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FILTRATIONS ON HIGHER CHOW GROUPS AND ARITHMETIC NORMAL FUNCTIONS

by

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To my father Cirilo Hernández Alvarado, my mother Paula Castillo González and my sister María Paulina Hernández Castillo

Abstract

We first recall the filtration $F^\bullet$ on the higher Chow group $\mathrm{CH}^r(X, m; \mathbb{Q})$ of a complex smooth projective variety X as done by J. Lewis (for $m = 0$), and separately by M. Saito / M. Asakura and explain the various invariants (Mumford-Griffiths and de Rham), as well as the notion of arithmetic normal functions due to M. Kerr and J. Lewis. As in the case of Griffiths' use of normal functions for $m = 0$ to detect interesting cycles, we do the same thing for the higher Chow groups.

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Chapter 1

Introduction

An important notion in transcendental algebraic geometry is that of explaining invariants on a complex smooth projective variety X by algebraically defined objects on it. In this context we are interested in studying Chow groups. We consider algebraic cycles on the variety and consider the group formed by them. An algebraic cycle of codimension r is a formal sum of codimension r irreducible subvarieties in X. Modulo an adequate equivalence relation, called rational equivalence, we get the Chow group of codimension r cycles on X. We expect, for instance, that the rational cohomology classes of type (r, r) of X should be generated by fundamental classes of rational elements in the Chow group of codimension r cycles on X. This is the famous Hodge conjecture.

Let X be a complex smooth projective variety. We would like to find good invariants to better understand the structure of $\mathrm{CH}^r(X; \mathbb{Q})$, the Chow group of codimension r cycles on X with rational coefficients. Consider the cycle class map

$$c_r : \mathrm{CH}^r(X; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r}(X, \mathbb{Q}(r))).$$

This map is expected to be surjective (Hodge Conjecture). Then we look at the kernel of this map and denote it by $\mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q})$. We get the Abel-Jacobi map

$$\mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-1}(X, \mathbb{Q}(r))).$$

This map is neither surjective nor injective when $r > 1$, as was proved by Griffiths ([22]) and Mumford ([41]) respectively; it is an isomorphism when

$r = 1$ (see [34],[36]). However these two maps give us the idea that if we look at the kernel of the Abel-Jacobi map, we could capture these cycles by higher extension groups in the category of mixed Hodge structures. This is unfortunately false because $\mathrm{Ext}^{\ell}_{\mathrm{MHS}} = 0$ for $\ell \geq 2$. Nevertheless, we still expect to define a filtration on $\mathrm{CH}^r(X;\mathbb{Q})$ with certain properties, which we will call a filtration of Bloch-Beilinson type, and find maps to Hodge theoretic invariants. In [35] we can find a construction of such a filtration by J. Lewis:

Theorem 1.1. Let X be a complex smooth projective variety X. Then, for all r, there is a descending filtration $\{F^j\mathrm{CH}^r(X;\mathbb{Q})\}$,

$$\begin{aligned}\mathrm{CH}^r(X;\mathbb{Q}) = F^0\mathrm{CH}^r(X;\mathbb{Q}) \supset F^1\mathrm{CH}^r(X;\mathbb{Q}) \supset \cdots \\ \cdots \supset F^j\mathrm{CH}^r(X;\mathbb{Q}) \supset F^{j+1}\mathrm{CH}^r(X;\mathbb{Q}) \supset \cdots\end{aligned}$$

which satisfies the following:

(i) $F^1\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$.

(ii) F^j is preserved under the action of correspondences.

(iii) $F^j\mathrm{CH}^r(X;\mathbb{Q}) \bullet F^{\ell}\mathrm{CH}^s(X;\mathbb{Q}) \subset F^{j+\ell}\mathrm{CH}^{r+s}(X;\mathbb{Q})$, under the intersection product.

(iv) Assume that the components of the diagonal are algebraic. Then

$$\Delta_X(2d-2r+\ell, 2r-\ell)_*|_{Gr^j_F\mathrm{CH}^r(X;\mathbb{Q})} = \begin{cases}\text{Identity} & \text{, if } \ell = j\\ 0 & \text{, otherwise}\end{cases}$$

where $Gr^j_F\mathrm{CH}^r(X;\mathbb{Q}) = F^j\mathrm{CH}^r(X;\mathbb{Q})/F^{j+1}\mathrm{CH}^r(X;\mathbb{Q})$.

(v) $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \ker\{\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-1}(X,\mathbb{Q}(r)))\}$.

(vi) $F^{r+1}\mathrm{CH}^r(X;\mathbb{Q}) = F^{r+2}\mathrm{CH}^r(X;\mathbb{Q}) = F^{r+3}\mathrm{CH}^r(X;\mathbb{Q}) = \cdots$

If we assume that

$$\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-1}(X,\mathbb{Q}(r)))$$

is injective for X smooth quasiprojective over $\bar{\mathbb{Q}}$, then Lewis also proves that $D^r(X;\mathbb{Q}) := \bigcap_{j\geq 0} F^j\mathrm{CH}^r(X;\mathbb{Q}) = 0$.

In [38] another filtration is defined. If we assume that the components of the diagonal are algebraic and $D^r(X;\mathbb{Q}) = 0$ then both filtrations are the same. Lewis and Saito define the space of Mumford-Griffiths invariants $\nabla J^{r,j}(X/\mathbb{C})$ and the space of de Rham invariants $\nabla DR^{r,j}(X/\mathbb{C})$ (see Chapter (5)) and construct maps between graded pieces of the filtration and these invariants

$$Gr_F^j \mathrm{CH}^r(X;\mathbb{Q}) \to \nabla J^{r,j}(X/\mathbb{C}),$$

$$Gr_F^j \mathrm{CH}^r(X;\mathbb{Q}) \to \nabla DR^{r,j}(X/\mathbb{C}).$$

Then they give conditions for which the kernel and image of these maps are "uncountably large".

The next natural step is to try to reproduce all these concepts and results for higher Chow groups, as well a generalization of Griffiths' use of normal functions to detect interesting cycles. This involves explaining a mountain of technical material, such as the aforementioned filtration for higher Chow groups $\mathrm{CH}^r(X,m;\mathbb{Q})$. This was done in [5] and we basically follow the ideas there. This involves M. Saito's theory of mixed Hodge modules. Then we recall the analogous definition of higher Mumford-Griffiths and de Rham invariants and the corresponding maps

$$Gr_F^j \mathrm{CH}^r(X,m;\mathbb{Q}) \to \nabla J^{r,m,j}(X/\mathbb{C}), \tag{1.1}$$

$$Gr_F^j \mathrm{CH}^r(X,m;\mathbb{Q}) \to \nabla DR^{r,m,j}(X/\mathbb{C}).$$

We also explain M. Saito's argument that the image of the two maps is the same (see [46]). That is, we explain that for a cycle in $\mathrm{CH}^r(X,m;\mathbb{Q})$ its image in $\nabla J^{r,m,j}(X/\mathbb{C})$ and its image in $\nabla DR^{r,m,j}(X/\mathbb{C})$ are the same, i.e. one vanishes if the other does. This answers a question posed in [38], where they state this equivalence as a conjecture for $m = 0$ and assume it in several theorems. Actually, we explain M. Saito's argument that both invariants factor through the space $\underline{\underline{E}}_{\infty}^{j,2r-m-j}(\eta)$. This space (defined in Chapter (6)) is given by hom of two mixed Hodge structures and provides a more natural way to deal with the Mumford-Griffiths invariant of a cycle.

Under some conjectural assumptions we prove some basic results about the image and kernel of (1.1). Then we recall a new filtration due to Kerr/Lewis ([31]), constructed using the theory of arithmetic normal functions. These "higher" normal functions coincide with classical normal functions in the case $m = 0$ and provide means to define a more geometrical filtration and help us to determine conditions to find indecomposable cycles, thus generalizing some earlier ideas of Griffiths on normal functions. New results and directions are explained in the final two chapters.

Chapter 2

Higher Chow groups

Let X be a quasiprojective variety over a field k. For $m \in \mathbb{N}$, define the "simplex" Δ^m by

$$\Delta^m = \operatorname{Spec}\left(\frac{k[t_0, ..., t_m]}{\sum_{j=0}^{m} t_j - 1}\right).$$

The codimension one faces of the simplex Δ^m are the $(m+1)$ linear hypersurfaces in Δ^m obtained by setting the coordinate $t_j = 0$ of t_j in Δ^m. By intersecting the codimension one faces, one gets codimension $(m-n)$-faces isomorphic to Δ^n for every $n < m$. These faces are parametrized by strictly increasing maps $\rho : \{1, \ldots, n\} \to \{1, \ldots, m\}$, which are characterized by the conditions p_i goes to $p_{\rho(i)}$, where $p_i = (0, \ldots, 1, \ldots, 0)$ are the vertices of the simplex, making Δ^m a simplicial set.

Put $Z^r(X \times \Delta^m) =$ set of cycles of $X \times \Delta^m$ of codimension r. If ξ is a cycle in $Z^r(X \times \Delta^m)$, and every irreducible component of ξ meets all faces $X \times \Delta^n$ in codimension at least r for $n < m$, we say that ξ meets $X \times \Delta^m$ properly. Set

$$Z^r(X, m) := \{\xi \in Z^r(X \times \Delta^m) |\ \xi \text{ meets } X \times \Delta^m \text{ properly}\}.$$

Let $\bar{\partial}_j : Z^r(X, m) \to Z^r(X, m-1)$ be the restriction map to the j-th codimension one face for $j = 0, \ldots, m$ and let $\partial_m = \sum_{j=0}^{m} (-1)^j \bar{\partial}_j$. The boundary map ∂_m satisfies $\partial_m \circ \partial_{m+1} = 0$.

Definition 2.1. The mth higher Chow group of X in codimension r, denoted by $\mathrm{CH}^r(X, m)$, is defined as the mth homology group of the complex $\{Z^r(X, m), \partial_m\}$.

If $r > m + \dim X$, then it is clear by the definition that $\mathrm{CH}^r(X, m) = 0$. The following proposition establishes functoriality for higher Chow groups (see [8]).

Proposition 2.2. Let X and Y be quasiprojective varieties over k, $\dim X = d_1$, $\dim Y = d_2$, and $f : X \to Y$ a morphism. If f is proper there is a morphism

$$f_* : Z^r(X, m) \to Z^{r-d}(Y, m),$$

for all r and all m, with $d = d_1 - d_2$. There is also for f flat and for all r and all m, a morphism

$$f^* : Z^r(Y, m) \to Z^r(X, m).$$

Using proposition (2.2), we can construct, for f proper, a push-forward morphism

$$f_* : \mathrm{CH}^r(X, m) \to \mathrm{CH}^{r-d}(Y, m),$$

and, for f flat, a pull-back morphism

$$f^* : \mathrm{CH}^r(Y, m) \to \mathrm{CH}^r(X, m),$$

for all r and all m, with $d = d_1 - d_2$.

Remark. For smooth varieties, the morphism f^* exists unconditionally on the level of Chow groups.

The notion of product is also defined for higher Chow groups, in fact we have (see [8] and the excellent explanation in [20]):

Proposition 2.3. Let X and Y be smooth quasiprojective varieties over k. There exists a well defined product

$$\mathrm{CH}^r(X, m) \otimes \mathrm{CH}^s(Y, n) \to \mathrm{CH}^{r+s}(X \times Y, m + n).$$

Moreover, this product induces an internal product

$$\mathrm{CH}^r(X, m) \otimes \mathrm{CH}^s(X, n) \to \mathrm{CH}^{r+s}(X, m + n).$$

Remark. (i) The internal product is only valid in the smooth case.

(ii) If ξ_1, ξ_2 are two cycles, we denote their product by $\xi_1 \bullet \xi_2$.

The product of cycles in higher Chow groups has a more natural description if we consider the cubical version. Set $\square^m = (\mathbb{P}^1_k - \{1\})^m$ with coordinates t_j. Codimension one faces on $\square^m$ are obtained by setting $t_j = 0, \infty$. Intersecting these faces gives us higher codimension faces. Let $C^r(X, m)$ be the free abelian group generated by subvarieties of $X \times \square^m$ of codimension r meeting all the faces of the cubes $\square^n$, $n < m$, again in the same codimension. There is a map

$$d_m = \sum_{j=1}^{m} (-1)^j (\partial_j^\infty - \partial_j^0),$$

where ∂_j^0 is the pullback to the face $t_j = 0$ and ∂_j^∞ is the pullback to the face $t_j = \infty$. The map d_m satisfies $d_m \circ d_{m+1} = 0$, effectively making $(C^r(X, m), d_m)$ a complex. Consider projections $\square^m \to \square^{m-1}$ of the form $(t_1, \ldots, t_m) \mapsto (t_1, \ldots, \hat{t_j}, \ldots t_m)$ ($\hat{t_j}$ means omit t_j). Those elements in $C^r(X, m)$ which are the pullback of cycles on $X \times \square^{m-1}$ via such projections are called degenerate cycles. We define $D^r(X, m)$ as the subgroup of $C^r(X, m)$ generated by all degenerate cycles. Then $\{D^r(X, m), d_m\}$ is a subcomplex of $\{C^r(X, m), d_m\}$. Let

$$Z^r_c(X, \bullet) = C^r(X, \bullet)/D^r(X, \bullet).$$

Proposition 2.4 ([32])**.** There is a quasi-isomorphism between $Z^r_c(X, \bullet)$ and $Z^r(X, \bullet)$, which induces an isomorphism

$$\mathrm{CH}^r(X, m) \simeq H_n(Z^r_c(X, \bullet)).$$

Because of this isomorphism, we can use any of the two constructions to define higher Chow groups. The isomorphism

$$\square^1 \times \square^{m-1} \simeq \square^m$$

induces an isomorphism

$$(X \times \square^m) \times (Y \times \square^n) \simeq (X \times Y) \times \square^{m+n}$$

for varieties X, Y over k. Thus the product of higher Chow groups has an explicit description in this setting. $\mathrm{CH}^*(X, *) = \bigoplus_{r,m} \mathrm{CH}^r(X, m)$ has a commutative graded ring structure such that $\xi_1 \bullet \xi_2 = (-1)^{mn} \xi_2 \bullet \xi_1$ for $\xi_1 \in \mathrm{CH}^*(X, m), \xi_2 \in \mathrm{CH}^*(X, n)$.

Proposition 2.5 (Projection formula)**.** Let X and Y be smooth quasiprojective varieties over k, $f : X \to Y$ a proper morphism, $\xi_1 \in \mathrm{CH}^*(X,*), \xi_2 \in \mathrm{CH}^*(Y,*)$. Then
$$f_*(\xi_1 \bullet f^*(\xi_2)) = f_*(\xi_1) \bullet \xi_2.$$

A consequence of the product structure for higher Chow groups is the existence of indecomposable elements. Take a smooth quasiprojective variety X over k. Then we have a map:

$$\Pi : \bigoplus_{r_1+r_2=r, m_1+m_2=m} \mathrm{CH}^{r_1}(X,m_1) \otimes \mathrm{CH}^{r_2}(X,m_2) \to \mathrm{CH}^r(X,m),$$

where $(r_1,m_1) \neq (0,0), (r_2,m_2) \neq (0,0)$. We say that an element of $\mathrm{CH}^r(X,m)$ is decomposable if it is in the image of Π. The space of indecomposable elements of $\mathrm{CH}^r(X,m)$ is the quotient $\mathrm{CH}^r(X,m)/\mathrm{Image}\Pi$. Now, let X be a smooth projective variety over k. Later we will use the subgroup of decomposables $\mathrm{CH}^r_{\mathrm{dec}}(X,m)$ given by the image of

$$\mathrm{CH}^{r-m}(X,0) \otimes \mathrm{CH}^1(X,1)^{\otimes m} \to \mathrm{CH}^r(X,m)$$

under the product for higher Chow groups and the subgroup of indecomposables $\mathrm{CH}^r_{\mathrm{ind}}(X,m)$ given by

$$\mathrm{CH}^r_{\mathrm{ind}}(X,m) := \mathrm{CH}^r(X,m)/\mathrm{CH}^r_{\mathrm{dec}}(X,m).$$

Another important result is the existence of a localization sequence ([9]).

Proposition 2.6. Let X be a quasiprojective variety over a field. If $Y \subset X$ is a closed subvariety of pure codimension r, then one has a localization

$$\ldots \to \mathrm{CH}^*(Y,m) \to \mathrm{CH}^{*+r}(X,m) \to \mathrm{CH}^{*+r}(X \setminus Y,m) \to \mathrm{CH}^*(Y,m-1) \to \ldots$$

Corollary 2.7 (Mayer-Vietoris sequence)**.** Let $X = U \cup V$ be a Zariski cover. Then we have a long exact sequence

$$\ldots \to \mathrm{CH}^*(U \cup V,m) \to \mathrm{CH}^*(U,m) \oplus \mathrm{CH}^*(V,m) \to \mathrm{CH}^*(U \cap V,m) \to \mathrm{CH}^*(U \cup V,m-1) \to \ldots$$

The existence of a product and functoriality on higher Chow groups are necessary to define the morphism induced by a “correspondence”.

Definition 2.8. Let X and Y be smooth projective varieties over k.

(i) A correspondence from X to Y is a cycle $\Gamma \in \mathrm{CH}^\ell(X \times Y, n)$.

(ii) The morphism induced by a correspondence $\Gamma \in \mathrm{CH}^\ell(X \times Y, n)$,

$$\Gamma_* : \mathrm{CH}^r(X, m) \to \mathrm{CH}^s(Y, m+n),$$

with $s = r + \ell - \dim(X)$, is defined by

$$\Gamma_*(\xi) = \pi_{Y*}(\pi_X^*(\xi) \bullet \Gamma)$$

for $\xi \in \mathrm{CH}^s(X, m)$, where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projections.

If we consider $\mathrm{CH}^r(X)$, the Chow group of codimension r algebraic cycles on a quasiprojective variety X, then higher Chow groups provide a generalization of these groups. Indeed we have

Proposition 2.9. Let X be a quasiprojective variety over a field k. Then $\mathrm{CH}^r(X, 0) \simeq \mathrm{CH}^r(X)$.

We will restrict our discussion to higher Chow groups without torsion and set $\mathrm{CH}^r(X, m; \mathbb{Q}) := \mathrm{CH}^r(X, m) \otimes \mathbb{Q}$.

Chapter 3

Mixed Hodge structures

In the following we introduce the definitions of Hodge structures, and more generally mixed Hodge structures. These are necessary since later we will define maps from (filtered) higher Chow groups to extensions of mixed Hodge modules. The cycles mapped in this way can be described by looking at a short exact sequence that involves Ext and hom of (polarizable) mixed Hodge structures.

Definition 3.1. A Hodge structure H of weight ℓ is a pair consisting of a $\mathbb{Z}-$module of finite type $H_{\mathbb{Z}}$ such that $H_{\mathbb{R}} := H_{\mathbb{Z}} \otimes \mathbb{R}$ is a finite dimensional vector space over $\mathbb{R}$, and a decreasing filtration $F^{\bullet}$ of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$ such that

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{\ell-p+1}}.$$

Setting $H^{p,q} := F^p \cap \overline{F^q}$, we get a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=\ell} H^{p,q}$$

where $H^{p,q} = \overline{H^{q,p}}$ (the bar denotes complex conjugation).

If we have a $\mathbb{Z}-$module of finite type $H_{\mathbb{Z}}$, such that $H_{\mathbb{R}}$ is a finite dimensional vector space over $\mathbb{R}$, then the existence of a decomposition of $H_{\mathbb{C}}$ like the one above, can be used to construct a filtration by setting $F^r = \bigoplus_{p\geq r} H^{p,\ell-p}$. This induces a Hodge structure of weight ℓ on $H_{\mathbb{Z}}$. Thus the existence of decompositions is equivalent to the existence of filtrations with the properties described.

Example 3.2. Let X be a smooth projective variety over $\mathbb{C}$. The classical example of a Hodge structure of weight ℓ is a consequence of the Hodge decomposition theorem (see [24] or [34]), which allows us to write

$$H^{\ell}_{\mathrm{DR}}(X,\mathbb{C}) = \bigoplus_{p+q=\ell} H^{p,q}(X).$$

Example 3.3. Another example is given by the Hodge structure of Tate $\mathbb{Z}(\ell)$ of weight -2ℓ defined by $\mathbb{Z}(\ell) := \mathbb{Z}$. It is the unique integral Hodge structure of weight -2ℓ on $\mathbb{Z}$, the decomposition is given by $\mathbb{Z}(\ell) = \mathbb{Z}(\ell)^{-\ell,-\ell}$.

Definition 3.4. A morphism of Hodge structures $f : H_1 \to H_2$ is a morphism of $\mathbb{Z}-$modules that preserves the filtration, i.e. $f(F^r H_{1\mathbb{C}}) \subset F^r H_{2\mathbb{C}}$ for all r.

We can define the direct sum of Hodge structures of the same weight in an obvious way. We can also define tensor product and hom. If H_1 is a Hodge structure of weight ℓ_1 such that $H_{1\mathbb{C}} = \bigoplus_{p+q=\ell_1} H_1^{p,q}$ and H_2 is a Hodge structure of weight ℓ_2 such that $H_{2\mathbb{C}} = \bigoplus_{p+q=\ell_2} H_2^{p,q}$, then $H_1 \otimes H_2$ is a Hodge structure of weight $\ell_1 + \ell_2$, with

$$(H_1 \otimes H_2)^{p,q} = \bigoplus_{p_1+p_2=p, q_1+q_2=q} H_1^{p_1,q_1} \otimes H_1^{p_2,q_2}.$$

$\hom(H_1, H_2)$ has weight $-\ell_1 + \ell_2$ and

$$\hom(H_1, H_2)^{p,q} = \{f : H_{1\mathbb{C}} \to H_{2\mathbb{C}} |\ f(H_1^{p_1,q_1}) \subset H_2^{p_1+p,q_1+q})\}.$$

In particular, $H_1 \otimes \mathbb{Z}(\ell)$ is a Hodge structure of weight $\ell_1 - 2\ell$ and the dual Hodge structure H_1^* has weight $-\ell_1$.

Definition 3.5. Let H be a Hodge structure of weight ℓ. A polarization of H is a nonsingular, bilinear form

$$S : H_{\mathbb{C}} \otimes H_{\mathbb{C}} \to \mathbb{C},$$

which is defined over $\mathbb{Q}$, such that:

(i) $S(x, y) = (-1)^{\ell} S(y, x)$.

(ii) $S(H^{p,q}, H^{r,s}) = 0$ unless $p = s$, $q = r$.

(iii) $i^{p-q}S(x,\bar{y})$ is a hermitian positive-definite bilinear form on $H^{p,q}$.

A polarizable Hodge structure is a Hodge structure that admits a polarization. Hodge structures form an abelian category with tensor products. We need polarized Hodge structures to get a semisimple category. In fact, if $G \subset H$ is a sub Hodge structure of a polarized Hodge structure H (i.e. G is a Hodge structure such that the inclusion is a morphism of Hodge structures) then G inherits a polarization from the one on G and $H_{\mathbb{Q}} = G_{\mathbb{Q}} \oplus G_{\mathbb{Q}}^{\perp}$.

Definition 3.6. A mixed Hodge structure H is a triple consisting of

(i) A $\mathbb{Z}-$module of finite type $H_{\mathbb{Z}}$, such that $H_{\mathbb{R}} := H_{\mathbb{Z}} \otimes \mathbb{R}$ is a finite dimensional vector space over $\mathbb{R}$.

(ii) An increasing filtration $W_{\bullet}$ of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes \mathbb{Q}$.

(iii) A decreasing filtration $F^{\bullet}$ of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$.

Furthermore, $F^{\bullet}$ induces a Hodge structure of weight ℓ on each of the graded pieces

$$Gr_W^{\ell} = W_{\ell}/W_{\ell-1}.$$

W is usually called the weight filtration and F the Hodge filtration. A morphism of mixed Hodge structures is a morphism of $\mathbb{Z}-$modules that preserves both filtrations. A graded-polarizable mixed Hodge structure is a mixed Hodge structure such that each graded piece Gr_W^{ℓ} is a polarizable Hodge structure. The category of mixed Hodge structures is abelian.

Deligne proved in [18] that for any complex variety X, $H^{\ell}(X,\mathbb{Q})$ carries a canonical and functorial mixed Hodge structure. This structure is the usual Hodge structure when X is smooth and projective. More generally we have the following ([18], [19]):

Theorem 3.7. Let U be a complex quasiprojective variety. Then $H^{\ell}(U,\mathbb{Z})$ has a canonical and functorial mixed Hodge structure such that

(i) If U is smooth projective, the mixed Hodge structure on $H^{\ell}(U)$ is the usual one.

