On Gromov-Witten Invariants, Hurwitz Numbers and Topological Recursion

by

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Abstract

In this thesis, we present expositions of Gromov-Witten invariants, Hurwitz numbers, topological recursion and their connections. By remodeling theory, open Gromov-Witten invariants of $\mathbb{C}^3$ *and* $\mathbb{C}^3/\mathbb{Z}_a$ *satisfy the topological recursion of Eynard-Orantin defined on framed mirror curve of toric Calabi-Yau three orbifold target spaces. Studies show that simple/orbifold Hurwitz numbers can be obtained using topological recursion with spectral curve given by Lambert/a-Lambert curve. Also, both the open Gromov-Witten invariants of toric Calabi-Yau three orbifold and the simple and double Hurwitz numbers can be formulated via Hodge integrals. We extend these connections by determining the relationship between the open Gromov-Witten invariants of* $\mathbb{C}^3/\mathbb{Z}_a$ *(with insertions of orbifold cohomology classes) and the full double Hurwitz numbers through referring to their Hodge integral formulations and to orbifold Riemann-Roch formula. By remodeling theory and mirror theorem for disk potentials, we make a conjecture relating a specific type of double Hurwitz numbers* $H_g(\nu(\gamma,2),\mu)$ *and topological recursion. We predict that these double Hurwitz numbers* $H_g(\nu(\gamma,2),\mu)$ *can be generated using topological recursion defined on the spectral curve* $ye^{-2y} - \tau x e^{-y} + x^2 = 0$.

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Introduction

Unification is a wonderful notion that researchers endeavor to achieve. A unified system may lead to great insights into each framework's component and may provide key to new door of studies. This thesis is about the connections among three seemingly unrelated topics: topological recursion, Hurwitz numbers and Gromov-Witten invariants.

Topological recursion first arose from the context of random matrix theory. This recursion developed by Eynard-Orantin originally provided a way to generate statistical solutions to matrix model problem using a geometric object called spectral curve. The solutions to matrix model can be obtained through topological recursion formula, which involves meromorphic differentials on the spectral curve. Moreover, topological recursion has also found surprising applications [**EO**] in various areas such as integrable systems, enumeration of maps, number theory, physics and algebraic geometry. In particular, topological recursion can be utilized to generate the enumerative quantities called Hurwitz numbers.

A Hurwitz number is an automorphic weighted count of covers of Riemann surfaces by Riemann surfaces. This quantity depends on the partitions of the degree of the covering map of Riemann surfaces. This quantity depends on the partitions of the degree of covering map of Riemann surfaces. The polynomiality of this in the partitions of covering map degree was discovered in [**GJ**], then it was realized that this characteristic of Hurwitz number can be formulated equivalently by integrals over moduli space of stable curves and of stable maps (called Hodge integrals). This description of Hurwitz number was found in [**ELSV**] and [**JPT**] through the process called localization technique on the moduli space of stable curves and of stable maps.

On a different aspect, localization technique was also applied in the context of topological string theory. In particular, it was used in order to get an expression for invariants called Gromov-Witten invariants. These invariants are related to topological string amplitudes. The invariant has enumerative interpretation as the number of holomorphic maps from Riemann surface to Calabi-Yau manifold. By localization, it was shown in [**KL**] and [**BC**] that the Gromov-Witten invariants can also be formulated in terms of Hodge integrals.

Starting with the idea that the chiral boson theory is related to topological recursion and that the B-model topological string is a specific type of chiral boson theory, conjectures were made in [**M**] and [**BKMP**] that topological string amplitude is related to topological recursion as well. The conjecture, named remodeling conjecture, asserts that the Gromov-Witten invariants of all toric Calabi-Yau three-orbifolds can be obtained using topological recursion, with spectral curve given by the framed mirror curve of the B-model. This was later proved in [**Chen**], [**Zhou**] and [**BCMS**] for invariants of $\mathbb{C}^3$, and further extended to Gromov-Witten invariants of all toric threefold in [**EO2**] and of orbifolds in [**FLZ**].

On the other hand, the connection between simple Hurwitz numbers (covering with only one arbitrary ramification profile over a branch point) and Gromov-Witten invariants of $\mathbb{C}^3$ can be expressed in terms of Hodge integrals. It was found in [**BM**] that the generating function for simple Hurwitz numbers can be recovered from generating function for Gromov-Witten invariants of $\mathbb{C}^3$ at large value of the framing parameter.

This relation motivated the formation of Bouchard-Marino conjecture, which claims that the topological recursion, with Lambert curve as spectral curve, produces the simple Hurwitz numbers. The Bouchard-Marino conjecture was later proved in [**EMS**] by identifying that the direct image of Laplace transform of cut-and-join equation recovers the topological recursion. The idea was further developed and proved in [**BHLM**], extending to orbifold Hurwitz numbers (special type of double covering/double Hurwitz numbers), where the associated spectral curve is the a-Lambert curve. The link was realized by referring to the formula of orbifold Hurwitz numbers and Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ in terms of Hodge integrals at large framing parameter, and to remodeling theory.

The goal of this study is to enlarge the connections by extending to the case of full double Hurwitz numbers (covering with exactly 2 arbitrary ramification profiles over 2 branchpoints). On the Gromov-Witten side, this amounts to including insertions of orbifold cohomology classes on the toric Calabi-Yau. The question is what is the relationship among topological recursion, Hurwitz numbers and Gromov-Witten invariants when we consider the full double Hurwitz numbers and when we include insertions in the Gromov-Witten theory. We examine the relationship between the Hodge integral formulas for double Hurwitz numbers and Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (with insertion of cohomology classes included). After this, we predict the associated spectral curve for double

Hurwitz numbers by extracting conditions from the generalized infinite framing limit and applying mirror symmetry theorem for disk potentials. We do this by considering the case of Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_2$, since we can reasonably determine the spectral curve for this case and the invariants of this were already determined in [**Ross**].

Besides from determining the relationship between double Hurwitz numbers and Gromov-Witten invariants and giving a new conjecture relating topological recursion and double Hurwitz numbers, another aim of this paper is to provide an exposition of each of topological recursion, Hurwitz numbers and Gromov-Witten invariants. We do this in a mix of mathematical and physical way, which means that the explanation of concepts are similar to physicists' way of illustrating things: we give intuitive explanations of ideas and sketch of some proofs. At the same time, we also attempt to maintain a degree of mathematical rigor. Some of the topics are highly abstract, such as moduli spaces, localization technique and other algebraic geometry topics. Explaining these things in detail would cause introduction of numerous prerequisites which may lead away from the main focus of the paper. In this regard, we introduce these subjects in an intuitive manner so as to get a clear big picture of this study.

We begin the thesis by providing background discussions about topological recursion, Hurwitz numbers, Hodge integrals, toric Calabi-Yau variety and Gromov-Witten invariants, which are the concepts necessary for the understanding the whole story of this study. The second chapter contains a section about simple Hurwitz numbers and Gromov-Witten invariants of $\mathbb{C}^3$ in terms of Hodge integrals. It is followed by the sections narrating the connections among the three main topics, namely the infinite framing limit (GromovWitten-Hurwitz), remodeling theory (GromovWitten-topological recursion) and Bouchard-Marino theory (Hurwitz-topological recursion). Then the next chapter involves the generalization of the topics in chapter two to the orbifold setting. Chapter three composes of topics regarding theorems expressing orbifold Hurwitz numbers and Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ via Hodge integrals, and their connection with topological recursion. In the fourth chapter, we provide new results by generalizing everything to full double Hurwitz numbers and Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ with insertions of orbifold cohomology classes included. Lastly, we give conclusion and some future research directions regarding the theme of the thesis.

1 Background

The first chapter is about the objects that play important role in main story of the paper. We give motivation, definition and examples for each topic. We start with topological recursion. It is then followed by Hurwitz numbers. As Hurwitz numbers and Gromov-Witten invariants can be formulated via Hodge integral, moduli space of stable curves and of stable maps and Hodge integrals are the topics in the third section. Before presenting Gromov-Witten invariants, toric variety is introduced in the fourth section. This is a special class of variety that plays essential role in topological string theory. As will be seen later, the three main topics in this paper can be sewn together if we consider a toric Calabi-Yau as the target space in topological string. Lastly, we discuss Gromov-Witten invariants in the last section of the first chapter.

1.1 Topological Recursion

The focus in this section is the tool that sprouted from the random matrix research program. First investigated thoroughly in [**EO**], the topological recursion exhibits universality in a way that it generates solution to many phenomena such as non-intersecting Brownian motions, partitions with Plancherel weight, topological string theory, etc. It also provides insights to some purely mathematical problems in enumeration of discrete surfaces or maps, Weil-Petersson volumes, intersection numbers and volumes of moduli spaces and Hurwitz numbers. Motivation behind this recursion is the topic in the next subsection.

1.1.1 Matrix Models

Random matrix theory has been a very active area of research in recent years. Remarkable applications of the theory were found in numerous studies, ranging from fluctuations in atomic resonances, quantum chaos to statistical properties of zeros of Riemann zeta function. The basic idea in the field of random matrix theory is that properties of a system can be modeled mathematically as matrix problems. As an example, there are studies [**K**], [**BK**] suggesting that the statistical behavior of nontrivial zeros of the Riemann zeta function resembles the statistical behavior of eigenvalues of large random matrices.

Statistical properties, such as eigenvalue distribution and ensembles, from a matrix model are encoded in the quantities called partition function Z and correlation function W. For instance, the partition function and correlation function for a Hermitian matrix model are

$$Z = \int e^{N \cdot Tr[V(M)]} dM, \tag{1}$$

$$W(x_1, .., x_n) = \left\langle Tr(\frac{1}{x_1 - M})...Tr(\frac{1}{x_n - M}) \right\rangle, \tag{2}$$

respectively, where N is the rank of Hermitian matrix M, V is a polynomial and x_i are variables in matrix model theory. (The names come from the study **[DGZ]** on representation of discrete $2D$ gravity in terms if random matrices, where the Z above is the partition function of $2D$ discrete gravity, as well as the generating function/formal matrix integral of random triangulations of $2D$ surfaces. The expectation value W of a product of n traces is the generating function for discrete surfaces with n holes and can be interpreted as averages of one-body operators in a free N fermion state). Developments in the field reveal that these have series expansion in inverse powers of rank N of matrices. As for the Hermitian matrix model, we have the expansions

$$F := log(Z) = \sum_{g=0}^{\infty} \frac{F_g}{N^{2g-2}}, \tag{3}$$

$$W(x_1, .., x_n) = \sum_{g=0}^{\infty} \frac{W_g(x_1, .., x_n)}{N^{2g-n-2}}, \tag{4}$$

where F is called *free energy*. Solving the series coefficients F_g, W_g (called "amplitudes") is essential in the study of random matrix theory. Many methods were developed to compute these quantities. One of the successful approaches is called *loop equations* method **[Kaz]**. A method was introduced in **[E]** to calculate the large N expansion of loop equations, which consists of solving the equations recursively in inverse power of N^2. To leading order, loop equations becomes an algebraic equation, and give rise to an algebraic curve $P(x, y)$ called *spectral curve*. As a result, loop equations can be written entirely in terms of geometric properties of spectral curve. In other words, the solution of loop equations depends only on the properties of the spectral curve, and not on the matrix model which gives that curve.

1.1.2 Eynard-Orantin Topological Recursion

The central object in the recursion that solves the matrix model is an algebraic curve $P(x, y)$,

$$\{P(x, y) = 0\} \subset \mathbb{C}^2, \tag{5}$$

called **spectral curve**. (Note that this can be generalized to $(\mathbb{C}^*)^2$ or $\mathbb{C} \times \mathbb{C}^*$)

Put differently, a spectral curve $\mathcal{S} = (\Sigma, x, y)$ is a data consisting of compact Riemann surface Σ and meromorphic functions x and y on Σ.

The main idea is that all the series coefficients W_g, F_g in the expansion of correlation function and free energy can be determined recursively from the geometric data of the spectral curve $\mathcal{S}$.

To write down the recursion, we need the following items from the spectral curve:

- Zeros a_i of the differential dx: $dx(a_i) = 0$.
  (We assume that all a_i are simple zeros)
- Two distinct points $z, \bar{z}$ in the vicinity of a_i, called *conjugated points*, such that $x(z) = x(\bar{z})$.
- *Canonical bilinear differential* $B(z_1, z_2)$, which is defined as a unique meromorphic differential on Σ with double pole at $z_1 = z_2$, no residue, no other pole and normalized such that

  $$\int_{A^\alpha} B(z_1, z_2) = 0, \tag{6}$$

  wherein (A^α, B_α) is a canonical basis of cycles on Σ.

  For instance, if Σ has genus 0, the canonical bilinear differential takes the form

  $$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{7}$$

- One forms, defined locally near each a_i:

$$dE_z(z_1) := \frac{1}{2}\int_{z'=z}^{\bar{z}} B(z_1, z'), \tag{8}$$

$$\eta(z) := (y(z) - y(\bar{z}))dx(z). \tag{9}$$

With these quantities from the spectral curve, it was found in [**EO**] that the amplitude W_g can be determined recursively on $2g+n$ using the formula called **topological recursion**:

$$W_g^{(n)}(z_1, J) = \sum Res_{z\to a_i}\frac{dE_z(z_1)}{\eta(z)}\left[W_{g-1}^{(n+1)}(z,\bar{z},J)+\sum_{h=0}^{g}\sum_{I\subset J} W_h^{(|I|+1)}(z,I)W_{g-h}^{(|J|-|I|+1)}(\bar{z},J\backslash I)\right]. \tag{10}$$

In the above expression, J means the variables $z_2, .., z_n$, and the finite sum is taken over residues at all zeros of dx. Also, the recursion starts with the initial values:

$$W_0^{(1)}(z) := 0, \qquad W_0^{(2)}(z_1, z_2) := B(z_1, z_2). \tag{11}$$

Here, the W_g's can be viewed as symmetric meromorphic differentials on Σ. So the topological recursion provides a way to generate infinite sequence of meromorphic differentials defined on Riemann surface Σ. On the other hand, the amplitude F_g can be calculated through the formula

$$F_g = \sum Res_{z\to a_i}\Phi(z)W_g^{(1)}(z), \tag{12}$$

where $\Phi(z)$ is an arbitrary antiderivative of ydx, i.e. $d\Phi = ydx$.

Besides from providing solutions to a matrix model, the topological recursion exhibits a lot of nice properties such as homogeneity and symplectic invariance on F_g and modularity on both amplitudes. As will be seen later, the topological recursion has connection as well to Gromov-Witten invariants from topological string and to Hurwitz numbers.

Examples:
(In the examples below, the Riemann surface Σ is $\mathbb{P}^1$. The origin of the first two examples can be found in [**EO**], [**TW**]):
1) The curve described by $y^2 = x$, called *Airy curve*, arises as the spectral curve in the statistical study of extreme eigenvalues of a random matrix. This can be used also to generate intersection numbers of stable classes on moduli space of curves [**Zhou2**], and the topological recursion can be used to prove Witten's conjecture [**W**].
2) The genus 0 hyperelliptical curve, $y^2 - 2 = x^3 - 3x$, is the spectral curve associated to the application of topological recursion to pure gravity Liouville field theory.
3) It will be shown later that the *Lambert curve*, $x = ye^{-y}$, and *a-Lambert curve*, $x^a = ye^{-ay}$, is the spectral curve that generates the simple Hurwitz numbers and orbifold Hurwitz numbers, respectively, using the topological recursion.

1.2 Hurwitz Numbers

As mentioned in the example above, topological recursion can be used to generate Hurwitz numbers. We introduce in this section the definition of Hurwitz numbers, which is one of the main topics of the thesis. We start with ramification on branched covering map of Riemann surfaces, and then Hurwitz number's definition via covering map.

1.2.1 Ramification

From complex analysis, holomorphic maps of Riemann surfaces can be given local expression of the form $z \to z^k$, with $k \in \mathbb{Z}^+$. Let $f : X \to Y$ be a branched covering map of Riemann surfaces X, Y and $V \subset Y$. When the local expression at a particular point $x \in f^{-1}(V)$ is $F(x) := x^k$, the positive integer $k := r_f(x)$ is called the **ramification order** of f at x. If $k \geq 2$, then the point $x \in X$ is a **ramification point** of f. A point $y \in Y$ is a **branch point** of f if it is the image of a ramification point.

