University of Alberta

Attitude Control of a Differentially Flat Underactuated Rigid Spacecraft

by

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Abstract

We consider the attitude control problem of a two-input underactuated rigid spacecraft. We initially assume a certain inertia property to hold and design flatness-based open-loop controls that steer the spacecraft from equilibrium-to-equilibrium in finite time. Using the Dynamic Extension Algorithm, a dynamic feedback is designed that input-output linearizes the system. A singularity in the dynamic feedback restricts its domain away from the equilibrium set. Nonetheless, it is proved that the dynamic controller tracks a flat output trajectory and brings the spacecraft to a neighbourhood of a desired equilibrium in finite time. The control law is then switched to an existing stabilizing controller. A quasi-static feedback is designed as an alternative to the dynamic feedback. Removing the initial inertia constraint, we show that the system is orbitally flat and use this property to design open- and closed-loop controls which require a non-zero angular velocity about the uncontrolled axis.

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Symbol	Description of symbol
$\|\cdot\|$	: standard Euclidean norm
$[f, g]$	: Lie bracket of vector fields f and g
$A \backslash B$	: complement of B in A
$\mathrm{int}(S)$	: interior of a set S
$\mathrm{ad}_f^k g$	: iterative Lie bracket $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$
$\mathrm{Br}(\mathcal{C})$	: set of brackets in the algebra $\mathcal{C}$
$\mathbb{C}$	: set of complex numbers
$\overline{\mathbb{C}}_+$	: set of complex numbers with non-negative real part
$\mathbb{C}_+$	: set of complex numbers with positive real part
$\mathbb{C}_-$	: set of complex numbers with negative real part
$\mathrm{cl}(S)$	: closure of a set S
$\mathrm{Lie}(D)$	: involutive closure of the smooth distribution D
$L_f h$	: Lie derivative of a function h along the vector field f
$L_f^k h$	: repeated Lie derivative $L_f^k h = L_f(L_f^{k-1} h)$
$\mathbb{R}$	: set of real numbers
$\mathbb{R}^n$	: n-dimensional Euclidean space
$(\mathbb{R}^n)^k$	: Cartesian product of k copies of $\mathbb{R}^n$
$\mathbb{R}_+$	: set of real numbers with positive real part
$\mathbb{R}_-$	: set of real numbers with negative real part
$\mathbb{S}^n$	: unit sphere in $\mathbb{R}^{n+1}$
S_m	: group of permutations on m elements
$\mathcal{R}^V(x_0, T)$	: set of reachable points from x_0 in time T
$\mathcal{R}^V(x_0, \leq T)$	: set of reachable points from x_0 in time at most T
$\mathrm{spec}(A)$	: the spectrum of the linear map A
$\mathrm{tr}(A)$	: trace of a linear map A
$\mathfrak{X}(\mathsf{M})$	: vector space of smooth vector fields on M
$\mathbb{Z}$	: set of integers

Chapter 1

Introduction

1.1 Problem Description and Background

In this thesis we consider the attitude control problem of an underactuated rigid spacecraft. The attitude of a spacecraft is its orientation in space, and hence, the attitude control problem involves directing the spacecraft to achieve a specified orientation. Spacecraft attitude control has many applications. For example, it is used to avoid solar or atmospheric damage to sensitive components, to point directional antennas and solar panels, and to orient rockets used for orbit manoeuvres. In real space applications, the attitude of a spacecraft is commonly controlled with jet thrusters and/or reaction wheels. Theoretically, the number of independent inputs required to completely control the attitude of a spacecraft is three, corresponding to the three degrees of freedom required to specify the orientation of the spacecraft. However, for real spacecraft, the number of control inputs exceeds this number in case of actuator failure and to have as much control authority as possible. From a practical and theoretical point of view, it is of interest to consider the attitude control problem when some of the actuators have failed, that is, when the spacecraft is underactuated. The case of one control input has received little attention because of reduced practical significance. Here, we are interested in the two-input case. Although this case has been considered by many researchers, it still remains a difficult problem to which no general solution has been obtained.

The attitude control problem of a two-input spacecraft has a longstanding history, beginning with the work of Meyer (1971) in which modeling and control issues were first formulated. Controllability of the fully actuated and underactuated spacecraft were first studied by Crouch (1984). Specifically, Crouch proved that the underactuated spacecraft is controllable and locally controllable about any equilibrium[1], provided it is asymmetric. Crouch obtained his controllability results using geometric control theory. Other works on the controllability of the underactuated spacecraft are those of Krishnan et al. (1992) and Kerai (1995). In both works, the authors showed that the two-input spacecraft is locally controllable about any equilibrium. Their results were obtained by applying Sussmann's well-known local controllability result (Sussmann, 1987).

Concerning stabilizability, the first asymptotically stabilizing controller for the two-input spacecraft was constructed by Crouch (1984) and involved Lie algebraic techniques as proposed by Hermes (1980). The stabilizing control law proposed by Crouch is a complicated algorithm yielding a piecewise constant control. An important discovery concerning the existence of stabilizing controllers for the underactuated spacecraft was first reported by Byrnes and Isidori (1991). Byrnes and Isidori proved that the two-input spacecraft cannot be asymptotically stabilized about an equilibrium using a continuously differentiable (C^1) time-invariant static or dynamic state feedback. Their result was obtained by applying directly Brockett's necessary condition for C^1-stabilization (Brockett, 1983). This implies that all techniques yielding smooth control laws such as, for instance, feedback linearization and backstepping, cannot be used to stabilize the two-input spacecraft. Later, from the work of Zabczyk (1989) in which Brockett's condition was generalized, the result of Byrnes and Isidori was further strengthened: the two-input rigid spacecraft cannot be asymptotically stabilized to an equilibrium using a continuous time-invariant control. This outcome is significant because it gives an example of a real physical system that is locally controllable about an equilibrium but cannot be asymptotically stabilized using a continuous state feedback (compare this result with the linear case in which controllability implies smooth stabilizability). As a consequence, any control asymptotically stabilizing the

[1]In general, controllability does not imply local controllability.

underactuated spacecraft about a desired orientation must be either discontinuous or time-varying. In the case of time-varying feedbacks, it was proved in the major work of Coron (1995) that a locally controllable system with state dimension greater than or equal to four can be asymptotically stabilized by a continuous time-varying feedback. It remained, however, to construct such feedbacks for the underactuated spacecraft.

The first attempt to use time-varying feedback to stabilize the two-input spacecraft was reported by Morin (1992). However, no stability proof is given and only simulations are provided. Next came the works of Walsh et al. (1994) and Morin et al. (1995) in which smooth time-varying feedbacks were constructed yielding asymptotic stability. However, due to the smoothness of the control laws, the rate of convergence to the origin is slow. To improve the rate of convergence to the origin, Coron and Kerai (1996) and Morin and Samson (1997) derived continuous time-varying feedbacks yielding exponential stability. The controller of Coron and Kerai (1996) involves switching periodically between two control laws, while the controller of Morin and Samson (1997) involves one simple expression.

The literature on the more general trajectory tracking problem for an underactuated spacecraft is scarce compared to that of the stabilizability problem. In particular, there has been no work reported on the asymptotic trajectory tracking problem for the full dynamics of an underactuated spacecraft. Some special cases, however, have been considered in which it is assumed that the spacecraft has an axis of symmetry. Tsiotras and Luo (1997, 2000) derived asymptotic tracking control laws for an axis-symmetric spacecraft assuming that the angular velocity about the axis of symmetry is null. Tsiotras (1999) proposed a method that approximates the original spacecraft system with a differentially flat system so that feasible trajectories can be generated and then used in the trajectory tracking problem. However, the asymptotic trajectory tracking problem for the general asymmetric case remains unsolved.

1.2 Thesis Contribution

In this thesis we consider the attitude control problem of a two-input spacecraft actuated by jet thrusters. The main contributions of the thesis are the following:

- Provided a certain principal moments of inertia condition is satisfied, we show that the two-input spacecraft is flat. We then design an open-loop control steering the spacecraft from one equilibrium to another. A closed-loop trajectory tracking dynamic feedback controller is designed using the Dynamic Extension Algorithm (DEA). The dynamic feedback steers the spacecraft to an arbitrary small neighbourhood of a desired equilibrium in finite time. The dynamic controller is composed with the control law of Morin and Samson (1997) that yields a desired equilibrium asymptotically stable. The two-phase controller yields an asymptotic trajectory tracking controller.

- We construct a quasi-static feedback control as an alternative control law to the dynamic controller.

- A time-scaling function is constructed such that the system is orbitally flat for general values of the principal moments of inertia (excluding the symmetric case). Open-loop controls based on an orbital flatness design are constructed and simulated. Closed-loop controls are also designed using the DEA on the time-scaled system.

The thesis work is motivated by the result of Rouchon (1992) which showed that the underactuated spacecraft satisfying the special inertia property is differentially flat. Although Rouchon's result has been known for some time now, there has not been a lot of work reported in the literature concerning this special case. We should, however, mention the technical report by Adam (2004) in which similar results to the open-loop design reported here were obtained.

1.3 Thesis Overview

The thesis is organized as follows:

Chapter 2: In this chapter we present the necessary background from nonlinear control theory. In particular, we review some well-known results concerning stabilizability and controllability, followed by feedback linearization. Next, differentially flat systems and the available techniques for controlling such systems are introduced. We end the chapter with an application of flatness-based control to a simplified model of a vertical take-off and landing aircraft.

Chapter 3: In this chapter we derive the mathematical model describing the attitude dynamics of a rigid spacecraft. We give a thorough discussion of how the attitude can be described using different kinematic parameterizations. We then analyze the fundamental properties of stabilizability and controllability for the two-input spacecraft model.

Chapter 4: In this chapter we show that under a certain condition on the principal moments of inertia of the underactuated spacecraft, the system is differentially flat. We then solve the open-loop motion planning problem for the flat underactuated spacecraft and present simulations in which the spacecraft is steered from one equilibrium configuration to another.

Chapter 5: In this chapter we design a dynamic feedback controller that can track a given reference trajectory such that the system is steered to an arbitrary small neighbourhood of a desired equilibrium in finite time. We then combine the dynamic controller with an existing asymptotically stabilizing controller to obtain an asymptotic trajectory tracking control law. Simulations are presented that show the state trajectory converging to the desired trajectory.

Chapter 6: In this chapter we consider the more general idea of orbital flatness with the purpose of extending the results of Chapters 4 and 5. We construct a time-scaling function such that the system in the new time-scale is flat for more general inertia values. We design a closed-loop dynamic controller based on the orbital flatness approach. The dynamic control law is obtained by applying the DEA to the time-scaled system.

Chapter 7: In this chapter we summarize the contributions of the thesis and discuss possible extensions and improvements.

Chapter 2

Theoretical Background

In this chapter we review some well-known results from nonlinear control theory. The subjects covered are stabilizability, controllability for affine systems, feedback linearization, differential flatness, and motion planning and trajectory tracking for flat systems. In particular, the link between flatness and dynamic feedback linearization is discussed. We end the chapter with an application of differential flatness to the control of a planar vertical take-off and landing aircraft (PVTOL). A detailed presentation of the material in this chapter can be found in Nijmeijer and van der Shaft (1990), Isidori (1995), Bullo and Lewis (2004), and Martin et al. (2003). The reader is encouraged to refer to Appendix A for standard notation and definitions from differential geometry.

2.1 Stabilizability

Consider the nonlinear control system having the form

$$\dot{x} = f(x, u), \tag{2.1}$$

with equilibrium point x_0 and corresponding constant control u_0, that is, $f(x_0, u_0) = 0$. We assume without loss of generality that $(x_0, u_0) = (0, 0)$. We also assume that the state x evolves on an open subset M of $\mathbb{R}^n$, that $u \in \mathbb{R}^m$, and that f is smooth map. The stabilizability problem for (2.1) is concerned with the existence of a state feedback $u = \alpha(x)$, with $\alpha(x_0) = u_0$, such that the closed-loop system

$$\dot{x} = f(x, \alpha(x))$$

has x_0 as a locally asymptotically stable equilibrium. It is desirable to obtain conditions on f that can be used to determine if a stabilizing feedback exists. A simple method to obtain such conditions is to analyze the linearization of (2.1) at (x_0, u_0). To this end, let

$$\dot{x} = Ax + Bu$$

denote the linearization of (2.1) about (x_0, u_0). If the pair (A, B) is controllable, then there exists a matrix K such that all the eigenvalues of $(A + BK)$ lie in $\mathbb{C}_-$. Applying the feedback $u = \alpha(x) = Kx$ to the original system (2.1) guarantees that the origin of the closed-loop system $\dot{x} = f(x, Kx)$ is locally asymptotically stable according to Lyapunov's indirect method (Lyapunov, 1892). Indeed, the linearization of $\dot{x} = f(x, Kx)$ is $\dot{x} = (A + BK)x$, which, by the choice of K, is asymptotically stable. Therefore, in the case that (A, B) is controllable there exists a smooth (even linear) state feedback asymptotically stabilizing the equilibrium (x_0, u_0) of (2.1). If (A, B) is not controllable and at least one uncontrollable eigenvalue lies in $\mathbb{C}_+$ then there does not exist a smooth feedback stabilizing (2.1). For if (A, B) is not controllable and there exists an uncontrollable eigenvalue $\lambda \in \mathbb{C}_+$ then the linearization of $\dot{x} = f(x, \alpha(x))$, for any smooth feedback $u = \alpha(x)$, will contain the unstable eigenvalue λ and thus, according to Lyapunov's indirect method, x_0 is not an asymptotically stable equilibrium for the system $\dot{x} = f(x, \alpha(x))$. Therefore, a necessary condition for there to exist a smooth asymptotically stabilizing feedback for (2.1) is that the linearized system should have no uncontrollable eigenvalues in $\mathbb{C}_+$.

An interesting situation occurs when the linearization contains a purely imaginary uncontrollable eigenvalue. In this case one cannot use the linearization to deduce the existence of a smooth stabilizing feedback since Lyapunov's indirect method gives no information when the linearization contains a purely imaginary eigenvalue. This situation is often referred to as the *critical case* for determining the existence of a smooth stabilizing controller. As an example, consider the system

$$\dot{x} = u^3.$$

Its linearization about $x_0 = 0$ is clearly uncontrollable and has the uncontrollable eigenvalue $\lambda = 0$. Thus, from the linearization we cannot deduce whether a smooth stabilizing feedback exists. However, the feedback $u = \alpha(x) = -x$ makes $x_0 = 0$ a globally asymptotically stable equilibrium.

We now state a well known necessary condition that, when applicable, provides an alternative method to determining if (2.1) is asymptotically stabilizable when the linearization fails to give any information.

Theorem 2.1.1 (Brockett's Necessary Condition). *If there exists a continuously differentiable (C^1) state feedback locally asymptotically stabilizing the system* (2.1) *at* $(x_0, 0)$ *then the map* f *is surjective onto a neighbourhood of the origin.*

In other words, if the equation $y = f(x, u)$ cannot be solved for (x, u) for y sufficiently close to the origin then the system cannot be locally asymptotically stabilized by a C^1-feedback. Theorem 2.1.1 is due to Brockett (1983), and as shown by Zabczyk (1989), it remains valid if all objects are assumed to be only continuous, even if the assumption of uniqueness of solutions is dropped (Orsi et al., 2003). Below we apply the theorem to the well known *nonholonomic integrator.*

Example 2.1.1 (Brockett's Example). Consider the system $\dot{x} = f(x, u)$ where $f(x, u) = (u_1, u_2, x_1u_2 - x_2u_1)$. It is clear that the linearization of the system at the origin is uncontrollable and has an uncontrollable eigenvalue $\lambda = 0$. Thus, we cannot conclude from the linearization that the system is smoothly stabilizable about the origin. However, since no point of the form $y = (0, 0, \epsilon)$ is in the range of f where $\epsilon \in \mathbb{R}$, the origin cannot be made an asymptotically stable equilibrium with a continuous control by Brockett's necessary condition. □

2.2 Controllability

The problem of deciding when a general control system is "controllable" is one of the very basic questions in systems theory. Loosely speaking, it is concerned with determining the set of states that a control system can reach. The problem was first rigorously studied for linear control systems during the early 1960's ((Kalman, 1960), (Kalman et al., 1963), (Kalman, 1963)) and resulted in sufficient and necessary conditions for linear controllability. There even exists an explicit expression for the set of reachable states. For nonlinear systems the theory of controllability is less complete, and is further complicated by the fact that different non-equivalent notions of controllability exist. In spite of this, nonlinear controllability continues to draw much attention because of its importance in control design and analysis. In the present section, we will briefly summarize some basic notions of nonlinear controllability and present some well known results for control-affine systems that are of interest to us.

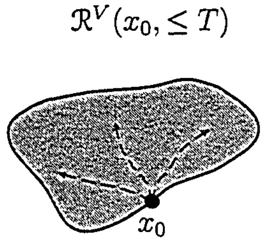
Before stating the basic notions of controllability we first review some standard notation and terminology. A *control-affine system* is governed by the differential equation

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} f_a(x(t))u_a(t), \tag{2.2}$$

where $x(t)$ belongs to an open subset M of $\mathbb{R}^n$, $f_0, f_1, \ldots, f_m$ are smooth vectors fields on M, and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. For compactness we usually write a control-affine system as a triple $\Sigma = (\mathsf{M}, \mathscr{C} = \{f_0, f_1, \ldots, f_m\}, \mathbb{R}^m)$. A *controlled trajectory* for (2.2) is a pair (c, u), both defined on some interval $[0, T]$, such that $c(t)$ is a solution curve of (2.2) when the control $u(t)$ is applied. We denote $\mathrm{Ctraj}(\Sigma, T)$ as the set of controlled trajectories for Σ on the interval $[0, T]$. For $x_0 \in \mathsf{M}$, a neighbourhood V of x_0, and $T > 0$ we denote

$$\begin{aligned} \mathcal{R}^V(x_0, T) &= \{x \in \mathsf{M} \mid (c, u) \in \mathrm{Ctraj}(\Sigma, T),\ c(t) \in V, \\ &\qquad\qquad c(0) = x_0 \text{ and } c(T) = x\}, \\ \mathcal{R}^V(x_0, \leq T) &= \bigcup_{t \in [0,T]} \mathcal{R}^V(x_0, t). \end{aligned}$$

In words, $\mathcal{R}^V(x_0, T)$ is the set of states reachable from x_0 in exactly time T by trajectories remaining in V, and $\mathcal{R}^V(x_0, \leq T)$ is the set of states reachable from x_0 in time at most T by trajectories remaining in V. With these preliminary definitions, we are now ready to introduce some basic notions of controllability.



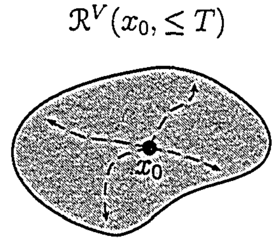


Figure. 2.1: Illustration of a system accessible from x_0 (left) and STLC from x_0 (right).

Definition 2.2.1. Let $\Sigma = (\mathsf{M}, \mathscr{C} = \{f_0, f_1, \ldots, f_m\}, \mathbb{R}^m)$ be a control-affine system, and let $x_0 \in \mathsf{M}$.

(i) The system Σ is *locally accessible* from x_0 if there exists a $T > 0$ such that $\mathrm{int}(\mathcal{R}^V(x_0, \leq t)) \neq \emptyset$ for all neighbourhoods V of x_0 and for all $t \in (0, T]$.

(ii) The system Σ is *controllable* from x_0 if, for each $x \in M$, there exists a $T > 0$ and $(c, u) \in \mathrm{Ctraj}(\Sigma, T)$ such that $c(0) = x_0$ and $c(T) = x$.

(iii) The system Σ is *small-time locally controllable (STLC)* from x_0 if there exists $T > 0$ such that $x_0 \in \mathrm{int}(\mathcal{R}^V(x_0, \leq t))$ for all neighbourhoods V of x_0 and for all $t \in (0, T]$.

Thus by definition, if a system is STLC from x_0 then it is possible to steer the system in any direction from x_0 in an arbitrary small time. Controllability, on the other hand, is obviously a weaker property since the time required to steer the system to its final configuration may be large. Figure 2.1 illustrates the concepts of accessibility and STLC. For linear systems, all of the controllability definitions given in Definition 2.2.1 are equivalent and can be checked using the Kalman controllability test. Below, we give an example to gain insight into the differences between these notions of controllability.

Example 2.2.1 (Bullo and Lewis (2004)). Consider the single-input system

$$\dot{x} = y^2$$
$$\dot{y} = u$$

defined on $\mathsf{M} = \mathbb{R}^2$ with $u \in \mathbb{R}$. Given any initial condition (x_0, y_0), y can be steered to any desired final position $y(T)$ with a control $u(t)$ satisfying $y(T) = y_0 + \int_0^T u(\tau)\, d\tau$. By inspection, $u(t) = \frac{1}{T}(-y_0 + y(T))$ is one such control. However, x cannot be steered to the "left" because its velocity is non-negative. Therefore the set of states reachable from (x_0, y_0) in at most time T will contain a non-empty interior but will not contain a neighbourhood of (x_0, y_0). Thus the system is accessible from any (x_0, y_0) but not STLC from (x_0, y_0). Since no point to the left of x_0 can be reached, the system is also not controllable. □

The previous example demonstrates that accessibility does not imply either STLC or controllability. On a moment's thought one can also conclude that controllability does not imply either STLC or accessibility. However, STLC implies accessibility and controllability in a local sense.

To state the controllability results that we are interested in, we must further introduce some notions and notation. Let $\mathfrak{X}(\mathsf{M})$ denote the vector space of smooth vector fields on M, and let $X, Y \in \mathfrak{X}(\mathsf{M})$. The *Lie bracket* of the vector fields X and Y, denoted $[X, Y]$, is the new vector field

$$[X, Y] = \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y.$$

The vector space $\mathfrak{X}(\mathsf{M})$ endowed with the Lie bracket is a Lie algebra (Appendix A). For a control-affine system $\Sigma = (\mathsf{M}, \mathscr{C} = \{f_0, f_1, \ldots, f_m\}, \mathbb{R}^m)$, define the *accessibility algebra* $\mathcal{C}$ as the smallest subalgebra of $\mathfrak{X}(\mathsf{M})$ that contains the family of vector fields $\mathscr{C}$, and the *accessibility distribution* C as the distribution generated by the accessibility algebra $\mathcal{C}$, that is,

$$C(x) = \operatorname{span}\{X(x) \mid X \in \mathcal{C}\}.$$

Since we will be explicitly computing C for the underactuated spacecraft in §3.4, we state the following lemma that gives a method for determining the generators of C.

Lemma 2.2.1 (Nijmeijer and van der Shaft (1990)). *Every element of C is a $\mathbb{R}$-linear combination of repeated Lie brackets of the form*

$$[X_k, [X_{k-1}, [\cdots, [X_2, X_1]]\cdots],$$

where $X_i \in \mathscr{C} = \{f_0, f_1, \dots, f_m\}$ *for* $i = 0, 1, \dots, k$ *and* $k \geq 0$.

We are now ready to state a well-known result concerning accessibility at a point $x_0 \in \mathsf{M}$.

Theorem 2.2.1 (Nijmeijer and van der Shaft (1990)). *A control-affine system* $\Sigma = (\mathsf{M}, \mathscr{C} = \{f_0, f_1, \dots, f_m\}, \mathbb{R}^m)$ *is accessible from* $x_0 \in \mathsf{M}$ *if* $\dim C(x_0) = n$. *If* Σ *is accessible then* $\dim C(x) = n$ *for* x *in an open and dense subset of* M.

Theorem 2.2.1 thus gives a characterization of accessible systems via the Lie algebra generated by the system's vector fields $f_0, f_1, \dots, f_m$. In general, however, the number of Lie bracket operations required to apply Theorem 2.2.1 is not known in advance. Nonetheless, the condition of the theorem is attractive because it involves algebraic operations than can be carried out with the help of symbolic mathematics software. The condition $\dim C(x_0) = n$ is known as the *Lie algebra rank condition* (LARC).

Example 2.2.2 (Example 2.2.1 continued). Using Theorem 2.2.1 we can easily show that the system considered in Example 2.2.1 is accessible from any point in $\mathsf{M} = \mathbb{R}^2$. We compute $[g, [f, g]] = (-2, 0)$. Thus, the vector fields $\{g, [g, [f, g]]\} \in C$ globally span $\mathbb{R}^2$, and therefore, $\dim C(x) = 2$ for all $x \in \mathsf{M}$. Hence, by Theorem 2.2.1 the system is accessible from any point in M. □

We finish the section with a sufficient condition for STLC given by Sussmann (1987). To state Sussmann's result we need some notation. A *bracket* of the accessibility algebra $\mathcal{C}$ is an element that cannot be written as a sum of other elements. For instance $[f_0, f_1]$ and f_0 are brackets but $[f_0, f_1] + f_0$ is not. Let $\mathrm{Br}(\mathcal{C})$ denote the set of brackets in the accessibility algebra $\mathcal{C}$, and denote B a typical element in $\mathrm{Br}(\mathcal{C})$. Let $|B|_a$ denote the number of times the vector field f_a appears in the bracket B for $a = 0, 1, \dots, m$. For instance, $|[f_1, [f_0, f_1]]|_1 = 2$. A bracket B is *bad* if $|B|_0$ is odd and $|B|_a$ is even for all $a \in \{1, \dots, m\}$. A bracket is *good* if it is not bad. For example, $[f_0, f_1]$ and $[f_0, [f_1, f_0]]$ are good brackets but f_0 and $[f_1, [f_1, f_0]]$ are bad brackets. For $\theta \in [1, +\infty) \cup \{+\infty\}$, the *$\theta$-degree* of a bracket B is defined by

$$\deg_\theta(B) = \begin{cases} \frac{1}{\theta}|B|_0 + \sum_{a=1}^{m} |B|_a, & \theta \in [1, +\infty), \\ \sum_{a=1}^{m} |B|_a, & \theta = +\infty. \end{cases}$$

Let S_m denote the the group of permutations on m elements. For a bracket B and $\sigma \in S_m$, define $\sigma(B)$ to be the bracket obtained by leaving the position of f_0 unchanged and switching the position of f_a with $f_{\sigma(a)}$ for $a = 1, \ldots, m$. Now define

$$\beta(B) = \sum_{\sigma \in S_m} \sigma(B).$$

Example 2.2.3. Consider the permutation $\{1,2\} \xrightarrow{\sigma_1} \{2,1\}$ and the bracket $B = [f_0, [f_1, [f_0, f_2]]]$. Then, $\sigma_1(B) = [f_0, [f_2, [f_0, f_1]]]$, and

$$\beta(B) = \sum_{\sigma \in S_2} \sigma(B) = [f_0, [f_1, [f_0, f_2]]] + [f_0, [f_2, [f_0, f_1]]].$$

We are now ready to state the sufficient condition for STLC obtained by Sussmann (1987).

Theorem 2.2.2 (Sussmann (1987)). *Consider the smooth control-affine system $\Sigma = (\mathsf{M}, \mathscr{C} = \{f_0, f_1, \ldots, f_m\}, \mathbb{R}^m)$ and suppose that $f_0(x_0) = 0$. Assume that Σ satisfies the LARC at x_0. Assume that there is a $\theta \in [1, +\infty) \cup \{+\infty\}$ such that, whenever $B \in Br(\mathcal{C})$ is a bad bracket then there are brackets $C_1, \ldots, C_k$ in $Br(\mathcal{C})$ such that*

(i) $\beta(B)(x_0) \in span\{C_j(x_0) \mid j \in \{1, \ldots, k\}\}$, and

(ii) $deg_\theta(C_j) < deg_\theta(B)$ for $j \in \{1, \ldots, k\}$.

Then Σ is STLC from x_0.

Theorem 2.2.2 says that if Σ is accessible at x_0 and if every bad bracket B evaluated at x_0 (or more exactly $\beta(B)(x_0)$) can be written as a linear combination of lower order brackets evaluated at x_0, then the system is STLC from x_0. It is important to mention that Theorem 2.2.2 has been generalized by Bianchini and Stefani (1993), however, Sussmann's result suffices for our purposes.

Example 2.2.4. The equations of motion of the inverted pendulum on a cart can be written as

$$\dot{x} = \begin{pmatrix} x_2 \\ 0 \\ x_4 \\ \sin(x_3) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\cos(x_3) \end{pmatrix} u = f(x) + g(x)u, \tag{2.3}$$

where x_1 denotes the cart's position with respect to some reference frame, x_2 the velocity of the cart, x_3 the angle the pendulum makes with the vertical, and x_4 the angular velocity of the pendulum. Let $x = (x_1, x_2, x_3, x_4)$ denote the state vector. The origin, $x_0 = 0$, corresponds to the unstable equilibrium. To prove that system (2.3) is STLC from x_0 we must first prove that it is accessible from x_0. Straightforward calculations show that the set of vector fields $\{g, [f,g], [f,[f,g]], [f,[f,[f,g]]]\} \subset C$ span $\mathbb{R}^4$ at x_0. Therefore, $\dim C(x_0) = 4$, and consequently by Theorem 2.2.1 the system is accessible from x_0. Now we apply Theorem 2.2.2 to show that the system is STLC from x_0. Setting $\theta = 1$ implies that every bad bracket has an odd θ-degree. The first bad bracket is the drift vector field, but since $f(x_0) = 0$, it can be trivially written as a linear combination of other brackets at x_0. The next bad bracket is $[g,[f,g]] = (0,0,0,-\sin(2x_3))$, which vanishes at the equilibrium x_0 and thus can be written as a linear combination of lower order brackets. Since $\text{span}\{g, [f,g], [f,[f,g]], [f,[f,[f,g]]]\}(x_0) = \mathbb{R}^4$, any bad bracket of order five or greater can be written as a linear combination of lower order brackets at x_0. This proves that (2.3) is STLC from x_0. □

2.3 Feedback Linearization

The problem of transforming a general nonlinear control system into a linear controllable system with a state feedback and coordinate change is known as the *feedback linearization problem* (FBLP). To be more precise, consider the nonlinear system

$$\dot{x} = f(x,u), \; x \in \mathsf{M} \subset \mathbb{R}^n, \; u \in \mathbb{R}^m, \tag{2.4}$$

where M is an open set. Let $x_0 \in \mathsf{M}$. The FBLP about x_0 is concerned with finding: (1) a dynamic state feedback having the form

$$\begin{aligned} \dot{\xi} &= a(x,\xi,v) \\ u &= b(x,\xi,v), \end{aligned} \tag{2.5}$$

where $\xi \in \mathbb{R}^q$ is the compensator state, $v \in \mathbb{R}^m$ is the new control input, and a and b are smooth functions locally defined on an open neighbourhood of

$\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^m$ containing (x_0, ξ_0, v_0), and (2) a coordinate change $z = \Phi(x, \xi)$ locally defined on an open neighbourhood of $\mathbb{R}^n \times \mathbb{R}^q$ containing (x_0, ξ_0), such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, b(x, \xi, v)) \\ \dot{\xi} &= a(x, \xi, v), \end{aligned}$$

is equivalent to a linear controllable system under the transformation Φ, that is, $\dot{z} = Fz + Gv$, where the pair (F, G) is controllable. If one can find a coordinate change and a dynamic feedback transforming (2.4) to a linear controllable system then we say that (2.4) is *dynamically feedback linearizable* (DFBL) at x_0. The control law (2.4) is called a *static state feedback* if $q = 0$, that is, if there are no compensator states. If (2.4) is linearizable with a static feedback then we say that it is *static feedback linearizable* (SFBL) at x_0.

2.3.1 Static Feedback Linearization (SFBL)

The FBLP was first posed and solved by Brockett (1978) for single-input control-affine systems and the restricted class of static feedbacks $u = \alpha(x) + v$. Later, Jakubczyk and Respondek (1980) obtained necessary and sufficient conditions for control-affine multi-input systems and the more general static feedbacks $u = \alpha(x) + \beta(x)v$, where $\beta(x)$ is locally invertible matrix (see also Hunt et al. (1983) where a slightly different formulation is used). To be consistent with the literature on feedback linearization we will denote a control-affine system by $\Sigma = (\mathsf{M}, \mathscr{C} = \{f, g_1, \ldots, g_m\}, \mathbb{R}^m)$, that is, f_0 is replaced by f and f_i by g_i for $i = 1, \ldots, m$. Let $x_0 = 0$ be an uncontrolled equilibrium point of Σ, that is, $f(x_0) = 0$. To state the necessary and sufficient conditions of Jacubczyk and Respondek, we first need to define the nested set of distributions

$$\begin{aligned} \mathcal{G}_0 &= \operatorname{span}\{g_1, \ldots, g_m\} \\ \mathcal{G}_1 &= \operatorname{span}\{g_1, \ldots, g_m, \operatorname{ad}_f g_1, \ldots, \operatorname{ad}_f g_m\} \\ \vdots &= \vdots \\ \mathcal{G}_i &= \operatorname{span}\{\operatorname{ad}_f^k g_j \mid 0 \leq k \leq i, 1 \leq j \leq m\} \end{aligned}$$

for $i = 0, 1, \ldots, n-1$, where $\mathrm{ad}_f g = [f, g]$ is the Lie bracket of f and g and $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$.

Theorem 2.3.1 (Jakubczyk and Respondek (1980)).
Let $\Sigma = (\mathsf{M}, \mathscr{C} = \{f, g_1, \ldots, g_m\}, \mathbb{R}^m)$ be a control-affine with $f(x_0) = 0$ and suppose that span$\{g_1(x_0) \ \cdots \ g_m(x_0)\} = m$. Then Σ can be transformed to a linear controllable system via a static state feedback and coordinate change about x_0 if and only if

(i) for each $0 \leq i \leq n-1$, the distribution $\mathcal{G}_i$ is involutive and has constant dimension near x_0;

(ii) the distribution $\mathcal{G}_{n-1}$ has dimension n.

Theorem 2.3.1 is an attractive result because it gives computable conditions for checking if a control-affine system is SFBL at x_0. Condition (ii) of the theorem is equivalent to checking that the system's Jacobian linearization about x_0 is controllable, and therefore, a necessary condition for SFBL is that the Jacobian linearization be controllable. Although Theorem 2.3.1 completely answers the question of whether a given control-affine system is SFBL, it does not directly give a procedure for constructing the static feedback and coordinate change. Fortunately, a procedure does indeed exist and involves constructing a "fictitious" m-dimensional smooth output map

$$y = h(x) = (h_1(x), h_2(x), \ldots, h_m(x)) \tag{2.6}$$

satisfying certain conditions. To introduce the procedure, we first give the following definition, where we denote $L_f s$ as the Lie derivative of the smooth function $s : \mathbb{R}^n \to \mathbb{R}$ along the vector field f.

Definition 2.3.1. A control-affine system $\Sigma = (\mathsf{M}, \mathscr{C} = \{f, g_1, \ldots, g_m\}, \mathbb{R}^m)$ with output (2.6) has *vector relative degree* $\{r_1, \ldots, r_m\}$ at a point $x_0 \in \mathsf{M}$ if $L_{g_j} L_f^k h_i(x) = 0$ for all $j = 1, \ldots, m$ and all $k = 0, 1, \ldots, r_i - 2$, for all $i = 1, \ldots, m$, and for all x in a neighbourhood of x_0, and the $m \times m$ matrix

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}$$

is nonsingular at x_0. The matrix $A(x)$ is called the *decoupling matrix* of the system.

We now state the following lemma relating relative degree and SFBL.

Lemma 2.3.1 (Isidori (1995)). *Consider the control-affine system* $\Sigma = (\mathsf{M}, \mathscr{C} = \{f, g_1, \ldots, g_m\}, \mathbb{R}^m)$ *with* $f(x_0) = 0$ *and suppose that* $span\{g_1(x_0) \ \cdots \ g_m(x_0)\} = m$. *Then, the system* Σ *is SFBL at* x_0 *if and only if there exists a neighbourhood* U *of* x_0 *and a smooth map* $h : U \to \mathbb{R}^m$, *such that the system has vector relative degree* $\{r_1, \ldots, r_m\}$ *at* x_0, *with*

$$r_1 + r_2 + \cdots + r_m = n.$$

Sketch of proof. We sketch the proof for sufficiency since it explicitly gives the coordinate change and static feedback. Thus, suppose that there exists an output $y = h(x) = (h_1(x), \ldots, h_m(x))$ with vector relative degree $\{r_1, \ldots, r_m\}$ at x_0 satisfying $\sum_{i=1}^m r_i = n$. Then, the condition that $L_{g_j} L_f^k h_i(x) = 0$ for all $j = 1, \ldots, m$ and all $k = 0, 1, \ldots, r_i - 2$, for all $i = 1, \ldots, m$, and for all x in a neighbourhood of x_0, implies that the input $u = (u_1, \ldots, u_m)$ does not explicitly appear in the derivatives of y_i until the r_i-th derivative. Calculating the r_i-th derivative of y_i and writing the result in matrix form gives

$$\begin{bmatrix} y_1^{(r_1)} \\ y_2^{(r_2)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ L_f^{r_2} h_2(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}}_{b(x)} + \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} u$$

$$= b(x) + A(x)u.$$

Since $A(x)$ is locally invertible about x_0, we can define the static state feedback

$$u(x) = \underbrace{-A^{-1}(x)b(x)}_{\alpha(x)} + \underbrace{A^{-1}(x)}_{\beta(x)} v. \tag{2.7}$$

A consequence of the condition $\sum_{i=1}^m r_i = n$ is that the map

$$z = \Phi(x) = \big(h_1(x), L_f h_1(x), \ldots, L_f^{r_1-1} h_1(x), \ldots, \\ h_m(x), L_f h_m(x), \ldots, L_f^{r_m-1} h_m(x)\big)$$

is a coordinate change defined locally about x_0. Applying the control law (2.7) to the original system and then writing the closed-loop system in z-coordinates yields a system in Brunovsky controller canonical form with controllability indices $\{r_1, \ldots, r_m\}$, that is, for each $i = 1, \ldots, m$,

$$\begin{aligned}
\dot{z}_1^i &= z_2^i \\
\dot{z}_2^i &= z_3^i \\
\vdots &= \vdots \\
\dot{z}_{r_i-1}^i &= z_{r_i}^i \\
\dot{z}_{r_i}^i &= v_i.
\end{aligned}$$

Thus, in z coordinates the system is controllable. ■

The "fictitious" output considered in Lemma 2.3.1 is not generally a real physical system output and only serves as a tool to design the static feedback and coordinate change. To avoid confusing with a real system output, one usually calls it a *linearizing output*. If, however, $y_r = h_r(x)$ is a real system output and the relative degree $\{r_1, \ldots, r_m\}$ of Σ with respect to y_r is well-defined, then one can apply the feedback (2.7) to obtain a input-output linear system. One can then design v to solve the output tracking problem provided that the internal dynamics are BIBS stable (Isidori, 1995).