(ii) If U is smooth with smooth compactification X, then $H^{\ell}(U)$ has weights in the range $[\ell, 2\ell]$ and $\mathrm{Im}(H^{\ell}(X) \to H^{\ell}(U)) = W_{\ell}H^{\ell}(U)$.

(iii) If U is complete then $H^\ell(U)$ has weights $\leq \ell$.

Example 3.8. Consider a compact Riemann surface X and a finite set of points $\Sigma \subset X$. Then the exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(U,\mathbb{Z}) \xrightarrow{\text{Residue}} H^0_{\deg 0}(\Sigma, \mathbb{Z}(-1)) \to 0$$

is an exact sequence of mixed Hodge structures, where $U = X - \Sigma$ and also $H^0_{\deg 0}(\Sigma, \mathbb{Z}(-1)) \simeq \mathbb{Z}(-1)^{|\Sigma|-1}$. In this case $W_1 H^1(U,\mathbb{Z}) = \mathrm{Im}(H^1(X,\mathbb{Z}) \to H^1(U,\mathbb{Z}))$ and $\mathrm{Gr}^2_W H^1(U,\mathbb{Z}) \simeq \mathbb{Z}(-1)^{|\Sigma|-1}$.

We denote the category of mixed Hodge structures by MHS. We will be interested in extensions of mixed Hodge structures. The following result of Carlson ([11]) is essential.

Theorem 3.9. Let H be a mixed Hodge structure. Then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H) = \frac{W_0 H_\mathbb{C}}{F^0 W_0 H_\mathbb{C} + W_0 H_\mathbb{Q}}.$$

The theorem in this form is proved by U. Jannsen in [26] following the proof of Carlson. Moreover by Carlson's description of $\mathrm{Ext}^1_{\mathrm{MHS}}$, since the functor $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), -)$ is right exact, one can show that $\mathrm{Ext}^j_{\mathrm{MHS}}(V, H) = 0$ for $j \geq 2$ and any MHS H, V, as was first established by Beilinson [6]. If we consider the category of graded-polarizable mixed Hodge structures, the theorem takes the following form stated in [5]:

Theorem 3.10. Let H be a graded-polarizable mixed Hodge structure. Then

$$\mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H) = \frac{W_{-1} H_\mathbb{C}}{W_{-1} H_\mathbb{C} \cap (F^0 W_0 H_\mathbb{C} + W_0 H_\mathbb{Q})} \hookrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H).$$

Proposition 3.11. Let H be a graded-polarizable mixed Hodge structure. Then

$$\mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H) = \frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{-1} H)}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr^0_W H)}$$

Proof. There is a natural map

$$\frac{W_{-1} H_\mathbb{C}}{F^0 W_{-1} H_\mathbb{C} + W_{-1} H_\mathbb{Q}} \to \frac{W_{-1} H_\mathbb{C}}{W_{-1} H_\mathbb{C} \cap (F^0 W_0 H_\mathbb{C} + W_0 H_\mathbb{Q})}$$

whose kernel is $F^0 Gr_W^0 H_{\mathbb{C}} \cap Gr_W^0 H_{\mathbb{Q}}$. By theorems (3.9), (3.10) we get an exact sequence

$$0 \to F^0 Gr_W^0 H_{\mathbb{C}} \cap Gr_W^0 H_{\mathbb{Q}} \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{-1}H) \to \mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H) \to 0.$$

On the other hand, by using the long exact sequence associated to the short exact sequence

$$0 \to W_{-1}H \to W_0 H \to Gr_W^0 H \to 0$$

we get the exact sequence

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^0 H) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{-1}H) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_0 H).$$

Then, using (3.9) again, we deduce that (also see [26])

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^0 H) = F^0 Gr_W^0 H_{\mathbb{C}} \cap Gr_W^0 H_{\mathbb{Q}}$$

and the proposition follows. $\square$

We define for any Hodge structure H:

$$\Gamma(H) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H),$$

$$J(H) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H).$$

Although we aren't going to get into the details of degeneration of Hodge structures, we will need to know what a variation of Hodge structure is. If X is a complex manifold, to any locally constant sheaf V of complex vector spaces (called a local system) we can associate a holomorphic vector bundle on X with a flat connection. The holomorphic vector bundle is given by $\mathcal{V} := \mathcal{O}_X \otimes V$ endowed with an integrable connection ∇, with space of horizontal sections V. Moreover, there is a bijective correspondence between isomorphism classes of holomorphic vector bundles equipped with a flat connection and isomorphism classes of local systems.

Definition 3.12. Let X be a complex manifold. A polarized variation of Hodge structure (or VHS) of weight ℓ over X consists of a local system $V_{\mathbb{Z}}$ over X of $\mathbb{Z}-$modules of finite rank such that:

(i) There is a decreasing filtration $\mathcal{F}^\bullet$ of $\mathcal{V} = \mathcal{O}_X \otimes V_{\mathbb{Z}}$ by holomorphic sub-bundles.

(ii) There exists a flat bilinear form $S : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$.

They satisfy:

1. the Griffiths transversality condition $\nabla(\mathcal{F}^p) \subset \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F}^{p-1}$;
2. For every $x \in X$, the $\mathbb{Z}-$module $V_{\mathbb{Z},x}$, with the filtration $F^\bullet_x$ induced by $\mathcal{F}^\bullet$ and the bilinear form S_x is a polarized Hodge structure of weight ℓ.

Example 3.13. Let $f : Y \to X$ be a smooth projective morphism between quasi-projective algebraic varieties over $\mathbb{C}$. Then the local system $R^m_{\text{prim}} f_* \mathbb{Z} \subset R^m f_* \mathbb{Z}$ consisting of the primitive cohomology classes in the fibers of f is a polarized variation of Hodge structure.

Chapter 4

Mixed Hodge modules

Fix an algebraically closed field k of characteristic 0. Let X be a smooth variety over k, of dimension n. Take $U \subset X$ an open affine subset. A differential operator of order $\leq r$ on U is a k-linear endomorphism $g : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ such that $[\hat{f}_r \dots [\hat{f}_1, [\hat{f}_0, g]] \dots] = 0$ for any $f_0, f_1, \dots f_r \in \mathcal{O}_X(U)$, where $\hat{f}_i$ is the operator of multiplication by f_i. Let $D(U)$ be the ring of differential operators on U. It is the union of all differential operators on U of all orders.

Proposition 4.1 ([7])**.** The functor $U \mapsto D(U)$ defines a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Definition 4.2. The sheaf of proposition (4.1) is called the sheaf of differential operators on X and is denoted by D_X. A D_X-module is a sheaf $\mathcal{M}$ of left D_X-modules which is quasi-coherent as $\mathcal{O}_X$-module.

Remark. Although we chose left D_X-modules in our definition, we could also have chosen right D_X-modules. However, we can go from one to the other by a well defined operation (as discussed in [3]), so we use only D_X modules like in the definition.

Example 4.3. $\mathcal{O}_X$ is clearly a D_X-module because the sheaf $\mathcal{O}_X$ is itself quasi-coherent.

It is possible to find (see [7]) for each $x \in X$ an affine neighbourhood U of x, functions $x_1, \dots, x_n$ on U and vector fields $\partial_1, \dots, \partial_n$ on U, with $\partial_i(x_j) = \delta_{ij}$, such that the tangent sheaf $\mathcal{T}_X$ (the definition can be found later in Chapter (5)) is generated by $\{x_i, \partial_j\}$. Then $D(U) = \mathcal{O}_X(U) \otimes k[\partial_1, \dots, \partial_n]$.

Example 4.4. When $X = \mathbb{C}^n$, $D_X(X)$ is the Weyl algebra over $\mathbb{C}$, denoted by D_n. D_n is the noncommutative $\mathbb{C}$-algebra generated by symbols $x_1, \ldots, x_n, \partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$ subject to the relations $[x_i, x_j] = [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij}$. An element $P \in D_n$ can be written in a unique way as

$$P = \sum \alpha_{I,J} x^{i_1} \cdots x^{i_n} \partial^{j_1} \cdots \partial^{j_n},$$

where $i_1, \cdots, i_n, j_1, \cdots, j_n \in \mathbb{N}, \alpha_{I,J} \in \mathbb{C}$ and $I, J \in \mathbb{N}^n$ correspond to $i_1, \cdots, i_n$ and $j_1, \cdots, j_n$ respectively. A differential operator of order r is then an operator P such that the maximum $j_1 + \cdots + j_n = r$.

We can define direct image and inverse image functors of D_X-modules by working in the derived category. Let $D^b(D_X)$ be the bounded derived category of D_X-modules, and $f : X \to Y$ a morphism of varieties. We denote the inverse image by $f^* : D^b(D_Y) \to D^b(D_X)$ and the direct image by $f_* : D^b(D_X) \to D^b(D_Y)$. By taking cohomology we get functors $\mathcal{H}^i : D^b(D_X) \to M(D_X)$, where $M(D_X)$ is the category of D_X-modules.

If D_X^r is the sheaf of differential operators of order $\leq r$, $\{D_X^\bullet\}$ is a filtration of D_X by coherent $\mathcal{O}_X$-modules such that $D_X^0 = \mathcal{O}_X$ and $D^i \cdot D^j \subset D^{i+j}$. Let $Gr_D(D_X) = \oplus D_X^i / D_X^{i-1}$. This sheaf is isomorphic to the cotangent bundle T^*X.

Definition 4.5. Let $\mathcal{M}$ be a D_X-module. A good filtration on $\mathcal{M}$ is an increasing filtration $\mathcal{M}^\bullet$ of $\mathcal{M}$ by $\mathcal{O}_X$-submodules such that

(i) $\mathcal{M} = \cup \mathcal{M}^i$.

(ii) $D_X^i \mathcal{M}^j \subset \mathcal{M}^{i+j}$.

(ii) Each $\mathcal{M}^j$ is a coherent $\mathcal{O}_X$-module and $D_X^1 \mathcal{M}^j = \mathcal{M}^{j+1}$ for $j \gg 0$.

Proposition 4.6 ([7],[10])**.** If $\mathcal{M}$ is a coherent D_X-module then $\mathcal{M}$ has a good filtration.

Let $\mathcal{M}$ be a coherent D_X-module with $\mathcal{M}^\bullet$ a good filtration on $\mathcal{M}$. Let $Gr_{\mathcal{M}}(\mathcal{M}) = \oplus \mathcal{M}^i / \mathcal{M}^{i-1}$. Then $Gr_{\mathcal{M}}(\mathcal{M})$ is a coherent $Gr_D(D_X)$-module. Thus $Gr_{\mathcal{M}}(\mathcal{M})$ has a support $\mathrm{supp}(Gr_{\mathcal{M}}(\mathcal{M}))$, which is a closed subvariety of T^*X.

Proposition 4.7 ([7]). $\text{supp}(Gr_{\mathcal{M}}(\mathcal{M}))$ doesn't depend on the filtration of $\mathcal{M}$.

Definition 4.8. $Ch(\mathcal{M}) := \text{supp}(Gr_{\mathcal{M}}(\mathcal{M}))$ is called the characteristic variety of $\mathcal{M}$.

Using the following important result (see [7]) we can define holonomic D_X-modules.

Theorem 4.9 (Bernstein inequality). Let $\mathcal{M} \neq 0$ be a coherent D_X-module. Then $\dim Ch(\mathcal{M}) \geq \dim X$.

Definition 4.10. A coherent D_X-module $\mathcal{M}$ is called holonomic if $\dim Ch(\mathcal{M}) \leq \dim X$.

Any locally free sheaf $\mathcal{F}$ on X induces a vector bundle V on X. An integrable (or flat) connection on V is a map

$$\nabla : V \to \Omega^1_X \otimes V$$

such that $\nabla \circ \nabla = 0$.

Proposition 4.11 ([3]). If $\mathcal{M}$ is a holonomic D_X-module, there exists an open dense subset $U \subset X$ such that $\mathcal{M}|_U$ induces an integrable connection.

Consider a vector bundle V with a connection ∇ on a smooth variety X. The pair (V, ∇) has regular singularities if there exists a smooth compactification $\bar{X}$ of X, such that $D = \bar{X} - X$ is a divisor with normal crossings and log connection

$$\nabla : \bar{V} \to \Omega^1_X(\log D) \otimes \bar{V}.$$

If $\mathcal{M}$ is a holonomic D_X-module, $\dim X = 1$, we say that $\mathcal{M}$ has regular singularities if there is a dense open set $U \subset X$ such that $\mathcal{M} \mid_U$ induces a connection with regular singularities. For X of arbitrary dimension, $\mathcal{M}$ has regular singularities if $\mathcal{H}^0(i_C^*\mathcal{M})$ has regular singularities for any smooth curve C with $i_C : C \to X$ (see [5]). $\text{MF}_{rh}(X)$ denotes the category of regular holonomic D_X-modules with a good filtration.

Now we define perverse sheaves and describe its relation with D_X-modules. Let X be an algebraic variety over $\mathbb{C}$. A sheaf $\mathcal{F}$ of $\mathbb{C}$ vector spaces is called constructible if X can be written as a finite union $X = \bigcup X_i$ of locally closed algebraic subvarieties X_i of X such that $\mathcal{F}|_{X_i}$ is finite and locally constant. Let $D^b(\mathbb{C}_X)$ be the derived category of bounded complexes of sheaves of $\mathbb{C}$ vector spaces. Given $K^\bullet \in D^b(\mathbb{C}_X)$, the Verdier dual complex is $\mathbb{D}(K^\bullet) := \mathbb{R}\hom_X(K^\bullet, \mathbb{C}_X[2n])$, where $\dim X = n$.

Definition 4.12. A complex $K^\bullet \in D^b(\mathbb{C}_X)$ is called a perverse sheaf if:

(i) The cohomology sheaves $\mathcal{H}^j(K^\bullet)$ are constructible and

$$\dim(\mathrm{supp}\mathcal{H}^j(K^\bullet)) \leq -j,$$

for all j.

(ii) $\dim(\mathrm{supp}\mathcal{H}^j(\mathbb{D}(K^\bullet))) \leq -j$, for all j.

The category of perverse sheaves is denoted by $\mathrm{Perv}(\mathbb{C}_X)$.

Given a algebraic variety X over $\mathbb{C}$ of dimension n, for any D_X-module $\mathcal{M}$ we have the de Rham complex:

$$\mathcal{M} \xrightarrow{d} \Omega^1_X \otimes \mathcal{M} \xrightarrow{d} \Omega^2_X \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega^n_X \otimes \mathcal{M}$$

where, for local coordinates $(z_1, \dots, z_n)$,

$$d(\omega \otimes m) = \sum_{j=1}^{n} (dz_j \wedge \omega) \otimes \partial_j m.$$

If we shift the complex n places to the left, i.e. $\Omega^j_X \otimes \mathcal{M}$ is put in degree $j-n$, and denote this by $[n]$, we get the de Rham functor from D_X-modules to complexes:

$$DR(\mathcal{M}) = \Omega^\bullet_X \otimes \mathcal{M}[n].$$

The following theorem is due to Kashiwara, Kawai and Mebkhout ([27],[28], [39]).

Theorem 4.13 (Riemann-Hilbert correspondence)**.** Let X be a smooth variety over $\mathbb{C}$. Then the de Rham functor $\mathcal{M} \mapsto DR(\mathcal{M})$ induces an equivalence of categories between the category of holonomic D_X-modules with regular singularities and the category of perverse sheaves $\mathrm{Perv}(\mathbb{C}_X)$.

Perverse sheaves can be defined for complexes of sheaves with coefficients in $\mathbb{Q}$. A weight filtration on an object $K^\bullet_{\mathbb{Q}}$ in $\mathrm{Perv}(\mathbb{Q}_X)$ is a finite increasing filtration $W_{\mathbb{Q}}$ of sub-perverse sheaves, the category of perverse sheaves with weight filtration is denoted by $\mathrm{PervW}(\mathbb{Q}_X)$. On the other hand, a weight filtration on a D_X-module $\mathcal{M}$ is a finite increasing filtration $W_{\mathcal{M}}$ of $\mathcal{M}$ by D_X-submodules. Let $\mathrm{MFW}_{rh}(X)$ denote the category of filtered (by a good filtration) regular holonomic D_X-modules with a weight filtration.

There is a natural functor from $\mathrm{MFW}_{rh}(X)$ to $\mathrm{PervW}(\mathbb{C}_X)$ (the category of weight filtered perverse sheaves in $\mathrm{Perv}(\mathbb{C}_X)$). It takes the pair $(\mathcal{M}, W_{\mathcal{M}})$ to the pair $(DR(\mathcal{M}), DR(W_{\mathcal{M}}))$. Now, let's consider objects of the form $(K^\bullet_{\mathbb{Q}}, W_{\mathbb{Q}}, \mathcal{M}, F, W, \alpha)$ where $K^\bullet_{\mathbb{Q}}$ is a perverse sheaf with weight filtration $W_{\mathbb{Q}}$, $\mathcal{M}$ is a holonomic D_X-module with regular singularities and weight filtration W, F is a good filtration on $\mathcal{M}$ and α is an isomorphism of filtered objects, i.e. $\alpha : (K^\bullet_{\mathbb{Q}} \otimes \mathbb{C}, W_{\mathbb{Q}} \otimes \mathbb{C}) \simeq (DR(\mathcal{M}), DR(W_{\mathcal{M}}))$. In the language of categories, we are taking elements in the fiber product $\mathrm{MFW}_{rh}(X;\mathbb{Q}) := \mathrm{PervW}(\mathbb{Q}_X) \times_{\mathrm{PervW}(\mathbb{C}_X)} \mathrm{MFW}_{rh}(X)$. In [44] M. Saito defined mixed Hodge modules:

Theorem 4.14 (M. Saito)**.** For any smooth variety X over $\mathbb{C}$ there exists an abelian category $\mathrm{MHM}(X)$ that is a full subcategory of $\mathrm{MFW}_{rh}(X;\mathbb{Q})$. $\mathrm{MHM}(X)$ is called the category of mixed Hodge modules.

The category of mixed Hodge modules contains a semi-simple full subcategory of modules of pure weight. These are called polarizable Hodge modules (defined in [43]). In the derived category $D^b(\mathrm{MHM}(X))$ all expected operations are defined: $f_*, f^*, f_!, f^!, \mathbb{D}$, etc. $\mathrm{MHM}(\mathrm{Spec}(\mathbb{C}))$ is isomorphic to the category of graded polarizable mixed Hodge structures.

For k a subfield of $\mathbb{C}$, the category $\mathrm{MHM}(X)$ of mixed Hodge modules of a smooth variety X over k is well defined. We have a natural functor

$$\mathrm{MFW}_{rh}(X) \to \mathrm{MFW}_{rh}(X_{\mathbb{C}}),$$

and $\mathrm{MHM}(X)$ is a full subcategory of the fibre product of $\mathrm{MHM}(X_{\mathbb{C}})$ and $\mathrm{MFW}_{rh}(X)$ over $\mathrm{MFW}_{rh}(X_{\mathbb{C}})$.

Example 4.15. Let $r \in \mathbb{Z}$. The D_X-module $\mathcal{O}_X$ has a good filtration defined by $F^{-r} = \mathcal{O}_X$, $F^s = 0$ for $s \neq -r$. Since $DR(\mathcal{O}_X)$ is quasi-isomorphic to $\mathbb{C}_X[\mathrm{dim}X]$, we get a mixed Hodge module $\mathbb{Q}_X(r)[\mathrm{dim}X]$ given by

$$(\mathbb{Q}_X(r)[\mathrm{dim}X], W, \mathcal{O}_X, F),$$

where $Gr_W^{\dim X-2r}(\mathbb{Q}_X(r)[\dim X]) = \mathbb{Q}_X(r)[\dim X]$. This is called the constant (or Tate) mixed Hodge module. We usually use the shifted module

$$\mathbb{Q}_X(r) = \mathbb{Q}_X(r)[\dim X][-\dim X].$$

Since we have defined mixed Hodge modules for smooth varieties, we should find a way to deal with varieties not necessarily smooth. What we would like is to have a category $M(X)$ for any separated algebraic variety X over a field k together with functors to the category of perverse sheaves over $X \times_k \mathbb{C}$, such that we have a dual functor $\mathbb{D}$, external product, pull-backs f^*, direct images f_* (for f a morphism between separated varieties), etc., in the bounded derived category $D^bM(X)$, and a constant object $\mathbb{Q}_X$, all compatible with the functor to perverse sheaves. Moreover, we would like $M(X)$ to be MHM(X) when X is smooth. This category always exists and is called the category of $\mathbb{Q}$-mixed sheaves on X (see [45], the definition is more technical that the one given here, but the idea is the same, we want forgetful functors to perverse sheaves and stability for pull-backs, duals, products, direct images, etc., and constant objects).

Now consider a morphism $f : X \to Y$ between separated varieties. The constant objects $\mathbb{Q}_X \in M(X)$, $\mathbb{Q}_Y \in M(Y)$ always exist and we have morphisms

$$\mathbb{Q}_Y \to f_*\mathbb{Q}_X,\ f_!\mathbb{D}\mathbb{Q}_X \to \mathbb{D}\mathbb{Q}_Y$$

which are called the restriction and Gysin morphisms respectively.

Proposition 4.16 ([45])**.** Let X be a purely n-dimensional smooth variety. Then

$$\mathbb{D}\mathbb{Q}_X = \mathbb{Q}_X(n)[2n].$$

Remark. If X is a separated variety and $n = \dim X$ we have a canonical morphism:

$$\mathbb{Q}_X(n)[2n] \to \mathbb{D}\mathbb{Q}_X. \tag{4.1}$$

Let $\pi : Y \to X$ be a proper morphism such that Y is smooth of pure dimension n and $\pi(Y) = X'$, where X' is the union of the irreducible components X_i of X such that $\dim X_i = \dim X$. Then (4.1) coincides with the composition of the restriction and Gysin morphisms:

$$\mathbb{Q}_X(n)[2n] \to \pi_*\mathbb{Q}_Y(n)[2n] \to \mathbb{D}\mathbb{Q}_X.$$

The following decomposition will be relevant later, when we construct filtrations.

Proposition 4.17 ([45])**.** Let $f : X \to Y$ be a proper morphism of smooth varieties. If $M \in MHM(X)$ is pure, we have a non canonical isomorphism

$$f_*M \simeq \bigoplus_i H^i f_*M[-i]$$

in $D^b\mathrm{MHM}(Y)$.

Remark. We should note that the proposition above is also valid when the varieties are not smooth. In that case we work in the category of mixed sheaves, where corresponding concepts of weight filtration and "pure" object are defined in complete analogy with the ones in mixed Hodge modules.

Corollary 4.18. Let $f : X \to Y$ be a proper smooth morphism of quasi-projective smooth varieties over $k \subset \mathbb{C}$. Then there is a Leray spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), R^q f_*M) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), M),$$

which degenerates at E_2, for any pure $M \in \mathrm{MHM}(X)$.

Proof. The existence of the Leray filtration on $\mathrm{Ext}^{p+q}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), M)$ such that

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), R^q f_*M)$$

is clear. Then we use the decomposition in proposition (4.17) and apply Deligne's criterion to conclude that the Leray spectral sequence associated to the Leray filtration degenerates at E_2. $\square$

The following theorem provides the link we need between higher Chow groups and mixed Hodge modules, namely the cycle map.

Theorem 4.19 (M. Saito)**.** Let X be a smooth variety over $k \subset \mathbb{C}$. Then there exists a cycle map

$$c_{r,m} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)).$$

Proof. First we construct the cycle map for $\mathrm{CH}^r(X;\mathbb{Q})$. Let $Z \subset X$ be a irreducible closed subvariety of X of codimension r, with $\dim X = n$. Then we can find a resolution of singularities for Z, i.e., we can find a smooth variety Z' and a proper morphism $g : Z' \to Z$ such that $g(Z') = Z$. By using the restriction morphism we have a map

$$\mathbb{Q}_Z(n-r)[2(n-r)] \to g_*\mathbb{Q}_{Z'}(n-r)[2(n-r)]. \tag{4.2}$$

Since Z' is smooth we can use proposition (4.16), so $\mathbb{Q}_{Z'}(n-r)[2(n-r)] = \mathbb{D}\mathbb{Q}_{Z'}$. Then we compose (4.2) with the Gysin map to get a morphism

$$\mathbb{Q}_Z(n-r)[2(n-r)] \to \mathbb{D}\mathbb{Q}_Z.$$

We can use again the restriction and Gysin morphisms for the inclusion of Z in X to get

$$\mathbb{Q}_X \to \mathbb{Q}_Z \to \mathbb{D}\mathbb{Q}_Z(-n+r)[2(-n+r)] \to \mathbb{D}\mathbb{Q}_X(-n+r)[2(-n+r)].$$

(here we omitted i_*, where i is the inclusion). Since X is smooth

$$\mathbb{D}\mathbb{Q}_X(-n+r)[2(-n+r)] = \mathbb{Q}_X(r)[2r],$$

so we have constructed for every cycle, an element of

$$\hom_{D^b\mathrm{MHM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(r)[2r]).$$

Then we have the cycle map, since

$$\hom_{D^b\mathrm{MHM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(r)[2r]) = \mathrm{Ext}^{2r}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)).$$

To prove that this map is well defined under rational equivalence we take a cycle W in $X \times \mathbb{P}^1$. We would like to prove that the element in $\hom_{D^b\mathrm{MHM}(X)}$ corresponding to $\pi_*(W \bullet X \times t)$ is independent of $t \in \mathbb{P}^1$, where $\pi : X \times \mathbb{P}^1 \to X$. But a consequence of the theory of mixed Hodge modules ([45]), is that we have a morphism from π_* to i_t^*, for $i_t : X \times t \to X \times \mathbb{P}^1$. So, it is enough to prove that $i_t^*(W \bullet X \times t)$ induces a morphism in $\hom_{D^b\mathrm{MHM}(X\times t)}$ such that: 1) it corresponds to the pullback of the morphism in $\hom_{D^b\mathrm{MHM}(X\times\mathbb{P}^1)}$ induced by $W \bullet X \times t$, 2) it is independent of t. This is a consequence of the fact that there exists a functorial morphism from i_t^* to the nearby cycle functor, and the morphism induced by $i_t^*(W \bullet X \times t)$ is precisely the one induced by the intersection of W with $X \times t$, and the nearby cycle functor is invariant by t (the details can be found in [45], section 8).

