be a function in $C^2(0, \infty)$. We define the operator $\mathcal{L}$ by

$$\mathcal{L}\psi(x) \ := \ \frac{1}{2}\sigma^2 x^2 \psi''(x) + \mu x \psi'(x) - u\,\psi'(x) - \lambda\psi(x). \tag{3.4}$$

For a function $v : (0, \infty) \to \mathbb{R}$ in $C^2(0, \infty)$, consider the Hamilton-Jacobi-Bellman (HJB) equation

$$\forall x > 0 : \qquad \inf_{0 \le u \le \overline{U}} \Big\{ \mathcal{L}\, v(x) + k\, u \ + h(x) \Big\} = 0\,. \tag{3.5}$$

We next state a sufficient condition for a policy to be optimal.

3.5 Theorem. *Let $v \in C^2(0, \infty)$ be an increasing and convex function such that $v(0+) = \frac{\beta}{\lambda}$. Suppose that v satisfies the HJB equation (3.5) for every $x \in (0, \infty)$, and the polynomial growth condition*

$$v(x) \ \le \ M(1 + x^{2n}), \tag{3.6}$$

for some constant $M > 0$. Then, for every $u \in \mathcal{A}(x)$,

$$v(x) \le J(x; u).$$

Moreover, the control u^v, defined by

$$u^v := arg \inf_{u \in [0, \overline{U}]} \Big\{ \mathcal{L}v(x) + k\, u \ + h(x) \Big\}, \tag{3.7}$$

satisfies

$$v(x) = J(x; u^v).$$

In other words, $\hat{u} := u^v$ is optimal control and $V := v$ is the value function for Problem 3.2.

Proof. Since v is twice continuously differentiable, we may apply Ito's formula to obtain

$$v(x) = e^{-\lambda T} v(X_T) - \int_0^T e^{-\lambda t} \left\{ \mathcal{L} v\big(X_t\big) \right\} dt - \int_0^T e^{-\lambda t} X_t v'(X_t) \sigma dW_t \,,$$

where $\mathcal{L}$ is defined in (3.4). According to the HJB equation (3.5), we have $\mathcal{L}v(x) + k\,u + h(x) \geq 0$. Hence

$$\begin{aligned} v(x) \quad &\leq \quad e^{-\lambda T} v(X_T) + \int_0^T e^{-\lambda t} h(X_t) dt + \int_0^T e^{-\lambda t} k\, u_t dt \\ &\qquad - \int_0^T e^{-\lambda t} X_t v'(X_t) \sigma dW_t \,. \end{aligned} \tag{3.8}$$

Let $0 < x < c < \infty$. We define $\tau_c := \inf\{t \geq 0 : X_t = c\}$. Then, for every $T \geq 0$, we have

$$\begin{aligned} v(x) \quad &\leq \quad e^{-\lambda (T \wedge \tau_c)} v(X_{T \wedge \tau_c}) + \int_0^{T \wedge \tau_c} e^{-\lambda t} h(X_t) dt + \int_0^{T \wedge \tau_c} e^{-\lambda t} k\, u_t dt \\ &\qquad - \int_0^{T \wedge \tau_c} e^{-\lambda t} X_t v'(X_t) \sigma dW_t \,. \end{aligned}$$

Taking conditional expectation

$$\begin{aligned} v(x) \quad &\leq \quad E_x \left[e^{-\lambda (T \wedge \tau_c)} v(X_{T \wedge \tau_c}) \right] + E_x \left[\int_0^{T \wedge \tau_c} e^{-\lambda t} h(X_t) dt \right] \\ &\qquad + E_x \left[\int_0^{T \wedge \tau_c} e^{-\lambda t} k\, u_t dt \right] - E_x \left[\int_0^{T \wedge \tau_c} e^{-\lambda t} X_t v'(X_t) \sigma dW_t \right] . \end{aligned} \tag{3.9}$$

We recall that $v \in C^2(0, \infty)$. Let $\eta > 0$ be an upper bound for $v'(X_t)$ for all $t \in [0, T \wedge \tau_c]$. Thus

$$E_x \left[\int_0^{T \wedge \tau_c} \left(e^{-\lambda t} X_t v'(X_t) \sigma \right)^2 dt \right] \leq c^2 \eta^2 \sigma^2 \int_0^T e^{-2\lambda t} dt < \infty.$$

Consequently, the above stochastic integral

$$\int_0^{T \wedge \tau_c} e^{-\lambda t} X_t v'(X_t) \sigma dW_t$$

is a square integrable martingale, and hence has expected value equal to zero.

Since v is continuous

$$\lim_{c\uparrow\infty} e^{-\lambda(T\wedge\tau_c)}v(X_{T\wedge\tau_c}) = e^{-\lambda T}v(X_T)\,, \quad P-a.s.$$

Hence

$$\lim_{c\uparrow\infty} E_x\left[e^{-\lambda(T\wedge\tau_c)}v(X_{T\wedge\tau_c})\right] = E_x\left[e^{-\lambda T}v(X_T)\right]\,.$$

On the other hand, by the Monotone Convergence Theorem,

$$\lim_{c\uparrow\infty} E_x\left[\int_0^{T\wedge\tau_c} e^{-\lambda t}h(X_t)dt\right] = E_x\left[\int_0^T e^{-\lambda t}h(X_t)dt\right]$$

and

$$\lim_{c\uparrow\infty} E_x\left[\int_0^{T\wedge\tau_c} e^{-\lambda t}k\,u_t dt\right] = E_x\left[\int_0^T e^{-\lambda t}k\,u_t dt\right]\,.$$

Taking the limit as $c\uparrow\infty$ in (3.9), we obtain

$$\begin{aligned} v(x) \;\; &\leq \;\; E_x\left[e^{-\lambda T}v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t}h(X_t)dt\right] \\ &\quad + E_x\left[\int_0^T e^{-\lambda t}k\,u_t dt\right]. \end{aligned} \tag{3.10}$$

Suppose $u \in \mathcal{A}$. In view of (3.2),

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T}(1+X_T^{2n})\right] = 0,$$

and the polynomial growth condition (3.6), we get

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T}v(X_T)\right] = 0.$$

Taking limit as $T\to\infty$ in (3.10), by the Monotone Convergence Theorem, we obtain the desired result, namely

$$v(x) \leq E_x \left[\int_0^\infty e^{-\lambda t} h(X_t) dt + \int_0^\infty e^{-\lambda t} k \, u_t \, dt \right] \; = \; J(x; u). \tag{3.11}$$

This proves the first part of this theorem.

Now we consider the second part of the theorem. Let X^v stand for the process generated by u^v. Since $u := u^v$ satisfies $\mathcal{L}v(x) + ku + h(x) = 0$, the inequality (3.10) becomes an equality:

$$v(x) \quad = \quad E_x \left[e^{-\lambda T} v(X_T^v) \right] + E_x \left[\int_0^T e^{-\lambda t} h(X_t^v) dt \right] + E_x \left[\int_0^T e^{-\lambda t} k \, u_t^v dt \right]. \tag{3.12}$$

We note that

$$E_x \left[\int_0^T e^{-\lambda t} h(X_t^v) dt \right] \; \leq \; v(x), \quad \forall T > 0.$$

Since u^v is bounded, we conclude that u^v is admissible. Hence, for $X = X^v$ and $u = u^v$, the inequality in (3.11) turns out to be an equality. This completes the proof. □

3.3 The explicit solution

We start this section constructing a candidate for solution to Problem 3.2. Next we verify that such candidate is indeed solution. Then we describe how the optimal debt policy works. We complete this section with a numerical illustration of the solution.

Construction of the solution

Our goal is to find a function that satisfies the conditions of Theorem 3.5. We note that the HJB equation (3.5) can be expressed equivalently as

$$\left\{\frac{1}{2}\sigma^2\, x^2\, v''(x) + \mu\, x\, v'(x) - \lambda\, v(x) + \inf_{u\,\in[0,\,\overline{U}]}\left[\Big(k - v'(x)\Big)\, u\right] + h(x)\right\} = 0. \tag{3.13}$$

By (3.7) in Theorem 3.5, our candidate for optimal control $\hat{u}$ has the following form:

$$\hat{u}(t) := \arg\inf_{u\in[0,\overline{U}]}\left[\Big(k - v'(X_t)\Big)\, u(t)\right] = \begin{cases} 0 & \text{if } v'(X_t) < k \\ \overline{U} & \text{if } v'(X_t) \geq k\ . \end{cases} \tag{3.14}$$

Consequently, solving the HJB equation (3.5) is also equivalent to solving

$$\frac{1}{2}\sigma^2\, x^2\, v''(x) + \mu\, x\, v'(x) - \lambda\, v(x) + h(x) = 0 \tag{3.15}$$

for $v'(x) < k$, and

$$\frac{1}{2}\sigma^2\, x^2\, v''(x) + \mu\, x\, v'(x) - \lambda\, v(x) + \Big(k - v'(x)\Big)\overline{U}\ + h(x) = 0 \tag{3.16}$$

for $v'(x) \geq k$. Thus a solution v of the HJB equation (3.5) defines the regions $\mathcal{C} = \mathcal{C}^v$ and $\Sigma = \Sigma^v$ by

$$\mathcal{C} := \left\{x > 0 : \frac{1}{2}\sigma^2\, x^2\, v''(x) + \mu\, x\, v'(x) - \lambda\, v(x) + h(x) = 0,\ v'(x) < k\right\}, \tag{3.17}$$

$$\Sigma := \left\{x > 0 : \frac{1}{2}\sigma^2\, x^2\, v''(x) + \mu\, x\, v'(x) - \lambda\, v(x) + \Big(k - v'(x)\Big)\overline{U}\ + h(x) = 0,\right.$$
$$\left. v'(x) \geq k\right\}. \tag{3.18}$$

We observe that the process $\hat{u}$ takes the value zero on $\mathcal{C}$, whereas it takes the value $\overline{U}$ on Σ. We conjecture that there exists a threshold $b \in (0, \infty)$ such that the government should intervene with $u = \overline{U}$ when the debt ratio $X \geq b$, and should not intervene when the debt ratio $X < b$. Accordingly, if v satisfies the HJB equation (3.5), we will call $\mathcal{C} = (0, b)$ the continuation region and $\Sigma = [b, \infty)$ the intervention region. Thus, it is natural to define the debt ratio ceiling as follows.

3.6 Definition. *Let v be a function that satisfies the HJB equation (3.5), and $\mathcal{C}$ the corresponding continuation region. If $\mathcal{C} \neq \emptyset$, the debt ratio ceiling b is defined by*

$$b \ := \ \sup\{x \in (0\,, \infty) \,|\, x \in \mathcal{C}\}.$$

Moreover, if v is equal to the value function, then b is said to be the optimal debt ceiling.

Thus, to obtain the optimal debt ceiling, we need to find the value function. To that end, we need some notation. Let ${}_1F_1$ denote the hypergeometric function defined by

$$ {}_1F_1(\theta, \eta, z) := 1 + \sum_{n=1}^{\infty} \frac{(\theta, n)}{(\eta, n)\, n!} z^n \,, \tag{3.19}$$

where

$$(a, n) := a(a+1)(a+2)\cdots(a+n-1) \quad \forall n \in \mathbb{N}.$$

The general solution of the HJB equation (3.5) has two parts depending on whether we consider the continuation region $\mathcal{C}$ or the intervention region Σ :

$$v(x) = \begin{cases} A_1\, x^{\gamma_1} + A_2\, x^{\gamma_2} + \alpha\, \zeta\, x^{2n} + \frac{\beta}{\lambda} & \text{if } x \in \mathcal{C} = (0, b) \\ f(x) & \text{if } x \in \Sigma = [b, \infty), \end{cases}$$

with

$$f(x) := \sum_{j=0}^{2n} \zeta_j \, x^j + B_1 \, x^{\gamma_2} \left(\frac{\sigma^2}{2\overline{U}} \right)^{\gamma_2} {}_1F_1 \left(-\gamma_2, c_2, -\frac{2\overline{U}}{\sigma^2 \, x} \right)$$

$$+ B_2 \left(\frac{2\overline{U}}{\sigma^2 \, x} \right)^{c_3} {}_1F_1 \left(c_3, 2 - c_2, -\frac{2\overline{U}}{\sigma^2 \, x} \right), \tag{3.20}$$

where A_1, A_2, B_1 and B_2 are constants to be found. Furthermore,

$$\tilde{\mu} := \mu - \frac{1}{2}\sigma^2 , \tag{3.21}$$

$$\gamma_1 := \frac{-\tilde{\mu} - \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} < 0, \tag{3.22}$$

$$\gamma_2 := \frac{-\tilde{\mu} + \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} > 0, \tag{3.23}$$

$$\zeta := \frac{1}{\lambda - \sigma^2 n(2n-1) - 2\mu n} > 0, \tag{3.24}$$

$$c_2 := 2 \left(1 - \gamma_2 - \frac{\mu}{\sigma^2} \right), \tag{3.25}$$

$$c_3 := \gamma_2 + 2\frac{\tilde{\mu}}{\sigma^2}, \tag{3.26}$$

$$\zeta_j := -\binom{2n}{j} \frac{\alpha \, (2n-j)! \, \overline{U}^{2n-j}}{\prod_{i=j}^{2n} \left(i\mu + i(i-1)\sigma^2/2 - \lambda \right)}, \quad \forall j \in \{2, 3, \ldots, 2n\}, \tag{3.27}$$

$$\zeta_1 := \frac{2\overline{U}}{\mu - \lambda} \zeta_2, \tag{3.28}$$

$$\zeta_0 := \frac{\beta + k \, \overline{U}}{\lambda} - \frac{\overline{U}}{\lambda} \zeta_1. \tag{3.29}$$

3.7 Remark. *From definition of ${}_1F_1$, we note that*

$$\lim_{x \to \infty} {}_1F_1 \left(\cdot \, , \cdot \, , -\frac{2\overline{U}}{\sigma^2 \, x} \right) = 1.$$

For references on hypergeometric functions and their relation to second order differential equations, see, for example, Bell (2004) and Kristensson (2010).

To guarantee the existence of the parameters $\{\zeta_j : j = 2, 3, \cdots, 2n\}$, in addition to (3.3), we need to assume

$$\lambda \neq j(j-1)\frac{\sigma^2}{2} + j\mu, \quad \forall j \in \{2, 3, \cdots, 2n-1\}. \tag{3.30}$$

For future reference, we state conditions that the parameters satisfy.

3.8 Lemma. *The following results are valid:*

(i) $\zeta = \zeta_{2n}/\alpha > 0,$

(ii) $\lambda > \mu,$

(iii) $\gamma_2 > 2n,$

(iv) $c_3 > 0.$

Proof. Part (i) follows immediately from (3.3). Part (iv) follows directly from the definition of γ_2, given in (3.23). For the other parts, see Lemma 2.13. □

We recall that Proposition 3.4 states $V(0+) = \frac{\beta}{\lambda}$. According to Theorem 3.5, it is required that the value function satisfies the polynomial growth condition (3.6). Furthermore, we conjecture that v is twice continuously differentiable. Then the five constants A_1, A_2, B_1, B_2 and b are found from the following five conditions:

$$v(0+) = \frac{\beta}{\lambda}, \tag{3.31}$$

$$v(x) \leq M(1 + x^{2n}), \tag{3.32}$$

$$v(b+) = v(b-), \tag{3.33}$$

$$v'(b+) = v'(b-), \tag{3.34}$$

$$v''(b+) = v''(b-). \tag{3.35}$$

Since $\gamma_1 < 0$, condition (3.31) implies $A_1 = 0$ in the equation for v. Moreover, in the lemma below we show that $B_1 = 0$.

3.9 Lemma. *Suppose f, defined in (3.20), is non-negative and satisfies the polynomial growth condition*

$$f(x) \leq M\,(1 + x^{2n}), \tag{3.32}$$

for some $M > 0$. Then $B_1 = 0$.

Proof. See Appendix B.1. □

Thus the candidate for value function is given by

$$v(x) = \begin{cases} A_2\, x^{\gamma_2} + \alpha\, \zeta\, x^{2n} + \frac{\beta}{\lambda} & \text{if } x \in \mathcal{C} = (0, b) \\ f_1(x) & \text{if } x \in \Sigma = [b, \infty), \end{cases} \tag{3.36}$$

with

$$f_1(x) \;\; := \;\; \sum_{j=0}^{2n} \zeta_j\, x^j \;+\; B_2 \left(\frac{2\overline{U}}{\sigma^2\, x}\right)^{c_3} {}_1F_1\Big(c_3, 2 - c_2, -\frac{2\overline{U}}{\sigma^2\, x}\Big),$$

where the remaining three parameters A_2, B_2 and b are found by solving the system of non-linear equations (3.33)-(3.35).

Summarizing, the candidate for value function is given by (3.36), and the candidate for optimal control is then determined by (3.14). Moreover, the candidate for optimal debt ceiling is b. Naturally, we expect b to depend on the underlying parameters of the model $(\mu, \sigma, \lambda, k, \alpha, \beta, n, \overline{U})$.

Verification of the solution

In this subsection, we are going to prove rigorously that the above candidate for optimal control is indeed the optimal control, and the above candidate for value function is indeed the value function.

3.10 Theorem. *Let A_2, B_2 and b be solution of the system of equations (3.33)-(3.35) such that $0 < b < \infty$. Suppose that*

$$A_2 \gamma_2 b^{\gamma_2 - 1} + 2n\alpha\zeta b^{2n-1} = k. \tag{3.37}$$

Let us define the function $V : (0, \infty) \to [0, \infty)$ by

$$V(x) = v(x) = \begin{cases} A_2\, x^{\gamma_2} + \alpha\, \zeta\, x^{2n} + \frac{\beta}{\lambda} & \text{if } x \in (0, b) \\ f_1(x) & \text{if } x \in [b, \infty), \end{cases} \tag{3.38}$$

with

$$f_1(x) \;\; := \;\; \sum_{j=0}^{2n} \zeta_j\, x^j \; + \; B_2 \left(\frac{2\overline{U}}{\sigma^2\, x} \right)^{c_3} {}_1F_1\Big(c_3, 2 - c_2, -\frac{2\overline{U}}{\sigma^2\, x} \Big) \, .$$

Furthermore, let us define the process $\hat{u}$ by

$$\hat{u}(t) := \begin{cases} 0, & \text{if } X_t < b \\ \overline{U}, & \text{if } X_t \geq b \ . \end{cases} \tag{3.39}$$

If

$$\forall x \in (0, \infty) : \quad V''(x) \; > \; 0, \tag{3.40}$$

then V is the value function, $\hat{u}$ is optimal debt policy, and b is the optimal debt ceiling.

Proof. To prove this theorem, it suffices to show that all conditions of Theorem 3.5 are satisfied. We observe that $V(0+) = \beta/\lambda$. By construction, we get immediately that $V \in C^2(0, \infty)$. In view of (3.40), V is strictly convex. Thus $V'(x)$ is strictly increasing. Next we show $V'(0+) = 0$. We note that, for every $x \in (0, b)$,

$$V'(x) = A_2 \gamma_2 x^{\gamma_2 - 1} + 2n\alpha\zeta x^{2n-1}.$$

It follows that $V'(0+) = 0$. Consequently, $V'(x) > 0$ for every $x > 0$. Hence V is strictly increasing.

Now we show that V satisfies the HJB equation (3.5). We observe that condition (3.37) is equivalent to $V'(b) = k$. Since $V'(x) > 0$ and $V''(x) > 0$ on $(0, \infty)$, we conclude that: $0 < V'(x) < k$ for every $x \in (0, b)$, and $V'(x) > k$ for every $x \in (b, \infty)$. By construction, V satisfies (3.15) on $(0, b)$, and (3.16) on (b, ∞). Hence V satisfies the HJB equation (3.5) for every $x \in (0, \infty)$. This also shows that indeed $\mathcal{C} = (0, b)$ and $\Sigma = [b, \infty)$. Regarding the value function, it remains to verify that V satisfies the polynomial growth condition (3.32). Since

$$\lim_{x \to \infty} {}_1F_1\left(c_3, 2 - c_2, -\frac{2\overline{U}}{\sigma^2 x}\right) = 1.$$

and $c_3 > 0$ (by Lemma 3.8), we have

$$\lim_{x \to \infty} B_2 \left(\frac{2\overline{U}}{\sigma^2 x}\right)^{c_3} {}_1F_1\left(c_3, 2 - c_2, -\frac{2\overline{U}}{\sigma^2 x}\right) = 0.$$

This implies that V is bounded above by a polynomial of degree $2n$ on the interval $[b, \infty)$. On the other hand, the function V is bounded above by $|A_2| b^{\gamma_2} + \alpha \xi b^{2n} + \beta / \lambda$ on the interval $(0, b)$. Hence V satisfies the polynomial growth condition (3.32). Consequently, by Theorem 3.5, V is the value function. By construction, Theorem 3.5 also implies that $\hat{u}$ is optimal debt policy. By Definition 3.6, the constant b is the optimal debt ceiling. This completes the proof of this theorem. $\square$

How the optimal debt policy works

Now we provide an explanation of how the optimal debt control process $\hat{u}$ works. When the actual debt ratio $\hat{X}_t$ is below the optimal debt ceiling b, the definition of optimal debt control given in equation (3.39) states that $\hat{u}(t) = 0$. That is, there is no need to generate primary surpluses in order to reduce the debt ratio. When the debt ratio reaches b and tries to cross it, the optimal control states that $\hat{u}(t) = \overline{U}$.

In other words, the government has to intervene to reduce the debt ratio with the maximum rate allowed $\overline{U}$.

Let us analyze the implications of the optimal debt policy described above.

Suppose that the actual debt ratio is below the optimal debt ceiling b. We notice that the government might or might not succeed in preventing the debt ratio from crossing b, the result depends on the value of the underlying parameters of the model, in particular on $\overline{U}$. Specifically, the higher the upper bound $\overline{U}$, the more likely the controlled debt ratio will remain below the optimal debt ceiling b (see Section 3.4). This results differs from the unbounded case presented in Chapter 2 in which the optimal policy implies that once the actual debt ratio is below or equal to its debt ceiling it will remain there all the time.

On the other hand, if the initial debt ratio x is strictly greater than the optimal debt ceiling b, the optimal debt policy in this chapter states that the government should reduce the debt ratio with the maximal rate $\overline{U}$. Certainly, there is no guarantee that the resulting debt ratio will equal the debt ceiling b immediately, it may take some time to accomplish that goal. Indeed, in Section 3.4 we present an example in which the expected time to reach the optimal debt ceiling is strictly positive. However, under a situation in which the government constraint is too severe, there is a positive probability that a country will never be able to reach the optimal debt ceiling (see Section 3.4). In contrast, in the unbounded model of Chapter 2, the optimal debt ceiling is reached immediately.

Illustration of the solution

According to Theorem 3.10, the solution of the problem involves the numerical solution of the system of three equations (3.33)-(3.35) with three unknowns: A_2, B_2 and b. Now we present an example to illustrate the solution.

3.11 Example. *Let us consider the parameter values:*

$$\mu = 0.05,\ \sigma = 0.05,\ \lambda = 0.7,\ k = 1,\ \overline{U} = 0.05,\ n = 2,\ \alpha = 1,\ \beta = 0.$$

Solving numerically the equations (3.33)-(3.35), we obtain

$$A_2 = -14.219, \quad B_2 = -5.09042 \times 10^{-75}, \textit{ and } \quad b = 54.2756\%.$$

One can verify that condition (3.37) is satisfied.

The graph of the corresponding value function and its first and second derivatives are shown in Figure **??**. We observe that the value function is increasing, convex and twice continuously differentiable. In particular, condition (3.40) is satisfied.

The optimal debt ratio ceiling is 54.2756%. Accordingly, our optimal debt policy takes the form

$$\hat{u}(t) := \begin{cases} 0, & \text{if } X_t < 54.2756\% \\ 0.05, & \text{if } X_t \geq 54.2756\% \ . \end{cases} \tag{3.41}$$

That is, if the actual debt ratio is below the level 54.2756%, the government should not intervene. Otherwise, the government should intervene with the maximal allowed rate of 0.05. If we measure the time in years, it means that the government is allowed to reduce the government debt ratio up to 5% during a whole year. Given such constraint, there is no guarantee that the resulting debt ratio after intervention of the government will remain below the optimal debt ceiling 54.2756%; it may be the case that sometimes the debt ratio is above that ceiling.

3.12 Remark. *We conjecture that conditions (3.37) and (3.40) in Theorem 3.10 are satisfied for every solution of the system of equations (3.33)-(3.35), provided that conditions (3.3) and (3.30) hold. Certainly, those conditions are verified not only for the example we present in this subsection, but also for all the numerical results*

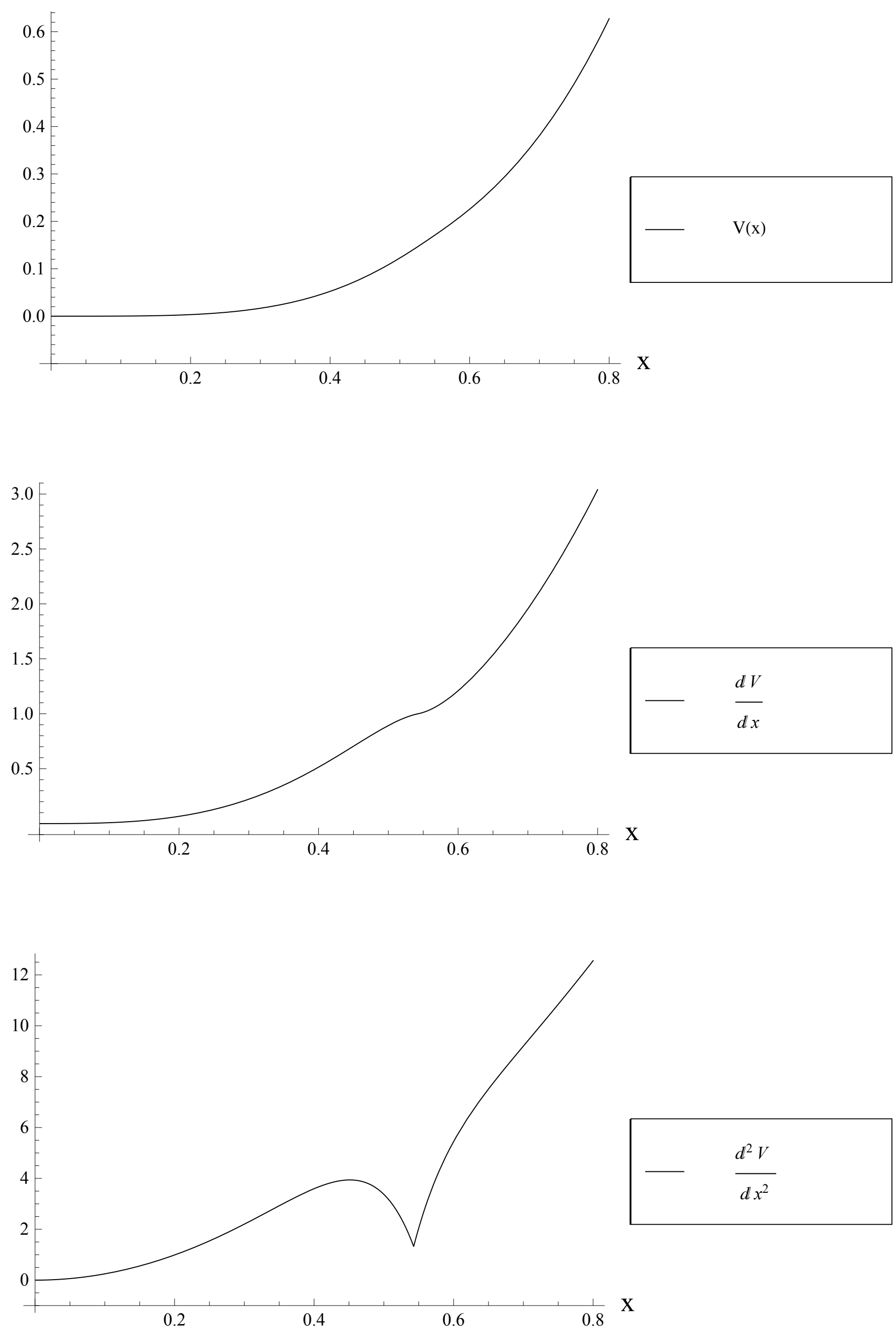

Figure 3.1: The value function V, and its first and second derivatives

presented in Section 3.4. However, since there is not an explicit solution of the system (3.33)-(3.35), we are not able to prove such conjecture.

3.4 Economic analysis

As we mentioned in the previous section, we can compute numerically the optimal debt ceiling for any set of parameter values $(\mu, \sigma, \lambda, k, \alpha, \beta, n, \overline{U})$. In this section we consider different parameter values in order to analyze the optimal debt policy. Specifically, we are going to make the following four analyzes:

1. Compare the results of the optimal debt policy with the policy of non-government intervention.
2. Compare the results of the optimal debt policy presented in this chapter with the policy derived in the unbounded model presented in Chapter 2.
3. Study the time to reach the optimal debt ceiling, assuming that the initial debt ratio is above the debt ceiling.
4. Analyze the effects of some parameters on the optimal debt ceiling.

Unless otherwise stated, the values of the parameters we will use in this section are given in the following example.

3.13 Example. *Let us consider the parameter values:*

$$\mu = 0.05,\ \sigma = 0.05,\ \lambda = 0.7,\ k = 1,\ \overline{U} = 0.01,\ n = 1,\ \alpha = 1,\ \beta = 0.$$

Solving numerically the equations (3.33)-(3.35), we obtain

$$A_2 = -510.12, \quad B_2 = -7.71446 \times 10^{-67}, \text{ and } \quad b = 31.04266\%.$$

The optimal debt policy versus non-intervention policy

We may wonder whether the results of the government intervention by means of the optimal debt policy is better than the case of non-intervention at all. The latter can be modelled as $\overline{U} = 0$, which implies that $u \equiv 0$, that is, $u(t) = 0$ for all $t \geq 0$. If the government never intervenes, then

$$X_t \;=\; x \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right\}.$$

From Example 2.3, the total cost function is

$$J(x;0) = \alpha\zeta x^{2n} + \frac{\beta}{\lambda}.$$

We recall that the value function V represents the minimum total cost, and is a function of the initial debt ratio. Thus the cost $J(x;0)$ should be greater than or equal to $V(x)$. Indeed, using the parameter values of Example 3.13, with $\overline{U} = 0.05$, in Figure 3.2 we plot these functions. We observe that the expected relation is satisfied. Hence it is better to intervene optimally than do not intervene at all. We point out that the benefits of intervention are bigger, the larger the initial debt ratio is.

The bounded model versus the unbounded model

Now we compare the results of this chapter to the ones obtained in Chapter 2.

In Figure 3.3 we compare the value function, given in equation (3.38), to the one of the unbounded model, given by equation (2.36). We observe that, as expected, the value function of the bounded case is above the value function of the unbounded case. We also note that the higher the initial debt ratio, the larger the difference between the two value functions. This means that for countries that have low levels of debt ratios, there is no significant difference in terms of total expected

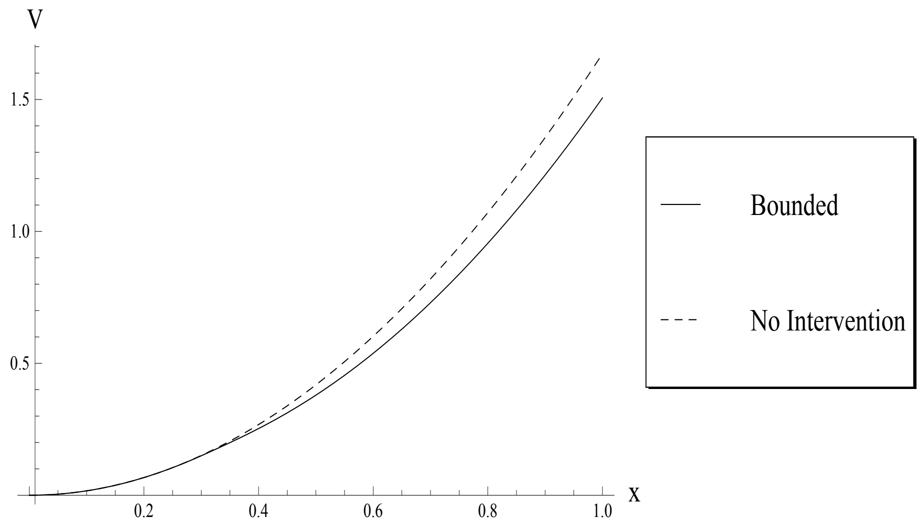

Figure 3.2: The optimal bounded debt policy and non-intervention policy

cost of having constraints in the intervention. For countries with high debt ratios, however, there is a significant difference in terms of the expected total cost.

In Table 3.1 we study the effects of changes in the maximal rate of intervention $\overline{U}$ on the optimal debt ceiling. We notice that the optimal debt ceiling in the bounded model is always smaller than the one in the unbounded case $\overline{U} = \infty$, given by equation (2.37). Moreover, we observe that as the maximal rate increases the optimal debt ceiling in the bounded model increases towards the optimal debt ceiling in the unbounded model. Furthermore, other things being equal, we conclude that countries with more constraints to control its debt ratio should have a lower optimal debt ceiling.

Time to reach the optimal debt ceiling

In this subsection, we consider a country whose initial debt ratio is larger than their corresponding optimal debt ceiling. We study the time to reach the optimal debt ceiling considering that the government follows the optimal debt policy $\hat{u}$.

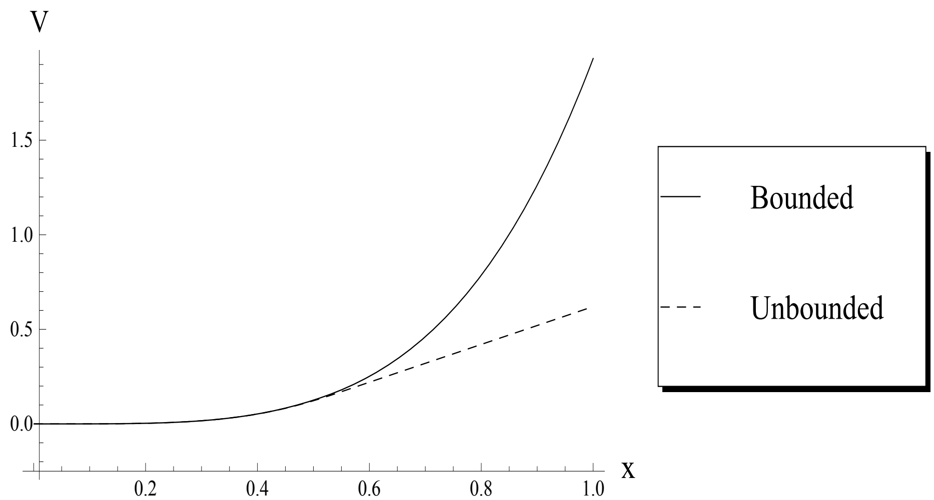

Figure 3.3: The value function for the bounded and unbounded cases

Table 3.1: Effect of $\overline{U}$ on the optimal debt ceiling

	$\overline{U} = 0.001$	$\overline{U} = 0.01$	$\overline{U} = 1$	$\overline{U} = 5$	$\overline{U} = \infty$
b	0.300024	0.310426	0.331213	0.331325	0.331352

The other parameters used in these computations come from Example 3.13. The unbounded case $\overline{U} = \infty$ is calculated using the explicit formula (2.37).