Suppose $f^{-1}(y) = \{x_1, ..., x_n\}$, then the unordered collection of integers $\eta(y) := \{r_f(x_1), .., r_f(x_n)\}$ is the **ramification profile** of f over y. When $\eta(y) = \{2, 1, 1, .., 1\}$, f has **simple ramification** or is **simply ramified** over y.

Let X, Y be Riemann surfaces, and $p_1, .., p_r, q_1, .., q_s \in Y$ be the branch points of $f : X \to Y$. A **Hurwitz cover** of Y is a covering map f such that:

i) f has ramification profile $\eta_1, .., \eta_s$ over $q_1, .., q_s \in Y$, respectively.
ii) f is simply ramified over $p_1, .., p_r \in Y$.

Note that (i) above implies that the degree of f is equal to the sum of elements of η_i $\forall i = 1, .., s$. In other words, η_i's are unordered partitions of deg(f). For compact Riemann surfaces, the number of simple ramification points can be determined using the Riemann-Hurwitz formula.

1.2.2 Hurwitz Numbers via Covering

Two covering maps $f_1 : X_1 \to Y, X_2 \to Y$ are said to be **automorphic** if $\exists$ isomorphism $g : X_1 \to X_2$ such that $f_2 \circ g = f_1$. Each covering map f has naturally associated automorphism group Aut(f).

A **Hurwitz number** $H_g(\eta_1, .., \eta_s)$ is a weighted count (up to isomorphism) of distinct Hurwitz covers f mapping genus g Riemann surface X to Riemann surface Y, with ramification profile η_i over $q_i \in Y$ for $i = 1, .., s$ and simple ramification over all other branch points. Each such cover is weighted by the inverse of the size $|Aut(f)|$ of the automorphism group.

Hurwitz number comes in two different types, depending if the covering space X of Y is connected or not. Connected covering space and Hurwitz covers mapping to $Y = \mathbb{P}^1$ will be considered in this thesis. In the case when there is only one arbitrary ramification profile η over the ∞ point in $\mathbb{P}^1$, we have **simple Hurwitz number**. It is a **double Hurwitz number** if there are exactly two arbitrary ramification profiles η_1, η_2 over the points $0, \infty$, respectively.

Moreover, it can be easier to realize that a Hurwitz number is finite through its definition via symmetric group S_ν. Focusing on simple Hurwitz number, consider the following quantities:

i) $\nu = \nu_1 + .. + \nu_\zeta$ as a partition of ν
ii) $m := \nu + \zeta + 2g - 2$
iii) $t_1, .., t_m$ as transpositions in S_ν
iv) σ as a permutation with ζ cycles of lengths $\nu_1, .., \nu_\zeta$.

When the product $t_1 \cdots t_m \cdot \sigma$ of permutations is equal to the identity permutation and the group generated by $t_1, .., t_m$ is transitive, then $(t_1, .., t_m, \sigma)$ is said to be *transitive factorization of identity*. Simple Hurwitz number $H_g(\nu_1, .., \nu_\zeta)$ is then $1/\nu!$ times the number of lists of transitive factorization of identity. In other words, $H_g(\nu)$ is $1/\nu!$ times the number of factorizations of a ν-cycle into a product of $\nu + 2g - 1$ transpositions. The equivalence between the above definitions of Hurwitz number can be realized by considering the monodromies about the branch points of the covering.

It will be presented in chapters two and three that simple and double Hurwitz numbers can be expressed in terms of *Hodge integrals*. In order to define these integrals, we need to know first what are moduli space of stable curves and stable maps, which bring us into the next section.

1.3 Moduli Space of Stable Curves, Stable Maps and Hodge Integrals

In order to introduce Hodge integrals, we begin with stable curves and stable maps. The notions of *stable curve* and *stable map* were introduced in order to compactify the moduli space of maps and of (rational and elliptic) curves.

A *pre-stable curve* is a compact connected curve which has only nodes as singularities. An automorphism $\ell : (C, z_1, .., z_n) \rightarrow (C', z_1', .., z_n')$ of pre-stable curves with n marked points is a morphism such that $\ell(z_i) = z_i'$ for all $i = 1, .., n$ (i.e., marked points are preserved). A pre-stable curve C is said to be a **stable curve** if its group of automorphisms is finite. This condition on automorphism group is equivalent to the condition that every genus 0 (resp. genus 1) component of pre-stable curve has at least 3 (resp. 1) special point (marked and nodal points) lying on it. The **moduli space** $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points is defined as the set of isomorphism classes of these stable curves; two stable curves being isomorphic if there is a holomorphic diffeomorphism between them. The moduli space of stable curves is a projective variety [**HM**].

Remark: The terms curve and Riemann surface in this thesis will be used interchangeably. In Hurwitz number context, we use the term curve most of the time, while in Gromov-Witten setting, we use the term Riemann surface.

With moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves in hand, the **Hodge bundle** E on $\overline{\mathcal{M}}_{g,n}$ is defined as the vector bundle whose fiber over a point is the space $H^0(C, \omega_C)$ of holomorphic sections of the canonical line bundle ω_C of C. The *Hodge classes* λ_i on $\overline{\mathcal{M}}_{g,n}$ are Chern classes of E, while the *descendent classes* $\overline{\psi_i}$ on $\overline{\mathcal{M}}_{g,n}$ is the first Chern class of the cotangent line bundle to the i-th marked point. The **Hodge integral** over $\overline{\mathcal{M}}_{g,n}$ is then defined as an integral involving Hodge classes λ_i and descendent classes $\overline{\psi_i}$.

It will be presented in chapter two that simple Hurwitz numbers can be written in terms of Hodge integrals over $\overline{\mathcal{M}}_{g,n}$; Whereas the double Hurwitz numbers can be formulated similarly, but with Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ of stable maps to *classifying space* $\mathcal{B}\mathbb{Z}_a$ of cyclic group $\mathbb{Z}_a = \{0, 1, .., a-1\}$.
(Classifying space $\mathcal{B}\mathbb{Z}_a$ of $\mathbb{Z}_a$ can be thought of as an orbifold $[pt/\mathbb{Z}_a]$, where $\mathbb{Z}_a$ acts trivially on a point)

A **stable map** $f : C \to D$ is a pseudo-holomorphic map from a pre-stable curve C of genus g with n marked points whose automorphism group is finite. Note that this does not imply that C is a stable curve. However, it is easy to show that it is equivalent to requiring that C has at least one stable component, and that for each component $C_i \subset C$, either f is non-constant or C_i is stable.

Two stable maps $f : C \to D$ and $f' : C' \to D$ are *isomorphic* if $\exists$ isomorphism $\ell : C \to C'$ such that $f' \circ \ell = f$. The **moduli space of stable maps** $\overline{\mathcal{M}}_{g,n}(D)$ to topological space D is the set of all isomorphism classes of stable maps from pre-stable curve C of genus g with n marked points to D. The moduli space of stable maps is locally a quotient of a smooth variety by a finite group, and its boundary is a *normal crossing divisor* [**HM**]. The moduli space of stable curves and of stable maps are always compact.

The Hodge bundle E and Hodge classes λ_i on moduli space $\overline{\mathcal{M}}_{g,n}(D)$ of stable maps from C to topological space D are defined in the same manner as the E and λ_i on moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. That is, *Hodge bundle* on $\overline{\mathcal{M}}_{g,n}(D)$ is the vector bundle whose fiber at a point is $H^0(C, \omega_C)$, and *Hodge classes* on $\overline{\mathcal{M}}_{g,n}(D)$ are Chern classes of E. The *descendent classes* ψ_i on $\overline{\mathcal{M}}_{g,n}(D)$ are defined by the pullback

$$\psi_i = \epsilon^*(\overline{\psi}_i) \tag{13}$$

via map $\epsilon : \overline{\mathcal{M}}_{g,n}(D) \to \overline{\mathcal{M}}_{g,n}$. If a stable map f maps a pre-stable curve C to an orbifold, such as $\mathcal{B}\mathbb{Z}_a$, then the theory of stable maps [**AGV**] requires that the source curve C must acquire a stack structure at marked and nodal points, where they locally look like $[pt/\mathbb{Z}_a]$. This source curve is called *twisted curve* in literature. Denote by $\mathcal{C}$ the twisted curve (source curve of stable maps to orbifold) with r marked points and n nodal points.

Let the vector

$$\gamma - \mu := (\gamma_1, .., \gamma_r, -\mu_1, .., -\mu_n) \tag{14}$$

denotes the special points of twisted curve $\mathcal{C}$, where $\gamma_i \in \mathbb{Z}_a \backslash \{0\}$ are the marked points with stack structure and $\mu_i \in \mathbb{Z}^+$ are the nodal points. (Here, μ_i is a positive integer, but in the vector $\gamma - \mu$, $-\mu_i$ is understood mod a, i.e. as an element of $\mathbb{Z}_a$).

Denote by $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ the moduli space of stable maps $\mathcal{F}$ from $\mathcal{C}$, with prescribed monodromy data $\gamma - \mu$, to orbifold $\mathcal{B}\mathbb{Z}_a$. The group action on $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ induces a $\mathbb{Z}_a$ action on Hodge bundle E, which gives a decomposition into subbundles. Let the representation U of group $\mathbb{Z}_a$ be given by

$$\phi^U : \mathbb{Z}_a \to \mathbb{C}^*, \qquad \phi^U(1) = e^{2\pi i/a}. \tag{15}$$

Then the **Hodge bundle** E^U, corresponding to representation U, on $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ is a vector bundle constructed by associating each map $[\mathcal{F}] \in \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ the U-summand of the $\mathbb{Z}_a$-representation $H^0(\mathcal{C}, \omega_{\mathcal{C}})$. The rank of E^U can be determined using the orbifold Riemann-Roch formula [**AGV**] (to be discussed in Appendix 1). The *Hodge classes* λ_i^U on $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ are Chern classes of E^U,

$$\lambda_i^U = c_i(E^U). \tag{16}$$

The *descendent classes* ψ on $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ are defined by pullback

$$\psi_i = \epsilon^*(\overline{\psi}_i) \tag{17}$$

via map $\epsilon : \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \to \overline{\mathcal{M}}_{g,n}$.

Then the **Hodge integral** over the moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ is the integral involving the Hodge classes λ_i^U and descendent classes ψ_i.

Note that when the group is trivial (when $a = 1$), the moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ of stable maps reduces to moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. The formulation of simple and double Hurwitz numbers via Hodge integrals was done through *localization technique* (to be discussed briefly in section 1.5) on moduli space of stable curves and of stable maps, respectively. On a completely different topic, similar technique on the same spaces was applied to the Gromov-Witten context. It will be presented later that the geometric invariant called Gromov-Witten invariants can be expressed as well in terms of Hodge integrals. Basically, Gromov-Witten invariant counts the number of holomorphic maps from Riemann surface Σ to Calabi-Yau manifold X. This thesis is mainly concerns with the case when the target space X is also a toric variety. To build up things carefully and to define Gromov-Witten invariants in turn, we introduce toric variety in the next section.

1.4 Toric Calabi-Yau Variety

Toric variety first arose in the study of torus embeddings. It can described equivalently in various ways using symplectic geometry, algebraic geometry, commutative algebra and gauged linear sigma model. For simplicity, we introduce in this section using the concept of projective spaces. It is followed by describing it in terms of cones and fans, which are combinatorial data of convex polytopes on a lattice. This combinatorial approach to toric variety is considered because this description will be useful later in the discussions of topological string B-model and spectral curve generating Gromov-Witten invariants. Then we define toric Calabi-Yau after the presentation of toric variety. Toric Calabi-Yau is an interest because this particular type of geometry gives surprisingly beautiful mathematical and physical theories.

1.4.1 Toric Variety

The **complex projective space** $\mathbb{P}^m$ is a quotient of complex manifold $\mathbb{C}^{m+1}$ defined as

$$\mathbb{P}^m = (\mathbb{C}^{m+1}\backslash\{0\})/(\mathbb{C}^*), \tag{18}$$

where the quotient by $\mathbb{C}^*$ means the identification

$$(z_1, ..., z_{m+1}) \sim (\lambda z_1, ...\lambda z_{m+1}), \tag{19}$$

wherein $\lambda \in \mathbb{C}^*$.

Generalizing the notion of $\mathbb{P}^m$, the **weighted complex projective space** $\mathbb{P}^m_{(\beta_1,...,\beta_{m+1})}$ is constructed by assigning *weight* $\beta_i \in \mathbb{N}$ to z_i coordinate of $\mathbb{C}^{m+1}$. That is,

$$\mathbb{P}^m_{(\beta_1,...,\beta_{m+1})} = (\mathbb{C}^{m+1} \backslash \{0\})/(\mathbb{C}^*), \tag{20}$$

where the weighted $\mathbb{C}^*$ action is given by the equivalence relation

$$(z_1, ..., z_{m+1}) \sim (\lambda^{\beta_1} z_1, ..., \lambda^{\beta_{m+1}} z_{m+1}). \tag{21}$$

Toric variety can be thought of as a further generalization of weighted complex projective space in which there are more than one weighted $\mathbb{C}^*$ action, with possibly negative weights.

Suppose

$$\underbrace{\mathbb{C}^* \times ... \times \mathbb{C}^*}_{\text{p}} \cong (\mathbb{C}^*)^p, \qquad p < m+1, \tag{22}$$

is an algebraic torus and $U \subset \mathbb{C}^{m+1}$ is invariant under the weighted action of a subgroup of $(\mathbb{C}^*)^p$, then the **toric variety** $\mathcal{T}$ is defined as

$$\mathcal{T} = (\mathbb{C}^{m+1} \backslash U)/(\mathbb{C}^*)^p. \tag{23}$$

All projective spaces and weighted projective spaces are clearly toric varieties. If X and Y are toric varieties, then so is $X \times Y$. Another example of toric variety is the tautological bundle $\mathcal{O}(-3)$ over $\mathbb{P}^2$.

Alternatively, toric variety can be described combinatorially using *cones* and *fans* over a lattice. The reader is referred to **[CLS]** for the proof of the equivalence of the two different descriptions of toric variety.

Let E be an integer lattice contained in the vector space $\mathcal{E}$. A **cone** $\mathfrak{C} \in \mathcal{E}$ is a set

$$\mathfrak{C} = \{a_1 v_1 + a_2 v_2 + ... + a_t v_t | a_i \geq 0\} \tag{24}$$

generated by the vectors $\{v_1, ..., v_t\} \in E$ with the condition that $\mathfrak{C} \cap (-\mathfrak{C}) = \{0\}$.

A **face** of a cone $\mathfrak{C}$ is the intersection of $\mathfrak{C}$ with one of the hyperplanes bounding $\mathfrak{C}$. A **fan** $\mathfrak{F}$ is a set of cones $\mathfrak{C}$ such that every face in $\mathfrak{F}$ is also a cone and the intersection of two cones in $\mathfrak{F}$ is a face of each.

Focusing on three-dimensional toric varieties, suppose $\mathfrak{F}$ is a fan in E. To each one dimensional cone in $\mathfrak{F}$, we assign a homogeneous coordinate w_i to each generator v_i, $i = 1, .., t$. From the resulting $\mathbb{C}^t$, we remove the set

$$Z_{\mathfrak{F}} = \bigcup_I \{(w_1, ..., w_t) : w_i = 0 \quad \forall i \in I\}, \tag{25}$$

where the union is taken over all sets $I \subseteq \{1, ..., t\}$ for which $\{w_i : i \in I\}$ does not belong to a cone in $\mathfrak{F}$. This means that the w_i's are allowed to vanish simultaneously only if the corresponding v_i's belong to the same cone. Then the toric variety $\mathcal{T}$ from the fan $\mathfrak{F}$ is given by

$$\mathcal{T}_\Sigma = \frac{\mathbb{C}^t \backslash Z_{\mathfrak{F}}}{G}, \tag{26}$$

where the group G is $(\mathbb{C}^*)^{t-3}$ acting by

$$(w_1, ..., w_t) \sim (\lambda^{Q_1^{(j)}} w_1, ..., \lambda^{Q_t^{(j)}} w_t), \qquad \sum_{i=1}^t Q_i^{(j)} v_i = 0, \tag{27}$$

wherein $\lambda \in \mathbb{C}^*$ and $j = 1, .., t-3$. The **toric weights** $Q_i^{(j)}$ are integers. For a fixed j, $Q_i^{(j)}$ are relatively prime integers.