Now consider an element in $\mathrm{CH}^r(X, m; \mathbb{Q})$, i.e. take a cycle in $X \times \Delta^m$ that meets all the faces $X \times \Delta^i$ properly and in the kernel of the restriction map to every face. Then it is possible to prove (see [45], Prop. 8.3 and Lemma 8.4) that this cycle Z induces a morphism

$$\mathbb{Q}_{|Z|} \to j_! \mathbb{D}\mathbb{Q}_U(-n+r)[2(-n+r)],$$

where $U = (X \times \Delta^m) \setminus (X \times \cup \Delta^i)$, $j : U \to X \times \Delta^m$ and $|Z|$ is the support of Z. We also have the identity

$$p_* j_! \mathbb{Q}_U = \mathbb{Q}_X[-m], \tag{4.3}$$

for $p : X \times \Delta^m \to X$. Then using the restriction morphism for p and the inclusion of $|Z|$ in $X \times \Delta^m$ we have the following composition

$$\mathbb{Q}_X \to p_* \mathbb{Q}_{X \times \Delta^m} \to p_* \mathbb{Q}_{|Z|} \to p_* j_! \mathbb{D}\mathbb{Q}_U(-n+r)[2(-n+r)].$$

Finally

$$p_* j_! \mathbb{D}\mathbb{Q}_U(-n+r)[2(-n+r)] = \mathbb{D}\mathbb{Q}_X(-n+r)[2(-n+r)-m] = \mathbb{Q}_X(r)[2r-m],$$

by (4.3) and smoothness of X. In conclusion we have constructed, for a cycle in $\mathrm{CH}^r(X, m; \mathbb{Q})$, an element of

$$\hom_{D^b\mathrm{MHM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(r)[2r-m]) = \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)).$$

which we can check is well defined by a similar argument to the case $m = 0$. $\square$

The cycle map is compatible with direct image for proper morphisms and pullback. If $f : X \to Y$ is proper, the direct image of an element in

$$\hom_{D^b\mathrm{MHM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(r)[2r-m])$$

is given by the restriction and Gysin morphisms for f, i.e.

$$\begin{aligned}\mathbb{Q}_Y \to f_* \mathbb{Q}_X \to f_* \mathbb{D}\mathbb{Q}_X(-d_1+r)[-2(d_1-r)-m] \to \\ \mathbb{D}\mathbb{Q}_Y(-d_1+r)[-2(d_1-r)-m] = \mathbb{Q}_Y(r-d)[2(r-d)-m]\end{aligned}$$

gives the element in

$$\hom_{D^b\mathrm{MHM}(Y)}(\mathbb{Q}_Y, \mathbb{Q}_Y(r-d)[2(r-d)-m])$$

where $d = \dim X - \dim Y$, $\dim X = d_1$. For general $f : X \to Y$, the pullback of an element in

$$\hom_{D^b\mathrm{MHM}(Y)}(\mathbb{Q}_Y, \mathbb{Q}_Y(r)[2r-m])$$

is given by the natural pullback of the morphism, so we get an element in

$$\hom_{D^b\mathrm{MHM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(r)[2r-m]).$$

The cycle map is also compatible with the product of cycles on smooth X. If we consider the product of two cycles $\xi_1 \bullet \xi_2$, with $\xi_1 \in \mathrm{CH}^r(X, m; \mathbb{Q}), \xi_2 \in \mathrm{CH}^s(X, n; \mathbb{Q})$, then the corresponding element in

$$\hom_{D^b\mathrm{MHM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(r+s)[2(r+s)-(m+n)]).$$

is given in the following way: the cycles ξ_1 and ξ_2 induce morphisms

$$\mathbb{Q}_X \to \mathbb{Q}_{|\xi_1|} \to \mathbb{D}\mathbb{Q}_{|\xi_1|}(-d+r)[-2(d-r)-m] \to \mathbb{Q}_X(r)[2r-m]$$

and

$$\mathbb{Q}_X \to \mathbb{Q}_{|\xi_2|} \to \mathbb{D}\mathbb{Q}_{|\xi_2|}(-d+s)[-2(d-s)-n] \to \mathbb{Q}_X(s)[2s-n]$$

respectively, with $\dim X = d$. Then the cycle $\xi_1 \times \xi_2$ in $\mathrm{CH}^{r+s}(X \times X, m+n; \mathbb{Q})$ corresponds to the natural external product of the two morphisms above

$$\begin{aligned} &\mathbb{Q}_X \otimes \mathbb{Q}_X \to \mathbb{Q}_{|\xi_1|} \otimes \mathbb{Q}_{|\xi_2|} \to \\ &\quad \mathbb{D}\mathbb{Q}_{|\xi_1|}(-d+r)[-2(d-r)-m] \otimes \mathbb{D}\mathbb{Q}_{|\xi_2|}(-d+s)[-2(d-s)-n] \to \\ &\qquad\qquad \mathbb{Q}_X(r)[2r-m] \otimes \mathbb{Q}_X(s)[2s-n] \end{aligned}$$

which is clearly equivalent to

$$\begin{aligned} &\mathbb{Q}_{X\times X} \to \mathbb{Q}_{|\xi_1\times\xi_2|} \to \mathbb{D}\mathbb{Q}_{|\xi_1\times\xi_2|}(-2d+r+s)[-2(d-r)-m-2(d-s)-n] \to \\ &\qquad\qquad \mathbb{Q}_{X\times X}(r+s)[2(r+s)-m-n] \end{aligned}$$

We can apply the pullback along the diagonal morphism to get the required morphism, which corresponds to the morphism induced by the intersection.

Chapter 5

Higher Mumford-Griffiths invariants

We first introduce the sheaf of relative differentials, which is a purely algebraic construction used to define the algebraic de Rham cohomology groups, which in turn allows us to define the space of higher Mumford-Griffiths invariants.

Definition 5.1. Let $f : A \to B$ be a morphism of rings. The B-module $\Omega_{B/A}$ is defined to be the free B-module generated by $\{db | b \in B\}$ divided out by the relations (i) $d(b_1 + b_2) = db_1 + db_2$, (ii) $d(b_1 b_2) = b_1 db_2 + b_2 db_1$, (iii) $da = 0$ if a comes from A. $\Omega_{B/A}$ is called the module of relative differentials of B over A.

Let $\delta : B \otimes_A B \to B$ be the map defined by $b_1 \otimes b_2 \mapsto b_1 b_2$, and let $I = \ker \delta$. I is a $B \otimes_A B$-module, I/I^2 is a $B \otimes_A B/I$-module, and therefore is a B-module (see [42]).

Proposition 5.2 ([42])**.** I/I^2 is isomorphic to $\Omega_{B/A}$.

Example 5.3 ([42])**.** Let $A = k$, k a field, $B = k[x_1, \ldots, x_n]$. Then $\Omega_{B/A}$ is a free B-module with generators $dx_1, \ldots, dx_n$ and

$$dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i$$

for all $g \in B$. More generally, if $B = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ then $\Omega_{B/A}$ is generated as B-module by $dx_1, \dots, dx_n$ with relations:

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j = 0.$$

We can now extend the definition of relative differentials to schemes (as defined in [42]). Let $f : X \to Y$ be a morphism of schemes. Let

$$\Delta : X \to X \times_Y X$$

be the diagonal morphism. X is isomorphic to its image $\Delta(X)$ by Δ.

Definition 5.4. Let $\mathscr{I}$ be the sheaf ideals of $\Delta(X)$. The sheaf of relative differentials of X over Y is defined to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2)$ on X.

The sheaf $\Omega_{X/Y}$ can also be defined locally. If $U = \mathrm{Spec}(B) \subset X, V = \mathrm{Spec}(A) \subset Y$ are open affine such that $f(U) \subset V$, then $\Omega_{X/Y}$ restricted to U is $\Omega_{B/A}$. There is a natural differential map $d : \mathcal{O}_X \to \Omega_{X/Y}$ and $\Omega_{X/Y}$ is quasi-coherent (see [25]).

Proposition 5.5 ([25])**.** Let X a variety over an algebraically closed field $k = \overline{k}$. X is a smooth variety over k if and only if $\Omega_{X/k}$ is a locally free sheaf of rank $n = \mathrm{dim}X$.

If X is a smooth variety over $k = \overline{k}$, then we define the tangent sheaf of X by $\mathcal{T}_X := \hom_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$. It is a locally free sheaf of rank $n = \mathrm{dim}X$. Thus we can associate to $\mathcal{T}_X$ a vector bundle over X.

In general we put $\Omega^\ell_{X/Y} = \wedge^\ell \Omega_{X/Y}$. In some constructions where the context is clear, we will denote the global sections of $\Omega^\ell_{X/Y}$ also by $\Omega^\ell_{X/Y}$. Then we have a complex $\Omega^\bullet_{X/Y}$ where the differential $d_\ell : \Omega^\ell_{X/Y} \to \Omega^{\ell+1}_{X/Y}$ is the natural morphism induced by d:

$$\Omega^\bullet_{X/Y} : 0 \to \mathcal{O}_X \to \Omega_{X/Y} \to \Omega^2_{X/Y} \to \dots$$

Let X and S be smooth quasiprojective varieties over k and $f : X \to S$ a proper smooth morphism.

Definition 5.6. The q-th relative de Rham cohomology sheaf of X over S is

$$\mathcal{H}^q_{DR}(X/S) := \mathbb{R}^q f_* \Omega^\bullet_{X/S}.$$

The sequence

$$0 \to f^*(\Omega^1_{S/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0,$$

is exact because f is smooth and we can define a canonical filtration on the complex $\Omega^\bullet_{X/k}$ by

$$F^j(\Omega^\bullet_{X/k}) = \text{image}(\Omega^{\bullet-j}_{X/k} \otimes_{\mathcal{O}_X} f^*(\Omega^j_{S/k}) \to \Omega^\bullet_{X/k}).$$

with graded pieces

$$Gr^j_F(\Omega^\bullet_{X/k}) = \frac{F^j(\Omega^\bullet_{X/k})}{F^{j+1}(\Omega^\bullet_{X/k})} = f^*(\Omega^j_{S/k}) \otimes_{\mathcal{O}_X} \Omega^{\bullet-j}_{X/S}.$$

There is a spectral sequence abutting to $\mathbb{R}^q f_* \Omega^\bullet_{X/k}$ with

$$\begin{aligned} E_1^{p,q} =& \mathbb{R}^{p+q} f_*(Gr^p_F(\Omega^\bullet_{X/k})) = \mathbb{R}^{p+q} f_*(f^*(\Omega^p_{S/k}) \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/S}) \\ \simeq& \Omega^p_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S). \end{aligned}$$

Let's consider, for every q, the complex $E_1^{\bullet,q}$:

$$0 \to \mathcal{H}^q_{DR}(X/S) \xrightarrow{d_1^{0,q}} \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S) \xrightarrow{d_1^{1,q}} \Omega^2_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S) \to \dots$$

Since the filtration is compatible with the exterior product and the sequence of hyperderived functors is multiplicative we have pairings

$$E_r^{p,q} \times E_r^{p_1,q_1} \to E_r^{p+p_1,q+q_1}$$

sending (e, e_1) to $e \cdot e_1$ where e and e_1 are sections of $E_r^{p,q}$ and $E_r^{p_1,q_1}$ respectively, over an open subset of S. Moreover

$$d_r(e \cdot e_1) = d_r(e) \cdot e_1 + (-1)^{p+q} e \cdot d_r(e_1).$$

Thus, in particular, we get

$$d_1^{p,q}(\omega \cdot e) = d\omega \cdot e + (-1)^p \omega \cdot d_1^{0,q} e$$

for ω a section of $\Omega^p_{S/k}$ and e a section of $\mathcal{H}^q_{DR}(X/S)$ over an open subset of S. Note that we took a section ω of $\Omega^p_{S/k}$ because we can consider $\Omega^p_{S/k}$ as a subcomplex of $E_1^{\bullet,0}$. Indeed, for $q=0$, the complex $E_1^{\bullet,0}$ is isomorphic to $\Omega^\bullet_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^0_{DR}(X/S)$ with the differential $d\otimes 1$, where d is the differentiation in $\Omega^\bullet_{S/k}$. Therefore $\nabla := d_1^{0,q}$ is a connection on $\mathcal{H}^q_{DR}(X/S)$ and clearly $d_1^{1,q}\circ d_1^{0,q}=0$, i.e. the connection is integrable. We have proved (see [29]):

Theorem 5.7 (Katz-Oda)**.** Let $f: X\to S$ be a proper smooth morphism of smooth quasiprojective varieties over k. There exists a canonical integrable connection $\nabla:\mathcal{H}^q_{DR}(X/S)\to\Omega^1_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)$ on the relative de Rham cohomology sheaf $\mathcal{H}^q_{DR}(X/S)$.

The connection ∇ is called the Gauss-Manin connection. One can extend ∇ to
$$\nabla:\Omega^\ell_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)\to\Omega^{\ell+1}_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)$$
by
$$\nabla(\omega\otimes e)=d\omega\otimes e+(-1)^\ell\omega\otimes\nabla e.$$
Now consider the short exact sequence
$$0\to Gr_F^{p+1}(\Omega^\bullet_{X/k})\to\frac{F^p(\Omega^\bullet_{X/k})}{F^{p+2}(\Omega^\bullet_{X/k})}\to Gr_F^p(\Omega^\bullet_{X/k})\to 0.$$

By applying the functor $\mathbb{R}^{p+q}f_*$ we get a connecting homomorphism from $\mathbb{R}^{p+q}f_*(Gr_F^p(\Omega^\bullet_{X/k}))$ to $\mathbb{R}^{p+q+1}f_*(Gr_F^{p+1}(\Omega^\bullet_{X/k}))$. In [29], Katz and Oda show that this connecting homomorphism is precisely the differential
$$d_1^{p,q}: E_1^{p,q}=\mathbb{R}^{p+q}f_*(Gr_F^p(\Omega^\bullet_{X/k}))\to E_1^{p+1,q}=\mathbb{R}^{p+q+1}f_*(Gr_F^{p+1}(\Omega^\bullet_{X/k})).$$
In particular, the Gauss-Manin connection $\nabla=d_1^{0,q}$ is obtained from the connecting homomorphism of the short exact sequence
$$0\to Gr_F^1(\Omega^\bullet_{X/k})\to\frac{\Omega^\bullet_{X/k}}{F^2(\Omega^\bullet_{X/k})}\to Gr_F^0(\Omega^\bullet_{X/k})\to 0.$$
when we apply $\mathbb{R}^qf_*$. Then one gets a commutative diagram
$$\begin{tikzcd}
\mathbb{R}^qf_*(Gr_F^0(\Omega^{\bullet\geq j}_{X/k})) \arrow[r, "\nabla"] \arrow[d] & \mathbb{R}^{q+1}f_*(Gr_F^1(\Omega^{\bullet\geq j}_{X/k})) \arrow[d] \\
\mathbb{R}^qf_*(Gr_F^0(\Omega^\bullet_{X/k})) \arrow[r, "\nabla"] & \mathbb{R}^{q+1}f_*(Gr_F^1(\Omega^\bullet_{X/k})).
\end{tikzcd}$$

Since

$$\mathbb{R}^q f_*(Gr_F^0(\Omega_{X/k}^{\bullet\geq j})) = \mathbb{R}^q f_*(f^*(\Omega_{S/k}^0) \otimes_{\mathcal{O}_X} \Omega_{X/S}^{\bullet\geq j}) \simeq \mathbb{R}^q f_* \Omega_{X/S}^{\bullet\geq j}$$

and

$$\mathbb{R}^{q+1} f_*(Gr_F^1(\Omega_{X/k}^{\bullet\geq j})) = \mathbb{R}^{q+1} f_*(f^*(\Omega_{S/k}^1) \otimes_{\mathcal{O}_X} \Omega_{X/S}^{\bullet-1\geq j-1}) \simeq \Omega_{S/k}^1 \otimes_{\mathcal{O}_S} \mathbb{R}^q f_* \Omega_{X/S}^{\bullet\geq j-1}$$

we obtain

$$\begin{tikzcd}
\mathbb{R}^q f_* \Omega_{X/S}^{\bullet\geq j} \arrow[r, "\nabla"] \arrow[d] & \Omega_{S/k}^1 \otimes_{\mathcal{O}_S} \mathbb{R}^q f_* \Omega_{X/S}^{\bullet\geq j-1} \arrow[d] \\
\mathbb{R}^q f_* \Omega_{X/S}^{\bullet} \arrow[r, "\nabla"] & \Omega_{S/k}^1 \otimes_{\mathcal{O}_S} \mathbb{R}^q f_* \Omega_{X/S}^{\bullet}
\end{tikzcd}$$

We define a filtration on the sheaves $\mathcal{H}_{DR}^q(X/S)$ by

$$F^j \mathcal{H}_{DR}^q(X/S) := \mathrm{image}(\mathbb{R}^q f_* \Omega_{X/S}^{\bullet\geq j} \to \mathbb{R}^q f_* \Omega_{X/S}^{\bullet}).$$

Then the diagram above shows that:

$$\nabla(F^j \mathcal{H}_{DR}^q(X/S)) \subset \Omega_{S/k}^1 \otimes_{\mathcal{O}_S} F^{j-1} \mathcal{H}_{DR}^q(X/S)$$

This property is called Griffiths transversality.

More generally, applying $\mathbb{R}^q f_*$ to the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & Gr_F^{p+1}(\Omega_{X/k}^{\bullet\geq r}) \arrow[r] \arrow[d, equal] & \frac{F^p(\Omega_{X/k}^{\bullet\geq r})}{F^{p+2}(\Omega_{X/k}^{\bullet\geq r})} \arrow[r] & Gr_F^p(\Omega_{X/k}^{\bullet\geq r}) \arrow[r] \arrow[d, equal] & 0 \\
& \Omega_{S/k}^{p+1} \otimes \Omega_{X/S}^{\bullet\geq r-p-1} & & \Omega_{S/k}^p \otimes \Omega_{X/S}^{\bullet\geq r-p} &
\end{tikzcd}$$

yields

$$\nabla : \Omega_{S/k}^p \otimes F^{r-p} \mathcal{H}_{DR}^q(X/S) \to \Omega_{S/k}^{p+1} \otimes F^{r-p-1} \mathcal{H}_{DR}^q(X/S) \qquad (5.1)$$

Now, let $X = X_K$ be a smooth projective variety defined over a subfield $K \subset \mathbb{C}$ that is say finitely generated over a subfield $k \subset K$. One should keep in mind the situation of a k-spread of smooth quasi-projective varieties $\mathcal{X} \to S$, where $k(S) = K$, and $X = \mathcal{X}_\eta$, where η is the generic point of S (see Chapter (6)).

Definition 5.8. The q-th algebraic de Rham cohomology group of X over k is given by

$$H^q_{DR}(X/k) := \mathbb{H}^q(X, \Omega^\bullet_{X/k}).$$

The Hodge filtration is given by

$$F^j H^q_{DR}(X/k) := \mathbb{H}^q(X, \Omega^{\bullet \geq j}_{X/k}).$$

Thus, using the constructions above we get the Gauss-Manin connection:

$$\nabla : H^q_{DR}(X/K) \to \Omega^1_{K/k} \otimes H^q_{DR}(X/K)$$

that can be extended to

$$\nabla : \Omega^\ell_{K/k} \otimes H^q_{DR}(X/K) \to \Omega^{\ell+1}_{K/k} \otimes H^q_{DR}(X/K)$$

Clearly, as in the relative case, we have:

(i) $\nabla^2 = 0$, i.e. the connection is integrable.

(ii) Griffiths transversality:

$$\nabla(F^j H^i_{DR}(X/K)) \subset \Omega^1_{K/k} \otimes F^{j-1} H^i_{DR}(X/K).$$

Definition 5.9. (i) The space of higher Mumford-Griffiths invariants of X_K/K, $\nabla J^{r,m,j}(X_K/K)$, is defined by the cohomology of:

$$\Omega^{j-1}_{K/k} \otimes F^{r-j+1} H^{2r-m-j}_{DR}(X_K/K) \xrightarrow{\nabla} \Omega^j_{K/k} \otimes F^{r-j} H^{2r-m-j}_{DR}(X_K/K) \xrightarrow{\nabla} \Omega^{j+1}_{K/k} \otimes F^{r-j-1} H^{2r-m-j}_{DR}(X_K/K)$$

(ii) The space of higher de Rham invariants of X_K/K, $\nabla DR^{r,m,j}(X_K/K)$, is defined by the cohomology of:

$$\Omega^{j-1}_{K/k} \otimes H^{2r-m-j}_{DR}(X_K/K) \xrightarrow{\nabla} \Omega^j_{K/k} \otimes H^{2r-m-j}_{DR}(X_K/K) \xrightarrow{\nabla} \Omega^{j+1}_{K/k} \otimes H^{2r-m-j}_{DR}(X_K/K).$$

Remark. By applying the global sections functor to (5.1), one has a variational version $\nabla J^{r,m,j}(X_S/S)$, and likewise $\nabla DR^{r,m,j}(X_S/S)$.

Chapter 6

Filtrations on higher Chow groups

Let X be a smooth projective variety over $\mathbb{C}$. Since there are a finite number of polynomials defining X, we can consider X as defined over a finitely generated extension K of $\bar{\mathbb{Q}}$, and call it X_K. Then there exist varieties $\mathcal{X}$ and S over $\bar{\mathbb{Q}}$ and a proper smooth morphism $f : \mathcal{X} \to S$ such that the fibre $\mathcal{X}_{\eta_S}$ over the generic point $\eta_S \in S$ is X_K. This is the process of taking a $\bar{\mathbb{Q}}$-spread of the variety (see [30]). Let's denote $f^{-1}(U)$ by X_U, for any open affine $U \subset S$. If we consider the category $\mathrm{MHM}(X_U)$ of mixed Hodge modules over X_U, we have the cycle map

$$c_{r,m} : \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X_U)}(\mathbb{Q}_{X_U}(0), \mathbb{Q}_{X_U}(r)).$$

Moreover, since f is smooth and proper we can use Corollary (4.18). Then there is the Leray spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^q f_* \mathbb{Q}_{X_U}(r)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{MHM}(X_U)}(\mathbb{Q}_{X_U}(0), \mathbb{Q}_{X_U}(r)), \tag{6.1}$$

which degenerates at E_2. Let's denote the canonical Leray filtration on $\mathrm{Ext}^{p+q}_{\mathrm{MHM}(X_U)}(\mathbb{Q}_{X_U}(0), \mathbb{Q}_{X_U}(r))$ by $F_L^\bullet$.

Now we can construct a filtration on X_U. Let

$$F^j\mathrm{CH}^r(X_U, m; \mathbb{Q}) := c_{r,m}^{-1}(F_L^j \,\mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X_U)}(\mathbb{Q}_{X_U}(0), \mathbb{Q}_{X_U}(r))).$$

Also, we get maps between the graded pieces of this filtration and the groups $E_2^{p,q}$. More explicitly, by using the cycle map and the degeneration of the spectral sequence (6.1), there are injective maps

$$c^j_{r,m} : Gr^j_F \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r)).$$

Note the filtration F^j and maps $c^j_{r,m}$ can be defined, besides for X_U, for any smooth variety defined over a field $k \subset \mathbb{C}$, which is finitely generated over $\bar{\mathbb{Q}}$.

