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COMPACTLY SUPPORTED SYMMETRIC MRA WAVELET FRAMES

by

Qun Mo

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Compactly supported symmetric MRA wavelet frames** submitted by **Qun Mo** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

Dr. Bin Han (Supervisor)

Dr. Jòachim Stőckler (University of Dortmund)

March 21, 2003

To my parents.

ABSTRACT

In wavelet analysis, high vanishing moments and symmetry are two highly desirable properties of a wavelet frame. In this thesis, we study wavelet frames with high vanishing moments and symmetry derived from refinable function vectors.

In chapter 1, we give a construction of a pair of dual wavelet frames with rd generators with vanishing moments n and m respectively. These generators are derived from any given pair of d-refinable function vectors in $(L_2(\mathbb{R}))^r$ with sum rule orders m and n respectively. To investigate the relation between vanishing moments and sum rule orders, a very useful new canonical form for the matrix mask of a refinable function vector is given.

In chapter 2, we prove that there always exists a tight wavelet frame with vanishing moments m which is derived from a given stable d-refinable function vector with sum rules of order m.

In chapter 3, for any B-spline of order m, we derive a tight wavelet frame with three symmetric generators with vanishing moments m.

In chapter 4, we give a necessary and sufficient condition on constructing a tight wavelet frame with two symmetric generators derived from a symmetric 2-refinable function. To construct such a tight wavelet frame, we need to split a 2×2 matrix of Laurent

polynomials symmetrically. A necessary and sufficient condition for the split is also given. In this chapter, a clear algorithm is given to guide the construction.

Finally, in chapter 5, we present a step by step algorithm to construct pairs of symmetric dual 2-wavelet frames from any pair of symmetric 2-refinable functions. Our algorithm can be easily implemented and yields all possible pairs of symmetric dual wavelet frames.

Several examples are provided to demonstrate the main results in chapters 1, 2, 4 and 5.

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1.1 Introduction

Basically, a wavelet system consists of a set of functions, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, which is an orthogonal basis, or a Riesz basis, or a frame (in a slightly different form) for $L_2(\mathbb{R})$. Each element of the family $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is defined as

$$\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k) \quad \forall j,k \in \mathbb{Z},$$

where ψ is a suitable compactly supported function called a **mother wavelet**.

The simplest example is given by $\psi := \chi_{[0,1/2)} - \chi_{[1/2,1)}$. Thus,

$$\psi_{j,k} = 2^{j/2}\chi_{[2^{-j}k,2^{-j}k+2^{-j-1})} - 2^{j/2}\chi_{[2^{-j}k+2^{-j-1},2^{-j}k+2^{-j})}.$$

It is easy to check that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthogonal basis of $L_2(\mathbb{R})$. (Note: This system is very close to Rademacher functions, which play an important role in probability theory.) Therefore, for each $f \in L_2(\mathbb{R})$, we have the following expansion,

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}2^{j/2}\big(\chi_{[2^{-j}k,2^{-j}k+2^{-j-1})} - \chi_{[2^{-j}k+2^{-j-1},2^{-j}k+2^{-j})}\big),$$

where

$$(1.1.1) \qquad c_{j,k} := 2^{j/2}\left[\int_{2^{-j}k}^{2^{-j}k+2^{-j-1}} f(x)dx - \int_{2^{-j}k+2^{-j-1}}^{2^{-j}k+2^{-j}} f(x)dx\right].$$

Every wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ has the following advantages.

1. Simplicity: Every element of a wavelet system is derived from a single function ψ or several functions by dilations and integer shifts.

2. Multiscale structure: By the definition of $\psi_{j,k}$, we know that for different j, $\psi_{j,k}$ corresponds to a different scale. This correspondence is clear in the previous example. It is easy to see that for different j, $c_{j,k}$ represents f in a different scale with the weight $2^{j/2}$. The weight $2^{j/2}$ is a good balance for the length of the interval $[2^{-j}k, 2^{-j}(k+1)]$: When j is very large, the length of the interval is very small. Thus, $c_{j,k}$ represents f in detail. Moreover, the weight $2^{j/2}$ shows that $c_{j,k}$ amplifies information about f corresponding to the scale j; i.e., for different j, $c_{j,k}$ adjusts information about f by a different scale. Thus, every wavelet system can describe the time-frequency locality very well. This property is highly desirable in many situations.

3. Efficient representation: Every wavelet system can efficiently describe each function $f \in L_2(\mathbb{R})$ such that $\int |\hat{f}(\xi)|^2(1+\xi^2)^\alpha d\xi < \infty$ for some $\alpha > 0$, i.e., only a few coefficients $c_{j,k}$ are large, and the other coefficients are very small. The following example will demonstrate this fact. Consider the previous wavelet system. Let $f = \chi_{[0,1/3]}$. Let us check the coefficients $c_{j,k}$. By definition (1.1.1), we know that when $k \leqslant -1$ or $k \geqslant 2^j/3$, we can easily see that $c_{j,k} = 0$; when $k \in [0, 2^j/3 - 1]$, we have

$$c_{j,k} = 2^{j/2}\left[\int_{2^{-j}k}^{2^{-j}(k+1/2)} 1dx - \int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} 1dx\right] = 0;$$

when $k \in (2^j/3 - 1, 2^j/3)$, i.e., when $k = \lfloor (2^j-1)/3 \rfloor$, we have

$$|c_{j,k}| \leqslant 2^{j/2}\min(2^{-j}, 1/3) \leqslant \min(2^{-j/2}, 2^{j/2}) = 2^{-|j|/2}.$$

Therefore, only a few $c_{j,k}$ are large, and the other coefficients $c_{j,k}$ are very small.

Wavelet analysis originates from many areas such as atomic decomposition in harmonic analysis, sub-band coding and windowed Fourier transform in engineering. The mother wavelet ψ plays an important role in a wavelet system. In 1986, Y. Meyer and S. Mallat introduced multiresolution analysis (MRA) to generally construct ψ. Let V_0 be a linear subspace of $L_2(\mathbb{R})$ and define $V_j := \{f(2^j\cdot) : f \in V_0\}$. If we have

(a) $\cdots \subset V_{-k} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots$.

(b) $\cap_{j\in\mathbb{Z}} V_j = 0$.

(c) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R})$.

(d) There exists a compactly supported function $\phi \in L_2(\mathbb{R})$ such that

$$V_0 = \overline{\operatorname{span}\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}}.$$

(e) There exists $W \subset L_2(\mathbb{R})$ and a compactly supported function $\psi \in L_2(\mathbb{R})$ such that

(e.1) $V_1 = V_0 \oplus W$,

(e.2) $W = \overline{\operatorname{span}\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}}$.

Then the compactly supported function ψ mentioned in (e) is a mother wavelet. For constructing an MRA, first we need to construct the kind of compactly supported function ϕ mentioned in (d). Notice that from $V_0 = \overline{\operatorname{span}\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}}$, we have $\phi \in V_0$. Then by the definition of V_1, we have $V_1 = \overline{\operatorname{span}\{\phi(2\cdot -k)\}_{k\in\mathbb{Z}}}$. By $\phi \in V_0 \subset V_1$, we have the following **refinement equation**:

$$\phi = 2\sum_{k\in\mathbb{Z}} a_k \phi(2\cdot -k), \tag{1.1.2}$$

where a is a sequence on $\mathbb{Z}$, called the **mask** for ϕ. Any scalar mask a in this thesis is assumed to be finitely supported and to satisfy $\sum_{k\in\mathbb{Z}} a_k = 1$. Any function $\phi \in L_2(\mathbb{R})$ satisfying a refinement equation is called a **refinable function**.

Once we get a refinable function ϕ, under mild assumptions on the mask a, we can have an MRA. Thus we can construct a wavelet system. In this context, we can understand how important a refinement equation is in wavelet theory.

As we mentioned before, as a wavelet system, a wavelet basis has many useful properties. However, in some situations, it is not good enough. For example, Daubechies proved that there does not exist any wavelet basis that is smooth, compactly supported, orthogonal and (anti)symmetric at the same time. Nevertheless, in some situations, we wish we could have all the properties above. Therefore, we have to make some tradeoffs to have all the properties. There are two natural methods for doing so. One method is to study wavelet frames instead of wavelet bases. We will discuss this method later. Another method, instead of deriving from one mother wavelet, is to a basis consisting of $\{\psi^{\ell}_{j,k}\}_{j,k\in\mathbb{Z},\ell=1,2,\cdots,L}$ derived from several suitable functions ψ^1, ψ^2, ..., ψ^L.

Geronimo et al. [17] constructed a wavelet basis $\{\psi^1_{j,k}, \psi^2_{j,k}\}_{j,k\in\mathbb{Z}}$ derived from two functions ψ^1 and ψ^2. This wavelet basis is smooth, compactly supported, orthogonal and (anti)symmetric. Their work motivates the study of **multiwavelet theory**.

Multiwavelet theory is a natural extension of the classical wavelet theory. From now on, by d we denote a dilation factor which is an integer such that $|d| > 1$. For a function $\psi \in L_2(\mathbb{R})$, we denote

$$\psi_{j,k} := |d|^{j/2}\psi(d^j \cdot -k), \qquad j \in \mathbb{Z}, k \in \mathbb{Z}.$$

A multiwavelet system is a system $\{\psi^{\ell}_{j,k}\}_{j,k\in\mathbb{Z},\ell=1,2,\cdots,L}$ of L_2-functions which is an orthogonal basis, or a Riesz basis, or a frame of $L_2(\mathbb{R})$.

As we have seen, multiwavelets offer more freedom than classical wavelets. Moreover, in the case of multiwavelets, if we want to let the mother wavelet functions ψ^1, ψ^2, $\ldots$, ψ^L to achieve reasonable smoothness, the support of the mother wavelet functions is much shorter than that of classical wavelets. This property, i.e., reasonable smoothness and short support, is very critical when wavelets are applied to solve some PDEs numerically.

As we can do in classical wavelet theory, we can construct an MRA by a **refinable function vector**. We say that an $r \times 1$ function vector $\phi = (\phi_1, \ldots, \phi_r)^T$ is a **d-refinable function vector**, where r is called the multiplicity, if ϕ satisfies the following **refinement equation**,

$$\phi = |d| \sum_{k \in \mathbb{Z}} a_k \phi(d \cdot -k), \tag{1.1.3}$$

where d is the **dilation factor**, and a is a finitely supported sequence of $r \times r$ complex-valued matrices on $\mathbb{Z}$, called the **(matrix) mask** for ϕ. Under an appropriate mild condition (see [2, 9, 27, 34, 35, 42, 43]) on a matrix mask a, there exists a unique normalized distributional solution of the refinement equation (1.1.3). The refinement equation as well as the various properties of its refinable function vector has been well studied in the literature; see [2, 9, 15, 16, 20, 27, 34, 35, 42, 43, 45] and references therein. When $r = 1$, the refinable function vector in (1.1.3) is a scalar function, so the refinement equation (1.1.2) in classical wavelet theory is a special case $r = 1$ and $d = 2$ of the general refinement equation (1.1.3).

As we mentioned previously, we have two methods to tradeoff a wavelet basis to be smooth, compactly supported, (anti)symmetric and orthogonal at the same time. One method is to study a multiwavelet basis instead of a wavelet basis. The other method is to study a wavelet frame. Now we will discuss wavelet frames.

