UNIVERSITY OF ALBERTA

DIFFERENTIAL EQUATIONS INVOLVING HYSTERESIS

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

IN

MATHEMATICS

DEPARTMENT OF MATHEMATICAL SCIENCES

Edmonton, Alberta
Spring 1998

0-612-29055-7

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis **Differential equations involving hysteresis** submitted by **Jana Kopfová** in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

This thesis is concerned with the theory of hysteresis operators and their coupling with different differential equations. Firstly, the concept of hysteresis operators is introduced and some of their most important properties are listed. Secondly, a uniqueness theorem for an ordinary differential equation with non-Lipschitz hysteresis boundary curves is proved, using a simple theorem from ordinary differential equations. Thirdly, a partial differential equation of parabolic type with hysteresis is studied by a semigroup approach, as pioneered by A.Visintin, and asymptotic stability of solutions is obtained via this approach.

Fourthly, a partial differential equation of parabolic type with hysteresis in the source term is considered. Existence of periodic solutions of this equation is proved using fixed point arguments.

Finally, a hyperbolic equation of first order with hysteresis is studied and an entropy condition is derived.

ACKNOWLEDGEMENT

First and foremost, I would like to thank my supervisor, Dr.J.Macki for introducing me to the subject of hysteresis operators. I am deeply indebted to him for his valuable guidance and constant encouragement throughout my Ph.D. program.

I owe a debt of gratitude to Dr.J.So for his willingness to discuss problems, for his advice and support throughout my stay at the University of Alberta. I am also grateful to Professors A.Visintin and N.Kenmochi for their helpful comments and fruitful discussions.

My sincere thanks also go to my husband Tomáš Kopf for his understanding, encouragment and his help with typing this thesis, and to my sons Matúš and Michal for their patience.

Contents

Introduction

0.1. What is hysteresis

The term hysteresis means to lag behind, and originates from ancient Greek. When speaking of hysteresis, one usually refers to a relation between two scalar time-dependent quantities $u(t)$ and $w(t)$ that cannot be expressed in terms of a single-valued function, but takes the form of loops like the one depicted in Fig.0.1.

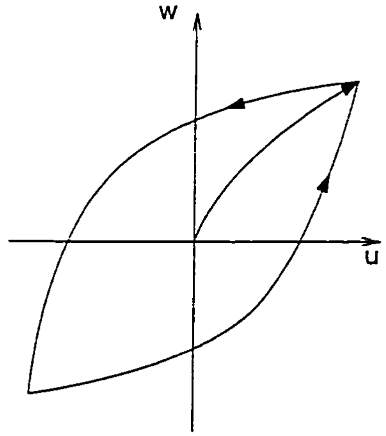

FIGURE 0.1. Hysteresis diagram.

Hysteresis diagrams like the one depicted above arise in different areas of science such as ferromagnetism, elastoplasticity, superconductivity, spin glasses, porous media filtration, thermostats, plasticity and shape memory alloys; numerous other examples can be added. Hysteresis diagrams are often related to each other by their appearance and shape, that is, the notion of hysteresis is essentially a phenomenological one. Thus, any analysis of the formal structures and common features of hysteresis that abstracts from the respective meanings of the involved quantities has, by its very nature, to be a mathematical one.

It seems that the term hysteresis was first used by Ewing in 1882 in his study of ferromagnetism. In 1887 Rayleigh proposed a model of ferromagnetic hysteresis, equivalent to what we call today the Prandtl-Ishlinskii model of play type. Another model was considered by Duhem around 1897. There were several other models developed, e.g. the Preisach model in 1935.

The mathematical development has lagged considerably behind the physical one. It was only in 1966 that hysteresis was given a first functional approach - due to an engineering student, Bouc. In 1970 Krasnosel'skii and co-workers proposed a mathematical formulation of the Prandtl-Ishlinskii model in terms of hysteresis operators, and later they conducted a systematic analysis of the mathematical properties of these operators. In the 1980's several applied mathematicians also began to study hysteresis models, mainly in connection with applications.

Mathematically, a hysteresis relationship between two functions u and w that are defined on some time interval $[0, t]$ and attain their values in some sets U and W, respectively, can be expressed as an operator equation with an operator $\mathcal{F}$:

$$w = \mathcal{F}[u]. \tag{0.1.1}$$

Hysteresis operators are characterized by two main properties:

(i) Memory: at any instant t, $w(t)$ depends on the previous evolution of u. We also assume that

$$\text{if } u_1 = u_2 \text{ in } [0, t], \text{ then } [\mathcal{F}(u_1)](t) = [\mathcal{F}(u_2)](t) (\text{causality}). \tag{0.1.2}$$

(ii) Rate independence: the output w is invariant with respect to changes of the time scale, formally

$$\mathcal{F}[u] \circ \phi = \mathcal{F}[u \circ \phi] \tag{0.1.3}$$

for all inputs u and all increasing functions ϕ mapping the considered time interval onto itself.

When only (i) is fulfilled, we speak of a Volterra operator.

At any instant t, the output $w(t)$ usually depends not only on $u|_{[0,t]}$, but also on the initial state of the system. Hence, the initial value $w_0 = w(0)$, or some equivalent information, must be prescribed. Therefore we write $\mathcal{F}(u, w_0)$.

For any such operator $\mathcal{F}(u, w_0)$ it is also sensible to require the semigroup property:

$$\begin{cases} \forall (u, w_0) \in \mathrm{Dom}(\mathcal{F}), \forall [t_1, t_2] \subset (0, T], \\ \text{setting } w(t_1) = [\mathcal{F}(u, w_0)](t_1), \text{ then} \\ [\mathcal{F}(u, w_0)](t_2) = [\mathcal{F}(u(t_1 + \cdot), w(t_1))](t_2 - t_1). \end{cases} \tag{0.1.4}$$

This has the following meaning: for any $t_1 \in (0, T]$, in order to evaluate $\mathcal{F}(u, w_0)(t_2)$ for $t_2 > t_1$, the information contained in $\mathcal{F}(u, w_0)(t_1)$ can replace that given by w_0 and the evolution of u in $[0, t_1]$. Among other things this implies that $t = 0$ is not a privileged instant. Krasnosel'skii and Pokrovskii call such an operator deterministic. However, the semigroup property can also have other forms, depending how the information on the initial conditions is represented.

The rate independence property offers the possibility of graphical representation of the operator, see Figure 0.1.

Many hysteresis operators also satisfy other typical properties:

1) Piecewise monotonicity:

$$\begin{cases} \forall (u, w_0) \in \mathrm{Dom}(\mathcal{F}), \forall [t_1, t_2] \subset [0, T], \\ \text{if } u \text{ is nondecreasing (resp. nonincreasing) in } [t_1, t_2], \\ \text{then so is } \mathcal{F}(u, w_0). \end{cases} \tag{0.1.5}$$

If $u, \mathcal{F}(u, w_0) \in W^{1,1}(0, T)$, then this can be described by a simple inequality:

$$\frac{du}{dt}\left[\frac{d}{dt}\mathcal{F}(u, w_0)\right] \geq 0 \qquad \text{a.e. in } (0, T). \tag{0.1.6}$$

2) Order preservation:

$$\begin{cases} \forall (u_1, w_{10}), (u_2, w_{20}) \in \mathrm{Dom}(\mathcal{F}), \forall t \in (0,T], \\ \text{if } u_1 \leq u_2 \text{ in } [0,t], \text{ and } w_{10} \leq w_{20}, \text{ then} \\ \mathcal{F}(u_1, w_{10})(t) \leq \mathcal{F}(u_2, w_{20})(t). \end{cases} \tag{0.1.7}$$

The piecewise monotonicity property is especially natural for rate independent operators, however, there also exist rate dependent operators which satisfy it. The following example is due to P.Krejčí (see [**37**], p.62): Let

$$[\mathcal{F}(u)](t) = tu(t) - \int_0^t u(\tau)d\tau. \tag{0.1.8}$$

Then for any $u \in W^{1,1}(0,T)$; $w := \mathcal{F}(u) \in W^{1,1}(0,T)$ and $w(t) = \int_0^t \tau \frac{du}{d\tau}(\tau)d\tau$ in $[0,T]$, so (0.1.6) is clearly satisfied. It can be easily checked that this operator is rate dependent.

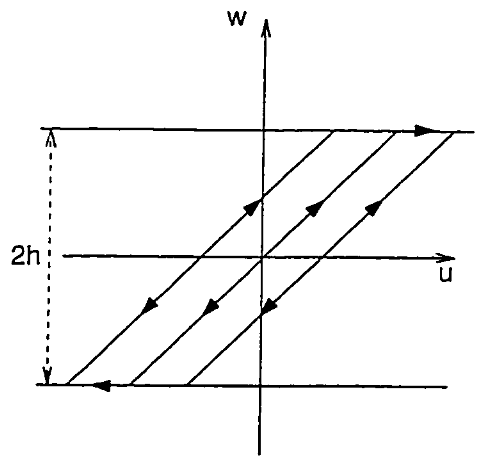

FIGURE 0.2. The stop operator, $\mathcal{G}$.

The stop operator, depicted in Figure 0.2, is an example of a hysteresis operator (i.e. an operator with memory, which is rate independent), which is piecewise monotone and is not order preserving. The stop operator can be defined first for any continuous, monotone input u as

$$w(t) = \mathcal{G}(u, w_0) = \begin{cases} \min\{h, u(t) - u_0 + w_0\} & \text{if } u(t) \text{ is nondecreasing} \\ \max\{-h, u(t) - u_0 + w_0\} & \text{if } u(t) \text{ is nonincreasing,} \end{cases} \tag{0.1.9}$$

where $h > 0$ and $|w_0| \leq h$. Then the output w can be defined for any piecewise monotone continuous input using the semigroup property (0.1.4) and then extended to any continuous input by continuity, see [**12**].
To see that the stop operator is not order preserving, consider

$$w_{10} = w_{20} = h, \qquad u_1(t) = 0, \qquad u_2(t) = \sin t, \qquad \text{for } 0 \leq t \leq \pi.$$

There also exist rate independent operators which are not piecewise monotone, e.g. the operator

$$\tilde{\mathcal{E}}(u, w_0) = -\mathcal{E}(u, w_0) = -[1 - \mathcal{G}(u, w_0)], \tag{0.1.10}$$

where $\mathcal{G}$ is the stop operator defined above and $\mathcal{E}$ is the play operator depicted in Figure 0.3.

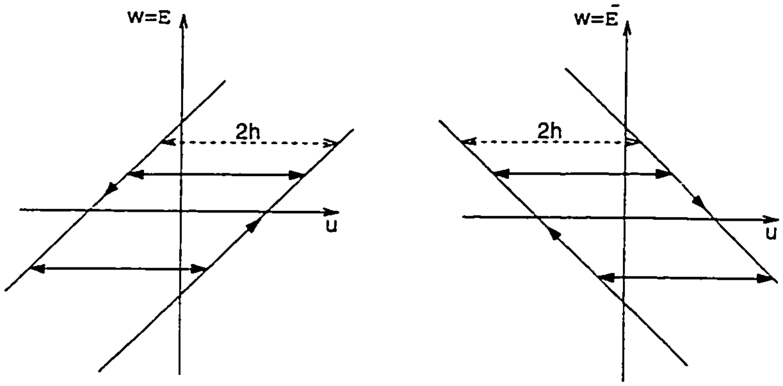

FIGURE 0.3. The play operator $\mathcal{E}$ and the operator $\tilde{\mathcal{E}} = -\mathcal{E}$.

Clearly $\tilde{\mathcal{E}}$ is a rate independent operator, which also satisfies the semigroup property and is not piecewise monotone, since $\mathcal{E}$ is.

The play can be thought of as a piston with plunger, of length $2h$ (Figure 0.4). The input is the plunger position $u(t)$, the output is the position of the center of the piston $w(t)$. Figure 0.3 shows the rules of motion.

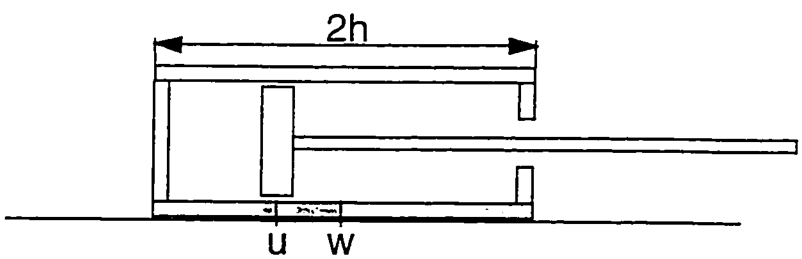

FIGURE 0.4. Play - a piston with plunger.

0.2. Definitions of hysteresis operators and their properties

0.2.1. Generalized play operator.

Let

$$\gamma_l, \gamma_r : \mathbb{R} \to \mathbb{R} \text{ be continuous nondecreasing functions with } \gamma_r \leq \gamma_l. \tag{0.2.1}$$

Now, given $w_0 \in \mathbb{R}$, we construct the hysteresis operator $\mathcal{E}(\cdot, w_0)$ as follows. Let u be any continuous, piecewise linear function on $\mathbb{R}^+$ such that u is linear on $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots$ We then define $w := \mathcal{E}(u, w_0) : \mathbb{R}^+ \to \mathbb{R}$ by

$$w(t) := \begin{cases} \min\{\gamma_l(u(0)), \max\{\gamma_r(u(0)), w_0\}\} & \text{if } t = 0, \\ \min\{\gamma_l(u(t)), \max\{\gamma_r(u(t)), w(t_{i-1})\}\} & \text{if } t \in (t_{i-1}, t_i], i = 1, 2, \ldots \end{cases} \tag{0.2.2}$$

Note that $w(0) = w_0$ only if $\gamma_r(u(0)) \le w_0 \le \gamma_l(u(0))$. If w_0 lies outside this interval, the actual initial value $w(0)$ of the operator is then obtained as a projection of w_0 onto the closest hysteresis boundary curve, see (0.2.2). In the rest of the thesis we do not distinguish between w_0 and $w(0)$, keeping the previous remarks always in mind.
As proved in Visintin [37], Section III.2, for any continuous piecewise linear functions u_1, u_2 on $\mathbb{R}^+$, with the notation $\epsilon_k := \mathcal{E}(u_k, w_{0k})$, $k = 1, 2$, we have the following inequality:

$$\max_{[t_1,t_2]} |\epsilon_1 - \epsilon_2| \le \max\left\{|\epsilon_1(t_1) - \epsilon_2(t_1)|, m_M\left(\max_{[t_1,t_2]} |u_1 - u_2|\right)\right\} \qquad (0.2.3)$$
$$\forall [t_1, t_2] \subset [0, T], T \in \mathbb{R}^+,$$

where for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and any constant $M > 0$, $|f|_M(h)$ denotes its local modulus of continuity:

$$|f|_M(h) := \sup\{|f(y_1) - f(y_2)| : y_1, y_2 \in [-M, M], |y_1 - y_2| \le h\} \qquad \forall h > 0, \qquad (0.2.4)$$

$$m_M(h) := \max\{|\gamma_l|_M(h), |\gamma_r|_M(h)\} \qquad \forall h, M > 0, \qquad (0.2.5)$$

and

$$M := \max\{|u_k(t)| : t \in [0, T], k = 1, 2\}. \qquad (0.2.6)$$

Hence $\mathcal{E}(\cdot, w_0)$ has a unique continuous extension, denoted by $\mathcal{E}(\cdot, w_0)$ again, to an operator

$$\mathcal{E} : C(\mathbb{R}^+) \times \mathbb{R} \to C(\mathbb{R}^+). \qquad (0.2.7)$$

This operator is called a generalized play, see Figure 0.5.

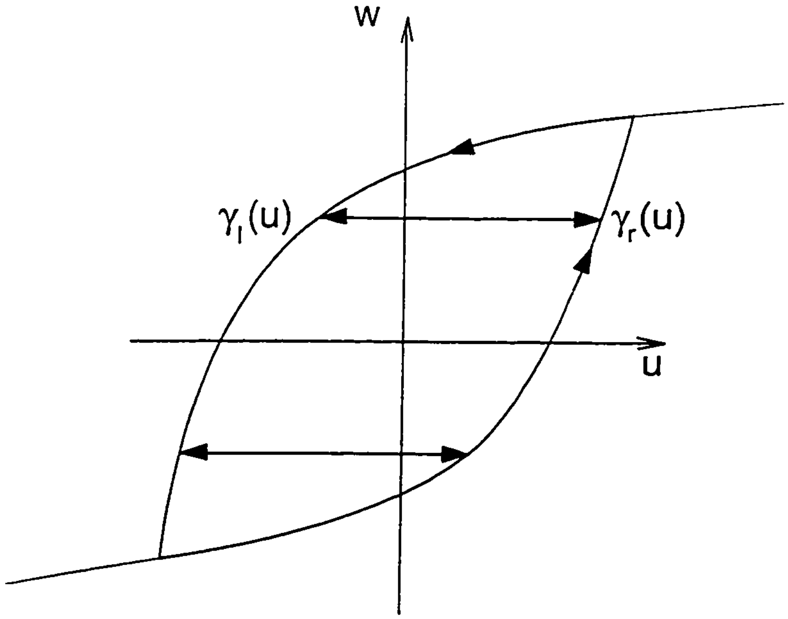

FIGURE 0.5. The generalized play.

The inequality (0.2.3) holds also for this extended operator, which is then uniformly continuous on bounded sets. If γ_l, γ_r are Lipschitz continuous, then $\mathcal{E}$ is also Lipschitz continuous.

THEOREM 0.1. *The generalized play operator defined above is rate independent, piecewise monotone and order preserving. If γ_l, γ_r are Lipschitz continuous, then for any $w_0 \in \mathbb{R}$, $\mathcal{E}(\cdot, w_0)$ operates and is bounded from $W^{1,p}(0,T)$ to $W^{1,p}(0,T)$ for any $p \in [1,\infty]$. It is also weakly continuous for any $p \in [1,\infty)$, weak star continuous for $p = \infty$.*

Moreover, denoting by L the larger of the Lipschitz constants of γ_l, γ_r, for any $(u, w_0) \in W^{1,1}(0,T) \times \mathbb{R}$,

$$\left|\frac{d}{dt}\mathcal{E}(u, w_0)\right| \leq L\left|\frac{du}{dt}\right| \qquad a.e. \ in \ (0,T). \tag{0.2.8}$$

Proofs of the above properties and many others can be found in Visintin [37], or in Brokate and Sprekels [3], where a different approach is taken, based on so called memory sequences, which we will not explain here in detail.

0.2.2. Generalized Prandtl-Ishlinskii operator of play type.

To define the generalized Prandtl-Ishlinskii operator of play type, let us assume that we are given a measure space $(\mathcal{P}, \mathcal{A}, \mu)$, where μ is a finite Borel measure. For μ-almost any $\rho \in \mathcal{P}$, let $(\gamma_{\rho l}, \gamma_{\rho r})$ be a pair of functions $\mathbb{R} \to \mathbb{R}$, satisfying (0.2.1), and for each $\rho \in \mathcal{P}$ let $w_{\rho 0} \in \mathbb{R}$, be a given initial value. Let $\mathcal{E}_\rho(\cdot, w_{\rho 0})$ be the generalized play operator corresponding to the couple $(\gamma_{\rho l}, \gamma_{\rho r})$. Then the operator defined as

$$\bar{\mathcal{E}}_\mu\left(\bar{u}, \{w_{\rho 0}\}_{\rho\in\mathcal{P}}\right) = \int_{\mathcal{P}} \mathcal{E}_\rho\left(\bar{u}, w_{\rho 0}\right) d\mu(\rho)$$

is a generalized Prandtl-Ishlinskii operator of play type. Intuitively, this operator is a weighted superposition of generalized plays with boundary curves $\gamma_{\rho l}$, $\gamma_{\rho r}$.

Let us denote by $M(\mathcal{P})$ the set of measurable functions $\mathcal{P} \to \mathbb{R}$. If the family $\{\gamma_{\rho l}, \gamma_{\rho r}; \rho \in \mathcal{P}\}$ is equicontinuous, then by the estimate (0.2.3), $\bar{\mathcal{E}}_\mu$ is also strongly continuous from $C(\mathbb{R}^+) \times M(\mathcal{P})$ to $C(\mathbb{R}^+)$. If μ is nonnegative, then $\bar{\mathcal{E}}_\mu$ is piecewise monotone and order preserving and satisfies a theorem similar to Theorem 0.1.

0.2.3. Delayed relay operator.

For any couple $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ with $\rho_1 < \rho_2$, we introduce the delayed relay operator

$$h_\rho : C^0([0,T]) \times \{-1,1\} \to BV(0,T) \cup C^0_r([0,T)), \tag{0.2.9}$$

where $C^0_r([0,T))$ denotes the space of functions right-continuous in $[0,T)$. For any $u \in C^0([0,T])$ and any $\xi = -1$ or 1, $h_\rho(u,\xi) = w : [0,T] \to \{-1,1\}$ is defined as follows:

$$w(0) = \begin{cases} -1 & \text{if } u(0) \leq \rho_1 \\ \xi & \text{if } \rho_1 < u(0) < \rho_2 \\ 1 & \text{if } u(0) \geq \rho_2, \end{cases} \tag{0.2.10}$$

for any $t \in (0,T]$, setting $X_t = \{\tau \in (0,T], \ u(\tau) = \rho_1 \text{ or } \rho_2\}$

$$w(t) = \begin{cases} w(0) & \text{if } X_t = \emptyset \\ -1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = \rho_1 \\ 1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = \rho_2, \end{cases} \tag{0.2.11}$$

see Figure 0.6.