If Σ is SFBL at x_0 then a smooth asymptotically stabilizing feedback defined about x_0 can be constructed as follows. Since the system in z coordinates, $\dot{z} = Fz + Gv$, is controllable, there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that the control $v = Kz$ makes $z_0 = 0$ an asymptotically stable equilibrium. Now, since the trajectories of $x(t)$ and $z(t)$ are the same up to a coordinate change, asymptotic stability of z_0 implies asymptotic stability of x_0. The asymptotically stabilizing controller in original coordinates is

$$u(x) = -A^{-1}(x)b(x) + A^{-1}(x)K\Phi(x).$$

2.3.2 Dynamic Feedback Linearization (DFBL)

The result of Jakubczyk and Respondek (1980) is important because it identifies a class of control-affine systems that are equivalent under static state feedback and coordinate change to a linear controllable system. However, the conditions of Theorem 2.3.1 are, not surprisingly, not generally satisfied. This shortcoming lead researchers to consider the more general dynamic feedbacks (2.5). At the present moment, there do not exist sufficient and necessary conditions for a control-affine system to be DFBL at an equilibrium. There has been, however, some progress in determining necessary or sufficient conditions for DFBL, see for instance the work of Charlet et al. (1989), Charlet et al. (1991), Sluis (1993). In particular, it is shown by Charlet et al. (1989) that dynamic feedback linearization is a multi-input phenomenon: a single-input control-affine system is SFBL if and only if it is DFBL. Another condition obtained by the same authors, and easily verified, is given by the following theorem.

Theorem 2.3.2 (Charlet et al. (1991)). *If a control-affine system is dynamic feedback linearizable at x_0 then its linear approximation at x_0 is controllable.*

The above necessary condition can be directly applied to driftless systems. Indeed, since a driftless system (excluding the case $m \geq n$) has an uncontrollable linearization at any equilibrium x_0 then it is not DFBL at x_0.

2.4 Differential Flatness

Differentially flat systems were first introduced by Fliess, Lévine, Martin, and Rouchon (1992a,b). Initially, flat systems were defined in a differential algebraic setting. Subsequently, flatness was defined in an infinite dimensional differential geometric setting where Lie-Bäcklund transformations were the main tool (see Fliess et al. (1993, 1994, 1999)). Application of flatness to real life control problems has increased since the introduction of the theory. Flatness has been applied to a planar vertical take-off and landing aircraft (PVTOL) (Martin et al., 1996), a car with n-trailers (Fliess et al., 1993), a towed cable system (Murray, 1996), gantry cranes (Lévine et al., 1997), magnetic bearings (Lévine et al., 1996), chemical reactor models (Rothfuß et al., 1996), and mobile wheeled robots (Oriolo et al., 2002) (see (Martin et al., 2003) for a catalogue of flat systems). Below we give the practical definition of flatness.

Definition 2.4.1. The control system

$$\dot{x} = f(x, u), \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \tag{2.8}$$

is said to be *differentially flat* or just *flat* if there exists smooth maps h, $\mathcal{A}$, and $\mathcal{B}$, with $\mathcal{A}$ and $\mathcal{B}$ locally surjective, defined on open neighbourhoods of $\mathbb{R}^n \times (\mathbb{R}^m)^{\rho+1}$, $(\mathbb{R}^m)^{r+1}$, and $(\mathbb{R}^m)^{r+2}$, respectively, such that

$$y = h(x, u, \dot{u}, \ldots, u^{(\rho)}) \tag{2.9a}$$

$$x = \mathcal{A}\big(y, \dot{y}, \ldots, y^{(r)}\big) \tag{2.9b}$$

$$u = \mathcal{B}\big(y, \dot{y}, \ldots, y^{(r)}, y^{(r+1)}\big), \tag{2.9c}$$

where ρ and r are positive integers, and the components of y are not related by a differential relation of the form

$$P\big(y, \dot{y}, \ldots, y^{(\kappa)}\big) = 0.$$

In other words, a system with m inputs is flat if there exists m output functions $y_1, \ldots, y_m$ that parameterize the state x and input u. This property is quite remarkable because it says that the state and input can be explicitly determined without integrating any differential equation. Therefore, given a feasible trajectory $y(t)$ for the flat output, the corresponding trajectories for the state and control are $x(t) = \mathcal{A}\big(y(t), \dot{y}(t), \ldots, y^{(r)}(t)\big)$, and $u(t) = \mathcal{B}\big(y(t), \dot{y}(t), \ldots, y^{(r)}(t), y^{(r+1)}(t)\big)$, respectively. The condition that y and its derivatives are not differentially related ensures that the components of y can be independently designed to parameterize the state and input.

Unfortunately, determining whether a system is flat is a difficult task, and presently the characterization of flat systems is still an open problem. There are, however, special cases where flatness has been completely characterized. We list these cases below:

1. Single-input control-affine systems: In this case, flatness is equivalent to static feedback linearization and the conditions of Theorem 2.3.1 can be applied.

2. Control-affine systems with $m = n - 1$: In this case the system is flat if and only if it is controllable (Charlet et al., 1989).

3. Control-affine systems with $n = 4$ and $m = 2$: Necessary and sufficient conditions were given by Pomet (1995).

4. Driftless systems with $m = 2$: Necessary and sufficient conditions were given by Martin and Rouchon (1994).

5. Driftless systems with $m = n - 2$: In this case the system is flat if and only if it is strongly accessible for almost every x (Martin and Rouchon, 1995).

We also mention the result of Rouchon (1994) giving a necessary condition, referred to as the "ruled manifold criterion", for flatness. In the next section we discuss the strong relationship that exists between differential flatness and dynamic feedback linearization.

2.4.1 Link Between Flatness and Dynamic Feedback Linearization

We begin the discussion relating flatness and dynamic feedback linearization by first considering what implications one can make when a system is DFBL. To this end, suppose that the control system $\dot{x} = f(x, u)$ is DFBL at x_0. Then there exists a dynamic feedback having the form (2.5) and a coordinate change $z = \Phi(x, \xi)$ such that the closed-loop system can be transformed under Φ to a linear controllable system $\dot{z} = Fz + Gv$. We can assume without loss of generality that the linear system is in Brunovsky controller canonical form, with controllability indices $\{r_1, \ldots, r_m\}$, that is, the coordinate change is such that $z = \left(z_1, \ldots, z_1^{(r_1-1)}, \ldots, z_m, \ldots, z_m^{(r_m-1)}\right)$ and

$$\begin{aligned} z_1^{(r_1)} &= v_1 \\ &\vdots \\ z_m^{(r_m)} &= v_m. \end{aligned}$$

By the invertibility of Φ we can write

$$\begin{bmatrix} x \\ \xi \end{bmatrix} = \Phi^{-1}(z)$$
$$u = b(\Phi^{-1}(z), v). \tag{2.10}$$

Define the output $y = (z_1, z_2, \ldots, z_m)$. Since $v_i = z_i^{(r_i)}$ for $i = 1, \ldots, m$, then from (2.10), x and u can be expressed as smooth functions of y and its derivatives, that is,

$$x = \mathcal{A}(y, \dot{y}, \ldots, y^{(r-1)})$$
$$u = \mathcal{B}(y, \dot{y}, \ldots, y^{(r)}).$$

This shows that a DFBL system is flat and that the flat outputs are given by the "head" components of the coordinate change. Now we consider the converse implication, that is, whether a flat system is DFBL. The answer is affirmative but in a generic sense, as will be explained below, and the dynamic feedback linearizing a flat system is a special type of feedback called an *endogenous dynamic feedback.*

Definition 2.4.2. The dynamic feedback

$$\dot{\xi} = a(x, \xi, v)$$
$$u = b(x, \xi, v),$$

is called *endogenous* if ξ can be expressed as a smooth function of $x, u, \dot{u}, \ldots,$ $u^{(\rho)}$, where ρ is a non-negative integer.

Therefore, an endogenous feedback does not introduce any new external variables since the compensator state ξ can be written in terms of the original variables, hence the name endogenous. The link between flatness and endogenous dynamic feedbacks is summarized in the following theorem. (Recall that a subset $A \subset B$ is *dense* in B if $\mathrm{cl}(A) = B$.)

Theorem 2.4.1 (Martin (1992); Fliess et al. (1999)). *Every differentially flat system can be dynamic feedback linearized in a dense open set with an endogenous dynamic feedback.*

Thus, flatness and dynamic feedback linearization are equivalent in a dense open set, that is, there will be singular points where a flat system is not DFBL. The dense open set may contain singular points where the coordinate change is undefined. It may so happen that such singular points are points of interest, in particular equilibrium points. One may argue then that knowing that a system is flat, from the point of view of closed-loop control, is useless if one is interested in controlling the system about a singular point. For instance, suppose that the driftless system $\dot{x} = \sum_{i=1}^{m} g_i(x)u_i$ $(m < n)$ is flat. Then the endogenous feedback given by Theorem 2.4.1 will not be defined in a neighbourhood of any equilibrium x_0 because the linearization at x_0 is not controllable (see Theorem 2.3.2). However, if one is interested in controlling the system near a singular point then the flatness property can allow one to steer the system to an arbitrary small neighbourhood of the singularity. Then, one can switch to a control law that is well-defined in a neighbourhood of the singularity. This is exactly what will be done for the underactuated spacecraft.

2.4.2 Open-loop Motion Planning

The motion planning problem can be stated as follows. Given $x_1, x_2 \in \mathsf{M}$, find a control u, defined on some interval $[t_1, t_2]$, so that the controlled trajectory (c, u) with $c(t_1) = x_1$ has the property that $c(t_2) = x_2$. Fortunately, the motion planning problem for a flat system is relatively simply. Indeed, to transfer the state of a flat system from x_1 to x_2, the flat output trajectory $y(t)$ can be designed so that it satisfies

$$\begin{aligned} x_1 &= \mathcal{A}\big(y(t_1), \dot{y}(t_1), \ldots, y^{(r)}(t_1)\big) \\ x_2 &= \mathcal{A}\big(y(t_2), \dot{y}(t_2), \ldots, y^{(r)}(t_2)\big). \end{aligned} \tag{2.11}$$

The conditions (2.11) impose constraints on $y(t), \dot{y}(t), \ldots, y^{(r)}(t)$ at the end-points $t = t_1$ and $t = t_2$. To simplify the problem of designing a flat output trajectory satisfying the above constraints, we can write the components of the flat output in terms of basis functions $\phi_j(t)$ (for example polynomials, splines, etc.),

$$y_i(t) = \sum_{j=1}^{N} a_{ij}\phi_j(t),$$

and then solve for the coefficients a_{ij} for $j = 1, \ldots, N$ so that (2.11) is satisfied for all $i = 1, \ldots, m$. This gives rise to a linear system of equations for each $i = 1, \ldots, m$, having the form

$$\begin{aligned} y_i(t_1) &= \sum_j a_{ij}\phi_j(t_1) & y_i(t_2) &= \sum_j a_{ij}\phi_j(t_2) \\ \dot{y}_i(t_1) &= \sum_j a_{ij}\dot{\phi}_j(t_1) & \dot{y}_i(t_2) &= \sum_j a_{ij}\dot{\phi}_j(t_2) \\ &\vdots & &\vdots \\ y_i^{(r)}(t_1) &= \sum_j a_{ij}\phi_j^{(r)}(t_1) & y_i^{(r)}(t_2) &= \sum_j a_{ij}\phi_j^{(r)}(t_2), \end{aligned} \tag{2.12}$$

or what is the same

$$\left[\begin{array}{cccc} \phi_1(t_1) & \phi_2(t_1) & \cdots & \phi_N(t_1) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(r)}(t_1) & \phi_2^{(r)}(t_1) & \cdots & \phi_N^{(r)}(t_1) \\ \hline \phi_1(t_2) & \phi_2(t_2) & \cdots & \phi_N(t_2) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(r)}(t_2) & \phi_2^{(r)}(t_2) & \cdots & \phi_N^{(r)}(t_2) \end{array}\right] \begin{bmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \\ \vdots \\ a_{iN} \end{bmatrix} = \left[\begin{array}{c} y_i(t_1) \\ \dot{y}_i(t_1) \\ \vdots \\ y_i^{(r)}(t_1) \\ \hline y_i(t_2) \\ \dot{y}_i(t_2) \\ \vdots \\ y_i^{(r)}(t_2) \end{array}\right]. \tag{2.13}$$

The only conditions on the basis functions is that for each $i = 1, \ldots, m$, the coefficient matrix in (2.13) must have full rank for there to exist a solution for the coefficients. In practice it may be convenient to make (2.13) a square linear system of equations, in which case we must set $N = 2(r+1)$. If we wish to include state configurations between the motion from x_1 to x_2 then the above procedure can be easily extended. As a matter of fact, suppose we wish to steer the system from an initial state x_1 at time t_1 to a final state x_M at time t_M while passing through the intermediate states $x_2, \ldots, x_{M-1}$ at times $t_2, \ldots, t_{M-1}$, respectively. We can mimic the above procedure and design the flat output trajectory satisfying

$$x_k = \mathcal{A}(y(t_k), \dot{y}(t_k), \ldots, y^{(r)}(t_k)), \quad \text{for } k = 1, \ldots, M.$$

The corresponding system of equations for each flat output component will contain $M(r+1)$ unknowns. In addition to state constraints, one can map input constraints to the flat output space via the map $u = \mathcal{B}(y, \dot{y}, \ldots, y^{(r+1)})$. One can also consider optimization problems as done, for example, by van Nieuwstadt and Murray (1998) and v. Löewis (2002).

A common choice for basis functions are the polynomials $\phi_1(t) = 1$, $\phi_2(t) = t$, …, $\phi_N(t) = t^{N-1}$. If (2.13) is a square linear system of equations, that is, $N = 2(r+1)$, then the coefficient matrix for the unknowns a_{ij} for $j = 1, \ldots, N$, can be shown to have the form

$$\left[\begin{array}{cccccc|ccc}
1 & t_1 & t_1^2 & t_1^3 & \cdots & t_1^r & t_1^{r+1} & \cdots & t_1^{2r+1} \\
0 & 1!\binom{1}{1} & 1!\binom{2}{1}t_1 & 1!\binom{3}{1}t_1^2 & \cdots & 1!\binom{r}{1}t_1^{r-1} & 1!\binom{r+1}{1}t_1^{r} & \cdots & 1!\binom{2r+1}{1}t_1^{2r} \\
0 & 0 & 2!\binom{2}{2} & 2!\binom{3}{2}t_1 & \cdots & 2!\binom{r}{2}t_1^{r-2} & 2!\binom{r+1}{2}t_1^{r-1} & \cdots & 2!\binom{2r+1}{2}t_1^{2r-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & r!\binom{r}{r} & r!\binom{r+1}{r}t_1 & \cdots & r!\binom{2r+1}{r}t_1^{r+1} \\
\hline
1 & t_2 & t_2^2 & t_2^3 & \cdots & t_2^r & t_2^{r+1} & \cdots & t_2^{2r+1} \\
0 & 1!\binom{1}{1} & 1!\binom{2}{1}t_2 & 1!\binom{3}{1}t_2^2 & \cdots & 1!\binom{r}{1}t_2^{r-1} & 1!\binom{r+1}{1}t_2^{r} & \cdots & 1!\binom{2r+1}{1}t_2^{2r} \\
0 & 0 & 2!\binom{2}{2} & 2!\binom{3}{2}t_2 & \cdots & 2!\binom{r}{2}t_2^{r-2} & 2!\binom{r+1}{2}t_2^{r-1} & \cdots & 2!\binom{2r+1}{2}t_2^{2r-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & r!\binom{r}{r} & r!\binom{r+1}{r}t_2 & \cdots & r!\binom{2r+1}{r}t_2^{r+1}
\end{array}\right]$$

Let Π denote the above coefficient matrix, and notice that each block of Π has dimension $(r+1) \times (r+1)$. A tedious calculation shows that

$$\det(\Pi) = \Big((1!2!\cdots r!)(t_2 - t_1)^{(r+1)}\Big)^2,$$

and therefore, Π is invertible if and only if $t_1 \neq t_2$. We can state the following.

Proposition 2.4.1. *Let $N = 2(r+1)$ in (2.13), and suppose that $t_1 \neq t_2$. Then there exist unique coefficients a_{ij}, $j = 1, \ldots, N$, solving (2.13) for arbitrary $y_i(t), \dot{y}_i(t), \ldots, y_i^{(r)}(t)$ at $t = t_1, t_2$, for $i = 1, \ldots, m$.*

Proof. Since $t_2 \neq t_1$ the matrix Π is invertible and thus we can uniquely solve for the coefficients a_{ij}, $j = 1, \ldots, N$. ∎

In principle then, the flatness property converts the difficult problem of motion planning and trajectory generation to a linear algebra problem. This allows the creation of efficient algorithms using computer algebra software. A Maple program was created that takes as input the arguments $t_1, \ldots, t_M$, and the conditions $y(t_1), \ldots, y^{(r)}(t_1), \ldots, y(t_M), \ldots, y^{(r)}(t_M)$, and returns a polynomial flat output trajectory steering the system to the desired states (see Appendix C).

2.4.3 Closed-loop Trajectory Tracking

An important problem in control theory is the state and input trajectory tracking problem which can be stated as follows. Given a state and input reference trajectory $x_d(t)$ and $u_d(t)$, respectively, design a controller $u(t)$ such that $\|x(t) - x_d(t)\|$, $\|u(t) - u_d(t)\| \to 0$ as $t \to \infty$. Tracking a state and input trajectory for flat systems, away from singularities, is straightforward. Indeed, suppose that the system $\dot{x} = f(x, u)$ is flat and that $y = h(x, u, \dot{u}, \ldots, u^{(\rho)})$ is a flat output. Let $y_d(t)$ be a flat output reference trajectory parameterizing the trajectories $x_d(t)$ and $u_d(t)$. Then, to track $x_d(t)$ and $u_d(t)$, it is enough to track the flat output reference trajectory $y_d(t)$. Now, to track $y_d(t)$, we can use an endogenous feedback guaranteed to exist by Theorem 2.4.1 to impose the linear input-output dynamics $y_i^{(r_i)} = v_i$ for $i = 1, \ldots, m$. Let $e_i = y_i - y_{d,i}$ and $\boldsymbol{e}_i = (e_i, \ldots, e_i^{(r_i-1)})$ for $i = 1, \ldots, m$ and set $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m) \in \mathbb{R}^{\sum_i r_i}$. Now assign

$$v := y_d^{(r)} + K\boldsymbol{e}, \tag{2.14}$$

where $K \in \mathbb{R}^{m \times \sum_i r_i}$ is a block matrix with each block $K_i \in \mathbb{R}^{1 \times r_i}$ chosen so that the linear controllable differential equation

$$e_i^{(r_i)} = K_i \boldsymbol{e}_i$$

is asymptotically stable. Now set $e = (e_1, \ldots, e_m)$. Since the system is flat there exists a smooth map $\mathcal{A}$ such that $x = \mathcal{A}(y, \ldots, y^{(r-1)})$. Therefore,

$$\begin{aligned} x &= \mathcal{A}\big(y, \dot{y}, \ldots, y^{(r-1)}\big) \\ &= \mathcal{A}\big(y_d + e, \dot{y}_d + \dot{e}, \ldots, y_d^{(r-1)} + e^{(r-1)}\big) \\ &= \mathcal{A}\big(y_d, \dot{y}_d, \ldots, y_d^{(r-1)}\big) + R(y_d, e) \\ &= x_d + R(y_d, e), \end{aligned}$$

where $R(y_d, e)$ is a higher order term such that $R \to 0$ as $e \to 0$ (Taylor's Theorem). Thus as $e \to 0$ we have that $\|x - x_d\| \to 0$, that is the state trajectory is tracked. A similar analysis can be done for u using the relation $u = \mathcal{B}(y, \dot{y}, \ldots, y^{(r)})$. Thus, for flat systems, tracking a reference trajectory $(x_d(t), u_d(t))$ is straightforward away from singular points of the maps $\mathcal{A}$ and $\mathcal{B}$.

It is important to emphasize that for a general system not necessarily flat, tracking an output trajectory will not guarantee that a state and input trajectory are tracked as well. It is the flatness property that allows one to reduce the state tracking problem to the lower dimensional output tracking problem.

2.4.4 The PVTOL Example

In this section we illustrate how flatness can be used to solve the open-loop motion planning and closed-loop trajectory tracking problems. The sample system that we consider is a simplified model of a planar vertical take-off and landing (PVTOL) aircraft ((Hauser et al., 1992), (Martin et al., 1996), (Olfati-Saber, 2002)). The PVTOL example is motivated by aircrafts that are able to take-off vertically such as helicopters and special military airplanes. An illustration of the PVTOL is shown in Figure 2.2. Let $\{\boldsymbol{i}, \boldsymbol{j}\}$ be a fixed inertial frame and $\{\boldsymbol{i}_b, \boldsymbol{j}_b\}$ be a body fixed frame with origin at the aircraft's center of mass (CM). The forces acting on the PVTOL are the two thrust forces, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, and gravity $m\boldsymbol{g}$, where m is the aircraft's mass. These forces are

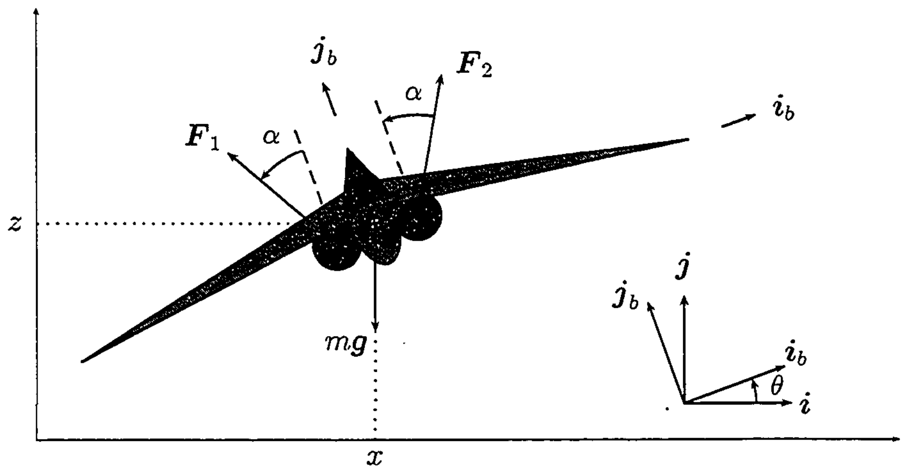


Figure. 2.2: Schematic of PVTOL aircraft

given by

$$\begin{aligned} \boldsymbol{F}_1 &= (-\sin\alpha \boldsymbol{i}_b + \cos\alpha \boldsymbol{j}_b)F_1 \\ \boldsymbol{F}_2 &= (\sin\alpha \boldsymbol{i}_b + \cos\alpha \boldsymbol{j}_b)F_2 \\ \boldsymbol{mg} &= -mG\boldsymbol{j}, \end{aligned}$$

where α is a constant angle and G denotes the acceleration due to gravity. The moment arms for the thrust forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are

$$\boldsymbol{r}_1 = -l\boldsymbol{i}_b - h\boldsymbol{j}_b, \qquad \boldsymbol{r}_2 = l\boldsymbol{i}_b - h\boldsymbol{j}_b,$$

respectively, where l and h denote the horizontal and vertical distances from the CM to where the thrust forces are applied. The equations of motion, obtained from Newton's second law, are

$$m\boldsymbol{a}_c = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{mg} \tag{2.15}$$

$$J\ddot{\theta}\boldsymbol{j}_b = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2, \tag{2.16}$$

where $\boldsymbol{a}_c$ is the acceleration of the CM, J is the moment of inertia about the CM along the $\boldsymbol{k}$-axis, and $\ddot{\theta}$ is the angular acceleration of the aircraft about the $\boldsymbol{i} \times \boldsymbol{j}$ axis. Projecting equations (2.15) and (2.16) onto the fixed frame using the transformation

$$\begin{aligned} \boldsymbol{i}_b &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{j}_b &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j}, \end{aligned}$$

yield the equations

$$m\ddot{x} = \sin\alpha\,\cos\theta\,(F_2 - F_1) - \cos\alpha\,\sin\theta\,(F_1 + F_2) \tag{2.17a}$$

$$m\ddot{z} = \sin\alpha\,\sin\theta\,(F_2 - F_1) + \cos\alpha\,\cos\theta\,(F_1 + F_2) - mG \tag{2.17b}$$

$$J\ddot{\theta} = (F_2 - F_1)\,(l\cos\alpha + h\sin\alpha). \tag{2.17c}$$

We make the following change of coordinates in the input space

$$u_1 = \frac{\cos\alpha}{m}(F_1 + F_2)$$
$$u_2 = \frac{l\cos\alpha + h\sin\alpha}{J}(F_2 - F_1),$$

and define the constant

$$\epsilon = \frac{J\,\sin\alpha}{m(l\cos\alpha + h\sin\alpha)}.$$

The control inputs u_1 and u_2 represent normalized vertical thrust and angular rolling torque, respectively. With these definitions, we obtain the following simplified equations of motion for the PVTOL

$$\begin{aligned} \ddot{x} &= -\sin\theta\,u_1 + \epsilon\cos\theta\,u_2 \\ \ddot{z} &= \cos\theta\,u_1 + \epsilon\sin\theta\,u_2 - G \\ \ddot{\theta} &= u_2. \end{aligned} \tag{2.18}$$

Writing the system in state space form with $X = (x, z, \dot{x}, \dot{z}, \theta, \dot{\theta})$ and $u = (u_1, u_2)$, we obtain

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{z} \\ -\sin\theta u_1 + \epsilon\cos\theta u_2 \\ \cos\theta u_1 + \epsilon\sin\theta u_2 - G \\ \dot{\theta} \\ u_2 \end{bmatrix}. \tag{2.19}$$

When $u = (u_1, u_2) = (0, 0)$, the system does not have any equilibrium points, as expected by physical reasoning. By inspection, an equilibrium point (X, u) must satisfy $\dot{x} = \dot{z} = \dot{\theta} = 0$, and $u_2 = 0$, which implies that $F_1 = F_2 =: F$. Since $u_2 = 0$ at an equilibrium then $u_1 \neq 0$. We are then lead to the equations

$$u_1 \sin\theta = 0$$
$$u_1\cos\theta - G = 0.$$

These are satisfied for $\theta = \theta_n = n\pi$ and $u_1 = \frac{G}{\cos\theta_n}$, where $n \in \mathbb{Z}$. From (2.17b) this implies that

$$2F\cos\alpha\cos\theta_n = mG,$$

which is just a balance of vertical forces. For example, when n is even, which corresponds to the aircraft being horizontal, each thrust force must be $F = \frac{mG}{2\cos\alpha}$. This will keep the aircraft at an equilibrium.

A control objective for the PVTOL is to transfer the aircraft position from an initial configuration $X_1 = (x_1, z_1, 0, 0, \theta_1, 0)$ at time t_1 to a final configuration $X_2 = (x_2, z_2, 0, 0, \theta_2, 0)$ at time t_2. We assume that the aircraft starts and stops at leveled flight, that is, $\theta_1 = \theta_2 = 0$. Without loss of generality we set $t_1 = 0$ and $t_2 = 10$ seconds. To accomplish the control objective we will use the flatness property of the system. For the PVTOL, a flat output is (Martin et al., 1996)

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x - \epsilon\sin\theta \\ z + \epsilon\cos\theta \end{bmatrix}. \tag{2.20}$$

To see this we start differentiating y:

$$\begin{aligned} \dot{y}_1 &= \dot{x} - \epsilon\dot{\theta}\cos\theta \\ \dot{y}_2 &= \dot{z} - \epsilon\dot{\theta}\sin\theta, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \ddot{y}_1 &= -u_1\sin\theta + \epsilon\dot{\theta}^2\sin\theta \\ \ddot{y}_2 &= u_1\cos\theta - G - \epsilon\dot{\theta}^2\cos\theta. \end{aligned} \tag{2.22}$$

From (2.22), it is not hard to see that

$$\ddot{y}_1\cos\theta + (\ddot{y}_2 + G)\sin\theta = 0. \tag{2.23}$$

and consequently

$$\tan\theta = \frac{-\ddot{y}_1}{\ddot{y}_2 + G}. \tag{2.24}$$

We can thus solve for θ, the solution being

$$\theta = \arctan\left(\frac{-\ddot{y}_1}{\ddot{y}_2 + G}\right). \tag{2.25}$$

From (2.24) we obtain the relationships

$$\begin{aligned} \sin\theta &= \frac{-\ddot{y}_1}{\sqrt{(\ddot{y}_2 + G)^2 + \ddot{y}_1{}^2}} \\ \cos\theta &= \frac{\ddot{y}_2 + G}{\sqrt{(\ddot{y}_2 + G)^2 + \ddot{y}_1{}^2}}. \end{aligned} \tag{2.26}$$

To obtain $\dot{\theta}$ in terms of y and its derivatives, we differentiate expression (2.25):

$$\begin{aligned}\dot{\theta} &= \frac{(\ddot{y}_2+G)^2}{(\ddot{y}_2+G)^2+\ddot{y}_1{}^2}\left(\frac{-y_1^{(3)}(\ddot{y}_2+G)+\ddot{y}_1 y_2^{(3)}}{(\ddot{y}_2+G)^2}\right)\\ &= \frac{-y_1^{(3)}(\ddot{y}_2+G)+\ddot{y}_1 y_2^{(3)}}{(\ddot{y}_2+G)^2+\ddot{y}_1{}^2}. \end{aligned}\tag{2.27}$$

From (2.20), (2.21), and (2.26), we obtain the remaining states in terms of y and its derivatives,

$$x = y_1+\epsilon\sin\theta = y_1-\epsilon\frac{\ddot{y}_1}{\sqrt{(\ddot{y}_2+G)^2+\ddot{y}_1{}^2}} \tag{2.28a}$$

$$z = y_2-\epsilon\cos\theta = y_2-\epsilon\frac{\ddot{y}_2+G}{\sqrt{(\ddot{y}_2+G)^2+\ddot{y}_1{}^2}} \tag{2.28b}$$

$$\dot{x} = \dot{y}_1+\epsilon\dot{\theta}\cos\theta = \dot{y}_1+\epsilon\frac{(\ddot{y}_2+G)[-y_1^{(3)}(\ddot{y}_2+G)+\ddot{y}_1 y_2^{(3)}]}{[(\ddot{y}_2+G)^2+\ddot{y}_1{}^2]^{3/2}} \tag{2.28c}$$

$$\dot{z} = \dot{y}_2+\epsilon\dot{\theta}\sin\theta = \dot{y}_2-\epsilon\frac{\ddot{y}_1[-y_1^{(3)}(\ddot{y}_2+G)+\ddot{y}_1 y_2^{(3)}]}{[(\ddot{y}_2+G)^2+\ddot{y}_1{}^2]^{3/2}}. \tag{2.28d}$$

To express u_1 in terms of the flat output and its derivatives we first solve for u_1 from the $\ddot{z}$ equation in (2.18) and then use (2.28d):

$$\begin{aligned}u_1 &= \frac{\ddot{z}+G-\epsilon u_2\sin\theta}{\cos\theta}\\ &= \frac{\ddot{y}_2+\epsilon(\cos\theta\,\dot{\theta}^2+\ddot{\theta}\sin\theta)+G-\epsilon\ddot{\theta}\sin\theta}{\cos\theta}\\ &= \frac{\ddot{y}_2+G}{\cos\theta}+\epsilon\dot{\theta}^2\\ &= \sqrt{(\ddot{y}_2+G)^2+\ddot{y}_1{}^2}+\epsilon\left[\frac{y_1^{(3)}(\ddot{y}_2+G)-\ddot{y}_1 y_2^{(3)}}{(\ddot{y}_2+G)^2+\ddot{y}_1^2}\right]^2.\end{aligned}$$

Finally, to obtain u_2 in terms of y and its derivatives we differentiate (2.27) yielding

$$u_2 = \frac{[\ddot{y}_1 y_2^{(4)}-y_1^{(4)}(\ddot{y}_2+G)][\ddot{y}_2+G)^2+\ddot{y}_1^2]}{[(\ddot{y}_2+G)^2+\ddot{y}_1^2]^2} - 2\frac{[(\ddot{y}_2+G)y_2^{(3)}+2\ddot{y}_1 y_1^{(3)}][\dot{y}_2 y_2^{(3)}-y_1^{(3)}(\ddot{y}_2+G]}{[(\ddot{y}_2+G)^2+\ddot{y}_1^2]^2}.$$

This shows that the PVTOL with output (2.20) is flat.

Since the system is flat, the motion planning constraints on the state $X(t)$ can now be imposed on the flat output. To transfer the system from X_1 to X_2, the flat output reference trajectory must satisfy

$$\begin{aligned} y_{d,1}(t_1) &= x_1 - \epsilon \sin\theta_1 \quad y_{d,2}(t_1) = z_1 + \epsilon\cos\theta_1 \\ y_{d,1}(t_2) &= x_2 - \epsilon \sin\theta_2 \quad y_{d,2}(t_2) = z_2 + \epsilon\cos\theta_2. \end{aligned} \tag{2.29}$$

For a smooth aircraft departure and landing, that is, to avoid a jerked initial and final motion, we will design $y_d(t)$ to satisfy the constraints

$$y_d^{(j)}(t_1) = 0, \qquad y_d^{(j)}(t_2) = 0, \qquad \text{for } j = 1, 2, 3. \tag{2.30}$$

Equations (2.29)-(2.30) impose eight conditions on each flat output component. We choose a polynomial reference trajectory for each flat output component having the form

$$y_{d,i}(t) = \sum_{k=0}^{N-1} a_{ik} t^k.$$

To obtain a square linear system of equations for the coefficients a_{ik} we set $N = 8$, resulting in the system

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1! & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2! & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3! & 0 & 0 & 0 & 0 \\ \hline 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & t_2^5 & t_2^6 & t_2^7 \\ 0 & 1! & 1!\binom{2}{1}t_2 & 1!\binom{3}{1}t_2^2 & 1!\binom{4}{1}t_2^3 & 1!\binom{5}{1}t_2^4 & 1!\binom{6}{1}t_2^5 & 1!\binom{7}{1}t_2^6 \\ 0 & 0 & 2! & 2!\binom{3}{2}t_2 & 2!\binom{4}{2}t_2^2 & 2!\binom{5}{2}t_2^3 & 2!\binom{6}{2}t_2^4 & 2!\binom{7}{2}t_2^5 \\ 0 & 0 & 0 & 3! & 3!\binom{4}{3}t_2 & 3!\binom{5}{3}t_2^2 & 3!\binom{6}{3}t_2^3 & 3!\binom{7}{3}t_2^4 \end{array}\right] \begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \\ a_{17} \end{bmatrix} = \begin{bmatrix} x_1 - \epsilon\sin\theta_1 \\ 0 \\ 0 \\ 0 \\ x_2 - \epsilon\sin\theta_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $i = 1$ and $t_1 = 10$ seconds. From Proposition 2.4.1, there exists unique coefficients a_{ik} solving the above linear equation. They are explicitly given by

$$
\begin{aligned}
a_{10} &= x_1 - \epsilon \sin\theta_1 \\
a_{11} &= a_{12} = a_{13} = 0 \\
a_{14} &= 35\,\frac{-x_1 + x_2 - \epsilon\ \sin(\theta_2) + \epsilon\ \sin(\theta_1)}{{t_2}^4} \\
a_{15} &= -84\,\frac{-x_1 + x_2 - \epsilon\ \sin(\theta_2) + \epsilon\ \sin(\theta_1)}{{t_2}^5} \\
a_{16} &= 70\,\frac{-x_1 + x_2 - \epsilon\ \sin(\theta_2) + \epsilon\ \sin(\theta_1)}{{t_2}^6} \\
a_{17} &= -20\,\frac{-x_1 + x_2 - \epsilon\ \sin(\theta_2) + \epsilon\ \sin(\theta_1)}{{t_2}^7}.
\end{aligned}
$$

The coefficients for the second flat output are identical to those above except x is replaced by z. To obtain a closed-loop controller that asymptotically tracks the flat output reference trajectory and thus transfers the system from X_1 to X_2, we input-output linearize the PVTOL dynamics using a dynamic feedback linearizing controller. To this end, let

$$
\begin{aligned}
\xi_1 &= u_1 - \epsilon\dot{\theta}^2 \\
\xi_2 &= \dot{\xi}_1.
\end{aligned}
$$

Then a straightforward calculation shows that

$$
\begin{aligned}
y_1^{(4)} &= \dot{\xi}_2 \sin\theta + 2\xi_2\dot{\theta}\cos\theta + \xi_1 u_2 \cos\theta - \xi_1\dot{\theta}^2\sin\theta \\
y_2^{(4)} &= \dot{\xi}_2 \cos\theta - 2\xi_2\dot{\theta}\sin\theta - \xi_1 u_2 \sin\theta - \xi_1\dot{\theta}^2\cos\theta.
\end{aligned}
\tag{2.31}
$$

Setting

$$
\begin{aligned}
y_1^{(4)} &= v_1 \\
y_2^{(4)} &= v_2,
\end{aligned}
$$

where $v = (v_1, v_2)$ is the new control input, and solving for u_2 and $\dot{\xi}_2$ from (2.31), yields the following dynamic controller

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= v_1 \sin\theta + v_2\cos\theta + \dot{\theta}^2\xi_1 \\
u_1 &= \xi_1 + \epsilon\dot{\theta}^2 \\
u_2 &= \frac{1}{\xi_1}(v_1\cos\theta - v_2\sin\theta - 2\dot{\theta}\xi_2).
\end{aligned}
\tag{2.32}
$$

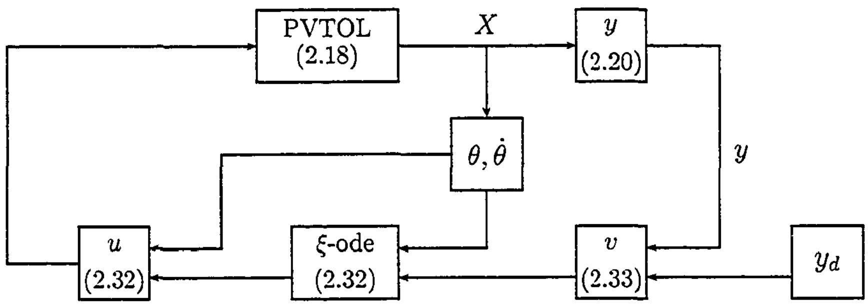


Figure. 2.3: Simulink model for the PVTOL

The auxiliary inputs are set to

$$v_j = y_{d,j}^{(4)} + \sum_{k=0}^{3} b_{jk} e_j^{(k)}, \qquad \text{for } j = 1, 2, \tag{2.33}$$

where $e_j = y_j - y_{d,j}$. This results in the linear tracking error dynamics

$$e_j^{(4)} - b_{j,3}\, e_j^{(3)} - b_{j,2}\, e_j^{(2)} - b_{j,1}\, e_j^{(1)} - b_{j,0} e_j = 0, \qquad \text{for } j = 1, 2, \tag{2.34}$$

which can be made asymptotically stable provided the coefficients $b_{j,k}$ are chosen appropriately.