A frame is a generalization of a basis. Figure 1.1 shows the difference between a basis and a frame. Figures 1.1.(a), 1.1.(b), 1.1.(c) and 1.1.(d) show a basis of $\mathbb{R}^2$, a frame of $\mathbb{R}^2$, an orthogonal basis of $\mathbb{R}^2$ and a tight frame of $\mathbb{R}^2$, respectively. A basis of $\mathbb{R}^2$ can represent each element of $\mathbb{R}^2$ uniquely, but a frame of $\mathbb{R}^2$ can have different representations for the same element of $\mathbb{R}^2$. Thus, a frame provides redundancy, which is useful in some situations. In wavelet applications, wavelet frames have already proved to be very useful for signal denoising and currently are being explored for image compression.

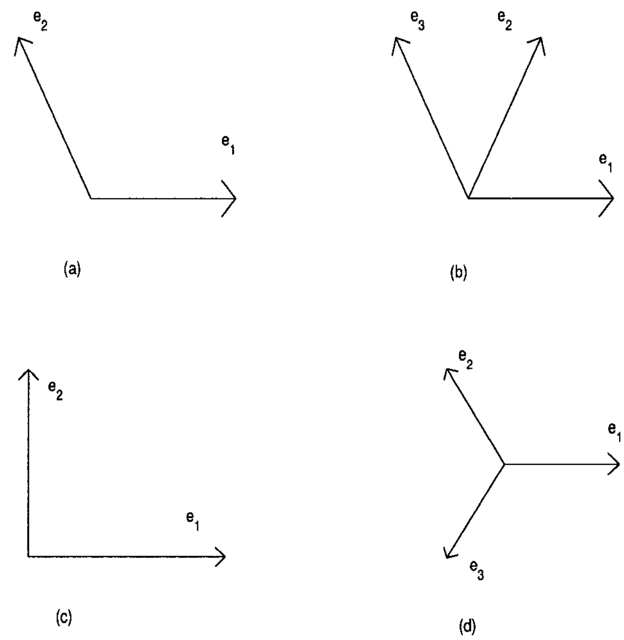

Figure 1.1: Bases and frames.

Let $\{\psi^1, \dots, \psi^L\}$ be a finite set of functions in $L_2(\mathbb{R})$. We say that $\{\psi^1, \dots, \psi^L\}$ generates a d-**wavelet frame** in $L_2(\mathbb{R})$ if there exist positive constants C_1 and C_2 such that

$$C_1\|f\|^2 \leqslant \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{j,k}\rangle|^2 \leqslant C_2\|f\|^2 \qquad \forall\, f \in L_2(\mathbb{R}), \tag{1.1.4}$$

where $\langle f, g\rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt$ for $f, g \in L_2(\mathbb{R})$ and $\|f\|^2 := \langle f, f\rangle$. In particular, when $C_1 = C_2$ in (1.1.4), we say that $\{\psi^1, \dots, \psi^L\}$ generates a **tight** d-**wavelet frame** in $L_2(\mathbb{R})$.

If both $\{\psi^1, \dots, \psi^L\}$ and $\{\widetilde{\psi}^1, \dots, \widetilde{\psi}^L\}$ generate d-wavelet frames in $L_2(\mathbb{R})$ and satisfy

$$\langle f, g\rangle = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j,k}\rangle\langle\widetilde{\psi}^\ell_{j,k}, g\rangle \qquad \forall\, f, g \in L_2(\mathbb{R}),$$

then we say that $\{\psi^1, \dots, \psi^L\}$ and $\{\widetilde{\psi}^1, \dots, \widetilde{\psi}^L\}$ generate a **pair of dual** d-**wavelet frames** in $L_2(\mathbb{R})$. A pair of dual wavelet frames is also called a "bi-frame" in the literature [40]. An important property of a wavelet system is its order of vanishing moments. A function f with enough decay is said to have **vanishing moments** of order m if

$$\int_{\mathbb{R}} t^k f(t)\, dt = 0 \qquad \forall\, k = 0, \ldots, m-1.$$

We say that $\{\psi^1, \ldots, \psi^L\}$ has **vanishing moments** of order m if $\psi^1, \ldots, \psi^L$ all have vanishing moments of order m. The order of vanishing moments of a wavelet system generated by $\{\psi^1, \ldots, \psi^L\}$ has a great impact on how efficient the wavelet system is in representing a function.

Wavelet frames can be characterized theoretically (see [19] and [39]), but those characterizations are not easy to verify in practice. People tried for years to find a good way to generally construct wavelet frames with useful properties. Recently, Daubechies, Han, Ron and Shen [13], and Chui, He and Stöckler [4] independently discovered a way to construct wavelet frames derived from refinable functions.

Inspired by their work, we are particularly interested, in this chapter, in obtaining pairs of dual wavelet frames derived from pairs of refinable function vectors. Define the Fourier transform to be $\widehat{f}(\xi) := \int_{\mathbb{R}} f(t) e^{-i\xi t}\, dt$ for $f \in L_1(\mathbb{R})$. Taking the Fourier transform on both sides of the refinement equation (1.1.3), we get

$$\widehat{\phi}(d\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi} \widehat{\phi}(\xi) = a(\xi)\widehat{\phi}(\xi).$$

Here the **symbol** of a matrix sequence a is defined to be

$$a(\xi) := \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}, \qquad \xi \in \mathbb{R}. \tag{1.1.5}$$

We can also use a matrix of Laurent polynomials as follows:

$$a(z) := \sum_{k\in\mathbb{Z}} a_k z^k, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Therefore other than a matrix sequence, we can use a symbol $a(\xi)$ or its Laurent polynomial matrix $a(z)$ to represent the corresponding mask.

A multiwavelet system is usually generated by several wavelet function vectors $\psi^\ell, \ell = 1, \ldots, L$, which are derived from a d-refinable function vector ϕ in the following way:

$$\widehat{\psi^\ell}(d\xi) = b^\ell(\xi)\widehat{\phi}(\xi), \qquad \ell = 1, \ldots, L,$$

where $b^\ell(\xi)$, $\ell = 1, \ldots, L$ are some appropriate matrices of 2π-periodic trigonometric polynomials.

When such $\{\psi^\ell : \ell = 1, \ldots, L\}$ generates a wavelet frame or a wavelet basis in $L_2(\mathbb{R})$, in the literature they are also called an "MRA wavelet frame" or an "MRA wavelet" basis, respectively (see [4, 11, 12, 13, 39]). Note that all the wavelet function vectors ψ^1, ..., ψ^L are derived from the space $V_1(\phi)$ generated by the refinable function vector $\phi = (\phi_1, \ldots, \phi_r)^T$, where the space $V_j(\phi)$ denotes the L_2-closed linear span of $\phi_\ell(d^j \cdot -k), k \in \mathbb{Z}, \ell = 1, \ldots, r$. Even in the scalar case $r = 1$, it is natural to study a wavelet system which is generated from wavelet functions derived from a general space $V_j(\phi)$ for $j \geqslant 1$ generated by the scalar refinable function ϕ. Deriving wavelet functions from the space $V_j(\phi)$ for some $j \geqslant 1$ is equivalent to deriving them from the space $V_1(\Phi)$, where Φ is a d-refinable function vector whose entries are $\phi(d^{j-1} \cdot -k), k = 0, \ldots, |d|^{j-1} - 1$. This fact naturally leads us to investigate MRA wavelet frames derived from a refinable function vector.

Tight wavelet frames and dual wavelet frames derived from a scalar refinable function (that is, multiplicity $r = 1$) have been recently studied in Chui, He and Stöckler [4], Daubechies, Han, Ron and Shen [13] for the case $d = 2$, and Daubechies and Han [12] for the general dilation factor. The construction of wavelet frames in [4, 12, 13] essentially uses the important fact that a univariate finitely supported mask a satisfies the sum rules of order m if and only if

$$(1 + e^{-i\xi} + \cdots + e^{-i(|d|-1)\xi})^m \mid a(\xi), \tag{1.1.6}$$

which means $a(\xi) = (1 + e^{-i\xi} + \cdots + e^{-i(|d|-1)\xi})^m p(\xi)$ for some 2π-periodic trigonometric polynomial $p(\xi)$. In the multiwavelet case, the definition of sum rules becomes much more complicated. In the univariate setting, Plonka (see [42]) proposed a very nice factorization technique of the symbol of a mask which generalizes the factorization in the scalar case as (1.1.6). However, in order to construct wavelet frames from a refinable function vector, Plonka's factorization cannot be directly used to generalize the construction in [4, 12, 13] to the multiwavelet setting. In this chapter, by employing the ideas from [4, 12, 13, 42] and the idea of Jordan canonical form of a matrix, we are going to overcome this problem to generalize the construction of dual wavelet frames and wavelet frames in [4, 12, 13] from the scalar case to the multiwavelet case. Such a generalization to the multiwavelet case is not trivial due to the complicated form of the sum rules and approximation order. In particular, this chapter closely follows the lines developed in Daubechies and Han [12] where pairs of dual d-wavelet frames derived from any two d-refinable scalar functions were obtained. In this chapter, we shall be able to generalize almost all the results in [12] on pairs of dual wavelet frames for the scalar case to the general multiwavelet case.

The outline of this chapter is as follows. In Section 1.2, we shall present a new canonical form of a matrix mask which is quite useful in the construction of dual wavelet frames from refinable function vectors. The new canonical form proposed in Section 1.2 can be easily generalized to multiple dimensions. Also, the new canonical form in Section 1.2 can preserve the symmetry of a matrix mask. In Section 1.3, by employing the new canonical form we gained in Section 1.2 and generalizing several results in [12], we shall discuss how to derive d-wavelet frames with the highest possible order of vanishing moments from refinable function vectors. First we shall discuss how to derive pairs of dual d-wavelet frames from any two refinable function vectors. Second, we shall discuss how to derive real-valued and symmetric dual d-wavelet frames from two real-valued and symmetric d-refinable function vectors. In fact, we shall show that if the symmetry centers of all the components in the two real-valued d-refinable function vectors differ only by half integers, then we can obtain a pair of dual d-wavelet frames whose generators are real-valued and are either symmetric or antisymmetric with the same symmetry center. Note that the condition on the symmetry centers of the two d-refinable function vectors is automatically satisfied when $d = 2$ and is almost a necessary condition in order to derive pairs of symmetric dual d-wavelet frames from two symmetric d-refinable function vectors. In the rest of Section 1.3, we shall investigate how to derive d-wavelet frames from a single d-refinable function vector. Finally, in Section 1.4, using the method in Section 1.3, we shall present a few examples of pairs of dual wavelet frames to illustrate the general procedure for constructing (real-valued and symmetric) pairs of dual wavelet frames in this chapter. All the results in this chapter were obtained through joint work with my supervisor, Bin Han, and have been published in *Advances in Computational Mathematics* ([23]).

