It is not easy to use the geometrical method to discover things. It is very difficult, but the elegance of the demonstrations after the discoveries are made is really very great. The power of the analytic method is that it is much easier to discover things and to prove things. But not in any degree of elegance. It's a lot of dirty paper, with x's and y's and crossed out cancellations and so on.

Richard P. Feynman,
Feynman's Lost Lecture

University of Alberta

Moving Mirrors and the Boulware State for Black Holes

by

Warren G. Anderson ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Moving Mirrors and the Boulware State for Black Holes** submitted by **Warren G. Anderson** in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

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Abstract

I argue that the state inside an initially empty box with mirrored walls being quasistatically lowered from infinity toward a black hole is the Boulware state for that black hole. Using an expression for the stress-energy tensor from a moving mirror, which I derive specifically for this purpose, I find the energy density inside a box moving with nearly uniform acceleration in Minkowski spacetime. I then invoke the use of a quantum equivalence principle to obtain the energy density of the Boulware state inside a box undergoing equivalent motion in a black hole background. This programme is successful in 1+1 dimensions, but encounters some technical difficulties in 3+1 dimensions.

Acknowledgements

The list of people to whom I owe a debt of gratitude is too long to write here, but there are a few for whom a lack of acknowledgement would comprise too great an injustice for me to knowingly commit.

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Chapter 1

Introduction

1.1 Historical Context

As little as 100 years ago, physicists thought the vacuum to be empty and uninteresting. After all, what could be interesting about nothing? Two major revolutions of modern physics, Einstein's general theory of relativity in 1915[1] and the introduction of quantum theory over most of the first half of the 20th century[2] have forever changed our notion of a vacuum, however.

In general relativity, interesting vacuum solutions arise because only part of the curvature, and hence gravitation, is coupled to sources (stress-energy). In this sense, it is not unlike electromagnetism, in which it is possible to have a field even in the absence of sources. However, there is a sense in which vacuum solutions are more generic to gravitation. Consider, for instance, an electron and positron, inspiralling toward one another, and emitting electromagnetic radiation. To properly model this situation, one would include distributional source terms for the electron and positron. An analogous situation in gravitation would be the inspiral of two black holes which emit gravitational waves. In this case, the nonlinearity of gravity precludes the use of distributional sources. The standard practice is to remove the singular points at the center of the black holes from the space-time altogether. The result is a complex and physically interesting system that we describe entirely in terms of properties of the vacuum.

Quantum theory makes the vacuum interesting in a different way. It turns out that if one quantizes a field, even one described by a field equation with no source terms (a free field equation) so that the classical solution vanishes everywhere, one finds that there is infinite energy contained in the quantum solution. This energy is typically renormalized away, but it is not meaningless. Rather, it is a mathematical manifestation of the physical reality that the quantum vacuum is filled with virtual particles which simply need some physical impetus to acquire reality.

While both gravitational and quantum vacua are interesting in their own right, much richer and more interesting phenomena occur when one considers quantum theory on a gravitational background. In these situations, the different notions of vacua can interact. The gravitational vacuum can cause virtual particles in the quantum vacuum to become real. The real particles then act as a source of stress-energy for the gravitational field, changing it. This bootstrapping process is thought to be responsible for the creation of at least some fraction of matter in the current universe.

One of the most remarkable and widely studied instances of the interaction of quantum and gravitational fields has been the creation of particles in the quantum vacuum by black holes. The most amazing feature of these particles is that they exhibit thermodynamical properties[3]. This discovery was presaged by the work of Bardeen, Carter and Hawking[4] in which they pointed out an analogy between laws governing certain properties of black holes and the laws of ordinary thermodynamics. In particular, the analogue of the second law of thermodynamics is Hawking's theorem that the surface area of a black hole is nondecreasing [5], i.e.

$$\frac{dA_{BH}}{d\tau} \geq 0. \tag{1.1}$$

It was based on this analogy between (1.1) and the second law of thermodynamics that Bekenstein[6] conjectured a generalized second law of thermodynamics (GSL): *The sum of the black hole entropy and the ordinary entropy in the black hole exterior never decreases.* More precisely, the GSL states that for any physical process

$$\delta S_{matter} + \frac{1}{4}\delta A_{BH} \geq 0, \tag{1.2}$$

(units $\hbar$=c=G=k=1), where S_{matter} is the entropy of the matter outside the black hole. In (1.2), $\frac{1}{4}A_{BH}$, one quarter of the black hole's surface area, plays the role of the entropy of the black hole. This correspondence between the surface area and entropy of a black hole has become firmly established in the context of black hole thermodynamics, beginning with Hawking's discovery of the thermal radiation emitted by a black hole[3].

Bekenstein[7] further argued that an entropy bound on matter was required in order for the GSL to hold. His argument relied on the following *gedankenexperiment*. Imagine that a box of linear dimension ℓ with reflecting walls is filled with ordinary matter of energy E_∞ and entropy S at a very large proper distance from a black hole. The box is then slowly (adiabatically) lowered toward the black hole of mass M. When the box is opened and the matter released into the black hole, the energy of the matter will have been reduced by the red-shift factor $\chi = (1-2M/r)^{1/2}$ so that the black hole's energy is increased by

$$E = \chi E_\infty. \tag{1.3}$$

Since the box can be lowered to approximately the proper distance ℓ (the dimension of the box) from the event horizon before releasing the energy into the black hole, the black hole energy can be increased by as little as $E \simeq (\ell/4M)E_\infty$. However, this will lead to a change in the black hole entropy of

$$\delta S_{BH} = \frac{1}{4}\delta A_{BH} = 8\pi ME. \tag{1.4}$$

After the box is emptied, it can be slowly pulled back out to infinity. But observe that, if $\ell < S/(2\pi E_\infty)$, then $\delta S_{BH} < S$ and the GSL will be violated. Therefore, Bekenstein concluded there was a bound on the entropy of matter with energy E that could be placed in a box of dimension ℓ,

$$S/E \leq 2\pi\ell. \tag{1.5}$$

Unruh and Wald[8] pointed out, however, that Bekenstein failed to consider quantum effects in his analysis. In particular, they pointed out the effect of acceleration radiation on the box as it is being lowered. Since, in the reference frame of the almost stationary (hence accelerated) box, the black hole is surrounded by a bath of thermal radiation, there will be an upward pressure on the box. In fact, when this is taken into account, Unruh and Wald demonstrate that a box of negligible height will float when the energy contained in the box, E, is exactly the same as the energy of the acceleration radiation displaced by the box. In order to lower the box further, one will have to do work against this buoyancy.

Equivalently, Unruh and Wald explained, in the frame of a freely falling (inertial) observer, the lowering process removes energy from the box not just through the classical red-shift, but also through quantum effects. Thus, before the box reaches the event horizon, all the energy it contained will be canceled by the lowering process. To lower the box further would induce a negative energy density in the box. This negative energy bubble in a sea of zero energy would float upward due to buoyancy, restoring the internal energy to zero.

Unruh and Wald went on to show that in order to minimize the entropy increase of the black hole, the box must be opened at this floating point. They further showed that the matter released at this point will contribute at least enough energy to the black hole to increase its entropy by an amount

$$\delta S_{BH} \geq S, \qquad (1.6)$$

where S is the entropy of the matter in the box. Thus, they concluded, the GSL will hold independent of the validity of (1.5).

There was some debate in the literature over the Unruh-Wald result, and it was in the context of this debate that it was first pointed out in [9] that the energy draining effects of lowering the box quasi-statically could be understood in terms of the negative energy density of the Boulware state which must occupy the interior of the box. The Boulware state is the quantum state that appears as empty as is possible (more on this in Section 2.5) for static observers, and is exactly empty for static observers infinitely far from the black hole. That this is the correct state inside the box in one space and one time dimension (henceforth denoted (1+1)D) was proven in [9], however the more interesting (3+1)D result has been delayed due to technical difficulties described herein.

1.2 This Thesis

The goals of this thesis are threefold. The first goal is to derive the quantum flux from a mirror moving with nearly uniform acceleration. While such a flux has been known for mirrors moving in space-times with one space and one time dimension for some time[10], this thesis is the first place such a result has been presented for three space and one time dimensions to my knowledge. The second goal is to investigate the possibility that through a suitable quantum equivalence principle these results for moving mirrors in Minkowski space-time can be used to deduce something about the quantum state in a box being quasi-statically lowered toward a black hole. The third goal is to explicitly construct the stress-energy tensor for a scalar field in this state and to show that it is that of the Boulware state on a Schwarzschild black hole background. As mentioned in the last Section, all of these goals have been explicitly accomplished in one space and one time dimension in a previous publication[9].

This thesis has been written with the assumption that the reader has at least a graduate level understanding of the pertinent physics. In particular, I assume an understanding of general relativity equivalent to a senior undergraduate course and of canonical quantum field theory (in Minkowski space-time) equivalent to a first year graduate level. I have endeavored in Chapter 2 to supply an elementary introduction to the parts of quantum field theory in curved space-times necessary to understand the contents of the remainder of the thesis. Readers who are familiar with quantum fields in curved space-times may want to skip most of that Chapter.

Throughout this thesis, whenever I am working in a specific number of dimensions, it will always be either 4 or 2. Most of the time, I will be considering Lorentzian metrics with signature (-,+,+,+) (or in 2 dimensions (-,+)). For these metrics, I will use the Misner, Thorne and Wheeler[11] sign conventions for metric and curvature. The 2 dimensional (one space and one time) space-times I will denote by (1+1)D, and the 4 dimensional (three space and one time) by (3+1)D. The only place I will consider non-Lorentzian metrics will be small parts of Sections 3.2 and 4.2, where I will be working in Euclidean spaces, with signature (+,+,+,+) (or in 2 dimensions (+,+)). In all parts of this thesis I will use the units in which $\hbar = G = c = k = 1$, where $\hbar$ is Planck's constant divided by 2π, G is the universal gravitational constant, c is the speed of light, and k is Boltzmann's constant.

The contents of this thesis are as follows: in Chapter 2 I provide some background material. In Chapter 3, I provide a (1+1)D calculation showing that the goals outlined above can be achieved in (1+1) dimensions in a way that should be easily generalized to (3+1) dimensions. In Chapter 4 I find the Wightman function for a spherical mirror expanding with nearly uniform acceleration in (3+1)D. In Chapter 5 I find the flux from such a mirror and show that it is in accord with previously known results in the appropriate limits. Chapter 6 contains the energy density calculation for a nearly uniformly accelerating box in (3+1)D and some discussion of the quantum equivalence principle and the Boulware state for a (3+1)D black hole. Finally, Chapter 7 summarizes the result of the previous Chapters and offers some concluding remarks. Details of the calculational technique used to obtain the results of

Chapter 5 are included in Appendix A.

Chapter 2

Background Material

2.1 Acceleration in Minkowski Space

In one of its many incarnations, Einstein's equivalence principle states that there is no local physical process by which one can determine whether one is subject to a uniform gravitational field or a uniform acceleration, because of "the complete physical equivalence of a gravitational field and of a corresponding acceleration of the reference system"[12]. The classic *gedanken-experiment* which he used to illustrate this is the elevator experiment[13]. Whether I feel the floor of an elevator push against my feet because it is preventing me from falling freely in the gravitational field or because it is accelerating upward in the absence of a gravitational field is irrelevant. The only physically significant fact is that I am being accelerated upward by the elevator.

In Chapters 3-6 I will be essentially using the elevator *gedankenexperiment* above to show that, at least in some contexts, quantum fields also obey the equivalence principle. To do this, I will need to discuss mirrors moving with both uniform acceleration and motion that is perturbed slightly from

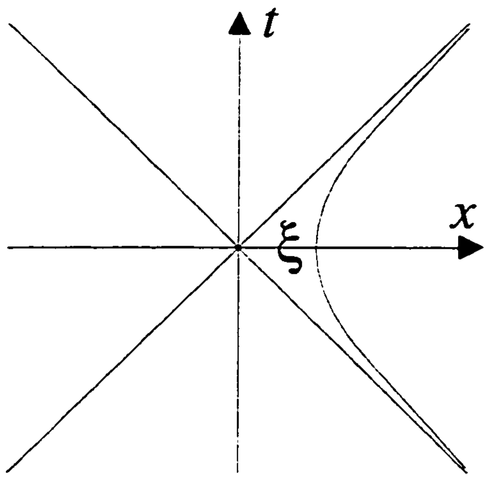

Figure 2.1: The hyperbola is the worldline of a particle following a trajectory parameterized by (2.6). Note that this trajectory asymptotically approaches the light cone from the origin.

uniform acceleration. In preparation for this discussion, I will discuss uniform acceleration and perturbations from it in 1+1 and 3+1 dimensions in this Section.

I will begin in 1+1 dimensions and in the absence of a gravitational field. The space-time metric will therefore be the standard Minkowski metric,

$$ds^2 = -dt^2 + dx^2. \tag{2.1}$$

Next, let me consider a particle moving along a space-time hyperbola. It has a trajectory parameterized by T according to the parametric equations

$$t = \xi \sinh(T), \qquad x = \xi \cosh(T). \tag{2.2}$$

The constant T is the distance of closest approach of the trajectory to the origin. The trajectory (2.6) is illustrated in Fig. 2.1.

The proper time τ for a particle with trajectory (2.6) is given by

$$d\tau = \sqrt{dt^2 - dx^2} = \alpha\sqrt{\cosh^2(T/\alpha) - \sinh^2(T/\alpha)}\, dT = \alpha dT. \tag{2.3}$$

In other words, the parameter T is proportional to the proper time of a particle with trajectory (2.6). Let me therefore express the trajectory parameter in terms of the proper time.

$$t = \xi \sinh(\tau/\alpha), \qquad x = \xi \cosh(\tau/\alpha). \tag{2.4}$$

The arbitrary constant α is introduced simply to keep the arguments of the hyperbolic trig functions dimensionless, so it must have dimensions of length (just as τ does).

I can use this to calculate the magnitude of the particle's 2-acceleration. It is given by

$$a = \sqrt{\left(\frac{d^2x}{d\tau^2}\right)^2 - \left(\frac{d^2t}{d\tau^2}\right)^2} = \frac{\xi}{\alpha^2}. \tag{2.5}$$

Thus, the particle's acceleration is expressed entirely in terms of the constants ξ and α and must itself be constant. There is one particular trajectory of this type which has an especially simple form. If I take the distance of closest approach to be equal to the arbitrary constant α, so that the trajectory has an acceleration given by

$$t = \alpha \sinh(\tau/\alpha), \qquad x = \alpha \cosh(\tau/\alpha), \tag{2.6}$$

then the acceleration is simply the inverse of the distance of closest approach, $a = 1/\alpha$. I will single this curve out for special attention in the next Chapter.

Next, consider a one parameter congruence of such constant acceleration particles, so that $\xi \geq 0$ in (2.4) becomes the congruence parameter. This congruence fills a portion of Minkowski space, as shown in Fig. 2.2. Eq.s (2.4) essentially define new coordinates (ξ, τ) on this part of Minkowski space. These coordinates, called *Rindler coordinates*[14], are particularly useful for

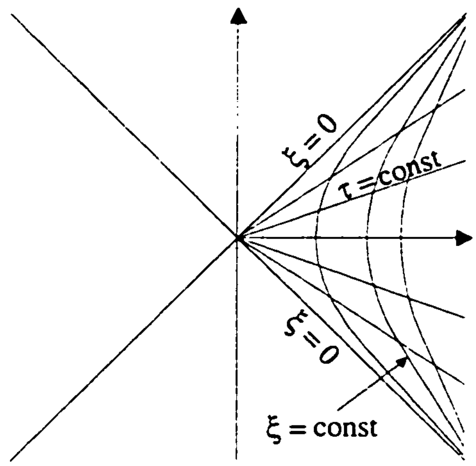

Figure 2.2: Rindler coordinates τ and ξ. Note that they only cover one quarter of Minkowski space, the *Rindler wedge.*

discussing accelerated motion. The portion of Minkowski space covered by these coordinates is called *Rindler space* or the *Rindler wedge.* Finally, I can rewrite the metric (2.1) in terms of the Rindler coordinates using the coordinate transformation (2.4). I get

$$ds^2 = -\frac{\xi^2}{\alpha^2}d\tau^2 + d\xi^2. \tag{2.7}$$

It comes as no surprise at this point that this metric is called the *Rindler metric.*

I shall also be discussing motion that is perturbed from uniform acceleration. Since the acceleration of the particle trajectory (2.6) is the inverse of α, I can perturb the acceleration by perturbing α, i.e. by letting $\alpha \to \alpha + \delta\alpha(\tau)$. Let me therefore consider now a particle with the trajectory

$$t = (\alpha + \delta\alpha(\tau))\sinh(\tau/\alpha), \qquad x = (\alpha + \delta\alpha(\tau))\cosh(\tau/\alpha). \tag{2.8}$$

For such a perturbed particle, the proper time is given by

$$dT = \sqrt{\left(1 + \frac{\delta\alpha}{\alpha}\right)^2 - \dot{\delta\alpha}^2}\, d\tau, \tag{2.9}$$

where $\dot{\delta\alpha}$ denotes the derivative of $\delta\alpha$ with respect to the parameter τ. If I assume, as I shall for the remainder of this thesis, that the perturbation is small, that is, that

$$\alpha^{n-1} \frac{d^n}{d\tau^n} \delta\alpha(\tau) << 1, \tag{2.10}$$

then, to first order in the perturbation I can rewrite (2.9) as

$$dT \approx \left(1 + \frac{\delta\alpha}{\alpha}\right) d\tau. \tag{2.11}$$

In other words, the proper time for such a particle is perturbed by an amount $(\delta\alpha/\alpha)\, d\tau$.

Using (2.11) I can now calculate the magnitude of the 2-acceleration for the particle trajectory in the same way I did in (2.5). Doing so, I obtain to first order in the perturbation

$$a(\tau) := a + \delta a(\tau) = \frac{1}{\alpha} + \left(\ddot{\delta\alpha} - \frac{\delta\alpha}{\alpha^2}\right), \tag{2.12}$$

where dots denote derivatives with respect to τ. Therefore, as predicted, the perturbation $\delta\alpha(\tau)$ in the trajectory leads to a perturbation in the acceleration

$$\delta a(\tau) = \ddot{\delta\alpha} - \frac{\delta\alpha}{\alpha^2}. \tag{2.13}$$

So far I have done everything in this Section in 1+1 dimensions. To extend it to 3+1 dimensions, however, is a simple matter. Consider a particle accelerating in (3+1)D Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.14}$$

If the acceleration is in the x direction, the trajectory is given by (2.6) augmented by y =constant and z =constant.

I can also consider a congruence of such particles in 3+1 dimensions. If that congruence is spanned by 2 parameters (apart from τ) and behaves nicely (e.g. is continuous in the two parameters), then the congruence will form a time-like 3-surface in Minkowski space. In other words, the particles can be thought of as points on a space-like 2-surface that is undergoing constant acceleration. For example, consider the set of trajectories parameterized by the angles θ and ϕ according to

$$x = \alpha\cosh(\tau/\alpha)\cos(\theta)\sin(\phi), \qquad y = \alpha\cosh(\tau/\alpha)\cos(\theta)\sin(\phi),$$
$$z = \alpha\cosh(\tau/\alpha)\cos(\phi), \qquad t = \alpha\sinh(\tau/\alpha), \tag{2.15}$$

where α is again constant.

The proper time for each particle described by (2.15) is again given by τ, which can therefore be thought of as the proper time for the entire surface. At a fixed proper time, the angles ϕ and θ parameterize a 2-sphere. An incremental change in τ changes the radius of this 2-sphere. Therefore, (2.15) can be thought of as a 2-sphere expanding radially with uniform acceleration. In other words, each point on the sphere is accelerating outward. By looking at a single point on the sphere, say $\phi = \pi/2$ and $\theta = 0$, I recover the 1+1 dimensional motion described above. I can therefore conclude that the radial acceleration of the sphere is given by $1/\alpha$ and that this is the closest approach of the points to the origin (i.e. the minimum radius of the 2-sphere), as is shown in Fig. 2.3.

I should point out that this is not the most general expression for a 2-sphere expanding with constant radial acceleration: instead of having $r =$

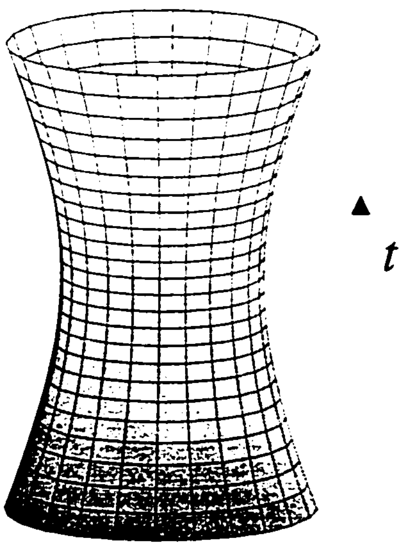

Figure 2.3: The worldsheet of a 2-sphere expanding radially with constant acceleration, as parameterized by (2.15). Note that in order to draw this 3-surface I have had to suppress one dimension. Thus, each point actually represents a circle.

$\sqrt{x^2+y^2+z^2} = \alpha\cosh(\tau/\alpha)$ in (2.15) I could have used $r = r_0 + \alpha\cosh(\tau/\alpha)$. Note that while (2.15) in invariant under boosts (which preserve the spacetime interval $x^2+y^2+z^2-t^2$), the generalization with $r_0 \neq 0$ is not. For this and other reasons I will restrict my attention to the special case (2.15) (i.e. $r_0 = 0$).

The generalization to non-uniform acceleration is equally simple. In particular, if I wish to perturb the motion of the surface (2.15) so as to preserve the 2-spheres, then I again simply replace α by $\alpha + \delta\alpha(\tau)$ in (2.15). I then obtain the parameterization of a 2-sphere expanding radially with perturbed acceleration $a(\tau)$ as given in (2.12).

It is worth mentioning that the perturbation breaks the Lorentz invariance of the problem. In this sense, the (3+1) dimensional case differs from the

(1+1) dimensional case. In (1+1) dimensions, I could always arrange for a given point to be at $\tau = 0$ by boosting to an appropriate frame. This is also true for the Lorentz invariant geometry of the unperturbed accelerating sphere (2.15). However, the perturbations pick out a unique frame, the one in which the perturbations are spherical (i.e. $\delta\alpha$ is a function only of τ). Thus, it will not be possible to assume that any point can generically be taken to be at $\tau = 0$. This will be a minor inconvenience in light of the comparative ease of working with such surfaces, however. They will be the primary object of interest in Chapters 4 and 5.

2.2 Quantum Fields in Curved Space-Time

Perhaps the single most important advance in modern physics has been the introduction and widespread adoption of quantum theory. The initial quantum revolution, at the beginning of the twentieth century, dealt mainly with the quantization of systems of particles. Interestingly, even at that time it was known that much of physics could more easily be described in terms of fields. However, the quantum theory of fields, or *quantum field theory*, was delayed in its development until the middle of the twentieth century. The largest obstacle contributing to this delay was the unavoidable occurrence of divergent quantities in the theory[2]. Today, these divergences are usually considered technical issues and there is an arsenal of regularization and renormalization techniques available to deal with them[15].

Nonetheless, quantum field theory is far from being *fait accompli*. Apart from any foundational issues that might remain unsettled, there is simply a dearth of actual quantum field theoretical results for physically interesting systems. This is in large part due to the fact that even with modern techniques available, such results involve laborious and often subtle calculations. Most of this thesis deals with exactly such calculations, and this Section is intended to provide some background against which to set them. Most of the material in this Section can be found in a more complete form in [16, 17, 18].