Specifically, we assume that $X(0) = x > b$ and define the stopping time

$$\begin{aligned}\tau &:= \inf\left\{t \in (0,\infty) : \hat{X}(t) \le b\right\}\\ &= \inf\left\{t \in (0,\infty) : x + \int_0^t \mu \hat{X}_s ds + \int_0^t \sigma \hat{X}_s\, dW_s - \int_0^t \hat{u}_s\, ds\ \le b\right\}.\end{aligned}$$

We recall that $\tilde{\mu} := \mu - \frac{1}{2}\sigma^2$. We need to distinguish two cases: $\overline{U} < \tilde{\mu}$ and $\overline{U} \ge \tilde{\mu}$.

3.14 Proposition. *Suppose* $(\log x - b) > 0$. *The following assertions are valid:*

(a) If $\overline{U} < \tilde{\mu}$, *then the distribution function of* τ *is defective. This means that* $P\{\tau < \infty\} < 1$, *or equivalently* $P\{\tau = \infty\} > 0$. *Hence* $E[\tau] = \infty$. *Furthermore,*

the probability that the debt ceiling will eventually be reached is bounded above by

$$P\{\tau < \infty\} \leq \exp\left\{\frac{2(\overline{U}-\tilde{\mu})}{\sigma^2}\Big(\log x - b\Big)\right\}.$$

(b) If $\overline{U} \geq \tilde{\mu}$*, then* $P\{\tau < \infty\} = 1$*. Furthermore, for* $\overline{U} > \tilde{\mu}$*, the expected value of* τ *is bounded below by*

$$0 \, < \, \left(\frac{\sigma}{\overline{U}-\tilde{\mu}}\right)\Big(\log x - b\Big) \, \leq \, E[\tau].$$

In addition, for $\overline{U} = \tilde{\mu}$*, we have* $E[\tau] = \infty$*.*

Proof. See Appendix B.2. □

Suppose that the initial debt ratio x satisfies $\log x > b$. Among other results, Proposition 3.14 above states that if the maximal rate of intervention $\overline{U}$ is big enough ($\overline{U} \geq \tilde{\mu}$) a government will succeed in reducing the debt ratio and reaching the optimal debt ceiling in a finite period of time with probability one. For the case $\overline{U} > \tilde{\mu}$, below we present an example to illustrate the value of $E[\tau]$.

3.15 Example. *Let us consider the parameter values:*

$$\mu = 0.03, \ \sigma = 0.05, \ \lambda = 0.7, \ k = 1, \ \overline{U} = 0.7, \ n = 1, \ \alpha = 1, \ \beta = 0.$$

Solving numerically the equations (3.33)-(3.35), we get

$$A_2 = -13449.1, \quad B_2 = -3.66206 \times 10^{-57}, \ \textit{and} \ \ b = 34.34186\%.$$

We obtain $E[\tau] = 1.6471$, with the 95% confidence interval $[1.6442, 1.6500]$. Here we have performed Monte Carlo simulations with $10,000$ sample paths and time steps equal to 0.001. The initial debt ratio we have considered is $X_0 = 1.41$. For a reference on Monte Carlo simulations see, for instance, Brandimarte (2002).

Now let us consider the opposite situation in which the maximal rate of intervention $\overline{U}$ is not big enough ($\overline{U} < \tilde{\mu}$). Proposition 3.14 states the there is a positive probability that the government does not reach the optimal debt ceiling b, even in the long run. As a result, the expected value of reaching the optimal debt ceiling in a finite period of time is infinite.

The effect of α, g, and σ on the optimal debt ceiling

We recall that α represents the importance of government debt. In Table 3.2 we observe that the more important the government debt, the lower the optimal debt ceiling. In other words, the more concerned is the government about its debt, the more control should be exerted. We note that this result holds for every value of the maximal rate of intervention $\overline{U}$.

Table 3.2: Effect of α, μ and σ on the optimal debt ceiling b

	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.3$
$\overline{U} = 0.01$	0.609820	0.310426	0.241035
$\overline{U} = 2.00$	0.662425	0.331283	0.254845
$\overline{U} = \infty$	0.662704	0.331352	0.254886
	$\mu = 0.5$	$\mu = 0.10$	$\mu = 0.14$
$\overline{U} = 0.01$	0.310426	0.26399	0.25309
$\overline{U} = 2.00$	0.331283	0.303408	0.282346
$\overline{U} = \infty$	0.331352	0.303466	0.282397
	$\sigma = 0.05$	$\sigma = 0.13$	$\sigma = 0.17$
$\overline{U} = 0.01$	0.310426	0.301363	0.295110
$\overline{U} = 2.00$	0.331283	0.349697	0.359007
$\overline{U} = \infty$	0.331352	0.350220	0.359954

The other parameters used in these computations come from Example 3.13. The unbounded case $\overline{U} = \infty$ is calculated using the explicit formula (2.37).

To analyze the effect of the rate of economic growth we recall that $\mu := r - g$.

In Table 3.2 we observe that the larger the rate of economic growth the bigger the optimal debt ceiling. In other words, countries with high economic growth are allowed to have a high debt ceiling, because the economic growth reduces the debt ratio. We point out that this results holds regardless of the value of the maximal rate of intervention $\overline{U}$.

In Table 3.2 we also show that the higher the debt volatility the larger the debt ceiling, for every level of maximal rate of intervention $\overline{U}$. Thus, for example, countries with a large proportion of their debt issue in foreign currencies should have a high debt ceiling.

We have also studied the effects of those parameters on the value function. We find that an increment in the importance of the debt α for the government, or in the debt volatility σ, implies an increase in the value function, thereby generating a bad result for the government. By contrast, an increase in the rate of economic growth, reduces the value function and, hence, improves the government welfare.

We would like to point out that all the qualitative results of this subsection coincide with the ones found in Chapter 2, the unbounded case $\overline{U} = \infty$ (see Table 3.2).

Summary of analysis

We have shown that the optimal debt policy under bounded intervention is better than the policy of non-intervention at all. On the other hand, as expected, the optimal debt policy under bounded intervention generates a worse result than the optimal policy in the unbounded case studied in Chapter 2.

In addition, we have studied the time to reach the optimal debt ceiling, when the initial debt ratio is above it. We have found that countries with strong constraints (low maximal rate of intervention $\overline{U}$) may not be able to reduce their debt debt ratio and reach the optimal debt ceiling. On the contrary, for countries with less constraints we have estimated (via Monte Carlo simulations) the finite expected time of reaching the optimal debt ceiling. Those results imply that the governments that succeed in reducing their debt ratio to the debt ceiling level do not do so immediately, but over some period of time.

Furthermore, we have analyzed the effects of some parameters on the optimal debt ceiling. We have shown that when the importance of debt decreases or the volatility increases, the optimal debt ceiling increases as well. The higher the rate of economic growth generates, the larger the optimal debt ceiling. The qualitative results of these findings coincide with the ones found in Chapter 2. However, the quantitative results differ. For example, the optimal debt ceiling in this chapter is smaller than the optimal debt ceiling in the unbounded case. Another important difference is that the optimal debt policies differ and, hence, produce different results. Suppose the initial debt ratio of a country is below the optimal debt ceiling. Then, as pointed out in Subsection 3.3, the controlled debt ratio may be sometimes above its corresponding optimal deb ceiling; whereas in the unbounded model, it will always be equal or less than the optimal debt ceiling (except perhaps at time zero).

3.5 Concluding remarks

In contrast to Chapter 2, in this chapter we consider that the ability of the government to produce primary surpluses to reduce its debt ratio is bounded.

In such theoretical framework, we solve the debt control problem and we are able to find an explicit solution for the optimal debt ceiling, which depends on key macro-financial variables, such as the interest rate on debt, debt volatility, and the rate of economic growth. Although, we do not obtain a formula for the optimal debt ceiling as in Chapter 2, we are still able to perform the corresponding economic analysis. Specifically, we show that when the importance of debt decreases or the volatility increases, the optimal debt ceiling increases. An increment in the rate of economic growth generates an increment in the optimal debt ceiling. These qualitative results coincide with the ones found in Chapter 2. However, the quantitative results differ. For example, the optimal debt ceiling in this chapter is smaller than the optimal debt ceiling in the unbounded case.

Moreover, we have found that the optimal debt control policy has a different impact on the debt ratio than the one in Chapter 2. Here it works as follows. If the actual debt ratio of a country is below the optimal debt ceiling, then it is optimal for the government not to intervene. Otherwise, the government should intervene at the maximal rate to reduce the debt ratio. Given the constraint on the generation of primary surpluses, the controlled debt ratio may be sometimes above the optimal debt ceiling. This result differs from the one obtained in the unbounded model, in which the debt ratio will always be equal or less than the optimal debt ceiling (except perhaps at time zero).

In particular, the optimal debt policy described above implies the following. If the initial debt ratio is strictly greater than the optimal debt ceiling, the optimal debt policy states that the government should reduce the debt ratio with the maximal rate. Certainly, there is no guarantee that the resulting debt ratio will equal the debt ceiling b immediately, it may take some time to accomplish that goal. Furthermore, we find, under some condition on the parameters, there is a positive probability that a country will never reach the optimal debt ceiling (see Section 3.4). In contrast, in the unbounded model of Chapter 2, an immediate reduction will take place to reach the optimal debt ceiling.

To summarize, the main contribution of this chapter is that we present for the first time a model that not only allows us to compute analytically the optimal debt ceiling (as a function of macro-financial variables), but also accounts for the constraints that governments face in reducing their debt ratio. We have shown that this constraint plays a key role in explaining, for instance, why some countries may fail to reduce their debt ratio to the level of the optimal debt ceiling in the very short term, or even in the long run.

Chapter 4

The optimal debt ceiling: unbounded intervention with non-constant parameters

In Chapter 2 we have solved the debt control problem, and obtained an explicit formula for the optimal debt ceiling, under the assumption that both the economic growth g and the interest rate r are constant parameters. In this chapter we generalize that model. We consider that the debt ratio X affects both the rate of interest r and the rate of economic growth g. The former is motivated by the fact that countries with higher debt are considered riskier and, as a result, the debt buyers charge a higher interest rate. The motivations for the latter are the changes on the productivity and liquidity of the economy that debt generates that, in turn, have an impact on the economic growth.

4.1 The model

Instead of considering g and r as constant parameters as in Chapter 2, now we assume that they are functions of the debt ratio X; namely

$$g(X) = g_0 + g_1 X,$$
$$r(X) = r_0 + r_1 X,$$

where $g_0 \in \mathbb{R}$, $g_1 \in \mathbb{R}$, $r_0 \in [0, \infty)$ and $r_1 \in [0, \infty)$ constants. That is, the bigger the debt ratio the larger the interest rate. On the other hand, when the debt ratio goes up the rate of economic growth goes up or down, depending on the sign of g_1. The specific values of these parameters depend on the characteristics of the country.

With the above considerations, we extend the debt dynamics in (2.2) to

$$X_t \;=\; x + \int_0^t (\rho X_s + \mu) X_s ds + \int_0^t \sigma X_s \, dW_s - Z_t, \tag{4.1}$$

where $\rho = r_1 - g_1 \in \mathbb{R}$, $\mu = r_0 - g_0 \in \mathbb{R}$ and $\sigma \in (0, \infty)$ are constants.

We note that if $\rho = 0$ we recover the debt dynamics given in equation (2.2). Thus throughout this chapter we assume $\rho \neq 0$.

Except for the above dynamics (4.1), the setting of the problem in this chapter is identical to the one in Chapter 2. To facilitate the exposition, we present again some key definitions.

As in Chapter 2, the government wants to select the control Z that minimizes the expected cost of having debt plus the expected cost of interventions:

$$J(x; Z) \;:=\; E_x \left[\int_0^\infty e^{-\lambda t} h(X_t) dt + \int_0^\infty e^{-\lambda t} k dZ_t \right].$$

Here $k \in (0, \infty)$ is the proportional (marginal) cost for reducing the debt, $\lambda \in (0, \infty)$ is the government's discount rate, and h is a cost function, which we assume nonnegative and convex, with $h(0) \geq 0$.

We next define an admissible control process.

4.1 Remark. *We require that*

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T} X_T^{2n}\right] \;=\; 0. \tag{4.2}$$

4.2 Definition. *Let $x \in (0,\infty)$. An $\mathbb{F}$-adapted, non-negative, and non-decreasing stochastic control process $Z : [0,\infty)\times\Omega \to [0,\infty)$, with sample paths that are left-continuous with right-limits, is called an admissible stochastic singular control if $J(x;Z) < \infty$ and (4.2) holds. The set of all admissible controls is denoted by $\mathcal{A}(x) = \mathcal{A}$. By convention, we set $Z_0 = 0$.*

4.3 Problem. *The government wants to select the control $Z \in \mathcal{A}$ that minimizes the functional J defined by*

$$J(x;Z) \;:=\; E_x\left[\int_0^\infty e^{-\lambda t} h(X_t)dt + \int_0^\infty e^{-\lambda t} k dZ_t\right].$$

Here $k > 0$ is the proportional (marginal) cost for reducing debt, h is the cost function defined by $h : (0,\infty) \to [0,\infty)$

$$h(y) = \alpha y^{2n} + \beta,$$

where α is the importance of debt to the government and n is associated with the aversion to debt. Furthermore, $\lambda > 0$ is the government's discount rate. As in Chapter 2, we will consider that the following condition on the discount rate is satisfied:

$$\lambda \;>\; \sigma^2 n(2n-1) + 2\mu n. \tag{4.3}$$

Since the model in this chapter is more general than the one in Chapter 2, we will see later that additional conditions on the discount rate are required for the existence of the optimal debt ceiling.

4.2 A verification theorem

The goal of this section is to state a sufficient condition for a debt control to be optimal.

We define the value function $V : (0, \infty) \to \mathbb{R}$ by

$$V(x) \;:=\; \inf_{Z \in \mathcal{A}} J(x; Z).$$

This function represents the smallest cost that can be achieved when the initial debt ratio is x and we consider all the admissible controls.

Now we define the Hamilton-Jacobi-Bellman (HJB) for Problem 4.3. Let $\psi : (0, \infty) \to \mathbb{R}$ be a function in $C^2(0, \infty)$. We define the operator $\mathcal{L}$ by

$$\mathcal{L}\psi(x) \;:=\; \frac{1}{2}\sigma^2 x^2 \psi''(x) + \rho x^2 \psi'(x) + \mu x \psi'(x) - \lambda \psi(x).$$

For a function $v : (0, \infty) \to \mathbb{R}$ in $C^2(0, \infty)$, we consider the HJB equation

$$\forall x > 0 : \quad \min\left\{\mathcal{L}v(x) + h(x), k - v'(x)\right\} = 0. \tag{4.4}$$

We observe that a solution v of the HJB equation defines the regions $\mathcal{C} = \mathcal{C}^v$ and $\Sigma = \Sigma^v$ by

$$\mathcal{C} = \mathcal{C}^v := \Big\{x \in (0, \infty) : \mathcal{L}v(x) + h(x) = 0 \;\text{ and }\; k - v'(x) > 0\Big\}, \tag{4.5}$$

$$\Sigma = \Sigma^v := \Big\{x \in (0, \infty) : \mathcal{L}v(x) + h(x) \geq 0 \;\text{ and }\; k - v'(x) = 0\Big\}. \tag{4.6}$$

We note that $\mathcal{C} \cup \Sigma = (0, \infty)$ and $\mathcal{C} \cap \Sigma = \emptyset$.

4.4 Definition. *Let v satisfy the HJB equation (4.4). An $\mathbb{F}$-adapted, non-negative, and non-decreasing control process Z^v, with $Z_0^v = 0$ and sample paths that are left-continuous with right-limits, is said to be associated with the function v above if the following three conditions are satisfied:*

$$(i) \quad X_t^v \;=\; x + \int_0^t (\rho X_s^v + \mu)\, X_s^v\, ds + \int_0^t \sigma X_s^v dW_s - Z_t^v, \forall t \in [0, \infty), P - a.s.,$$

$$(ii) \quad X_t^v \in \bar{\mathcal{C}}, \quad \forall t \in (0, \infty),\; P - a.s.,$$

$$(iii) \quad \int_0^\infty I_{\{X_t^v \in \mathcal{C}\}} dZ_t^v = 0, \;\; P - a.s..$$

Here I_A denotes the indicator function of the event $A \subset [0, \infty)$.

4.5 Remark. *According to Definitions 4.2 and 4.4, if an associate control Z^v satisfies $J(x; Z^v) < \infty$, then it is admissible.*

Now we state a lemma that will be used in the proof of the Verification Theorem 4.7 below.

4.6 Lemma. *Suppose that v is increasing and satisfies the HJB equation (4.4). Let Z^v be the control associated to v, and X^v the process generated by Z^v. Then*

$$\int_0^T e^{-\lambda t} v'(X_t^v) d(Z^v)_t^c = \int_0^T e^{-\lambda t} k \, d(Z^v)_t^c \tag{4.7}$$

and

$$v(X_t^v) - v(X_{t+}^v) = k(Z_{t+}^v - Z_t^v), \quad \forall t \in \Delta. \tag{4.8}$$

Proof. Similar to the proof of Lemma 2.10. □

4.7 Theorem. *Let $v \in C^2(0, \infty)$ be an increasing and convex function on $(0, \infty)$ with $v(0+) < \infty$. Suppose that v satisfies the Hamilton-Jacobi-Bellman equation (4.4)* *for every $x \in (0, \infty)$, and there exists $d \in (0, \infty)$ such that the region $\mathcal{C}$ associated with v is $\mathcal{C}^v = (0, d)$. Then, for every $Z \in \mathcal{A}(x)$:*

$$v(x) \leq J(x; Z).$$

If the stochastic control Z^v associated with v is admissible, then

$$v(x) = J(x; Z^v).$$

In other words, $\hat{Z} = Z^v$ is the optimal control and $V = v$ is the value function for Problem 4.3.

Proof. This proof is similar to the proof of Theorem 2.11 in Chapter 2. For the sake of completeness, we present it. Since v is twice continuously differentiable, and v' and v'' are bounded functions, we may apply an appropriate version of Ito's formula. Thus, according to Meyer (1976) or Chapter 4 of Harrison (1985),

$$\begin{aligned} v(x) \;=\; & E_x\left[e^{-\lambda T} v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t} v'(X_t) dZ_t^c\right] - E_x\left[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t\right] \\ & - E_x\left[\int_0^T e^{-\lambda t}\left\{\frac{1}{2}\sigma^2 X_t^2 v''(X_t) + \rho X_t^2 v'(X_t) + \mu X_t v'(X_t) - \lambda v(X_t)\right\} dt\right] \\ & - E_x\left[\sum_{\substack{t \in \Delta \\ 0 \leq t < T}} e^{-\lambda t}\left\{v(X_{t+}) - v(X_t)\right\}\right]. \end{aligned} \tag{4.9}$$

Since v satisfies the HJB equation (4.4), we have $\mathcal{L}v(x) + h(x) \geq 0$ and $v'(x) \leq k$ for all $x \in (0, \infty)$. Thus

$$-\int_0^T e^{-\lambda t}\Big\{\frac{1}{2}\sigma^2 X_t^2 v''(X_t) + \rho X_t^2 v'(X_t) + \mu X_t v'(X_t) - \lambda v(X_t)\Big\} dt \;\leq\; \int_0^T e^{-\lambda t} h(X_t) dt. \tag{4.10}$$

$$\int_0^T e^{-\lambda t} v'(X_t) dZ_t^c \;\leq\; \int_0^T e^{-\lambda t} k dZ_t^c \tag{4.11}$$

and

$$v(X_t) - v(X_{t+}) \;\le\; k(Z_{t+} - Z_t) \quad \forall t \in \Delta. \tag{4.12}$$

Hence

$$\begin{aligned} v(x) \;&\le\; E_x\left[e^{-\lambda T} v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t} h(X_t)dt\right] + E_x\left[\int_0^T e^{-\lambda t} k dZ_t^c\right] \\ &\quad + E_x\left[\sum_{\substack{t\in\Delta \\ 0\,\le\, t\,<\,T}} e^{-\lambda t} k\left\{Z_{t+} - Z_t\right\}\right] - E_x\left[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t\right] \\ &= E_x\left[e^{-\lambda T} v(X_T)\right] + E_x\left[\int_0^T e^{-\lambda t} h(X_t)dt + \int_0^T e^{-\lambda t} k dZ_t\right] \\ &\quad - E_x\left[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t\right]. \end{aligned} \tag{4.13}$$

Suppose $Z \in \mathcal{A}$. From (4.2), v' bounded, and the linear growth of v on the interval $\Sigma^v = [d, \infty)$, we have

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T} v(X_T)\right] = 0$$

and

$$E_x\left[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t\right] = 0.$$

In addition, letting $T \to \infty$, by the Monotone Convergence Theorem,

$$\lim_{T\to\infty} E_x\left[\int_0^T e^{-\lambda t} h(X_t)dt\right] = E_x\left[\int_0^\infty e^{-\lambda t} h(X_t)dt\right]$$

and

$$\lim_{T\to\infty} E_x\left[\int_0^T e^{-\lambda t} k dZ_t\right] = E_x\left[\int_0^\infty e^{-\lambda t} k dZ_t\right].$$

This proves the first part of this theorem.

Now we consider the second part of the theorem. Let X^v be the process generated by Z^v. We recall that we are assuming $\mathcal{C}^v = (0, d)$. We also note that, since

$\mathcal{L}v(x) + h(x)$ is a continuous function for every $x \in (0, \infty)$, using (4.5), we have $\mathcal{L}v(x) + h(x) = 0$ for every $x \leq d$. These remarks, and condition (ii) of Definition 4.4, yield

$$\int_0^T e^{-\lambda t}\mathcal{L}v(X_t)dt = -\int_0^T e^{-\lambda t}h(X_t^v)dt.$$

That is, (4.10) turns into an equality for Z^v. Moreover, by Lemma 4.6, the inequalities (4.11) and (4.12) become equalities as well. As a result, (4.13) is an equality for Z^v, namely

$$v(x) = E_x\left[e^{-\lambda T}v(X_T^v)\right] + E_x\left[\int_0^T e^{-\lambda t}h(X_t^v)dt + \int_0^T e^{-\lambda t}kdZ_t^v\right]. \tag{4.14}$$

Since the process X_t^v is bounded above by $\max\{x, d\}$, $v(0+)$ is bounded and v is continuous on $(0, d]$, we conclude that the process $v(X_t^v)$ is bounded. Hence

$$\lim_{T\to\infty} E_x\left[e^{-\lambda T}v(X_T^v)\right] = 0.$$

Letting $T \to \infty$, by the Monotone Convergence Theorem, we conclude

$$v(x) = E_x\left[\int_0^\infty e^{-\lambda t}h(X_t^v)dt + \int_0^\infty e^{-\lambda t}kdZ_t^v\right] = J(x; Z^v).$$

We observe that, in particular, this shows that Z^v is admissible. This completes the proof of this theorem. □

4.3 The explicit solution

4.8 Definition. *Let v be a function that satisfies the HJB equation (4.4), and $\mathcal{C}$ the corresponding continuation region. If $\mathcal{C} \neq \emptyset$, the debt ratio ceiling b is*

$$b := \sup\{x \in (0, \infty) \,|\, x \in \mathcal{C}\}.$$

Moreover, if v is equal to the value function, then b is said to be the optimal debt ceiling.

Thus, to obtain the optimal debt ceiling, we need to find the value function. The Hamilton-Jacobi-Bellman equation (4.4) in the continuation region $\mathcal{C} = (0, b)$ implies

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \rho x^2 v'(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta,$$

and the Hamilton-Jacobi-Bellman equation (4.4) in the intervention region $\Sigma = [b, \infty)$ implies

$$v(x) = v(b) + k(x - b).$$

We get the differential equation

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \rho x^2 v'(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta, \quad \text{if } x < b \tag{4.15}$$

and

$$v'(x) = k, \qquad \text{if} \quad x \geq b. \tag{4.16}$$

The general solution of (4.15)-(4.16) takes different forms depending on whether $\rho > 0$ or $\rho < 0$. Accordingly, we need some definitions.

$$H(x) := H_1(x) I_{\{\rho > 0\}} \; + \; H_2(x) I_{\{\rho < 0\}}, \tag{4.17}$$

$$G(x) := G_1(x) I_{\{\rho > 0\}} \; + \; G_2(x) I_{\{\rho < 0\}}, \tag{4.18}$$

where

$$H_1(x) := x^{\widetilde{\gamma}} \exp\{-2\rho x/\sigma^2\}\, U\big(\widetilde{\gamma} + 2\mu/\sigma^2,\, \gamma + 1,\, 2\rho x/\sigma^2\big), \tag{4.19}$$

$$H_2(x) := x^{\widetilde{\gamma}}\, U\big(\widetilde{\gamma},\, \gamma + 1,\, -2\rho x/\sigma^2\big), \tag{4.20}$$

$$G_1(x) := x^{\widetilde{\gamma}} \exp\{-2\rho x/\sigma^2\}\, L\big(-\widetilde{\gamma} - 2\mu/\sigma^2,\, \gamma,\, 2\rho x/\sigma^2\big), \tag{4.21}$$

$$G_2(x) := x^{\widetilde{\gamma}}\, L\big(-\widetilde{\gamma},\, \gamma,\, -2\rho x/\sigma^2\big). \tag{4.22}$$

Here

$$\gamma = \frac{\sqrt{(\sigma^2 - 2\mu)^2 + 8\lambda\sigma^2}}{\sigma^2} = \sqrt{1 + 4\frac{\mu^2}{\sigma^4} + 4\frac{(2\lambda - \mu)}{\sigma^2}} \; > 1, \tag{4.23}$$

$$\widetilde{\gamma} = \frac{(\sigma^2 - 2\mu) + \sqrt{(\sigma^2 + 2\mu)^2 + 8(\lambda - \mu)\sigma^2}}{2\sigma^2} \; > 1, \tag{4.24}$$

U and L stand for the U hipergeometric function and the generalized Laguerre polynomial, respectively. In Appendix C we present the definitions of these functions. For references on hypergeometric functions and its relation to second order differential equations, see, for example, Bell (2004), Hochstadt (1986) and Kristensson (2010).

Inequalities (4.23) and (4.24) above follow from $\lambda > \mu$, which in turn follows from (4.3). Then the function U above is well-defined (see Appendix C.4). Hence H_1 and H_2 are well-defined as well.

The following remark states some inequalities on the parameters, which are consequences of $\lambda > \mu$.

4.9 Remark. *The following results are valid:*

$$\widetilde{\gamma} + 2\frac{\mu}{\sigma^2} = \frac{(\sigma^2 + 2\mu) + \sqrt{(\sigma^2 + 2\mu)^2 + 8(\lambda - \mu)\sigma^2}}{2\sigma^2} \; > 0, \tag{4.25}$$

$$\widetilde{\gamma} - \gamma = \frac{(\sigma^2 - 2\mu) - \sqrt{(\sigma^2 - 2\mu)^2 + 8\lambda\sigma^2}}{2\sigma^2} \; < 0. \tag{4.26}$$

Using the previous remark, we establish some properties of the functions H and G.

4.10 Remark. *The following results are valid:*

$$H_1(0+) = +\infty, \;\; \textit{and} \;\; G_1(0+) = 0, \tag{4.27}$$

$$H_2(0+) = +\infty, \;\; \textit{and} \;\; G_2(0+) = 0. \tag{4.28}$$

Hence $H(0+) = +\infty$ *and* $G(0+) = 0$.

Proof. *Let us show (4.27). To prove* $H_1(0+) = +\infty$, *suppose first* $\gamma+1 \notin \{3, 4, 5, \cdots\}$. *To simplify the notation, let* $c_1 = \widetilde{\gamma}+2\mu/\sigma^2$, $c_2 = \gamma+1$, *and* $c_3 = 2\rho/\sigma^2$. *By definition of* $H_1(x)$ *(see Appendix C.4)*

$$H_1(x) := x^{\widetilde{\gamma}} e^{-c_3\, x} \, \frac{\Gamma(1-c_2)}{\Gamma(c_1-c_2+1)} \; {}_1F_1(c_1; c_2; c_3\, x)$$

$$+ \, x^{\widetilde{\gamma}-\gamma} e^{-c_3\, x} \, \frac{\Gamma(c_2-1)}{\Gamma(c_1)} \; {}_1F_1(c_1-c_2+1; 2-c_2; c_3\, x)\,. \tag{4.29}$$

Taking the limit as $x \to 0+$, *the result follows from (4.24), Remark 4.9, and the fact that*

$$\lim_{x\to 0+} \; {}_1F_1(\cdot\,,\cdot\,, c_3\, x) = 1. \tag{4.30}$$

The proof for the case $\gamma + 1 \in \{3, 4, 5, \cdots\}$ *is similar. Next let us show* $G_1(0+) = 0$. *By definition (see Appendix C.5),*

$$G_1(x) = x^{\widetilde{\gamma}} e^{-c_3 x} \binom{c_2 - c_1 - 1}{-c_1} \; {}_1F_1(c_1, c_2, c_3\, x). \tag{4.31}$$

Taking the limit as $x \to 0+$ *in the above inequality, the desired results follows from (4.24) and (4.30). This establishes (4.27). Similarly, one can prove (4.28).* □

The general solution of (4.15)-(4.16), the candidate for value function, is defined in terms of the functions H and G:

$$v(x) = \begin{cases} \frac{\beta}{\lambda} + \sum_{m=0}^{\infty} \zeta_{2n+m} x^{2n+m} + A\, H(x) + B\, G(x) & \text{if} \quad x < b \\ \\ kx + D & \text{if} \quad x \geq b, \end{cases}$$

where A, b, B, and D are constants to be determined, and

$$\zeta_{2n} = \frac{\alpha}{\lambda - n(2n-1)\sigma^2 - 2n\mu} > 0\,, \tag{4.32}$$

$$\zeta_{2n+m} = \frac{\binom{2n-1+m}{2n-1} \alpha \, \rho^m \, m!}{\prod_{j=0}^{m} \Big(\lambda - (2n+j)(2n+j-1)\sigma^2/2 - (2n+j)\mu \Big)}, \qquad \forall m \in \{1, 2, 3, \cdots\}. \tag{4.33}$$

The parameter ζ_{2n} exists by condition (4.3). To guarantee the existence of the parameters $\{\zeta_{2n+m} : m = 1, 2 \cdots\}$ we need to impose the following set of conditions:

$$\lambda \neq (2n+j)(2n+j-1)\sigma^2/2 + (2n+j)\mu, \quad \forall j \in \{1, 2, \cdots\}. \tag{4.34}$$

According to Theorem 4.7, it is required that the value function satisfies $V(0+) < \infty$. Then, by Remark 4.10, we must have $A = 0$. To find the remaining three constants B, b, and D, we conjecture that v is twice continuously differentiable. Hence v satisfies the following system of three equations:

$$v(b+) = v(b-), \tag{4.35}$$

$$v'(b+) = v'(b-), \tag{4.36}$$

$$v''(b+) = v''(b-). \tag{4.37}$$

We proceed with the solution of the parameters B, b, and D. The candidate for value function is given by

$$v(x) = \begin{cases} \frac{\beta}{\lambda} + \sum_{j=2n}^{\infty} \zeta_j x^j + B\, G(x) & \text{if } x < b \\ kx + D & \text{if } x \geq b. \end{cases} \tag{4.38}$$

Taking the first and second derivatives, it follows that

$$v'(x) = \begin{cases} \sum_{j=2n}^{\infty} j\zeta_j x^{j-1} + B\, G'(x) & \text{if } x < b \\ k & \text{if } x \geq b, \end{cases}$$

and

$$v''(x) = \begin{cases} \sum_{j=2n}^{\infty} j(j-1)\zeta_j x^{j-2} + B\, G''(x) & \text{if } x < b \\ 0 & \text{if } x \geq b. \end{cases}$$

Using Eqs. (4.36)-(4.37), we obtain the equation that determines b:

$$G'(b) \sum_{j=2n}^{\infty} j(j-1)\zeta_j b^{j-2} + \Big(k - \sum_{j=2n}^{\infty} j\zeta_j b^{j-1}\Big) G''(b) = 0, \tag{4.39}$$

where $G'(b)$ is assumed to be different from zero.

From (4.36), we find B in terms of b, namely

$$B = \frac{1}{G'(b)} \Big(k - \sum_{j=2n}^{\infty} j\zeta_j b^{j-1}\Big). \tag{4.40}$$

Using the previous two constants, and Eq. (4.35), we obtain

$$D = \frac{\beta}{\lambda} + \sum_{j=2n}^{\infty} \zeta_j b^j + B\, G(b) - k\, b. \tag{4.41}$$

Hence, from Eqs. (4.35)-(4.37), we obtain the constants b, B and D, as a function of the parameters $(\rho, k, n, \lambda, \mu, \sigma, \alpha, \beta)$.

Summarizing, the candidate for value function is given by (4.38), and the candidate for optimal control is then determined by Definition 4.4. Moreover, the candidate for optimal debt ceiling is determined by (4.39).

To complete this section, we are going to prove rigorously that the above candidate for optimal control is indeed the optimal control, and the above candidate for value function is indeed the value function.

4.11 Theorem. *Consider $\rho \neq 0$. Suppose (4.3) and (4.34). Let the parameter $b > 0$ be the solution of*

$$G'(b) \sum_{j=2n}^{\infty} j(j-1)\zeta_j\, b^{j-2} + \Big(k - \sum_{j=2n}^{\infty} j\zeta_j\, b^{j-1}\Big) G''(b) = 0, \tag{4.42}$$

with G defined in (4.17). Moreover, let the parameters, ζ_{2n}, $\{\zeta_{2n+m}\}_{m=1}^{\infty}$, B, and D be given by (4.32), (4.33), (4.40), and (4.41), respectively. Suppose that

$$2\rho bk - (\lambda - \mu)k + 2n\alpha b^{2n-1} > 0. \tag{4.43}$$

Let us define the function $V : (0, \infty) \to [0, \infty)$ by

$$V(x) = v(x) = \begin{cases} \frac{\beta}{\lambda} + \sum_{j=2n}^{\infty} \zeta_j \, x^j + B \, G(x) & \text{if } x < b \\ \\ kx + D & \text{if } x \geq b. \end{cases} \tag{4.44}$$

If

$$\forall x \in (0, b) : \quad V''(x) \, > \, 0, \tag{4.45}$$

then V is the value function, and b is the optimal debt ceiling. Furthermore, the optimal debt control is the process $\hat{Z}$ given by

$$(i) \quad \hat{X}_t \; = \; x + \int_0^t (\rho \hat{X}_s + \mu) \hat{X}_s ds + \int_0^t \sigma \hat{X}_s dW_s - \hat{Z}_t, \quad \forall t \in [0, \infty), \; P - a.s.,$$

$$(ii) \quad \hat{X}_t \in [0, b], \quad \forall t \in (0, \infty), \quad P - a.s.,$$

$$(iii) \quad \int_0^{\infty} I_{\{\hat{X}_t \in (0,b)\}} d\hat{Z}_t = 0, \quad P - a.s..$$

Proof. To prove this theorem, it suffices to show that all conditions of Theorem 4.7 are satisfied. By construction, we get immediately that $V \in C^2\big(0, \infty\big)$ and $V(0+)$ is bounded. Regarding the value function, it remains to verify that V is increasing, convex and satisfies the HJB equation (4.4). Let us show that V is convex. We observe that $V''(x) = 0$, for every $x > b$. On the other hand, v is also convex on $(0, b]$ by (4.45). The continuity of v'' yields $V''(b) = 0$. This shows that V is convex on $(0, \infty)$ and strictly convex on $(0, b)$. Next we show that V is increasing. Taking the first derivative, we have $V'(x) = k > 0$ for every $x > b$. On the other hand, due to V is strictly convex on $(0, b)$, $V'(0+) = 0$ [this follows from $\tilde{\gamma} > 1$] and $V'(b) = k$, we conclude that $0 < V'(x) < k$ on $(0, b)$. Hence V is strictly increasing on $(0, \infty)$.