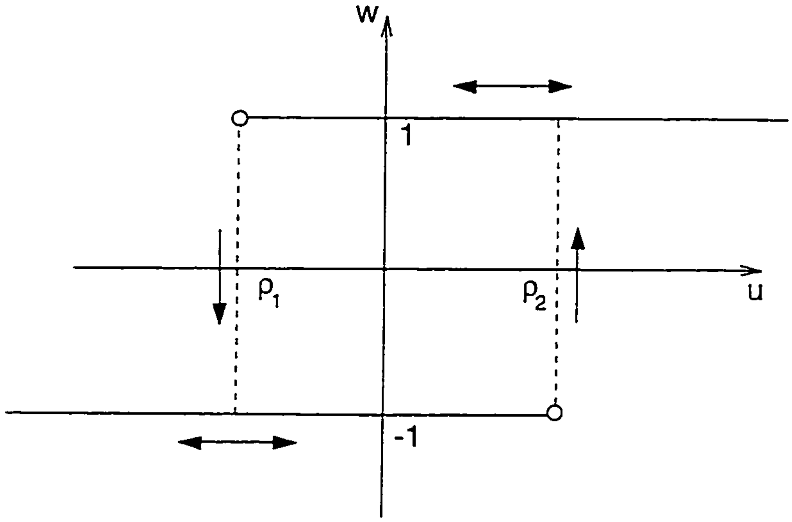

FIGURE 0.6. The relay operator.

The delayed relay operator is a rate independent, piecewise monotone, order preserving and discontinuous hysteresis operator (in any sense).

Especially for applications to differential equations, it is convenient to extend the delayed relay operator to a maximal monotone graph, that is to fill the jumps in the corresponding graph with vertical segments.
So we introduce a multivalued operator, which we denote by k_ρ. For any $u \in C_0([0,T])$ and any $\xi \in [-1,1]$, $w \in k_\rho(u,\xi)$ if and only if w is measurable in $(0,T)$,

$$\begin{cases} \text{if } u(t) \neq \rho_1, \rho_2, & \text{then } w \text{ is constant in a neighbourhood of } t \\ \text{if } u(t) = \rho_1, & \text{then } w \text{ is nonincreasing in a neighbourhood of } t \\ \text{if } u(t) = \rho_2, & \text{then } w \text{ is nondecreasing in a neighbourhood of } t, \end{cases} \tag{0.2.12}$$

$$w(0) \in \begin{cases} \{-1\} & \text{if } u(0) < \rho_1 \\ [-1,\xi] & \text{if } u(0) = \rho_1 \\ \{\xi\} & \text{if } \rho_1 < u(0) < \rho_2 \\ [\xi,1] & \text{if } u(0) = \rho_2 \\ \{1\} & \text{if } u(0) > \rho_2, \end{cases} \tag{0.2.13}$$

$$w(t) \in \begin{cases} \{-1\} & \text{if } u(t) < \rho_1 \\ [-1,1] & \text{if } \rho_1 \leq u(t) \leq \rho_2 \\ \{1\} & \text{if } u(t) > \rho_2. \end{cases} \tag{0.2.14}$$

The behaviour of k_ρ is outlined in Figure 0.7. Note that the graph of k_ρ in the (u, w)-plane includes the whole rectangle $[\rho_1, \rho_2] \times [-1, 1]$. This operator is called a completed delayed relay operator.

$k_\rho : C^0[0,T]) \times [-1,1] \to \mathcal{P}(BV(0,T))$ (here $\mathcal{P}$ denotes the power set) .

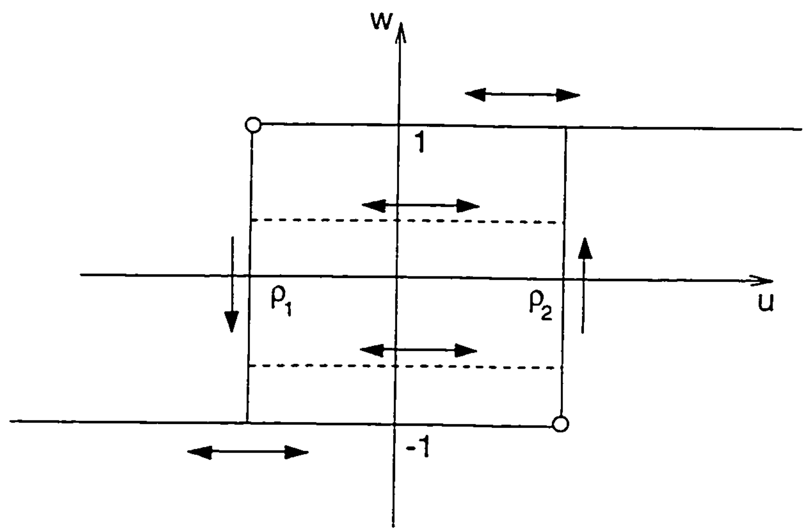

FIGURE 0.7. The completed delayed relay operator.

0.2.4. Preisach operator.

The set of possible thresholds of delayed relay operators forms the so called Preisach (half) plane

$$\mathcal{P} = \{\rho = (\rho_1, \rho_2) \in \mathbb{R}^2, \rho_1 < \rho_2\}. \qquad (0.2.15)$$

We denote by $\mathcal{R}$ the family of Borel measurable functions from $\mathcal{P}$ into $\{-1, 1\}$ and by $\{\xi_\rho\}$ a generic element of $\mathcal{R}$. We fix any finite (signed) Borel measure μ over P, and introduce the corresponding Preisach operator

$$\mathcal{H}_\mu : C^0([0,T]) \times \mathcal{R} \to L^\infty(0,T) \cap C^0_r([0,T])$$

$$[\mathcal{H}_\mu(v,\xi)](t) = \int_P [h_\rho(v,\xi_\rho)](t) \mathrm{d}\mu(\rho) \qquad \forall t \in [0,T].$$

This model has very nice geometric properties.

THEOREM 0.2. *For any finite Borel measure μ over $\mathcal{P}$, the operator $\mathcal{H}_\mu$ is causal and rate independent, so it is a (possibly discontinuous) hysteresis operator. If $\mu \geq 0$, then $\mathcal{H}_\mu$ is piecewise monotone and order preserving. If μ is a finite Borel measure over $\mathcal{P}$ and if*

$$\exists \delta > 0 : \mu(\{(\rho_1, \rho_2) \in \mathcal{P} : \rho_1 - \rho_2 \leq \delta\}) = 0,$$

then $\mathcal{H}_\mu : C^0([0,T]) \times \mathcal{R} \to BV(0,T)$.

$$\text{If} \qquad \xi^v_\rho = \begin{cases} 1 & \text{if } \rho_1 + \rho_2 < 0, \\ -1 & \text{if } \rho_1 + \rho_2 > 0, \end{cases}$$

and

$$\widetilde{S} := \{z : \rho \to [h_\rho(u, \xi_\rho^v)](T) : u \in C^0([0,T])\},$$

then for $\xi \in \widetilde{S}$ *we have*

$$\mathcal{H}_\mu(.,\xi) : C^0([0,T]) \to C^0([0,T])$$

and $\mathcal{H}_\mu$ *is strongly continuous if and only if*

$$\mu(\mathbb{R} \times \{r\}) = \mu(\{r\} \times \mathbb{R}) = 0, \qquad \forall r \in \mathbb{R}.$$

Moreover $\mathcal{H}_\mu$ *is Lipschitz continuous with Lipschitz constant equal to L if and only if*

$$\sup_{B \in \mathcal{B}} \mu(\mathcal{N}(B,\epsilon)) \le L\epsilon \qquad \forall \epsilon > 0,$$

where for $\sigma_1 = \frac{\rho_1 - \rho_2}{\sqrt{2}}$, $\sigma_2 = \frac{\rho_1 + \rho_2}{\sqrt{2}}$ *we define*

$$\mathcal{N}(B,\epsilon) := \{(\sigma_1, \sigma_2 + \alpha) \in \mathbb{R}^+ \times \mathbb{R} : (\sigma_1, \sigma_2) \in B, |\alpha| \le \epsilon\}$$

and $\mathcal{B}$:= *set of maximal antimonotone graphs in* $\mathcal{P}$.
We introduce the notation:

$$R_i(\lambda_1, \lambda_2) := \{\rho \in \mathcal{P} : \lambda_1 \le \rho_i \le \lambda_2\}, \qquad i = 1, 2.$$

$$k(\zeta) := 2\sup\{\mu(R_i(\lambda_1, \lambda_2)) : 0 \le \lambda_2 - \lambda_1 \le \zeta, i = 1, 2\} \qquad \forall \zeta \in \mathbb{R}.$$

If $\quad k(\zeta) \le C\zeta \quad \forall \zeta \ge 0 \quad$ *for some constant* $C > 0$,

then, for any $u \in W^{1,1}(0,T)$,

$$\left|\frac{d}{dt}[\mathcal{H}_\mu(u)]\right| \le C\left|\frac{du}{dt}\right| \qquad a.e. \text{ in } (0,T).$$

Hence $\mathcal{H}_\mu$ *maps* $W^{1,p}(0,T)$ *into itself for any* $p \in [1,\infty]$, *and is affinely bounded. It is also weakly continuous for any* $p \in [1,\infty)$, *weak star continuous for* $p = \infty$.

REMARK 0.1. Under some hypothesis $\mathcal{H}_\mu$ is actually strongly continuous for $1 \le p < \infty$ and Lipschitz continuous on bounded subsets of $W^{1,1}(0,T)$.

For the proof of Theorem 0.2, see [37], Chapter IV; for the proof of the above remark see [4]; p.60-61.

There are also special relations between various types of hysteresis operators. For example it is clear that the generalized play operator is a special case of the generalized Prandtl-Ishlinskii operator of play type. The following theorem shows a relation between the generalized play, the generalized Prandtl-Ishlinskii operator of play type and the Preisach operator.

THEOREM 0.3. *Let* μ *be a nonnegative Borel measure over* $\mathcal{P}$. *Assume that the support of* μ *is confined to the graph* Γ *of a curve with equation* $\rho_2 = p(\rho_1)$, *where*

$$p : \mathbb{R} \to \mathbb{R} \text{ is continuous and strictly increasing}, \quad p(\rho_1) > \rho_1 \quad \forall \rho_1 \in \mathbb{R}. \qquad (0.2.16)$$

Denote by μ_i *the projection of* μ *onto the* ρ_i*-axis, that is,*

$$\mu_i((a,b)) := \mu(\{(\rho_1, \rho_2 = p(\rho_1)) : a < \rho_i < b\}) \qquad \forall (a,b) \subset \mathbb{R}, i = 1, 2;$$

define $\widetilde{v}(< 0)$ *by the condition* $p(\widetilde{v}) + \widetilde{v} = 0$, *and set*

$$\gamma_l(u) := \mu_1((\widetilde{v}, u)), \qquad \gamma_r := \mu_2((-\widetilde{v}, u)) \qquad \forall u \in \mathbb{R}, \qquad (0.2.17)$$

with the convention that $\mu_i((\beta,\alpha)) := -\mu_i((\alpha,\beta))$ *if* $\alpha < \beta$.

Then γ_l *and* γ_r *fulfill (0.2.1), and the corresponding generalized play operator* $\mathcal{E}(\cdot,0)$ *is equivalent to the Preisach operator* $\mathcal{H}_\mu(\cdot,\xi^v)$. *Moreover,*

$$\gamma_l^{-1}(0) = -\gamma_r^{-1}(0) = \tilde{v}. \tag{0.2.18}$$

Conversely, let the curves γ_l, γ_r *be absolutely continuous, and fulfil conditions (0.2.1) and (0.2.18) (here the latter constitutes the definition of* $\tilde{v}$*). Then the corresponding generalized play is equivalent to a Preisach operator, whose measure* μ *is supported by the graph* Γ *of a curve with equation* $\rho_2 = p(\rho_1)$.

A similar theorem holds also for a generalized Prandtl-Ishlinskii operator of play type:

THEOREM 0.4. *Let* (I,ν) *be a finite measure space. For any* $\tau \in I$, *let* μ_τ *be a finite nonnegative Borel measure over* $\mathcal{P}$, *having support confined to a curve* Γ_τ *of equation* $\rho_2 = p_\tau(\rho_1)$ *fulfillig (0.2.16). Assume that for any Borel set* $A \subset \mathbb{R}^2$ *the function* $\tau \to \mu_\tau(A)$ *is* ν*-measurable in* I, *and set* $\mu(A) := \int_I \mu_\tau(A)d\nu(\tau)$.

Then for any $\tau \in I$, $\mathcal{H}_{\mu_\tau}(\cdot,\xi^v)$ *is equivalent to the generalized play operator* $\mathcal{E}_\tau(\cdot,0)$ *corresponding to* γ_l^τ *and* γ_r^τ *defined in (0.2.17). Hence the averaged operator* $\mathcal{H}_\mu(\cdot,\xi^v) := \int_I \mathcal{H}_{\mu_\tau}(\cdot,\xi^v)d\nu(\tau)$ *is equivalent to the generalized Prandtl-Ishlinskii operator* $\tilde{\mathcal{E}}_\nu(\cdot,0) := \int_I \mathcal{E}_\tau(\cdot,0)d\nu(\tau)$.

The proofs of theorems 0.3 and 0.4 can be found in **[37]**.

The mathematical studies of hysteresis are mainly confined to two main areas. The first one studies different models of hysteresis and their properties. Since 1970, when Krasnosel'skii and co-workers proposed a mathematical formulation of the Prandtl-Ishlinskii model in terms of hysteresis operators, Krasnosel'skii, Pokrovskii and others conducted a systematic analysis of the mathematical properties of these operators. In the period 1970-1980 they published a number of papers, which formed the basis for the 1983 monograph **[12]** of Krasnosel'skii and Pokrovskii (translated into English in 1989).

Most of the results about play and stop operators are due to Krasnosel'skii and Pokrovskii, however properties of those operators were studied also by Krejčí **[19]**, **[20]**, Brokate **[3]**, Krejčí and Lovicar **[22]** and by Visintin **[35]**.

Mathematical aspects of the Preisach model were dealt with by Krasnosel'skii and Pokrovskii. Properties of the Preisach operator were analyzed by Brokate and Visintin **[5]**, using an approach different from that of Krejčí **[17]**.

Agreement between calculations based on the Preisach model and experimental results on ferromagnetic materials is often poor. Nevertheless, this model allows a qualitative understanding of many magnetic processes, and is widely used. Several physically justified modifications have also been proposed, to get a better quantitative agreement with measurements. For instance, in **[6]** Della Torre proposed replacing the relation $M = \mathcal{H}(H)$ by $M = \mathcal{H}(H + \alpha M)$, where α is a positive constant - this is known as the moving Preisach model. Here M is the magnetization of the material and H is the applied magnetic field. See also **[7]**, **[30]**, **[31]**, for example. Several generalizations of the Preisach model are discussed in detail in the recent monograph of Mayergoyz **[28]**.

There are other hysteresis models which have been studied, e.g. the Duhem model.

So far only a few phenomena exhibiting hysteresis have been given a mathematical formulation. Many open problems remain. For instance, we do not yet have any satisfactory vectorial model of ferromagnetic hysteresis.

The other area of research is more closely related to applications and consists of studying differential equations involving hysteresis operators of different types.

Ordinary differential equations coupled with hysteresis nonlinearities were studied e.g. by V.I.Borzdyko, [2], A.A.Vladimirov, M.A.Krasnosel'skii and V.V.Chernorutskii, [38]. This coupling leads to interesting mathematical problems in the theory of nonlinear oscillations, see e.g. [11], [26], [27].

Hysteretic constitutive laws in continuum mechanics formulated in terms of hysteresis operators lead in a natural way to partial differential equations coupled with hysteresis operators, where the former represents the balance laws for mass, momentum and internal energy.
These equations can be investigated by means of few fundamental methods:

(i) Formulation of the problem as a system of variational inequalities;

(ii) Formulation as a differential equation containing an accretive operator and then application of the theory of nonlinear semigroups of contractions. This approach was used mainly by Visintin [36] and [37] and more recently also by Little [23] and Little and Showalter [24]. See also [25] and [29].

(iii) Formulation as a fixed point problem, and use of an appropriate fixed point theorem;

(iv) Approximation, a priori estimates, and passage to the limit by means of compactness techniques.

As long as the hysteresis operator has suitable continuity properties and does not appear in the principal part of the differential equation, the existence theory is not too difficult; in this case usually enough compactness can be extracted from the equation.

Once the hysteresis operator occurs in the principal part of the differential equation, the mathematical analysis becomes considerably more difficult. Then the necessary compactness has to be recovered from special structural properties of the hysteresis operator. A.Visintin and P.Krejčí have discovered that the analytical property of strict monotonicity and the geometrical property of strict convexity of the hysteresis loops have such a smoothing effect.

P.Krejčí obtained several important results for quasilinear and semilinear hyperbolic equations, see e.g. [14], [16], [21]:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = g \qquad \text{in } (a,b)\times(0,T); \tag{0.2.19}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \mathcal{F}(u) = g \qquad \text{in } (a,b)\times(0,T); \tag{0.2.20}$$

where $\mathcal{F}$ denotes a Prandtl-Ishlinskii operator of stop type. He studied asymptotic behaviour of solutions of those equations and proved existence and uniquenes of the periodic solution. His approach is essentially based on energy estimates. He observed that the strict convexity of the individual hysteresis loops implies the continuity of the solutions and thus prevents the formation of shock waves, in addition the theory of hysteresis potentials can be used to show that the speed of propagation is finite. For a detailed description of his work, see the recent monograph [21].

The mathematical theory of parabolic equations with hysteresis is closely connected with the name of A.Visintin, who investigated them from several different points of view. For the quasilinear PDE with hysteresis

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(x,t) \qquad \text{in } Q \qquad (0.2.21)$$

$$w(x,t) = [\mathcal{F}(u(\cdot,x); w_0(x))](t) \qquad (x,t) \in Q \qquad (0.2.22)$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \qquad (0.2.23)$$

$$u(0,\cdot) = u_0(x) \qquad \text{in } \Omega \qquad (0.2.24)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, Ω bounded, $Q = \Omega \times (0,\infty)$. A.Visintin obtained existence and regularity results using the piecewise monotonicity of $\mathcal{F}$ as his main tool. To obtain uniqueness results, a fundamental inequality due to M.Hilpert, see [8], was employed, using L^1 - techniques. A problem of this sort arises as a simplified model of scalar ferromagnetism, for instance. The differential equation is obtained from Maxwell's equations by neglecting the displacement current term, assuming linear relations between the electric field, the electric displacement and the electric current density.

For the semilinear PDE with hysteresis

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = f, \qquad \text{in } Q, \qquad (0.2.25)$$

coupled with initial and boundary conditions, similar results hold without the piecewise monotonicity assumption. Assuming Lipschitz continuity of $\mathcal{F}$, a uniqueness result can be obtained. On the other hand, if $\mathcal{F} = \mathcal{H}_\mu$, the solution of the problem may not be unique, see [37], Section XI.5 and also [1]. This equation can represent e.g. a distribution of thermostats.

Numerical treatment of equations containing hysteresis nonlinearities can be found in [32], [33], [34].

Of special interest, as far as applications are concerned, is the analysis of the asymptotic behaviour and periodicity of solutions in time. To our knowledge there are so far only two papers dealing with this problem, [10], [9] . In [10] they consider the equation (0.2.21) with $\mathcal{F}$ a generalized play and their proof of asymptotic stability as well as of the existence of periodic solutions relies on the special properties of this operator. [9] deals with the question of asymptotic behaviour of solutions of (0.2.25) also only in the special case when $\mathcal{F}$ is assumed to be a generalized play.

This thesis is organized as follows. In "Uniqueness theorem for a Cauchy problem with hysteresis", the Cauchy problem for an ordinary differential equation coupled with a hysteresis operator is studied. Under physically reasonable assumptions on the forcing term, uniqueness of solutions is shown without assuming Lipschitz continuity of hysteresis curves. The result is true for any kind of hysteresis operators with monotone curves of motion.

In the second chapter "Semigroup approach to the question of stability for a partial differential equation with hysteresis" the parabolic equation (0.2.21) is studied and the asymptotic behaviour of the solution as $t \to \infty$ is investigated. A semigroup approach is used to show stability in $L^1(\Omega)$. Although this approach gives us slightly weaker results than those in [10] in the case when the hysteresis operator is a generalized play, it enables us to get stability results also for the Prandtl-Ishlinskii operator of play type and for some discontinuous hysteresis operators.

In chapter three we consider the question of existence of periodic solutions and asymptotic behaviour for a parabolic PDE with hysteresis in the source term, (0.2.25). We prove existence of periodic solutions of (0.2.25) with a general hysteresis operator and give two different proofs of this result. We also prove an asymptotic result for solutions of (0.2.25), using ideas due to P.Krejčí (see [**37**], p. 287).