For simplicity, I will be dealing throughout this thesis exclusively with massless scalar fields. The standard action for such a field ϕ in an n-dimensional space-time with (contravariant) metric $g^{\mu\nu}$ is

$$S \equiv -\frac{1}{2}\int \sqrt{-g}d^nx\mathcal{L} = \frac{1}{2}\int \sqrt{-g}d^nx(g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \xi R\phi^2), \tag{2.16}$$

where g is the determinant of the metric, R is the scalar curvature of the space-time, $;\mu$ indicates the space-time covariant derivative with respect to the coordinate x^μ and ξ is (at this point) an arbitrary constant. There are, in fact, only two values of ξ that are generally considered to be interesting. For *minimal coupling* $\xi = 0$ while for *conformal coupling* $\xi = \frac{1}{4}[(n-2)/(n-1)]$. I will be using both types of coupling in the coming Chapters, so, for the time being, I will let ξ be arbitrary.

Demanding that the action be stationary with respect to variations in the field ϕ, that is, that

$$\frac{\delta S}{\delta\phi} = 0, \tag{2.17}$$

I obtain the scalar field (wave) equation

$$\mathcal{D}\phi := \Box\phi - \xi R\phi := g^{\mu\nu}\phi_{;\mu\nu} - \xi R\phi = 0. \tag{2.18}$$

Typically, I will be interested in a scalar field which satisfies (2.18) in a given domain Ω in the space-time and behaves in some specified way on the boundary $\partial\Omega$ of that domain. In that case, I will need to augment (2.71) with a boundary condition (BC). Thus, just as in ordinary quantum mechanics, the problem is formulated in terms of a boundary value problem (BVP). It will therefore usually admit an infinite (possibly uncountable) set of eigenfunction solutions, or *modes*.

It is possible to define an inner product on the set of solutions of the BVP. Given a space-like hypersurface Σ in the domain Ω with future directed unit normal vector n^μ, the inner product of two solutions $\phi_1(x)$ and $\phi_2(x)$ is given by

$$(\phi_1, \phi_2) = -i \int_\Sigma \{\phi_1(x)\,(\partial_\mu \phi_2^*(x)) - (\partial_\mu \phi_1(x))\,\phi_2^*(x)\}\, n^\mu \sqrt{-h}\, d\Sigma, \qquad (2.19)$$

where $*$ signifies complex conjugation and $\sqrt{-h}\, d\Sigma$ is the invariant volume element of the hypersurface Σ.

There will exist a set of modes which form an orthonormal set with respect to the inner product (2.19), i.e., every pair of modes ϕ_{ω_1} and ϕ_{ω_2} satisfy

$$(\phi_{\omega_1}, \phi_{\omega_2}) = \delta_{\omega_1\omega_2}, \qquad (\phi_{\omega_1}^*, \phi_{\omega_2}^*) = -\delta_{\omega_1\omega_2}, \qquad (\phi_{\omega_1}, \phi_{\omega_2}^*) = 0. \qquad (2.20)$$

and that span the space of solutions of the BVP. Thus, any solution ϕ of the BVP can be decomposed in terms of these modes. In particular, if ϕ_ω are all the modes of positive norm (i.e. $(\phi_\omega, \phi_\omega) > 0$), then the field may be expanded as

$$\phi = \Sigma_\omega [a_\omega \phi_\omega(x) + a_\omega^\dagger \phi_\omega^*(x)], \qquad (2.21)$$

with numerical coefficients a_ω and $a_\omega^\dagger$.

The formalism developed thus far is suitable for canonical quant-iz-a-tion[19]. The first step in this quantization is the definition of the momentum conjugate to ϕ. According to Lagrangian theory the momentum is given by

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}}. \tag{2.22}$$

To begin the quantization I raise ϕ and π to the status of operators $\hat{\phi}$ and $\hat{\pi}$ on a Hilbert space (which I shall specify shortly). I then impose the *equal time commutation relations*

$$[\hat{\phi}(x), \hat{\pi}(x')] = i\delta(x, x'), \tag{2.23}$$

where x and x' are any two points on a space-like hypersurface Σ and $\delta(x, x')$ is the generalized Dirac distribution satisfying

$$\int_{\partial\Omega} d^n x \sqrt{-h} f(x) \delta(x, x') \equiv f(x'). \tag{2.24}$$

The decomposition of the field (2.21) suggests a similar decomposition for the field operator

$$\hat{\phi} = \Sigma_\omega (\hat{a}_\omega \phi_\omega(x) + \hat{a}^\dagger_\omega \phi^*_\omega). \tag{2.25}$$

The commutation relations (2.23) then imply that $\hat{a}$ and $\hat{a}^\dagger$ must satisfy

$$[\hat{a}_{\omega_1}, \hat{a}^\dagger_{\omega_2}] = \delta_{\omega_1 \omega_2}. \tag{2.26}$$

The algebra of the operators $\hat{a}$ and $\hat{a}^\dagger$ is reminiscent of the algebra of creation and annihilation operators for a simple harmonic oscillator (SHO) (c.f. [20], pp. 182–183). In the SHO case, they operate on energy eigenstates to create and annihilate quanta of energy. In the case of the scalar field, these operators must create and annihilate quanta of the field. In other words, they create and annihilate particles. The Hilbert space upon which they (and by extension $\hat{\phi}$ and $\hat{\pi}$) act, therefore, must be the space of particle representations, or *Fock space.* In particular, the zero particle or vacuum state is defined by

$$a_\omega|0\rangle = 0 \qquad \forall\omega, \tag{2.27}$$

So far, everything has proceeded much in the way that it would in flat (Minkowski) space-time, but this similarity is deceiving. The crucial observation is that the BVP does not admit just one choice of a complete set of modes. Indeed, there are an infinite number of choices, even in flat space-time. However, flat space-time has a symmetry group (the Poincaré group) under which it is invariant. The time-like orbits of this group, which turn out to be the time-like geodesics of the space-time, provide a family of preferred observers (actually, an infinite number of equivalent families). I can therefore use the proper time of these preferred observers to define a preferred time coordinate. This, in turn, allows me to divide the modes into positive and negative frequency sets and to identify the positive frequency modes with the positive normed modes ϕ_ω and the negative frequency modes with their conjugates ϕ^*_ω. This choice of positive normed modes is invariant under the Poincaré group, and therefore defines a preferred vacuum (2.27) which is empty for all inertial observers at every point in the space-time.

In general, however, there will be no symmetry group of the space-time. A state which appears to be a vacuum for one inertial observer may not be a vacuum for any other inertial observer, and there may be no way to distinguish a preferred "vacuum" state. In fact, in many cases, it would be surprising if there were, since external interactions with the space-time geometry tend to change the particle content in QFT[21]. Thus, if I consider a space-time which is asymptotically flat at early and late times, but has some gravitational evolution in between, I would expect the particle content might be different for inertial observers in the early and late asymptotic regions, since the intervening gravitational effects might very well create or destroy particles.

In light of the fact that particle content is not covariant, but depends on the arbitrary choice of positive normed modes (which is equivalent to specifying a preferred time coordinate), it is preferable to avoid using the notion of particles when possible. This is especially true in the context of general relativity, where one might be tempted to identify particle content with matter content and hence as a source of gravitation. Clearly, this would be a dangerous temptation since the definition of a particle is not covariant.

Of course, the true source for gravitation is the stress-energy tensor $T_{\mu\nu}$ (SET). For the classical field, it is given by the variation of the action with respect to the contravariant metric

$$\begin{aligned} T_{\mu\nu}(x) &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}(x)} \\ &= (1-2\xi)\phi_{;\mu}\phi_{;\nu} + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu} g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - 2\xi\phi_{;\mu\nu} \\ &\quad + \frac{2}{n}\xi g_{\mu\nu}\phi\Box\phi + \xi\left[R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \frac{2(n-1)}{n}\xi R g_{\mu\nu}\right]\phi^2 \end{aligned} \tag{2.28}$$

The quantum version would thus presumably be the same with the fields ϕ elevated to the status of operators. However, this also turns $T_{\mu\nu}$ into an operator, which raises a number of issues. First of all, to use the operator $\hat{T}_{\mu\nu}$ in Einstein's field equations, I would have to be able to express them in operator form. This would require a curvature operator, and I am not sure how to formulate one in a sensible way (in particular, there are difficulties with renormalization of such operators[22]). In fact, if I did I might reasonably claim to have a quantum theory of gravity. Unfortunately, such a theory is not known at this time. The solution to this dilemna is to use the expectation value of the SET, $\langle \hat{T}_{\mu\nu} \rangle$ in the classical field equations. This is essentially the semi-classical approximation of ordinary quantum mechanics (c.f. [20], Ch. 11) as applied to fields on a classical background geometry.

There is a second difficulty, however. It turns out that the expectation values of the field operators $\hat{\phi}(x)$ are typically distributional. Since the SET is quadratic in ϕ, and there is in general no way to multiply distributions in a meaningful way, one might suppose that the expectation value of the SET is ill-defined. Fortunately, this difficulty can also be circumvented by taking the distributions to be at different points. For example, while $\delta(x)\delta(x)$ is ill-defined, $\delta(x)\delta(x')$ is not. By taking the field operators to be at different points in the space-time, one hopes that any divergences that may arise may be identified and cancelled with other divergences before a final finite coincidence limit is taken. This process is known as point-splitting, and is one of many regularization methods currently known to be effective. It is the only regularization method I will be using in this thesis.

Thus, what I will be calculating throughout this thesis is the expectation value of the two point operator

$$\begin{aligned} \langle \hat{T}_{\mu\nu} \rangle(x, x') \;\; := \;\; & \frac{1}{2} \Bigg\{ (1 - 2\xi) \langle \hat{\phi}_{;\mu}(x) \hat{\phi}_{;\nu'}(x') \rangle \\ & + \left(2\xi - \frac{1}{2} \right) g_{\mu\nu} g^{\alpha\beta} \langle \hat{\phi}_{;\alpha}(x) \hat{\phi}_{;\beta'}(x') \rangle \\ & - 2\xi \langle \hat{\phi}_{;\mu\nu}(x) \hat{\phi}(x') \rangle + \frac{2}{n} \xi g_{\mu\nu} \langle \hat{\phi}(x') \Box \hat{\phi}(x) \rangle \end{aligned}$$

$$+\xi\left[R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\frac{2(n-1)}{n}\xi Rg_{\mu\nu}\right]\langle\phi(x)\phi(x')\rangle$$

$$\left.+x\leftrightarrow x'\right\}. \tag{2.29}$$

where $x \leftrightarrow x'$ denotes terms identical to all the proceeding terms but with x and x' interchanged. At the end, I will take the coincidence limit $x' \to x$ to find the expectation value of the SET,

$$\langle\hat{T}_{\mu\nu}\rangle(x)=\lim_{x'\to x}\langle\hat{T}_{\mu\nu}(x,x')\rangle. \tag{2.30}$$

The expectation value of the SET, (2.30), has some interesting properties which I shall exploit in later discussions. The first is that, like the SET for classical matter, (2.30) must satisfy the covariant conservation equations

$$g^{\nu\alpha}\langle\hat{T}_{\mu\nu}\rangle_{;\alpha}\equiv 0. \tag{2.31}$$

This is a necessary condition for using this object in the Einstein field equations.

A more interesting property is apparent when the field in question is conformally coupled ($\xi = 1/6$ in 3+1 dimensions). For this coupling the classical SET is traceless ($T^{\mu}{}_{\mu} = 0$). However, the quantization breaks the conformal invariance of the SET and induces an anomalous trace which depends only on the spin of the field being quantized and the space-time, but not on the actual state or field[23]. In 1+1 dimensions this anomalous trace is

$$\langle T^{\mu}{}_{\mu}\rangle=\frac{1}{24\pi}R, \tag{2.32}$$

while in 3+1 dimensions the expression is somewhat more complicated

$$\langle T^{\mu}{}_{\mu}\rangle=\frac{1}{2880\pi^2}\left(-C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}-(R_{\alpha\beta}R^{\alpha\beta}-\frac{1}{3}R^2)+\Box R\right). \tag{2.33}$$

A comprehensive explanation of the rather complicated history of the conformal anomaly can be found in [16] (in particular, see their bibliography of the relevant literature on pp. 335 – 336). As I shall explain shortly, one can draw some interesting conclusions about fields quantized in curved backgrounds from this anomaly.

While in theory I have provided enough background to obtain $\langle T_{\mu\nu}\rangle$ at this point, in practice it is notoriously difficult to explicitly find the modes and carry out the construction (2.29). However, it is often possible to bypass some of these difficulties by working with Green's functions. I will be using a variety of Green's functions in this thesis. The most fundamental Green's functions are the Wightman functions $G^+(x,x')$ and $G^-(x,x')$, defined by

$$G^+(x,x') = G^-(x',x) := \langle \hat{\phi}(x)\hat{\phi}(x')\rangle = \Sigma_\omega \phi_\omega(x)\phi^*_\omega(x'), \qquad (2.34)$$

and called the positive and negative frequency Wightman functions respectively. In terms of these functions, the Feynman Green's function, or Feynman propagator is defined to be

$$iG_F(x,x') := \theta(x^0 - x'^0)G^+(x,x') + \theta(x'^0 - x^0)G^-(x,x'), \qquad (2.35)$$

where $\theta(x)$ is the Heaviside distribution

$$\theta(x) := \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \qquad (2.36)$$

The Feynman propagator is a complex function. Its real and imaginary parts are both very useful in their own rights. The imaginary part is proportional to the Hadamard Green's function

$$G^{(1)}(x,x') = -2\mathrm{Im}(G_F(x,x')), \qquad (2.37)$$

which is defined to be

$$G^{(1)}(x,x') := \langle \hat{\phi}(x)\hat{\phi}(x') + \hat{\phi}(x')\hat{\phi}(x)\rangle. \quad (2.38)$$

This makes it particularly suitable for calculating the stress-energy tensor via (2.29), since

$$\begin{aligned} \langle \hat{T}_{\mu\nu}\rangle(x,x') &= \frac{1}{2}\Big\{(1-2\xi)\nabla_\mu\nabla_{\nu'} + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu} g^{\alpha\beta}\nabla_\alpha\nabla_{\beta'}\rangle \\ &\quad -\xi(\nabla_\mu\nabla_\nu + \nabla_{\mu'}\nabla_{\nu'}) + \frac{1}{n}\xi g_{\mu\nu}(\Box + \Box') \\ &\quad -\xi\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{2(n-1)}{n}\xi R g_{\mu\nu}\right]\Big\} G^{(1)}(x,x') \end{aligned} \quad (2.39)$$

The real part is the negative of the average of the advanced and retarded Green's functions

$$\mathrm{Re}(G_F(x,x')) = \frac{1}{2}(G_R(x,x') + G_A(x,x')), \quad (2.40)$$

which are defined to be

$$G_R(x,x') := -\theta(x^0 - x'^0)(G^+(x,x') - G^-(x,x')), \quad (2.41)$$

$$G_A(x,x') := \theta(x'^0 - x^0)(G^+(x,x') - G^-(x,x')), \quad (2.42)$$

respectively.

By applying the differential operator $\mathcal{D}$ defined in (2.18) to the various Green's functions, one discovers that they are not all truly Green's functions in a strict mathematical sense. Recall that a mathematical Green's function satisfies the differential equation

$$\mathcal{D}\mathcal{G}(x,x') = \sqrt{-g}\,\delta(x,x'), \quad (2.43)$$

and the requisite boundary conditions. While this is true for G_R and G_A, G_F is in fact the negative of a Green's function and $G^{(1)}$ and $G^{\pm}$ are actually homogeneous solutions, i.e.

$$\mathcal{D}\mathcal{G}(x, x') = 0. \tag{2.44}$$

Nonetheless, they are generally all referred to as Green's functions in the literature and I shall do so here.

One might wonder about the efficacy of introducing Green's functions, since they are all defined in terms of the field modes. If the modes are available, surely it is most efficient to calculate the SET directly. The mitigating factor, however, is that the Green's functions can sometimes be constructed without the need to solve for the modes directly.

In particular, if the space-time is static (there exists a time-like coordinate t such that the metric is invariant under the transformations $t \to t+$constant and $t \to -t$) then the operator $\mathcal{D}$ can be made elliptical by replacing t by iw. For elliptical operators, there is a unique Green's function $G_E(x, x')$. A variety of techniques are known by which $G_E(x, x')$ can be found. By replacing w in G_E by $-it$, one of the Lorentzian Green's functions for $\mathcal{D}$ is recovered. The only question is which.

The answer is supplied by examining the behaviour of the Euclidean Green's function in the limit $x \to x'$. In this limit, the Green's function can be thought of as being a local function. Locally, all manifolds are flat, so the dominant behaviour of the Green's functions will be that of the flat space-time Green's function.

For simplicity let me restrict my attention to (3+1)D, and without loss of generality, let me take x' to be the origin. Then the dominant behaviour

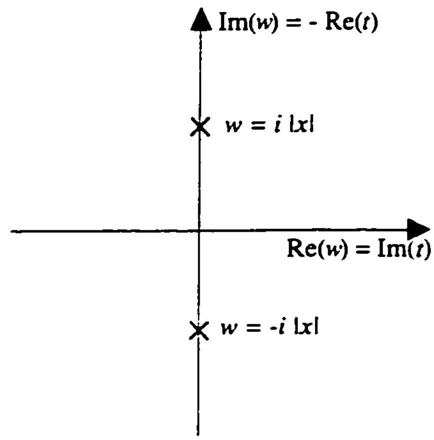

Figure 2.4: The singularities of the Euclidean Green's function (2.45) are marked by ×'s on the imaginary w axis.

of the Green's functions near the origin can easily be found (by, for instance, extending to 4-dimensions the arguments on p. 910 of [24])

$$G_E(x,0) \sim \frac{1}{|x|^2}. \tag{2.45}$$

Now, observe that, for a fixed spatial position $\mathbf{x}$ ($\neq 0$), (2.45) is singular at two points, $w = \pm i|\mathbf{x}|$ as shown in Fig. 2.4.

What happens to these singularities in the Lorentzian domain? The transformation $w \rightarrow -it$ rotates the axes in Fig. 2.4 by $\pi/2$ so that the singularities now lie on the real t axis, as shown in Fig. 2.5. In order for this transformation to be analytic, however, the axis cannot pass through the singularities. The correct way to think of this transformation, therefore, is as the limit of the rotation $\mathrm{Re}(w) \rightarrow \mathrm{Re}(t)$, as shown in Fig. 2.5.

While I will not prove it explicitly here, careful analysis of the pole $t = |\mathbf{x}|$ shows that its residue is given by $G^+(x,x')$, and likewise the residue of $t = -|\mathbf{x}|$ is given by $G^-(x,x')$ (c.f. [17], p. 75). Thus, as shown in

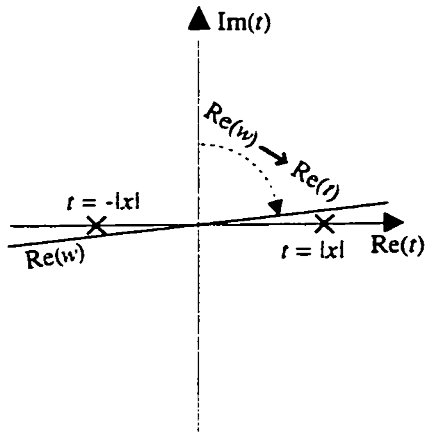

Figure 2.5: In order for the transformation $w \to it$ to be analytic, the real axis must rotate without intersecting the poles (denoted by $\times$). This means that the transformation is really the limit of the rotation shown here.

Fig. 2.5, the analytic continuation of G_E will contain the future directed (upper semi-circle) contribution of G^+ and the past directed (lower semi-circle) contribution of G^-. Inspection of (2.35) reveals that this is precisely the content of the Feynman propagator. Therefore, the analytic extension of G_E is G_F,

$$G_F(t, \mathbf{x}; t', \mathbf{x}') = -i \lim_{\substack{w \to -it \\ w' \to -it'}} G_E(w, \mathbf{x}; w', \mathbf{x}'). \tag{2.46}$$

Finally, let me briefly mention thermal states. So far, I have discussed Green's functions, which involve expectation values in some state, without specifying the state. Usually, one is interested in some pure state. However, at least for static space-times, everything I've done can be easily generalized to mixed states. In particular, for a thermal state with temperature T the expectation value is replaced by an ensemble average (c.f. [20], pp.378 – 381)

$$\langle \phi \rangle \to \mathrm{Tr}(\rho\phi)/\mathrm{Tr}(\rho), \tag{2.47}$$

where Tr denotes a trace and the density operator ρ is, in its simplest form,

$$\rho := e^{-\beta H}, \tag{2.48}$$

where $\beta := 1/T$ and H is the Hamiltonian of the thermal system.

The Green's functions for thermal states have a most remarkable property. Consider for instance

$$\begin{aligned} G^+(x,x') &= \frac{Tr(e^{-\beta H}\phi(x)\phi(x'))}{\mathrm{Tr}(e^{-\beta H})} \\ &= \frac{Tr(e^{-\beta H}\phi(x)e^{\beta H}e^{-\beta H}\phi(x'))}{\mathrm{Tr}(e^{-\beta H})}. \end{aligned} \tag{2.49}$$

Applying the Heisenberg equations of motion,

$$\phi(t+\Delta t,\mathbf{x}) = e^{iH\Delta t}\phi(t,\mathbf{x})e^{-iH\Delta t}, \tag{2.50}$$

to (2.49) I can rewrite it as

$$\begin{aligned} G^+(t,\mathbf{x};t',\mathbf{x}') &= \frac{\mathrm{Tr}(e^{-\beta H}\phi(t',\mathbf{x}')\phi(t+i\beta,\mathbf{x}))}{\mathrm{Tr}(e^{-\beta H})}, \\ &= G^-(t+i\beta,\mathbf{x};t',\mathbf{x}'). \end{aligned} \tag{2.51}$$

In particular, it is clear from (2.38) that

$$G^{(1)}(t+i\beta,\mathbf{x};t',\mathbf{x}') = G^{(1)}(t,\mathbf{x};t',\mathbf{x}'). \tag{2.52}$$

This relation will be useful in identifying thermal states when I encounter them in Section 2.4.

2.3 Perturbations of Green's Functions

In Section 2.2, I discussed quantum field theory in terms of a boundary value problem. However, even in Minkowski space-time, there are many interesting boundary configurations for which explicit solutions have not been constructed. Of particular interest in this thesis will be Dirichlet boundaries (mirrors) undergoing non-uniform acceleration. While these boundaries are well understood in (1+1)D [10], there has been relatively little known about them in (3+1)D.