1.2 A new canonical form of a matrix mask and the order of sum rules

In this section, we shall discuss a new canonical form of a matrix mask with a certain order of sum rules. The order of sum rules of a mask is closely related to the approximation order of its refinable function vector and the order of vanishing moments of a wavelet system derived from such a refinable function vector. See the work [2, 20, 27, 34, 35, 42, 43] and references therein for a detailed discussion on approximation order and sum rules. We say that a mask a with multiplicity r satisfies the **sum rules** of order m with respect to the lattice $d\mathbb{Z}$ if there exists a $1 \times r$ row vector $y(\xi)$ of 2π-periodic trigonometric polynomials such that $y(0) \neq 0$ and for $k = 0, \ldots, |d| - 1$,

$$(1.2.7) \qquad [y(d\cdot)a(\cdot)]^{(j)}(2\pi k/d) = \delta_k y^{(j)}(0), \qquad j = 0, \ldots, m-1,$$

where δ is the **Dirac sequence** such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$, and $y^{(j)}(\xi)$ denotes the jth derivative of $y(\xi)$ for all $j \in \mathbb{Z}$. By the Leibniz differentiation formula, for $k = 0, \dots, |d| - 1$, it is easy to rewrite (1.2.7) into the following form

$$\sum_{\ell=0}^{j} \binom{j}{\ell} d^{j-\ell} y^{(j-\ell)}(0) a^{(\ell)}(2\pi k/d) = \delta_k y^{(j)}(0), \quad j = 0, \dots, m-1, \tag{1.2.8}$$

where $\binom{j}{\ell} := \frac{j!}{\ell!(j-\ell)!}$.

Consider condition (1.2.8) for the special case $r = 1$. For $k = 0, \dots, |d|-1$, when $j = 0$, condition (1.2.8) becomes $y(0)a(2\pi k/d) = \delta_k y(0)$. Notice that $y(0) \neq 0$, we have $a(2\pi k/d) = 0$ for $k = 1, \dots, |d| - 1$. Similarly, check condition (1.2.8) for $j = 1, \dots, m-1$, we have $a^{(j)}(2\pi k/d) = 0$ for $k = 1, \dots, |d|-1$. On another hand, if we have $a^{(j)}(2\pi k/d) = 0$ for $j = 0, \dots, m-1$, $k = 1, \dots, |d| - 1$, then (1.2.8) is satisfied. Therefore, we get the conclusion that when $r = 1$, i.e., in the scalar case, the mask $a(\xi)$ has sum rules of order m if and only if

$$\left(1 + e^{-i\xi} + \cdots + e^{-i(|d|-1)\xi}\right)^m \mid a(\xi). \tag{1.2.9}$$

Unfortunately, the factorization (1.2.9) is no longer true for general multiplicity r. For instance, consider a function vector $\phi := (\phi_1, \phi_2)^T$ where ϕ_1 and ϕ_2 are two piecewise polynomials and $\phi_1 := \chi_{[-1,0)}(t+1)^2(-2t+1) + \chi_{[0,1)}(t-1)^2(2t+1)$, $\phi_2 := t(t+1)^2\chi_{[-1,0)} + t(t-1)^2\chi_{[0,1)}$. The function vector ϕ is the well known Hermite cubics and it satisfies the following refinement equation

$$\phi = \begin{bmatrix} 1/2 & 3/4 \\ -1/8 & -1/8 \end{bmatrix} \phi(2\cdot+1) + \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \phi(2\cdot) + \begin{bmatrix} 1/2 & -3/4 \\ 1/8 & -1/8 \end{bmatrix} \phi(2\cdot-1).$$

Hence the mask $a(\xi)$ is defined as

$$a(\xi) := \begin{bmatrix} (e^{i\xi} + 2 + e^{-i\xi})/4 & 3(e^{i\xi} - e^{-i\xi})/8 \\ (-e^{i\xi} + e^{-i\xi})/16 & (-e^{i\xi} + 4 - e^{-i\xi})/16 \end{bmatrix}.$$

Although the mask $a(\xi)$ satisfies the sum rules of order 4 with a row vector $y(\xi) = [1, e^{i\xi}/3 + 1/2 - e^{-i\xi} + e^{-i2\xi}/6]$, we have $(1+e^{-i\xi}) \nmid a(\xi)$. The factorization (1.2.9) is no longer true for the case $r > 1$.

However, we still can get a factorization similar to (1.2.9) by employing the idea of Jordan canonical form. For an $r \times r$ matrix $U(\xi)$ of 2π-periodic trigonometric polynomials, we say that $U(\xi)$ is **invertible** if the inverse of $U(\xi)$ is also a matrix of 2π-periodic trigonometric polynomials. In Section 1.1, we introduced the MRA generated by ϕ. As before, let V_0 be the L_2-closed linear span of $\phi_1(\cdot - k), \ldots, \phi_r(\cdot - k)$, $k \in \mathbb{Z}$. Notice that V_0 is also the L_2-closed linear span of $\widetilde{\phi}_1(\cdot - k), \ldots, \widetilde{\phi}_r(\cdot - k)$, $k \in \mathbb{Z}$ where $\widehat{\widetilde{\phi}} := (\widehat{\widetilde{\phi}_1}, \ldots, \widehat{\widetilde{\phi}_r})^T := U(\xi)\widehat{\phi}(\xi)$ for an invertible $U(\xi)$. So when we are talking about an MRA generated by ϕ, there is no big difference between ϕ and $\widetilde{\phi}$. Define $\widetilde{a}(\xi) := U(d\xi)a(\xi)U(\xi)^{-1}$, then $\widehat{\widetilde{\phi}}(d\xi) = \widetilde{a}(\xi)\widehat{\widetilde{\phi}}(\xi)$. So $\widetilde{a}(\xi)$ is the matrix mask of the refinable function vector $\widetilde{\phi}$. Since $\widetilde{a}(\xi) = U(d\xi)a(\xi)U(\xi)^{-1}$ for some invertible $U(\xi)$, we have some choices for $\widetilde{a}$ and we hope we can get some special form for $\widetilde{a}$ when $U(\xi)$ is suitable.

Before going further, let us go through the following Lemma.

Lemma 1.1. *Let $U(\xi)$ be an invertible $r \times r$ matrix of 2π-periodic trigonometric polynomials. If a finitely supported mask a with multiplicity r satisfies the sum rules of order m in* (1.2.7) *with a $1 \times r$ row vector $y(\xi)$ of 2π-periodic trigonometric polynomials, then the finitely supported new mask $\widetilde{a}$, defined by $\widetilde{a}(\xi) := U(d\xi)a(\xi)U(\xi)^{-1}$, also satisfies the sum rules of order m in* (1.2.7) *with the new row vector $\widetilde{y}(\xi)$ of 2π-periodic trigonometric polynomials given by $\widetilde{y}(\xi) = y(\xi)U(\xi)^{-1}$.*

Proof: We need to check that $[\widetilde{y}(d\cdot)\widetilde{a}(\cdot)]^{(j)}(2\pi k/d) = \delta_k \widetilde{y}^{(j)}(0)$ for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$. Since

$$\widetilde{y}(d\xi)\widetilde{a}(\xi) = y(d\xi)U(d\xi)^{-1}U(d\xi)a(\xi)U(\xi)^{-1} = y(d\xi)a(\xi)U(\xi)^{-1},$$

we only need to verify that for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$,

$$[y(d\cdot)a(\cdot)U(\cdot)^{-1}]^{(j)}(2\pi k/d) = \delta_k [y(\cdot)U(\cdot)^{-1}]^{(j)}(0). \tag{1.2.10}$$

By assumption, $[y(d\cdot)a(\cdot)]^{(j)}(2\pi k/d) = \delta_k y^{(j)}(0)$ for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$, plus the Leibniz differentiation formula, (1.2.10) holds for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$. ∎

Now we are in the position to give the following result about factorization of a matrix mask which generalizes the scalar case in (1.2.9).

Theorem 1.1. *Let a be a finitely supported mask on $\mathbb{Z}$ with multiplicity r. Then a satisfies the sum rules of order m with respect to the lattice $d\mathbb{Z}$ if and only if there exists an invertible $r \times r$ matrix $U(\xi)$ of 2π-periodic trigonometric polynomials such that $U(d\xi)a(\xi)U(\xi)^{-1}$ takes the canonical form*

$$\begin{bmatrix} \left(1 + e^{-i\xi} + \cdots + e^{-i(|d|-1)\xi}\right)^m P_{1,1}(\xi) & \left(1 - e^{-i|d|\xi}\right)^m P_{1,2}(\xi) \\ P_{2,1}(\xi) & P_{2,2}(\xi) \end{bmatrix} \tag{1.2.11}$$

where $P_{1,1}, P_{1,2}, P_{2,1}$ and $P_{2,2}$ are some $1 \times 1, 1 \times (r-1), (r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of 2π-periodic trigonometric polynomials respectively and $P_{1,1}(0) = d^{-m}$.

Proof: Since a satisfies the sum rules of order m, by definition, there exists a $1 \times r$ row vector $y(\xi)$ of 2π-periodic trigonometric polynomials such that $y(0) \neq 0$ and (1.2.7) holds. Let U_1 be an $r \times r$ invertible matrix of numbers such that $y(0)U_1 = [1, 0, \ldots, 0]$. Define the trigonometric polynomials $p_\ell, \ell = 1, \ldots, r$, by $[p_1(\xi), \ldots, p_r(\xi)] = y(\xi)U_1$. Since $p_1(0) = 1$, there exist 2π-periodic trigonometric polynomials $f_\ell(\xi), \ell = 2, \ldots, r$, such that $f_\ell^{(j)}(0) = [p_\ell/p_1]^{(j)}(0)$ for all $j = 0, \ldots, m-1$ and $\ell = 2, \ldots, r$, i.e., $[p_\ell(\cdot) - f_\ell(\cdot)p_1(\cdot)]^{(j)}(0) = 0$ for all $j = 0, \ldots, m-1$ and $\ell = 2, \ldots, r$.

Take $U(\xi) = \begin{bmatrix} 1 & f_2(\xi) & \cdots & f_r(\xi) \\ 0 & & I_{r-1} & \end{bmatrix} U_1^{-1}$. Define $\widetilde{y}(\xi) := y(\xi)U(\xi)^{-1} = [p_1(\xi), \cdots, p_r(\xi)] \begin{bmatrix} 1 & -f_2(\xi) & \ldots & -f_r(\xi) \\ 0 & & I_{r-1} & \end{bmatrix}$. It is evident that $U(\xi)$ is invertible, $\widetilde{y}_1(0) = 1$, and $\widetilde{y}_\ell^{(j)}(0) = [p_\ell - f_\ell p_1]^{(j)}(0) = 0$ for all $\ell = 2, \ldots, r$ and $j = 0, \ldots, m-1$, where $[\widetilde{y}_1(\xi), \ldots, \widetilde{y}_r(\xi)] = \widetilde{y}(\xi)$. Let $\widetilde{a}(\xi) = U(d\xi)a(\xi)U(\xi)^{-1}$. By Lemma 1.1, $\widetilde{a}$ satisfies the sum rules of order m; that is, for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d| - 1$,

$$[\widetilde{y}(d\cdot)\widetilde{a}(\cdot)]^{(j)}(2\pi k/d) = \delta_k \widetilde{y}^{(j)}(0). \tag{1.2.12}$$

Since $\widetilde{y}_\ell^{(j)}(0) = 0$ for all $\ell = 2, \ldots, r$ and $j = 0, \ldots, m-1$, we observe that $[\widetilde{y}_\ell(d\cdot)]^{(j)}(2\pi k/d) = 0$ for all $\ell = 2, \ldots, r, j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$. So by matrix multiplication, for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$, (1.2.12) is equivalent to

$$[\widetilde{y}_1(d\cdot)\widetilde{a}_{1,1}(\cdot)]^{(j)}(2\pi k/d) = \delta_k \widetilde{y}_1^{(j)}(0) \tag{1.2.13}$$

and

$$[\widetilde{y}_1(d\cdot)\widetilde{a}_{1,\ell}(\cdot)]^{(j)}(2\pi k/d) = 0, \tag{1.2.14}$$

where $\widetilde{a}_{1,\ell}(\xi)$ denotes the $(1,\ell)$-entry of the matrix $\widetilde{a}(\xi)$. Since $\widetilde{y}_1(0) = 1$, by the Leibniz differentiation formula, (1.2.13) implies $\widetilde{a}_{1,1}(0) = 1$ and $\widetilde{a}_{1,1}^{(j)}(2\pi k/d) = 0$ for all $j = 0, \ldots, m-1$ and $k = 1, \ldots, |d|-1$; that is, $\widetilde{a}_{1,1}(0) = 1$ and $\left(1+e^{-i\xi}+\cdots+e^{-i(|d|-1)\xi}\right)^m \mid \widetilde{a}_{1,1}(\xi)$. Similarly, (1.2.14) implies $\widetilde{a}_{1,\ell}^{(j)}(2\pi k/d) = 0$ for all $j = 0, \ldots, m-1$, $\ell = 2, \ldots, r$ and $k = 0, \ldots, |d|-1$; that is, $(1 - e^{-i|d|\xi})^m \mid \widetilde{a}_{1,\ell}(\xi)$ for all $\ell = 2, \ldots, r$. So $\widetilde{a}(\xi)$ can be written in the form of (1.2.11).