Examples:

1) The projective space $\mathbb{P}^2$ is toric variety with one dimensional cones generated by the vectors $(1,0), (0,1), (-1,-1)$, and the set $Z_{\mathfrak{F}}$ is simply $\{0\}$. The $\mathbb{C}^*$ quotient is made by the identification $(w_1, w_2, w_3) \sim (\lambda^1 w_1, \lambda^1 w_2, \lambda^1 w_3)$, where w_i is the homogeneous coordinate associated to generating vector v_i, and the weights are $(1,1,1)$ since $1(1,0) + 1(0,1) + 1(-1,-1) = 0$.

2) The weighted projective space $\mathbb{P}^2_{(3,1,2)}$ is another example of toric variety wherein the generating vectors are $(1,0),(-1,2),(-1,-1)$ and $Z_{\mathfrak{F}} = \{0\}$. The weights are $(3,1,2)$ since $3(1,0)+1(-1,2)+2(-1,-1)=0$.

1.4.2 Toric Calabi-Yau Threefold

Calabi-Yau manifolds are significant in string theory since supersymmetry conditions [**CHSW**] in the theory requires that the extra spatial dimensions are compactified in a form of a Calabi-Yau manifold, which is a Kahler manifold with vanishing first Chern class or trivial canonical bundle. Now that the definition of toric variety has built up, the Calabi-Yau condition on it is simply stated in the following theorem:

Toric Calabi-Yau Condition:
A toric variety X is Calabi-Yau variety if only if:

- The vectors generating the cones defining X all lie in the same hyperplane $\iff$ The sum of the toric weights vanishes.

Proof of this can be found in [**B**]. This condition implies that the toric varities $\mathbb{P}^2$ and $\mathbb{P}^2_{(3,1,2)}$ are not Calabi-Yau, whereas the complex manifold $\mathbb{C}^n$ is a (rather trivial) example of toric Calabi-Yau. In this paper, we are mainly focused on **toric Calabi-Yau threefold**, which is a toric Calabi-Yau of complex dimension 3

More examples of toric Calabi-Yau:
1) Another example is the resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$. Its cones are generated by the vectors $v_1=(1,0,1), v_2=(-1,0,1), v_3=(0,1,1), v_4=(0,-1,1)$ that are all lying in the plane $z=1$. The toric weights are (1,1,-1,-1) since $1\cdot v_1+1\cdot v_2-1\cdot v_3-1\cdot v_4=0$.
2) The bundle $\mathcal{O}(-3)\to\mathbb{P}^2$, with vectors generating its cones are $v_1=(-1,-1,1), v_2=(1,0,1), v_3=(0,1,1), v_4=(0,0,1)$, is an example of toric Calabi-Yau threefold with toric weights $(1,1,1,-3)$.

Referring to the Calabi-Yau condition above, the vectors generating the cones of three dimensional toric variety all lie in a two dimensional plane. Toric Calabi-Yau threefold has diagrammatic representation developed in [**CLS**] that encodes the degeneration of its fibers. The toric diagram Γ of toric Calabi-Yau threefold X is the dual graph of two dimensional graph $\tilde{\Gamma}$ called **fan triangulation**, which is formed by the intersection of fan of X with plane on which the v_i's lie.

Example diagram:

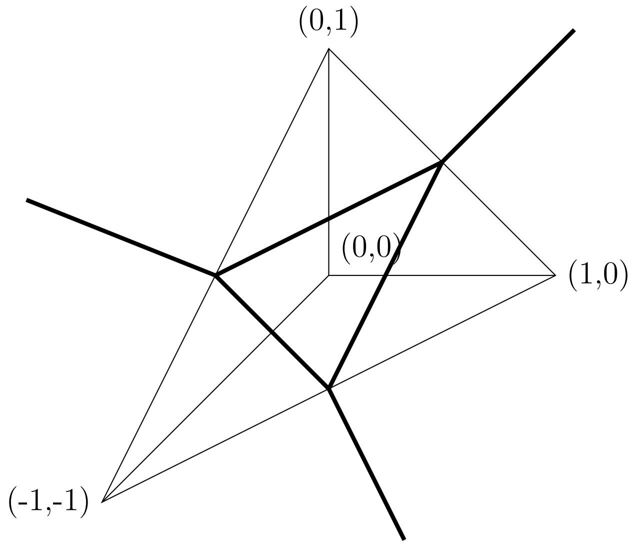

Figure 1 (Toric diagram): For toric Calabi-Yau threefold $\mathcal{O}(-3) \to \mathbb{P}^2$ (bundle over $\mathbb{P}^2$), the thick planar graph corresponds to the toric diagram, while the triangle with vertices $(-1,-1),(1,0),(0,-1)$ is the graph for the fan triangulation of this toric Calabi-Yau generated by the cones $(-1,-1,1),(1,0,1),(0,-1,1),(0,0,1)$.

1.5 Gromov-Witten Invariants

Now that the concept of toric Calabi-Yau has been established, it is time to introduce Gromov-Witten invariants. We discuss in this section its origin from string theory and then followed by its definition. We begin with the story of string theory.

1.5.1 String Theory

The goal of string theory is to combine general relativity and quantum theory into a single unified framework. This ambitious endeavor has introduced an entirely new physical picture into theoretical physics and new mathematics that has surprised even the mathematicians. String theory originated from the *dual models*, which was a candidate for theory of hadrons or strongly interacting particles. Physicists wanted to construct a scattering amplitude for hadrons obeying the stringent requirements of the S-matrix theory of strong interactions (*Bootstrap program*), which includes the duality hypothesis between s- and t-channel amplitudes. The desired scattering amplitude formula was discovered 'accidentally', and it was later realized that the physical model behind such formula is actually a relativistic string. Imposing quantum mechanical laws on these relativistic strings turns out to give surprising results, such as the prediction of spacetime six extra dimensions, as well as the natural emergence of graviton (particle that mediates gravitational interaction) in the theory, to mention a few. The spacetime dimensionality problem is reconciled through the notion of *compactification* that puts six of the extra spatial dimensions on a small six real dimensional space. Physical conditions require these extra six dimensions to be a Calabi-Yau manifold X [**CHSW**].

Propagating string in spacetime generates a surface called *worldsheet*, and this takes a form of a Riemann surface Σ. Quantum vibrational modes of the string give rise to species of particles. In string theory, the integral over the holomorphic maps $\varphi : \Sigma \to X$ defines a two dimensional quantum field theory called *sigma model*. Shifting the operators on sigma model worldsheet (in order to obtain the right physical quantum spins) gives rise to a special version of the theory called *topological string theory*. This shift or 'twist' in the theory comes into two types, called *A*- and *B-model*, depending on the shifting sector of the operators. These two models can be compactified on two different Calabi-Yau manifolds giving rise to the same physics. These Calabi-Yau manifolds are said to be **mirror**, and the relationship or duality between the two physical theories is called **mirror symmetry**. Through this duality, statements in Calabi-Yau manifold can be translated into equivalent statements in the mirror manifold, allowing mathematicians and physicists to solve intractable problems on the other mirror Calabi-Yau manifold.
The Gromov-Witten invariant has origin from topological string theory, namely in Witten's work [**W2**] on integrals in two dimensional gravity with enumerative meaning of counting instantons (non-trivial solutions of equations of motion) on X of topological string. The object of interest in Gromov-Witten theory is a holomorphic map $\varphi : \Sigma \to X$. The number of such maps is equivalent to the Gromov-Witten invariant$\in \mathbb{Q}$, which exhibits invariance under complex deformations on X. These invariants can be gathered into *Gromov-Witten potential* F_g. This potential physically gives the genus g amplitude of topological string partition function with $\mathcal{N} = 2$ sigma model defined on X whose twisting yields the A-model topological string.

1.5.2 Definition of Gromov-Witten Invariant and Localization

This thesis is mainly focused on *open* Gromov-Witten invariants. The term *open* pertains to the domain *bordered* (pre-stable) Riemann surface Σ, which is Riemann surface whose boundary consists of finitely many disjoint simple closed curves.

Open Gromov-Witten invariant counts the number of holomorphic maps $\varphi : \Sigma \to X$ from genus g bordered Riemann surface Σ to Calabi-Yau threefold (or three orbifold) X, such that $\varphi(\partial\Sigma) \subset \mathcal{L}$, where $\mathcal{L}$ is a Lagrangian submanifold of X; The Lagrangian $\mathcal{L}$ is a submanifold of Calabi-Yau X such that its dimension is half of the real dimension of X and the Kahler form vanishes on $\mathcal{L}$. If X is a toric Calabi-Yau, there is a special class of $\mathcal{L}$ called *toric branes* [**AV**] that have topology $S^1 \times \mathbb{R}^2$. In this case, the holes of bordered Riemann surface map around nontrivial cycles in the toric brane, with prescribed winding numbers. In the diagram of toric Calabi-Yau X, the toric brane projects to points in the edges of toric diagram.

There is a formal definition of Gromov-Witten invariant in algebraic geometry, wherein it can be expressed through cohomology classes on Calabi-Yau manifold X. With this sophisticated definition of Gromov-Witten invariants for toric Calabi-Yau target spaces, it was shown via localization technique that they can be written in terms of Hodge integrals. *Localization* was first introduced and defined precisely in [**AB**] as a tool in equivariant cohomology theory. The basic idea of localization is that if there is a group acting on a variety (or stack) with isolated fixed points, cohomology classes on it 'localizes' to classes on the fixed point locus. This is in turn allows calculation of global integral on variety in terms of local information at a finite set of points.

Localization examples:

1) As shown in [**AB**], if a torus T acts on a manifold M with isolated fixed point set F, then the integral of any (*equivariant*) closed form $\Omega \in M$ can be expressed as

$$\int_M \Omega = \sum_{p \in F} \frac{\omega|_p}{e^T(n_p)}, \tag{28}$$

where $\Omega|_p$ is the restriction of closed form Ω to a fixed point p and $e^T(n_p)$ is the (equivariant) Euler class of normal bundle n_p to p in M.

2) Example for Gromov-Witten invariants:

The moduli space $\overline{\mathcal{M}}_{g,r}(X)$ of stable maps φ from Riemann surface Σ of genus g, with r-marked points $p_1, .., p_r$, to Calabi-Yau manifold X, consists of components of different dimension. The lower bound to the local dimension is called *virtual dimension*. In the case when all local dimensions are the same, the theory is said to be *unobstructed*. From the deformation theory of stable maps, the virtual dimension can be calculated using Riemann-Roch formula in the case when the theory is obstructed. Define the *evaluation map*

$$ev_i : (\overline{\mathcal{M}}_{g,r}(X)) \to X \tag{29}$$

by sending a stable map φ to $\varphi(p_i)$. Then the **Gromov-Witten invariant** $G_g(\alpha_1..\alpha_r)$ is given by the integral involving cohomology classes $\alpha_i \in H^*(X)$,

$$G_g(\alpha_1...\alpha_r) = \int_{[\overline{\mathcal{M}}_{g,r}(X)]^{vir}} ev_1^*(\alpha_1)...ev_r^*(\alpha_r), \tag{30}$$

where $[\overline{\mathcal{M}}_{g,r}(X)]^{vir} \in H_{2D}(\overline{\mathcal{M}}_{g,r}(X), \mathbb{Q})$ means virtual fundamental class of $\overline{\mathcal{M}}_{g,r}(X)$, where D is the expected dimension of $\overline{\mathcal{M}}_{g,r}(X)$. See section 7.1.4 of [**CK**] for complete definition of virtual fundamental class and the clarification about how this integral gives the number of holomorphic maps from Riemann surface to Calabi-Yau manifold. Via localization technique, it was shown [**KL**] that these invariants for toric Calabi-Yau target spaces with toric branes $\mathcal{L}$ can be expressed in terms of Hodge integrals and of finite set $\mu := (\mu_1, ..\mu_n)$ denoting winding numbers of the n holes of genus g Riemann surface Σ, as in (34) (to be presented shortly).

When the target space is an orbifold $\mathfrak{X}$, the cohomology of $\mathfrak{X}$ that is involved in the Gromov-Witten invariant theory is called *Chen-Ruan cohomology* $H^*_{CR}(\mathfrak{X})$. This is the type of cohomology that is sufficient [**CR**], [**S**] for orbifolds, rather than the orbifold de Rham cohomology (sufficient in the sense that this enlarges the orbifold de Rham cohomology by keeping track of the automorphisms that the cohomology classes might have). This is defined as the cohomology of a related orbifold $I\mathfrak{X}$, called *inertia orbifold*. Inertia orbifold $I\mathfrak{X}$ of $\mathfrak{X} := \mathbb{C}^3/\mathbb{Z}_a$ is the union of $X^{\gamma_i} \subset \mathbb{C}^3$ that are fixed under the action of $\gamma_i \in \mathbb{Z}_a$ on $\mathbb{C}^3$. Let $\mathbb{1}_{\gamma_i}$ be the cohomology classes on $H^*_{CR}(\mathfrak{X})$ or the generators of $\mathbb{C}_{\gamma_i}$, where $\bigoplus_{\gamma_i \in \mathbb{Z}_a} \mathbb{C}_{\gamma_i} := H^*_{CR}(\mathfrak{X})$. Then the Gromov-Witten invariant of orbifold $\mathfrak{X}$ is defined as

$$G_g(\mathbb{1}_1^{m_1}...\mathbb{1}_{a-1}^{m_{a-1}}) = \int_{[\overline{\mathcal{M}}_{g,\sum m_i}(X)]^{vir}} (ev_1^*(\mathbb{1}_1))^{m_1}...(ev_r^*(\mathbb{1}_{a-1}))^{m_{a-1}}, \tag{31}$$

where $(ev_i^*(\mathbb{1}_i))^{m_i}$ means union of m_i $ev_i^*(\mathbb{1}_i)$. Here, we also let $\sum_{j=1}^r \gamma_j = \sum_{i=1}^{a-1} im_i$. By localization process in **[BC]**, **[Ross]**, it was shown that this expression for Gromov-Witten invariant of orbifold $\mathfrak{X}$ is equivalent to expression (76).

2 Simple Hurwitz Numbers and Open Gromov-Witten Invariants of $\mathbb{C}^3$

This chapter contains the beginning of the connections. Simple Hurwitz numbers via Hodge integrals is the first topic, followed by the discussion of Gromov-Witten invariants of $\mathbb{C}^3$. The other parts are about the connections among simple Hurwitz numbers, Gromov-Witten invariants of $\mathbb{C}^3$ and topological recursion.

2.1 ELSV Formula and Open Gromov-Witten Invariants of $\mathbb{C}^3$

The polynomiality of simple Hurwitz number $H_g(\mu)$ in partition μ of the Hurwitz cover degree was discovered in **[GJ]**. Simple Hurwitz number can be calculated using the formula

$$H_g(\mu) = \frac{(2g-2+n+d)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} P_{g,n}(\mu), \tag{32}$$

where g is the genus of the domain of Hurwitz cover of degree $d = \sum_{i=1}^n \mu_i$ and $P_{g,n}$ is a symmetric polynomial satisfying the following properties:

- $deg(P_{g,n}) = 3g-3+n$
- $P_{g,n}$ does not have any term of degree less than $2g+n-3$
- The sign of the coefficient of a monomial of degree d is $(-1)^{d-(3g-3+n)}$

It was later realized that the polynomial above can be expressed in terms of integral over the moduli space of stable curves of genus g. The equivalent formulation of simple Hurwitz number was discovered when localization technique was applied on the moduli space of stable curves.