A simulation of the PVTOL with the dynamic controller (2.32) and (2.33) was performed in Simulink. The simulation file is setup as shown in Figure 2.3. It is assumed that the system starts at the configuration $X_1 = (100, 50, 0, 0, 0, 0)$, and that desired final configuration is the origin. To show the robustness properties of the controller in the presence of initial condition errors, the *actual* initial configuration is set to $\tilde{X}_1 = (120, 40, 0, 0, 0, 0)$. The result of the simulation is shown in Figure 2.4. The tracking error poles for both reference trajectories were set to $\{-1, -2, -3, -4\}$. The simulation shows that the flatness-based controller tracks the desired state and input trajectories, and steers the system to the desired configuration.

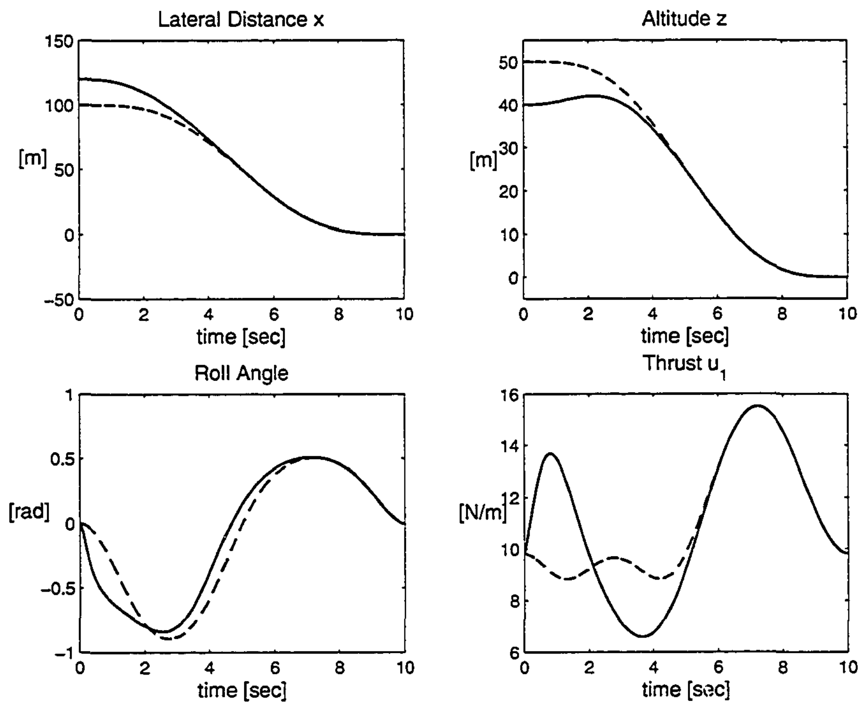


Figure. 2.4: Trajectory tracking for the PVTOL. The reference trajectories (dashed) and the closed-loop trajectories (solid) for x, z, θ and u_1.

Chapter 3

Modeling and Analysis

The attitude dynamics of a spacecraft can be described using a rigid body model undergoing pure rotation about a fixed point. By modeling the spacecraft as a rigid body we neglect any relative motion between any two points of the spacecraft, hence neglecting the effects of flexible components. Even in the presence of flexible components, a rigid body model is still needed in the full model of the attitude dynamics. Two equations are needed to describe the attitude dynamics of a rigid body. One equation describes the motion of the body without considering the effect of the forces acting on the body. The second equation describes the effects of forces on the motion of the body. The former equation is called the *kinematic equation* and the latter is the *dynamic equation*. We derive these equations in the present chapter. We then proceed to analyze the fundamental properties of controllability and stabilizability for the underactuated spacecraft. We show that the underactuated spacecraft is locally controllable about any equilibrium, but that no equilibrium can be made asymptotically stable using a continuous time-invariant feedback control. For a more complete discussion of rigid body mechanics and spacecraft attitude control see the texts by Goldstein (2002), Murray et al. (2000), and Wertz (1978).

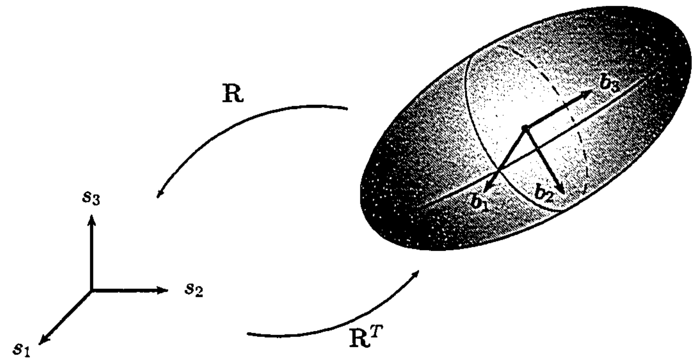


Figure. 3.1: Determination of the orientation of a rigid body by a spatial and body frame.

3.1 Mechanics of a Rigid Body

3.1.1 Kinematic model

The orientation of a rigid body fixed in space and free to rotate can be described by relating a set of fixed body axes to an inertial reference frame. Let $S = \{s_1, s_2, s_3\}$ be an inertial reference frame and let $B = \{b_1, b_2, b_3\}$ be a frame fixed in the body and that rotates with the body. If the orientation of B relative to S is known then the orientation of the body relative to S will be known. Henceforth, we call the frame S the *spatial frame* and B the *body frame*. Both frames are assumed to be right-handed and orthonormal. The setup is schematically shown in Figure 3.1. Note that since we are only concerned with the attitude of the rigid body and not in its translation, we can place the origin of the frame S to coincide with the origin of the body frame B. The orientation of the body frame B relative to the spatial frame S is obtained by expressing the body basis vectors in terms of the spatial bases vectors, that is, for $j = 1, 2, 3$, we write

$$\boldsymbol{b}_j = (\boldsymbol{s}_1 \cdot \boldsymbol{b}_j)\boldsymbol{s}_1 + (\boldsymbol{s}_2 \cdot \boldsymbol{b}_j)\boldsymbol{s}_2 + (\boldsymbol{s}_3 \cdot \boldsymbol{b}_j)\boldsymbol{s}_3,$$

where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$. Define the matrix

$$\mathbf{R} := (\mathbf{R}_{ij}) = \boldsymbol{s}_i \cdot \boldsymbol{b}_j,$$

that is, $\mathbf{R}$ is the matrix whose j^{th} column contains the components of $\boldsymbol{b}_j$ in the frame S. By construction, the matrix $\mathbf{R}$ describes the orientation of the rigid body with respect to the spatial frame. The columns of $\mathbf{R}$ form an orthonormal set of vectors, and since S and B were assumed to be right-handed frames (or more generally, have the same orientation), $\mathbf{R}$ belongs to the *special orthogonal group of* $\mathbb{R}^3$ defined as

$$\mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}_{3\times 3},\ \det(\mathbf{R}) = +1\}.$$

The set SO(3) is also commonly referred to as the *group of rotations in* $\mathbb{R}^3$, and the matrix $\mathbf{R}$ is called a *rotation matrix.* As the body rotates, the rotation matrix $\mathbf{R}$ changes with time. This motion defines a map $R(t)$ from an interval $I \subset \mathbb{R}$ to SO(3).

For a given vector $\boldsymbol{x} \in \mathbb{R}^3$, we have the relationship

$$\mathbf{R}\boldsymbol{x}_B = \boldsymbol{x}_S,$$

where $\boldsymbol{x}_B$ and $\boldsymbol{x}_S$ denote the coordinates of $\boldsymbol{x}$ in the body and spatial frame, respectively. Thus $\mathbf{R}$ can be interpreted as the linear map taking B-coordinates into S-coordinates. For instance, the coordinate vector $\boldsymbol{x}_B = (1, 0, 0)$ is mapped to $(\boldsymbol{b}_1)_S$. Now let $\boldsymbol{x}$ be an arbitrary vector *fixed* in the body and suppose that the body is rotating. Then $\boldsymbol{x}_B$ is constant but $\boldsymbol{x}_S$ changes with time. We can write this as

$$\mathbf{R}(t)\boldsymbol{x}_B = \boldsymbol{x}_S(t). \tag{3.1}$$

Differentiating (3.1) with respect to time yields

$$\frac{\mathrm{d}\boldsymbol{x}_S}{\mathrm{d}t} = \dot{\mathbf{R}}(t)\boldsymbol{x}_B, \tag{3.2}$$

which can be re-written, using the identity $\boldsymbol{x}_B = \mathbf{R}^T(t)\boldsymbol{x}_S(t)$, as

$$\frac{\mathrm{d}\boldsymbol{x}_S}{\mathrm{d}t} = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)\boldsymbol{x}_S(t). \tag{3.3}$$

The matrix $\dot{\mathbf{R}}\mathbf{R}^T$ has an important property, namely, that it is skew-symmetric. Indeed, differentiating the expression $\mathbf{R}\mathbf{R}^T = \mathbf{I}_{3\times 3}$ yields $\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = 0$, which implies that $\dot{\mathbf{R}}\mathbf{R}^T = -(\dot{\mathbf{R}}\mathbf{R}^T)^T$. Let $\mathfrak{so}(3) \subset \mathbb{R}^{3\times 3}$ denote the set of skew-symmetric matrices of dimension three (Lee, 2003). It is not hard to show that $\mathfrak{so}(3)$ is a vector space of dimension three and that a basis is

$$\left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

Therefore, $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ via the invertible linear map

$$S : \mathbb{R}^3 \to \mathfrak{so}(3)$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \mapsto S(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{3.4}$$

A straightforward calculation shows that the cross product of any two vectors $\boldsymbol{\omega}$ and $\boldsymbol{x}$ can be written as $\boldsymbol{\omega} \times \boldsymbol{x} = S(\boldsymbol{\omega})\boldsymbol{x}$. Thus, as the body rotates there exists a unique vector $\boldsymbol{\omega}_S(t) := S^{-1}(\dot{\mathbf{R}}(t)\mathbf{R}^T(t))$ such that

$$\frac{\mathrm{d}\boldsymbol{x}_S}{\mathrm{d}t} = S(\boldsymbol{\omega}_S(t))\boldsymbol{x}_S(t) = \boldsymbol{\omega}_S(t) \times \boldsymbol{x}_S(t). \tag{3.5}$$

The unique vector $\boldsymbol{\omega}(t)$ is called the *instantaneous angular velocity* of the body. It corresponds to the instantaneous angular velocity of the body with respect to the spatial frame. Now, equating (3.2) with (3.5) yields the equation

$$S(\boldsymbol{\omega}_S)\boldsymbol{x}_S = \dot{\mathbf{R}}\boldsymbol{x}_B.$$

Using the identity $S(\boldsymbol{\omega}_S)\boldsymbol{x}_S = \mathbf{R}S(\boldsymbol{\omega}_B)\boldsymbol{x}_B$ (a rotation matrix preserves orientation), we can write

$$\mathbf{R}S(\boldsymbol{\omega}_B)\boldsymbol{x}_B = \dot{\mathbf{R}}\boldsymbol{x}_B. \tag{3.6}$$

Since $\boldsymbol{x}$ was arbitrary, (3.6) must hold for all points on the body. We thus obtain the fundamental relation

$$\dot{\mathbf{R}} = \mathbf{R}S(\boldsymbol{\omega}_B). \tag{3.7}$$

Equation (3.7) is the kinematic equation of the attitude of a rigid body. It gives the rate of change of the orientation of the body frame B with respect to the spatial frame S in terms of the current body frame orientation, $\mathbf{R}$, and the relative angular velocity between the frames, $\boldsymbol{\omega}$. Henceforth, we will denote the angular velocity of the body in the B-frame simply as $\boldsymbol{\omega}$, thus dropping the B subscript.

3.1.2 Dynamic Model

The fundamental equation governing the rotational dynamics of a rigid body in an inertial coordinate system is

$$\left(\frac{\mathrm{dL}}{\mathrm{d}t}\right)_S = \boldsymbol{\tau}, \tag{3.8}$$

where the subscript S denotes differentiation in an inertial reference frame S, $\mathbf{L}$ denotes the angular momentum of the rigid body about the center of mass, and $\boldsymbol{\tau}$ is the total external moment acting on the system. Equation (3.8) is the angular dynamics analogue of Newton's second law "$\mathbf{F} = \dot{\mathbf{p}}$", where $\mathbf{p} = m\mathbf{v}$ is the linear momentum vector. The angular momentum vector satisfies the equation

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega}, \tag{3.9}$$

where $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the inertia tensor of the body about the center of mass and $\boldsymbol{\omega}$ is the body angular velocity. The inertia tensor is a symmetric and positive-semidefinite linear map with respect to the standard inner-product on $\mathbb{R}^3$. A consequence of the symmetry of $\mathbb{I}$ is that there exists a basis of orthonormal eigenvectors for $\mathbb{I}$, and the eigenvalues of $\mathbb{I}$ are all real. The eigenvalues of the inertia tensor $\mathbb{I}$, denoted $I_1, I_2, I_3 \in \mathbb{R}$, are called the *principal moments of inertia* of the rigid body, and the eigenvectors corresponding to the principal inertias are called the *principal axes* of the rigid body. In Cartesian coordinates, the principal moments of inertia of a rigid body, with mass density $\rho(x, y, z)$, about its center of mass are given by the formulas (Tenenbaum, 2004)

$$\begin{aligned} I_1 &= \int_V (y^2 + z^2)\rho(x, y, z)\, dV \\ I_2 &= \int_V (x^2 + z^2)\rho(x, y, z)\, dV \\ I_3 &= \int_V (x^2 + y^2)\rho(x, y, z)\, dV, \end{aligned} \tag{3.10}$$

where $V \subset \mathbb{R}^3$ is a compact set representing the volume occupied by the rigid body.

Let $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ be the principal axes of the body with origin at the body's center of mass and fixed to rotate with the body. By definition, the matrix representation of the inertia tensor in the body frame takes the form

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

If $\boldsymbol{\omega} = \omega_1 \boldsymbol{b}_1 + \omega_2 \boldsymbol{b}_2 + \omega_3 \boldsymbol{b}_3$ is the body angular velocity in the body frame, then from (3.9) the angular momentum vector can be written in the body frame as

$$\mathbf{L} = I_1 \omega_1 \boldsymbol{b}_1 + I_2 \omega_2 \boldsymbol{b}_2 + I_3 \omega_3 \boldsymbol{b}_3.$$

Since B is a rotating frame, the time rate of change of $\mathbf{L}$ as seen from the spatial frame is

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_B + \boldsymbol{\omega} \times \mathbf{L} = \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_B + \boldsymbol{\omega} \times \mathbb{I}\boldsymbol{\omega}. \tag{3.11}$$

Note that since $\mathbf{L}$ is not fixed in the body $\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_B$ is generally non-zero. Therefore, combining (3.8) and (3.11), the dynamical equation of the attitude of a rigid body in the body frame is

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_B + \boldsymbol{\omega} \times \mathbb{I}\boldsymbol{\omega} = \boldsymbol{\tau}. \tag{3.12}$$

Equation (3.12) can be written in component form as

$$\begin{aligned} I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) &= \tau_1 \\ I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) &= \tau_2 \\ I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) &= \tau_3. \end{aligned} \tag{3.13}$$

Equations (3.13) are known as *Euler's equations.* They represent the effect of the external torques acting on the body. Integrating these equations will give the curve $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t), \omega_3(t))$ representing the body's angular velocity in the body frame. One can then substitute $\boldsymbol{\omega}$ into (3.7) and integrate to obtain the orientation of the rigid body with respect to the spatial frame.

3.1.3 Equations of Motion for an Underactuated Spacecraft

In this thesis, we are interested in studying the control problem for the case of an underactuated spacecraft with two independent control torques. Without loss of generality we can set $\tau_3 = 0$ in (3.13). Applying the feedback

$$\tau_1 = (I_3 - I_2)\omega_2\omega_3 + I_1 u_1$$
$$\tau_2 = (I_1 - I_3)\omega_1\omega_3 + I_2 u_2,$$

where u_1 and u_2 are the new control inputs, (3.13) can be written as

$$\begin{aligned} \dot\omega_1 &= u_1 \\ \dot\omega_2 &= u_2 \\ \dot\omega_3 &= \alpha\omega_1\omega_2, \end{aligned} \tag{3.14}$$

where

$$\alpha = \frac{I_1 - I_2}{I_3}. \tag{3.15}$$

Note that if $\alpha = 0$ then $\boldsymbol{b}_3$ is an axis of symmetry, that is, slicing the rigid body through a plane intersecting the center of mass and parallel to $\boldsymbol{b}_3$ will result in two identical mass distributed bodies. Using the formulas (3.10) of the principal moments of inertia, we can determine a bound on the magnitude of α. Indeed, using (3.10) and the short-hand notation $\rho = \rho(x, y, z)$ we have that

$$\begin{aligned} |\alpha| &= \left|\frac{I_1 - I_2}{I_3}\right| \\ &= \frac{\left|\int_V (y^2 - x^2)\rho\, dV\right|}{\int_V (y^2 + x^2)\rho\, dV} \\ &\le \frac{\int_V |y^2 - x^2|\, \rho\, dV}{\int_V (y^2 + x^2)\rho\, dV} \\ &\le 1. \end{aligned}$$

Thus $\alpha \le 1$. The parameter α plays a crucial role in the flatness analysis of the underactuated spacecraft (see §4.1).

Collecting (3.7) and (3.14), the attitude dynamics of an underactuated spacecraft with torque inputs applied along the principal axes $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are

$$\begin{aligned} \dot\omega_1 &= u_1 \\ \dot\omega_2 &= u_2 \\ \dot\omega_3 &= \alpha\omega_1\omega_2 \\ \dot{\mathbf{R}} &= \mathbf{R}\, S(\omega). \end{aligned} \tag{3.16}$$

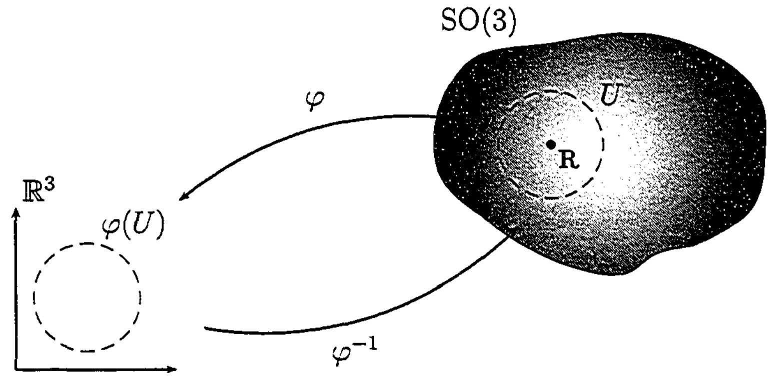


Figure. 3.2: Smooth manifold SO(3) with parameterization φ^{-1}.

The state space of system (3.16) is the six dimensional smooth manifold $\mathsf{M} = \mathbb{R}^3 \times \mathsf{SO}(3)$. Working with the full state space M to derive control laws for system (3.16) is the ideal situation since then one could obtain global results. However, obtaining $\mathbf{R}$ from the kinematic equation requires the integration of nine scalar equations. If, however, it is possible to represent $\mathbf{R}$ with fewer than nine parameters, then the kinematic equation will be equivalent to a system with fewer than nine parameters. It is then of interest to find useful parameterizations for the manifold SO(3).

3.2 Parameterization of SO(3)

The group of rotations SO(3) is a real smooth manifold of dimension three (Lee, 2003). This means that locally, SO(3) is homeomorphic to $\mathbb{R}^3$. To be more precise, about each point $\mathbf{R} \in \mathsf{SO}(3)$ there exists an open neighbourhood U of $\mathbf{R}$ and a homeomorphism $\varphi : U \to \mathbb{R}^3$ (a homeomorphism is a continuous bijection whose inverse is also continuous). The pair (U, φ) is called a *coordinate chart* about $\mathbf{R}$, and the inverse map $\varphi^{-1} : \varphi(U) \to U$ is called a *local parameterization* about $\mathbf{R}$. Figure 3.2 illustrates these ideas.

The minimum number of parameters to locally parameterize SO(3) is three. However, Stuelpnagel (1964) showed that it is topologically impossible for a three-dimensional parameterization of SO(3) to be both global and nonsingular. Consequently, any three-dimensional parameterization will have orientations where the transformed differential equation becomes singular or there will exist rotations that cannot be represented by the parameters. It is interesting to point out that Hopf (1940) showed that five is the minimum number of parameters to globally parameterize SO(3) in a 1-1 manner.

Popular parameterizations of SO(3) are Euler angles, quaternions, and Rodrigues parameters, just to name a few. There is no rule to choosing a parameterization, but certain factors should be taken into account. Depending on the application, factors to consider are: (1) the number of parameters needed, (2) the form of the transformed differential equation, and (3) the numerical accuracy in the integration of the new equations. In the following sections we will describe three parameterizations that are of interest to us, namely, Euler angles, quaternions, and Rodrigues parameters.

3.2.1 Euler Angles

The Euler angles parameterization is perhaps the most popular local parameterization of SO(3). It is physically intuitive and straight forward to derive. The idea is to bring the spatial frame S to the body frame B by making three successive rotations about a combination of the successive coordinates axes. More precisely, start with a frame S_0 aligned with the spatial frame S. Rotate the frame S_0 about the z-axis of S_0 by an angle ψ and denote the new frame by S_1 and $\mathbf{R}_\psi$ the corresponding rotation matrix. Next, rotate the frame S_1 about the y-axis of S_1 by an angle θ and denote the new frame by S_2 and $\mathbf{R}_\theta$ the corresponding rotation matrix. Lastly, rotate the frame S_2 about the x-axis of S_2 by an angle ϕ and denote $\mathbf{R}_\phi$ the corresponding rotation matrix. The angles ψ, θ, ϕ, known as *Euler angles*, are chosen so that the final frame S_2 coincides with the body frame B. The Euler angles parameterization just described is more indicatively known as a 3-2-1 (or ZYX) Euler angles parameterization and is given the name of *Fick angles.* The 3-2-1 parameterization is frequently used in aerospace applications as it corresponds to the roll-pitch-raw angles of an aircraft.

The rotation matrix $\mathbf{R}$ in terms of the Euler angles ψ, θ, and ϕ is

$$\mathbf{R}_{\psi\theta\phi} = \mathbf{R}_\psi \mathbf{R}_\theta \mathbf{R}_\phi = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi s_\theta c_\phi \\ s_\psi c_\theta & c_\psi c_\phi + s_\psi s_\theta s_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix},$$

where the shorthand notation $s_x = \sin x$ and $c_x = \cos x$ has been used and

$$\mathbf{R}_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}.$$

If $\psi, \phi \in (-\pi, \pi)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ then the map $(\psi, \theta, \phi) \mapsto \mathbf{R}_{\psi\theta\phi}$ has an inverse and is given by

$$\mathbf{R} \mapsto (\operatorname{atan2}(\mathbf{R}_{21}, \mathbf{R}_{11}), \operatorname{atan2}(-\mathbf{R}_{31}, \sqrt{\mathbf{R}_{11}^2 + \mathbf{R}_{21}^2}), \operatorname{atan2}(\mathbf{R}_{32}, \mathbf{R}_{33})),$$

where $\operatorname{atan2}(y, x)$ is the "smart" arctangent function which uses the sign of both x and y to determine the quadrant in which the resulting angle lies. The atan2 function is explicitly given by

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right), & \text{for } x > 0 \text{ and } y > 0, \\ \pi + \arctan(\frac{y}{x}), & \text{for } x < 0 \text{ and } y > 0, \\ -\pi + \arctan(\frac{y}{x}), & \text{for } x < 0 \text{ and } y < 0, \\ \frac{\pi}{2}, & \text{for } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2}, & \text{for } x = 0 \text{ and } y < 0. \end{cases} \tag{3.17}$$

Therefore, the map $(\psi, \theta, \phi) \mapsto \mathbf{R}_{\psi\theta\phi}$ defined for $\psi, \phi \in (-\pi, \pi)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is a local parameterization of SO(3) about the identity rotation $\mathbf{I}_{3\times 3}$.

The kinematic equation $\dot{\mathbf{R}} = \mathbf{R}\, S(\omega)$ in terms of the Euler angles can be written as

$$\begin{aligned} \dot{\phi} &= \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi) \tan\theta \\ \dot{\theta} &= \omega_2 \cos\phi - \omega_3 \sin\phi \\ \dot{\psi} &= (\omega_2 \sin\phi + \omega_3 \cos\phi) \sec\theta \end{aligned} \tag{3.18}$$

Equation (3.18) can be obtained by solving for $\dot{\psi}, \dot{\theta}, \dot{\phi}$ from the equation $\dot{\mathbf{R}}_{\psi\theta\phi} = \mathbf{R}_{\psi\theta\phi} S(\boldsymbol{\omega})$. From (3.18) we observe that a singularity in the kinematic equation occurs when $\theta = \pm\frac{\pi}{2}$. As mentioned previously, the singularity is a consequence of the topological fact that SO(3) cannot be globally parameterized with three parameters.

3.2.2 Quaternions

Quaternions can be used to represent rotations in $\mathbb{R}^3$ in a similar manner as complex numbers on the unit circle can be used to represent rotations on the plane. Let $\mathbb{Q} = \mathbb{C} \times \mathbb{C}$ (considered a real vector space) and define on $\mathbb{Q}$ the bilinear product

$$(a, b) \cdot (c, d) = (ac - \bar{d}b, da + b\bar{c}), \qquad a, b, c, d \in \mathbb{C},$$

where $\bar{a}$ denotes the complex conjugate of a. With these definitions, the space $(\mathbb{Q}, \cdot)$ is a four-dimensional algebra over $\mathbb{R}$ called the algebra of *quaternions*. A basis for $\mathbb{Q}$ is given by

$$\mathbf{1} = (1, 0), \quad \mathbf{i} = (i, 0), \quad \mathbf{j} = (0, 1), \quad \mathbf{k} = (0, i).$$

Straightforward calculations show that this basis satisfies

$$\begin{aligned}
&\mathbf{1} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{1} = \mathbf{q}, \qquad \text{for all } \mathbf{q} \in \mathbb{Q},\\
&\mathbf{i} \cdot \mathbf{j} = -\mathbf{j} \cdot \mathbf{i} = \mathbf{k},\\
&\mathbf{j} \cdot \mathbf{k} = -\mathbf{k} \cdot \mathbf{j} = \mathbf{i},\\
&\mathbf{k} \cdot \mathbf{i} = -\mathbf{i} \cdot \mathbf{k} = \mathbf{j}\\
&\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = -\mathbf{1}.
\end{aligned}$$

We usually write a quaternion $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ as simply $\mathbf{q} = (q_0, q_1, q_2, q_3)$ and say that q_0 is the scalar component and $\boldsymbol{q} = (q_1, q_2, q_3)$ the vector component of $\mathbf{q}$. The product of two quaternions $\mathbf{q}$ and $\mathbf{p}$ satisfies

$$\begin{aligned}
\mathbf{q} \cdot \mathbf{p} &= (q_0, \boldsymbol{q}) \cdot (p_0, \boldsymbol{p})\\
&= (q_0 p_0 - \boldsymbol{q} \cdot \boldsymbol{p}, q_0\boldsymbol{p} + p_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}).
\end{aligned}$$

We define the inner product of $\mathbf{q}$ and $\mathbf{p}$ as $\langle \mathbf{q}, \mathbf{p} \rangle = q_0 p_0 + \boldsymbol{q} \cdot \boldsymbol{p}$, and the norm of $\mathbf{q}$ is $\|\mathbf{q}\| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. Before describing how quaternions are used to describe rotations, we first need to state the following classic theorem due to Euler.

Theorem 3.2.1 (Euler). *Any orientation* $\mathbf{R} \in SO(3)$ *is equivalent to a rotation about a fixed axis* $\boldsymbol{a} \in \mathbb{R}^3$ *through an angle* $\Psi \in [-\pi, \pi)$.

The vector $\boldsymbol{a}$ is called the *Euler axis* and the angle Ψ the *Euler angle*. Given an Euler axis and Euler angle pair, $\boldsymbol{a}$ and Ψ, we would like to write the corresponding rotation matrix in terms of $\boldsymbol{a}$ and Ψ. It turns out that the corresponding rotation matrix for an Euler axis and angle pair is given by $e^{\hat{\boldsymbol{a}}\Psi}$, where we have used the standard notation $S(\boldsymbol{a}) = \hat{\boldsymbol{a}}$, where $S(\boldsymbol{a})$ is given by (3.4) and $e^{\mathbf{A}}$ is the matrix exponential of a matrix $\mathbf{A}$. To verify that $e^{\hat{\boldsymbol{a}}\Psi}$ is indeed a rotation matrix, we must show that $(e^{\hat{\boldsymbol{a}}\Psi})^{-1} = (e^{\hat{\boldsymbol{a}}\Psi})^T$ and that $\det e^{\hat{\boldsymbol{a}}\Psi} = +1$. Since $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}}^T$ commute, that is, $\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^T = \hat{\boldsymbol{a}}^T\hat{\boldsymbol{a}}$, and that $e^{\hat{\boldsymbol{a}}^T} = (e^{\hat{\boldsymbol{a}}})^T$, it follows that

$$e^{\hat{\boldsymbol{a}}}(e^{\hat{\boldsymbol{a}}})^T = e^{\hat{\boldsymbol{a}}}e^{\hat{\boldsymbol{a}}^T} = e^{\hat{\boldsymbol{a}}+\hat{\boldsymbol{a}}^T} = e^0 = \mathbf{I}.$$

Therefore, $(e^{\hat{\boldsymbol{a}}\Psi})^{-1} = (e^{\hat{\boldsymbol{a}}\Psi})^T$, and thus $\det e^{\hat{\boldsymbol{a}}\Psi} = \pm 1$. Now, since the determinant and the exponential map are continuous functions and $\det e^{\mathbf{0}} = \mathbf{I}$, it follows that $\det e^{\hat{\boldsymbol{a}}\Psi} = +1$. Therefore, $e^{\hat{\boldsymbol{a}}\Psi} \in \mathrm{SO}(3)$ and thus $e^{\hat{\boldsymbol{a}}\Psi}$ *is* the rotation matrix corresponding to $\boldsymbol{a}$ and Ψ.

There exists an elegant formula for $e^{\hat{\boldsymbol{a}}\Psi}$. To derive this formula, we first state the following lemma, which can be proved by induction, concerning the powers of a skew-symmetric matrix.

Lemma 3.2.1. *Given* $\hat{\boldsymbol{a}} \in \mathfrak{so}(3)$ *then*

$$\hat{\boldsymbol{a}}^{2k+1} = (-1)^k \|\boldsymbol{a}\|^{2k} \hat{\boldsymbol{a}}, \qquad \text{for } k = 0, 1, 2, \ldots$$

Assuming $\|\boldsymbol{a}\| = 1$ and applying Lemma 3.2.1 to the series expansion of $e^{\hat{\boldsymbol{a}}\Psi}$ yields

$$\begin{aligned} e^{\hat{\boldsymbol{a}}\Psi} &= \mathbf{I} + \left(\Psi - \frac{\Psi^3}{3!} + \frac{\Psi^5}{5!} - \cdots\right)\hat{\boldsymbol{a}} + \left(\frac{\Psi^2}{2!} - \frac{\Psi^4}{4!} + \frac{\Psi^6}{6!} - \cdots\right)\hat{\boldsymbol{a}}^2 \\ &= \mathbf{I} + \hat{\boldsymbol{a}}\sin\Psi + \hat{\boldsymbol{a}}^2(1 - \cos\Psi). \end{aligned} \tag{3.19}$$

If $\|\boldsymbol{a}\| \neq 1$, then performing a similar calculation yields

$$e^{\hat{a}\Psi} = \mathbf{I} + \frac{\hat{a}}{\|\boldsymbol{a}\|}\sin\Psi + \frac{\hat{a}^2}{\|\boldsymbol{a}\|}(1-\cos\Psi).$$

Given an Euler axis and Euler angle pair, $\boldsymbol{a}$ and Ψ, with $\|\boldsymbol{a}\| = 1$, we define the associated unit quaternion as

$$\mathbf{q} = (\cos(\Psi/2), \boldsymbol{a}\sin(\Psi/2)).$$

Conversely, given a unit quaternion $\mathbf{q} = (q_0, \boldsymbol{q})$, the corresponding Euler axis and Euler angle pair are calculated to be

$$\Psi = 2\cos^{-1} q_0 \quad \text{and} \quad \boldsymbol{a} = \begin{cases} \frac{1}{\sin(\Psi/2)}\boldsymbol{q}, & \text{if } \Psi \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

Equations (3.19) and (3.20) can be used to derive the expression for the rotation matrix $\mathbf{R}$ in terms of a quaternion. Indeed, writing Ψ and $\boldsymbol{a}$ in terms of $\mathbf{q}$ via equation (3.20), substituting the result into (3.19) and then simplifying yields

$$\mathbf{R}_{\mathbf{q}} = \begin{bmatrix} 2q_0^2 + 2q_1^2 - 1 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 + 2q_2^2 - 1 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + q_0q_1 & 2q_0^2 + 2q_3^2 - 1 \end{bmatrix}.$$

The kinematic equation $\dot{\mathbf{R}} = \mathbf{R}\,S(\boldsymbol{\omega})$ in terms of a quaternion can be written as

$$\dot{\mathbf{q}} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}\mathbf{q}. \tag{3.21}$$

The above ODE can be derived using the relationship $\dot{\mathbf{R}}_{\mathbf{q}} = \mathbf{R}_{\mathbf{q}}\,S(\boldsymbol{\omega})$.

An advantage of using quaternions to represent rotations is that if $\mathbf{q}_1$ represents a rotation between frames A and B, and $\mathbf{q}_2$ a rotation between frames B and C, then the rotation between frames A and C is

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2.$$

In other words, quaternions can be multiplied to yield composite rotations just like rotation matrices can. This is similar to the situation in which complex numbers on the unit circle, $e^{i\theta}$, are used to represent rotations on the plane. Indeed, if θ_1 and θ_2 correspond to distinct rotations on the plane, then the composite rotation obtained by rotating through an angle $\theta_1 + \theta_2$ is given by

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}.$$

Another advantage of using quaternions to represent rotations is that they globally parameterize SO(3), albeit in a two-to-one manner since $\mathbf{q}$ and $-\mathbf{q}$ correspond to the same rotation matrix. However, a disadvantage of using quaternions is that four parameters are used instead of the minimum three.

3.2.3 Rodrigues Parameters

Given a unit quaternion $\mathbf{q} = (q_0, q_1, q_2, q_3) \in \mathbb{S}^3$, where $\mathbb{S}^3$ denotes the unit sphere in $\mathbb{R}^4$, we define on the hemisphere $q_0 > 0$ the *Rodrigues vector*

$$X = (x_1, x_2, x_3) = \left(\frac{q_1}{q_0}, \frac{q_2}{q_0}, \frac{q_3}{q_0} \right).$$

The components of the Rodrigues vector are the *Rodrigues parameters.* The map $\mathbf{q} \mapsto X$ is a diffeomorphism with inverse given by

$$(x_1, x_2, x_3) \mapsto \left(\frac{1}{\sqrt{1+\|x\|^2}}, \frac{x_1}{\sqrt{1+\|x\|^2}}, \frac{x_2}{\sqrt{1+\|x\|^2}}, \frac{x_3}{\sqrt{1+\|x\|^2}} \right), \quad (3.22)$$

where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$. The rotation matrix in terms of the Rodrigues parameters is given by

$$\mathbf{R}_X = \begin{bmatrix} \frac{1+x_1^2-x_2^2-x_3^2}{1+\|X\|^2} & \frac{2x_1^2x_2^2-2x_3}{1+\|X\|^2} & \frac{2x_1^2x_3^2+2x_2}{1+\|X\|^2} \\ \frac{2x_1^2x_2^2+2x_3}{1+\|X\|^2} & \frac{1+x_2^2-x_1^2-x_3^2}{1+\|X\|^2} & \frac{2x_2^2x_3^2-2x_1}{1+\|X\|^2} \\ \frac{2x_1^2x_3^2-2x_2}{1+\|X\|^2} & \frac{2x_2^2x_3^2+2x_1}{1+\|X\|^2} & \frac{1+x_3^2-x_1^2-x_2^2}{1+\|X\|^2} \end{bmatrix},$$

and the kinematic equation $\dot{\mathbf{R}} = \mathbf{R}\,S(\omega)$ is

$$\dot{x}_1 = \frac{1}{2}\left(\omega_1 + \omega_3 x_2 - \omega_2 x_3 + (\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)\, x_1\right)$$

$$\dot{x}_2 = \frac{1}{2}\left(\omega_2 - \omega_3 x_1 + \omega_1 x_3 + (\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)\, x_2\right)$$

$$\dot{x}_3 = \frac{1}{2}\left(\omega_3 + \omega_2 x_1 - \omega_1 x_2 + (\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)\, x_3\right).$$

Parameterization	Notation	Advantages	Disadvantages
Rotation matrix	$\mathbf{R} = \mathbf{R}_{ij}$	No singularities No trigonometric functions	Six redundant parameters
Euler angles	ψ, θ, ϕ	No redundant parameters Physical interpretation	Trigonometric functions Singularities
Quaternions	q_0, q_1, q_2, q_3	No singularities No trigonometric functions	One redundant parameter No physical interpretation
Rodrigues parameters	x_1, x_2, x_3	No redundant parameters No trigonometric functions	Infinite for 180° rotation

Table. 3.1: Comparison of SO(3) parameterizations

Table 3.1 compares the three parameterizations considered in this section. As mentioned previously, which parameterization one chooses may depend on the application and the available analysis techniques. In this thesis, the majority of the control design will be done using the Euler angles parameterization.