In the fourth chapter we give a detailed introduction to the topic of hyperbolic equations of first order of the form

$$u_t + [\phi(u)]_x = 0, \qquad u(0) = u_0. \tag{0.2.26}$$

We show how (0.2.26) can have a multiple solutions even if the data ϕ and u_0 are smooth. We define a generalized solution of (0.2.26) and discuss various entropy conditions, under which the generalized solution of (0.2.26) is unique. Then we consider a hyperbolic equation with hysteresis

$$u_t + w_t + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu = f, \tag{0.2.27}$$

which was studied by Visintin, [**37**], using a semigroup approach. We sketch his results and derive an entropy condition for (0.2.27), which solves an open problem stated in Visintin's book, [**37**].

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CHAPTER 1

Uniqueness theorem for a Cauchy problem with hysteresis

1.1. Introduction

In this chapter we consider the question of uniqueness of solutions for the Cauchy problem for an ordinary differential equation coupled with a hysteresis operator without assuming that the operator is Lipschitz continuous. The equation we consider is

$$\frac{du}{dt} + \mathcal{F}(u) = f \qquad \text{in } (0,T) \tag{1.1.1}$$

$$u(0) = u_0. \tag{1.1.2}$$

The existence of a solution of (1.1.1)-(1.1.2) for e.g. f continuous is well known, see [5]. As pointed out by Visintin (see [4], p. 324), uniqueness was an open problem for $\mathcal{F}$ not Lipschitz continuous, while it was known that Lipschitz continuity guarantees uniqueness. Using simple techniques from the theory of differential equations, we are able to prove uniqueness even in the non-Lipschitz case. This is done first for $f = 0$, then extended to the more general case. We show that under physically reasonable assumptions on f we do have uniqueness. As was shown recently by V.Chernorutskii and D.Rachinskii in [1], our assumptions are necessary. In [1] they constructed a specific continuous right hand side, oscillatory in every neighborhood of 0, for which there is nonuniqueness.

1.2. Preliminaries

In this section we state a uniqueness theorem for the Cauchy problem without the hysteresis operator, which will be useful in the sequel. The result is classical, a detailed proof is given in [2], Section III.6.

THEOREM 1.1. *Let $U(t,u)$ be a continuous real-valued function for $t_0 \leq t \leq t_0 + a$, $|u - u_0| \leq b$, which is nonincreasing with respect to u (for fixed t). Then the initial value problem*

$$\frac{du}{dt} = U(t,u) \tag{1.2.1}$$

$$u(t_0) = u_0 \tag{1.2.2}$$

has at most one solution on any interval $[t_0, t_0 + \epsilon]$, $\epsilon > 0$.

1.3. Main result

Using the theorem from the previous section, we prove a uniqueness theorem for (1.1.1)-(1.1.2), when $\mathcal{F}$ is a generalized play operator.

THEOREM 1.2. *Suppose that $\mathcal{F}(u, w_0)$ is a generalized play operator with hysteresis boundary curves γ_l and γ_r and $f(t) \equiv 0$. Then the solution of the Cauchy problem (1.1.1)-(1.1.2) is unique.*

PROOF. From (1.1.1) we have

$$\frac{du}{dt} = -\mathcal{F}(u). \tag{1.3.1}$$

Suppose that initially $w(0) = 0$. Then

$$\frac{du}{dt}(0) = 0$$

and therefore all the points

$$\mathcal{S} = \{(u, w); w = 0, a \leq u \leq b, \text{ where } \gamma_l(a) = 0 \text{ and } \gamma_r(b) = 0\}$$

are equilibria. The solution of (1.1.1)-(1.1.2) with $w(0) = 0$ is unique, $u(t) \equiv 0$. This can be proved as follows: Assume that for some $t_1 > 0$ we have $w(t_1) > 0$ and put $t_0 := \max\{t \in [0, t_1], w(t) = 0\}$. Then in $[t_0, t_1]$ the function u is decreasing, hence w is nonicreasing, which is a contradiction. Similarly for $w(t_1) < 0$.

Suppose now $w(0) > 0$. Then at $(u_0, w(0))$

$$\frac{du}{dt}(0) = -w(0) < 0, \tag{1.3.2}$$

thus u is decreasing on a right neighborhood of $t = 0$. We have three possibilities. Either $(u_0, w(0))$ lies inside the hysteresis region or on γ_r or on γ_l. In the first two cases $\mathcal{F}$ stays constant on some interval $[0, t_1)$; from (1.3.1) we have $u = -w(0)t + c_0$ and since $u(0) = u_0$, $u = -w(0)t + u_0$ for $t \in [0, t_1]$, and u is decreasing until (u, w) hits the hysteresis boundary γ_l. The second possibility is that $(u_0, w(0))$ lies on γ_l. Here we again have from (1.3.1) that u is decreasing and (u, w) moves on the curve γ_l. Therefore u must satisfy the equation

$$\frac{du}{dt} + \gamma_l(u) = 0$$
$$u(0) = u_0$$

which by Theorem 1.1 has a unique solution and is approaching the equilibrium $(a, 0)$ as $t \to \infty$. The case $w(0) < 0$ can be handled analogously, using the uniqueness of solutions to the problem:

$$\frac{du}{dt} + \gamma_r(u) = 0$$
$$u(0) = u_0.$$

□

REMARK 1.1. From the above analysis we are actually able to prove more: Except in the trivial case when $w(0) = 0$, solutions of (1.1.1)-(1.1.2) are stable and they converge to points $(a, 0)$ or $(b, 0)$ as $t \to \infty$, depending on the initial value $w(0)$.

REMARK 1.2. Using the same methods as above we can prove uniqueness for (1.1.1)-(1.1.2) for $\mathcal{F}$ a generalized Prandtl-Ishlinskii operator of play type or any other hysteresis operator with monotone curves of motion. In this case all the points $(u_0, 0)$, $a \leq u_0 \leq b$, where $\gamma_l(a) = \gamma_r(b) = 0$, and γ_l, γ_r are here the hysteresis boundary curves; are equilibria.

THEOREM 1.3. *Suppose there exists an $\epsilon > 0$ such that $f(t) > w(0)$ or $f(t) < w(0)$ on $[0, \epsilon)$. Then there exists at most one solution of (1.1.1)-(1.1.2) on $[0, \epsilon)$.*

PROOF. By contradiction. Let there exist two different solutions on $[0, \epsilon)$. We can assume without loss of generality that $w(0) = 0, u_0 = 0$ and that $\gamma_r(u)$ is such that $\gamma_r(0) = 0$ and that the point $(0, 0)$ is a point where the curve $\gamma_r(u)$ is non-Lipschitz. For $0 \leq t < \epsilon$, the motions $(u_i(t), w_i(t))$, $i = 1, 2$ must lie respectively on one of the curves $w = \gamma_r(u)$ or $w = 0$, see Figure 1.1. This means that each $u_i(t)$ must satisfy either

$$\frac{du}{dt} = -\gamma_r(u) + f(t) \tag{1.3.3}$$

or

$$\frac{du}{dt} = -w(0) + f(t) = f(t) \tag{1.3.4}$$

for $0 \leq t < \epsilon$. But each of these has unique solution by Theorem 1.1, so we can assume u_1 solves (1.3.3) and u_2 solves (1.3.4).

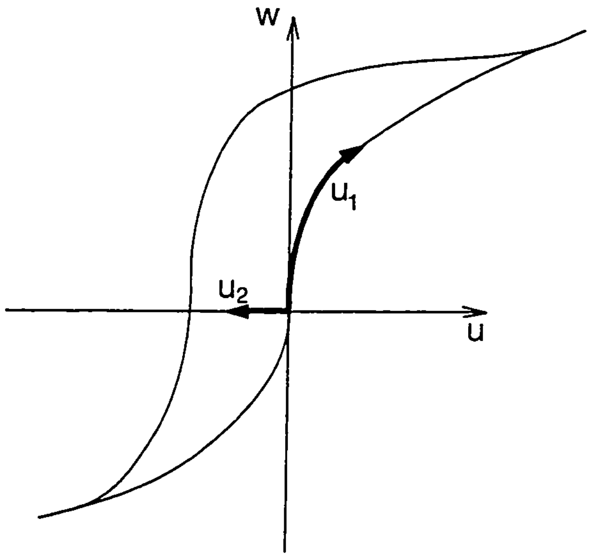

FIGURE 1.1. The two solutions in the proof of Theorem 1.3.

Look carefully at what this means for $0 \leq t < \epsilon$:

for u_1 : $\quad \dfrac{du_1}{dt} = -\gamma_r(u_1) + f(t) \geq 0 \quad$ i.e. $f(t) \geq \gamma_r(u_1) > 0$

for u_2 : $\quad \dfrac{du_2}{dt} = f(t) \leq 0 \quad$ i.e. $f(t) \leq 0$,

a contradiction. □

REMARK 1.3. As the above proof suggests, we can actually assume less than $f(t)$ has constant sign on $[0, \epsilon)$. It suffices to have assumptions which will guarantee that the solution pair $(u(t), w(t))$ will move either on $\gamma_r(u)$ or inside the hysteresis region, on the line $w = 0$ (see Figure 1.1). A little thought shows that we need only exclude the only other alternative to the motions depicted in Figure 1.1: in every interval $(0, \epsilon)$ the motion leaves $\gamma_r(u)$ and returns to it via a horizontal segment, see Figure 1.2.

In this case there would exist a sequence $t_n \to 0$, with
$Q_n = (u(t_{2n}), w(t_{2n}))$ a minimum of $u(t)$ on γ_r and $P_n = (u(t_{2n+1}), w(t_{2n+1}))$ a maximum of $u(t)$ on the horizontal segment, $t_{2n+1} < t_{2n}$. Then we must have

$$f(t_{2n+1}) = \gamma_r(u(t_{2n})) = f(t_{2n}). \tag{1.3.5}$$

This is the basically the case from the paper of V. Chernorutskii and D.Rachinskii, [1], when nonuniqueness occurs. This can be excluded, for example, by assuming that f' exists on $[0, \epsilon)$, is continuous at $t = 0$ and $f'(0) \neq 0$.

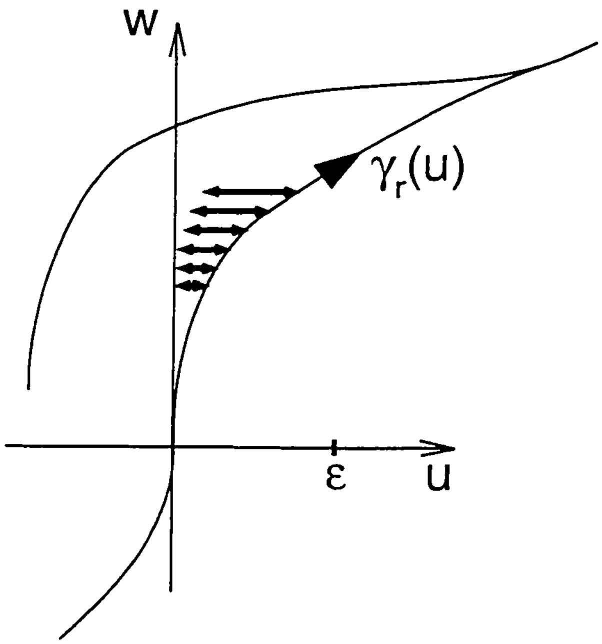

FIGURE 1.2. The case when nonuniqueness can occur.

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CHAPTER 2

Semigroup approach to the question of stability for a partial differential equation with hysteresis

2.1. Introduction

This chapter considers the following evolution problem on $Q = \Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, Ω bounded :

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(x,t) \qquad \text{in } Q \tag{2.1.1}$$

$$w(x,t) = [\mathcal{F}(u(\cdot,x); w_0(x))](t) \qquad (x,t) \in Q \tag{2.1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{2.1.3}$$

$$u(0,\cdot) = u_0(x) \qquad \text{in } \Omega. \tag{2.1.4}$$

Here w represents a hysteresis operator, which we consider to be either a possibly discontinuous generalized play or a possibly discontinuous generalized Prandtl-Ishlinskii operator of play type, which includes some cases of possibly discontinuous Preisach operators.

Existence and uniqueness for (2.1.1)-(2.1.4) have already been proven under reasonable hypotheses, see e.g. Visintin's book **[11]**.

Of special interest, as far as applications are concerned, is the analysis of the asymptotic behaviour of solutions in time. To our knowledge there is so far only one paper dealing with this problem, **[6]**. They consider the case where $\mathcal{F}$ is a generalized play and their proof of asymptotic stability relies on the special properties of this operator. In the present chapter, we are able, using a semigroup approach combined with ideas from **[10]**, to prove asymptotic stability of the solutions of (2.1.1)-(2.1.4), when $\mathcal{F}$ is a generalized play or generalized Prandtl-Ishlinskii operator of play type, which is of more interest for applications. Our approach enables us to deal with the continuous and the discontinuous case at the same time. As far as we know, so far there does not seem to be any result about the asymptotic behaviour of our system for the discontinuous case.

In Section 2.2 we recall a theorem of Wittbold **[10]**. Section 2.3 contains the main result of the paper together with its proof. The main result is first formulated for an equation with zero right-hand side, then extended to the general case.

2.2. Preliminary results

In this section we state the theorem from **[10]** which inspired this chapter. For completness we will also include the proof because of some misprints in the original.

THEOREM 2.1. *(Wittbold [10]) Let A be an m- and T-accretive operator in $L^1(\Omega)$, i.e.*

$$R(I + \lambda A) = L^1(\Omega) \qquad \forall \lambda \geq 0 \tag{2.2.1}$$

and

$$\|(J_\lambda^A u - J_\lambda^A \tilde{u})^+\|_1 \leq \|(u - \tilde{u})^+\|_1 \qquad \forall \lambda \geq 0, \quad u, \tilde{u} \in L^1(\Omega), \tag{2.2.2}$$

where

$$J_\lambda^A u = (I + \lambda A)^{-1} u \qquad \text{and } R \text{ denotes the range.} \tag{2.2.3}$$

Suppose that $A^{-1}0 = \{0\}$ and let $u_0 \in \overline{D(A)}$. Then the following holds:
If there exist a stationary subsolution and a stationary supersolution of

$$u_t + Au \ni 0, \qquad u(0) = u_0 \tag{2.2.4}$$

in $L^1(\Omega)$, then the solution u of (2.2.4) is stable, i.e. $\|u(x,t)\|_1 \to 0$ as $t \to \infty$.

DEFINITION 2.1. A stationary supersolution of (2.2.4) is defined to be a function $v \in L^1(\Omega)$ satisfying

$$u_0 \leq v \qquad \text{a.e. on } \Omega \tag{2.2.5}$$

and

$$(I + \lambda A)^{-1} v \leq v \qquad \text{a.e. on } \Omega, \forall \lambda > 0. \tag{2.2.6}$$

A stationary subsolution of (2.2.4) is defined in the same way with reversed inequalities.

REMARK 2.1. Note that if $v \in D(A)$ and A is single valued, then (2.2.5)-(2.2.6) is equivalent to

$$v \geq u_0 \qquad \text{a.e. on } \Omega, \tag{2.2.7}$$

$$Av \geq 0 \qquad \text{a.e. on } \Omega. \tag{2.2.8}$$

Also note that if A is an accretive operator, (2.2.7)-(2.2.8) imply (2.2.5)-(2.2.6).

PROOF. Without loss of generality we may assume that
$u_0 \geq 0$. Let v be a supersolution of (2.2.4) corresponding to $u_0 \in \overline{D(A)}$.
We will first show that we can also always assume that $v \in \overline{D(A)}$.
The resolvent identity

$$(\tilde{\lambda} I + A)^{-1} - (\tilde{\mu} I + A)^{-1} = (\tilde{\mu} - \tilde{\lambda})(\tilde{\mu} I + A)^{-1}(\tilde{\lambda} I + A)^{-1}$$

gives us

$$\lambda(I + \lambda A)^{-1} - \mu(I + \mu A)^{-1} = \left(\frac{\lambda - \mu}{\lambda\mu}\right) \lambda\mu(I + \lambda A)^{-1}(I + \mu A)^{-1},$$

where we used the notation $\lambda = 1/\tilde{\lambda}, \mu = 1/\tilde{\mu}$,

thus

$$J_\lambda^A v = J_\mu^A \left(\frac{\mu}{\lambda} v + \frac{\lambda - \mu}{\lambda} J_\lambda^A v\right) \qquad \forall \lambda, \mu > 0. \tag{2.2.9}$$

The T-accretivity of A implies that J_λ^A is order preserving,
i.e. $u \leq v \Rightarrow J_\lambda^A u \leq J_\lambda^A v$. This is true because of the following :

Suppose $u \le v$, then $u - v \le 0$, so $(u-v)^+ = 0$. From the T-accretivity of A we get

$$\|(J_\lambda^A u - J_\lambda^A v)^+\|_1 \le 0,$$

which implies
$$(J_\lambda^A u - J_\lambda^A v)^+ = 0,$$
thus
$$J_\lambda^A u \le J_\lambda^A v.$$

Since v is a stationary supersolution, we have $J_\lambda^A v \le v$. Therefore for $\lambda > \mu > 0$ we get

$$\frac{\lambda-\mu}{\lambda} J_\lambda^A v \le \frac{\lambda-\mu}{\lambda} v$$
$$\frac{\mu}{\lambda} v + \frac{\lambda-\mu}{\lambda} J_\lambda^A v \le \frac{\lambda-\mu}{\lambda} v + \frac{\mu}{\lambda} v = v$$

and (2.2.9) gives us now that $J_\lambda^A v \le J_\mu^A v \le v$ for $\lambda > \mu > 0$ as v is a supersolution. Hence, by the dominated convergence theorem, $w = \|\cdot\|_1 - \lim_{\lambda\to 0^+} J_\lambda^A v$ exists and $w \in \overline{D(A)}$.

We will show that w is also a supersolution. The order preservation property and (2.2.9) also give us

$$J_\lambda^A v = J_\mu^A \left(\frac{\mu}{\lambda} v + \frac{\lambda-\mu}{\lambda} J_\lambda^A v\right) \ge J_\mu^A \left(\frac{\mu}{\lambda} J_\lambda^A v + \frac{\lambda-\mu}{\lambda} J_\lambda^A v\right) = J_\mu^A J_\lambda^A v \quad (2.2.10)$$
$$\text{a.e. on } \Omega, \forall \lambda, \mu > 0.$$

Passing to the limit in the last inequality with $\lambda \to 0^+$ yields $w \ge J_\mu^A w$. Moreover $u_0 \le v$ implies $J_\lambda^A u_0 \le J_\lambda^A v$ for any $\lambda > 0$ (order preservation). As $\|\cdot\|_1 - \lim_{\lambda\to 0^+} J_\lambda^A u_0 = u_0$ (because $u_0 \in \overline{D(A)}$, see e.g. [9], p.71), a passage to the limit in the inequality shows us that $u_0 \le w$ and therefore w is also a supersolution.

As a consequence we may assume without loss of generality that $v \in \overline{D(A)}$, so we may consider the solution of (2.2.4) corresponding to the initial value v, i.e. $S^A(.)v$, the semigroup motion through v.

