University of Alberta

Banach spaces and topology

by

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To my wife

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Contents

Chapter 1

Introduction

The thesis consists of three independent parts. The first part, Chapter 2, is based on the paper - *Jan Rychtář: Pointwise uniformly rotund norms*, accepted for publication in the Proceedings of the American Mathematical Society. The second part, Chapter 3, is based on the paper *Jan Rychtář: On biorthogonal systems and Mazur's intersection property*, Bulletin of Australian Mathematical Society, vol. 69 (2004), pages 107-111. The third part, Chapter 4, is based on the paper *Jan Rychtář: On Gâteaux differentiability of convex functions in WCG spaces*, accepted for publication in Canadian Mathematical Bulletin. Only minor changes of format or presentation were made to the above papers to fit them better into this thesis. Appendices contain additional material and remarks not covered in the final versions of the above papers.

The main concept of this thesis is a concept of renorming. Renorming a Banach space means replacing the original norm by a norm that generates the same topology. Usually, and this is the reason why renorming is so important, the new norm has certain additional "desirable" properties, for example a certain type of convexity or smoothness. We refer to [7], [10], [12] and [18] as to excellent books and survey articles on renorming.

An important tool for introducing a new equivalent norm is a notion of

Markushevich basis. This relatively weak notion of basis allows us to use coordinates which is particularly useful when we are introducing a formula for a new equivalent norm.

Chapter 2 studies topological properties of compact sets. The main idea is that topological properties of a compact set K influence geometrical properties of $C(K)$ and $M(K) = C(K)^*$. And, since K is in a natural way a subspace of $(C(K)^*, w^*)$, properties of K are influenced by properties of $C(K)$ and $M(K)$.

The problem of finding a property of K that can be characterized by a convexity property of $C(K)$ goes back to Dashiell and Lindenstrauss, [6]. They investigated renorming by a strictly convex norm. Since then, a lot of work has been done trying to find a necessary and sufficient condition on K for $C(K)$ to admit an equivalent strictly convex norm (see e.g. [5, Notes for Chapters 6 and 7]). However, all "reasonable" properties were either too strong - like carrying a strictly positive measure, see [2] and [5, Chapter 6]; or too weak - like having a property ccc, see [3] and [5, Chapter 7]. It is unclear, whether an existence of an equivalent strictly convex norm on $C(K)$ implies some "natural" topological property of K at all.

The work of Rosenthal, [17], suggested that stronger convexity property of norm on $C(K)$ (being uniformly convex in every direction) can imply topological property of K (having property ccc) - see Appendix A.2 for details. This led to the characterization of compacts carrying a strictly positive measure by renorming a space $C(K)$ by a pointwise uniformly rotund norm, which is probably the most significant result of the Chapter 2 and the only known result of this type.

There are several natural topological properties of compact sets from the view of functional analysis. The fundamental ones are being Eberlein compact, i.e. homeomorphic to a weakly compact set in a Banach space, and being uniform Eberlein compact, i.e. homeomorphic to a weakly compact set in a Hilbert space. The work of Amir and Lindenstrauss, [1], Benyamini and Starbird,[4], and Fabian, Godefroy and Zizler, [9], clearly illustrates the relationship between smoothness of a Banach space X, convexity of its dual X^* and the type of compactness of the dual unit ball in X^* equipped with weak* topology.

In Chapter 3 we study biorthogonal systems. In particular, we are trying to find connections between the following three properties of a Banach space X: the existence of a shrinking Markushevich basis, being an Asplund space, and having an equivalent Fréchet smooth norm. It was known that all the above properties are equivalent if a dual unit ball of X^* is a Corson compact in weak* topology, see [18]. The main result of this chapter reads that the spirit of the above equivalence is preserved even when we drop the condition on the dual unit ball. More precisely, existence of a subspace of X having one of the above property implies an existence of a (possibly different, yet "of the same size") subspace of X having all of the above properties simultaneously. We refer to [7] and [18] for more results on this subject.

We also use biorthogonal systems to show that a "big" subspace with a Fréchet smooth norm can be renormed and the resulting norm can be extended to the whole space to a norm with Mazur's intersection property. Norms with the above property were extensively studied in [15] and [11]. In [13] authors used biorthogonal systems and results from [11] to find a sufficient condition for a Banach space to admit an equivalent norm with Mazur's intersection property.

In Chapter 4 we study the differentiability of convex functions on a Banach space X. Our aim is to prove that the set of Gâteaux differentiability points of any convex function is "big". Heuristically, the bigger the set of differentiability points the "better" the function and the easier to handle such function, for example for a purpose of finding a minima or maxima. It is known (see e.g. [16] and [8]) that the set of Gâteaux differentiability points of convex functions on a separable or even weakly compactly generated space is norm dense. This reads that for every $x_0 \in X$ and every closed ball $B \subset X$ there is a point of differentiability in the set $x_0 + B$. Our main result extends naturally this notion of density - instead of using balls, we use any weakly compact convex symmetric and linearly dense set B. Thus our result extends naturally a result of Klee, see [14], from separable spaces - where he used compact sets - to spaces that are weakly compactly generated.

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Chapter 2

Pointwise uniformly rotund norms[1]

2.1 Definitions

Let X be a Banach space. If F is a closed, weak* dense subspace of X^*, then a norm $\|.\|$ on X is said to be *F-uniformly rotund* (UR^F) if $\lim_{n\to\infty} f(x_n - y_n) = 0$ for every $f \in F$ and every $x_n, y_n \in X$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. The norm is called *pointwise uniformly rotund (p-UR)* if it is UR^F for some weak* dense $F \subset X^*$, (see [20],[19]). In particular, the norm on $X = Y^*$ is called *weak* uniformly rotund*, if it is UR^Y with the canonical embedding $Y \subset X^* = Y^{**}$. The norm $\|.\|$ is called *uniformly rotund in every direction (URED)* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for every $x_n, y_n \in X$ such that $\|x_n\| = \|y_n\| = 1$, $\lim_{n\to\infty} \|x_n + y_n\| = 2$, and $x_n - y_n \in \mathrm{span}\{z_0\}$ for some $z_0 \in X$.

A measure μ on a compact space K is said to be *strictly positive* if $\mu(U) > 0$ for every nonempty open set $U \subset K$. A compact space K is called a *uniform Eberlein compact* if K is homeomorphic to a weakly compact set in a Hilbert

[1] A version of this chapter has been accepted for publication in Proceedings of the American Mathematical Society.

space, [3]. A family $\mathfrak{N}$ of subsets of a compact space K is said to be a *network* if every open set in K is a union of members of $\mathfrak{N}$. A compact space K is *descriptive* if there are closed sets $A_n \subset K$ and a network $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ such that, for all $n \in \mathbb{N}$, $\mathfrak{N}_n$ consists of relatively open and pairwise disjoint sets in A_n, [18, Lemma 3.1]. A compact space (K, τ) is *fragmentable*, if there is a metric ϱ on K such that for every $\varepsilon > 0$ and every nonempty subset $M \subset K$ there exists a τ-open set $\Omega \subset K$ such that $M \cap \Omega$ is nonempty and has ϱ-diameter less then ε, [7],[16]. A Banach space X is *weakly compactly generated* if there is a weakly compact set $K \subset X$ such that $X = \overline{\operatorname{span}}\, K$. For unexplained terms used in this paper we refer to [7] and [9].

2.2 Results

Clearly, every p-UR norm is URED. URED norms are used in fixed point theory, see e.g. [5]. It turned out that p-UR norms can be used in characterizing some properties of compact spaces as follows.

Theorem 2.1 *Let K be a compact space. The following are equivalent.*

(1) *The space $C(K)$ of continuous functions on K admits an equivalent pointwise uniformly rotund norm.*

(2) *K carries a strictly positive Radon probability.*

Theorem 2.2 *Let K be a compact space. The following are equivalent.*

(1) *The space $C(K)^*$ admits a pointwise uniformly rotund (in general non-dual) norm.*

(2) *The space $L_1(\mu)$ is separable for every Radon probability μ on K.*

The following theorems show how are other properties of compact sets related to p-UR renorming of $C(K)^*$.

Theorem 2.3 *If K is a descriptive compact space, then $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.*

Theorem 2.4 *There exists a non-descriptive compact space K such that $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.*

Theorem 2.5 *If K is a fragmentable compact space, then $C(K)^*$ admits an equivalent pointwise uniformly rotund norm. Consequently, the space $L_1(\mu)$ is separable for every Radon probability μ on a fragmentable compact K.*

It also turned out that renormings by pointwise uniformly rotund norms are important in the class of L_1-spaces.

Theorem 2.6 *Let μ be a finite measure. Then $L_1(\mu)$ admits an equivalent pointwise uniformly rotund norm if and only if $L_1(\mu)$ is separable.*

Finally, the next theorem shows how is p-UR renorming of a Banach space X related to the topology of weakly compact subsets X.