3.3 Stabilizability Analysis

In this section we investigate the stabilizability properties of the underactuated spacecraft. Henceforth, we will use the kinematic equations from the Euler angles parameterization developed in §3.2.1. The resulting system is

$$\dot{\omega}_1 = u_1 \tag{3.23a}$$

$$\dot{\omega}_2 = u_2 \tag{3.23b}$$

$$\dot{\omega}_3 = \alpha\omega_1\omega_2 \tag{3.23c}$$

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \tag{3.23d}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \tag{3.23e}$$

$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta. \tag{3.23f}$$

We assume that $\alpha \neq 0$ since otherwise ω_3 is unaffected by the control and thus the system is not stabilizable or controllable. Let $x = (\omega_1, \omega_2, \omega_3, \phi, \theta, \psi)$ denote the state variable of (3.23) defined on the open set

$$\mathsf{M} = \{x \in \mathbb{R}^6 \mid \omega_1, \omega_2, \omega_2 \in \mathbb{R}, \psi, \phi \in (-\pi, \pi), \theta \in (-\pi/2, \pi/2)\}. \tag{3.24}$$

From (3.23e) and (3.23f) we observe that the system is at an equilibrium if and only if $\omega_2 = \omega_3 = 0$. From (3.23d) this implies that $\omega_1 = 0$. Therefore, the set of equilibrium points of the underactuated spacecraft are

$$\{(\omega_1, \omega_2, \omega_3, \phi, \theta, \psi) \in \mathsf{M} \;:\; \omega_1 = \omega_2 = \omega_3 = 0\}.$$

In words, all possible orientations are equilibrium points when the system is uncontrolled and has zero velocity. If gravity were present this would not be the case as a non-zero control would be required to counter the affect of gravity. We start the stabilizability analysis by studying the system's linear approximation about an equilibrium point. Let $x_e = (0, 0, 0, \phi_e, \theta_e, \psi_e)$ be an equilibrium point of (3.23), and let $\Delta x = x - x_e$. The linear approximation of (3.23) about x_e is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \sin(\phi_e)\tan(\theta_e) & \cos(\phi_e)\tan(\theta_e) & 0 & 0 & 0 \\ 0 & \cos(\phi_e) & -\sin(\phi_e) & 0 & 0 & 0 \\ 0 & \sin(\phi_e)\sec(\theta_e) & \cos(\phi_e)\sec(\theta_e) & 0 & 0 & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u. \tag{3.25}$$

By inspection, the only eigenvalue of the linearization is $\lambda = 0$ with algebraic multiplicity $m_\lambda = 6$. The controllability matrix of the linear approximation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \tan(\theta_e)\sin(\phi_e) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\phi_e) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(\phi_e)\sec(\theta_e) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the linearization of the spacecraft about any equilibrium is uncontrollable and the uncontrollable eigenvalue is at the origin. Therefore, the two-input spacecraft cannot be asymptotically stabilized to an equilibrium using linear feedback. We are thus in the critical case (see §2.1) of determining whether the system can be asymptotically stabilized using C^1-feedback.

Using Brockett's necessary condition (Theorem 2.1.1), it was shown by Byrnes and Isidori (1991) that the two-input spacecraft cannot be asymptotically stabilized about any equilibrium using continuous time-invariant static or dynamic state feedback. Due to its importance, we state and prove this result.

Theorem 3.3.1 (Byrnes and Isidori (1991)). *The two-input spacecraft* (3.23) *cannot be asymptotically stabilized to any equilibrium using continuous time-invariant static or dynamic state feedback control.*

Proof. The proof is by contradiction. Suppose that the two-input spacecraft can be asymptotically stabilized to the equilibrium x_e by a continuous time-invariant control. Let $u = (u_1(x), u_2(x))$ be such a control. Then, by Brockett's condition, the closed-loop vector field

$$F(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \alpha\omega_1\omega_2 \\ \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \\ \omega_2 \cos\phi - \omega_3 \sin\phi \\ (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta \end{bmatrix}$$

must be onto a neighbourhood of the origin. However, no point of the form $(0, 0, \delta, 0, 0, 0)$, for δ arbitrarily small, is in the range of F, which is a contradiction. Hence there does not exist a continuous static time-invariant feedback asymptotically stabilizing (3.23). In the case of dynamic feedback, suppose that q is the dimension of the compensator state. Then no point of the form $(0, 0, \delta, 0, 0, 0, \ldots, 0) \in \mathbb{R}^{6+q}$ is in the image of the closed-loop vector field, which is a contradiction. Thus, (3.23) cannot be asymptotically stabilized using dynamic state time-invariant feedback either. This completes the proof. ∎

The above result demonstrates that any controller that locally asymptotically stabilizes the two-input spacecraft must necessarily be time-varying and/or discontinuous in the state. This does not, however, rule out the possibility that there exists a control that brings the spacecraft to an arbitrarily small neighbourhood of an equilibrium.

3.4 Controllability Analysis

We begin the controllability analysis by studying the accessibility of (3.23) from any point in M. For this, let f denote the drift vector field, and g_1 and

g_2 the control vector fields of (3.23), that is,

$$f(x) = \begin{bmatrix} 0 \\ 0 \\ \alpha\omega_1\omega_2 \\ \omega_1 + (\omega_2 \sin\phi + \omega_3\cos\phi)\tan\theta \\ \omega_2\cos\phi - \omega_3\sin\phi \\ (\omega_2\sin\phi + \omega_3\cos\phi)\sec\theta \end{bmatrix}, \; g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Proposition 3.4.1. *The underactuated spacecraft* (3.23) *is accessible for every* $x \in \mathsf{M}$ *if and only if* $\alpha \neq 0$.

Proof. We must show that that accessibility algebra of the system has full dimension at any $x \in \mathsf{M}$ provided $\alpha \neq 0$. Using Lemma 2.2.1 the elements of the accessibility algebra will be linear combinations of elements of the form

$$[X_k, [X_{k-1}, [\cdots, [X_2, X_1]]\cdots],$$

where $X_i \in \mathscr{C} = \{f, g_1, g_2\}$ for $i = 1, \ldots, k$ for $k \geq 0$. The relevant Lie brackets are

$$\begin{aligned}
X_1 &= [f, g_2] = (0, 0, -\alpha\omega_2, -1, 0, 0) \\
X_2 &= [f, g_1] = (0, 0, -\alpha\omega_1, -\tan\theta\sin\phi, -\cos\phi, -\sin\phi\sec\theta) \\
X_3 &= [[f, g_1], g_2] = (0, 0, \alpha, 0, 0, 0) \\
X_4 &= [f, [[f, g_1], g_2]] = (0, 0, 0, -\alpha\tan\theta\cos\phi, \alpha\sin\phi, -\alpha\cos\phi\sec\theta).
\end{aligned}$$

Arranging the vector fields f, g_1, g_2, and the X_i, for $i = 1, \ldots, 4$, as columns of a matrix and then taking the determinant yields $-\frac{\alpha^2}{\cos\theta}$. Therefore, if $\alpha \neq 0$ then the accessibility algebra will always span a six dimensional space and thus (3.23) is accessible at any $x \in \mathsf{M}$. If $\alpha = 0$ then ω_3 will be constant for all time and thus the set of states reachable from any $x \in \mathsf{M}$ will contain an empty interior. This completes the proof. ∎

Hence by definition, the set of points that can be reached from any point in M will contain a non-empty interior provided the uncontrolled axis is not an axis of symmetry. To determine if an equilibrium will be in the interior of the reachable set we directly apply Sussmann's sufficient condition, Theorem 2.2.2.

Proposition 3.4.2. *The underactuated spacecraft* (3.23) *is small-time locally controllable at any equilibrium if and only if* $\alpha \neq 0$.

Proof. If $\alpha = 0$ then the system cannot be small-time locally controllable since it is not accessible. Now suppose $\alpha \neq 0$ and choose $\theta = 1$. We must show that every bad bracket can be written as a linear combination of lower-order brackets at an equilibrium. Since $\theta = 1$, every bad bracket must have an odd degree. The first bad bracket is the drift vector field, but since it vanishes at any equilibrium it can be trivially written as a linear combination of any bracket. The only third order brackets that are bad are $[[f, g_1], g_1]$ and $[[f, g_2], g_2]$, and these can be shown to be identically zero. Since $\text{span}\{f, g_1, g_2, X_1, X_2, X_3, X_4\}(x) = \mathbb{R}^6$ for all $x \in \mathsf{M}$ then bad brackets of order five or greater can be written as a linear combination of lower-order brackets. This completes the proof. ∎

The controllability analysis above demonstrates that when the underactuated spacecraft is at an equilibrium it will be possible to steer it in any direction we wish. However, even though the underactuated spacecraft possess this strong controllability property at an equilibrium, it cannot be asymptotically stabilized to an equilibrium using continuous state feedback control as shown in Theorem 3.3.1. This result is radically different to the situation for linear systems. As a matter of fact, if a linear system is controllable then there exists a smooth (even linear) control that will asymptotically stabilize the system. Nonlinear systems do not enjoy this useful property. Examples of nonlinear systems that are locally controllable about an equilibrium but fail to be stabilizable have been known for some time ((Sussmann, 1979), (Brockett, 1972), (Aeyels, 1985)). As another example, it can be easily shown that Brockett's system (Example 2.1.1) is small-time locally controllable about any equilibrium, and yet it is not possible to asymptotically stabilize it using continuous state feedback control. The non-existence of locally stabilizing continuous controls for locally controllable system has lead to the investigation of broadening the class of controls used for stabilization, in particular, time-varying and/or discontinuous controllers. Coron (1995) showed that if a general system having the form $\dot{x} = f(x, u)$ (with the state dimension greater than or equal to four) satisfies a controllability and accessibility property at the origin then the system is locally smoothly stabilizable to the origin in small time using a periodic time-varying feedback law. Similarly, Clarke et al. (1997) showed that if a system having the form $\dot{x} = f(x, u)$ is "asymptotically controllable" to the origin then it can be globally stabilized by a discontinuous control.

Chapter 4

Flatness and Open-loop Motion Planning

We begin the chapter by proving via direct calculation that the underactuated spacecraft with principal inertias satisfying

$$\alpha = \frac{I_1 - I_2}{I_3} = -1$$

is flat. A similar calculation first appeared in Rouchon (1992) for the different case $\alpha = 1$ using a 3-1-2 Euler angles parameterization. Although both cases $\alpha = \pm 1$ are similar, it has not been explicitly shown that flatness can be deduced for both cases of α using a single Euler angles parameterization. However, the dependence of flatness on α clearly demonstrates that flatness is a geometric property of the underactuated spacecraft and hence should not depend on the kinematic parameterization used.

The expressions obtained for the state and input in terms of the flat output and its derivatives contain a singularity. Nonetheless, we are able to design a flat output reference trajectory such that the spacecraft is steered from a given initial equilibrium to a final desired equilibrium while avoiding the singularity. In general, the presence of singularities in the expressions for the state and input is a common phenomenon for flat systems. In fact, Luca and Benedetto (1993) show that the presence of singularities for nonholonomic flat systems is structural. For "simple" singularities, as in Luca and Oriolo (2002) and Oriolo et al. (2002), one may be able to develop criteria on the reference trajectory that guarantees that the system will not reach the singularity. In special cases, one can also take advantage of the geometry of the system to "blow up" the singularity, see for instance Fliess et al. (1993) and Rouchon et al. (1993).

4.1 Flatness of the Underactuated Spacecraft

In this section we show by calculation that the underactuated spacecraft is flat for the special case $\alpha = -1$. The flatness property will be shown using a 3-2-1 Euler angles parameterization discussed in §3.2.1. For convenience, we rewrite the full dynamic and kinematic equations:

$$\dot{\omega}_1 = u_1 \tag{4.1a}$$

$$\dot{\omega}_2 = u_2 \tag{4.1b}$$

$$\dot{\omega}_3 = \alpha\omega_1\omega_2 \tag{4.1c}$$

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \tag{4.1d}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \tag{4.1e}$$

$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta. \tag{4.1f}$$

Let $x = (\omega_1, \omega_2, \omega_3, \phi, \theta, \psi)$ denote the state vector of system (4.1). Recall that equations (4.1) are defined on the open set

$$\mathsf{M} = \left\{x \in \mathbb{R}^6 \mid \omega_1, \omega_2, \omega_2 \in \mathbb{R}, \psi, \phi \in (-\pi, \pi), \theta \in (-\pi/2, \pi/2)\right\}.$$

In general, finding a flat output for a system is a difficult task. However, from the structure of equations (4.1) we can deduce that if a flat output exists then it must necessarily depend on ψ. To see this, notice that ψ does not directly affect the dynamic behaviour of the states ω_1, ω_2, ω_3, θ, and ϕ. From this observation we state the following.

Proposition 4.1.1. *If $y = h(x)$ is a flat output of system (4.1) then it explicitly depends on ψ.*

Proof. Suppose that $y = h(x)$ does not depend on ψ. Then the derivative of y of any order will not contain ψ and consequently ψ cannot be written as a function of y and its derivatives. Therefore, $y = h(x)$ cannot be a flat output. ■

Next, we state and prove the main result of this section.

Proposition 4.1.2. *The two-input spacecraft* (4.1) *with* $\alpha = -1$ *is flat, and a flat output is* $y = (\theta, \psi)$.

Proof. We must show that the state x and input u can be expressed in terms of y and a finite number of its derivative. Denote (y_1, y_2) as the components of the output. By definition of y we can write that $\theta = y_1$ and $\psi = y_2$. Now, notice that (4.1d) can be written as

$$\dot{\phi} = \omega_1 + \dot{y}_2 \sin y_1,$$

from which we obtain that

$$\omega_1 = \dot{\phi} - \dot{y}_2 \sin y_1. \tag{4.2}$$

From (4.1e) and (4.1f) we obtain that

$$\omega_2 = \dot{y}_1 \cos\phi + \dot{y}_2 \sin\phi \, \cos y_1 \tag{4.3a}$$

$$\omega_3 = -\dot{y}_1 \sin\phi + \dot{y}_2 \cos\phi \, \cos y_1. \tag{4.3b}$$

Now we need to express ϕ in terms of y and its derivatives. From (4.3b) we obtain

$$\begin{aligned} \dot{\omega}_3 &= -\ddot{y}_1 \sin\phi + \ddot{y}_2 \cos\phi \cos y_1 - \dot{y}_2\dot{y}_1 \cos\phi \sin y_1 - \dot{\phi}(\dot{y}_1 \cos\phi + \dot{y}_2 \sin\phi \cos y_1) \\ &= -\ddot{y}_1 \sin\phi + \ddot{y}_2 \cos\phi \cos y_1 - \dot{y}_2\dot{y}_1 \cos\phi \sin y_1 - \dot{\phi}\omega_2. \end{aligned} \tag{4.4}$$

From (4.1c), (4.2), (4.3a), and (4.4) we obtain

$$c_\phi[(\alpha - 1)\dot{y}_1\dot{y}_2 s_{y_1} + \ddot{y}_2 c_{y_1}] - s_\phi(\ddot{y}_1 - \alpha \dot{y}_1^2 s_{y_1} c_{y_1}) = \dot{\phi}(\alpha + 1)\omega_2, \tag{4.5}$$

where we have used the short-hand notation $s_x = \sin x$ and $c_x = \cos x$. If $\alpha = -1$ then the right-hand side of (4.5) vanishes and we can thus solve for ϕ, the solution being

$$\phi = \operatorname{atan2}(r_2, r_1), \tag{4.6}$$

where we have defined

$$\begin{aligned} r_1 &= \ddot{y}_1 + \dot{y}_2^2 \sin y_1 \cos y_1 \\ r_2 &= \ddot{y}_2 \cos y_1 - 2\dot{y}_1\dot{y}_2 \sin y_1, \end{aligned} \tag{4.7}$$

and atan2 is the "smart" arctangent function defined in (3.17). To finish, we differentiate (4.2) and (4.3a) to obtain expressions for u_1 and u_2 in terms of y and its derivatives. This proves that (4.1) is flat with flat output $y = (\theta, \psi)$ when $\alpha = -1$. ∎

The explicit expressions for the state x and input u in terms of y and its derivatives are

$$\omega_1 = \frac{\dot{r}_2 r_1 - \dot{r}_1 r_2}{r_1^2 + r_2^2} - \dot{y}_2 \sin y_1 \tag{4.8a}$$

$$\omega_2 = \frac{r_1 \dot{y}_1 + r_2 \dot{y}_2 \cos y_1}{(r_1^2 + r_2^2)^{\frac{1}{2}}} \tag{4.8b}$$

$$\omega_3 = \frac{r_1 \dot{y}_2 \cos y_1 - \dot{y}_1 r_2}{(r_1^2 + r_2^2)^{\frac{1}{2}}} \tag{4.8c}$$

$$\phi = \operatorname{atan2}(r_2, r_1) \tag{4.8d}$$

$$\theta = y_1 \tag{4.8e}$$

$$\psi = y_2 \tag{4.8f}$$

$$u_1 = \frac{(\ddot{r}_2 r_1 - \ddot{r}_1 r_2)(r_1^2 + r_2^2) - 2(r_1 \dot{r}_1 + r_2 \dot{r}_2)(\dot{r}_2 r_1 - \dot{r}_1 r_2)}{(r_1^2 + r_2^2)^2} - \ddot{y}_2 \sin y_1 - \dot{y}_2 \dot{y}_1 \cos y_1 \tag{4.8g}$$

$$u_2 = \frac{(r_1 \ddot{y}_1 - r_2 \dot{y}_2 \dot{y}_1 \sin y_1 + r_2 \ddot{y}_2 \cos y_1)(r_1^2 + r_2^2)}{(r_1^2 + r_2^2)^{\frac{3}{2}}} + \frac{(r_1 \dot{r}_2 - r_2 \dot{r}_1)(r_1 \dot{y}_2 \cos y_1 - r_2 \dot{y}_1)}{(r_1^2 + r_2^2)^{\frac{3}{2}}}, \tag{4.8h}$$

where

$$\dot{r}_1 = y_1^{(3)} + \dot{y}_2\ddot{y}_2 \sin(2\,y_1) + \dot{y}_2^2\dot{y}_1 \cos(2\,y_1)$$

$$\dot{r}_2 = (y_2^{(3)} - 2\dot{y}_2\dot{y}_1^2)\cos y_1 - (3\ddot{y}_2\dot{y}_1 + 2\dot{y}_2\ddot{y}_1)\sin y_1$$

$$\ddot{r}_1 = y_1^{(4)} + (\ddot{y}_2^2 + \dot{y}_2 y_2^{(3)} - 2\dot{y}_1^2\dot{y}_2^2)\sin(2\,y_1) + (4\dot{y}_1\dot{y}_2\ddot{y}_2 + \dot{y}_1^2\ddot{y}_1)\cos(2\,y_1)$$

$$\ddot{r}_2 = (2\dot{y}_2\dot{y}_1^3 - 4y_2^{(3)}\dot{y}_1 - 5\ddot{y}_2\ddot{y}_1 - 2\dot{y}_2 y_1^{(3)})\sin y_1 + (y_2^{(4)} - 5\ddot{y}_2\dot{y}_1^2 - 6\dot{y}_2\ddot{y}_1\dot{y}_1)\cos y_1.$$

From the proof of the previous proposition, it is not hard to see that any output of the form $y = h(\theta, \psi)$ where $(\theta, \psi) \mapsto h(\theta, \psi)$ is a diffeomorphism is also a flat output of system (4.1). Indeed, if h is invertible then there exists a map h^{-1} such that $(\theta, \psi) = h^{-1}(y_1, y_2)$. We can then perform the same procedure as in Proposition 4.1 with (θ, ψ) replaced by $h^{-1}(y_1, y_2)$.

When $\alpha = -1$ the principal inertias satisfy $I_1 + I_3 - I_2 = 0$. Using the formulas for the principal moments of inertia (3.10), it is straightforward to show that this condition is equivalent to

$$\int_V y^2 \rho(x, y, z)\, dV = 0. \tag{4.9}$$

Since the smooth function $f : V \to \mathbb{R}$ given by $f(x, y, z) = y^2\rho(x, y, z)$ is non-negative, (4.9) implies that the set

$$\{(x, y, z) \in V \mid f(x, y, z) \neq 0\} = \{(x, y, z) \in \mathbb{R}^3 \mid y \neq 0\} \cap \{(x, y, z) \in V \mid \rho(x, y, z) \neq 0\},$$

has (Lebesgue) measure zero (Rudin, 1987). This implies that $\{(x, y, z) \in V \mid \rho(x, y, z) \neq 0\}$ has measure zero. Geometrically, this situation corresponds to a two-dimensional body in space, and hence can be seen as the limiting case of a "real" three-dimensional body occupying a volume. Nonetheless, $\alpha = \pm 1$ is still a case of interest since it may be used to better understand the more general case $\alpha \neq \pm 1$.

Let $x = \mathcal{A}(y, \dot{y}, \ddot{y}, y^{(3)})$ be the map given by the right-hand side of (4.8a)-(4.8f) and $u = \mathcal{B}(y, \dot{y}, \ddot{y}, y^{(3)}, y^{(4)})$ the map given by the right hand side of

(4.8g)-(4.8h), that is,

$$\mathcal{A}(y,\dot{y},\ddot{y},y^{(3)}) = \begin{bmatrix} \frac{\dot{r}_2 r_1 - \dot{r}_1 r_2}{r_1^2+r_2^2} - \dot{y}_2 \sin y_1 \\ \frac{r_1\dot{y}_1 + r_2\dot{y}_2\cos y_1}{(r_1^2+r_2^2)^{\frac{1}{2}}} \\ \frac{r_1\dot{y}_2\cos y_1 - \dot{y}_1 r_2}{(r_1^2+r_2^2)^{\frac{1}{2}}} \\ \operatorname{atan2}(r_2, r_1) \\ y_1 \\ y_2 \end{bmatrix} \tag{4.10}$$

$$\mathcal{B}(\tilde{y}) = \begin{bmatrix} \frac{(\ddot{r}_2 r_1 - \ddot{r}_1 r_2)(r_1^2+r_2^2) - 2(r_1\dot{r}_1 + r_2\dot{r}_2)(\dot{r}_2 r_1 - \dot{r}_1 r_2)}{(r_1^2+r_2^2)^2} - \ddot{y}_2\sin y_1 - \dot{y}_2\dot{y}_1\cos y_1 \\ \frac{(r_1\ddot{y}_1 - r_2\dot{y}_2\dot{y}_1\sin y_1 + r_2\ddot{y}_2\cos y_1)(r_1^2+r_2^2) + (r_1\dot{r}_2 - r_2\dot{r}_1)(r_1\dot{y}_2\cos y_1 - r_2\dot{y}_1)}{(r_1^2+r_2^2)^{\frac{3}{2}}} \end{bmatrix}, \tag{4.11}$$

where $\tilde{y} = (y, \dot{y}, \ldots, y^{(4)})$. The maps $\mathcal{A}$ and $\mathcal{B}$ have a singularity when $r_1^2+r_2^2 = 0$. A straightforward calculation using the dynamic equations (4.1) shows that

$$r_1^2 + r_2^2 = (u_2 - \omega_1\omega_3)^2. \tag{4.12}$$

Therefore, $r_1^2+r_2^2 = 0$ if and only if $u_2 - \omega_1\omega_3 = 0$. Due to the importance of the expression $r_1^2 + r_2^2$ in subsequent analysis, we define the mapping $r : \mathbb{R}^2 \to \overline{\mathbb{R}}_+$ as

$$r(y) = r(y_1, y_2) = r_1^2 + r_2^2 = (\ddot{y}_1 + \dot{y}_2^2\cos y_1 \sin y_1)^2 + (\ddot{y}_2\cos y_1 - 2\dot{y}_1\dot{y}_2\sin y_1)^2. \tag{4.13}$$

Notice that when $\dot{y} = \ddot{y} = 0$ then $r(y_1, y_2) = 0$. This situation can occur, for instance, when the system is at an equilibrium.

4.2 Open-loop Motion Planning

In this section we consider the problem of designing an open-loop control that steers the flat underactuated spacecraft (4.1) ($\alpha = -1$) from an initial equilibrium $x_1 = (0, 0, 0, \phi_1, \theta_1, \psi_1)$ at time $t = t_1$ to a final equilibrium $x_2 = (0, 0, 0, \phi_2, \theta_2, \psi_2)$ at time $t = t_2$, that is, we want to solve the motion planning problem for the particular case of rest-to-rest motion. Since the system is flat, to solve the motion planning problem we must design the flat output reference trajectory $y_d(t)$ so that it satisfies

$$x_1 = \mathcal{A}\big(y_d(t_1), \dot{y}_d(t_1), \ddot{y}_d(t_1), y_d^{(3)}(t_1)\big)$$
$$x_2 = \mathcal{A}\big(y_d(t_2), \dot{y}_d(t_2), \ddot{y}_d(t_2), y_d^{(3)}(t_2)\big),$$

where $\mathcal{A}$ is given by (4.10). Once a reference trajectory is constructed satisfying the above constraints, the control steering the system from x_1 to x_2 will be given by

$$u(t) = \mathcal{B}\big(y_d(t), \dot{y}_d(t), \ddot{y}_d(t), y_d^{(3)}(t), y_d^{(4)}(t)\big),$$

where $\mathcal{B}$ is given by (4.11). If the maps $\mathcal{A}$ and $\mathcal{B}$ do not contain a singularity at the end-points, then it would be sufficient to only impose conditions on $y_d(t)$ up to its fourth order derivative. However, since the maps $\mathcal{A}$ and $\mathcal{B}$ are undefined when $r(y_d(t)) = 0$ (recall (4.13)), the flat output reference trajectory must also be designed so that when $r(y_d(t^*)) = 0$ for some $t^* \in [t_1, t_2]$, the maps $\mathcal{A}$ and $\mathcal{B}$ can be prolonged into sufficiently smooth mappings at t^*, for example, at least continuous. Furthermore, since the system will be steered from one equilibrium to another, for a smooth initial and final motion the control $u(t)$ must be zero at the times t_1 and t_2. More precisely, since $u(t) = 0$ for $t < t_1$ and $t > t_2$, we must set $u(t_1) = u(t_2) = 0$ to obtain continuous controls at t_1 and t_2. We could also impose the constraints $\dot{u}(t_1) = \dot{u}(t_2) = 0$ for an even smoother motion. These extra design constraints will impose extra conditions on $y_d(t)$. As will be shown, a de L'Hôpital's Rule analysis can be performed to yield conditions on $y_d(t)$ so that all the constraints are satisfied. The idea of using a de L'Hôpital's Rule analysis to construct an open-loop control the steers the spacecraft from rest-to-rest has been previously considered in the work of Adam (2004) and discussed in Oriolo et al. (2002) for the open-loop design of wheeled mobile robots. We take a similar approach as in Adam (2004) but obtain lower order conditions on $y_d(t)$ at the end-points.

We begin the design of the flat output reference trajectory, $y_d(t)$, by first considering the aforementioned conditions $u(t_1) = u(t_2) = 0$. As mentioned previously, these conditions are necessary so that the system undergoes a smooth motion from the initial equilibrium to the final equilibrium. From (4.8g)-(4.8h), we observe that if $r_1(t^*) = r_2(t^*) = 0$ for some $t^* \in [t_1, t_2]$, then the controls are undefined, specifically, the indeterminate form of the type $\frac{0}{0}$ results. If $\dot{y}_d(t_i) \neq 0$ and $\ddot{y}_d(t_i) \neq 0$, for $i = 1, 2$, then in general, $u(t_i) \neq 0$ from (4.8g)-(4.8h). However, if

$$\dot{y}_d(t_i) = \ddot{y}_d(t_i) = 0, \qquad \text{for } i = 1, 2, \tag{4.14}$$

then $r(y_d(t_i)) = 0$, for $i = 1, 2$, and consequently $u(t_i)$ will be undefined. In this case, we can perform a de L'Hôpital's rule analysis to design the higher order derivatives of $y_d(t)$ at t_1 and t_2 so that $\lim_{t \to t_i} u(t)$ exists and is equal to zero. To this end, let us first define

$$q_1 = (\ddot{r}_2 r_1 - \ddot{r}_1 r_2)(r_1^2 + r_2^2) - 2(r_1 \dot{r}_1 + r_2 \dot{r}_2)(\dot{r}_2 r_1 - \dot{r}_1 r_2)$$

$$q_2 = (r_1^2 + r_2^2)^2.$$

Assuming (4.14) holds then

$$\begin{aligned}
\lim_{t \to t_i} u_1(t) &= \lim_{t \to t_i} \left(\frac{q_1}{q_2} - \ddot{\psi} \sin\theta - \dot{\psi}\dot{\theta} \cos\theta \right) \\
&= \lim_{t \to t_i} \left(\frac{q_1}{q_2} \right) \\
&\overset{\text{L'HR}}{=} \lim_{t \to t_i} \left(\frac{q_1^{(4)}}{q_2^{(4)}} \right) \\
&= \lim_{t \to t_i} \left(\frac{2\left(\dot{r}_1^2 + \dot{r}_2^2\right)\left(r_2^{(3)}\dot{r}_1 - r_1^{(3)}\dot{r}_2\right)}{6\left(\dot{r}_1^2 + \dot{r}_2^2\right)^2} \right. \\
&\qquad \left. - \frac{3\left(r_2^{(2)}\dot{r}_1 - r_1^{(2)}\dot{r}_2\right)\left(\dot{r}_1 r_1^{(2)} + \dot{r}_2 r_2^{(2)}\right)}{6\left(\dot{r}_1^2 + \dot{r}_2^2\right)^2} \right),
\end{aligned}$$

where L'HR refers to de L'Hôpital's Rule for determining limits of the form $\frac{0}{0}$ (de L'Hôpital's Rule was applied four times). Substituting the expressions for $r_i^{(j)}$ for $i = 1, 2$ and $j = 1, 2, 3$ (see Appendix B), and with the assumption of

(4.14), we obtain

$$\lim_{t\to t_i} u_1(t) = \lim_{t\to t_i}\left(\frac{2\cos\theta\left(\psi^{(5)}\theta^{(3)}-\theta^{(5)}\psi^{(3)}\right)}{6\left(\left(\theta^{(3)}\right)^2+\left(\psi^{(3)}\cos\theta\right)^2\right)} - \frac{3\cos\theta\left(\psi^{(4)}\theta^{(3)}-\theta^{(4)}\psi^{(3)}\right)\left(\theta^{(3)}\theta^{(4)}+\psi^{(3)}\psi^{(4)}\right)}{6\left(\left(\theta^{(3)}\right)^2+\left(\psi^{(3)}\cos\theta\right)^2\right)^2}\right).$$

From the above equation we observe that by setting $y_d^{(3)}(t_i)\neq 0$, and $y_d^{(4)}(t_i) = y_d^{(5)}(t_i) = 0$, then the above limit exists, and is equal to zero. A similar analysis can be done for the control $u_2(t)$. For convenience set

$$q_3 = (r_1\ddot{\theta} - r_2\dot{\psi}\dot{\theta}\sin\theta + r_2\ddot{\psi}\cos\theta)(r_1^2+r_2^2) + (r_1\dot{r}_2 - r_2\dot{r}_1)(r_1\dot{\psi}\cos\theta - r_2\dot{\theta})$$

$$q_4 = (r_1^2+r_2^2)^{\frac{3}{2}}.$$

Then,

$$\begin{aligned}
\lim_{t\to t_i} u_2(t) &= \lim_{t\to t_i}\left(\frac{q_3}{q_4}\right) \\
&\overset{\text{L'HR}}{=} \lim_{t\to t_i}\left(\frac{q_3^{(2)}}{q_4^{(2)}}\right) \\
&= \frac{1}{3}\lim_{t\to t_i}\frac{q_3^{(2)}}{\frac{(r_1\dot{r}_1+r_2\dot{r}_2)^2+(r_1^2+r_2^2)(\dot{r}_1^2+\dot{r}_2^2+r_1\ddot{r}_1+r_2\ddot{r}_2)}{(r_1^2+r_2^2)^{\frac{1}{2}}}} \qquad \text{(expanding } q_4^{(2)}\text{)} \\
&= \frac{1}{3}\lim_{t\to t_i}\frac{q_3^{(2)}(r_1^2+r_2^2)^{\frac{1}{2}}}{(r_1\dot{r}_1+r_2\dot{r}_2)^2+(r_1^2+r_2^2)(\dot{r}_1^2+\dot{r}_2^2+r_1\ddot{r}_1+r_2\ddot{r}_2)} \\
&= \underbrace{\frac{1}{3}\left(\lim_{t\to t_3}\frac{q_3^{(2)}}{(r_1\dot{r}_1+r_2\dot{r}_2)^2+(r_1^2+r_2^2)(\dot{r}_1^2+\dot{r}_2^2+r_1\ddot{r}_1+r_2\ddot{r}_2)}\right)}_{A}\times \\
&\qquad \underbrace{\left(\lim_{t\to t_i}(r_1^2+r_2^2)^{\frac{1}{2}}\right)}_{B}.
\end{aligned}$$

The last equality holds if each limit A and B exist (limit law for products). It is clear that B exists and is equal to zero. To determine if A exists we first set

$$\tilde{q}_4 = (r_1\dot{r}_1+r_2\dot{r}_2)^2+(r_1^2+r_2^2)(\dot{r}_1^2+\dot{r}_2^2+r_1\ddot{r}_1+r_2\ddot{r}_2).$$

Then, assuming (4.14) holds and applying de L'Hôpital's Rule twice, we obtain

$$
\begin{aligned}
A &= \lim_{t\to t_i} \frac{q_3^{(2)}}{\tilde{q}_4} \\
&\overset{\text{L'HR}}{=} \lim_{t\to t_i} \frac{q_3^{(4)}}{\tilde{q}_4^{(2)}} \\
&= 2\lim_{t\to t_i} \frac{(\dot{r}_1)^3\theta^{(3)} + \dot{r}_1\theta^{(3)}(\dot{r}_2)^2 + \cos\theta\left((\dot{r}_2)^3\psi^{(3)} + \dot{r}_2\psi^{(3)}(\dot{r}_1)^2\right)}{(\dot{r}_1^2+\dot{r}_2^2)^2} \\
&= 2\lim_{t\to t_i} \frac{\left(\theta^{(3)}\right)^4 + 2\left(\theta^{(3)}\right)^2\left(\psi^{(3)}\cos\theta\right)^2 + \left(\psi^{(3)}\cos\theta\right)^4}{\left(\left(\theta^{(3)}\right)^2 + \left(\psi^{(3)}\cos\theta\right)^2\right)^2} \\
&= 2\lim_{t\to t_i} \frac{\left(\left(\theta^{(3)}\right)^2 + \left(\psi^{(3)}\cos\theta\right)^2\right)^2}{\left(\left(\theta^{(3)}\right)^2 + \left(\psi^{(3)}\cos\theta\right)^2\right)^2} \\
&= 2.
\end{aligned}
$$

Since A and B exist, and $B = 0$, then $\lim_{t\to t_i} u_2(t) = 0$. Therefore, if the flat output reference trajectory $y_d(t)$ satisfies (4.14), $y_d^{(3)}(t_i) \neq 0$, and $y_d^{(4)}(t_i) = y_d^{(5)}(t_i) = 0$, for $i = 1, 2$, then the control inputs $u_1(t)$ and $u_2(t)$ will tend to zero at $t = t_1, t_2$. We can therefore define the input,

$$
u(t) = \begin{cases} \mathcal{B}\left(y_d(t), \dot{y}_d(t), \ddot{y}_d(t), y_d^{(3)}(t), y_d^{(4)}(t)\right), & t \in (t_1, t_2) \\ 0, & t = t_1, t_2. \end{cases} \tag{4.15}
$$

By construction, the input (4.15) is continuous as $t \to t_1^+$ and as $t \to t_2^-$. If the reference trajectory is such that $r(y_d(t)) > 0$ for $t \in (t_1, t_2)$, that is the singularity is avoided, then the control (4.15) is smooth on (t_1, t_2).