Theorem 1. (ELSV Formula) *Let C be pre-stable curve of genus g with n-marked points, and h be the Hurwitz cover $h : C \to \mathbb{P}^1$ of degree d with ramification profile $\mu := \{\mu_1, .., \mu_n\}$ over $\infty \in \mathbb{P}^1$, i.e. $d = \sum_{i=1}^n \mu_i$. Also, denote $\overline{\mathcal{M}}_{g,n}$ as the moduli space of genus g stable curves with n marked points, and λ_i, $\overline{\psi}_i$ as the Hodge classes and descendent classes on $\overline{\mathcal{M}}_{g,n}$, respectively. Then the simple Hurwitz number $H_g(\mu_1, .., \mu_n)$, counting weighted Hurwitz covers h of degree d from curve C of genus g to $\mathbb{P}^1$ with ramification μ over ∞ and simple ramification elsewhere, can be expressed as*

$$H_g(\mu) = \frac{(2g-2+n+d)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\sum_{j=0}^{g} (-1)^j \lambda_j}{\prod_{i=1}^{n} (1 - \mu_i \overline{\psi}_i)}, \tag{33}$$

for $g = 0, n \geq 3$ or $g \geq 1$.

Proof. The reader is referred to the paper **[ELSV]** for the proof of the formula. □

It will be demonstrated later than this way of expressing simple Hurwitz number is going to be useful for relating it to topological recursion and Gromov-Witten invariants. On a different note, it was mentioned earlier that localization technique on $\overline{\mathcal{M}}_{g,n}$ can be applied as well in Gromov-Witten context. Through this, Gromov-Witten invariants of $\mathbb{C}^3$ can be expressed also via Hodge integrals.

Theorem 2. *The open Gromov-Witten invariant G_g of $\mathbb{C}^3$, counting holomorphic maps $\varphi : \Sigma \to \mathbb{C}^3$, can be formulated in terms of the Hodge integral and of the finite set $\mu := (\mu_1, .., \mu_n)$ denoting the winding numbers of the n holes of the genus g bordered Riemann surface Σ:*

$$\begin{aligned} G_g(\mu_1, .., \mu_n) = \frac{(-1)^{g+n}}{|Aut(\mu)|} (f(f+1))^{n-1} \prod_{i=1}^{n} \frac{\prod_{j=1}^{\mu_i} (\mu_i f + j)}{\mu_i!} \\ \times \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f)}{\prod_{i=1}^{n} (1 - \mu_i \overline{\psi}_i)}. \end{aligned} \tag{34}$$

Here, the parameter f is called the framing of open Gromov-Witten invariants, (framing will be discussed in section 2.5). As in the Hurwitz context, $\overline{\psi}_i$ is the descendent classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of Σ, and

$$\Lambda_g^\vee(y) := y^g \sum_{i=0}^{g} (-1/y)^i \lambda_i \tag{35}$$

wherein λ_i are the Hodge classes on $\overline{\mathcal{M}}_{g,n}$.

Proof. See [**KL**]. $\square$

2.2 Infinite Framing Limit of Open Gromov-Witten Invariants of $\mathbb{C}^3$

The explicit formulas for simple Hurwitz numbers (33) and Gromov-Witten invariants of $\mathbb{C}^3$ (34) exhibit similar features. They only differ by the prefactors, numerator in the integrand and the quantity μ has a different meaning in each theory. Also, the framing f only exists on the Gromov-Witten side. It turns out that the expression for simple Hurwitz number will be recovered by sending f to infinity in the Gromov-Witten theory. We show here a direct calculation that leads to this infinite framing limit in [**BM**].

Theorem 3. *Let*

$$F_g(x_1, .., x_n) = \sum_{\mu_i \in \mathbb{Z}^+} G_g(\mu_1, .., \mu_n) \prod_{i=1}^{n} x_i^{\mu_i}, \tag{36}$$

$$N_g(x_1, .., x_n) = \sum_{\mu_i \in \mathbb{Z}^+} \frac{1}{(2g-2+n+d)!} H_g(\mu_1, .., \mu_n) \prod_{i=1}^{n} x_i^{\mu_i}. \tag{37}$$

be the generating function for open Gromov-Witten invariants of $\mathbb{C}^3$ and for simple Hurwitz numbers, respectively. Then these two are related by

$$N_g(x_1, .., x_n) = \lim_{f \to \infty} \left((-1)^n f^{2-2g-n} F_g \left(\frac{x_1}{f}, .., \frac{x_n}{f} \right) \right). \tag{38}$$

Proof. Consider the following factors when $f \to \infty$ in the Gromov-Witten invariant formula (34):

- Clearly, the factor $(f(f+1))^{n-1}$ becomes f^{2n-2}.

- At large f,

$$\prod_{i=1}^{n} \frac{\prod_{j=1}^{\mu_i}(\mu_i f + j)}{\mu_i!} \tag{39}$$

goes to

$$f^{\sum_{j=1}^{n}\mu_j} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!}. \tag{40}$$

- For the Hodge integral, $\Lambda_g^\vee(-f-1)\Lambda_g^\vee(f) \to \Lambda_g^\vee(-f)\Lambda_g^\vee(f)$. Then by Mumford relation [**Mum**],

$$\Lambda_g^\vee(-f)\Lambda_g^\vee(f) = (-1)^g f^{2g}. \tag{41}$$

Using the definition in (35), the Hodge integral becomes

$$(-1)^g f^{2g} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\sum_{j=0}^{g}(-1)^j \lambda_j}{\prod_{i=1}^{n}(1-\mu_i \overline{\psi}_i)}. \tag{42}$$

Now, to fix the prefactors we just construct the generating function (37) for simple Hurwitz numbers. Also, the formal variable has to be rescaled by $x_i \to x_i/f$ in order to cancel the factor $f^{\sum_{j=1}^{n}\mu_j}$. Collecting everything, we then have the relation

$$N_g(x_1,..,x_n) = \lim_{f\to\infty}\left((-1)^n f^{2-2g-n} F_g\left(\frac{x_1}{f},..,\frac{x_n}{f}\right)\right). \tag{43}$$

□

This link between open Gromov-Witten invariants of $\mathbb{C}^3$ and simple Hurwitz numbers motivates the conjecture of Bouchard-Mariño, which claims that the simple Hurwitz numbers can be constructed recursively through Eynard-Orantin topological recursion. The conjecture is based on the remodeling theory, which is the topic in the next section.

2.3 Remodeling Theory

As mentioned earlier, topological string theory comes into two types, A- and B-model, which are related by the mirror symmetry. Topological string amplitudes in A-model are related to Gromov-Witten invariants. A formalism was developed in [**AKMV**] called *topological vertex*, which computes these A-model amplitudes on toric Calabi-Yau threefold. However, this technique is not applicable to all points on the underlying Kahler moduli space. Specifically, orbifold and conifold points are not suited for this formalism.

This is where the *remodeling theory* comes into play, for it enables the calculation of all topological string amplitudes of toric Calabi-Yau orbifolds.

Throughout this section, $\mathbb{C}^3$ will be considered as the target toric Calabi-Yau threefold. We discuss first the B-model before presenting the meaning of remodeling theory for $\mathbb{C}^3$ and role of Eynard-Orantin topological recursion in this formalism.

2.3.1 Mirror Symmetry for Toric Calabi-Yau Threefold

In the case in which the Calabi-Yau threefold (or three orbifold) X is a toric, the mirror target space $\tilde{X}$ in the B-model [**HIVf**] is a family of hypersurfaces in $\mathbb{C}^2(\xi_1, \xi_2) \times (\mathbb{C}^*)^2(x, y)$,

$$\xi_1 \xi_2 = P_X(x, y), \tag{44}$$

where $P_X(x, y)$ is the Newton polynomial corresponding to the polytope formed by the fan triangulation of toric Calabi Yau X in the A-model. The zero locus $P_X(x, y) = 0$ is called the **mirror curve** of the B-model (or of X).

As a curve embedded on $\mathbb{C}^* \times \mathbb{C}^*$, the mirror curve has reparameterization group $SL(2, \mathbb{Z})$ which acts as

$$(x, y) \to (x^a y^b, x^c y^d), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{45}$$

Under mirror symmetry [**AV**],[**AKV**], it turns out that:

- The mirror curve of the toric Calabi-Yau threefold X can be visualized as a thickening of the toric diagram Γ_X of X such that the number of closed loops in Γ_X corresponds to genus of the mirror curve, while the toric branes in Γ_X corresponds to points on the mirror curve.
- The formal parameters x_i in the generating function (36) of the A-model open amplitudes (open Gromov-Witten invariants) correspond to choice of projection of mirror curve of B-model onto $\mathbb{C}^*$.

- Choosing a different projection is analogous to changing the parameterization of the mirror curve. Thus, this change leads to moving the location of toric brane on the toric diagram Γ_X.

- There exists a (one-parameter) subgroup of reparameterization group of the mirror curve of B-model which fixes the position of the toric brane on the toric diagram Γ_X in A-model side. The action of this subgroup is given by a reparameterization called **framing transformation**:

$$(x, y) \to (xy^{-f}, y), \qquad f \in \mathbb{Z}. \tag{46}$$

The integer f is called **framing** of the toric brane (or Gromov-Witten invariants) on the A-model, which is an ambiguity in the computation of open string amplitudes. Therefore, fixing the location and framing of the toric brane on A-model will amount to fixing the parameterization of the mirror curve of B-model.

Example:
1) The rays for the fan of $\mathbb{C}^3$ can be taken to be $(0,0,1),(0,1,1),(1,0,1)$. So the fan triangulation of $\mathbb{C}^3$ is a triangle whose vertices are $(0,0),(0,1),(1,0)$. The Newton polynomial, in x, y variables, associated to this triangle is $1-y-x$, where the coefficients were chosen by convention. Therefore, the mirror curve of $\mathbb{C}^3$ is

$$1 - y - x = 0. \tag{47}$$

Applying the framing transformation (46), we get the framed mirror curve

$$-y^{f+1} + y^f - x = 0. \tag{48}$$

2) If we have the toric Calabi-Yau orbifold $\mathbb{C}^3/\mathbb{Z}_a$, where $\mathbb{Z}_a$ acts as

$$(z_1, z_2, z_3) \to (\alpha z_1, \alpha^s z_2, \alpha^{-s-1} z_3), \qquad s \in \mathbb{Z}, \quad \alpha = e^{2\pi i/a}, \tag{49}$$

then its rays can be taken to be $(0,0,1),(0,1,1),(a,-s,1)$. It follows that the Newton polynomial associated to its fan triangulation is

$$1 - y - x^a y^{-s} = 0, \tag{50}$$

where the signs where chosen by convention. Framing this will result to the framed mirror curve of $\mathbb{C}^3/\mathbb{Z}_a$

$$y^{s+af}(1-y) - x^a = 0. \tag{51}$$

3) As mentioned in subsection 1.4.2, $\mathcal{O}(-3) \to \mathbb{P}^2$ is an example of toric Calabi-Yau. Its rays are $(-1,-1,1),(1,0,1),(0,1,1),(0,0,1)$. So the fan triangulation of this has vertices $(-1,-1),(1,0),(0,1)$. Then the associated Newton polynomial is

$$1 - x - y - \frac{1}{xy} = 0. \tag{52}$$

Framing this gives the framed mirror of $\mathcal{O}(-3) \to \mathbb{P}^2$,

$$1 - xy^{-f} - y - \frac{1}{xy^{-f+1}} = 0. \tag{53}$$

2.3.2 Remodeling Theory for $\mathbb{C}^3$

Topological string B-model on mirror toric Calabi-Yau threefold is part of a larger physical class called *chiral boson theory* on a (*quantum*) Riemann surface. In [**ADKMV**],[**M**], it was argued that the topological recursion from the matrix models generates the amplitudes of the chiral boson theory, as the recursion depends only on the Riemann surface. Following this line of thought, it was claimed in [**M**] that the B-model amplitudes can be extracted as well from the topological recursion. Developing this B-model formalism, including the framing context, leads to the remodeling conjecture in [**BKMP**], which was proved independently in the open sector in [**Chen**] and [**Zhou**], closed sector in [**BCMS**] for the target space $\mathbb{C}^3$, and in full generality in [**EO2**] and [**FLZ**]:

Remodeling theory for $\mathbb{C}^3$: The differentials $d_1..d_n F_g(x_1,..,x_n)$ of the generating function for Gromov Witten invariants (A-model amplitudes) of $\mathbb{C}^3$ satisfy the topological recursion, wherein the associated spectral curve is the framed mirror curve of a toric Calabi-Yau threefold $X := \mathbb{C}^3$. In addition, the recursion starts with the fundamental one form

$$dF_0 = log(y)\frac{dx}{x}. \tag{54}$$

Note that in the above theory, we assumed the case when the *mirror map*, transformation relating quantities in A- and B-model, is trivial. (Mirror map is trivial for genus 0 curve). So the amplitudes are the same in both topological string models. The precise statement of the remodeling conjecture is that the B-model amplitude satisfy the topological recursion. Hence, mirror map has to be determined for nontrivial mirror map so that the Gromov-Witten invariants defined on A-model can be obtained. Moreover, differentials of the generating function are the object that can be constructed recursively since the quantity that obeys the topological recursion are meromorphic differentials. Another important thing is the crucial difference that the mirror curve lives on $\mathbb{C}^* \times \mathbb{C}^*$, rather than $\mathbb{C}^2$. In turn, the one form $\eta(z)$ that is needed in the recursion takes the form

$$\eta(z) = (log\{y(z)\} - log\{y(\bar{z})\})\frac{dx(z)}{x(z)} \tag{55}$$

instead of $\eta(z) = [y(z) - y(\bar{z})]dx(z)$ in (9), reflecting the notion that the symplectic form on $\mathbb{C}^* \times \mathbb{C}^*$ is

$$\frac{dx}{x} \wedge \frac{dy}{y}. \tag{56}$$

2.4 Bouchard-Mariño Theory

Taking into consideration the remodeling theory in the last section and the infinite framing limit in theorem 3, which relates simple Hurwitz numbers and open Gromov-Witten invariants of $\mathbb{C}^3$, we are naturally led into thinking that simple Hurwitz numbers should be related as well to the topological recursion.

The question out from this is what is the spectral curve that will be used for recursive construction of simple Hurwitz numbers. Recall from the infinite framing statement that the generating function for simple Hurwitz numbers can be recovered from the generating function for open Gromov-Witten invariants by sending $f \to \infty$. As claimed in [**BM**], the spectral curve for this recursion can be obtained by taking $f \to \infty$ in the framed mirror curve in (48) from the Gromov-Witten side.

The infinite framing limit in (43) implies that the variables x, y must be rescaled. By (43), it is clear that x should be reparameterized to x/f. For the variable y, consider the initial value (54),

$$dF_0 = log(y)\frac{dx}{x}, \tag{57}$$

and the infinite framing limit (43). It follows that we should have the rescaling

$$y \to 1 - \frac{y}{f}. \tag{58}$$

This is because when $f \to \infty$, $dF_0 = log(1 - \frac{y}{f})\frac{dx}{x}$ has leading order of

$$-\frac{y}{f}\frac{dx}{x}, \tag{59}$$

and the $-1/f$ above will cancel out the factor $(-1)^n f^{2-2g-n}$ in the infinite framing limit (43) for all g and n, since

$$\eta(z) \to \frac{-1}{f}\eta(z) \quad \Rightarrow \quad W_g^{(n)} \to (-1)^n f^{2g+n-2} W_g^{(n)}. \tag{60}$$

So reparameterizing the variables in framed mirror curve (48) as

$$x \to \frac{x}{f}, \quad y \to 1 - \frac{y}{f}, \tag{61}$$

the framed mirror curve becomes

$$\frac{x}{f} = \frac{y}{f}\Big(1 - \frac{y}{f}\Big)^f. \tag{62}$$

Taking the limit at large f, we obtain the *Lambert curve*

$$x = ye^{-y}. \tag{63}$$

Bouchard-Mariño Theory: The differentials $d_1..d_nN_g(x_1,..x_n)$ of the generating functions for simple Hurwitz numbers can be constructed recursively using the topological recursion, with the spectral curve given by the Lambert curve $x = ye^{-y}$ and the fundamental one form $dN_0 = ydx/x$.

This was conjectured in [**BM**] by the reasoning above and through comparing the simple Hurwitz numbers calculations using the conjecture and its formula via Hodge integrals. The conjecture was later proved in [**EMS**] by first calculating the Laplace transform of the *cut-and-join equation* (which expresses Hurwitz number of a given genus and profile in terms of profiles modified by either cutting a part into two pieces or joining two parts into one). Then identifying that the direct image of this Laplace transform is the topological recursion.