We have, using (2.2.9), (2.2.4) and the order preservation property, the following estimate

$$(J_\lambda^A)^n v = J_\lambda^A((J_\lambda^A)^{n-1} v) = J_\mu^A \left(\frac{\mu}{\lambda}(J_\lambda^A)^{n-1} v + \frac{\lambda-\mu}{\lambda}(J_\lambda^A)^n v\right) \ge$$
$$\ge J_\mu^A \left(\frac{\mu}{\lambda}(J_\lambda^A)^n v + \frac{\lambda-\mu}{\lambda}(J_\lambda^A)^n v\right) = J_\mu^A((J_\lambda^A)^n v)$$
$$\text{a.e. on } \Omega, \quad \forall \mu, \lambda > 0, n \in N.$$

Applying this estimate with $\lambda = \frac{t}{n}$ and passing to the limit with $n \to \infty$ yields

$$S^A(t)v \ge J_\mu^A(S^A(t)v) \qquad \text{a.e. on } \Omega, \forall t > 0, \mu > 0.$$

If we iterate this last inequality n times, we obtain for $\mu = \frac{s}{n}$

$$S^A(t)v \ge (J_{\frac{s}{n}}^A)^n S^A(t)v \qquad \text{a.e. on } \Omega, \forall t, s > 0$$

and thus, in the limit as $n \to \infty$

$$S^A(t)v \ge S^A(s)S^A(t)v = S^A(t+s)v \qquad \text{a.e. on } \Omega, \forall t, s > 0. \qquad (2.2.11)$$

Futhermore, since J_λ^A is order preserving, $v \geq u_0 \geq 0$ actually implies $J_\lambda^A v \geq J_\lambda^A u_0 \geq J_\lambda^A 0$. If we iterate this n times and evaluate at $\lambda = \frac{t}{n}$, $t > 0$, we obtain

$$(J_{\frac{t}{n}}^A)^n v \geq (J_{\frac{t}{n}}^A)^n u_0 \geq (J_{\frac{t}{n}}^A)^n 0$$

and thus as $n \to \infty$

$$S^A(t)v \geq S^A(t)u_0 \geq S^A(t)0 = 0 \qquad \text{a.e. on } \Omega, t > 0, \tag{2.2.12}$$

where we used the fact that $S^A(t)0$ is the unique solution of (2.2.4) with 0 initial condition and obviously 0 is such a solution. So by (2.2.11), (2.2.12)

$$S^A(t)v \geq S^A(t+s)v \geq 0 \qquad \text{a.e. on } \Omega, \text{ for any } t > 0$$

and these together imply that $v_\infty = \|\cdot\|_1 - \lim_{t\to\infty} S^A(t)v$ exists. It is well-known that $v_\infty \in A^{-1}0$ and thus, by assumption, $v_\infty = 0$. Also by (2.2.12)

$$0 \leq u(t) = S^A(t)u_0 \leq S^A(t)v \qquad \text{a.e. on } \Omega, \forall t > 0, \tag{2.2.13}$$

so $\|S^A(t)u_0\|_1 \to 0$ as $t \to \infty$, i.e. stability. □

2.3. A semigroup approach for a PDE with a hysteresis term

Consider now the following PDE with hysteresis :

$$\frac{\partial}{\partial t}(u + w) - \Delta u = f \qquad \text{in } \Omega \times (0,T) = Q, \tag{2.3.1}$$

where Ω is a bounded subset of $\mathbb{R}^N$, and the hysteresis relation $w = \mathcal{E}(u, w_0)$ represents a generalized play, defined in Introduction, with the only difference that (0.2.1) is replaced with a more general assumption

$$\begin{gathered}\gamma_r, \gamma_l \text{ are maximal monotone (possibly multivalued) functions.} \\ \text{and} \qquad \inf \gamma_r(u) \leq \sup \gamma_l(u), \qquad \forall u \in \mathbb{R}\end{gathered} \tag{2.3.2}$$

and the hysteresis relation is assumed to hold pointwise in space :

$$w(x,t) = [\mathcal{E}(u(x,.), w_0(x))](t) \qquad \text{in } [0,T], \text{ a.e. in } \Omega. \tag{2.3.3}$$

As pointed out by Visintin [**11**], the system (2.3.1), (2.3.3) is formally equivalent to

$$\begin{aligned} \frac{\partial u}{\partial t} + \xi - \Delta u &= f && \text{in } Q \\ \frac{\partial w}{\partial t} - \xi &= 0 && \text{in } Q \\ \xi &\in \phi(u, w) && \text{in } Q, \end{aligned} \tag{2.3.4}$$

where

$$\phi(u,w) = \begin{cases} +\infty & \text{if } w < \inf \gamma_r(u) \\ \bar{\mathbb{R}}^+ & \text{if } w \in \gamma_r(u) \backslash \gamma_l(u) \\ \{0\} & \text{if } \sup \gamma_r(u) < w < \inf \gamma_l(u) \\ \bar{\mathbb{R}}^- & \text{if } w \in \gamma_l(u) \backslash \gamma_r(u) \\ -\infty & \text{if } w > \sup \gamma_l(u) \\ \bar{\mathbb{R}} & \text{if } w \in \gamma_l(u) \cap \gamma_r(u). \end{cases} \tag{2.3.5}$$

$\bar{\mathbb{R}} := [-\infty, +\infty]$, $\bar{\mathbb{R}}^+ := [0, +\infty]$ and $\bar{\mathbb{R}}^- := [-\infty, 0]$.
We can write the Cauchy problem for (2.3.4) coupled with homogeneous Dirichlet boundary conditions as

$$\frac{\partial U}{\partial t} + AU \ni F \qquad \text{in } Q \tag{2.3.6}$$

$$U(0) = U_0 \qquad \text{in } \Omega \tag{2.3.7}$$

$$\text{where} \quad U = \begin{pmatrix} u \\ w \end{pmatrix}, \qquad F = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

$$A\begin{pmatrix} u \\ w \end{pmatrix} = AU = \left\{ \begin{pmatrix} \xi - \Delta u \\ -\xi \end{pmatrix}, \xi \in \varphi(U) \cap \mathbb{R} \right\}$$

and

$$D(A) = \left\{ U = \begin{pmatrix} u \\ w \end{pmatrix}; \inf \gamma_r(u) \le w \le \sup \gamma_l(u) \text{ a.e. on } \Omega, U \in L^1(\Omega, \mathbb{R}^2), \right.$$

$$\left. u \in W_0^{1,1}(\Omega), -\Delta u \in L^1(\Omega) \right\} \tag{2.3.8}$$

We have the following theorem, see [11], p.234 :

THEOREM 2.2. *Assume that γ_l, γ_r are maximal monotone, satisfy (2.3.2), and are affinely bounded, that is, there exist constants $C_1, C_2 > 0$, such that $\forall v \in \mathbb{R}, \forall z \in \gamma_h(v)$*

$$\|z\| \le C_1\|v\| + C_2 \quad (h = l, r) \tag{2.3.9}$$

Then the operator A defined above is m- and T-accretive in $L^1(\Omega, \mathbb{R}^2)$.

The following theorem (Visintin, [11]) stems from the m-accretivity of the operator A and from general results of the theory of nonlinear semigroups of contractions, for details see e.g. [1], [2], [3], [4], [9], or [12]. The integral solution (in the sense of Benilan) is defined as follows:

DEFINITION 2.2. u is called an integral solution of the Cauchy problem:

$$\frac{du}{dt} + A(u(t)) \ni f \qquad \text{in } (0, T) \tag{2.3.10}$$

$$u(0) = u_0, \tag{2.3.11}$$

where $A : D(A) \subset B \to B$, B a (real) Banach space, is a (possibly nonlinear and multivalued) m-accretive operator, if
(i) $u : [0, T] \to B$ is continuous,
(ii) $u(t) \in \overline{D(A)}$ for any $t \in [0, T]$,
(iii) (2.3.11) holds and

$$\|u(t_2) - v\|_B^2 \le \|u(t_1) - v\|_B^2 + 2\int_{t_1}^{t_2} \langle u(\tau) - v, f(\tau) - z \rangle_s d\tau \tag{2.3.12}$$

$$\forall v \in D(A), \forall z \in A(v), \forall [t_1, t_2] \subset [0, T]. \tag{2.3.13}$$

Here the semi-inner product $\langle ., . \rangle_s : B^2 \to \mathbb{R}$ is defined by

$$\langle u, v \rangle_s := \lim_{\lambda \to 0} \frac{\|u + \lambda v\|_B^2 - \|u\|_B^2}{2\lambda} \qquad \forall u, v \in B. \tag{2.3.14}$$

THEOREM 2.3. *The Cauchy problem (2.3.6)-(2.3.7) has one and only one integral solution* $U : [0,T] \to L^1(\Omega, \mathbb{R}^2)$, *which depends continuously on the data* u_0, w_0, f. *Moreover, if* $f \in BV(0,T; L^1(\Omega))$ *and* $-\Delta u_0 \in L^1(\Omega)$, *then* U *is Lipschitz continuous.*

2.4. Main result

In this section we first state our main theorem, the proof of which will be very similar to the proof of Theorem 2.1. Before proving it, however, we will prove two lemmas which will be needed in the proof of the theorem.

THEOREM 2.4. *Suppose all conditions of Theorem 2.2 are satisfied. i.e. the operator* A *is m- and T-accretive in* $L^1(\Omega, \mathbb{R}^2)$:

$$R(I + \lambda A) = L^1(\Omega, \mathbb{R}^2)$$

and

$$\|(J_\lambda^A U - J_\lambda^A \tilde{U})^+\|_* \leq \|(U - \tilde{U})^+\|_*, \tag{2.4.1}$$

where

$$\|U\|_* = \int_\Omega (|u(x)| + |w(x)|)dx$$

denotes the norm in $L^1(\Omega, \mathbb{R}^2)$. *Suppose also that* $u_0 \in \overline{D(A)}$. *Then there exists* $w_\infty(x)$ *dependent on* x *only, such that for the solution* $U = \begin{pmatrix} u \\ w \end{pmatrix}$ *of*

$$\frac{\partial U}{\partial t} + AU \ni 0 \tag{2.4.2}$$

$$U(0) = \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} \tag{2.4.3}$$

the following holds:

$$\|\cdot\|_1 - \lim_{t \to \infty} u(x,t) = 0 \text{ and}$$

$$\|\cdot\|_1 - \lim_{t \to \infty} w(x,t) = w_\infty(x).$$

LEMMA 2.5. $A^{-1}0 = \left\{ \begin{pmatrix} 0 \\ w(x) \end{pmatrix}, \text{ such that } \inf \gamma_r(0) \leq w(x) \leq \sup \gamma_l(0) \right\}.$

PROOF. By direct computation.

Clearly $\begin{pmatrix} u \\ w \end{pmatrix} \in A^{-1}0$ if and only if the following is satisfied :

$$\xi - \Delta u = 0$$
$$-\xi = 0,$$

$$\text{which implies} \qquad \xi = 0,$$

which is satisfied by any $w(x)$, if $\inf \gamma_r(u(x)) \leq w(x) \leq \sup \gamma_l(u(x))$. We then must have

$$-\Delta u = 0 \qquad \text{on } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Obviously $u = 0$ is the unique solution of the above equation, so the assertion follows. □

LEMMA 2.6. *If $u_0 \in L^\infty(\Omega)$ there exist a stationary supersolution and a stationary subsolution of equation (2.3.6). Moreover, those can be chosen so that they belong to $\overline{D(A)}$.*

PROOF. We will show the existence of a stationary supersolution. The stationary subsolution can be found analogously. First note that by the last sentence of Remark 2.1 we can look for solutions of

$$\begin{aligned} \xi - \Delta u &\geq 0 && \text{a.e. on } \Omega \\ -\xi &\geq 0 && \text{a.e. on } \Omega \\ u &\geq u_0 && \text{a.e. on } \Omega \\ w &\geq w_0 && \text{a.e. on } \Omega, \end{aligned} \tag{2.4.4}$$

we can choose $\xi = 0$ and the second equation will be satisfied. From the first and the third one we require

$$-\Delta u \geq 0 \qquad \text{a.e. on } \Omega \tag{2.4.5}$$

$$u \geq u_0 \qquad \text{a.e. on } \Omega. \tag{2.4.6}$$

Since we only require $u_0 \in L^\infty(\Omega)$, such u can be easily found, in fact $u = \|u_0\|_{L^\infty}$ satisfies

$$\begin{aligned} -\Delta u &= 0 && \text{a.e. on } \Omega \\ u &\geq u_0 && \text{a.e. on } \Omega. \end{aligned}$$

The fourth equation in (2.4.4) is satisfied by any $w \geq w_0$. Then $\begin{pmatrix} u \\ w \end{pmatrix}$ will be the supersolution which we seek if we choose w such that $\inf \gamma_r(u) < w < \sup \gamma_l(u)$. The last assertion in the lemma is obvious. □

PROOF OF THE MAIN THEOREM. Suppose first that $u_0 \in L^\infty(\Omega)$. We will write

$$J_\lambda^A U_1 \leq J_\lambda^A U_2$$

if and only if $J_\lambda^{A_1} U_1 \leq J_\lambda^{A_1} U_2$ and $J_\lambda^{A_2} U_1 \leq J_\lambda^{A_2} U_2$, where

$$U_1 = \begin{pmatrix} u_1 \\ w_1 \end{pmatrix} \quad U_2 = \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \quad A\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} A_1(u,w) \\ A_2(u,w) \end{pmatrix} = \begin{pmatrix} \xi - \Delta u \\ -\xi \end{pmatrix}.$$

We will first prove that the resolvent J_λ^A is order preserving in the above defined sense : Suppose $U_1 \leq U_2$, i.e. $u_1 \leq u_2$ and $w_1 \leq w_2$, then

$$\begin{aligned} & u_1 - u_2 \leq 0 && w_1 - w_2 \leq 0 \\ \text{i.e.} \quad & (u_1 - u_2)^+ = 0 && (w_1 - w_2)^+ = 0. \end{aligned}$$

Therefore by (2.4.1) we have

$$\|(J_\lambda^A U_1 - J_\lambda^A U_2)^+\|_* = \int_\Omega |(J_\lambda^{A_1} U_1 - J_\lambda^{A_1} U_2)^+| + |(J_\lambda^{A_2} U_1 - J_\lambda^{A_2} U_2)^+| dx \leq 0$$

from which it follows that a.e. on Ω we have $J_\lambda^{A_i} U_1 - J_\lambda^{A_i} U_2 \leq 0$, $i = 1, 2$, i.e. $J_\lambda^A U_1 \leq J_\lambda^A U_2$ and J_λ^A is order preserving.

We may consider the solution of (2.4.2) corresponding to the initial value V, i.e. $S^A(.)V$, the semigroup motion through V. We have by the resolvent identity and the order preservation property that

$$(J_\lambda^A)^n V = J_\lambda^A((J_\lambda^A)^{n-1}V) = J_\mu^A(\frac{\mu}{\lambda}(J_\lambda^A)^{n-1}V + \frac{\lambda-\mu}{\lambda}(J_\lambda^A)^n V) \geq J_\mu^A((J_\lambda^A)^n V)$$
$$\text{a.e. on } \Omega, \forall \mu, \lambda > 0, n \in N.$$

Applying this estimate with $\lambda = \frac{t}{n}$ and passing to the limit as $n \to \infty$ yields:

$$S^A(t)V \geq J_\mu^A S^A(t)V \qquad \text{a.e. on } \Omega, \forall t > 0, \mu > 0.$$

If we iterate this last inequality n times, we obtain for $\mu = \frac{s}{n}$

$$S^A(t)V \geq (J_{\frac{s}{n}}^A)^n S^A(t)V \qquad \text{a.e. on } \Omega, \forall t, s > 0$$

and thus, in the limit $(n \to \infty)$

$$S^A(t)V \geq S^A(s)S^A(t)V = S^A(t+s)V \qquad \text{a.e. on } \Omega, \forall t, s > 0. \tag{2.4.7}$$

Furthermore, since J_λ^A is order preserving

$$V \geq U_0 \geq 0 \qquad \text{implies} \qquad J_\lambda^A V \geq J_\lambda^A U_0 \geq J_\lambda^A 0$$

and also

$$S^A(t)V \geq S^A(t)U_0 \geq S^A(t)0 = \begin{pmatrix} 0 \\ w_0(x) \end{pmatrix} \geq 0. \tag{2.4.8}$$

The last estimate together with the monotonicity in (2.4.7) implies that $V_\infty = \|\cdot\|_1 - \lim_{t\to\infty} S^A(t)$ exists and $V_\infty \in A^{-1}0$. As (2.4.8) gives us

$$0 \leq S^A(t)U_0 \leq S^A(t)V \qquad \text{a.e. on } \Omega, \forall t > 0,$$

it follows that $\|S^A(t)U_0\|_1$ exists and $S^A(t)u \to 0, S^A(t)w \to w(x)$ as $t \to \infty, w_0(x) \leq w(x)$.

The statement for arbitrary u_0 now follows from the density of $L^\infty(\Omega)$ in $\overline{D(A)}$ and from the continuous dependence on initial conditions, see Theorem IX.2.5 in [11]. □

THEOREM 2.7. *Consider the equation (2.3.1) with a generalized Prandtl-Ishlinskii operator of play type, where for μ-almost any $\rho \in \mathcal{P}$, $\gamma_{\rho l}$ and $\gamma_{\rho r}$ fulfill (2.3.2). With the notation*

$$P_{\rho h} = \{u \in \mathbb{R} : card(\gamma_{\rho h}(u)) > 1\} \quad \mu - a.e. \ in \ \mathcal{P}, h = l, r,$$

we assume that

$$D = \Big\{ U = \begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R} \times L^1(\mathcal{P}) :$$
$$\begin{pmatrix} u \\ w_\rho \end{pmatrix} \text{ are such that } \inf \gamma_{\rho r}(u) \leq w_\rho \leq \sup \gamma_{\rho l}(u), \mu - \ a.e. \ in \ \mathcal{P} \Big\} \neq \emptyset$$

and

$$\begin{cases} \exists A \in \mathcal{A} : \mu(A) = 0, \ and \ \forall \rho', \rho'' \in \mathcal{P} \backslash A, \\ if \ \rho' \neq \rho'' \ then \ P_{\rho' h} \cap P_{\rho'' h} = \emptyset \quad (h = l, r). \end{cases} \tag{2.4.9}$$

Then the conclusions of the previous theorem remain true.

The proof is analogous to the proof of Theorem 2.4. By Visintin [11], p.234, the operator A_μ under assumption (2.4.9) is m- and T-accretive in the space $L^1(\Omega;\mathbb{R}\times L^1(\mathcal{P}))$ coupled with the norm

$$\|U\|_{**} = \int_\Omega \left(|u(x)| + \int_{\mathcal{P}} |w_\rho(x)| d\mu(\rho) \right) dx$$

for all $U = (u, w) \in L^1(\Omega, R \times L^1(\mathcal{P}))$.

In order to extend the stability results in Theorems 2.4 and 2.7 to equations with a nonzero right hand side, we give two theorems with proofs, from which the results immediately follow. The first one is proved for X a Hilbert space (and thus A a maximal monotone operator) in [8], but we did not find the corresponding version for an arbitrary Banach space anywhere else in the literature. Even though the idea of the proof is similar, we present it here for completeness.

THEOREM 2.8. *Assume that* $A : D(A) \subset X \to X$, X *a Banach space, is an m-accretive operator and* $f \in L^1(\mathbb{R}^+, X)$. *Let* $S^A(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0$ *be the semigroup generated by* A. *If* $\forall x \in \overline{D(A)}$, $S^A(t)x$ *converges strongly (weakly) as* $t \to \infty$, *then every integral solution* $z(t)$ *of*

$$\frac{\partial z}{\partial t} - Az \ni f(x,t)$$

converges strongly (weakly) as $t \to \infty$.

PROOF. Define $f_n : \mathbb{R}^+ \to X$ as follows

$$f_n(t) = \begin{cases} f(t) & \text{for a.e. } t \in (0, nT) \\ 0 & \text{for } t \geq nT, \end{cases}$$

where $n \in N$ and $T > 0$ is a fixed number. For each $n \in N$, denote by $z_n(t)$, $t \geq 0$ the integral solution of the Cauchy problem

$$\frac{\partial z_n(t)}{\partial t}(t) + A(z_n(t)) \ni f_n(t) \qquad t > 0$$
$$z_n(0) = z(0).$$

Clearly, $z_n(t) = S^A(t) z_n(nT)$ for all $t \geq nT$. Now, from the assumption that $S^A(t)x$ converges strongly (resp. weakly) we can see that for each $n \in N$ there exists a $p_n \in X$ such that

$$\lim_{t\to\infty} z_n(t) = p_n \qquad \text{strongly (resp. weakly).} \tag{2.4.10}$$

We now note that

$$\|u(t) - v(t)\| \leq \|u(0) - v(0)\| + \int_0^t \|f(\theta) - g(\theta)\| d\theta \tag{2.4.11}$$

where $u(t)$, resp. $v(t)$ are solutions of

$$\frac{\partial u}{\partial t} + Au \ni f \qquad \frac{\partial v}{\partial t} + Av \ni g$$
$$u(x,0) = u(0) \qquad v(x,0) = v(0)$$

respectively. This is a special case of a result from [9], p.42. (2.4.11) implies

$$\|z_n(t) - z_m(t)\| \leq \int_{mT}^{nT} \|f(s)\| ds \quad \forall t \geq 0, n > m, \tag{2.4.12}$$

and we get from (2.4.10) that

$$\|p_n - p_m\| \le \int_{mT}^{nT} \|f(s)\|ds, \qquad n > m.$$

Therefore, there exists a $p \in X$, such that

$$p_n \to p \qquad \text{strongly in } X. \tag{2.4.13}$$

On the other hand, since z_n and z are integral solutions of

$$\frac{\partial z_n}{\partial t}(t) + A(z_n(t)) \ni f_n(t) \qquad \frac{\partial z}{\partial t}(t) + Az(t) \ni f(t)$$
$$z_n(0) = z(0) \qquad z(0) = z(0)$$

respectively, we have by (2.4.11) that

$$\|z(t) - z_n(t)\| \le \int_{nT}^{t} \|f(s)\|ds, \qquad \forall t \ge nT \tag{2.4.14}$$

Now, using (2.4.10), (2.4.13), (2.4.14) and the following decomposition

$$z(t) - p = [z(t) - z_n(t)] + [z_n(t) - p_n] + [p_n - p]$$

we find that $\lim_{t\to\infty} z(t) = p$ in the strong (resp. weak) topology. This completes the proof.

NOTE 2.1. It is then a standard result that $p \in A^{-1}0$.

□

THEOREM 2.9. *Let $f(x,t)$ be such that*

$$\lim_{t\to\infty} f(x,t) = f_\infty \text{ in } X, \qquad (f(x,t) - f_\infty) \in L^1(\mathbb{R}^+, X), \tag{2.4.15}$$

and let u be a solution of

$$\frac{\partial u}{\partial t} + Au = f(x,t)$$
$$u(x,0) = u_0(x),$$

where A is an m-accretive operator in X, and $\forall x \in X$, $S^A(t)x$ converges strongly as $t \to \infty$.