To establish the HJB equation (4.4), let us define $L(x) := \mathcal{L}V(x) + h(x)$, for every $x \in (0, \infty)$. We note that, since V is $C^2(0, \infty)$ and h is polynomial, L is continuous on $(0, \infty)$. We proceed by considering cases. First, we consider the case $x < b$. Then, by construction of the candidate for value function, $L(x) = 0$, for $x < b$. Moreover, we showed above that $0 < V'(x) < k$, for all $x \in (0, b)$. Thus $L(x) = 0$ and $V'(x) < k$ for every $x < b$. In other words, the HJB equation (4.4) is satisfied for every $x < b$. Since L is continuous, we note that $L(b) = L(b-) = 0$. Now we consider the case $x \geq b$. By construction of the candidate for value function, we have $V'(x) = k$, for every $x \geq b$. It only remains to show $L(x) \geq 0$ for every $x \geq b$. By definition of L, for every $x \geq b$:

$$L(x) = \rho x^2 k + \mu\, x\, k - \lambda(k\, x + D) + \alpha x^{2n} + \beta.$$

Hence $L \in C^2(b, \infty)$. Computing the first and second derivatives on (b, ∞),

$$L'(x) = 2\rho x k - (\lambda - \mu)k + 2n\alpha x^{2n-1}\,,$$

$$L''(x) = 2\rho k + 2n(2n-1)\alpha x^{2n-2},$$

$$L'''(x) = 2n(2n-1)(2n-2)\alpha x^{2n-3} \geq 0.$$

Since $L(b) = 0$, to prove that $L(x) \geq 0$ on $[b, \infty)$, it is enough to show that $L(x) \geq L(b)$, for every $x > b$. Thus we just need to prove that $L'(x) > 0$ on (b, ∞). To this end, it suffices to prove $L''(x) > 0$ for $x > b$, and $L'(b+) > 0$. The latter follows directly from (4.43). For the former, using (4.43) to justify the last inequality below, we get

$$L''(x) \geq L''(b+) = 2\rho k + 2n(2n-1)\alpha b^{2n-2} \geq 1/b\Big(2\rho b k + 2n(2n-1)\alpha b^{2n-1}\Big)$$

$$> 1/b\Big(2\rho b k + 2n\alpha b^{2n-1}\Big) > 1/b\Big(2\rho b k - (\lambda - \mu)k + 2n\alpha b^{2n-1}\Big) > 0.$$

Thus we have established that $V'(x) = k$ and $L(x) \geq 0$, for every $x \geq b$. Therefore,

Table 4.1: Basic parameter values

ρ	μ	σ	λ	k	n	α	β
0.05	0.05	0.05	0.7	1.0	1.0	1.0	0.0

V satisfies the HJB equation (4.4) for every $x \in (0, \infty)$. From (4.5), we observe that the corresponding continuation region is $\mathcal{C} = (0, b)$.

From Theorem 4.7, we conclude that $\hat{Z}$ is optimal debt control. Moreover, by Definition 4.8, b is the optimal debt ceiling. □

4.12 Example. *Using the parameter values in Table 4.1, the optimal debt ceiling is* 0.3145*. The value function is given by:*

$$V(x) = \sum_{m=0}^{\infty} \zeta_{2+m} x^{2+m} \; - \; 17721.1174\, x^{11.1635}\, e^{-40x} \sum_{j=0}^{\infty} \frac{(51.1635, j)}{(62.327, j)} \frac{1}{j!} (40x)^j,$$

for $0 < x < 0.3145$*, and*

$$V(x) = x - 0.143669,$$

for $x \geq 0.3145$*. Here*

$$\zeta_{2+m} \quad = \quad \frac{0.05^m (m+1)!}{\prod_{j=0}^{m} \left(0.5975 - 0.05375 j - 0.00125 j^2\right)}, \quad \forall m \in \{0, 1, 2, 3, \ldots\},$$

$(a, 0) := 1$ *and* $(a, j) := a(a+1)(a+2)\cdots(a+j-1)$, *for* $j \in \{1, 2, 3, \ldots\}$.

We note that conditions (4.43) and (4.45) stated in Theorem 4.11 are satisfied. Indeed,

$$2\rho b k - (\lambda - \mu)k + 2n\alpha b^{2n-1} = 0.0050\,,$$

and Figure 4.1 shows the second derivative of the value function is strictly positive for values less than the optimal debt ceiling 0.3145*.*

4.13 Remark. *We conjecture that conditions (4.43) and (4.45) of Theorem 4.11 are*

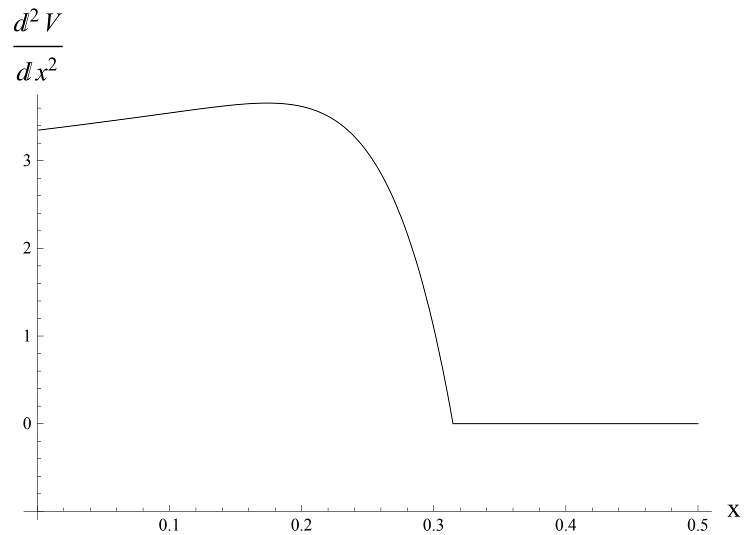

Figure 4.1: The second derivative of the value function V

satisfied as long as (4.3) and (4.34) are met. Certainly, they are satisfied not only in the above example, but also in all the numerical solutions in the next section. As opposed to the model in Chapter 2, here we do not know the explicit formulas for the values of the parameters A, b and D. This is the reason why we cannot prove such conjecture.

4.4 Economic analysis

We present some numerical computations to analyze the effects of the parameters on the optimal debt ceiling. The basic parameters that we use are given in Table 4.1.

In Table 4.2 row $b(\rho = 0)$ shows the values for the optimal debt ceiling for $\rho = 0$, which are calculated using the explicit formula given in (2.37). In row $b(\rho = -0.05)$ we present the values for the optimal debt ceiling that corresponds to $\rho = -0.05$, which are solutions of (4.42) with $G = G_1$. Similarly, row $b(\rho = 0.05)$ shows the values for the optimal debt ceiling corresponding to $\rho = 0.05$, which

Table 4.2: Effect of α, μ and σ on the optimal debt ceiling b

	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1.0$	$\alpha = 1.2$	$\alpha = 1.4$
$b(\rho = -0.05)$	0.745428	0.513658	0.350468	0.289242	0.246234
$b(\rho = 0)$	0.662704	0.473360	0.331352	0.276127	0.236680
$b(\rho = 0.05)$	0.599230	0.439947	0.314573	0.264360	0.227973
	$\mu = 0.02$	$\mu = 0.04$	$\mu = 0.05$	$\mu = 0.08$	$\mu = 0.10$
$b(\rho = -0.05)$	0.371974	0.357253	0.350468	0.331690	0.319987
$b(\rho = 0)$	0.350203	0.337368	0.331352	0.314312	0.303466
$b(\rho = 0.05)$	0.331507	0.320041	0.314573	0.298820	0.288659
	$\sigma = 0.03$	$\sigma = 0.05$	$\sigma = 0.07$	$\sigma = 0.09$	$\sigma = 0.12$
$b(\rho = -0.05)$	0.345860	0.350468	0.355514	0.360696	0.368501
$b(\rho = 0)$	0.327626	0.331352	0.335763	0.340483	0.347774
$b(\rho = 0.05)$	0.311519	0.314573	0.318405	0.322661	0.329419

See Table 4.1 for the values of the other parameters used in these computations.

are solutions of (4.42) with $G = G_2$. Thus, for instance, the optimal debt ceiling 0.745428 is calculated using $\rho = -0.05$, $\alpha = 0.5$, and the other parameters are specified in Table 4.1.

We observe that the optimal debt ceiling b is a decreasing function of α, an increasing function of $\mu = g - r$ (hence an increasing function of economic growth g and a decreasing function of the interest rate r), and an increasing function of volatility σ. Therefore, the qualitative effects of the economic parameters α, g, r and σ on the optimal debt ceiling b are the same in this chapter and Chapter 2.

Considering Chapters 2 and 4 together, we have solved the problem for $\rho \in \mathbb{R}$. In this general setting, in Table 4.2 we observe that the the value of $\rho = r_1 - g_1$ and the optimal debt ceiling b are inversely related. That is, the higher the ρ the lower the optimal debt ceiling, and the other way around. Let us analyze the economic meaning of this result.

By definition of ρ, large values of ρ are associated with large values of r_1 relative to g_1. That is, a situation in which the effect of the interest rate is dominant.

Thus countries in which the debt has a strong effect on the interest rate, but a weak effect on the rate of economic growth, should have a low optimal debt ceiling.

On the other hand, small values of ρ are related to large values of g_1 relative to r_1. In this case the effect on the economic growth rate is dominant. Hence countries in which debt has a strong positive effect on the economic growth, but a weak effect on the interest rate, should have a high optimal debt ceiling.

4.5 Concluding remarks

This chapter complements Chapter 2. In that chapter we assume that the interest rate and the economic growth are constant parameters. In this chapter, we consider that both the interest rate and the rate of economic growth depend on the debt ratio. The former is justified by the fact that countries with higher debt are considered riskier and, as a result, the debt buyers charge a higher interest rate. The motivations for the latter are the changes on the productivity and liquidity of the economy that debt generates that, in turn, have an impact on the economic growth.

In this more general setting we still succeed in solving the stochastic debt control problem, and finding an explicit solution for the optimal debt ceiling. The cost of this generalization is that we no longer have an explicit formula for the optimal debt ceiling as in Chapter 2. However, this does not preclude us from performing the economic analysis, which is based on numerical solutions.

We find that the qualitative effects of the economic parameters α, g, r and σ on the optimal debt ceiling b are the same in this chapter and Chapter 2. As expected, the quantitative effects are different.

The economic contributions of this chapter on the study of the debt ceiling are two novel conclusions. First, other things being equal, we find that countries with a stronger positive link between debt and economic growth should have a higher optimal debt ceiling. Second, other things being equal, countries with a stronger positive relation between debt and interest rate on debt should have a lower optimal debt ceiling.

Chapter 5

The optimal currency debt portfolio

Although the order of magnitude of the debt of a country can be of trillions of dollars, and the government debt portfolio is in general the largest in the country, the theoretical literature has paid almost no attention to currency government debt portfolios.

To the best of our knowledge, Licandro and Masoller (2000) and Giavazzi and Missale (2004) are the only references for this problem. These approaches have the following limitations: (1) the debt dynamics is not realistic because they consider only one period models, (2) the jumps in the exchange rates are not considered explicitly, and (3) the role of debt aversion is not included. Thus important elements of the currency debt analysis have not been included.

To the best of our knowledge, for the first time in the literature, we present a continuous-time model for government debt management that includes debt aversion and jumps in the exchange rates. We obtain explicitly the optimal currency debt portfolio and optimal debt payments. We use this model to show that the behavior of developing countries of reducing their proportion of foreign debt in their debt portfolios is consistent with a high debt aversion. Moreover, we show that an extremely high debt aversion can lead to issue only government debt in local currency. That is, it would be optimal for such a country to have no debt in foreign currency.

We model the currency debt problem as a stochastic control model in continuous time with infinite horizon. We believe that is the suitable framework to study debt portfolios. In fact, Bolder (2003) shows that the government debt problem can be conceptualized in this manner. However, his problem is different from ours. He assumes that the government issues only local currency debt and his goal is to find the optimal proportion of the different terms of debt. In contrast, our focus is on finding the optimal currency debt composition. That is why we do not consider debt in different terms; instead, we assume that the government issues bonds in different currencies. (Besides, the consideration of different terms makes the problem intractable from an analytical perspective. That is why Bolder 2003 has to pursue a simulation approach.)

To get a realistic debt dynamics, we extend the model of government debt for a single currency, presented in macroeconomic textbooks (see Blanchard and Fischer 1989, for example), to a multicurrency setting. We model the exchange rates dynamics as stochastic process that present jumps, and thus consider a more general framework than Zapatero (1995) and Cadenillas and Zapatero (1999). We succeed in finding a stochastic differential equation for government debt, in which its current value depends on the present and past values of variables such as the interest rates, exchange rates, debt payments and the proportions of debt in different currencies.

The running cost, or what we call here the debt disutility of the government, depends on both the debt payments and the debt itself. The existence of the former cost comes from the fact that, in order to get additional positive fiscal results to repay debt, the government has to cut spendings or increase taxes. The rationale for considering the latter cost stems from the fact that the government acknowledges that having high debt can lead to debt problems, which can end up in a debt crisis, for instance. Since we are interested in analyzing the effects of debt aversion on the optimal currency debt portfolio, we have to consider a disutility function in which the debt aversion is a parameter such that: the greater the debt aversion, the higher the disutility generated by the debt itself and debt payments. In our model, the disutility function presents constant relative risk aversion (CRRA) in debt payments and the debt itself. We have two reasons for such choice. First, we have a unique parameter that fully characterizes government debt aversion. Second, our model with constant relative risk aversion is a generalization of the quadratic function, which is the disutility function most widely used in different areas of economics (see, for example, Kydland and Prescott 1977, Taylor 1979, Cadenillas and Zapatero 1999, and Cadenillas et al. 2013).

The goal of the government debt manager is to choose both the currency debt portfolio and the payments that minimize the expected total cost (disutility).

We succeed in solving the problem explicitly. Thus, we perform some comparative statics to analyze the effects of some parameters (such as the debt aversion and the size and frequency of the jumps of the exchange rates) on the optimal currency debt portfolio and the optimal debt payments. We find that higher debt aversion and jumps in the exchange rates lead to a lower proportion of optimal debt in foreign currency.

5.1 The government debt model

In this section we derive the stochastic differential equation for the debt of a country and, based on it, we then state the government currency debt control problem. Our goal here is to obtain a realistic debt dynamics, that considers debt in a finite number of foreign currencies, and includes jumps in the exchange rates.

The debt ratio dynamics

The government debt is defined by

$$X(t) \ := \ \text{gross public debt expressed in local currency at time } t.$$

The gross public debt is the cumulative total of all government borrowings less repayments. That is, it includes the central and local government debt, and the domestic and external debt.

In this subsection, our goal is to extend the following version for one single currency debt presented in macroeconomic textbooks (see, for instance, Blanchard and Fischer 1989):

$$X(t) \ = \ X(0) + \int_0^t r_0 X(s) ds - \int_0^t p(s) ds, \tag{5.1}$$

where r_0 is the (continuous) interest rate, and p stands for the process of debt rate payments.

We consider a government that issues bonds in local and m foreign currencies. Let $\Lambda_0(t)$ denote the number of bonds held in local currency at time t, and $\Lambda_j(t)$ the number of bonds held in foreign currency j for $j \in \{1, \cdots, m\}$. The prices of the bonds are denoted by $R_j(t)$ for $j \in \{0, 1, \cdots, m\}$. Thus, $\Lambda_j(t)R_j(t)$ is the amount of debt held in currency j. For instance, if $j = 1$ represents Euros, and this currency is a foreign currency for the country, then $\Lambda_1(t)R_1(t)$ is the amount of debt in Euros at time t.

We require m exchange rates to express the total debt in local currency. For $j \in \{1, \cdots, m\}$, let $Q_j(t)$ be the exchange rate of the currency j with respect to the local currency. To be more precise,

$$Q_j(t) \ := \ \text{local currency units per unit of foreign currency } j \text{ at time } t.$$

Then the total debt in terms of local currency X can be written as

$$X(t) = \Lambda_0(t)R_0(t) + \sum_{j=1}^{m} \Lambda_j(t)R_j(t)Q_j(t) \tag{5.2}$$

or, equivalently,

$$X(t) = Z_0(t) + \sum_{j=1}^{m} Z_j(t)Q_j(t),$$

where $Z_i(t) := \Lambda_i(t)R_i(t)$, $\forall i \in \{0, 1, \cdots, m\}$.

To find the debt dynamics, given the evolution of the number of bonds in each currency, we require the dynamics of the price of the bonds and the dynamics of the exchange rates.

Since we are interested in studying a currency debt portfolio, in our model the source of randomness will come from the exchange rates. We will assume that the exchange rates follow a process driven by Brownian motions and Poisson processes. For technical reasons, we need to specify a suitable probability space in which these processes are defined.

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$, which is the P-augmentation of the filtration generated by both an m-dimensional Brownian motion $W = \{W_1, \cdots, W_m\}$, and an independent m-dimensional Poisson process $N = \{N_1, \cdots, N_m\}$ with corresponding intensities $\{\lambda_1, \cdots, \lambda_m\}$.

Let $Q = \{(Q_1(t), \cdots, Q_m(t)), t \geq 0)\}$ be an $\mathbb{F}$-adapted process. Following Jeanblanc-Picqué and Pontier (1990), Cadenillas (2002) and Guo and Xu (2004), we generalize the model in Zapatero (1995), and Cadenillas and Zapatero (1999), and consider the following multidimensional setting for the exchange rates. For $j \in \{1, 2, \cdots, m\}$:

$$dQ_j(t) = \mu_j Q_j(t)dt + \sigma_j Q_j(t)dW(t) + \widetilde{\varphi}_j Q_j(t^-)d\tilde{N}(t)\,, \tag{5.3}$$

with initial exchange rates $Q_j(0) = q_j > 0$. Here σ_j and $\widetilde{\varphi}_j$ are the j-th row of the $m \times m$-matrices $\sigma = [\sigma_{ij}]$ and $\varphi = [\varphi_{ij}]$, respectively. The parameter $\mu_j \in (-\infty, \infty)$. Throughout the paper A^T denotes the transpose of matrix A. We will assume that $\sigma\sigma^T$ and the family of matrices $\{\varphi_j \varphi_j^T : j \in \{1, \cdots, m\}\}$ are all positive definite, where φ_j stands for the j-th column of matrix φ.

Furthermore, we denote by $\tilde{N}$ the compensated Poisson process. That is, for $j \in \{1, \cdots, m\}$:

$$\tilde{N}_j(t) := N_j(t) - \lambda_j t \,,$$

where $\lambda_j > 0$ and N_j is a Poisson process. The left continuous version of any process $Y(t)$ is denoted by $Y(t^-)$. We note that the exchange rate Q_j has right-continuous with left-limits paths, a property that is determined by the stochastic integral with respect to the jump process. For more information about stochastic differential equations like (5.3), see for example, Ikeda and Watanabe (1981), Protter (2004), Cont and Tankov (2004), and Applebaum (2009).

We point out that Cadenillas and Zapatero (1999) assume that, in the absence of government interventions in the exchange markets, the exchange rate follows a geometric Brownian motion. That is, they assume equation (5.3) without the component that corresponds to the compensated Poisson process $\tilde{N}$.

5.1 Remark. *Equation (5.3) implies that the process Q_j jumps at time $t > 0$ if and only if some of the Poisson processes in $N = \{N_1, \cdots, N_m\}$ do so. That is why Q_j is right-continuos with left-limits. Moreover, since the number of jumps of a Poisson process is finite on each finite interval $[0, t]$, then the sample paths of both the process $Q_j(\cdot)$ and its left-continuous version $Q_j(\cdot^-)$ are bounded on any finite interval $[0, t]$.*

The model for the exchange rates, equation (5.3), is appropriate to describe sudden depreciations (or devaluations) of the local currency. The time of the jumps in the exchange rates are random, driven by Poisson processes. For instance, if a Poisson event occurs in the exchange rate j, $j \in \{1, 2, \cdots, m\}$, then it generates an effect on all the exchange rates via φ_{ij}. The occurrence of this event is random, but we know that the intensity of the event is given by λ_j. This parameter measures the frequency of the jumps in the exchange rate. The information about the size of the jumps is contained in the matrix φ. In particular, if the element in the diagonal φ_{jj} is positive, this will be consistent with empirical currency crises (such as Mexico in 1994, Asia in 1997, and Russia in 1998) in which the exchange rate went up dramatically. Thus, if a jump in the exchange rate j occurs, then an increase in the rate of depreciation (or devaluation) of size $\varphi_{jj} > 0$ takes place. For example, $\varphi_{jj} = 0.3$ means that the exchange rate went up unexpectedly 30%. Thus, our model accounts for the fact that a government that has foreign currencies in its debt portfolio faces a risk of depreciation (or devaluation) of its local currency. Since we also want to analyze the effects of the size of jumps on the optimal currency debt portfolio, the parameters in matrix φ give us a precise, direct and intuitive measure of the magnitude of the jumps.

We assume that the prices of the bonds R_j satisfy

$$dR_j(t) = R_j(t)\, r_j\, dt, \quad \forall j \in \{0, 1, \cdots, m\}, \tag{5.4}$$

where $R_j(0) = 1$, and $r_j \in (0, \infty)$ is the (continuous) interest rate on debt issued in currency j. We also assume that the process of the prices of the bonds $R = \{(R_0(t), R_1(t), \cdots, R_m(t)), t \geq 0)\}$ is $\mathbb{F}$-adapted.

In Appendix B.3, we derive the debt dynamics for a discrete time model with one foreign currency. The continuous-time version with m foreign currencies is

$$X(t) \quad = \quad X(0) + \int_0^t \Lambda_0(s^-) dR_0(s) + \sum_{j=1}^{m} \int_0^t \Lambda_j(s^-) d\big(R_j(s) Q_j(s)\big) - \int_0^t p(s) ds. \tag{5.5}$$

We assume that the process of rate payment p and the m-dimensional process $\Lambda =$

$(\Lambda_1, \cdots, \Lambda_m)$ are $\mathbb{F}$-adapted and right-continuous with left-limits. In addition, we assume p non-negative, and the technical condition $E_x[\int_0^t p(s)ds] < \infty$ for every $t > 0$. Here $E_x[Y]$ denotes the expected value of the random variable Y given that $X(0) = x$.

Using both the dynamics of the exchange rates (5.3) and the dynamics of the price of the bonds (5.4) in equation (5.5), we obtain the stochastic differential equation for the debt process X:

$$
\begin{aligned}
dX(t) \quad = \quad & r_0 X(t)dt + \sum_{j=1}^{m} \big(r_j + \mu_j - r_0\big)\Lambda_j(t)R_j(t)Q_j(t)dt \\
& + \sum_{j=1}^{m} \Lambda_j(t^-)R_j(t)Q_j(t^-)\widetilde{\varphi}_j d\tilde{N}(t) - p(t)dt \\
& + \sum_{j=1}^{m} \Lambda_j(t)R_j(t)Q_j(t)\sigma_j dW(t). \qquad (5.6)
\end{aligned}
$$

We point out that since the process R_j is continuous, its left-continuous version coincides with the process itself, and hence they are interchangeable.

The debt portfolio vector process $\pi = (\pi_1, \cdots, \pi_m)$ is defined for $j \in \{1, \cdots, m\}$ by

$$
\pi_j(t) := \frac{\Lambda_j(t)R_j(t)Q_j(t)}{X(t)}, \quad \forall X(t) > 0; \qquad (5.7)
$$

from which we deduce that

$$
\pi_j(t^-) = \frac{\Lambda_j(t^-)R_j(t)Q_j(t^-)}{X(t^-)}, \quad \forall X(t^-) > 0. \qquad (5.8)
$$

For completeness, we define $\pi(t) := 0$ if $X(t) = 0$, and $\pi(t^-) := 0$ if $X(t^-) = 0$.

Here $\pi \in \mathbb{R}^m$ is $\mathbb{F}$-adapted and π_j represents the proportion of debt issued in foreign currency $j \in \{1, \cdots, m\}$. Obviously, the proportion of debt in local currency $\pi_0 \in \mathbb{R}$ is given by $\pi_0(t) = 1 - \sum_{j=1}^{m} \pi_j(t)$.

Expressing equation (5.6) in a compact manner, we have the following government debt dynamics:

$$\begin{aligned} dX(t) &= X(t)\, r_0\, dt \;+\; X(t)\pi^T(t) b\, dt \;+\; X(t)\pi^T(t)\sigma\, dW(t) \\ &\quad + X(t^-)\pi^T(t^-)\varphi\, d\tilde{N}(t) \;-\; p(t)\, dt, \end{aligned} \tag{5.9}$$

where $X(0) = x > 0$, $b = r + \mu - r_0 I$, with $\mu = (\mu_1, \cdots, \mu_m)$ and $r = (r_1, \cdots, r_m)$, with $\mathbb{1}$ a vector of ones in $\mathbb{R}^m$. We require the following technical assumption: $E_x[\int_0^t \pi^T \sigma\sigma^T \pi ds] < \infty$ for every $t > 0$.

If we set $\pi_j(t) = 0$ for all $j \in \{1, \cdots, m\}$ and $t \geq 0$ in equation (5.9), we recover the dynamics of debt in one single currency given by equation (5.1). Thus equation (5.9) is indeed an extension to the multi-currency debt dynamics.

We state our result in the next proposition.

5.2 Proposition. *The stochastic differential equation for the government debt dynamics is given by*

$$\begin{aligned} dX(t) &= X(t)\, r_0\, dt \;+\; X(t)\pi^T(t) b\, dt \;+\; X(t)\pi^T(t)\sigma dW(t) \\ &\quad + X(t^-)\pi^T(t^-)\varphi\, d\tilde{N}(t) \;-\; p(t)\, dt\,, \end{aligned} \tag{5.10}$$

where $X(0) = x > 0$*;* $b = r + \mu - r_0 \mathbb{1}$*;* $\pi = (\pi_1, \cdots, \pi_m)$ *is the vector of proportions of debt in foreign currencies, and* p *is the debt payment rate process, i.e., the debt payment rate expressed in local currency.*

We recall that φ_j denote the j-th column of matrix φ. If we impose the technical condition that for every $s \geq 0$

$$1 + \pi^T(s^-)\varphi_j \;>\; 0\,, \quad \forall j \in \{1, 2, \cdots, m\}\,, \tag{5.11}$$

the above linear stochastic differential equation for the debt X possesses a unique explicit solution. Indeed, an application of Ito's formula to $X(t)/\xi(t)$ gives

$$X(t) = \xi(t)\Big(x - \int_0^t p(s)\xi(s)^{-1}ds\Big)\,, \quad \forall t \geq 0\,. \tag{5.12}$$

Here

$$\xi(t) := \exp\Big\{\int_0^t \beta(s)ds + \int_0^t \pi^T(s)\sigma dW(s) + \sum_{j=1}^m \int_0^t \log\Big(1+\pi^T(s^-)\varphi_j\Big)dN_j(s)\Big\}\,,$$

where

$$\beta(s) := r_0 + \pi^T(s)\,b - \sum_{j=1}^m \lambda_j \pi^T(s)\varphi_j - \frac{1}{2}\pi^T(s)\sigma\sigma^T\pi(s),$$

is the unique solution of the homogenous equation

$$d\xi(t) \;=\; \xi(t)r_0\,dt \,+\, \xi(t)\pi^T(t)b\,dt \,+\, \xi(t)\pi^T(t)\sigma dW(t) + \,\xi(t^-)\pi^T(t^-)\varphi d\tilde{N}(t)\,,$$

with initial condition $\xi(0) = 1$. Here $\log a$ stands for the natural logarithm of $a > 0$. We note that the process ξ can be interpreted as the debt dynamics with initial value equal to one, and without any debt payments.

We now discuss the nature of the jumps of the debt process. Suppose the Poisson event N_i occurs at the random time J. Then, equation (5.12) implies that $\frac{X(J)}{X(J^-)} = \frac{\xi(J)}{\xi(J^-)}$. Since

$$\begin{aligned}
\frac{\xi(J)}{\xi(J^-)} &= \frac{\exp\Big\{\int_0^J \beta(s)ds + \int_0^J \pi^T(s)\sigma dW(s) + \int_0^J \log\Big(1+\pi^T(s^-)\varphi_i\Big)dN_i(s)\Big\}}{\exp\Big\{\int_0^{J^-} \beta(s)ds + \int_0^{J^-} \pi^T(s)\sigma dW(s) + \int_0^{J^-} \log\Big(1+\pi^T(s^-)\varphi_i\Big)dN_i(s)\Big\}} \\
&= \Big(1+\pi^T(J^-)\varphi_i\Big),
\end{aligned}$$

we have

$$X(J) = X(J^-)\big(1+\pi^T(J^-)\varphi_i\big). \tag{5.13}$$

By (5.11), the debt values before and after the jump $X(J^-)$ and $X(J)$, respectively, have the same sign. In particular, if the value before the jump is positive the debt process will never jump to a negative or zero value. Notice that the jump in X can be upward or downward.

From Proposition 5.2, the debt at time t, $X(t)$, depends on the following variables: interest rates of the bonds $(r_0, r_1, \cdots, r_m)$, exchange rate depreciation $(\mu_1, \cdots, \mu_m)$, portfolio currency composition process $\pi = (\pi_1, \cdots, \pi_m)$, and debt payment rate process p. Moreover, it also depends on the realizations of the random components of the exchange rates, i.e, the Brownian motion and the Poisson process, and their corresponding parameters σ, φ, and $(\lambda_1, \cdots, \lambda_m)$. Thus we have a realistic debt dynamics.

From the perspective of a developing country, the foreign interest rates and the randomness of the exchange rates are given. Furthermore, for a debt policy maker, the local interest rate is essentially exogenous. On the other hand, the debt manager can exert control on the debt portfolio process π and, to some degree, on the debt payment rate process p. They are precisely the control variables in our government debt problem that we will describe in the next subsection.

The debt problem

In reality there exists a debt problem as long as public debt is positive. Thus, we consider the following stopping time

$$\Theta(\omega) := \inf\{s \geq 0 : X(s^-, \omega) \leq 0\} . \tag{5.14}$$

If $\Theta(\omega)$ is finite, in view of (5.11) and (5.13), we must have $X_\Theta(\omega) = X_{\Theta^-}(\omega) = 0$. We impose $X_t(\omega) = 0$ for every $t > \Theta(\omega)$. Thus, the dynamics of X is given by equation (5.10) for every $t \in [0, \Theta)$, and $X(t) = 0$ for every $t \geq \Theta$. We observe that $X(t) > 0$ for every $t \in [0, \Theta)$.

Now we turn our attention to the running costs, or the disutility function of the government. This cost depends on the debt payments and the level of the existing debt. The existence of the former cost comes from the fact that, in order to get positive fiscal results, countries have to cut spendings and increase taxes. On the other hand, the cost linked to the existing debt exists because high debt can lead to debt problems in the future, a default or a debt crisis, for example.

We observe that the disutility function most widely used in different areas of economics is quadratic (see, for example, Kydland and Prescott 1977, Taylor 1979, Cadenillas and Zapatero 1999, and Cadenillas et al. 2013). This function represents an agent with risk aversion. More precisely, it has constant relative risk aversion (CRRA) equal to 1 (see Remark 5.3 below). Since we are interested in analyzing the effects of debt aversion on the optimal currency debt portfolio, we consider a function in which the debt aversion is a parameter. Specifically, for $\gamma \in (0, \infty)$, we define the cost (or loss) function by

$$h(x, p) := \alpha x^{\gamma+1} + p^{\gamma+1}, \tag{5.15}$$

where x represents the public debt and p the debt payment rate. Here, $\alpha \in (0, \infty)$ is a parameter that represents the importance that the government gives to the existing debt x relative to the debt rate payment p. The function h has the property of CRRA equal to γ in x and p. Thus, the parameter γ represents the aversion of the debt manager with respect to the existing debt and the debt payment. That is, for a given debt level and debt payment, the bigger the parameter γ the higher the disutility of the government. For instance, countries that have never had a default or have never suffered a severe debt crisis (such as Canada and USA) have a lower γ than countries that have experienced serious debt problems (such as Argentina and Greece). Thus our specification of disutility function not only provides us with a unique parameter that characterizes fully the debt aversion of the government, but also generalizes the most common disutility function used in economics, namely, the quadratic function.

5.3 Remark. *For a utility function $u(y)$ the relative risk aversion is defined by $-yu''(y)/u'(y)$ (see, for instance, MasCollel et al. 1995 or Pratt 1964). Similarly, for a disutility function $d(y)$ we define the relative risk aversion as $yd''(y)/d'(y)$.*

Since $X(t) = 0$ for every $t \geq \Theta$, there is no debt problem after Θ. So we impose $p(t) = 0$ for every $t \geq \Theta$. Thus the total cost after the debt becomes zero is null. Hence

$$E_x \left[\int_{\Theta}^{\infty} e^{-\delta t} \Big(\alpha X_t^{\gamma+1} + p_t^{\gamma+1} \Big) dt \right] = 0.$$

Here $\delta > 0$ represents the discount rate.

As we discussed above, we will assume that the debt policy maker can exert control on both the currency debt portfolio and the debt payment rate. We provide below the formal definition of control process.

5.4 Definition. *Let $u : [0, \infty) \times \Omega \to \mathbb{R}^m \times \mathbb{R}$ be a process defined by $u(t, \omega) := \big(\pi(t, \omega), p(t, \omega)\big)$, where π is a portfolio debt process and p is a payment rate process, which are right-continuous with left-limits and adapted to $\mathbb{F}$. For a given $x > 0$, the process (π, p) will be called an admissible control process if it satisfies:*

$$(i) \qquad E_x \Big[\int_0^{\Theta} e^{-\delta t} h\big(X_t, p_t\big) dt \Big] \; < \; \infty \, , \tag{5.16}$$

$$(ii) \qquad \Big(1 + \pi^T(s^-) \varphi_j \Big) \; > \; 0, \quad \forall j \in \{1, \cdots, m\} \, , \quad \forall s \geq 0 \, , \tag{5.17}$$

$$(iii) \qquad E_x \Big[\int_0^t \Big(1 + \pi^T(s^-) \varphi_j \Big)^{2(\gamma+1)} ds \Big] < \infty, \quad \forall j \in \{1, \cdots, m\}, \quad \forall t \geq 0. \tag{5.18}$$

The set of all admissible controls will be denoted by $\mathcal{A}(x) = \mathcal{A}$.

5.5 Remark. *We observe that condition (5.16) implies $E_x \Big[\int_0^{\Theta} e^{-\delta t} X_t^{\gamma+1} \, dt \Big] < \infty$. As a result*

$$\lim_{t \to \infty} E_x \Big[e^{-\delta t} X_t^{\gamma+1} I_{\{t \leq \Theta\}} \Big] = 0. \tag{5.19}$$

To complete this section, we state the debt problem.

5.6 Problem. *Consider the debt dynamics given in Proposition 5.2. We want to select the admissible control* $\hat{u} = (\hat{\pi}, \hat{p})$ *that minimizes the performance functional given by*

$$J(x; u) = J(x; \pi, p) := E_x \left[\int_0^{\Theta} e^{-\delta t} \Big(\alpha X_t^{\gamma+1} + p_t^{\gamma+1} \Big) dt \right]. \tag{5.20}$$

The control $\hat{u} = (\hat{\pi}, \hat{p})$ *will be called optimal debt control.*

From a mathematical point of view, Problem 5.6 is a stochastic control problem with jumps. This theory has been studied and/or applied, for instance, in Jeanblanc-Picqué and Pontier (1990), Cadenillas (2002), Guo and Xu (2004), and Oksendal and Sulem (2008).

For future reference we present the following result.