Theorem 2.7 *If a Banach space X admits an equivalent pointwise uniformly rotund norm, then every weakly compact subset of X is a uniform Eberlein compact.*

2.3 Proofs

Proof of Theorem 2.7. By Šmulyan's type theorem [5, Theorem 2.6.7], if the norm $\|.\|$ on a Banach space X is UR^F, then the limit

$$\lim_{t\to 0} \frac{\|f+tg\|^* - \|f\|^*}{t} \tag{2.1}$$

exists for every $g \in X^*, \|g\|^* = 1$ and is uniform in $f \in F, \|f\|^* = 1$, where $\|.\|^*$ is the dual norm to $\|.\|$. In particular, the norm $\|.\|^*$ is uniformly Gâteaux smooth on F. By [8], the dual unit ball B_{F^*} is a uniform Eberlein compact in weak* topology of F^*. Hence, by [2], F is a subspace of weakly compactly generated space $C(B_{F^*})$.

For a given weak* dense subspace $F \subset X^*$, let an operator $T : X \to F^*$ be given by $T = r \circ i$, where $i : X \to X^{**}$ is the canonical inclusion and $r : X^{**} \to F^*$ is the canonical restriction. The operator T is one-to-one and $\sigma(X, X^*) - \sigma(F^*, F)$ continuous. Since B_{F^*} is a uniform Eberlein compact in $\sigma(F^*, F)$ topology, $T(K)$ is a uniform Eberlein compact for every weakly compact set $K \subset X$. Hence K is a uniform Eberlein compact and the proof of the Theorem 2.7 is finished. □

Note, that if F admits a uniformly Gâteaux smooth norm then F^* admits a weak* uniformly rotund norm (see [5, Theorem 2.6.7]), and thus the norm $|||.|||$ on X defined by

$$|||x|||^2 = \|x\|^2 + \|Tx\|^2$$

is an equivalent UR^F norm.

Proof of Theorem 2.1. Let μ be a strictly positive Radon probability measure on K. Then the identity map $I : C(K) \to L_2(\mu)$ is one-to-one and with a dense range. Thus the norm $|||.|||$ defined on $C(K)$ by

$$|||f|||^2 = \|f\|^2 + \|If\|^2_{L_2(\mu)}$$

is an equivalent UR^F norm, where $F = \overline{\mathrm{span}}\left[I^*(L_2(\mu))\right] \subset C(K)^*$.

Conversely, if $C(K)$ admits an equivalent UR^F norm, then F is a subspace of a weakly compactly generated space (see the proof of Theorem 2.7). Thus $\ell_1(\Gamma)$ is not a subspace of F for any uncountable set Γ, see [9, Chapter 11]. By [17, Lemma 1.3], there is a Radon probability μ on K such that $F \subset L_1(\mu) \subset C(K)^*$. Note that the measure μ is strictly positive as F is weak* dense in $C(K)^*$. This concludes the proof of Theorem 2.1. □

Proof of Theorem 2.6. If $L_1(\mu)$ is separable, then it admits an equivalent p-UR norm with the same proof as of [5, Corollary 2.6.9]. Assume that $L_1(\mu)$ is nonseparable and admits an equivalent UR^F norm. We claim that F is norm separable. This means that $L_1(\mu)^*$ is weak* separable, which is a contradiction with [9, Theorem 11.3].

To prove our claim, let us identify $L_1(\mu)^* \cong L_\infty(\mu)$ with $C(\Omega)$ where Ω is a Stonian space for measure μ (see [4, Appendix B] for details). Since the measure μ is finite, the space $L_1(\mu)^*$ admits an equivalent weak* uniformly rotund norm. By Theorem 2.1, Ω carries a strictly positive probability measure. In particular, Ω has a property ccc, that is every collection of pairwise disjoint open sets of Ω is countable. Thus we only need to prove the following fact, which is a version of [17, Theorem 4.5. (a) and Proposition 4.7].

Fact 2.8 *Let Ω be a compact space with a property ccc and $X \subset C(\Omega)$ be isomorphic to a subspace of weakly compactly generated space. Then X is separable.*

Proof. By [7, Theorem 7.2.2], there exists a Markushevich basis of X, i.e. a biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ such that $\overline{\mathrm{span}}\{x_\gamma; \gamma \in \Gamma\} = X$

and $\{f_\gamma; \gamma \in \Gamma\}$ separates points of X. We may and do assume that $\|x_\gamma\| = 3$. By [10], there exists a decomposition of $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ such that, for every $n \in \mathbb{N}$,

$$\emptyset \neq \overline{\{x_\gamma; \gamma \in \Gamma_n\}}^{\sigma(X^{**},X^*)} \setminus \{x_\gamma; \gamma \in \Gamma_n\} \subset B_{X^{**}}. \tag{2.2}$$

Take n such that Γ_n is uncountable and define open sets $U_\gamma \subset \Omega$ by

$$U_\gamma = \{\omega \in \Omega; |x_\gamma(\omega)| > 2\} \text{ for } \gamma \in \Gamma_n.$$

Since Ω has ccc, there is a sequence $\{\gamma_i\}_{i=1}^{\infty} \subset \Gamma_n$ such that $\bigcap_{i=1}^{\infty} U_{\gamma_i} \neq \emptyset$, [17, Lemma 4.2]. Thus there is $\omega \in \bigcap_{i=1}^{\infty} U_{\gamma_i}$ such that $|x_{\gamma_i}(\omega)| > 2$ for every $i \in \mathbb{N}$, a contradiction with (2.2). Thus X is separable. This concludes the proof of Fact 2.8 and the proof of Theorem 2.6 is complete. □

Remark. After a submittion of the original paper to Proceedings of the American Mathematical Society, we have learned that Theorem 2.6 was proved by a different method in [6]. See also Appendix A for another proof.

Proof of Theorem 2.2. Theorem follows easily from Theorem 2.6 and the Kakutani's Theorem, see e.g. [15]. □

Proof of Theorem 2.3. Let $\|.\|_1$ be the canonical dual norm on $C(K)^*$. Fix the family $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ given by the definition of descriptivity of K. Consider $\mathfrak{N} \subset C(K)^{**}$ by the action $N(\mu) = \mu(N)$ for $N \in \mathfrak{N}$ and $\mu \in C(K)^*$. Let $F = \overline{\operatorname{span}}\,\mathfrak{N}$. We will show that there is an equivalent dual UR^F norm on $C(K)^*$.

We claim that F is weak* dense in $C(K)^{**}$. Indeed, $\mu(G) = 0$ for all open $G \subset K$ whenever $\mu(N) = 0$ for all $N \in \mathfrak{N}$, as $\mathfrak{N}$ is a σ-isolated network consisting of relatively open pairwise disjoint sets.

Define a norm $\|.\|$ on $C(K)^*$ in four steps, similarly as in [18, Proof of Theorem 3.3]. First, for every $n \in \mathbb{N}$, define a convex function F_n on $C(K)^*$ by

$$F_n(\mu)^2 = \sum_{N \in \mathfrak{N}_n} |\mu|(N)^2.$$

The function F_n is weak* lower semi-continuous on $C(A_n)^*$. Second, for every $n, m \in \mathbb{N}$, define a weak* lower semi-continuous seminorm $\|.\|_{m,n}$ on $C(K)^*$ by

$$\|\mu\|_{m,n}^2 = \inf\Big\{\|\mu - u\|_1^2 + m^{-1}F_n(u)^2; u \in C(A_n)^*\Big\}.$$

Third, define an equivalent dual norm on $C(K)^*$ by

$$\|\mu\|_+^2 = \|\mu\|_1^2 + \sum_{m,n \in \mathbb{N}} 2^{-m-n}\|\mu\|_{m,n}^2.$$

Claim 2.9

$$\lim_{\omega \to \infty} (\mu_\omega - \nu_\omega)(N) = 0, \tag{2.3}$$

for all $n \in \mathbb{N}$, $N \in \mathfrak{N}_n$ *and all positive measures* $\mu_\omega, \nu_\omega \in C(K)^*, \omega \in \mathbb{N}$ *such that* $\|\mu_\omega\|_1 \leq 1$, $\|\nu_\omega\|_1 \leq 1$, *and*

$$\lim_{\omega \to \infty} 2\|\mu_\omega\|_+^2 + 2\|\nu_\omega\|_+^2 - \|\mu_\omega + \nu_\omega\|_+^2 = 0. \tag{2.4}$$

Once the claim is proved, finally define a norm $\|.\|$ by

$$\|\mu\|^2 = \inf\Big\{\|\mu_1\|_+^2 + \|\mu_2\|_+^2 \,;\; \mu_i \in M(K), \mu_i \geq 0, \mu = \mu_1 - \mu_2\Big\}. \tag{2.5}$$

Using the compactness argument, it follows from the weak* lower semicontinuity of $\|.\|_+$ that the infimum in (2.5) is attained for every $\mu \in C(K)^*$ and that the norm $\|.\|$ is an equivalent dual norm on $C(K)^*$. Thus (2.3) holds whenever $\|\mu_\omega\| = 1 = \|\nu_\omega\|$ and $\lim_{\omega \to \infty} \|\mu_\omega + \nu_\omega\| = 2$. Hence, the norm $\|.\|$ is UR^F.