3 Orbifold Hurwitz Numbers and Open Gromov-Witten Invariants of $\mathbb{C}^3/\mathbb{Z}_a$

The goal of this chapter is to extend the ideas from the previous chapter to orbifold setting. First section in this chapter is about double Hurwitz numbers in terms of Hodge integrals. It is then be followed by Gromov-Witten invariants. The target space that is considered is the toric Calabi-Yau orbifold $\mathfrak{X} := \mathbb{C}^3/\mathbb{Z}_a$, i.e. toric Calabi Yau threefold quotiented by finite group $\mathbb{Z}_a$. Then connections among these, together with topological recursion, are the topics in the remaining sections of this chapter.

3.1 JPT Formula

In section 2.1, simple Hurwitz numbers $H_g(\mu := (\mu_1, .., \mu_n))$ has an expression, called ELSV formula, in terms of Hodge integrals. This was obtained using localization on moduli space $\overline{\mathcal{M}}_{g,n}$ of genus g pre-stable curves with n marked points. Extending this process to moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)$ of stable maps from twisted curve $\mathcal{C}$ to classifying space of the cyclic group $\mathbb{Z}_a$ gives a relationship between double Hurwitz number $H_g(\nu, \mu)$ and Hodge integral. This was the main result in **[JPT]**, and we present in this section the necessary ingredients in the relation proved in **[JPT]**, without explaining the proof through localization. The *orbifold Hurwitz number* (to be defined shortly) is just a special type of double Hurwitz numbers, and so similar result applies for this case.

Let $\mathcal{C}$ be a twisted curve of genus g, and h be the Hurwitz cover $h : C \to \mathbb{P}^1$ of degree d with non-simple ramification profiles ν, μ over $0, \infty \in \mathbb{P}^1$. For $\mu_j \in \mathbb{Z}^+$ and $\gamma_i \in \mathbb{Z}_a \backslash \{0\}$, suppose the partitions of degree d into elements of ν and μ are as follows:

$$d = \sum_{i=1}^{r+t} \nu_i, \qquad \nu := \{\gamma_1, .., \gamma_r, a, .., a\}, \tag{64}$$

$$d = \sum_{j=1}^{n} \mu_j, \qquad \mu := \{\mu_1, .., \mu_n\}, \tag{65}$$

where the number of a in the ramification profile ν is

$$t := \frac{d - \sum \gamma_i}{a}, \tag{66}$$

and we define $\sum_{i=1}^{a-1} im_i = \sum_{j=1}^{r} \gamma_j$; i.e., there are $m_i \in \mathbb{N}$ number of i in the vector $\gamma := (\gamma_1, .., \gamma_r)$.

Denote $H_g(\nu(\gamma, a), \mu)$ as the double Hurwitz number counting weighted Hurwitz covers h of degree d from curve $\mathcal{C}$ of genus g to $\mathbb{P}^1$, with ramifications ν, μ as above.

The **orbifold Hurwitz number** $H_g(\nu(a), \mu)$ is a special type of double Hurwitz number wherein $\gamma = \emptyset$ in (64). Take note that $a \neq 1$ here so it is still a particular case of double Hurwitz number. For $a = 1$, we recover simple Hurwitz numbers.

Theorem 4. (JPT Formula) *The double Hurwitz number $H_g(\nu(\gamma, a), \mu)$, counting weighted Hurwitz covers h of degree d from twisted curve (Riemann surface) C of genus g to $\mathbb{P}^1$, with ramifications ν, μ as in* (64),(65)*, can be expressed in terms of Hodge integral as*

$$\begin{aligned} H_g(\nu(\gamma, a), \mu) = & \frac{(2g - 2 + r + n + \frac{d}{a} - \sum_{i=1}^{a-1} \frac{im_i}{a})!}{|Aut(\gamma)||Aut(\mu)|} \\ & \times \quad a^{1-g-\sum_{i=1}^{a-1} \frac{im_i}{a} + \sum_{i=1}^{n} \langle \frac{\mu_i}{a} \rangle} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \frac{\mu_i}{a} \rfloor}}{\lfloor \frac{\mu_i}{a} \rfloor !} \\ & \times \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)} \frac{\sum_{j \geq 0} (-a)^j \lambda_j^U}{\prod_{i=1}^{n} (1 - \mu_i \psi_i)}. \end{aligned} \tag{67}$$

The integer and fractional parts of a number $\theta \in \mathbb{Q}$ above is given by $\theta = \lfloor \theta \rfloor + \langle \theta \rangle$. The above formula was proved in **[JPT]** by considering virtual localization technique on the moduli space of stable maps to stack $\mathbb{P}^1[a]$. Take note that the formula is valid provided that the following conditions are satisfied:

$$d - \sum_{i=1}^{a-1} i m_i = 0 \ (\text{mod } a) \qquad (\textit{nonemptiness}), \tag{68}$$

$$d - \sum_{i=1}^{a-1} i m_i \geq 0 \qquad (\textit{nonnegativity}), \tag{69}$$

$$\forall i \neq j, \quad \gamma_i + \gamma_j \leq a \qquad (\textit{boundedness}). \tag{70}$$

Consequently, the orbifold Hurwitz number (or when $\gamma = \emptyset$ but $a \neq 1$) can be written as

$$H_g(\nu(a), \mu) = \frac{(2g - 2 + n + \frac{d}{a})!}{|Aut(\mu)|} a^{1-g+\sum_{i=1}^{n} \langle \frac{\mu_i}{a} \rangle} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \frac{\mu_i}{a} \rfloor}}{\lfloor \frac{\mu_i}{a} \rfloor !} \int_{\overline{\mathcal{M}}_{g,-\mu}(B\mathbb{Z}_a)} \frac{\sum_{j \geq 0} (-a)^j \lambda_j^U}{\prod_{i=1}^{n} (1 - \mu_i \psi_i)}. \tag{71}$$

When $a = 1$ ($\gamma = \emptyset$ in turn), the formula for double Hurwitz number in terms of Hodge integral reduces to ELSV formula,

$$H_g(\mu) = \frac{(2g - 2 + n + d)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\sum_{j=0}^{g} (-1)^j \lambda_j}{\prod_{i=1}^{n} (1 - \mu_i \psi_i)}, \tag{72}$$

which expresses the simple Hurwitz number as an integral over the moduli space of stable curves.

3.2 Open Gromov-Witten Invariants of $\mathbb{C}^3/\mathbb{Z}_a$

We present in this section the results in [**BC**] and [**Ross**], which relate the open orbifold Gromov-Witten invariants $\mathbb{C}^3/\mathbb{Z}_a$ and Hodge integrals.

Consider the toric Calabi-Yau orbifold $\mathfrak{X} := \mathbb{C}^3/\mathbb{Z}_a$ as the target space, where group $\mathbb{Z}_a$ acts on $(z_1, z_2, z_3) \in \mathbb{C}^3$ as

$$(z_1, z_2, z_3) \to (\epsilon^1 z_1, \epsilon^s z_2, \epsilon^{-s-1} z_3), \tag{73}$$

with $\epsilon = e^{2\pi i/a}$, and toric weights $w := (w_1, w_2, w_3) = (1, s, -s-1)$, $s \in \mathbb{Z}$.

This orbifold is treated as an open chart of a global quotient of resolved conifold, i.e. $\mathfrak{X} \subset [\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_a]$. By this identification, the $\mathbb{C}^*$ action on the resolved conifold must lift the canonical action on $\mathbb{P}^1$, descend to the quotient, be compatible with the properties of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and with the anti-holomorphic involution defined on it. It was argued in [**AC**] that only the Calabi-Yau action (action in which the sum of weights vanishes) satisfies these conditions. Choosing fractional weights (t_1, t_2, t_3) by convention, the weights for the induced $\mathbb{C}^*$ action on the resolved conifold can be taken to be

$$(t_1, t_2, t_3) := \left(\frac{1}{a}, f, -f - \frac{1}{a}\right), \tag{74}$$

where the parameter $f \in \mathbb{Z}$ is the framing.

Through localization, open Gromov-Witten theory can be viewed as theory of stable maps wherein the fixed loci on the moduli space of stable maps consist of maps, $\varphi : \Sigma \to \mathfrak{X}$, from twisted Riemann surface of genus g, with n disks attached at n distinct twisted nodes and r marked points, to orbifold $\mathfrak{X}$. This map contracts the compact Riemann surface to the origin, and sends the attached disk d_i to the i-th axis with disk boundary winding around the intersection of i-th axis and Lagrangian submanifold $\mathcal{L}_i \subset \mathfrak{X}$. The fixed locus is encoded by (i) winding number $\mu_i \in \mathbb{Z}^+$, $i = 1, ...n$, of a disk d_i, and by (ii) twisting $k_i \in \mathbb{Z}_a$ of the nodes attaching the corresponding disk d_i. The case that we are going to consider is the *effective* action of $\mathbb{Z}_a$, which means that the Lagrangian submanifold $\mathcal{L} \subset \mathfrak{X}$ intersects the z_1-axis of $\mathfrak{X}$ where the action of $\mathbb{Z}_a$ is effective.

Also, it was shown that in order for the locus to be nonempty, the following relation must hold:

$$\mu_i \equiv k_i \ (\mathrm{mod}\ a), \qquad \forall i = 1, .., n. \tag{75}$$

Taking into account the objects described above, and the contribution from each combinatorial data [**KL**],[**BC**] of the fixed locus in localization, then the open orbifold Gromov-Witten invariant of $\mathfrak{X} := \mathbb{C}^3/\mathbb{Z}_a$ (with m_j insertions of classes $\mathbb{1}_j$ of orbifold cohomology $H^*_{CR}(\mathfrak{X})$ of $\mathfrak{X}$) has the following form:

$$G_g(\mathbb{1}_1^{m_1}...\mathbb{1}_{a-1}^{m_{a-1}},\mu) = \frac{a^n}{|Aut(\gamma)||Aut(\mu)|}\prod_{j=1}^{n}\delta_0^{(j)}\delta_1^{(j)}\delta_2^{(j)}D_{k_j}(d_i,a)$$
$$\times\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)}\frac{e^{eq}(\mathbb{E}_1^{\vee}(1/a)\oplus\mathbb{E}_s^{\vee}(f)\oplus\mathbb{E}_{-s-1}^{\vee}(-f-1/a))}{\prod_{j=1}^{n}(\frac{1}{\mu_j}-\psi_j)}. \qquad (76)$$

Here, $\delta_i = t_i$ if $kw_i \equiv 0 \pmod{a}$ and 1 otherwise. D_{k_i} is called the *disk function*, defined as

$$D_{k_i}(\mu_i,a) := \left(\frac{1}{\mu_i}\right)^{2\left\lfloor\frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle-\frac{gcd(k_i,a)}{a}\right\rfloor-\left\lfloor\frac{\mu_i}{a}\right\rfloor+1}\frac{1}{\left\lfloor\frac{\mu_i}{a}\right\rfloor!}\frac{\Gamma(\mu_i t_1+\langle\frac{k_j\alpha_2}{a}\rangle+\frac{\mu_i}{a})}{\Gamma(\mu_i t_1-\langle\frac{k_j\alpha_1}{a}\rangle+1)}. \qquad (77)$$

The numerator in the integrand is the equivariant Euler class of three copies of the dual of w_i-character sub-bundles of Hodge bundle $\mathbb{E}_{w_i}(t_i)$. ψ_i are the descendent classes on moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)$. Note that the m_i will appear on the right side of (76) upon integration over $\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)$ (as will be seen in section 4.1).

3.3 Remodeling Theory for $\mathbb{C}^3/\mathbb{Z}_a$

For the first connection, topological recursion and Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ are also related by the remodeling theory. The work in [**BKMP**] extends the remodeling theory for $\mathbb{C}^3$ in [**M**] to the case when the target space is the orbifold $\mathbb{C}^3/\mathbb{Z}_a$. This was proved in [**FLZ**]. In the case where the mirror map is trivial, the remodeling theory simply states that:

Remodeling theory for $\mathbb{C}^3/\mathbb{Z}_a$: The differentials $d_1..d_nF_g(x_1,..,x_n)$ of the generating function for open Gromov Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (with or without insertion of orbifold cohomology classes $\mathbb{1}_j$) can be obtained recursively using Eynard-Orantin topological recursion, with spectral curve given by the framed mirror curve of $\mathbb{C}^3/\mathbb{Z}_a$ and fundamental one form $dF_0 = \log(y)dx/x$.

3.4 Infinite Framing Limit of Open Gromov-Witten Invariants $\mathbb{C}^3/\mathbb{Z}_a$

In order to relate all the topological recursion, Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (without insertion of cohomology classes) and orbifold Hurwitz numbers, we first need to extend the infinite framing (theorem 3) for Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$. The extension to orbifold case was shown [**BHLM**], and it is stated in the next theorem.

Theorem 5. *If we set the generating function for orbifold Hurwitz numbers and open Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (without insertion of cohomology classes) as*

$$N_g(x_1,..,x_n) = \sum_{\mu_i\in\mathbb{Z}^+} \frac{1}{(2g-2+n+\frac{d}{a})!}\left(H_g(\nu(a),\mu)\prod_{i=1}^{n} x_i^{\mu_i}\right). \tag{78}$$

,

$$F_g(x_1,..,x_n) = \sum_{\mu_i\in\mathbb{Z}^+}\left((-1)^{g-1+\sum_{i=1}^{n}\langle\frac{-\mu_i(s+1)}{a}\rangle}G_g(\mu)\prod_{i=1}^{n} x_i^{\mu_i}\right), \tag{79}$$

respectively, then these are related by

$$N_g(x_1,..,x_n) = \lim_{f\to\infty}\left((-1)^n f^{2-2g-n}F_g\left(\frac{x_1}{f^{1/a}},..,\frac{x_n}{f^{1/a}}\right)\right). \tag{80}$$

This will be shown later as a particular case of the first theorem in next chapter. By remodeling theory, the differential of F_g is related to topological recursion, then this theorem suggests that the orbifold Hurwitz numbers can be generated as well through the recursion. The spectral curve will then be obtained by reparameterizing the framed mirror curve (51) of $\mathbb{C}^3/\mathbb{Z}_a$ and then taking its limit as $f\to\infty$.

Using the same reasoning in section 2.4, variables x, y are reparameterized as

$$x \to \frac{x}{f^{1/a}}, \qquad y \to 1-\frac{y}{f}. \tag{81}$$

This will in turn transform the framed mirror curve (51) into

$$x^a = y\left(1-\frac{y}{f}\right)^{af}\left(1-\frac{y}{f}\right)^{s}. \tag{82}$$

Taking the limit as $f \to \infty$, we get the *a-Lambert-curve*,

$$x^a = ye^{-ay}, \tag{83}$$

as the spectral curve for the differentials of generating function for orbifold Hurwitz numbers.

Theorem 6. *The differentials $d_1..d_n N_g(x_1,..x_n)$ of the generating function (78) for orbifold Hurwitz numbers can be obtained recursively through topological recursion, with the spectral curve given by the a-Lambert curve $x^a = ye^{-ay}$ and fundamental one form $dN_0 = ydx/x$.*

Proof. Complete proof can be found in [**BHLM**]. This was proved by identifying that the Laplace transform of the cut-and-join equation, after Galois averaging and restricting the principal part, changes into Eynard-Orantin topological recursion defined on the a-Lambert curve. □

4 Generalization to Double Hurwitz Numbers

This chapter is about the new developments in the study. It will be shown that previous results can be obtained as particular cases of the generalization made in this part of the thesis.

4.1 Generalized Infinite Framing Limit

In section 2.2, as the framing parameter f goes to infinity, the generating function $N_g(x_1,..,x_n)$ for simple Hurwitz numbers $H_g(\mu)$ can be recovered from the generating function $F_g(x_1,..,x_n)$ for Gromov-Witten invariants of $\mathbb{C}^3$:

$$N_g(x_1,..,x_n) = \lim_{f\to\infty}\left((-1)^n f^{2-2g-n} F_g\left(\frac{x_1}{f},..,\frac{x_n}{f}\right)\right). \tag{84}$$

The following question is how are the Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ and the double Hurwitz numbers $H_g(\nu(\gamma,a),\mu)$ are related. A calculation in **[BHLM]** shows that the generating function $N_g(x_1,..,x_n)$ for the orbifold Hurwitz numbers $H_g(\nu(a),\mu)$ and the generating function $F_g(x_1,..,x_n)$ for Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (without insertion of cohomology classes) also exhibit the infinite framing limit but with the rescaling $x_i \to x_i/f^{1/a}$:

$$N_g(x_1,..,x_n) = \lim_{f\to\infty}\left((-1)^n f^{2-2g-n} F_g\left(\frac{x_1}{f^{1/a}},..,\frac{x_n}{f^{1/a}}\right)\right). \tag{85}$$

We extend these notions to the full double Hurwitz numbers and to the case when there are insertions of cohomology classes in the Gromov-Witten side, i.e. when $\gamma \neq \emptyset$. We will show that the next statement is true by explicit calculations.