Now we investigate the relationship between the initial and final configurations $x_1 = (0, 0, 0, \phi_1, \theta_1, \psi_1)$ and $x_2 = (0, 0, 0, \phi_2, \theta_2, \psi_2)$, respectively, with $y_d(t)$ and its derivatives at $t = t_i$ for $i = 1, 2$. For compatibility with the initial and final configuration of the spacecraft with those of the flat output components we must set

$$
y_d(t_1) = (\theta_1, \psi_1) \quad \text{and} \quad y_d(t_2) = (\theta_2, \psi_2). \tag{4.16}
$$

The conditions that must be imposed on $y_d(t)$ so that it is compatible with the known initial value ϕ_1 and the desired final value ϕ_2 can be determined using the expression relating ϕ with the flat output, namely (4.8d). However, since we have imposed the conditions $\dot{y}_d(t_i) = \ddot{y}_d(t_i) = 0$, we cannot use (4.8d) directly because we have an indeterminate form of the type $\frac{0}{0}$. Therefore, we must look at the limiting behaviour of ϕ and apply de L'Hôpital's Rule. For now, assume that $|\phi_i| < \frac{\pi}{2}$. Then,

$$\begin{aligned}
\lim_{t\to t_i} \phi &= \lim_{t\to t_i} \arctan\left(\frac{r_2}{r_1}\right) \\
&= \arctan\left(\lim_{t\to t_i} \frac{r_2}{r_1}\right) \\
&\overset{\text{L'HR}}{=} \arctan\left(\lim_{t\to t_i} \frac{\dot{r}_1}{\dot{r}_2}\right) \\
&= \arctan\left(\lim_{t\to t_i} \frac{\psi^{(3)}\cos\theta}{\theta^{(3)}}\right) \\
&= \arctan\left(\frac{\psi^{(3)}(t_i)\cos(\theta(t_i))}{\theta^{(3)}(t_i)}\right).
\end{aligned}$$

Hence, for compatibility with the initial value ϕ_1 and to assign the final value ϕ_2, it is necessary that for $i = 1, 2$,

$$\phi_i = \arctan\left(\frac{\psi^{(3)}(t_i)\cos(\theta(t_i))}{\theta^{(3)}(t_i)}\right),$$

or equivalently

$$\psi^{(3)}(t_i) = \frac{\theta^{(3)}(t_i)}{\cos(\theta(t_i))}\tan(\phi_i).$$

If $\frac{\pi}{2} \le |\phi_i| < \pi$ then the same type of calculation can be performed using the definition of atan2 given in (3.17). The resulting expression for $\psi^{(3)}(t_i)$ for all cases of $|\phi_i| < \pi$ is

$$\psi^{(3)}(t_i) = \begin{cases} \pm\delta, & \text{for } \phi_i = \pm\frac{\pi}{2} \text{ where } \delta > 0, \\ \frac{\theta^{(3)}(t_i)}{\cos(\theta(t_i))}\tan(\phi_i), & \text{otherwise.} \end{cases} \tag{4.17}$$

Although (4.17) is necessary for compatibility with the initial angle ϕ_1 and the reference trajectory, and for assigning the final value ϕ_2, we must also take care in choosing the sign of $\theta^{(3)}(t_i)$ for $i = 1, 2$. For instance, if $\phi_1 = \phi_2 = 45°$ then $\theta^{(3)}(t_1) > 0$ but $\theta^{(3)}(t_2) < 0$. The difference in the sign is because when the system arrives at the angle $\phi_2 = 45°$ it does so with direction pointing in the third-quadrant. Figure 4.1 shows the appropriate sign of $\theta^{(3)}(t_i)$ for the different values of ϕ_i corresponding to the four quadrants in $\mathbb{R}^2$.

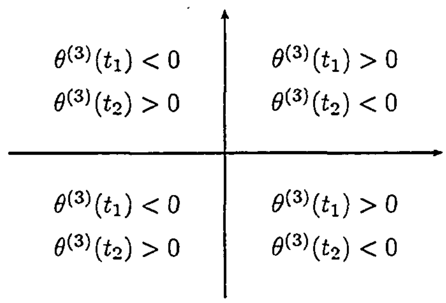


Figure. 4.1: Sign of $\theta^{(3)}(t_i)$ to ensure that ϕ_i is appropriately assigned, for $i = 1, 2$.

Finally, we must analyze the behaviour of $\omega_1, \omega_2, \omega_3$ in terms of the reference trajectory. Thus far, the conditions on the derivatives of $y_d(t)$ at $t = t_i$ are $y_d^{(j)}(t_i) = 0$ for $j = 1, 2, 4, 5$ and $y_d^{(3)}(t_i) \neq 0$, for $i = 1, 2$. Now, for ω_1 we have

$$\begin{aligned}
\lim_{t\to t_i} \omega_1(t) &= \lim_{t\to t_i} \left(\frac{\dot{r}_2 r_1 - \dot{r}_1 r_2}{r_1^2 + r_2^2} - \dot{y}_2 \sin y_1 \right) \\
&= \lim_{t\to t_i} \left(\frac{\dot{r}_2 r_1 - \dot{r}_1 r_2}{r_1^2 + r_2^2} \right) \\
&\overset{\text{L'HR}}{=} \lim_{t\to t_i} \left(\frac{\ddot{r}_2 r_1 - \ddot{r}_1 r_2}{2 r_1 \dot{r}_1 + 2 r_2 \dot{r}_2} \right) \\
&\overset{\text{L'HR}}{=} \lim_{t\to t_i} \left(\frac{\dddot{r}_2 \dot{r}_1 - \dddot{r}_1 \dot{r}_2}{2\dot{r}_1^2 + 2\dot{r}_2^2} \right) \\
&= \lim_{t\to t_i} \left(\frac{\cos(\theta)(\psi^{(4)}\theta^{(3)} - \theta^{(4)}\psi^{(3)})}{2(\theta^{(3)})^2 + 2(\psi^{(3)})^2} \right) \\
&= 0,
\end{aligned}$$

since $y_d^{(4)}(t_i) = 0$ and $y_d^{(3)}(t_i) \neq 0$ for $i = 1, 2$. Similarly, for ω_2 we have

$$\lim_{t\to t_i} \omega_2(t) = \lim_{t\to t_i} \left(\frac{r_1 \dot{y}_1 + r_2 \dot{y}_2 \cos y_1}{(r_1^2 + r_2^2)^{\frac{1}{2}}} \right)$$

$$\overset{\text{L'HR}}{=} \lim_{t\to t_i} \left(\frac{\dot{r}_1 \dot{y}_1 + r_1 \ddot{y}_1 \dot{r}_2 \dot{y}_2 \cos y_1 + r_2 \ddot{y}_2 \cos y_1 - \dot{y}_1 r_2 \dot{y}_2 \sin y_1}{\frac{1}{2}(2r_1\dot{r}_1 + 2r_2\dot{r}_2)^{-\frac{1}{2}}} \right)$$

$$= 2 \lim_{t\to t_i} \Big[(\dot{r}_1 \dot{y}_1 + r_1 \ddot{y}_1 \dot{r}_2 \dot{y}_2 \cos y_1 + r_2 \ddot{y}_2 \cos y_1 - \dot{y}_1 r_2 \dot{y}_2 \sin y_1) (2r_1\dot{r}_1 + 2r_2\dot{r}_2)^{\frac{1}{2}} \Big]$$

$$= 0,$$

since $\dot{y}_d(t_i) = \ddot{y}_d(t_i) = 0$ for $i = 1, 2$. Finally, for ω_3 we have

$$\lim_{t\to t_i} \omega_3(t) = \lim_{t\to t_i} \left(\frac{r_1 \dot{y}_1 \cos y_1 - \dot{y}_1 r_2}{(r_1^2 + r_2^2)^{\frac{1}{2}}} \right)$$

$$\overset{\text{L'HR}}{=} \lim_{t\to t_i} \left(\frac{\dot{r}_1 \dot{y}_1 \cos y_1 + r_1 \ddot{y}_1 \cos y_1 - r_1 \dot{y}_1^2 \sin y_1 - \ddot{y}_1 r_2 - \dot{y}_1 \dot{r}_2}{\frac{1}{2}(2r_1\dot{r}_1 + 2r_2\dot{r}_2)^{-\frac{1}{2}}} \right)$$

$$= 2 \lim_{t\to t_i} \Big[(\dot{r}_1 \dot{y}_1 \cos y_1 + r_1 \ddot{y}_1 \cos y_1 - r_1 \dot{y}_1^2 \sin y_1 - \ddot{y}_1 r_2 - \dot{y}_1 \dot{r}_2) (2r_1\dot{r}_1 + 2r_2\dot{r}_2)^{\frac{1}{2}} \Big]$$

$$= 0,$$

since $\dot{y}_d(t_i) = \ddot{y}_d(t_i) = 0$ for $i = 1, 2$. Therefore, if $y_d^{(j)}(t_i) = 0$ for $j = 1, 2, 4, 5$ and $y_d^{(3)}(t_i) \neq 0$, for $i = 1, 2$, then the spacecraft is steered from rest-to-rest.

In summary, to steer the spacecraft from an initial equilibrium $x_1 = (0, 0, 0, \phi_1, \theta_1, \psi_1)$ to a final equilibrium $x_2 = (0, 0, 0, \phi_2, \theta_2, \psi_2)$, a reference trajectory $y_d(t)$ can be designed satisfying the following conditions:

C1: $y_d(t_1) = (\theta_1, \phi_1)$ and $y_d(t_2) = (\theta_2, \phi_2)$

C2: $y_d^{(j)}(t_i) = 0$ for $j = 1, 2, 4, 5$ and $i = 1, 2$

C3: $y_d^{(3)}(t_i)$ must satisfy (4.17) and the sign of $\theta^{(3)}(t_i)$ must satisfy the conditions in Figure 4.1.

From C1-C3 we observe that there are a total of twelve conditions (six at each end-point of the motion) on each component of the flat output reference trajectory $y_d(t)$.

Now we design a flat output reference $y_d(t) = (y_{d,1}(t), y_{d,2}(t)) = (\theta_d(t), \psi_d(t))$ satisfying the conditions C1-C3 using the procedure discussed in §2.4.2. We choose a polynomial reference trajectory having the form

$$y_{d,i}(t) = \sum_{k=0}^{N-1} a_k^i t^k,$$

for $i = 1, 2$. To impose a square linear system of equations for the coefficients a_k^i we set $N = 12$. The resulting linear system of equations for $i = 1$ is

$$\left[\begin{array}{cccc|ccc} 1 & t_1 & \cdots & t_1^5 & t_1^6 & \cdots & t_1^{11} \\ 0 & 1!\binom{1}{1} & \cdots & 1!\binom{5}{1}t_1^4 & 1!\binom{6}{1}t_1^5 & \cdots & 1!\binom{11}{1}t_1^{10} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 5!\binom{5}{5} & 5!\binom{6}{5}t_1 & \cdots & 5!\binom{11}{5}t_1^6 \\ \hline 1 & t_2 & \cdots & t_2^5 & t_2^6 & \cdots & t_2^{11} \\ 0 & 1!\binom{1}{1} & \cdots & 1!\binom{5}{1}t_2^4 & 1!\binom{6}{1}t_2^5 & \cdots & 1!\binom{11}{1}t_2^{10} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 5!\binom{5}{5} & 5!\binom{6}{5}t_2 & \cdots & 5!\binom{11}{5}t_2^6 \end{array}\right] \begin{bmatrix} a_0^1 \\ a_1^1 \\ a_2^1 \\ \vdots \\ a_{10}^1 \\ a_{11}^1 \end{bmatrix} = \left[\begin{array}{c} \theta_1 \\ 0 \\ 0 \\ \theta_{1,d}^{(3)} \\ 0 \\ 0 \\ \hline \theta_2 \\ 0 \\ 0 \\ \theta_{2,d}^{(3)} \\ 0 \\ 0 \end{array}\right].$$

From Proposition 2.4.1, the resulting linear system of equations will have a unique solution for the a_k^i for $k = 0, \ldots, N-1$. Assuming $t_1 = 0$ (without loss of generality), the non-zero coefficients for the $y_{d,1}(t) = \theta_d(t)$ trajectory are

$$
\begin{aligned}
a_0^1 &= \theta_1 \\
a_3^1 &= \frac{1}{6}\,\theta_{1,d}^{(3)} \\
a_6^1 &= \frac{1}{t_2^6}\left(-\frac{28}{3}\,\theta_{1,d}^{(3)}t_2^3 + 462(\theta_2-\theta_1) - \frac{14}{3}\,\theta_{2,d}^{(3)}t_2^3\right) \\
a_7^1 &= \frac{1}{t_2^7}\left(35\,\theta_{1,d}^{(3)}t_2^3 + 1980(\theta_1-\theta_2) + 22\,\theta_{2,d}^{(3)}t_2^3\right) \\
a_8^1 &= \frac{1}{t_2^8}\left(-56\,\theta_{1,d}^{(3)}t_2^3 + 3465(\theta_2-\theta_1) - \frac{83}{2}\,\theta_{2,d}^{(3)}t_2^3\right) \\
a_9^1 &= \frac{1}{t_2^9}\left(\frac{140}{3}\,\theta_{1,d}^{(3)}t_2^3 + 3080(\theta_1-\theta_2) + \frac{235}{6}\,\theta_{2,d}^{(3)}t_2^3\right) \\
a_{10}^1 &= \frac{1}{t_2^{10}}\left(-20\,\theta_{1,d}^{(3)}t_2^3 + 1386(\theta_2-\theta_1) - \frac{37}{2}\,\theta_{2,d}^{(3)}t_2^3\right) \\
a_{11}^1 &= \frac{1}{t_2^{11}}\left(\frac{7}{2}\,\theta_{1,d}^{(3)}t_2^3 + 252(\theta_1-\theta_2) + \frac{7}{2}\,\theta_{2,d}^{(3)}t_2^3\right),
\end{aligned}
\tag{4.18}
$$

where $\theta_{i,d}^{(3)} = \theta_d^{(3)}(t_i)$ for $i = 1, 2$. Replacing θ with ψ in the above expressions for the coefficients gives the non-zero coefficients for the ψ trajectory.

4.3 Simulation: Open-loop Motion Planning

In this section, we simulate the open-loop control, designed using the algorithm of the previous section (§4.2), that steers the spacecraft from rest-to-rest starting from the initial orientation $\phi_1 = 120°$, $\theta_1 = 80°$, $\psi_1 = -120°$ to the final orientation $\phi_2 = -120°$, $\theta_2 = -80°$, $\psi_2 = 120$. Setting $\theta_d^{(3)}(t_1) = -3.5$ and $\theta_d^{(3)}(t_2) = 0.2$ results in (via equation (4.17)) $\psi_d^{(3)}(t_1) = 34.9$ and $\psi_d^{(3)}(t_2) = 2.0$. Although there are an infinite number of choices for $\theta^{(3)}(t_i)$ and $\psi^{(3)}(t_i)$, this choice ensures that the singularity $r(y_d(t)) = 0$ is not reached in the interval (t_1, t_2) (see §4.4). The reference trajectory $y_d(t)$ that results from this choice of parameters is shown in Figure 4.2, and it satisfies $r(y_d(t)) > 0$ for $t \in (t_1, t_2)$, as shown in Figure 4.3. Using the constructed flat output reference trajectory one can then use (4.8g)-(4.8h) to obtain the open-loop controls that steer the system from the given orientation to the final desired orientation. The result of an open-loop simulation is shown in Figure 4.4. As one can observe, the open-loop control steers the system from the initial to the final equilibrium.

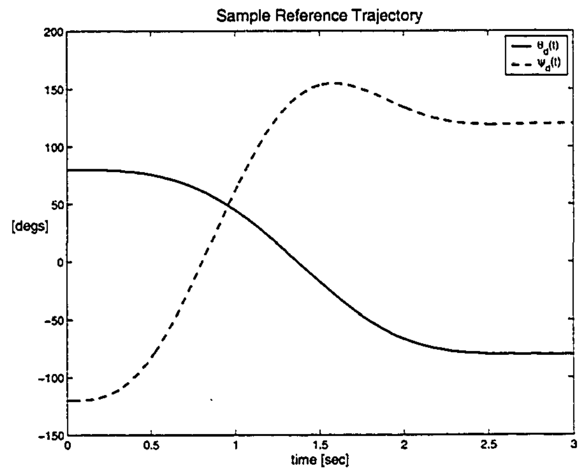


Figure. 4.2: Flat output reference trajectory for parameter values $\theta_d^{(3)}(t_1) = -3.5$, $\theta_d^{(3)}(t_2) = 0.2$, $\psi_d^{(3)}(t_1) = 34.9$ and $\psi_d^{(3)}(t_2) = 2.0$.

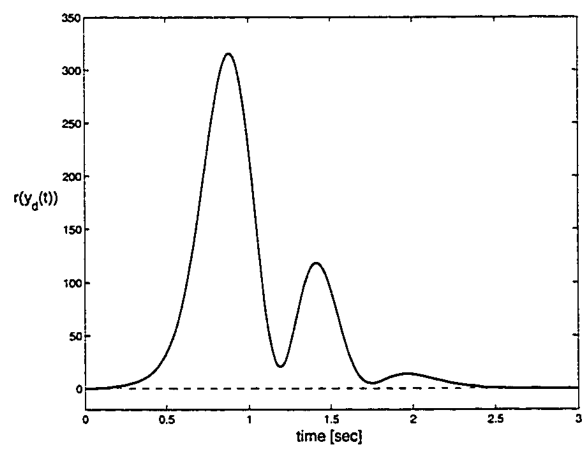


Figure. 4.3: Singularity $r(y_d(t))$ for parameter values $\theta_d^{(3)}(t_1) = -3.5$, $\theta_d^{(3)}(t_2) = 0.2$, $\psi_d^{(3)}(t_1) = 34.9$ and $\psi_d^{(3)}(t_2) = 2.0$, the trajectory $y_d(t)$ satisfies $r(y_d(t)) > 0$ for $t \in (t_1, t_2)$.

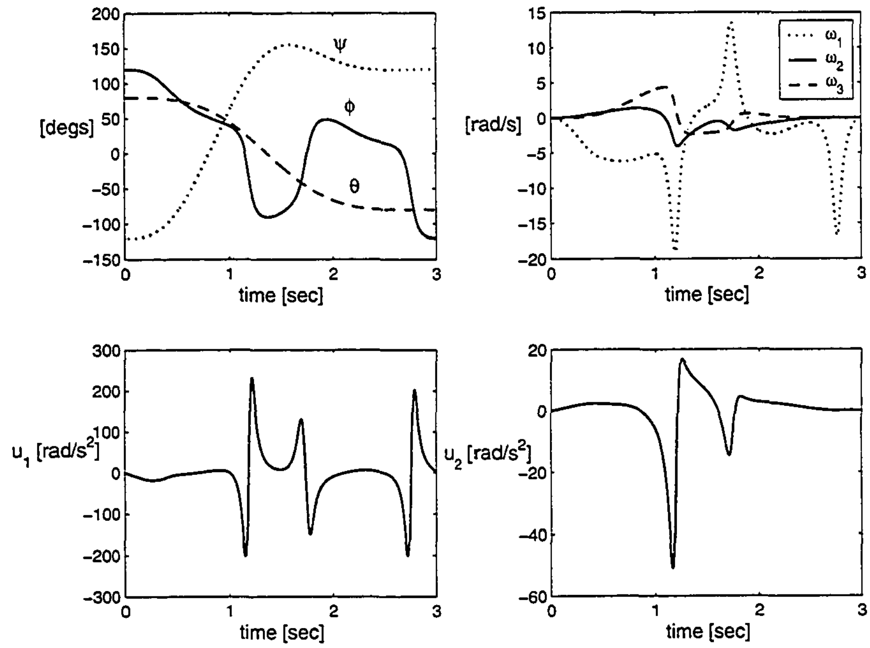


Figure. 4.4: Open-loop control: Initial angles are $\phi_1 = 120°, \theta_1 = 80°, \psi_1 = -120°$, final angles are $\phi_2 = -120°, \theta_2 = -80°, \psi_2 = 120°$, and trajectory parameters are $y_d(t_1) = (-3.5, 34.9)$ and $y_d(t_2) = (0.2, 2.0)$.

In practice, the performance of the open-loop control will be reduced by external disturbances, modeling errors, and the uncertainty of the initial condition. For instance, suppose the actual initial condition of the spacecraft is such that $\phi_1 = 125°$, $\theta_1 = 85°$, and $\psi_1 = -115°$, instead of the assumed initial conditions $\phi_{d,1} = 120°$, $\theta_{d,1} = 80°$, and $\psi_{d,1} = -120°$. Then, applying the same open-loop control yields a final orientation $\phi_2 = -150°$, $\theta_2 = -81°$, and $\psi_2 = 150°$, as shown in Figure 4.5. The error in the final Euler angles are $\phi_2 - \phi_{d,2} = -30°, \theta_2 - \theta_{d,2} = 1°, \psi_2 - \psi_{d,2} = 30°$. Thus, if an open-loop control is used, a small initial deviation in the orientation can cause a large deviation from the desired orientation. However, the open-loop control design is important because it generates a feasible state trajectory from one equilibrium to another and constructs the corresponding nominal control. Furthermore, the resulting reference trajectory can be used in a feedback trajectory tracking controller.

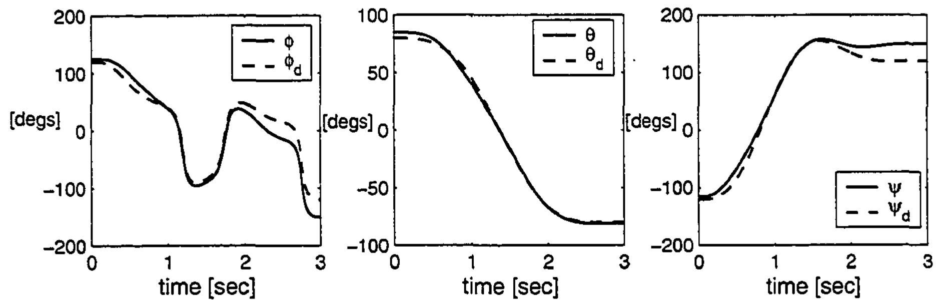


Figure. 4.5: Open-loop control with initial state error. Error in final angles are $\phi_2 - \phi_{d,2} = -30°$,$\theta_2 - \theta_{d,2} = 1°$,$\psi_2 - \psi_{d,2} = 30°$.

4.4 Avoiding the Singularity

In the open-loop design of the previous section, the reference trajectory $y_d(t)$ was designed so that

$$r(y_d(t)) > 0, \quad \text{for } t \in (t_1, t_2). \tag{4.19}$$

Condition (4.19) guarantees that the open-loop controls are smooth functions of time for $t \in (t_1, t_2)$. Condition (4.19) was satisfied by appropriately assigning the values

$$y_d^{(3)}(t_1) = (\theta_d^{(3)}(t_1), \psi_d^{(3)}(t_1))$$
$$y_d^{(3)}(t_2) = (\theta_d^{(3)}(t_2), \psi_d^{(3)}(t_2)).$$

Indeed, from (4.18) we observe that once the initial and final angle of θ are given, the values of $y_d^{(3)}(t_1)$ and $y_d^{(3)}(t_2)$ completely determine the reference trajectory and thus determine whether (4.19) is satisfied. Since $\psi_d^{(3)}(t_i)$ is determined by $\theta_d^{(3)}(t_i)$ via equation (4.17), condition (4.19) really only depends on two parameters, namely, $\theta_d^{(3)}(t_1)$ an $\theta_d^{(3)}(t_2)$. Ideally, one would like to know in advance if there exists values for $\theta_d^{(3)}(t_1)$ and $\theta_d^{(3)}(t_2)$ so that (4.19) is satisfied. A direct approach to solving this problem would involve determining how $\theta_d^{(3)}(t_i)$ affects the solution of the nonlinear equation

$$r(y_d(t)) = \left(\ddot{y}_{d,1}(t) + \dot{y}_{d,2}^2(t)\sin(y_{d,1}(t))\cos(y_{d,1}(t))\right)^2 + \left(\cos(y_{d,1}(t))\ddot{y}_{d,2}(t) - 2\dot{y}_{d,1}(t)\dot{y}_{d,2}(t)\sin(y_{d,1}(t))\right)^2 = 0 \tag{4.20}$$

for $t \in (t_1, t_2)$. Once a solution t^* for (4.20) is found (if it exists at all), we can assign $\theta_d^{(3)}(t_1)$ and $\theta_d^{(3)}(t_2)$ so that $t^* \notin (t_1, t_2)$, and thus ensure that (4.19) is satisfied. However, the complexity of equation (4.20) is such that a closed form solution cannot be found, and therefore, we are forced to consider numerical methods to obtain values for $\theta_d^{(3)}(t_1)$ and $\theta_d^{(3)}(t_2)$. Let

$$z = \left(\theta_d^{(3)}(t_1), \theta_d^{(3)}(t_2), \psi_d^{(3)}(t_1), \psi_d^{(3)}(t_2)\right),$$

and $\epsilon > 0$ be a small positive number. The problem of avoiding the singularity is solved if we can find a point z which belongs to the set

$$\Omega = \left\{z \in \mathbb{R}^4 \mid C_{\text{eq}}(z) = 0,\ \Phi(z,t) \leq 0 \text{ for } t \in (t_1, t_2)\right\},$$

where C_{eq} is the relationship between $\theta^{(3)}(t_i)$ and $\psi^{(3)}(t_i)$ given by (4.17), $\Phi(z,t) = \epsilon - r(y_d(t))$, where $y_d(t)$ is written in terms of the coefficients (4.18). Finding a point z in Ω will ensure that the singularity is not hit. To obtain a point $z \in \Omega$, we employ the Matlab command `fseminf` which solves a semi-infinite constrained optimization problem. To be precise, `fseminf` solves the constrained optimization problem

$$\min_z \left\{F(z) \mid C_{\text{eq}}(z) = 0,\ \Phi(z,t) \leq 0 \text{ for } t \in (t_1, t_2)\right\},$$

where F is some function we wish to minimize, $C_{\text{eq}}(z) = 0$ is a constraint on z, and $\Phi(z,t)$ is a time-varying constraint on z. In our case, the function F can be chosen to be a constant since we are just concerned with finding a point z satisfying the constraints and not necessarily minimizing a certain cost function. We can thus set $F(z) = 1$ for simplicity. Note that once a point $z^* \in \Omega$ is found (assuming Ω is non-empty), then any other point arbitrarily close to z^* and in Ω also solves the singularity problem since $C_{\text{eq}}(z)$ and $\Phi(z,t)$ are continuous functions of z. This allows some flexibility in designing the flat output reference trajectory while remaining away from the singularity. For the specific trajectory considered in §4.3, the solution to the constrained optimization problem returned the value

$$z^* = (\theta_d^{(3)}(t_1), \theta_d^{(3)}(t_2), \psi_d^{(3)}(t_1), \psi_d^{(3)}(t_2)) = (-3.5, 0.2, 34.9, 2.0),$$

which was the point used in the previous section.

As mentioned at the beginning of this chapter, the presence of singularities in the expressions for the state and input of a flat system is common. Presently, there is no general method to avoid the singularities in the open-loop design and thus one must study each case separately.

Chapter 5

Flatness-based Closed-loop Trajectory Tracking

In this chapter we design a trajectory tracking controller steering the two-input spacecraft asymptotically to a desired equilibrium. The first phase of the controller consists in designing a flat output tracking controller using the *Dynamic Extension Algorithm* (DEA) of Zhan et al. (1991). The dynamic controller brings the spacecraft to an arbitrarily small neighbourhood of a desired final equilibrium in finite time. Once the system has reached a neighbourhood of the final equilibrium, the control law is switched from the dynamic controller to the time-varying controller of Morin and Samson (1997). The two-phase controller yields an asymptotic tracking controller for large motions of the spacecraft. The flatness property of the spacecraft allows us to prove that the two-phase controller asymptotically tracks the state trajectory. In the feedback control design we assume that the Euler angles and angular velocities are measured.

5.1 Dynamic Feedback Linearization of the Underactuated Spacecraft

In many cases, a system may fail to have a well-defined relative degree at a point x_0 (§2.3.1), which occurs when the decoupling matrix is singular at x_0. In this case the Dynamic Extension Algorithm (DEA) can be used to modify the original system with a dynamic state feedback so that the extended system has a well-defined relative degree and possibly, after a number of iterations of the DEA, the extended system is feedback linearizable. Unfortunately, there is no guarantee that the DEA will be successful in producing a feedback linearizable system, and thus one can only hope that the DEA will succeed. Even for a flat system the DEA may not be successful even though a dynamic feedback does indeed exist (Theorem §2.4.1). From Theorem 3.3.1 we know that it will not be possible to dynamic feedback linearize the underactuated spacecraft about x_0 using the DEA (or using any other method that constructs a dynamic feedback). Indeed, either the algorithm will yield a system with the sum of the vector relative degree not equal to the dimension of the extended state or the decoupling matrix of the extended system will be singular at the equilibrium of interest. However, as will be shown, it is still possible to apply the DEA on the spacecraft to construct a dynamic feedback that is well-defined away from the desired equilibrium.

5.1.1 Dynamic Extension Algorithm (DEA)

Consider the control-affine system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i(x), \quad x \in \mathsf{M} \subset \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\ y &= h(x), \end{aligned} \tag{5.1}$$

where M is an open set and $h : \mathsf{M} \to \mathbb{R}^m$ is a smooth map. Suppose that the decoupling matrix $A(x)$ has constant rank in a neighbourhood of $x_0 \in \mathsf{M}$ and that it is strictly less than m (the number of inputs). Denote $a_i(x)$, for $i = 1, \dots, m$, the ith row of the matrix $A(x)$. Then, because $A(x)$ is singular, there exist a row of $A(x)$ that can be written as a linear combination of other rows of $A(x)$. By possibly rearranging the outputs, we can assume that it is the pth row which can be written as a linear combination of the first $p-1$ rows. More precisely, there exists $p-1$ smooth functions $c_1(x), \dots, c_{p-1}(x)$ such that

$$a_p(x) = \sum_{i=1}^{p-1} c_i(x)a_i(x).$$

Also, we can assume that there exists integers i_0 and j_0 such that $c_{i_0}(x)$ is not identically zero and

$$a_{i_0 j_0}(x_0) = L_{g_{j_0}} L_f^{r_{i_0}-1} h_{i_0}(x_0) \neq 0.$$

Define the dynamic state feedback

$$\begin{aligned} u_j &= v_j, \quad \text{for } j \neq j_0 \\ u_{j_0} &= \frac{1}{a_{i_0 j_0}(x)} \Big(p(x) + q(x)\xi - \sum_{\substack{j=1 \\ j \neq j_0}}^{m} a_{i_0 j}(x) v_j \Big) \\ \dot{\xi} &= v_{j_0}, \end{aligned} \tag{5.2}$$

where $p(x)$ and $q(x)$ are any smooth functions such that $p(x_0) = 0$ and $q(x_0) = 1$. Applying the dynamic feedback (5.2) to the original system (5.1) defines a new system with state $(x, \xi) \in \mathsf{M} \times \mathbb{R}$ and input $v = (v_1, \ldots, v_m)$. If the decoupling matrix of the extended system has constant rank and strictly less than m, then re-iterate the procedure on the extended system, otherwise the algorithm stops. Note that since $p(x_0) = 0$, the point $(x_0, 0)$ is an equilibrium of the extended system after applying the DEA. Also, the two functions p and q can be chosen to simplify the structure of the composite system and can affect the possibility of continuing the algorithm. Below we give an example illustrating the algorithm.

Example 5.1.1 (Nijmeijer and van der Shaft (1990)). Consider the system

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2 \\ y &= h(x) \end{aligned} \tag{5.3}$$

where $f(x) = (0, x_3, 0)$, $g_1(x) = (1, e^{x_1}, 0)$, $g_2(x) = (0, 0, 1)$, and $h(x) = (x_1, x_2)$. The decoupling matrix of the system is

$$A(x) = \begin{bmatrix} L_{g_1} h_1 & L_{g_2} h_1 \\ L_{g_1} h_2 & L_{g_2} h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{x_1} & 0 \end{bmatrix},$$

which has constant rank one for all $x \in \mathbb{R}^3$. Therefore, the system does not have a well-defined relative degree. Now consider applying the DEA with

$p(x) = 0$, $q(x) = e^{x_1}$, $i_0 = 2$ and $j_0 = 1$. The resulting dynamic state feedback is

$$\begin{aligned} \dot{\xi} &= v_1 \\ u_1 &= \xi \\ u_2 &= v_2. \end{aligned} \tag{5.4}$$

Applying (5.4) to (5.3) yields a system with state $\tilde{x} = (x, \xi) \in \mathbb{R}^4$, drift vector field $\tilde{f}(\tilde{x}) = (\xi, e^{x_1}\xi + x_3, 0, 0)$, and control vector fields $\tilde{g}_1(\tilde{x}) = (0, 0, 0, 1)$ and $\tilde{g}_2(\tilde{x}) = (0, 0, 1, 0)$. The decoupling matrix of the extended system is

$$\tilde{A}(\tilde{x}) = \begin{bmatrix} L_{\tilde{g}_1} L_{\tilde{f}} h_1 & L_{\tilde{g}} L_{\tilde{f}} h_1 \\ L_{\tilde{g}_1} L_{\tilde{f}} h_2 & L_{\tilde{g}} L_{\tilde{f}} h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{x_1} & 1 \end{bmatrix},$$

which has full rank for all $(x, \xi) \in \mathbb{R}^4$. Therefore, applying the input (2.7) to the extended system input-output linearizes the original system (5.4) with a dynamic feedback. □

Suppose that the DEA has been iterated q times to yield an extended system with state dimension $n+q$ and that the relative degree of the extended system $\{\tilde{r}_1, \ldots, \tilde{r}_m\}$ satisfies

$$\sum_{i=1}^{m} \tilde{r}_i = n + q. \tag{5.5}$$

Then the extended system can be completely linearized using static feedback, or equivalently, the original system can be completely linearized using dynamic state feedback.

5.1.2 DEA Applied to the Two-input Spacecraft

In this section we apply the DEA to the underactuated spacecraft to construct a dynamic feedback that tracks a flat output reference trajectory. Recall that the dynamic equations for the two-input spacecraft using a 3-2-1 Euler angles parameterization are

$$\dot{\omega}_1 = u_1 \tag{5.6a}$$

$$\dot{\omega}_2 = u_2 \tag{5.6b}$$

$$\dot{\omega}_3 = \alpha\omega_1\omega_2 \tag{5.6c}$$

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \tag{5.6d}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \tag{5.6e}$$

$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta. \tag{5.6f}$$

Let $\Sigma = (\mathsf{M}, \mathscr{C} = \{f, g_1, g_2\}, \mathbb{R}^2)$ denote the control-affine system (5.6), where f denotes the drift vector field, g_1 and g_2 the control vector fields, and $x = (\omega_1, \omega_2, \omega_3, \phi, \theta, \psi)$ the state. Recall that the dynamics (5.6) are defined on the open set

$$\mathsf{M} = \left\{x \in \mathbb{R}^6 \mid \omega_1, \omega_2, \omega_2 \in \mathbb{R}, \psi, \phi \in (-\pi, \pi), \theta \in (-\pi/2, \pi/2)\right\}.$$

We take as output the flat output

$$y = h(x) = (\theta, \psi). \tag{5.7}$$

The decoupling matrix of system (5.6) with output (5.7) is

$$A(x) = \begin{bmatrix} L_{g_1}L_f h_1 & L_{g_2}L_f h_1 \\ L_{g_1}L_f h_2 & L_{g_2}L_f h_2 \end{bmatrix} = \begin{bmatrix} 0 & \cos\phi \\ 0 & \frac{\sin\phi}{\cos\theta} \end{bmatrix},$$

which has constant rank one for all $x \in \mathsf{M}$, and thus the spacecraft dynamics with output (5.7) does not have a well-defined relative degree at any $x \in \mathsf{M}$, and in particular at any equilibrium x_0. The hypothesis for the DEA are satisfied and hence we apply the algorithm to the spacecraft.

1^{st} **Iteration:** The second row of the decoupling matrix $A(x)$ can be written as a linear combination of the first row, with the coefficient $c_1(x) = \frac{\sin\phi}{\cos\phi\cos\theta}$ which is not identically zero. Also, the a_{12} entry of $A(x)$ satisfies $a_{12}(x_0) \neq 0$. Let $p_1(x)$ and $q_1(x)$ be any two smooth functions satisfying the hypothesis of the DEA. Then the feedback (5.2) is

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= \frac{1}{\cos\phi}(p_1(x) + q_1(x)\xi_1) \\ \dot{\xi}_1 &= v_2, \end{aligned} \tag{5.8}$$

where v_1 and v_2 are the new inputs and ξ_1 the compensator state. We can make a simplification to the feedback (5.8) by cancelling the $\cos\phi$ term by setting $p_1(x) = \tilde{p}(x)\cos\phi$ and $q_1(x) = \tilde{q}(x)\cos\phi$, where $\tilde{p}$ and $\tilde{q}$ are smooth functions. Let $\tilde{\Sigma}$ denote the extended control-affine system with state $\tilde{x} = (x, \xi_1) \in \mathsf{M} \times \mathbb{R}$, obtained by applying (5.8) to the original system Σ. A straightforward calculation shows that the decoupling matrix for $\tilde{\Sigma}$ is

$$A_1(\tilde{x}) = \begin{bmatrix} L_{\tilde{g}_1}L_{\tilde{f}}^2 h_1 & L_{\tilde{g}_2}L_{\tilde{f}}^2 h_1 \\ L_{\tilde{g}_1}L_{\tilde{f}}^2 h_2 & L_{\tilde{g}_2}L_{\tilde{f}}^2 h_2 \end{bmatrix} = \begin{bmatrix} \cos\phi\left(\frac{\partial\tilde{p}}{\partial\omega_1} + \frac{\partial\tilde{q}}{\partial\omega_1}\xi_1 - \omega_3\right) & \tilde{q}(x)\cos\phi \\ \frac{\sin\phi}{\cos\theta}\left(\frac{\partial\tilde{p}}{\partial\omega_1} + \frac{\partial\tilde{q}}{\partial\omega_1}\xi_1 - \omega_3\right) & \tilde{q}(x)\frac{\sin\phi}{\cos\theta} \end{bmatrix}.$$

The matrix $A_1(\tilde{x})$ has constant rank one for all $x \in \mathsf{M}$ and thus we can perform another iteration of the DEA. However, we can simplify the design of the dynamic feedback by designing the functions $\tilde{p}$ and $\tilde{q}$ such that

$$\frac{\partial\tilde{p}}{\partial\omega_1} + \frac{\partial\tilde{q}}{\partial\omega_1}\xi_1 - \omega_3 = 0.$$

Indeed, setting

$$\begin{aligned} \tilde{p}(x) &= \omega_1\omega_3 \\ \tilde{q}(x) &= \cos\theta \end{aligned} \tag{5.9}$$

yields the decoupling matrix

$$A_1(\tilde{x}) = \begin{bmatrix} 0 & \cos\phi\cos\theta \\ 0 & \sin\phi \end{bmatrix}. \tag{5.10}$$

The matrix $A_1(\tilde{x})$ has constant rank one for all $x \in \mathsf{M}$ and therefore we can re-iterate the DEA on the extended system $\tilde{\Sigma}$.