Then $\lim_{t\to\infty} u(x,t)$ exists and $\lim_{t\to\infty} u(x,t) = p + u_\infty$, where u_∞ is such that $Au_\infty = f_\infty$ and $p \in A^{-1}0$.

PROOF. Note that $u - u_\infty$ satisfies the following equation :

$$\frac{\partial}{\partial t}(u - u_\infty) + A(u - u_\infty) = f - f_\infty$$

and $f - f_\infty \in L^1(\mathbb{R}^+, X)$. We can therefore apply Theorem 2.8, which says that

$$\lim_{t\to\infty} (u - u_\infty) = p, \quad p \in A^{-1}0 \qquad \text{in } L^1(\Omega).$$

and the statement follows. □

Applying the previous results to our hysteresis problem, we get

THEOREM 2.10. *Suppose that all conditions of either Theorem 2.4 or Theorem 2.7 are satisfied . Suppose also that $f(x,t)$ is a given right-hand side satisfying (2.4.15) with $X = L^1(\Omega)$. Then for the solution of*

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} + Au = f(x,t)$$

the following is true: $\lim_{t\to\infty} u(x,t) = u_\infty(x)$, *where* $u_\infty(x)$ *is such that* $Au_\infty(x) = f_\infty$ *and* $\lim_{t\to\infty} w(x,t)$ *exists in* $L^1(\Omega)$.

REMARK 2.2. All statements in this chapter remain true if we replace $-\Delta u$ by a more general elliptic operator

$$Au = -\sum_{l,m=1}^{N} \frac{\partial}{\partial x_l}\left(a_{lm}\frac{\partial u}{\partial x_m}\right) + \sum_{l=1}^{N} \frac{\partial}{\partial x_l}(b_l u) + cu$$

$a_{lm}, b_l \in C^1(\overline{\Omega}), c \in L^\infty(\Omega)$ and for some constant $\alpha > 0$

$$\sum_{l,m=1}^{N} a_{lm}\psi_l\psi_m \geq \alpha\|\psi\|^2 \;\forall\psi \in \mathbb{R}^n \text{ a.e. in } \Omega,$$

with $c \geq 0$ and some coefficient condition (see e.g. [5]), which will guarantee the uniqueness of the corresponding elliptic problem with Dirichlet boundary data..

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CHAPTER 3

About periodic solutions and asymptotic behaviour of a PDE with hysteresis in the source term

3.1. Introduction

In this chapter we consider the following model equation

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = f \qquad \text{in } Q, \tag{3.1.1}$$

coupled with initial and boundary conditions;
where $\mathcal{F} : M(\Omega; C^0([0,\infty))) \to M(\Omega; C^0([0,\infty)))$ is a continuous operator with memory, $M(\Omega; C^0([0,\infty)))$ denotes the Fréchet space of (strongly) measurable functions $\Omega \to C^0([0,\infty))$ and f is a given function. Here we fix an open bounded set $\Omega \subset \mathbb{R}^N (N \geq 1)$ of Lipschitz class, denote by $\partial\Omega$ the boundary of Ω, and set $Q := \Omega \times (0,\infty), \Sigma := \partial\Omega \times (0,\infty)$.

The existence and uniqueness of solutions of (3.1.1) is well known and we present those results in the first section of this chapter.

We study the question of existence of periodic solutions of (3.1.1) as well as asymptotic behaviour of solutions as $t \to \infty$. To our knowledge there are so far only two papers dealing with such problems, [1] and [5]. In [1] they investigated the asymptotic behaviour (as $t \to \infty$) of both the solution of (3.1.1) and the corresponding memory term $\mathcal{F}(u)$. They showed that under some assumptions on the hysteresis boundary curves there exists $u_\infty \in H_0^1(\Omega) \cap W_{\mathrm{loc}}^{1,p}(\Omega)$, $\forall p \in [1,\infty)$, such that $u(.,t) \to u_\infty$ weakly in $H_0^1(\Omega)$, $w(x,t) = \mathcal{F}(u(x,t)) \to -\Delta u_\infty$ strongly in $L^p(\Omega), \forall p \in [1,\infty)$, and a.e. in Ω as $t \to \infty$. They assume $\mathcal{F}$ is a generalized play operator and their proof of asymptotic stability relies on the specific properties of this operator. The question of periodic solutions of (3.1.1) was considered by Xu Longfeng in [5], but only in a special case, where $\mathcal{F}$ is assumed to be a specific type of hysteresis operator. In this chapter we prove the existence of a periodic solution of (3.1.1) with a general hysteresis operator. We give two different proofs of this result. The idea of the first one is based on the ideas given in [5], but we have different assumptions, which we think are more physically reasonable. The second proof is based on a variation of ideas P.Krejčí used in his papers, see e.g. [3]. The last section contains an asymptotic result for (3.1.1).

3.2. A parabolic problem

We denote by $M(\Omega; C^0([0,T]))$ the Fréchet space of (strongly) measurable functions $\Omega \to C^0([0,T])$, see e.g. the appendix in [7]. Let

$$\mathcal{F} : M(\Omega; C^0([0,T])) \quad \to \quad M(\Omega; C^0([0,T])) \tag{3.2.1}$$

be a causal and strongly continuous operator. We fix a relatively open subset Γ_1 of $\partial\Omega$, and set

$$V := H^1_{\Gamma_1}(\Omega) := \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \Gamma_1\} \tag{3.2.2}$$

where γ_0 denotes the trace operator. Thus if $\Gamma_1 = \emptyset$, then $V = H^1(\Omega)$; if $\Gamma_1 = \partial\Omega$, then $V = H^1_0(\Omega)$. We identify the space $L^2(\Omega)$ with its dual $L^2(\Omega)'$. As V is a dense subspace of $L^2(\Omega)$, $L^2(\Omega)'$ can be identified with a subspace of V'. So we get

$$V \subset L^2(\Omega) = L^2(\Omega)' \subset V', \tag{3.2.3}$$

with continuous, dense and compact injections. We define the operator $A : V \to V'$, $u \mapsto Au$ as follows :

$$_{V'}\langle Au, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v dx \quad \forall v \in V; \tag{3.2.4}$$

hence $Au = -\Delta u$ in $\mathcal{D}'(\Omega)$,
where $\mathcal{D}(\Omega) = \{\phi; \phi$ infinitely differentiable on Ω and with compact support in $\Omega\}$ and $\mathcal{D}'(\Omega) =$ dual of $\mathcal{D}(\Omega) =$ space of distributions on Ω. We assume that

$$u_0, w_0 \in L^2(\Omega), \quad f \in L^2(0,T;V'). \tag{3.2.5}$$

PROBLEM 3.1. To find $u \in M(\Omega; C^0([0,T])) \cap L^2(0,T;V)$ such that $\mathcal{F}(u) \in L^2(Q)$ and

$$\iint_Q \left(-u\frac{\partial v}{\partial t} + \nabla u \cdot \nabla v + \mathcal{F}(u)v\right) dxdt = \\ = \int_0^T {}_{V'}\langle f, v\rangle_V dt + \int_\Omega u_0(x)v(x,0)dx \tag{3.2.6}$$

$$\forall v \in L^2(0,T;V) \cap H^1(0,T;L^2(\Omega)), \qquad v(\cdot,T) = 0, \text{ a.e. in } \Omega.$$

INTERPRETATION. Equation (3.2.6) yields

$$\frac{\partial u}{\partial t} + Au + \mathcal{F} = f \qquad \text{in } \mathcal{D}'(0,T;V'). \tag{3.2.7}$$

By comparing the terms of this equation, we see that $\frac{\partial u}{\partial t} \in L^2(0,T;V')$, thus $u \in L^2(0,T;V) \cap H^1(0,T;V')$ and (3.2.7) holds in V' a.e. in $(0,T)$. The functions of this space admit time traces in $L^2(\Omega)$. Hence, integrating by parts in (3.2.6) and using (3.2.7), we get

$$u(x,0) = u_0(x) \qquad \text{in } L^2(\Omega) \quad \text{(in the sense of traces)}. \tag{3.2.8}$$

Let us now interpret (3.2.7) for $V = H^1_{\Gamma_1}(\Omega)$. Let $\Gamma_2 := \Gamma/\Gamma_1$, fix any

$$f_1 \in L^2(Q), \qquad f_2 \in L^2(\Gamma_2 \times (0,T)), \tag{3.2.9}$$

and define $f \in L^2(0,T;L^2(\Omega)) \bigoplus L^2(0,T;V')$ by

$$_{V'}\langle f(t), v\rangle_V := \int_\Omega f_1(x,t)v(x)dx + \int_{\Gamma_2} f_2(\sigma,t)\gamma_0 v(\sigma)d\sigma \\ \forall v \in V, \text{ a.e. in } (0,T).$$

Then (3.2.6) corresponds to the differential equation

$$\frac{\partial}{\partial t}u - \Delta u + \mathcal{F}(u) = f_1 \qquad \text{in } \mathcal{D}'(\Omega), \text{ a.e. in } (0,T), \tag{3.2.10}$$

coupled with the boundary conditions

$$\gamma_0 u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{3.2.11}$$

$$\frac{\partial u}{\partial \nu} = f_2 \qquad \text{in } \mathcal{D}'(\Gamma_2 \times (0,T)), \tag{3.2.12}$$

where $\frac{\partial}{\partial \nu}$ denotes the exterior normal derivative.

The following theorem is proved in [7] :

THEOREM 3.1. *Assume that (3.2.1)-(3.2.3) hold. Let $\mathcal{F}$ be affinely bounded, in the sense that*

$$\exists L \in \mathbb{R}^+, \exists g \in L^2(\Omega); \forall v \in M(\Omega, C^0([0,T])); \tag{3.2.13}$$

$$\|[\mathcal{F}(v)](x,\cdot)\|_{C^0([0,T])} \le L\|v(x,\cdot)\|_{C^0([0,T])} + g(x) \ \text{a.e. in } \Omega.$$

Moreover, let

$$f = f_1 + f_2, \quad f_1 \in L^2(\Omega), \quad f_2 \in W^{1,1}(0,T,V'), \quad u_0 \in V, \quad w_0 \in L^2(\Omega). \tag{3.2.14}$$

Then Problem 3.1 has at least one solution such that

$$u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; V), \tag{3.2.15}$$

$$\mathcal{F}(u) \in L^2(\Omega; C^0([0,T])). \tag{3.2.16}$$

If $\mathcal{F}$ has also the global Lipschitz continuity property

$$\exists L > 0; \forall t \in (0,T], \quad \forall v_1, v_2 \in L^2(\Omega; C^0([0,t])), \tag{3.2.17}$$

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^2(\Omega;C^0([0,t]))} \le L\|v_1 - v_2\|_{L^2(\Omega;C^0([0,t]))}, \tag{3.2.18}$$

then Problem 3.1 has only one solution satisfying (3.2.15).

3.3. Periodic solutions

We consider the question of existence of periodic solutions for (3.1.1) coupled with suitable boundary conditions. Here f will be a given function ω−periodic in t.

We will make use of various subsets of the following assumptions:
(A1) Global Lipschitz continuity:

$$\exists K > 0; \forall t \in (0,\infty), \quad \forall v_1, v_2 \in L^2(\Omega; C^0([0,t])),$$

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^2(\Omega;C^0([0,t]))} \le K\|v_1 - v_2\|_{L^2(\Omega;C^0([0,t]))}$$

(A2) Monocyclicity:
If $u(x,t)$ is ω−periodic in t, then $[\mathcal{F}(u)](x,t+\omega) = [\mathcal{F}(u)](x,t)$ for all $t \ge \omega, x \in \Omega$.
(A3) Affine boundedness:

$$\exists K_1 \in \mathbb{R}^+, \exists g \in L^2(\Omega); \forall v \in M(\Omega, C^0([0,\infty)));$$

$$\|[\mathcal{F}(v)](x,\cdot)\|_{C^0([0,\infty))} \le K_1\|v(x,\cdot)\|_{C^0([0,\infty))} + g(x) \text{ a.e. in } \Omega.$$

(A4) Saturation:

$$|\mathcal{F}(u)| \le C,$$

where C is some positive constant.
(A5) $\mathcal{F}$ is odd, i.e. $\mathcal{F}(u) = -\mathcal{F}(-u)$, for all $u \in L^2(\Omega; C^0([0,T]))$.

REMARK 3.1. The term monocyclicity was introduced by M.A.Krasnosel'skii and A.V.Pokrovskii in [2]. The least $\delta > 0$ such that the identity

$$[\mathcal{F}(u)](x, t + \omega) = [\mathcal{F}(u)](x, t) \qquad (t \geq \delta)$$

holds is called a periodicity stabilization time of the output. If, for any periodic input, this time does not exceed the value of one period, then the operator is monocyclic. More details as well as the proof of the fact that the generalized play operator, and therefore also the generalized Prandtl-Ishlinskii operator of play type, is monocyclic can be found in [2]. The property (A4) is physically sensible for many problems. (A5) is often satisfied in applications, especially to elastoplasticity.

Let $\omega > 0$ and let B be a Banach space. A measurable function $u : \mathbb{R}^+ \to B$ is called ω-periodic if

$$u(t + \omega) = u(t)$$

for almost all $t \in \mathbb{R}^+$. By $L^2_\omega(0, \infty; B)$ we denote the Banach space of all (classes of) ω-periodic functions $u : (0, \infty) \to B$ for which $u|_{(0,\omega)} \in L^2(0, \omega; B)$. The norm is given by

$$\|u|_{(0,\omega)}\|_{L^2((0,\omega);B)} = \left(\int_0^\omega \|u(x,t)\|^2_B dt\right)^{\frac{1}{2}}.$$

We can similarly define other spaces of functions, ω-periodic in t, for more details see e.g. [6].

Define $D = H^{1,2}_\omega(Q) \cap G$, where $G = \overline{\{u \in C^\infty_\omega(\overline{Q}), u = 0 \text{ on } \partial\Omega, t \in \mathcal{R}\}}$ in $H^1_\omega(Q)$.

We will prove the following theorem:

THEOREM 3.2. *There exists $K > 0$ such that if $f \in L^2_\omega(0, \infty, L^2(\Omega))$ is given and $\mathcal{F}$ satisfies the assumptions (A1) and (A2) and at least one of the assumptions (A3) with $K_1 < K$, (A4), then there exists $u \in D$, which is periodic and satisfies the equation (3.1.1) almost everywhere in Q for $t \geq \omega$.*

REMARK 3.2. The operator $-\Delta$ in the equation (3.1.1) can be replaced by any symmetric and uniformly elliptic operator.

PROOF. To prove the Theorem we will need the following lemma, for the proof see [6], Theorem III. 1.3.1.

LEMMA 3.3. *Suppose that $f \in L^2_\omega(0, \infty; L^2(\Omega))$. Then there exists a unique periodic solution of the equation*

$$\frac{\partial u}{\partial t} - \Delta u = f \qquad \text{in } Q, \tag{3.3.1}$$

which satisfies the Dirichlet boundary condition

$$u = 0 \qquad \text{on } \Sigma,$$

such that $u \in D$. Moreover, there exists a positive constant K_2 such that

$$|u|_D \leq K_2 |f|_{L^2_\omega(0,\infty;L^2(\Omega))}.$$

The main tool in our proof will be the homotopy version of the Leray-Schauder fixed point theorem:

THEOREM 3.4. *Let B be a Banach space, $T : B\times[0,1] \to B$ a compact mapping such that*
(i) $T(x,0) = 0$ for all $x \in B$,
(ii) there exists a constant M such that $|x|_B \le M$ for all $(x,\sigma) \in B\times[0,1]$ satisfying $x = T(x,\sigma)$.
Then the mapping T_1 of B into itself given by $T_1x = T(x,1)$ has a fixed point.

We introduce the Banach space $B = L^2(\Omega, C^0[0,\infty))$. It can be easily seen from property (A3) or (A4) and property (A2) that for $\forall v \in B$

$$\mathcal{F}(v) \in L^2\left(\Omega, C^0([0,\infty)\right) \cap L^2\left(\Omega, C^0_\omega([\omega,\infty))\right).$$

For any $\sigma \in [0,1]$ and $t \ge \omega$ we consider the equation

$$\frac{\partial u}{\partial t} - \Delta u = -\sigma\mathcal{F}(v) + \sigma f, \qquad v \in B.$$

By Lemma 3.3, the above equation has for any $\sigma \in [0,1]$ and any $v \in B$ a unique solution $\tilde{u} \in D$.
Let $u \in H^1_\omega\left(0,\infty; H^1_0(\Omega)\right) \cap H^1_\omega\left(0,\infty; H^2(\Omega)\right)$ be an extension of $\tilde{u}$ to $(0,\infty)$.
By interpolation, see e.g. [4], we have

$$D \subset H^1_\omega(Q) \subset L^2\left(\Omega; C^0_\omega[0,\infty)\right) \tag{3.3.2}$$

with continuous injections and the last one is also compact.
If we denote by $T : T(v,\sigma) = u$, by the above it follows that $T : B \times [0,1] \to B$.
We shall show that all the assumptions of the Leray-Schauder theorem are satisfied for the mapping T. Obviously, for $\forall v \in B$, $T(v,0) = 0$, so assumption (i) is satisfied.
To show that T is a compact mapping: For $\forall\sigma \in [0,1]$, $v_1, v_2 \in B$, letting

$$T(v_1,\sigma) = u_1$$
$$T(v_2,\sigma) = u_2,$$

we get

$$\frac{\partial}{\partial t}(u_1 - u_2) - \Delta(u_1 - u_2) = -\sigma[\mathcal{F}(v_1) - \mathcal{F}(v_2)] \qquad \forall t \ge \omega. \tag{3.3.3}$$

Multiplying the last equation by $u_1 - u_2$ and integrating over Ω, we get after integration by parts

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_1 - u_2\|^2_{L^2(\Omega)} + \int_\Omega [\nabla(u_1 - u_2)]^2 dx =$$
$$= -\sigma\int_\Omega (u_1 - u_2)[\mathcal{F}(v_1) - \mathcal{F}(v_2)]dx$$

After integrating this in t over $[\omega, 2\omega]$, we have

$$\frac{1}{2}\left\{\|u_1(2\omega)-u_2(2\omega)\|^2_{L^2(\Omega)}-\|u_1(\omega)-u_2(\omega)\|^2_{L^2(\Omega)}\right\}+\int_\omega^{2\omega}\int_\Omega[\nabla(u_1-u_2)]^2dxdt\leq$$
$$\leq|\sigma|\int_\omega^{2\omega}\int_\Omega|u_1-u_2||\mathcal{F}(v_1)-\mathcal{F}(v_2)|dxdt\leq$$
$$\leq\|u_1-u_2\|_{L^2(\Omega;L^2(\omega,2\omega))}\omega^{\frac{1}{2}}\left\{\int_\Omega\sup_t[\mathcal{F}(v_1)-\mathcal{F}(v_2)]^2dx\right\}^{\frac{1}{2}}\leq$$
$$\leq L\omega^{\frac{1}{2}}\|v_1-v_2\|_B\|u_1-u_2\|_{L^2(\Omega,L^2(\omega,2\omega))}. \quad (3.3.4)$$

Because u_i, $i=1,2$ are periodic in t with period ω, the difference of the first two terms on the left-hand side of the last inequality is zero. Moreover, using the Poincaré inequality to estimate the last term on the left-hand side we get

$$\mu_1\|u_1-u_2\|^2_{L^2(\Omega,L^2(\omega,2\omega))}\leq\omega^{\frac{1}{2}}L\|u_1-u_2\|_{L^2(\Omega,L^2(\omega,2\omega))}\|v_1-v_2\|_B.$$

Thus

$$\|u_1-u_2\|_{L^2(\Omega,L^2(\omega,2\omega))}\leq\frac{L\omega^{\frac{1}{2}}}{\mu_1}\|v_1-v_2\|_B. \quad (3.3.5)$$

We also get from the inequality (3.3.4) using equivalent norms on the space $H^1_0(\Omega)$ that

$$\|u_1-u_2\|_{L^2(\omega,2\omega;H^1(\Omega))}\leq LR\omega^{\frac{1}{2}}\|v_1-v_2\|_B. \quad (3.3.6)$$

If we now multiply (3.3.3) by $\frac{\partial}{\partial t}(u_1-u_2)$ and integrate over Ω, we get

$$\int_\Omega\left[\frac{\partial(u_1-u_2)}{\partial t}\right]^2dx-\frac{\partial}{\partial t}\int_\Omega[\nabla(u_1-u_2)]^2dx\leq$$
$$\leq|\sigma|\int_\Omega|\mathcal{F}(v_1)-\mathcal{F}(v_2)|\left|\frac{\partial(u_1-u_2)}{\partial t}\right|dx.$$

After integrating in t over $[\omega,2\omega]$, using estimates similar to those used above, we get

$$\left\|\frac{\partial}{\partial t}(u_1-u_2)\right\|_{L^2(\Omega,L^2(\omega,2\omega))}\leq L\omega^{\frac{1}{2}}\|v_1-v_2\|_B. \quad (3.3.7)$$

It follows from (3.3.5), (3.3.6) and (3.3.7) that

$$\|u_1-u_2\|_{H^1(\Omega)}\leq R_1\|v_1-v_2\|_B \quad (3.3.8)$$

and that T is completely continuous with respect to v because of the compact imbedding (3.3.2).