Proof of Claim 2.9. Fix $n \in \mathbb{N}$ and $N \in \mathfrak{N}_n$. From (2.4) and a convexity argument,

$$\lim_{\omega \to \infty} 2\|\mu_\omega\|_{m,n}^2 + 2\|\nu_\omega\|_{m,n}^2 - \|\mu_\omega + \nu_\omega\|_{m,n}^2 = 0, \tag{2.6}$$

for every $m \in \mathbb{N}$. From a compactness argument, for every $\omega, m \in \mathbb{N}$, there are positive measures $u_\omega^{m,n}, v_\omega^{m,n} \in C(A_n)^*$ such that

$$\|\mu_\omega\|_{m,n}^2 = \|\mu_\omega - u_\omega^{m,n}\|_1^2 + m^{-1}F_n(u_\omega^{m,n})^2, \text{ and} \tag{2.7}$$

$$\|\nu_\omega\|_{m,n}^2 = \|\nu_\omega - v_\omega^{m,n}\|_1^2 + m^{-1}F_n(v_\omega^{m,n})^2. \tag{2.8}$$

Consequently,

$$F_n(u_\omega^{m,n}) \leq m\|\mu_\omega\|_{m,n} \leq m\|\mu_\omega\|_1 \leq m$$

and similarly $F_n(v_\omega^{m,n}) \leq m$. By passing to a subsequence, we may assume that

$$\lim_{\omega \to \infty} \|\mu_\omega\|_{m,n} = d_{m,n} = \lim_{\omega \to \infty} \|\nu_\omega\|_{m,n}.$$

The sequence $\{\|\mu\|_{m,n}\}_{m=1}^\infty$ is nonincreasing for every measure $\mu \in C(K)^*$. Thus there is $d_n = \lim_{m \to \infty} d_{m,n}$. Choose $\varepsilon > 0$ and let $m_0 \in \mathbb{N}$ be such that $d_{m_0,n} < d_n + \varepsilon$. We will estimate $|(\mu_\omega - \nu_\omega)(N)|$ by

$$\left|(\mu_\omega - u_\omega^{m_0,n})(N)\right| + \left|(u_\omega^{m_0,n} - v_\omega^{m_0,n})(N)\right| + \left|(v_\omega^{m_0,n} - \nu_\omega)(N)\right|.$$

By a convexity argument and (2.6), (2.7), (2.8),

$$\lim_{\omega \to \infty} 2F_n(u_\omega^{m_0,n})^2 + 2F_n(v_\omega^{m_0,n})^2 - F_n(u_\omega^{m_0,n} + v_\omega^{m_0,n})^2 = 0.$$

Since $u_\omega^{m_0,n}$ and $v_\omega^{m_0,n}$ are positive measures, by a convexity argument again

$$\lim_{\omega \to \infty} \left|(u_\omega^{m_0,n} - v_\omega^{m_0,n})(N)\right| = 0.$$

In order to estimate $|(\mu_\omega - u_\omega^{m_0,n})(N)|$, consider a measure

$$u = \mu_\omega \restriction_N + u_\omega^{m_0,n} \restriction_{K\setminus N}$$

in the definition of $\|\mu_\omega\|_{m,n}$, where $\mu \restriction_A$ means the restriction of μ on $A \subset K$. We get

$$\begin{aligned}
\|\mu_\omega\|_{m,n}^2 &\leq \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 + m^{-1} F_n(\mu_\omega \restriction_N + u_\omega^{m_0,n} \restriction_{K\setminus N})^2 \\
&\leq \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 + m^{-1} \big(F_n(\mu_\omega \restriction_N) + F_n(u_\omega^{m_0,n} \restriction_{K\setminus N})\big)^2 \\
&\leq \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 + m^{-1} \big(\mu_\omega(N) + F_n(u_\omega^{m_0,n})\big)^2 \\
&\leq \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 + m^{-1}(1+m_0)^2.
\end{aligned}$$

Thus, for all $m \in \mathbb{N}$,

$$\begin{aligned}
\limsup_{\omega\to\infty} \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 &\geq \lim_{\omega\to\infty} \|\mu_\omega\|_{m,n}^2 - m^{-1}(1+m_0)^2, \\
\limsup_{\omega\to\infty} \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 &\geq d_{m,n}^2 - m^{-1}(1+m_0)^2, \text{ and} \\
\limsup_{\omega\to\infty} \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1^2 &\geq d_n^2.
\end{aligned}$$

For all $\omega \in \mathbb{N}$ we have

$$\begin{aligned}
|(\mu_\omega - u_\omega^{m_0,n})(N)| &\leq \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_N\|_1 \\
&= \|\mu_\omega - u_\omega^{m_0,n}\|_1 - \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1 \\
&\leq \|\mu_\omega\|_{m_0,n} - \|(\mu_\omega - u_\omega^{m_0,n}) \restriction_{K\setminus N}\|_1.
\end{aligned}$$

Thus

$$\liminf_{\omega\to\infty} \big|(\mu_\omega - u_\omega^{m_0,n})(N)\big| \leq d_{m_0,n} - d_n \leq \varepsilon.$$

The same estimate holds for $|(\nu_\omega - v_\omega^{m_0,n})(N)|$. The proof of Claim 2.9 and Theorem 2.3 is now complete. $\square$

Proof of Theorem 2.4. First, we prove the following claim, which is a version of [11, Theorem 7.1].

Claim 2.10 *Suppose that on a tree T there is an increasing function $\varrho : T \to \mathbb{R}$ which is constant on no strictly increasing sequence in T. Then there is an equivalent dual p-UR norm on $C_0(T)^*$.*

Proof of Claim 2.10. The space $C_0(T)^*$ can be identified with $\ell_1(T)$ with the canonical dual norm $\|\mu\|_1 = \sum_{t\in T} |\mu(t)|$. Let us define T^+ as the set of successors and T_0 as the set of all $t \in T^+$ such that $\varrho(t) > \varrho(t^-)$. We may modify the function ϱ so that it takes rational values at all points of T_0.

We will show, that there is an equivalent dual UR^F norm where

$$F = \overline{\text{span}}\Big\{\{\{s\}; s \in T^+\} \cup \{[s,\infty); s \in T_0\} \cup \{T\}\Big\} \subset \ell_\infty(T).$$

We claim that F is weak*-dense in $C_0(T)^{**}$. To prove it, let $\mu \in C(T)^*$ be such that $\mu(f) = 0$ for all $f \in F$. We want to show that $\mu(\{t\}) = 0$ for all $t \in T$. Choose $t \in T$ and put $A(t) = \{u; u \in (t,\infty), \varrho(u) = \varrho(t)\}$ and $B(t) = \min\{u \in (t,\infty); \varrho(u) > \varrho(t)\}$. We have

$$(t,\infty) = \bigcup_{u\in A(t)} \{u\} \cup \bigcup_{u\in B(t)} [u,\infty).$$

The union above is a union of disjoint open sets and $|\mu|$ is non zero at most on countable many of them. Hence $\mu\big((t,\infty)\big) = 0$. Thus $\mu\big([t,\infty)\big) = 0$ for all $t \in T^+$. Since

$$(0,t] = T \setminus \bigcup_{s\leq t} \Big(\bigcup_{r\in s^+\setminus(0,t]} [r,\infty)\Big),$$

we have that $\mu\big((0,t]\big) = 0$ for all $t \in T$. Every limit element $t \in T$ is a limit of a sequence (of elements of T_0), thus $\mu((0,t)) = 0$ for all $t \in T$. Hence $\mu\big(\{t\}\big) = 0$ for all $t \in T$.

For every $q \in \mathbb{Q}$, the wedges $[s, \infty)$, with $s \in T_0$ and $\varrho(s) = q$ are disjoint, so we can define an equivalent dual norm on $C(T)^*$ by

$$\|\mu\|_+^2 = \|\mu\|_1^2 + \sum_{s \in T^+} \|\mu \restriction_{\{s\}} \|_1^2 + \sum_{q \in \mathbb{Q}} c_q \Big(\sum_{s \in T_0 \cap \varrho^{-1}(q)} \|\mu \restriction_{[s,\infty)} \|_1^2 \Big),$$

where c_q are some positive constants.