Suppose that a mask $\widetilde{a}(\xi) = U(d\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.11) where $U(\xi)$ is invertible. Since $\widetilde{a}_{1,1}(0) = 1$, by [12, Lemma 2.3] or Lemma 1.4 in this chapter, there exists a 2π-periodic trigonometric polynomial $\widetilde{y}_1$ such that $\widetilde{y}_1(0) = 1$ and

$$[\widetilde{y}_1(\cdot) - \widetilde{y}_1(d\cdot)\widetilde{a}_{1,1}(\cdot)]^{(j)}(0) = 0 \qquad \forall\, j = 0, \ldots, m-1.$$

Take $\widetilde{y}(\xi) = [\widetilde{y}_1(\xi), 0, \ldots, 0]$. Since $\widetilde{a}$ takes the form of (1.2.11), equations (1.2.13) and (1.2.14) are satisfied for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, |d|-1$. Hence we have equation (1.2.12), i.e.,

$$[\widetilde{y}(d\cdot)\widetilde{a}(\cdot)]^{(j)}(2\pi k/d) = \delta_k \widetilde{y}^{(j)}(0) \qquad \forall\, j = 0, \ldots, m-1;\ k = 0, \ldots, |d|-1.$$

Therefore, $\widetilde{a}$ must satisfy the sum rules of order m. Consequently, since $a(\xi) = U(d\xi)^{-1}\widetilde{a}(\xi)U(\xi)$, the mask a must satisfy the sum rules of order m by Lemma 1.1. ∎

The above theorem generalizes the factorization (1.2.9) in the scalar case $r=1$ to the general multiwavelet case. Moreover, to describe a wavelet system derived from a refinable function vector ϕ, we need the following theorem. To give a simpler and easier understanding, in the following theorem, we put the condition " 1 is a simple eigenvalue of $a(0)$, and d^k is not an eigenvalue of $a(0)$ for all $k \in \mathbb{N}$" on the matrix mask a of the refinable function vector ϕ. As we will see it later, such a condition can be replaced by a much weaker condition.

Theorem 1.2. *Let a be a finitely supported matrix mask on $\mathbb{Z}$ with multiplicity r. Suppose that a satisfies the sum rules of order m with respect to the lattice $d\mathbb{Z}$, 1 is a simple eigenvalue of $a(0)$, and d^k is not an eigenvalue of $a(0)$ for all $k \in \mathbb{N}$. Then for any nonnegative integer n, there exists an invertible $r \times r$ matrix $U(\xi)$ of 2π-periodic trigonometric polynomials such that $U(d\xi)a(\xi)U(\xi)^{-1}$ takes the following canonical form*

$$\begin{bmatrix} (1+e^{-i\xi}+\cdots+e^{-i(|d|-1)\xi})^m P_{1,1}(\xi) & (1-e^{-i|d|\xi})^m P_{1,2}(\xi) \\ (1-e^{-i\xi})^n P_{2,1}(\xi) & P_{2,2}(\xi) \end{bmatrix}, \tag{1.2.15}$$

where $P_{1,1}, P_{1,2}, P_{2,1}$ and $P_{2,2}$ are some $1\times 1, 1\times(r-1), (r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of 2π-periodic trigonometric polynomials respectively and $P_{1,1}(0)=d^{-m}$. Moreover, if $\phi=(\phi_1,\ldots,\phi_r)^T$ satisfies the refinement equation with the dilation factor d and a mask taking the form of (1.2.15), then $\widehat{\phi}_\ell^{(j)}(0)=0$ for all $\ell=2,\ldots,r$ and $j=0,\ldots,n-1$; that is, all the functions $\phi_2,\ldots,\phi_r$ have vanishing moments of order n.

Proof: Since a satisfies the sum rules of order m, by Theorem 1.1, there exists an invertible $r\times r$ matrix $U_1(\xi)$ of 2π-periodic trigonometric polynomials such that $\widetilde{a}(\xi)=U_1(d\xi)a(\xi)U_1(\xi)^{-1}$ takes the form of (1.2.11). By the assumption and $\widetilde{a}(0)=U_1(0)a(0)U_1(0)^{-1}$, we deduce that 1 is a simple eigenvalue of $\widetilde{a}(0)$ and d^k is not an eigenvalue of $\widetilde{a}(0)$ for all $k\in\mathbb{N}$.

Take $U_2(\xi)=\begin{bmatrix} 1 & 0 \\ g(\xi) & I_{r-1} \end{bmatrix}$, where $g(\xi)=[g_2(\xi),\ldots,g_r(\xi)]^T$ with $g_2(\xi)$, ..., $g_r(\xi)$ being some 2π-periodic trigonometric polynomials to be determined.

We now compute the matrix $b(\xi) := U_2(d\xi)\widetilde{a}(\xi)U_2(\xi)^{-1}$ as follows:

$$b(\xi) = \begin{bmatrix} 1 & 0 \\ g(d\xi) & I_{r-1} \end{bmatrix} \begin{bmatrix} \widetilde{a}_{1,1}(\xi) & \widetilde{a}_{1,2}(\xi) \\ \widetilde{a}_{2,1}(\xi) & \widetilde{a}_{2,2}(\xi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -g(\xi) & I_{r-1} \end{bmatrix} =$$

$$\begin{bmatrix} \widetilde{a}_{1,1}(\xi) - \widetilde{a}_{1,2}(\xi)g(\xi) & \widetilde{a}_{1,2}(\xi) \\ g(d\xi)\left(\widetilde{a}_{1,1}(\xi) - \widetilde{a}_{1,2}(\xi)g(\xi)\right) - \widetilde{a}_{2,2}(\xi)g(\xi) + \widetilde{a}_{2,1}(\xi) & \widetilde{a}_{2,2}(\xi) + g(d\xi)\widetilde{a}_{1,2}(\xi) \end{bmatrix}.$$

Since $(1 + e^{-i\xi} + \ldots + e^{-i(|d|-1)\xi})^m \mid \widetilde{a}_{1,1}(\xi)$ and $(1 - e^{-i|d|\xi})^m \mid \widetilde{a}_{1,2}(\xi)$, $b(\xi)$ takes the form of (1.2.15) if and only if for all $j = 0, \ldots, n-1$,

$$[g(d\cdot)\widetilde{a}_{1,1}(\cdot) - \widetilde{a}_{2,2}(\cdot)g(\cdot) + \widetilde{a}_{2,1}(\cdot) - g(d\cdot)\widetilde{a}_{1,2}(\cdot)g(\cdot)]^{(j)}(0) = 0. \tag{1.2.16}$$

That is, by the Leibniz differentiation formula, for every $j = 0, \ldots, n-1$,

$$\begin{aligned} &\sum_{k=0}^{j} \binom{j}{k} [\widetilde{a}_{1,1}^{(j-k)}(0)d^k g^{(k)}(0) - \widetilde{a}_{2,2}^{(j-k)}(0)g^{(k)}(0)] \\ &- \sum_{k=0}^{j} \binom{j}{k} d^k g^{(k)}(0) \sum_{\ell=0}^{j-k} \binom{j-k}{\ell} \widetilde{a}_{1,2}^{(j-k-\ell)}(0)g^{(\ell)}(0) \\ &= -\widetilde{a}_{2,1}^{(j)}(0). \end{aligned} \tag{1.2.17}$$

Note that $\widetilde{a}_{1,2}(0) = 0$. The term $g^{(j)}(0)$ does not essentially appear in the sum

$$\sum_{k=0}^{j} \binom{j}{k} d^k g^{(k)}(0) \sum_{\ell=0}^{j-k} \binom{j-k}{\ell} \widetilde{a}_{1,2}^{(j-k-\ell)}(0)g^{(\ell)}(0)$$

since when $k = j$, then $\sum_{\ell=0}^{j-k} \binom{j-k}{\ell} \widetilde{a}_{1,2}^{(j-k-\ell)}(0)g^{(\ell)}(0) = \widetilde{a}_{1,2}(0)g(0) = 0$, and when $k = 0$, then the term $g^{(j)}(0)$ in the sum $\sum_{\ell=0}^{j-k} \binom{j-k}{\ell} \widetilde{a}_{1,2}^{(j-k-\ell)}(0)g^{(\ell)}(0)$ has the coefficient $\widetilde{a}_{1,2}(0) = 0$. Consequently, since 1 is a simple eigenvalue of $\widetilde{a}(0)$ and d^k is not an eigenvalue of $\widetilde{a}(0)$ for all $k \in \mathbb{N}$, we define the $(r-1) \times 1$ column vectors $c_j, j = 0, \ldots, n-1$, of numbers by the following recursive formula:

$$c_0 := -[\widetilde{a}_{1,1}(0)I_{r-1} - \widetilde{a}_{2,2}(0)]^{-1}\widetilde{a}_{2,1}(0) = -[I_{r-1} - \widetilde{a}_{2,2}(0)]^{-1}\widetilde{a}_{2,1}(0)$$

since $\widetilde{a}_{1,1}(0) = 1$, and for $j = 1, \ldots, n-1$, define

$$\begin{aligned} c_j := [d^j I_{r-1} - \widetilde{a}_{2,2}(0)]^{-1} \Big[&- \widetilde{a}_{2,1}^{(j)}(0) - \sum_{k=0}^{j-1} \binom{j}{k} \left(\widetilde{a}_{1,1}^{(j-k)}(0)d^k c_k - \widetilde{a}_{2,2}^{(j-k)}(0)c_k \right) \\ &+ \sum_{k=0}^{j-1} \binom{j}{k} d^k c_k \sum_{\ell=0}^{j-k-1} \binom{j-k}{\ell} \widetilde{a}_{1,2}^{(j-k-\ell)}(0)c_\ell \Big]. \end{aligned}$$

Since 1 is a simple eigenvalue of $\widetilde{a}(0)$ and d^k is not an eigenvalue of $\widetilde{a}(0)$ for all $k \in \mathbb{N}$, we deduce that $d^j I_{r-1} - \widetilde{a}_{2,2}(0)$ is invertible for all $j = 0, 1, 2, \ldots$. So $c_j, j = 0, \ldots, n-1$, are well defined. Consequently, there exists an $(r-1) \times 1$ column vector $g(\xi) = [g_2(\xi), \ldots, g_r(\xi)]^T$ of 2π-periodic trigonometric polynomials such that $g^{(j)}(0) = c_j$ for all $j = 0, \ldots, n-1$. By the choice of $c_j, j = 0, \ldots, n-1$, using the Leibniz differentiation formula, one can easily verify that (1.2.16) holds. Therefore, setting $U(\xi) = U_2(\xi)U_1(\xi)$, we see that $U(d\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.15).