2nd Iteration: The second row of $A_1(\tilde{x})$ can be written as a linear combination of the first row, with coefficient $c_1(x) = \frac{\sin\phi}{\cos\phi\cos\theta}$ which is not identically zero. Also, the $\tilde{a}_{12}$ entry of $A_1(\tilde{x})$ satisfies $\tilde{a}_{12}(x_0, \xi_0) \neq 0$. Let $p_2(\tilde{x})$ and $q_2(\tilde{x})$ be two smooth functions satisfying the hypothesis of the DEA. Then the feedback (5.2) is

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= \frac{1}{\cos\phi\cos\theta}(p_2(\tilde{x}) + q_2(\tilde{x})\xi_2) \\ \dot{\xi}_2 &= w_2, \end{aligned} \tag{5.11}$$

where w_1 and w_2 are the new inputs and ξ_2 is the compensator state. Similar to the first iteration, we can simplify the control (5.11) by cancelling the $\cos\phi\cos\theta$ term by setting $p_2(\tilde{x}) = \bar{q}(\bar{x})\cos\phi\cos\theta$ and $q_2(\tilde{x}) = \bar{p}(\bar{x})\cos\phi\cos\theta$, where $\bar{p}$ and $\bar{q}$ are smooth functions. Let $\bar{\Sigma}$ denote the extended control-affine system with state $\bar{x} = (x, \xi_1, \xi_2) \in \mathsf{M} \times \mathbb{R}^2$, obtained by applying (5.11) to the system $\tilde{\Sigma}$. The decoupling matrix for $\bar{\Sigma}$ is

$$A_2(\bar{x}) = \begin{bmatrix} L_{\bar{g}_1} L_{\bar{f}}^3 h_1 & L_{\bar{g}_2} L_{\bar{f}}^3 h_1 \\ L_{\bar{g}_1} L_{\bar{f}}^3 h_2 & L_{\bar{g}_2} L_{\bar{f}}^3 h_2 \end{bmatrix} = \begin{bmatrix} \cos\theta\left(\frac{\partial\bar{p}}{\partial\omega_1}\cos\phi - \xi_1\sin\phi + \xi_2\frac{\partial\bar{q}}{\partial\omega_1}\cos\phi\right) & \bar{q}(\bar{x})\cos\phi\cos\theta \\ \frac{\partial\bar{p}}{\partial\omega_1}\sin\phi + \xi_1\cos\phi + \xi_2\frac{\partial\bar{q}}{\partial\omega_1}\sin\phi & \bar{q}(\bar{x})\sin\phi \end{bmatrix}.$$

A straightforward calculation shows that

$$\det A_2(\bar{x}) = -\xi_1\bar{q}(\bar{x})\,\cos\theta.$$

Therefore, A_2 does not have constant rank in a neighbourhood of the equilibrium $(x_0, 0, 0) \in \mathsf{M} \times \mathbb{R}^2$ for any choice of the functions $\bar{q}$ and $\bar{p}$. Consequently, we cannot continue applying the DEA to the extended system $\bar{\Sigma}$. However, away from the set

$$\Lambda = \{(x, \xi_1, \xi_2) \in \mathsf{M} \times \mathbb{R}^2 \mid \xi_1 = 0\}, \tag{5.12}$$

the decoupling matrix $A_2(\bar{x})$ is nonsingular and the extended system $\bar{\Sigma}$ has a well-defined vector relative degree $\{\bar{r}_1, \bar{r}_2\} = \{4, 4\}$. Since $\bar{r}_1 + \bar{r}_2 = 8$, the composite system $\bar{\Sigma}$ is static state feedback linearizable on the set $(\mathsf{M} \times \mathbb{R}^2)\backslash\Lambda$. Now, if we set $\bar{q}(\bar{x}) = 1$ and $\bar{p}(\bar{x}) = 0$ then the resulting dynamic feedback constructed by the DEA is

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= w_2 \\
u_1 &= w_1 \\
u_2 &= \omega_1\omega_3 + \xi_1 \cos\theta.
\end{aligned} \tag{5.13}$$

Applying (5.13) to the original system (5.6) yields the control-affine system

$$\dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{g}_1 w_1 + \bar{g}_2 w_2, \tag{5.14}$$

where

$$\bar{f}(\bar{x}) = \begin{bmatrix} 0 \\ \omega_1\omega_3 + \xi_1 \cos\theta \\ \alpha\omega_1\omega_2 \\ \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \\ \omega_2 \cos\phi - \omega_3 \sin\phi \\ (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta \\ \xi_2 \\ 0 \end{bmatrix}, \bar{g}_1(\bar{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \bar{g}_2(\bar{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By construction of the dynamic feedback (5.13), we obtain that

$$\begin{bmatrix} y_1^{(4)} \\ y_2^{(4)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_{\bar{f}}^4 h_1(\bar{x}) \\ L_{\bar{f}}^4 h_2(\bar{x}) \end{bmatrix}}_{b(\bar{x})} + A_2(\bar{x}) w.$$

The feedback linearizing the input-output dynamics is

$$w = A_2^{-1}(\bar{x})(-b(\bar{x}) + v) = \begin{bmatrix} -\frac{\sin\phi}{\xi_1 \cos\theta} & \frac{\cos\phi}{\xi_1} \\ \frac{\cos\phi}{\cos\theta} & \sin\phi \end{bmatrix} \left(- \begin{bmatrix} L_{\bar{f}}^4 h_1(\bar{x}) \\ L_{\bar{f}}^4 h_2(\bar{x}) \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right), \tag{5.15}$$

where $v = (v_1, v_2)$ is an auxiliary input. Notice that the dynamic feedback is undefined when $\xi_1 = 0$.

The coordinate transformation $z = \Phi(x, \xi)$ putting the system into a linear controllable one is

$$z_1 = h_1 = \theta \tag{5.16a}$$

$$z_2 = L_{\bar{f}} h_1 = \omega_2 \cos\phi - \omega_3 \sin\phi \tag{5.16b}$$

$$z_3 = L_{\bar{f}}^2 h_1 = -\tan\theta(\omega_2 \sin\phi + \omega_3 \cos\phi)^2 + \xi_1 \cos\phi\cos\theta \tag{5.16c}$$

$$z_4 = L_{\bar{f}}^3 h_1 \tag{5.16d}$$

$$z_5 = h_2 = \psi \tag{5.16e}$$

$$z_6 = L_{\bar{f}} h_2 = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta \tag{5.16f}$$

$$z_7 = L_{\bar{f}}^2 h_2 = 2\tan\theta\,(\omega_2\cos\phi - \omega_3\sin\phi)(\omega_2\sin\phi + \omega_3\cos\phi)\sec\theta + \xi_1\sin\phi \tag{5.16g}$$

$$z_8 = L_{\bar{f}}^3 h_2, \tag{5.16h}$$

where $L_{\bar{f}}^3 h_1$ and $L_{\bar{f}}^3 h_2$ have been suppressed as their exact expressions are not important. Notice that the new coordinates are given by the flat output and its derivatives. A straightforward calculation shows that

$$\det \frac{\partial \Phi}{\partial(x,\xi)} = -\xi_1 \cos\theta,$$

and therefore $\Phi(x,\xi)$ is locally invertible on the set $(\mathsf{M}\times\mathbb{R}^2)\backslash\Lambda$.

To summarize, the dynamic feedback (5.13), (5.15) and transformation (5.16) linearize the underactuated spacecraft (5.6) on the set $(\mathsf{M}\times\mathbb{R}^2)\backslash\Lambda$.

5.1.3 Analysis of the Dynamic Feedback and Coordinate Change

In this section we take a closer look at the the dynamic feedback and coordinate change constructed in the previous section. Since the dynamic controller (5.15) has a singularity at $\xi_1 = 0$, we need to analyze the dependence of ξ_1 on the z-coordinates (or equivalently on the flat output and its derivatives). To obtain the expression for ξ_1 in terms of z we first notice that (5.16c) and (5.16g) can be written as

$$\begin{aligned} z_3 &= -z_6^2 \sin z_1 \cos z_1 + \xi_1 \cos\phi \cos z_1 \\ z_7 &= \frac{2 z_2 z_6 \sin z_1 + \xi_1 \sin\phi \cos z_1}{\cos z_1}. \end{aligned} \tag{5.17}$$

If $\xi_1 \neq 0$ then (5.17) can be solved for ξ_1 and ϕ, the solutions being

$$\phi = \text{atan2}\big(z_7 \cos z_1 - 2z_2 z_6 \sin z_1, z_3 + z_6^2 \cos z_1 \sin z_1\big) \tag{5.18}$$

$$\xi_1 = \frac{\sqrt{(z_3 + z_6^2 \cos z_1 \sin z_1)^2 + (z_7 \cos z_1 - 2z_2 z_6 \sin z_1)^2}}{\cos z_1}. \tag{5.19}$$

Notice that (5.18) is just the expression of ϕ in terms of the flat output and its derivatives (compare with (4.8d)). As a matter of fact, the expressions for r_1 and r_2 given by (4.7) can be written in the z coordinates as

$$r_1 = z_3 + z_6^2 \cos z_1 \sin z_1$$
$$r_2 = z_7 \cos z_1 - 2z_2 z_6 \sin z_1.$$

Equation (5.19) can then be written as

$$\xi_1 = \frac{\sqrt{r_1^2 + r_2^2}}{\cos z_1}. \tag{5.20}$$

Thus the same singularity, $r_1^2 + r_2^2 = 0$, in the expressions for the state and input, (4.8) , are present in the dynamic feedback. This fact is consistent with the result of Martin (1992) (Martin et al., 1997, see Remark after Theorem 2).

Equation (5.20) is important because it tells us precisely how to design the flat output reference trajectory so that the singularity is avoided. Indeed, for the dynamic controller to be well-defined, the actual flat output trajectory $y(t)$ must not reach the singularity $r(y(t)) = 0$ (Recall that in (4.13) we defined the mapping $r = r_1^2 + r_2^2$.). If the reference trajectory $y_d(t)$ is designed so that $r(y_d(t)) > 0$ and the dynamic controller is designed so that $\|y(t) - y_d(t)\|$ is sufficiently small, then since r is a continuous function we will have $r(y(t)) > 0$.

Remark 5.1.0. The relation (5.19) can also be obtained using (5.13) and (4.12).

5.1.4 Trajectory Tracking

By Theorem 3.3.1, the dynamic controller cannot be used to steer the system to an equilibrium *asymptotically.* This negative result can be seen explicitly by the expression (5.15); if as $t \to \infty$ the system approaches an equilibrium then $\xi_1 \to 0$ and the control (5.15) becomes unbounded. However, we will show that the control (5.15) can still be used to steer the spacecraft to an arbitrarily small neighbourhood of the final equilibrium at a pre-determined time t_2 and avoid the singularity $r(y(t)) = 0$, provided the tracking error is kept sufficiently small.

Applying (5.15) to the extended system $\bar{\Sigma}$ (5.14) yields the linear input-output dynamics

$$y_1^{(4)} = v_1$$
$$y_2^{(4)} = v_2.$$

Let $x(t_1) = x_1$ and $x(t_2) = x_2$ be an initial and final equilibrium point, respectively, of the underactuated spacecraft. Since the flat output parameterizes the whole state, we can use the motion planning algorithm of §4.2 to construct a flat output reference trajectory $y_d(t) = (y_{d,1}(t), y_{d,2}(t))$ such that

$$x_1 = \mathcal{A}\big(y_d(t_1), \dot{y}_d(t_1), \ddot{y}_d(t_1), y_d^{(3)}(t_1)\big)$$
$$x_2 = \mathcal{A}\big(y_d(t_2), \dot{y}_d(t_2), \ddot{y}_d(t_2), y_d^{(3)}(t_2)\big),$$

and $r(y_d(t)) > 0$. Thus, to steer the system from x_1 to x_2 we can use the dynamic feedback to track the flat output reference trajectory $y_d(t)$. This can be done using the auxiliary inputs v_1 and v_2. Let $e_i = y_i - y_{d,i}$ denote the tracking error for $i = 1, 2$. We then set

$$v_i = y_{d,i}^{(4)} + \sum_{j=0}^{3} c_j^i e_i^{(j)}, \tag{5.21}$$

where the coefficients c_j^i for $j = 0, \ldots 3$ are chosen so that the linear differential equation

$$e_i^{(4)} - c_3^i e_i^{(3)} - c_2^i e_i^{(2)} - c_1^i e_i^{(1)} - c_0^i e = 0$$

is exponentially stable for $i = 1, 2$. Therefore, with the dynamic control (5.13), (5.15) and (5.21), the tracking error converges exponentially to zero provided that the system remains away from $r(y(t)) = 0$. As the next result demonstrates, if the tracking error $e(t) = (e_1(t), e_2(t))$ is kept sufficiently small then $r(y(t)) > 0$.

Lemma 5.1.1. *If the reference trajectory* $y_d(t)$ *is such that* $r(y_d(t)) > 0$ *and the output tracking error is kept sufficiently small on the interval* $[t_1, t_2]$*, then the actual output trajectory* $y(t)$ *satisfies* $r(y(t)) > 0$ *on* $[t_1, t_2]$*, that is, the system will not reach the singularity on the interval* $[t_1, t_2]$*.*

Proof. Since the function r is continuous, for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|y(t) - y_d(t)\| = \|e(t)\| < \delta$ then $\|r(y(t)) - r(y_d(t))\| < \epsilon$, which implies that

$$r(y_d(t)) - \epsilon < r(y(t)).$$

If $\epsilon = \min_{t\in[t_1,t_2]} r(y_d(t))$ then $r(y(t)) > 0$ for $t \in [t_1, t_2]$, provided that $\|e(t)\|$ is kept sufficiently small. ∎

The norm of the tracking error, $\|e(t)\|$, depends explicitly on the closed-loop eigenvalues and the initial tracking error $e(0)$. Therefore, if the closed-loop eigenvalues are chosen sufficiently far into the left-hand complex plane and the initial tracking error is not too large then the flat output will not reach the singularity. A similar continuity argument can be used to show that at time t_2 of the motion, the actual state of the system can be made sufficiently close to the desired equilibrium.

Proposition 5.1.1. *Suppose we apply the dynamic controller* (5.13), (5.15), (5.21) *to the spacecraft dynamics* (5.6) *with a reference trajectory* $y_d(t)$ *satisfying* $r(y_d(t)) > 0$ *on* $[t_1, t_2]$*. Then the actual final state of the spacecraft at time* t_2*,* $x(t_2)$*, can be made arbitrarily close to the desired equilibrium* x_2 *provided the tracking error is kept sufficiently small.*

Proof. From Lemma 5.1.1, the actual output satisfies $r(y(t)) > 0$ which implies that the controller is well-defined on the interval $[t_1, t_2]$. From §2.4.3, the actual state at time t_2 satisfies

$$\begin{aligned} x(t_2) &= \mathcal{A}\big(y_d(t_2), \dot{y}_d(t_2), \ddot{y}_d(t_2), y_d^{(3)}(t_2)\big) + R(y_d(t_2), e(t_2)) \\ &= x_2 + R(y_d(t_2), e(t_2)), \end{aligned}$$

where $\|R(y_d(t), e(t))\| \to 0$ as $e(t) \to 0$. Therefore,

$$\|x(t_2) - x_2\| = \|R(y_d(t_2), e(t_2))\|$$

can be made arbitrarily small by choosing the eigenvalues of the tracking errors as far left in the complex plane as necessary. ■

Proposition 5.1.1 shows that the dynamic controller (5.13), (5.15) and (5.21) can in fact steer the system from an initial equilibrium x_1 to an arbitrarily small neighbourhood of the final equilibrium x_2 while remaining bounded. A simulation in Matlab is performed to illustrate the tracking performance of the dynamic controller. It is assumed that the spacecraft is at the equilibrium $x_1 = (0, 0, 0, 120°, 60°, -120°)$ at $t_1 = 0$ and we wish to steer it to the final equilibrium $x_2 = (0, 0, 0, -140°, -60°, 150°)$ at time $t_2 = 3$. To test the tracking performance of the dynamic controller, we set the actual initial condition to $\tilde{x}_1 = (1, -1, 1, 100°, 80°, -140°)$. The closed-loop tracking eigenvalues are chosen as $\{-2, -3, -4, -5\}$. The flat output reference trajectory $y_d(t)$ was designed (using the algorithm of §4.2) so that

$$x_2 = \mathcal{A}\big(y_d(t_2), \dot{y}_d(t_2), \ddot{y}_d(t_2), y_d^{(3)}(t_2)\big),$$

and $\xi_1(t_2) = 0.2611$. The results of the simulation are shown in Figure 5.1. As one can observe, the dynamic controller tracks the reference trajectory on the interval $[t_1, t_2]$ while remaining bounded.

If we continue applying the dynamic controller for $t > t_2$ then $\xi_1 \approx 0$ and the dynamic control will become unbounded. To avoid this we can switch to a controller that is well defined in a neighbourhood of the final equilibrium and that asymptotically stabilizes the system. From Theorem 3.3.1, an asymptotically stabilizing controller must necessarily be time-varying or discontinuous. However, since the spacecraft is STLC from any equilibrium x_0 (Proposition 3.4.2) there exists a time-varying periodic controller asymptotically stabilizing the system about x_0 (Coron, 1995). Various time-varying and discontinuous asymptotically stabilizing controllers have been proposed in the literature (Krishnan et al., 1994), (Morin et al., 1995), (Coron and Kerai, 1996), (Morin and Samson, 1997), (Tian, 2002). In particular, Morin and Samson (1997) designed a continuous time-varying exponentially stabilizing controller. Their controller is designed using homogeneous techniques on

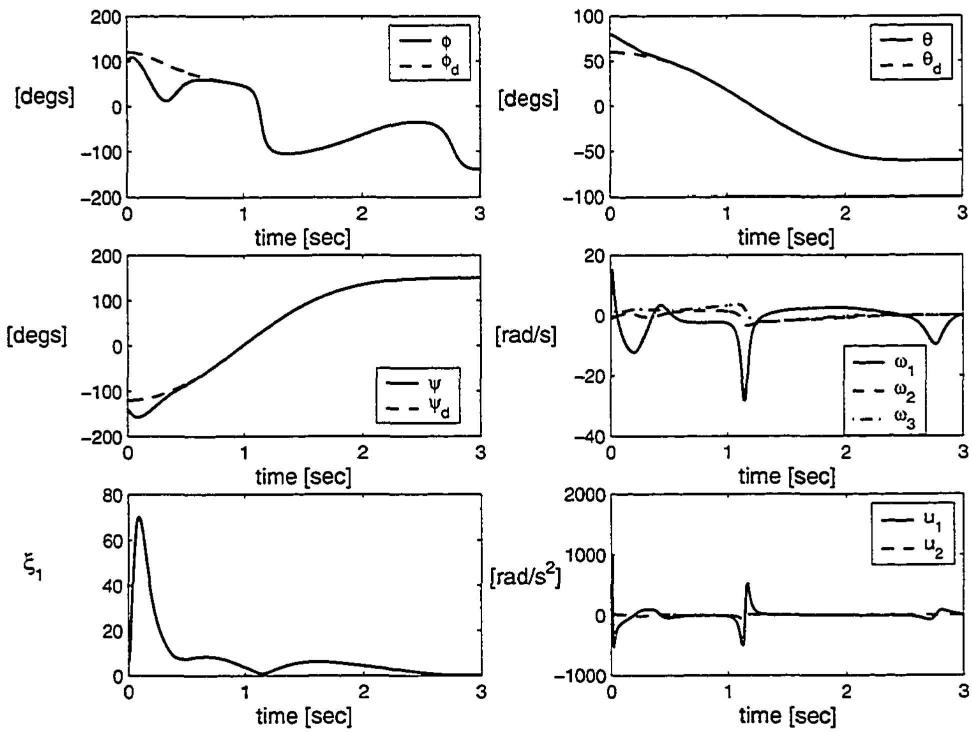


Figure. 5.1: Tracking a reference trajectory with dynamic controller (5.13), (5.15) and (5.21). The initial tracking error is $e(0) = (20°, -20°)$. The dynamic controller causes the error $e(t)$ to decay exponentially on $[t_1, t_2]$.

the spacecraft dynamics expressed in Rodrigues parameters. The controller is valid for all values of α except the axis-symmetric case, that is, the $\alpha = 0$ case. We choose to augment our tracking controller with their controller to obtain an asymptotic tracking controller.

5.2 Morin-Samson Stabilizing Controller

Morin and Samson (1997) designed a time-varying exponentially stabilizing control law for the spacecraft dynamics using the Rodrigues parameterization. Recall that the attitude dynamics of the underactuated spacecraft using the Rodrigues parameterization (§3.2.3) are

$$\begin{aligned}
\dot{\omega}_1 &= u_1 \\
\dot{\omega}_2 &= u_2 \\
\dot{\omega}_3 &= \alpha\omega_1\omega_3 \\
\dot{x}_1 &= \frac{1}{2}\left(\omega_1 + \omega_3 x_2 - \omega_2 x_3 + \left(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3\right) x_1\right) \\
\dot{x}_2 &= \frac{1}{2}\left(\omega_2 - \omega_3 x_1 + \omega_1 x_3 + \left(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3\right) x_2\right) \\
\dot{x}_3 &= \frac{1}{2}\left(\omega_3 + \omega_2 x_1 - \omega_1 x_2 + \left(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3\right) x_3\right).
\end{aligned} \tag{5.22}$$

Let $X = (x_1, x_2, x_3)$ denote the Rodrigues parameters. It is assumed that $\alpha \neq 0$, since otherwise ω_3 is unaffected by the control and thus the system would not be stabilizable. To state the main result of Morin and Samson (1997), we first need to make two definitions. For a complete treatment of homogeneous systems see (Kawzki, 1990) and (Hermes, 1991).

Definition 5.2.1. A *dilation* δ_λ^r is a map $\delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\delta_\lambda^r(x_1, \cdots, x_n) = (\lambda^{r_1} x_1, \cdots, \lambda^{r_n} x_n),$$

where $1 \leq r_1 \leq \cdots \leq r_n$ are integers and $\lambda > 0$.

Definition 5.2.2. A *homogeneous norm* associated with a dilation δ_λ^r is a function $\rho_p^r : \mathbb{R}^n \to \mathbb{R}$ of the form

$$\rho_p^r(x) = \left(\sum_{j=1}^{n} |x_j|^{p/r_j}\right)^{1/p}$$

where $p \in \mathbb{R}$ and $p > 0$.

We are now ready to state the main result of Morin and Samson (1997).

Theorem 5.2.1 (Morin and Samson (1997)). *Consider the functions*

$$\begin{aligned} s_1(X, \omega_3, t) &= -k_1 x_1 - \rho(X, \omega_3)\sin(t/\epsilon) \\ s_2(X, \omega_3, t) &= -k_2 x_2 + \frac{1}{\rho(X, \omega_3)}(x_3 + \omega_3)\sin(t/\epsilon) \end{aligned} \tag{5.23}$$

with ρ*, of class* C^1 *on* $\mathbb{R}^4 \backslash \{0\}$*, a homogeneous norm associated with the dilation* $\delta_\lambda^r(X, \omega_3, t) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 \omega_3, t)$ *and the following time-varying continuous feedback*

$$\begin{aligned} u_1 &= -k_3(\omega_1 - s_1(X, \omega_3, t)) + c_1\omega_2\omega_3 \\ u_2 &= -k_4(\omega_2 - s_2(X, \omega_3, t)) + c_2\omega_1\omega_3, \end{aligned} \tag{5.24}$$

where $c_1 = \frac{I_2 - I_3}{I_1}$ *and* $c_2 = \frac{I_3 - I_1}{I_2}$*. Then, for any positive parameters* k_1 *and* k_2*, there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0]$ *and large enough parameters* $k_3 > 0$ *and* $k_4 > 0$*, the feedback* (5.23)-(5.24) *locally exponentially stabilizes the origin of* (5.22).

Theorem 5.2.1 ensures that for small enough values of ϵ and large enough values of k_3 and k_4, (5.23)-(5.24) make the origin of (5.22) locally exponentially stable. If one is interested in stabilizing an equilibrium $(0, X_0)$ different from the origin then one can replace X with $X - X_0$ in (5.23)-(5.24).

If we assume that the Euler angles are measured then to use the Morin-Samson controller we must find the coordinate change from Euler angles to Rodrigues parameters. Recall from §3.2.3 that the Rodrigues parameters are defined in terms of the quaternion $\mathbf{q} = (q_0, q_1, q_2, q_3)$ via the formula

$$X = (x_1, x_2, x_3) = \left(\frac{q_1}{q_0}, \frac{q_2}{q_0}, \frac{q_3}{q_0}\right). \tag{5.25}$$

Also recall that the rotation matrix in terms of the Euler angles and the quaternions are

$$\mathbf{R}_{\psi,\theta,\phi} = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi s_\theta c_\phi \\ s_\psi c_\theta & c_\psi c_\phi + s_\psi s_\theta s_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

$$\mathbf{R}_\mathbf{q} = 2 \begin{bmatrix} q_0^2 + q_1^2 - \frac{1}{2} & q_1 q_2 - q_0 q_3 & q_1 q_3 + q_0 q_2 \\ q_1 q_2 + q_0 q_3 & q_0^2 + q_2^2 - \frac{1}{2} & q_2 q_3 - q_0 q_1 \\ q_1 q_3 - q_0 q_2 & q_2 q_3 + q_0 q_1 & q_0^2 + q_3^2 - \frac{1}{2} \end{bmatrix},$$

respectively. For $q_0 > 0$, we can solve for the quaternion $\mathbf{q} = (q_0, q_1, q_2, q_3)$ in terms of $\mathbf{R}_\mathbf{q}$, the solution being

$$\begin{aligned} q_0 &= \sqrt{\frac{\operatorname{tr}\mathbf{R} + 1}{4}} \\ q_1 &= \frac{\mathbf{R}_{32} - \mathbf{R}_{23}}{4q_0} \\ q_2 &= \frac{\mathbf{R}_{13} - \mathbf{R}_{31}}{4q_0} \\ q_3 &= \frac{\mathbf{R}_{21} - \mathbf{R}_{12}}{4q_0}. \end{aligned} \tag{5.26}$$

We can then use $\mathbf{R}_{\psi,\theta,\phi}$ and (5.26) to obtain the change of coordinates from Euler angles to quaternions:

$$\begin{aligned} q_0 &= \sqrt{\frac{c_\psi c_\theta + c_\psi c_\phi + s_\psi s_\theta s_\phi + c_\theta c_\phi + 1}{4}} \\ q_1 &= \frac{c_\theta s_\phi + c_\psi s_\phi - s_\psi s_\theta c_\phi}{4q_0} \\ q_2 &= \frac{s_\psi s_\phi + c_\psi s_\theta c_\phi + s_\theta}{4q_0} \\ q_3 &= \frac{s_\psi c_\theta + s_\psi c_\phi - c_\psi s_\theta s_\phi}{4q_0}. \end{aligned} \tag{5.27}$$

Finally, we use the map (5.25) and (5.27) to obtain the change of coordinates from Euler angles to Rodrigues parameters:

$$\begin{aligned} x_1 &= \frac{c_\theta s_\phi + c_\psi s_\phi - s_\psi s_\theta c_\phi}{c_\psi c_\theta + c_\psi c_\phi + s_\psi s_\theta s_\phi + c_\theta c_\phi + 1} \\ x_2 &= \frac{s_\psi s_\phi + c_\psi s_\theta c_\phi + s_\theta}{c_\psi c_\theta + c_\psi c_\phi + s_\psi s_\theta s_\phi + c_\theta c_\phi + 1} \\ x_3 &= \frac{s_\psi c_\theta + s_\psi c_\phi - c_\psi s_\theta s_\phi}{c_\psi c_\theta + c_\psi c_\phi + s_\psi s_\theta s_\phi + c_\theta c_\phi + 1}. \end{aligned} \tag{5.28}$$

For future reference, let $(\psi, \theta, \phi) \mapsto (x_1, x_2, x_3) = \text{Eul2X}(\psi, \theta, \phi)$ denote the map given by (5.28).

5.3 Simulation: Asymptotic Tracking

In this section we combine the dynamic controller (5.13), (5.15),(5.21) with the Morin-Samson controller (5.23)-(5.24) to asymptotically track a flat output reference trajectory that takes the system from an initial equilibrium x_1 at time t_1 to a final equilibrium x_2 asymptotically. From the analysis in §5.1.4 we can state the following result. (We denote $y = h(x) = (\theta, \psi)$ as the flat output.)

Proposition 5.3.1. *Let $y_d(t)$ be a flat output reference trajectory such that $r(y_d(t)) > 0$ on $[t_1, t_2]$,*

$$x_1 = \mathcal{A}\big(y_d(t_1), \ldots, y_d^{(3)}(t_1)\big)$$
$$x_2 = \mathcal{A}\big(y_d(t_2), \ldots, y_d^{(3)}(t_2)\big),$$

and $y_d(t) = h(x_2)$ for $t > t_2$, where x_1 and x_2 are equilibria of the spacecraft. Then the controller

$$u = \begin{cases} (5.13),(5.15),(5.21) & \text{for } t \in [t_1, t_2], \\ (5.23),(5.24) & \text{for } t > t_2, \end{cases} \tag{5.29}$$

makes $y_d(t)$ an asymptotically stable reference trajectory and the spacecraft is steered from an initial condition $x(t_1)$ to x_2 asymptotically, provided the output tracking errors are kept sufficiently small.

Proof. From Proposition 5.1.1, the dynamic controller is well-defined on $[t_1, t_2]$ and will cause $\|y(x(t)) - y_d(t)\|$ to decay exponentially to zero on $[t_1, t_2]$. If

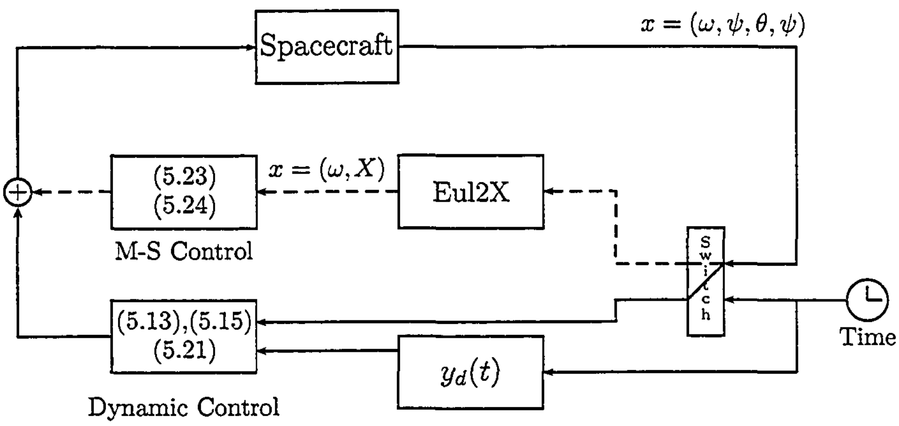


Figure. 5.2: Simulink file implementing the two-phase controller (5.29).

the initial state error $\|x(t_1) - x_1\|$ is not too large and the output tracking errors are kept sufficiently small, then at $t = t_2$ the state of the system, $x(t_2)$, will be in an arbitrarily small neighbourhood of the final desired equilibrium x_2. In particular, the final state can be brought to the region of attraction of the Morin-Samson controller (5.23),(5.24). At $t = t_2$ we can then switch to the Morin-Samson control which asymptotically stabilizes the equilibrium x_2. This completes the proof. ■

A Matlab simulation is performed illustrating the performance of the two-phase controller. The structure of the simulation file is shown in Figure 5.2. The state of the spacecraft and time are fed into a "switch" block. The switch block is designed such that if $t \in [t_1, t_2]$ then the state signal is sent to the dynamic controller, otherwise the state signal is sent to the Eul2X block where the change of coordinates (5.28) is applied to the Euler angles and the resulting state signal $x = (\omega, X)$ is sent to the Morin-Samson control. In the simulation, it is assumed that the spacecraft is at the equilibrium $x_1 = (0, 0, 0, 120°, 60°, -120°)$ at $t_1 = 0$ and it is desired to steer the spacecraft to the final equilibrium $x_2 = (0, 0, 0, -140°, -60°, 150°)$ at time $t_2 = 3$. The actual initial condition of the system is set to $\tilde{x}_1 = (1, -1, 1, 100°, 80°, -140°)$. The closed-loop tracking eigenvalues are chosen as $\{-2, -3, -4, -5\}$ for each component of the reference trajectory. The flat output reference trajectory

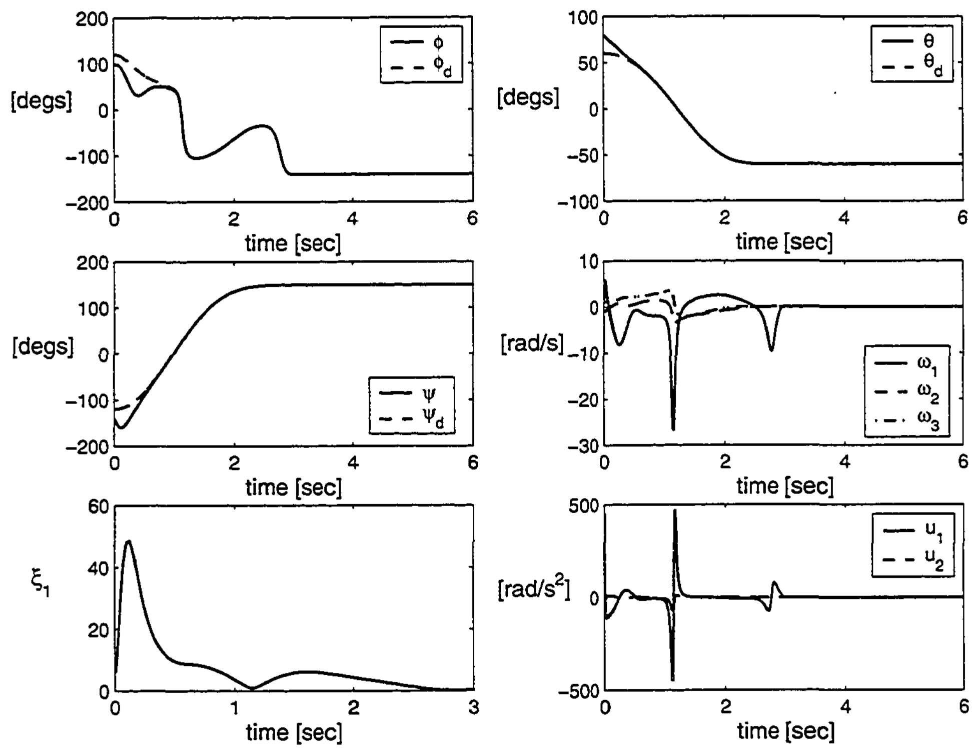


Figure. 5.3: Asymptotic trajectory tracking with two-phase control (5.29). Initial state error is $\Delta x = (1, -1, 1, -20°, 20°, -20°)$.

$y_d(t)$ was designed (using the algorithm of §4.2) so that

$$x_2 = \mathcal{A}\big(y_d(t_2), \dot{y}_d(t_2), \ddot{y}_d(t_2), y_d^{(3)}(t_2)\big),$$

and $\xi_1(t_2) = 0.2611$. The gains in the Morin-Samson controller were chosen to be $k_1 = 120$, $k_2 = 120$, $k_3 = 100$, and $k_4 = 100$, and $\epsilon = 0.3$.

The results of the simulation are shown in Figure 5.3. At time t_2, the actual compensator state is $\xi_1 = 0.2613$, agreeing with the desired value up to three significant digits. The minimum value of ξ_1 on $[t_1, t_2]$ is 0.22 and occurs at approximately $t = 2.8$ seconds.

5.4 Robustness to Parameter Uncertainty

In this section we test via simulation the robustness of controller (5.29) to uncertainty in the parameter α. Thus far, we have assumed that $\alpha = -1$. We perform a simulation with the same initial conditions and control parameters

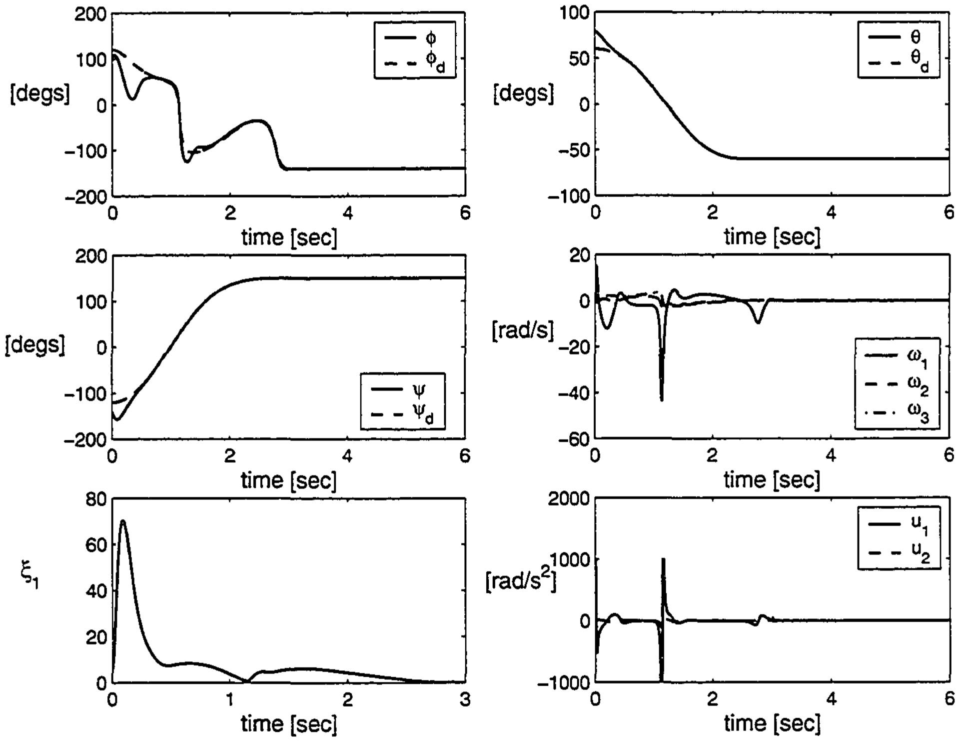


Figure. 5.4: Robustness to parameter uncertainty: Asymptotic tracking with two-phase control (5.29) for $\alpha = -0.95$.

as in the previous section and set $\alpha = -0.95$ in the spacecraft model. This value of α corresponds to a 5% error. The result of the simulation is shown in Figure 5.4. As one can observe from Figure 5.4 the controller (5.29) is still able to asymptotically track the desired reference trajectory in the presence of parameter uncertainty.