Much of the original work presented in this thesis addresses the question of non-uniformly accelerating mirrors in (3+1)D. In preparation for the discussions in Chapters 3 and 4, I will review in this Section perturbation theory for Green's functions of Section 2.2. The method I discuss here is essentially identical to that proposed by Ford and Vilenkin[25], although it was rediscovered independently for use in the work presented in later Chapters of this thesis and related publications[26]. I begin by considering the boundary value problem for the unperturbed modes, ϕ_ω, in the (n+1) dimensional Lorentzian domain Ω,

$$\left.\begin{array}{rcll} \Box\phi_\omega(x) & = & 0, & x \in \Omega, \\ \phi_\omega(x) & = & 0, & x \in \partial\Omega. \end{array}\right\} \tag{2.53}$$

where $\partial\Omega$ is a time-like n dimensional boundary of the domain Ω, and is given by the equation $f(x) = \alpha$ for some constant α. Perturbations which preserve the spatial geometry of the boundary can be written as $\delta\alpha(\tau)$, and the perturbed path of the mirror is therefore $f(x) = \alpha + \delta\alpha(\tau)$, where τ is the proper time for points on the unperturbed boundary. This small change in the boundary will cause the modes to change slightly to $\phi_\omega(x) + \delta\phi_\omega(x)$, which will satisfy the same BVP,

$$\left.\begin{array}{rcll} \Box(\phi_\omega + \delta\phi_\omega)(x) & = & 0, & x \in \Omega, \\ (\phi_\omega + \delta\phi_\omega)(x) & = & 0, & x \in \partial\Omega. \end{array}\right\} \tag{2.54}$$

Expanding the boundary condition of (2.54) about $f(x) = \alpha$ and noting that the outward normal vector to this surface is $n_\alpha = -\partial f/\partial x^\alpha$ I have to first order in the perturbation

$$\phi_\omega(x) - \frac{\partial \phi_\omega}{\partial n}(x)\ \delta\alpha + \delta\phi_\omega(x) = 0, \qquad x \in \partial\Omega, \tag{2.55}$$

where $\partial_n := g^{\alpha\beta}\frac{n_\alpha}{|n|}\partial_\beta$ is the outward unit normal derivative. Thus, from (2.53), (2.54) and (2.55) the BVP for the perturbation of the mode is

$$\left.\begin{array}{rcll} \Box\ \delta\phi_\omega(x) & = & 0, & x \in \Omega, \\ \delta\phi_\omega(x) & = & \frac{\partial \phi_\omega}{\partial n}(x)\ \delta\alpha, & x \in \partial\Omega. \end{array}\right\} \tag{2.56}$$

Such inhomogeneous BVPs can often be solved using Green's second identity (c.f. [24], p. 485), which, for suitably well behaved functions f and g in a domain Ω asserts

$$\int_\Omega \sqrt{-g}d^{n+1}x\,[f\Box g - g\Box f] = \int_{\partial\Omega} \sqrt{|h|}d^n x \left[f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right], \tag{2.57}$$

where $\partial\Omega$ is the boundary of Ω, and h is the determinant of the induced metric on the boundary. If I take g to be $\delta\phi_\omega$ then (2.57) has the form

$$\int_\Omega \sqrt{-g}d^{n+1}x\ \delta\phi_\omega(x)\ \Box f(x) = \int_{\partial\Omega} \sqrt{|h|}d^n x \left[f\frac{\partial}{\partial n}\delta\phi_\omega - \frac{\partial \phi_\omega}{\partial n}\ \delta\alpha\frac{\partial f}{\partial n}\right] \tag{2.58}$$

Clearly, I would have a solution of the BVP (2.56) if I could find a solution f to the homogeneous boundary value problem

$$\Box_x f(x,x') = \delta^{n+1}(x - x'), \qquad x \in \Omega \tag{2.59}$$

$$f(x,x') = 0, \qquad x \in \partial\Omega \tag{2.60}$$

since in that case (2.58) reduces to

$$\delta\phi_\omega(x) = \int_\Omega \sqrt{|h|}d^n x'\ \frac{\partial \phi_\omega}{\partial n}(x')\ \delta\alpha(x')\ \frac{\partial f}{\partial n}(x,x'). \tag{2.61}$$

The BVP (2.60-2.61) has a number of solutions. However, if I want $\delta\phi_\omega$ to be a causal response to the boundary perturbation $\delta\alpha$ then I would like $f(x,x')$ to have support only when the point x' is in the causal past of the point x. As is evident from (2.42) in Section 2.2, this condition is satisfied by the retarded Green's function, $G_R(x,x')$, for the unperturbed boundary. Thus, I can obtain the required perturbation by taking $f(x,x') = G_R(x,x')$ in (2.61).

Having obtained the perturbation in modes of the scalar field ϕ, I can now construct the corresponding perturbation of any of the Lorentzian Green's functions of Section 2.2. For example, I will explicitly construct the perturbation in $G^+(x,x')$ here. The perturbation of any other Green's function proceeds similarly.

First, I need the perturbation of the field operator. Recall from Section 2.2 that the field operator is defined in terms of the field modes by

$$\hat{\phi}(x) \equiv \sum_\omega (\hat{a}_\omega \phi_\omega(x) + \hat{a}^\dagger_\omega \phi^*_\omega(x)), \tag{2.62}$$

The perturbation of the field operator is related to the perturbation of the modes by

$$\widehat{\delta\phi}(x) = \sum_\omega (\hat{a}_\omega \delta\phi_\omega(x) + \hat{a}^\dagger_\omega \delta\phi^*_\omega(x)), \tag{2.63}$$

to first order.

Now, suppose that I have a state $|0\rangle$ which I define to be the vacuum. Then, from Section 2.2 I know that the positive frequency Wightman function for that state will be

$$G^+(x',x'') = \langle 0|\hat{\phi}(x')\hat{\phi}(x'')|0\rangle. \tag{2.64}$$

Thus, the first order perturbation of $G^+(x', x'')$ is given in terms of the mode perturbations by

$$\begin{aligned} \delta G^+(x', x'') &= \langle 0|\widehat{\delta\phi}(x')\hat{\phi}(x'') + \hat{\phi}(x')\widehat{\delta\phi}(x'')|0\rangle \\ &= \sum_\omega \big[\langle 0|\hat{a}_\omega\hat{\phi}(x'')|0\rangle\delta\phi_\omega(x') \\ &\quad +\langle 0|\hat{\phi}(x')\hat{a}^\dagger_\omega|0\rangle\delta\phi^*_\omega(x'')\big] \end{aligned} \quad (2.65)$$

where I have used the fact that $\hat{a}_\omega|0\rangle = \langle 0|\hat{a}^\dagger_\omega = 0$. Replacing f by G_R in (2.61) and then inserting the result into (2.65) I get

$$\begin{aligned} \delta G^+(x', x'') = \int_{\partial\Omega} \sqrt{-h}d^n x\ \delta\alpha(x)\times & \\ \Big[\partial_n\langle 0|\textstyle\sum_\omega[\hat{a}_\omega\phi_\omega(x)]\hat{\phi}(x'')|0\rangle\partial_n G_R(x', x) & \\ + \partial_n\langle 0|\hat{\phi}(x')\textstyle\sum_\omega[\hat{a}^\dagger_\omega\phi_\omega(x)]|0\rangle\partial_n G_R(x'', x)\Big], & \end{aligned} \quad (2.66)$$

where I have interchanged the order of integration and summation. Again recalling $\hat{a}_\omega|0\rangle = \langle 0|\hat{a}^\dagger_\omega = 0$ and the definition of $G^+(x', x'')$, I can rewrite (2.66) in the simple form

$$\begin{aligned} \delta G^+(x', x'') = \int_{\partial\Omega} \sqrt{-h}d^n x\ \delta\alpha(x)\ [\partial_n G^+(x, x'')\partial_n G_R(x', x) & \\ + \partial_n G^+(x', x)\partial_n G_R(x'', x)] & \end{aligned} \quad (2.67)$$

Thus, the perturbation of $G^+(x', x'')$ is given in terms of an integral of its normal derivative over the unperturbed boundary. An almost identical argument would have produced for the perturbation of the Hadamard Green's function

$$\begin{aligned} \delta G^{(1)}(x', x'') = \int_{\partial\Omega} \sqrt{-h}d^n x\ \delta\alpha(x)\ \Big[\partial_n G^{(1)}(x, x'')\partial_n G_R(x', x) & \\ + \partial_n G^{(1)}(x', x)\partial_n G_R(x'', x)\Big] & \end{aligned} \quad (2.68)$$

Eq.s (2.68) and (2.67) will be used in Chapters 3 and 4 respectively.

2.4 Rindler and Minkowski States

To illustrate some of the concepts presented in Section 2.2, and for future reference, consider Minkowski space-time. For simplicity, I will work in 1+1 dimensions, although the conclusions I can draw will be quite general.

As mentioned in Section 2.2, in Minkowski space-time there is a natural choice of vacuum state associated with inertial observers and invariant under the Poincaré group. In particular, in terms of the standard Minkowski coordinates (2.1), which are associated with orbits of this group, the scalar field equation is

$$(-\partial_t^2 + \partial_x^2)\phi(x) = 0. \tag{2.69}$$

Although one can easily solve for the modes of this equation, I will instead find the Feynman propagator by way of the Euclidean Green's function. Letting $t \to iw$, (2.69) becomes Laplace's equation. The Green's function for the free field is well known (c.f. [24], p. 911),

$$G_E(x, x') = \frac{1}{2\pi} \ln |x - x'|_E, \tag{2.70}$$

where $|\cdot|_E$ means the Euclidean distance (e.g. $|x|_E^2 = w^2 + x^2$).

As per (2.46), the analytic continuation of this function

$$G_F(x, x') = \frac{i}{2\pi} \ln |x - x'|_M, \tag{2.71}$$

is the Feynman propagator. Note that G_F is a complex function because the Minkowski norm $|x|_M^2 = -t^2 + x^2$ can be negative. Using (2.37) I can find the Hadamard Green's function

$$G^{(1)}_{\text{Minkowski}}(x, x') = \frac{1}{2\pi} \ln \left| |x - x'|_M \right|, \tag{2.72}$$

where the outer bars $|\cdot|$ denote absolute value.

Eq. (2.72) immediately presents me with a problem. If I apply (2.39) to it I will get a divergent stress-energy tensor. It would therefore seem that Minkowski space would be quantum mechanically unstable. However, this is not what is observed in nature. In weak gravity and in the absence of other interactions, the vacuum appears stable. Recalling that the zero level of energy is arbitrary, I *define* the stress-energy of the Minkowski vacuum to vanish,

$$\langle M|T_{\mu\nu}|M\rangle \equiv 0. \tag{2.73}$$

All other stress-energies in Minkowski space are then measured relative to this zero level. This procedure, called renormalization, can be implemented in actual calculations by subtracting the Minkowski state contribution at the level of the Green's functions. Thus, the renormalized Hadamard Green's function is given by

$$G^{(1)}_{\rm ren}(x,x') = G^{(1)}(x,x') - G^{(1)}_{\rm Minkowski}(x,x'). \tag{2.74}$$

This distinguishes the Minkowski vacuum from all other vacuum states in Minkowski space. To denote this distinction, I will call the Minkowski vacuum the *ground state* for Minkowski space-time.

Notice that in curved space-times there is, as stated earlier in this Chapter, generally no natural choice of preferred state with which to perform such a renormalization. Furthermore, even if a preferred state can be defined (I will give an explicit example of one in Section 2.5), one would not expect $\langle T_{\mu\nu}\rangle$ to vanish for it, as is evidenced by the conformal anomaly (2.33). Thus, there is no ground state and renormalization must be carried out by other means. I will not deal with renormalization in curved space-time here, the interested reader is referred to [16].

Thus far, everything I have done has been standard flat space-time QFT. However, I am already in a position to see that things are not as simple as they seem. By simply using the Rindler coordinates in the covariant expression (2.72) I find for the Hadamard's Green function

$$G^{(1)}(x,x') = \frac{1}{4\pi}\ln|\xi^2+\xi'^2-2\xi\xi'\cosh((\tau-\tau')/\alpha)|. \qquad (2.75)$$

The most important feature to notice is that the Green's function is invariant under the transformation $\tau \to \tau + i2\pi\alpha$. It appears, therefore, to be a Green's function for a thermal state with temperature $T = 1/2\pi\alpha$.

What is this thermal state? Clearly it is the Minkowski vacuum: the only thing I have done is change coordinates. In particular, the new time coordinate τ in which (2.75) has an imaginary periodicity is the proper time of an accelerated observer. Thus, I conclude that to such an observer the Minkowski vacuum appears to contain a thermal spectrum of particles with characteristic temperature

$$T = \frac{1}{2\pi\alpha} = \frac{a}{2\pi}. \qquad (2.76)$$

Detailed analysis of the sort found in [16] bears this out. In other words, a state which appears empty to an inertial observer appears filled with a thermal gas as seen by accelerated observers. Conversely, the state which appears empty to accelerating observers (called the *Rindler vacuum*) cannot appear empty to an inertial observer.

Since I have defined the Minkowski vacuum to be the ground state, and therefore to have a vanishing quantum SET, I must conclude that the Rindler vacuum has a finite stress-energy contribution. The expectation value of the SET in the Rindler state in 3+1 dimensions has been calculated by Candelas and Deutsch [27]. For a conformally coupled scalar field it is

$$\langle T_{\mu\nu}\rangle = -\frac{a^4}{480\pi^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{2.77}$$

The analogous expression in 1+1 dimensions is

$$\langle T_{tt}\rangle = -\frac{a^2}{24\pi}, \qquad \langle T_{xx}\rangle = -\frac{a^2}{24\pi}. \tag{2.78}$$

Probably the most remarkable feature of (2.77) and (2.78) is that they have energy densities less than the ground state energy density. This is peculiar but generic feature of quantum states, and shall appear again in this thesis.

Another noteworthy feature of (2.77) becomes apparent if one observes that energy density of an isotropic bath of thermal scalar radiation is given by (c.f. [28], p108)

$$\rho = \frac{\pi^2}{30}T^4. \tag{2.79}$$

If I substitute (2.76) into this energy density, it is exactly the negative of the Rindler vacuum energy density. In fact, the whole SET (2.77) is exactly the negative of the SET for an isotropic bath of thermal scalar radiation. This suggests that if one put an isotropic bath of thermalized Rindler particles into the Rindler vacuum that one would recover the Minkowski vacuum, and that this is why the Minkowski vacuum looks like a thermal state to an accelerating observer. While this is true for conformally coupled scalar fields, it is not true for fields of greater spin. This issue would seem to bear further analysis, but I will not explore it further in this thesis.

While the existence of thermal radiation and a negative energy density in the Rindler vacuum might at first seem odd, they can both be understood intuitively as having to do with the fact that Rindler coordinates only cover part of Minkowski space (see Fig. 2.2). Firstly, it is well known that mixed (including thermal) states can be obtained by tracing out parts of the density matrix for pure states. The fact that Rindler space-time is only a part of Minkowski space-time results is analogous to such a tracing process. Likewise, by excluding modes (and the energy carried by them) which extend outside the Rindler wedge when forming the Rindler state, I am inducing a negative energy density with respect to the Minkowski (vacuum) state.

It is worth pointing out that while the choice of the Minkowski vacuum as the ground state for flat space-time might seem simply to be a matter of gauge, it is not when gravity is considered. When gravity is neglected, it is only the energy difference between states that is physically significant. Since all observers, regardless of which state they consider to be vacuum, agree on what this energy difference is, they all arrive at the correct physical predictions.

However, choosing a different state to be the ground state would lead to different physical predictions when gravitation is concerned. This is because stress-energy is the source for gravitation. In other words, the gravitational field cares not only about the difference between the energy of two states, but also about what the absolute energy of each state is. To illustrate, consider the example of inertial and accelerated observers in Minkowski space-time. If a Rindler state, which is the vacuum state for an accelerated observer, were the ground state, then the Minkowski vacuum, which has a positive energy density when compared to the Rindler state, would truly have a positive energy. Thus, there would be quantum corrections to Minkowski space-time, that is, the classical vacuum space-time would not be quantum mechanically stable.

This vacuum stability argument and the fact that observers with different accelerations at the same event have different vacua, which means that there is not a unique choice of Rindler state, lead one to the conclusion that the Minkowski state, which is the vacuum for all inertial observers, is the correct ground state for gravitational calculations. In this respect, the bath of thermal radiation seen by accelerating observers in the Minkowski state is an artifact of these observers measuring energy with respect to the vacuum of his non-inertial (accelerated) frame. These particles lack "gravitational reality" since they cannot contribute to the gravitational field. Only the particle content of a state with respect to the ground state is "real".

As I mentioned earlier in this Section, in curved space-times there is not, in general, a preferred state. However, if the space-time has a symmetry group whose orbits are time-like (that is, if the space-time is stationary), then invariance under the motions of this group can be used as a criterion for preferred states. Unfortunately, these states do not possess many of the nice properties of the Minkowski vacuum. In particular, they will not be free of particle content for all inertial (or, if one prefers, geodesic) observers, nor will it be possible to define a ground state for the purpose of renormalization, as mentioned above. Nonetheless, I will demonstrate at least one case below in which such a state will play a role that in some ways analogous to the Minkowski vacuum.

2.5 Black Holes and Their Quantum States

One of the most interesting applications of QFT in curved space-time deals with quantum fields in a black hole background. Arguably the single most exciting discovery made in this context is Hawking radiation. Hawking showed[3] that black holes, which are classically absolutely dark and cold because no particles can escape them, actually evaporate into clouds of thermal radiation through quantum processes.

Before describing how this happens, let me review some basic elements of black hole theory. The simplest black hole model, and the only one that I will be concerned with in this thesis, is the Schwarzschild black hole. The metric describing this black hole is the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (2.80)$$

where, in the standard Schwarzschild coordinates, $f(r) = 1 - \frac{2m}{r}$. The coordinate ranges are $-\infty < t < \infty$, $0 < r < \infty$, $0 \leq \phi < 2\pi$ and $0 \leq \theta \leq \pi$. It is clear from (2.80) that the Schwarzschild space-time is spherically symmetric. Thus, the space-time can be foliated by a 2-parameter family of 2-spheres. ϕ and θ are just the familiar angular coordinates on these spheres. The radial coordinate r is related to the area of these 2-spheres in the same way as in flat space, $A = 4\pi r^2$. The constant m is the mass of the black hole. Finally, the Schwarzschild time coordinate t is proportional to proper time for a static observer. These observers, which remain at a fixed radius r, must accelerate to avoid falling into the black hole with an acceleration

$$a = \frac{f'}{2\sqrt{f}}(r), \qquad (2.81)$$

where the prime denotes differentiation with respect to r.

Three of the foliating 2-spheres are of special interest. The first is the limiting sphere $r \to \infty$. In this limit, $f(r) \to 1$ and (2.80) looks like Minkowski space written in the standard spherical coordinates r, θ and ϕ. This is called the asymptotically flat region. The second is the 2-sphere $r = 2m$. At this radius, called the Schwarzschild radius, the function $f(r)$ vanishes and the metric (2.80) is ill-defined. This has a number of consequences, but in particular, it means that the acceleration (2.81) diverges there. Thus, nothing can remain static at this sphere (or inside it, for $r < 2m$): everything in this region must fall inward, eventually arriving at the center of the black hole. The sphere at $r = 2m$ is called the event horizon. Finally, the limit sphere $r \to 0$ is at the center of the space-time. It has no area, and it is clear that (2.80) is also ill-defined there. This pathology of the metric is mirrored in the curvature, which also diverges at $r = 0$. This is the central singularity of the space-time, where all infalling matter eventually accrues.

A clearer understanding of the Schwarzschild metric is provided by the space-time diagram Fig. 2.6. In order to provide a two dimensional representation of the space-time I have had to suppress the θ and ϕ coordinates, so each point on the diagram actually represents a two sphere. As with the standard space-time diagrams of Minkowski space, I have drawn Fig. 2.6 so that radial light (and other massless particles) follow the 45° diagonal lines. One usually envisions only an exterior and interior region for the black hole, which correspond to the right hand and upper wedge respectively. However, the space-time can be analytically extended as shown in Fig. 2.6 to include two other areas. Note that four copies of the Schwarzschild coordinates,

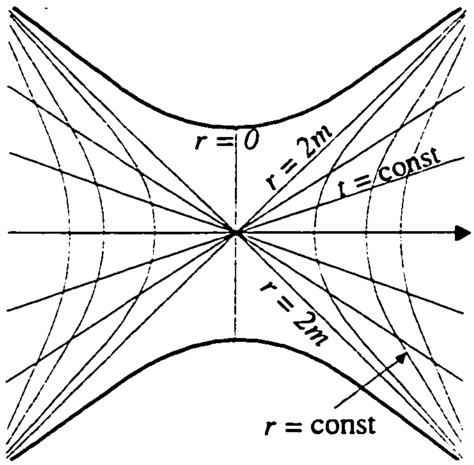

Figure 2.6: This space-time diagram of the Schwarzschild black hole shows the event horizon ($r = 2m$) and the singularity ($r = 0$). Massless particles travel on the 45° diagonal line.

which behave badly on $r = 2m$, are required to cover the entire space-time.

One is immediately struck by the resemblance of the Rindler coordinates in Fig. 2.2, which represent uniformly accelerating observers in Minkowski space-time, and the Schwarzschild coordinates in Fig. 2.6, which represent uniformly accelerating observers in a black hole space-time. One would suspect, then, that an attempt to quantize a field using these coordinates would lead to a state very like the Rindler state. This is, in fact, the case. Such a state was described by Boulware[29] and is called the Boulware state. It is characterized by the boundary condition that it appear empty to static observers at $r \to \infty$ and by the requirement that the state itself be static.

The natural question at this point is whether there is also a state which is analogous to the Minkowski state in flat space-time. Recalling the special status of the Minkowski vacuum and observing the rule that there is generally no such preferred states in curved space-times, it seems unlikely. Fortunately, however, the Schwarzschild black hole is an exception to this rule. It is evident from (2.80) that the Schwarzschild space-time is invariant under the time translation transformation $t \to t$+constant. Thus, it has a symmetry group for which the t-coordinate curves are orbits (the technical term for this symmetry is stationarity). Quantum states which are invariant under the transformation $t \to t +$ constant have the same symmetry as the space-time. As mentioned in Section 2.4, they are therefore in some sense more natural to this space-time, and thus preferred.

It turns out that there is a unique quantum state on the Schwarzschild space-time which is invariant under time translations and is everywhere non-singular. This is called the Hartle-Hawking-Israel state[30]. It is characterized by the boundary condition that it appears static and non-singular to inertial observers at the event horizon. In fact, the Hartle-Hawking-Israel state contains only the irremovable vacuum polarization contribution at the horizon, so not only is it non-singular, it is as empty as a quantum state could be there. Thus, it shares with the Minkowski state some semblance of the claim to being the vacuum state for inertial observers.

The Hartle-Hawking-Israel state lacks many of the pleasing characteristics of the Minkowski vacuum, however. For instance, it is not generally quite possible for a quantum state to be empty where the curvature does not vanish. This is most easily seen by noting that the conformal anomaly (2.33) does not generally vanish unless the curvature does. Physically, this is simply a manifestation of the fact that an external field will create a stress in the vacuum. This process is known as vacuum polarization. Nor does the Hartle-Hawking-Israel state possess only the vacuum polarization contribution (which one might describe as "as close as possible to vacuum") for all inertial observers in all regions. At $r \to \infty$, the vacuum polarization vanishes, but inertial observers (which coincide with static observers in this limit), see a thermal bath of radiation.

Nonetheless, the relationship between the Hartle-Hawking-Israel (HHI) and Boulware states closely parallels the relationship between the Minkowski and Rindler states in flat space-time. Like the Rindler state, the Boulware state has a negative energy density when compared to the appropriate inertial state, in this case the HHI state. Furthermore, as in Minkowski space-time, the inertial (HHI) state appears to be a thermal state to accelerated observers, at least in the asymptotically flat region $r \to \infty$. Therefore, since the HHI (inertial) state is the natural state for a Schwarzschild black hole, and since this state appears to be a thermal state to static observers at $r \to \infty$, the black hole appears to be in equilibrium with thermal radiation there. While this, by itself, is not a derivation of the Hawking radiation emitted by such a black hole, it is at least suggestive.

Unfortunately, calculations for black hole quantum states in 3+1 dimensions are notoriously difficult to do. To illustrate, therefore, let me consider a 1+1 dimensional black hole with metric

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} \\ f(r_0) &= 0, f(\infty) = 1, f'(r_0) = 2\kappa \end{aligned} \tag{2.82}$$

where r_0 is the horizon radius, f' denotes df/dr, and κ is a constant.

Let me further introduce null coordinates u and v defined by

$$dv := dt + \frac{dr}{f(r)} := -n_a dx^a,$$

$$du \quad := \quad dt - \frac{dr}{f(r)} := -l_a dx^a, \tag{2.83}$$

where indices $a, b, c, \ldots$ range over 0, 1. The expectation value for the stress-energy tensor of a massless field on the background (2.82) can be written in the form

$$T^{ab} = \frac{1}{2} T^c_c g^{ab} + E(l^a l^b + n^a n^b) + F l^a l^b, \tag{2.84}$$

where the unspecified functions $T^c_c(r), E(r)$ and $F(r)$ correspond to vacuum polarization, an isotropic radiation field and a net outward flux respectively.