Let $\mu_n, \nu_n \in C_0(T)^*$ be positive elements such that $\|\mu_n\| \leq 1$, $\|\nu_n\| \leq 1$ and

$$\lim_{n \to \infty} 2\|\mu_n\|_+^2 + 2\|\nu_n\|_+^2 - \|\mu_n + \nu_n\|_+^2 = 0.$$

A standard convexity argument shows that

$$\lim_{n \to \infty} (\mu_n - \nu_n)(T) = 0, \lim_{n \to \infty} (\mu_n - \nu_n)(s) = 0,$$

for all $s \in T^+$, and

$$\lim_{n \to \infty} (\mu_n - \nu_n)\big([s, \infty)\big) = 0,$$

for any $s \in T_0$. Thus the norm $\|.\|$ defined by (2.5) is UR^F. This concludes the proof of Claim 2.10. □

Now, let Λ be a tree defined in [11, Section 10] and K be its Alexandroff compactification. Then $C(K)^*$ admits an equivalent dual p-UR norm by Claim 2.10. The space $C(K)^*$ does not admit any equivalent dual locally uniformly rotund norm, since $C(K)$ does not admit an equivalent Fréchet smooth norm [11, Corollary 10.9]. Thus K is not a descriptive compact space by [18, Corollary 4.9]. The proof of Theorem 2.4 is complete. □

Proof of Theorem 2.5. Let K be a fragmentable compact. By [7, Theorem 5.1.9 and Proof of Theorem 5.1.12(iii)], there is a family $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}_n$ of subsets of K such that

1. $\mathfrak{U}$ is a separating family, i.e. if $x \neq y \in K$ then there is $U \in \mathfrak{U}$ such that $\# U \cap \{x, y\} = 1$.

2. $\mathfrak{U}$ is a network.

3. for every $n \in \mathbb{N}$, $\mathfrak{U}_n$ is an open partitioning, i.e. $\mathfrak{U}_n = \{U_\xi; \xi < \xi_n\}$ is well ordered such that U_ξ is contained and is relatively open in $K \setminus (\bigcup_{\eta<\xi} U_\eta)$ for every $\xi < \xi_n$ and $K = \bigcup_{\xi<\xi_n} U_\xi$.

4. for every $U \in \mathfrak{U}_{n+1}$ there is $V \subset \mathfrak{U}_n$ such that $\overline{U} \subset V$.

As $\mathfrak{U}_n$ is an open partitioning, it follows that

$$\sum_{U \in \mathfrak{U}_n} \mu(U) = \mu(K).$$

Define equivalent norms on $C(K)^*$

$$\|\mu\|_+^2 = |\mu|^2(K) + \sum_{n=1}^{\infty} 2^{-n} \sum_{U \in \mathfrak{U}_n} |\mu|^2(U)$$

and

$$\|\mu\|^2 = \inf\{\|\mu_1\|_+^2 + \|\mu_2\|_+^2; \mu_i \in C(K)^*, \mu_i \geq 0, \mu = \mu_1 - \mu_2\}. \tag{2.9}$$

From a definition of a norm $\|.\|_+$ it follows that $\|\mu\|^2 = \|\mu^+\|_+^2 + \|\mu^-\|_+^2$. Let $F = \overline{\text{span}}\{U; U \in \mathfrak{U}\} \subset C(K)^{**}$. We will show that the norm $\|.\|$ is UR^F. Note that $F \subset C(K)^{**}$ is weak* dense. Indeed, assume $\mu(U) = 0$ for all $U \in \mathfrak{U}$ and let $G \subset K$ be an open set. Since $\mathfrak{U}$ is a network, we have $G = \bigcup_{n=1}^{\infty} (\bigcup \mathfrak{U}'_n)$, where for every $n \in \mathbb{N}$, $\mathfrak{U}'_n$ is a subfamily of $\mathfrak{U}_n$. Moreover, by the condition (4), we may assume that $\mathfrak{U}'_n \cap \mathfrak{U}'_m = \emptyset$ for $m \neq n$. Thus

$$\mu(G) = \mu\Big(\bigcup_{n=1}^{\infty}(\bigcup \mathfrak{U}'_n)\Big) = \sum_{n=1}^{\infty}\mu(\bigcup \mathfrak{U}'_n) = \sum_{n=1}^{\infty}\sum_{U\in\mathfrak{U}'_n}\mu(U) = 0,$$

where the third equality hold as $\mathfrak{U}'_n$'s are relatively open partitioning. Thus, by a convexity argument, the norm $\|.\|$ is UR^F.

The proof of Theorem 2.5 is complete. □

2.4 Remarks

For any finite measure μ, the space $L_1(\mu)$ admits an equivalent URED norm by [13], see also [5, Theorem 2.7.16]. Consequently, by Theorem 2.6, nonseparable $L_1(\mu)$ admits an equivalent URED norm and no p-UR norm. This is connected to [20, Problem 1]. Moreover, every weakly compact subset of $L_1(\mu)$ is a uniform Eberlein compact [1, Section 4]. Thus the converse of Theorem 2.7 does not hold even in WCG spaces. This is connected to [1, Problem 2.9].

There are fragmentable compact spaces such that $C(K)^*$ admits no dual strictly convex norm (e.g. $[0, \omega_1]$, see [5, Theorem 7.5.2]) and thus no dual p-UR norm (cf. Theorem 2.5). It was proved in [21, Theorem 2] that $L_1(\mu)$ is separable for every Radon probability μ on a compact subset of first Baire class. Thus split interval $S(I)$ is a nonfragmentable compact satisfying the conclusion of Theorem 2.5. By [13] and Kakutani's Theorem, $C(K)^*$ admits an equivalent URED norm for every compact K. The space $C([0,1]^{[0,1]})^*$ does not admit an equivalent p-UR norm, as $L_1(\lambda)$ is nonseparable, where λ is a product of Lebesgue measures on [0,1].

By [18], if $C(K)^*$ admits a dual weak* locally uniformly rotund norm, then K is descriptive. Thus by Theorem 2.3, $C(K)^*$ admits an equivalent dual p-UR norm. By [7, Theorem 5.3.1], if $C(K)^*$ admits a dual strictly convex norm, then

K is fragmentable and thus, by Theorem 2.5, $C(K)^*$ admits an equivalent p-UR norm. We do not know if $C(K)^*$ admits an equivalent dual p-UR norm.

As shown in [14], there is a reflexive Banach space that does not admit any equivalent norm that is uniformly rotund in every direction. Thus this space does not admit any equivalent p-UR norm, although it admits an equivalent dual locally uniformly rotund norm.

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Chapter 3

On biorthogonal systems and Mazur's intersection property[1]

3.1 Definitions

A *biorthogonal system* in a Banach space X is a subset $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ such that $f_\gamma(x_{\gamma'}) = \delta_{\gamma\gamma'}$ for $\gamma, \gamma' \in \Gamma$. The biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ in X is called *fundamental* if $X = \overline{\text{span}}\{x_\gamma; \gamma \in \Gamma\}$. A *Markushevich basis* is a fundamental biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ in X such that $\{f_\gamma\}_{\gamma\in\Gamma}$ separates points of X. A Markushevich basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ is called *shrinking* if $X^* = \overline{\text{span}}\{f_\gamma; \gamma \in \Gamma\}$. In this note we use Γ as a cardinal number.

A Banach space X is said to be an *Asplund space*, if every separable subspace of X has a separable dual. A Banach space X has *Mazur's intersection property* if every bounded closed convex set can be represented as an intersection of closed balls. A *density* of a topological space is the least cardinality of a dense set. We refer to [2] for undefined terms used in this paper.

[1] A version of this chapter has been published in Bulletin of the Australian Mathematical Society, vol. 69 (2004), pages 107-111.

3.2 Introduction

It is known, [9, Theorem 7.18, Theorem 7.12], that if a dual unit ball of a Banach space X is a Corson compact, then $\text{dens}\, X = w^*\text{-dens}\ X^*$ and the following are equivalent.

(i) X has a shrinking Markushevich basis,

(ii) X is an Asplund space,

(iii) X admits a Fréchet smooth norm.

Let us remark that if a norm on X is Fréchet smooth, then X has Mazur's intersection property, [1, Proposition 4.5].

When we do not assume that the dual unit ball is a Corson compact, then the above equivalence is no longer true. For example, the Banach space $C(K)$, where K is a Kunen's compact (see e.g. [8] and [5]), is an Asplund space without a shrinking Markushevich basis and without Mazur's intersection property, [6].

3.3 Main Result

The aim of this note is to prove a theorem in the spirit of equivalences above but without assuming anything about a dual unit ball.

Theorem 3.1 *Let E be a Banach space. Then the following are equivalent.*

(i) There is a space $Y \subset E$ with a shrinking Markushevich basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$.

(ii) There is an Asplund space $X \subset E$ with $densX = w^*\text{-}densX^* = \Gamma$.

(iii) There is a subspace $Z \subset E$ that admits a Fréchet smooth norm and such that $densZ = w^*\text{-}densZ^* = \Gamma$.

Moreover, if one from the above occurs with $\Gamma = densE^*$, then *(iv) E admits a norm with the Mazur's intersection property.*

Remark. The condition $\operatorname{dens} E = \operatorname{dens} E^*$ is necessarily for renorming with Mazur's intersection property due to [3].