5.5 Linearization with a Quasi-static Feedback

In this section we consider the input-output feedback linearization problem of the underactuated spacecraft using a *quasi-static state* feedback. These special state feedbacks were introduced by Delaleau and Fliess (1992), and it was shown in Delaleau and Rudolph (1998) that they can be used to linearize a flat system. More precisely, Delaleau and Rudolph (1998) show that any flat system can be transformed to a linear controllable system using a quasi-static feedback. Two advantages of quasi-static feedbacks over dynamic feedbacks is that one can consider systems in which the derivatives of the input appear, and no extra dynamics are added in the controller. Here we give a brief explanation of how a quasi-static feedback is constructed and then apply the procedure to construct a quasi-static feedback for the underactuated spacecraft.

Consider the control system

$$\dot{x} = f(x, u, \dot{u}, \ldots, u^{(\rho)}), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{5.30}$$

Suppose that there exists a feedback control u that preserves the state x, that is, no extra dynamics are added, and that renders the closed-loop behaviour of the system to be a linear controllable system

$$\dot{x} = Ax + Bv, \tag{5.31}$$

where the matrices A and B are of appropriate size and $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$. In general, if such a feedback exists it will depend on the derivatives of v, that is, it will be of the form

$$u_i = \varphi_i\big(x, v, \dot{v}, \ldots, v^{(r_i)}\big), \quad \text{for } i = 1, \ldots, m. \tag{5.32}$$

We call such a feedback a *quasi-static state* feedback. Below we give a more formal definition.

Definition 5.5.1. The control input u and the closed-loop input v are related by a *quasi-static feedback of the state* x, if for some non-negative integer r_0:

(i) the feedback law reads

$$u_i = \varphi_i(x, v, \dot{v}, \cdots, v^{(r_0)}), \qquad i = 1, \ldots, m, \tag{5.33}$$

(ii) this feedback is invertible

$$v_i = \tilde{\varphi}_i(x, u, \dot{u}, \cdots, u^{(r_0)}), \qquad i = 1, \ldots, m,$$

(iii) and x is the state of the closed-loop system with new input v, that is, the state x is preserved.

Thus, a quasi-static feedback preserves the state of the system and involves only a finite number of derivatives of the closed-loop control v. In the case that the feedback (5.33) is used to track a known reference trajectory, then the time derivatives of v can be expressed in terms of the derivatives of the reference trajectory and therefore no numerical differentiation is required. Below we give an example of how to construct a quasi-static feedback.

Example 5.5.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_1 & \qquad y_1 &= x_1 \\ \dot{x}_2 &= x_3 + e^{x_1} u_1 & \qquad y_2 &= x_2. \\ \dot{x}_3 &= u_2 \end{aligned}$$

To obtain a quasi-static feedback we begin by differentiating the first component of the output:

$$\dot{y}_1 = -x_1 + u_1.$$

Setting $\dot{y}_1 = v_1$ we obtain the feedback

$$u_1 = v_1 + x_1. \tag{5.34}$$

Now differentiate y_2:

$$\begin{aligned} \dot{y}_2 &= x_3 + e^{x_1} u_1 \\ &= x_3 + e^{x_1}(v_1 + x_1). \end{aligned}$$

Continuing to differentiate y_2 we obtain

$$\begin{aligned} \ddot{y}_2 &= u_2 + \dot{x}_1 e^{x_1}(v_1 + x_1) + e^{x_1}(\dot{v}_1 + \dot{x}_1) \\ &= u_2 + v_1 e^{x_1}(v_1 + x_1) + e^{x_1}(\dot{v}_1 + v_1). \end{aligned}$$

Setting $\ddot{y}_2 = v_2$ we obtain the feedback

$$u_2 = v_2 - e^{x_1}(\dot{v}_1 + v_1^2 + v_1 + x_1). \tag{5.35}$$

Therefore, with the quasi-static feedback (5.34), (5.35) the input-output dynamics have the form,

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \ddot{y}_2 &= v_2, \end{aligned}$$

that is, we have input-output linearized the system. The inputs v_1 and v_2 can then be designed to asymptotically track an output reference trajectory. □

Using the same method as in the above example, we derive a quasi-static feedback for the underactuated spacecraft (5.6). We do all the calculations symbolically and then at the end display explicit expressions for the quasi-static control law. We start by differentiating the flat output until the input appears:

$$\ddot{y}_1 = \ddot{\theta} = \underbrace{L_f^2 h_1(x)}_{b(x)} + \underbrace{L_{g_2} L_f h_1(x)}_{a(x)} u_2 \tag{5.36}$$

$$\ddot{y}_2 = \ddot{\psi} = L_f^2 h_2(x) + L_{g_2} L_f h_2(x) u_2. \tag{5.37}$$

Setting $\ddot{y}_1 = v_1$ we obtain the feedback

$$u_2 = \frac{v_1 - b(x)}{a(x)}. \tag{5.38}$$

Rewriting (5.37) using (5.38) yields an expression having the form

$$\ddot{y}_2 = b_1(x) + a_1(x) v_1, \tag{5.39}$$

for suitable functions b_1 and a_1. Computing $y_2^{(3)}$ using (5.39) yields

$$y_2^{(3)} = L_f b_1(x) + L_{g_2} b_1(x) u_2 + L_f a_1(x) v_1 + a_1(x) \dot{v}_1. \tag{5.40}$$

Substituting (5.38) into (5.40) we obtain an expression having the form

$$y_2^{(3)} = b_2(x) + a_2(x) v_1 + a_1(x) \dot{v}_1, \tag{5.41}$$

for suitable functions a_2 and b_2. Since u_1 has not appeared we continue differentiating y_2, yielding

$$y_2^{(4)} = b_3(x) + a_3(x) v_1 + a_4(x) v_1^2 + a_5(x) \dot{v}_1 + a_1(x) \ddot{v}_1 + (c_{11}(x) + c_{12}(x) v_1) u_1,$$

where a_3, a_4, a_5, b_3, c_{11}, and c_{12} are appropriately defined functions. Setting $y_2^{(4)} = v_2$ we obtain the feedback

$$u_1 = \frac{v_2 - b_3(x) - a_3(x) v_1 - a_4(x) v_1^2 - a_5 \dot{v}_1 - a_1 \ddot{v}_1}{c_{11}(x) + c_{12}(x) v_1}. \tag{5.42}$$

Therefore, with the quasi-static feedback (5.38) and (5.42) we obtain the following linear input-output dynamics:

$$\ddot{y}_1 = v_1$$
$$y_2^{(4)} = v_2.$$

If we wish to track a reference trajectory $y_d(t)$, then the auxiliary inputs v_1 and v_2 can be designed so that the error dynamics $e_1 = y_1 - y_{1,d}$ and $e_2 = y_2 - y_{2,d}$ are asymptotically stable. This can be accomplished by setting

$$v_1 = \ddot{y}_{1,d} + c_0(y_1 - y_{1,d}) + c_1(\dot{y}_1 - \dot{y}_{1,d}), \tag{5.43}$$

$$v_2 = y_{2,d}^{(4)} + d_0(y_2 - y_{2,d}) + d_1(\dot{y}_2 - \dot{y}_{2,d}) + d_2(\ddot{y}_2 - \ddot{y}_{2,d}) + d_3(y_2^{(3)} - y_{2,d}^{(3)}), \tag{5.44}$$

and choosing the coefficients c_i and d_k so that the polynomials $p_1(\lambda) = \lambda^2 - c_1\lambda - c_0$ and $p_2(\lambda) = \lambda^4 - d_3\lambda^3 - d_2\lambda^2 - d_1\lambda_1 - d_0$ have all their roots in $\mathbb{C}_-$.

Notice that (5.38) and (5.42) are of the form

$$u_1 = \varphi_1(x, v, \dot{v}, \ddot{v})$$
$$u_2 = \varphi_2(x, v, \dot{v}, \ddot{v}),$$

where φ_1 and φ_2 are given by the right-hand side of (5.38) and (5.42), respectively. Thus, the inputs u_1 and u_2 can be written as functions of the state x, the auxiliary input v and a finite number of derivatives of v. Using (5.38) and (5.42), we can also derive expressions for v_1 and v_2 in terms of x, u, and a finite number of derivatives of u, that is, we can invert φ_1 and φ_2 for v_1 and v_2. To this end, we first note that by inspection

$$v_1 = b(x) + a(x)u_1 \tag{5.45}$$

$$v_2 = b_3(x) + a_3(x)v_1 + a_4(x)v_1^2 + a_5(x)\dot{v}_1 + a_1\ddot{v}_1 + c_1(x)u_1. \tag{5.46}$$

From (5.45) it can be shown that

$$\dot{v}_1 = p_1(x) + q_1(x)u_1 + q_2(x)u_2 + a(x)\dot{u}_1$$

$$\ddot{v}_1 = p_2(x) + q_3(x)u_1 + q_4(x)u_2 + q_5(x)u_1^2 + q_6(x)u_1u_2 + q_7(x)u_2^2 + q_8(x)\dot{u}_1 + q_2(x)\dot{u}_2 + a(x)\ddot{u}_1, \tag{5.47}$$

for suitable functions $p_i(x)$ and $q_k(x)$ for $i = 1, 2$ and $k = 1, \ldots, 8$. We can then substitute (5.47) into (5.46) to obtain expressions of the form

$$v_1 = \tilde{\varphi}_1(x, u, \dot{u}, \ddot{u})$$
$$v_2 = \tilde{\varphi}_2(x, u, \dot{u}, \ddot{u}).$$

Therefore, we can transform the controls u to the controls v, and vice-versa, without having to change coordinates for x.

To compute u_2, (5.38), the closed-loop input v_1 is required. From (5.43), v_1 can be computed directly from the reference trajectory $y_{1,d}$ and from the relations $y_1 = \theta$ and $\dot{y}_1 = \omega_2 \cos\phi - \omega_3 \sin\phi$. We thus obtain

$$v_1 = \ddot{y}_{1,d} + c_0(\theta - y_{1,d}) + c_1(\omega_2 \cos\phi - \omega_3 \sin\phi - \dot{y}_{1,d}). \tag{5.48}$$

To compute u_1, (5.42), we require v_2, v_1, $\dot{v}_1$, and $\ddot{v}_1$. However, to compute v_2, (5.44), we need $\ddot{y}_2$ and $y_2^{(3)}$. Now, from (5.39) and (5.41) we see that $\ddot{y}_2$ and $y_2^{(3)}$ are completely determined by x, v_1 and $\dot{v}_1$. The expression for v_1 is given by (5.48), therefore we just need $\dot{v}_1$ and $\ddot{v}_1$:

$$\begin{aligned} \dot{v}_1 &= y_{1,d}^{(3)} - c_0(\dot{y}_1 - \dot{y}_{1,d}) - c_1(\ddot{y}_1 - \ddot{y}_{1,d}) \\ &= y_{1,d}^{(3)} - c_0(\dot{y}_1 - \dot{y}_{1,d}) - c_1(v_1 - \ddot{y}_{1,d}), \end{aligned} \tag{5.49}$$

and

$$\begin{aligned} \ddot{v}_1 &= y_{1,d}^{(4)} - c_0(\ddot{y}_1 - \ddot{y}_{1,d}) - c_1(\dot{v}_1 - y_{1,d}^{(3)}) \\ &= y_{1,d}^{(4)} - c_0(v_1 - \ddot{y}_{1,d}) - c_1(\dot{v}_1 - y_{1,d}^{(3)}). \end{aligned} \tag{5.50}$$

Therefore, u_1 and u_2 can be explicitly expressed in terms of x and y_d and derivatives of y_d (up to order four) without adding any further dynamics or differentiations of the states. After some simplifications, the explicit expressions for u_1, (5.42), and u_2, (5.38), are

$$u_1 = \frac{c_\theta^4 c_\phi^3 v_2 - \beta_1 - c_\theta \beta_2 v_1 - c_\theta^2 \beta_3 v_1^2 - c_\phi c_\theta^2 \beta_4 \dot{v}_1 - c_\phi^2 c_\theta^3 \beta_5 \ddot{v}_1}{c_\theta^3 c_\phi \big(\tan\theta(\omega_2 s_\phi + \omega_3 c_\phi)^2 + v_1\big)}, \tag{5.51}$$

$$u_2 = \frac{v_1 + \tan\theta(\omega_2 \sin\phi + \omega_3 \cos\phi)^2 + \omega_1 \omega_3 \cos\phi}{\cos\phi}, \tag{5.52}$$

where $\beta_1 = \beta_1(x)$ and $\beta_2 = \beta_2(x)$ are smooth functions (too long to display) globally defined on M, and

$$\beta_3 = 3\sin\phi\sin\theta(\cos^2\phi + 1)$$
$$\beta_4 = 4\sin\phi\sin\theta\omega_2(\cos^2\phi + 1) + 4\cos^3\phi\sin\theta\omega_3 + 2\omega_1\cos\theta$$
$$\beta_5 = \sin\phi,$$

and v_1, $\dot{v}_1$, $\ddot{v}_1$, and v_2 are given by (5.48), (5.49), (5.50), and (5.44), respectively. The symbolic expressions for the functions a, b, a_i and b_j, for $i = 1, \ldots, 5$ and $j = 1, 2, 3$, β_1, β_2, c_{11}, and c_{22} are

$$a = L_{g_2}L_f h_1 \qquad b = L_f h_1$$
$$a_1 = \frac{L_{g_2}L_f h_2}{a} \qquad b_1 = L_f h_2 - \frac{b}{a}L_{g_2}L_f h_2$$
$$a_2 = \frac{L_{g_2}b_1}{a} + L_f a_1 \qquad b_2 = L_f b_1 - \frac{b}{a}L_{g_2}b_1$$
$$a_3 = L_{g_2}b_2 a + L_f a_2 - \frac{b}{a}L_{g_2}a_2 \qquad b_3 = L_f b_2 - \frac{b}{a}L_{g_2}b_2$$
$$a_4 = \frac{L_{g_2}a_2}{a} \qquad \beta_1 = b_3\cos^4(\theta)\cos^3(\phi)$$
$$a_5 = a_2 + L_f a_1 \qquad \beta_2 = a_3\cos^3(\theta)\cos^3(\phi)$$
$$c_{11} = L_{g_1}b_2 \qquad c_{12} = L_{g_1}a_2.$$

From (5.51) and (5.52), we observe that the quasi-static feedback has a singularity when $\cos\phi = 0$. Therefore, to use the quasi-static feedback, the angle ϕ must be restricted to the set $(-\pi/2, \pi/2)$. As a consequence, the reference trajectory $y_d(t)$ must be designed such that the angle ϕ does not leave the allowable interval. Unfortunately, constructing such a reference trajectory imposes restrictive motions on the spacecraft that do not exist if the dynamic controller constructed in §5.1.4 is used. For instance, with the quasi-static feedback, the reference trajectory in §5.3 cannot be tracked since $\phi \notin (-\pi/2, \pi/2)$. In this sense, the dynamic controller is preferred over the quasi-static feedback. However, the quasi-static feedback control has the advantage that no new extra states are added into the system.

A simulation implementing the quasi-static feedback (5.51) and (5.52) is performed in Matlab. It is assumed that the spacecraft is at the equilibrium $x_1 = (0, 0, 0, 80°, 75°, -120°)$. The actual initial condition is set to $\tilde{x}_1 =$

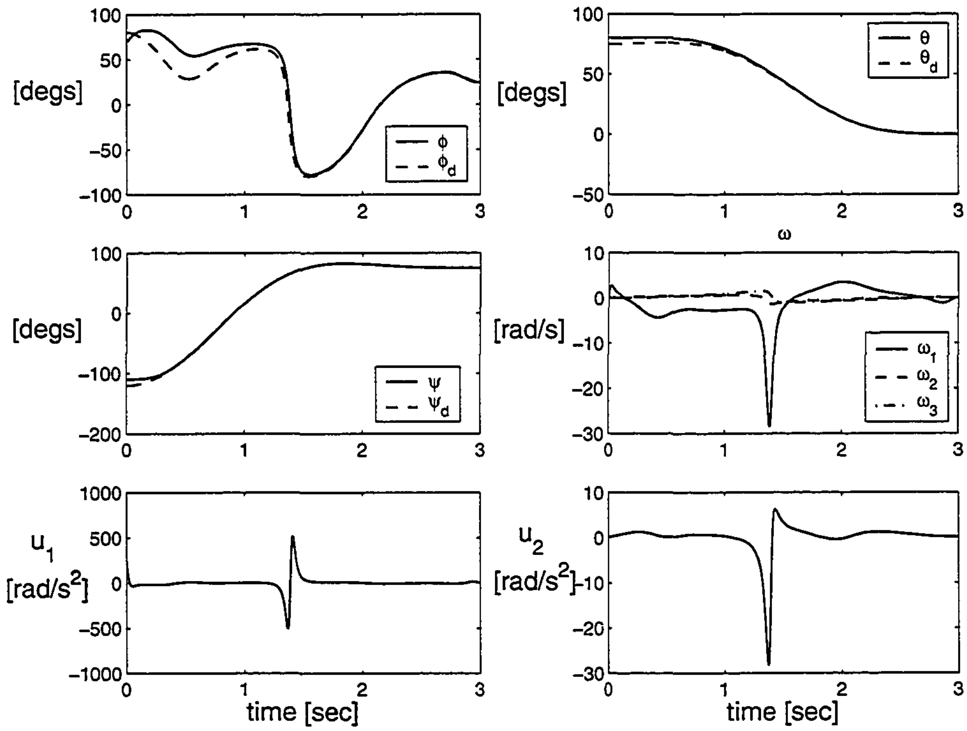


Figure. 5.5: Closed-loop tracking with the quasi-static feedback (5.51) and (5.52). The initial state error is $\Delta x = (0, 0, 0, -10°, 5°, 10°)$.

$(0, 0, 0, 70°, 80°, -110°)$. The desired equilibrium is $x_2 = (0, 0, 0, 25°, 0°, 75°)$. The eigenvalues of the tracking dynamics are set to $\{-1.2 - 1.2i, -1.2 + 1.2i\}$ for y_1 and $\{-6 - 3.6i, -6 + 3.6i, -4.8 - 6i, -4.8 + 6i\}$ for y_2. The results of the simulation are shown in Figure 5.5. As one can observe the quasi-static feedback tracks the desired reference trajectory. However, it was found in the simulations that if the initial state error was large enough then the initial control effort would cause the angle ϕ to leave the set $(-\pi/2, \pi/2)$. A solution to this problem is to decrease the gains of the output error tracking dynamics. However, by decreasing the gains the tracking performance of the controller will also decrease. This gives rise to a trade-off between the tracking performance and the applicability of the quasi-controller.

Chapter 6

Orbital Flatness and Open-loop Motion Planning

In this chapter we consider extending the results of Chapters 4 and 5 by relaxing the condition that $\alpha = -1$. It is not known if the spacecraft is flat for $\alpha \neq \pm 1$ but it has been known for some time now that it is *orbitally flat* (Regahi, 1995). Orbital flatness is a generalization of flatness by allowing time-scaling. With the help of the Dynamic Extension Algorithm we construct a time-scaling function and show that the system in the new time-scale is flat for values of α different than $\alpha = -1$. Open-loop and closed-loop controls are constructed for the time-scaled system and simulations are provided illustrating the performance of the controllers.

6.1 Orbital Flatness

Orbitally flat systems were introduced by Fliess et al. (1999). The main ingredient introduced in orbital flatness is a time-scaling function, $\gamma : \mathsf{M} \to \mathbb{R}$, satisfying $0 < \gamma(x) < \infty$, where $\mathsf{M} \subset \mathbb{R}^n$ denotes the state space and $x \in \mathsf{M}$ the state vector. A new time τ is introduced via the formula

$$\tau(t) = \tau_0 + \int_{t_0}^{t} \frac{1}{\gamma(x(r))} \,\mathrm{d}r,$$

or equivalently

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\gamma(x(t))}.$$

The condition that $0 < \gamma(x) < \infty$ ensures that the new time τ is a strictly monotone increasing function with respect to the original time t, and therefore, the state trajectory and stability are preserved in the new time scale. A system having the form

$$\dot{x} = f(x, u), \quad x \in \mathsf{M}, \quad u \in \mathbb{R}^m \tag{6.1}$$

is written in the new time-scale as

$$x' = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma(x) f(x, u), \tag{6.2}$$

where the prime superscript denotes differentiation in the new time-scale, that is, $x' = \frac{\mathrm{d}x}{\mathrm{d}\tau}$. System (6.1) is said to be *orbitally flat* if the new time-scaled system (6.2) is flat. From (6.2), we observe that the time-scaling function adds an extra parameter for the design of a control law. Clearly, any flat system is orbitally flat by setting $\gamma(x) = 1$. However, the converse does not hold as the next example shows.

Example 6.1.1 (Sampei and Furuta (1986)). Consider the system $\dot{x} = f(x) + g(x)u$ where $x = (x_1, x_2, x_3)$, $f(x) = (x_2 e^{x_3}, x_3 e^{x_3}, 0)$, and $g(x) = (0, 0, 1)$. Recall from §2.4 that a single-input system is flat if and only if it is SFBL. It can be easily checked that this system is not static state feedback linearizable and hence it is not flat. Consider the time-scaling function $\gamma(x) = e^{-x_3}$. The system in the new time scale is

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \gamma(x) f(x) + \gamma(x) g(x) u = \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v$$

where $v = e^{x_3} u$. The system in the new time-scale is in Brunovsky controller canonical form and thus it is feedback linearizable. Consequently, the original system is orbitally flat. □

The idea of using state dependent time-scaling functions to study the feedback linearization problem was first considered and solved for single-input systems by Sampei and Furuta (1986). Their conditions rely on solving a set of partial differential equations (PDEs) to obtain the time-scaling function. Later, Respondek (1998) obtained necessary and sufficient conditions for single-input systems in terms of the system's vector fields $f(x)$ and $g(x)$. Guay (2001) obtained necessary and sufficient conditions for multi-input systems using an exterior calculus approach.

6.2 Time-scaling Function

In this section we construct a time-scaling function for the two-input spacecraft with the help of the Dynamic Extension Algorithm (DEA) (see §5.1.1). The main idea of the design is to impose conditions on the time-scaling function $\gamma(x)$ so that the DEA can be iterated sufficiently many times to yield a dynamic feedback linearizable system. Let $\gamma(x)$ denote the time-scaling function to be designed, where $x = (\omega_1, \omega_2, \omega_3, \phi, \theta, \psi)$ is the spacecraft's state vector. In the new time-scale the system can be written as

$$\omega_1' = v_1 \tag{6.3a}$$

$$\omega_2' = v_2 \tag{6.3b}$$

$$\omega_3' = \gamma(x)\alpha\omega_1\omega_2 \tag{6.3c}$$

$$\phi' = \gamma(x)(\omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta) \tag{6.3d}$$

$$\theta' = \gamma(x)(\omega_2 \cos\phi - \omega_3 \sin\phi) \tag{6.3e}$$

$$\psi' = \gamma(x)((\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta), \tag{6.3f}$$

where we have defined

$$\begin{aligned} v_1 &= \gamma(x)u_1 \\ v_2 &= \gamma(x)u_2. \end{aligned} \tag{6.4}$$

We choose as output

$$y = h(x) = (\theta, \psi). \tag{6.5}$$

Let $x_e = (0, 0, 0, \phi_e, \theta_e, \psi_e)$ denote an equilibrium of system (6.3). A straightforward calculation shows that the decoupling matrix of (6.3) with output (6.5) is

$$A(x) = \begin{bmatrix} \frac{\partial\gamma}{\partial\omega_1}(\omega_2\cos\phi - \omega_3\sin\phi) & \gamma(x)\cos\phi + \frac{\partial\gamma}{\partial\omega_2}(\omega_2\cos\phi - \omega_3\sin\phi) \\ \frac{\partial\gamma}{\partial\omega_1}(\omega_2\sin\phi + \omega_3\cos\phi)\sec\theta & \left(\gamma(x)\sin\phi + \frac{\partial\gamma}{\partial\omega_2}(\omega_2\sin\phi + \omega_3\cos\phi)\right)\sec\theta \end{bmatrix}.$$

To satisfy the hypothesis of the DEA, we must design $\gamma(x)$ so that the matrix $A(x)$ has constant rank one in a neighbourhood of x_e. By inspection, if $\frac{\partial\gamma}{\partial\omega_1} \neq 0$ in a neighbourhood of x_e, then $A(x)$ will not have constant rank one. Therefore, to apply the DEA, $\gamma(x)$ must satisfy $\frac{\partial\gamma}{\partial\omega_1} = 0$, that is, $\gamma(x)$ must not depend on ω_1. We make a further simplification by assuming that $\gamma(x)$ does not depend on ω_2 also. In this case, the decoupling matrix becomes

$$A(x) = \begin{bmatrix} 0 & \gamma(x)\cos\phi \\ 0 & \gamma(x)\frac{\sin\phi}{\cos\theta} \end{bmatrix},$$

which has constant rank one in a neighbourhood of x_e. We then apply the dynamic feedback given by the DEA, (5.2), to the system (6.3) resulting in a system with state $(x,\xi) \in \mathsf{M} \times \mathbb{R}$. Calculating the determinant of the decoupling matrix, denoted $A_1(x,\xi)$, of the extended time-scaled system yields

$$\det(A_1(x,\xi)) = -\frac{q_1(x)\gamma(x)\left(\omega_2\gamma(x)(\alpha+1) + \alpha\omega_2\omega_3\frac{\partial\gamma}{\partial\omega_3} + \omega_3\frac{\partial\gamma}{\partial\phi}\right)}{\cos\theta},$$

where $q_1(x)$ is a smooth function satisfying $q_1(x) \neq 0$. To apply the DEA once again it is necessary that the above determinant vanish in a neighbourhood of x_e. This can only occur, however, if and only if $\gamma(x)$ is a solution to the PDE

$$\omega_2\gamma(x)(\alpha+1) + \alpha\omega_2\omega_3\frac{\partial\gamma}{\partial\omega_3} + \omega_3\frac{\partial\gamma}{\partial\phi} = 0. \tag{6.6}$$

Assuming $\gamma(x)$ depends only on ω_3, (6.6) simplifies to

$$\gamma(\omega_3)(\alpha+1) + \alpha\omega_3\frac{\mathrm{d}\gamma}{\mathrm{d}\omega_3} = 0,$$

which is equivalent to

$$\frac{\mathrm{d}\gamma}{\gamma} = -\frac{\alpha+1}{\alpha}\frac{\mathrm{d}\omega_3}{\omega_3}. \tag{6.7}$$

Integrating (6.7) yields

$$\ln(\gamma) = -\frac{\alpha+1}{\alpha}\ln(\omega_3) = \ln\left(\omega_3^{-\frac{\alpha+1}{\alpha}}\right),$$

which, is equivalent to

$$\gamma(\omega_3) = \omega_3^{-\frac{\alpha+1}{\alpha}}. \tag{6.8}$$

The function (6.8) is not a time-scaling function in a strict sense since it does not satisfy the condition $0 < \gamma(\omega_3) < \infty$, except in the case that $\omega_3 > 0$. Notice that if $\alpha = -1$ then $\gamma(\omega_3) = 1$ and consequently $\mathrm{d}\tau = \mathrm{d}t$, that is, the time is preserved. A time-scaling function similar to (6.8) was obtained by Rudolph (2003).

With $\gamma(\omega_3)$ defined as in (6.8), it is straightforward to show that system (6.3) is flat with the output (6.5). From (6.3e) and (6.3f), we can solve for ω_2 and ω_3:

$$\omega_2 = (\psi' \cos\phi \cos\theta - \theta' \sin\phi)^{(-\alpha-1)}(\theta' \cos\phi + \psi' \cos\theta \sin\phi) \tag{6.9}$$

$$\omega_3 = (\psi' \cos\phi \cos\theta - \theta' \sin\phi)^{-\alpha}. \tag{6.10}$$

From (6.3d), we can write

$$\omega_1 = (\phi' - \psi' \sin\theta)\omega_3^{\left(\frac{\alpha+1}{\alpha}\right)} = (\phi' - \psi' \sin\theta)(\psi' \cos\phi \cos\theta - \theta' \sin\phi)^{-(\alpha+1)}. \tag{6.11}$$

Notice that if $\alpha = -1$ then (6.9), (6.10), and (6.11) all simplify to the expressions obtained in §4.1. Differentiating (6.10) and equating the result to $\gamma(\omega_3)\alpha\omega_1\omega_2$, where ω_1 and ω_2 are given by (6.11) and (6.9), respectively, yields after simplification the equation

$$-\cos\phi(\psi'' \cos\theta - 2\psi'\theta' \sin\theta) + \sin\phi(\theta'' + (\psi')^2 \sin\theta \cos\theta) = 0.$$

We can then solve for ϕ from the last equation, the solution being

$$\phi = \text{atan2}(\psi'' \cos\theta - 2\psi'\theta' \sin\theta, \theta'' + (\psi')^2 \sin\theta \cos\theta). \tag{6.12}$$

The relation (6.12) has the same form as the one obtained in (4.6). The inputs v_1 and v_2 can then be obtained by differentiating (6.11) and (6.9), respectively. This shows that system (6.3) with scaling function defined by (6.8) is flat, which implies that the original spacecraft is orbitally flat. We have thus proved the following:

Proposition 6.2.1. *The two-input spacecraft with time-scaling function* $\gamma(\omega_3) = \omega_3^{-\frac{\alpha+1}{\alpha}}$ *is orbitally flat provided* α *and* ω_3 *are such that* $0 < \gamma(\omega_3) < \infty$.

The explicit expressions for the state and input in terms of the flat output for the time-scaled system are

$$\omega_1 = s^{-(\alpha+1)}\left(\phi' - y_2' \sin y_1\right) \tag{6.13a}$$

$$\omega_2 = s^{-(\alpha+1)} p \tag{6.13b}$$

$$\omega_3 = s^{-\alpha} \tag{6.13c}$$

$$\phi = \operatorname{atan2}(r_2, r_1) \tag{6.13d}$$

$$\theta = y_1 \tag{6.13e}$$

$$\psi = y_2 \tag{6.13f}$$

$$v_1 = (\alpha+1)\left(\phi' - y_2' s_{y_1}\right)^2 s^{-(\alpha+2)} p + s^{-(\alpha+1)}(\phi'' - y_2'' s_{y_1} - y_1' y_2' c_{y_1}) \tag{6.13g}$$

$$\begin{aligned} v_2 = (-\alpha-1)s^{-(\alpha+2)}&\left[c_\phi(y_2'' c_{y_1} - y_1' y_2' s_{y_1} - y_1'\phi') - s_\phi(y_2'\phi' c_{y_1} + y_1'')\right]p \\ &+ s^{-(\alpha+1)}\left[c_\phi(y_1'' + y_2'\phi' c_{y_1}) + s_\phi(y_2'' \cos v_1 - y_1'\phi' - y_2' y_1' s_{y_1})\right] \end{aligned} \tag{6.13h}$$

where

$$\begin{aligned} r_1 &= y_1'' + (y_2')^2 \sin y_1 \cos y_2 \\ r_2 &= y_2'' \cos y_1 - 2y_2'' y_1' \sin y_2 \\ \cos\phi &= r_1(r_1^2 + r_2^2)^{-\frac{1}{2}} \\ \sin\phi &= r_2(r_1^2 + r_2^2)^{-\frac{1}{2}} \\ p &= y_1' \cos\phi + \psi' \cos y_1 \sin\phi \\ s &= y_2' \cos y_1 \cos\phi - y_1' \sin\phi \\ \phi' &= \frac{r_2' r_1 - r_1' r_2}{r_1^2 + r_2^2} \\ \phi'' &= \frac{(r_2'' r_1 - r_1'' r_2)(r_1^2 + r_2^2) - 2(r_1' r_1 + r_2' r_2)(r_2' r_1 - r_1' r_2)}{(r_1^2 + r_2^2)^2}. \end{aligned}$$

Let $(y, \ldots, y''') \mapsto \mathcal{A}(y, \ldots, y''')$ be the map given by the right-hand side of (6.13a)-(6.13f), and $(y, \ldots, y'''') \mapsto \mathcal{B}(y, \ldots, y'''')$ the map given by the right-hand side of (6.13g)-(6.13h). We observe that both $\mathcal{A}$ and $\mathcal{B}$ contain a singularity when $r_1^2 + r_2^2 = 0$.

Since $\gamma(\omega_3)$ is a time-scaling function only if $\omega_3 > 0$, an open- or closed-loop control obtained via an orbital flatness design can only be applied to a spacecraft that is continuously spinning about its uncontrolled axis. In this situation, the control would have to be designed to maintain $\omega_3(t) > 0$ for $t > 0$.

6.3 Simulation: Open-loop

In this section we design and simulate an open-loop orbital flatness-based controller that transfers the spacecraft from an initial configuration $x_1 = (\omega_1^1, \omega_2^1, \omega_3^1, \phi_1, \theta_1, \psi_1)$ at t_1 with $\omega_1^1 > 0$ to a final desired configuration $x_2 = (\omega_1^2, \omega_2^2, \omega_3^2, \phi_2, \theta_2, \psi_2)$ at t_2 with $\omega_1^2 > 0$. Since the system is orbitally flat, the open-loop control in the t-scale can be obtained by first designing the open-loop control in the τ-scale and then scaling the resulting control into the real time t. To design the open-loop controls in the τ-scale, the flat output reference trajectory, $y_d(\tau)$, must be designed so that

$$\begin{aligned} x_1 &= \mathcal{A}(y_d(\tau_1), \ldots, y_d'''(\tau_1)) \\ x_2 &= \mathcal{A}(y_d(\tau_2), \ldots, y_d'''(\tau_2)), \end{aligned} \tag{6.14}$$

where $\tau_1 = t_1$ and

$$\tau_2 = t_1 + \int_{t_1}^{t_2} \frac{1}{\gamma(\omega_3(r))}\, \mathrm{d}r = t_1 + \int_{t_1}^{t_2} [\omega_3(r)]^{\frac{\alpha+1}{\alpha}}\, \mathrm{d}r,$$

are the corresponding times in the τ-scale. From (6.14), we observe that we must impose conditions on $y_d(\tau_i)$ up to third-order, for $i = 1, 2$. By inspection, the zero-order conditions are

$$y_d(\tau_i) = (y_{d,1}(\tau_i), y_{d,2}(\tau_i)) = (\theta_i, \psi_i), \tag{6.15}$$

for $i = 1, 2$. From (6.3e) and (6.3f), the first-order conditions on $y_d(\tau)$ are

$$\begin{aligned} y_{d,1}'(\tau_i) &= (\omega_3^i)^{-\frac{\alpha+1}{\alpha}} (\omega_2^i \cos\phi_i - \omega_3^i \sin\phi_i) \\ y_{d,2}'(\tau_i) &= (\omega_3^i)^{-\frac{\alpha+1}{\alpha}} \left(\frac{\omega_2^i \sin\phi_i + \omega_3^i \cos\phi_i}{\cos\theta_i} \right) \end{aligned} \tag{6.16}$$

for $i = 1, 2$. Next, for compatibility with the value of ϕ_i for $i = 1, 2$, we need to impose second-order conditions on $y_d(\tau)$, as one can observe from (6.12). Thus, the second-order conditions on $y_d(\tau)$ must satisfy

$$\phi_i = \text{atan2}(\psi_i'' \cos\theta_i - 2\psi_i'\theta_i' \sin\theta_i, \theta_i'' + (\psi_i')^2 \sin\theta_i \cos\theta_i), \tag{6.17}$$

for $i = 1, 2$. Finally, from (6.11), for compatibility with ω_1^i for $i = 1, 2$, we need to impose third-order conditions on $y_d(\tau)$ since ϕ' contains third-order derivatives of the flat output. Thus, the third-order conditions on $y_d(\tau)$ must satisfy

$$\phi_i' = \left.\frac{r_2' r_1 - r_1' r_2}{r_1^2 + r_2^2}\right|_{t=t_i}, \tag{6.18}$$

where r_1' and r_2' are given in Appendix B. Since we are imposing conditions up to third-order, there are a total of eight conditions (four at each time end-point) for each component of the flat output. As done in previous open-loop designs, we choose a polynomial trajectory

$$y_{d,i}(\tau) = \sum_{k=1}^{N} a_k^i \tau^{k-1},$$

with $N = 2(r+1) = 8$, where r denotes the highest order imposed on the reference trajectory, in this case three. Solving for the coefficients a_k^i satisfying the constraints (6.15)-(6.18) yields the reference trajectory $y_d(\tau)$ parameterizing the open-loop input that transfers the spacecraft from x_1 to x_2, namely

$$v(\tau) = \mathcal{B}(y_d(\tau), y_d'(\tau), y_d''(\tau), y_d'''(\tau)).$$

The controls in the t-scale are then given by

$$u(t) = \frac{1}{\gamma(\omega_3(t))} v(\tau(t)) = [\omega_3(t)]^{\frac{\alpha+1}{\alpha}} \mathcal{B}(y_d(\tau(t)), \ldots, y_d'''(\tau(t))), \tag{6.19}$$

where $\omega_3(t)$ is written in terms of the flat output trajectory, that is, $\omega_3(t)$ is given by (6.13c). The results of an open-loop simulation using the control (6.19) is shown in Figure 6.2 for the case $\alpha = 1/2$. The simulation model is shown in Figure 6.1. The initial and final conditions were chosen as $x_1 = (-2, -2, 3, 120°, 65°, -135°)$ and $x_2 = (0, 0, 3.3, -100°, 0, 100°)$, respectively.