As discussed in Section 2.2, for a massless scalar field in (1+1)D, T^c_c is given by the trace anomaly,

$$T^c_c = \frac{1}{24\pi} R = -\frac{1}{24\pi} f'', \tag{2.85}$$

where R is the curvature scalar for metric (2.82) and f' denotes the derivative of f with respect to r. One can obtain the remaining components from the conservation law, $T^{ab}{}_{;a} = 0$, where ; denotes covariant differentiation. In terms of E and F the conservation law takes the form

$$\begin{aligned} F'(r) &= 0, \\ E'(r) &= -\frac{1}{4} f(r) \frac{d}{dr} T^c_c(r). \end{aligned} \tag{2.86}$$

The specific state with respect to which the expectation value of the stress-energy tensor is taken is given by the boundary conditions which are imposed on (2.86).

As mentioned earlier in this Chapter, I will be interested in two quantum states here. The Boulware state appears empty, apart from the irremovable vacuum polarization represented by (2.85), to stationary observers, and exactly empty to stationary observers at $r = \infty$. This is expressed by the boundary condition $T^{ab} \to 0$ as $r \to \infty$. This condition and the conservation equations (2.86) imply

$$\begin{aligned} E &= E_B \equiv \frac{1}{48\pi}\left(\frac{1}{2}ff'' - \frac{1}{4}f'^2\right), \\ F &= F_B \equiv 0. \end{aligned} \tag{2.87}$$

When (2.85) and (2.87) are substituted into (2.84) the stress-energy takes the form of a stationary fluid with energy density and pressure

$$\rho_B = \frac{1}{24\pi}\left(f'' - \frac{f'^2}{4f}\right), \tag{2.88}$$

$$P_B = -\frac{1}{24\pi}\frac{f'^2}{4f}, \tag{2.89}$$

respectively.

The Hartle-Hawking-Israel state is the one which is appropriate for an eternal black hole inside a cavity with reflecting walls, in thermal equilibrium with its own radiation. It appears empty (modulo vacuum polarization) to free-falling observers at the horizon. This corresponds to the boundary condition that the stress-energy be regular on both the past and future event horizons. By imposing this boundary condition on equations (2.86) E and F take the form

$$\begin{aligned} E \equiv E_{HHI} &= \frac{1}{48\pi}\left(\frac{1}{2}ff'' + \kappa^2 - \frac{1}{4}f'^2\right) \\ F \equiv F_{HHI} &= 0 \end{aligned} \tag{2.90}$$

Thus, the expectation value of the stress-energy in the Hartle-Hawking-Israel state also takes the form of a stationary fluid with energy density and pressure

$$\rho_{HHI} = \frac{1}{24\pi}\left(f'' - \frac{4\kappa^2 - f'^2}{4f}\right), \tag{2.91}$$

$$P_{HHI} = -\frac{1}{24\pi}\frac{4\kappa^2 - f'^2}{4f}, \tag{2.92}$$

respectively. Notice that as $r \to \infty$ we have $P_{HHI} \approx \rho_{HHI} \approx \kappa^2/24\pi$. This is the thermodynamical equation of state for black-body radiation at temperature

$$T \equiv T_{BH} = \kappa/2\pi. \tag{2.93}$$

T_{BH} is referred to as the temperature of the black hole, since it is the characteristic temperature of the radiation seen by distant static observers.

Chapter 3

A (1+1)D Calculation

3.1 Introduction

The purpose of this chapter is to review the results first obtained in [9]. These results indicate that one can calculate the energy density for the Boulware state of a quantum field in a Schwarzschild black hole background by considering the fluxes of quantum energy emitted by moving mirrors in Minkowski space-time. This review will serve a two-fold purpose. In the first instance, it provides a template for the more difficult (3+1)D result. In the second, it demonstrates that the heuristic arguments I use involving a quantum equivalence principle to relate the flux of radiation from a moving mirror to the Boulware state for a black hole have validity, at least in (1+1)D.

Let me begin with the general form of the argument. It is well established that, when quantum effects are considered, a mirror experiencing nonuniform acceleration will radiate two fluxes of energy proportional to the change in acceleration, $dE/d\tau \propto da/d\tau$[10] (I will re-derive this result in the next section). One of these fluxes will be in the direction of the increase in the acceleration of the mirror, and in the case of a scalar field will have negative energy. The other, in the opposite direction, will have positive energy.

Now consider the situation from the point of view of an inertial observer watching an empty (apart from Casimir contributions, which I will systematically ignore throughout this thesis) mirrored box accelerate from left to right in Minkowski space. If the box increases its acceleration by a small amount, two fluxes of energy will enter the box. The flux from the rear (left) wall will be negative and the flux from the front (right) wall will be positive. However, even if I keep the proper length of the box constant, these fluxes will not be equal. As the box accelerates, it will undergo Lorentz contraction as viewed by the inertial observer. The rear wall will therefore be forced to accelerate, and change its acceleration, at a higher rate than the front wall, and will thus emit a larger flux. As a result, the inertial observer sees a negative energy density developing inside the box.

On the other hand, I now consider the situation from the point of view of an observer inside the box, accelerating with it. There is no reason for this observer to suspect that the quantum state inside the box is changing. Indeed, if the change in acceleration is sufficiently small (i.e. the acceleration process is adiabatic), the state inside the box will continue to be the vacuum state for his frame. Thus, with respect to what he sees as the empty vacuum state inside the box, the space-time outside the box becomes filled with a positive energy fluid.

These apparently disparate descriptions of the same scenario are easily reconciled once one realizes that the quantum state inside the box, the Rindler vacuum $|R\rangle$, is different from the quantum state outside the box, the Minkowski vacuum$|M\rangle$. Recall that the renormalized energy density of the Rindler state is lower than that of the Minkowski state. Therefore, the inertial observer, whose natural vacuum is the Minkowski state outside the box, sees the Rindler state inside the box as having negative energy density. Conversely, the accelerating observer, whose natural vacuum is the Rindler state, sees the box as being empty but the exterior as having a net positive energy density.

Let me now turn to the case of a rigid box being lowered toward a black hole. Both the top and bottom reflecting walls will undergo a change of acceleration when lowered. The positive energy flux from both mirrors will be toward the horizon, the negative energy fluxes away from the horizon. Thus, as the box is lowered, positive energy will flow from the mirror at the top of the box into the box's interior, while at the same time, negative energy will flow from the bottom mirror into the box. But for a box of fixed proper length, the change in acceleration during lowering is larger at the bottom than at the top. Therefore, the flux from the bottom mirror will be larger, and there is a net negative energy flow into the box. The interior of a box which is initially empty will consequently acquire a negative energy density through the lowering process.

However, just as in Minkowski space, an observer inside the box, being lowered with it would have no reason to suspect that the state inside the box was changed. The state inside the box would continue to be the vacuum state as observed by this observer. As discussed in Section 2.4, the vacuum state as seen by a static observer near $r \to \infty$ is the Boulware state. The negative energy density developing inside the box due to the quantum fluxes from the mirrors should, therefore, just be the energy density of the Boulware state.

It would be difficult, even in (1+1)D, to perform the analysis in the black hole space-time which would confirm that the Boulware state does indeed develop inside the box as it is being lowered. However, according to the equivalence principle there should be no *local* difference between a box being accelerated in Minkowski space and one being quasi-statically lowered into a black hole from the point of view of an observer inside the box.

The extent to which an equivalence principle holds for quantum states is complicated by the fact that they are inherently non-local objects. This is because a choice of positive normed modes is equivalent to a preferred choice of time at every point in space. Fortunately for me, static states like the Boulware state are exceptional in this sense. Because they exist for an infinite time, they are characterized by a continuous spectrum of modes. This allows one to localize the state in an arbitrarily small region of space, such as a box. Thus, there is reason to believe that the equivalence principle may apply in this case, and I will be able to derive the energy of the Boulware state from arguments in Minkowski space-time. I will show explicitly in this Chapter that these expectations are fulfilled.

In the next Section, I will re-derive the Fulling-Davies[10] result for a moving mirror in (1+1)D in a way that will be extensible to the (3+1) dimensional case. In the remaining sections of this chapter, I will make the heuristic statements above precise. Using the fluxes from moving mirrors in (1+1)D Minkowski space, I will calculate the energy density that develops in a box of fixed proper length as it is accelerated non-uniformly. Next, I will use the equivalence principle to translate this result into the context of a black hole space-time. Finally, I will show that the energy density of the state inside the box as calculated by this method is precisely that of the Boulware state in (1+1)D.

3.2 Moving Mirrors in (1+1)D

The standard derivation of the stress-energy tensor (SET) associated with a moving mirror in (1+1)D is that of Fulling and Davies[10]. Unfortunately, this elegant result involves the conformal invariance of the wave equation in (1+1)D and can therefore not be extended to higher dimension. However, as noted in Ford and Vilenkin[25] and in Section 2.3, given a boundary geometry for which the SET, or an appropriate Green's function, is known, one may use perturbation methods to extend the results to nearby boundary geometries in any dimension.

To illustrate the perturbation method, which I will be using in Chapter 4 and to obtain the equivalent to the Fulling-Davies result for nearly uniformly accelerating mirrors in (3+1)D, I will re-derive the Fulling-Davies result in this Section using it.

I begin with the unperturbed Green's function. The world history of a uniformly accelerating mirror in (1+1)D is given by $|x|_M = \alpha$, where $|\cdot|_M$ denotes the Minkowski norm (e.g. $|x|_M = \sqrt{x^2 - t^2}$). Thus, the Feynman propagator for a Dirichlet boundary condition at the mirror is a solution of

$$\Box G_F(x,x') = \delta(x-x'), \qquad G_F(x,x') = 0 \ \ \forall x \in \{x : |x|_M = \alpha\}. \quad (3.1)$$

The equivalent problem in the Euclidean sector is

$$\nabla^2 G_E(x,x') = \delta(x-x'), \qquad G_E(x,x') = 0 \ \ \forall x \in \{x : |x|_E = \alpha\}, \quad (3.2)$$

where $|\cdot|_E$ is the positive semi-definite Euclidean norm ($|x|_E = \sqrt{x^2+y^2}$).

Thus, I need only solve Laplace's equation in 2 dimensions with a Dirichlet boundary condition on the circle $|x|_E = \alpha$ to obtain the unperturbed propagator. This is easily solved using the method of images. The elementary solution for the Laplacian was found in (2.70)

$$E(x, x') = \frac{1}{4\pi} \ln \left(|x - x'|_E^2 \right). \tag{3.3}$$

This is the "potential" at x due to a unit point charge at x'. To impose the boundary condition (3.2) we simply add a second charge of strength $-|x'|^2/\alpha^2$

$$\tilde{E}(x, x') = -\frac{1}{4\pi} \ln \left(\frac{|x'|_E^2}{\alpha^2} |x - \tilde{x}'|_E^2 \right). \tag{3.4}$$

where

$$\tilde{x}' := \frac{\alpha^2}{|x'|_E^2} x'. \tag{3.5}$$

is the image point to x'. Wick rotating back, I get the Feynman propagator

$$G_F(x, x') = \frac{1}{4\pi} \left[\ln(|x - x'|_M^2) - \ln(\frac{|x'|_M^2}{\alpha^2} |x - \tilde{x}'|_M^2) \right]. \tag{3.6}$$

The Feynman propagator is easily separated into real and imaginary parts using the standard identity

$$\ln(x) = \ln|x| + i\pi\theta(-x). \tag{3.7}$$

Recalling (2.37) and (2.40), therefore, I get

$$G^{(1)}(x, x') = \frac{1}{2\pi} \left[\ln \left(\left| |x - x'|_M^2 \right| \right) - \ln \left(\left| \frac{|x'|_M^2}{\alpha^2} |x - \tilde{x}'|_M^2 \right| \right) \right], \tag{3.8}$$

$$G_R(x, x') = \frac{1}{2}\theta(t' - t) \left[\theta \left(-|x - x'|_M^2 \right) - \theta \left(-|x - \tilde{x}'|_M^2 \right) \right]. \tag{3.9}$$

where the outer $|\cdot|$'s inside the ln's denote absolute values.

I can now calculate the perturbed Hadamard function $\delta G^{(1)}(x, x')$ using the perturbation formula (2.67). First I need the normal derivatives of $G^{(1)}$ and G_R on the boundary. Using the Rindler coordinates (2.7) I have

$$|x'|^2_M = \xi'^2, \tag{3.10}$$

$$|x - x'|^2_M = \xi^2 + \xi'^2 - 2\xi\xi' \cosh((\tau - \tau')/\alpha), \tag{3.11}$$

$$|x - \tilde{x}'|^2_M = \xi^2 + \frac{\alpha^4}{\xi^2} - 2\alpha^2 \frac{\xi}{\xi'} \cosh((\tau - \tau')/\alpha). \tag{3.12}$$

Clearly, since the boundary is parameterized by τ, the outward normal direction (with respect to a point outside the sphere) is the $-\xi$ direction. It is therefore straightforward to calculate the normal derivatives of G_R and $G^{(1)}$ on the boundary. They are

$$-\frac{\partial}{\partial \xi} G^{(1)}\bigg|_{\xi=\alpha} = -\frac{1}{\pi\alpha}\left[\frac{\alpha^2 - \xi'^2}{\alpha^2 + \xi'^2 - 2\alpha\xi' \cosh((\tau - \tau')/\alpha)}\right], \tag{3.13}$$

$$\begin{aligned}-\frac{\partial}{\partial \xi} G_R\bigg|_{\xi=\alpha} &= -\theta(\xi'' \sinh(\tau''/\alpha) - \alpha \sinh(\tau/\alpha))\left(\frac{\xi''^2 - \alpha^2}{\alpha}\right) \\ &\quad \times \delta\left(-\alpha^2 - \xi'^2 + 2\alpha\xi' \cosh((\tau - \tau')/\alpha)\right).\end{aligned} \tag{3.14}$$

Eq. (3.14) can be simplified considerably by recalling the identity

$$\delta(f(x)) = \frac{\delta\left(x - f^{-1}(0)\right)}{|f'(f^{-1}(0))|}. \tag{3.15}$$

Thus, the Dirac distribution in (3.14) can be written

$$\delta\left(-\alpha^2 - \xi'^2 + 2\alpha\xi' \cosh((\tau - \tau')/\alpha)\right) = \left|\frac{\alpha}{\alpha^2 - \xi''^2}\right| \delta(\tau - \tau''_R) \tag{3.16}$$

where

$$\tau''_R := \tau'' - \alpha \cosh^{-1}\left(\frac{\alpha^2 + \xi''^2}{2\alpha\xi''}\right). \tag{3.17}$$

Furthermore, observing that

$$\xi'' \sinh(\tau''/\alpha) - \alpha \sinh(\tau''_R/\alpha) = \left(\frac{\xi''^2 - \alpha^2}{2\xi''}\right)(\cosh(\tau''/\alpha) - \sinh(\tau''/\alpha)), \tag{3.18}$$

and that for all τ''

$$\cosh(\tau''/\alpha) > \sinh(\tau''/\alpha), \tag{3.19}$$

$$\xi''^2 > \alpha^2, \tag{3.20}$$

I can eliminate the Heaviside step function in (3.14) and write

$$-\frac{\partial}{\partial \xi} G_R \bigg|_{\xi=\alpha} = -\delta(\tau - \tau''_R). \tag{3.21}$$

The choice of the subscript R for τ''_R may lead one to think that it is a retarded quantity, and this is indeed the case. It is the proper time at which the mirror's world-history intersects the past null cone from x'', as illustrated in Fig. 3.1.

Eq.s (3.21) and (3.13) are sufficient to allow me to calculate the perturbation in the Hadamard Green's function due to the perturbation of the boundary. Substituting (3.13) and (3.21) into (2.68) and performing the trivial integration I find

$$\delta G^{(1)} = \delta\alpha(\tau''_R)\, \frac{\xi''}{\pi\, \alpha} \times \tag{3.22}$$
$$\left[\frac{\alpha^2 - \xi'^2}{\xi''(\xi'^2 - \alpha^2) - \xi'\left(\alpha^2 \exp\left(\frac{\tau' - \tau''}{\alpha}\right) + \xi''^2 \exp\left(-\frac{\tau' - \tau''}{\alpha}\right)\right)}\right]$$
$$+\, x' \leftrightarrow x''$$

where $x' \leftrightarrow x''$ denotes a term the same as the preceding with ξ' and ξ'' interchanged and likewise for τ' and τ''.

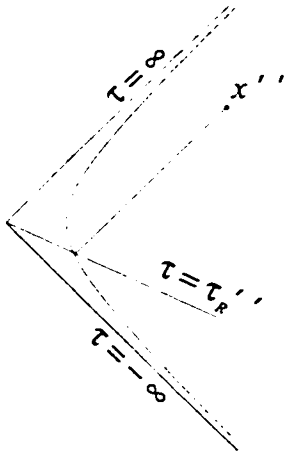

Figure 3.1: A geometric interpretation of τ_R''. The hyperbola is the world-history of the mirror. The line joining x'' to the mirror is a null line. τ_R'' is the proper time at which the mirror and the event x'' have the same retarded time.

In Fig. 3.1 it is evident that any flux from the mirror will travel along the null rays $u := t - x =$ constant. In anticipation of calculating the flux, therefore, I express (3.22) in terms of the null coordinates u and $v := x + t$. Noting that

$$\xi^2 = -uv, \tag{3.23}$$

$$\tau = \alpha \ln\sqrt{-\frac{u}{v}}, \tag{3.24}$$

$$\tau_R'' = \alpha \ln\left(-\frac{\alpha}{u''}\right) \tag{3.25}$$

I get

$$\delta G^{(1)} = \frac{\delta\alpha(\tau_R''(u''))}{\alpha\pi}\left[\frac{\alpha^2 + u'v'}{\alpha^2(1 - \frac{u'}{u''}) + v'(u'' - u')}\right] + x' \leftrightarrow x'', \tag{3.26}$$

Next, I wish to calculate the quantum flux normal to the mirror's world history, as measured in the mirror's frame. Observing that $a^\mu v_\mu = 0$, where

a^μ is the acceleration 4-vector of the mirror and v^μ its 4-velocity. I find that the normal flux is given by the expression

$$\langle M|\,\text{Flux}\,|M\rangle = \langle M|\,T_{\mu\nu}\,|M\rangle v^\mu \frac{a^\nu}{a}, \tag{3.27}$$

where $a := \sqrt{g_{\mu\nu}a^\mu a^\nu}$ is the magnitude of a^ν. Thus, once I have $\langle M|\,T_{\mu\nu}\,|M\rangle$, I will have the necessary flux. However, recall from Section 2.2 that

$$\langle M|\,T_{\mu\nu}\,|M\rangle(x) = \frac{1}{2}\lim_{\substack{x''\to x\\ x'\to x}} \left(\partial_{\mu'}\partial_{\nu''} - \frac{1}{2}g_{\mu\nu}g^{\alpha'\beta''}\partial_{\alpha'}\partial_{\beta''}\right) G^{(1)}(x', x''). \tag{3.28}$$

It is obvious upon inspection of (3.26) that $\langle M|\,T_{vv}\,|M\rangle = 0$. Furthermore, in Minkowski space-time the conformal anomaly vanishes which implies $\langle M|\,T_{uv}\,|M\rangle = 0$. Hence I need only calculate

$$\langle M|\,T_{uu}\,|M\rangle = -\frac{1}{12\pi u^2}\left(\alpha^2\frac{\partial^3}{\partial\tau^3}\delta\alpha - \frac{\partial}{\partial\tau}\delta\alpha\right). \tag{3.29}$$

Substituting (3.29) into (3.27) and observing from the results of Section 2.1 that (to 0^{th} order in the perturbation) $v^u\frac{a^u}{a} = \frac{u^2}{\alpha^2}$, I get for the flux

$$\langle M|\,\text{Flux}\,|M\rangle = -\frac{1}{12\pi\alpha^2}\left(\alpha^2\frac{\partial^3}{\partial\tau^3}\delta\alpha - \frac{\partial}{\partial\tau}\delta alpha\right) \tag{3.30}$$

But, to first order in the perturbation, I have from Section 2.1

$$\frac{da}{d\tau} = \frac{1}{\alpha^2}\left(\alpha^2\frac{\partial^3}{\partial\tau^3}\delta\alpha - \frac{\partial}{\partial\tau}\delta\alpha\right) \tag{3.31}$$

Thus, I have for the flux

$$\langle M|\,\text{Flux}\,|M\rangle = -\frac{1}{12\pi}\frac{da}{d\tau}, \tag{3.32}$$

which agrees with the result of Fulling and Davies[10]. This is exactly the result I need to prove the state inside the box has the energy density of the Boulware state in (1+1)D. This proof will occupy the remainder of this chapter.

3.3 Accelerating Mirrors and the Boulware State

In this Section I will use the results of the previous Section to calculate the energy density in a box being lowered quasi-statically toward a (1+1)D black hole (2.82). For this space-time, a is given at a proper distance z (defined by $dz := dr/\sqrt{f}$) from the horizon by

$$a = \frac{1}{\chi}\frac{d\chi}{dz} = \frac{f'}{2\sqrt{f}}, \tag{3.33}$$

where prime denotes differentiation with respect to the Schwarzschild radius r. Thus, the magnitude of the energy flux from a mirror is

$$\frac{dE}{d\tau} = -\frac{1}{24\pi}\left(f'' - \frac{f'^2}{2f}\right)\frac{dz}{d\tau}. \tag{3.34}$$

Now, let me consider a rigid box with reflecting walls being lowered adiabatically toward the black hole. I will assume the top and bottom walls are rigidly separated by a proper length ℓ which is much less than the radius of the black hole. I will consider the energy flux at an arbitrary fixed surface (a point, in this case) labeled by z_i, ($z_B < z_i < z_T$), in the interior of the box. Consider the flux from, say, the top mirror of the box. In terms of the proper time τ_i for stationary observer at z_i the flux at the top will have the form

$$\frac{dE}{d\tau_i}(z_T) = -\frac{1}{24\pi}\left(f'' - \frac{f'^2}{2f}\right)_T \frac{dz_T}{d\tau_i}, \tag{3.35}$$

where the subscripts i and T denote quantities at z_i and the top of the box respectively. In particular, the energy E in this expression is that relative to a momentarlyy stationary observer at z_T. If one blue-shifts the energy to that seen by a momentarily stationary observer at z_i then the flux becomes

$$\frac{dE}{d\tau_i}(z_i) = -\frac{1}{24\pi}\frac{\sqrt{f(z_T)}}{\sqrt{f(z_i)}}\left(f'' - \frac{f'^2}{2f}\right)_T \frac{dz_T}{d\tau_i}. \tag{3.36}$$

The net rate of change of energy as measured by a comoving observer at z_i will be (3.36) plus the energy flux from the bottom of the box, red-shifted to correspond to energy as measured at z_i,

$$\left.\frac{dE}{d\tau}\right|_{\rm net}(z_i) = \frac{1}{24\pi}\left\{\frac{\sqrt{f(z_T)}}{\sqrt{f(z_i)}}\left(f'' - \frac{f'^2}{2f}\right)_T - \frac{\sqrt{f(z_B)}}{\sqrt{f(z_i)}}\left(f'' - \frac{f'^2}{2f}\right)_B\right\}\frac{dz}{d\tau_i}, \tag{3.37}$$

where the subscript B denotes quantities at the bottom of the box. In obtaining (3.37) I have taken advantage of the fact that since the proper length of the box, ℓ, is assumed constant, $z_B = z_T + \ell$, and therefore $dz_B = dz_T = dz$. Note that while this result depends hold only for one particular point in the box, $z = z_i$, if the box is sufficiently small this dependence will be negligible and the rate of change of energy can be assumed to be approximately independent of the point within the box at which the energy is measured (i.e. the point z_i where the Killing vector $\partial_t/\sqrt{f(z_i)}$ is normalized).