Suppose $\widehat{\phi}(d\xi) = a(\xi)\widehat{\phi}(\xi)$ with $a(\xi)$ taking the form of (1.2.15). Due to the special form of $a(\xi)$ in (1.2.15), we deduce that $a(0) = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$ and $a^{(j)}(0) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ for all $j = 0, \ldots, n-1$, where $*$ denotes some unknown matrix or number. We use induction to prove that $\widehat{\phi}^{(j)}(0) = [*, 0, \ldots, 0]^T$ for all $j = 0, \ldots, n-1$. When $j = 0$, we have $\widehat{\phi}(0) = a(0)\widehat{\phi}(0)$. Since 1 is the simple eigenvalue of $a(0)$ and $a(0) = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$, we must have $\widehat{\phi}(0) = [*, 0, \ldots, 0]^T$, where $*$ denotes a number. Suppose that $\widehat{\phi}^{(k)}(0) = [*, 0, \ldots, 0]^T$ for all $k = 0, \ldots, j-1$ with $0 < j \leqslant n-1$. Then by the Leibniz differentiation formula, $d^j \widehat{\phi}^{(j)}(0) = \sum_{k=0}^{j} \binom{j}{k} a^{(j-k)}(0)\widehat{\phi}^{(k)}(0)$. So

$$[d^j I_r - a(0)]\widehat{\phi}^{(j)}(0) = \sum_{k=0}^{j-1} \binom{j}{k} a^{(j-k)}(0)\widehat{\phi}^{(k)}(0).$$

Since $a^{(j-k)}(0) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ and by the induction hypothesis, for all $k = 0, \ldots, j-1$, $\widehat{\phi}^{(k)}(0) = [*, 0, \ldots, 0]^T$, we have $a^{(j-k)}(0)\widehat{\phi}^{(k)}(0) = [*, 0, \ldots, 0]^T$ for $k = 0, \ldots, j-1$. Since $d^j I_r - a(0)$ is invertible for $j > 0$, we have

$$\widehat{\phi}^{(j)}(0) = [d^j I_r - a(0)]^{-1} \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} (d^j - 1)^{-1} & 0 \\ 0 & * \end{bmatrix} \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix},$$

where $*$ denotes some number or matrix. By induction, we conclude that

$\widehat{\phi}_\ell^{(j)}(0) = 0$ for all $j = 0, \ldots, n-1$ and $\ell = 2, \ldots, r$ which is equivalent to saying that $\phi_2, \ldots, \phi_r$ have vanishing moments of order n. ■

Theorem 1.2 gives us a clear canonical form of a matrix mask a having sum rules of order m. Moreover, we can use this canonical form to characterize the vanishing moments of a function derived from the refinable function vector ϕ associated with its matrix mask a.

Corollary 1.2. Let $\phi \in (L_2(\mathbb{R}))^r$ be a refinable function vector with its matrix mask a and the matrix mask a takes the canonical form (1.2.15). Suppose $\psi \in L_2(\mathbb{R})$ is defined by $\widehat{\psi}(d\xi) = b(\xi)\widehat{\phi}(\xi)$ where b is a $1 \times r$ vector of trigonometric polynomials. Then for an integer $\ell \leqslant n$, ψ has vanishing moments of order ℓ if and only if $b_1^{(j)}(0) = 0$ for all $j = 0, \ldots, \ell - 1$ where b_1 is the first component of b.

Although Theorems 1.1 and 1.2 are stated in the univariate case, from the proofs of these two theorems, we know that they can be generalized to the multivariate case as follows.

Theorem 1.3. *Let a be a finitely supported matrix mask on $\mathbb{Z}^s$ with multiplicity r. Then a satisfies the sum rules of order m with respect to the lattice $M\mathbb{Z}^s$ if and only if there exists an invertible $r \times r$ matrix $U(\xi)$ of 2π-periodic trigonometric polynomials such that the mask $\widetilde{a}(\xi) = U(M^T\xi)a(\xi)U(\xi)^{-1}$ satisfies*

$$\widetilde{a}_{1,1}(0) = 1, \ \partial^\mu \widetilde{a}_{1,1}(2\pi k) = 0 \quad \forall k \in (M^T)^{-1}\mathbb{Z}^s \backslash \mathbb{Z}^s, \ |\mu| < m,$$
$$\partial^\mu \widetilde{a}_{1,\ell}(2\pi k) = 0 \quad \forall k \in (M^T)^{-1}\mathbb{Z}^s, \ |\mu| < m, \ \ell = 2, \ldots, r.$$

Theorem 1.4. *Let $\phi = (\phi_1, \ldots, \phi_r)^T \in (L_2(\mathbb{R}^s))^r$ satisfy the refinement equation $\widehat{\phi}(M^T\xi) = a(\xi)\widehat{\phi}(\xi)$ with the dilation matrix M and its matrix mask a. Suppose that a satisfies the sum rules of order m with respect to the lattice $M\mathbb{Z}^s$, i.e., there exists a $1 \times r$ row vector $v(\xi)$ of 2π-periodic trigonometric polynomials such that $v(0) \neq 0$ and*

$$\partial^\mu [v(M^T\cdot)a(\cdot)](2\pi k) = \delta_k \partial^\mu v(0) \quad \forall k \in ((M^T)^{-1}\mathbb{Z}^s)/\mathbb{Z}^s, \ |\mu| < m.$$

If $v(0)\widehat{\phi}(0) \neq 0$, then for any nonnegative integer n, there exists an invertible $r \times r$ matrix $U(\xi)$ of 2π-periodic trigonometric polynomials such that the mask $\widetilde{a}(\xi) = U(M^T\xi)a(\xi)U(\xi)^{-1}$ satisfies

$$\widetilde{a}_{1,1}(0) = 1,\ \partial^\mu \widetilde{a}_{1,1}(2\pi k) = 0 \quad \forall k \in (M^T)^{-1}\mathbb{Z}^s \backslash \mathbb{Z}^s,\ |\mu| < m,$$

$$\partial^\mu \widetilde{a}_{1,\ell}(2\pi k) = 0 \quad \forall k \in (M^T)^{-1}\mathbb{Z}^s,\ |\mu| < m,\ \ell = 2, \dots, r,$$

$$\partial^\mu \widetilde{a}_{\ell,1}(2\pi k) = 0 \quad \forall k \in \mathbb{Z}^s,\ |\mu| < n,\ \ell = 2, \dots, r.$$

Moreover, if the matrix mask a satisfies the above conditions, then $\partial^\mu \widehat{\phi}_\ell(0) = 0$ for all $\ell = 2, \dots, r$ and $|\mu| < n$, that is, all the functions $\phi_2, \dots, \phi_r$ have vanishing moments of order n.

As mentioned in Section 1.1, symmetry is very important in many applications of wavelet systems. In order to construct symmetric pairs of dual d-wavelet frames from symmetric d-refinable function vectors, in the rest of this section, let us discuss how symmetry can be preserved in our canonical form of a matrix mask in Theorems 1.1 and 1.2.

For a function ϕ on $\mathbb{R}$, if $\phi(c-x) = \pm\overline{\phi(x)}$, then we say that ϕ is **symmetric** about the point $c/2$ and the **symmetry center** of ϕ is $c/2$. Especially, if $\phi(c-x) = -\overline{\phi(x)}$, then we say that ϕ is **antisymmetric** about the point $c/2$ and the **symmetry center** of ϕ is $c/2$. It is easy to see that $\phi(c-x) = \pm\overline{\phi(x)}$ for all $x \in \mathbb{R}$ if and only if $\overline{\widehat{\phi}(\xi)} = \pm e^{ic\xi}\widehat{\phi}(\xi)$ for all $\xi \in \mathbb{R}$. Similarly, ϕ is a real-valued function, that is, $\overline{\phi(x)} = \phi(x)$ for all $x \in \mathbb{R}$, if and only if $\overline{\widehat{\phi}(\xi)} = \widehat{\phi}(-\xi)$ for all $\xi \in \mathbb{R}$.

We say that a matrix mask a with multiplicity r is **symmetric** if there exist $c_1, \dots, c_r \in \mathbb{R}$ and $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 1\}$ such that

$$\overline{a(\xi)} = S(d\xi)a(\xi)S(\xi)^{-1} \text{ with } S(\xi) = \operatorname{diag}(\varepsilon_1 e^{ic_1\xi}, \dots, \varepsilon_r e^{ic_r\xi}). \tag{1.2.18}$$

It is evident that a is a real-valued matrix mask if and only if $\overline{a(-\xi)} = a(\xi)$. By a simple computation, we have the following result.

Lemma 1.3. *Let $\phi = (\phi_1, \dots, \phi_r)^T$ be a d-refinable function vector with a finitely supported mask a on $\mathbb{Z}$. If*

$$(1.2.19)\qquad \begin{aligned} \phi_j(c_j - x) = \varepsilon_j \overline{\phi(x)} \qquad & \forall\, x \in \mathbb{R},\ j = 1, \dots, r \\ \textit{for some}\quad c_j \in \mathbb{R} \quad & \textit{and} \quad \varepsilon_j \in \{-1, 1\}, \end{aligned}$$

then $\overline{\widehat{\phi}(\xi)} = S(\xi)\widehat{\phi}(\xi)$ and $\widehat{\phi}(d\xi) = S(d\xi)^{-1}\overline{a(\xi)}S(\xi)\widehat{\phi}(\xi)$, where

$$S(\xi) := \operatorname{diag}(\varepsilon_1 e^{ic_1\xi}, \dots, \varepsilon_r e^{ic_r\xi}).$$

Conversely, if a is symmetric and (1.2.18) *is satisfied, then $\overline{\widehat{\phi}(\xi)} = S(\xi)\widehat{\phi}(\xi)$ and* (1.2.19) *holds.*

Now by the following result, we can see how the symmetry of a matrix mask can be preserved in our canonical form of a matrix mask.