Theorem 7. *Let $x = (x_1,..,x_n)$ and $\tau = (\tau_1,..,\tau_{a-1})$ be formal parameters. If*

$$\begin{aligned} N_g(x,\tau) = \sum_{\mu_i\in\mathbb{Z}^+}\sum_{m_j\in\mathbb{N}} &\frac{1}{(2g-2+r+n+\frac{d}{a}-\sum_{i=1}^{a-1}\frac{im_i}{a})!} \\ &\times (H_g(\nu(\gamma,a),\mu)\prod_{i=1}^{n} x_i^{\mu_i}\prod_{j=1}^{a-1}\tau_j^{m_j}) \end{aligned} \tag{86}$$

and

$$\begin{aligned} F_g(x,\tau) = \sum_{\mu_i\in\mathbb{Z}^+}\sum_{m_j\in\mathbb{N}} &((-1)^{g-1+\sum_{i=1}^{n}\langle\frac{-\mu_i(s+1)}{a}\rangle-\sum_{i=1}^{a-1}m_i\langle\frac{-i(s+1)}{a}\rangle} \\ &\times G_g(\mathbb{1}_1^{m_1}...\mathbb{1}_{a-1}^{m_{a-1}},\mu)\prod_{i=1}^{n}x_i^{\mu_i}\prod_{j=1}^{a-1}\tau_j^{m_j}) \end{aligned} \tag{87}$$

denotes the generating functions for (full) double Hurwitz numbers and open Gromov Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (with insertion of orbifold cohomology classes), respectively. Then we have the generalized infinite framing limit,

$$N_g(x,\tau) = \lim_{f\to\infty}\left((-1)^n f^{2-2g-n} F_g\left(\frac{x}{f^{1/a}}, \frac{\tau}{f^{\left\langle\frac{is}{a}\right\rangle+\left\langle-\frac{i(s+1)}{a}\right\rangle}}\right)\right), \tag{88}$$

where the rescaling of variables denotes

$$\frac{x}{f^{1/a}} := (\frac{x_1}{f^{1/a}}, ..., \frac{x_n}{f^{1/a}}), \tag{89}$$

$$\frac{\tau}{f^{\left\langle\frac{is}{a}\right\rangle+\left\langle-\frac{i(s+1)}{a}\right\rangle}} := \left(\frac{\tau_1}{f^{\left\langle\frac{1\cdot s}{a}\right\rangle+\left\langle-\frac{1\cdot(s+1)}{a}\right\rangle}}, \frac{\tau_2}{f^{\left\langle\frac{2\cdot s}{a}\right\rangle+\left\langle-\frac{2\cdot(s+1)}{a}\right\rangle}}, ..., \frac{\tau_{a-1}}{f^{\left\langle\frac{(a-1)\cdot s}{a}\right\rangle+\left\langle-\frac{(a-1)\cdot(s+1)}{a}\right\rangle}}\right). \tag{90}$$

Proof. The proof of this statement is similar to the procedure done in [**BHLM**] in proving (85), wherein we rewrite the open Gromov-Witten invariant expression into a form that is closer to the Hurwitz number formula in terms of Hodge integral and then we will evaluate the limit when framing $f \to \infty$. Clearly, when there are no insertions of orbifold cohomology classes then all m_i's vanish, which means that the generalized infinite framing limit (87) reduces to the infinite framing limit (85) established in [**BHLM**]

Let us obtain first the equivalent expressions for the following factors in open Gromov-Witten invariant formula for $\mathbb{C}^3/\mathbb{Z}_a$ (76):

- The factor $\delta_0\delta_1\delta_2$ can be written as $(1/a)^{\delta_{\langle\frac{k_i}{a}\rangle,0}} f^{\delta_{\langle\frac{k_i s}{a}\rangle,0}} (-f-\frac{1}{a})^{\delta_{\langle\frac{-k_i(s+1)}{a}\rangle,0}}$, due to the definition of δ_i. So taking the products of $\delta_0^{(i)}\delta_1^{(i)}\delta_2^{(i)}$ over i, we get

$$\left(\frac{1}{a}\right)^{\sum_{i=1}^n \delta_{\langle\frac{k_i}{a}\rangle,0}} f^{\sum_{i=1}^n \delta_{\langle\frac{k_i s}{a}\rangle,0}} \left(-f-\frac{1}{a}\right)^{\sum_{i=1}^n \delta_{\langle\frac{-k_i(s+1)}{a}\rangle,0}}. \tag{91}$$

- The disk function D_{k_i} becomes

$$D_{k_i}(\mu_i, f) = \left(\frac{1}{\mu_i}\right)^{2\left\lfloor\frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle-\frac{gcd(k_i,a)}{a}\right\rfloor-\left\lfloor\frac{\mu_i}{a}\right\rfloor+1} \frac{1}{\left\lfloor\frac{\mu_i}{a}\right\rfloor!}\frac{\Gamma(\mu_i f+\langle\frac{k_i(-s-1)}{a}\rangle+\frac{\mu_i}{a})}{\Gamma(\mu_i f-\langle\frac{k_i s}{a}\rangle+1)} \tag{92}$$

$$= \left(\frac{1}{\mu_i}\right)^{2\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor - \left\lfloor\frac{\mu_i}{a}\right\rfloor + 1} \frac{1}{\left\lfloor\frac{\mu_i}{a}\right\rfloor!}(\mu_i)^{\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor}$$
$$\times \prod_{j=1}^{\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor} (f - \frac{1}{\mu_i}\langle\frac{k_i s}{a}\rangle + \frac{j}{\mu_i}) \tag{93}$$

$$= \left(\frac{1}{\mu_i}\right)^{\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor - \left\lfloor\frac{\mu_i}{a}\right\rfloor + 1} \frac{1}{\left\lfloor\frac{\mu_i}{a}\right\rfloor!} \prod_{j=1}^{\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor} (f\mu_i - \langle\frac{k_i s}{a}\rangle + j) \tag{94}$$

We have used equation (17) in [**BC**] in moving from (92) to (93).

- From [**CC**], the equivariant Euler class in the integrand takes the form:

$$e^{eq}(\mathbb{E}^\vee_{\alpha^1}(1/a)\oplus\mathbb{E}^\vee_{\alpha^s}(f)\oplus\mathbb{E}^\vee_{\alpha^{-s-1}}(-f-\frac{1}{a})) = \Lambda_g^{\vee,\alpha}(\frac{1}{a})\Lambda_g^{\vee,\alpha^s}(f)\Lambda_g^{\vee,\alpha^{-s-1}}(-f-\frac{1}{a}), \tag{95}$$

where

$$\Lambda_g^{\vee,\alpha^t}(\omega) := \omega^{rk(\mathbb{E}_{\alpha^t})}\sum_{i=0}^{rk(\mathbb{E}_{\alpha^t})}(-1/\omega)^i\lambda_{i,\alpha^t}, \tag{96}$$

with $\mathbb{E}_{\alpha^t}$ as Hodge bundle corresponding to $\mathbb{Z}_a$ representation given by

$$\varphi_{\alpha^t}:\mathbb{Z}_a\to\mathbb{C}^*, \qquad \varphi_{\alpha^t}(1) = e^{\frac{2\pi i t}{a}}, \tag{97}$$

and $\lambda_{i,\alpha^t} := c_i(\mathbb{E}_{\alpha^t})$ are its Chern classes c_i.

Collecting the above equivalent expressions for each factor in (76), we obtain

$$G_g(\mathbb{1}_1^{m_1}...\mathbb{1}_{a-1}^{m_{a-1}},\mu) = \frac{a^{n-\sum_{i=1}^n \delta_{\langle\frac{k_i}{a}\rangle,0}}}{|Aut(\gamma)||Aut(\mu)|} f^{\sum_{i=1}^n \delta_{\langle\frac{k_i s}{a}\rangle,0}} \left(-f-\frac{1}{a}\right)^{\sum_{i=1}^n \delta_{\langle\frac{-k_i(s+1)}{a}\rangle,0}}$$
$$\times \prod_{i=1}^n \frac{1}{\mu_i^{\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor - \left\lfloor\frac{\mu_i}{a}\right\rfloor}\left\lfloor\frac{\mu_i}{a}\right\rfloor!}$$
$$\times \prod_{j=1}^{\left\lfloor \frac{\mu_i}{a}+\langle\frac{k_i s}{a}\rangle - \frac{gcd(k_i,a)}{a}\right\rfloor} (f\mu_i - \langle\frac{k_i s}{a}\rangle + j)$$
$$\times \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)} \frac{\Lambda_g^{\vee,\alpha}(\frac{1}{a})\Lambda_g^{\vee,\alpha^s}(f)\Lambda_g^{\vee,\alpha^{-s-1}}(-f-\frac{1}{a})}{\prod_{i=1}^n(1-\mu_i\psi_i)}. \tag{98}$$

Taking into account the dependancy of winding numbers μ_i and twisting k_i of attachment points, then $\exists I_i \in \mathbb{Z}, i = 1, ..., n$ such that $(\mu_i - k_i)/a = I_i$. Hence, $\langle \frac{k_i \lambda}{a} \rangle = \langle \frac{\mu_i \lambda}{a} \rangle$ for any $\lambda \in \mathbb{Z}$. Also, since the twisting k_i is constrained by $0 < k_i < a$, then $gcd(k_i, a) = gcd(\mu_i, a)$. Setting $t_{eff} := a/gcd(\mu_i, a)$, the open orbifold Gromov-Witten invariant becomes:

$$G_g(\mathbb{1}_1^{m_1} ... \mathbb{1}_{a-1}^{m_{a-1}}, \mu) = \frac{a^{n - \sum_{i=1}^n \delta_{\langle \frac{\mu_i}{a} \rangle, 0}}}{|Aut(\gamma)||Aut(\mu)|} f^{\sum_{i=1}^n \delta_{\langle \frac{\mu_i s}{a} \rangle, 0}} \left(-f - \frac{1}{a}\right)^{\sum_{i=1}^n \delta_{\langle \frac{-\mu_i(s+1)}{a} \rangle, 0}}$$
$$\times \prod_{i=1}^n \frac{1}{\mu_i^{\left\lfloor \frac{\mu_i}{a} + \langle \frac{\mu_i s}{a} \rangle - \frac{1}{t_{eff}} \right\rfloor - \left\lfloor \frac{\mu_i}{a} \right\rfloor} \left\lfloor \frac{\mu_i}{a} \right\rfloor !} \prod_{j=1}^{\left\lfloor \frac{\mu_i}{a} + \langle \frac{\mu_i s}{a} \rangle - \frac{1}{t_{eff}} \right\rfloor} (f\mu_i - \langle \frac{\mu_i s}{a} \rangle + j)$$
$$\times \int_{\overline{\mathcal{M}}_{g, \gamma - \mu}(B\mathbb{Z}_a)} \frac{\Lambda_g^{\vee, \alpha}(\frac{1}{a}) \Lambda_g^{\vee, \alpha^s}(f) \Lambda_g^{\vee, \alpha^{-s-1}}(-f - \frac{1}{a})}{\prod_{i=1}^n (1 - \mu_i \psi_i)}. \tag{99}$$

We now proceed to the evaluation of limit when the framing f goes to infinity.

We first consider the numerator in Hodge integral in Gromov-Witten invariant (99). Using the definition in (96), this has leading order term at large f expansion as

$$\Lambda_g^{\vee, \alpha}(\frac{1}{a}) \Lambda_g^{\vee, \alpha^s}(f) \Lambda_g^{\vee, \alpha^{-s-1}}(-f - \frac{1}{a}) \simeq \left(a^{-rk(\mathbb{E}_\alpha)} \sum_{i=0}^{rk(\mathbb{E}_\alpha)} (-a)^i \lambda_{i, \alpha} \right) f^{rk(\mathbb{E}_{\alpha^s})} (-f)^{rk(\mathbb{E}_{\alpha^{-s-1}})} \tag{100}$$

Through orbifold Riemann-Roch formula **[JPT]**, we have the following rank of Hodge bundles over $\overline{\mathcal{M}}_{g, \gamma - \mu}(B\mathbb{Z}_a)$ (More details on Appendix):

$$rk(\mathbb{E}_\alpha) = g - 1 + \sum_{i=1}^n \langle -\frac{\mu_i}{a} \rangle + \sum_{i=1}^{a-1} \frac{i m_i}{a}, \tag{101}$$

$$rk(\mathbb{E}_{\alpha^s}) = g - 1 + \sum_{i=1}^n \langle -\frac{\mu_i s}{a} \rangle + \sum_{i=1}^{a-1} m_i \left\langle \frac{is}{a} \right\rangle, \tag{102}$$

$$rk(\mathbb{E}_{\alpha^{-s-1}}) = g - 1 + \sum_{i=1}^n \langle \frac{\mu_i (s+1)}{a} \rangle + \sum_{i=1}^{a-1} m_i \left\langle -\frac{i(s+1)}{a} \right\rangle, \tag{103}$$

Thus, the Hodge integral in (99) has the following leading order term as $f \to \infty$:

$$\begin{aligned}(-1)^{g-1+\sum_{i=1}^n \langle \frac{\mu_i(s+1)}{a}\rangle + \sum_{i=1}^{a-1} m_i \left\langle -\frac{i(s+1)}{a}\right\rangle} a^{1-g-\sum_{i=1}^n \langle -\frac{\mu_i}{a}\rangle - \sum_{i=1}^{a-1}\frac{im_i}{a}} \\ \times f^{2g-2+\sum_{i=1}^n(\langle -\frac{\mu_i s}{a}\rangle + \langle \frac{\mu_i(s+1)}{a}\rangle) + \sum_{i=1}^{a-1} m_i \left(\left\langle \frac{is}{a}\right\rangle + \left\langle -\frac{i(s+1)}{a}\right\rangle\right)} \int_{\overline{\mathcal{M}}_g(B\mathbb{Z}_a)} \frac{\sum_{j\geq 0}(-a)^j \lambda_j}{\prod_{i=1}^n (1-\mu_i\psi_i)}.\end{aligned} \tag{104}$$

For the third line in (99), the leading order term is

$$\prod_{i=1}^n \frac{\mu_i^{\left\lfloor \frac{\mu_i}{a}\right\rfloor}}{\left\lfloor \frac{\mu_i}{a}\right\rfloor !} f^{\left\lfloor \frac{\mu_i}{a} + \langle \frac{\mu_i s}{a}\rangle - \frac{1}{t_{eff}}\right\rfloor}, \tag{105}$$

while for the first and second lines, we have

$$\frac{a^{n-\sum_{i=1}^n \delta_{\langle \frac{\mu_i}{a}\rangle,0}}}{|Aut(\gamma)||Aut(\mu)|} f^{\sum_{i=1}^n \delta_{\langle \frac{\mu_i s}{a}\rangle,0}} (-f)^{\sum_{i=1}^n \delta_{\langle \frac{-\mu_i(s+1)}{a}\rangle,0}}. \tag{106}$$

We then obtain the following overall exponents:

- For f:

$$\begin{aligned}&\sum_{i=1}^n \left(\langle -\frac{\mu_i s}{a}\rangle + \langle \frac{\mu_i(s+1)}{a}\rangle + \delta_{\langle \frac{\mu_i s}{a}\rangle,0} + \delta_{\langle -\frac{\mu_i(s+1)}{a}\rangle,0} + \left\lfloor \frac{\mu_i}{a} + \langle \frac{\mu_i s}{a}\rangle - \frac{1}{t_{eff}}\right\rfloor \right) \\ &\qquad + 2g-2+\sum_{i=1}^{a-1} m_i \left(\left\langle \frac{is}{a}\right\rangle + \left\langle -\frac{i(s+1)}{a}\right\rangle\right) \\ &= 2g-2+n+\sum_{i=1}^n \frac{\mu_i}{a} + \sum_{i=1}^{a-1} m_i \left(\left\langle \frac{is}{a}\right\rangle + \left\langle -\frac{i(s+1)}{a}\right\rangle\right),\end{aligned} \tag{107}$$

  where we have used the equality

$$\begin{aligned}\sum_{i=1}^n (\langle -\frac{\mu_i s}{a}\rangle + \langle \frac{\mu_i(s+1)}{a}\rangle + \delta_{\langle \frac{\mu_i s}{a}\rangle,0} + \delta_{\langle -\frac{\mu_i(s+1)}{a}\rangle,0} \\ + \left\lfloor \frac{\mu_i}{a} + \langle \frac{\mu_i s}{a}\rangle - \frac{1}{t_{eff}}\right\rfloor) = n + \sum_{i=1}^n \frac{\mu_i}{a}\end{aligned} \tag{108}$$

  from section 3.2 of [**BHLM**]. Note that $\left\langle \frac{is}{a}\right\rangle + \left\langle -\frac{i(s+1)}{a}\right\rangle$ has two possible values, $2-\frac{i}{a}$ or $1-\frac{i}{a}$, depending on the numerical values of i, s, a.