Let me now assume the length of the box is small compared to other relevant length scales in the problem. This will not be true close to the horizon and will therefore prevent me from considering the floating point of the box. Nonetheless, when the box is far enough from the horizon and ℓ is small,

$$\frac{dF}{dz} \simeq \frac{F(z+\ell) - F(z)}{\ell}, \tag{3.38}$$

for any function $F(z)$. Therefore, I rewrite (3.37):

$$\left.\frac{dE}{d\tau}\right|_{\rm net}(z_i) \simeq \frac{\ell}{24\pi\sqrt{f}}\left\{\frac{d}{dz}\left[\sqrt{f}\left(f'' - \frac{f'^2}{2f}\right)\right]\right\}\frac{dz}{d\tau_i}. \tag{3.39}$$

The total energy is the sum of the energies entering the box as it is lowered from infinity to z.

$$E_{net}(z) \simeq \frac{\ell}{24\pi}\int_{\tau_i(z_i=\infty)}^{\tau_i(z_i=z)} \frac{1}{\sqrt{f}}\left\{\frac{d}{dz}\left[\sqrt{f}\left(f'' - \frac{f'^2}{2f}\right)\right]\right\}\frac{dz}{d\tau_i}d\tau_i$$
$$\simeq \frac{\ell}{24\pi}\left(f'' - \frac{f'^2}{4f}\right)_z . \qquad (3.40)$$

But this is just the energy of the Boulware state (2.88). Thus, I conclude that the energy of the matter content of the box is properly measured with respect to the Boulware vacuum energy.

3.4 Conclusion

In the previous Section I have shown that the effect of the acceleration radiation on the energy density inside the box is exactly the same as obtained by considering the vacuum state of the interior of the box to be the Boulware state. I have therefore concluded that throughout the quasi-static change of acceleration, the box has maintained the Boulware state inside it.

As stated in Section 3.1, that the state in the box's interior is always the Boulware state is to be expected on general grounds. The interior vacuum is initially the Boulware state (the vacuum for a stationary observer at infinity) and is invariant under the adiabatic (quasi-static) process of lowering. Thus, there is no reason for the box *not* to maintain the in-vacuum.

It may be surprising that the acceleration of an empty box in flat space-time can be used to obtain the energy density of the Boulware state in a curved space-time. However, this is not difficult to understand. As I mentioned, it is expected that the state inside an adiabatically lowered box is the Boulware state. However, a small enough box sees the gravitational field of the black hole as being essentially homogeneous. The equivalence principle then implies that local phenomena inside such a box are insufficient to determine whether it is accelerating in a gravitational field or in Rindler space. The staticity of the Boulware state implies that it may be localized inside the box so that the equivalence principle applies to it.

While the validity of an equivalence principle for quantum states is a matter of some debate[31], I can apparently implement it in this case. Using it, I have demonstrated how to obtain the Boulware energy density simply by considering the energy balance between two accelerating mirrors in 1+1 dimensional Rindler space. The remainder of this thesis will be involved with exploring this equivalence in the more physically interesting case of (3+1)D.

Chapter 4

The Perturbed Green's Function in (3+1)D

4.1 Introduction

The goal in this chapter is to find the quantum flux emitted normal to a mirror with nonuniform acceleration in 3+1 dimensions. Such a flux has been found for a plane mirror perturbed from rest by Ford and Vilenkin [25]. However, my ultimate goal is to reproduce the Boulware state energy density calculation of Chapter 3 in 3+1 dimensions. For this purpose, I require the flux from a mirror perturbed from an arbitrary constant acceleration, with which to simulate the quasi-static (adiabatic) lowering process.

As in Chapter 3, I will consider the perturbation of a mirror which is initially accelerating uniformly. The required Green's functions for a uniformly accelerating plane mirror have been obtained by Candelas and Deutsch [27], but they are extremely complex. This is possibly connected to the fact that almost all null rays will intersect the mirror an infinite number of times for this mirror geometry. Because of the complexity of this result, the perturbation calculation would be a formidable task.

However, for a spherical mirror expanding with uniform acceleration as given by (2.15), a simple calculation demonstrates that every light ray intersects the mirror exactly once except for those impinging on the surface tangentially, which are confined to the surface forever afterward [32]. This results in a remarkable simplicity for the Green's functions for this geometry, and I will therefore use this as a starting point for the perturbation calculation.

4.2 The Unperturbed Green's Function

I begin in this Section with the unperturbed case, i.e. a spherical mirror expanding with uniform acceleration. Here, and throughout the rest of this thesis, I will restrict my consideration to the special geometry described by (2.15). The Green's function for such a mirror was originally calculated by Frolov and Serebriany [32], and I reproduce the calculation here only for completeness.

Although the quantum stress energy is calculated using the Hadamard Green's function, as in Chapter 3 I will begin by finding the Feynman propagator. This is again accomplished by first rotating to the Euclidean sector and then using the method of images to obtain the Euclidean Green's function. The Feynman propagator is then obtained by a Wick rotation back to the Lorentzian sector.

Although it is not necessary, I will explicitly use the Minkowski gauge

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2, \tag{4.1}$$

in this Section for clarity. In terms of these coordinates, points on the surface of the 2-sphere expanding with uniform acceleration satisfy the condition

$$|x|^2 = x^2 + y^2 + z^2 - t^2 = \alpha^2, \qquad (4.2)$$

where α is the inverse of the acceleration. Green's functions $G(x, x')$ for a reflecting sphere which undergoes the motion (4.2) therefore satisfy the wave equation

$$\Box_x G(x, x') \equiv ({\partial_x}^2 + {\partial_y}^2 + {\partial_z}^2 - {\partial_t}^2)G(x, x') = -\delta^4(|x - x'|). \qquad (4.3)$$

and vanish on the mirror's surface as described by (4.2).

Note that Eq. (4.2) is reminiscent of the equation for a 3 dimensional sphere in Euclidean 4-space when t^2 is replaced by $-t^2$. Clearly, therefore, under the Wick rotation $t \to iT$, the problem of finding $G(x, x')$ is transformed to that of finding the unique Green's function for the Laplace operator

$$\triangle_x G_E(x, x') \equiv ({\partial_x}^2 + {\partial_y}^2 + {\partial_z}^2 + {\partial_T}^2)G_E(x, x') = -\delta^4(|x - x'|). \qquad (4.4)$$

which vanishes on the 3-sphere

$$x^2 + y^2 + z^2 + T^2 = \alpha^2, \qquad (4.5)$$

in Euclidean 4-space. This elliptic boundary value problem (BVP) is easily solved using the method of images. Recall that the elementary solution for the operator $\triangle_x$ in (4.4) is

$$\begin{aligned} E(x, x') &= -\frac{1}{4\pi^2}\frac{1}{|x - x'|^2} \\ &\equiv -\frac{1}{4\pi^2}\frac{1}{(x - x')^2 + (y - y')^2 + (z - z')^2 + (T - T')^2}. \end{aligned} \qquad (4.6)$$

Just as in the (1+1)D calculation, if I want the potential to vanish on the sphere (4.5) I must add a counter charge of strength $-\alpha^2/|x'|^2$ at the image point $(\alpha^2/|x'|^2)x'$, i.e.

$$G_E(x,x') = -\frac{1}{4\pi^2}\left(\frac{1}{|x-x'|^2} - \frac{\alpha^2}{|x'|^2}\frac{1}{\left|x-\frac{\alpha^2}{|x'|^2}x'\right|^2}\right). \tag{4.7}$$

This, therefore, is the solution to the BVP (4.4) and (4.5).

Having obtained the solution in the Euclidean sector I now need to transform back to the Lorentzian sector in order to obtain the Lorentzian Green's functions. As discussed in Section 2.2, the Feynman propagator is obtained by performing a Wick rotation on the Euclidean Green's function. Thus, the Feynman propagator corresponding to (4.7) is

$$G_F(x,x') = -\frac{1}{4\pi^2}\lim_{\epsilon\to 0}\left(\frac{1}{|x-x'|^2 - i\epsilon} - \frac{\alpha^2}{|x'|^2}\frac{1}{\left|x-\frac{\alpha^2}{|x'|^2}x'\right|^2 - i\epsilon}\right). \tag{4.8}$$

Recalling the distributional identity

$$\lim_{\epsilon\to 0}\frac{1}{x\mp i\epsilon} = P\left(\frac{1}{x}\right) \pm i\pi\delta(x), \tag{4.9}$$

where P denotes the principle part of the an integral and $\delta(x)$ is the Dirac delta distribution, I can rewrite (4.8) in the more useful form

$$\begin{aligned} G_F(x,x') &= -\frac{1}{4\pi^2}P\left(\frac{1}{|x-x'|^2} - \frac{\alpha^2}{|x'|^2}\frac{1}{\left|x-\frac{\alpha^2}{|x'|^2}x'\right|^2}\right) \\ &\quad -\frac{i}{4\pi}\left(\delta(|x-x'|^2) - \frac{\alpha^2}{|x'|^2}\delta\left(\left|x-\frac{\alpha^2}{|x'|^2}x'\right|^2\right)\right). \end{aligned} \tag{4.10}$$

Thus, following Frolov and Serebriany[32], I have obtained the Feynman propagator for a spherical mirror expanding with uniform acceleration in 3+1 dimension. Since, as discussed in Section 2.2, all other Green's functions can be obtained from the Feynman propagator, this is all I need to begin the perturbation analysis.

4.3 Propagation of Retarded Signals

I can now use the perturbation procedure outlined in Section 2.3 and (4.10) to find the perturbed Hadamard function for the non-uniformly accelerating mirror.

To begin with, I have to find the two Green's functions appearing in the right hand side of (2.67) corresponding to this geometry. Using Eq (2.35) I find that

$$G^+(x,x'') = -\frac{1}{4\pi^2}\lim_{\epsilon\to 0}\left(\frac{1}{|x-x''|^2+2i(t-t'')\epsilon}\right. \tag{4.11}$$
$$\left.-\frac{a^2}{|x''|^2}\frac{1}{\left|x-\frac{a^2}{|x''|^2}x''\right|^2+2i\left(t-\frac{a^2}{|x''|^2}t''\right)\epsilon}\right).$$

For the time being I will suppress the ϵ to make G^+ easier to write with the understanding that I will reinstate it should I encounter a singularity in evaluating (2.67). Hence,

$$G^+(x,x'') = -\frac{1}{4\pi^2}\left(\frac{1}{|x-x''|^2}-\frac{a^2}{|x''|^2}\frac{1}{\left|x-\frac{a^2}{|x''|^2}x''\right|^2}\right) \tag{4.12}$$

will appear to be symmetric in the points x and x'', although it will in fact be Hermitian (that is, it will satisfy the condition $G^*(x',x'') = G(x'',x')$).

I can also obtain $G_R(x',x)$ using (2.40). It is

$$G_R(x',x) = \frac{1}{2\pi}\theta(t'-t)\left[\delta(|x-x'|^2)-\frac{a^2}{|x'|^2}\delta\left(\left|x-\frac{a^2}{|x'|^2}x'\right|^2\right)\right]. \tag{4.13}$$

Having obtained G^+ and G_R, it is now simply a matter of substituting them into (2.67) and performing the requisite integration. While the calculation is straightforward, it is not short. Recall that the two terms of (2.67) describe signals propagating from a retarded source $\delta\alpha(x)\,\partial_n G^+(x,\cdot)$ (a source on the past null cone) to points x'' and x' respectively.

To facilitate matters, I will spend the remainder of this section analyzing integrals of the form

$$I = \int_{|x|=\alpha} \sqrt{|h|}d^3x\ f(x)\ \partial_n G_R(x', x), \tag{4.14}$$

with the retarded Green's function as given in (4.13). This integral represents the propagation of a signal when the source $f(x)$ is on a sphere expanding with uniform acceleration.

Consider the integral (4.14). From Eq.(4.2) the outward (with respect to a point outside the mirror) normal to the mirror's surface is given by $n^\mu = -x^\mu/\alpha$. Thus,

$$\begin{aligned}\partial_n G_R(x', x) &= -\frac{x^\mu}{\alpha}\partial_\mu G_R(x, x')\\ &= \frac{1}{2\pi\alpha}\left\{-t\delta(t'-t)\left[\delta(|x-x'|^2) - \frac{\alpha^2}{|x'|^2}\delta\left(\left|x-\frac{\alpha^2}{|x'|^2}x'\right|^2\right)\right]\right.\\ &\quad + \theta(t'-t)\left[2(|x|^2 - x\cdot x')\,\delta'(|x-x'|^2)\right.\\ &\quad\left.\left. -2\left(|x|^2 - \frac{\alpha^2}{|x'|^2}x\cdot x'\right)\frac{\alpha^2}{|x'|^2}\delta\left(\left|x-\frac{\alpha^2}{|x'|^2}x'\right|^2\right)\right]\right\}.\end{aligned} \tag{4.15}$$

Now, observing that on the surface $|x|^2 = \alpha^2$

$$\begin{aligned}\left|x - \frac{\alpha^2}{|x'|^2}x'\right|^2 &= \alpha^2 + \frac{\alpha^4}{|x'|^2} - 2\frac{\alpha^2}{|x'|^2}x\cdot x'\\ &= \frac{\alpha^2}{|x'|^2}|x-x'|^2\Big|_{|x|^2=\alpha^2},\end{aligned} \tag{4.16}$$

and recalling the identities

$$\delta(kx) = \frac{1}{|k|}\delta(x), \tag{4.17}$$

$$\delta'(kx) = \frac{1}{k|k|}\delta'(x), \tag{4.18}$$

where k is a constant, I find that the term proportional to $\delta(t'-t)$ in (4.15) vanishes on the surface and I have

$$\partial_n G_R(x',x)\big|_{|x|^2=\alpha^2} = -\frac{\theta(t'-t)}{\alpha\pi}(\alpha^2-|x'|^2)\delta'(\alpha^2+|x'|^2-2x\cdot x')\Big|_{|x|^2=\alpha^2}. \tag{4.19}$$

Thus (4.14) takes the simple form

$$I = -\frac{(\alpha^2-|x'|^2)}{\alpha\pi}\int_{|x|^2=\alpha^2}\sqrt{|h|}d^3x\ \theta(t'-t)f(x)\delta'(\alpha^2+|x'|^2-2x\cdot x'). \tag{4.20}$$

At this point it will be necessary to make a choice of coordinates. I will begin with cylindrical coordinates which are related to the Minkowski coordinates (4.1) by $x \to r\cos(\phi)$, $y \to r\sin(\phi)$, $z \to z$, $t \to t$. Due to the spherical symmetry of the geometry, the z-axis can always be aligned such that $r' = 0$, giving me $x\cdot x' = zz' - tt'$.

Now, consider what the integral (4.20) represents. I am considering null signals propagating from a sphere expanding with uniform acceleration. Thus, I'm integrating over the intersection of the past light cone of the point x' and the world history of the sphere. As shown in Fig. 4.1, the intersection of these two 3 dimensional surfaces is a 2 dimensional surface. This 2-surface has a particularly simple geometric form, it lies in a 3 dimensional plane.

I can take advantage of this geometric simplicity to facilitate the integration. I define the new coordinate w so that the 3-plane in which the

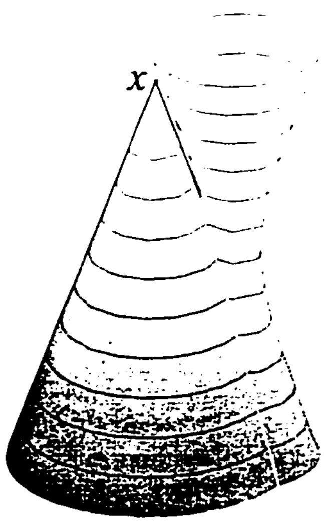

Figure 4.1: The intersection of the world history of a mirror expanding with uniform acceleration (the hyperboloid) and the past light cone of a point x. This figure is dimensionally reduced: each point represents a compactified circle.

integration surface lies is at w = const. Explicitly, this coordinate is

$$w := \frac{z'}{\alpha}z - \frac{t'}{\alpha}t \quad \rightarrow \quad z = \frac{\alpha}{z'}w + \frac{t'}{z'}t. \tag{4.21}$$

In terms of these coordinates the Minkowski line element (4.2) is

$$ds^2 = dr^2 + r^2 d\phi^2 + \frac{\alpha}{z'^2}dw^2 + 2\frac{\alpha t'}{z'^2}dw\, dt + \frac{(t'^2 - z'^2)}{z'^2}dt^2. \tag{4.22}$$

Projecting this metric onto the surface of the sphere ($|x|^2 = \alpha^2$) by imposing $r = \sqrt{\alpha^2 + t^2 - z^2}$ I obtain an induced metric on the surface of the mirror. All I need for this calculation is the invariant surface element for the induced metric, which is given by the simple expression,

$$d^3x = \frac{\alpha^2}{z'}dt\, d\phi\, dw. \tag{4.23}$$

In terms of these (w, ϕ, t) coordinates (4.20) becomes

$$I = \frac{(\alpha^2 - |x'|^2)}{2\pi z'} \times \tag{4.24}$$

$$\int_{-\infty}^{\infty} dt\ \theta(t'-t) \int_0^{2\pi} d\phi \int_{-w'_-(t)}^{w'_+(t)} dw\ f(x)\ \partial_w \delta(\alpha^2 + |x'|^2 - 2\alpha w),$$

where the limits $w'_\pm(t) \equiv (-t't \pm z'\sqrt{(a^2+t^2)})/\alpha$ on w arise from the restriction that $-\sqrt{\alpha^2+t^2} < z < \sqrt{\alpha^2+t^2}$.

Performing the w integration in (4.25) by parts I get

$$\begin{aligned} I &= \frac{(\alpha^2 - |x'|^2)}{2\pi z'} \int_{-\infty}^{\infty} dt\ \theta(t'-t) \int_0^{2\pi} d\phi \\ &\qquad \left\{ \left[f(w)\delta(\alpha^2 + |x'|^2 - 2\alpha)w \right]_{w'_-(t)}^{w'_+(t)} \right. \qquad (4.25) \\ &\qquad \left. + \frac{1}{2\alpha}\theta(w - w'_-)\theta(w'_+ - w)\partial_w f \Big|_{w=\frac{\alpha^2+|x'|^2}{2\alpha}} \right\}, \end{aligned}$$

where the Heaviside distributions in the last term of (4.25) ensure that the Dirac distribution is evaluated only for values of w between $w'_-(t)$ and $w'_+(t)$.

Since $w'_\pm$ are actually functions of t, the limits $w'_\pm$ correspond to limits in t, defined by

$$w'_+(t'_+) = w'_-(t'_-) := \frac{\alpha^2 + |x'|^2}{2\alpha}. \qquad (4.26)$$

The limits $t'_\pm$ have a remarkably simple geometric meaning. They are the Minkowski times where the future and past light cones of the point x' first intersect the world history of the sphere, as shown in Fig. 4.2.

Thus, the second term in (4.25) is simply

$$I_2 = \frac{\alpha^2 - |x'|^2}{4\pi\alpha z'} \int_0^{2\pi} d\phi \int_{-\infty}^{t_-} \partial_w f(x)|_{w=\frac{\alpha^2+|x'|^2}{2\alpha}}. \qquad (4.27)$$

The first term is also relatively simple. Recalling again the identity

$$\delta(f(x)) = \frac{\delta(x - f^{-1}(0))}{|f'(x)|}. \qquad (4.28)$$

I have

$$\delta(\alpha^2 + |x'|^2 - 2\alpha w'_\pm) = -\frac{1}{2}\frac{\sqrt{\alpha^2 + {t'_\pm}^2}\,\delta(t - t'_\pm)}{(\pm z't - t'\sqrt{\alpha^2 + {t'_\pm}^2})}. \qquad (4.29)$$

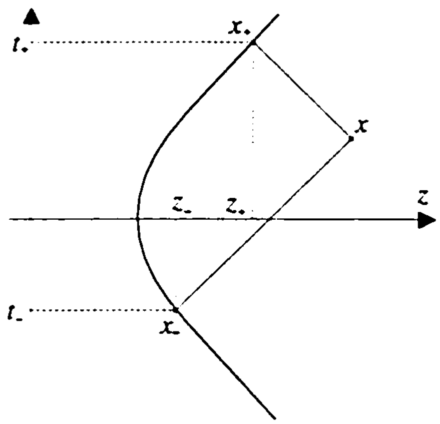

Figure 4.2: For every point x outside the mirror I define the advanced point x_+ and the retarded point x_- as the points of intersection of the light cone from x (represented by long dashes) and the world history of the mirror (represented by the hyperbola) in the $z-t$ plane. These are the points of intersection in the future and past of x that are closest to x. The coordinates of the points $x_\pm$ are $t = t_\pm$, $z = z_\pm$ and $r = 0$.

It is obvious from the definition of $t'_\pm$ that

$$\sqrt{\alpha^2 + {t'_\pm}^2} = z'_\pm, \tag{4.30}$$

as seen in Fig. 4.2. Also, a quick calculation yields

$$t'z'_\pm \mp z't_\pm = \frac{1}{2}(\alpha^2 - |x'|^2). \tag{4.31}$$

Putting (4.27 - 4.31) into (4.25) I get for the integral (4.14)

$$\boxed{I = -\frac{1}{2\pi z'}\int_0^{2\pi} d\phi \left[\frac{\alpha^2 - |x'|^2}{2\alpha}\int_{-\infty}^{t'_-} dt\, \partial_w f + z'_- f\right]_{w=\frac{\alpha^2+|x'|^2}{2\alpha}}} \tag{4.32}$$

This formula describes the propagation of any source $f(x)$ from the surface of a uniformly expanding mirror to the exterior of the mirror. Its simple form indicates that this geometry, in which all but tangentially incident light rays intersect the surface of the mirror at most once, will simplify considerably the perturbation calculation. In the next Section, I apply (4.32) to the problem at hand, that is, to the perturbation of the Wightman function.