Theorem 1.5. *Let a be a matrix mask with multiplicity r satisfying all the conditions in Theorem 1.2. If a is symmetric and satisfies* (1.2.18), *then there exists an invertible $r \times r$ matrix $U(\xi)$ of 2π-periodic trigonometric polynomials such that $b(\xi) := U(d\xi)a(\xi)U(\xi)^{-1}$ takes the form of* (1.2.15) *and the mask b is symmetric. Moreover, if a is a real-valued mask, that is $\overline{a(-\xi)} = a(\xi)$, then both U and b are real-valued sequences; that is, $\overline{U(-\xi)} = U(\xi)$ and $\overline{b(-\xi)} = b(\xi)$.*

Proof: Since a satisfies the sum rules of order m, by definition, there exists a $1 \times r$ row vector $v(\xi)$ of 2π-periodic trigonometric polynomials such that $v(0) \neq 0$ and

$$(1.2.20)\qquad \begin{aligned} & [v(d\cdot)a(\cdot)]^{(j)}(2\pi k/d) = \delta_k v^{(j)}(0) \\ & \forall\, j = 0, \dots, m-1;\ k = 0, \dots, |d| - 1. \end{aligned}$$

By (1.2.18), we have $\overline{v(d\xi)}S(d\xi)a(\xi) = \overline{v(d\xi)a(\xi)}S(\xi)$. Therefore, it follows from (1.2.20) that

$$[\overline{v(d\cdot)}S(d\cdot)a(\cdot)]^{(j)}(2\pi k/d) = [\overline{v(d\cdot)a(\cdot)}S(\cdot)]^{(j)}(2\pi k/d) = \delta_k[\overline{v(\cdot)}S(\cdot)]^{(j)}(0)$$

for all $j = 0, \dots, m-1$ and $k = 0, \dots, |d|-1$. Since (1.2.20) implies $v(0)a(0) = v(0)$, it follows from (1.2.18) that $\overline{v(0)}S(0)a(0) = \overline{v(0)}S(0)$. Since 1 is a simple eigenvalue of $a(0)$ and $S(0) = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_r)$, we must have $\varepsilon\overline{v(0)}S(0) = v(0)$ for some $|\varepsilon| = 1$.

Take $y(\xi) = v(\xi) + \varepsilon\overline{v(\xi)}S(\xi)$. Then $y(0) = 2v(0) \neq 0$ and it is easy to see that (1.2.7) holds. Denote $[y_1(\xi), \dots, y_r(\xi)] = y(\xi)$. Then $y_\ell(\xi) = \varepsilon\varepsilon_\ell e^{ic_\ell\xi}\overline{y_\ell(\xi)}$ for all $\ell = 1, \dots, r$. Without loss of generality, we may assume that $y_1(0) = 1$. Otherwise, since $y(0) \neq 0$, there is a permutation matrix U_0 such that $[y(0)U_0]_1 \neq 0$, where $[y(0)U_0]_1$ denotes the first component of $y(0)U_0$. Then we can replace $y(\xi)$ by $y(\xi)U_0/[y(0)U_0]_1$ and replace $a(\xi)$ by $U_0^{-1}a(\xi)U_0$. Clearly, $U_0^{-1}a(\xi)U_0$ is still symmetric and by Lemma 1.1 satisfies the sum rules of order m.

As in the proof of Theorem 1.1, since $y_1(0) = 1$, there exist 2π-periodic trigonometric polynomials $f_2, \dots, f_r$ such that

$$(1.2.21) \qquad [y_\ell(\cdot) - f_\ell(\cdot)y_1(\cdot)]^{(j)}(0) = 0 \qquad \forall\, \ell = 2, \dots, r;\ j = 0, \dots, m-1.$$

Note that

$$\begin{aligned} y_\ell(\xi) - \varepsilon_1\varepsilon_\ell e^{i(c_\ell - c_1)\xi}\overline{f_\ell(\xi)}y_1(\xi) &= \varepsilon\varepsilon_\ell e^{ic_\ell\xi}\overline{y_\ell(\xi)} - \varepsilon_1\varepsilon_\ell e^{i(c_\ell - c_1)\xi}\overline{f_\ell(\xi)}\varepsilon\varepsilon_1 e^{ic_1\xi}\overline{y_1(\xi)} \\ &= \varepsilon\varepsilon_\ell e^{ic_\ell\xi}\overline{[y_\ell(\xi) - f_\ell(\xi)y_1(\xi)]}. \end{aligned}$$

Let $\widetilde{f}_\ell(\xi) = f_\ell(\xi)/2 + \varepsilon_1\varepsilon_\ell e^{i(c_\ell - c_1)\xi}\overline{f_\ell(\xi)}/2$, for $\ell = 2, \dots, r$. Then (1.2.21) still holds with f_ℓ being replaced by $\widetilde{f}_\ell$ for $\ell = 2, \dots, r$. Define $U_1(\xi) = \begin{bmatrix} 1 & \widetilde{f}(\xi) \\ 0 & I_{r-1} \end{bmatrix}$ with $\widetilde{f}(\xi) = [\widetilde{f}_2(\xi), \dots, \widetilde{f}_r(\xi)]$. As in the proof of Theorem 1.1, $\widetilde{a}(\xi) := U_1(d\xi)a(\xi)U_1(\xi)^{-1}$ must take the form of (1.2.11). Since $\widetilde{f}_\ell(\xi) = \varepsilon_1\varepsilon_\ell e^{i(c_\ell - c_1)\xi}\overline{\widetilde{f}_\ell(\xi)}$, it is easy to verify that $\overline{U_1(\xi)} = S(\xi)U_1(\xi)S(\xi)^{-1}$. Consequently,

$$\begin{aligned} \overline{\widetilde{a}(\xi)} &= \overline{U_1(d\xi)}\,\overline{a(\xi)}\,\overline{U_1(\xi)^{-1}} \\ &= S(d\xi)U_1(d\xi)S(d\xi)^{-1}S(d\xi)a(\xi)S(\xi)^{-1}S(\xi)U_1(\xi)^{-1}S(\xi)^{-1} \\ &= S(d\xi)U_1(d\xi)a(\xi)U_1(\xi)^{-1}S(\xi)^{-1} = S(d\xi)\widetilde{a}(\xi)S(\xi)^{-1}. \end{aligned}$$

So $\widetilde{a}$ is symmetric.

Let $g(\xi) = [g_2(\xi), \ldots, g_r(\xi)]^T$ be given in the proof of Theorem 1.2 such that (1.2.16) is satisfied. Define $S_1(\xi) := \mathrm{diag}(c_2 e^{ic_2\xi}, \ldots, c_r e^{ic_r\xi})$. Since $\overline{\widetilde{a}(\xi)} = S(d\xi)\widetilde{a}(\xi)S(\xi)^{-1}$, we have

$$\overline{\widetilde{a}_{1,1}(\xi)} = e^{i(d-1)c_1\xi}\widetilde{a}_{1,1}(\xi), \qquad \overline{\widetilde{a}_{1,2}(\xi)} = \varepsilon_1 e^{idc_1\xi}\widetilde{a}_{1,2}(\xi)S_1(\xi)^{-1}$$

and

$$\overline{\widetilde{a}_{2,1}(\xi)} = \varepsilon_1 e^{-ic_1\xi}S_1(d\xi)\widetilde{a}_{2,1}(\xi), \qquad \overline{\widetilde{a}_{2,2}(\xi)} = S_1(d\xi)\widetilde{a}_{2,2}(\xi)S_1(\xi)^{-1}.$$

Let $\widetilde{g}(\xi) = \varepsilon_1 e^{ic_1\xi}S_1(\xi)^{-1}\overline{g(\xi)}$. Then

$$\begin{aligned}
&\widetilde{g}(d\xi)\widetilde{a}_{1,1}(\xi) + \widetilde{a}_{2,1}(\xi) - \widetilde{a}_{2,2}(\xi)\widetilde{g}(\xi) - \widetilde{g}(d\xi)\widetilde{a}_{1,2}(\xi)\widetilde{g}(\xi) \\
&= \varepsilon_1 e^{idc_1\xi}S_1(d\xi)^{-1}\overline{g(d\xi)}e^{-i(d-1)c_1\xi}\overline{\widetilde{a}_{1,1}(\xi)} + \varepsilon_1 e^{ic_1\xi}S_1(d\xi)^{-1}\overline{\widetilde{a}_{2,1}(\xi)} - \\
&\quad S_1(d\xi)^{-1}\overline{\widetilde{a}_{2,2}(\xi)}S_1(\xi)\varepsilon_1 e^{ic_1\xi}S_1(\xi)^{-1}\overline{g(\xi)} - \\
&\quad \varepsilon_1 e^{idc_1\xi}S_1(d\xi)^{-1}\overline{g(d\xi)}\varepsilon_1 e^{-idc_1\xi}\overline{\widetilde{a}_{1,2}(\xi)}S_1(\xi)\varepsilon_1 e^{ic_1\xi}S_1(\xi)^{-1}\overline{g(\xi)} \\
&= \varepsilon_1 e^{ic_1\xi}S_1(d\xi)^{-1}\overline{g(d\xi)}\,\overline{\widetilde{a}_{1,1}(\xi)} + \varepsilon_1 e^{ic_1\xi}S_1(d\xi)^{-1}\overline{\widetilde{a}_{2,1}(\xi)} - \\
&\quad \varepsilon_1 e^{ic_1\xi}S_1(d\xi)^{-1}\overline{\widetilde{a}_{2,2}(\xi)}\,\overline{g(\xi)} - \varepsilon_1 e^{ic_1\xi}S_1(d\xi)^{-1}\overline{g(d\xi)}\,\overline{\widetilde{a}_{1,2}(\xi)}\,\overline{g(\xi)} \\
&= \varepsilon_1 e^{ic_1\xi}S_1(d\xi)^{-1}\overline{\left[g(d\xi)\widetilde{a}_{1,1}(\xi) + \widetilde{a}_{2,1}(\xi) - \widetilde{a}_{2,2}(\xi)g(\xi) - g(d\xi)\widetilde{a}_{1,2}(\xi)g(\xi)\right]}.
\end{aligned}$$

Therefore, (1.2.16) still holds with g being replaced by $(g+\widetilde{g})/2$. Let

$$U_2(\xi) = \begin{bmatrix} 1 & 0 \\ g(\xi)/2 + \widetilde{g}(\xi)/2 & I_{r-1} \end{bmatrix}.$$

As in the proof of Theorem 1.2, $b(\xi) = U_2(d\xi)\widetilde{a}(\xi)U_2(\xi)^{-1}$ must take the form of (1.2.15). Since $g(\xi) + \widetilde{g}(\xi) = \varepsilon_1 e^{ic_1\xi}S_1(\xi)^{-1}\overline{[g(\xi) + \widetilde{g}(\xi)]}$, it is easy to check that $\overline{U_2(\xi)} = S(\xi)U_2(\xi)S(\xi)^{-1}$ and therefore, one can verify that $\overline{b(\xi)} = S(d\xi)b(\xi)S(\xi)^{-1}$. So the mask b is symmetric. Take $U(\xi) = U_2(\xi)U_1(\xi)$. Then $b(\xi) = U(d\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.15) and is symmetric.