5.7 Lemma. *Let* ρ *be a positive real number, and let* π *be a constant vector in* $\mathbb{R}^m$ *such that*

$$(1 + \pi^T \varphi_j) > 0, \quad \forall j \in \{1, \cdots, m\}.$$

Consider the following debt policy: $\pi(t) = \pi$ *and* $p(t) = \rho X(t)$ *for every* $t \geq 0$. *Then*

$$E_x \Big[e^{-\delta t} X_t^{\gamma+1} \Big] = x^{\gamma+1} \exp \Big\{ - \big[(\gamma+1)\rho + \zeta \big] t \Big\}, \tag{5.21}$$

where

$$\begin{aligned} \zeta \quad := \quad & \delta - 1/2\, (\gamma+1)\, \gamma \pi^T \sigma \sigma^T \pi - \sum_{j=1}^{m} \lambda_j \Big\{ (1 + \pi^T \varphi_j)^{\gamma+1} - 1 - (\gamma+1) \pi^T \varphi_j \Big\} \\ & - (\gamma+1) \big(\pi^T b + r_0 \big) . \end{aligned}$$

Proof. See Appendix B.4. □

To illustrate our framework, let us consider two examples. In the first one the debt manager chooses not to issue debt in foreign currencies and to pay only the interest rate of their current debt. The second example is a more general version of the first one.

5.8 Example. *Suppose the debt manager chooses the following debt policy: $\tilde{\pi}(t) = 0$ and $\tilde{p}(t) = r_0 X(t)$ for every $t \geq 0$. According to equation (5.10), the debt at every point in time equals the initial debt. That is, $\forall t \geq 0$,*

$$X(t) = x\,.$$

This implies that $X(t) > 0$ for all $t \geq 0$, and hence $\Theta = \infty$. Thus the total discounted government cost (disutility) is

$$J(x; \tilde{\pi}, \tilde{p}) = \left(\frac{\alpha + r_0^{\gamma+1}}{\delta}\right) x^{\gamma+1}\,. \tag{5.22}$$

We point out that the debt policy $(\tilde{\pi}, \tilde{p})$ is admissible if and only if $\delta > 0$. Since the latter condition is satisfied, this policy is admissible.

5.9 Example. *Let π_c be an arbitrary constant vector in $\mathbb{R}^m$ whose components are positive, and such that condition (5.17) is satisfied. Let ρ be a given positive real number. Suppose the debt manager considers the following debt policy: $\tilde{\pi}(t) = \pi_c \in \mathbb{R}^m$ and $\tilde{p}(t) = \rho X(t)$ for every $t \geq 0$. Then, considering this debt policy in equation (5.10), the dynamics of the debt becomes*

$$dX(t) \;=\; (r_0 - \rho)X(t)dt \;+\; X(t)\pi_c^T b\,dt \;+\; X(t)\pi_c^T \sigma dW(t) \;+\; X(t^-)\pi_c^T \varphi d\tilde{N}(t)\,,$$

with $X_0 = x$. We note that the above stochastic differential equation (SDE) has the form of equation (5.10), except for the last term. Consequently, using equation (5.12) with $p(s) = 0$ for $s \geq 0$, the solution to the above SDE is given by

$$X(t) = x\exp\left\{\int_0^t \beta\,ds + \int_0^t \pi_c^T \sigma dW(s) + \sum_{j=1}^m \int_0^t \log\left(1 + \pi_c^T \varphi_j\right) dN_j(s)\right\}, \tag{5.23}$$

with

$$\beta := r_0 - \rho + \pi_c^T b - \sum_{j=1}^{m} \lambda_j \pi_c^T \varphi_j - \frac{1}{2}\pi_c^T \sigma\sigma^T \pi_c.$$

We observe that $X(t) > 0$ for every $t \geq 0$. Hence $\Theta = \infty$. Then the discounted government disutility is

$$\begin{aligned} J(x;\pi_c,p) &= E_x\left[\int_0^\infty e^{-\delta t} h(X_t)dt\right] \\ &= E_x\left[\int_0^\infty e^{-\delta t}\Big(\alpha X_t^{\gamma+1} + \rho^{\gamma+1} X_t^{\gamma+1}\Big)dt\right] \\ &= (\alpha+\rho^{\gamma+1})\int_0^\infty e^{-\delta t} E_x\left[X_t^{\gamma+1}\right] dt \\ &= (\alpha+\rho^{\gamma+1})x^{\gamma+1}\int_0^\infty e^{-\big((\gamma+1)\rho+\zeta\big)t} dt \\ &= \begin{cases} \frac{(\alpha+\rho^{\gamma+1})x^{\gamma+1}}{(\gamma+1)\rho+\zeta} & \text{if } (\gamma+1)\rho+\zeta > 0, \\ \infty & \text{if } (\gamma+1)\rho+\zeta \leq 0. \end{cases} \end{aligned}$$

To get the fourth equality above, we have used the result computed in Lemma 5.7, with

$$\begin{aligned} \zeta &:= \delta - 1/2\,(\gamma+1)\gamma\pi_c^T\sigma\sigma^T\pi_c - \sum_{j=1}^{m}\lambda_j\Big\{(1+\pi_c^T\varphi_j)^{(\gamma+1)} - 1 - (\gamma+1)\pi_c^T\varphi_j\Big\} \\ &\quad - (\gamma+1)\big(\pi_c^T b + r_0\big). \end{aligned}$$

We note that (5.18) is satisfied because π_c is a constant vector. Thus the debt policy given in Example 5.9 is admissible if and only if $(\gamma+1)\rho+\zeta > 0$. It would be interesting to compare those policies with the optimal debt policy that we will obtain in Section 5.3. Certainly, we expect this type of arbitrary debt policies not to be, in general, optimal. We will confirm this fact in Section 5.4.

5.2 A verification theorem

The main purpose of this section is to state a sufficient condition that an optimal solution of the debt problem must satisfy. The value function is a key instrument to achieve that goal.

We define the value function $V : (0, \infty) \to \mathbb{R}$ by

$$V(x) \;:=\; \inf_{(\pi,\,p)\,\in\,\mathcal{A}(x)} J(x; \pi, p). \tag{5.24}$$

This represents the smallest expected cost that can be achieved when the initial debt is $x > 0$ and we consider all the admissible debt controls.

5.10 Proposition. *The value function V is non-negative and homogeneous of degree $\gamma + 1$. Therefore, it is increasing, convex and $V(0+) = 0$.*

Proof. See Appendix B.5. □

We require some notation to define the Hamilton-Jacobi-Bellman equation. Let $g : (0, \infty) \to \mathbb{R}$ be a function in $C^2(0, \infty)$. For $\pi \in \mathbb{R}^m$ and $p \in \mathbb{R}$, let us define the operator $\mathcal{L}(\pi, p)$ by

$$\begin{aligned} \mathcal{L}(\pi, p)g(x) \;:=\;& \tfrac{1}{2}\pi^T\sigma\sigma^T\pi x^2 g''(x) + \left(\pi^T bx + r_0 x - p\right) g'(x) - \delta g(x) \\ &+ \textstyle\sum_{j=1}^m \lambda_j \Big(g(x + \pi^T\varphi_j x) - g(x) - g'(x)\pi^T\varphi_j x\Big), \end{aligned} \tag{5.25}$$

where we recall that φ_j is the j-th column of matrix φ.

For a function $v : (0, \infty) \to \mathbb{R}$ in $C^2(0, \infty)$, consider the Hamilton-Jacobi-Bellman (HJB) equation

$$\forall x > 0: \quad \min_{(\pi,p)} \Big\{\mathcal{L}(\pi, p)v(x) + h(x, p)\Big\} = 0. \tag{5.26}$$

Next we state a sufficient condition for a debt policy to be optimal.

5.11 Theorem. *Let* $v \in C^2(0,\infty)$ *be an increasing and convex function on* $(0,\infty)$ *with* $v(0+) = 0$*. Suppose that* v *satisfies the HJB equation (5.26) for every* $x \in (0,\infty)$*, and the polynomial growth condition*

$$v(x) \leq C(1 + x^{\gamma+1}), \tag{5.27}$$

for some constant C*. Then, for every* $x \in (0,\infty)$*, we have the following two results.*

(a) For every $(\pi, p) \in \mathcal{A}(x)$*:*

$$v(x) \leq J(x; \pi, p).$$

(b) Suppose that the stochastic control $\hat{u} = (\hat{\pi}, \hat{p})$*, defined by*

$$\hat{u} = (\hat{\pi}, \hat{p}) := \underset{(\pi,p)\in\mathcal{A}}{\operatorname{argmin}} \Big\{ \mathcal{L}(\pi, p) v(x) + h(x, p) \Big\}, \tag{5.28}$$

is admissible for $X = \hat{X}$ *and* $\theta = \hat{\Theta}$*. Then*

$$v(x) = J(x; \hat{\pi}, \hat{p}).$$

In other words, $(\hat{\pi}, \hat{p})$ *is the optimal debt control and* $V = v$ *is the value function for Problem 5.6. Here* $\hat{X}$ *is the debt process generated by the control* $(\hat{\pi}, \hat{p})$*, and* $\hat{\Theta}$ *is the corresponding stopping time defined in (5.14).*

Proof. Let (π, p) be an admissible control process, whose corresponding debt dynamics is given by equation (5.10). Let τ be a stopping time such that $\tau \leq \Theta$, and let us consider the non-negative process $\{X_{t\wedge\tau} : t \geq 0\}$. Since v is twice continuously differentiable, an application of Ito's formula

$$e^{-\delta(t\wedge\tau)} v\big(X_{t\wedge\tau}\big) = v(X_0) + \int_0^{t\wedge\tau} e^{-\delta s} \Big(\mathcal{L}(\pi_s, p_s) v(X_s)\Big) ds + M(t) + \sum_{j=1}^{m} M_j(t), \tag{5.29}$$

where $\mathcal{L}$ is the operator defined in (5.25), and

$$M(t) := \int_0^{t\wedge\tau} e^{-\delta s} v'(X_s)\pi_s^T \sigma X_s dW(s),$$
$$M_j(t) := \int_0^{t\wedge\tau} e^{-\delta s}\Big(v\left(X_{s^-} + \pi_{s^-}^T \varphi_j X_{s^-}\right) - v(X_{s^-})\Big) d\tilde{N}_j(s),$$

for every $j \in \{1, 2, \cdots, m\}$.

Let a and b be real numbers such that $0 < a < X_0 = x < b < \infty$. We define $\tau_a := \inf\{s \geq 0 : X_{s^-} \leq a\}$, $\tau_b := \inf\{s \geq 0 : X_{s^-} \geq b\}$, and $\tau_{ab} := \tau_a \wedge \tau_b$. We observe that $\tau_{ab} \leq \Theta$. Hence, for $s < \tau_{ab}$ we have that both X_s and X_{s^-} belong to the interval $[a\,, b]$. In view of (5.11) and (5.13), if $\tau_a < \infty$, then we have $0 < X_{\tau_a} \leq a$.

From now on, we set $\tau = \tau_{ab}$. We claim that the above stochastic integrals M and M_j are martingales. Indeed, since $X_s \in [a, b]$ for all $s \in [0, \tau_{ab})$, and v' is continuous, there exists a real number $\xi > 0$ such that:

$$E_x\Big[\int_0^{t\wedge\tau_{ab}} e^{-2\delta s}\Big(v'(X_s)\Big)^2 \pi_s^T \sigma\sigma^T \pi_s X_s^2\, ds\Big] \;\leq\; \xi E_x\Big[\int_0^{t} \pi_s^T \sigma\sigma^T \pi_s\, ds\Big] \;<\; \infty\,.$$

Hence the integral with respect to the Brownian motion above is a martingale. Similarly, we prove the other part of the claim, that is, M_j is a martingale. It suffices to show that for each $j \in \{1, \cdots, m\}$

$$E_x\Big[\int_0^{t\wedge\tau_{ab}} e^{-2\delta s}\lambda_j\Big(v\big(X_{s^-} + \pi_{s^-}^T \varphi_j X_{s^-}\big) - v(X_{s^-})\Big)^2 ds\Big] \;<\; \infty\,.$$

Indeed, recalling that $X_{s^-} \in [a, b]$ for all $s \in [0, \tau_{ab})$, and v is continuous, we have that $E_x\big[\int_0^{t\wedge\tau_{ab}} e^{-2\delta s}\lambda_j\Big(v(X_{s^-})\Big)^2 ds\big]$ is bounded. Using equations (5.18) and (5.27), we conclude that $E_x\big[\int_0^{t\wedge\tau_{ab}} e^{-2\delta s}\lambda_j\Big(v\big(X_{s^-} + \pi_{s^-}^T \varphi_j X_{s^-}\big)\Big)^2 ds\big]$ is also bounded. Hence the previous inequality holds. This proves that the integrals with respect to the compensated Poisson process are martingales. This completes the proof of the claim. Consequently, for every $t \geq 0$, we have $E_x[M(t)] = 0$, and $E_x[M_j(t)] = 0$ for each $j \in \{1, \cdots, m\}$.

Taking expectations in equation (5.29),

$$E_x\Big[e^{-\delta(t\wedge\tau_{ab})}v(X_{t\wedge\tau_{ab}})\Big] = v(x) + E_x\Big[\int_0^{t\wedge\tau_{ab}} e^{-\delta s}\Big(\mathcal{L}(\pi_s,p_s)v(X_s)\Big)ds\Big].$$

Since the HJB equation (5.26) implies that, for every $s \in [0,\tau)$,

$$\mathcal{L}(\pi_s,p_s)v(X_s) \;\geq\; -h(X_s,p_s), \tag{5.30}$$

it follows that

$$v(x) \;\leq\; E_x\Big[e^{-\delta(t\wedge\tau_{ab})}v(X_{t\wedge\tau_{ab}})\Big] + E_x\Big[\int_0^{t\wedge\tau_{ab}} e^{-\delta s}h(X_s,p_s)ds\Big]. \tag{5.31}$$

Letting $b \uparrow \infty$, we have $\tau_b \uparrow \infty$, and hence $\tau_{ab} \uparrow \tau_a$. By the Monotone Convergence Theorem,

$$\lim_{b\to\infty} E_x\Big[\int_0^{t\wedge\tau_{ab}} e^{-\delta s}h(X_s,p_s)ds\Big] = E_x\Big[\int_0^{t\wedge\tau_a} e^{-\delta s}h(X_s,p_s)ds\Big].$$

On the other hand, since v is continuous, letting $b \uparrow \infty$, we obtain

$$v(X_{t\wedge\tau_{ab}}) \;\rightarrow\; v(X_{t\wedge\tau_a})$$

and

$$e^{-\delta(t\wedge\tau_{ab})} = e^{-\delta(t\wedge\tau_a)}.$$

Hence

$$\begin{aligned}\lim_{b\to\infty} E_x\Big[e^{-\delta(t\wedge\tau_{ab})}v(X_{t\wedge\tau_{ab}})\Big] &= E_x\Big[\lim_{b\to\infty} e^{-\delta(t\wedge\tau_{ab})}v(X_{t\wedge\tau_{ab}})\Big]\\ &= E_x\Big[e^{-\delta(t\wedge\tau_a)}v(X_{t\wedge\tau_a})\Big].\end{aligned}$$

Consequently, taking the limit as $b \uparrow \infty$ in inequality (5.31), we get

$$v(x) \le E_x\Big[e^{-\delta(t\wedge\tau_a)}v(X_{t\wedge\tau_a})\Big] + E_x\Big[\int_0^{t\wedge\tau_a} e^{-\delta s}h(X_s,p_s)ds\Big]. \tag{5.32}$$

Now letting $a \downarrow 0$, we have $\tau_a \uparrow \Theta$. Recalling that $v(0+) = 0$, taking the limit as $a \downarrow 0$, and proceeding as in the case $b \uparrow \infty$, we obtain

$$\lim_{a\to 0} E_x\Big[\int_0^{t\wedge\tau_a} e^{-\delta s}h(X_s,p_s)ds\Big] = E_x\Big[\int_0^{t\wedge\Theta} e^{-\delta s}h(X_s,p_s)ds\Big],$$

and

$$\lim_{a\to 0} E_x\Big[e^{-\delta(t\wedge\tau_a)}v(X_{t\wedge\tau_a})\Big] = E_x\Big[\lim_{a\to 0} e^{-\delta(t\wedge\tau_a)}v(X_{t\wedge\tau_a})\Big] = E_x\Big[e^{-\delta t}v(X_t)I_{\{t<\Theta\}}\Big].$$

Taking the limit as $a \downarrow 0$ in inequality (5.32), we get

$$v(x) \le E_x\Big[e^{-\delta t}v(X_t)I_{\{t<\Theta\}}\Big] + E_x\Big[\int_0^{t\wedge\Theta} e^{-\delta s}h(X_s,p_s)ds\Big].$$

We note that (5.19) and (5.27) imply

$$\lim_{t\to\infty} E_x\Big[e^{-\delta t}v(X_t)I_{\{t<\Theta\}}\Big] = 0.$$

Now letting $t \to \infty$, by the Monotone Convergence Theorem, we conclude that

$$v(x) \le E_x\Big[\int_0^{\Theta} e^{-\delta s}h(X_s,p_s)ds\Big]. \tag{5.33}$$

This proves part (a) of this Theorem.

Let us show part (b). Since $(\hat{\pi},\hat{p})$ satisfies (5.28), the HJB equation (5.26) implies that the inequality (5.30) becomes an equality when $(\pi_s,p_s) = (\hat{\pi}_s,\hat{p}_s)$ and $X(s) = \hat{X}(s)$. Consequently, inequality (5.33) becomes an equality as well. This completes the proof of this theorem. □

5.3 The explicit solution

At the beginning of this section we are going to make conjectures to obtain a candidate for optimal debt control and a candidate for value function. At the end of this section, we are going to apply Theorem 5.11 to prove rigorously that the candidate for optimal control is indeed the optimal control, and the candidate for value function is indeed the value function.

We want to find a control (π, p) and the corresponding function v that satisfy the conditions of Theorem 5.11. According to equation (5.28) in that theorem,

$$\hat{\pi} = \underset{\pi}{\operatorname{argmin}}\Big[\frac{1}{2}x^2\pi^T\sigma\sigma^T\pi v''(x) + x\pi^T\, bv'(x) + \sum_{j=1}^{m}\lambda_j\Big(v\big(x+\pi^T\varphi_j x\big) - v'(x)\pi^T\varphi_j x\Big)\Big],$$

$$\hat{p} = \underset{p}{\operatorname{argmin}}\Big[-pv'(x) + h(x,p)\Big].$$

Let us conjecture that v is strictly convex. Then, if $\hat{\pi} \in \mathbb{R}^m$ satisfies the following equation, it is a global minimum.

$$\pi^T\sigma\sigma^T x^2 v''(x) + b^T x v'(x) + \sum_{j=1}^{m}\lambda_j\Big(v'\left(x+\pi^T\varphi_j x\right)\varphi_j^T x - v'(x)\varphi_j^T x\Big) = 0. \tag{5.34}$$

Given that h is strictly convex, if $\hat{p} \in \mathbb{R}_+$ solves the following equation, it is a global minimum.

$$-v'(x) + \frac{\partial h(x,p)}{\partial p} = 0\,.$$

Thus, if we know the value function v, we can characterize the candidate for optimal debt control $(\hat{\pi}, \hat{p})$. Based on both the proof of Proposition 5.10 and the form of the disutility function h, we conjecture that the value function for $\gamma \in (0, \infty)$ is given by

$$v(x) = Kx^{\gamma+1}\,, \tag{5.35}$$

for every $x > 0$ and some constant $K > 0$.

By means of this conjecture, using (5.34), we characterize $\hat{\pi} \in \mathbb{R}^m$ as the vector that solves the following equation in π:

$$\gamma\sigma\sigma^T\pi + b + \sum_{j=1}^{m} \lambda_j \varphi_j \Big\{(1+\pi^T\varphi_j)^\gamma - 1\Big\} \;=\; 0\,; \tag{5.36}$$

and for $\hat{p} \in \mathbb{R}$ we have:

$$\hat{p}(t) = K^{\frac{1}{\gamma}} \hat{X}(t)\,, \tag{5.37}$$

where $\hat{X}$ is the debt dynamics generated by the debt policy $(\hat{\pi}, \hat{p})$. Since all the entries in σ and φ are constants, we observe that the process $\hat{\pi}$ is indeed a constant vector in $\mathbb{R}^m$.

To complete the specification of the candidates, it remains to characterize the constant K. By Theorem 5.11, we know that for the function v and the process $\hat{u} = (\hat{\pi}, \hat{p})$ to be suitable candidates for solutions, they should satisfy the HJB equation (5.26). Considering it along with equation (5.28), we must have

$$\mathcal{L}(\hat{\pi}, \hat{p})v(x) + h(x, \hat{p}) = 0. \tag{5.38}$$

After simplifying the previous equation, we obtain the equivalent form:

$$\gamma K^{\frac{\gamma+1}{\gamma}} + \hat{\zeta} K - \alpha = 0\,, \tag{5.39}$$

where

$$\begin{aligned}\hat{\zeta} := \delta - \sum_{j=1}^{m} \lambda_j \Big\{(1+\hat{\pi}^T\varphi_j)^{\gamma+1} - 1 - (\gamma+1)\hat{\pi}^T\varphi_j\Big\} - 1/2\,(\gamma+1)\gamma\hat{\pi}^T\sigma\sigma^T\hat{\pi} \\ - (\gamma+1)\big(\hat{\pi}^T b + r_0\big).\end{aligned} \tag{5.40}$$

Here we emphasize that the choice of K in (5.39) guarantees that $v(x) = Kx^{\gamma+1}$

satisfies the HJB equation (5.26) for $(\hat{\pi}, \hat{p})$. Let us prove that indeed $K > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(z) := \gamma z^{\frac{\gamma+1}{\gamma}} + \hat{\zeta} z - \alpha \,.$$

We note that f is strictly convex on $(0, \infty)$. Since $f(0^+) = -\alpha < 0$ and $f(+\infty) = +\infty$, then there exists a unique real number $K > 0$ such that $f(K) = 0$, as required.

To complete this section, we are going to prove rigorously that for $\gamma \in (0, \infty)$ the above candidate for optimal debt control (5.36)-(5.37) is indeed the optimal control, and the above candidate for value function (5.35) is indeed the value function. For such proof, we observe from (5.39) that

$$(\gamma + 1)K^{1/\gamma} + \hat{\zeta} = \frac{\alpha}{K} + K^{1/\gamma} > 0, \tag{5.41}$$

because $K > 0$ and $\alpha > 0$.

5.12 Theorem. *Suppose $\hat{\pi} \in \mathbb{R}^m$ satisfies:*

$$(i) \qquad \gamma \sigma \sigma^T \hat{\pi} + b + \sum_{j=1}^{m} \lambda_j \varphi_j \left\{ \left(1 + \hat{\pi}^T \varphi_j\right)^{\gamma} - 1 \right\} = 0 \,, \tag{5.42}$$

$$(ii) \qquad 1 + \hat{\pi}^T \varphi_j > 0, \quad \textit{for every } j \in \{1, \cdots, m\}. \tag{5.43}$$

Let $\hat{p}$ be the process defined by

$$\hat{p}(t) := K^{\frac{1}{\gamma}} \hat{X}(t), \tag{5.44}$$

where K is the unique positive real solution to equation (5.39), and $\hat{X}$ denotes the debt process generated by the debt control $\hat{u} = (\hat{\pi}, \hat{p})$, which is given by

$$\hat{X}_t = x \exp \left\{ \beta\, t + \hat{\pi}^T \sigma\, W(t) + \sum_{j=1}^{m} \log\left(1 + \hat{\pi}^T \varphi_j\right) N_j(t) \right\},$$

with

$$\beta := r_0 - K^{1/\gamma} + \hat{\pi}^T b - \sum_{j=1}^{m} \lambda_j \hat{\pi}^T \varphi_j - \frac{1}{2}\hat{\pi}^T \sigma\sigma^T \hat{\pi}.$$

Then $\hat{u} = (\hat{\pi}, \hat{p})$ *is optimal, and* $V(x) = v(x) = Kx^{\gamma+1}$ *is the value function for Problem 5.6.*

Proof. To prove this theorem, it suffices to show that all conditions of Theorem 5.11 are satisfied. Regarding the candidate for value function, we get immediately that $v \in C^2\big(0,\infty\big)$ and $v(0+) = 0$. Moreover, since $K > 0$, v is increasing and strictly convex on its domain. On the other hand, by construction, we observe that the candidate v also satisfies the HJB equation (5.26),

$$\mathcal{L}(\hat{\pi}, \hat{p})v(x) + h(x, \hat{p}) = 0.$$

Taking $C = K$, condition (5.27) also holds. It remains to verify that conditions (5.16) and (5.18) are satisfied for $(\hat{\pi}, \hat{p})$. Since $\hat{\pi}$ is a constant vector, condition (5.18) is immediate. To show condition (5.16), we will use the result in Lemma 5.7, equation (5.21), with $\rho = K^{1/\gamma}$. We note that, since $(\hat{\pi}, \hat{p})$ is one case of the class of debt policies given in Example 5.9, we know that $\hat{\Theta} = \infty$. Thus,

$$\begin{aligned} E_x\Big[\int_0^{\hat{\Theta}} e^{-\delta t} h(\hat{X}_t, \hat{p}_t)dt\Big] \\ &= E_x\Big[\int_0^{\infty} e^{-\delta t}\Big(\alpha X_t^{\gamma+1} + K^{(\gamma+1)/\gamma}\hat{X}_t^{\gamma+1}\Big)dt\Big] \\ &= \Big(\alpha + K^{(\gamma+1)/\gamma}\Big)\int_0^{\infty} E[e^{-\delta t}\hat{X}_t^{\gamma+1}]dt \\ &= \Big(\alpha + K^{(\gamma+1)/\gamma}\Big)x^{\gamma+1}\int_0^{\infty} \exp\Big\{-\Big((\gamma+1)K^{\frac{1}{\gamma}} + \hat{\zeta}\Big)t\Big\} \\ &< \infty, \end{aligned}$$

where the last inequality follows from (5.41), that is, $(\gamma+1)K^{1/\gamma} + \hat{\zeta} > 0$. Hence, $(\hat{\pi}, \hat{p})$ is admissible.

By virtue of Theorem 5.11, $(\hat{\pi}, \hat{p})$ is optimal and $v(x) = Kx^{\gamma+1}$ is the value function for Problem 5.6. □

We observe that the explicit solution for the debt process $\hat{X}$ given in Theorem 5.12 is an extension of a geometric Brownian motion to include jumps. This model is general enough to reflect the debt evolution in reality. For instance, if $\beta > 0$, then the theoretical trend generated by the model is consistent with the recent trend of most countries in which debt increases over time. Moreover, it is also a generalization of the basic stochastic model for the debt evolution presented in Greiner and Fincke (2009).

As an application of Theorem 5.12, we present below two particular cases in which the government debt portfolio can be obtained explicitly.

5.13 Remark. *Let $\gamma = 1$. Then the value function is $V(x) = Kx^2$, and the optimal debt control is given by*

$$\begin{aligned}\hat{\pi}(s) &= [\sigma\sigma^T + \sum_{j=1}^{m} \lambda_j \varphi_j \varphi_j^T]^{-1}(r_0 \mathbb{1} - r - \mu)\,,\\ \hat{p}(s) &= K\hat{X}(s)\,.\end{aligned}$$

Here $\hat{X}$ is the debt process generated by the debt policy $(\hat{\pi}, \hat{p})$. The parameter K is given by

$$K := \frac{\sqrt{c^2 + 4\alpha} - c}{2} > 0\,,$$

where

$$c := \delta - 2r_0 + b^T\,\Gamma^{-1}\,b\,,$$

with $\Gamma = [\sigma\sigma^T + \sum_{j=1}^{m} \lambda_j \varphi_j \varphi_j^T]$.

5.14 Remark. *Suppose that there are no jumps in the exchange rates. Then the* *value function is* $V(x) = Kx^{\gamma+1}$*, and the optimal debt control is given by*

$$\hat{\pi}(s) = \frac{(\sigma\sigma^T)^{-1}(r_0\mathbb{1} - r - \mu)}{\gamma},$$

$$\hat{p}(s) = K^{1/\gamma}\hat{X}(s).$$

Here $\hat{X}$ *is the debt process generated by the debt policy* $(\hat{\pi}, \hat{p})$*. The parameter* K *is given by (5.39) with*

$$\hat{\zeta} := \delta - \frac{1}{2}(\gamma+1)\gamma\hat{\pi}^T\sigma\sigma^T\hat{\pi} - (\gamma+1)(\hat{\pi}^T b + r_0).$$

If $\gamma = 1$*, then we have an explicit formula for* K*, namely*

$$K := \frac{\sqrt{\hat{\zeta}^2 + 4\alpha} - \hat{\zeta}}{2} > 0.$$

5.4 Economic analysis

In this section we analyze the effects of some parameters on both the optimal debt control and the value function. To simplify the analysis and facilitate the interpretation of the implications of the model, in this section we will consider two currencies: the local and one foreign currency. In other words, throughout this section $m = 1$.

We recall that the source of the jumps in our model comes from a Poisson process, whose jumps have size one. The parameter φ allows us to introduce arbitrary sizes of the jump. In other words, anytime the Poisson process jumps, the effect on the exchange rate depreciation (or devaluation) is φ. Thus, to be consistent with empirical currency crises, we are going to assume $\varphi > 0$.

Optimal solution with two currencies

Under the previous considerations, we have the following straightforward corollary of Theorem 5.12.

5.15 Corollary. *Let* $m = 1$ *and* $\gamma \in (0, \infty)$. *Suppose* $\hat{\pi}_1 \in \mathbb{R}$ *satisfies:*

$$(i) \qquad \gamma\sigma^2\hat{\pi}_1 + b + \lambda\varphi\Big\{\big(1 + \hat{\pi}_1\varphi\big)^{\gamma} - 1\Big\} = 0\,, \tag{5.45}$$

$$(ii) \qquad 1 + \hat{\pi}_1\varphi > 0\,. \tag{5.46}$$

Let $\hat{p}$ *be the process defined by*

$$\hat{p}(t) = K^{\frac{1}{\gamma}}\hat{X}(t), \tag{5.47}$$

where $\hat{X}$ *denotes the debt process generated by the debt control* $\hat{u} = (\hat{\pi}_1, \hat{p})$, *and* K *is the unique positive real solution to equation*

$$\gamma K^{\frac{\gamma+1}{\gamma}} + \hat{\zeta}_1\, K - \alpha = 0\,,$$

where

$$\hat{\zeta}_1 := \delta - \lambda\Big\{(1 + \hat{\pi}_1\varphi)^{\gamma+1} - 1 - (\gamma+1)\hat{\pi}_1\varphi\Big\} - 1/2(\gamma+1)\gamma\sigma^2\hat{\pi}_1^2 - (\gamma+1)\big(\hat{\pi}_1 b + r_0\big).$$

Then $(\hat{\pi}_1, \hat{p})$ *is the optimal debt control, and* $V(x) = v(x) = Kx^{\gamma+1}$ *is the value function for Problem 5.6.*

Proof. Set $m = 1$ in Theorem 5.12. □

We know that developing countries hold positive proportions of foreign currency in their debt portfolio. With this fact in mind, we provide below a proposition which states a sufficient and necessary condition for a positive solution $\hat{\pi}_1 \in \mathbb{R}$ to exist.

5.16 Proposition. *Suppose all the assumptions in Corollary 5.15 are satisfied. Then we have $b < 0$ if and only if there exists a unique $\hat{\pi}_1 > 0$ such that (5.45) and (5.46) are satisfied. In either case $\hat{\pi}_1 \in (0, \bar{\eta})$, where $\bar{\eta} := \frac{\lambda\varphi - b}{\gamma\sigma^2} > 0$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(z) := \gamma\sigma^2 z + \lambda\varphi\big(1 + z\varphi\big)^\gamma - (\lambda\varphi - b)\,.$$

Note first that $f(0) = b$ and $f(+\infty) = +\infty$. Suppose $b < 0$. Since f is strictly increasing and continuous, there exists a unique $\hat{\pi}_1 > 0$ such that $f(\hat{\pi}_1) = 0$. Now suppose that there exists a unique $\hat{\pi}_1 > 0$ such that (5.45) and (5.46) are satisfied. Then, from

$$-b = f(\hat{\pi}_1) - b = \gamma\sigma^2\hat{\pi}_1 + \lambda\varphi\big(1 + \hat{\pi}_1\varphi\big)^\gamma - \lambda\varphi > 0,$$

we conclude that $b < 0$. To show the $\bar{\eta}$ is an upper bound, observe that condition (5.45) implies

$$\gamma\sigma^2(\bar{\eta} - \hat{\pi}_1) = (\lambda\varphi - b) - \gamma\sigma^2\hat{\pi}_1 = \lambda\varphi(1 + \hat{\pi}_1\varphi)^\gamma > 0.$$

□

Proposition 5.16 says that to capture the reality of developing countries we need to assume $b < 0$. Otherwise, we do not get the empirical fact that the proportion of foreign debt in the government debt portfolio is positive. Accordingly, from now on, we assume $b = r + \mu - r_0 < 0$. That is, the interest rate of domestic currency debt r_0 is greater than the interest rate of foreign currency debt r plus the rate of depreciation (or devaluation) of the exchange rate μ.

Optimal policy versus arbitrary debt policies

We compare numerically the optimal debt policy we have obtained with the ones given in Examples 5.8 and 5.9 of Section 2. We will use the parameter values in Table 5.1 for the numerical computations.

Table 5.1: Parameter values

φ	r_0	r	σ	δ	α	λ	b
0.2	0.09	0.05	0.05	0.9	0.1	1	-0.02

Specifically, we will consider three debt policies chosen arbitrarily, and denoted by $(\pi^{(i)}, p^{(i)})$, with $i \in \{1,2,3\}$. For every $t \geq 0$ we have $\pi^{(1)}(t) = 0$, $\pi^{(2)}(t) = 50\%$, $\pi^{(3)}(t) = 100\%$, and $p^{(i)}(t) = r_0 X(t)$, for $i \in \{1,2,3\}$. The corresponding disutilities are given by $J^{(i)}\big(x\big) := J\big(x; \pi^{(i)}, p^{(i)}\big)$. We will compare them with the optimal debt policy $(\hat{\pi}, \hat{p})$ given in Corollary 5.15. We recall that the value function V represents the minimum disutility, and is a function of the initial debt. Hence, the cost $J(x; u^{(i)})$ must be greater than or equal to $V(x)$.

Table 5.2: Disutility values of some debt policies and value function

	$J^{(1)}(x)$	$J^{(2)}(x)$	$J^{(3)}(x)$	$V(x)$
$\gamma = 0.8$	$0.11617\, x^{1.8}$	$0.11484\, x^{1.8}$	$0.11543\, x^{1.8}$	$0.11443\, x^{1.8}$
$\gamma = 1.0$	$0.11388\, x^{2.0}$	$0.11272\, x^{2.0}$	$0.11421\, x^{2.0}$	$0.10889\, x^{2.0}$
$\gamma = 1.2$	$0.11264\, x^{2.2}$	$0.11166\, x^{2.2}$	$0.11426\, x^{2.2}$	$0.10225\, x^{2.2}$

See Table 5.1 for the values of the other parameters used in these computations.

Table 5.2 shows that indeed $J^{(i)}(x) > V(x)$ for $i \in \{1,2,3\}$. As shown in Table 5.3, the optimal currency debt portfolio and the optimal payments are sensitive to the degree of debt aversion γ. In particular, the higher the degree of debt aversion, the lower the proportion of foreign currency debt in the government portfolio. Indeed, this result will be proved rigorously in the next subsection.

Table 5.3: The optimal solution

	$\hat{\pi}(t)$	$\hat{p}(t)$	$V(x)$
$\gamma = 0.8$	0.59459	$0.06655\, X(t)$	$0.11443\, x^{1.8}$
$\gamma = 1.0$	0.47058	$0.10889\, X(t)$	$0.10889\, x^{2.0}$
$\gamma = 1.2$	0.38936	$0.14952\, X(t)$	$0.10225\, x^{2.2}$

See Table 5.1 for the values of the other parameters used in these computations.