Lemma 6.1. Let $f : \mathcal{X} \to S$ be as above and $\eta_S \in S$ the generic point. Then

$$\mathrm{CH}^r(\mathcal{X}_{\eta_S}, m) = \varinjlim_{U \subset S} \mathrm{CH}^r(X_U, m).$$

Proof. First we have

$$\varinjlim_{U \subset S} Z^r(X_U \times \Delta^m) = Z^r(\mathcal{X}_{\eta_S} \times \Delta^m).$$

Every cycle meeting $X_U \times \Delta^m$ properly has limit in $Z^r(\mathcal{X}_{\eta_S}, m)$, i.e.

$$\varinjlim_{U \subset S} Z^r(X_U, m) = Z^r(\mathcal{X}_{\eta_S}, m).$$

Since $\mathrm{CH}^r(X_U, m)$ is the homology of

$$Z^r(X, m+1) \xrightarrow{\partial_{m+1}} Z^r(X, m) \xrightarrow{\partial_m} Z^r(X, m-1),$$

we can take the limit to get

$$\varinjlim_{U \subset S} \mathrm{CH}^r(X_U, m) = \mathrm{CH}^r(\mathcal{X}_{\eta_S}, m).$$

□

Theorem 6.2 (Asakura [5])**.** Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. There exists a descending filtration $F^\bullet$ on $\mathrm{CH}^r(X, m; \mathbb{Q})$, which satisfies the following:

(i) $F^0\mathrm{CH}^r(X, m; \mathbb{Q}) = \mathrm{CH}^r(X, m; \mathbb{Q})$.

(ii) $F^j\mathrm{CH}^r(X, m; \mathbb{Q}) \bullet F^\ell\mathrm{CH}^s(X, n; \mathbb{Q}) \subset F^{j+\ell}\mathrm{CH}^{r+s}(X, m+n; \mathbb{Q})$, under the intersection product. (Not specifically stated in [5].)

(iii) F^j is preserved under the action of correspondences.

(iv) Assume the components of the diagonal class are algebraic. Then

$$\Delta_X(2d-2r+m+\ell, 2r-m-\ell)_*|_{Gr_F^j \mathrm{CH}^r(X,m;\mathbb{Q})} = \begin{cases} \text{Identity} & \text{, if } \ell = j \\ 0 & \text{, otherwise} \end{cases}$$

where $Gr_F^j \mathrm{CH}^r(X,m;\mathbb{Q}) = F^j \mathrm{CH}^r(X,m;\mathbb{Q})/F^{j+1}\mathrm{CH}^r(X,m;\mathbb{Q})$.

(v) $F^{r+1}\mathrm{CH}^r(X,m;\mathbb{Q}) = F^{r+2}\mathrm{CH}^r(X,m;\mathbb{Q}) = \ldots$

Proof. It is instructive to sketch some details. We already showed how to define a filtration on X_U. To get a filtration on the variety X we perform two more steps. First, we can define a filtration on X_K using the fact that $X_K \simeq \mathcal{X}_{\eta_S} \simeq \varprojlim X_U$ (here the limit is over all $U \subset S$ affine open). Then

$$F^j \mathrm{CH}^r(X_K;\mathbb{Q}) := \varinjlim_{U \subset S} F^j \mathrm{CH}^r(X_U;\mathbb{Q}),$$

where the limit is taken over all open affine subsets of S (this is justified by the previous lemma). Finally we set

$$F^j \mathrm{CH}^r(X;\mathbb{Q}) := \varinjlim_{K \subset \mathbb{C}} F^j \mathrm{CH}^r(X_K;\mathbb{Q}),$$

where K is finitely generated over $\overline{\mathbb{Q}}$.

(i) Follows from the definition.

(ii) Let's take two cycles $\xi_1 \in F^j \mathrm{CH}^r(X,m;\mathbb{Q}), \xi_2 \in F^\ell \mathrm{CH}^s(X,n;\mathbb{Q})$. We can spread out ξ_1 and ξ_2 to cycles $\bar{\xi_1} \in F^j \mathrm{CH}^r(X_U,m;\mathbb{Q}), \bar{\xi_2} \in F^\ell \mathrm{CH}^s(X_U,n;\mathbb{Q})$. From the construction of the filtration we have the following injective map

$$c^j_{r,m} : Gr_F^j \mathrm{CH}^r(X_U,m;\mathbb{Q}) \to \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r)),$$

so that $\bar{\xi_1}$ induces a morphism in

$$\hom_{D^b\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r)[j]).$$

Similarly, $\bar{\xi_2}$ induces a morphism in

$$\hom_{D^b\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2s-n-\ell} f_* \mathbb{Q}_{X_U}(s)[\ell]).$$

Since the product of cycles corresponds to the natural product of morphisms, $\xi_1 \bullet \xi_2$ induces a morphism in

$$\hom_{D^b\mathrm{MHM}(U)}(\mathbb{Q}_U(0) \otimes \mathbb{Q}_U(0), R^{2r-m-j}f_*\mathbb{Q}_{X_U}(r)[j] \otimes R^{2s-n-\ell}f_*\mathbb{Q}_{X_U}(s)[\ell]).$$

There is a map

$$R^{2r-m-j}f_*\mathbb{Q}_{X_U}(r) \otimes R^{2s-n-\ell}f_*\mathbb{Q}_{X_U}(s) \to R^{2(r+s)-m-n-j-\ell}f_*\mathbb{Q}_{X_U}(r+s)$$

(the product is compatible with the product of filtered modules, using the forgetful functor from the category of mixed Hodge modules to the category of filtered regular holonomic D_U-modules). Therefore we get an element of

$$\hom_{D^b\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2(r+s)-m-n-j-\ell}f_*\mathbb{Q}_{X_U}(r+s)[j+\ell])$$

which gives us an element in $F_L^{j+\ell}\operatorname{Ext}^{2(r+s)-m-n}_{\mathrm{MHM}(X_U)}(\mathbb{Q}_{X_U}(0), \mathbb{Q}_{X_U}(r))$, thus $\bar{\xi_1} \bullet \bar{\xi_2}$ is in $F^{j+\ell}\mathrm{CH}^{r+s}(X_U, m+n; \mathbb{Q})$.

(iii) Consider a correspondence $\Gamma \in \mathrm{CH}^\ell(X \times Y, n)$. We can find proper smooth morphisms $f : \mathcal{X} \to S$,$g : \mathcal{Y} \to W$ such that for open sets $U \subset S, V \subset W$, Γ can be represented by $\bar{\Gamma} \in \mathrm{CH}^\ell(X_U \times Y_V, n)$. Take $\xi \in F^j\mathrm{CH}^r(X, m; \mathbb{Q})$ with corresponding spread $\bar{\xi} \in F^j\mathrm{CH}^r(X_U, m; \mathbb{Q})$. Then $\bar{\xi}$ maps to an element of

$$\begin{aligned}\operatorname{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j}f_*\mathbb{Q}_{X_U}(r)) = \\ \hom_{D^b\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j}f_*\mathbb{Q}_{X_U}(r)[j]). \quad (6.2)\end{aligned}$$

Similarly $\bar{\Gamma}$ maps to an element of

$$\begin{aligned}\operatorname{Ext}^0_{\mathrm{MHM}(U\times V)}(\mathbb{Q}_{U\times V}(0), R^{2\ell-n}(f\times g)_*\mathbb{Q}_{X\times Y_{U\times V}}(\ell)) = \\ \hom_{D^b\mathrm{MHM}(U\times V)}(\mathbb{Q}_{U\times V}(0), R^{2\ell-n}(f\times g)_*\mathbb{Q}_{X\times Y_{U\times V}}(\ell)). \quad (6.3)\end{aligned}$$

Since Γ_* is defined by

$$\Gamma_*(\xi) = \pi_{Y*}(\pi_X^*(\xi) \bullet \Gamma),$$

we can apply $\pi_{X_U}^*$ to the morphism defined by $\bar{\xi}$ in (6.2), then take the product with the morphism defined by $\bar{\Gamma}$ in (6.3), and finally apply π_{Y_V*} to get an element in

$$\begin{aligned}\operatorname{Ext}^j_{\mathrm{MHM}(V)}(\mathbb{Q}_V(0), R^{2(r+\ell-d)-(m+n)-j}g_*\mathbb{Q}_{Y_V}(r+\ell-d)) = \\ \hom_{D^b\mathrm{MHM}(V)}(\mathbb{Q}_V(0), R^{2(r+\ell-d)-(m+n)-j}g_*\mathbb{Q}_{Y_V}(r+\ell-d)[j]).\end{aligned}$$

The property follows.

(iv) Suppose the components of the diagonal class $[\Delta_X] \in H^{2d}(X \times X, \mathbb{Q})$ are algebraic, i.e. for the decomposition

$$H^{2d}(X \times X, \mathbb{Q}) = \bigoplus_{p+q=2d} H^p(X) \otimes H^q(X)$$

we have $[\Delta_X] = \sum[\Delta(p,q)]$. Then we can find a decomposition $[\bar{\Delta}_{X_U}] = \sum[\bar{\Delta}(p,q)]$ where $\bar{\Delta}(p,q) \in \mathrm{CH}^d(X_U \times X_U; \mathbb{Q})$, $\bar{\Delta}_{X_U} \in \mathrm{CH}^d(X_U \times X_U; \mathbb{Q})$. If we take $\xi \in Gr_F^j \mathrm{CH}^r(X, m; \mathbb{Q})$ with corresponding spread given by $\bar{\xi} \in Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q})$, it is clear by (iii), that $\bar{\Delta}(p,q)_*$ acts as the identity on $\bar{\xi}$ precisely when $p = 2d - 2r + m - j$ because $\bar{\Delta}(p,q)$ maps to an element of

$$\mathrm{Ext}^0_{\mathrm{MHM}(U\times U)}(\mathbb{Q}_{U\times U}(0), R^p f_* \mathbb{Q}_{X_U} \otimes R^q f_* \mathbb{Q}_{X_U}(d)).$$

(v) Consider the operation L_X of intersecting with a hyperplane section of X. Then we have an induced map

$$L_X^{d-2r+m+j} : Gr_F^j \mathrm{CH}^r(X, m; \mathbb{Q}) \to Gr_F^j \mathrm{CH}^{d-r+m+j}(X, m; \mathbb{Q}).$$

This is because the product is preserved by the filtration, using (ii). Moreover, since the product is compatible with the cycle map we have the following commutative diagram:

$$\begin{tikzcd}
Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r)) \arrow[d] \\
Gr_F^j \mathrm{CH}^{d-r+m+j}(X_U, m; \mathbb{Q}) \arrow[r] & \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2d-2r+m+j} f_* \mathbb{Q}_{X_U}(d-r)).
\end{tikzcd}$$

The right vertical arrow is an isomorphism because of the hard Lefschetz theorem. If $j - r > 0$ then $d - r + m + j > m + d$. Therefore

$$Gr_F^j \mathrm{CH}^{d-r+m+j}(X, m; \mathbb{Q}) = 0$$

and as a result $Gr_F^j \mathrm{CH}^r(X, m; \mathbb{Q}) = 0$. $\square$

The graded pieces of the filtration we just constructed can be mapped to the space of higher Mumford-Griffiths invariants (we closely follow [5]).

Theorem 6.3. Let X be a smooth projective variety over $\mathbb{C}$. Then there exists a map
$$Gr_F^j \mathrm{CH}^r(X,m;\mathbb{Q}) \to \nabla J^{r,m,j}(X/\mathbb{C}).$$

Proof. The variety X is defined over K, a finitely generated extension of $\bar{\mathbb{Q}}$, such that $X = X_K \otimes \mathbb{C}$. Take a spread of X_K, i.e. a morphism $f : \mathcal{X} \to S$ proper and smooth such that $X_K = \mathcal{X}_{\eta_S}$ for the generic point η_S of S. Using the same notation we have been using, there is an injective map

$$c^j_{r,m} : Gr_F^j \mathrm{CH}^r(X_U, m;\mathbb{Q}) \to \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r)).$$

Recall that we can map any mixed Hodge module to its corresponding filtered regular holonomic D_U-module. In particular, $\mathbb{Q}_U(0)$ maps to $\mathcal{O}_U$ and $R^{2r-m-j} f_*\mathbb{Q}_{X_U}(r)$ to

$$\mathbb{R}^{2r-m-j} f_* \Omega^\bullet_{X_U/U}(r) = \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r).$$

Then we have a natural map

$$\mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_*\mathbb{Q}_{X_U}(r)) \to \mathrm{Ext}^j_{\mathrm{MF_{rh}}(U)}(\mathcal{O}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)).$$

Consider the Kozul resolution

$$0 \to D_U(-d) \otimes \wedge^d \mathcal{T}_U \to \ldots \to D_U(-1) \otimes \mathcal{T}_U \to D_U \to \mathcal{O}_U \to 0,$$

with $d = \dim X$. By applying hom in the category $\mathrm{MF_{rh}}(U)$ we get the sequence

$$\begin{aligned} 0 \to \hom(\mathcal{O}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)) &\to \hom(D_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)) \\ &\to \hom(D_U(-1) \otimes \mathcal{T}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)) \to \ldots \end{aligned}$$

Since, in general, $\mathrm{Ext}^p_{\mathrm{MF_{rh}}(U)}(D_U(\ell), \mathcal{M}) = 0$ for any D_U-module $\mathcal{M}$ and for all $p \geq 1$, we can calculate $\mathrm{Ext}^j_{\mathrm{MF_{rh}}(U)}(\mathcal{O}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r))$ as the cohomology of

$$\begin{aligned} &\hom(D_U(-j+1) \otimes \wedge^{j-1}\mathcal{T}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)) \to \\ &\qquad \hom(D_U(-j) \otimes \wedge^{j}\mathcal{T}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)) \to \\ &\qquad\qquad \hom D_U(-j-1) \otimes \wedge^{j+1}\mathcal{T}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)) \end{aligned}$$

On the other hand

$$\begin{aligned}&\hom(D_U(-\ell)\otimes\wedge^\ell\mathcal{T}_U,\mathcal{H}^q_{DR}(X_U/U)(r))\\&=\hom(D_U\otimes\wedge^\ell\mathcal{T}_U,\mathcal{H}^q_{DR}(X_U/U)(r-\ell))\\&\simeq\Omega^\ell_{U/\bar{\mathbb{Q}}}\otimes F^{r-\ell}\mathcal{H}^q_{DR}(X_U/U),\end{aligned}$$

for any ℓ and q. If we apply the global sections functor we get:

$$\Gamma(U,\Omega^j_{U/\bar{\mathbb{Q}}}\otimes F^{r-j}\mathcal{H}^{2r-m-j}_{DR}(X_U/U))=\Omega^j_{U/\bar{\mathbb{Q}}}\otimes F^{r-j}H^{2r-m-j}_{DR}(X_U/U).$$

So, $\mathrm{Ext}^j_{\mathrm{MF}_{\mathrm{rh}}(U)}(\mathcal{O}_U,\mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r))$ maps to $\nabla J^{r,m,j}(X_U/U)$. Thus we have a map

$$Gr^j_F\mathrm{CH}^r(X_U,m;\mathbb{Q})\to\nabla J^{r,m,j}(X_U/U).$$

Furthermore,

$$\begin{aligned}\varinjlim_{U\subset S}\Omega^j_U\otimes F^{r-j}H^{2r-m-j}_{DR}(X_U/U)&=\Omega^j_{\eta/\bar{\mathbb{Q}}}\otimes F^{r-j}H^{2r-m-j}_{DR}(X_\eta/\eta)\\&\simeq\Omega^j_{K/\bar{\mathbb{Q}}}\otimes F^{r-j}H^{2r-m-j}_{DR}(X_K/K).\end{aligned}$$

Then we can apply the limit over $U\subset S$ and then the limit over all K to get the result. $\square$

By considering the natural map $\nabla J^{r,m,j}(X/\mathbb{C})\to\nabla DR^{r,m,j}(X/\mathbb{C})$ by forgetting the Hodge filtration, we also have a map

$$Gr^j_F\mathrm{CH}^r(X,m;\mathbb{Q})\to\nabla DR^{r,m,j}(X/\mathbb{C}).$$

In general, higher Mumford-Griffiths invariants and de Rham invariants are difficult to describe. For that reason we would like to factor the previous maps through a more workable space. The first step is the construction of a short exact sequence whose first and last term are extensions in the category of mixed Hodge structures, a category that is well known.

Lemma 6.4. Let Y be a smooth quasiprojective variety over $k\subset\mathbb{C}$, $M\in\mathrm{MHM}(Y)$ and $g:Y\to\mathrm{Spec}(k)$ be the natural morphism. Then there exists a short exact sequence

$$\begin{aligned}0\to\mathrm{Ext}^1_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0),R^{q-1}g_*M)\to\\\mathrm{Ext}^q_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0),M)\to\hom_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0),R^qg_*M)\to0\end{aligned}$$

Proof. By Corollary (4.18) there is the Leray spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^q g_* M) \Rightarrow \operatorname{Ext}^{p+q}_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), M),$$

which degenerates at E_2. Then, if L is the Leray filtration on $\operatorname{Ext}^{p+q}_{\mathrm{MHM}(Y)}$,

$$E_2^{p,q} = Gr_L^p \operatorname{Ext}^{p+q}_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), M).$$

On the other hand, MHM(Spec(k)) is a subcategory of the category of mixed Hodge structures, and we know that $\operatorname{Ext}^\ell_{\mathrm{MHS}} = 0$ for all $\ell \geq 2$, and similarly for MHM(Spec(k)). Then

$$\begin{aligned} L^1 \operatorname{Ext}^q_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), M) &= Gr_L^1 \operatorname{Ext}^q_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), M) \\ &= \operatorname{Ext}^1_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^{q-1} g_* M). \end{aligned}$$

Since

$$\hom_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^q g_* M) = Gr_L^0 \operatorname{Ext}^q_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), M),$$

we get that $\hom_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^q g_* M)$ is the quotient of $\operatorname{Ext}^q_{\mathrm{MHM}(Y)}(\mathbb{Q}_Y(0), M)$ and $\operatorname{Ext}^1_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^{q-1} g_* M)$, and this implies that we have the short exact sequence required. □

Since we will we passing to spreads when working with complex smooth projective varieties, the following proposition is what we need:

Proposition 6.5. Let $f : X \to S$ be a proper smooth morphism of quasiprojective smooth varieties over $k \subset \mathbb{C}$. Then there exists a short exact sequence

$$0 \to \underline{E}_\infty^{j,2r-m-j} \to E_\infty^{j,2r-m-j} \to \underline{\underline{E}}_\infty^{j,2r-m-j} \to 0$$

where

$$E_\infty^{j,2r-m-j} = \operatorname{Ext}^j_{\mathrm{MHM}(S)}(\mathbb{Q}_S(0), R^{2r-m-j} f_* \mathbb{Q}_X(r)),$$

$$\underline{E}_\infty^{j,2r-m-j} = \frac{\operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{-1} H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r)))}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^0 H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r)))}$$

and

$$\underline{\underline{E}}_\infty^{j,2r-m-j} = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(S, R^{2r-m-j} f_* \mathbb{Q}(r))).$$

Proof. In the previous lemma, we set $M = R^{2r-m-j} f_* \mathbb{Q}_X(r)$ and $q = j$ to get a sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^{j-1} g_* R^{2r-m-j} f_* \mathbb{Q}_X(r)) \\ \to \mathrm{Ext}^j_{\mathrm{MHM}(S)}(\mathbb{Q}_S(0), R^{2r-m-j} f_* \mathbb{Q}_X(r)) \\ \to \hom_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^j g_* R^{2r-m-j} f_* \mathbb{Q}_X(r)) \to 0. \end{aligned}$$

But MHM(Spec(k)) is a subcategory of PMHS, the category of graded polarizable mixed Hodge structures. The mixed Hodge module $\mathbb{Q}_{\mathrm{Spec}(k)}(0)$ is isomorphic to $\mathbb{Q}(0)$, $R^{j-1} g_* R^{2r-m-j} f_* \mathbb{Q}_X(r) \simeq H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r))$ and $R^j g_* R^{2r-m-j} f_* \mathbb{Q}_X(r) \simeq H^j(S, R^{2r-m-j} f_* \mathbb{Q}(r))$, because $g : S \to \mathrm{Spec}(k)$ has constant fibre S. Thus,

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^{j-1} g_* R^{2r-m-j} f_* \mathbb{Q}_X(r)) \\ &\simeq \mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r))), \end{aligned}$$

$$\begin{aligned} &\hom_{\mathrm{MHM}(\mathrm{Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^j g_* R^{2r-m-j} f_* \mathbb{Q}_X(r)) \\ &\simeq \hom_{\mathrm{PMHS}}(\mathbb{Q}(0), H^j(S, R^{2r-m-j} f_* \mathbb{Q}(r))). \end{aligned}$$

Finally,

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r))) \\ &= \frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{-1} H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r)))}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr^0_W H^{j-1}(S, R^{2r-m-j} f_* \mathbb{Q}(r)))} \end{aligned}$$

by Proposition (3.11). □

Now we can use the short exact sequence we just constructed and relate it to the filtration on higher Chow groups. Take a complex smooth projective variety X, and consider a spread given by a proper smooth morphism $f : \mathcal{X} \to S$ of varieties over $\bar{\mathbb{Q}}$ such that $X_K = \mathcal{X}_{\eta_S}$, where K is the field of definition of X of finite transcendence degree over $\bar{\mathbb{Q}}$, η_S is the generic point of S and $\mathcal{X}_{\eta_S} = f^{-1}(\eta_S)$. Then, for $U \subset S, X_U = f^{-1}(U)$, we have an injective map

$$c^j_{r,m} : Gr^j_F \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r)).$$

Using the notation of Proposition (6.5), this means we have a map

$$Gr^j_F \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to E^{j, 2r-m-j}_\infty$$

because

$$E_\infty^{j,2r-m-j} = \operatorname{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j} f_* \mathbb{Q}_{X_U}(r))$$

for $f : X_U \to U$. An immediate consequence is the following property for our filtration.

Corollary 6.6. Let X be a complex smooth projective variety and $F^\bullet$ the descending filtration on $\mathrm{CH}^r(X, m; \mathbb{Q})$ of Theorem (6.2). Then $F^0 = F^1$ when $m \geq 1$.

Proof. Take a spread of X as before. If $j = 0$ then $\underline{E}_\infty^{j,2r-m-j} = 0$ and $E_\infty^{j,2r-m-j} \simeq \underline{\underline{E}}_\infty^{j,2r-m-j}$ by Proposition (6.5). Since we have an injective map

$$Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to E_\infty^{j,2r-m-j},$$

then we have an injective map

$$Gr_F^0 \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(U, R^{2r-m} f_* \mathbb{Q}(r))).$$

We know that $H^0(U, R^{2r-m} f_* \mathbb{Q}(r))$ is isomorphic to the cycles in $H^{2r-m}(X_t, \mathbb{Q}(r))$ invariant under the action of the monodromy group $\pi(U,t)$, for $t \in U$ and X_t the fibre by $f : X_U \to U$. We can write this as

$$H^0(U, R^{2r-m} f_* \mathbb{Q}(r)) \simeq H^{2r-m}(X_t, \mathbb{Q}(r))^{\pi(U,t)}.$$

On the other hand (see [26])

$$\begin{aligned} &\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r))) = \\ &\qquad F^0 W_0 H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)) \otimes \mathbb{C} \cap W_0 H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)). \end{aligned}$$

But $H^{2r-m}(X_t, \mathbb{Q}(r))^{\pi(U,t)}$ is a pure Hodge structure by the work of Deligne ([18]) so that we can drop the W_0. Then

$$F^0 H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)) \otimes \mathbb{C} \simeq F^r H^{2r-m}(X_t, \mathbb{C})^{\pi(U,t)}.$$

Using the fact that X_t is projective and a weight argument, $F^r H^{2r-m}(X_t, \mathbb{C}) = 0$. Therefore

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r))) = 0.$$

This means that $Gr_F^0 \mathrm{CH}^r(X_U, m; \mathbb{Q}) = 0$ and $F^0 = F^1$. $\square$

Now we can factor the map from the filtration on higher Chow groups to the de Rham invariants through $\underline{\underline{E}}_{\infty}^{j,2r-m-j}$. First, we need the following technical lemma.

Lemma 6.7. (i) Let X be a complex analytic variety and V a complex local system over X. If $\mathcal{V} = \mathcal{O}_X \otimes_{\mathbb{C}} V$, then the complex $\Omega_X^\bullet(\mathcal{V}) = \Omega_X^\bullet \otimes \mathcal{V}$ is a resolution of sheaves of $\mathcal{V}$.