4.4 The Perturbed Green's Function

I am now in a position to evaluate (2.67). I denote the perturbation of the boundary by $\delta a(x)$, so that the perturbed boundary is given by $|x| = a + \delta a(x)$. For the first term in (2.67) (for instance) I need to evaluate (4.14) with f of the form

$$f = \delta a(x)\ \partial_n G^+(x, x''), \tag{4.33}$$

that is, by virtue of the identity (4.32), I need to evaluate

$$\begin{aligned} I \ = \ & -\frac{1}{2\pi z'}\int_0^{2\pi} do \left[\frac{a^2-|x'|^2}{2a}\int_{-\infty}^{t'_-} dt\, \partial_w \left(\delta a(x)\ \partial_n G^+(x, x'')\right) \right. \\ & \left. + z'_-\ \delta a(x)\ \partial_n G^+(x, x'')\right]_{w=\frac{a^2+|x'|^2}{2a}} . \end{aligned} \tag{4.34}$$

Here x' and x'' will be the arguments of the Green's function perturbation and x represent the integration variable on the surface. From (4.12) I can find the normal derivative of $G^+(x, x'')$. It is

$$\begin{aligned} \partial_n G^+(x, x'') &= -\frac{x^a}{a}\partial_a G^+(x, x'') \\ &= \frac{1}{4\pi^2 a}\left[\frac{2|x|^2 - x\cdot x''}{|x-x''|^4} - \frac{a^2}{|x''|^2}\frac{2|x|^2 - \frac{a^2}{|x''|^2}x\cdot x''}{|x - \frac{a^2}{|x''|^2}x''|^4}\right]. \end{aligned} \tag{4.35}$$

Evaluating this on the surface of the mirror $|x|^2 = a^2$ I get

$$\partial_n G^+(x, x'') = \frac{1}{2\pi^2 a}\frac{a^2 - |x''|^2}{|x-x''|^4}\bigg|_{|x|^2=a^2} . \tag{4.36}$$

Up to this point, the calculation of the perturbation of the positive frequency Wightman function has been entirely general. However, for simplicity I will restrict my attention to spherical perturbations of the mirror. Such perturbations depend only on the proper time τ of points on the surface of the mirror, which is related to their Minkowski time t by

$$\tau = \alpha \ \mathrm{arctanh}\left(\frac{t}{\sqrt{\alpha^2 + t^2}}\right). \tag{4.37}$$

Note that τ is independent of w, and that therefore

$$\partial_w \delta\alpha(\tau) = 0. \tag{4.38}$$

Also, to repeat the calculation of Chapter 3 I need only calculate the flux of quantum radiation normal to the mirror. This component of the stress energy cannot depend on the azimuthal angle separating the points x' and x''. Thus, for my purposes I need not calculate $\delta G^+(x', x'')$ in all generality: it will suffice to restrict my attention to the special case $r'' = 0$. As a result, I have the following identities:

$$x \cdot x'' \,\Big|_{w = \frac{\alpha^2 + |x'|^2}{2\alpha}} = \frac{z''}{2z'}(\alpha^2 + |x'|^2 + 2t't) - t''t. \tag{4.39}$$

and

$$\partial_w \left[\frac{1}{\alpha^2 + |x''|^2 - 2x \cdot x'')^2}\right]_{|x|=\alpha} = \frac{4\alpha z'' z'^2}{[(\alpha^2 + |x''|^2 + 2t''t)z' - (\alpha^2 + |x'|^2 + 2tt')z'']^3}. \tag{4.40}$$

Putting (4.36) into (4.34), using (4.39), (4.40) and (4.38) and performing the trivial ϕ integration yields

$$I = \frac{z'(|x''|^2 - \alpha^2)}{2\pi^2 \alpha} \times$$

$$\left[2z''(\alpha^2 - |x'|^2)\int_{-\infty}^{t'_-} dt \frac{\delta\alpha(\tau)}{[(\alpha^2 + |x''|^2 + 2tt'')z' - (\alpha^2 + |x'|^2 + 2tt')z'']^3} + z'_- \frac{\delta\alpha(\tau(t'_-))}{[(\alpha^2 + |x''|^2 + 2t''t'_-)z' - (\alpha^2 + |x'|^2 + 2t't'_-)z'']^2]}\right]. \tag{4.41}$$

This can be simplified somewhat by noting that

$$\frac{1}{[(\alpha^2 + |x''|^2 + 2t''t'_-)z' - (\alpha^2 + |x'|^2 + 2t't'_-)z'']^2} = \frac{(t' - z')^2}{z'[\alpha^2 + (t' - z')(t'' + z'')]^2[(t' - z') - (t'' - z'')]^2}. \tag{4.42}$$

and recalling the definition of z'_-, (4.30), so that (4.41) becomes

$$\begin{aligned} I &= -\frac{(|x'|^2 - \alpha^2)(|x''|^2 - \alpha^2)z'z''}{\pi^2\alpha} \times \\ & \int_{-\infty}^{t'_-} dt \frac{\delta\alpha(\tau(t))}{[(\alpha^2 + |x''|^2 + 2t''t)z' - (\alpha^2 + |x'|^2 + 2t't)z']^3} \\ & + \frac{(|x''|^2 - \alpha^2)}{4\pi^2\alpha z'} \frac{(t' - z')(\alpha^2 + (t' - z')^2)\delta\alpha(t'_-)}{[\alpha^2 + (t' - z')(t'' + z'')]^2[(t' - z') - (t'' - z'')]^2}. \end{aligned} \tag{4.43}$$

Recall that from (2.67) the perturbation of the Wightman function is just (4.43) and a like term with $x' \leftrightarrow x''$. However, before combining terms, I wish to make modifications to (4.43) which will simplify its form somewhat. The first is the introduction of the null coordinates

$$v := t + z, \qquad u := t - z. \tag{4.44}$$

Since I have taken both x' and x'' to be in the $z - t$ plane, this implies that $|x'|^2 = -u'v'$ and $|x''|^2 = -u''v''$. In terms of these coordinates

$$t = \frac{1}{2}\left(u - \frac{\alpha^2}{u}\right), \qquad \Rightarrow \qquad dt = \frac{1}{2}\left(1 + \frac{\alpha^2}{u^2}\right) du. \tag{4.45}$$

and $t'_- = u'$, $t''_- = u''$ on the mirror. The second simplification has to do with a partial coincidence limit. I will argue in the next Chapter that the flux should be dominated by the $u-u$ component of the stress-energy. This component is expressed entirely in terms of derivatives with respect to u, and I will therefore take $v'' = v' = v$ immediately. With the null coordinates (4.44) and this partial coincidence, I find after a bit of algebra that I can write the perturbation (2.67) in the form

$$\boxed{\begin{aligned}\delta G^+(x', x'') = & -\frac{(u'v+\alpha^2)(u''v+\alpha^2)}{\pi^2\alpha(u''-u')}\times \\ & \left[\frac{(u'-v)(u''-v)}{(u''-u')}\int_{u'}^{u''} A(u)du \right. \\ & \left. -(u''-v)^2 A(u'') - (u'-v)^2 A(u')\right],\end{aligned}} \tag{4.46}$$

where

$$\boxed{A(u) := \frac{u(u^2+\alpha^2)\,\delta\alpha(\tau(u))}{(\alpha^2+uv)^3(u-v)^3}.} \tag{4.47}$$

There are two noteworthy properties of (4.46). The first is that it can be shown that δG^+ is finite in the coincidence limit $x'' \to x'$. This might be intuitively understood by recalling that any renormalization that is necessary will have been performed at the level of the unperturbed Green's function. With the divergent Minkowski contribution already removed, one would expect all subsequent contributions to be finite.

Examination of (4.46) reveals a second amazing feature of this perturbation. As seen in Fig. 4.1, the retarded Green's function propagates signals from sources anywhere on the past null cone of the point of interest, and the intersection of that past null cone with the world history of the mirror (time-like hyperboloid) is non-compact. Yet (4.46) has support only on the compact portion of this intersection between u' and u''. The mathematical reason for this is that the first term in the expression (4.43) for $I(x', x'')$, which does not have compact support itself, is odd under interchange of x' and x''. Thus, for the Hermitian function $G^+(x', x'') = I(x', x'') + I(x'', x')$, if a perturbation contributes to the signal at x' and at x'', these contributions cancel. I cannot help but feel that there is some physical explanation for this, but I do not know of one at this time.

The compactness of the support of (4.46) will have an important consequence when I calculate $\langle T_{uu} \rangle$. Recall that the SET involves a coincidence limit $x'' \to x'$. In this limit, the support of (4.46) will be a single point. In other words, while amplitudes from the entire history of the mirror in the light-like past of a point might be expected to contribute to the flux at that point, it will turn out that the flux will arise from perturbations only at a *single instant of time*. Furthermore, this single time corresponds to the perturbation closest to the point at which the flux is evaluated.

The explicit calculation of $\langle T_{uu} \rangle$ which will bear this out will be the topic of the next Chapter.

Chapter 5

Quantum Flux from a Moving Mirror

5.1 Introduction

In the last Chapter I derived the perturbation of the Hadamard Green's function $\delta G^{(1)}$ due to a perturbation in the motion of a spherical mirror expanding with uniform acceleration. In this Chapter, I derive the quantum energy flux normal to the mirror due to this perturbation. As in Eq. (3.27), we wish to calculate

$$\langle \text{Flux} \rangle = T_{\mu\nu} \frac{a^{\mu}}{a} v^{\nu}. \tag{5.1}$$

However, recall that I was able to take both x' and x'' to be in the $z - t$ plane. This means that I will need to calculate the flux in the $z - t$ plane, which is the same flux I calculated for the (1+1)D case. Therefore I again have

$$\langle \text{Flux} \rangle = \langle \delta T_{uu} \rangle \frac{u^2}{\alpha^2} + \langle \delta T_{vv} \rangle \frac{v^2}{\alpha^2}. \tag{5.2}$$

Now, unlike 1+1 dimensions, where $\langle T_{uu} \rangle$ was the only non-vanishing component, in 3+1 dimensions it is not immediately obvious that $\langle T_{vv} \rangle$ vanishes. Indeed, even for perturbations of plane mirrors from rest one finds that $\langle T_{vv}\rangle$ does not vanish. In fact, for the minimal stress-energy tensor, even a plane mirror at rest has a non-vanishing $\langle T_{vv}\rangle$[25]. In addition, for the spherical mirror there are additional terms that enter due to the curvature of the surface[33]. To repeat the (1+1)D calculation, however, what I want is the flux due to the motion of the mirror, rather than these other extraneous contributions.

This raises the question of how such a flux is identified. In electromagnetism, one would decompose the field into near and far field pieces and use the far field for this purpose. For my problem, this sort of decomposition also seems to make sense. It would be reasonable to suppose that those effects tied to the boundary would vanish more rapidly than a flux in the far field. Furthermore, in the far field, any $\langle T_{vv}\rangle$ contribution to the flux should die off more quickly than the $\langle T_{uu}\rangle$ contribution, since there is no source in the past v direction. This is born out explicitly in the case of plane mirrors perturbed from rest, where the dominant far field contributions die off like

$$\lim_{z\to\infty}\langle T_{uu}\rangle \sim z^{-1}, \qquad \lim_{z\to\infty}\langle T_{vv}\rangle \sim z^{-3}, \tag{5.3}$$

for both minimal and conformal SETs. If this line of reasoning is correct the dominant flux is simply

$$\langle \text{Flux}\rangle = \langle \delta T_{uu}\rangle \frac{u^2}{\alpha^2}. \tag{5.4}$$

For this reason, I will turn my focus to finding $\langle T_{uu}\rangle$ at this point.

As noted in Chapter 2, the two distinct stress-energy tensors (SETs) that are frequently used are both special cases of (2.28): for the minimal SET one sets $\xi = 0$ and for the conformal SET one takes $\xi = \frac{1}{6}$. Note that for either SET $R_{\mu\nu}$ and R both vanish in flat space-time. It is therefore apparent from (2.28) that the $u-u$ component of the minimal SET is just proportional to $\partial_{u'}\partial_{u''}G^{(1)}(x',x'')$, while the same component of the conformal SET also depends on $(\partial_{u'}\partial_{u'}+\partial_{u''}\partial_{u''})G^{(1)}(x',x'')$. By calculating both of these quantities, we will have the $u-u$ component of both SETs at our disposal.

The last point to which I want to draw attention in this Section is that there is an imaginary part of $\delta G^{+}(x',x'')$ that I have been suppressing. However, as can be seen from (2.38) and (2.34), the Hadamard Green's function is simply the positive (or negative) frequency Wightman function symmetrized in its two arguments,

$$\delta G^{(1)}(x',x'') = \delta G^{+}(x',x'') + \delta G^{-}(x'',x'). \tag{5.5}$$

Furthermore, the Wightman function of the perturbed mirror (4.46) is Hermitian in x' and x''. Thus, the imaginary parts will cancel when I form $\delta G^{(1)}$ and I will be left with twice the real part of δG^{+}, which is all I really calculated in (4.46). As a result, I only need to differentiate $\delta G^{+}(x',x'')$ as indicated in the previous paragraph in order to obtain the required component of both SETs. The next Section deals with this "simple task".

5.2 The Method of Means and Differences

Unfortunately, even with the relatively simple form of (4.47), it is a long and difficult calculation to take the required derivatives (and coincidence limits) in order to find the required δT_{uu}'s. The task is made somewhat simpler, however, by employing what I will call the "method of means and differences".

The central idea is as follows: given any function $F(x)$ of a single variable x, denote the difference of F at points $x = x'$ and $x = x''$ by

$$\Delta F(x) := F(x'') - F(x'). \tag{5.6}$$

Then $\Delta F(x)$ can be expanded in terms of a power series in $\Delta x = (x'' - x')$,

$$\Delta F(x) = \overline{F'(x)}\Delta x - \frac{1}{12}\overline{F^{(3)}(x)}(\Delta x)^3 + \frac{1}{120}\overline{F^{(5)}(x)}(\Delta x)^5 + O((\Delta x)^7), \tag{5.7}$$

where $F^{(n)}(x)$ is short for $\frac{d^n}{dx^n}F$ and

$$\overline{F(x)} := \frac{1}{2}(F(x') + F(x'')), \tag{5.8}$$

is the mean of the values of F at x' and x''. Thus, for example, the integral in (4.46), which has the form,

$$B(u', u'') := \frac{1}{\Delta t}\int_{u'}^{u''} A(u)du \tag{5.9}$$

is expanded according to (5.8) into

$$B(u', u'') = \overline{A(u)} - \frac{1}{12}\overline{A''}(\Delta u)^2 + \frac{1}{120}\overline{A^{(4)}}(\Delta u)^4 + O((\Delta u)^6). \tag{5.10}$$

I will defer straightforward but tedious application of the method of means and differences to (4.46) to Appendix A, and simply use the results here. In particular, then, from (A.41) and (A.42) I get the following expressions for the derivatives of $\delta G^+(x', x'')$

$$\boxed{\begin{aligned}\lim_{\substack{u''\to u\\ u'\to u}}\left[\frac{\partial^2}{\partial u'\partial u''}\,\delta G^+\right] = -\frac{v^2}{\alpha\pi^2}\Bigg\{ & \frac{1}{60}\mu^2\nu^2\frac{\partial^4 A}{\partial u^4} \\ & -\frac{1}{12}\left[\mu\nu^2 + 2\mu^2\nu\right]\frac{\partial^3 A}{\partial u^3} \\ & -\frac{1}{12}\left[4\mu^2 + 8\mu\nu + \nu^2\right]\frac{\partial^2 A}{\partial u^2} \\ & -\frac{1}{2}\left[\nu + 2\mu\right]\frac{\partial A}{\partial u} - \frac{1}{2}A\Bigg\},\end{aligned}} \tag{5.11}$$

$$\lim_{\substack{u''\to u \\ u'\to u}} \left[\overline{\frac{\partial^2}{\partial u^2}\delta G^+} \right] = -\frac{v^2}{\alpha\pi^2} \left\{ \begin{aligned} & -\frac{1}{40}\mu^2\nu^2\frac{\partial^4 A}{\partial u^4} \\ & -\frac{1}{12}\left[\mu\nu^2 + 3\mu^2\nu\right]\frac{\partial^3 A}{\partial u^3} \\ & -\frac{1}{6}\left[3\mu^2 + 4\mu\nu\right]\frac{\partial^2 A}{\partial u^2} \\ & -\mu\frac{\partial A}{\partial u} \end{aligned} \right\}. \tag{5.12}$$

where

$$\begin{aligned} \mu &:= \left(u + \frac{\alpha^2}{v}\right), \\ \nu &:= (u - v), \end{aligned} \tag{5.13}$$

and $A(u)$ is defined in (4.47).

Expressions (5.11) and (5.12) are sufficient to calculate either the minimal or conformal $\langle T_{uu}\rangle$ component. As I pointed out would be the case at the end of Chapter 4, a most remarkable feature of these expressions is that they are entirely local. This contradicts the naive expectation that a given space-time event x should receive fluxes from every perturbation that is in its causal past, which is manifestly non-local (see Fig. 4.1). As mentioned in Chapter 4, I do not know of a *physical* explanation for this. Regardless of the actual cause, this fortunate simplification of the flux and lends itself to the repetition of the (1+1)D calculation in Chapter 3.

This concludes the Section on the use of the method of means and differences. In the next Section, I will explicitly calculate the flux in terms of the acceleration of the mirror.

5.3 The Quantum Flux

While, as mentioned in the previous Section, the forms of (5.11) and (5.12) are remarkably simple, they are not terribly useful for repeating the calculation of the Boulware energy density. In order to do that, I would like to express the quantum fluxes in terms of the acceleration, as was done in the (1+1)D case. I will therefore need to express T_{uu} in terms of the perturbation of the acceleration, δa, or at least in terms of the mirror perturbation $\delta\alpha$. Since both minimal and conformal $\langle T_{uu}\rangle$'s will be useful, at least in checking my results, I will calculate both of them. However, the $u-u$ component of the conformal SET $\langle T_{uu}\rangle^{\rm conf}$ will be of special interest since it is both simpler in form and interpretation. In particular, because the conformal SET vanishes identically in the case of uniform acceleration [32], the flux contains contributions only from the perturbed SET which allows a cleaner comparison with the (1+1)D result.

To begin with, recall (2.39) which gives for the $u-u$ component of the conformal SET

$$\langle T_{uu}\rangle = \frac{1}{6}\lim_{\substack{u''\to u\\ u'\to u}}\left[2\partial_{u'}\partial_{u''}G^{(1)}(x',x'') - \overline{\partial_u{}^2 G^{(1)}(x',x'')}\right]. \tag{5.14}$$

But, according to (2.38),

$$G^{(1)}(x',x'') = G^{+}(x',x'') + G^{+}(x'',x'), \tag{5.15}$$

which, since δG^{+} is Hermitian, means that I can write

$$\begin{aligned}\langle \delta T_{uu}\rangle^{\rm conf} &= -\frac{1}{3}\lim_{\substack{u''\to u\\ u'\to u}}\left\{2\left[\frac{\partial^2(\delta G^{+})}{\partial u'\partial u''}\right] - \left[\overline{\frac{\partial^2(\delta G^{+})}{\partial u^2}}\right]\right\}, \\ &= -\frac{v^2}{3\pi^2\alpha}\left\{-\frac{\mu^2\nu^2}{120}\frac{\partial^4 A}{\partial u^4} - \frac{\mu\nu}{12}(\mu+\nu)\frac{\partial^3 A}{\partial u^3}\right. \end{aligned} \tag{5.16}$$

$$-\frac{1}{6}\left((\mu+\nu)^2+2\mu\nu\right)\frac{\partial^2 A}{\partial u^2}-(\mu+\nu)\frac{\partial A}{\partial u}-A\Big\}.$$

where μ and ν are defined in (5.13) and $A(u)$ is defined as in (4.47).

$$A := \frac{u(u^2+\alpha^2)\delta\alpha}{v^3\mu^2\nu^3}. \tag{5.17}$$

Likewise, for the minimal component I have

$$\begin{aligned}\langle\delta T_{uu}\rangle^{\text{min}} &= -\lim_{\substack{u''\to u\\ u'\to u}}\left[\frac{\partial^2(\delta G^+)}{\partial u'\partial u''}\right],\\ &= -\frac{v^2}{\pi^2\alpha}\Big\{\frac{\mu^2\nu^2}{60}\frac{\partial^4 A}{\partial u^4}-\frac{\mu\nu}{12}(2\mu+\nu)\frac{\partial^3 A}{\partial u^3}\\ &\quad -\frac{1}{12}\left((2\mu+\nu)^2+4\mu\nu\right)\frac{\partial^2 A}{\partial u^2}-\frac{1}{2}(2\mu+\nu)\frac{\partial A}{\partial u}-\frac{1}{2}A\Big\}.\end{aligned} \tag{5.18}$$

Again, it is easiest to expand this in more than one step. Defining

$$H := u(u^2+\alpha^2) \tag{5.19}$$

so that

$$A(u) = \frac{H(u)}{v^3\mu^3\nu^3}, \tag{5.20}$$

I get

$$\begin{aligned}\langle\delta T_{uu}\rangle^{\text{conf}} &= \frac{1}{180\alpha\pi^2 v\mu^3\nu^3}\Big[\mu^2\nu^2 H^{(4)}-2\mu\nu(\mu+\nu)H^{(3)}\\ &\quad +2(\mu^2+4\mu\nu+\nu^2)H''\\ &\quad -12(\mu+\nu)H'+24H\Big],\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}\langle\delta T_{uu}\rangle^{\text{min}} &= \frac{1}{30\alpha\pi^2 v\mu^5\nu^3}\Big[\mu^4\nu^2 H^{(4)}-\mu^3\nu(2\mu+7\nu)H^{(3)}\\ &\quad +\mu^2(2\mu^2+13\mu\nu+32\nu^2)H''\\ &\quad -6\mu(2\mu^2+7\mu\nu+15\nu^2)H'\\ &\quad +12(2\mu^2+5\mu\nu+10\mu\nu)H\Big].\end{aligned} \tag{5.22}$$

Now, recall (2.13) where I found that

$$\delta a = -\frac{\delta\alpha}{\alpha^2} + \ddot{\delta\alpha}, \tag{5.23}$$

where dots denote differentiation with respect to the proper time of the (unperturbed) mirror, τ. Thus, I need to convert derivatives with respect to u to those with respect to τ. The two coordinates are related by

$$u \equiv (t - z) = -\alpha e^{-\tau/\alpha}. \tag{5.24}$$

But note that by defining

$$\sigma := u^2 + \alpha^2, \tag{5.25}$$

I have that

$$H = u\sigma\,\delta\alpha, \tag{5.26}$$

and

$$\frac{\partial}{\partial\sigma} = \frac{1}{2u}\frac{\partial}{\partial u} = -\frac{\alpha}{2u^2}\frac{\partial}{\partial\tau}. \tag{5.27}$$

Therefore,

$$\frac{\partial^2}{\partial\sigma^2}(u\,\delta\alpha) = \frac{\alpha^2}{4u^3}\,\delta a. \tag{5.28}$$

Thus, the derivatives of $H(u)$ with respect to u (denoted by the primes) are explicitly

$$\begin{aligned} H' &= (\sigma + 2u^2)\delta\alpha + u\sigma(\delta\alpha)', \\ &= 2u(u\,\delta\alpha) + 2u\sigma\frac{\partial}{\partial\sigma}(u\,\delta\alpha), \end{aligned} \tag{5.29}$$

$$\begin{aligned} H'' &= 6u\,\delta\alpha + 2(\sigma + 2u^2)(\delta\alpha)' + u\sigma(\delta\alpha)'', \\ &= 2(u\,\delta\alpha) + 2(\sigma + 4u^2)\frac{\partial}{\partial\sigma}(u\,\delta\alpha) + \frac{\alpha^2\sigma}{u}da, \end{aligned} \tag{5.30}$$

$$H^{(3)} = 6\delta\alpha + 18u(\delta\alpha)' + 3(\sigma + 2u^2)(\delta\alpha)'' + u\sigma(\delta\alpha)^{(3)},$$

$$= 24u\frac{\partial}{\partial\sigma}(u\,\delta\alpha) + 6\alpha^2\,da - \frac{\sigma\alpha^3}{u^2}\frac{d}{d\tau}(da). \tag{5.31}$$

$$H^{(4)} = 24(\delta\alpha)' + 36u(\delta\alpha)'' + 4(\sigma + 2u^2)(\delta\alpha)^{(3)} + u\sigma(\delta\alpha)^{(4)},$$
$$= 24\frac{\partial}{\partial\sigma}(u\,\delta\alpha) + 12\frac{\alpha^2}{u}\,da + 2\frac{\alpha^3}{u^3}(\sigma - 4u^2)\frac{d}{d\tau}(da) + \frac{\alpha^4\sigma}{u^3}\frac{d^2}{d\tau^2}(\delta\alpha). \tag{5.32}$$

It is now a straightforward matter to substitute (5.26) and (5.29-5.32) into (5.21) and (5.22) respectively to find the $u-u$ components of the conformal and minimal SETs respectively. For $\langle T_{uu}\rangle^{\rm min}$ this results in a rather long expression. Furthermore, I will only be using this quantity as a check, rather than to repeat the (1+1)D box calculation. I will therefore not write the full expression here.

In contrast, the conformal component has a comparatively compact form

$$\langle\delta T_{uu}\rangle^{\rm conf} = \frac{1}{360\pi^2\alpha v\omega^6}\Bigg[4\kappa^2(u\,\delta\alpha) + 2\frac{\zeta\kappa^2\alpha}{u^2}\frac{\partial}{\partial\tau}(u\,\delta\alpha) + 2\frac{\alpha^2\kappa^2\sigma}{u}\,\delta a + \frac{\alpha^4\omega^4\sigma}{u^3}\,\ddot{\delta a} - 2\frac{\alpha^3\omega}{u^3}(2\lambda u\zeta + \zeta^4 - 4\alpha^2u^2)\,\dot{\delta a}\Bigg]. \tag{5.33}$$

where

$$\begin{aligned} \lambda &:= \frac{\alpha^2 - v^2}{v}, \\ \omega &:= u^2 + \lambda u - \alpha^2, \\ \kappa^2 &:= \lambda^2 + 4\alpha^2, \\ \zeta &:= u^2 - \alpha^2. \end{aligned} \tag{5.34}$$

Note that I have expressed (5.33) as much as possible in terms of the perturbation the acceleration δa rather than the perturbation of the mirror $\delta\alpha$. Using (5.23) one could easily express this quantity entirely in terms of $\delta\alpha$.

however, in order to repeat the box argument of Chapter 3 it will be more convenient to think in terms of acceleration.