After illustrating that the government gets the best result by following the optimal debt policy given in Corollary 5.15, we proceed with the economic analysis.

Economic results

We will state and prove two economic results. We will use the Intermediate Value Theorem to prove them.

ECONOMIC RESULT 1 *This economic result shows the effects of jumps in the exchange rates on the optimal currency debt portfolio. Let* $b < 0$. *Let* $\hat{\pi}(\lambda, \varphi)$ *represent the optimal proportion of foreign currency debt when the intensity of the Poisson process* N *is* λ, *and the size of the jumps is* φ. *Then*

(i) For every $\varphi > 0$, $\lambda_1 > \lambda_0$ implies $\hat{\pi}(\lambda_1, \varphi) < \hat{\pi}(\lambda_0, \varphi)$,

(ii) For every $\lambda > 0$, $\varphi_1 > \varphi_0$ implies $\hat{\pi}(\lambda, \varphi_1) < \hat{\pi}(\lambda, \varphi_0)$.

The Economic Result 1 implies that countries with debt in a foreign currency, whose exchange rate shows recurrent and big depreciations (or devaluations), will reduce its proportion of this type of debt in their portfolio in favor of debt in local currency.

Proof. Economic Result 1. (i) Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(z, \lambda) := \gamma\sigma^2 z + \lambda\varphi\big(1 + z\varphi\big)^{\gamma} - (\lambda\varphi - b)\,.$$

We note that f is strictly increasing and continuous in each of its arguments. By Corollary 5.15, and definition of $\hat{\pi}(\lambda_0, \varphi)$, we have $f\big(\hat{\pi}(\lambda_0, \varphi), \lambda_0\big) = 0$. Since f is strictly increasing in λ,

$$f\big(\hat{\pi}(\lambda_0, \varphi), \lambda_1\big) > f\big(\hat{\pi}(\lambda_0, \varphi), \lambda_0\big) = 0.$$

On the other hand, we observe that

$$f\big(0, \lambda_1\big) = b < 0.$$

Applying the Intermediate Value Theorem to $f(\cdot, \lambda_1)$, there exists a unique $z_0 \in \big(0, \hat{\pi}(\lambda_0, \varphi)\big)$ such that $f(z_0, \lambda_1) = 0$. By the uniqueness of the solution given in Proposition 5.16, we must have $z_0 = \hat{\pi}(\lambda_1, \varphi)$. Hence the claim holds.

(ii) This claim can be established in a similar manner. □

For a special case, we provide below the explicit effect of λ and φ on the optimal proportion of foreign debt.

5.17 Remark. *Let us consider $\gamma = 1$ and $m = 1$. Then, using Remark 5.13 , we have*

$$\frac{\partial \hat{\pi}}{\partial \lambda} = \frac{b\varphi^2}{(\sigma^2 + \lambda\varphi^2)^2} < 0\,,$$

$$\frac{\partial \hat{\pi}}{\partial \varphi} = \frac{2\lambda\varphi b}{(\sigma^2 + \lambda\varphi^2)^2} < 0.$$

ECONOMIC RESULT 2 *This economic assertion is concerned with the effects of debt aversion on the optimal currency debt portfolio. Let $b < 0$. For $\gamma \in (0, \infty)$, let $\hat{\pi}(\gamma)$ be the optimal proportion of foreign currency debt when the degree of debt aversion is γ. Then $\hat{\pi}(\cdot)$ is strictly decreasing. Moreover, $\lim_{\gamma \to \infty} \hat{\pi}(\gamma) = 0$.*

The Economic Result 2 states the role played by the degree of debt aversion on the optimal currency debt portfolio. We observe that an increase in debt aversion leads to a decrease in the proportion of foreign currency in the portfolio, and hence an increase of the proportion of debt in local currency. In other words, an increase in the degree of debt aversion discourages countries from borrowing debt in foreign currency. In addition, it is interesting to note that an extreme debt aversion leads to decide not to borrow in foreign currency.

Proof. (Economic Result 2). Let $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be defined by

$$f(z, \gamma) := \gamma\sigma^2 z + \lambda\varphi\big(1 + z\varphi\big)^{\gamma} - (\lambda\varphi - b)\,.$$

Consider $\gamma_1 > \gamma_0$. We note first that, for each fixed $z > 0$ we have $f(z, \gamma_1) > f(z, \gamma_0)$ and $\lim_{\gamma\to\infty} f(z, \gamma) = \infty$. We also note that for each fixed $\gamma > 0$, the function $f(\cdot, \gamma)$ is strictly increasing. Moreover, by definition of $\hat{\pi}(\gamma_0)$, $f\big(\hat{\pi}(\gamma_0), \gamma_0\big) = 0$. Then

$$f\big(\hat{\pi}(\gamma_0), \gamma_1\big) > f\big(\hat{\pi}(\gamma_0), \gamma_0\big) = 0.$$

On the other hand, $f(0, \gamma_1) = b < 0$. Now applying the Intermediate Value Theorem to $f(\cdot, \gamma_1)$, there exists a unique real number $z \in \big(0, \hat{\pi}(\gamma_0)\big)$ such that $f\big(z, \gamma_1\big) = 0$. By the uniqueness of the solution given in Proposition 5.16, we must have $z = \hat{\pi}(\gamma_1)$. Hence,

$$0 < \hat{\pi}(\gamma_1) < \hat{\pi}(\gamma_0).$$

To prove the second claim, we note that from Proposition 5.16, we have $0 < \hat{\pi}_1(\gamma) < \bar{\eta}$. The proof concludes after observing that $\lim_{\gamma\to\infty} \bar{\eta} = \lim_{\gamma\to\infty} \frac{\lambda\varphi - b}{\gamma\sigma^2} = 0$. □

We show below that, in a special case, the effect of the degree of debt aversion can be computed explicitly.

5.18 Remark. *Suppose $m = 1$ and no jumps. Using Remark 5.14, we get*

$$\frac{d\hat{\pi}}{d\gamma} = \frac{b}{\gamma^2\sigma^2} < 0,$$

$$\lim_{\gamma\to\infty} \hat{\pi}(\gamma) = \lim_{\gamma\to\infty} -\frac{b}{\gamma\sigma^2} = 0.$$

5.5 Concluding remarks

We have made two contributions in this chapter. On the theoretical side, we have developed for the first time a model for government debt control (that includes jumps in the exchange rates and debt aversion), to find explicitly the optimal currency debt portfolio and optimal debt payments. On the applied side, this model provides a rigorous explanation of the consistent reduction in the proportion of foreign currency in government debt portfolios in favor of local currency. Specifically, we have found that high debt aversion and jumps in the exchange rates explain that behavior.

Chapter 6

The optimal fiscal stabilization fund bands

The global and fiscal crises have led to more interest in countercyclical fiscal policies to mitigate the negative consequences of a crisis. That is why governments have created stabilization funds, which is a mechanism of fiscal policy used to save money in the good economic times to be used in the bad economic times. The stabilization fund is managed through a band, which determines the lower and upper bound for the fund. However, this band is unique, i.e. it does not depend on the state of the economy.

Natural questions arise: what are the optimal bands for a stabilization fund? Should the bands depend on the economic cycles? In other words, do we expect that governments should have one band for recessions and a different band for expansions? To answer these questions, we present the first theoretical model for the government stabilization fund.

We model the optimal management of the stabilization fund as a stochastic singular problem with economic cycles (or regime switching). This type of stochastic control problems have been studied, for instance, in Sotomayor (2008).

We consider a government that wants to control the stabilization fund by depositing money and withdrawing money from the fund. Following Sotomayor (2008), we consider that the fund without intervention of the government follows a geometric Brownian motion with parameters modulated by a Markov chain which is the regime switching process. In this setting, the Brownian motion accounts for the minor and continuous uncertain movements, and the Markov chain models the uncertainty of the long-term macroeconomic conditions. We assume that there is a target of the fund for each regime of the economy. Having funds either above or below such targets is not desirable. Indeed, the cost of having too much stabilization funds (above the target) is due to the fact that the funds can be used in the present to pay public debt or invest on any other government program with high social and/or private return. Having too little funds (below the target) generates a cost associated with the fear of the government of facing the need of resources when the economy is going through bad economic times. These costs are known as running costs and are modelled by a loss function.

On the other hand, the government can intervene in order to modify the level of the fund by increasing or decreasing it. These interventions have costs whose level depend on the state of the economy and whether we are increasing or decreasing the fund. Indeed, in good economic times it is cheaper to increase the fund than in recessions. In addition, for every regime of the economy, we consider that costs of increasing the fund are bigger that the costs of decreasing the fund. That is, we have asymmetric costs of government intervention. The goal of the government is to find the optimal control of dynamic interventions (decreasing or increasing the fund) that minimizes the expected total cost given by the loss function plus the cost of the interventions.

We succeed in finding explicit solution for the stochastic singular control problem with regime switching described above. We obtain explicit solutions for the bands that characterize the optimal control, that we call optimal bands. These bands depend on the state of the economy, i.e. the optimal band for recession is different from the optimal band for expansions. These bands depend on the parameters of the model such as the cost of increasing and decreasing the fund, the volatility of the fund, and the length of recession and expansions. Moreover, we derive a recommendation for the management of the stabilization fund based on the optimal bands.

6.1 The model

We assume that the stabilization fund is driven by a Brownian motion that generates minor and continuous uncertain movements, and a Markov chain that models the uncertainty of the long-term macroeconomic conditions. It is assumed that the government has information of both sources of uncertainty from the beginning of the fund dynamics up to the present time. Following Sotomayor (2008), let us consider a complete probability space $(\Omega, \mathcal{F}, P)$, a one-dimensional Brownian motion W, and an observable continuous-time, stationary, finite-state Markov chain $\epsilon = \{\epsilon_t, t \geq 0\}$. Let $\mathcal{S} = \{0, 1, 2, \cdots, N\}$ denote the state-space of the Markov chain. In this set up ϵ represents the regime of the economy and N the number of regimes. The processes W and ϵ are assumed to be independent. In addition, we assume that the Markov chain has a strongly irreducible generator $Q = [q_{ij}]_{N\times N}$, where $q_{ii} := -\lambda_i < 0$ and $\sum_{j\in\mathcal{S}} q_{ij} = 0$ for every regime of the economy $i \in \mathcal{S}$. We denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the P-augmentation of the filtration generated by a one-dimensional Brownian motion and the Markov chain, that is, $\mathcal{F}_t := \sigma\{W_s, \epsilon_s : 0 \leq s \leq t\}$, for every $t \geq 0$.

The state variable is the ratio of the level of the stabilization fund divided by the GDP of the country, that will be denoted by $X = \{X_t, t \in [0, \infty)\}$. Such ratio fund will be called simply *fund*. We assume that $X = \{X_t, t \in [0, \infty)\}$ is an $\mathbb{F}$-adapted stochastic process that follows the dynamics

$$X_t \;=\; x + \int_0^t \mu_{\epsilon(t)}\, X_s\, ds + \int_0^t \sigma_{\epsilon(t)}\, X_s\, dW_s - U_t + L_t\,, \tag{6.1}$$

where the coefficients $(\mu_i\,, i \in \mathcal{S})$, and $(\sigma_i\,, i \in \mathcal{S})$ are strictly positive real numbers. We note that the process in (6.1) without intervention ($L \equiv U \equiv 0$) is a geometric Brownian motion with regime switching, where $(\mu_i\,, i \in \mathcal{S})$ is the set of rates of growth of the stabilization fund for each regime of the economy, and $(\sigma_i\,, i \in \mathcal{S})$ are the corresponding volatilities. Sometimes, the government intervenes to withdraw money from the fund or to deposit money in the fund. The process $U = \{U_t, t \in [0,\infty)\}$ represents the cumulative withdrawals, and $L = \{L_t, t \in [0,\infty)\}$ the cumulative deposits in the fund. Hence L and U are formally $\mathbb{F}$-adapted, non-negative, and non-decreasing stochastic control processes from $[0,\infty) \times \Omega$ to $[0,\infty)$. Moreover, we assume that these interventions to manage the fund generate costs that depend on the regime of the economy. For a given regime of the economy $i \in \mathcal{S}$, κ_i^L represents the cost of increasing the fund in one unit, and κ_i^U is the cost of decreasing the fund in one unit.

The government wants to select the fund control (L, U) that minimizes the functional J defined by

$$J(x, i\,; L, U) \;:=\; E_{x,i}\left[\int_0^\infty e^{-\delta t} h(X_t, i)\, dt + \int_0^\infty e^{-\delta t}\, \kappa^L_{\epsilon(t)}\, dL_t + \int_0^\infty e^{-\delta t}\, \kappa^U_{\epsilon(t)}\, dU_t\right], \tag{6.2}$$

where $E_{x,i}$ represents the expectation conditioned on the initial values $X_0 = x$ and $\epsilon_0 = \epsilon(0) = i$. Here $\delta \in (0,\infty)$ is the discount rate; the proportional cost parameters $(\kappa_i^L\,, i \in \mathcal{S})$ and $(\kappa_i^U\,, i \in \mathcal{S})$ are non-negative with $\kappa_i^L + \kappa_i^U > 0$; and h is a cost or loss function, which we assume nonnegative and convex. Specifically, we will assume $h(x, i) = \alpha(x - \rho_i)^2$, where $\rho_i > 0$ is the fund target for the regime $i \in \mathcal{S}$. The rationale for this form of h is as follows. The cost of having too much stabilization funds (above the target) is due to the fact that funds can be used in the present to pay public debt or invest on any other government program with high social and/or private return. On the other hand, given the existence of macroeconomic fluctuations, having too little stabilization funds (below the target) generates a cost associated with the fear of the government of facing the need of resources when the economy is going through bad economic times. Clearly, the perceived likelihood of this event increases when the actual level of the fund decreases.

We next provide the definition of an admissible fund control process.

6.1 Remark. *We require that*

$$\lim_{T\to\infty} E_{x,i}\left[e^{-\delta T} X_T^2\right] \;=\; 0. \tag{6.3}$$

We will use condition (6.3) in the proof of Theorem 6.6 below.

6.2 Definition. *Let $x \in (0,\infty)$. Let L and U be two stochastic processes $\mathbb{F}$-adapted, nonnegative, and nondecreasing from $[0,\infty)\times\Omega$ to $[0,\infty)$, with sample paths that are left-continuous with right-limits. The pair (L,U) is called an admissible stochastic singular control if $J(x,i\,;L\,,U) < \infty$ and (6.3) holds. By convention, we set $U_0 = L_0 = 0$. The set of all admissible controls is denoted by $\mathcal{A}(x,i) = \mathcal{A}(i)$.*

Given the nature of the problem, we obviously have $(\kappa_i^L > 0\,, i \in \mathcal{S})$, i.e. there are always costs for increasing the fund. However, $(\kappa_i^U \geq 0\,, i \in \mathcal{S})$, i.e. the intervention to decrease the level of the fund could be zero. We will assume these conditions on the cost parameters.

6.3 Problem. *The government wants to select the admissible stochastic singular control $Z = (L\,,U)$ that minimizes the functional J defined in (6.2).*

From a mathematical point of view, Problem 6.3 is a stochastic singular control problem with regime switching. The theory of stochastic singular control with regime switching has been studied, for instance, in Sotomayor (2008). For other references regarding this topic see, for example, Cadenillas et al. 2013, Sotomayor and Cadenillas (2009), and Sotomayor and Cadenillas (2011).

6.2 A verification theorem

We define the value function $V : (0,\infty) \times \mathcal{S} \to \mathbb{R}$ by

$$V(x,i) := \inf_{Z \in \mathcal{A}(x,i)} J(x,i;Z).$$

This represents the smallest cost that can be achieved when we consider all the admissible controls for the initial fund $x > 0$ and the regime $i \in \mathcal{S}$.

We have formulated the fund management problem as a stochastic singular control problem. Let $Y = \{Y(t) : t \geq 0\}$ be non-negative, non-decreasing, and left-continuous with right-limits. We define

$$\Delta_Y := \{t \geq 0 : Y_t \neq Y_{t+}\},$$

the set of times when Y has a discontinuity. The set Δ_Y is countable because Y is non-decreasing and, hence, can jump only a countable number of times during $[0,\infty)$. We will denote the discontinuous part of Y by Y^d, that is $Y_t^d := \sum_{0 \leq s < t,\, s \in \Delta_Y} (Y_{s+} - Y_s)$. The continuous part of Y will be denoted by Y^c, that is $Y_t^c = Y_t - Y_t^d$.

Given the nature of the government interventions through the processes L and U, we consider $\Delta_L \cap \Delta_U = \emptyset$. That is, interventions by increasing and decreasing the stabilization fund at the same point in time are not allowed.

To define the HJB for our problem, we introduce some notation. Let $\psi : (0,\infty) \times \mathcal{S} \to \mathbb{R}$. For each $i \in \mathcal{S}$ we define the operator $\mathcal{L}_i$ by

$$\mathcal{L}_i \psi(x,i) := \frac{1}{2} {\sigma_i}^2 x^2 \psi''(x,i) + \mu_i x \psi'(x,i) - \delta \psi(x,i). \tag{6.4}$$

Let $v : (0,\infty) \times \mathcal{S} \to \mathbb{R}$. Consider the Hamilton-Jacobi-Bellman (HJB) equation:

$$\forall i \in \mathcal{S},\ \forall x > 0: \quad \min\Big\{\mathbb{L}_i^v(x,i),\ \ \Big(k_i^L + v'(x,i)\Big)\Big(k_i^U - v'(x,i)\Big)\Big\} = 0, \tag{6.5}$$

where

$$\mathbb{L}_i^v(x,i) := \mathcal{L}_i v(x,i) - \lambda_i v(x,i) + \sum_{j\neq i} q_{ij} v(x,j) + h(x,i) \tag{6.6}$$

This equation is equivalent to the variational inequalitites (see Bensoussan and Lions 1982 for a classical reference):

$$\begin{aligned} \mathbb{L}_i^v(x,i) &\geq 0, \\ \Big(k_i^L + v'(x,i)\Big)\Big(k_i^U - v'(x,i)\Big) &\geq 0, \\ \mathbb{L}_i^v(x,i)\Big(k_i^L + v'(x,i)\Big)\Big(k_i^U - v'(x,i)\Big) &= 0. \end{aligned}$$

For each $i \in \mathcal{S}$, we observe that a solution v of the HJB equation defines the regions $\mathcal{C}(i) = \mathcal{C}^v(i)$ and $\Sigma(i) = \Sigma^v(i)$ as follows:

$$\mathcal{C}(i) := \Big\{x > 0 : \mathbb{L}_i^v(x,i) = 0 \text{ and } \Big(k_i^L + v'(x,i)\Big)\Big(k_i^U - v'(x,i)\Big) > 0\Big\}, \tag{6.7}$$

$$\Sigma(i) := \Big\{x > 0 : \mathbb{L}_i^v(x,i) \geq 0 \text{ and } \Big(k_i^L + v'(x,i)\Big)\Big(k_i^U - v'(x,i)\Big) = 0\Big\}. \tag{6.8}$$

In turn, the region $\Sigma(i)$ can be split into two disjoint sets, namely

$$\Sigma_1(i) = \Sigma_1^v(i) := \Big\{x > 0 : \mathbb{L}_i^v(x,i) \geq 0 \text{ and } \Big(k_i^L + v'(x,i)\Big) = 0\Big\}, \tag{6.9}$$

$$\Sigma_2(i) = \Sigma_2^v(i) := \Big\{x > 0 : \mathbb{L}_i^v(x,i) \geq 0 \text{ and } v'(x,i) = k_i^U\Big\}. \tag{6.10}$$

It is possible to construct a control process associated with v in the following manner.

6.4 Definition. *Let v satisfy the HJB equation (6.5), and for each $i \in \mathcal{S}$ let $0 < \alpha_i < \beta_i < \infty$ be such that $\Sigma_1(i) = (0, \alpha_i]$, $\mathcal{C}(i) = \big(\alpha_i,\, \beta_i\big)$ and $\Sigma_2(i) = [\beta_i, \infty)$. In addition, let L^v and U^v be two $\mathbb{F}$-adapted, non-negative, and non-decreasing stochastic processes from $[0,\infty) \times \Omega$ to $[0, \infty)$, with sample paths that are left-continuous with right-limits, and $U_0^v = L_0^v = 0$. The control $Z^v = (L^v, U^v)$ is said to be associated with the function v if the following conditions are satisfied:*

$$(i) \quad X_t^v \;=\; x + \int_0^t \mu_{\epsilon(s)} X_s^v ds + \int_0^t \sigma_{\epsilon(s)} X_s^v dW_s - U_t^v + L_t^v, \quad \forall t \in [0, \infty), P - a.s.,$$

$$(ii) \quad X_t^v \in \overline{\mathcal{C}(\epsilon_t)}, \quad Lebesgue\ a.s.\ t \in [0, \infty),\ P - a.s.,$$

$$(iii) \quad \int_0^\infty I_{\{\alpha_{\epsilon(t)} < X_t^v\}} dL_t^v = 0, \;\; P - a.s.,$$

$$(iv) \quad \int_0^\infty I_{\{X_t^v < \beta_{\epsilon(t)}\}} dU_t^v = 0, \;\; P - a.s..$$

Here X^v is the stochastic process generated by the control $Z^v = (L^v, U^v)$.

Now we state a lemma to be used in the proof of the verification Theorem 6.6 below.

6.5 Lemma. *Suppose that v satisfies the HJB equation (6.5). Let Z^v be the control associated with v, and X^v the process generated by $Z^v = (L^v, U^v)$. Then*

$$\int_0^T e^{-\delta t} v'(X_t^v, \epsilon_t) d(U^v)_t^c \;=\; \int_0^T e^{-\delta t} k_{\epsilon_t}^U \, d(U^v)_t^c, \tag{6.11}$$

$$\int_0^T e^{-\delta t} v'(X_t^v, \epsilon_t) d(L^v)_t^c \;=\; -\int_0^T e^{-\delta t} k_{\epsilon_t}^L \, d(L^v)_t^c, \tag{6.12}$$

$$v(X_t^v, \epsilon_t) - v(X_{t+}^v, \epsilon_t) \;=\; k_{\epsilon_t}^U (U_{t+}^v - U_t^v), \quad \forall t \in \Delta_U, \tag{6.13}$$

$$v(X_t^v, \epsilon_t) - v(X_{t+}^v, \epsilon_t) \;=\; k_{\epsilon_t}^L (L_{t+}^v - L_t^v), \quad \forall t \in \Delta_L. \tag{6.14}$$

Proof. Similar to the proof of Lemma 2.10. □

We next state a sufficient condition for a policy to be optimal.

6.6 Theorem. *For $i \in \mathcal{S}$, let $v(\cdot, i) \in C^1(0,\infty)$ and $v(\cdot, i) \in C^2\Big((0,\infty) - N_i\Big)$, where $N_i = \cup_{j \neq i}\{\alpha_j, \beta_j\}$ are finite sets. Let $v(\cdot, i)$ be a convex function on $(0,\infty)$, with $v(0+, i)$ bounded. Suppose that v satisfies the HJB equation (6.5). In addition, suppose that there exists $0 < \alpha_i < \beta_i < \infty$, $i \in \mathcal{S}$ such that $\mathcal{C}(i) = \mathcal{C}^v(i) = \big(\alpha_i, \beta_i\big)$, and consider the stochastic control $Z^v = (L^v, U^v)$ associated with v. Then, for every $(L, U) \in \mathcal{A}(x, i)$:*

$$v(x, i) \ \leq \ J(x, i; L, U), \quad \forall i \in \mathcal{S}.$$

Furthermore, the stochastic control $Z^v = (L^v, U^v)$ satisfies

$$v(x, i) \ = \ J(x, i; L^v, U^v), \quad \forall i \in \mathcal{S}.$$

In other words, $(L, U) = (L^v, U^v)$ is optimal control and $V = v$ is the value function for Problem 6.3.

Proof. Let $\{X_t : t \geq 0\}$ be a modulated Markov process given by:

$$X_t = x + \int_0^t \mu_{\epsilon_s} X_s ds + \int_0^t \sigma_{\epsilon_s} X_s dW_s - U_t + L_t.$$

Let us define the function f by $f(t, x, i) = e^{-\delta t} v(x, i)$ for $x > 0$ and $i \in \mathcal{S}$. Let $\bar{b}$ be a real number satisfying $X_0 < \bar{b} < +\infty$, and define $\tau_b := \{t \geq 0 : X_t \geq \bar{b}\}$. Applying Ito's formula for a modulated Markov process (see, for instance, Sotomayor and

Cadenillas 2011), for every $t \in [0, \infty)$, we have:

$$
\begin{aligned}
&e^{-\delta(t\wedge\tau_b)}v(X_{t\wedge\tau_b}, \epsilon_{t\wedge\tau_b}) = \\
&\qquad v(X_0,\epsilon_0) + \int_0^{t\wedge\tau_b} e^{-\delta s}\mathbb{L}^v_{\epsilon_s}(X_s,\epsilon_s)ds - \int_0^{t\wedge\tau_b} e^{-\delta s}h(X_s,\epsilon_s)ds \\
&\qquad + \int_0^{t\wedge\tau_b} e^{-\delta s}v_x(X_s,\epsilon_s)X_s\sigma_{\epsilon_s}dW_s - \int_0^{t\wedge\tau_b} e^{-\delta s}v_x(X_s,\epsilon_s)dU^c_s \\
&\qquad + \int_0^{t\wedge\tau_b} e^{-\delta s}v_x(X_s,\epsilon_s)dL^c_s + M^f_{t\wedge\tau_b} - M^f_0. \\
&\qquad + \sum_{\substack{s\in\Delta \\ 0\le s<t\wedge\tau_b}} e^{-\delta s}\Big(v(X_{s^+},\epsilon_s) - v(X_s,\epsilon_s)\Big).
\end{aligned} \tag{6.15}
$$

where $\Delta := \Delta_L + \cup\Delta_U$, and $\{M^f_{t\wedge\tau_b} : t \geq 0\}$ is a square integrable martingale, due to the fact that v is bounded on $[0, t\wedge\tau_b]$.

The HJB equation (6.5) implies

$$
\int_0^{t\wedge\tau_b} e^{-\delta s}\mathbb{L}^v_{\epsilon_s}(X_s,\epsilon_s)ds \geq 0. \tag{6.16}
$$

The HJB equation also implies that $-k^L_{\epsilon_s} \leq v_x(X_s,\epsilon_s) \leq k^U_{\epsilon_s}$ for every $X_s \in (0,\infty)$. We note that, in particular, $v_x(\cdot, i)$ is bounded. Hence

$$
\int_0^{t\wedge\tau_b} e^{-\delta s}v_x(X_s,\epsilon_s)dU^c_s \;\leq\; \int_0^{t\wedge\tau_b} e^{-\delta s}k^U_{\epsilon_s}dU^c_s \tag{6.17}
$$

and

$$
\int_0^{t\wedge\tau_b} e^{-\delta s}v_x(X_s,\epsilon_s)dL^c_s \;\geq\; -\int_0^{t\wedge\tau_b} e^{-\delta s}k^L_{\epsilon_s}dL^c_s. \tag{6.18}
$$

Moreover, $X_t - X_{t^+} = U_{t^+} - U_t > 0$ for every $t \in \Delta_U$, and $X_{t^+} - X_t = L_{t^+} - L_t > 0$ for every $t \in \Delta_L$. Consequently,

$$
v(X_t,\epsilon_t) - v(X_{t^+},\epsilon_t) \;\leq\; k^U_{\epsilon_t}(U_{t+} - U_t), \quad \forall t \in \Delta_U \tag{6.19}
$$

and

$$-k^L_{\epsilon_t}(L_{t^+}-L_t) \;\le\; v(X_{t^+},\epsilon_t)-v(X_t,\epsilon_t), \quad \forall t\in\Delta_L. \tag{6.20}$$

Using (6.16)-(6.20) and taking conditional expectation, we obtain:

$$\begin{aligned}
v(x,i) \;&\le\; E_{x,i}\Big[e^{-\delta(t\wedge\tau_b)}v(X_{t\wedge\tau_b},\epsilon_{t\wedge\tau_b})\Big]+E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}h(X_s,\epsilon_s)ds\Big]\\
&\quad + E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}k^L_{\epsilon_s}dL^c_s\Big]+E_{x,i}\Big[\sum_{\substack{s\in\Delta_L\\ 0\le s<t\wedge\tau_b}}e^{-\delta s}k^L_{\epsilon_s}(L_{s^+}-L_s)\Big]\\
&\quad + E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}k^U_{\epsilon_s}dU^c_s\Big]+E_{x,i}\Big[\sum_{\substack{s\in\Delta_U\\ 0\le s<t\wedge\tau_b}}e^{-\delta s}k^L_{\epsilon_s}(U_{s^+}-U_s)\Big]\\
&\quad - E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta t}v_x(X_s,\epsilon_s)X_s\sigma_{\epsilon_s}dW_s\Big]\\
&= E_{x,i}\Big[e^{-\delta(t\wedge\tau_b)}v(X_{t\wedge\tau_b},\epsilon_{t\wedge\tau_b})\Big]-E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}v_x(X_s,\epsilon_s)X_s\sigma_{\epsilon_s}dW_s\Big]\\
&\quad + E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}k^L_{\epsilon_s}dL_s\Big]+\;E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}k^U_{\epsilon_s}dU_s\Big]\\
&\quad + E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}h(X_s,\epsilon_s)ds\Big].
\end{aligned} \tag{6.21}$$

Since X is bounded on $[0,t\wedge\tau_b]$ and v_x is bounded, we have

$$E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}v_x(X_s,\epsilon_s)X_s\sigma_{\epsilon_s}dW_s\Big]=0. \tag{6.22}$$

Letting $\bar{b}\uparrow+\infty$, we have $\tau_b\uparrow+\infty$ and hence $t\wedge\tau_b\uparrow t$. By the Monotone Convergence Theorem,

$$\lim_{b\uparrow+\infty}E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}h(X_s,\epsilon_s)ds\Big]=E_{x,i}\Big[\int_0^{t}e^{-\delta s}h(X_s,\epsilon_s)ds\Big], \tag{6.23}$$

$$\lim_{b\uparrow+\infty}E_{x,i}\Big[\int_0^{t\wedge\tau_b}e^{-\delta s}k^U_{\epsilon_s}dU_s\Big]=E_{x,i}\Big[\int_0^{t}e^{-\delta s}k^U_{\epsilon_s}dU_s\Big]. \tag{6.24}$$

and

$$\lim_{b\uparrow+\infty} E_{x,i}\Big[\int_0^{t\wedge\tau_b} e^{-\delta s}k^L_{\epsilon_s}dL_s\Big] = E_{x,i}\Big[\int_0^t e^{-\delta s}k^L_{\epsilon_s}dL_s\Big]. \tag{6.25}$$

On the other hand, since v is continuous, letting $b\uparrow+\infty$, we obtain

$$e^{-\delta(t\wedge\tau_b)}v(X_{t\wedge\tau_b},\epsilon_{t\wedge\tau_b}) \;\to\; e^{-\delta t}v(X_t,\epsilon_t) \tag{6.26}$$

Hence

$$\begin{aligned}\lim_{b\to\infty} E_{x,i}\Big[e^{-\delta(t\wedge\tau_b)}v(X_{t\wedge\tau_b},\epsilon_{t\wedge\tau_b})\Big] &= E_{x,i}\Big[\lim_{b\to\infty} e^{-\delta(t\wedge\tau_b)}v(X_{t\wedge\tau_b},\epsilon_{t\wedge\tau_b})\Big]\\ &= E_{x,i}\Big[e^{-\delta t}v(X_t,\epsilon_t)\Big]. \end{aligned}\tag{6.27}$$

Consequently, taking the limit as $b\to+\infty$ in (6.21), we get

$$\begin{aligned} v(x,i) \;\le E_{x,i}\Big[e^{-\delta t}v(X_t,\epsilon_t)\Big] + E_{x,i}\Big[\int_0^t e^{-\delta s}h(X_s,\epsilon_s)ds\Big]\\ +\; E_{x,i}\Big[\int_0^t e^{-\delta s}k^L_{\epsilon_s}dL_s\Big] +\; E_{x,i}\Big[\int_0^t e^{-\delta s}k^U_{\epsilon_s}dU_s\Big]. \end{aligned}\tag{6.28}$$

Now suppose that $(L,U)\in\mathcal{A}(x,i)$. From (6.3) and the linear growth of v on the interval $[\beta_i,\infty)$, we have

$$\lim_{t\to+\infty} E_{x,i}\Big[e^{-\delta t}v(X_t,\epsilon_t)\Big] = 0.$$

Letting $t\to+\infty$, the Monotone Convergence Theorem implies

$$v(x,i) \;\le\; E_{x,i}\Big[\int_0^\infty e^{-\delta s}h(X_s,\epsilon_s)ds + \int_0^\infty e^{-\delta s}k^L_{\epsilon_s}dL_s + \int_0^\infty e^{-\delta s}k^U_{\epsilon_s}dU_s\Big].$$

This shows the first part of this theorem.

To show the second part, we observe that (6.15) holds for the control $Z^v = (L^v,U^v)$ associated with v. The resulting controlled process X^{Z^v} will be denoted by X^v. By condition (ii) in Definition 6.4 we know that $X_t^v \in \overline{\mathcal{C}(\epsilon_t)}$ except on a subset of $[0,\infty)$ that has Lebesgue measure zero. Consequently, $\mathbb{L}^v_{\epsilon_s}(X_s^v, \epsilon_s) = 0$ Lebesgue a.s. on $[0,\infty)$. Thus (6.16) turns out to be an equality for $Z^v = (L^v, U^v)$. By Lemma 6.5, the inequalities (6.17)-(6.20) become equalities as well. As a result, (6.28) is an equality for $Z^v = (L^v, U^v)$:

$$E_{x,i}\Big[e^{-\delta t} v(X_t^v, \epsilon_t)\Big] = v(x,i) - E_{x,i}\Big[\int_0^t e^{-\delta s} h(X_s^v, \epsilon_s) ds\Big] \\ - E_{x,i}\Big[\int_0^t e^{-\delta s} k^L_{\epsilon_s} dL_s^v\Big] - E_{x,i}\Big[\int_0^t e^{-\delta s} k^U_{\epsilon_s} dU_s^v\Big]. \quad (6.29)$$

Finally, since X^v is bounded by $\max\{x, \beta_1, \cdots, \beta_N\}$, $v(0+, i)$ is finite and $v(\cdot, i)$ is continuous, we conclude

$$\lim_{t\to+\infty} E_{x,i}\Big[e^{-\delta t} v(X_t^v, \epsilon_t)\Big] = 0.$$

Letting $t \to \infty$ in (6.29), the Monotone Convergence Theorem gives the desired result:

$$v(x,i) = E_{x,i}\Big[\int_0^\infty e^{-\delta s} h(X_s^v, \epsilon_s) ds\Big] + E_{x,i}\Big[\int_0^\infty e^{-\delta s} k^L_{\epsilon_s} dL_s^v\Big] + E_{x,i}\Big[\int_0^\infty e^{-\delta s} k^U_{\epsilon_s} dU_s^v\Big].$$

We note that, in particular, this shows that $Z^v = (L^v, U^v)$ is admissible. This completes the proof of the theorem. □

6.3 The explicit solution

In this section we construct a candidate for solution to Problem 6.3. Afterwards, we prove that this candidate is indeed a solution. In addition, we discuss how the optimal solution works. Finally, we present some numerical examples to illustrate the optimal solution.