(ii)(Deligne [17]) Let X be a smooth variety over $\mathbb{C}$ and V a local system over X equipped with a flat connection with regular singular points. Let $\Omega_X^\bullet(\mathcal{V})$ be as in (i). Then, we have an isomorphism between the algebraic and analytic hypercohomology

$$\mathbb{H}^\ell(X, \Omega_X^\bullet(\mathcal{V})) \simeq \mathbb{H}^\ell(X, \Omega_{X^{an}}^\bullet(\mathcal{V}))$$

for all ℓ.

Proposition 6.8. Let $f : X \to S$ be a smooth proper morphism of smooth quasi-projective varieties over a field $k \subset \mathbb{C}$. Assume S is an affine variety. There is an injective map

$$\Phi^{r,m,j} : \underline{\underline{E}}_{\infty}^{j,2r-m-j} \hookrightarrow \nabla DR^{r,m,j}(X/S) \otimes_k \mathbb{C}$$

Proof. We can use part (i) of the previous lemma applied to the local system $R^{2r-m-j} f_*\mathbb{C}$ over S to conclude that $\Omega_{S/k}^\bullet \otimes_{\mathcal{O}_S} R^{2r-m-j} f_*\mathbb{C}$ is a resolution of $R^{2r-m-j} f_*\mathbb{C}$. Then

$$H^j(S/\mathbb{C}, R^{2r-m-j} f_*\mathbb{C}) = \mathbb{H}^j(S/\mathbb{C}, \Omega_{S^{an}}^\bullet \otimes_{\mathcal{O}_S} R^{2r-m-j} f_*\mathbb{C}).$$

Using (ii) of the previous lemma,

$$\mathbb{H}^j(S/\mathbb{C}, \Omega_{S^{an}}^\bullet \otimes_{\mathcal{O}_S} R^{2r-m-j} f_*\mathbb{C}) \simeq \mathbb{H}^j(S/\mathbb{C}, \Omega_{S/\mathbb{C}}^\bullet \otimes_{\mathcal{O}_S} R^{2r-m-j} f_*\mathbb{C}).$$

We can now use the fact that $\mathcal{O}_S \otimes_{\mathbb{C}} R^{2r-m-j} f_*\mathbb{C} \simeq \mathbb{R}^{2r-m-j} f_*\Omega_{X/\mathbb{C}}^\bullet$ to get

$$H^j(S/\mathbb{C}, R^{2r-m-j} f_*\mathbb{C}) = \mathbb{H}^j(S, \Omega_{S/k}^\bullet \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega_{X/S}^\bullet) \otimes_k \mathbb{C}.$$

The complex

$$\Omega_{S/k}^\bullet \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega_{X/S}^\bullet$$

has differential ∇, the Gauss-Manin connection on sheaves. The term $E_2^{p,q}$ of the second spectral sequence abutting to

$$\mathbb{H}^j(S, \Omega_{S/k}^\bullet \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega_{X/S}^\bullet)$$

is given by
$$E_2^{p,q} = H^q_\nabla(H^p(S, \Omega^\bullet_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}))$$
where $H^q_\nabla(-)$ means the cohomology of

$$H^p(S, \Omega^{q-1}_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}) \xrightarrow{\nabla}$$
$$H^p(S, \Omega^{q}_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}) \xrightarrow{\nabla}$$
$$H^p(S, \Omega^{q+1}_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S})$$

By hypothesis, S is affine. The sheaves $\Omega^q_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}$ are coherent. Therefore
$$H^p(S, \Omega^\bullet_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}) = 0$$
for $p > 0$. Then $E_2^{p,q} = 0$ for $p > 0$ and we can calculate the hypercohomology of $\Omega^q_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}$ by the terms $E_2^{0,q}$ of the spectral sequence. Specifically,
$$\mathbb{H}^j(S, \Omega^\bullet_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S})$$
is the cohomology of

$$H^0(S, \Omega^{j-1}_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}) \xrightarrow{\nabla}$$
$$H^0(S, \Omega^{j}_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S}) \xrightarrow{\nabla}$$
$$H^0(S, \Omega^{j+1}_{S/k} \otimes_{\mathcal{O}_S} \mathbb{R}^{2r-m-j} f_*\Omega^\bullet_{X/S})$$

But this is precisely the definition of $\nabla DR^{r,m,j}(X/S)$. On the other hand,

$$\underline{\underline{E}}_\infty^{j,2r-m-j} = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(S, R^{2r-m-j} f_*\mathbb{Q}(r)))$$

maps naturally to $H^j(S/\mathbb{C}, R^{2r-m-j} f_*\mathbb{C})$. In conclusion we have

$$\underline{\underline{E}}_\infty^{j,2r-m-j} \hookrightarrow H^j(S/\mathbb{C}, R^{2r-m-j} f_*\mathbb{C}) \simeq \nabla DR^{r,m,j}(X/S) \otimes_k \mathbb{C}.$$

□

Using the previous proposition, and the map from $Gr^j_F \mathrm{CH}^r(X_U, m; \mathbb{Q})$ to the space $E_\infty^{j,2r-m-j}$, we get a map

$$Gr^j_F \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \underline{\underline{E}}_\infty^{j,2r-m-j} \hookrightarrow \nabla DR^{r,m,j}(X_U/U) \otimes_k \mathbb{C}.$$

Unfortunately, we cannot relate the map above to our map from the graded pieces $Gr^j_F \mathrm{CH}^r(X_U, m; \mathbb{Q})$ to $\nabla DR^{r,m,j}(X_U/U)$, which was constructed by showing that $E_\infty^{j,2r-m-j}$ maps to $\nabla J^{r,m,j}(X_U/U)$ and then using the natural map from the Mumford-Griffiths invariants to the de Rham invariants. What we are going to do next, it is to construct a new map to the space of Mumford-Griffiths invariants that factors through $\underline{\underline{E}}_\infty^{j,2r-m-j}$ as required. Later, we will see this one coincides with the first map we constructed.

Theorem 6.9. Let $f : X \to S$ be a proper morphism of smooth varieties over $k \subset \mathbb{C}$. Assume S is projective, $U \subset S$ is affine open and f is smooth over U. Set $X_U = f^{-1}(U)$. Then there exists an injection

$$\underline{\underline{E}}_\infty^{j,2r-m-j} \hookrightarrow \nabla J^{r,m,j}(X_U/U).$$

Proof. As a consequence of Hironaka's theorem, $D = S \setminus U$ is a normal crossings divisor because S is projective and $Y = f^{-1}(D)$ is a normal crossings divisor as well. Then $X_U = X \setminus Y$ and by Deligne,

$$F^r H^{2r-m}(X_U, \mathbb{Q}(r)) = F^r \mathbb{H}^{2r-m}(X, \Omega_X^\bullet(\log Y)) = \mathbb{H}^{2r-m}(X, F^r \Omega_X^\bullet(\log Y)).$$

We have a canonical Leray filtration $\mathcal{L}^\bullet$ on $F^r H^{2r-m}(X_U, \mathbb{Q}(r))$ with graded pieces

$$Gr^j_{\mathcal{L}} F^r H^{2r-m}(X_U, \mathbb{Q}(r)) = F^r H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)).$$

Consider a filtered locally free $\mathcal{O}$-module V^i on $S \setminus D$ underlying a variation of Hodge structure whose fibre V^i_s at $s \in S \setminus D$ is $H^i(X_s, \mathbb{C})$. Since D is a divisor with normal crossings on S we can find the Deligne extension $\tilde{V}^i$ of V^i such that the eigenvalues of the residue of the connection are contained in $[0, 1)$. If $F^\bullet$ denotes the Hodge filtration on V^i, then we can extend it to $\tilde{V}^i$. The logarithmic de Rham complex is defined by

$$DR_{\log}(\tilde{V}^i) = \Omega_S^\bullet(\log D) \otimes_{\mathcal{O}} \tilde{V}^i$$

with $F^r DR_{\log}(\tilde{V}^i)$ defined by $\Omega^j_S(\log D) \otimes_{\mathcal{O}} F^{r-j}\tilde{V}^i$. We will use $V^i = \mathcal{H}^i_{DR}(X_U/U)$. A consequence of the work of M. Saito in [46] is that:

$$Gr^j_{\mathcal{L}} \mathbb{H}^{2r-m}(X, F^r \Omega_X^\bullet(\log Y)) = \mathbb{H}^j(F^r DR_{\log}(\tilde{V}^{2r-m-j})).$$

Then by using the restriction to U we get a morphism

$$F^r H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)) \to \mathbb{H}^j(U, F^r DR_{\log}(\tilde{V}^{2r-m-j})),$$

where the RHS can be computed in the Zariski topology by GAGA. The last term is $\mathbb{H}^j(U, F^r DR(\mathcal{H}_{DR}^{2r-m-j}(X_U/U)))$ when we take $V^{2r-m-j} = \mathcal{H}_{DR}^{2r-m-j}(X_U/U)$, i.e. $\nabla J^{r,m,j}(X_U/U)$. Since there is a natural morphism

$$\underline{\underline{E}}_\infty^{j,2r-j} = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r))) \hookrightarrow F^r H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r))$$

we get a map

$$\underline{\underline{E}}_\infty^{j,2r-m-j} \to \nabla J^{r,m,j}(X_U/U).$$

Moreover, by [46] we have a natural filtration $L^\bullet$ on $\mathbb{H}^{2r-m}(X_U, F^r \Omega^\bullet_{X_U})$ compatible with the Leray filtration on $\mathbb{H}^{2r-m}(X, F^r \Omega^\bullet_X(\log Y))$ such that

$$Gr_L^j \mathbb{H}^{2r-m}(X_U, F^r \Omega^\bullet_{X_U}) = \mathbb{H}^j(U, F^r DR_{\log}(\tilde{V}^{2r-m-j})).$$

Since the Hodge filtration is strict, the composition

$$\mathbb{H}^{2r-m}(X, F^r \Omega^\bullet_X(\log Y)) \to \mathbb{H}^{2r-m}(X_U, F^r \Omega^\bullet_{X_U}) \to \mathbb{H}^{2r-m}(X_U, \Omega^\bullet_{X_U})$$

is injective and in consequence the morphism

$$\mathbb{H}^{2r-m}(X, F^r \Omega^\bullet_X(\log Y)) \to \mathbb{H}^{2r-m}(X_U, F^r \Omega^\bullet_{X_U})$$

is injective. Therefore

$$F^r H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)) \to \nabla J^{r,m,j}(X_U/U)$$

is injective as well. $\square$

Using the map

$$Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to E_\infty^{j,2r-m-j}$$

and the short exact sequence in proposition (6.5), we get the morphism

$$Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \nabla J^{r,m,j}(X_U/U)$$

which factors through $\underline{\underline{E}}_\infty^{j,2r-m-j}$.

We also have a well-defined filtration $L_0^\bullet$ on $\mathbb{H}^{2r-m}(X_U, \Omega^\bullet_{X_U})$ with graded pieces

$$Gr_{L_0}^j \mathbb{H}^{2r-m}(X_U, \Omega^\bullet_{X_U}) = \mathbb{H}^j(U, DR_{\log}(\tilde{V}^{2r-m-j})).$$

Then, setting $V^{2r-m-j} = \mathcal{H}_{DR}^{2r-m-j}(X_U/U)$, we get

$$Gr_{L_0}^j \mathbb{H}^{2r-m}(X_U, \Omega^\bullet_{X_U}) \simeq \nabla DR^{r,m,j}(X_U/U).$$

Now, consider the morphism

$$\mathbb{H}^{2r-m}(X_U, F^r\Omega^\bullet_{X_U}) \to \mathbb{H}^{2r-m}(X_U, \Omega^\bullet_{X_U}).$$

This morphism is compatible with the filtrations $L^\bullet$ (defined in the proof above) and $L_0^\bullet$ respectively. Therefore we have a map

$$\nabla J^{r,m,j}(X_U/U) \to \nabla DR^{r,m,j}(X_U/U).$$

and since the composition

$$\mathbb{H}^{2r-m}(X, F^r\Omega^\bullet_X(\log Y)) \to \mathbb{H}^{2r-m}(X_U, F^r\Omega^\bullet_{X_U}) \to \mathbb{H}^{2r-m}(X_U, \Omega^\bullet_{X_U})$$

is injective we have a diagram

$$\begin{tikzcd}
\underline{\underline{E}}_\infty^{j,2r-m-j} \arrow[r, hook] \arrow[dr, hook] & \nabla J^{r,m,j}(X_U/U) \arrow[d] \\
& \nabla DR^{r,m,j}(X_U/U).
\end{tikzcd}$$

Actually, we arrive the following:

Theorem 6.10 (M. Saito)**.** Let $f : X \to S$ be a proper morphism of smooth projective varieties over $k \subset \mathbb{C}$. Let $U \subset S$ be affine open and assume f is smooth over U. Set $X_U = f^{-1}(U)$. Then the image of a cycle $\bar{\xi} \in Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q})$ in $\nabla J^{r,m,j}(X_U/U)$ and its image in $\nabla DR^{r,m,j}(X_U/U)$ are equivalent to each other. i.e. one vanishes if the other does.

We constructed a map $Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \nabla J^{r,m,j}(X_U/U)$ in Proposition (6.3) using the forgetful functor from the category $\mathrm{MHM}(U)$ to the category $\mathrm{MF}_{\mathrm{rh}}(U)$ to get a morphism

$$\mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j}f_*\mathbb{Q}_{X_U}(r)) \to \mathrm{Ext}^j_{\mathrm{MF}_{\mathrm{rh}}(U)}(\mathcal{O}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r)).$$

and we proved that $\mathrm{Ext}^j_{\mathrm{MF}_{\mathrm{rh}}(U)}(\mathcal{O}_U, \mathcal{H}^{2r-m-j}_{DR}(X_U/U)(r))$ is isomorphic to the cohomology of a complex whose global sections are $\nabla J^{r,m,j}(X_U/U)$. Finally we used the map

$$Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \mathrm{Ext}^j_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-j}f_*\mathbb{Q}_{X_U}(r)).$$

This construction coincides with the map of Proposition (6.9). Indeed, it follows from the discussion in 2.4 of [46] that the (global) extension group

$\mathrm{Ext}^{2r-m}(\mathcal{O}_U, \mathcal{O}_U(r))$ is isomorphic to $\mathbb{H}^{2r-m}(X_U, F^r\Omega^\bullet_{X_U})$. The filtration $L^\bullet$ on $\mathbb{H}^{2r-m}(X_U, F^r\Omega^\bullet_{X_U})$ and the Leray filtration $F_L^\bullet$ on $\mathrm{Ext}^j(\mathcal{O}_U, \mathcal{O}_U(r))$ coming from the Leray filtration on the extension group of the corresponding mixed Hodge modules coincide and therefore the maps are the same.

Note that the map $\nabla J^{r,m,j}(X_U/U) \to \nabla DR^{r,m,j}(X_U/U)$ is not, in general, injective. Suppose that X_U is the product of two varieties, i.e. it is of the form $X_U = X \times_k U$ with corresponding morphism $f : X \times_k U \to U$. Then $H^p_{DR}(X_U/U) \simeq H^p_{DR}(X/k)$ for all p and the Gauss-Manin connection is just differentiation along U. Then $\nabla J^{r,m,j}(X_U/U)$ is given by the cohomology of

$$\begin{aligned}\Omega^{j-1}_{U/k} \otimes F^{r-j+1} H^{2r-m-j}_{DR}(X/k) \xrightarrow{d\otimes 1} \\ \Omega^{j}_{U/k} \otimes F^{r-j} H^{2r-m-j}_{DR}(X/k) \xrightarrow{d\otimes 1} \\ \Omega^{j+1}_{U/k} \otimes F^{r-j-1} H^{2r-m-j}_{DR}(X/k)\end{aligned}$$

and $\nabla DR^{r,m,j}(X_U/U)$ by the cohomology of

$$\Omega^{j-1}_{U/k}\otimes H^{2r-m-j}_{DR}(X/k) \xrightarrow{d\otimes 1} \Omega^{j}_{U/k}\otimes H^{2r-m-j}_{DR}(X/k) \xrightarrow{d\otimes 1} \Omega^{j+1}_{U/k}\otimes H^{2r-m-j}_{DR}(X/k).$$

Thus, for example when $k = \mathbb{C}$, $H^{2r-m-j}_{DR}(X/k) \simeq H^{2r-m-j}(X, \mathbb{C})$ and we can find elements in $\nabla J^{r,m,j}(X_U/U)$ that are trivial in $\nabla DR^{r,m,j}(X_U/U)$ because of the Hodge filtration.

Chapter 7

Invariants for some special cases

The first part of this section deals with detecting cycles with non-trivial Mumford-Griffiths invariant. From here on we will denote any product of the form $X \times_k \mathrm{Spec}(L)$ just by $X \times_k L$ or $X \times L$ if the context is clear, for X a variety over k and $k \subset L \subset \mathbb{C}$. Let X be a smooth quasiprojective variety over $k \subset \mathbb{C}$. Consider the natural morphism $g : X \to \mathrm{Spec}(k)$. Then we can use lemma (6.4) to get a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MHM(Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^{2r-m-1}g_*\mathbb{Q}_X(r)) \\ \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)) \to \\ \hom_{\mathrm{MHM(Spec}(k))}(\mathbb{Q}_{\mathrm{Spec}(k)}(0), R^{2r-m}g_*\mathbb{Q}_X(r)) \to 0 \end{aligned}$$

The category $\mathrm{MHM(Spec}(k))$ is a subcategory of the category of polarizable mixed Hodge structures, and we have the isomorphisms

$$\begin{aligned} \mathbb{Q}_{\mathrm{Spec}(k)}(0) &\simeq \mathbb{Q}(0), \\ R^{2r-m-1}g_*\mathbb{Q}_X(r) &\simeq H^{q-1}(X, \mathbb{Q}(r)), \\ R^{2r-m}g_*\mathbb{Q}_X(r) &\simeq H^q(X, \mathbb{Q}(r)). \end{aligned}$$

Then we can rewrite the short exact sequence as

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(X, \mathbb{Q}(r))) \to \\ \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X, \mathbb{Q}(r))) \to 0 \end{aligned} \quad (7.1)$$

We previously constructed a morphism

$$c_{r,m} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)).$$

Then we have a map

$$\mathrm{CH}^r(X, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X, \mathbb{Q}(r))).$$

Conjecture 7.1 (Conjecture BE-HDG)**.** Let W be a smooth quasiprojective variety over $\mathbb{C}$ that can be expressed as $W = W_0 \times_{\bar{\mathbb{Q}}} \mathbb{C}$ where W_0 is defined over $\bar{\mathbb{Q}}$. Then

$$\mathrm{CH}^r(W, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(W, \mathbb{Q}(r)))$$

is surjective.

Beilinson [6] conjectured the surjectivity of

$$\mathrm{CH}^r(X, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X, \mathbb{Q}(r)))$$

for any complex smooth quasiprojective variety X. However this was disproved by Jannsen [26]. For X projective defined over a subfield k of $\mathbb{C}$ and $m = 0$ this becomes the famous Hodge conjecture

Conjecture 7.2 (HC)**.**

$$\mathrm{CH}^r(X, \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r}(X, \mathbb{Q}(r)))$$

is surjective.

We denote the kernel of the map

$$\mathrm{CH}^r(X, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X, \mathbb{Q}(r)))$$

by $\mathrm{CH}^r_{\mathrm{hom}}(X, m; \mathbb{Q})$. The Abel-Jacobi map is given by

$$AJ : \mathrm{CH}^r_{\mathrm{hom}}(X, m; \mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(X, \mathbb{Q}(r))),$$

using (7.1). Now we can state the Bloch-Beilinson conjecture:

Conjecture 7.3 (BBC)**.** For X smooth projective defined over $\bar{\mathbb{Q}}$, AJ is injective.

If we assume the two previous conjectures, then Conjecture BE-HDG is true. This was proved by M. Saito [47] and using different methods by Kerr-Lewis [31] and de Jeu-Lewis [16].

Proposition 7.4. Conjecture BE-HDG follows from HC and BBC.

Take a complex smooth projective variety X such that $X = X_K \times \mathbb{C}$, where X_K is defined over K, a finitely generated extension of $\bar{\mathbb{Q}}$. Given a spread of X by a morphism $f : \mathcal{X} \to S$ proper and smooth such that $X_K = \mathcal{X}_{\eta_S}$ for the generic point η_S of S, we constructed a map

$$Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \underline{\underline{E}}_\infty^{j,2r-m-j} = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r))).$$

where $X_U = f^{-1}(U)$.

Proposition 7.5. Assume Conjecture BE-HDG and that the components of the diagonal class of X are algebraic. Then

$$F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \underline{\underline{E}}_\infty^{j,2r-m-j}$$

is surjective.

Proof. Let $\mathcal{L}$ denote the canonical Leray filtration on $H^{2r-m}(X_U, \mathbb{Q}(r))$ such that

$$Gr_{\mathcal{L}}^j H^{2r-m}(X_U, \mathbb{Q}(r)) = H^j(U, R^{2r-m-j} f_* \mathbb{Q}(r)).$$

We have a map

$$h : \mathrm{CH}^r(X_U, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X_U, \mathbb{Q}(r))) \to H^{2r-m}(X_U, \mathbb{Q}(r))$$

such that $h(F^j \mathrm{CH}^r(X, m; \mathbb{Q})) \subseteq \mathcal{L}^j H^{2r-m}(X_U, \mathbb{Q}(r))$. A correspondence $\Gamma_0 \in \mathrm{CH}^d(X \times_{\bar{\mathbb{Q}}} X, 0)$ can be spread out to a correspondence
$\Gamma \in \mathrm{CH}^d(X_U \times_U X_U, 0)$ and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^r(X_U, m; \mathbb{Q}) \arrow[r, "h"] \arrow[d, "\Gamma_*"] & H^{2r-m}(X_U, \mathbb{Q}(r)) \arrow[d, "h(\Gamma)_*"] \\
\mathrm{CH}^r(X_U, m; \mathbb{Q}) \arrow[r, "h"] & H^{2r-m}(X_U, \mathbb{Q}(r)).
\end{tikzcd}$$

Now let $\Gamma_0 = \Delta_X$. The Chow-Künneth decomposition of Δ_X leads to isomorphisms

$$\mathrm{CH}^r(X_U, m; \mathbb{Q}) \simeq \bigoplus_{j \geq 0} Gr_F^j \mathrm{CH}^r(X_U, m; \mathbb{Q}),$$

$$H^{2r-m}(X_U,\mathbb{Q}(r)) \simeq \bigoplus_{j\geq 0} H^j(U, R^{2r-m-j}f_*\mathbb{Q}(r)),$$

where the latter is an isomorphism of MHS (the latter holds regardless of whether Δ_X has a Chow-Künneth decomposition, by using cohomological Künneth decomposition), and $Gr_F^j\mathrm{CH}^r(X,m;\mathbb{Q})$ is identified with $\Delta_X(2d-2r+m+j, 2r-m-j)_*\mathrm{CH}^r(X,m;\mathbb{Q})$. This induces a commutative diagram,

$$\begin{tikzcd}
\mathrm{CH}^r(X_U,m;\mathbb{Q}) \arrow[r, two heads] \arrow[d, "\wr"] & \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X_U,\mathbb{Q}(r))) \arrow[d, "\wr"] \\
\bigoplus_{j\geq 0} Gr_F^j\mathrm{CH}^r(X_U,m;\mathbb{Q}) \arrow[r] & \bigoplus_{j\geq 0}\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(U, R^{2r-m-j}f_*\mathbb{Q}(r)))
\end{tikzcd}$$

for which the proposition follows by applying BE-HDG. $\square$

Corollary 7.6. Let X be a complex smooth projective variety. Let us assume $\underline{\underline{E}}_\infty^{j,2r-m-j} \neq 0$. Assume BE-HDG and that the components of the diagonal class are algebraic. Then the image of the map

$$Gr_F^j\mathrm{CH}^r(X_K,m;\mathbb{Q}) \to \nabla J^{r,m,j}(X_K/K)$$

is not zero.

Proof. By Theorem (6.9) there is an injection $\underline{\underline{E}}_\infty^{j,2r-m-j} \hookrightarrow J^{r,m,j}(X_U/U)$. Since $\underline{\underline{E}}_\infty^{j,2r-m-j} \neq 0$ we can use the previous proposition to get a cycle in $Gr_F^j\mathrm{CH}^r(X_U,m;\mathbb{Q})$ with non trivial Mumford-Griffiths invariant. To get the result we take the limit for $U \subset S$ affine open. $\square$

Our next discussion involves detecting cycles with trivial Mumford-Griffiths invariant, and towards this goal we now consider a special product situation. [31] also discusses this situation and there is some overlap in results; however we feel that the arguments presented there are not too accessible for the reader. Rather, our intention here, besides completeness, is to present a more "user-friendly" approach. If X is a smooth projective variety over a field $k \subset \mathbb{C}$, we can take any smooth projective variety S over k and let $f : X \times S \to S$ be the morphism given by the projection. For $U \subset S$ open, $X_U = f^{-1}(U) = X \times U$. Let $K = k(S)$ and $X_K = X \times_k K$. Then we have an isomorphism

$$\mathrm{CH}^r(X_K,m;\mathbb{Q}) \simeq \varinjlim \mathrm{CH}^r(X \times U, m;\mathbb{Q}).$$

and we can define a complex variety by setting $X_{\mathbb{C}} := X \times_k \mathbb{C}$.