3.4 Proof

Proof. Implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). If Y has a shrinking Markushevich basis, then Y admits a Fréchet differentiable norm [2, Theorem 11.23]. Thus it is an Asplund space [2, Theorem 8.24]. It remains to show that w^*-$\operatorname{dens} Y^* = \operatorname{dens} Y = \Gamma$. Let $\{g_\alpha; \alpha \in A\} \subset Y^*$ be a weak* dense set. As the basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ is shrinking, we may assume without loss of generality that $\{g_\alpha; \alpha \in A\} \subset \operatorname{span}\{f_\gamma; \gamma \in \Gamma\}$. For a contradiction, assume that $|A| < \Gamma$. Thus there is $\Gamma' < \Gamma$ such that

$$\{g_\alpha; \alpha \in A\} \subset \operatorname{span}\{f_\gamma; \gamma \in \Gamma'\}.$$

Hence, for $\gamma \in \Gamma \setminus \Gamma'$ and all $\alpha \in A$

$$|(f_\gamma - g_\alpha)(x_\gamma)| = 1,$$

a contradiction with the density of $\{g_\alpha; \alpha \in A\}$.

Implication (i) $\Rightarrow$ (iv). Due to [6, Theorem 2.4], to show that E admits a norm with the Mazur's intersection property, it is enough to construct a fundamental biorthogonal system $\{q_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset E^* \times E$. As we assume that $Y \subset E$ has a shrinking Markushevich basis, that is a fundamental biorthogonal system $\{f_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset Y^* \times Y$, we only need to show the following.

Lemma 3.2 *Let E be a Banach space with $\operatorname{dens} E^* = \Gamma$ and $Y \subset E$ be a closed subspace. Assume that there is a fundamental biorthogonal system $\{f_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset Y^* \times Y$. Then there is a fundamental biorthogonal system $\{q_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset E^* \times E$.*

Proof. By a relabeling and rescaling, we may have a fundamental system $\{f_\gamma^n, x_\gamma^n\}_{\gamma\in\Gamma,n\in\mathbb{N}} \subset Y^*\times Y$ such that for every $\gamma \in \Gamma$, $\lim_n \|f_\gamma^n\| = 0$. By the Hahn-Banach theorem, consider $f_\gamma^n \in E^*$. Let $\{g_\gamma\}_{\gamma\in\Gamma}$ be a dense set of $B_{E^*} \cap Y^\perp$.

We claim, that $A = \{g_\gamma + f_\gamma^n\}_{\gamma\in\Gamma,n\in\mathbb{N}}$ is linearly dense in E^*. Indeed, let $G \in E^{**}$ be such that $G(f) = 0$ for every $f \in A$. Then $G(g_\gamma) = \lim_n G(g_\gamma + f_\gamma^n) = 0$ and thus $G \in (Y^\perp)^\perp = Y^{**}$. Hence $G = 0$ as $\{f_\gamma^n\}_{\gamma\in\Gamma,n\in\mathbb{N}}$ are linearly dense in Y^*.

Hence $\{g_\gamma + f_\gamma^n, x_\gamma^n\}_{\gamma\in\Gamma,n\in\mathbb{N}} \subset E^* \times E$ is a fundamental biorthogonal system.

□

Remark. As $c_0(\Gamma) \subset C([0,\Gamma])$, Lemma 3.2 provides a direct proof of the fact that there is a fundamental biorthogonal system $\{f_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset C([0,\Gamma])^* \times C([0,\Gamma])$. Thus $C([0,\Gamma])$ admits a norm with Mazur's intersection property, see also [6, Lemma 3.5].

It remains to prove the implication (ii) $\Rightarrow$ (i).

The proof goes in the spirit of [7, Theorem 1.a.5] and [4]. We will use the concept of the Jayne-Rogers selector, see [1, Chapter 1]. The Jayne-Rogers selection map $\mathcal{D}^X$ on an Asplund space X is a multi-valued map that satisfies the following.

(i) $\mathcal{D}^X(x) = \{D_n^X(x); n \in \mathbb{N}\} \cup D_\infty^X(x) \subset X^*$,

(ii) D_n^X, for $n \in \mathbb{N}$, are continuous functions from X to X^*,

(iii) $D_\infty^X(x) = \lim_{n\to\infty} D_n^X(x)$ for every $x \in X$,

(iv) $D_\infty^X(x)(x) = \|x\|^2 = \|D_\infty^X(x)\|^2$,

(v) $X^* = \overline{\operatorname{span}}\, \mathcal{D}^X(X)$.

Such selector exists by [1, Theorem 1.5.2].

In order to construct $Y \subset X$ we will define, by a transfinite induction, vectors $x_{\alpha+1} \in X$, subspaces $Y_\alpha \subset X$ and subsets $F_\alpha \subset X^*$, for all $\alpha < \Gamma$. Put $Y_0 = 0$ and $F_0 = 0$ and pick arbitrary nonzero $x_1 \in (F_0)_\perp = \{x \in X; f(x) = 0 \text{ for all } f \in F_0\}$. Then put $Y_1 = \text{span}\{x_1\}$, and $F_1 = \{\mathcal{D}^X(x); x \in Y_1\}$. Let Y_α and F_α for $\alpha < \Gamma$ have been chosen. Notice that $\text{dens} Y_\alpha < \Gamma$ and thus $\text{dens} F_\alpha \leq \aleph_0 . \text{dens} Y_\alpha < \Gamma$. Thus F_α is not w^*-dense and we can pick a nonzero vector $x_{\alpha+1} \in (F_\alpha)_\perp$. Set $Y_{\alpha+1} = \text{span}\{Y_\alpha \cup \{x_{\alpha+1}\}\}$ and $F_{\alpha+1} = \{\mathcal{D}^X(x); x \in Y_{\alpha+1}\}$.

If $\alpha \leq \Gamma$ is a limit ordinal, define $Y_\alpha = \overline{\text{span}} \bigcup_{\beta<\alpha} Y_\beta$ and $F_\alpha = \{\mathcal{D}^X(x), x \in Y_\alpha\}$.

Put $Y = \overline{\text{span}} \bigcup_{\alpha<\Gamma} Y_\alpha$. We will show that Y has a shrinking Markushevich basis $\{x_{\alpha+1}, f_{\alpha+1}\}_{\alpha<\Gamma}$, where $\{x_{\alpha+1}\}_{\alpha<\Gamma}$ have been already chosen and their biorthogonals $f_{\alpha+1}$ will be defined by projections.

Clearly $Y = \overline{\text{span}}\{x_{\alpha+1}; a < \Gamma\}$. Let us define projections $P_\alpha : Y \to Y_\alpha$ for all $\alpha \leq \Gamma$. First define projections $\widetilde{P}_\alpha : \text{span}\{x_{\alpha+1}; \alpha < \Gamma\} \to Y_\alpha$ by letting $P_\alpha(x_\beta) = x_\beta$ if $\beta \leq \alpha$ and 0 otherwise. $\widetilde{P}_\alpha$ are well defined and once we show that they all have norm 1, they will extend naturally onto desired projections on Y.

Take $x \in \text{span}\{x_{\alpha+1}; \alpha < \Gamma\}$ and fix $\alpha \leq \Gamma$. Then by the properties of the Jayne-Rogers selector and due to the choice of $\{x_{\alpha+1}; \alpha < \Gamma\}$ we have

$$\|\widetilde{P}_\alpha(x)\|^2 = D^X_\infty(\widetilde{P}_\alpha(x))(\widetilde{P}_\alpha(x)) = D^X_\infty(\widetilde{P}_\alpha(x))(x) \leq$$

$$\leq \|x\|.\left\|D^X_\infty(\widetilde{P}_\alpha(x))\right\| = \|x\|.\|\widetilde{P}_\alpha(x)\|.$$

Thus $\|\widetilde{P}_\alpha\| = 1$.

Pick $f_{\alpha+1} \in Y^*$, for $\alpha < \Gamma$, such that $\|f_{\alpha+1}\| = 1$ and $f_{\alpha+1} \in (P_{\alpha+1} - P_\alpha)^* Y^*$. Clearly $\{x_{\alpha+1}, f_{\alpha+1}\}_{\alpha<\Gamma}$ is a biorthogonal system.

We will show that the projection $\{P_\alpha\}_{\alpha<\Gamma}$ are shrinking. From that it follows that $\overline{\operatorname{span}}\,\{f_{\alpha+1}; \alpha < \Gamma\} = Y^*$.