The construction of the solution

We want to find a function v and a control $Z^v = (L^v, U^v)$ that satisfies the condition of Theorem 6.6. In particular, v must satisfy the HJB equation (6.5). Let $i \in \mathcal{S}$. We conjecture that $v(\cdot, i)$ is convex. Let

$$a_i := \sup\{x > 0 : v_x(x,i) = -k_i^L\},$$

and

$$b_i := \sup\{x > 0 : v_x(x,i) < k_i^U\}.$$

Then $a_i < b_i$. Suppose $0 < a_i < b_i < \infty$. Since $v(x,i)$ should satisfy the HJB equation (6.5), we have $-k_i^L \leq v_x(x,i) \leq k_i^U$. Then, by convexity, we have

$$\begin{aligned} v_x(x,i) &= -k_i^L, \quad \forall x \in (0, a_i], & (6.30)\\ v_x(x,i) &> -k_i^L, \quad \forall x \in (a_i, \infty), & (6.31) \end{aligned}$$

and

$$\begin{aligned} v_x(x,i) &< k_i^U, \quad \forall x \in (0, b_i), & (6.32)\\ v_x(x,i) &= k_i^U, \quad \forall x \in [b_i, \infty). & (6.33) \end{aligned}$$

Thus

$$\begin{aligned} v_x(x,i) &= -k_i^L, & \forall x \in (0, a_i], && (6.34)\\ -k_i^L < v_x(x,i) &< k_i^U, & \forall x \in (a_i, b_i), && (6.35)\\ v_x(x,i) &= k_i^U, & \forall x \in [b_i, \infty). && (6.36) \end{aligned}$$

If v satisfies the HJB equation (6.5), the above inequality implies that

$$\mathbb{L}_i^v(x,i) = 0, \quad \forall x \in (a_i, b_i). \tag{6.37}$$

This leads to the conclusion that the continuation region and intervention regions are given by:

$$\mathcal{C}(i) = (a_i, b_i), \tag{6.38}$$

$$\Sigma_1(i) = (0, a_i], \tag{6.39}$$

$$\Sigma_2(i) = [b_i, \infty). \tag{6.40}$$

The above conjecture about the continuation region motivates the following definition of *optimal stabilization fund band*.

6.7 Definition. *Let $i \in \mathcal{S}$. Let v be the value function and $\mathcal{C}(i)$ the corresponding continuation region for $i \in \mathcal{S}$. If $\mathcal{C}(i) \neq \emptyset$ and bounded, we define*

$$a_i := \inf\{x > 0 : x \in \mathcal{C}(i)\},$$

$$b_i := \sup\{x > 0 : x \in \mathcal{C}(i)\}.$$

The closed interval $[a_i, b_i] \subset (0, \infty)$ is said to be the optimal stabilization fund band for regime $i \in \mathcal{S}$.

For the sake of simplicity, from now on we assume $\mathcal{S} = \{1, 2\}$. Thus we have two bands $[a_1, b_1]$ and $[a_2, b_2]$. The regime $i = 1$ will represent the good economic time for the fund, while the regime $i = 2$ the bad one. In contrast to the bad economic time, in the good economic time the fund increases steadily, and the cost to increase the fund is low. For these reasons, we will consider $k_1^L \leq k_2^L$, $k_1^U \leq k_2^U$, $\sigma_1 \leq \sigma_2$, $\mu_1 \geq \mu_2$, $\rho_1 \geq \rho_2 > 0$. To differentiate the regimes of the economy we assume $(\mu_1, k_1^L, k_1^U, \sigma_1, \rho_1) \neq (\mu_2, k_2^L, k_2^U, \sigma_2, \rho_2)$.

Even though we have the above conditions on the parameters, we do not know the relations among the band parameters a_1, a_2, b_1, and b_2. We observe that there are six cases:

1. $0 < a_2 < a_1 < b_1 < b_2$,
2. $0 < a_2 < a_1 < b_2 < b_1$,
3. $0 < a_2 < b_2 < a_1 < b_1$,
4. $0 < a_1 < a_2 < b_2 < b_1$,
5. $0 < a_1 < a_2 < b_1 < b_2$,
6. $0 < a_1 < b_1 < a_2 < b_2$.

By symmetry, we need to consider only cases 1, 2 and 3. Indeed, case 4 can be handle as case 1, by interchanging a_1 for a_2 and b_1 for b_2. Similarly, cases 5 and 6 follow from cases 2 and 3, respectively.

Case 1**:** $0 < a_2 < a_1 < b_1 < b_2$

Accordingly, we will consider five intervals:

(i) $x \in (0,\, a_2] = \Sigma_1(1) \cap \Sigma_1(2)$,

(ii) $x \in (a_2,\, a_1] = \Sigma_1(1) \cap \mathcal{C}(2)$,

(iii) $x \in (a_1,\, b_1) = \mathcal{C}(1) \cap \mathcal{C}(2)$,

(iv) $x \in [b_1,\, b_2) = \Sigma_2(1) \cap \mathcal{C}(2)$,

(v) $x \in [b_2,\, \infty[= \Sigma_2(1) \cap \Sigma_2(2)$.

Using equations (6.34) and (6.36)-(6.40), we define the candidate for value function for each of the above cases.

(i) For $x \in (0,\, a_2]$, we define $v(x, 1) = D_1 - k_1^L x$ and $v(x, 2) = D_2 - k_2^L x$.

(ii) For $x \in (a_2, a_1]$, we define $v(x,1) = D_1 - k_1^L x$. The function $v(x,2)$ is defined to be the solution of

$$\frac{1}{2}\sigma_2^2 x^2 v''(x,2) + \mu_2 x v'(x,2) - (\delta + \lambda_2)v(x,2) + \lambda_2(D_1 - k_1^L x) + \alpha(x - \rho_2)^2 = 0. \tag{6.41}$$

Solving the above ODE, we obtain

$$v(x,2) = A_1 x^{\gamma_1} + A_2 x^{\gamma_2} + \eta_0 + \eta_1 x + \eta_2 x^2, \tag{6.42}$$

where

$$\eta_0 = \frac{\alpha\rho_2^2 + \lambda_2 D_1}{\delta + \lambda_2}, \tag{6.43}$$

$$\eta_1 = -\frac{\lambda_2 k_1^L + 2\alpha\rho_2}{\delta + \lambda_2 - \mu_2}, \tag{6.44}$$

$$\eta_2 = \frac{\alpha}{\delta + \lambda_2 - \sigma_2^2 - 2\mu_2}, \tag{6.45}$$

and $\gamma_1 < 0 < \gamma_2$ are the real roots of $\phi_2(y) = \frac{1}{2}\sigma_2^2 y^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)y - (\lambda_2 + \delta)$.

(iii) For $x \in (a_1, b_1)$, the functions $v(x,1)$ and $v(x,2)$ are defined to be the solutions of the following system of ODEs:

$$\frac{1}{2}\sigma_1^2 x^2 v''(x,1) + \mu_1 x v'(x,1) - (\delta + \lambda_1)v(x,1) + \lambda_1 v(x,2) + \alpha(x - \rho_1)^2 = 0, \tag{6.46}$$

$$\frac{1}{2}\sigma_2^2 x^2 v''(x,2) + \mu_2 x v'(x,2) - (\delta + \lambda_2)v(x,2) + \lambda_2 v(x,1) + \alpha(x - \rho_2)^2 = 0. \tag{6.47}$$

Solving the above system, we obtain

$$v(x,1) = \sum_{j=1}^{4} x^{\widetilde{\gamma}_j}\widetilde{A}_j + \xi_{12}x^2 + \xi_{11}x + \xi_{10}, \tag{6.48}$$

$$v(x,2) = \sum_{j=1}^{4} x^{\widetilde{\gamma}_j}\widetilde{B}_j + \xi_{22}x^2 + \xi_{21}x + \xi_{20}, \tag{6.49}$$

where, for all $i, j \in \{1, 2\}, i \neq j$,

$$\xi_{i0} = \frac{\alpha\lambda_i\rho_j^2 + (\delta + \lambda_j)\alpha\rho_i^2}{(\delta + \lambda_i)(\delta + \lambda_j) - \lambda_i\lambda_j}, \tag{6.50}$$

$$\xi_{i1} = -\frac{2\alpha\rho_i(\delta + \lambda_j - \mu_j) + 2\alpha\lambda_i\rho_j}{(\delta + \lambda_j - \mu_j)(\delta + \lambda_i - \mu_i) - \lambda_i\lambda_j}, \tag{6.51}$$

$$\xi_{i2} = \frac{\alpha(\delta + \lambda_j + \lambda_i - \sigma_j^2 - 2\mu_j)}{(\delta + \lambda_j - \sigma_j^2 - 2\mu_j)(\delta + \lambda_i - \sigma_i^2 - 2\mu_i) - \lambda_i\lambda_j}, \tag{6.52}$$

and $\widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \widetilde{\gamma}_3 < \widetilde{\gamma}_4$ are the roots of the characteristic equation $\phi_1(\gamma)\phi_2(\gamma) = \lambda_1\lambda_2$, with $\phi_i(y) = \frac{1}{2}\sigma_i^2 y^2 + (\mu_i - \frac{1}{2}\sigma_i^2)y - (\lambda_i + \delta)$, $i \in \{1, 2\}$. Furthermore, for $j \in \{1, 2, 3, 4\}$,

$$\widetilde{B}_j = -\frac{1}{\lambda_1}\widetilde{A}_j\phi_1(\widetilde{\gamma}_j). \tag{6.53}$$

(iv) For $x \in [b_1, b_2)$, we define $v(x, 1) = F_1 + k_1^U x$. The function $v(x, 2)$ is defined to be the solution of

$$\frac{1}{2}\sigma_2^2 x^2 v''(x, 2) + \mu_2 x v'(x, 2) - (\delta + \lambda_2)v(x, 2) + \lambda_2(F_1 + k_1^U x) + \alpha(x - \rho_2)^2 = 0. \tag{6.54}$$

Solving the above ODE, we obtain

$$v(x, 2) = A_3 x^{\gamma_1} + A_4 x^{\gamma_2} + \tilde{\eta}_0 + \tilde{\eta}_1 x + \tilde{\eta}_2 x^2, \tag{6.55}$$

where

$$\tilde{\eta}_0 = \frac{\alpha\rho_2^2 + \lambda_2 F_1}{\delta + \lambda_2}, \tag{6.56}$$

$$\tilde{\eta}_1 = \frac{\lambda_2 k_1^U - 2\alpha\rho_2}{\delta + \lambda_2 - \mu_2}, \tag{6.57}$$

$$\tilde{\eta}_2 = \frac{\alpha}{\delta + \lambda_2 - \sigma_2^2 - 2\mu_2}. \tag{6.58}$$

In addition, $\gamma_1 < 0 < \gamma_2$ are the real roots of $\phi_2(y) = \frac{1}{2}\sigma_2^2 y^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)y - (\lambda_2 + \delta)$.

(v) For $x \in [b_2, \infty)$, we define $v(x,1) = F_1 + k_1^U x$ and $v(x,2) = F_2 + k_2^U x$.

We note that there are sixteen unknowns: the parameters a_1, a_2, b_1, b_2, and the coefficients A_1, A_2, A_3, A_4, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{A}_4$, D_1, D_2, F_1, and F_2. We conjecture that $v(\cdot,1) \in C^2(0,\infty)$ and $v(\cdot,2) \in C^2\Big((0,\infty) - \{a_1, b_1\}\Big)$. Thus, the sixteen unknowns can be found from the following system of sixteen equations:

$$\begin{aligned}
\text{For } i \in \{1,2\}: \quad & v(a_i^+, i) = D_i - k_i^L a_i, & v_x(a_i^+, i) = -k_i^L, \\
& v_{xx}(a_i^+, i) = 0, & v(b_i^-, i) = F_i + k_i^U b_i, \\
& v_x(b_i^-, i) = k_i^U, & v_{xx}(b_i^-, i) = 0, \\
& v(a_1^-, 2) = v(a_1+, 2), & v_x(a_1^-, 2) = v_x(a_1^+, 2), \\
& v(b_1^-, 2) = v(b_1^+, 2), & v_x(b_1^-, 2) = v_x(b_1^+, 2).
\end{aligned} \tag{6.59}$$

The solution of the above system determines the candidate for value function $V = v$, and the candidate for optimal stabilization fund band $[a_i, b_i]$ for each regime $i \in \{1,2\}$. The candidate for optimal fund control $Z^v = (L^v, U^v)$ is the control associated with v given in Definition 6.4.

Case 2: $0 < a_2 < a_1 < b_2 < b_1$

Accordingly, we will consider five intervals:

(i) $x \in (0,\, a_2] = \Sigma_1(1) \cap \Sigma_1(2)$,

(ii) $x \in (a_2,\, a_1] = \Sigma_1(1) \cap \mathcal{C}(2)$,

(iii) $x \in (a_1,\, b_2) = \mathcal{C}(1) \cap \mathcal{C}(2)$,

(iv) $x \in [b_2,\, b_1) = \mathcal{C}(1) \cap \Sigma_2(2)$,

(v) $x \in [b_1, \infty[= \Sigma_2(1) \cap \Sigma_2(2)$.

Using equations (6.34) and (6.36)-(6.40), we define the candidate for value function for each of the above cases.

(i) For $x \in (0, a_2]$, we define $v(x, 1) = D_1 - k_1^L x$ and $v(x, 2) = D_2 - k_2^L x$.

(ii) For $x \in (a_2, a_1]$, we define $v(x, 1) = D_1 - k_1^L x$. The function $v(x, 2)$ is defined to be the solution of

$$\frac{1}{2}\sigma_2^2 x^2 v''(x, 2) + \mu_2 x v'(x, 2) - (\delta + \lambda_2) v(x, 2) + \lambda_2 (D_1 - k_1^L x) + \alpha (x - \rho_2)^2 = 0. \tag{6.60}$$

Solving the above ODE, we obtain $v(x, 2)$ given by (6.42).

(iii) For $x \in (a_1, b_2)$, the functions $v(x, 1)$ and $v(x, 2)$ are defined to be the solutions of the following system of ODEs:

$$\frac{1}{2}\sigma_1^2 x^2 v''(x, 1) + \mu_1 x v'(x, 1) - (\delta + \lambda_1) v(x, 1) + \lambda_1 v(x, 2) + \alpha (x - \rho_1)^2 = 0, \tag{6.61}$$

$$\frac{1}{2}\sigma_2^2 x^2 v''(x, 2) + \mu_2 x v'(x, 2) - (\delta + \lambda_2) v(x, 2) + \lambda_2 v(x, 1) + \alpha (x - \rho_2)^2 = 0. \tag{6.62}$$

Solving the above system, we obtain $v(x, 1)$ and $v(x, 2)$ given by (6.48) (6.49), respectively.

(iv) For $x \in [b_2, b_1)$, we define $v(x, 2) = F_2 + k_2^U x$. The function $v(x, 1)$ is defined to be the solution of

$$\frac{1}{2}\sigma_1^2 x^2 v''(x, 1) + \mu_1 x v'(x, 1) - (\delta + \lambda_1) v(x, 1) + \lambda_1 (F_2 + k_2^U x) + \alpha (x - \rho_1)^2 = 0. \tag{6.63}$$

Solving the above ODE, we obtain

$$v(x, 1) = A_3 x^{\gamma_1} + A_4 x^{\gamma_2} + \overline{\eta}_0 + \overline{\eta}_1 x + \overline{\eta}_2 x^2, \tag{6.64}$$

where

$$\overline{\eta}_0 = \frac{\alpha\rho_1^2 + \lambda_1 F_2}{\delta + \lambda_1}, \tag{6.65}$$

$$\overline{\eta}_1 = \frac{\lambda_1 k_2^U - 2\alpha\rho_1}{\delta + \lambda_1 - \mu_1}, \tag{6.66}$$

$$\overline{\eta}_2 = \frac{\alpha}{\delta + \lambda_1 - \sigma_1^2 - 2\mu_1}. \tag{6.67}$$

(v) For $x \in [b_1, \infty)$, we define $v(x,1) = F_1 + k_1^U x$ and $v(x,2) = F_2 + k_2^U x$.

We note that there are sixteen unknowns: the parameters a_1, a_2, b_1, b_2, and the coefficients A_1, A_2, A_3, A_4, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{A}_4$, D_1, D_2, F_1, and F_2. We conjecture that $v(\cdot,1) \in C^2\Big((0,\infty) - \{b_2\}\Big)$ and $v(\cdot,2) \in C^2\Big((0,\infty) - \{a_1\}\Big)$. Thus, the sixteen unknowns can be found from the following system of sixteen equations:

$$\begin{aligned}
\text{For } i \in \{1,2\}: \quad & v(a_i^+, i) = D_i - k_i^L a_i, && v_x(a_i^+, i) = -k_i^L, \\
& v_{xx}(a_i^+, i) = 0, && v(b_i^-, i) = F_i + k_i^U b_i, \\
& v_x(b_i^-, i) = k_i^U, && v_{xx}(b_i^-, i) = 0, \\
& v(a_1^-, 2) = v(a_1+, 2), && v_x(a_1^-, 2) = v_x(a_1^+, 2), \\
& v(b_2^-, 1) = v(b_2^+, 1), && v_x(b_2^-, 1) = v_x(b_2^+, 1).
\end{aligned} \tag{6.68}$$

The solution of the above system determines the candidate for value function $V = v$, and the candidate for optimal stabilization fund band $[a_i, b_i]$ for each regime $i \in \{1,2\}$. The candidate for optimal fund control $Z^v = (L^v, U^v)$ is the control associated to v given in Definition 6.4.

Case 3: $0 < a_2 < b_2 < a_1 < b_1$

Accordingly, we will consider the following intervals:

(i) $x \in (0,\, a_2] = \Sigma_1(1) \cap \Sigma_1(2)$,

(ii) $x \in (a_2,\, b_2] = \Sigma_1(1) \cap \mathcal{C}(2)$,

(iii) $x \in [b_2,\, a_1) = \Sigma_1(1) \cap \Sigma_2(2)$,

(iv) $x \in [a_1,\, b_1) = \mathcal{C}(1) \cap \Sigma_2(2)$,

(v) $x \in [b_1,\, \infty[= \Sigma_2(1) \cap \Sigma_2(2)$.

Using equations (6.34) and (6.36)-(6.40), we define the candidate for value function for each of the above cases.

(i) For $x \in (0,\, a_2]$, we define $v(x,1) = D_1 - k_1^L x$ and $v(x,2) = D_2 - k_2^L x$.

(ii) For $x \in (a_2, b_2]$, we define $v(x,1) = D_1 - k_1^L x$. The function $v(x,2)$ is defined to be the solution of

$$\frac{1}{2}\sigma_2^2 x^2 v''(x,2) + \mu_2 x v'(x,2) - (\delta + \lambda_2)v(x,2) + \lambda_2(D_1 - k_1^L x) + \alpha(x - \rho_2)^2 = 0. \tag{6.69}$$

Solving the above ODE, we obtain $v(x,2)$ given by (6.42).

(iii) For $x \in (b_2,\, a_1]$, we define $v(x,1) = D_1 - k_1^L x$ and $v(x,2) = F_2 + k_2^U x$.

(iv) For $x \in [a_1,\, b_1)$, we define $v(x,2) = F_2 + k_2^U x$. The function $v(x,1)$ is defined to be the solution of

$$\frac{1}{2}\sigma_1^2 x^2 v''(x,1) + \mu_1 x v'(x,1) - (\delta + \lambda_1)v(x,1) + \lambda_1(F_2 + k_2^U x) + \alpha(x - \rho_1)^2 = 0. \tag{6.70}$$

Solving the above ODE, we obtain $v(x,1)$ given by (6.64).

(v) For $x \in [b_1, \infty)$, we define $v(x,1) = F_1 + k_1^U x$ and $v(x,2) = F_2 + k_1^U x$.

We note that there are twelve unknowns: the parameters a_1, a_2, b_1, b_2, and the coefficients A_1, A_2, A_3, A_4, D_1, D_2, F_1 and F_2. We conjecture that $v(\cdot,1) \in C^2(0,\infty)$ and $v(\cdot,2) \in C^2(0,\infty)$. Thus, the twelve unknowns can be found from the following system of twelve equations:

$$\text{For } i \in \{1,2\}: \quad v(a_i^+, i) = D_i - k_i^L a_i, \qquad v_x(a_i^+, i) = -k_i^L,$$
$$v_{xx}(a_i^+, i) = 0, \qquad v(b_i^-, i) = F_i + k_i^U x, \tag{6.71}$$
$$v_x(b_i^-, i) = k_i^U, \qquad v_{xx}(b_i^-, i) = 0.$$

The solution of the above system determines the candidate for value function $V = v$, and the candidate for optimal stabilization fund band $[a_i, b_i]$ for each regime $i \in \{1,2\}$. The candidate for optimal fund control $Z^v = (L^v, U^v)$ is the control associated to v given in Definition 6.4.

Verification of the solution

In the previous subsection, we have established the candidates for solution for Problem 6.3 depending on the cases. We observe that, for the value function candidates to be well-defined, we need to assume some conditions:

$$(\delta + \lambda_1 - \sigma_1^2 - 2\mu_1) \; > \; 0, \tag{6.72}$$

$$(\delta + \lambda_2 - \sigma_2^2 - 2\mu_2) \; > \; 0, \tag{6.73}$$

$$(\delta + \lambda_1 - \sigma_1^2 - 2\mu_1)(\delta + \lambda_2 - \sigma_2^2 - 2\mu_2) \; \neq \; \lambda_1\lambda_2. \tag{6.74}$$

We note that the first two conditions resemble condition (2.5).

In this subsection, we will prove first that the candidate for value function v is indeed the value function. Then we will show that the candidate for optimal control $Z^v = (L^v, U^v)$ is indeed the optimal control. We need the following lemma.

6.8 Lemma. *Let $i \in \mathcal{S}$. Let v be a candidate for value function such that $v_{xx}(x,i) \geq 0$ for every $x \in (0,\infty)$. Recall that $\mathbb{L}_i^v(x,i)$ is defined by (6.6). Suppose that the* *partial derivatives of* $\mathbb{L}_i^v(x,i)$ *satisfy the inequalities*

$$\frac{\partial \mathbb{L}_i^v(a_i-,i)}{\partial x} = -(\mu_i - \delta - \lambda_i)k_i^L + \lambda_i v_x(a_i, 3-i) + 2\alpha(a_i - \rho_i) \ \leq \ 0,$$
$$\frac{\partial \mathbb{L}_i^v(b_i+,i)}{\partial x} = (\mu_i - \delta - \lambda_i)k_i^U + \lambda_i v_x(b_i, 3-i) + 2\alpha(b_i - \rho_i) \ \geq \ 0,$$

for every $i \in \{1,2\}$*. Then*

$$\mathbb{L}_i^v(x,i) \geq 0, \quad \forall x \in (0, a_i] \cup [b_i, \infty).$$

Proof. See Appendix B.6. □

6.9 Theorem. *(Case 1) Let* a_1*,* a_2*,* b_1*,* b_2*,* A_j*,* $\widetilde{A}_j$*,* $j \in \{1,2,3,4\}$*,* D_i *and* F_i*,* $i \in \{1,2\}$ *be the solution of the system of equations (6.59) such that* $0 < a_2 < a_1 < b_1 < b_2 < \infty$*. Let* $\widetilde{B}_j$ *be given by equation (6.53). Let us define* $v : (0,\infty) \times \mathcal{S} \to [0,\infty)$ *by*

$$v(x,1) = \begin{cases} D_1 - k_1^L x & \text{if } x \in \big(0, a_1\big] \\ \sum_{j=1}^4 x^{\widetilde{\gamma}_j} \widetilde{A}_j + \xi_{12}x^2 + \xi_{11}x + \xi_{10} & \text{if } x \in \big(a_1, b_1\big) \\ F_1 + k_1^U x & \text{if } x \in \big[b_1, \infty\big) \end{cases}$$

and

$$v(x,2) = \begin{cases} D_2 - k_2^L x & \text{if } x \in \big(0, a_2\big] \\ A_1 x^{\gamma_1} + A_2 x^{\gamma_2} + \eta_2 x^2 + \eta_1 x + \eta_0 & \text{if } x \in \big(a_2, a_1\big] \\ \sum_{j=1}^4 \widetilde{B}_j x^{\widetilde{\gamma}_j} + \xi_{22}x^2 + \xi_{21}x + \xi_{20} & \text{if } x \in \big(a_1, b_1\big) \\ A_3 x^{\gamma_1} + A_4 x^{\gamma_2} + \widetilde{\eta}_2 x^2 + \widetilde{\eta}_1 x + \widetilde{\eta}_0 & \text{if } x \in \big[b_1, b_2\big) \\ F_2 + k_2^U x & \text{if } x \in \big[b_2, \infty\big). \end{cases}$$

Suppose that

$$\forall i \in \{1,2\}, \quad \forall x \in (a_i, b_i): \quad v_{xx}(x,i) \;>\; 0, \tag{6.75}$$

$$-(\mu_2 - \delta - \lambda_2)k_2^L + 2\alpha(a_2 - \rho_2) - \lambda_2 k_1^L \;\leq\; 0, \tag{6.76}$$

$$(\mu_2 - \delta - \lambda_2)k_2^U + 2\alpha(b_2 - \rho_2) + \lambda_2 k_1^U \;\geq\; 0, \tag{6.77}$$

and

$$-(\mu_1 - \delta - \lambda_1)k_1^L + \lambda_1\big(A_1\gamma_1 a_1^{\gamma_1 - 1} + A_2\gamma_2 a_1^{\gamma_2 - 1} + 2\eta_2 a_1 + \eta_1\big) + 2\alpha(a_1 - \rho_1) \;\leq 0, \tag{6.78}$$

$$(\mu_1 - \delta - \lambda_1)k_1^U + \lambda_1\big(A_3\gamma_1 b_1^{\gamma_1 - 1} + A_4\gamma_2 b_1^{\gamma_2 - 1} + 2\widetilde{\eta}_2 b_1 + \widetilde{\eta}_1\big) + 2\alpha(b_1 - \rho_1) \;\geq 0. \tag{6.79}$$

Then $V = v$ is the value function for Problem 6.3. In addition, $[a_1, b_1]$ and $[a_2, b_2]$ are the optimal bands for the good economic time and bad economic time, respectively.

Proof. It is enough to verify that all the conditions for v in Theorem 6.3 are satisfied. By definition, conditions (6.59) imply $v(\cdot, 1) \in C^2(0,\infty)$, $v(\cdot,2) \in C^1(0,\infty)$ and $v(\cdot,2) \in C^2\Big((0,\infty) - \{a_1, b_1\}\Big)$. Let $i \in \{1,2\}$. Condition (6.75) implies $v_x(x,i)$ is strictly increasing on (a_i, b_i). Noting that $v_x(x,i) = -k_i^L$ on $(0, a_i)$ and $v_x(x,i) = k_i^U$ on $[b_i, \infty)$, condition (6.75) implies that $v_{xx}(x,i) \geq 0$ for every x in $(0,\infty)$. Hence $v(\cdot, i)$ is convex on $(0,\infty)$. Moreover, $v(0+, i) = D_i < \infty$.

It remains to verify that v satisfies the HJB equation (6.5). Let $i \in \{1,2\}$. By construction, v satisfies $\mathbb{L}_i^v(x,i) = 0$ on (a_i, b_i). Now we observe that

$$-k_i^L < v_x(x,i) < k_i^U \quad \forall x \in (a_i, b_i),$$

due to (6.75), $v_x(x,i) = -k_i^L$ on $(0, a_i]$ and $v_x(x,i) = k_i^U$ on $[b_i, \infty)$. As a result, $\Big(k_i^L + v_x(x,i)\Big)\Big(k_i^U - v_x(x,i)\Big) > 0$ on (a_i, b_i). Consequently, v satisfies the HJB equation on (a_i, b_i). On the other hand, conditions (6.75)-(6.79) are precisely the assumptions in Lemma 6.8. Thus $\mathbb{L}_i^v(x,i) \geq 0$ on $(0, a_i] \cup [b_i, \infty)$. In addition, by definition, v satisfies $\left(k_i^L + v_x(x,i)\right) = 0$ on $(0, a_i]$ and $\left(k_i^U - v_x(x,i)\right) = 0$ on $[b_i, \infty)$. Thus v satisfies the HJB equation on $(0, a_i] \cup [b_i, \infty)$. Hence the HJB equation (6.5) is satisfied. As a consequence, $\mathcal{C}(i) = (a_i, b_i)$.

Since v is the actual value function, the closed intervals $[a_1, b_1]$ and $[a_2, b_2]$ are the optimal bands by Definition 6.7. □

6.10 Theorem. *(Case 2) Let a_1, a_2, b_1, b_2, A_j, $\widetilde{A}_j$, $j \in \{1,2,3,4\}$, D_i and F_i, $i \in \{1,2\}$ be the solution of the system of equations (6.68) such that $0 < a_2 < a_1 < b_2 < b_1 < \infty$. Let $\widetilde{B}_j$ be given by equation (6.53). Let us define $v : (0,\infty) \times \mathcal{S} \to [0,\infty)$ by*

$$v(x,1) = \begin{cases} D_1 - k_1^L x & \text{if } x \in \big(0, a_1\big] \\ \sum_{j=1}^4 x^{\widetilde{\gamma}_j} \widetilde{A}_j + \xi_{12} x^2 + \xi_{11} x + \xi_{10} & \text{if } x \in \big(a_1, b_1\big) \\ F_1 + k_1^U x & \text{if } x \in \big[b_1, \infty\big) \end{cases}$$

and

$$v(x,2) = \begin{cases} D_2 - k_2^L x & \text{if } x \in \big(0, a_2\big] \\ A_1 x^{\gamma_1} + A_2 x^{\gamma_2} + \eta_2 x^2 + \eta_1 x + \eta_0 & \text{if } x \in \big(a_2, a_1\big] \\ \sum_{j=1}^4 \widetilde{B}_j x^{\widetilde{\gamma}_j} + \xi_{22} x^2 + \xi_{21} x + \xi_{20} & \text{if } x \in \big(a_1, b_2\big) \\ A_3 x^{\gamma_1} + A_4 x^{\gamma_2} + \overline{\eta}_2 x^2 + \overline{\eta}_1 x + \overline{\eta}_0 & \text{if } x \in \big[b_2, b_1\big) \\ F_2 + k_2^U x & \text{if } x \in \big[b_1, \infty\big). \end{cases}$$

Suppose that

$$\forall i \in \{1,2\}, \quad \forall x \in (a_i, b_i): \quad v_{xx}(x,i) \; > 0, \tag{6.80}$$

$$-(\mu_2 - \delta - \lambda_2) k_2^L + 2\alpha(a_2 - \rho_2) - \lambda_2 k_1^L \;\leq\; 0, \tag{6.81}$$

$$(\mu_1 - \delta - \lambda_1) k_1^U + 2\alpha(b_1 - \rho_1) + \lambda_1 k_2^U \;\geq\; 0, \tag{6.82}$$

and

$$-(\mu_1-\delta-\lambda_1)k_1^L+\lambda_1\big(A_1\gamma_1 a_1^{\gamma_1-1}+A_2\gamma_2 a_1^{\gamma_2-1}+2\eta_2 a_1+\eta_1\big)+2\alpha(a_1-\rho_1) \quad \leq 0, \tag{6.83}$$

$$(\mu_2-\delta-\lambda_2)k_2^U+\lambda_1\Big(\sum_{j=1}^{4}\widetilde{A}_j\widetilde{\gamma}_j b_2^{\widetilde{\gamma}_j-1}+2\xi_{12}b_2+\xi_{11}\Big)+2\alpha(b_2-\rho_2) \quad \geq 0. \tag{6.84}$$

Then $V=v$ is the value function for Problem 6.3. In addition, $[a_1,b_1]$ and $[a_2,b_2]$ are the optimal bands for the good economic time and bad economic time, respectively.

Proof. Similar to the proof of Theorem 6.9. □

6.11 Theorem. *(Case 3) Let a_1, a_2, b_1, b_2, A_j, $j\in\{1,2,3,4\}$, D_i and F_i, $i\in\{1,2\}$, be the solution of the system of equations (6.71) such that $0<a_2<b_2<a_1<b_1<\infty$. Let us define $v:(0,\infty)\times\mathcal{S}\to[0,\infty)$ by*

$$v(x,1)=\begin{cases} D_1-k_1^L x & \textit{if } x\in\big(0,a_1\big] \\ A_1x^{\gamma_1}+A_2x^{\gamma_2}+\eta_2x^2+\eta_1x+\eta_0 & \textit{if } x\in\big(a_2,a_1\big] \\ F_1+k_1^U x & \textit{if } x\in\big[b_1,\infty\big) \end{cases}$$

and

$$v(x,2)=\begin{cases} D_2-k_2^L x & \textit{if } x\in\big(0,a_2\big] \\ A_3x^{\gamma_1}+A_4x^{\gamma_2}+\widetilde{\eta}_2x^2+\widetilde{\eta}_1x+\widetilde{\eta}_0 & \textit{if } x\in\big[b_1,b_2\big) \\ F_2+k_2^U x & \textit{if } x\in\big[b_2,\infty\big). \end{cases}$$

Suppose that

$$\forall i \in \{1,2\}, \quad \forall x \in (a_i, b_i): \quad v_{xx}(x,i) \;\; > 0, \tag{6.85}$$

$$-(\mu_2 - \delta - \lambda_2)k_2^L + 2\alpha(a_2 - \rho_2) - \lambda_2 k_1^L \;\leq\; 0, \tag{6.86}$$

$$(\mu_1 - \delta - \lambda_1)k_1^U + 2\alpha(b_1 - \rho_1) + \lambda_1 k_2^U \;\geq\; 0, \tag{6.87}$$

$$-(\mu_1 - \delta - \lambda_1)k_1^L + 2\alpha(a_1 - \rho_1) + \lambda_1 k_2^U \;\leq\; 0, \tag{6.88}$$

$$(\mu_2 - \delta - \lambda_2)k_2^U + 2\alpha(b_2 - \rho_2) - \lambda_2 k_1^L \;\geq\; 0. \tag{6.89}$$

Then $V = v$ is the value function for Problem 6.3. In addition, $[a_1, b_1]$ and $[a_2, b_2]$ are the optimal bands for the good economic time and bad economic time, respectively.

Proof. Similar to the proof of Theorem 6.9. □

Next we state that the candidate for optimal fund control is indeed the optimal control.

6.12 Theorem. *Let v be the value function for Problem 6.3, with $\mathcal{C}^v(i) = [a_i, b_i]$ for $i \in \{1,2\}$. Suppose that the process associated to v, $Z^v = (L^v, U^v)$, exists. That is, $Z^v = (L^v, U^v)$ is solution of the following Skorokhod problem:*

$$(i) \quad X_t^v \;=\; x + \int_0^t \mu_{\epsilon(s)} X_s^v ds + \int_0^t \sigma_{\epsilon(s)} X_s^v dW_s - U_t^v + L_t^v, \quad \forall t \in [0,\infty), P-a.s.,$$

$$(ii) \quad X_t^v \in [a_{\epsilon_t}, b_{\epsilon_t}], \quad Lebesgue\ a.s.\ t \in [0,\infty),\ P-a.s.,$$

$$(iii) \quad \int_0^\infty I_{\{a_{\epsilon(t)} < X_t^v\}} dL_t^v = 0, \quad P-a.s.,$$

$$(iv) \quad \int_0^\infty I_{\{X_t^v < b_{\epsilon(t)}\}} dU_t^v = 0, \quad P-a.s..$$

Then $Z^v = (L^v, U^v)$ is optimal control for Problem 6.3.

Proof. This follows directly from Theorem 6.6. □

From a mathematical point of view, to the best of our knowledge, we provide the first explicit solution of a stochastic singular control problem with regime switching when the control intervenes in two manners, namely increasing and decreasing the state variable. Thus we extend Sotomayor and Cadenillas (2011), that considers the dividend policy problem in which the manager only decreases the state variable. The more general setting that we have in this Chapter is relevant for the problem of fund management, but at the same time it poses a more difficult mathematical and computational problem.