Eq. (5.33) is one of the primary results of this thesis. With (5.4) (and the assumption that $\langle T_{vv}\rangle$ is irrelevant for this calculation) it gives me the flux of conformal quantum energy emitted normal to the surface of a spherical mirror when the rate of expansion is perturbed from constant acceleration. In the next Section, various checks are made to ensure that this expression and (5.22) are in agreement with previously known results.

5.4 Checks and Balances

In this Section, I present three checks that the results above make sense. The first, and easiest, has to do with the fall off of the flux at future null infinity, which in these coordinates corresponds to $v = \infty$. It is simple to verify that

$$\lim_{v\to\infty} \lambda \sim -v, \tag{5.35}$$

$$\lim_{v\to\infty} \omega \sim -uv, \tag{5.36}$$

$$\lim_{v\to\infty} \kappa^2 \sim v^2, \tag{5.37}$$

from which I get

$$\lim_{v\to\infty} \langle \text{Flux}\rangle^{\text{conf}} = \frac{1}{360\pi^2\alpha^3 uv^2}\left[4u\,\delta\alpha + 2\frac{\zeta\alpha}{u^2}\frac{\partial}{\partial\tau}(u\,\delta\alpha) + 2\frac{\sigma\alpha^2}{u}\,\delta a - 4\frac{\alpha^3\zeta}{u}\,\dot{\delta a} + \frac{\alpha^4\sigma}{u}\ddot{\delta a}\right]. \tag{5.38}$$

Thus, the flux falls off as v^{-2}, which is the expected rate for a spherical source.

The next check answers the following question: are the results above for an arbitrary perturbation compatible with those for a fixed perturbation, one with $\delta\dot{\alpha} = 0$? By taking $\delta\dot{\alpha} = 0$, I am simply redefining the minimum radius of the 3-sphere $\alpha \to \alpha + \delta\alpha$, so I can compare my results to those obtained by Frolov and Serebriany [32] in the constant acceleration case.

It is quickly obvious that this sort of compatibility exists for $\langle \delta T_{uu} \rangle$ as calculated in (5.33). Note that Frolov and Serebriany show that for fixed acceleration the conformal SET vanishes identically. This is also true for (5.33), in which the first two terms give canceling contributions. Such null checks $(0 = 0)$ are somewhat less than satisfying, however. There are many paths to a null result.

The minimal SET, however, does not vanish for constant acceleration. Although, as I mentioned above, $\langle \delta T_{uu} \rangle^{\min}$ for a time dependent perturbation is messy, it is very simple for the constant perturbation case. I get

$$\langle \delta T_{uu} \rangle^{\min} = -\frac{2\alpha v^2(uv - 3\alpha^2)}{\pi^2(uv + \alpha^2)^5}\,\delta\alpha \tag{5.39}$$

On the other hand, starting with the minimal SET for constant acceleration as calculated by Frolov and Serebriany [32]

$$\langle T_{uu} \rangle^{\min} = \frac{-(\alpha)^2 v^2}{\pi^2(uv + \alpha^2)^4}, \tag{5.40}$$

and perturbing by letting $\alpha \to \alpha + \delta\alpha$, it is straightforward to see that to first order in $\delta\alpha$ the stress-energy component changes by exactly (5.39) as well. Thus, the minimal stress-energy tensor result is also compatible with the Frolov-Serebriany result for constant perturbation.

The final check I will perform is a comparison to the results of Ford and Vilenkin [25]. The Ford-Vilenkin results are for a planar mirror perturbed from rest. Thus, in order to compare I must consider the results I've obtained for vanishing velocity and acceleration and for infinite radius (the case in which the mirror becomes approximately flat). The first requirement, vanishing velocity, is easily achieved by considering $\langle \delta T_{uu} \rangle$ only in the null future of the point of stationarity for the accelerating mirror, i.e., only on the null line $u = -\alpha$, which implies $v = 2z - \alpha$. Next, I need to consider the stress-energy at a finite distance x from the mirror. Thus, I let $z = \alpha + x$. Finally, recalling that the radius of the sphere at the stationary point is α and that the acceleration of the mirror is $1/\alpha$, I observe that the other two requirements (vanishing acceleration and infinite radius) are simultaneously achieved in the $\alpha \to \infty$ limit.

Performing this series of steps I find that the SET components (5.21) and (5.22) become

$$\lim_{\alpha\to\infty} \left[\langle \delta T_{uu} \rangle^{\text{conf}}_{u=-\alpha} \right] = -\frac{1}{1440\pi^2 x^3} \left((\delta\alpha)'' + 2x(\delta\alpha)^{(3)} + 2x^2(\delta\alpha)^{(4)} \right), \quad (5.41)$$

and

$$\lim_{\alpha\to\infty} \left[\langle \delta T_{uu} \rangle^{\text{min}}_{u=-\alpha} \right] = -\frac{1}{480\pi^2 x^5} \Big(30\delta\alpha + 45x(\delta\alpha)' + 32x^2(\delta\alpha)'' + 14x^3(\delta\alpha)^{(3)} + 4x^4(\delta\alpha)^{(4)} \Big), \quad (5.42)$$

respectively, where the derivatives of $\delta\alpha$ are with respect to u.

On the other hand, by noting that

$$T_{uu} = \frac{1}{4} \left(T_{zz} + T_{tt} - 2T_{tz} \right), \quad (5.43)$$

I can compose the same stress-energy components from those provided in by Ford and Vilenkin in [25]. Upon doing so, I obtain exact agreement with (5.41) and (5.42).

There are two conclusions I can reach from these checks, especially the comparisons with the Ford-Vilenkin results. The first is that the perturbation method I've outlined does give results which are reasonable and is the same as the Ford-Vilenkin perturbation method, as I mentioned in Section 2.4. The second is that the $\alpha \to \infty$ limit does correspond to the case of a planar mirror. I will take advantage of this correspondence in the next Chapter.

Chapter 6

The Boulware State for Black Holes

6.1 The Flat Mirror Limit

Having obtained in Section 5.3 the quantum flux from a mirror with non-uniform acceleration in 3+1 dimensions, I can now repeat the arguments from Section 3.3. This should allow me to calculate the energy density of the Boulware state for a Schwarzschild black hole as measured by an inertial observer. Given this component of the stress-energy tensor I can obtain an analytic expression for the entire tensor.

First, however, I must identify which parts of the flux (5.33) are relevant for this analysis. In particular, it is known that even for static spherical mirrors, the stress-energy contains terms due to the curvature of the mirror[33]. Presumably the Boulware state is independent of these terms, and it would be advantageous to remove them *a priori*. In order to do this, I need to the radius of the sphere to be arbitrarily large. This will be the case for very early ($\tau \to -\infty$) and very late ($\tau \to \infty$) proper times for the spherical mirror. Note that we cannot simply take $\alpha \to \infty$, because although this does correspond to large radius, it also corresponds to vanishing acceleration which is not true for mirrors hovering quasi-statically above black holes.

In terms of the null coordinates u and v appearing in (5.33), the early proper time limit $\tau \to -\infty$ is simply $u \to -\infty$. In this limit, I find the dominant behaviour of the quantities defined in (5.34) to be

$$\omega \sim \xi^2 \sim \kappa^2 \sim u^2. \tag{6.1}$$

Substituting (6.1) into (5.33) I get

$$\begin{aligned}\langle T_{uu}\rangle &= \frac{1}{360\pi^2\alpha v u^6}[2\omega u(\delta\alpha + \alpha\dot{\delta\alpha} + \alpha^2\delta\alpha) \\ &\qquad - 2\alpha^3 u(u^2 + 2\lambda u - 4\alpha^2)\dot{\delta\alpha} + \alpha^4 u^3\ddot{\delta\alpha}]. \end{aligned}\tag{6.2}$$

From (6.2), and recalling from (5.4) that

$$\langle \text{Flux}\rangle = \frac{u^2}{\alpha^2}\langle T_{uu}\rangle. \tag{6.3}$$

I read off the dominant contribution to the $\langle$Flux$\rangle$ in the limit $u \to -\infty$ to be

$$\langle \text{Flux}\rangle \sim \frac{1}{360\pi^2 u v}[\alpha\ddot{\delta a} - 2\dot{\delta a}]. \tag{6.4}$$

Now, notice that (6.4) vanishes in the limit $u \to -\infty$. This is to be expected since, in this limit, the mirror is receding from a static external observer at the speed of light, and the flux is therefore infinitely red-shifted. To correct for this, I must multiply by the appropriate blue-shift factor. Since the flux has units of Energy/Area/Time, and the area is unaffected by the (orthogonal) velocity, I need two correction factors, one for the energy

$$E_{\rm rec} = \frac{d\tau_{\rm em}}{d\tau_{\rm rec}}E_{\rm em}. \tag{6.5}$$

and one for the inverse proper time

$$\tau_{rec} = \frac{d\tau_{rec}}{d\tau_{em}}\tau_{em}, \tag{6.6}$$

where the labels rec and em denote quantities for the receiver and emitter respectively.

Observing that the blue-shift factor is given by

$$\frac{d\tau_{em}}{d\tau_{rec}} = \frac{\sqrt{-uv}}{\alpha} \tag{6.7}$$

I find, therefore, that the flux in the frame of the emitter is

$$\langle \text{Flux} \rangle = -\frac{1}{360\pi^2\alpha^2}[\alpha\ddot{\delta a} - 2\dot{\delta a}]. \tag{6.8}$$

Thus, recalling that the unperturbed acceleration is simply $1/\alpha$, I have for the flux from a planar mirror perturbed from uniform acceleration a by an amount δa in $3+1$ dimensions

$$\langle \text{Flux} \rangle = -\frac{a}{360\pi^2}[\ddot{\delta a} - 2a\dot{\delta a}]. \tag{6.9}$$

(at least to the extent that such a mirror is approximated by a large sphere).

In the next Section, I will use this expression to calculate the energy density inside a box being lowered quasi-statically toward a black hole. This calculation in 1+1 dimensions (see Section 3.3) yielded the energy density of the Boulware state for the black hole.

6.2 The Quasi-Static Box in (3+1)D

It is now a straightforward matter to use (6.9) to repeat the arguments of Section 3.3 where I was able to obtain the density for the Boulware state for Schwarzschild space-time in 1+1 dimensions. First, consider the relative magnitudes of the two terms in (6.9). This is most easily done by expanding

$$\frac{d}{d\tau} = \frac{dz}{d\tau}\frac{d}{dz} = v\frac{d}{dz}, \tag{6.10}$$

where v is the proper 3-velocity with respect to a static observer. For quasi-static lowering (which I am assuming here), the velocity v must be taken to be small $v << 1$. But $\ddot{a} \sim v^2$, while $\dot{a} \sim v$. Thus, for quasi-static lowering, $\ddot{a}$ will be negligible compared to $\dot{a}$, in (6.9) and I can write

$$\langle \text{Flux} \rangle = \frac{1}{90\pi^2} a^2 \dot{\delta a}. \tag{6.11}$$

Next, I invoke the equivalence principle, and identify the acceleration a to be the acceleration of a static observer outside a black hole. As in Section 3.3, if I write the metric in the standard Schwarzschild coordinates

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{6.12}$$

the acceleration is given in terms of the Schwarzschild metric function $f(r)$ by

$$a = \frac{f'}{2\sqrt{f}}, \tag{6.13}$$

where the prime denotes a derivative with respect to the radial coordinate r.

Thus, lowering a mirror a small proper distance $dz = dr/\sqrt{f}$ changes its acceleration by the amount

$$\begin{aligned} da &= \frac{da}{dz}dz = \sqrt{f}\frac{da}{dr}dz \\ &= \frac{1}{2}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right) dz. \end{aligned} \tag{6.14}$$

Therefore, I can rewrite (6.11) as

$$\langle \mathrm{Flux} \rangle = \frac{1}{1440\pi^2} \frac{f'^2}{f} \left(f'' - \frac{1}{2} \frac{f'^2}{f} \right) \frac{dz}{d\tau}, \tag{6.15}$$

Now, if I consider a box with mirrored walls, then there will be a flux of the form (6.15) entering the box from the bottom wall and similarly a flux of the opposited sign from the top wall. As in Section 3.3, the positive energy flux will be blue shifted and the negative energy flux will be red shifted. Thus, the net rate of energy deposition into the box per unit mirror area, as measured at some point z_i within the box, is given by

$$\begin{aligned} \langle \frac{de}{d\tau} \rangle_{\mathrm{net}} &= -\frac{1}{1440\pi^2} \left\{ \left[\frac{f'^2}{f} \left(f'' - \frac{1}{2} \frac{f'^2}{f} \right) \sqrt{f} \right]_T \right. \\ &\qquad \left. - \left[\frac{f'^2}{f} \left(f'' - \frac{1}{2} \frac{f'^2}{f} \right) \sqrt{f} \right]_B \right\} \frac{1}{\sqrt{f_i}} \frac{dz}{d\tau} \end{aligned} \tag{6.16}$$

where e denotes the energy per unit area.

If the proper distance between the top and bottom walls is small enough, I can express the difference in the braces in (6.16) as a derivative multiplied by ℓ,

$$\langle \frac{de}{d\tau} \rangle_{\mathrm{net}} = -\frac{\ell}{1440\pi^2} \left[\frac{f'^2}{f} \left(f'' - \frac{1}{2} \frac{f'^2}{f} \right) \right] \frac{1}{\sqrt{f}} \frac{dz}{d\tau}. \tag{6.17}$$

To convert the rate of change of energy per unit area to the rate of change of energy density, $\langle \rho \rangle$, I need to divide by ℓ

$$\langle \frac{d\rho}{d\tau} \rangle = -\frac{1}{1440\pi^2} \left[\frac{f'^2}{f} \left(f'' - \frac{1}{2} \frac{f'^2}{f} \right) \right] \frac{1}{\sqrt{f}} \frac{dz}{d\tau}. \tag{6.18}$$

To calculate the total change in the energy density as the box is lowered, then, I need to integrate (6.18) over the history of the lowering process. The energy density inside the box when it has been lowered from asymptotic infinity to a proper distance z from the black hole horizon is

$$\langle\rho\rangle = -\frac{1}{1440\pi^2}\int_\infty^z dz \frac{1}{\sqrt{f}}\frac{d}{dz}\left[\frac{f'^2}{\sqrt{f}}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right)\right]. \tag{6.19}$$

By rewriting the derivative in the integrand

$$\begin{aligned}\frac{d}{dz}\left[\frac{f'^2}{\sqrt{f}}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right)\right] &= \sqrt{f}\frac{d}{dz}\left[\frac{f'^2}{f}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right)\right] \\ &+ \frac{d\sqrt{f}}{dz}\frac{f'^2}{f}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right).\end{aligned} \tag{6.20}$$

I can rewrite (6.19) in the form

$$\langle\rho\rangle = -\frac{1}{1440\pi^2}\left\{\left[\frac{f'^2}{f}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right)\right]_\infty^z + \frac{1}{2}\int_\infty^z dz\sqrt{f}\frac{f'^3}{f^2}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right)\right\} \tag{6.21}$$

Then, noting that

$$\frac{1}{4}\frac{d}{dr}\left[\frac{f'^4}{f^2}\right] = \frac{f'^3}{f^2}\left(f'' - \frac{1}{2}\frac{f'^2}{f}\right). \tag{6.22}$$

and that for the Schwarzschild metric

$$\lim_{r\to\infty} f'(r) = \lim_{r\to\infty} f''(r) = 0, \tag{6.23}$$

I find

$$\langle\rho\rangle = -\frac{1}{1440\pi^2}\frac{f'^2}{f}\left(f'' - \frac{3}{8}\frac{f'^2}{f}\right). \tag{6.24}$$

Thus, the energy density which accrues inside the box due to the quantum fluxes from the mirrors during the quasi-static lowering process is given by (6.24). But, as I argued in previous Chapters, and as I showed explicitly in 1+1 dimensions, this might be related to the energy density of the Boulware state for the black hole. In the next Section, I will discuss this result and its implications.

6.3 A Quantum Equivalence Principle?

If (6.24) truly represents the $\langle T_{tt}\rangle$ component of the Boulware stress-energy tensor for the Schwarzschild black hole, it is now straightforward to obtain the other components as well. The first step is noting that if the Boulware state is to be a vacuum state for static observers at all Schwarzschild times t then it must be invariant under time reversal. This in turn implies that T_{it}, where i is a spatial index, must vanish identically. Similarly, the spherical symmetry of the space-time implies that components having a single θ or ϕ index must also vanish.

Thus, I am lead to the conclusion that the Boulware state stress-energy tensor must be diagonal. If (6.24) represents one of the diagonal components, I am left with three to find. I can accomplish this by using the conformal anomaly equation (2.33) which provides one independent equation and the covariant conservation equations (2.31) which provide two more. Thus, by solving two differential and one algebraic equations, I would have the entire Boulware state SET.

However, there is good reason to believe that (6.24) is *not* the energy density of the Boulware SET. The evidence is provided by the same equivalence principle I have been invoking toward the opposite end. The argument goes as follows: if a quantum equivalence principle holds then no local measurement can distinguish between the interior of a box being lowered quasi-statically toward a black hole and one being moved through the exact same range of accelerations in exactly the same way in Minkowski space-time. Since I've argued that the lowering process should preserve the vacuum state for accelerated observers, the Boulware state, inside the box lowered toward a black hole, it follows that the equivalent quasi-static process in Minkowski space-time should preserved the vacuum state for uniformly accelerated observers in that space-time as well, which is the Rindler state. Therefore, although (6.24) was developed derived specifically for Schwarzschild space-time, through the equivalence principle I should be able to recover results for the Rindler state from it.

In (1+1)D, this equivalence is manifest for at least the energy densities of these SETs in (1+1)D. Recall that the box lowering process in Chapter 3 gave me the Boulware energy density (2.88),

$$\langle B|\rho|B\rangle^{(1+1)} = \frac{1}{24\pi}\left(f'' - \frac{f'^2}{4f}\right). \tag{6.25}$$

while the Rindler energy density in (1+1)D was given by (2.78),

$$\langle R|\rho|R\rangle^{(1+1)} = -\frac{a^2}{24\pi}. \tag{6.26}$$

where $|B\rangle$ and $|R\rangle$ are the Boulware and Rindler states respectively.

Now, if I write the Rindler metric (2.7) in the Schwarzschild form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}. \tag{6.27}$$

I find that f is linear in r for the Rindler metric, so that $f'' = 0$ for it. Also, for every metric of the form (6.27) the acceleration is given by $a = f'/(2\sqrt{f})$. Substituting from these expressions for f' and f'' into (6.25) I see that it does indeed simplify to (6.26) for Minkowski space-time, and all is well.

In (3+1)D, however, this argument runs into trouble. In this case, I have calculated for the energy density of a box being quasi-statically lowered toward a black hole the energy density (6.24),

$$\langle \mathrm{Box}|\rho|\mathrm{Box}\rangle^{(3+1)} = -\frac{1}{1440\pi^2}\frac{f'^2}{f}\left(f'' - \frac{3}{8}\frac{f'^2}{f}\right). \tag{6.28}$$

where $|\text{Box}\rangle$ indicates the state inside the box being lowered toward the black hole. However, the Rindler energy density in (3+1)D was given by (2.77),

$$\langle R|\rho|R\rangle^{(3+1)} = -\frac{a^4}{480\pi^2}. \tag{6.29}$$

This time, by substituting $f'/(2\sqrt{f}) = a$ and $f'' = 0$ into (6.28), I obtain *twice* the *negative* of (6.29), i.e.

$$\langle \text{Box}|\rho|\text{Box}\rangle^{(3+1)} = \frac{a^4}{240\pi^2} = -2\langle R|\rho|R\rangle^{(3+1)}. \tag{6.30}$$

Thus, I am left with two possible conclusions. The first is that there is not a quantum equivalence principle as I argued there should be. In that case there would be no reason to expect that the energy density inside the box should be the Rindler energy density when I transform from the black hole scenario to the flat space-time scenario. On the other hand, in this case there would be no reason to expect that I could get information about the Boulware state from considering a box accelerating in Minkowski space-time in the first place. The concurrence of the box calculation with the Boulware energy density in (1+1)D would then have to be seen as a misleading coincidence, and one would conclude that trying to repeat the calculation in (3+1)D had been an exercise doomed from the beginning.

The second possibility is that the equivalence principle holds but that I have not applied it properly in (3+1)D. If this is true, one must ask in what sense the application has differed in (3+1)D from the successful application in (1+1)D. I will examine some possibilities in Chapter 7.

Chapter 7

Conclusions

Let me begin this Chapter with a summary of the original work presented in previous Chapters. In Chapter 3, I showed that in (1+1)D the energy density inside a box with reflecting walls which is initially empty and an infinite distance from a black hole and then is lowered quasi-statically toward the black hole was the energy density of the Boulware state. This, I reasoned, was natural because the static state which is empty (ignoring Casamir contributions, as I have throughout this thesis) of particles at $r = \infty$ and which will be unaffected by quasi-static processes is the Boulware state. Since the quasi-static lowering process preserves the Boulware state inside the box, the energy density at any finite proper distance from the horizon of the black hole should be the Boulware energy density.

The second interesting feature of the calculation in Chapter 3 was that I was able to deduce the Boulware energy density inside the box by considering only quantum effects from a single moving mirror in Minkowski space-time. This implied that there was a quantum equivalence principle at work which allowed me to make a connection between quantum states for uniformly accelerating observers regardless of the background metric (gravitational field).

To repeat this calculation in (3+1)D I needed to know the quantum SET for a moving mirror in (3+1)D. This SET was not previously known, but I have presented a set of calculational tools for obtaining it in the limit of small deviations from uniform acceleration in Chapters 4 and 5 and Appendix A. However, I have calculated only the one component (the $u-u$ component) explicitly, since that was all that was needed in (1+1)D. Using this result for the moving mirror, I repeated the accelerating box arguments in Chapter 6. However, I found there that my expression for what I expected to be the Boulware energy density was suspect, since, unlike the (1+1)D result, it did not reproduce to the Rindler energy density in the appropriate limit. I have also outlined how one would go about finding an analytic form for the entire Boulware SET given the correct energy density.

The question to be answered at this point is "where does one go from here?" One might conclude that there is no quantum equivalence principle and simply abandon this line of inquiry. Indeed, as mentioned earlier in this thesis, the existence of a quantum equivalence principle is debatable.

While I have outlined reasons in Chapter 3 why I think there should be such a principle for interior states of quasi-static boxes in static space-times, I have no direct evidence for one in (3+1)D at this point. Indeed, there is at least one significant reason to suspect that if there is some sort of equivalence, it is not as straightforward as claiming that the SET should be the same for all such boxes. That reason is the existence of the conformal anomaly (2.33). Clearly, the SETs cannot be identical for boxes in Minkowski and Schwarzschild space-times, since the conformal anomaly vanishes in the former but not the latter, which means that the traces of the SETs are not the same!

Nonetheless, Chapter 3 does seem to imply an equivalence principle for such boxes in (1+1)D. Furthermore, this equivalence involves only an identification of the *classical accelerations* of uniformly accelerating observers in different static space-times rather than the actual quantum states. Also, the result in Chapter 6 for (3+1)D fails to satisfy the same equivalence by only an overall factor of 2. It seems, therefore, reasonable to suspect that there is simply some error in the specific application of this principle in (3+1)D.