Regarding the kernel of the map from $Gr_F^j \mathrm{CH}^r(X_K, m; \mathbb{Q})$ to $\nabla J^{r,m,j}(X_K/K)$, we can get cycles in this kernel by considering cycles coming from the product of two varieties. Specifically, consider two smooth projective varieties X and S over $k \subset \mathbb{C}$ and assume $m \geq 1$. We have a map

$$F^j \mathrm{CH}^r(X \times S, m; \mathbb{Q}) \to \underline{\underline{E}}_\infty^{j,2r-m-j} = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(S, R^{2r-m-j} f_* \mathbb{Q}(r)))$$

with kernel in $\underline{E}_\infty^{j,2r-m-j}$ because of the short exact sequence

$$0 \to \underline{E}_\infty^{j,2r-m-j} \to E_\infty^{j,2r-m-j} \to \underline{\underline{E}}_\infty^{j,2r-m-j}.$$

Since f is given by the projection $f : X \times S \to S$ and both X and S are projective , then

$$\begin{aligned} &\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(S, R^{2r-m-j} f_* \mathbb{Q}(r))) = \\ &\qquad\qquad \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(S, \mathbb{Q}) \otimes H^{2r-m-j}(X, \mathbb{Q}(r))) = 0 \end{aligned}$$

using a weight argument. Also since

$$\underline{E}_\infty^{j,2r-m-j} = \frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{-1}(H^{j-1}(S, \mathbb{Q}) \otimes H^{2r-m-j}(X, \mathbb{Q}(r))))}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^0(H^{j-1}(S, \mathbb{Q}) \otimes H^{2r-m-j}(X, \mathbb{Q}(r))))},$$

we have

$$\underline{E}_\infty^{j,2r-m-j} = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{j-1}(S, \mathbb{Q}) \otimes H^{2r-m-j}(X, \mathbb{Q}(r)))$$

because we are working with pure Hodge structures. Thus we have a map

$$F^j \mathrm{CH}^r(X \times S, m; \mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), H^{j-1}(S, \mathbb{Q}) \otimes H^{2r-m-j}(X, \mathbb{Q}(r))\right).$$

Now let $N_H^1 H^{j-1}(S, \mathbb{Q})$ be the largest subHodge structure lying in the intersection $F^1 H^{j-1}(S, \mathbb{C}) \cap H^{j-1}(S, \mathbb{Q})$ and $N_H^{r-j+1} H^{2r-m-j}(X, \mathbb{Q}(r))$ be the largest subHodge structure lying in $F^{r-j+1} H^{2r-m-j}(X, \mathbb{C}) \cap H^{2r-m-j}(X, \mathbb{Q}(r))$. We have a morphism

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), H^{j-1}(S, \mathbb{Q}) \otimes H^{2r-m-j}(X, \mathbb{Q}(r))\right) \\ &\qquad \to \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), \frac{H^{j-1}(S, \mathbb{Q})}{N_H^1 H^{j-1}(S, \mathbb{Q})} \otimes \frac{H^{2r-m-j}(X, \mathbb{Q}(r))}{N_H^{r-j+1} H^{2r-m-j}(X, \mathbb{Q}(r))}\right) \end{aligned}$$

induced by projection and we get a map

$$F^j \mathrm{CH}^r(X \times S, m; \mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), \frac{H^{j-1}(S, \mathbb{Q})}{N_H^1 H^{j-1}(S, \mathbb{Q})} \otimes \frac{H^{2r-m-j}(X, \mathbb{Q}(r))}{N_H^{r-j+1} H^{2r-m-j}(X, \mathbb{Q}(r))}\right).$$

Proposition 7.7. Suppose the map

$$F^j\mathrm{CH}^r(X\times S,m;\mathbb{Q})\to \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0),\frac{H^{j-1}(S,\mathbb{Q})}{N^1_H H^{j-1}(S,\mathbb{Q})}\otimes\frac{H^{2r-m-j}(X,\mathbb{Q}(r))}{N^{r-j+1}_H H^{2r-m-j}(X,\mathbb{Q}(r))}\right)$$

is not zero. Then, for $K=k(S)$, the kernel of the map

$$\phi: Gr^j_F\mathrm{CH}^r(X_K,m;\mathbb{Q})\to\nabla J^{r,m,j}(X_K/K).$$

is non trivial.

Proof. Let U be an open subset of S. Let i denote the inclusion $i:U\hookrightarrow S$. Then we have a commutative diagram.

$$\begin{tikzcd}
F^j\mathrm{CH}^r(X\times S,m;\mathbb{Q}) \arrow[r, "i^*"] \arrow[d] & F^j\mathrm{CH}^r(X\times U,m;\mathbb{Q}) \arrow[d] \\
\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H^{j-1}(S,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))) \arrow[r] & \mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0),H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))),
\end{tikzcd} \tag{7.2}$$

where

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0),H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))\\ &\qquad\simeq\frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),W_{-1}(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))))}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0),Gr^0_W(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))))}\end{aligned}$$

We will show that under our hypothesis the image of the map

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H^{j-1}(S,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))\\ &\qquad\to\frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),W_{-1}(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))))}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0),Gr^0_W(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))))}\end{aligned}\tag{7.3}$$

is non trivial. This is indeed true because of the following. First of all, observe $H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))$ is a product of mixed Hodge structures and in general for two mixed Hodge structures H_1,H_2:

$$W_n(H_1\otimes H_2)=\bigoplus_{p+q=n}W_pH_1\otimes W_qH_2.$$

Therefore

$$W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes H^{2r-m-j}(X,\mathbb{Q}(r))\hookrightarrow W_{-1}(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))).$$

Moreover

$$W_{j-1}H^{j-1}(U,\mathbb{Q}) = \mathrm{Im}(H^{j-1}(S,\mathbb{Q}) \to H^{j-1}(U,\mathbb{Q})) \simeq \frac{H^{j-1}(S,\mathbb{Q})}{H^{j-1}_{S\setminus U}(S,\mathbb{Q})}$$

and

$$\mathrm{Im}\left(H^{j-1}(S,\mathbb{Q}) \to \varinjlim_{U\subset S} H^{j-1}(U,\mathbb{Q})\right) \simeq \frac{H^{j-1}(S,\mathbb{Q})}{N^1H^{j-1}(S,\mathbb{Q})},$$

where $N^1H^{j-1}(S,\mathbb{Q})$ is the filtration by coniveau. In general $N^1H^{j-1}(S,\mathbb{Q}) \subseteq N^1_HH^{j-1}(S,\mathbb{Q})$. Thus by assumption there exists an element in $F^j\mathrm{CH}^r(X\times S,m;\mathbb{Q})$ whose image in

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H^{j-1}(S,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))\\ &\qquad\to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))\end{aligned}$$

is not zero. Next, observe that the quotient

$$\frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))}{\hom_{\mathrm{MHS}}(\mathbb{Q}(0),Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))))}$$

comes from the short exact sequence

$$\begin{aligned}0 &\to W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes H^{2r-m-j}(X,\mathbb{Q}(r))\\ &\qquad\to W_0(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))\\ &\qquad\qquad\to Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r)))\to 0\end{aligned}$$

after taking Ext, because $W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes H^{2r-m-j}(X,\mathbb{Q}(r)) = W_{-m-1}\big(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))\big)$. Then we get a map

$$\begin{aligned}\mathcal{E}_2 &:= \hom_{\mathrm{MHS}}(\mathbb{Q}(0),Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q})\otimes H^{2r-m-j}(X,\mathbb{Q}(r))))\\ &\qquad\to \mathcal{E}_3 := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes H^{2r-m-j}(X,\mathbb{Q}(r))).\end{aligned}$$

Similarly we have a map

$$\begin{aligned}\mathcal{E}_0 &:= \hom_{\mathrm{MHS}}(\mathbb{Q}(0),Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q})\otimes N_H^{r-j+1}H^{2r-m-j}(X,\mathbb{Q}(r))))\\ &\to \mathcal{E}_1 := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),W_{j-1}\left(H^{j-1}(U,\mathbb{Q})\right)\otimes N_H^{r-j+1}H^{2r-m-j}(X,\mathbb{Q}(r))).\end{aligned}$$

In the commutative diagram

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r] \arrow[d] & \mathcal{E}_1 \arrow[d] \\ \mathcal{E}_2 \arrow[r] & \mathcal{E}_3 \end{tikzcd}$$

the left vertical arrow is an equality because the natural inclusion

$$\begin{aligned} &\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q}) \otimes N_H^{r-j+1} H^{2r-m-j}(X,\mathbb{Q}(r)))) \\ &\qquad \subseteq \hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q}) \otimes H^{2r-m-j}(X,\mathbb{Q}(r)))) \end{aligned}$$

is an equality. This relies on the fact that $F^j H^{j-1}(U,\mathbb{C}) = 0$ and that the maximum weight of $H^{j-1}(U,\mathbb{Q})$ is $2j-2$. Indeed, let us untwist everything by $\mathbb{Q}(-r)$. We are looking at

$$\begin{aligned} &\left[Gr_W^{2r,2r-m-1}\left(H^{j-1}(U,\mathbb{Q}) \otimes H^{2r-m-j}(X,\mathbb{Q})\right)\right]^{r,r} \\ &\qquad\qquad \subset \left[W_{m+j}\big(H^{j-1}(U)\big) \otimes H^{2r-m-j}(X)\right]^{r,r}. \end{aligned}$$

From the short exact sequence:

$$0 \to W_{m+j-1} \to W_{m+j} \to Gr_W^{m+j} \to 0,$$

together with

$$\left[W_{m+j-1}\left(H^{j-1}(U)\right) \otimes H^{2r-m-j}(X)\right]^{r,r} = 0,$$

it follows that

$$\left[Gr_W^{2r,2r-m-1}\left(H^{j-1}(U,\mathbb{Q}) \otimes H^{2r-m-j}(X,\mathbb{Q})\right)\right]^{r,r} \subset \left[V \otimes H^{2r-m-j}(X)\right]^{r,r},$$

where V is a pure HS of weight $m+j$ (and also recall that $F^j V_{\mathbb{C}} = 0$). So in terms of type (r,r) we have:

$$V_{\mathbb{C}}^{j-1,m+1} \otimes H^{r-j+1,r-m-1} \oplus \cdots \oplus V_{\mathbb{C}}^{m+1,j-1} \otimes H^{r-m-1,r-j+1}.$$

This is contained in $V_{\mathbb{C}} \otimes F^{r-j+1} H^{2r-j-m}(X)$ precisely when $r-j+1 \leq r-m-1$, i.e. $j \geq m+2$. But recall from Theorem (3.7) that weight$(H^{j-1}(U)) \leq 2j-2$. Thus $m+j \leq 2j-2$ and the desired equality follows.

What all this shows is that

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_W^{0,-m-1}(H^{j-1}(U,\mathbb{Q}) \otimes H^{2r-m-j}(X,\mathbb{Q}(r))))$$

is contained in

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), W_{j-1}H^{j-1}(U,\mathbb{Q}) \otimes N_H^{r-j+1}H^{2r-m-j}(X,\mathbb{Q}(r)))$$

and, by assumption, this means that (7.3) is not trivial. Then we use the diagram (7.2) and conclude that $\ker\phi$ is not trivial after taking the limit over $U \subset S$. □

In the last proposition the assumption that the map

$$F^j\mathrm{CH}^r(X\times S,m;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), \frac{H^{j-1}(S,\mathbb{Q})}{N_H^1H^{j-1}(S,\mathbb{Q})} \otimes \frac{H^{2r-m-j}(X,\mathbb{Q}(r))}{N_H^{r-j+1}H^{2r-m-j}(X,\mathbb{Q}(r))}\right)$$

is not zero is a harder result. If we work with the category of MHS over $\mathbb{R}$, the natural situation that comes to mind is the Hodge $\mathcal{D}$-conjecture, i.e. the surjectivity of

$$\mathrm{CH}^r(Y,m;\mathbb{R}) \to H_{\mathcal{D}}^{2r-m}(Y,\mathbb{R}(r)).$$

While this conjecture is not true in general, it is expected to hold for $k \subset \bar{\mathbb{Q}}$ for any smooth projective variety Y over k, in particular for $X \times S$ defined over such k.

Let's take two complex smooth projective varieties X and S. When $m \geq 1$:

$$\begin{aligned} H_{\mathcal{D}}^{2r-m}(X\times S,\mathbb{R}(r)) &\simeq \frac{H^{2r-m-1}(X\times S,\mathbb{C})}{F^rH^{2r-m-1}(X\times S,\mathbb{C}) + H^{2r-m-1}(X\times S,\mathbb{R}(r))} \\ &\underset{\simeq}{\xrightarrow{\pi}} \frac{H^{2r-m-1}(X\times S,\mathbb{R}(r-1))}{\pi(F^rH^{2r-m-1}(X\times S,\mathbb{C}))} \end{aligned} \tag{7.4}$$

where $\pi : \mathbb{C} = \mathbb{R}(r) \oplus \mathbb{R}(r-1) \to \mathbb{R}(r-1)$ is the projection. Then we can use the Hodge and Künneth decompositions to get a map

$$H_{\mathcal{D}}^{2r-m}(X\times S,\mathbb{R}(r)) \xrightarrow{\alpha} H^{r-j,r-m}(X) \otimes H^{j-1,0}(S)$$

By using the real regulator r, defining β as the cup product with a class in $[\beta] \in H^{s-j+1,s}(S)$ and using $H^{s,s}(S) \simeq \mathbb{C}$ where $s = \dim S$, we get the map

$$\begin{aligned} \mathrm{CH}^r(X\times S,m;\mathbb{Q}) \xrightarrow{r} H_{\mathcal{D}}^{2r-m}(X\times S,\mathbb{R}(r)) \xrightarrow{\alpha} H^{r-j,r-m}(X)\otimes H^{j-1,0}(S) \\ \xrightarrow{\beta} H^{r-j,r-m}(X). \end{aligned}$$

In the case $m = 0$ we use the cycle class map and the Hodge and Künneth decompositions and define the last morphism similarly to β above to get

$$\begin{aligned}\mathrm{CH}^r(X \times S, 0; \mathbb{Q}) \to H^{2r-m}(X \times S, \mathbb{C}) \to H^{r-j,r-m}(X) \otimes H^{j-1,0}(S) \\ \to H^{r-j,r-m}(X).\end{aligned}$$

Thus for any m we have constructed

$$\mathrm{CH}^r(X \times S, m; \mathbb{Q}) \to H^{r-j,r-m}(X). \tag{7.5}$$

The following definition is introduced by J. Lewis in [33].

Definition 7.8. (i) $H^{\{r,j,m\}}(X) =$ complex subspace of $H^{r-j,r-m}(X)$ generated by the image of (7.5) in $H^{r-j,r-m}(X)$ over all smooth projective S.

(ii) $H_N^{\{r-j,r-m\}}(X) =$ complex subspace of $H^{r-j,r-m}(X)$ generated by the Hodge projected image

$$N^{r-j} H^{2r-m-j}(X, \mathbb{Q}) \otimes \mathbb{C} \to H^{r-j,r-m}(X)$$

where $N^\bullet$ denotes the filtration by coniveau.

One has $H^{\{r,j,0\}}(X) \subseteq H_N^{\{r-j,r\}}(X)$. Under the assumption of the hard Lefschetz conjecture, one can show that $H^{\{r,j,0\}}(X) = H_N^{\{r-j,r\}}(X)$.

Theorem 7.9 (Lewis)**.** *Let X be a complex smooth projective variety and $m \leq 2$. Then:*

(i) $H^{\{r-m,j-m,0\}}(X) \subset H^{\{r,j,m\}}(X)$.

(ii) $H^{\{r,j,m\}}(X)/H^{\{r-m,j-m,0\}}(X) \neq 0$ *and* $j - m \geq 1 \Rightarrow \mathrm{CH}^r_{\mathrm{ind}}(X, m; \mathbb{Q})$ *is uncountable. Moreover, there are an uncountable number of indecomposables in the kernel of*

$$\mathrm{CH}^r(X, m; \mathbb{Q}) \to H_{\mathcal{D}}^{2r-m}(X, \mathbb{Q}(r)).$$

Remark. The theorem is also valid for $m \geq 3$ but needs an extra conjectural condition. It can be found in [33].

Note that $H^{\{r,j,m\}}(X) \neq 0$ is the "real" analogue to the hypothesis:

$$F^j \mathrm{CH}^r(X \times S, m; \mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), \frac{H^{j-1}(S,\mathbb{Q})}{N^1_H H^{j-1}(S,\mathbb{Q})} \otimes \frac{H^{2r-m-j}(X,\mathbb{Q}(r))}{N_H^{r-j+1} H^{2r-m-j}(X,\mathbb{Q}(r))}\right)$$

has non trivial image, in Proposition (7.7). Under their respective assumptions, in Proposition (7.7) we find elements in the kernel of

$$Gr_F^j \mathrm{CH}^r(X_K, m; \mathbb{Q}) \to \nabla J^{r,m,j}(X_K/K).$$

and in Theorem (7.9) we find elements in the kernel of

$$\mathrm{CH}^r(X, m; \mathbb{Q}) \to H_{\mathcal{D}}^{2r-m}(X, \mathbb{Q}(r)).$$

When $m = 1$, Collino and Fakhruddin (see [15]) prove

$$H^{\{r,j,1\}}(X)/H^{\{r-1,j-1,0\}}(X) \neq 0$$

for the Jacobian $X = J(C)$ of a generic hyperelliptic curve C of genus $g \geq 3$, $r = g$ and all j such that $2 \leq j \leq g-1$.

Chapter 8

Arithmetic normal functions

The filtration we have defined can be compared to a more geometrical filtration defined on higher Chow groups. We need to introduce the concept of arithmetic normal functions first, which provide a generalization to classical normal functions. Then we will describe conditions to find indecomposable elements in higher Chow groups using the topological invariants defined by arithmetic normal functions.

Let's take a complex smooth projective variety X. If X is defined over K, a finitely generated extension field of $\bar{\mathbb{Q}}$, we denote this by X_K and consider a spread of X, given by a smooth projective morphism $\rho : \mathcal{X} \to S$ of smooth projective varieties over $\bar{\mathbb{Q}}$ such that $X = \mathcal{X}_\eta \times \mathbb{C}$, where $\eta \in S$ is the generic point and $X_K = \mathcal{X}_\eta$ is the fibre by ρ.

We constructed a filtration $F^\bullet$ on $\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$ and well-defined injective maps

$$c^j_{r,m} : Gr^j_F \mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \hookrightarrow E^{j,2r-m-j}_\infty(\eta).$$

The space $E^{j,2r-m-j}_\infty(\eta)$ fits in the short exact sequence

$$0 \to \underline{E}^{j,2r-m-j}_\infty(\eta) \to E^{j,2r-m-j}_\infty(\eta) \to \underline{\underline{E}}^{j,2r-m-j}_\infty(\eta) \to 0,$$

where

$$\underline{\underline{E}}^{j,2r-m-j}_\infty(\eta) = \Gamma(H^j(\eta, R^{2r-m-j}\rho_*\mathbb{Q}(r))),$$

$$\underline{E}^{j,2r-m-j}_\infty(\eta) = \frac{J(W_{-1}H^{j-1}(\eta, R^{2r-m-j}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr^0_W H^{j-1}(\eta, R^{2r-m-j}\rho_*\mathbb{Q}(r)))}.$$

(Recall the notation introduced in Chapter (3): $\Gamma(H) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H)$, $J(H) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H)$ for any MHS H.)
Here we explicitly make reference to the generic point and use the definition:

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(\eta, R^{2r-m-j}\rho_*\mathbb{Q}(r))) := \varinjlim_{U\subset S} \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(U, R^{2r-m-j}\rho_*\mathbb{Q}(r)))$$

and similarly

$$\underline{\underline{E}}_\infty^{j,2r-m-j}(\eta) := \varinjlim_{U\subset S} \underline{\underline{E}}_\infty^{j,2r-m-j}.$$

We call $[\xi]_j$ to the class of $\xi \in F^j\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$ in $\underline{\underline{E}}_\infty^{j,2r-m-j}(\eta)$ under the composition

$$Gr_F^j\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \hookrightarrow E_\infty^{j,2r-m-j}(\eta) \to \underline{\underline{E}}_\infty^{j,2r-m-j}(\eta).$$

Also, we have a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(\mathcal{X}_\eta, \mathbb{Q}(r))) \to \\ \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\eta)}(\mathbb{Q}_{\mathcal{X}_\eta}(0), \mathbb{Q}_{\mathcal{X}_\eta}(r)) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))) \to 0 \end{aligned}$$

and if the image of ξ in $\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r)))$ is zero using the map

$$\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\eta)}(\mathbb{Q}_{\mathcal{X}_\eta}(0), \mathbb{Q}_{\mathcal{X}_\eta}(r)) \tag{8.1}$$

then ξ maps to an element of $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(\mathcal{X}_\eta, \mathbb{Q}(r)))$. We call this map AJ. The kernel of (8.1) is denoted by $\mathrm{CH}^r_{\mathrm{hom}}(\mathcal{X}_\eta, m; \mathbb{Q})$.

A complex subvariety $\mathcal{T} \subset S(\mathbb{C})$ is called very general if no rational function $f \in \bar{\mathbb{Q}}(S)^*$ has $f|_\mathcal{T} \equiv 0$. Equivalently, the minimal field of definition L of $\mathcal{T}$ has $\mathrm{trdeg}(L/\bar{\mathbb{Q}}) = \mathrm{codim}_S(\mathcal{T})$. Let $S[j]$ be the set of $j-1$ dimensional very general subvarieties of $S(\mathbb{C})$ and $\eta_\mathcal{T} := \varprojlim V$ over $V \subset \mathcal{T}$ affine Zariski open over L ($\mathcal{T}$ means $\mathcal{T}_L$).

Definition 8.1. If $[\xi]_0 = [\xi]_1 = \ldots = [\xi]_{j-1} = 0$, the j^{th} arithmetic normal function

$$\nu_\xi^j : S[j] \to \coprod_{\mathcal{T}\in S[j]} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(\mathcal{X}_{\eta_\mathcal{T}}, \mathbb{Q}(r)))$$

associated to ξ is given by $\nu_\xi^j(\mathcal{T}) := AJ(\xi|_{\mathcal{X}_{\eta_\mathcal{T}}})$.

In particular, when $j = 1$, ν_ξ^1 is defined in very general complex points of $S(\mathbb{C})$, i.e. all the generic points corresponding to different embeddings of $\bar{\mathbb{Q}}(S) \simeq K$ in $\mathbb{C}$. In this case and when $m = 0$, ν_ξ^1 is essentially a classical normal function. To better understand this idea, we look at the commutative diagram

$$\begin{tikzcd} \mathcal{X}_{\mathcal{T}} \arrow[r, hook] \arrow[d, "\rho_{\mathcal{T}}"] & \mathcal{X}(\mathbb{C}) \arrow[d, "\rho"] \\ \mathcal{T} \arrow[r, hook] & S(\mathbb{C}) \end{tikzcd}$$

and the induced

$$\begin{tikzcd} 0 \arrow[r] & \underline{E}_\infty^{j,2r-m-j}(\rho) \arrow[r] \arrow[d] & E_\infty^{j,2r-m-j}(\rho) \arrow[r] \arrow[d] & \underline{\underline{E}}_\infty^{j,2r-m-j}(\rho) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{E}_\infty^{j,2r-m-j}(\rho_{\mathcal{T}}) \arrow[r] & E_\infty^{j,2r-m-j}(\rho_{\mathcal{T}}) \arrow[r] & \underline{\underline{E}}_\infty^{j,2r-m-j}(\rho_{\mathcal{T}}) \arrow[r] & 0 \end{tikzcd}$$

Then a cycle $\xi \in Gr_F^j \mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$ has a value $\nu_\xi^j(\mathcal{T}) \in \underline{E}_\infty^{j,2r-m-j}(\rho_{\mathcal{T}})$. This follows from the existence of the map

$$Gr_F^j \mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \to E_\infty^{j,2r-m-j}(\rho)$$

and the fact that $\underline{\underline{E}}_\infty^{j,2r-m-j}(\rho_{\mathcal{T}}) = 0$. Indeed, we have the following version of the weak Lefschetz theorem (see [1]):

Proposition 8.2. Let G be a locally constant sheaf on a n dimensional nonsingular complex affine variety Y. Then $H^i(Y, G) = 0$ for $i > n$, and $H^i(Y, G) \to H^i(H, G)$ is injective for $i < n$ and any general affine hyperplane section H.