How the optimal policy works

We provide a detailed explanation on how the optimal policy in Theorem 6.12 works. Let $i \in \mathcal{S}$ be the initial regime of the economy. Suppose $x > b_i$, then conditions (i), (ii) and (iv) in that Theorem imply that the control process U^v jumps from $U_0^v = 0$ to $U_{0+}^v = x - b_i$ and, as a result, the fund jumps from $X_0^v = x$ to $X_{0+}^v = b_i$. Similarly, if $x < a_i$, we have $L_0^v = 0$ and $L_{0+}^v = a_i - x$, and hence the fund jumps from $X_0^v = x$ to $X_{0+}^v = a_i$. That is, if the initial value of the fund is outside the optimal band $[a_i, b_i]$, then the government intervention takes the fund to the frontier of the band. In general, if at any point in time $t \geq 0$ we have $X_t^v \notin [a_i, b_i]$, the optimal policy works in a similar manner. More precisely, $X_{t+}^v = a_i$ or $X_{t+}^v = b_i$, depending on whether $X_t^v < a_i$ or $X_t^v > b_i$, respectively.

Now suppose that the regime of the economy is $j \in \mathcal{S}$, and $X_t^v \in (a_j, b_j)$. Then, condition (iv) in Theorem 6.12 implies that U^v and L^v remain constant. Hence there is no need to either withdraw from or deposit money in the fund. However, if the fund reaches b_j and tries to cross it, condition (ii) implies that the government has to intervene in order to prevent the fund from crossing b_j from below. Similarly, if the debt ratio reaches a_j from above and tries to cross it, condition (ii) implies that the government has to intervene in order to prevent the fund from crossing a_j from above. Thus, during the regime j, we have $X_t^v \in [a_j, b_j]$. When the economy switches to regime $i \neq j$, from that point in time up to the end of the duration of regime i, the government has to guarantee that $X_t^v \in [a_i, b_i]$. The government intervenes in the way described in the last part of the previous paragraph.

In summary, if at any point in time the regime of the economy is i and the fund of a country is inside the optimal band $[a_i, b_i]$, then no intervention is required. If the fund is equal to b_i or a_i, then control should be exerted to prevent the fund from being outside the band $[a_i, b_i]$. If the initial fund is outside the band $[a_i, b_i]$, then the government should intervene to bring immediately the fund to the level a_i (respectively, b_i) if the fund was below a_i (respectively, if the fund was above b_i).

Numerical solutions

A stated in Theorems 6.9, 6.10 and 6.11, the solution of the stabilization fund problem involves the numerical solution of either sixteen equations or twelve equations. To illustrate the optimal solution, we present two examples, one with $\rho_1 = \rho_2$ and other with $\rho_1 > \rho_2$.

Although we have solved the stabilization fund problem for $k_i^U \geq 0$, it is difficult to justify the case $k_i^U > 0$, i.e. the existence of cost of interventions of the government to reduce the fund. For this reason, from now on we consider $k_i^U = 0$ for every $i \in \{1, 2\}$.

6.13 Example. *The case $\rho_1 = \rho_2 = \rho$. Let us consider the following parameter values for the good economic time,*

$$\mu_1 = 0.02, \qquad \sigma_1 = 0.1, \qquad \lambda_1 = 0.05, \qquad k_1^L = 0.5, \qquad \rho = 0.10,$$

and

$$\mu_2 = 0.00, \qquad \sigma_2 = 0.3, \qquad \lambda_2 = 0.05, \qquad k_2^L = 0.7, \qquad \delta = 0.07,$$

for the bad economic time. Solving numerically the system of equations (6.59), we find

$$D_1 = 0.0540817, \quad D_2 = 0.0768086, \quad F_1 = 0.0126754, \quad F_2 = 0.0245567,$$

$$\widetilde{A}_1 = -1.19707 \times 10^{-11}, \quad \widetilde{A}_2 = -0.00022609, \quad \widetilde{A}_3 = 205.862, \quad \widetilde{A}_4 = -90.3111,$$

$$A_1 = -0.000118369, \quad A_2 = -36.0011, \quad A_3 = 0.000138696, \quad A_4 = -37.2756,$$

and

$$a_2 = 0.0319876, \quad a_1 = 0.0598183, \quad b_1 = 0.116299, \quad b_2 = 0.148094.$$

We have verified that conditions (6.76)-(6.79) are satisfied. Condition (6.75) is also satisfied, as illustrated in Figure 6.1.

In the previous example, we note that $0 < a_2 < a_1 < \rho < b_1 < b_2 < \infty$. That is, in the good economic time $i = 1$ the government can keep the fund closer to the optimal size of the fund target ρ than in the bad economic time $i = 2$.

6.14 Example. *The case $\rho_1 > \rho_2$. Let us consider the same parameter values as in Example 6.13 with $\rho_1 = 0.30$ and $\rho_2 = 0.10$. Solving numerically the system of equations (6.71), we obtain*

$$D_1 = 0.182525,\ D_2 = 0.147313,\ F_1 = 0.0475033,\ F_2 = 0.0883877,$$

$$A_1 = -5.93289 \times 10^{-7},\ A_2 = -17.448,\ A_3 = -0.0000232071,\ A_4 = -16.0334,$$

and

$$a_2 = 0.0326229,\ b_2 = 0.173974,\ a_1 = 0.219245,\ b_1 = 0.335791.$$

We have verified that conditions (6.86)-(6.89) are satisfied. Condition (6.85) is also satisfied, as illustrated in Figure 6.2.

In the preceding example, we observe that $0 < a_2 < \rho_2 < b_2 < a_1 < \rho_1 < b_1 < \infty$. That is, the optimal bands do not intersect. More precisely, the optimal band in the good economic time $[a_1, b_1]$ is above the optimal band in the bad economic time $[a_2, b_2]$.

For Examples 6.13 and 6.14 the graph of the value function and the first and second derivatives are given in Figures 6.1 and 6.2, respectively. Now we are going to make observations that are valid for either of the above examples. First, we recall that conditions (6.75) and (6.85) are satisfied. Moreover, we observe that for each $i \in \{1, 2\}$ the function $V(\cdot, i)$ is bounded, decreasing and convex. In particular, it is constant on $[b_i, \infty)$. This is clearly a consequence of the zero cost of withdrawals. Here is the explanation. Any initial level of the fund $x > b_i$ can be driven to the level b_i at no cost, and after this intervention the optimal strategy can be followed. As a result, $V(x, i) = V(b_i, i)$ for every $x \geq b_i$.

Furthermore, in the examples above we have $\rho_i < b_i$. That is, despite the fact that we do not have cost for withdrawals, we do not obtain $\rho_i = b_i$. What is the rationale for $\rho_i < b_i$? The interpretation is that putting the value of the fund closer to ρ_i from above (and hence closer to a_i) makes more likely future intervention through costly deposits to keep the fund level above a_i. Since our optimization problem is of infinite horizon, such potential future deposits will generate costs that have to be taken into account. Consequently, it is possible to have b_i above ρ_i. As a result, the optimal level of the fund ρ_i, and the upper level of the optimal stabilization fund band b_i are two different things. Hence withdrawing money from the fund until the optimal target value is attained is not optimal.

Another conclusion we draw from these examples is that $b_i - \rho_i < \rho_i - a_i$. This is clearly a consequence of the asymmetric costs that the fund manager faces. Since making deposits to increase the fund involves costs and there is no cost for withdrawals, the part below ρ_i is the harder side for the fund manager. Hence it is optimal that the distance from a_i to ρ_i is greater than the distance from b_i to ρ_i.

On the other hand, comparing the value functions in Figures 6.1 and 6.2, we

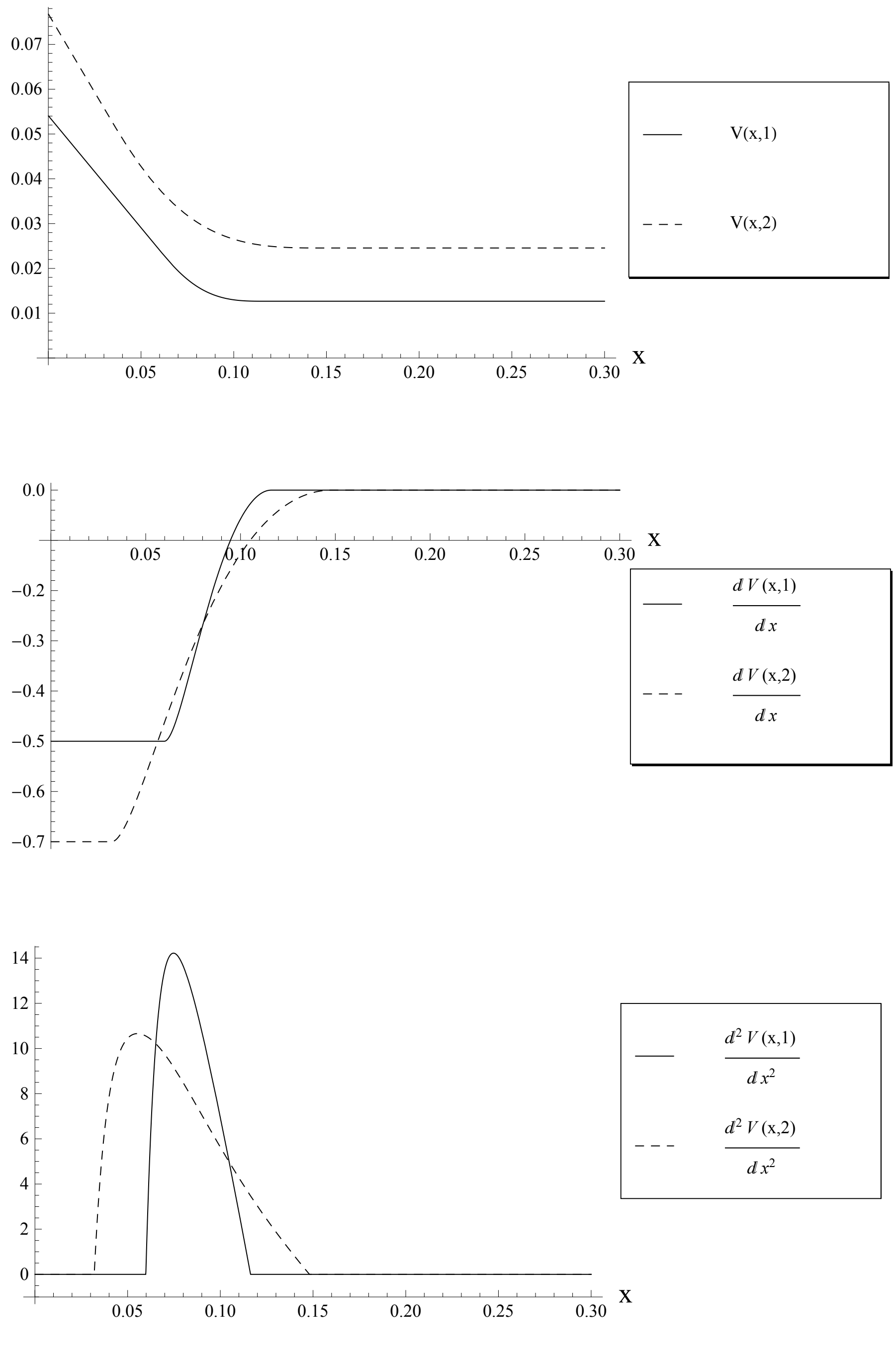

Figure 6.1: The value function and its derivatives for $a_2 < a_1 < b_1 < b_2$.

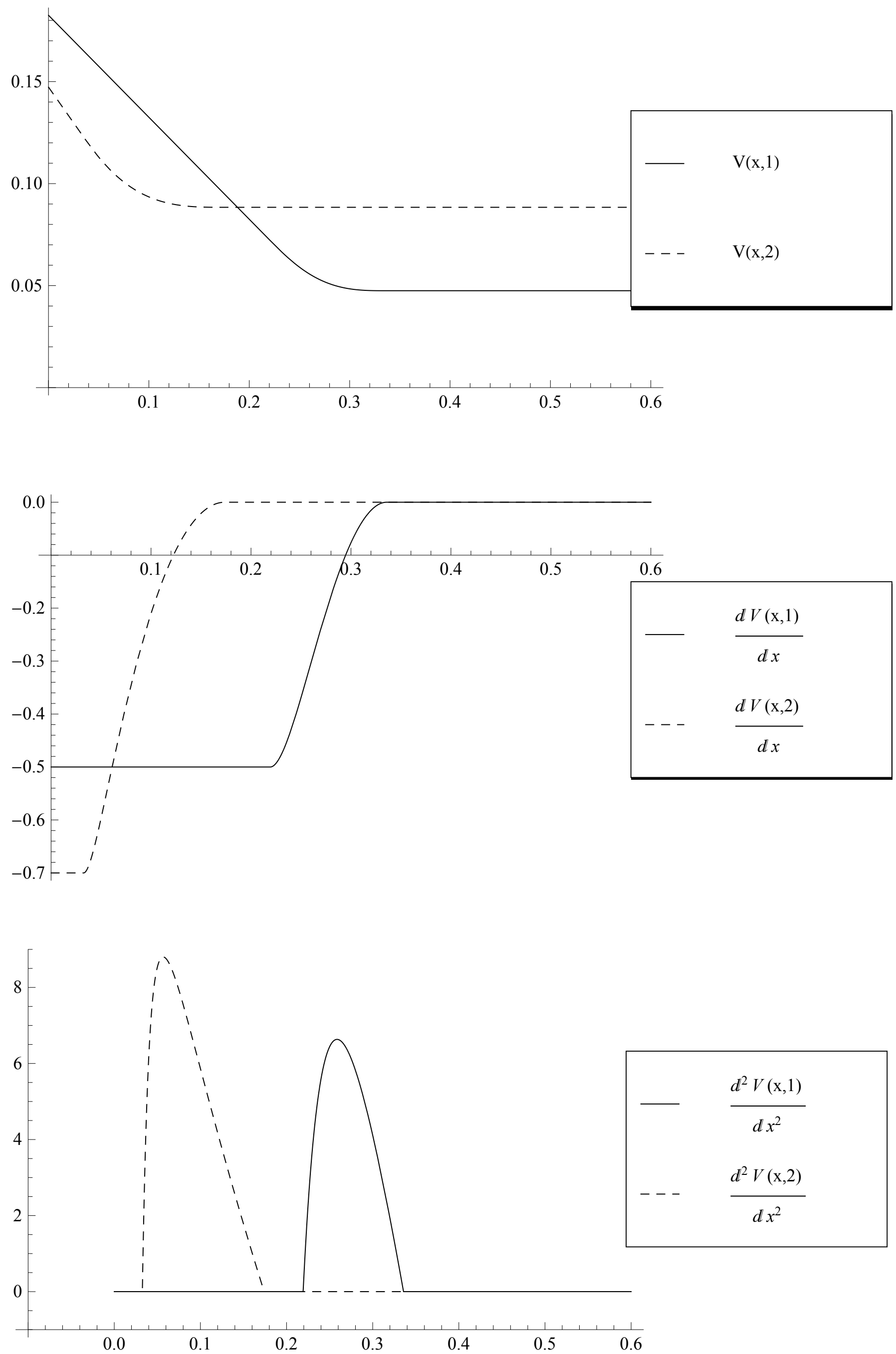

Figure 6.2: The value function and its derivatives for $a_2 < b_2 < a_1 < b_1$.

observe distinct behaviour. For the case $\rho_1 = \rho_2$ we have $V(x,1) < V(x,2)$ for every $x > 0$. However, this does not hold for $\rho_1 > \rho_2$. For the former case, since the target is the same for both economic times, it is natural that starting at the good economic time generates a lower expected total cost regardless of the initial value of the stabilization fund. On the contrary, if $\rho_1 > \rho_2$ the relation of $V(x,1)$ and $V(x,2)$ depends on the values of x. The pattern is as follows. For values of the initial fund close to the target ρ_2 (relative to ρ_1), we have $V(x,2) < V(x,1)$, while for values of the initial fund close to ρ_1 (relative to ρ_2), we have $V(x,1) < V(x,2)$.

6.15 Remark. *Provided that (6.72)-(6.74) are met, we conjecture that:*

(i) for any solution of the system of equations (6.59) such that $0 < a_2 < a_1 < b_1 < b_2 < \infty$ *the conditions (6.75)-(6.79) are satisfied;*

(ii) for any solution of the system of equations (6.68) such that $0 < a_2 < a_1 < b_2 < b_1 < \infty$ *the conditions (6.80)-(6.84) are satisfied;*

(iii) for any solution of the system of equations (6.71) such that $0 < a_2 < b_2 < a_1 < b_1 < \infty$ *the conditions (6.85)-(6.89) are satisfied.*

Since we do not have explicit solutions for the non-linear systems (6.59), (6.68) and (6.71), one cannot prove such conjectures. Certainly, the corresponding conjectures are verified not only for the previous example, but for all examples we present in this chapter.

6.4 Economic analysis of the solution

We study the impact of a strong recession on the optimal stabilization fund bands. We present two examples to illustrate the effect of a strong recession. The first corresponds to the case $\rho_1 = \rho_2$ and the second to $\rho_1 > \rho_2$.

For the case $\rho_1 = \rho_2$, we present Example 6.16 below to compare to Example 6.13. They have the same parameter values for the good economic time. However, the bad economic time represented in Example 6.16 is worse than in Example 6.13. Specifically, we have lower rate of growth of the fund and higher costs of increasing the fund.

6.16 Example. *The case $\rho_1 = \rho_2 = \rho$. Let us consider the following parameter values for the good economic time,*

$$\mu_1 = 0.02, \qquad \sigma_1 = 0.1, \qquad \lambda_1 = 0.05, \qquad k_1^L = 0.5, \qquad \rho = 0.10,$$

and

$$\mu_2 = -0.03, \qquad \sigma_2 = 0.3, \qquad \lambda_2 = 0.05, \qquad k_2^L = 1.0, \qquad \delta = 0.07$$

for the bad economic time. Solving the system of equations (6.59), we obtain

$$D_1 = 0.0581168, \quad D_2 = 0.09909841, \quad F_1 = 0.0162267, \quad F_2 = 0.0320564,$$

$$\widetilde{A}_1 = -1.3888 \times 10^{-11}, \ \widetilde{A}_2 = -0.000374, \ \widetilde{A}_3 = -37.609, \ \widetilde{A}_4 = -83.3821,$$

$$A_1 = -0.00004878, \quad A_2 = -15.2941, \quad A_3 = 0.0004952, \quad A_4 = -17.5279,$$

and

$$a_2 = 0.0176954, \quad a_1 = 0.0607365, \quad b_1 = 0.117222, \quad b_2 = 0.155095.$$

Comparing Example 6.16 to Example 6.13, we observe that the band for the good economic time changes slightly, while the band for the bad economic times change significantly, especially the lower bound. Indeed, the lower bound is now 1.76% compared to 3.19%. This is a consequence of the higher cost of interventions to increase the fund.

Now we turn to the case $\rho_1 > \rho_2$. We will compare Example 6.17 below to Example 6.14. For the good economic time they have the same parameter values.

Regarding the bad economic time, the parameter values in Examples 6.17 are the same as in Example 6.16, except for the values of the targets ρ_1 and ρ_2. That is, the bad economic time represented in Example 6.17 is worse than the bad economic time in Example 6.14.

6.17 Example. *The case $\rho_1 > \rho_2$. Let us consider the same parameter values as in Example 6.16 with $\rho_1 = 0.30$ and $\rho_2 = 0.10$. Solving the system of equations (6.71), we obtain*

$$D_1 = 0.178234, \quad D_2 = 0.14906, \quad F_1 = 0.043212, \quad F_2 = 0.07808872,$$

$$A_1 = -5.9329 \times 10^{-7}, \; A_2 = -17.448, \; A_3 = -0.0000499, \; A_4 = -16.3863,$$

and

$$a_2 = 0.0179603, \quad b_2 = 0.168314, \quad a_1 = 0.219245, \quad b_1 = 0.335791.$$

We note that the band for the good economic time is essentially the same in Examples 6.17 and 6.14. However, the band for the bad economic time changes significantly. In fact, the lower bound is now 1.79% compare to 3.26%. As discussed above, this is a consequence of the higher cost of intervention to increase the fund.

From Examples 6.16 and 6.17 we have the following conclusion: the stronger a recession, the closer to zero is the lower bound of the optimal band that corresponds to a recession, with minor impact on the optimal band for the good economic time. This is consistent with reality. In fact governments impose the lower limit of their bands equal to zero, because they are assuming implicitly huge costs to increase the fund in a bad economic time. Thus, governments are doing the right thing only in the (really) bad economic time. (We recall that they set a unique band regardless of the regime of the economy.) However, they need to realize that in the good economic time the lower bound of the band must be strictly positive and significant. Governments should take advantage of the lower cost of increasing the fund in the good economic time so that they are prepared for the bad economic time. Indeed, in Examples 6.13 and 6.16, the lower bound for the good economic time is 6% approximately. For a more saver government that considers $\rho_1 > \rho_2$, as represented in Examples 6.14 and 6.17, this is even more evident. This is one of the main normative contributions of this research.

6.5 Concluding remarks

Due to global and fiscal crises some governments have created stabilization funds as a mechanism of fiscal policy to save money in the good economic times to be used in the bad economic times. In this Chapter we study the optimal management of a government stabilization fund.

We model the problem of managing the stabilization fund as a stochastic singular control with economic cycles (regime switching). We assume that the stabilization fund is driven by a Brownian, that generates minor and continuous uncertain movements, and a Markov chain that accounts for the long-term macroeconomic conditions. There are two types of costs. The costs of having a stabilization fund away from the fund target and the costs of the intervention of the government to keep the stabilization fund close to the fund target.

We succeed in finding explicit solution for the optimal policy, and explicit optimal bands for the optimal management of the stabilization fund. The optimal bands are functions of the underlying parameters, such as the duration of the regimes, cost of increasing the fund, the rate of growth of the fund and the volatility of the fund. In particular, we obtain that the optimal band depends on the regime. We have computed for the first time optimal bands for the management of government stabilization funds through a model that integrates good and bad economic times. This is one of the main contributions of this research.

Moreover, we have found an optimal fund control that implies the following recommendation for fund management: given a regime of the economy i, if at any point in time the actual fund of a country is inside the corresponding optimal band $[a_i, b_i]$, then no intervention is required. If the fund is equal to either a_i or b_i, then intervention is required to prevent the fund from going away from the band. If the fund is outside the band $[a_i, b_i]$, then the government should intervene to take the fund to the frontier of the optimal band $[a_i, b_i]$. Hence, during the duration of the regime i, the resulting controlled fund remains in the band $[a_i, b_i]$, except perhaps at time zero or at the time in which the change of regime took place. It is instructive to analyze the case $a_2 < b_2 < a_1 < b_1$. In the good time the fund is kept inside $[a_1, b_1]$. When the bad economic time arrives, the government decreases the fund to b_2, withdrawing from the fund at least $(a_1 - b_2) > 0$ to mitigate the bad situation. During regime 2, the government keeps the fund inside $[a_2, b_2]$. Thus the optimal fund policy works as expected: the saving collected in the good economic time is used in the bad economic time.

In the economic analysis section, we have studied the effects of a strong recession on the optimal bands. We have found that the effects on the band for bad economic time $[a_2, b_2]$ are significant. In particular, the value of a_2 becomes close to zero. This is in line with reality in which countries impose a zero lower bound. However, we point out that they impose only a unique band regardless of the regime of the economy. On the contrary, we have seen that the bands depend on the regime of the economy. In particular, having two bands that do not intersect each other implies that the fund is playing a crucial role to help the economy in the bad economic time.

To summarize, in this chapter we have computed optimal bands for the management of the stabilization fund in a model that integrates good economic time and bad economic time. We have also derived a practical optimal policy of fund management based on such bands.

Chapter 7

Conclusions

In this research we develop mathematical models for optimal government management. Specifically, we study two problems related to optimal government debt control and a problem on the optimal management of the stabilization funds. We apply techniques of classical and singular stochastic control in order to solve them. This research is policy-oriented, we focus on providing insights that are useful to policy-makers.

First, we present for the first time in the literature theoretical models to study rigorously the optimal debt ceiling. In such models the government wants to control its debt by imposing an upper bound or ceiling on its debt-to-GDP ratio. We assume that debt generates a cost for the country, and this cost is an increasing and convex function of debt ratio. The government can intervene to reduce its debt ratio, but there is a cost generated by this reduction. The goal of the government is to find the optimal control that minimizes the expected total cost. We obtain an explicit solution for the government debt problem, that allows us to calculate the optimal debt ceiling as a function of macro-financial variables. In one of the models we even find an explicit formula for optimal government debt ceiling. In all the models, we derive a practical policy for optimal debt management in terms of the optimal government debt ceiling.

Second, we develop a theoretical model for optimal currency government debt portfolio and debt payments, which allows both government debt aversion and jumps in the exchange rates. We obtain first a realistic stochastic differential equation for public debt, and then solve explicitly the optimal currency debt problem. We show that higher debt aversion and jumps in the exchange rates lead to a lower proportion of optimal debt in foreign currencies. Furthermore, we show that for a government with extreme debt aversion it is optimal not to issue debt in foreign currencies. To the best of our knowledge, this is the first theoretical model that provides a rigorous explanation of why developing countries have reduced consistently their proportion of foreign debt in their debt portfolios.

Third, we study the stabilization fund. We present the first theoretical model for computing the optimal bands for the government stabilization fund, in a model that integrates good economic times and bad economic times. We consider that deviations from the fund target generate costs for the government. On the other hand, the interventions of the government to keep the fund close to the fund target generates costs as well. We allow the fund target to depend on the regime of the economy. The objective of the government is to minimize the expected total cost. We obtain explicitly the optimal bands, which depend on the regime of the economy. Furthermore, we derive a practical recommendation for the management of the stabilization fund based on the optimal bands.

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Appendix A

A.1 Proof of Remark 2.5

Proof. Let us define

$$d\xi_t = \xi_t \mu dt + \xi_t \sigma dW_t, \quad \xi_0 = 1.$$

Let $f(X, \xi) = X/\xi$. Applying Ito's Lemma for jump processes (see Shreve 2004, p. 489 or Rogers and Williams 2001, p. 394) to f, we obtain

$$\begin{aligned} f(X_t, \xi_t) &= f(X_0, \xi_0) - \int_0^t \frac{1}{\xi_s} dZ_s^c - \sum_{0 \leq s < t} \Big(\frac{X_s}{\xi_s} - \frac{X_{s+}}{\xi_s} \Big) \\ &= X_0 - \int_0^t \frac{1}{\xi_s} dZ_s^c - \sum_{0 \leq s < t} \frac{1}{\xi_s} \Big(Z_{s+} - Z_s \Big). \end{aligned}$$

Hence

$$X_t = \xi_t \Big(x - \int_{[0,t)} \frac{1}{\xi_s} dZ_s \Big) = \xi_t x - \int_{[0,t)} \frac{\xi_t}{\xi_s} dZ_s.$$

This in turn implies

$$X_t \leq \xi_t x = x \exp \Big\{ \big(\mu - 1/2\sigma^2 \big) t + \sigma W_t \Big\}.$$

Taking conditional expectation on both sides of the above inequality, we get

$$E_x \big[e^{-\lambda t} X_t^{2n} \big] \leq x^{2n} \exp \Big\{ - \Big(\lambda - \sigma^2 n(2n-1) - 2\mu n \Big) t \Big\}.$$

By 2.5, taking the limit as $t \to \infty$, the proof follows. □

A.2 Proof of Proposition 2.7

Proof. The value function V is nonnegative because J is nonnegative.

Consider $x_1 < x_2$, and $Z^{(2)} \in \mathcal{A}(x_2)$. Since h is strictly increasing, $Z^{(1)} \in \mathcal{A}(x_1)$. Then

$$V(x_1) \leq J(x_1, Z^{(2)}) < J(x_2, Z^{(2)}).$$

Hence $V(x_1) \leq V(x_2)$. Thus, V is increasing.

Consider $x_1 \leq x_2$ with corresponding controls $Z^{(1)} \in \mathcal{A}(x_1)$ and $Z^{(2)} \in \mathcal{A}(x_2)$. Let $\gamma \in [0,1]$. We define $Z^{(3)} := \gamma Z^{(1)} + (1-\gamma)Z^{(2)}$ and $x_3 := \gamma x_1 + (1-\gamma)x_2$. We denote by $X^{(j)}$ the trajectory that starts at x_j and is determined by the control $Z^{(j)}$, for $j = 1, 2, 3$. Since Eq. (2.2) is linear, we observe that for every $t \geq 0$:

$$X_t^{(3)} = \gamma X_t^{(1)} + (1-\gamma)X_t^{(2)}.$$

Since h is a convex function,

$$\begin{aligned} \int_0^\infty e^{-\lambda t} h(X_t^{(3)})dt &\leq \int_0^\infty e^{-\lambda t}\Big(\gamma h(X_t^{(1)}) + (1-\gamma)h(X_t^{(2)})\Big)dt \\ &= \gamma\Big(\int_0^\infty e^{-\lambda t} h(X_t^{(1)})dt\Big) + (1-\gamma)\Big(\int_0^\infty e^{-\lambda t} h(X_t^{(2)})dt\Big). \end{aligned}$$

As a consequence, we obtain

$$J(x_3; Z^{(3)}) \leq \gamma J(x_1; Z^{(1)}) + (1-\gamma)J(x_2; Z^{(2)}).$$

Hence

$$\begin{aligned} V\Big(\gamma x_1 + (1-\gamma)x_2\Big) &\le J\Big(\gamma x_1 + (1-\gamma)x_2, \gamma Z^{(1)} + (1-\gamma)Z^{(2)}\Big) \\ &= J(x_3; Z^{(3)}) \\ &\le \gamma J\big(x_1, Z^{(1)}\big) + (1-\gamma)J\big(x_2, Z^{(2)}\big). \end{aligned}$$

Consequently,

$$V\Big(\gamma x_1 + (1-\gamma)x_2\Big) \le \gamma V(x_1) + (1-\gamma)V(x_2),$$

which shows that V is convex.

It remains to prove the last assertion. Let $Z \in \mathcal{A}(x)$. Since

$$\int_0^\infty e^{-\lambda t}\beta dt \le E_x\left[\int_0^\infty e^{-\lambda t}\left\{\alpha X_t^{2n} + \beta\right\}dt + \int_0^\infty e^{-\lambda t}k dZ_t\right],$$

we have $\frac{\beta}{\lambda} \le J(x;Z)$, which yields $\frac{\beta}{\lambda} \le V(x)$. Hence $\frac{\beta}{\lambda} \le V(0+)$. On the other hand, by Example 2.3, condition (2.5) implies

$$V(x) \le J(x;0) = \frac{\alpha x^{2n}}{\lambda - \sigma^2 n(2n-1) - 2\mu n} + \frac{\beta}{\lambda},$$

from which we conclude that $V(0+) \le \frac{\beta}{\lambda}$, as required. This completes the proof. □

A.3 Proof of Lemma 2.13

We will use Lemmas A.1, A.2 and A.3 below to prove Lemma 2.13.

We recall $\tilde{\mu} := \mu - \frac{1}{2}\sigma^2$, by definition given in Eq. (3.21). Let us consider the constant

$$\bar{y} := 2n^2\sigma^2 + 2\tilde{\mu}n = \sigma^2 n(2n-1) + 2\mu n.$$

We observe that $\lambda > \bar{y}$, because we are assuming condition (2.5).

Let us consider the function $f : [-\frac{\tilde{\mu}^2}{2\sigma^2}, \infty) \to \mathbb{R}$ defined by

$$f(y) := \frac{\sqrt{\tilde{\mu}^2 + 2y\sigma^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2}.$$

A.1 Lemma. *(i) f is strictly increasing on its domain $[-\frac{\tilde{\mu}^2}{2\sigma^2}, \infty)$,*

(ii) $f(\bar{y}) = 2n$ *if* $\tilde{\mu} + 2n\sigma^2 \geq 0$,

(iii) $f(0) > 4n$ *if* $\tilde{\mu} + 2n\sigma^2 < 0$,

(iv) $f(\mu) = 1$ *if* $\mu > 0$,

(v) $f(0) \geq 1$ *if* $\mu \leq 0$.

Proof. (i) Taking the first derivative of f, we obtain

$$f'(y) = \frac{1}{\sqrt{\tilde{\mu}^2 + 2y\sigma^2}} > 0\,, \quad \text{for} \quad y \in \Big(-\frac{\tilde{\mu}^2}{2\sigma^2}, \infty\Big).$$

This establishes the desired result.

(ii) We have

$$\begin{aligned} f(\bar{y}) &= \frac{\sqrt{\tilde{\mu}^2 + 2(2n^2\sigma^2 + 2\tilde{\mu}n)\sigma^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} \\ &= \frac{\sqrt{\tilde{\mu}^2 + 4n^2\sigma^4 + 4\tilde{\mu}\sigma^2 n}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} \\ &= \frac{\sqrt{(\tilde{\mu} + 2n\sigma^2)^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} \\ &= \frac{\tilde{\mu} + 2n\sigma^2}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} \\ &= 2n, \end{aligned}$$

where we used $\tilde{\mu} + 2n\sigma^2 \geq 0$ to get the fourth equality above.

(iii) First of all, note that $\tilde{\mu} + 2n\sigma^2 < 0$ implies that $\tilde{\mu}$ is negative. To be more precise, it implies $\tilde{\mu} < -2n\sigma^2$, or equivalently $-2\frac{\tilde{\mu}}{\sigma^2} > 4n$. Hence,

$$f(0) = \frac{\sqrt{\tilde{\mu}^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} = \frac{|\tilde{\mu}|}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} = -2\frac{\tilde{\mu}}{\sigma^2} > 4n.$$

(iv) We have

$$\begin{aligned} f(\mu) &= \frac{\sqrt{\tilde{\mu}^2 + 2\mu\sigma^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} \\ &= \frac{\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\mu\sigma^2}}{\sigma^2} - \frac{(\mu - \frac{1}{2}\sigma^2)}{\sigma^2} \\ &= \frac{\sqrt{(\mu + \frac{1}{2}\sigma^2)^2}}{\sigma^2} - \frac{(\mu - \frac{1}{2}\sigma^2)}{\sigma^2} \\ &= \frac{(\mu + \frac{1}{2}\sigma^2)}{\sigma^2} - \frac{(\mu - \frac{1}{2}\sigma^2)}{\sigma^2} \\ &= 1\,, \end{aligned}$$

where we used $\mu > 0$ to obtain the fourth equality above.

(v) If $\mu \leq 0$, then $\tilde{\mu} \leq -\frac{1}{2}\sigma^2 \leq 0$. Thus,

$$f(0) = \frac{\sqrt{\tilde{\mu}^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} = \frac{|\tilde{\mu}|}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} = -2\frac{\tilde{\mu}}{\sigma^2} = 1 - 2\frac{\mu}{\sigma^2} \;\geq\; 1.$$

This completes the proof. □

A.2 Lemma. *If*

(i) $$2\lambda(\sigma^2 n + \mu)^2 \;\geq\; (\lambda + \sigma^2 n)\Big(\lambda\sigma^2 + 2\mu\sigma^2 n + 2\mu^2 - \sigma^2\mu\Big),$$

then

(ii) $$\lambda\bar{y} \;\geq\; \lambda(\lambda - \mu) + \mu\bar{y}.$$

Proof. We will work with the left-hand side and the right-hand side of (i) separately.