Now, $\forall v\in B$, $\sigma_1,\sigma_2\in[0,1]$, let

$$T(v,\sigma_1)=u^1$$
$$T(v,\sigma_2)=u^2.$$

We have

$$\frac{\partial}{\partial t}(u^1-u^2)-\Delta(u^1-u^2)=(\sigma_1-\sigma_2)[f-\mathcal{F}(v)] \qquad \forall t\geq\omega.$$

By Lemma 3.3 and the compact imbedding (3.3.2) we get the estimate

$$\|u^1 - u^2\|_B \le |\sigma_1 - \sigma_2|\widetilde{K}\left[\|f\|_{L^2(\Omega,L^2(\omega,2\omega))} + \|\mathcal{F}(v)\|_{L^2(\Omega,L^2(\omega,2\omega))}\right].$$

Hence, T is uniformly continuous with respect to σ. Now, T is completely continuous with respect to v and uniformly continuous with respect to σ, and thus T is a compact mapping of $B \times [0,1] \to B$.

To show (ii): For $\forall\sigma \in [0,1]$, let $T(u,\sigma) = u$, i.e.

$$\frac{\partial u}{\partial t} - \Delta u = -\sigma\mathcal{F}(u) + \sigma f, \qquad \forall t \ge \omega.$$

Multiplying by u and integrating over Ω, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} + \int_\Omega (\nabla u)^2 dx \le \int_\Omega \mathcal{F}(u)u dx + \int_\Omega f u dx$$

After integrating the last inequality in t over $[\omega, 2\omega]$,and using the periodicity of u in t and the Poincaré inequality as before, we get

$$\begin{aligned}\widetilde{K}\|u\|^2_{L^2(\Omega,L^2(\omega,2\omega))} \le \|f\|_{L^2(\Omega,L^2(\omega,2\omega))}\|u\|_{L^2(\Omega,L^2(\omega,2\omega))} \\ +\|u\|_{L^2(\Omega,L^2(\omega,2\omega))}\|\mathcal{F}(u)\|_{L^2(\Omega,L^2(\omega,2\omega))}.\end{aligned} \tag{3.3.9}$$

The last term in (3.3.9) can be now estimated by assumption (A4) by

$$\|u\|_{L^2(\Omega,L^2(\omega,2\omega))}\|\mathcal{F}(u)\|_{L^2(\Omega,L^2(\omega,2\omega))} \le C_1\|u\|_{L^2(\Omega,L^2(\omega,2\omega))}.$$

Then we have all together that

$$\|u\|_{L^2(\Omega,L^2(\omega,2\omega))} \le \widetilde{C} = \text{constant}.$$

By the compact imbedding (3.3.2), also

$$\|u\|_B \le C_2.$$

So all assumptions of the Leray-Schauder fixed point theorem are satisfied, thus there exists

$$u \in D$$

such that $T(u,1) = u$. This is the ω-periodic solution of (3.3.1).

If instead of (A4) we assume (A3), as was done by Xu Longfeng in [5], then we get the same result, estimating the last term in (3.3.9) by assumption (A3), but we need to assume also that the constant K_1 in (A3) is small enough. This was done in [5] only for a special kind of hysteresis operator.

□

THEOREM 3.5. *Suppose that $f \in L^2_\omega(0,\infty,L^2(\Omega))$ and $\mathcal{F}$ satisfies the assumptions (A1), (A2), (A4) and (A5). Then there exists $u \in H^1_\omega(0,\infty,L^2(\Omega)) \cap L^2_\omega(0,\infty,H^1_0(\Omega))$, which is periodic and satisfies the equation (3.1.1) almost everywhere in Q for $t \ge \omega$.*

We assume that $\Omega = [0,\pi]^N$.

PROOF. We use a variation of an approach used by P. Krejčí in [3], based on the classical Galerkin method. Put

$$w_j(t) = \begin{cases}\sin\frac{2\pi}{\omega}jt & j = 1,2,\dots\\ \cos\frac{2\pi}{\omega}jt & j = 0,-1,-2,\dots\end{cases}$$

and $w_{jk_1\dots k_N}(x,t) = w_j(t)\sin k_1x_1 \sin k_2x_2 \dots \sin k_Nx_N$, j an integer, k_i, $i = 1,\dots,N$ natural numbers. Fix an integer $m \geq 1$. We want to determine real numbers $v_{jk_1\dots k_N}$, $j = -m,\dots,m$; $k_i = 1,\dots,m$; $i = 1,\dots,N$ in order that the function

$$u_m(x,t) = \sum_{j=-m}^{m} \sum_{k_1,\dots,k_N=1}^{m} v_{jk_1\dots k_N} w_{jk_1\dots k_N}(x,t)$$

satisfies the system

$$\int_\omega^{2\omega}\int_\Omega \left(\frac{\partial u_m}{\partial t} - \Delta u_m\right) w_{jk_1\dots k_N} dx dt + \int_\omega^{2\omega}\int_\Omega \mathcal{F}(u_m) w_{jk_1\dots k_N} dx dt = \int_\omega^{2\omega}\int_\Omega f w_{jk_1\dots k_N} dx dt \tag{3.3.10}$$

where $j = -m,\dots,m$; $k_i = 1,\dots,m$; $i = 1,\dots,N$.

We derive first some apriori estimates. In what follows we denote by C any positive constant independent of m. Let us multiply (3.3.10) by $-w_{-jk_1\dots k_N}\left(\frac{2\pi j}{\omega}\right)$ and sum over $j, k_1, \dots, k_N$. We obtain

$$\int_\omega^{2\omega}\int_\Omega \left(\frac{\partial u_m}{\partial t}\right)^2 dx dt + \int_\omega^{2\omega}\int_\Omega \mathcal{F}(u_m)\frac{\partial u_m}{\partial t} dx dt = \int_\omega^{2\omega}\int_\Omega f\frac{\partial u_m}{\partial t} dx dt,$$

because the term

$$\int_\omega^{2\omega}\int_\Omega \nabla u_m \nabla\left(\frac{\partial u_m}{\partial t}\right) dx dt = \int_\omega^{2\omega}\int_\Omega \frac{\partial}{\partial t}(\nabla u_m)^2 = 0,$$

where the last equality holds because of the ω-periodicity of the function u_m in t.

So we have

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega\times(\omega,2\omega))} = -\int_\omega^{2\omega}\int_\Omega \mathcal{F}(u_m)\frac{\partial u_m}{\partial t} dx dt + \int_\omega^{2\omega}\int_\Omega f\frac{\partial u_m}{\partial t} dx dt \leq$$
$$\leq \left(\|f\|_{L^2(\Omega\times(\omega,2\omega))} + \tilde{K}\right)\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega\times(\omega,2\omega))},$$

where we used the assumption (A4).
Therefore

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega\times(\omega,2\omega))} \leq C. \tag{3.3.11}$$

By multiplying (3.3.10) by $w_{jk_1\dots k_N}$ and summing over $j, k_1, \dots, k_N$ we get

$$\int_\omega^{2\omega}\int_\Omega \left(\frac{\partial u_m}{\partial t} - \Delta u_m\right) u_m dx dt + \int_\omega^{2\omega}\int_\Omega \mathcal{F}(u_m) u_m dx dt = \int_\omega^{2\omega}\int_\Omega f u_m dx dt.$$

Now the first term on the left hand side of the last inequality is zero, again because of the ω-periodicity of u_m in t, and we get

$$\int_\omega^{2\omega}\int_\Omega (\nabla u_m)^2 dx dt \leq \left(\|f\|_{L^2(\Omega\times(\omega,2\omega))} + \tilde{K}\right)\|u_m\|_{L^2(\Omega\times(\omega,2\omega))} \leq$$
$$\leq C\|u_m\|_{L^2(\omega,2\omega;H_1^0(\Omega))}.$$

Therefore we have

$$\|u_m\|_{L^2(\omega,2\omega;H_1^0(\Omega))} \leq C. \tag{3.3.12}$$

Equation (3.3.10) is an algebraic one in $\mathbb{R}^{(2m+1)m^N}$ of the type

$$\mathcal{Z}(U) = \epsilon F, \tag{3.3.13}$$

where

$$U = \{v_{jk_1\dots k_N}, j = -m, ..., m; k_i = 1, ..., m; i = 1, 2, ..., N\}$$
$$F = \left\{\int\int f w_{jk_1\dots k_N}; j = -m, ..., m; k_i = 1, ..., m; i = 1, ..., N\right\}$$
$$\epsilon = 1.$$

Let us vary the value of ϵ from 1 to 0. From the inequalities (3.3.11) and (3.3.12) (which are independent of ϵ) it follows that the equation (3.3.13) has no solution on the boundary of a sufficiently large ball

$$B_\rho(0) = \{U \in \mathbb{R}^{(2m+1)m^N}; \sum_{j,k_i} |v_{jk_1\dots k_N}|^2 \le \rho^2\},$$

where ρ is independent of $\epsilon \in [0,1]$. Thus, we can define the topological degree $\alpha(\mathcal{Z}(.) - \epsilon F, B_\rho(0), 0)$ of the mapping $\mathcal{Z}(.) - \epsilon F$ with respect to $B_\rho(0)$ and the point 0. Since $\mathcal{F}$ is an odd mapping, by assumption (A5) $\mathcal{Z}$ is also odd, hence $\alpha(\mathcal{Z}, B_\rho(0), 0)$ is different from 0. By a homotopy argument with respect to ϵ we conclude that there exists at least one solution u_m of (3.3.10).

The whole sequence $\{u_m, m = 1, 2, ...\}$ is bounded in the space

$$\{u \in L^2(\omega, 2\omega; H_0^1(\Omega)) : u_t \in L^2(\omega, 2\omega; L^2(\Omega))\}$$

It follows from the compact imbedding

$$L^2_\omega(0,\infty; H_0^1(\Omega)) \cap H^1_\omega(0,\infty; L^2(\Omega)) \subset L^2(\Omega; C^0_\omega[0,\infty))$$

and the above that

$$u_m \rightharpoonup u \qquad \text{in } L^2_\omega(0,\infty; H_0^1(\Omega)) \cap H^1_\omega(0,\infty; L^2(\Omega))$$
$$u_m \to u \qquad \text{in } L^2(\Omega; C^0_\omega[0,\infty)).$$

Now because of the assumptions (A1) and (A2) this implies

$$\mathcal{F}(u_m) \to \mathcal{F}(u) \qquad \text{in } L^2(\Omega; C^0_\omega[\omega,\infty)),$$

thus u is a weak periodic solution of our equation for $t \ge \omega$.

□

3.4. An asymptotic result

We consider the model equation (3.1.1) coupled with initial and boundary conditions, where $\mathcal{F}$ is a continuous operator with memory, and f is a given function. Here we do not require $\mathcal{F}$ to be rate independent, but applications to hysteresis are our main concern.

THEOREM 3.6. *Let all the assumptions of Theorem 3.1 be satisfied for any* $T \in (0,\infty)$ *and* $f = 0$ *in* $\mathbb{R}^+ \times \Omega$. *Suppose also that*

$$w_0, \Delta u_0 \in L^2(\Omega),$$

and $\mathcal{F}$ *is piecewise monotonicity preserving (or, more briefly, piecewise monotone).*

Then there exist positive constants C_1, C_2, K_1 such that, for any solution u of (3.1.1) with zero Dirichlet boundary data, we have

$$\int_\Omega \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 \right\}(x,t)dx \le C_1 e^{-C_2 t}. \tag{3.4.1}$$

This then implies that

$$u_\infty = \lim_{t\to\infty} u(.,t) \tag{3.4.2}$$

exists and that the following estimate holds:

$$\|u_\infty - u(.,t)\|_{L^1} \le \frac{K_1}{C_2} e^{-C_2 t}. \tag{3.4.3}$$

PROOF. By Theorem 3.1 we know that there exists a unique solution of (3.1.1) such that

$$u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T;V),$$
$$\mathcal{F}(u) \in L^2(\Omega; C^0([0,T])),$$

for any $T \in [0,\infty)$. Combining results of Proposition X.1.4 and Proposition IX.1.2 in [7], we have the following regularity of the solution :

$$u \in H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)),$$
$$\mathcal{F}(u) \in H^1(0,T;L^2(\Omega)),$$

for any $T \in [0,\infty)$.

We can now formally differentiate the equation (3.1.1) with respect to t and get

$$\frac{\partial^2 u}{\partial t^2} - \Delta\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}(\mathcal{F}(u)) = 0. \tag{3.4.4}$$

Now we do the following things: We multiply (3.4.4) by $\frac{\partial u}{\partial t}$ and get after integration over Ω:

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx\right] + \int_\Omega \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \le 0, \tag{3.4.5}$$

where we used the piecewise monotonicity property of the hysteresis operator. Let L be the Lipschitz constant for $\mathcal{F}$, and let K denote a constant which will be specified later. Choose $\alpha > \frac{L^2 K}{4}$, multiply (3.4.5) by α and get

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\Omega \alpha\left(\frac{\partial u}{\partial t}\right)^2 dx\right] + \int_\Omega \alpha\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \le 0. \tag{3.4.6}$$

We now multiply (3.4.4) by $\frac{\partial^2 u}{\partial t^2}$ and again integrate over Ω:

$$\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}\right)^2 dx + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \le \int_\Omega \left|\frac{\partial}{\partial t}\mathcal{F}(u)\right|\left|\frac{\partial^2 u}{\partial t^2}\right| dx \le$$
$$\le L\int_\Omega \left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial^2 u}{\partial t^2}\right| dx \le \int_\Omega \left(\frac{\partial^2 u}{\partial t^2}\right)^2 dx + \frac{L^2}{4}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx,$$

where we used the piecewise Lipschitz continuity of the hysteresis operator with Lipschitz constant L. The last inequality gives us:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \leq \frac{L^2}{4}\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2 dx. \tag{3.4.7}$$

Adding (3.4.6) and (3.4.7) gives us:

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_{\Omega}\left(\alpha\left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right)dx\right] \leq$$

$$\leq -\int_{\Omega}\left(\alpha\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 - \frac{L^2}{4}\left|\frac{\partial u}{\partial t}\right|^2\right)dx.$$

Using the equivalent norm in $H_0^1(\Omega)$, we have the following estimate for some constant K:

$$-\alpha\left\|\nabla\left(\frac{\partial u}{\partial t}\right)\right\|^2_{L^2(\Omega)} \leq -\frac{\alpha}{K}\left\|\nabla\left(\frac{\partial u}{\partial t}\right)\right\|^2_{L^2(\Omega)} - \frac{\alpha}{K}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}.$$

So we get all together:

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\int_{\Omega}\left\{\alpha\left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right\}dx\right\} \leq$$

$$\leq -\int_{\Omega}\left\{\frac{\alpha}{K}\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 + \left(\frac{\alpha}{K} - \frac{L^2}{4}\right)\left|\frac{\partial u}{\partial t}\right|^2\right\}dx \leq$$

$$\leq -\min\left\{\frac{\alpha}{K}, \frac{\left(\frac{\alpha}{K} - \frac{L^2}{4}\right)}{\alpha}\right\}\int_{\Omega}\left\{\alpha\left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right\}dx.$$

Note that $\frac{\alpha}{K} - \frac{L^2}{4} > 0$, because of our condition on α. Therefore Gronwall's lemma implies that:

$$\int_{\Omega}\left\{\alpha\left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right\}(x,t)dx \leq$$

$$\leq e^{-2Ct}\int_{\Omega}\left\{\alpha\left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right\}(x,0)dx.$$

The estimate (3.4.1) now follows.

To show (3.4.2), note first that (3.4.1) implies

$$\int_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2 dx \leq C_1 e^{-C_2 t}$$

and using Hölder's inequality we also have, since Ω is bounded,

$$\int_{\Omega}\left|\frac{\partial u}{\partial t}\right| dx \leq K_1 e^{-C_2 t}. \tag{3.4.8}$$

(3.4.8) implies that

$$\left|\frac{\partial u}{\partial t}\right| \in L^1\left(0,\infty; L^1(\Omega)\right),$$

and also

$$\left|\frac{\partial}{\partial t}\left(\int_{\Omega} u(x,t)dx\right)\right| \in L^1(0,\infty).$$

Therefore $\lim_{t\to\infty} \int_{\Omega} u(x,t)dx$ exists.

It also follows from (3.4.8) that for $t < s$

$$\int_{\Omega} |u(x,s) - u(x,t)|dx \leq \frac{K_1}{C_2}\left(e^{-C_2 t} - e^{-C_2 s}\right). \tag{3.4.9}$$

Hence the system $\{u(.,t)\}_{t>0}$ is fundamental in $L^1(\Omega)$, which is a complete space. Therefore we can conclude that
$u_\infty = \lim_{t\to\infty} u(.,t)$ exists and it also follows from (3.4.9) that

$$\|u_\infty - u(.,t)\|_{L^1} \leq \frac{K_1}{C_2} e^{-C_2 t}. \tag{3.4.10}$$

□

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CHAPTER 4

Entropy condition for a quasilinear hyperbolic equation with hysteresis

4.1. Introduction

In this chapter we study a hyperbolic equation of first order of the form

$$u_t + [\phi(u)]_x = 0, \qquad u(0) = u_0 \tag{4.1.1}$$

and the corresponding quasilinear hyperbolic equation with hysteresis

$$\frac{\partial}{\partial t}(u + w) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu = f, \tag{4.1.2}$$

where $w = \mathcal{F}(u)$ represents hysteresis.

It is well known that even for ϕ and u_0 smooth, (4.1.1) can have more than one solution. We present a criterion - a so called entropy condition - which selects a unique solution. We first explain the results of Kružkov [3], whose condition is equivalent to the so called E condition introduced by Olejnik [4], which in turn extends the classical entropy condition. In the first section we give an overview of their results and a discussion aimed a providing a feeling for the type of problems we wish to study.

It was expected, see [6], that the integral solution of (4.1.2), (for definition, see the end of section 2.3), for which existence was proved in [6] using the semigroup approach, and which is unique by construction, fulfils a condition of the type introduced by Kružkov. To derive such an entropy condition for the integral solution of (4.1.2) was posed as an open problem in Visintin's book and we present a solution to this problem in the last section.

4.2. Entropy conditions and uniqueness of solutions for a hyperbolic equation of first order

We will study the equation

$$u_t + \sum_{i}^{N} (\phi_i(u))_{x_i} = 0, \qquad \text{for } t > 0, x \in \mathbb{R}^N, \tag{4.2.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^N$ and denote by $\phi = (\phi_1, ..., \phi_N) : \mathbb{R} \to \mathbb{R}^N$ a continuous function with $\phi(0) = 0$.
Consider first the case $N = 1$.

If ϕ is a smooth function, we can rewrite (4.2.1) as

$$u_t + \phi'(u) u_x = 0$$

Consider the characteristics, defined by the following equation: (for simplicity, the projections of characteristics on the (x, t) plane are still called characteristics)

$$\frac{dx}{dt} = \phi'(u).$$

If $u(x, t)$ solves (4.2.1), then along a characteristic

$$\frac{d}{dt}u(x(t), t) = u_x\frac{dx}{dt} + u_t = u_x\phi'(u) + u_t = 0,$$

so u is constant along characteristics and it follows that characteristics have constant slope. In other words, the characteristics are straight lines with parametric velocity $\phi'(u)$ along these lines.

Assume now for convenience that $\phi''(u) > 0$. If $u(x, 0) = u_0(x)$ and $u_0(x)$ is decreasing - then there are points $x_1 < x_2$ with $\phi'(u_0(x_1)) > \phi'(u_0(x_2))$ - and the characteristics starting at $(x_1, 0)$ and $(x_2, 0)$ will intersect at a point P for $t > 0$, see Figure 4.1.

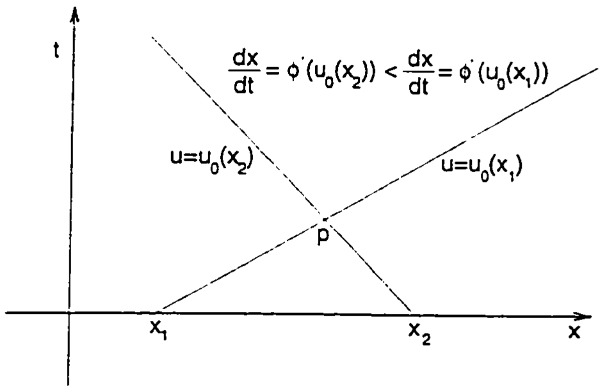

FIGURE 4.1. Characteristics intersect.

At the point P a continuous solution is overdetermined, since different characteristics meet there and each carries a different value of u. It turns out that the solution must be discontinuous. (We also can easily see that when $\phi''(u) > 0$, $u(x, t)$ is globally defined and continuous if and only if $u_0(x)$ is nondecreasing and continuous).

The above conclusion is independent of the smoothness properties of ϕ and $u_0(x)$. No matter how smooth the initial data, the solution may still have discontinuities. This is the most important feature of quasilinesr hyperbolic equations and an essential difference from linear hyperbolic equations. It is this phenomenon that leads to special difficulties.