Let $\alpha \leq \Gamma$ be a fixed limit ordinal and set $Z = P_\alpha Y$. Let $f \in Z^*$ be arbitrary. We need to show that there exist a sequence of ordinals $\beta_n \to \alpha$ and $g_n \in P^*_{\beta_n} Z^*$ such that $g_n \to f$ in Z^*. Fix $\varepsilon > 0$. Denote $\mathcal{D}^Z$ the restriction of $\mathcal{D}^X$ on Z, that is $D^Z_k(z) = D^X_k(z)|_Z$ for all $z \in Z$. Clearly $\mathcal{D}^Z$ is the Jayne-Rogers selection map for Z. As $Z \subset X$ is an Asplund space, $Z^* = \overline{\operatorname{span}}\,\mathcal{D}^Z(Z)$. Thus

$$\Big\| f - \Big(\sum_{i=1}^{n} D^Z_{k_i}(z_i) + \sum_{i=n+1}^{m} D^Z_\infty(z_i)\Big)\Big\| < \varepsilon,$$

where $k_i \in \mathbb{N}$, for $i = 1, \ldots, n$ and $z_i \in Z$, for $i = 1, \ldots, m$. Because D^Z_∞ is a pointwise limit of D^Z_n, there are $k_i \in \mathbb{N}, i = n+1, \ldots, m$ such that

$$\Big\| f - \sum_{i=1}^{m} D^Z_{k_i}(z_i)\Big\| < \varepsilon.$$

Because D^Z_n are continuous, there is $\beta < \alpha$ such that

$$\Big\| f - \sum_{i=1}^{m} D^Z_{k_i}(z'_i)\Big\| < \varepsilon,$$

for $z'_i \in P_\beta Z$.

Thus it remains to show that $\mathcal{D}^Z\big(P_\beta(Z)\big) \subset P^*_\beta Z^*$ for $\beta < \alpha$. Let $z \in P_\beta Z$. By the choice of $\{x_{\alpha+1}; \alpha < \Gamma\}$ we know that $\mathcal{D}^Z(z)(x_\gamma) = 0$ for $\gamma > \beta$. Thus

$$P^*_\beta\big(\mathcal{D}^Z(z)\big)(x) = \mathcal{D}^Z(z)(P_\beta x) = \mathcal{D}^Z(z)(x),$$

for all $x \in Z$, and it was exactly what we needed to prove. $\square$

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Chapter 4

On Gâteaux differentiability of convex functions in WCG spaces[1]

4.1 Introduction

The well known Mazur's theorem says that a continuous convex function f on a separable Banach space X is Gâteaux differentiable on a dense G_δ set, [4, Theorem 8.14]. A function f on X is said to be *Gâteaux differentiable* at $x \in X$ if there is $F \in X^*$ such that

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = F(h),$$

for all $h \in X$. A Banach space is called a *weak Asplund space* if every continuous convex function f on it is Gâteaux differentiable at the points of a dense G_δ set. It is known that weakly compactly generated spaces are weak Asplund spaces, [3, Theorem 1.3.4]. Recall that a Banach space X is called *weakly compactly generated* (WCG) if there is a weakly compact set $K \subset X$ such that $\overline{\operatorname{span}}\, K = X$.

[1] A version of this chapter has been accepted for publication in Canadian Mathematical Bulletin.

It is proved in [5] that, for a separable Banach space X, the set of points of Gâteaux differentiability of a convex continuous function f is even bigger than dense in the following sense. If $K \subset X$ is a norm compact convex symmetric set such that $\overline{\operatorname{span}}\, K = X$ and $x_0 \in X$, then there is $x \in x_0 + K$, a point of Gâteaux differentiability of f. A set $C \subset X$ is called *symmetric* if $-C = C$.

4.2 Main result

We will extend the above result to weakly compact set in WCG spaces.

Theorem 4.1 *Let X be a WCG space and K be a weakly compact convex symmetric set such that $\overline{\operatorname{span}}\, K = X$. Let f be a continuous convex function on X and $x_0 \in X$. Then there is $x \in x_0 + K$ such that f is Gâteaux differentiable at x.*

4.3 Definitions

Let us define terms used in the proof. For a closed convex symmetric set C let μ_C denote a *Minkowski functional* of C defined by

$$\mu_C(x) = \inf\{\lambda > 0;\ x \in \lambda C\}.$$

It is known that $\mu_C : X \to \mathbb{R} \cup \{\infty\}$ is a convex lower semicontinuous function. A function $f : X \to \mathbb{R} \cup \{\infty\}$ is said to be *lower semicontinuous* if its level sets $\{x \in X; f(x) \leq r\}$ are closed for every $r \in \mathbb{R}$. This is equivalent to saying that the *epigraph* of f,

$$\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R};\ f(x) \leq r\},$$

is closed in $X \times \mathbb{R}$. Thus the epigraph of a convex lower semicontinuous function is a closed convex set. The *subdifferential*, $\partial f(x)$, of f at $x \in X$ is the set of all $\varphi \in X^*$ such that

$$\varphi(y-x) \leq f(y) - f(x),$$

for all $y \in X$. A functional $\varphi \in X^*$ is called a *supporting functional* for a set K at a point $k_0 \in K$ if

$$\varphi(k_0) = \sup\{\varphi(k);\ k \in K\}.$$

A function $f : X \to \mathbb{R}$ is called a *Gâteaux smooth bump* if it is a Gâteaux differentiable function with a bounded support. A system $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma} \subset X \times X^*$ is called a *Markushevich basis* for X if $x_\beta^*(x_\gamma) = \delta_{\beta\gamma}$ (the Kronecker's delta) for all $\beta, \gamma \in \Gamma$, $\overline{\mathrm{span}}\{x_\gamma; \gamma \in \Gamma\} = X$, and if for every $0 \neq x \in X$ there is $\gamma \in \Gamma$ such that $x_\gamma^*(x) \neq 0$. A norm $\|.\|$ on X is called *strictly convex*, if $x = y$ whenever

$$2\|x\| = 2\|y\| = \|x+y\|.$$

4.4 Proof

Proof of Theorem 4.1. The proof will be divided into three steps. First we will show that there is a "smooth" weakly compact set $L \subset K$.

Lemma 4.2 *There is a weakly compact convex symmetric set $L \subset 2^{-1}K$ such that if $\varphi, \psi \in X^*$ are supporting functionals of L at a point $l \in L$ such that $\varphi(l) = \psi(l)$, then $\varphi = \psi$.*

Second, we will use a variational principle to touch the graph of f by a "smooth" function. We may assume that $f(x_0) = -1$. By the continuity of f, we may assume that $|f(x) - f(x_0)| < 1$, for $\|x - x_0\| \leq 1$. Let g be a function on X defined by

$$\begin{aligned} g(x) &= -f(x); \text{ for } \|x - x_0\| \le 1, \\ &= \infty; \text{ for } \|x - x_0\| > 1. \end{aligned}$$

Then g is lower semicontinuous and $g > 0$. Set $u_L(x) = \mu_L(x - x_0)$.

Lemma 4.3 *There is a Gâteaux smooth function $v : X \to \mathbb{R}$ and a point $x \in X$ such that $x \in x_0 + 2L \subset x_0 + K$, $0 < \|x - x_0\| < 1$ and $g + u_L - v$ attains its minimum at x.*

Finally, we will show that f is Gâteaux differentiable at x.

Lemma 4.4 *Let V denote a Gâteaux derivative of v at x. Then there is $\alpha \in \mathbb{R}\setminus\{0\}$ such that $\varphi + V$ is a supporting functional for $x_0 + \alpha L$, for all $\varphi \in \partial f(x)$. Consequently, f is Gâteaux differentiable at x.*

Proof of Lemma 4.2. Let $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset K \times X^*$ be a Markushevich basis of X, see [4, Theorem 11.12]. There is a one-to-one operator $T : X^* \to c_0(\Gamma)$ defined by

$$T(x^*) = \big(x^*(x_\gamma)\big)_{\gamma\in\Gamma}.$$

Let $\{e_\gamma\}_{\gamma\in\Gamma}$ denote the standard unit vector basis of $\ell_1(\Gamma)$. The dual operator $T^* : \ell_1(\Gamma) \to X^{**}$ satisfies

$$T^*(e_\gamma)(x^*) = e_\gamma(Tx^*) = x^*(x_\gamma),$$

for all $\gamma \in \Gamma$. Thus $T^*(e_\gamma) = x_\gamma$ and $T^*(B_{\ell_1(\Gamma)}) \subset K$. Moreover T^* is a weak*-weak continuous operator from $c_0(\Gamma)^*$ to X.