The most notable difference between the (1+1)D and (3+1)D results from this perspective is that, while in (1+1)D $\langle T_{vv} \rangle$ vanishes identically, and therefore cannot contribute to the flux, this is not the case in (3+1)D. Instead, in Chapter 5 I used the heuristic argument that the non-vanishing $\langle T_{vv} \rangle$ was likely a boundary effect rather than a flux due to the *motion* of the mirror. Since I believe that it is the motion (acceleration) of the boundaries that is responsible for the preservation of the Boulware state inside the box, rather than the specific nature of the boundary itself, I ignored this term in calculating what I thought to be the Boulware energy density. However, upon examining the results of Candelas and Deutsch [27] for a uniformly accelerating plane mirror in (3+1)D, one finds that the far field SET is simply the Rindler SET. Thus, there seems to be some real sense in which the entire SET (including the $v-v$ component) are related to the motion of the mirror.

The only way to determine with certainty what parts of the SET are related to motion and which parts are boundary terms independent of the motion is to calculate the remaining components of $\langle T_{\mu\nu} \rangle$ for a non-uniformly accelerating mirror in flat space-time. This work is currently in progress[26]. There are other issues to be resolved as well. One pertains to the fact that there is a difference between the superimposed quantum states of two single mirrors and the quantum state for a pair of mirrors. While it seems that the superimposed single mirrors capture the dominant contribution from the non-uniform acceleration, at least in (1+1)D, it is clear that the quantum state for a pair of mirrors will also contain Casimir[34] like contributions. Investigation is currently underway to understand how these contributions separate in (1+1)D with an eye toward determining whether one can also expect the superimposed single mirrors to give the dominant motion contribution in (3+1)D[26].

Unfortunately, the calculation of these components is likely to prove quite time consuming, but results are expected to be in hand within a year of the time of writing of this thesis. It remains to be seen whether they will provide the necessary clues to unravel this fascinating problem. One thing is certain, however. There remains much work to be done in resolving the status of this one small piece of the quantum equivalence puzzle.

Appendix A

The Method of Means and Differences

With the identities (5.7) and (5.10), I will be able to express the required derivatives in the appropriate power series. But first, I will need to get the required derivatives. This is best done in small steps. Recall that I am trying to differentiate

$$\begin{aligned} \delta G^{+}(x', x'') &= -\frac{(u'v + \alpha^2)(u''v + \alpha^2)}{\pi^2 \alpha (u'' - u')} \times \\ &\quad \left[\frac{(u' - v)(u'' - v)}{(u'' - u')} \int_{u'}^{u''} A(u) du \right. \\ &\quad \left. -(u'' - v)^2 A(u'') - (u' - v)^2 A(u') \right], \end{aligned} \tag{A.1}$$

where $A(u)$ is a function whose specific form will not matter for the discussion in this Appendix.

Let me therefore define

$$E(x', x'') := (v - x')(v - x'')B(x', x'') - \overline{(v - x)^2 A(x)}, \tag{A.2}$$

and

$$H(x', x'') := \frac{E(x', x'')}{(\Delta t)^2}, \tag{A.3}$$

and finally

$$W(x', x'') := (x' + b)(x'' + b)H(x', x''). \tag{A.4}$$

Clearly, $\delta G^{+}(u', u'')$ from (A.1) has the same functional form as $W(x', x'')$. Thus, I need to find $\partial_{x'}\partial_{x''}W(x', x'')$ and $\overline{(\partial_x)^2 W}$ on order to find the two required SET components.

I begin by calculating the first derivatives of $E(x', x'')$.

$$\frac{\partial E}{\partial x'} = (v-x'')^2\frac{B(x', x'')}{\Delta x}+(v-x')(2x''-v-x')\frac{A(x')}{\Delta x}-\frac{1}{2}(v-x')^2A'(x'), \tag{A.5}$$

and

$$\frac{\partial E}{\partial x''} = -(v-x')^2\frac{B(x', x'')}{\Delta x}-(v-x')(2x'-v-x'')\frac{A(x'')}{\Delta x}-\frac{1}{2}(v-x'')^2A'(x''). \tag{A.6}$$

Using either of these and the definition of $B(x', x'')$ I can find the mixed partial derivative

$$\frac{\partial^2 E}{\partial x'\partial x''} = -2\frac{(v-x')(v-x'')}{(\Delta x)^2}C(x', x'') - \frac{\Delta[(v-x)A(x)]}{\Delta t} \tag{A.7}$$

and the other two second derivatives

$$\begin{aligned}
\frac{\partial^2 E}{\partial x'^2} &= 2(v-x'')^2\frac{B}{(\Delta x)^2} + \left[-2\frac{(v-x')^2}{(\Delta x)^2} + 4\frac{(v-x')}{\Delta x} - 3\right]A(x') \\
&\quad + \left[-\frac{(v-x')^2}{\Delta x} + 3(v-x')\right]A'(x') - \frac{1}{2}(v-x')^2A''(x') \qquad (A.8)\\
\frac{\partial^2 E}{\partial x''^2} &= 2(v-x')^2\frac{B}{(\Delta x)^2} + \left[-2\frac{(v-x'')^2}{(\Delta x)^2} + 4\frac{(v-x'')}{\Delta x} - 3\right]A(x'') \\
&\quad + \left[-\frac{(v-x'')^2}{\Delta x} + 3(v-x'')\right]A'(x'') - \frac{1}{2}(v-x'')^2A''(x'')
\end{aligned} \tag{A.9}$$

Furthermore, I can find the first derivatives of $H(x', x'')$ in terms of $E(x', x'')$

$$\frac{\partial H}{\partial x'} = 2\frac{E}{(\Delta x)^3} + \frac{1}{(\Delta x)^2}\frac{\partial E}{\partial x'}, \tag{A.10}$$

$$\frac{\partial H}{\partial x''} = -2\frac{E}{(\Delta x)^3} + \frac{1}{(\Delta x)^2}\frac{\partial E}{\partial x''}, \tag{A.11}$$

and the second derivatives

$$\frac{\partial^2 H}{\partial x' \partial x''} = -6\frac{E}{(\Delta x)^4} + \frac{2}{(\Delta x)^3}\Delta\left(\frac{\partial E}{\partial x}\right) + \frac{1}{(\Delta x)^2}\frac{\partial^2 E}{\partial x' \partial x''}, \tag{A.12}$$

$$\frac{\partial^2 H}{\partial x'^2} = 6\frac{E}{(\Delta x)^4} + \frac{4}{(\Delta x)^3}\frac{\partial E}{\partial x'} + \frac{1}{(\Delta x)^2}\frac{\partial^2 E}{\partial x'^2}, \tag{A.13}$$

$$\frac{\partial^2 H}{\partial x''^2} = 6\frac{E}{(\Delta x)^4} + \frac{4}{(\Delta x)^3}\frac{\partial E}{\partial x''} + \frac{1}{(\Delta x)^2}\frac{\partial^2 E}{\partial x''^2}. \tag{A.14}$$

Using Eq.s (A.10-A.14), I find for the second derivatives of $W(x', x'')$ in terms of $E(x', x'')$ and its derivatives,

$$\begin{aligned}\frac{\partial^2 W}{\partial x' \partial x''} &= -\frac{E}{(\Delta x)^2} - 6(x''+b)(x'+b)\frac{E}{(\Delta x)^4} + \frac{2}{(\Delta x)^3}\Delta((x+b)^2 E') \\ &\quad -\frac{2}{(\Delta x)^2}\overline{(x+b)E'} + \frac{(x''+b)(x'+b)}{(\Delta x)^2}\frac{\partial^2 E}{\partial x' \partial x''}\end{aligned} \tag{A.15}$$

$$\begin{aligned}\frac{\partial^2 W}{\partial x'^2} &= (x'+b+\Delta x)\left[\frac{2}{(\Delta x)^3}\left(2+3\frac{x'+b}{\Delta x}\right)E\right. \\ &\quad \left.+\frac{2}{(\Delta t)^2}\left(1+2\frac{2(x'+b)}{\Delta x}\right)\frac{\partial E}{\partial x'} + \frac{(x'+b)}{(\Delta x)^2}\frac{\partial^2 E}{\partial x'^2}\right]\end{aligned} \tag{A.16}$$

$$\begin{aligned}\frac{\partial^2 W}{\partial x''^2} &= (x''+b-\Delta x)\left[\frac{2}{(\Delta x)^3}\left(-2+3\frac{x''+b}{\Delta x}\right)E\right. \\ &\quad \left.+\frac{2}{(\Delta x)^2}\left(1-2\frac{2(x''+b)}{\Delta x}\right)\frac{\partial E}{\partial x''} + \frac{(x''+b)}{(\Delta x)^2}\frac{\partial^2 E}{\partial x''^2}\right]\end{aligned} \tag{A.17}$$

Adding Eq.s (A.16) and (A.17) and dividing by two I get the averaged second derivative

$$\overline{\frac{\partial^2 W}{(\partial x)^2}} = \frac{(x'+b)(x''+b)}{(\Delta t)^2}\overline{\frac{\partial^2 E}{\partial x^2}} - \frac{2}{(\Delta x)^3}\Delta\left((x+b)^2\frac{\partial E}{\partial x}\right) - \frac{1}{\Delta x}\Delta\left(\frac{\partial E}{\partial x}\right)$$

$$+\frac{6}{(\Delta x)^2}\overline{\left((x+b)\frac{\partial E}{\partial x}\right)} - \frac{6}{(\Delta x)^4}\overline{(x+b)^2}E - \frac{E}{(\Delta x)^2} \qquad (A.18)$$

Eq.s (A.15) and (A.18) provide the expressions I need to evaluate the desired SET components in terms of $E(x', x'')$, but they must still be converted to expressions involving ΔA, $\overline{A}$, and $C(x', x'')$ to make the coincidence limit feasible. It will be easier to convert, however, if there is a degree of uniformity in the terms involving derivatives of E. Ideally, I would like these equations to involve only $\partial^2 E/\partial x'\partial x''$, $\overline{\partial^2 E/\partial x^2}$, $\Delta(\partial E/\partial x)$, $\overline{\partial E/\partial x}$, and E itself. This can be arranged by the application of the following identities to (A.15) and (A.18)

$$x' + b = \overline{(x+b)} - \frac{1}{2}\Delta x, \qquad (A.19)$$

$$x'' + b = \overline{(x+b)} + \frac{1}{2}\Delta x, \qquad (A.20)$$

$$\Delta\left((x+b)\frac{\partial E}{\partial x}\right) = \overline{(x+b)}\,\Delta\left(\frac{\partial E}{\partial x}\right) + \Delta x\,\overline{\frac{\partial E}{\partial x}}, \qquad (A.21)$$

$$\overline{(x+b)\frac{\partial E}{\partial x}} = \overline{(x+b)}\,\overline{\frac{\partial E}{\partial x}} + \frac{1}{4}\Delta x\,\Delta\left(\frac{\partial E}{\partial x}\right). \qquad (A.22)$$

$$\Delta\left((x+b)^2\frac{\partial E}{\partial x}\right) = \left[(x''+b)(x'+b) + \frac{(\Delta x)^2}{2}\right]\Delta\left(\frac{\partial E}{\partial x}\right) + 2\,\Delta x\,\overline{(x+b)}\,\overline{\frac{\partial E}{\partial x}}. \qquad (A.23)$$

Upon substituting (A.19-A.23) into (A.15) and (A.18) I obtain expressions for $\partial^2 W/\partial x'\partial x''$ and $\overline{\partial^2 W/\partial x^2}$ that depend only on E, $\Delta(\partial E/\partial x)$, $\overline{\partial E/\partial x}$, $\partial^2 E/\partial x'\partial x''$, and $\overline{\partial^2 E/\partial x^2}$. I will not write these expressions explicitly here, but rather I will now concentrate of the forms of these derivatives of $E(x', x'')$. I have already written down the explicit forms of $E(x', x'')$ and $\partial^2 E/\partial x' \partial x''$ explicitly in (A.2) and (A.7) respectively. It is a straightforward matter to calculate

$$\begin{aligned}
\Delta\left(\frac{\partial E}{\partial x}\right) &= -2\overline{(v-x)^2}\frac{C}{\Delta x} + \Delta[(v-x)A] \\
&\quad -\Delta x\overline{A} - \frac{1}{2}\Delta[(v-x)^2 A'], && \text{(A.24)} \\
\overline{\frac{\partial E}{\partial x}} &= -\overline{(v-x)}C + \frac{1}{2\Delta x}\Delta[(v-x)^2 A] + \overline{(v-x)A} \\
&\quad - \frac{\Delta x}{4}\Delta A - \frac{1}{2}\overline{(v-x)^2 A'}, && \text{(A.25)}
\end{aligned}$$

from (A.5) and (A.6), and

$$\begin{aligned}
2\overline{\frac{\partial^2 E}{\partial x^2}} &= 4\overline{(v-x)^2}\frac{C}{(\Delta x)^2} - \frac{2}{\Delta x}\Delta[(v-x)A] - 4\bar{A} \\
+ &\quad \frac{1}{\Delta x}\Delta[(v-x)^2 A'] + 6\overline{(v-x)A'} - \overline{(v-x)^2 A''}, && \text{(A.26)}
\end{aligned}$$

from (A.8) and (A.9).

The next step is to rewrite the expressions above so that they contain only $\overline{A^{(n)}}$ and $\Delta A^{(n)}$. This is easily done by using the following identities

$$\begin{aligned}
\overline{(v-x)A} &= \overline{(v-x)}\bar{A} - \frac{1}{4}\Delta x\,\Delta A, && \text{(A.27)} \\
\overline{(v-x)^2 A} &= \overline{(v-t)^2}\bar{A} - \frac{1}{2}\overline{(v-x)}\,\Delta x\,\Delta A, && \text{(A.28)} \\
\Delta[(v-x)A] &= \overline{(v-x)}\,\Delta A - \Delta x\,\bar{A}, && \text{(A.29)} \\
\Delta[(v-x)^2 A] &= \overline{(v-x)^2}\Delta A - 2\Delta x\,\overline{(v-x)}\,\bar{A}. && \text{(A.30)}
\end{aligned}$$

Using (A.27 - A.30) I can rewrite the means and differences of the derivatives of $E(x', x'')$ as follows:

$$E = (v-x')(v-x'')C + \frac{1}{2}\overline{(v-x)}\,\Delta A - \frac{1}{2}(\Delta x)^2\,\bar{A}, \qquad \text{(A.31)}$$

$$\Delta\left(\frac{\partial E}{\partial x}\right) = -\frac{2}{\Delta x}\overline{(v-x)^2}\,C + \overline{(v-x)}\,\Delta A - 2\Delta x\,\bar{A} - \frac{1}{2}\overline{(v-x)^2}\,\Delta(A') + \Delta x\,\overline{(v-x)}\,\overline{A'}. \tag{A.32}$$

$$\overline{\frac{\partial E}{\partial x}} = -\overline{(v-x)}\,C + \frac{1}{2}\frac{1}{\Delta x}\overline{(v-x)^2}\,\Delta A - \frac{1}{2}\Delta x\,\Delta A - \frac{1}{2}\overline{(v-x)^2}\,\overline{A'} + \frac{1}{4}\overline{(v-x)}\,\Delta x\,\Delta(A'), \tag{A.33}$$

$$\frac{\partial^2 E}{\partial x'\partial x''} = -\frac{2}{\Delta x}(v-x')(v-x'')\,C - \frac{1}{\Delta x}\,\Delta A + \bar{A} \tag{A.34}$$

$$\overline{\frac{\partial^2 E}{\partial x^2}} = \frac{2}{(\Delta x)^2}\overline{(v-x)^2}\,C - \frac{1}{\Delta x}\overline{(v-x)}\,\Delta A - \bar{A} + \frac{1}{2}\frac{1}{\Delta x}\overline{(v-x)^2}\,\Delta(A') + 2\overline{(v-x)}\,\overline{A'} - \frac{3}{4}\,\Delta x\,\Delta(A') - \frac{1}{2}\overline{(v-x)^2}\overline{A''} + \frac{1}{4}\overline{(v-x)}\,\Delta x\,\Delta(A'') \tag{A.35}$$

Eq.s (A.31 - A.35) are in forms which are conducive to expanding in power series of the form (5.7). Using (5.7) and (5.10) I get the following series expressions for them

$$E = \left(-\frac{1}{12}\overline{(v-x)^2}\,\overline{A''} + \frac{1}{2}\overline{(v-x)}\,\overline{A'} - \frac{1}{2}\bar{A}\right)(\Delta x)^2 + \left(\frac{1}{120}\overline{(v-x)^2}\,\overline{A^{(4)}} - \frac{1}{24}\overline{(v-x)}\,\overline{A^{(3)}} - \frac{1}{12}\overline{A''}\right)(\Delta x)^4 + O((\Delta x)^6). \tag{A.36}$$

$$\Delta\left(\frac{\partial E}{\partial x}\right) = \left(-\frac{1}{3}\overline{(v-x)^2}\,\overline{A''} + 2\overline{(v-x)}\,\overline{A'} - 2\bar{A}\right)\Delta x + \left(\frac{1}{40}\overline{(v-x)^2}\,\overline{A^{(4)}} - \frac{1}{12}\overline{(v-x)}\,\overline{A^{(3)}}\right)(\Delta x)^3 + O((\Delta x)^5). \tag{A.37}$$

$$\overline{\frac{\partial E}{\partial x}} = \left(-\frac{1}{24}\overline{(v-x)^2}\,\overline{A^{(3)}} + \frac{1}{3}\overline{(v-x)^2}\,\overline{A''} - \frac{1}{2}\overline{A'}\right)(\Delta x)^2 + \left(\frac{1}{240}\overline{(v-x)^2}\,\overline{A^{(5)}} - \frac{7}{240}\overline{(v-x)}\,\overline{A^{(4)}} + \frac{1}{24}\overline{A^{(3)}}\right)(\Delta x)^4 \tag{A.38}$$

$$+O((\Delta x)^6).$$

$$\frac{\partial^2 E}{\partial x' \partial x''} = \left(\frac{1}{6}\overline{(v-x)^2}\,\overline{A''} - \overline{(v-x)}\,\overline{A'} + \bar{A}\right) + \left(-\frac{1}{60}\overline{(v-x)^2}\,\overline{A^{(4)}} + \frac{1}{12}\overline{(v-x)}\,\overline{A^{(3)}} - \frac{1}{12}\overline{A''}\right)(\Delta x)^2 + O((\Delta x)^4), \tag{A.39}$$

$$\overline{\frac{\partial^2 E}{\partial x^2}} = \left(\frac{1}{6}\overline{(v-x)^2}\,\overline{A''} + \overline{(v-x)}\,\overline{A'} - \bar{A}\right) + \left(-\frac{1}{40}\overline{(v-x)^2}\,\overline{A^{(4)}} + \frac{1}{3}\overline{(v-x)}\,\overline{A^{(3)}} - \frac{3}{4}\overline{A''}\right)(\Delta x)^2 + O((\Delta x)^4). \tag{A.40}$$

Using Eq.s (A.36-A.40) and (A.19-A.23) I can write $\partial^2 W/\partial x'\partial x''$ from (A.15) and $\overline{\partial}^2 W/\partial x^2$ from (A.18) as power series

$$\frac{\partial W}{\partial x' \partial x''} = -\frac{1}{60}\overline{(x+b)^2}\,\overline{(v-x)^2}\,\overline{A^{(4)}} - \frac{1}{12}\left[\overline{(x+b)}\,\overline{(v-x)^2} - 2\overline{(x+b)^2}\,\overline{(v-x)}\right]\overline{A^{(3)}} - \frac{1}{12}\left[4\overline{(x+b)^2} - 8\overline{(x+b)}\,\overline{(v-x)} + \overline{(v-x)^2}\right]\overline{A''} + \frac{1}{2}\left[\overline{(v-x)} - 2\overline{(x+b)}\right]\overline{A'} - \frac{1}{2}\bar{A}, \tag{A.41}$$

$$\overline{\frac{\partial^2 W}{\partial x^2}} = -\frac{1}{40}\overline{(x+b)^2}\,\overline{(v-x)^2}\,\overline{A^{(4)}} - \frac{1}{12}\left[\overline{(x+b)}\,\overline{(v-x)^2} - 3\overline{(x+b)^2}\,\overline{(v-x)}\right]\overline{A^{(3)}} - \frac{1}{6}\left[3\overline{(x+b)^2} - 4\overline{(x+b)}\,\overline{(v-x)}\right]\overline{A''} - \overline{(x+b)}\,\overline{A'} \tag{A.42}$$

These are the required derivatives. I will now be able to use these identities to find the $u-u$ components of both the canonical and conformal stress-energy tensors for a scalar field outside a spherical mirror expanding with nearly uniform acceleration.

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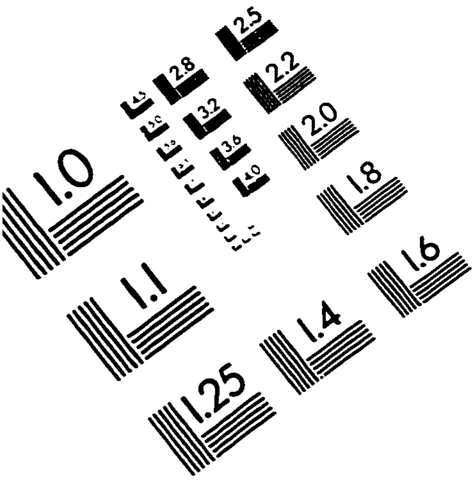
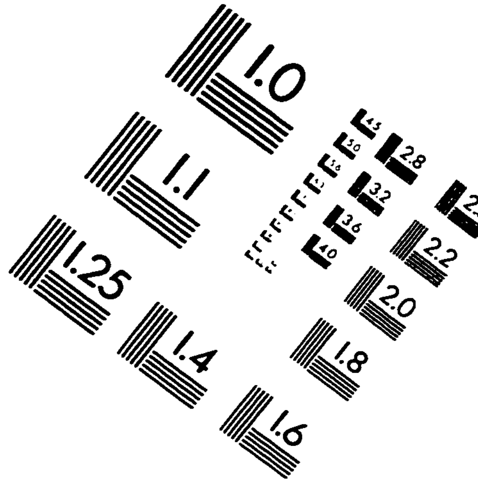
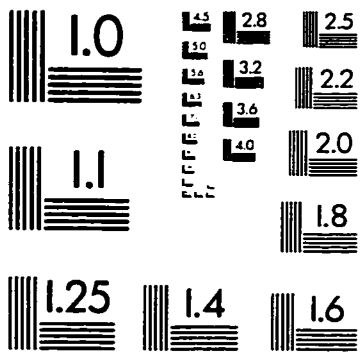
IMAGE EVALUATION TEST TARGET (QA-3)







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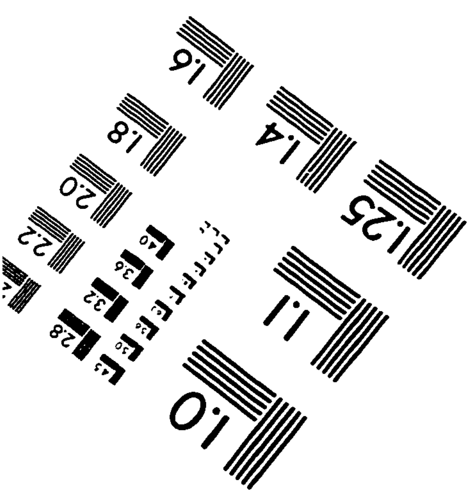

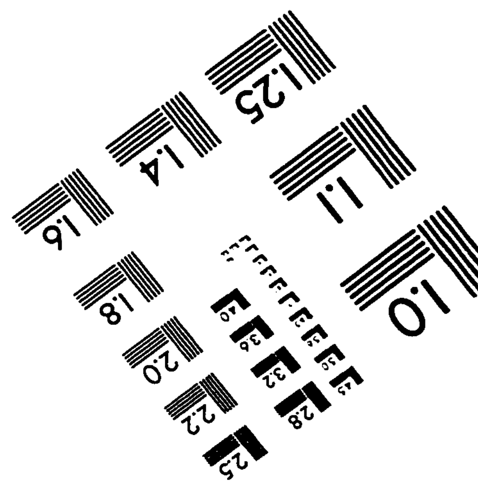

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