- For a:

$$
\begin{aligned}
1-g+n-\sum_{i=1}^{n}\Big(\langle -\frac{\mu_i}{a}\rangle+\delta_{\langle \frac{\mu_i}{a}\rangle,0}\Big)-\sum_{i=1}^{a-1}\frac{im_i}{a} \\
=1-g+\sum_{i=1}^{n}\langle \frac{\mu_i}{a}\rangle-\sum_{i=1}^{a-1}\frac{im_i}{a}
\end{aligned} \tag{109}
$$

Here, we used the fact that

$$
\langle \frac{-\mu_i}{a}\rangle=\begin{cases}1-\langle \frac{\mu_i}{a}\rangle & \langle \frac{\mu_i}{a}\rangle\neq 0\\ 0 & \langle \frac{\mu_i}{a}\rangle=0\end{cases} \tag{110}
$$

- For -1:

Including the minus one factor, $(-1)^{g-1+\sum_{i=1}^{n}\langle \frac{-\mu_i(s+1)}{a}\rangle-\sum_{i=1}^{a-1}m_i\left\langle \frac{-i(s+1)}{a}\right\rangle}$, in (87), we have the following overall exponent of -1:

$$
2g-2+\sum_{i=1}^{n}\Big(\langle -\frac{\mu_i(s+1)}{a}\rangle+\langle \frac{\mu_i(s+1)}{a}\rangle+\delta_{\langle -\frac{\mu_i(s+1)}{a}\rangle,0}\Big)=2g-2+n \tag{111}
$$

Piecing everything together, the leading term of

$$
(-1)^{g-1+\sum_{i=1}^{n}\langle \frac{-\mu_i(s+1)}{a}\rangle-\sum_{i=1}^{a-1}m_i\left\langle \frac{-i(s+1)}{a}\right\rangle}G_g(\mathbb{1}_1^{m_1}...\mathbb{1}_{a-1}^{m_{a-1}},\mu) \tag{112}
$$

at large f expansion is

$$
\begin{aligned}
&\frac{(-1)^n f^{2g-2+n+\sum_{i=1}^{n}\frac{\mu_i}{a}+\sum_{i=1}^{a-1}m_i\left(\left\langle \frac{is}{a}\right\rangle+\left\langle -\frac{i(s+1)}{a}\right\rangle\right)}}{|Aut(\gamma)||Aut(\mu)|}\\
&\times\quad a^{1-g-\sum_{i=1}^{a-1}\frac{im_i}{a}+\sum_{i=1}^{n}\langle \frac{\mu_i}{a}\rangle}\prod_{i=1}^{n}\frac{\mu_i^{\lfloor \frac{\mu_i}{a}\rfloor}}{\lfloor \frac{\mu_i}{a}\rfloor!}\\
&\times\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)}\frac{\sum_{j\geq 0}(-a)^j\lambda_j^U}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}.
\end{aligned} \tag{113}
$$

Now, referring to the formula for double Hurwitz numbers in (67) and to the generating functions in equations (86) and (87), together with the rescaling of formal parameters, then the statement in the theorem clearly follows. □

4.2 New Spectral Curve

Since the double Hurwitz numbers and Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ are related through the generalized infinite framing limit in the last section and the remodeling theory for the same orbifold implies the recursive structure of Gromov-Witten invariants, then the double Hurwitz numbers must be related to the topological recursion as well. The new question is what is the associated spectral curve for the full double Hurwitz numbers. To do this, we have to discuss first the topological string B-model side.

As mentioned in subsection 2.3.1, the mirror curve captures the geometry of the B-model side. The mirror curve expression can be extracted from the fan of the toric Calabi-Yau threefold.

Consider the target orbifold $\mathfrak{X} := \mathbb{C}^3/\mathbb{Z}_a$, where $\mathbb{Z}_a$ acts on $\mathbb{C}^3$ as

$$(z_1, z_2, z_3) \to (\alpha z_1, \alpha^s z_2, \alpha^{-s-1} z_3), \qquad \alpha = e^{2\pi i/a}, \qquad s \in \mathbb{Z}. \tag{114}$$

As mentioned earlier, the rays for the fan can be taken to be

$$(0,0,1), \quad (0,1,1), \quad (a,-s,1). \tag{115}$$

This fan intersects the plane $z = 1$ forming a polytope with (planar) vertices $(0,0), (0,1), (a,-s)$. Then the mirror curve is given by the Newton polynomial associated to this polytope:

$$1 + y + x^a y^{-s} = 0. \tag{116}$$

Now, we are interested in the general case when there are insertions of orbifold cohomology classes on the target space $\mathbb{C}^3/\mathbb{Z}_a$. The curve above is the mirror curve of $\mathbb{C}^3/\mathbb{Z}_a$ when there are no insertions of cohomology classes. To include the insertions, and in turn to determine the spectral curve for the full double Hurwitz numbers, we have to consider this time the lattice points (u_i, v_i) on the edge and inside the polytope. So the full mirror curve of $\mathbb{C}^3/\mathbb{Z}_a$ (with insertions of orbifold cohomology classes included) takes the form

$$1 + y + x^a y^{-s} + \sum_i \rho_i x^{u_i} y^{v_i} = 0. \tag{117}$$

Remark: The signs in the mirror curves above are not necessarily all positive. The signs will be chosen in accordance with the mirror symmetry theorem that will be presented shortly.

Now, to get the spectral curve for Hurwitz theory, recall that the Gromov-Witten invariants and Hurwitz numbers are related via infinite framing. So in order to obtain the spectral curve, we frame first the mirror curve (117), reparameterized it, and then take the limit as framing f goes to infinity. The remaining thing to be determined beforehand are the coefficients ρ_i in the mirror curve (117). We can determined ρ_i by implementing the following theorem as a consequence of mirror symmetry:

Theorem 8. Mirror symmetry for disk potentials
The disk potential F_0, defined as

$$F_0(x, \tau_1, .., \tau_{a-1}) := \sum_{\mu \in \mathbb{Z}^+} \sum_{m_j \in \mathbb{N}} \left(G_0(\mathbb{1}_1^{m_1} ... \mathbb{1}_{a-1}^{m_{a-1}}, \mu) x^\mu \prod_{j=1}^{a-1} \tau_j^{m_j} \right), \tag{118}$$

of toric Calabi-Yau X in A-model and the disk potential $\mathcal{F}_0$,

$$\mathcal{F}_0(x, y) = \int \frac{\log(y)}{x} dx, \tag{119}$$

of the mirror $\tilde{X}$ in B-model are equivalent up to a change in variables (y can be expressed in terms of x, ρ, and we need to find a relation between ρ_i and τ_i).

Hence, all we have to do in order to completely determine the expression of the full mirror curve of $\mathbb{C}^3/\mathbb{Z}_a$ (and in turn the spectral curve for double Hurwitz numbers) is to equate the disk potentials of A- and B-models.

Note that in previous studies or chapters, the spectral curve for simple and orbifold Hurwitz numbers were determined without referring to the above theorem since in those cases, only the vertices in the polytope were considered in the Newton polynomial. To determine the spectral curve for the Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ (with insertions of orbifold cohomology classes) and the full double Hurwitz numbers, we have to consider as well the lattice points on the edge and inside the polytope.

The equality of the disk potential $\mathcal{F}_0(x, \tau_1, .., \tau_{a-1})$ in the B-model and the integral of data from the mirror curve was established in [**FJ**], [**Hosono**]. As similar in the A-model, the disk potential in B-model gives the topological string amplitude on mirror target space $\tilde{X}$. It has been known that the B-model disk potential can be calculated by integrating the holomorphic 3-form Ω on $\tilde{X}$:

$$\mathcal{F}_0(x, \tau_1, .., \tau_{a-1}) = \int \Omega. \tag{120}$$

It was shown in [**FJ**], [**Hosono**] that the holomorphic 3-form Ω is equivalent to 1-differential $\log(y)d(\log(x))$ on mirror curve.

4.2.1 Case: Orbifold $\mathbb{C}^3/\mathbb{Z}_2$ and $s = 0$

Going back to the mirror curve, we consider the target orbifold $\mathbb{C}^3/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on coordinate of $\mathbb{C}^3$ as

$$(z_1, z_2, z_3) \to (-z_1, z_2, -z_3), \qquad i.e., s = 0 \ \ in \ \ (73). \tag{121}$$

The reason for choosing this case is because the rescaling of parameter τ in the infinite framing limit (88) is simple or does not depend on $i = 1, .., a-1$ when $s = 0$, and the open Gromov-Witten invariants were already determined explicitly in [**Ross**] in this case. More explicitly, when $s = 0$ and the group is $\mathbb{Z}_2$, conclusion in theorem 7 becomes

$$F_g(x_1, .., x_n, \tau) = \lim_{f \to \infty} \left((-1)^n f^{2-2g-n} F_g \left(\frac{x_1}{f^{1/2}}, .., \frac{x_n}{f^{1/2}}, \frac{\tau}{f^{\langle -\frac{1}{2} \rangle}} \right) \right), \tag{122}$$

which implies a simpler rescaling

$$\tau \to \frac{\tau}{f^{\langle -\frac{1}{2} \rangle}} = \frac{\tau}{f^{1/2}} \tag{123}$$

for the variable τ.

For the effective action of $\mathbb{Z}_2$, the generating function [**Ross**] for genus 0 open Gromov-Witten invariants or the A-model disk potential of $\mathbb{C}^3/\mathbb{Z}_2$ is given by

$$F_0(x,\tau) = \sum_{j\in\mathbb{Z}^+} \left(\frac{(-1)^{j+1}}{2j^2}\cos(j\tau)x^{2j} + \frac{2(-1)^j}{(2j-1)^2}\sin[\tau(2j-1)/2]x^{2j-1} \right). \tag{124}$$

In the B-model on mirror of $\mathbb{C}^3/\mathbb{Z}_2$, let us consider the mirror curve of the form

$$1 - y + x^2 - \rho x = 0, \tag{125}$$

where the signs were chosen so that the disk potentials will be equivalent. Applying the mirror symmetry for disk potentials (theorem 8), we get a chord function

$$\rho = 2\sin(\tau/2). \tag{126}$$

The above value of ρ works since if the B-model disk potential is

$$\begin{aligned} \mathcal{F}_0(x, y(x,\tau)) &= \int \log(1 + x^2 - 2\sin(\tau/2)x)\frac{dx}{x} \\ = (\frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{18}x^6 + ..) &+ (-x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + ..)\tau \\ &+ (\frac{-1}{4}x^2 + \frac{1}{4}x^4 - \frac{1}{4}x^6 + ..)\tau^2 + \mathcal{O}(\tau^3), \end{aligned} \tag{127}$$

then one can check that the above expression is equivalent to A-model disk potential (124).

Now, to get the spectral curve, we will just frame the mirror curve, reparameterize the variables, and then evaluate the limit when $f \to \infty$.

Framing the mirror curve via $x \to xy^{-f}$ and then reparameterizing the variables by

$$x \to \frac{x}{f^{1/2}}, \tag{128}$$

$$y \to 1 - \frac{y}{f}, \tag{129}$$

$$\tau \to \frac{\tau}{f^{1/2}}, \tag{130}$$

the mirror curve (125) becomes

$$-2f^{1/2}\sin\left(\frac{\tau}{2f^{1/2}}\right)x\left(1-\frac{y}{f}\right)^{-f}+x^2\left(1-\frac{y}{f}\right)^{-2f}+y=0. \tag{131}$$

At large f, this becomes the spectral curve for the generating function for double Hurwitz numbers $H_g(\nu(\gamma,2),\mu)$:

$$ye^{-2y}-\tau xe^{-y}+x^2=0. \tag{132}$$

We end this chapter by giving the following conjecture:

Conjecture: The differentials $d_1..d_{n+1}N_g(x_1..x_n,\tau)$ of the generating function

$$\begin{aligned} N_g(x_1,..,x_n,\tau) = \sum_{\mu_i\in\mathbb{Z}^+} & \frac{1}{(2g-2+r+n+\frac{d}{2}-\frac{m}{2})!} \\ & \times(H_g(\nu(\gamma,2),\mu)\prod_{i=1}^{n}x_i^{\mu_i}\tau^m) \end{aligned} \tag{133}$$

for the double Hurwitz numbers $H_g(\nu(\gamma,2),\mu)$ satisfy the topological recursion of Eynard-Orantin defined on the spectral curve $ye^{-2y}-\tau xe^{-y}+x^2=0$, with fundamental one form $dN_0=ydx/x$.

5 Conclusion and Future Directions

The last chapter of this thesis extended the relations among topological recursion, Hurwitz numbers and Gromov-Witten invariants. In summary, we make the following conclusions based on the calculations from the previous chapter:

- The double Hurwitz numbers' generating function $N_g(x,\tau)$ can be recovered from infinite framing limit of the generating function $F_g(x,\tau)$ for Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_a$ via:

$$N_g(x,\tau)=\lim_{f\to\infty}\left((-1)^n f^{2-2g-n}F_g\left(\frac{x}{f^{1/a}},\frac{\tau}{f^{\langle\frac{is}{a}\rangle+\langle-\frac{i(s+1)}{a}\rangle}}\right)\right). \tag{134}$$

- Considering the generalized infinite framing limit above and the remodeling theory, we conjecture that:

  Double Hurwitz numbers $H_g(\nu(\gamma,2),\mu)$ can be generated recursively using topological recursion, with spectral curve given by the a-Lambert curve with an additional term $\tau x e^{-y}$:

  $$ye^{-2y} - \tau x e^{-y} + x^2 = 0. \tag{135}$$

  More precisely, the differentials $d_1..d_{n+1}N_g(x_1..x_n,\tau)$ of the generating function for double Hurwitz numbers $H_g(\nu(\gamma,2),\mu)$ satisfy the topological recursion defined on the above spectral curve, with fundamental one form $dN_0 = ydx/x$.

In addition to these new things in the topological recursion-Hurwitz-Gromo-Witten study, we give the following prospective:

One of the main results of this thesis is giving a conjecture that the double Hurwitz numbers $H_g(\nu(\gamma,2),\mu)$ can be generated recursively via topological recusion with spectral curve given by the spectral curve (132). This conjecture was made by referring to the remodeling theory for $\mathbb{C}^3/\mathbb{Z}_a$ and generalized infinite framing limit. Proof of this conjecture, extending the proof in [**BHML**], would be desirable, confirming the validity of the idea produced in this study.

Another thing is that the expression for mirror curve was determined experimentally. Specifically, the signs were chosen so that the disk potentials in A- and B- models will match exactly. A systematic rule or additional theorem for determining the coefficients in mirror curve will be useful, since it would be a tedious task to experimentally determine the exact form of mirror curve for arbitrary group $\mathbb{Z}_a$. If possible, this would enable us to determine the spectral curve for an arbitrary double Hurwitz numbers $H_g(\nu(\gamma,a),\mu)$, i.e. for any value of toric weight s and group $\mathbb{Z}_a$.