Since

$$\underline{\underline{E}}_\infty^{j,2r-m-j}(\rho_{\mathcal{T}}) = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(\eta_{\mathcal{T}}, R^{2r-m-j}\rho_{\mathcal{T}*}\mathbb{Q}(r)))$$

and $\dim \eta_{\mathcal{T}} = j - 1$ because $\eta_{\mathcal{T}}$ is a limit of affines then $\underline{\underline{E}}_\infty^{j,2r-m-j}(\rho_{\mathcal{T}}) = 0$. We have a map

$$\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\eta)}(\mathbb{Q}_{\mathcal{X}\eta}(0), \mathbb{Q}_{\mathcal{X}\eta}(r)) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r)))$$

Recall the filtration on $\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$ is induced by a Leray filtration on $\mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\eta)}(\mathbb{Q}_{\mathcal{X}\eta}(0), \mathbb{Q}_{\mathcal{X}\eta}(r))$ and this induces a natural filtration F_L on

$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r)))$. This filtration coincides with the Leray filtration $\mathcal{L}$ on $H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))$, i.e.

$$F_L^j \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))) = \mathcal{L}^j \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))),$$

where

$$\mathcal{L}^j \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), \mathcal{L}^j H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))).$$

Therefore we have a map

$$Gr_F^j \mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \to \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^j(\eta, R^{2r-m-j}\rho_*\mathbb{Q}(r)))$$

where the last term is $\hom_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_{\mathcal{L}}^j H^{2r-m}(X_\eta, \mathbb{Q}(r)))$. Since we assumed $[\xi]_0 = [\xi]_1 = \ldots = [\xi]_{j-1} = 0$, then the class of $\xi|_{\mathcal{X}_{\eta\mathcal{T}}}$ in the graded piece $Gr_{\mathcal{L}}^\ell \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_{\eta\mathcal{T}}, \mathbb{Q}(r)))$ vanishes for $0 \le \ell \le j-1$. For $\ell \ge j$ we use Proposition (8.2) to show that
$Gr_{\mathcal{L}}^\ell \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_{\eta\mathcal{T}}, \mathbb{Q}(r))) = 0$.This proves that $\xi|_{\mathcal{X}_{\eta\mathcal{T}}}$ maps to zero in $\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X_\eta, \mathbb{Q}(r)))$ and ν_ξ^j is well defined. Moreover its value is in $\underline{E}_\infty^{j,2r-m-j}(\rho_\mathcal{T})$ as described in the diagram above and more generally (see [31]):

Proposition 8.3. Assume $[\xi]_0 = [\xi]_1 = \ldots = [\xi]_{j-1} = 0$. If $\nu_\xi^{j-1} = 0$ and $j \le m-2$, then ν_ξ^j factors through

$$\coprod_{\mathcal{T} \in S[j]} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{j-1}(\eta, R^{2r-m-j}\rho_*\mathbb{Q}(r))).$$

Actually, $[\xi]_j =: \delta(\nu_\xi^j)$ is the "topological invariant" of ν_ξ^j. Indeed there is the following result that relies on the explicit description of the higher Abel-Jacobi map.

Theorem 8.4 (Kerr-Lewis [31])**.** $[\xi]_j$ depends only on ν_ξ^j. Hence if $\nu_\xi^j = 0$, then $[\xi]_j = 0$.

This allows us to define a new filtration without making reference to our filtration $F^\bullet$.

Definition 8.5. Set $\Lambda^0\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) = \mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$, $\Lambda^1\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(\mathcal{X}_\eta, m; \mathbb{Q})$ and

$$\Lambda^j\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) := \{\xi \in \mathrm{CH}^r_{\mathrm{hom}}(\mathcal{X}_\eta, m; \mathbb{Q}) | \nu_\xi^1 = \ldots = \nu_\xi^{j-1} = 0\}.$$

Theorem 8.6 (Kerr-Lewis). $F^\ell \subseteq \Lambda^\ell$ for all ℓ, with equality for $\ell = 0, \dots, m+2$.

Actually under the assumption of the general Hodge conjecture and assuming X is defined over $\bar{\mathbb{Q}}$ it can be proved that both filtrations are the same (see [37]).

8.1 Some results on indecomposability

In Chapter (2) we define, for any smooth projective variety Y over a field k, the space of decomposable elements of $\mathrm{CH}^r(Y, m; \mathbb{Q})$ as the image of the map

$$\Pi : \bigoplus_{r_1+r_2=r, m_1+m_2=m} \mathrm{CH}^{r_1}(Y, m_1; \mathbb{Q}) \otimes \mathrm{CH}^{r_2}(Y, m_2; \mathbb{Q}) \to \mathrm{CH}^r(Y, m; \mathbb{Q}),$$

where $(r_1, m_1) \neq (0,0), (r_2, m_2) \neq (0,0)$. Let $\Xi(r, m; \mathbb{Q})(Y) := \mathrm{Image}\Pi$. We also introduced the subgroup of decomposables $\mathrm{CH}^r_{\mathrm{dec}}(Y, m; \mathbb{Q})$ by considering the elements in the image of

$$\mathrm{CH}^{r-m}(Y, 0; \mathbb{Q}) \otimes \mathrm{CH}^1(Y, 1; \mathbb{Q})^{\otimes m} \to \mathrm{CH}^r(Y, m; \mathbb{Q})$$

under the product for higher Chow groups and

$$\mathrm{CH}^r_{\mathrm{ind}}(Y, m; \mathbb{Q}) := \mathrm{CH}^r(Y, m; \mathbb{Q}) / \mathrm{CH}^r_{\mathrm{dec}}(Y, m; \mathbb{Q}).$$

As our results really only matter in the situation where $r \geq m$, we will assume this throughout this section. Recall that $\mathrm{CH}^{r-m}(Y, 0; \mathbb{Q}) \simeq \mathrm{CH}^{r-m}(Y; \mathbb{Q})$. Let X be a complex smooth projective variety and $Y \subset X$ a proper subvariety. The following commutative diagram is introduced in [16]:

$$\begin{tikzcd}
\frac{\mathrm{CH}^r(X \setminus Y, m+1; \mathbb{Q})}{\mathrm{CH}^r(X, m+1; \mathbb{Q})} \arrow[r] \arrow[d] & \mathrm{CH}^r_Y(X, m; \mathbb{Q})^0 \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathrm{CH}^r_{\mathrm{hom}}(X, m; \mathbb{Q}) \arrow[d, "AJ"] \\
\Gamma(H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))) \arrow[r] & \Gamma(H^{2r-m}_Y(X, \mathbb{Q}(r))^0) \arrow[r] & J\left(\frac{H^{2r-m-1}(X, \mathbb{Q}(r))}{H^{2r-m-1}_Y(X, \mathbb{Q}(r))}\right),
\end{tikzcd}$$

where

$$H^{2r-m}_Y(X, \mathbb{Q}(r))^0 := \ker\left(H^{2r-m}_Y(X, \mathbb{Q}(r)) \to H^{2r-m}(X, \mathbb{Q}(r))\right)$$

and

$$\mathrm{CH}^r_Y(X,m;\mathbb{Q})^0 := \\ \ker\left(\mathrm{CH}^r_Y(X,m;\mathbb{Q}) \to H^{2r-m}(X,\mathbb{Q}(r))\right) = \alpha^{-1}\mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q}).$$

Then we can put (see [16])

$$N^1\mathrm{CH}^r(X,m;\mathbb{Q}) := \varinjlim_{Y} \alpha(\ker \underline{\beta}),$$

where $Y \subset X$ ranges over all pure codimension one algebraic subsets of X. We have inclusions

$$\mathrm{CH}^r_{\mathrm{dec}}(X,m;\mathbb{Q}) \subseteq \Xi(r,m;\mathbb{Q})(X) \subseteq N^1\mathrm{CH}^r(X,m;\mathbb{Q}).$$

The first inclusion is clear by definition and the second one is a consequence of the work of de Jeu and Lewis in [16]. They are equalities in the case $(r,m) = (2,1)$; for $m > 1$, the first equality is in general strict, but it is not clear if equality holds in the second inclusion.

From the Lefschetz decomposition on cohomology with rational coefficients on a smooth complex projective variety Y and for $p \leq \dim Y$

$$H^p(Y,\mathbb{Q}) = H^p_{\mathrm{prim}}(Y,\mathbb{Q}) \oplus L\cdot H^{p-2}(Y,\mathbb{Q}) = \bigoplus_{2q\leq p} L^q H^{p-2q}_{\mathrm{prim}}(Y,\mathbb{Q}),$$

where $L : H^\ell(Y,\mathbb{Q}) \to H^{\ell+2}(Y,\mathbb{Q})$ is the operator defined by the cup product with the Kähler class.

Applying this to the family $\rho : \mathcal{X} \to S$ over the generic point $\eta \in S$ and assuming $2r-m-j \leq \dim \mathcal{X}_\eta$, we get a decomposition

$$\begin{aligned}\underline{\underline{E}}^{j,2r-m-j}_\infty(\eta) &= \Gamma\big(H^j(\eta, R^{2r-m-j}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))\big) \bigoplus \Gamma\big(H^j(\eta, L\cdot R^{2r-m-j-2}\rho_*\mathbb{Q}(r))\big) \\ &= \bigoplus_{2q\leq 2r-m-j} \Gamma\left(H^j(\eta, L^q R^{2r-m-j-2q}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))\right).\end{aligned} \tag{8.2}$$

So, for example when $m = j = 1, r = 2$:

$$\underline{\underline{E}}^{1,2}_\infty(\eta) \simeq \Gamma\big(H^1(\eta, R^2_{\mathrm{prim}}\rho_*\mathbb{Q}(2))\big) \bigoplus \Gamma\big(H^1(\eta,\mathbb{Q}(2))\big),$$

and in the case $j = 1$ and $r \geq m$ arbitrary,

$$\underline{\underline{E}}_{\infty}^{1,2r-m-1}(\eta) = \Gamma\big(H^1(\eta, R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))\big) \bigoplus \Gamma\big(H^1(\eta, LR^{2r-m-3}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))\big) \bigoplus \cdots .$$

We define

$$\Gamma_0 := \Gamma\big(H^1(\eta, R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))\big).$$

Let

$$\pi : \underline{\underline{E}}_{\infty}^{1,2r-m-1}(\eta) \to \Gamma_0$$

be the projection.

Proposition 8.7. Suppose $\Gamma(H^0(\eta, R^{2\ell}\rho_*\mathbb{Q})) \simeq \mathbb{Q}$ for $0 < \ell < r$ and set $X_K = \mathcal{X}_\eta$ and $X/\mathbb{C} = X_K \times_K \mathbb{C}$. Assume that $2r - m - 1 \leq \dim X/\mathbb{C}$. Then

$$\xi \in \Xi(r, m; \mathbb{Q})(X/\mathbb{C}) \Rightarrow \pi \circ \delta(\nu_\xi) = 0.$$

Proof. Let $\xi \in \Xi(r, m; \mathbb{Q})(X/\mathbb{C})$. We can write:

$$\mathrm{CH}^r(X/\mathbb{C}, m; \mathbb{Q}) = \varinjlim_{U/\bar{K}} \mathrm{CH}^r(X_{\bar{K}} \times U, m; \mathbb{Q}),$$

where $U/\bar{K}$ is an extension of finite type. Therefore by an argument similar to the proof of Lemma 5.1 in [36] and after collecting the coefficients of the polynomials defining the cycle, $\xi \in \Xi(r, m; \mathbb{Q})(X_L)$ for some finite extension L/K. Since $K = \bar{\mathbb{Q}}(S) \subset L = \bar{\mathbb{Q}}(S_0)$ for some S and S_0, we can find a finite and proper map $\kappa : S_0 \to S$ such that, on cohomology of locally constant systems, $\kappa_* \circ \kappa^* = \times N$, where $N = \deg \kappa$. This implies that if we pullback ν_ξ to over S_0 and show that $\pi \circ \delta(\nu_\xi) = 0$, then that is also the case over S. Therefore we can assume $\xi \in \Xi(r, m; \mathbb{Q})(X_K)$. Since $F^0 = F^1$ for $m \geq 1$, the map

$$\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q}) \to Gr_F^1\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$$

is well defined. Let's suppose $\xi = \xi_1 \bullet \xi_2$, with $\xi_1 \in \mathrm{CH}^{r_1}(\mathcal{X}_\eta, 0; \mathbb{Q}), \xi_2 \in \mathrm{CH}^{r_2}(\mathcal{X}_\eta, m; \mathbb{Q})$, where $r_1 + r_2 = r$, r_1, $m \geq 1$. Consider the map

$$\mathrm{CH}^{r_2}(\mathcal{X}_\eta, m; \mathbb{Q}) \to Gr_F^1\mathrm{CH}^{r_2}(\mathcal{X}_\eta, m; \mathbb{Q}) \to E_{\infty}^{1,2r_2-m-1}(\eta) \to \underline{\underline{E}}_{\infty}^{1,2r_2-m-1}(\eta)$$

and image of ξ_2 in $\underline{\underline{E}}_{\infty}^{1,2r_2-m-1}(\eta) \simeq \Gamma\big(H^1(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))\big)$; if it is zero there, it is in $J\big(H^0(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))\big)$. We also have a map

$$\mathrm{CH}^{r_1}(\mathcal{X}_\eta, 0; \mathbb{Q}) \to Gr_F^0\mathrm{CH}^{r_1}(\mathcal{X}_\eta, 0; \mathbb{Q}) \to \Gamma\big(H^0(\eta, R^{2r_1}\rho_*\mathbb{Q}(r_1))\big),$$

so ξ_1 maps to $\Gamma\big(H^0(\eta, R^{2r_1}\rho_*\mathbb{Q}(r_1))\big)$. Recall we have a product structure in extension groups and hom $=\mathrm{Ext}^0$. If $\xi_2 \in \Gamma\big(H^1(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))\big) = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2)))$ then the image of $\xi_1 \bullet \xi_2$ is in

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r_1}\rho_*\mathbb{Q}(r_1))) \otimes \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))) \subset \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(\eta, R^{2r-m-1}\rho_*\mathbb{Q}(r))).$$

This coincides with $[\xi]_1 = \delta(\nu_\xi)$.

By assumption $\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r_1}\rho_*\mathbb{Q}(r_1))) \simeq \mathbb{Q}(r_1)$, so the product lies in

$$\mathbb{Q}(r_1) \otimes \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))) \subset \Gamma\big(H^1(\eta, L^{r_1}\cdot R^{2r_2-m-1}\rho_*\mathbb{Q}(r))\big).$$

This is precisely in the complement of Γ_0 and the proposition follows in this case. If $\xi_2 \in J\big(H^0(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))\big)$ then the image of $\xi_1 \bullet \xi_2$ is in

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r_1}\rho_*\mathbb{Q}(r_1))) \otimes \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r_2-m-1}\rho_*\mathbb{Q}(r_2))) \subset \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r-m-1}\rho_*\mathbb{Q}(r))).$$

The last term is precisely the kernel of the map to $\underline{\underline{E}}_\infty^{1,2r-m-1}(\eta)$, so $\delta(\nu_\xi)=0$ trivially. If $\xi_2 \in F^\ell\mathrm{CH}^{r_2}(\mathcal{X}_\eta, m;\mathbb{Q})$ where $\ell \geq 2$ or $\xi_1 \in F^\mu\mathrm{CH}^{r_1}(\mathcal{X}_\eta, 0;\mathbb{Q})$ where $\mu \geq 1$ then $\xi_1 \bullet \xi_2 \in F^{\ell+\mu}\mathrm{CH}^r(\mathcal{X}_\eta, m;\mathbb{Q})$ with $\ell+\mu \geq 2$. This clearly implies $[\xi]_1 = 0$. Finally, if $\xi_1 \in \mathrm{CH}^{r_1}(\mathcal{X}_\eta, m_1;\mathbb{Q})$ where $m_1 \geq 1$ and $\xi_2 \in \mathrm{CH}^{r_2}(\mathcal{X}_\eta, m_2;\mathbb{Q})$ where $m_2 \geq 1$ then $\xi_1 \bullet \xi_2 \in F^\ell\mathrm{CH}^r(\mathcal{X}_\eta, m;\mathbb{Q})$ with $\ell \geq 2$ and $[\xi]_1 = 0$. □

Proposition 8.8. Let $t \in S(\mathbb{C})$ be a very general point (given by an embedding $K \subset \mathbb{C}$) and $\mathcal{X}_t = X_K \times \mathbb{C}$. Suppose $H^{2\ell}(\mathcal{X}_t, \mathbb{Q}) \simeq \mathbb{Q}$ for $0 < \ell < r$ and $2r-m-1 \leq \dim \mathcal{X}_t$. Then

$$AJ(\xi_t) \in AJ(\Xi(r,m;\mathbb{Q})(\mathcal{X}_t))\ \forall \text{ very general } t \in S(\mathbb{C}) \Rightarrow \pi \circ \delta(\nu_\xi) = 0.$$

Proof. Let's take a cycle $\omega \in \Xi(r,m;\mathbb{Q})(\mathcal{X}_t)$. Suppose $\omega = \omega_1 \bullet \omega_2$, where $\omega_1 \in \mathrm{CH}^{r_1}(\mathcal{X}_t, 0;\mathbb{Q})$, $\omega_2 \in \mathrm{CH}^{r_2}(\mathcal{X}_t, m;\mathbb{Q})$, where $r_1 + r_2 = r$, r_1, $m \geq 1$. We have maps

$$\mathrm{CH}^{r_1}(\mathcal{X}_t, 0;\mathbb{Q}) \to \Gamma\big(H^{2r_1}(\mathcal{X}_t, \mathbb{Q}(r_1))\big),$$
$$\mathrm{CH}^{r_2}(\mathcal{X}_t, m;\mathbb{Q}) \to J\big(H^{2r_2-m-1}(\mathcal{X}_t, \mathbb{Q}(r_2))\big).$$

Since $H^{2r_1}(\mathcal{X}_t, \mathbb{Q}(r_1)) \simeq \mathbb{Q}(r_1)$, using the maps above:

$$AJ(\omega) \in \mathbb{Q}(r_1) \otimes J\big(H^{2r_2-m-1}(\mathcal{X}_t, \mathbb{Q}(r_2))\big) \subset J\big(L^{r_1} \cdot H^{2r_2-m-1}(\mathcal{X}_t, \mathbb{Q}(r))\big).$$

There is a short exact sequence

$$0 \to R^{2r-m-1}\rho_*\mathbb{Q} \to \mathcal{O}_S\left(\coprod_{t\in S} \frac{H^{2r-m-1}(\mathcal{X}_t, \mathbb{C})}{F^r H^{2r-m-1}(\mathcal{X}_t, \mathbb{C})}\right) \to \mathcal{J} \to 0,$$

where $\mathcal{J}$ is the sheaf of normal functions. We can compute $\delta(\nu_\omega)$ by using the connecting morphism

$$H^0(S, \mathcal{J}) \xrightarrow{\delta} H^1(S, R^{2r-m-1}\rho_*\mathbb{Q}).$$

From the decomposition

$$\begin{aligned} H^1(S, R^{2r-m-1}\rho_*\mathbb{Q}) =& H^1(S, R^{2r-m-1}_{\text{prim}}\rho_*\mathbb{Q}) \bigoplus H^1(S, LR^{2r-m-3}_{\text{prim}}\rho_*\mathbb{Q}) \bigoplus \cdots \\ & \bigoplus H^1(S, L^{r_1} \cdot R^{2r_2-m-1}\rho_*\mathbb{Q}), \end{aligned}$$

and using that $AJ(\omega) \in J\big(L^{r_1} \cdot H^{2r_2-m-1}(\mathcal{X}_t, \mathbb{Q}(r))\big)$ and δ we conclude that $\pi \circ \delta(\nu_\omega) = 0$. If $\omega_1 \in \mathrm{CH}^{r_1}(\mathcal{X}_t, m_1; \mathbb{Q})$ where $m_1 \geq 1$ and $\omega_2 \in \mathrm{CH}^{r_2}(\mathcal{X}_t, m_2; \mathbb{Q})$ where $m_2 \geq 1$ then $\omega_1 \bullet \omega_2 \in F^2\mathrm{CH}^r(\mathcal{X}_t, m; \mathbb{Q})$. But $F^2\mathrm{CH}^r(\mathcal{X}_t, m; \mathbb{Q}) \subset \ker AJ$ (see Proposition (8.10) below), thus $\delta(\nu_\omega) = 0$. $\square$

Take our smooth complex projective variety $X = \mathcal{X}_\eta \times \mathbb{C}$. If $H^{2\ell}(X; \mathbb{Q})^{\pi_1(S)} \simeq \mathbb{Q}$ for $0 < \ell < r$, since $H^0(\eta, R^{2\ell}\rho_*\mathbb{Q})$ is isomorphic to the cycles in $H^{2\ell}(X; \mathbb{Q})$ invariant under the action of the monodromy group $\pi_1(S)$, we get the isomorphism $\Gamma(H^0(\eta, R^{2\ell}\rho_*\mathbb{Q})) \simeq \mathbb{Q}$, the assumption in Proposition (8.7). We also get $H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r)) \simeq \mathbb{Q}$ as required in the next proposition.

Let W be a complex smooth quasi-projective variety. In [16], R. de Jeu and J. Lewis give conditions for the surjectivity of

$$\mathrm{CH}^r(W, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(W, \mathbb{Q}(r))).$$

In particular they show it is always true for $r = m = 1$ (also see [4]). This implies that

$$\mathrm{CH}^1(\mathcal{X}_\eta, 1; \mathbb{Q}) \to \Gamma\big(H^1(\eta, R^0\rho_*\mathbb{Q}(1))\big)$$

is surjective. Quite generally, a version of the Beilinson-Hodge conjecture implies that for all m,

$$\mathrm{CH}^m(\mathcal{X}_\eta, m; \mathbb{Q}) \to \Gamma\big(H^1(\eta, R^{m-1}\rho_*\mathbb{Q}(m))\big)$$

is surjective ([16]). We will now assume that $2r - m - 1 \leq \dim \mathcal{X}_\eta$, $m = 1$, and that

$$H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r)) \simeq \mathbb{Q}.$$

Note that the map

$$\mathrm{CH}^{r-m}(\mathcal{X}_\eta; \mathbb{Q}) \to \Gamma\big(H^0(\eta, R^{2r-2m}\rho_*\mathbb{Q}(r-m))\big) \simeq \mathbb{Q}(r-m)$$

is surjective. Finally, suppose that $F^2\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ (by [16] Corollary 6.9, this is the case under a generalization of the Beilinson-Hodge conjecture, see Proposition (8.11) below.)