Let a norm $\|.\|$ on $c_0(\Gamma)$ be (a strictly convex) Day's norm (see [2, Theorem II.7.3] and let $B \subset \ell_1(\Gamma)$ be its dual unit ball. Put $L = T^*(B)$. We may assume that $\|.\|$ is small enough to have $2L \subset K$. Clearly L is a symmetric convex set. As T^* is weak*-weak continuous, L is weakly compact. Now assume that $\varphi, \psi \in X^*$ are supporting functionals of L at $l \in L$ such that $\varphi(l) = \psi(l)$. We claim that $\varphi = \psi$. Pick $b_0 \in B$ such that $T^*(b_0) = l$ and put $x = T(\varphi)$ and $y = T(\psi)$. Then for all $b \in B$

$$b(x) = b(T(\varphi)) = \varphi(T^*(b)) \leq \varphi(l) = b_0(x).$$

Thus $x, y \in c_0(\Gamma)$ are supporting functionals of B at b_0. Moreover

$$\|x\| = \sup\{b(x); b \in B\} = b_0(x) = b_0(y) = \|y\|, \quad \text{and}$$

$$2\|x\| = \|x\| + \|y\| = b_0(x+y) \leq \|x+y\| \leq \|x\| + \|y\|.$$

Thus $x = y$ as the norm $\|.\|$ is strictly convex. Hence, as T is one-to-one, $\varphi = \psi$.

$\square$

Proof of Lemma 4.3. We will use the Deville-Godefroy-Zizler version of the Borwein-Preiss smooth variational principle, see [1] and [2, Theorem 2.3].

Theorem 4.5 *Let X be a Banach space that admits a Lipschitzian bump function which is Gâteaux differentiable. Then for every lower semicontinuous bounded below function F on X and every $\varepsilon > 0$, there exist $x \in X$ and a function $G : X \to \mathbb{R}$, which is Lipschitzian and Gâteaux differentiable on X and such that $\|G\| = \sup\{|G(x)|; x \in X\} < \varepsilon$, $\|G'\| < \varepsilon$ and $F + G$ attains its minimum on X.*

We can use it, as X admits a Gâteaux smooth norm [4, Theorem 11.20] and thus it admits a Lipschitzian Gâteaux smooth bump. Let us fix $\varepsilon \in (0, 1/4)$. To assure that a point x we get by the variational principle is different from x_0, we will first modify the function $g+u_L$. Let $x_1 \in X$ be such that

$$(g+u_L)(x_1) < (g+u_L)(x_0) + \varepsilon/4.$$

Let $v_1 : X \to \mathbb{R}$ be a continuous Gâteaux smooth bump function such that $\|v_1\| < \varepsilon/2$ and

$$(g+u_L-v_1)(x_1) < (g+u_L-v_1)(x_0) - \varepsilon/4.$$

By applying the variational principle with $\varepsilon' = \varepsilon/8$ on $g+u_L-v_1$, we get a Gâteaux smooth function v_2, $\|v_2\| < \varepsilon/8$ and a point $x \in X$, such that $g+u_L-(v_1+v_2)$ attains its minimum at x. Thus

$$\begin{aligned}(g+u_L-v_1-v_2)(x) &\leq (g+u_l-v_1)(x_1) - v_2(x_1)\\ &< (g+u_L-v_1-v_2)(x_0) < \infty.\end{aligned}$$

It means that $x \neq x_0$, $g(x) < \infty$, and thus $0 < \|x-x_0\| < 1$. Put $v = v_1 + v_2$. Then $\|v\| < \varepsilon$ and thus $g(x) - v(x) > -\varepsilon$. We claim that $u_L(x) < 1 + 3\varepsilon < 2$. Indeed, if we assume a contrary, then

$$1 + 2\varepsilon \leq u_L(x) - \varepsilon < (g+u_L-v)(x) \leq (g+u_L-v)(x_0) \leq 1+\varepsilon,$$

a contradiction. Thus $x \in x_0 + 2L \subset x_0 + K$. □

Proof of Lemma 4.4. As f is a continuous convex function, $\partial f(x) \neq \emptyset$ and we only need to show that there is only one $\varphi \in \partial f(x)$, see [6]. For the rest of the proof we will assume without loss of generality that $g+u_L-v = g-(v-u_L)$ attains its minimum at $x=0$, $g(0)=0$ and $g(0)-(v-u_L)(0) = 0$. In particular, $0 < \|x_0\| < 1$ and $u_L(0) = v(0)$.

Pick any $\varphi \in \partial f(0)$. Let $\delta > 0$ be small enough to have $g(ty) = -f(ty) < \infty$ for $y \in S_X, |t| < \delta$. Then

$$-\varphi(ty) \geq -f(ty) = g(ty) \geq (v - u_L)(ty).$$

Let V be a Gâteaux derivative of v at 0. Then

$$v(ty) = v(0) + V(ty) + o_y(t),\ t \to 0,$$

for all $y \in S_X, |t| < \delta$, where $o_y(t)$ is a function (depending on y), such that $o_y(t)/t \to 0$, as $t \to 0$. Thus

$$(\varphi + V)(ty) + o_y(t) \leq u_L(ty) - v(0), t \to 0. \tag{4.1}$$

From that it follows that

$$(\varphi + V)(ty) \leq u_L(ty) - v(0) = u_L(ty) - u_L(0), \tag{4.2}$$

for all $y \in S_X$ and all $t \in \mathbb{R}$. Indeed, if (4.2) does not hold, then there is $y_0 \in S_X$, $0 \neq t_0 \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that

$$(\varphi + V)(t_0 y_0) - \varepsilon_0 > u_L(t_0 y_0) - v(0).$$

By a convexity of u_L, we may assume that $0 < |t_0| < \delta$. Because

$$(\varphi + V)(0) = 0 = (u_L - v)(0), \tag{4.3}$$

one has that for all $t \in (0, |t_0|]$

$$u_L(ty_0) - v(0) \leq t\,\frac{u_L(t_0 y_0) - v(0)}{t_0} < t\,\frac{(\varphi + V)(t_0 y_0) - \varepsilon_0}{t_0},$$

a contradiction with (4.1).

Notice that (4.2) says that $(\varphi + V) \in \partial u_L(0)$ and thus $(\varphi + V)(x_0) = u_L(0)$ as u_L is linear on half-lines emanating from x_0.

Thus, by (4.2) and (4.3), $(\varphi + V)$ is a support functional of $x_0 + v(0)L$ at the point $x = 0$. Indeed, by an assumption $u_L(0) = v(0)$ and thus $0 \in x_0 + v(0)L$. Moreover $(\varphi + V)(0) = 0$ and by (4.2)

$$(\varphi + V)(z) \leq u_L(z) - v(0) \leq 0,$$

for all $z \in x_0 + v(0)L$. Equivalently, $(\varphi + V)$ is a support functional of $v(0)L$ at $-x_0$ with $(\varphi + V)(-x_0) = -u_L(0)$. Because $x_0 \neq 0$, $v(0) = u_L(0) \neq 0$, by Lemma 4.2, there is only one support functional ψ of $v(0)L$ at $-x_0$ with $\psi(0) = -u_L(0)$. Thus there is only one $\varphi \in \partial f(0)$. This concludes the proof of Lemma 4.4 and the proof of the Theorem 4.1. □

4.5 Bibliography

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Chapter 5

General Discussion and Conclusions

The area of renorming Banach spaces is a very rich and active area of functional analysis. There are numerous methods used to introduce a new equivalent norm on a given Banach space. Many of these methods rely on facts and theorems of classical analysis, measure theory, topology and set theory.

Using renorming techniques, one can get results that are of interest of researches working in the area of Banach spaces as well as ones working in other fields of mathematics, such as topology or optimization. Clearly, results connected to optimization can find application even outside mathematics, for example in economics.

As shown in the Chapter 2, there is a direct connection between weak topology of a Banach space and an existence of certain convex renorming of that space. Also, there is a duality argument, hidden in the proofs, that connects convexity of Banach space with a smoothness of its dual. The main contribution of Chapter 2 is the characterization of compact spaces carrying a strictly positive measure.

The main contribution of Chapter 3 is the proof of the fact that every Asplund space X such that $\text{dens}X = w^* - \text{dens}X^*$ contains a subspace with shrinking Markushevich basis. Also, a relatively simple Lemma 3.2 on extending fundamental biorthogonal systems was useful as it allowed us to "extend" a Fréchet smooth norm from a subspace to a norm having a Mazur's intersection property (only slightly weaker than Fréchet smoothness) on the whole space.

In the Chapter 4 we showed that the set of Gâteaux differentiability points of convex functions on a weakly compactly generated Banach space is big - it is dense in certain topology (see Appendix C). Any result about the size of the set of differentiability points is important for optimization.

There still several open problems and question related to the subject of the thesis. Let us mention few of them.

Problem 1. Is there a reflexive Banach space that admits an equivalent URED norm but no p-UR norm?

Problem 2. Does result similar to the Theorem 3.1 (ii) $\Rightarrow$ (i) hold without an assumption $\text{dens}X = w^* - \text{dens}X^*$? (for example under some additional set-theoretical assumptions.)

Problem 3. Is the set of points of Gâteaux differentiability of convex function on WCG spaces even bigger than described in theorem 4.1? (see Appendix C for precise formulation.)

Appendix A

More on p-UR

A.1 Short proof of Theorem 2.6

In this section we will present a short proof of Theorem 2.6. We thank Professors W. B. Johnson and G. Pisier for suggesting this type of the proof.