Calculating Hurwitz numbers is not a straightforward task to do, it would be great to formulate arbitrary Hurwitz numbers, i.e. with more than two ramification profiles, in terms of a more generalized formula involving Hodge integrals. If possible, of course, it would not make the Hurwitz numbers calculation easier, but it would lead to possibility of extending the conjecture made here. Paralleling the extension in Gromov-Witten side, a formulation of Gromov-Witten invariants of an arbitrary orbifold would be a necessity in order to relate any Hurwitz numbers with Gromov-Witten invariants. These extensions would likely involved a more sophisticated localization technique on a moduli space having a complicated yet richer structure.

On the other context, as the Hurwitz numbers and Gromov-Witten invariants can be extracted from the topological recursion, which originated from matrix model theory, we are led into thinking that these two objects may be represented as matrix models. A research about matrix models for these would be great and may shed more light on the structure of Hurwitz numbers and Gromov-Witten invariants. In turn, the matrix model for Gromov-Witten invariants would hopefully say something deeper about the nature of topological string theory.

References

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Appendix: Rank of Hodge Bundle over Moduli Space of Stable Maps

We present here the ideas and theorems in [**JPT**] that are needed in the calculation of rank of Hodge bundle over the moduli space of stable maps. Refer to [**JPT**] for detailed explanation of the concepts that will be described in the following parts of this appendix.

A.1 Preliminaries

Let C be a twisted curve of genus g with stack structure at points $p_1, ..., p_r$ determined by the monodromy data $\gamma_i \in \mathbb{Z}_a \backslash \{0\}$, where $i = 1, ..., r$, respectively. (γ_i's here are the same as the γ_i's in section 3.1). Also, let $\mathbb{P}^1[a]$ be the projective line with stack point of order a at 0. At point 0, $\mathbb{P}^1[a]$ is treated as a quotient stack $\mathbb{C}/\langle\zeta_a\rangle$, where ζ_a is the ath-root of unity and $\langle\zeta_a\rangle \subset \mathbb{C}^*$.

Denote $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ as the stack of stable maps

$$f : (C, p_1, ..., p_r) \to \mathbb{P}^1[a], \tag{136}$$

where $\gamma := (\gamma_1, ..., \gamma_r)$ is a vector of nontrivial elements of $\mathbb{Z}_a$ and μ is a (unordered) partition of the map's degree $d \geq 1$ with parts μ_j and length n. The compactified moduli space $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ parametrizes the maps $[f] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ for which relative conditions over $\infty \in \mathbb{P}^1[a]$ are given by the partition μ.

A.2 Localization

The symmetric product $Sym^k(\mathbb{P}^1) := (\mathbb{P}^1)^k/S_k$ is defined as the quotient space of k-fold cartesian product $(\mathbb{P}^1)^k$ by the group action of symmetric group S_k of k symbols.

The work in **[FP]** and lemma 3 in **[JPT]** imply the existence of the restricted branch morphism

$$br_0 : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu) \to Sym^k(\mathbb{P}^1) \tag{137}$$

Let the $\mathbb{C}^*$ action on $\mathbb{P}^1$ be defined by

$$\eta \cdot (z_0, z_1) = (z_0, \eta z_1), \qquad \eta \in \mathbb{C}^*, (z_0, z_1) \in \mathbb{P}^1. \tag{138}$$

This group action on $\mathbb{P}^1$ lifts canonically to $\mathbb{C}^*$ actions on $\mathbb{P}^1[a], \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ and $Sym^k(\mathbb{P}^1)$. In the localization procedure in [JPT], if we consider the stack

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} := \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)^{\mathbb{C}^*} \cap br_0^{-1}(r[0]), \tag{139}$$

where $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)^{\mathbb{C}^*}$ is the $\mathbb{C}^*$-fixed locus on $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ and $r[0]$ is the $\mathbb{C}^*$-fixed point on $Sym^k(\mathbb{P}^1)$, then the strong restriction on branching will impose the following structures on maps $[f : C \to \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_0^{\mathbb{C}^*}$:

- $C = C_0 \cup \amalg_{j=1}^n C_j$.
- C_0 intersects C_j at a node q_j.
- The map $f^C|_{C_j} : C_j^C \to \mathbb{P}^1$ is a $\mathbb{C}^*$-fixed cover of degree μ_j for $j > 0$.
- $f|_{C_0}$ is a constant map from curve of genus g to $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$.

Moreover, the stack structure at $q_j \in C_j$ is of type $\mu_j \in \mathbb{Z}_a$, whereas at $q_j \in C_0$ where C_j is attached is of the opposite type $-\mu_j$. The constant map

$$f|_{C_0} : (C, p_1, ..., p_r, q_1, ..., q_n) \to [0/\mathbb{Z}_a] \tag{140}$$

is an element of the moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ of stable maps from (twisted) stable curve C, with stack structures determined by the vector $\gamma - \mu := (\gamma_1, ..., \gamma_r, -\mu_1, ..., -\mu_n)$, to classifying space $\mathcal{B}\mathbb{Z}_a$.

A.3 Hodge Bundle

The following idea will be used in the calculation of rank of Hodge bundle:

Suppose $f^*T_{\mathbb{P}^1[a]}(-\infty)$ is a locally free sheaf of rank 1 over C, then the space $H^1(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ yields the dual $(\mathbb{E}^U)^\vee$ of the Hodge bundle $\mathbb{E}^U$ over $\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)$ corresponding to irreducible representation U of $\mathbb{Z}_a$, given by

$$\phi^U : \mathbb{Z}_a \to \mathbb{C}^*, \qquad \phi^U(1) \to e^{\frac{2\pi i}{a}}. \tag{141}$$

The Hodge bundle over moduli space of stable maps can be obtained in the following way:

Let G be a finite group and R its irreducible representation, given by $\phi^R : G \to \mathbb{C}^*$. Denote the classifying space of G by $\mathcal{B}G$. By associating to each map

$$[\mathcal{F} : C \to \mathcal{B}G] \in \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \tag{142}$$

the R-summand of G-representation $H^0(C, \omega_{\mathfrak{C}})$, we then obtain the Hodge bundle

$$\mathbb{E}^R \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G). \tag{143}$$

So if $[\mathfrak{F} : C \to \mathcal{B}\mathbb{Z}_a] \in \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$, then the Hodge bundles $\mathbb{E}^U, (\mathbb{E}^U)^\vee$ are vector bundles over $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ with fibers

$$\mathbb{E}^U|_{\mathfrak{F}} \simeq H^0(C, \omega_C), \qquad (\mathbb{E}^U)^\vee|_{\mathfrak{F}} \simeq H^1(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty)), \tag{144}$$

respectively.

Note that there is a relation between Hodge bundles $\mathbb{E}^R, \mathbb{E}^U$ differing by irreducible representation of group. Consider the exact sequence associated with representation R

$$0 \to K \to G \xrightarrow{\phi^R} Im(\phi^R) \simeq \mathbb{Z}_a \to 0. \tag{145}$$

By construction, $R \simeq (\phi^R)^*(U)$. The homomorphism ϕ^R induces a morphism

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a) \tag{146}$$

by sending a principal G-bundle to its quotient by K. Then lemma 5 in [**JPT**] states that $\mathbb{E}^R \simeq \rho^*(\mathbb{E}^U)$.

A.4 Orbifold Riemann Roch Formula

Let $I_1, ..., I_r$ be the cyclic isotropy groups associated to stack points $p_1, ..., p_r$ of curve C. The isotropy group I_i is a subgroup of $\mathbb{Z}_a$ generated by the element γ_i of order a_i. This group is identified with the a_ith roots of unity that acts on $T_{p_i}C$ as

$$I_i \xrightarrow{\sim} \langle \zeta_{a_i} \rangle \subset \mathbb{C}^* \tag{147}$$

Suppose E is a locally free sheaf over the curve C, then I_i also acts on $E|_{p_i}$. Let

$$E|_{p_i} = \bigoplus_{0 \le s \le a_i - 1} V_s^{\bigoplus e_s} \tag{148}$$

be the direct sum decomposition of $E|_{p_i}$ into irreducible representation V_s of $\mathbb{Z}_{a_i}$ with associated character

$$\phi^s : I_i \to \mathbb{C}^*, \qquad \phi^s(\zeta_{a_i}) = \zeta_{a_i}^s. \tag{149}$$

Define the *age* of E at p_i as

$$age_{p_i}(E) = \sum_{0 \le s \le a_i - 1} \frac{s}{a_i} e_s. \tag{150}$$

The orbifold Riemann Roch formula [**AGV**] is given by

$$\chi(C, E) = rk(E)(1 - g) + deg(E) - \sum_{i=1}^{r} age_{p_i}(E). \tag{151}$$

The Euler characteristic χ of $f^*T_{\mathbb{P}^1[a]}(-\infty)$ can be calculated using the above formula. By the quotient presentation of $\mathbb{P}^1[a]$, the character of $f^*T_{\mathbb{P}^1[a]}(-\infty)$ at p_i is

$$\zeta_{a_i} \to \zeta_{a_i}^{\frac{\gamma_i a_i}{a}} = \zeta_a^{\gamma_i}. \tag{152}$$

So the $age_{p_i}(f^*T_{\mathbb{P}^1[a]}(-\infty))$ is γ_i/a. Also, $deg(f^*T_{\mathbb{P}^1[a]}(-\infty)) = d/a$. Therefore,

$$\chi(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) = 1 - g + \frac{d}{a} - \sum_{i=1}^{r} \frac{\gamma_i}{a}. \tag{153}$$

A.5 Rank of Hodge Bundle

The Euler characteristic χ can be expressed as alternating sum of sizes of cohomology groups:

$$\chi(C, E) = \sum (-1)^i rk(H^i(C, E)). \tag{154}$$

As $rk(f^*T_{\mathbb{P}^1[a]}(-\infty)) = 1$, then $H^i(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) = 0$ for all $i > 1$ by Grothendieck's vanishing theorem. Hence,

$$\begin{aligned}\chi(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) = 1 - g + \frac{d}{a} - \sum_{i=1}^{r} \frac{\gamma_i}{a} = rk(H^0(C, f^*T_{\mathbb{P}^1[a]}(-\infty))) \\ -rk(H^1(C, f^*T_{\mathbb{P}^1[a]}(-\infty))) - rk(H^0(q_j, f^*T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})),\end{aligned} \tag{155}$$

where the last term above is the $H^1 - H^0$ contribution from nodal points q_j's.

Taking note of (144) and the additivity of cohomology groups, it follows that

$$\begin{aligned}rk(\mathbb{E}^U) = rk((\mathbb{E}^U)^\vee) = rk(H^1(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))) = g - 1 - \frac{d}{a} + \sum_{i=1}^{r} \frac{\gamma_i}{a} \\ +rk(H^0(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))) + \sum_{j=1}^{n} rk(H^0(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty))) \\ -\sum_{j=1}^{n} rk(H^1(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty))) - rk(H^0(q_j, f^*T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})).\end{aligned} \tag{156}$$

The space $H^0(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ is the subspace of $T_{\mathbb{P}^1[a]}(-\infty)_{[0/\mathbb{Z}_a]}$ consisting of vectors invariant under the action of the image of the monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$. This imply that $H^0(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ vanishes for nontrivial monodromy representation, otherwise it is 1-dimensional. The trivial monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$ is possible when

$$\gamma = \emptyset, \qquad \forall j, \mu_j = 0 \bmod a. \tag{157}$$

Since

$$deg(f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = \frac{\mu_j}{a}, \tag{158}$$

then

$$H^k(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = H^k\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right). \tag{159}$$

Therefore, we have the following

$$rk(H^0(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty))) = rk\left(H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right)\right) = \left\lfloor \frac{\mu_j}{a} \right\rfloor + 1, \tag{160}$$

$$rk(H^1(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty))) = rk\left(H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right)\right) = 0. \tag{161}$$

In addition, the contribution from the nodal point q_j should be considered. In the case $\mu_j \neq 0 (\bmod a)$, q_j is a stack point and

$$H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j}) = 0 \tag{162}$$

because there is no invariant section. If $\mu_j = 0 (\bmod a)$, then $H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})$ is 1-dimensional. $H^1(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})$ vanishes as well for dimension reasons.

Therefore, for nontrivial monodromy representation, the rank of the Hodge bundle takes the form

$$rk(\mathbb{E}^U) = g - 1 - \frac{d}{a} + \sum_{i=1}^{r} \frac{\gamma_i}{a} + \sum_{j=1}^{n} \left(\left\lfloor \frac{\mu_j}{a} \right\rfloor + 1 \right) - \sum_{\mu_j = 0 \bmod a} 1, \tag{163}$$

$$rk(\mathbb{E}^U) = g - 1 + \sum_{i=1}^{r} \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 (\bmod a)} \left(1 - \left\langle \frac{\mu_j}{a} \right\rangle \right). \tag{164}$$

(The last term in 163 is the contribution from the nodes). Otherwise, the rank is g.

A.6 Rank of Hodge Bundle Corresponding to Different Representations

In completing the generalization of infinite framing limit to full double Hurwitz numbers, we need to determine the rank of the Hodge bundle corresponding to different $\mathbb{Z}_a$ representation. Recall in section 4.1 that the Hodge bundle $\mathbb{E}_{\alpha^\ell}$ corresponds to $\mathbb{Z}_a$ representation given by

$$\varphi_{\alpha^\ell} : \mathbb{Z}_a \to \mathbb{C}^*, \qquad \varphi_{\alpha^\ell}(1) = e^{\frac{2\pi i \ell}{a}}. \tag{165}$$

Paralleling with the relations in subsection A.3, consider the following:

$$\begin{array}{ccc} \rho^*(\mathbb{E}_\alpha) \simeq \mathbb{E}_{\alpha^\ell} & \longleftarrow & \mathbb{E}_\alpha \\ \downarrow & & \downarrow \\ \rho : \overline{\mathcal{M}}_{g,\pi_{\alpha^\ell}(\gamma-\mu)}(\mathcal{B}\mathbb{Z}_a) & \longrightarrow & \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \end{array}$$

where $\pi_{\alpha^\ell}(\gamma - \mu) := u \circ \varphi_{\alpha^\ell}$, and

$$u : \mathbb{C}^* \to \mathbb{Z}_a, \qquad e^{\frac{2\pi i \ell\delta}{a}} \mapsto \ell\delta (\mathrm{mod}\, a). \tag{166}$$

It follows that the rank of Hodge bundle $\mathbb{E}_{\alpha^\ell}$ corresponding to representation φ_{α^ℓ} is given by

$$rk(\mathbb{E}_{\alpha^\ell}) = g - 1 + \sum_{i=1}^{r} \frac{\pi_{\alpha^\ell}(\gamma_i)}{a} + \sum_{\mu_j \neq 0 (\bmod a)} \left(1 - \left\langle \frac{\pi_{\alpha^\ell}(\mu_j)}{a} \right\rangle \right). \tag{167}$$

Also note that the monodromy data in the calculation is given by $\gamma := \{\gamma_1, .., \gamma_r\}$ in such a way that there are m_i number of $i \in \mathbb{Z}_a \backslash \{0\}$, i.e. $\sum_{i=1}^{r} \gamma = \sum_{i=1}^{a-1} i m_i$. Combining all together, we have the rank of Hodge bundles over $\overline{\mathcal{M}}_{g,\gamma-\mu}(B\mathbb{Z}_a)$ corresponding to different representations of $\mathbb{Z}_a$:

$$rk(\mathbb{E}_\alpha) = g - 1 + \sum_{i=1}^{n} \langle -\frac{\mu_i}{a} \rangle + \sum_{i=1}^{a-1} \frac{i m_i}{a}, \tag{168}$$

$$rk(\mathbb{E}_{\alpha^s}) = g - 1 + \sum_{i=1}^{n} \langle -\frac{\mu_i s}{a} \rangle + \sum_{i=1}^{a-1} m_i \left\langle \frac{is}{a} \right\rangle, \tag{169}$$

$$rk(\mathbb{E}_{\alpha^{-s-1}}) = g - 1 + \sum_{i=1}^{n} \langle \frac{\mu_i (s+1)}{a} \rangle + \sum_{i=1}^{a-1} m_i \left\langle -\frac{i(s+1)}{a} \right\rangle. \tag{170}$$