For the reasons given above, we shall generalize the notion of solution for equations of the form (4.2.1):

DEFINITION 4.1. A bounded, measurable function $u(x, t)$ is called a weak solution of the initial value problem (4.2.1) with the initial condition $u(x, 0) = u_0(x)$ with bounded and measurable initial data $u_0(x)$, provided that

$$\int_0^T \int_{\mathbb{R}^N} (uf_t + \sum_i^N \phi_i(u)f_{x_i})dxdt + \int_{\mathbb{R}^N} u_0 f dx = 0 \qquad (4.2.2)$$

holds for all $f \in C_0^1((0, \infty) \times \mathbb{R}^N)$.

Note that if (4.2.2) holds for all $f \in C_0^1((0,\infty)\times\mathbb{R}^N)$, and if u is in $C^1((0,T)\times\mathbb{R}^N)$, then u is a classical solution (this is easy to see, using integration by parts).

In our effort to solve initial value problems which are not solvable classically, we are led to extend the class of solutions. In doing this, we run the risk of losing uniqueness. That this concern is well-founded follows from the next example.

EXAMPLE 4.1. (see [5]): Consider the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with the initial condition

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0. \end{cases}$$

For each $\alpha \geq 1$, this problem has a solution u_α defined by

$$u_\alpha(x,t) = \begin{cases} 1 & \text{if } 2x < (1-\alpha)t \\ -\alpha & \text{if } (1-\alpha)t < 2x < 0 \\ \alpha & \text{if } 0 < 2x < (\alpha-1)t \\ -1 & \text{if } (\alpha-1)t < 2x. \end{cases}$$

Thus our problem has a continuum of solutions (see Figure 4.2).

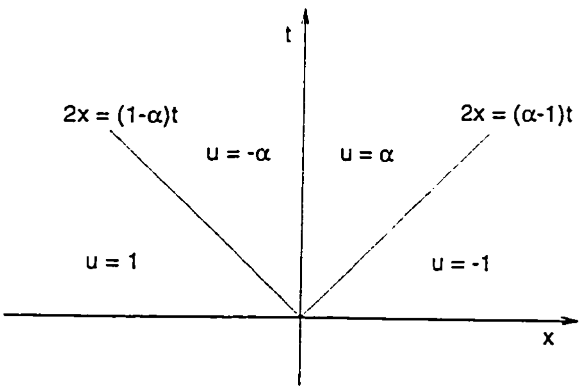

FIGURE 4.2. Continuum of solutions in Example 4.1.

Equations of the above form arise in the physical sciences and so we must have some mechanism to pick out the "physically relevant" solution. Thus, we are led to impose an a-priori condition on solutions which distinguishes the "correct" one from the others.

In the case of the equation when $N = 1$

$$u_t + [\phi(u)]_x = 0, \qquad \text{with } \phi'' > 0,$$

there is a unique solution which satisfies the "entropy" condition

$$\frac{u(x+a,t) - u(x,t)}{a} \leq \frac{E}{t} \qquad \forall a > 0, \forall t > 0, \tag{4.2.3}$$

where E is independent of x, t and a.

This condition implies that if we fix $t > 0$ and let x increase from $-\infty$ to $+\infty$, then we can only jump down, as we cross a discontinuity - hence the reason for the word "entropy".

If we return to the previous example, then we see that (4.2.3) is satisfied only when $\alpha = 0$.

It is worth noting that as the above shows, the loss of uniqueness occurs in the class of piecewise smooth functions. Uniqueness was lost not because we admitted as solutions functions which were unnecessarily "wild" from a regularity point of view; it is deeper than that. The "good" and "bad" shocks are indistinguishable from the point of view of regularity.

Condition (4.2.3) is a bit strange when it is encountered for the first time. Assume that $\phi''(u) > 0$. Since the characteristics are straight lines, if (x,t) is any point with $t > 0$, we let $y(x,t)$ denote the unique point on the x-axis which lies on the characteristic through (x,t). Since u is constant along characteristics, and $t\phi'(u) = x - y$, we see that u must be given implicitly by

$$u(x,t) = u_0(y(x,t)) = u_0(x - t\phi'(u(x,t))).$$

Now if u_0 is a differentiable function, then we can invoke the implicit function theorem and solve this last equation for u, provided that t is sufficiently small. We find

$$u_x(x,t) = \frac{u_0'(y)}{1 + u_0'(y)\phi''(u)t}.$$

Thus if $u_0'(y) = 0$, then $u_x(x,t) = 0$, and if $u_0' \geq 0$, then

$$u_x \leq \frac{u_0'}{u_0'\phi''(u)t} = \frac{1}{\phi''t} \leq \frac{E}{t},$$

where $E = \frac{1}{\mu}$, $\mu = \inf \phi''$, so (4.2.3) is quite natural.

So far we considered only the case $\phi''(u) > 0$. O.A.Olejnik [4] gives a uniqueness condition for (4.2.1), $N = 1$, now called an E condition, without any restriction on $\phi \in C^1$ as follows:

We consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial \phi(u,x,t)}{\partial x} = 0. \tag{4.2.4}$$

We introduce the notation

$$u(x+0,t) = u_+(x,t)$$
$$u(x-0,t) = u_-(x,t)$$
$$l(u) = \frac{\phi(u_+,x,t) - \phi(u_-,x,t)}{u_+ - u_-}(u - u_+) + \phi(u_+,x,t)).$$

Consider the straight line $w = l(u)$ in the uw-plane, which joints the points $(u_+, \phi(u_+,x,t))$ and $(u_-, \phi(u_-,x,t))$. We shall say that the generalized solution $u(x,t)$ of (4.2.4) satisfies condition E if at all points of discontinuity of $u(x,t)$ (except possibly a finite number of them), the following condition is satisfied: when $u_+ > u_-$, $l(u) \leq \phi(u,x,t)$ for all u in $[u_-, u_+]$, while when $u_+ < u_-$, $l(u) \geq \phi(u,x,t)$ for all u in $[u_+, u_-]$.

It is easy to see that if the function $\phi(u,x,t)$ is such that $\phi_{uu} \neq 0$, then condition E is identical with (4.2.3), namely $u_+ < u_-$ if $\phi_{uu} > 0$, and $u_+ > u_-$ if $\phi_{uu} < 0$.

We have the following

THEOREM 4.1. *(Olejnik [4]): A weak solution of (4.2.4) with $u(x,0) = u_0(x)$, which satisfies the condition E, is unique.*

A different approach to the question of existence of a unique solution of (4.2.4), $N \geq 1$, was given by Kružkov [3]. He defines a generalized solution of (4.2.4) as follows:

DEFINITION 4.2. A bounded measurable function $u(x,t)$ is called a generalized solution of (4.2.4) with $u(x,0) = u_0(x)$ in $Q_T = [0,T] \times \mathbb{R}^N$ if

1) for any constant k and any smooth function $f(x,t) \geq 0$ the following inequality holds:

$$\iint_{Q_T} \Big\{ |u(x,t) - k| f_t + \\ + [\text{sign}\ (u(x,t) - k)] \sum_{i=1}^{N} [\phi_i(u(x,t), x, t) - \phi_i(k, x, t)] f_{x_i} \Big\} \, dx dt \geq 0; \quad (4.2.5)$$

2) there exists a set $\mathcal{E}$ of measure zero on $[0,T]$ such that for $t \in [0,T] \backslash \mathcal{E}$ the function $u(x,t)$ is defined almost everywhere in $\mathbb{R}^N$, and for any ball

$$K_r = \{|x| \leq r\} \subset \mathbb{R}^N$$

$$\lim_{\substack{t \to 0 \\ t \in [0,T] \backslash \mathcal{E}}} \int_{K_r} |u(x,t) - u_0(x)| dx = 0.$$

Since the smooth function $f \geq 0$ is arbitrary, it is obvious that inequality (4.2.5) for $k = \pm \sup |u(x,t)|$ implies (4.2.4). But Definition 4.2 also contains a condition which characterizes the permissible discontinuities of solutions. This condition is expecially easy to visualize when the generalized solution is a piecewise smooth function in some neighborhood of the point of discontinuity; in this case, using integration by parts and the fact that f was chosen arbitrarily, we obtain from (4.2.5) that for any constant k, along the surface of discontinuity we have

$$\begin{aligned} &|u_+ - k| \cos(\nu, t) + \text{sign}\ (u_+ - k)[\phi(u_+, x, t) - \phi(k, x, t)] \cos(\nu, x) \leq \\ \leq &|u_- - k| \cos(\nu, t) + \text{sign}\ (u_- - k)[\phi(u_-, x, t) - \phi(k, x, t)] \cos(\nu, x), \end{aligned} \quad (4.2.6)$$

where ν is the normal vector to the surface of discontinuity at the point (x,t) and u_+, u_- are the one-sided limits of the generalized solution at the point (x,t) from the positive and negative side of the surface of discontinuity, respectively. It can be shown that in the case $N = 1$ (4.2.6) is equivalent to condition E introduced above (we just need to express $\cos(\nu,t)$, $\cos(\nu,x)$ by using (4.2.4) and choose $k = u \in [u_-, u_+]$).

Kružkov shows that there exists a unique generalized solution of (4.2.4) in the sense of Definition 4.2.

Inspired by the results of Kružkov, Crandall in his paper [1] treats the Cauchy problem for the equation

$$u_t + \sum_{i=1}^{N} (\phi_i(u))_{x_i} = 0, \qquad t > 0, \quad x \in \mathbb{R}^N$$

from the point of view of semigroups of nonlinear transformations.

The following notation will be used whenever it is meaningful:

$$\phi = (\phi_1, ..., \phi_N) : \mathbb{R} \to \mathbb{R}^N \tag{4.2.7}$$

$$[\phi(v)]_x = \sum_{i=1}^{N} (\phi_i(v(x)))_{x_i} \qquad \text{if } v : \mathbb{R}^N \to \mathbb{R} \tag{4.2.8}$$

$$f_x = (f_{x_1}, ... f_{x_N}) \qquad \text{if } f : \mathbb{R}^N \to \mathbb{R} \tag{4.2.9}$$

$$ab = \sum_{i=1}^{N} a_i b_i \qquad \text{if } a, b \in \mathbb{R}^N. \tag{4.2.10}$$

Given $\phi_i(v)$ with $\phi_i(0) = 0$, $i = 1, ..., N$, he defines

$$Av = \sum_{i=1}^{N} (\phi_i(v))_{x_i}, \qquad v \in D(A)$$

as the closure of A_0 given in the next definition.

DEFINITION 4.3. A_0 is the operator in $L^1(\mathbb{R}^N)$ defined by: $v \in D(A_0)$ and $w \in A_0(v)$ if $v, w \in L^1(\mathbb{R}^N)$, $\phi(v) \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} [\text{sign } (v(x) - k)]\{(\phi(v(x)) - \phi(k))f_x(x) + w(x)f(x)\}dx \geq 0 \tag{4.2.11}$$

for every $k \in \mathbb{R}$, and every $f \in C_0^\infty(\mathbb{R}^N)$ such that $f \geq 0$.

LEMMA 4.2. *(Crandall, [1]) Let $\phi \in C^1$ and A_0 be as given by Definition 4.3. If $v \in C_0^1(\mathbb{R}^N)$, then $v \in D(A_0)$ and $A_0 v = \{[\phi(v)]_x\}$.*

PROOF. Let $v \in C_0^1(\mathbb{R}^N)$. Then since $\phi(0) = 0$, $\phi(v) \in C_0^1(\mathbb{R}^N) \subseteq L^1(\mathbb{R}^N)$. For $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ and $f \in C_0^\infty(\mathbb{R}^N)$, integration by parts shows that

$$\int_{\mathbb{R}^N} (\Phi'(v)[\phi(v)]_x) f dx = \int_{\mathbb{R}^N} \left(\int_k^{v(x)} \Phi'(s)\phi'(s)ds \right)_x f(x)dx =$$

$$= - \int_{\mathbb{R}^N} \left(\int_k^{v(x)} \Phi'(s)\phi'(s)ds \right) f_x(x)dx.$$

Next choose $\Phi(s) = \Phi_p(s - k)$, where

$$\Phi_p(s) = \begin{cases} -s & \text{if } s \leq -\frac{1}{p} \\ \left(\frac{1}{2}\right) s^2 + \frac{1}{2} l & \text{if } |s| \leq \frac{1}{p} \\ s & \text{if } s \geq \frac{1}{p} \end{cases}$$

and let $p \to \infty$ to obtain

$$\int_{\mathbb{R}^N} [\text{sign}(v(x) - k)]\{(\phi(v(x)) - \phi(k))f_x(x) + [\phi(v)]_x f(x)\}dx = 0.$$

This shows $v \in D(A_0)$ and $[\phi(v)]_x \in A_0 v$. Finally, asssume $v \in D(A_0) \cap L^\infty(\mathbb{R}^N)$ and $w \in A_0 v$. Then using successively $k = \|v\|_\infty + 1$ and $k = -(\|v\|_\infty + 1)$ in (4.2.11) shows that $w = [\phi(v)]_x$ in the sense of distributions. Hence $A_0 v = \{w\}$, that is A_0 is single-valued on bounded functions. The proof is complete. □

The lemma shows that A extends A_0 from $C_0^1(\mathbb{R}^N)$. Crandall then shows that $A =$ the closure of A_0 is an m-accretive operator, thus generates a semigroup of contractions $S(t)$, and $S(t)u_0$ is the (unique) integral solution of (4.2.4). Then he shows that this solution constructed by the method of semigroups satisfies indeed the entropy condition introduced by Kružkov:

THEOREM 4.3. *(Crandall) Let S be the semigroup of contractions generated by A. Let u, $v \in \overline{D(A)}$ and $t \geq 0$. Then if $v \in L^\infty(\mathbb{R}^N)$*

$$\int_0^T \int_{\mathbb{R}^N} \{|S(t)v(x) - k|f_t + [sign(S(t)v(x) - k)][\phi(S(t)v(x)) - \phi(k)]f_x\}dxdt \geq 0.$$

for every $f(x,t) \in C_0^\infty((0,T) \times \mathbb{R}^N)$ such that $f \geq 0$ and every $k \in \mathbb{R}$ and $T > 0$.

PROOF. Let $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u_\epsilon(t)$ satisfy

$$\begin{aligned} \epsilon^{-1}(u_\epsilon(t) - u_\epsilon(t-\epsilon)) + A_0 u_\epsilon(t) &= 0 \qquad && t \geq 0 \\ u_\epsilon(t) &= v && t < 0. \end{aligned}$$

Let $u_\epsilon(x,t) = u_\epsilon(t)(x)$. By the definition of A_0:

$$\begin{aligned} \int_{\mathbb{R}^N} \{&\text{sign } (u_\epsilon(x,t) - k)(\phi(u_\epsilon(x,t)) - \phi(k))f_x(x,t) + \\ &+ \epsilon^{-1}[\text{sign } (u_\epsilon(x,t) - k)](u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t))f(x,t)\}dx \geq 0 \end{aligned} \tag{4.2.12}$$

for every $k \in \mathbb{R}$ and non-negative $f \in C_0^\infty((0,T) \times \mathbb{R}^N)$.
Let $h_\epsilon(x,t) = [\text{sign } (u_\epsilon(x,t) - k)](u_\epsilon(x,t) - k) = |u_\epsilon(x,t) - k|$. Notice that

$$\begin{aligned} &(u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t))[\text{sign } (u_\epsilon(x,t) - k)] = \\ &= (u_\epsilon(x,t-\epsilon) - k)[\text{sign}(u_\epsilon(x,t) - k)] - (u_\epsilon(x,t) - k)[\text{sign}(u_\epsilon(x,t) - k)] \leq \\ &\leq h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t). \end{aligned} \tag{4.2.13}$$

Using (4.2.13) and integrating (4.2.12) over $0 \leq t \leq T$ yields

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \{&[\text{sign } (u_\epsilon(x,t) - k)](\phi(u_\epsilon(x,t)) - \phi(k))f_x(x,t) + \\ &+\epsilon^{-1}(h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t))f(x,t)\}dxdt \geq 0. \end{aligned} \tag{4.2.14}$$

Now

$$\begin{aligned} &\epsilon^{-1} \int_0^T \int_{\mathbb{R}^N} \{(h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t))f(x,t)\}dxdt = \\ &= \epsilon^{-1}\left(\int_0^\epsilon \int_{\mathbb{R}^N} h_\epsilon(x,t-\epsilon)f(x,t)dxdt - \int_{T-\epsilon}^T \int_{\mathbb{R}^N} h_\epsilon(x,t)f(x,t)dxdt\right) + \\ &\qquad + \int_0^{T-\epsilon} \int_{\mathbb{R}^N} h_\epsilon(x,t)(\epsilon^{-1})(f(x,t+\epsilon) - f(x,t))dxdt \end{aligned}$$

The first and the second integrals vanish for ϵ small enough since f is in $C_0^\infty((0,T) \times \mathbb{R}^N)$. The convergence $u_\epsilon(x,t) \to S(t)v(x)$ in $L^1(\mathbb{R}^N)$, uniformly in t as $\epsilon \to 0$, implies that the third term tends to

$$\int_0^T \int_{\mathbb{R}^N} |S(t)v(x) - k|f_t(x,t)dxdt$$

as $\epsilon \downarrow 0$. So the Theorem follows letting $\epsilon \downarrow 0$ in (4.2.14). □

4.3. Quasilinear hyperbolic equation with hysteresis

Let b_j and c be given smooth functions. In this section we consider the equation

$$\frac{\partial}{\partial t}(u+w) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu = f \qquad \text{in } Q$$

and couple it with the hysteresis relation

$$w(x,t) = [\mathcal{E}(u(x,.), w_0(x)](t) \qquad \text{in } [0,T], \text{ a.e. in } \Omega,$$

where $\mathcal{E}$ is a generalized play operator, as considered in Section 2.3. This system is formally equivalent to

$$\begin{cases} \frac{\partial u}{\partial t} + \xi + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu = f & \text{in } Q \\ \frac{\partial w}{\partial t} - \xi = 0 & \text{in } Q \\ \xi \in \phi(u,w) & \text{in } Q, \end{cases} \tag{4.3.1}$$

where $\phi(u,w)$ was defined in Section 2.3. To simplify the discussion, we assume that $\{b_j \in C^1(\overline{\Omega})\}_{j=1,\dots,N}$, $\sum_{j=1}^{N} b_j \nu_j = 0$ a.e. on $\partial\Omega$ and $c \in L^\infty(\Omega)$, where $\overrightarrow{\nu}$ denotes a field normal to $\partial\Omega$.

By introducing the following operators

$$\begin{aligned} D(A) &: = \{U := (u,w) \in \mathbb{R}^2 : \inf \gamma_r(u) \le w \le \sup \gamma_l(u)\} \\ A(U) &: = \{(\xi, -\xi) : \xi \in \phi(U) \cap \mathbb{R}\} \qquad \forall U \in D(A) \\ B(u) &: = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu \\ R(U) &: = (B(u), 0) \end{aligned}$$

and by setting

$$U := (u,w), \qquad U_0 := (u_0, w_0), \qquad F := (f, 0)$$

the Cauchy problem for the system (4.3.1) can be written in the form

$$\begin{gathered} \frac{\partial U}{\partial t} + A(U) + R(U) \ni F \qquad \text{in } Q \\ U(0) = U_0. \end{gathered} \tag{4.3.2}$$

This approach can be easily extended to the case in which $\mathcal{E}$ is replaced by a generalized Prandtl-Ishlinskii operator of play type.

Then we have the following theorem (stated in Visintin [6]):

THEOREM 4.4. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \ge 1)$ of Lipschitz class. Let $L^1(\Omega;\mathbb{R}^2)$ be endowed with the norm*

$$\|U\|_{L^1(\Omega;\mathbb{R}^2)} : = \int_\Omega (|u(x)| + |w(x)|)\, dx \qquad \forall U : = (u,w) \in L^1(\Omega;\mathbb{R}^2).$$

Define the operator R as

$$R(U) := (Bu, 0) \qquad \forall U \in D(R) := \{U \in L^1(\Omega;\mathbb{R}^2) : Bu \in L^1(\Omega)\},$$

A is defined for

$$\begin{cases} \gamma_l, \gamma_r \text{ maximal monotone (possibly multivalued) functions:} \\ \mathbb{R} \to P(\tilde{\mathbb{R}}), \text{ such that } \inf \gamma_r(u) \le \sup \gamma_l(u) \quad \forall u \in \mathbb{R}. \end{cases}$$

$$\tilde{\mathbb{R}} := [-\infty, +\infty].$$

Also assume that γ_l, γ_r *are affinely bounded, that is, there exist constants* $C_1, C_2 > 0$, *such that* $\forall v \in \mathbb{R}, \forall z \in \gamma_h(v)$

$$\|z\| \le C_1 \|v\| + C_2 \quad (h = l, r)$$

and for any $(u, w) \in \mathbb{R}^2$ *we have*

$$\phi(u, w) = \begin{cases} +\infty & \text{if } w < \inf \gamma_r(u) \\ \tilde{\mathbb{R}}^+ & \text{if } w \in \gamma_r(u) \backslash \gamma_l(u) \\ \{0\} & \text{if } \sup \gamma_r(u) < w < \inf \gamma_l(u) \\ \tilde{\mathbb{R}}^- & \text{if } w \in \gamma_l(u) \backslash \gamma_r(u) \\ -\infty & \text{if } w > \sup \gamma_l(u) \\ \tilde{\mathbb{R}} & \text{if } w \in \gamma_l(u) \cap \gamma_r(u). \end{cases} \tag{4.3.3}$$

Take any $U_0 := (u_0, w_0) \in L^1(\Omega; \mathbb{R}^2)$, *such that* $U_0 \in D(A)$ *a.e. in* Ω, *and any* $f \in L^1(\Omega \times (0,T))$. *Then the Cauchy problem (4.3.2) has one and only one integral solution* $U : [0,T] \to L^1(\Omega, \mathbb{R}^2)$, *(see definition 2.2) which depends continuously on the data* u_0, w_0, f. *Moreover, if* $f \in BV(0,T; L^1(\Omega))$ *and* $Ru_0 \in L^1(\Omega)$, *then* U *is Lipschitz continuous.*

A similar statement is true for a generalized Prandtl-Ishlinskii operator of play type. It was conjectured (see [6]), that the integral solution from Theorem 4.4 fulfils a condition of the type introduced by Kružkov. The next Theorem establishes this in a precise form.