When a is a real-valued mask, one can replace every matrix $V(\xi)$ of 2π-periodic trigonometric polynomials in the above proof by the corresponding matrix $[V(\xi) + \overline{V(-\xi)}]/2$. That is, for every sequence in the above proof, we can ignore the imaginary part of the sequence. So, both U and b are real-valued sequences. ■

Finally, I would like to point out that in Theorems 1.2 and 1.5, the condition "1 is a simple eigenvalue of $a(0)$ and d^k is not an eigenvalue of $a(0)$ for all $k \in \mathbb{N}$" can be changed into a much weaker condition "$y(0)\widehat{\phi}(0) \neq 0$", where y satisfies (1.2.7). The reason is as follows. Use all the same notations as in the proof of Theorem 1.2 and define $\widehat{\widetilde{\phi}}(\xi) := U_1(\xi)\widehat{\phi}(\xi)$, $b(\xi) := U_2(d\xi)U_1(d\xi)a(\xi)U_1(\xi)^{-1}U_2(\xi)^{-1}$ and $\widehat{\eta}(\xi) := U_2(\xi)U_1(\xi)\widehat{\phi}(\xi)$. In the proof of Theorem 1.2, the main idea is to construct some suitable $g(\xi)$. In order to construct such $g(\xi)$, one needs to solve some linear equations in (1.2.17) for $g^{(j)}(0)$, $j = 0, \ldots, n-1$. The condition "1 is a simple eigenvalue of $a(0)$ and d^k is not an eigenvalue of $a(0)$ for all $k \in \mathbb{N}$" is to guarantee that one has a unique solution $g^{(j)}(0)$, $j = 0, \ldots, n-1$, to the linear system in (1.2.17). The new condition that $y(0)\widehat{\phi}(0) \neq 0$ can guarantee that (1.2.17) has a solution but may not be unique. The reason is as follows. By Lemma 1.1, $\widetilde{y}(0)\widehat{\widetilde{\phi}}(0) = y(0)\widehat{\phi}(0) \neq 0$. By the proof of Theorem 1.1, we know that $\widetilde{y}(0) = [\widetilde{y}_1(0), 0, \ldots, 0]$. Hence $\widehat{\widetilde{\phi}}_1(0) \neq 0$. Notice that

$$\widehat{\eta}(\xi) = U_2(\xi)\widehat{\widetilde{\phi}}(\xi) = [\widehat{\widetilde{\phi}}_1(\xi), \widehat{\widetilde{\phi}}_2(\xi) + \widehat{\widetilde{\phi}}_1(\xi)g_2(\xi), \ldots, \widehat{\widetilde{\phi}}_r(\xi) + \widehat{\widetilde{\phi}}_1(\xi)g_r(\xi)]^T.$$

Then by the Leibniz differentiation formula and $\widehat{\widetilde{\phi}}_1(0) \neq 0$, one can prove that there exists an $(r-1) \times 1$ column vector $g(\xi)$ of 2π-periodic trigonometric polynomials such that for every $j = 0, \ldots, n-1$,

$$[\widehat{\widetilde{\phi}}_\ell(\cdot) + \widehat{\widetilde{\phi}}_1(\cdot)g_\ell(\cdot)]^{(j)}(0) = 0 \qquad \forall\, \ell = 2, \ldots, r;\ j = 0, \ldots, n-1.$$

So for every $j = 0, \ldots, n-1$,

$$\widehat{\eta}^{(j)}(0) = [\widehat{\eta}_1^{(j)}(0), 0, \ldots, 0]^T. \tag{1.2.22}$$

By (1.2.22), it follows from the refinable equation $\widehat{\eta}(d\xi) = b(\xi)\widehat{\eta}(\xi)$ and the Leibniz differentiation formula that for every $j = 0, \ldots, n-1$,

$$b^{(j)}(0)[1, 0, \ldots, 0]^T = [b_{1,1}^{(j)}(0), 0, \ldots, 0]^T.$$

So $b(\xi)$ must take the form in (1.2.15). However, it is much easier to solve the linear system in (1.2.17) to obtain $g(\xi)$ than using the above procedure which involves computing the derivatives of $\widehat{\tilde{\phi}}$ at the origin.

1.3 Construction of multiwavelet frames

Since Section 1.2 gave an elegant canonical form of the matrix mask with certain order of sum rules, in this section, by applying the results in Section 1.2, we can investigate how to construct multiwavelet frames with certain vanishing moments from a given refinable function vector.

1.3.1 Algorithm for constructing pairs of dual wavelet frames from refinable function vectors

In this subsection, I shall generalize the construction of pairs of dual wavelet frames in [4, 12, 13] for the case $r = 1$ to the multiwavelet case with a general dilation factor.

By $\left(L_2(\mathbb{R})\right)^r$ we denote the set of all $r \times 1$ column vectors of functions in $L_2(\mathbb{R})$. Given a matrix A, we denote A^T the transpose of A and A^* the transpose of the complex conjugate of A.

By generalizing the results in [4, 12, 13], we have the following theorem.

Theorem 1.6. *Let ϕ and $\tilde{\phi}$ be two $r \times 1$ d-refinable function vectors in $\left(L_2(\mathbb{R})\right)^r$ with dilation factor d and finitely supported masks a and b, respectively. Suppose that there are $r \times r$ matrices $\Theta, a^1, \ldots, a^L, b^1, \ldots, b^L$ of 2π-periodic trigonometric polynomials such that*

$$\widehat{\phi}(0)^*\Theta(0)\widehat{\tilde{\phi}}(0) = 1, \qquad a(0)^*\Theta(0)\widehat{\tilde{\phi}}(0) = \Theta(0)\widehat{\tilde{\phi}}(0), \tag{1.3.23}$$

$$a^\ell(0)\widehat{\phi}(0) = b^\ell(0)\widehat{\tilde{\phi}}(0) = 0 \qquad \forall\, \ell = 1, \ldots, L \tag{1.3.24}$$

and

$$(1.3.25)\qquad \begin{bmatrix} a^1(\xi)^* & \cdots & a^L(\xi)^* \\ a^1(\xi+\frac{2\pi}{d})^* & \cdots & a^L(\xi+\frac{2\pi}{d})^* \\ \vdots & \ddots & \vdots \\ a^1\Big(\xi+\frac{2\pi(|d|-1)}{d}\Big)^* & \cdots & a^L\Big(\xi+\frac{2\pi(|d|-1)}{d}\Big)^* \end{bmatrix} \begin{bmatrix} b^1(\xi) \\ b^2(\xi) \\ \vdots \\ b^L(\xi) \end{bmatrix} = M(\xi)$$

where

$$(1.3.26)\qquad M(\xi) := \begin{bmatrix} \Theta(\xi) - a(\xi)^*\Theta(d\xi)b(\xi) \\ -a(\xi+\frac{2\pi}{d})^*\Theta(d\xi)b(\xi) \\ \vdots \\ -a\Big(\xi+\frac{2\pi(|d|-1)}{d}\Big)^*\Theta(d\xi)b(\xi) \end{bmatrix}.$$

Define $r\times 1$ wavelet function vectors $\psi^1,\dots,\psi^L,\widetilde{\psi}^1,\dots,\widetilde{\psi}^L$ as follows

$$(1.3.27)\qquad \widehat{\psi^\ell}(d\xi) = a^\ell(\xi)\widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\widetilde{\psi}^\ell}(d\xi) = b^\ell(\xi)\widehat{\widetilde{\phi}}(\xi), \qquad \ell = 1,\dots,L.$$

Then $\{\psi^1,\dots,\psi^L\}$ and $\{\widetilde{\psi}^1,\dots,\widetilde{\psi}^L\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbb{R})$.

Proof: Note that (1.3.24) implies that $\widehat{\psi^1}(0) = \cdots = \widehat{\psi^L}(0) = \widehat{\widetilde{\psi}^1}(0) = \cdots = \widehat{\widetilde{\psi}^L}(0) = 0$. Since both ϕ and $\widetilde{\phi}$ are compactly supported d-refinable functions in $(L_2(\mathbb{R}))^r$ with finitely supported matrix masks, by [21, Theorem 2.3], there exists a positive constant C such that

$$\sum_{\ell=1}^{L}\sum_{m=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \Big[|\langle f,\psi^\ell_{m;j,k}\rangle|^2 + |\langle f,\widetilde{\psi}^\ell_{m;j,k}\rangle|^2\Big] \leqslant C\|f\|^2 \qquad \forall\, f\in L_2(\mathbb{R})$$

where $\psi^\ell_{m;j,k} := |d|^{j/2}\psi^\ell_m(d^j\cdot - k)$ and ψ^ℓ_m denotes the m-th entry in the column vector ψ^ℓ.

By slightly modifying the original proof in [12] (the first version) for the scalar case, we generalize the proof in [12] to the multiwavelet case.

Note that for $\ell = 1,\dots,L$, we have

$$\widehat{\psi^\ell_{j,k}}(\xi) = |d|^{-j/2}e^{-ikd^{-j}\xi}\widehat{\psi^\ell}(d^{-j}\xi).$$

By the Plancherel theorem and the Parseval identity, we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\langle f,\psi^\ell_{j,k}\rangle^T\langle\widetilde{\psi}^\ell_{j,k},g\rangle\\
=&\sum_{k\in\mathbb{Z}}\frac{|d|^j}{4\pi^2}\int_{\mathbb{T}}\sum_{\alpha\in\mathbb{Z}}\widehat{f}\big(d^j(\xi+2\pi\alpha)\big)\widehat{\psi^\ell}(\xi+2\pi\alpha)^*e^{ik\xi}d\xi\times\\
&\qquad\int_{\mathbb{T}}\sum_{\beta\in\mathbb{Z}}\overline{\widehat{g}\big(d^j(\xi+2\pi\beta)\big)}\ \widehat{\widetilde{\psi}^\ell}(\xi+2\pi\beta)e^{-ik\xi}d\xi\\
=&\frac{|d|^j}{2\pi}\int_{\mathbb{T}}\sum_{\alpha\in\mathbb{Z}}\widehat{f}\big(d^j(\xi+2\pi\alpha)\big)\widehat{\psi^\ell}(\xi+2\pi\alpha)^*\times\\
&\qquad\sum_{\beta\in\mathbb{Z}}\overline{\widehat{g}\big(d^j(\xi+2\pi\beta)\big)}\ \widehat{\widetilde{\psi}^\ell}(\xi+2\pi\beta)d\xi \qquad (1.3.28)\\
=&\frac{|d|^j}{2\pi}\int_{\mathbb{R}}\sum_{\alpha\in\mathbb{Z}}\widehat{f}\big(d^j(\xi+2\pi\alpha)\big)\widehat{\psi^\ell}(\xi+2\pi\alpha)^*\overline{\widehat{g}(d^j\xi)}\ \widehat{\widetilde{\psi}^\ell}(\xi)\,d\xi\\
=&\frac{|d|^j}{2\pi}\int_{\mathbb{R}}\sum_{m=0}^{|d|-1}\sum_{k\in\mathbb{Z}}\widehat{f}\big(d^j(\xi+2\pi m+d2\pi k)\big)\widehat{\phi}\Big(\frac{\xi+2\pi m}{d}+2\pi k\Big)^*\times\\
&\qquad a^\ell\Big(\frac{\xi+2\pi m}{d}\Big)^*\overline{\widehat{g}(d^j\xi)}\ b^\ell\Big(\frac{\xi}{d}\Big)\widehat{\widetilde{\phi}}\Big(\frac{\xi}{d}\Big)\\
=&\frac{|d|^j}{2\pi}\int_{\mathbb{R}}\sum_{m=0}^{|d|-1,}h\Big(\frac{\xi+2\pi m}{d}\Big)a^\ell\Big(\frac{\xi+2\pi m}{d}\Big)^*\overline{\widehat{g}(d^j\xi)}\ b^\ell\Big(\frac{\xi}{d}\Big)\widehat{\widetilde{\phi}}\Big(\frac{\xi}{d}\Big)\,d\xi
\end{aligned}
$$

where

$$h(\xi)=\sum_{k\in\mathbb{Z}}\widehat{f}\big(d^{j+1}(\xi+2\pi k)\big)\widehat{\phi}(\xi+2\pi k)^*. \qquad (1.3.29)$$