Subtracting $(2\lambda\mu^2 + 2\lambda\mu\sigma^2 n + \lambda\sigma^4 n)$ from the left-hand side of (i), we have

$$\begin{aligned}
&2\lambda(\sigma^2 n + \mu)^2 - (2\lambda\mu^2 + 2\lambda\mu\sigma^2 n + \lambda\sigma^4 n)\\
&\qquad = 2\lambda(\sigma^4 n^2 + 2\mu\sigma^2 n + \mu^2) - (2\lambda\mu^2 + 2\lambda\mu\sigma^2 n + \lambda\sigma^4 n)\\
&\qquad = 2\lambda\sigma^4 n^2 + 4\lambda\mu\sigma^2 n + 2\lambda\mu^2 - 2\lambda\mu^2 - 2\lambda\mu\sigma^2 n - \lambda\sigma^4 n\\
&\qquad = 2\lambda\sigma^4 n^2 + 2\lambda\mu\sigma^2 n - \lambda\sigma^4 n\\
&\qquad = \sigma^2\lambda(2\sigma^2 n^2 + 2\mu n - \sigma^2 n)\\
&\qquad = \sigma^2\lambda\Big(\sigma^2 n(2n-1) + 2\mu n\Big)\\
&\qquad = \sigma^2\lambda\,\bar{y}.
\end{aligned}$$

Subtracting the same quantity from the right-hand side of (i), we obtain

$$\begin{aligned}
&(\lambda + \sigma^2 n)\Big(\lambda\sigma^2 + 2\mu\sigma^2 n + 2\mu^2 - \sigma^2\mu\Big) - (2\lambda\mu^2 + 2\lambda\mu\sigma^2 n + \lambda\sigma^4 n)\\
&\qquad = \lambda^2\sigma^2 + 2\lambda\mu\sigma^2 n + 2\lambda\mu^2 - \lambda\sigma^2\mu + \lambda\sigma^4 n + 2\mu\sigma^4 n^2 + 2\mu^2\sigma^2 n\\
&\qquad\quad - \mu\sigma^4 n - (2\lambda\mu^2 + 2\lambda\mu\sigma^2 n + \lambda\sigma^4 n)\\
&\qquad = \lambda^2\sigma^2 - \lambda\mu\sigma^2 + 2\mu\sigma^4 n^2 + 2\mu^2\sigma^2 n - \mu\sigma^4 n\\
&\qquad = \lambda\sigma^2(\lambda - \mu) + \sigma^2\mu\Big(2\sigma^2 n^2 + 2\mu n - \sigma^2 n\Big)\\
&\qquad = \lambda\sigma^2(\lambda - \mu) + \sigma^2\mu\Big(\sigma^2 n(2n-1) + 2\mu n\Big)\\
&\qquad = \lambda\sigma^2(\lambda - \mu) + \sigma^2\mu\bar{y}\\
&\qquad = \sigma^2\Big(\lambda(\lambda - \mu) + \mu\bar{y}\Big).
\end{aligned}$$

Since we are subtracting the same quantity from both sides of inequality (i), we get the following equivalent expression

$$\sigma^2\lambda\,\bar{y} \;\geq\; \sigma^2\Big(\lambda(\lambda - \mu) + \mu\bar{y}\Big).$$

Recalling that $\sigma > 0$, we see that inequality (ii) is satisfied. $\square$

A.3 Lemma. *Let us consider the function* $\phi : [-\frac{\tilde{\mu}^2}{2\sigma^2}\,, \infty) \to \mathbb{R}$ *defined by*

$$\phi(p) \;:=\; 2p - \tilde{\mu} - \sqrt{\tilde{\mu}^2 + 2p\sigma^2}.$$

Then,

(i) ϕ *is strictly increasing on the interval* $\big[\max\{\mu,\, 0\}\,, \infty\big)$,

(ii) $\phi(\mu) = 0$ *if* $\mu > 0$,

(iii) $\phi(0) = 0$ *if* $\mu \le 0$.

Proof. Before proceeding with the specific proof of (i)-(iii), we observe that for every $p \in \big(-\frac{\tilde{\mu}^2}{2\sigma^2}\,, \infty\big)$:

$$\begin{aligned} \phi'(p) &= 2 - \frac{\sigma^2}{\big(\tilde{\mu}^2 + 2p\sigma^2\big)^{\frac{1}{2}}}, \\ \phi''(p) &= \frac{\sigma^4}{\big(\tilde{\mu}^2 + 2p\sigma^2\big)^{\frac{3}{2}}} > 0. \end{aligned}$$

Since $\phi'' > 0$, we conclude that ϕ' is strictly increasing on the open interval above. We notice that $\tilde{\mu}^2 + 2\mu\sigma^2 = (\mu - \frac{1}{2}\sigma)^2 + 2\mu\sigma^2 = (\mu + \frac{1}{2}\sigma)^2$. If $\mu > 0$, we obtain

$$\phi'(\mu) = 2 - \frac{\sigma^2}{\big(\tilde{\mu}^2 + 2\mu\sigma^2\big)^{\frac{1}{2}}} = 2 - \frac{\sigma^2}{\mu + \frac{1}{2}\sigma^2} = \frac{2\mu}{(\mu + \frac{1}{2}\sigma^2)} > 0.$$

On the other hand, if $\mu \le 0$ we obtain

$$\phi'(0) = 2 - \frac{\sigma^2}{\sqrt{\tilde{\mu}^2}} = 2 - \frac{\sigma^2}{|\tilde{\mu}|} = 2 + \frac{\sigma^2}{\big(\mu - \frac{1}{2}\sigma^2\big)} = \frac{2\mu}{\big(\mu - \frac{1}{2}\sigma^2\big)} \ge 0.$$

(i) To prove (i), we proceed by considering cases. First, we consider the case $\mu > 0$. Then, we need to show that ϕ is strictly increasing on $[\mu, \infty)$. To this end, all we need to show is that $\phi'(p) > 0$ for all $p > \mu$. Indeed, since ϕ'' is strictly positive, we obtain for every $p > \mu$:

$$\phi'(p) > \phi'(\mu) > 0.$$

This shows that if $\mu > 0$, then ϕ is strictly increasing on the interval $[\mu, \infty) = [\max(\mu, 0), \infty)$. Now we consider the case $\mu \leq 0$. We have to show that ϕ is strictly increasing on $[0, \infty)$. Using again that ϕ'' is strictly positive, we have for every $p > 0$:

$$\phi'(p) > \phi'(0) \geq 0.$$

This shows that if $\mu \leq 0$, then ϕ is strictly increasing on the interval $[0, \infty) = \big[\max\{\mu, 0\}, \infty)$. Hence in both cases ϕ is strictly increasing on $\big[\max\{\mu, 0\}, \infty)$.

(ii) Let us show the second assertion. Since $\mu > 0$ and $\tilde{\mu}^2 + 2\mu\sigma^2 = (\mu + \frac{1}{2}\sigma^2)^2$,

$$\phi(\mu) = 2\mu - \tilde{\mu} - \sqrt{\big(\mu + \frac{1}{2}\sigma^2\big)^2} = 2\mu - \tilde{\mu} - \big(\mu + \frac{1}{2}\sigma^2\big) = 0.$$

(iii) Now we consider the third assertion. We note that $\sqrt{\tilde{\mu}^2} = -\tilde{\mu}$ because $\tilde{\mu} \leq 0$ due to $\mu \leq 0$. Consequently,

$$\phi(0) = -\tilde{\mu} - \sqrt{\tilde{\mu}^2} = 0.$$

This completes the proof. □

Proof of Lemma 2.13.

Proof. We observe that Eq. (2.26) can be written as $\gamma_2 = f(\lambda)$.

(i) It follows immediately from the definition of ζ (given in Eq. 2.27) and condition (2.5).

(ii) We need to distinguish two cases. Consider first the case $\mu > 0$. From

$$\bar{y} := \sigma^2 n(2n-1) + 2\mu n,$$

defined at the beginning of this Appendix, it follows obviously that $\bar{y} > \mu$. This result and $\lambda > \bar{y}$ (recall the beginning of this Appendix) imply $\lambda > \mu$. Let us consider now the case $\mu \leq 0$. From Section 2.1, we recall that $\lambda \in (0, \infty)$. Under this condition, we have $\lambda > 0 \geq \mu$. This proves that $\lambda > \mu$ for every value of μ.

(iii) To prove (iii) of Lemma 2.13, we apply Lemma A.1. We need to distinguish two cases. Consider first the case $\tilde{\mu}+2n\sigma^2 \geq 0$. Then, Lemma A.1 (ii) implies $f(\bar{y}) = 2n$. Since f is strictly increasing, we obtain $f(\lambda) > f(\bar{y})$. Hence $\gamma_2 > 2n$. Now we consider the case $\tilde{\mu} + 2n\sigma^2 < 0$. Then, Lemma A.1 (iii) yields $f(0) > 4n$. Again, since f is strictly increasing, we conclude $\gamma_2 = f(\lambda) > f(0) > 4n$. This completes the proof of this assertion.

(iv) Aiming at a contradiction, suppose $\lambda \leq \sigma^2 n(\gamma_2 - 1) + \gamma_2\mu$. Then

$$(\sigma^2 n + \mu)\gamma_2 \;\geq\; \lambda + \sigma^2 n > 0,$$

which implies $\sigma^2 n + \mu > 0$, because $\gamma_2 > 1$. Based on this observation, we can express $\lambda \leq \sigma^2 n(\gamma_2 - 1) + \gamma_2\mu$ equivalently as

$$\gamma_2 \;\geq\; \frac{\lambda + \sigma^2 n}{\sigma^2 n + \mu}.$$

Using the definition of γ_2, we have the following equivalent inequalities:

$$\begin{aligned} \frac{\sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} - \frac{\tilde{\mu}}{\sigma^2} = \gamma_2 \;&\geq\; \frac{\lambda + \sigma^2 n}{\sigma^2 n + \mu}, \\ \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2} \;&\geq\; \frac{\sigma^2(\lambda + \sigma^2 n)}{\sigma^2 n + \mu} + \tilde{\mu}. \end{aligned} \tag{A.1}$$

We recall $\tilde{\mu} := \mu - \frac{1}{2}\sigma^2$. We note that

$$\begin{aligned} \frac{\sigma^2(\lambda + \sigma^2 n)}{\sigma^2 n + \mu} + \tilde{\mu} \;&=\; \frac{\sigma^2(\lambda + \sigma^2 n) + (\mu - \frac{1}{2}\sigma^2)(\sigma^2 n + \mu)}{\sigma^2 n + \mu} \\ &=\; \frac{\lambda\sigma^2 + \frac{1}{2}\sigma^4 n + \mu^2 + \mu\sigma^2(n - \frac{1}{2})}{\sigma^2 n + \mu} > 0. \end{aligned}$$

The above inequality follows from $\lambda > \sigma^2 n(2n - 1) + 2n\mu$. Indeed, let $H(\mu) := \lambda\sigma^2 + 1/2\sigma^4 n + \mu^2 + \mu\sigma^2(n - 1/2)$. Since H is convex the global minimum is achieved at μ^*, where $H'(\mu^*) = 0$. Then, we conclude that $H(\mu) \geq H(\mu^*) = 1/16\sigma^4\big(12n^2 + 4n - 1\big) > 0$.

The above inequality together with (A.1) imply

$$\begin{aligned}
\tilde{\mu}^2 + 2\lambda\sigma^2 &\geq \Big(\frac{\sigma^2(\lambda+\sigma^2 n)}{\sigma^2 n + \mu} + \tilde{\mu}\Big)^2, \\
2\lambda\sigma^2 &\geq \Big(\frac{\sigma^2(\lambda+\sigma^2 n)}{\sigma^2 n + \mu} + \tilde{\mu}\Big)^2 - \tilde{\mu}^2, \\
2\lambda &\geq \sigma^2\frac{(\lambda+\sigma^2 n)^2}{(\sigma^2 n+\mu)^2} + 2\frac{(\lambda+\sigma^2 n)}{(\mu+\sigma^2 n)}\tilde{\mu}\,, \\
2\lambda(\sigma^2 n+\mu)^2 &\geq (\lambda+\sigma^2 n)\Big(\sigma^2(\lambda+\sigma^2 n) + 2(\mu+\sigma^2 n)\tilde{\mu}\Big) \\
&= (\lambda+\sigma^2 n)\Big(\sigma^2(\lambda+\sigma^2 n) + 2(\mu+\sigma^2 n)(\mu - \frac{1}{2}\sigma^2)\Big) \\
&= (\lambda+\sigma^2 n)\Big(\sigma^2\lambda + \sigma^4 n + 2\left[\mu^2 - \frac{\mu}{2}\sigma^2 + \mu\sigma^2 n - \frac{1}{2}\sigma^4 n\right]\Big) \\
&= (\lambda+\sigma^2 n)\Big(\lambda\sigma^2 + 2\mu\sigma^2 n + 2\mu^2 - \mu\sigma^2\Big).
\end{aligned}$$

Hence

$$2\lambda(\sigma^2 n+\mu)^2 \geq (\lambda+\sigma^2 n)\Big(\lambda\sigma^2 + 2\mu\sigma^2 n + 2\mu^2 - \mu\sigma^2\Big).$$

By virtue of Lemma A.2, we have $\lambda\bar{y} \geq \lambda(\lambda-\mu)+\mu\,\bar{y}$. Grouping terms and factoring $(\lambda-\mu)$, we get the equivalent form

$$\big(\bar{y}-\lambda)(\lambda-\mu) \geq 0.$$

Since $\lambda > \mu$, it follows that $\bar{y} \geq \lambda$. Thus, we obtain a contradiction. Therefore, we must have $\lambda > \sigma^2 n(\gamma_2 - 1) + \gamma_2\mu$. This completes the proof of inequality (iv).

(v) We will apply Lemma A.3 to prove this inequality. Accordingly, we distinguish two cases. Consider first the case $\mu > 0$. Since $\phi(\mu) = 0$ and $\lambda > \mu$, we conclude that $\phi(\lambda) > \phi(\mu) = 0$, because ϕ is strictly increasing on $[\mu, \infty)$. Now consider the case $\mu \leq 0$. From $\phi(0) = 0$, and $\lambda > 0$, it follows $\phi(\lambda) > \phi(0) = 0$, because

ϕ is strictly increasing on $[0, \infty)$. Consequently, $\phi(\lambda) = 2\lambda - \tilde{\mu} - \sqrt{\tilde{\mu}^2 + 2\lambda\sigma^2} > 0$ for every μ, as was to be shown. □

A.4 Proof of Lemma 2.14

Proof. We recall that for every $x \in (0, \infty)$:

$$L(x) := \frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - \lambda v(x) + h(x).$$

Using Eq. (2.32), we have

$$L(x) = \mu x k - \lambda(kx + D) + \alpha x^{2n} + \beta \quad \text{for} \quad x \in [b, \infty).$$

We observe that L is (infinitely) differentiable on (b, ∞). Taking the first derivative, for every $x \in (b, \infty)$ we obtain:

$$L'(x) = (\mu - \lambda)k + 2\alpha n x^{2n-1}.$$

Hence

$$L'(b+) = (\mu - \lambda)k + 2\alpha n b^{2n-1}.$$

By continuity of v', condition (2.30), we have

$$k = v'(b) = 2\alpha\zeta n b^{2n-1} + B\gamma_2 b^{\gamma_2 - 1}.$$

From Eqs. (2.34) and (2.27), we have

$$\begin{aligned}
L'(b+) &= (\mu - \lambda)\Big\{2\alpha\zeta n b^{2n-1} + B\gamma_2 b^{\gamma_2 - 1}\Big\} + 2\alpha n b^{2n-1} \\
&= (\mu - \lambda)\Big\{2\alpha\zeta n b^{2n-1} - 2\alpha\zeta n \frac{(2n-1)}{\gamma_2(\gamma_2 - 1)} b^{2n-\gamma_2}\gamma_2 b^{\gamma_2 - 1}\Big\} + 2\alpha n b^{2n-1} \\
&= (\mu - \lambda)\Big\{2\alpha\zeta n b^{2n-1} - 2\alpha\zeta n \frac{(2n-1)}{(\gamma_2 - 1)} b^{2n-1}\Big\} + 2\alpha n b^{2n-1}
\end{aligned}$$

$$= 2\alpha n b^{2n-1}\Big\{(\mu-\lambda)\zeta\Big(1-\frac{2n-1}{\gamma_2-1}\Big)+1\Big\}$$

$$= 2\alpha n b^{2n-1}\Big\{(\lambda-\mu)\zeta\Big(\frac{2n-\gamma_2}{\gamma_2-1}\Big)+1\Big\}$$

$$= 2\alpha n b^{2n-1}\Big\{\frac{(\lambda-\mu)}{(\lambda-\sigma^2 n(2n-1)-2\mu n)}\frac{(2n-\gamma_2)}{(\gamma_2-1)}+1\Big\}$$

$$= 2\alpha n b^{2n-1}\Big\{\frac{(\lambda-\mu)(2n-\gamma_2)+(\gamma_2-1)\big(\lambda-\sigma^2 n(2n-1)-2\mu n\big)}{(\lambda-\sigma^2 n(2n-1)-2\mu n)(\gamma_2-1)}\Big\}.$$

We notice that the numerator of the term in curly brackets above can be written in a convenient manner; namely

$$\begin{aligned}
(\lambda-\mu)(2n-\gamma_2)+(\gamma_2-1)\big(\lambda-\sigma^2 n(2n-1)-2\mu n\big) \\
&= 2\lambda n-\lambda\gamma_2-2\mu n+\mu\gamma_2+\lambda\gamma_2-\sigma^2 n(2n-1)\gamma_2 \\
&\quad -2\mu n\gamma_2-\lambda+\sigma^2 n(2n-1)+2\mu n \\
&= \lambda(2n-1)-\mu\gamma_2(2n-1)-\sigma^2 n(\gamma_2-1)(2n-1) \\
&= (2n-1)\big(\lambda-\sigma^2 n(\gamma_2-1)-\mu\gamma_2\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
L'(b+) &= 2\alpha n b^{2n-1}\frac{(2n-1)\big(\lambda-\sigma^2 n(\gamma_2-1)-\mu\gamma_2\big)}{(\lambda-\sigma^2 n(2n-1)-2\mu n)(\gamma_2-1)} \\
&= \frac{2\alpha n(2n-1)b^{2n-1}\,\zeta\,\big(\lambda-n\sigma^2(\gamma_2-1)-\mu\gamma_2\big)}{(\gamma_2-1)} > 0,
\end{aligned}$$

because $\zeta>0$, $\gamma_2>1$ and $\lambda-n\sigma^2(\gamma_2-1)-\mu\gamma_2>0$, by Lemma 2.13. □

Appendix B

B.1 Proof of Lemma 3.9

Proof. Let
$$G(x) := x^{\gamma_2}\left(\frac{\sigma^2}{2\overline{U}}\right)^{\gamma_2} {}_1F_1\left(-\gamma_2, c_2, -\frac{2\overline{U}}{\sigma^2\, x}\right)$$
and
$$H(x) := \left(\frac{2\overline{U}}{\sigma^2\, x}\right)^{c_3} {}_1F_1\left(c_3, 2-c_2, -\frac{2\overline{U}}{\sigma^2\, x}\right).$$
The function f defined in (3.20) can be written as

$$f(x) = \sum_{j=0}^{2n} \zeta_j\, x^j + B_1 G(x) + B_2 H(x).$$

By Lemma 3.8, we note that $\gamma_2 > 2n$ and $c_3 > 0$. Then, Remark 3.7 implies

$$\lim_{x\to\infty} B_2 H(x) = 0.$$

Consequently, there exists $\bar{x} > b$, such that for all $x > \bar{x}$, we have $-1 < B_2 H(x) < 1$. Thus

$$\underline{f}(x) := \sum_{j=0}^{2n} \zeta_j\, x^j + B_1 G(x) - 1 < f(x) < \overline{f}(x) := \sum_{j=0}^{2n} \zeta_j\, x^j + B_1 G(x) + 1.$$

Since $\gamma_2 > 2n$, using Remark 3.7, it follows that G does not satisfy any polynomial growth condition of degree $2n$. That is, for all $M_2 > 0$ there exists x_{M_2} such that $G(x) > M_2(1 + x^{2n})$ for all $x > x_{M_2}$.

Now, for a contradiction, suppose $B_1 > 0$. Then $\underline{f}$ does not satisfy the polynomial growth condition (3.32). This in turn implies that f does not satisfy (3.32) either. This contradiction implies that $B_1 \leq 0$. Next we show $B_1 \geq 0$. For a contradiction, suppose $B_1 < 0$. Then there exists x big enough such that $\overline{f}(x) < 0$. However, this contradicts the fact that f is non-negative. As a result $B \geq 0$. Combining these two results, we conclude that $B_1 = 0$. □

B.2 Proof of Proposition 3.14

Proof. By definition of the optimal debt policy $\hat{u}$, we can rewrite τ as

$$\tau = \inf \left\{ t \in (0, \infty) : x + \int_0^t \mu X_s ds + \int_0^t \sigma X_s \, dW_s \leq b + t\overline{U} \right\}.$$

Using the explicit solution of a geometric Brownian motion,

$$\begin{aligned} \tau &= \inf \left\{ t \in (0, \infty) : x \exp\{\tilde{\mu}\, t + \sigma\, W_t\} \leq b + t\overline{U} \right\} \\ &= \inf \left\{ t \in (0, \infty) : -\tilde{\mu}\, t + \sigma\, V_t \geq \log x - \log(b + t\overline{U}) \right\}, \end{aligned}$$

where $V_t := -W_t$ is a Brownian motion as well.

Let us define the stopping time

$$\begin{aligned} \theta &:= \inf \left\{ t \in (0, \infty) : -\tilde{\mu}\, t + \sigma\, V_t \geq \log x - b - t\overline{U} \right\} \\ &= \inf \left\{ t \in (0, \infty) : -(\tilde{\mu} - \overline{U})\, t + \sigma\, V_t \geq \log x - b \right\}. \end{aligned}$$

Since the process $-(\tilde{\mu} - \overline{U})\, t + \sigma V_t$ is a Brownian motion with drift, we may apply the methods in Section 8.4 of Ross (1996) and Section 7.5 of Karlin and

Taylor (1975) to show (a) and (b) below:

(a) If $\overline{U} < \tilde{\mu}$, then the distribution function of θ is defective. This means that $P\{\theta < \infty\} < 1$ or, equivalently, $P\{\theta = \infty\} > 0$. Hence $E[\theta] = \infty$. More precisely,

$$P\left\{\theta < \infty\right\} = \exp\left\{\frac{2(\overline{U} - \tilde{\mu})}{\sigma^2}\Big(\log x - b\Big)\right\}.$$

(b) If $\overline{U} \geq \tilde{\mu}$, then $P\{\theta < \infty\} = 1$. Furthermore, for $\overline{U} > \tilde{\mu}$, the expected value of θ is

$$E[\theta] \;=\; \left(\frac{\sigma}{\overline{U} - \tilde{\mu}}\right)\Big(\log x - b\Big).$$

In addition, for $\overline{U} = \tilde{\mu}$, we have $E[\theta] = \infty$.

The proof is complete after noting that $\theta \leq \tau$, because $\log x - \log(b + t\overline{U}) \geq \log x - (b + t\overline{U})$. Hence $P\{\tau < \infty\} \;\leq\; P\{\theta < \infty\}$, and $E[\theta] \;\leq\; E[\tau]$.

□

B.3 Debt dynamics in discrete time

To motivate the derivation of the multi-currency debt dynamics in continuous time, in this appendix we present a discrete-time example. Before doing so, we discuss the connection between the price of a bond and the interest charges on the corresponding debt. Suppose that at time t the government issues one unit of a bond in local currency. That is, the debt issued is $R_0(t)$. We want to calculate the interest charges of this debt at time $t + \Delta t$. From equation (5.4) we know that $R_0(t + \Delta t) - R_0(t) \;=\; R_0(t)\big(e^{r_0 \Delta t} - 1\big) \;\approx\; R_0(t)\, r_0 \Delta t$. Thus, the difference of the prices of the bond in local currency represents the interest charges on debt in local currency. Now suppose that the government issues one unit of a bond in foreign currency j instead. We want to calculate the interest charges of this debt at time

$t + \Delta t$ expressed in local currency. In this case, the answer is given by the difference $R_j(t + \Delta t)Q_j(t + \Delta t) - R_j(t)Q_j(t)$.

Suppose that $m = 1$ and the change of bond prices and debt payments occur only at times $t \in \{0, 1, 2\}$. By definition of total debt given in equation (5.2), at time $t = 0$,

$$X(0) = \Lambda_0(0)R_0(0) + \Lambda_1(0)R_1(0)Q_1(0).$$

Considering the prices of the bonds, the exchange rates, and possible payments, the debt at time $t = 1$ is:

$$X(1) = \Lambda_0(0)R_0(1) + \Lambda_1(0)R_1(1)Q_1(1) - p(1).$$

We note that the debt at time $t = 1$ is determined by the debt issued at time 0, i.e. $\Lambda_0(0)$ and $\Lambda_1(0)$, the new prices of the bonds $R_0(1)$ and $R_1(1)$, and the exchange rate $Q_1(1)$. Moreover, the payment $p(1)$ reduce the total debt. Based on the previous two equations, we can recast the total debt at time $t = 1$ as

$$\begin{aligned} X(1) &= X(0) + \Lambda_0(0)\big[R_0(1) - R_0(0)\big] + \Lambda_1(0)\big[R_1(1)Q_1(1) - R_0(0)Q_0(0)\big] \\ &\quad - p(1). \end{aligned} \tag{B.1}$$

This form states that the total debt at time 1 is the sum of the initial total debt $X(0)$, the interest of debt in local currency $\Lambda_0(0)[R_0(1) - R_0(0)]$, and the interest of debt in the foreign currency expressed in local currency $\Lambda_1(0)[R_1(1)Q_1(1) - R_0(0)Q_0(0)]$, minus the debt payment $p(1)$.

At time $t = 1$ the number of bonds in both currencies need not be the same as at time 0. In fact, using equation (5.2) we have

$$X(1) = \Lambda_0(1)R_0(1) + \Lambda_1(1)R_1(1)Q_1(1),$$

where we recall that $\Lambda_0(1)$ and $\Lambda_1(1)$ are the number of bonds in local currency and foreign currency held at time 1, respectively. Considering the prices of the bonds, the exchange rates, and possible payments, the total debt at time $t = 2$ is

$$X(2) = \Lambda_0(1)R_0(2) + \Lambda_1(1)R_1(2)Q_1(2) - p(2).$$

Combining the previous two equations, we arrive at

$$\begin{aligned} X(2) \quad &= \quad X(1) + \Lambda_0(1)\big[R_0(2) - R_0(1)\big] + \Lambda_1(1)\big[R_1(2)Q_1(2) - R_1(1)Q_1(1)\big] \\ &\quad - p(2). \end{aligned} \tag{B.2}$$

Using equations (B.1) and (B.2), we see that the total debt at time $t = 2$ can be written as

$$\begin{aligned} X(2) \quad &= \quad X(0) + \sum_{i=1}^{2} \Lambda_0(i-1)\big[R_0(i) - R_0(i-1)\big] \\ &\quad + \sum_{i=1}^{2} \Lambda_1(i-1)\big[R_1(i)Q_1(i) - R_1(i-1)Q_1(i-1)\big] - \sum_{i=1}^{2} p(i). \end{aligned} \tag{B.3}$$

B.4 Proof of Lemma 5.7

Proof. According to equations (5.10) and (5.12), the solution of the SDE that corresponds to this particular control (π, p) is given by

$$X(t) = x \exp\left\{ \int_0^t \beta ds + \int_0^t \pi^T \sigma dW(s) + \sum_{j=1}^{m} \int_0^t \log\left(1 + \pi^T \varphi_j\right) dN_j(s) \right\},$$

where

$$\beta := r_0 - \rho + \pi^T b - \sum_{j=1}^{m} \lambda_j \pi^T \varphi_j - \frac{1}{2} \pi^T \sigma \sigma^T \pi\,.$$

Since all the entries in σ, φ, and π are constants,

$$X(t) = x \exp\left\{ \beta t + \pi^T \sigma W(t) + \sum_{j=1}^{m} N_j(t) \log\left(1 + \pi^T \varphi_j\right) \right\} .$$

Thus, using the fact that $N = \{N_j : j = 1, \cdots, m\}$ and $W = \{W_j : j = 1, \cdots, m\}$ are independent, we have

$$E_x\left[e^{-\delta t} X_t^{\gamma+1}\right] = x^{\gamma+1} \exp\left\{\left[(\gamma+1)\beta - \delta\right] t\right\} E_1 \, E_2,$$

with

$$E_1 = E_x\Big[\exp\left\{(\gamma+1)\pi^T \sigma W(t)\right\}\Big],$$

$$E_2 = E_x\Big[\exp\Big\{\sum_{j=1}^{m} N_j(t) \log(1 + \pi^T \varphi_j)^{\gamma+1}\Big\}\Big].$$

Computing the above expected values, we obtain

$$E_x\Big[\exp\{(\gamma+1)\pi^T \sigma W(t)\}\Big] = \exp\Big\{1/2\,(\gamma+1)^2 \pi^T \sigma \sigma^T \pi \, t\Big\};$$

and for $j \in \{1, \cdots, m\}$:

$$E_x\Big[\exp\Big\{N_j(t) \log(1 + \pi^T \varphi_j)^{\gamma+1}\Big\}\Big] = \exp\Big\{\lambda_j t\Big((1 + \pi^T \varphi_j)^{\gamma+1} - 1\Big)\Big\}.$$

Consequently,

$$E_x\Big[e^{-\delta t} X_t^{\gamma+1}\Big] = x^{\gamma+1} \exp\Big\{ - \Big((\gamma+1)\rho + \zeta\Big) t\Big\}.$$

□

B.5 Proof of Proposition 5.10

Proof. V is non-negative by definition. To show the homogeneity property of V, we note that equation (5.12) implies that $\forall \nu > 0$:

$$J(\nu\, x; \pi, \nu\, p) = \nu^{\gamma+1} J(x; \pi, p).$$

Consequently,

$$\begin{aligned} V(\nu\, x) \quad &:= \quad \inf_{(\pi,\, \nu\, p)\, \in\, \mathcal{A}} J(\nu\, x; \pi, \nu\, p) \; = \; \nu^{\gamma+1} \inf_{(\pi,\, \nu\, p)\, \in\, \mathcal{A}} J(x; \pi, p) \\ &= \quad \nu^{\gamma+1} \inf_{(\pi,\; p)\, \in\, \mathcal{A}} J(x; \pi, p) \; = \; \nu^{\gamma+1}\, V(x). \end{aligned}$$

Taking $\nu = 1/x$ in the above equation, we obtain $V(x) = x^{\gamma+1} V(1)$. Since $x^{\gamma+1}$ is strictly increasing and convex, the proposition follows. □

B.6 Proof of Lemma 6.8

Proof. We recall that $v(\cdot, 1) \in C^2\Big((0, \infty) - \{a_2, b_2\}\Big)$. and $v(\cdot, 2) \in C^2\Big((0, \infty) - \{a_1, b_1\}\Big)$. Let $i \in \{1, 2\}$. We note that $v_{xx}(\cdot, i)$ is continuous on $\{a_i, b_i\}$, which implies that $\mathbb{L}_i^v(x, i)$ is continuous on $\{a_i, b_i\}$ as well. Since $\mathbb{L}_i^v(x, i) = 0$ for all $x \in (a_i, b_i)$, we have $\mathbb{L}_i^v(a_i, i) = 0 = \mathbb{L}_i^v(b_i, i)$.

By definition of $\mathbb{L}_i^v$, we obtain $\partial^2 \mathbb{L}_i^v(x, i)/\partial x^2 = \lambda_i v_{xx}(x, 3 - i) + 2\alpha > 0$ for every $x \in (0, a_i) \cup (b_i, \infty)$. Furthermore,

$$\frac{\partial \mathbb{L}_i^v(x, i)}{\partial x} = \lambda_i v_x(x, 3 - i) - k_i^L(\mu_i - \delta - \lambda_i) + 2\alpha(x - \rho_i), \quad \forall x \in (0, a_i),$$

$$\frac{\partial \mathbb{L}_i^v(x, i)}{\partial x} = \lambda_i v_x(x, 3 - i) + k_i^U(\mu_i - \delta - \lambda_i) + 2\alpha(x - \rho_i), \quad \forall x \in (b_i, \infty).$$

Since $\mathbb{L}_i^v(b_i+, i) = \mathbb{L}_i^v(b_i, i) = 0$, to prove that $\mathbb{L}_i^v(x, i) \geq 0$ on (b, ∞), it is enough to show that $\mathbb{L}_i^v(x, i) \geq \mathbb{L}_i^v(b_i+, i)$. Thus we just need to prove that $\partial \mathbb{L}_i^v(x, i)/\partial x > 0$

on (b_i, ∞). Since $\mathbb{L}_i^v$ is strictly convex on (b_i, ∞), all we need is $\partial \mathbb{L}_i^v(b+, i)/\partial x \geq 0$, which is precisely the second condition of the Lemma. In a similar manner, we consider the interval $(0, a_i)$. Since $\mathbb{L}_i^v$ is strictly convex on $(0, a_i)$, we have $\partial \mathbb{L}_i^v(x, i)/\partial x < \partial \mathbb{L}_i^v(a_i-, i)/\partial x, \forall x \in (0, a_i)$. From the first condition of the Lemma, that states $\partial \mathbb{L}_i^v(a_i-, i)/\partial x \leq 0$, we deduce that $\mathbb{L}_i^v(x, i) \geq \mathbb{L}_i^v(a_i-, i) = 0 \, \forall x \in (0, a_i)$. This completes the proof of the Lemma. □

Appendix C

Special functions

General references for this Appendix are Bell (2004) and Hochstadt (1986).

C.1 Gamma function

For complex numbers with strictly positive real part, it is defined by

$$\Gamma(s) := \int_0^\infty e^{-t}\, t^{s-1} dt\,; \quad Re(s) > 0.$$

It is extended (by analytical continuation) to all complex numbers except the non-positive integers.

C.2 Binomial coefficient

For x non-negative integer and y non-integer, the Gamma function is defined by

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}. \tag{C.1}$$

Otherwise,

$$\binom{x}{y} = \begin{cases} (-1)^y \binom{-x+y-1}{y} & \text{if } y \geq 0, \\ (-1)^{x-y} \binom{-y-1}{x-y} & \text{if } y \geq x, \\ 0 & \text{otherwise.} \end{cases} \tag{C.2}$$

C.3 Hypergeometric function: ${}_1F_1(a; b; z)$

It is defined by

$${}_1F_1(a; b; z) := \sum_{n=0}^{\infty} \frac{(a, n)}{(b, n)} \frac{1}{n!} z^n , \tag{C.3}$$

where $(u, 0) := 1$ and

$$(u, n) := u(u + 1)(u + 2) \ldots (u + n - 1).$$

C.4 U hypergeometric function: $U(a; b; z)$

For $b \notin \mathbb{Z}$, the `U hypergeometric function` is defined by

$$U(a; b; z) := {}_1F_1(a; b; z) \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} + {}_1F_1(a - b + 1; 2 - b; z) \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b}. \tag{C.4}$$

For $b \in \{2, 3, 4, \cdots\}$,

$$U(a; b; z) := {}_1F_1(a; b; z) \lim_{u \to b} \frac{\Gamma(1 - u)}{\Gamma(a - u + 1)} + {}_1F_1(a - b + 1; 2 - b; z) \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b}. \tag{C.5}$$

Thus $U(a; b; z)$ is defined for all $b \in \mathbb{R}$ except on the set $\{0, -1, -2, \cdots\}$. We note that if a is a non-positive integer, then $\Gamma(a) = \infty$. In this case, we define $1/\Gamma[a] = 0$.

C.5 Laguerre polynomials

The `generalized Laguerre polynomials` can be defined in terms of the hypergeometric function ${}_1F_1$:

$$L(c; \alpha; x) := \binom{\alpha + c}{c} \; {}_1F_1(-c, \alpha + 1, x)\,. \tag{C.6}$$