THEOREM 4.5. *Let the assumptions of Theorem 4.4 hold and let* $F = (f, 0) = (0, 0)$. *Assume also that the hysteresis operator is symmetric around* $w = u$. *Let* $A_0 U = A(U) + R(U)$ *on* $D(A_0)$, *and let* $S(t) = (S_1(t), S_2(t))$ *be the corresponding semigroup of contractions.*

Let $v \in \overline{D(A)}$ *and* $t \ge 0$. *Then if* $v = (v_1, v_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$:

$$\int_0^T \int_\Omega |S_1(t)v_1(x) - k| f_t(x,t) dx dt + \int_0^T \int_\Omega |S_2(t)v_2(x) - k| f_t(x,t) dx dt +$$
$$+ \int_0^T \int_\Omega \left\{ \sum_{j=1}^N b_j |S_1(t)v_1(x) - k| \frac{\partial}{\partial x_j} f(x,t) - c|S_1(t)v_1(x) - k| f(x,t) - \right.$$
$$\left. - [sign(S_1(t)v_1(x) - k)] k \left(\sum_{j=1}^N \frac{\partial}{\partial x_j} b_j + c \right) f(x,t) \right\} dx dt \ge 0$$

for every $f(x,t) \in C_0^\infty((0,T) \times \Omega)$ *such that* $f \ge 0$ *and every* $k \in \mathbb{R}$ *and* $T > 0$.

REMARK 4.1. The hysteresis operator can be also discontinuous, e.g. the relay operator, as long as the boundary curves γ_l, γ_r satisfy the symmetry relation given above.

PROOF. Let $v \in \overline{D(A_0)} \cap L^\infty(\Omega, \mathbb{R}^2)$ and $u_\epsilon(t)$, $w_\epsilon(t)$ satisfy:

$$\left.\begin{array}{r}\frac{u_\epsilon(t)-u_\epsilon(t-\epsilon)}{\epsilon} + \xi + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u_\epsilon(t)) + c u_\epsilon(t) = 0 \\ \frac{w_\epsilon(t)-w_\epsilon(t-\epsilon)}{\epsilon} - \xi = 0\end{array}\right\} \text{ for } t \geq 0, \tag{4.3.4}$$

$$\begin{pmatrix} u_\epsilon(t) \\ w_\epsilon(t) \end{pmatrix} = v \qquad \text{for } t < 0. \tag{4.3.5}$$

If $k \in \mathbb{R}$ is any constant, then we have

$$-\left(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j k + ck\right) = -k\left(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j + c\right) \tag{4.3.6}$$

We get from the second equation in (4.3.4) that

$$\xi = \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon},$$

which we can put into the first equation in (4.3.4). Adding the resulting equation with (4.3.6) gives us

$$\frac{u_\epsilon(t) - u_\epsilon(t-\epsilon)}{\epsilon} + \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon} + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}[b_j(u_\epsilon(t) - k)] +$$
$$+ [c(u_\epsilon(t) - k)] + k\left(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j + c\right) = 0.$$

Let $u_\epsilon(x,t) = u_\epsilon(t)(x)$ and $w_\epsilon(x,t) = w_\epsilon(t)(x)$. Multiply the last equation by $[\text{sign}(u_\epsilon(x,t) - k)]$ and integrate over Ω to get:

$$\int_\Omega \Bigg\{ [\text{sign}(u_\epsilon(x,t) - k)] \left[\frac{u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t)}{\epsilon} + \frac{w_\epsilon(x,t-\epsilon) - w_\epsilon(x,t)}{\epsilon}\right] -$$
$$- [\text{sign}(u_\epsilon(x,t) - k)] \Bigg[\sum_{j=1}^{N} \frac{\partial}{\partial x_j}[b_j(u_\epsilon(x,t) - k)] + [c(u_\epsilon(x,t) - k)] +$$
$$+ k\left(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j + c\right)\Bigg]\Bigg\} dx = 0$$

As before, let $h_\epsilon(x,t) = (u_\epsilon(x,t) - k)[\text{sign } (u_\epsilon(x,t) - k)] = |u_\epsilon(x,t) - k|$. Recall that

$$(u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t))[\text{sign } (u_\epsilon(x,t) - k)] =$$
$$= (u_\epsilon(x,t-\epsilon) - k)[\text{sign } (u_\epsilon(x,t) - k)] - (u_\epsilon(x,t) - k)[\text{sign } (u_\epsilon(x,t) - k)] \leq$$
$$\leq h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t). \tag{4.3.7}$$

Also

$$(w_\epsilon(x,t-\epsilon) - w_\epsilon(x,t))[\text{sign } (u_\epsilon(x,t) - k)] \leq$$
$$\leq (w_\epsilon(x,t-\epsilon) - w_\epsilon(x,t))[\text{sign } (w_\epsilon(x,t) - k)]. \tag{4.3.8}$$

This last inequality is true because of the following: The only way it could fail would be if either:

$\text{sign}(u_\epsilon(x,t)-k)=1, \text{sign}(w_\epsilon(x,t)-k)=-1$, and $w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t)>0$, so
$u_\epsilon(x,t)>k$, $w_\epsilon(x,t)<k$ and $w_\epsilon(x,t-\epsilon)>w_\epsilon(x,t)$

or

$\text{sign}(u_\epsilon(x,t)-k)=-1, \text{sign}(w_\epsilon(x,t)-k)=1$, and $w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t)<0$, so
$u_\epsilon(x,t)<k$, $w_\epsilon(x,t)>k$, and $w_\epsilon(x,t-\epsilon)<w_\epsilon(x,t)$

It can be easily seen from Figures 4.3a and 4.3b that these situations are not possible because of the properties of the hysteresis operator; thus (4.3.8) must be true.

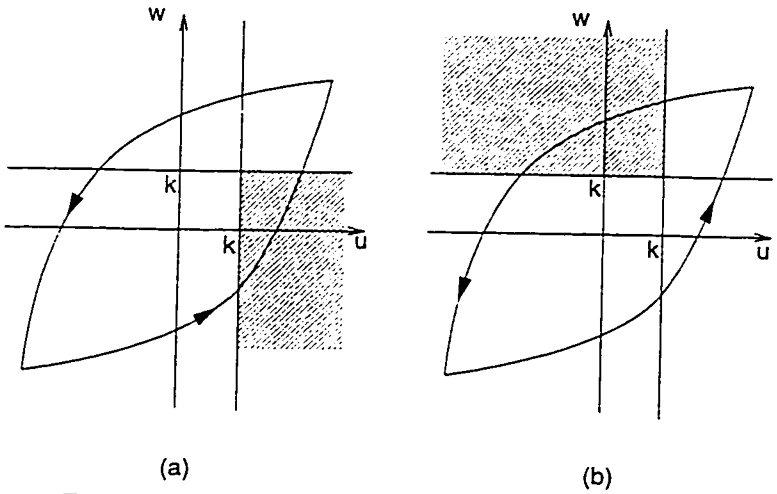

FIGURE 4.3. Possible situations in the proof of (4.3.8).

If we introduce the notation $g_\epsilon(x,t)=(w_\epsilon(x,t)-k)[\text{sign}\,(w_\epsilon(x,t)-k)]=|w_\epsilon(x,t)-k|$, then, similarly, we get

$$\begin{aligned}(w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t))[\text{sign}\,(u_\epsilon(x,t)-k)] &\le \\ &\le (w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t))[\text{sign}\,(w_\epsilon(x,t)-k)] = \\ = (w_\epsilon(x,t-\epsilon)-k)[\text{sign}\,(w_\epsilon(x,t)-k)] &- (w_\epsilon(x,t)-k)[\text{sign}\,(w_\epsilon(x,t)-k)] \le \\ &\le g_\epsilon(x,t-\epsilon)-g_\epsilon(x,t). \quad (4.3.9)\end{aligned}$$

Therefore

$$\int_\Omega \Bigg\{ \left[\frac{(h_\epsilon(x,t-\epsilon)-h_\epsilon(x,t))}{\epsilon} + \frac{(g_\epsilon(x,t-\epsilon)-g_\epsilon(x,t))}{\epsilon} \right] - \\ - [\text{sign}(u_\epsilon(x,t)-k)] \left[\sum_{j=1}^N \frac{\partial}{\partial x_j}[b_j(u_\epsilon(x,t)-k)] + [c(u_\epsilon(x,t)-k)] + \\ +k\left(\sum_{j=1}^N \frac{\partial}{\partial x_j} b_j + c \right) \right] \Bigg\} dx \ge 0.$$

Now we can multiply by any $f \geq 0$, $f(x,t) \in C_0^\infty((0,T) \times \Omega)$ and integrate over $[0,T]$ to get the following inequality:

$$0 \leq \epsilon^{-1} \int_0^T \int_\Omega \{[(h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t)] f(x,t)\} \, dxdt +$$
$$+ \epsilon^{-1} \int_0^T \int_\Omega \{[g_\epsilon(x,t-\epsilon) - g_\epsilon(x,t)] f(x,t)\} \, dxdt -$$
$$- \int_0^T \int_\Omega \left\{ \sum_{j=1}^N \frac{\partial}{\partial x_j} b_j(|u_\epsilon(x,t) - k|) f(x,t) + c|u_\epsilon(x,t) - k| f(x,t) + \right.$$
$$\left. +[\text{ sign } (u_\epsilon(x,t) - k)] k \left(\sum_{j=1}^N \frac{\partial}{\partial x_j} b_j + c \right) f(x,t) \right\} dxdt. \quad (4.3.10)$$

Now

$$\epsilon^{-1} \int_0^T \int_\Omega \{[h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t)] f(x,t)\} \, dxdt =$$
$$= -\epsilon^{-1} \left(\int_0^\epsilon \int_\Omega \{h_\epsilon(x,t-\epsilon) f(x,t)\} \, dxdt - \int_{T-\epsilon}^T \int_\Omega \{h_\epsilon(x,t) f(x,t)\} dxdt \right) +$$
$$+ \int_0^{T-\epsilon} \int_\Omega \left\{ h_\epsilon(x,t) \epsilon^{-1} (f(x,t+\epsilon) - f(x,t)) \right\} dxdt.$$

The first and the second integrals vanish for ϵ small enough, since f is in $C_0^\infty((0,T) \times \Omega)$. The convergence $u_\epsilon(x,t) \to S_1(t)v_1(x)$ in $L^1(\Omega)$, uniformly in t as $\epsilon \to 0$, implies that the third term tends to

$$\int_0^T \int_\Omega |S_1(t)v_1(x) - k| f_t(x,t) dxdt \qquad \text{as } \epsilon \downarrow 0.$$

By a similar argument, using the convergence $w_\epsilon(x,t) \to S_2(t)v_2(x)$ in $L^1(\Omega)$, we have that

$$\int_0^{T-\epsilon} \int_\Omega \left\{ g_\epsilon(x,t) \epsilon^{-1} (f(x,t+\epsilon) - f(x,t)) \right\} dxdt \qquad \text{tends to}$$

$$\int_0^T \int_\Omega |S_2(t)v_2(x) - k| f_t(x,t) dxdt \qquad \text{as } \epsilon \downarrow 0.$$

If we now let $\epsilon \downarrow 0$ in (4.3.10), we get

$$\int_0^T \int_\Omega |S_1(t)v_1(x) - k| f_t(x,t) dxdt + \int_0^T \int_\Omega |S_2(t)v_2(x) - k| f_t(x,t) dxdt +$$
$$+ \int_0^T \int_\Omega \left\{ \sum_{j=1}^N b_j |S_1(t)v_1(x) - k| \frac{\partial}{\partial x_j} f(x,t) - c|S_1(t)v_1(x) - k| f(x,t) - \right.$$
$$\left. -[\text{sign}(S_1(t)v_1(x) - k)] k \left(\sum_{j=1}^N \frac{\partial}{\partial x_j} b_j + c \right) f(x,t) \right\} dxdt \geq 0,$$

which is the claim of our theorem. □

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General Discussion and Conclusions

In this thesis we studied various differential equations with hysteresis nonlinearities.

In the introductory chapter we defined some basic hysteresis operators and listed their most important properties. For a more detailed discussion regarding definitions of hysteresis operators and many of their other properties, the reader should consult the monograph of M.A.Krasnosel'skii and A.V.Pokrovskii [5], as well as recent books by A.Visintin [7], M.Brokate and J.Sprekels [1], and P.Krejčí [6], and other references given therein.

In the first chapter we studied an ordinary differential equation

$$\frac{du}{dt} + \mathcal{F}(u) = f \qquad \text{in } (0,T) \tag{5.0.1}$$

$$u(0) = u_0. \tag{5.0.2}$$

The existence of a solution for f continuous, and uniqueness for $\mathcal{F}$ Lipschitz continuous, is well known (see [8]). We presented a proof of the uniqueness of the solution of (5.0.1)-(5.0.2), assuming that f is continuous and $f(t) > w(0)$ or $f(t) < w(0)$ in a right neighborhood of $t = 0$, without the assumption of a Lipschitz condition on $\mathcal{F}$. Our proof was based on a well-known theorem from ordinary differential equations. V.Chernorutskii and D.Rachinskii in [2] showed that for a special f, oscillatory in every neighborhood of 0, there exists more than one solution of (5.0.1)-(5.0.2).

The question of uniqueness of solutions of (5.0.1)-(5.0.2) is closely related to the uniqueness problem for a PDE with hysteresis in the source term:

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = f \qquad \text{in } Q, \tag{5.0.3}$$

$$u(0) = u_0, \tag{5.0.4}$$

coupled with appropriate boundary conditions. If $\mathcal{F}$ is (globally) Lipschitz continuous in $C^0([0,T])$, then the solution of (5.0.3)-(5.0.4) is unique, see [7]. On the other hand, if $\mathcal{F}$ is discontinuous, then the solution may not be unique, see the counterexample given in Visintin [7], Section XI.5.

The question of uniqueness for (5.0.3)-(5.0.4) with $\mathcal{F}$ continuous, but not Lipschitz continuous, remains an open problem even in the case $f = 0$ and $\mathcal{F}$ a continuous generalized play. Different techniques would be needed here.

In the second chapter we studied the parabolic partial differential equation with hysteresis:

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(x,t) \qquad \text{in } Q \qquad (5.0.5)$$

$$w(x,t) = [\mathcal{F}(u(\cdot,x); w_0(x))](t) \qquad (x,t) \in Q \qquad (5.0.6)$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \qquad (5.0.7)$$

$$u(0,\cdot) = u_0(x) \qquad \text{in } \Omega. \qquad (5.0.8)$$

Existence and uniqueness for (5.0.5)-(5.0.8) was proved by Visintin (see [7]), using a semigroup approach and also by means of approximation by implicit time discretization, a priori estimates and passage to the limit using compactness.

Asymptotic behaviour of solutions of (5.0.5) was first considered by N.Kenmochi and A.Visintin in [3]. They proved the asymptotic stability of solutions of (5.0.5) for $\mathcal{F}$ a generalized continuous play. We used a semigroup approach, which enabled us to consider the continuous and discontinuous cases at the same time, to prove asymptotic stability of solutions of (5.0.5) in L^1-spaces for $\mathcal{F}$ a generalized play or generalized Prandtl-Ishlinskii operator of play type.

In [3] N.Kenmochi and A.Visintin also studied the question of existence of periodic solutions of (5.0.5) and they proved existence and uniqueness of periodic solutions if f is periodic in t and $\mathcal{F}$ is a continuous generalized play operator. The same result can be proved for $\mathcal{F}$ a generalized Prandtl-Ishlinskii operator of play type, using a similar approach, i.e. using a fixed point argument combined with our asymptotic result for (5.0.5).

In the third chapter we studied the equation

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = f \qquad \text{in } Q, \qquad (5.0.9)$$

$$w(x,t) = [\mathcal{F}(u(\cdot,x); w_0(x))](t) \qquad (x,t) \in Q \qquad (5.0.10)$$

$$u(0,\cdot) = u_0(x) \qquad \text{in } \Omega, \qquad (5.0.11)$$

coupled with some boundary conditions. Existence of solutions of (5.0.9) as well as uniqueness for $\mathcal{F}$ Lipschitz continuous was proved by A.Visintin (see [7]).
N.Kenmochi and A.Visintin in [4] considered the question of asymptotic behaviour of solutions of (5.0.9) in the case when $\mathcal{F}$ is a generalized play operator. Their proof of asymptotic stability strongly depends on the specific properties of the generalized play.

We established results about asymptotic behaviour of solutions of (5.0.9) for a general hysteresis operator $\mathcal{F}$ at the end of chapter 3, which are true for some general hysteresis operators with properties often satisfied in applications.

Another idea for investigating the stability of (5.0.9) would be to try the idea of A.Ljapunov to consider functions or functionals which are decreasing along solutions of the equation.

Assume that $\mathcal{F}$ is a hysteresis functional with boundary curves γ_l, γ_r and a single family of inside hysteresis curves, i.e. the pair (u, w) increases or decreases inside the hysteresis loop on the same curve.

For (5.0.9) a Ljapunov functional may be defined as follows:

$$V : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R} \qquad \text{such that} \tag{5.0.12}$$

$$V(u,w) = \frac{1}{2}\int_\Omega (\nabla u)^2 dx + \int_\Omega \int_{u_0}^{u} \mathcal{F}(s) ds dx. \tag{5.0.13}$$

Since $\mathcal{F}(s)$ is a multivalued operator, we specify the meaning of $\int_{u_0}^{u} \mathcal{F}(s)ds$, for a given $x \in \Omega$ as:

$$\int_{u_0}^{u} \mathcal{F}(s)ds = \int_{u_0}^{u_1} \gamma_1(s)ds + \int_{u_1}^{u_2} \gamma_r(s)ds + \int_{u_2}^{u} \gamma_2(s)ds. \tag{5.0.14}$$

(see Figure 5.4),

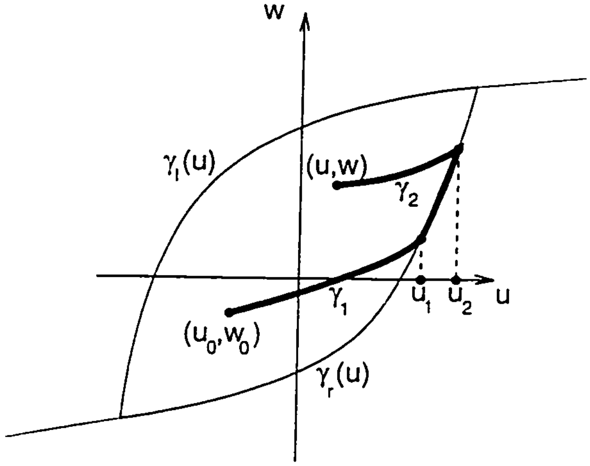

FIGURE 5.4. The hysteresis curves γ_1, γ_r and γ_2.

where γ_1, γ_r, γ_2 are hysteresis curves, on which the pair (u, w) would move from (u_0, w_0) to (u, w) if w first increases only and then w decreases only. Note that, depending on (u, w) and the hysteresis curves, we may have γ_1, γ_r or γ_2 empty. Then

$$\frac{d}{dt}V(u,w) = \int_\Omega \nabla u \nabla u_t dx + \int_\Omega \gamma_i(s) u_t dx = \int_\Omega u_t(-\Delta u + \mathcal{F}(u))dx =$$
$$-\int_\Omega (-\Delta u + \mathcal{F}(u))^2 dx \le 0, \tag{5.0.15}$$

where

$$i = \begin{cases} r & \text{if } w \in \gamma_r \\ 1 & \text{if } w \in \gamma_1 \\ 2 & \text{if } w \in \gamma_2. \end{cases} \tag{5.0.16}$$

So V is actually a Ljapunov functional. We are still working on the details of this approach.

The third chapter was also concerned with the question of the existence of periodic solutions of (5.0.9) for f periodic in time. We gave two different proofs of an existence result under quite general and physically reasonable assumptions on $\mathcal{F}$. We used a Galerkin method and a fixed point argument, which guarantee only existence of periodic solutions. The question of uniqueness of periodic solutions of (5.0.9) remains open, except for a very special result obtained by Xu Longfeng [9].

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