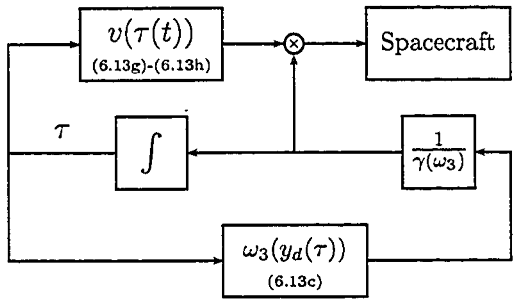


Figure. 6.1: Simulink Model - Open-loop orbital flatness implemented in the original time t.

The final and initial times in the τ-scale were chosen to be $\tau_1 = 0$ and $\tau_2 = 8.0$, which correspond to the initial and final times

$$t_1 = 0$$
$$t_2 = \int_{\tau_1}^{\tau_2} [\omega_3(\tau)]^{-\frac{\alpha+1}{\alpha}} \, d\tau = 2.0321.$$

The coefficients a_k^i for each flat output component were chosen so that the condition $\omega_3(t) > 0$ was satisfied, thus ensuring that the open-loop control remains bounded.

6.4 Closed-loop Control

In this section, we design a dynamic feedback linearizing controller for the time-scaled system (6.3). The dynamic feedback is constructed using the same procedure as in §5.1.2 for the case $\alpha = -1$. The controller is designed so that when $\alpha = -1$ the resulting expression for the dynamic extension simplifies to (5.13).

Consider the time-scaled system (6.3) with time-scaling function (6.8). Recall that the decoupling matrix of the system with flat output (6.5) is

$$A(x) = \begin{bmatrix} 0 & \gamma(x)\cos\phi \\ 0 & \gamma(x)\frac{\sin\phi}{\cos\theta} \end{bmatrix}.$$

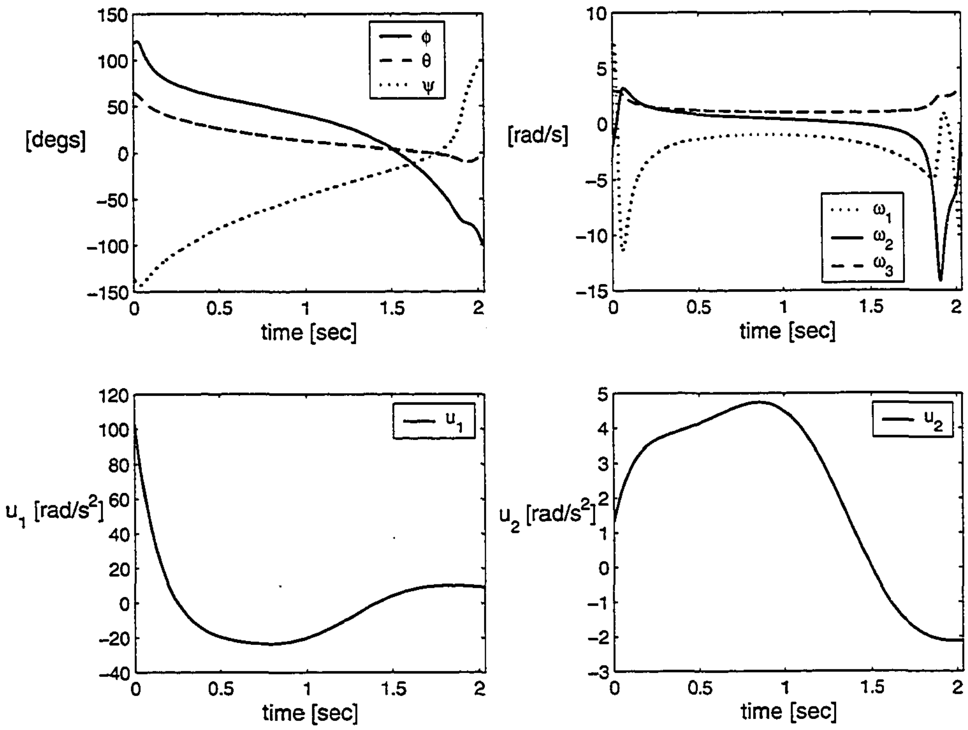


Figure. 6.2: Open-loop simulation with orbital flatness-based control with $\alpha = 1/2$. Initial and final conditions are $x_1 = (-2, -2, 3, 120°, 65°, -135°)$ and $x_2 = (0, 0, 3.3, -100°, 0, 100°)$, respectively.

Define the dynamic feedback

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= p(x) + q(x)\xi_1 \cos\theta \\ \dot{\xi}_1 &= w_2, \end{aligned} \tag{6.20}$$

where $p(x)$ and $q(x)$ are smooth functions yet to be designed and $w = (w_1, w_2)$ is the new control input. Let $\tilde{\Sigma} = (\tilde{f}, \{\tilde{g}_1, \tilde{g}_2\}, \mathbb{R}^2\})$ be the control-affine system obtained by applying the dynamic feedback (6.20) to the time-scaled system (6.3). Let $\tilde{x} = (x, \xi_1)$ denote the new extended state. The decoupling matrix of the extended system $\tilde{\Sigma}$ is

$$A_1(\tilde{x}) = \begin{bmatrix} L_{\tilde{g}_1} L_{\tilde{f}}^2 h_1 & L_{\tilde{g}_2} L_{\tilde{f}}^2 h_1 \\ L_{\tilde{g}_1} L_{\tilde{f}}^2 h_2 & L_{\tilde{g}_2} L_{\tilde{f}}^2 h_2 \end{bmatrix} = \begin{bmatrix} \frac{\cos\phi}{\omega_3}\Gamma(\tilde{x}) & \gamma(\omega_3) q(x) \cos\phi\cos\theta \\ \frac{\sin\phi}{\omega_3 \cos\theta}\Gamma(\tilde{x}) & \gamma(\omega_3) q(x) \sin\phi \end{bmatrix},$$

where

$$\Gamma(\tilde{x}) = \omega_3^{-\frac{1}{\alpha}} \left(\frac{\partial p}{\partial \omega_1} + \xi_1 \cos\theta \frac{\partial q}{\partial \omega_1} \right) - (\alpha+1)\omega_2^2 \gamma(\omega_3)^2 - \omega_3^{-\frac{2}{\alpha}}. \tag{6.21}$$

Note that by construction of $\gamma(\omega_3)$ in §6.2, the matrix $A_1(\tilde{x})$ has constant rank one (this can also be deduced by inspection). Since the matrix $A_1(\tilde{x})$ satisfies the conditions for the DEA we can proceed to perform a second iteration of the DEA. However, before proceeding to the next iteration, we can simplify the design of the dynamic feedback by imposing conditions on $p(x)$ and $q(x)$ such that $\Gamma(\tilde{x}) = 0$. Suppose that $\frac{\partial q}{\partial \omega_1} = 0$ and we want to solve $\Gamma(\tilde{x}) = 0$. Then from (6.21), the function $p(x)$ must satisfy

$$\mathrm{d}p = \left[(\alpha+1)\omega_2^2 \gamma(\omega_3)^2 \omega_3^{\frac{1}{\alpha}} + \omega_3^{-\frac{1}{\alpha}} \right] \mathrm{d}\omega_1. \tag{6.22}$$

Integrating both sides of (6.22) yields

$$p(x) = \left[(\alpha+1)\omega_2^2 \gamma(\omega_3)^2 \omega_3^{\frac{1}{\alpha}} + \omega_3^{-\frac{1}{\alpha}} \right] \omega_1. \tag{6.23}$$

Therefore, with $p(x)$ defined by (6.23) and choosing $q(x) = 1$, the decoupling matrix for the extended system $\tilde{\Sigma}$ simplifies to

$$A_1(\tilde{x}) = \begin{bmatrix} 0 & \gamma(\omega_3)\cos\phi\cos\theta \\ 0 & \gamma(\omega_3)\sin\phi \end{bmatrix}.$$

Compare the similarities between the decoupling matrix $A_1(\tilde{x})$ with the one given by (5.10) and the function $p(x)$ with the one obtained in (5.9). Now consider the dynamic feedback

$$w_1 = s_1$$
$$w_2 = \xi_2$$
$$\dot{\xi}_2 = s_2$$

applied to the extended system $\tilde{\Sigma}$, where $s = (s_1, s_2)$ is the new control input. Let $\bar{\Sigma} = \{\bar{f}, \{\bar{g}_1, \bar{g}_2\}, \mathbb{R}^2\}$ denote the new extended system with state $\bar{x} = (x, \xi_1, \xi_2) \in \mathsf{M} \times \mathbb{R}^2$. The decoupling matrix of $\bar{\Sigma}$ is

$$A_2(\bar{x}) = \begin{bmatrix} L_{\bar{g}_1} L_{\bar{f}}^3 h_1 & L_{\bar{g}_2} L_{\bar{f}}^3 h_1 \\ L_{\bar{g}_1} L_{\bar{f}}^3 h_2 & L_{\bar{g}_2} L_{\bar{f}}^3 h_2 \end{bmatrix} = \begin{bmatrix} -\frac{\xi_1 \cos\theta}{\omega_3^2} \Gamma_2(x) & \gamma(\omega_3) \cos\phi \cos\theta \\ -\frac{\xi_1}{\omega_3^2} \Gamma_3(x) & \gamma(\omega_3) \sin\phi \end{bmatrix},$$

where

$$\Gamma_2(x) = (\alpha + 1)\omega_2 \omega_3^{-\frac{2+\alpha}{\alpha}} \cos\phi + \omega_3^{-\frac{2}{\alpha}} \sin\phi$$

$$\Gamma_3(x) = (\alpha + 1)\omega_2 \omega_3^{-\frac{2+\alpha}{\alpha}} \sin\phi + \omega_3^{-\frac{2}{\alpha}} \cos\phi.$$

A direct calculation shows that

$$\det A_2(\bar{x}) = -\xi_1 \gamma(\omega_3)^3 \cos\theta,$$

and thus, the relative degree of the extended system $\bar{\Sigma}$ is $\{4, 4\}$ away from the set

$$\Lambda = \{(x, \xi_1, \xi_2) \in \mathsf{M} \times \mathbb{R}^2 \mid \xi_1 = 0\}.$$

Therefore, with the dynamic feedback

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= s_2 \\ v_1 &= s_1 \\ v_2 &= \left[(\alpha + 1)\omega_2^2 \gamma(\omega_3)^2 \omega_3^{\frac{1}{\alpha}} + \omega_3^{-\frac{1}{\alpha}}\right] \omega_1 + \xi_1 \cos\theta, \end{aligned} \tag{6.24}$$

the time-scaled two-input spacecraft (6.3) is dynamic feedback linearizable on the set $(\mathsf{M} \times \mathbb{R}^2) \backslash \Lambda$. It is worth mentioning that when $\alpha = -1$, the dynamic feedback (6.24) reduces to the dynamic feedback (5.13).

By construction, applying the dynamic feedback (6.24) to the time-scaled system (6.3) yields

$$\begin{bmatrix} y_1^{(4)} \\ y_2^{(4)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_{\bar{f}}^4 h_1(\bar{x}) \\ L_{\bar{f}}^4 h_2(\bar{x}) \end{bmatrix}}_{b(\bar{x})} + A_2(\bar{x}) s.$$

Away from the set Λ, we can define the input

$$s = A_2^{-1}(\bar{x})(-b(\bar{x}) + w),$$

which yields the linear input-output dynamics

$$\begin{aligned} y_1^{(4)} &= w_1 \\ y_2^{(4)} &= w_2. \end{aligned} \tag{6.25}$$

The control input w can then be designed to track a desired reference trajectory $y_d(t) = (y_{d,1}(t), y_{d,2}(t))$. Note that since we defined the input (6.4), the control inputs that must be applied to the time-scaled system are

$$u = \frac{1}{\gamma(\omega_3)} v = \omega_3^{\frac{\alpha+1}{\alpha}} v, \tag{6.26}$$

where v is given by (6.24). It is important to emphasize that the closed-loop control (6.26) is defined only when $\omega_3 > 0$ and $\xi_1 \neq 0$. The condition $\omega_3 > 0$ arises from the constraints imposed on the time-scaling function $\gamma(\omega_3)$ and the condition $\xi_1 \neq 0$ arises from the dynamic feedback control design.

As done in §5.1.3, we can derive a relationship for the extended state ξ_1 in terms of the linearizing coordinates z given by

$$\begin{aligned} z_1 &= h_1(x) & z_5 &= h_2(x) \\ z_2 &= L_{\bar{f}} h_1(x) & z_6 &= L_{\bar{f}} h_2(x) \\ z_3 &= L_{\bar{f}}^2 h_1(x) & z_7 &= L_{\bar{f}}^2 h_2(x) \\ z_4 &= L_{\bar{f}}^3 h_1(x) & z_8 &= L_{\bar{f}}^3 h_2(x). \end{aligned}$$

The expressions for z_3 and z_7 take the form

$$\begin{aligned} z_3 &= -z_6^2 \sin z_1 \cos z_1 + \xi_1 \gamma(\omega_3) \cos\phi \cos z_1 \\ z_7 &= \frac{2 z_2 z_6 \sin z_1 + \xi_1 \gamma(\omega_3) \sin\phi \cos z_1}{\cos z_1}, \end{aligned} \tag{6.27}$$

which are similar to the ones obtained in (5.17). Solving (6.27) for ϕ and ξ_1 yields

$$\phi = \operatorname{atan2}\big(z_7 \cos z_1 - 2z_2 z_6 \sin z_1,\, z_3 + z_6^2 \cos z_1 \sin z_1\big) \tag{6.28}$$

$$\xi_1 = \frac{\sqrt{(z_3 + z_6^2 \cos z_1 \sin z_1)^2 + (z_7 \cos z_1 - 2z_2 z_6 \sin z_1)^2}}{\gamma(\omega_3)\cos z_1}. \tag{6.29}$$

As in §5.1.3, we observe that the singularity $r_1^2 + r_2^2 = 0$ in the maps $\mathcal{A}$ and $\mathcal{B}$ is equivalent to the singularity $\xi_1 = 0$.

6.5 Simulation: Trajectory Tracking

In this section we present simulation results of the dynamic feedback (6.24) applied to the time-scaled system (6.3). The value of α in the simulation is set to

$$\alpha = \frac{1}{2}.$$

The initial and final conditions of the spacecraft are

$$x_1 = (-2.0, -2.0, 3.0, 120°, 65°, -135°) \quad \text{and} \quad x_2 = (0, 0, 3.3, -90°, 0, 0),$$

at $t_1 = 0$ and $t_2 = 8$ seconds, respectively,. To illustrate the tracking performance of the dynamic feedback we set the actual initial condition to $\tilde{x}_1 = (-1, -1, 3.5, 130°, 55°, -125°)$. The tracking error eigenvalues (6.25) are chosen to be $\{-1.5+0.9i, -1.5-0.9i, -1.2+0.6i, -1.2-0.6i\}$ for each flat output trajectory. The results of the simulation are shown in Figure 6.3. As one can observe, the dynamic controller tracks the reference trajectory on the interval $[t_1, t_2]$.

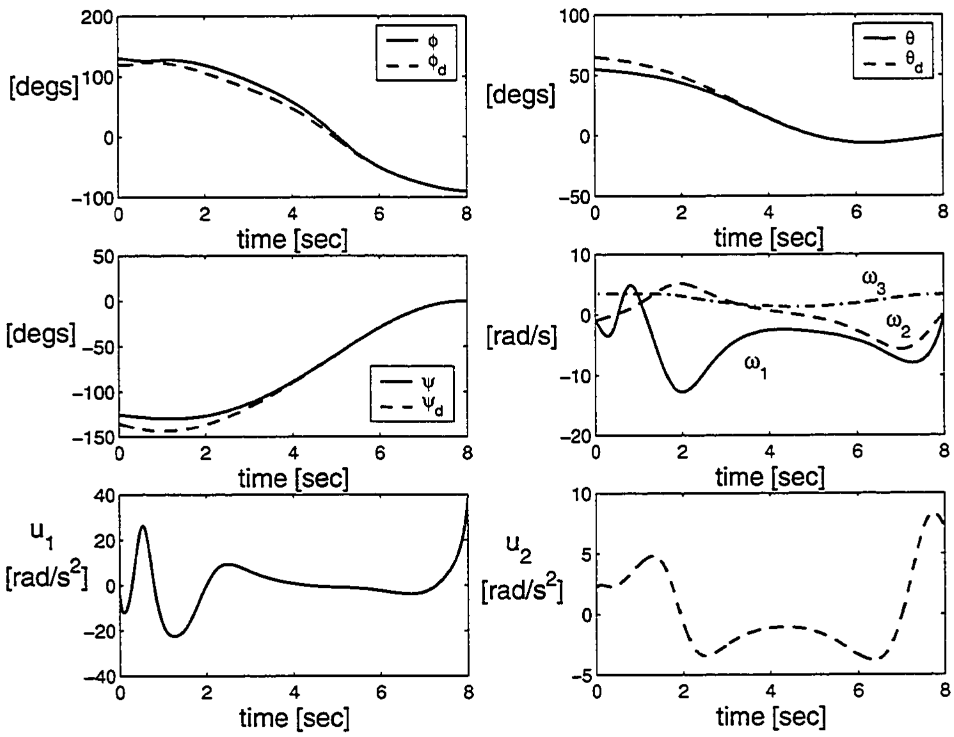


Figure. 6.3: Closed-loop simulation with orbital flatness-based control on the time-scaled system (6.3) with $\alpha = 1/2$. The assumed initial condition is $x_1 = (-2, -2, 3, 120°, 65°, -135°)$ and the final desired condition is $x_2 = (0, 0, 3.3, -90°, 0, 0°)$. The actual initial condition is set to $\tilde{x}_1 = (-1, -1, 3.5, 130°, 55°, -125°)$.

Chapter 7

Summary and Future Work

7.1 Summary

In this thesis, we considered the attitude control problem for a two-input rigid spacecraft. We first analyzed the fundamental properties of local controllability and stabilizability. It was shown that the two-input spacecraft is locally controllable about any equilibrium, but that no equilibrium can be made asymptotically stable using a continuous time-invariant control. Motivated by the work of Rouchon (1992), it was shown that the two-input spacecraft is flat provided a particular geometric condition is satisfied involving the principal moments of inertia of the spacecraft. As is common for flat systems, the expressions of the state and input in terms of the flat output contain a singularity at an equilibrium. A detailed analysis of the specific singularity was performed and it was proved that by appropriately assigning the high-order derivatives of the flat output trajectory, an open-loop continuous control can be designed that steers the spacecraft from an initial to a final desired equilibrium in finite time.

We then proceeded to design a closed-loop state tracking controller. By virtue of the flatness property, the closed-loop state tracking problem is reduced to the lower dimensional output tracking problem. To design an output tracking controller, the Dynamic Extension Algorithm (DEA) was employed to construct a dynamic state feedback that input-output linearized the system. As expected, a singularity in the dynamic state feedback restricts its domain away from the equilibrium point set of the system. Nonetheless, it was proved that the dynamic controller can be used to track a state trajectory and bring the spacecraft to an arbitrarily small neighbourhood of a desired final equilibrium in finite time by rendering the flat output error tracking dynamics exponentially stable. Since the dynamic controller is undefined on the system's equilibrium point set, to obtain an asymptotic state tracking controller the control law must be switched as the system approaches the equilibrium. At a pre-determined time, in which the state can be made arbitrarily close to the desired equilibrium, the control law is switched to the exponentially stabilizing controller of Morin and Samson (1997). The two phase controller, valid for large motions of the spacecraft, ensures that the state trajectory asymptotically converges to the desired equilibrium.

In the last chapter of the thesis, an analysis was performed concerning the orbital flatness property of the satellite. We derive a time-scaling function such that the system in the new time-scale is flat. The validity of a control design based on the orbital flatness property is restricted to a spacecraft that continuously spins about its uncontrolled axis. An open-loop control was constructed steering the system from one configuration to another. The open-loop control was obtained by first performing the design in the new time-scale and then re-scaling the controls. A closed-loop controller was also constructed for the time-scaled system using the DEA. The control law was implemented on the time-scaled system and a simulation is included illustrating the tracking performance of the controller.

7.2 Future Work

A natural extension to the work in this thesis would be to construct a flat output for the general case $\alpha \neq \pm 1$, if it exists at all. In particular, a flat output for a singular free kinematic parameterization, such as the quaternions, would be desirable. This seems to be a challenging task as no general method exists for constructing a flat output or determining whether a system is flat. We conjecture here that the two-input spacecraft is not flat for the general case $\alpha \neq \pm 1$.

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Appendix A

Definitions from Differential Geometry

We have included this appendix to review basic notions and definitions from differential geometry since many of the topics we talk about in the thesis assume a basic knowledge of this theory. For a complete introduction to differential geometry see the text by Lee (2003).

Henceforth, let M be an open set of the n-dimensional Euclidean space $\mathbb{R}^n$. A function $h : \mathsf{M} \to \mathbb{R}$ is *smooth at* $x \in \mathsf{M}$ if the function is continuous and its partial derivatives of any order exist and are continuous in a neighbourhood of x. The function h is smooth if it is smooth for all $x \in \mathsf{M}$. The set of all smooth functions on M is denoted by $C^\infty(\mathsf{M})$. A map $\Phi : \mathsf{M} \to \mathbb{R}^n$ is *smooth* if each component of Φ is a smooth function. A *diffeomorphism* is a smooth bijective map that has a smooth inverse.

For each $x \in \mathsf{M}$ we define $\mathsf{T}_x\mathsf{M} = \{(x, v) \mid v \in \mathbb{R}^n\}$ as the *tangent space of* M *at* $x \in \mathsf{M}$. A *tangent vector* at x is an element of $\mathsf{T}_x\mathsf{M}$, and will be denoted by v_x. Geometrically, v_x is the vector v with initial point at x. The tangent space $\mathsf{T}_x\mathsf{M}$ is a real vector space and is clearly isomorphic to $\mathbb{R}^n$, $\mathsf{T}_x\mathsf{M} \simeq \mathbb{R}^n$. The *tangent bundle* of M, denoted TM, is the disjoint union of all the tangent spaces $\mathsf{T}_x\mathsf{M}$:

$$\mathsf{TM} = \coprod_{x \in \mathsf{M}} \mathsf{T}_x\mathsf{M}.$$

Since we are working locally on the open set M, we can think of TM as being equivalent to the open set $\mathsf{M} \times \mathbb{R}^n$, and thus TM is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$.

property that $X(x) \in \mathsf{T}_x\mathsf{M}$. Thus, a vector field X assigns to each $x \in \mathsf{M}$ a tangent vector in $\mathsf{T}_x\mathsf{M}$. A vector field can be written as a $n \times 1$ column vector

$$X(x) = \begin{pmatrix} X_1(x) \\ X_2(x) \\ \vdots \\ X_n(x) \end{pmatrix},$$

where the X_i are the components of X. Let $\mathfrak{X}(\mathsf{M})$ denote the set of all smooth vector fields on M. It is straightforward to show that the set $\mathfrak{X}(\mathsf{M})$ is a $\mathbb{R}$-vector space under pointwise addition and scalar multiplication. An *integral curve* of the vector field X at x is a smooth curve $\gamma : (-\epsilon, \epsilon) \to \mathsf{M}$ such that $\gamma'(t) = X(\gamma(t))$ and $\gamma(0) = x$. Given a smooth function $h : \mathsf{M} \to \mathbb{R}$ and a vector field $X : \mathsf{M} \to \mathsf{TM}$, we define the *Lie derivative* of h along X by the formula

$$L_X h(x) = \frac{\partial h}{\partial x} X(x).$$

Geometrically, the Lie derivative of h along X is the rate of change of h along the integral curves of X.

Definition A.1. A *Lie algebra* is a real vector space V endowed with a bilinear map called the *bracket*, usually denoted by $V \times V \ni (X, Y) \mapsto [X, Y] \in V$, that satisfies

(i) anti-symmetry: for all $v_1, v_2 \in V$

$$[v_1, v_2] = -[v_2, v_1],$$

(ii) Jacobi identity: for all $v_1, v_2, v_3 \in V$

$$[v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0.$$

An example of a Lie algebra is $(\mathbb{R}^n, \times)$, where $\times$ is the cross product. If V is a Lie algebra, a linear subspace $W \subset V$ is called a *Lie subalgebra* of V if it is closed under the bracket operation. Thus, a Lie subalgebra is a Lie algebra with the bracket operation inherited from the parent space.

Given two smooth vector fields X and Y defined on M, we define the *Lie bracket* of X and Y, usually denoted as $[X, Y]$, as the smooth vector field

$$[X, Y] = \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y.$$

Geometrically, the Lie bracket $[X, Y]$ is the directional derivative of the vector field Y along the integral curves of X. In many cases we will use the notation $\mathrm{ad}_X Y = [X, Y]$ which allows us to define iterated Lie brackets via the relation $\mathrm{ad}_X^k Y = [X, \mathrm{ad}_X^{k-1} Y]$ for any $k \geq 1$ setting $\mathrm{ad}_X^0 Y = Y$. It can be verified that the Lie bracket is anti-symmetric and satisfies the Jacobi identity, and thus $(\mathfrak{X}(\mathsf{M}), [\cdot, \cdot])$ is a Lie algebra.

Definition A.2. A *distribution* D on M is an assignment to each $x \in \mathsf{M}$ a linear subspace $D(x) \subset \mathsf{T}_x\mathsf{M}$. We will say that D is a *smooth distribution* if for each $x_0 \in \mathsf{M}$ there exists a neighbourhood U_0 of x_0 and smooth vector fields $X_1, \ldots, X_d$ defined on U_0 such that $D(x) = \mathrm{span}\{X_1(x), \ldots, X_d(x)\}$ for each $x \in U_0$.

The vector fields $X_1, \ldots, X_d$ in the above definition are called the *local generators* of D about x_0. A point $x_0 \in \mathsf{M}$ is a *regular* point of a distribution D if there exists a neighbourhood U_0 of x_0 such that for all $x \in U_0$ we have that the $\dim(D(x)) = d$ for some constant integer d. We say that the vector field $X \in D$ if $X(x) \in D(x)$ for all $x \in \mathsf{M}$.

Definition A.3. A smooth distribution D on M is *involutive* if for any pair of vector fields $X_1, X_2 \in D$, $[X_1, X_2]$ is also a vector field in D, i.e., if it is closed under the Lie bracket.

From the above definition it follows that if $\mathcal{S}$ is a subalgebra of the Lie algebra $(\mathfrak{X}(\mathsf{M}), [\cdot, \cdot])$ then $S(x) = \mathrm{span}\{X(x) : X \in \mathcal{S}\}$ is an involutive distribution. The *involutive closure* of a smooth distribution D, denoted $\mathrm{Lie}(D)$, is the smallest (with respect to inclusion) involutive distribution containing D. One can show that

$$\mathrm{Lie}(D) = \bigcap_{i \in I} D_i,$$

where D_i is an involutive distribution containing D for each $i \in I$.

Appendix B

Expressions for r_1 and r_2

$$r_1 = \ddot{\theta} + \frac{1}{2}\dot{\psi}^2 \sin(2\theta)$$

$$r_2 = \ddot{\psi}\cos(\theta) - 2\dot{\psi}\dot{\theta}\sin(\theta)$$

$$\dot{r}_1 = \theta^{(3)} + \dot{\psi}\ddot{\psi}\sin(2\,\theta) + \dot{\psi}^2\dot{\theta}\cos(2\,\theta)$$

$$\dot{r}_2 = \psi^{(3)}\cos(\theta) - 3\ddot{\psi}\dot{\theta}\sin(\theta) - 2\dot{\psi}\ddot{\theta}\sin(\theta) - 2\dot{\psi}\dot{\theta}^2\cos(\theta)$$

$$\ddot{r}_1 = \theta^{(4)} + (\ddot{\psi}^2 + \dot{\psi}\psi^{(3)} - 2\dot{\theta}^2\dot{\psi}^2)\sin(2\,\theta) + (4\dot{\theta}\dot{\psi}\ddot{\psi} + \dot{\psi}^2\ddot{\theta})\cos(2\,\theta)$$

$$\ddot{r}_2 = (2\dot{\psi}\dot{\theta}^3 - 4\psi^{(3)}\dot{\theta} - 5\ddot{\psi}\ddot{\theta} - 2\dot{\psi}\theta^{(3)})\sin(\theta) + (\psi^{(4)} - 5\ddot{\psi}\dot{\theta}^2 - 6\dot{\psi}\ddot{\theta}\dot{\theta})\cos(\theta)$$

$$\begin{aligned} r_1^{(3)} &= \theta^{(5)} + (3\ddot{\psi}\psi^{(3)} + \dot{\psi}\psi^{(4)} - 12\dot{\psi}\dot{\theta}^2\ddot{\psi} - 6\dot{\psi}^2\ddot{\theta}\dot{\theta})\sin(2\,\theta) \\ &\quad + (6\dot{\psi}\dot{\theta}\psi^{(3)} + 6\ddot{\psi}^2\dot{\theta} + \dot{\theta}^2\theta^{(3)} + 6\dot{\psi}\ddot{\theta}\ddot{\psi} - 4\dot{\psi}^2\dot{\theta}^3)\cos(2\,\theta) \end{aligned}$$

$$\begin{aligned} r_2^{(3)} &= (7\ddot{\psi}\dot{\theta}^3 - 5\psi^{(4)}\dot{\theta} - 9\psi^{(3)}\ddot{\theta} + 12\dot{\psi}\ddot{\theta}\dot{\theta}^2 - 7\ddot{\psi}\theta^{(3)} - 2\dot{\psi}\theta^{(4)})\sin(\theta) \\ &\quad + (\psi^{(5)} - 9\psi^{(3)}\dot{\theta}^2 - 8\dot{\psi}\theta^{(3)}\dot{\theta} - 21\ddot{\psi}\dot{\theta}\ddot{\theta} - 6\dot{\psi}\ddot{\theta}^2 + 2\dot{\psi}\dot{\theta}^4)\cos(\theta) \end{aligned}$$

$$r_1^{(4)} = \theta^{(6)} + \Pi_1\sin(2\,\theta) + \Pi_2\cos(2\,\theta)$$

$$r_2^{(4)} = \psi^{(6)}\cos\theta + \Omega_1\sin(\theta) + \Omega_2\cos(\theta)$$

$$r_1^{(5)} = \theta^{(7)} + \left(\dot{\Pi}_1 - 2\dot{\theta}\Pi_2\right)\sin(2\theta) + \left(\dot{\Pi}_2 + 2\dot{\theta}\Pi_1\right)\cos(2\theta)$$

$$r_2^{(5)} = \psi^{(7)}\cos\theta + \left(\dot{\Omega}_1 - \psi^{(6)}\dot{\theta} - \Omega_2\dot{\theta}\right)\sin\theta + \left(\Omega_1\dot{\theta} + \dot{\Omega}_2\right)\cos\theta.$$

where

$$
\begin{aligned}
\Pi_1 &= 4\ddot{\psi}\psi^{(4)} + \dot{\psi}\psi^{(5)} - 24\ddot{\psi}^2\dot{\theta}^2 - 48\dot{\psi}\dot{\theta}\ddot{\psi}\ddot{\theta} - 24\dot{\psi}\dot{\theta}^2\psi^{(3)} \\
&\quad - 6\dot{\psi}^2\ddot{\psi}^2 - 8\dot{\psi}^2\dot{\theta}\theta^{(3)} + 3\left(\psi^{(3)}\right)^2 + 8\dot{\psi}^2\dot{\theta}^4
\end{aligned}
$$

$$
\begin{aligned}
\Pi_2 &= 8\dot{\psi}\theta^{(3)}\ddot{\psi} + 2\left(\dot{\psi}\psi^{(4)} - 12\dot{\psi}\dot{\theta}^2\ddot{\psi} - 6\dot{\psi}^2\dot{\theta}\ddot{\theta} + 3\ddot{\psi}\psi^{(3)}\right)\dot{\theta} + 18\ddot{\psi}\dot{\theta}\psi^{(3)} \\
&\quad + 12\ddot{\psi}^2\ddot{\theta} + 12\dot{\psi}\ddot{\theta}\psi^{(3)} + \\
&\quad 6\dot{\psi}\dot{\theta}\psi^{(4)} - 12\dot{\psi}^2\dot{\theta}^2\ddot{\theta} - 8\dot{\psi}\dot{\theta}^3\ddot{\psi} + \dot{\psi}^2\theta^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_1 &= -6\psi^{(5)}\dot{\theta} - 14\psi^{(4)}\ddot{\theta} - 16\psi^{(3)}\theta^{(3)} + 16\psi^{(3)}\dot{\theta}^3 + 54\ddot{\psi}\dot{\theta}^2\ddot{\theta} + 20\dot{\psi}\theta^{(3)}\dot{\theta}^2 \\
&\quad + 30\dot{\psi}\ddot{\theta}^2\dot{\theta} - 9\ddot{\psi}\theta^{(4)} - 2\dot{\psi}\theta^{(5)} - 2\dot{\psi}\dot{\theta}^5
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2 &= \left(-5\psi^{(4)}\dot{\theta} - 9\psi^{(3)}\ddot{\theta} + 7\ddot{\psi}\dot{\theta}^3 + 12\dot{\psi}\ddot{\theta}\dot{\theta}^2 - 7\ddot{\psi}\theta^{(3)} - 2\dot{\psi}\theta^{(4)}\right)\dot{\theta} \\
&\quad - 39\psi^{(3)}\dot{\theta}\ddot{\theta} - 27\ddot{\psi}\ddot{\theta}^2 \\
&\quad - 29\ddot{\psi}\dot{\theta}\theta^{(3)} - 9\psi^{(4)}\dot{\theta}^2 + 2\ddot{\psi}\dot{\theta}^4 + 8\dot{\psi}\dot{\theta}^3\ddot{\theta} - 8\dot{\psi}\theta^{(4)}\dot{\theta} - 20\dot{\psi}\theta^{(3)}\ddot{\theta}
\end{aligned}
$$

Appendix C

Maple Program: Polynomial Interpolation

```
trajectory_coeffs:=proc(times, conds)

local k, m, n, c, y, Eqns, i, j, Full_Eqns, A;

# Description of arguments:
# 1. times: the argument "times" is a list containing the
# times in which one imposes the conditions on the
# reference trajectory. For instance, if one wants a
# trajectory to start at t1 and end at tM, and one puts
# conditions ONLY at the end points t1 and tM then
# times:=[t1,tM]. If one imposes conditions on
# intermediate times t2,t3,...,t(M-1) between t1 and tM,
# then times:=[t1,t2,t3,...,tM].
# 2. conds: the "conds" argument is a list containing the
# conditions at each time t1,t2,...,tM. "conds" is a list
# of lists with conds[i] containing the conditions on the
# reference trajectory at time t_i, i=1,2,...,M. So for
# instance, if we put m conditions only at the end-point
# t1 and tM, then conds:=[[c1,c2,...,cm],[d1,d2,...,dm]];
# where c_k are the conditions at time t1 and d_k are the
# conditions at time tM. If there are intermediate points,
# then conds:=[[...],[...],[...],...,[...]];, etc.
# The OUTPUT of the file is a list in which the first element
# is the the polynomial reference trajectory y and the
# second element is the coefficient matrix Pi.

######################## START ##########################

# number of conditions for each time and the degree of
# polynomial
k:=nops(times);       # Number of times where we have put
                      # conditions, i.e, M.
m:=nops(conds[1]);    # At each t_i it is assumed that
                      # we have placed the same number
                      # of conditions.
n:= k*m - 1;          # The required degree for the
                      # polynomial.
```

```
# create the array which will hold the coefficients
c:= array(1..k*m);

# create the polynomial symbolically
y:= unapply( sum('c[i+1]*t^i', 'i'=0..n), t);

# setup the arrays that will hold the linear equations
Eqns:=array(1..k);
for i from 1 to k do
   Eqns[i]:= array(1..m);
end do;

# setup equations, first for the 0-derivative conditions
for i from 1 to k do
   Eqns[i][1]:= y(times[i]) = conds[i][1];
end do;

# continue setting up equations for the derivatives
for i from 1 to k do
  for j from 1 to m-1 do
    Eqns[i][j+1]:= eval(diff(y(t),t\$j),t=times[i]) =
    conds[i][j+1];
  end do;
end do;

# create a list called Full_Eqns which stores all k*m
# linear equations
Full_Eqns:=[pop(Eqns[1])];
for i from 2 to k do
  Full_Eqns:=[pop(Full_Eqns), pop(Eqns[i])];
end do;

# form the coefficient matrix
A:=matrix(k*m,k*m,[]);
for j from 1 to k*m do
  for i from 1 to k*m do
    A[i,j]:= coeff(lhs(Full_Eqns[i]), c[j]);
  end do;
end do;

# now return the desired trajectory

if det(A) = 0 then
  print("Cannot solve for the coefficients, the
         coefficient matrix is singular.");
  RETURN(evalm(A));
else
  sol_c:=map(simplify, solve({'Full_Eqns[j]'\$'j'=1..k*m},
             {'c[j]'\$'j'=1..k*m}));
  # return polynomial and coefficient matrix
  RETURN([subs(sol_c, y(t)), evalm(A)]);
end if;

end;
```