Proposition 8.9. Assume that $2r-2 \leq \dim \mathcal{X}_\eta$, that $H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r)) \simeq \mathbb{Q}$ and that $F^2\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$. Then

$$0 \neq \xi \in \mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})/N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \Rightarrow \pi \circ \delta(\nu_\xi) \neq 0.$$

Proof. Since $\mathrm{CH}^r_{\mathrm{dec}}(\mathcal{X}_\eta, 1; \mathbb{Q})$ consists of the image of the map

$$\mathrm{CH}^{r-1}(\mathcal{X}_\eta; \mathbb{Q}) \otimes \mathrm{CH}^1(\mathcal{X}_\eta, 1; \mathbb{Q}) \to \mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}),$$

the hypotheses above imply that the image of the group of decomposables in $\underline{\underline{E}}_\infty^{1,2r-2}(\eta)$ is $\mathbb{Q}(r-1) \otimes \Gamma\big(H^1(\eta, R^0\rho_*\mathbb{Q}(1))\big)$ and this map is surjective (here we only consider the map from $\mathrm{CH}^{r-1}(\mathcal{X}_\eta; \mathbb{Q})$ to Gr_F^0 and from $\mathrm{CH}^1(\mathcal{X}_\eta, 1; \mathbb{Q})$ to Gr_F^1 since maps to other pieces of the filtration don't really go to $\underline{\underline{E}}_\infty^{1,2r-2}(\eta)$ as we showed in the proof of Proposition (8.7) above). Therefore, if we work with $\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ modulo the decomposables $\mathrm{CH}^r_{\mathrm{dec}}(\mathcal{X}_\eta, 1; \mathbb{Q})$, its image in $\underline{\underline{E}}_\infty^{1,2r-2}(\eta)$ (modulo the image of the decomposables) can be made to lie in Γ_0. Since $\mathrm{CH}^r_{\mathrm{dec}}(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$, then this also shows that the image of $\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ modulo $N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ lies in Γ_0. Because $F^2\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ and $\xi \notin N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$, the image of the cycle $\xi \in \mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ in $E_\infty^{1,2r-2}(\eta)$ is not zero. Thus we have two possibilities: $[\xi]_1 \neq 0$ or $[\xi]_1 = 0$. If $[\xi]_1 \neq 0$ (modulo the image of N^1) then $\pi \circ \delta(\nu_\xi) \neq 0$. If $[\xi]_1 = 0$, then ξ maps to an element of

$$\underline{E}_\infty^{1,2r-2}(\eta) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r)))$$

Also,

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r))) \simeq \mathbb{C}/\mathbb{Q}$$

because $H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r)) \simeq \mathbb{Q}$. However, that the map

$$\mathrm{CH}^1(\mathcal{X}_\eta, 1; \mathbb{Q}) \to \Gamma(H^1(\eta, R^0\rho_*\mathbb{Q}(1)))$$

is surjective (with kernel in $\mathbb{C}/\mathbb{Q}$, see [16]) implies that any element in $J(H^0(\eta, R^0\rho_*\mathbb{Q}(1)))$, which is the kernel of the map

$$E_\infty^{1,0} \to \Gamma(H^1(\eta, R^0\rho_*\mathbb{Q}(1))),$$

comes from a cycle in $\mathrm{CH}^1(\mathcal{X}_\eta, 1; \mathbb{Q})$. We also have

$$J(H^0(\eta, R^0\rho_*\mathbb{Q}(1))) \simeq \mathbb{C}/\mathbb{Q}.$$

Therefore any element in $\underline{E}_\infty^{1,2r-2}(\eta)$ comes from a decomposable cycle, so $[\xi]_1$ cannot be zero when ξ is in $\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})/N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$. $\square$

Regarding the inclusion $F^2\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$ required for Proposition (8.9), we have the following property of the filtration $F^\bullet$ on higher Chow groups which relates F^2 with the higher Abel-Jacobi map AJ:

Proposition 8.10. Let X be a smooth complex projective variety.

$$F^2 \subseteq \ker AJ : \mathrm{CH}^r_{\mathrm{hom}}(X, m; \mathbb{Q}) \to J(H^{2r-m-1}(X, \mathbb{Q}(r))).$$

Proof. By Theorem (8.6), $F^2 = \Lambda^2$, where $\Lambda^\bullet$ is the filtration defined in (8.5). Then it is clear by the definition of arithmetic normal functions (8.1) that $F^2 \subset \ker AJ$. $\square$

Remark. An alternative proof can be given without reference to the theory of arithmetic normal functions. Indeed, consider X as defined over a field K, $X_K \simeq \mathcal{X}_\eta$, $\eta \in S$ the generic point and $\rho : \mathcal{X} \to S$. There is an injective map

$$Gr^1_F\mathrm{CH}^r(X_U, m; \mathbb{Q}) \hookrightarrow \mathrm{Ext}^1_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-1}\rho_*\mathbb{Q}_{X_U}(r))$$

for $U \subset S$. Since MHM(Spec($\mathbb{C}$)) is isomorphic to the category of graded polarizable mixed Hodge structures, we can use the inclusion of the generic point to get a functor from MHM(U) to MHS such that $\mathbb{Q}_U(0)$ maps to $\mathbb{Q}(0)$ and $R^{2r-m-1}\rho_*\mathbb{Q}_{X_U}(r)$ to $H^{2r-m-1}(\mathcal{X}_\eta, \mathbb{Q}(r))$. Thus we have a map

$$\mathrm{Ext}^1_{\mathrm{MHM}(U)}(\mathbb{Q}_U(0), R^{2r-m-1}\rho_*\mathbb{Q}_{X_U}(r)) \to \mathrm{Ext}^1(\mathbb{Q}(0), H^{2r-m-1}(\mathcal{X}_\eta, \mathbb{Q}(r))).$$

Now it is clear that an element in F^2 map to zero under AJ.

Then we use the last proposition with the next result to get the last condition required for (8.9). Let X be a complex smooth projective variety. Consider the map

$$\lim c_{r,m} : \lim \mathrm{CH}^r(U, m; \mathbb{Q}) \to \lim \Gamma(H^{2r-m}(U, \mathbb{Q}(r))),$$

where the limits are taken over open subsets of X.

Proposition 8.11 ([16], Cor. 6.9)**.** Assume the Hodge conjecture and let $r \geq m$. Then

$$\frac{\lim \Gamma(H^{2r-m}(U, \mathbb{Q}(r)))}{\mathrm{im}(\lim c_{r,m})} = 0 \,\text{implies}\; \ker AJ \subset N^1\mathrm{CH}^r(X, m-1; \mathbb{Q}).$$

Summarizing:

Theorem 8.12. Let $X_K = \mathcal{X}_\eta$ and $X/\mathbb{C} = X_K \times_K \mathbb{C}$. Assume that $2r - m - 1 \leq \dim X_K$. Suppose $\Gamma(H^0(\eta, R^{2\ell}\rho_*\mathbb{Q})) \simeq \mathbb{Q}$ for $0 < \ell < r$, $H^0(\eta, R^{2r-2}\rho_*\mathbb{Q}(r)) \simeq \mathbb{Q}$ and that $F^2\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq N^1\mathrm{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q})$. Then

(i) $\xi \in \Xi(r, m; \mathbb{Q})(X_K) \Rightarrow \pi([\xi]) = \pi \circ \delta(\nu_\xi) = 0$.

(ii) $0 \neq \xi \in \mathrm{CH}^r(X_K, 1; \mathbb{Q})/N^1\mathrm{CH}^r(X_K, 1; \mathbb{Q}) \Rightarrow \pi([\xi]) = \pi \circ \delta(\nu_\xi) \neq 0$.

In particular, when $r = 2$ and $m = 1$:

Corollary 8.13. Let $X_K = \mathcal{X}_\eta$, with $\dim X_K \geq 2$. Suppose that $H^2(X; \mathbb{Q})^{\pi_1(S)} \simeq \mathbb{Q}$. Also assume that $F^2\mathrm{CH}^2(\mathcal{X}_\eta, 1; \mathbb{Q}) \subseteq \mathrm{CH}^2_{\mathrm{dec}}(\mathcal{X}_\eta, 1; \mathbb{Q})$. Then

$$0 \neq \xi \in \mathrm{CH}^2_{\mathrm{ind}}(X_K, 1; \mathbb{Q}) \Leftrightarrow \pi([\xi]) = \pi \circ \delta(\nu_\xi) \neq 0.$$

8.2 Griffiths infinitesimal invariant and normal functions

This section, which is also discussed in the work of [13], is included here for completeness. Let's take a complex smooth projective variety X, defined over a finitely generated extension field K of $\bar{\mathbb{Q}}$, with $\bar{\mathbb{Q}}$-spread given by a smooth projective morphism $\rho : \mathcal{X} \to S$ of smooth projective varieties over $\bar{\mathbb{Q}}$ such that $X = \mathcal{X}_\eta \times \mathbb{C}$, where $\eta \in S$ is the generic point and $X_K = \mathcal{X}_\eta$ is the fibre by ρ.

We have the topological invariant $\delta(\nu_\xi) \in \underline{\underline{E}}_\infty^{1,2r-m-1}(\eta)$ defined for a cycle $\xi \in F^1\mathrm{CH}^r(\mathcal{X}_\eta, m; \mathbb{Q})$. Take $U \subset S$ affine open. Since η is obtained after taking a limit we focus our attention in the invariants over $X_U = \rho^{-1}(U)$. We have a map

$$\underline{\underline{E}}_\infty^{1,2r-m-1} \hookrightarrow \nabla J^{r,m,1}(X_U/U)$$

where

$$\underline{\underline{E}}_\infty^{1,2r-m-1} = \Gamma(H^1(U, R^{2r-m-1}\rho_*\mathbb{Q}(r))).$$

Moreover, by considering $\Gamma(H^1(U, R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{Q}(r)))$ using the decomposition

$$\underline{\underline{E}}_\infty^{1,2r-m-1}(\eta) = \bigoplus_{2q \le 2r-m-1} \Gamma(H^j(\eta, L^q R^{2r-m-1-2q}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))).$$

we get a map

$$\Gamma(H^1(U, R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{Q}(r))) \hookrightarrow \nabla\Gamma J$$

where $\nabla\Gamma J$ is the reduced invariant

$$\nabla\Gamma J := \frac{\ker \nabla : H^0(U, \Omega^1_U \otimes F^{r-1}R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C}) \to H^0(U, \Omega^2_U \otimes F^{r-2}R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C})}{\nabla(H^0(U, \mathcal{O}_U \otimes F^r R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C}))}$$

Associated to any normal function we have the Griffiths infinitesimal invariant (introduced in [23], see also [21], [12]). We can define a reduced Griffiths infinitesimal invariant of the normal function ν_ξ denoted by $\delta_G(\nu_\xi)$ and lying in $\Gamma\nabla J$, where $\Gamma\nabla J := H^0(U, \nabla J)$ and

$$\nabla J := \frac{\ker \nabla : \Omega^1_U \otimes F^{r-1}R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C} \to \Omega^2_U \otimes F^{r-2}R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C}}{\nabla(\mathcal{O}_U \otimes F^r R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C})}.$$

There is a natural map $\nabla\Gamma J \to \Gamma\nabla J$ and we want to prove $\nabla\Gamma J \simeq \Gamma\nabla J$. We need the following:

Assumption 8.14. For a fixed choice of r and m:

$$R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C} \cap \left(\mathcal{O}_U \otimes F^r R^{2r-m-1}_{\mathrm{prim}}\rho_*\mathbb{C}\right) = 0.$$

Since the kernel of

$$\mathcal{O}_U \otimes F^r R^{2r-m-1}_{\text{prim}} \rho_* \mathbb{C} \to \Omega^1_U \otimes F^{r-1} R^{2r-m-1}_{\text{prim}} \rho_* \mathbb{C}$$

is precisely

$$R^{2r-m-1}_{\text{prim}} \rho_* \mathbb{C} \cap \left(\mathcal{O}_U \otimes F^r R^{2r-m-1}_{\text{prim}} \rho_* \mathbb{C}\right),$$

we get a short exact sequence

$$0 \to \mathcal{O}_U \otimes F^r R^{2r-m-1}_{\text{prim}} \rho_* \mathbb{C} \to (\Omega^1_U \otimes F^{r-1} R^{2r-m-1}_{\text{prim}} \rho_* \mathbb{C})_{\ker \nabla} \to \nabla J \to 0.$$

If we apply the global sections functor we get a short exact sequence again because the functor is right exact as U is affine; it is always left exact.

Proposition 8.15. Under Assumption (8.14) $\nabla \Gamma J \simeq \Gamma \nabla J$.

The equivalence of these invariants (defined in a slightly different way) appears in [13]. This is relevant because a normal function with nontrivial Griffiths invariant gives us a nontrivial Mumford-Griffiths invariant. For instance, in [40] we can find an example of a normal function with nonzero Griffiths invariant. Of course any cycle in $Gr^j_F \mathrm{CH}^r(X, m; \mathbb{Q})$ inducing a nontrivial class in $\underline{\underline{E}}^{j,2r-m-j}_{\infty}$ gives rise to a nontrivial Mumford-Griffiths invariant.

8.3 A Griffiths type theorem on normal functions

Let $Z \subset \mathbb{P}^4$ be a smooth threefold of degree ≥ 4. Suppose ℓ_1, ℓ_2 are two distinct lines in Z. We can choose a $\mathbb{P}^3$ containing both lines, more precisely : $\mathbb{P}^3$ will contain the span of $\ell_1,\ \ell_2$, with equality if $\ell_1 \cap \ell_2 = \emptyset$. Let $X_0 := \mathbb{P}^3 \cap Z$ and assume it is smooth. Even though a general algebraic surface in $\mathbb{P}^3$ of degree $n \geq 4$ contains no lines, we can still find examples of smooth surfaces containing lines, hence the situation described with X_0 smooth occurs, such as when $Z,\ \ell_1,\ \ell_2$ are very general. Indeed, let $f : \mathcal{Z} \to \mathbb{C}$ be the family of Calabi-Yau threefolds whose fibre $\mathcal{Z}_t$ over $t \in \mathbb{C}$ is given by $V(F_t) \subset \mathbb{P}^4$ where

$$F_t(z_0, z_1, z_2, z_3, z_4) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5t z_0 z_1 z_2 z_3 z_4.$$

This family is known as the Dwork pencil of quintics. Let $H \simeq \mathbb{P}^3$ be the symmetric hyperplane in $\mathbb{P}^4$ given by $H := V(z_0 + z_1 + z_2 + z_3 + z_4)$. Set $Y_t := \mathcal{Z}_t \cap H$ and consider the pencil of surfaces $\{Y_t\}$.

Theorem 8.16 (J. Xie)**.** The base locus of the pencil $\{Y_t\}$ contains 15 lines.

A proof of this fact can be found in [48]. A line in Y_t outside the base locus is called an additional line. J. Xie also shows:

Theorem 8.17 (J. Xie)**.** Let E_t be the set of additional lines on the non singular surface Y_t. Then $E_t \neq \emptyset$ if and only if $t = 0, 2$ or 2τ, where τ is a root of $\tau^4 + \tau + 1$. The additional lines are $|E_0| = 20$, $|E_2| = 40$ and $|E_{2\tau}| = 60$. Furthermore, none of the surfaces Y_0,Y_2 and $Y_{2\tau}$ is isomorphic to the Fermat quintic surface.

Thus, the total lines in the surfaces Y_0,Y_2 and $Y_{2\tau}$ are 35, 55 and 75 respectively. It is well known that the Fermat quintic surface has 75 lines. However it is not isomorphic to $Y_{2\tau}$.

Denote by $[\ell_1]$, $[\ell_2]$ the fundamental classes of the lines ℓ_1, ℓ_2 in $H^2(X_0, \mathbb{Q}(1))$ respectively.

Lemma 8.18. $[\ell_1], [\ell_2]$ are independent in $H^2(X_0, \mathbb{Q}(1))$.

Proof. Since ℓ_1 and ℓ_2 are two distinct lines then either $\ell_1 \cap \ell_2 = \emptyset$ or 1. It follows that $\langle \ell_1, \ell_2 \rangle_{X_0} \geq 0$. Let us assume that $\ell_1 \sim_{\text{hom}} \ell_2$ on X_0 and put $[\ell] = [\ell_1] = [\ell_2]$, where $\ell \simeq \mathbb{P}^1$. Then $\langle \ell, \ell \rangle_{X_0} \geq 0$ and by the adjunction formula:

$$\mathcal{O}_\ell(-2) = \mathcal{O}_{\mathbb{P}^1}(-2) = \Omega_{\mathbb{P}^1} = \Omega_\ell = \mathcal{O}_\ell(\ell) \otimes \Omega^2_{X_0} = \mathcal{O}_\ell(\ell) \otimes \mathcal{O}_{X_0}(d-3).$$

This cannot happen if $d \geq 3$, and we are assuming $d \geq 4$, otherwise $-2 \geq 0$. $\square$

Consider a pencil $\{X_t\}_{t \in \mathbb{P}^1}$ of hyperplane sections of Z containing X_0 and a very general X_1 as members. Further, let $\mathcal{X}$ be the blowup of Z along the base locus of the pencil. Then for very general $t \in \mathbb{P}^1$, $\operatorname{Pic}(X_t) \otimes \mathbb{Q} \simeq \mathbb{Q}$. If $\ell_1 \sim_{\text{rat}} \ell_2$ on $\mathcal{X}$ we can find $\xi \in \mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q})$ such that $\operatorname{div}(\xi) = \ell_1 - \ell_2$. This follows from the description, for any complex smooth projective variety X, of $\mathrm{CH}^r(X, 1)$ as the homology of the middle term in the complex

$$\bigoplus_{\mathrm{cd}_X Y = r-2} K_2(\mathbb{C}(Y)) \xrightarrow{T} \bigoplus_{\mathrm{cd}_X Y = r-1} \mathbb{C}(Y)^\times \xrightarrow{\text{div}} z^r(X),$$

where T is the Tame symbol, div is the divisor map and $z^r(X)$ is the free abelian group generated by subvarieties of codimension r in X. Moreover, $K_0(\mathbb{C}(Y)) \simeq \mathbb{Z}$, $K_1(\mathbb{C}(Y)) = \mathbb{C}(Y)^\times$ and $K_2(\mathbb{C}(Y))$ is generated by symbols. Thus

$$\mathrm{CH}^r(X,1) = \frac{\{\sum_j (f_j, Z_j) : \mathrm{cd}_X Z_j = r-1, f_j \in \mathbb{C}(Z_j)^\times, \sum_j \mathrm{div}(f_j) = 0\}}{\text{Image (Tame symbol)}}.$$

See [36] or [40] for details. Since $F^0\mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q}) = F^1\mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q})$ we have a map

$$Gr_F^1 \mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q}) \to \Gamma(H^1(\mathbb{P}^1 \setminus 0, R^2\rho_*\mathbb{Q}(2))).$$

where $\rho : \mathcal{X} \to \mathbb{P}^1$ is the natural projection. Moreover, we have a decomposition

$$\Gamma(H^1(\mathbb{P}^1 \setminus 0, R^2\rho_*\mathbb{Q}(2))) = \Gamma(H^1(\mathbb{P}^1 \setminus 0, R^2_{\mathrm{prim}}\rho_*\mathbb{Q}(2))) \bigoplus \Gamma(H^1(\mathbb{P}^1 \setminus 0, \mathbb{Q}(2))).$$

Then $\Gamma(H^1(\mathbb{P}^1 \setminus 0, \mathbb{Q}(2))) = 0$, since $\mathbb{P}^1 \setminus 0 \simeq \mathbb{C}$ and $H^1(\mathbb{C}, \mathbb{Q}) = 0$. If we take the limit over $U \subset \mathbb{P}^1 \setminus 0$ open affine we obtain the map

$$Gr_F^1 \mathrm{CH}^2(\mathcal{X}_\eta, 1; \mathbb{Q}) \to \Gamma(H^1(\eta, R^2\rho_*\mathbb{Q}(2))),$$

where ξ maps to Γ_0 (see the notation in Section (8.1)). This image is not trivial; moreover $H^2_{\mathrm{alg}}(X_\eta; \mathbb{Q})^{\pi_1(U,t)} \simeq \mathbb{Q}$ if $d \geq 4$. Then we get an indecomposable in $\mathrm{CH}^2(\mathcal{X}_\eta, 1; \mathbb{Q})$ by Proposition (8.7) (or Corollary (8.13)). We also get a regulator indecomposable by Proposition (8.8). That this image is not trivial goes as follows. We have a localization sequence

$$\ldots \to \mathrm{CH}^2(\mathcal{X}, 1; \mathbb{Q}) \to \mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q}) \to \mathrm{CH}^1(X_0; \mathbb{Q}) \to \ldots$$

and corresponding diagram

$$\begin{tikzcd}
\mathrm{CH}^2(\mathcal{X}, 1; \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{CH}^1(X_0; \mathbb{Q}) \arrow[d] \\
\Gamma(H^3(\mathcal{X}, \mathbb{Q}(2))) \arrow[r] & \Gamma(H^3(\mathcal{X} \setminus X_0, \mathbb{Q}(2))) \arrow[r] & \Gamma(H^2(X_0, \mathbb{Q}(1)))
\end{tikzcd}$$

For $\mathcal{X}$ is projective, $\Gamma(H^3(\mathcal{X}, \mathbb{Q}(2))) = 0$ and the map $\Gamma(H^3(\mathcal{X} \setminus X_0, \mathbb{Q}(2))) \to \Gamma(H^2(X_0, \mathbb{Q}(1)))$ is injective. The morphism $\mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q}) \to \mathrm{CH}^1(X_0; \mathbb{Q})$

is the divisor map and the image of ξ is precisely $\ell_1 - \ell_2$. But $[\ell_1] - [\ell_2] \neq 0$ in $H^2(X_0, \mathbb{Q}(1))$ and consequently ξ maps to a nonzero element in the group $\Gamma(H^3(\mathcal{X} \setminus X_0, \mathbb{Q}(2)))$. Because of the (noncanonical) decomposition

$$H^3(\mathcal{X} \setminus X_0, \mathbb{Q}(2)) \simeq \bigoplus_{p+q=3} H^p(\mathbb{P}^1 \setminus 0, R^q\rho_*\mathbb{Q}(2))$$

there is an injection

$$\Gamma(H^1(\mathbb{P}^1 \setminus 0, R^2\rho_*\mathbb{Q}(2))) \hookrightarrow \Gamma(H^3(\mathcal{X} \setminus X_0, \mathbb{Q}(2))).$$

Moreover, $H^p(\mathbb{P}^1 \setminus 0, R^q\rho_*\mathbb{Q}(2)) = 0$ for $p > 1$ because $\mathbb{P}^1 \setminus 0 \simeq \mathbb{C}$ and $H^3(\mathcal{X} \setminus X_0, \mathbb{Q}(2))$ is isomorphic to

$$H^0(\mathbb{P}^1 \setminus 0, R^3\rho_*\mathbb{Q}(2)) \oplus H^1(\mathbb{P}^1 \setminus 0, R^2\rho_*\mathbb{Q}(2)).$$

This shows that the image of

$$Gr_F^1 \mathrm{CH}^2(\mathcal{X} \setminus X_0, 1; \mathbb{Q}) \to \Gamma(H^1(\mathbb{P}^1 \setminus 0, R^2\rho_*\mathbb{Q}(2))).$$

is not trivial and we are done.

Theorem 8.19. Let $Z \subset \mathbb{P}^4$ be a smooth threefold of degree ≥ 4. Suppose ℓ_1, ℓ_2 are two distinct lines in Z and assume given a smooth $X_0 = \mathbb{P}^3 \cap Z$ where $\mathbb{P}^3$ contains both lines (such as in the case where Z, ℓ_1, ℓ_2 are very general). Let $\mathcal{X}$ be the blow-up of Z along a very general hyperplane section of X_0, and assume that $\ell_1 \sim_{\mathrm{rat}} \ell_2$ on $\mathcal{X}$. Then there exists a hyperplane X in Z such that $\mathrm{CH}^2(X, 1; \mathbb{Q})$ contains a (regulator) indecomposable cycle.

So in summary, $\ell_1 \sim_{\mathrm{rat}} \ell_2$ on $\mathcal{X}$ leads to a normal function ν_ξ with nontrivial topological invariant. This is quite possible in the case $d := \deg Z = 4$, but for $d \geq 5$, and general enough pencil $\{X_t\}_{t \in \mathbb{P}^1}$ containing X_0 and a very general X_1, this cannot happen as the regulator is known to be trivial on $\mathrm{CH}^2(X_1, 1)$ (Müller-Stach [40], Chen and Lewis [14]). Thus for example in the case $d = 5$, $\ell_1 \not\sim_{\mathrm{rat}} \ell_2$. This can be thought of as a weak version of Griffiths famous theorem on normal functions and the nontriviality of the Griffihs group on such X.

8.4 New directions

The ideas presented in this thesis were first motivated by the results of J. Lewis and S. Saito in [38]. Working with a filtration on Chow groups in the case $m = 0$, they were able to determine conditions for which the kernel and image of the map

$$\phi^{r,j} : Gr_F^j \mathrm{CH}^r(X;\mathbb{Q}) \to \nabla J^{r,j}(X/\mathbb{C}),$$

are "uncountably" large. Of particular interest to us are the invariants $H_{\bar{\mathbb{Q}}}^{r-j,r}(X)$ defined for any complex smooth projective variety X. Assuming the components of the diagonal are algebraic and $j \geq 2$, $H_{\bar{\mathbb{Q}}}^{r-j,r}(X) \neq 0$ implies that there are an uncountable number of classes in $\ker \phi^{r,j}$. (Theorem 7.2 in [38].) For example, for X_o an abelian variety over $\bar{\mathbb{Q}}$ of dimension d the invariant $H_{\bar{\mathbb{Q}}}^{r-j,r}(X)$ is not trivial when $2 \leq j \leq r \leq d$ and $X_o = X \times_{\bar{\mathbb{Q}}} \mathbb{C}$. This is the kind of result we would like to reproduce in the case $m \geq 1$. Our future work would involve to define new invariants that give us the same type of conclusion.

In the use of normal functions to detect indecomposables, we would like to get an analogue to Proposition (8.7) with $\Xi(r, m; \mathbb{Q})(X/\mathbb{C})$ replaced by $N^1\mathrm{CH}^r(X/\mathbb{C}, 1; \mathbb{Q})$. This will certainly require new methods different to the ones used here. Also, we proved results for the first normal function only, i.e. we set $j = 1$ in Definition (8.1). One can ask, what is the relation between indecomposables and higher normal functions, i.e. for $j > 1$? A similar result to Theorem (8.12) is what would be interesting to find. In the last section, we could determine conditions to find indecomposables in certain surfaces arising from smooth threefolds of degree ≥ 4 in $\mathbb{P}^4$. It would be interesting to find indecomposable higher Chow cycles when $m \geq 2$ using this technique.

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