Theorem 2.6 *Let μ be a finite measure. Then $L_1(\mu)$ admits an equivalent pointwise uniformly rotund norm if and only if $L_1(\mu)$ is separable.*

Proof. Assume that $L_1(\mu)$ admits an equivalent UR^F norm. We show that F is separable. By the proof of the Theorem 2.7 we know that $F \subset Z$ where Z is a weakly compactly generated subspace. We also know that $F \subset L_\infty(\mu) = L_1(\mu)^*$. Since $L_\infty(\mu)$ is injective, see [9], the identity map $id : F \to L_\infty(\mu)$ can be extended to the map $\widetilde{id} : Z \to L_\infty(\mu)$. It yields the existence of a weakly compactly generated space $X = \widetilde{id}(Z)$ such that

$$F \subset X \subset L_\infty(\mu).$$

Now we can use [9, Theorem 4.8] to conclude that X is separable. However, we show it directly as follows. Let $K \subset X$ be a weakly compact convex set such that $\overline{\text{span}}\, K = X$. Since the identity operator

$$Id : L_\infty(\mu) \to L_1(\mu)$$

is integral and therefore compact, see [3, Corollary 6, p.109], K is separable. Indeed, K is weakly compact in X, hence weakly compact in $L_\infty(\mu)$. Thus $Id(K)$ is weakly compact in $L_1(\mu)$. In particular, $Id(K)$ is weakly closed and thus norm closed. Hence $Id(K)$ is norm compact and separable. Since $Id : K \to Id(K)$ is a homeomorphisms, K is separable.

Consequently, X and F are separable. We conclude in the same way as in the Chapter 2 that $L_1(\mu)$ is separable, since it is a weakly compactly generated space. □

A.2 URED versus p-UR

It is easy to see that any pointwise uniformly rotund norm is URED. In this section we will discuss the reverse implication, i.e. under what conditions on a Banach space the existence of an equivalent URED norm implies the existence of an equivalent pointwise uniformly rotund norm.

Let us remark, that many standard URED renormings (see [2]) actually produce p-UR norms. The only exception known to the author is the URED renorming of the space $L_1(\mu)$ done in [7].

Let us observe, that if the space $C(K)$ admits an equivalent URED norm, then $c_0(\Gamma)$ for uncountable set Γ is not isomorphic to a subspace of $C(K)$ (by [2, Proposition 2.7.9]) and thus K is ccc and moreover every weakly compact subset of $C(K)$ is separable (see [9, Theorem 4.5.(a)]). In particular, every weakly compact subset of such $C(K)$ is metrizable (see [5, p.417]) and hence a uniform Eberlein compact.

We note, that, assuming Continuum Hypothesis (see [6]), there is a compact space K such that K has ccc and $C(K)$ does not admit any equivalent strictly convex norm (see [1, Theorem 1.7]). In particular, the condition ccc does not imply the existence of URED norm on $C(K)$. Moreover, the above space $C(K)$ is another example showing that the converse of the Theorem 2.7 is not true in general.

On the other hand, a compact space K carries a strictly positive measure, if K is ccc and $C(K)$ is isomorphic to a conjugate space (see [9, Theorem 4.1]), thus we have the following.

Theorem A.1 *Assume that $C(K)$ is isomorphic to a conjugate Banach space. Then $C(K)$ admits an equivalent URED norm if and only if $C(K)$ admits an equivalent p-UR norm.*

Let us note that we do not know, whether the above equivalence is true for every space $C(K)$. However, the following holds (we refer to [10] for the original statement and more details.)

Theorem A.2 *Let X be a Banach space with an unconditional Schauder basis $\{x_\gamma\}_{\gamma\in\Gamma}$. Then the following is equivalent.*

(a) *X^* admits an equivalent URED norm.*

(b) *X^* admits an equivalent dual UR^X norm.*

It is known that the above result does not hold for an arbitrary Banach space X even if we do not require a dual renorming. It is enough to assume space $X = C(K)$ for $K = [0,1]^\Gamma$, where Γ has a cardinality continuum. By Kakutani's Theorem and result on URED renorming by Kutzarova ([7]), X^* admits an equivalent URED norm. On the other hand, the space $L_1(K,\lambda)$, where λ is the product Lebesgue measure is not separable, hence $C(K)^*$ does not admit an equivalent p-UR norm by Theorem 2.2.

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Appendix B

More on Mazur's intersection property

B.1 Definitions

Let X be a Banach space with a density character $\aleph$ and μ be the minimal ordinal of cardinality $\aleph$. A transfinite sequence of bounded linear projections $\{P_\alpha; 0 \leq \alpha \leq \mu\}$ of X is called a *projectional resolution of identity (PRI)* on X if

(i) $\|P_\alpha\| = 1$ for all $\alpha > 0$,

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min\{\alpha,\beta\}}$, $P_0 = 0$, $P_\mu =$ Identity,

(iii) the density character of $P_\alpha X$ is less than or equal to $\max\{\aleph_0, |\alpha|\}$ for all α, and

(iv) the map $\alpha \to P_\alpha x$ is continuous from the ordinal segment $[0, \mu]$ in its order topology into X in its norm topology for every $x \in X$.

A point f in a dual unit ball B_{X^*} is called a weak* denting point if it is contained in weak* slices of B_{X^*} of arbitrarily small diameter. Precisely, if for every $\varepsilon > 0$ there exist $x \in X$ and $\delta > 0$ such that $f \in S(x, \delta) \equiv \{g \in B_{X^*}; g(x) > \delta\}$, and $\operatorname{diam} S(x, \delta) < \varepsilon$.

As shown in [1], a Banach space X has a norm with Mazur's intersection property if and only if the weak* denting points of the unit ball B_{X^*} of X^* are norm dense in the unit sphere of X^*.

B.2 Theorem

Theorem B.1 *Let X be a Banach space with PRI $\{P_\alpha\}_{\alpha \leq \aleph_1}$ and such that every subspace of X has the Mazur intersection property. Then X has a shrinking Markushevich basis. In particular, X is WCG and has an equivalent Fréchet norm.*

Proof. We will show, that $\{P_\alpha^*\}_{\alpha \leq \aleph_1}$ is a PRI on X^*. Let $\mu \leq \aleph_1$ be a limit ordinal. Set $Y = P_\mu X$ and choose $x^* \in S_{Y^*}$ and $\varepsilon > 0$. As Y has Mazur's intersection property, there is a weak* denting point $f \in S_{Y^*}$ such that $\|x^* - f\|^* \leq \varepsilon$. Thus there is $x \in S_Y$ and $\delta \in (0, f(x))$ such that $\operatorname{diam}\{g \in B_{Y^*}, g(x) > \delta\} < \varepsilon$.

Set $\delta' = 2^{-1}(f(x) - \delta)$ and pick $\alpha_0 < \mu$ and $x' \in P_{\alpha_0} X$ such that $\|x - x'\| < \delta'$. Then $f(x') > \delta + \delta'$, and $g(x) > \delta$ provided $g(x') > \delta + \delta'$. Thus $\operatorname{diam}\{g \in B_{Y^*}, g(x') > \delta + \delta'\} < \varepsilon$.

For $\alpha > \alpha_0$, $P_\alpha^* f(x') = f(x') > \delta + \delta'$. Thus $\|P_\alpha^* f - f\|^* \leq \varepsilon$ and $\|P_\alpha^* f - x^*\|^* < 2\varepsilon$. Hence $\{P_\alpha^*\}_{\alpha \leq \aleph_1}$ is a PRI on X^*.

For every $\alpha < \aleph_1$, $Y = (P_{\alpha+1} - P_\alpha)(X)$ is separable and with Mazur intersection property. Thus $Y^* = (P_{\alpha+1}^* - P_\alpha^*)(X^*)$ is separable and there is a shrinking Markushevich bases $\{(x_\alpha^n, f_\alpha^n)\}_{n \in \mathbb{N}}$ on Y. Thus $\{(x_\alpha^n, f_\alpha^n)\}_{\alpha < \aleph_1, n \in \mathbb{N}}$ is a shrinking Markushevich bases on X. □

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Appendix C

More on Gâteaux differentiability

Given a convex symmetric set C in a Banach space space X, consider the following topology τ_C on X: The set $A \subset X$ is τ_C-open iff for every $a \in A$ there exists $\alpha > 0$ such that $a + \alpha C \subset A$.

Note that a choice $C = B_X$, the unit ball of X, gives exactly the norm topology on X.

Theorem 4.1 reads as follows. Given any weakly compact convex symmetric set $K \subset X$, such that $\overline{\operatorname{span}}\, K = X$, the set of points of Gâteaux differentiability of any convex function f on X is τ_K-dense in X.

An interesting and still open question is, whether under the above assumptions the set of Gâteaux differentiability points of convex function is τ_K-G_δ, or at least contains a dense τ_K-G_δ set.