Note that (1.3.25) can be rewritten as follows:

$$\sum_{\ell=1}^{L}a^\ell(\xi)^*b^\ell(\xi)=\Theta(\xi)-a(\xi)^*\Theta(d\xi)b(\xi)$$

and

$$\sum_{\ell=1}^{L}a^\ell\Big(\xi+\frac{2\pi m}{d}\Big)^*b^\ell(\xi)=-a\Big(\xi+\frac{2\pi m}{d}\Big)^*\Theta(d\xi)b(\xi),\qquad m=1,\cdots,|d|-1.$$

Therefore, we have

$$
\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\langle f,\psi_{j,k}^{\ell}\rangle^{T}\langle\widetilde{\psi}_{j,k}^{\ell},g\rangle\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{m=0}^{|d|-1}h\Big(\frac{\xi+2\pi m}{d}\Big)\sum_{\ell=1}^{L}a^{\ell}\Big(\frac{\xi+2\pi m}{d}\Big)^{*}b^{\ell}\Big(\frac{\xi}{d}\Big)\widehat{\widetilde{\phi}}\Big(\frac{\xi}{d}\Big)\,d\xi\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}h\Big(\frac{\xi}{d}\Big)\Big[\Theta\Big(\frac{\xi}{d}\Big)-a\Big(\frac{\xi}{d}\Big)^{*}\Theta(\xi)b\Big(\frac{\xi}{d}\Big)\Big]\widehat{\widetilde{\phi}}\Big(\frac{\xi}{d}\Big)\,d\xi-\\
&\quad\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{m=1}^{|d|-1}h\Big(\frac{\xi+2\pi m}{d}\Big)a\Big(\frac{\xi+2\pi m}{d}\Big)^{*}\Theta(\xi)b\Big(\frac{\xi}{d}\Big)\widehat{\widetilde{\phi}}\Big(\frac{\xi}{d}\Big)\,d\xi\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}h\Big(\frac{\xi}{d}\Big)\widehat{\widetilde{\eta}}\Big(\frac{\xi}{d}\Big)\,d\xi-\\
&\quad\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{m=0}^{|d|-1}h\Big(\frac{\xi+2\pi m}{d}\Big)a\Big(\frac{\xi+2\pi m}{d}\Big)^{*}\widehat{\widetilde{\eta}}(\xi)\,d\xi\\
=&\frac{|d|^{j+1}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j+1}\xi)}h(\xi)\widehat{\widetilde{\eta}}(\xi)\,d\xi-\\
&\quad\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{m=0}^{|d|-1}h\Big(\frac{\xi+2\pi m}{d}\Big)a\Big(\frac{\xi+2\pi m}{d}\Big)^{*}\widehat{\widetilde{\eta}}(\xi)\,d\xi
\end{aligned}
$$

where $\widehat{\widetilde{\eta}}(\xi)=\Theta(\xi)\widehat{\widetilde{\phi}}(\xi)$.

Using a similar argument as in (1.3.28), we deduce that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\langle f,\phi_{j,k}\rangle^{T}\langle\widetilde{\eta}_{j,k},g\rangle\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{k\in\mathbb{Z}}\widehat{f}\big(d^{j}(\xi+2\pi k)\big)\widehat{\phi}(\xi+2\pi k)^{*}\widehat{\widetilde{\eta}}(\xi)\,d\xi\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{k\in\mathbb{Z}}\widehat{f}\big(d^{j}(\xi+2\pi k)\big)\widehat{\phi}\Big(\frac{\xi+2\pi k}{d}\Big)^{*}a\Big(\frac{\xi+2\pi k}{d}\Big)^{*}\widehat{\widetilde{\eta}}(\xi)\,d\xi\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{m=0}^{|d|-1}\sum_{k\in\mathbb{Z}}\widehat{f}\left(d^{j+1}\Big(\frac{\xi+2\pi m}{d}+2\pi k\Big)\right)\times\\
&\quad\widehat{\phi}\Big(\frac{\xi+2\pi m}{d}+2\pi k\Big)^{*}a\Big(\frac{\xi+2\pi m}{d}\Big)^{*}\widehat{\widetilde{\eta}}(\xi)\,d\xi\\
=&\frac{|d|^{j}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j}\xi)}\sum_{m=0}^{|d|-1}h\Big(\frac{\xi+2\pi m}{d}\Big)a\Big(\frac{\xi+2\pi m}{d}\Big)^{*}\widehat{\widetilde{\eta}}(\xi)\,d\xi
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\langle f,\phi_{j+1,k}\rangle^T\langle\widetilde{\eta}_{j+1,k},g\rangle\\
=&\frac{|d|^{j+1}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j+1}\xi)}\sum_{k\in\mathbb{Z}}\widehat{f}(d^{j+1}(\xi+2\pi k))\widehat{\phi}(\xi+2\pi k)^*\widehat{\widetilde{\eta}}(\xi)\,d\xi\\
=&\frac{|d|^{j+1}}{2\pi}\int_{\mathbb{R}}\overline{\widehat{g}(d^{j+1}\xi)}h(\xi)\widehat{\widetilde{\eta}}(\xi)\,d\xi
\end{aligned}
$$

where the function $h(\xi)$ is defined in (1.3.29). Hence

$$
\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{m=1}^{r}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{m;j,k}\rangle\langle\widetilde{\psi}^{\ell}_{m;j,k},g\rangle\\
=&\sum_{m=1}^{r}\sum_{k\in\mathbb{Z}}\langle f,\phi_{m;j+1,k}\rangle\langle\eta_{m;j+1,k},g\rangle-\sum_{m=1}^{r}\sum_{k\in\mathbb{Z}}\langle f,\phi_{m;j,k}\rangle\langle\eta_{m;j,k},g\rangle
\end{aligned}
$$

for all $f,g\in L_2(\mathbb{R})$.

Note that for all $f\in L_2(\mathbb{R})$,

$$
\lim_{j\to-\infty}\sum_{m=1}^{r}\sum_{k\in\mathbb{Z}}\left[|\langle f,\phi_{m;j,k}\rangle|^2+|\langle\eta_{m;j,k},f\rangle|^2\right]=0.
$$

By (1.3.23), it is well known (see [14]) that $\sum_{m=1}^{r}\sum_{k\in\mathbb{Z}}\langle f,\phi_{m;j,k}\rangle\eta_{m;j,k}$ goes to f in the L_2 norm as j goes to ∞. Therefore,

$$
\sum_{\ell=1}^{L}\sum_{m=1}^{r}\sum_{j,k\in\mathbb{Z}}\langle f,\psi^{\ell}_{m;j,k}\rangle\langle\widetilde{\psi}^{\ell}_{m;j,k},g\rangle=\langle f,g\rangle
$$

which completes the proof. ∎

From the above proof, we see that the condition $a(0)^*\Theta(0)\widehat{\widetilde{\phi}}(0)=\Theta(0)\widehat{\widetilde{\phi}}(0)$ in Theorem 1.6 can be replaced by $b(0)^*\Theta(0)^*\widehat{\phi}(0)=\Theta(0)^*\widehat{\phi}(0)$. Moreover, as been pointed out by Chui and Stöckler ([7]) later, the condition

$$
a(0)^*\Theta(0)\widehat{\widetilde{\phi}}(0)=\Theta(0)\widehat{\widetilde{\phi}}(0)
$$

in (1.3.23) is a direct conclusion of (1.3.24) and (1.3.25).

Theorem 1.6 can be easily generalized to the multidimensional spaces by the same argument as in the proof of Theorem 1.6. The only little difference can be fixed by the following argument. Let M be an $s \times s$ integer matrix such that all its eigenvalues are greater than one in modulus. Suppose that $\phi = (\phi_1, \dots, \phi_r)^T \in \left(L_2(\mathbb{R}^s)\right)^r$ is compactly supported and $\widehat{\phi}(M^T\xi) = a(\xi)\widehat{\phi}(\xi)$ for some $r \times r$ matrix $a(\xi)$ of 2π-periodic trigonometric polynomials. Then it was proved in Han [21] that there exists $\alpha > 0$ such that $\int_{\mathbb{R}^s}(1 + \|\xi\|^2)^\alpha |\widehat{\phi}_m(\xi)|^2\, d\xi < \infty$ and $(1 + \|\xi\|)^\alpha \widehat{\phi}_m \in L_\infty$ for all $m = 1, \dots, r$. Moreover, by Han [21], we know that for any $\psi = (\psi_1, \dots, \psi_r)^T$ which is defined by $\widehat{\psi}(M^T\xi) = b(\xi)\widehat{\phi}(\xi)$ for some $r \times r$ matrix $b(\xi)$ of 2π-periodic trigonometric polynomials, if $\int_{\mathbb{R}^s} \psi_m(t)\, dt = 0$ for all $m = 1, \dots, r$, then there exists a positive constant C such that $\sum_{m=1}^r \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^s} |\langle f, \psi_{m;j,k}\rangle|^2 \leqslant C\|f\|^2$ for all $f \in L_2(\mathbb{R}^s)$, where $\psi_{m;j,k} := |\det M|^{j/2}\psi_m(M^j \cdot - k)$.

1.3.2 Existence and construction of pairs of dual wavelet frames with high vanishing moments

As mentioned in Section 1.1, the order of vanishing moments is a very important property for wavelet frames. In this subsection, I shall demonstrate that from any two d-refinable function vectors in $\left(L_2(\mathbb{R})\right)^r$, by Theorem 1.6, one can obtain a pair of dual d-wavelet frames with the highest possible orders of vanishing moments. Also, I shall apply Theorem 1.6 with the particular choice $L = d$. Given any two matrix masks a and b, in order to derive dual d-wavelet frames by Theorem 1.6 with $L = d$, we need construct matrices $\Theta, a^1, \dots, a^d, b^1, \dots, b^d$ of 2π-periodic trigonometric polynomials such that all the conditions in Theorem 1.6 are satisfied and all the corresponding wavelet functions $\psi^1, \dots, \psi^d, \widetilde{\psi}^1, \dots, \widetilde{\psi}^d$ defined in (1.3.27) have the highest possible orders of vanishing moments.

The following lemma generalizes a result in [12] which plays an important role in constructing pairs of dual d-wavelet frames from any two d-refinable function vectors.

Lemma 1.4. *Let d be a dilation factor. Let $A(\xi)$ and $B(\xi)$ be two 2π-periodic trigonometric polynomials such that $A(0) = B(0) = 1$. Then for any positive integer n, there exists a 2π-periodic trigonometric polynomial $\theta(\xi)$ such that*

$$\theta(0) = 1, \quad (1 - e^{-i\xi})^n \mid [\theta(\xi)A(\xi) - \theta(d\xi)B(\xi)]. \tag{1.3.30}$$

Moreover, when A and B have real coefficients, then so does θ.

Proof: We define coefficients c_j by $c_0 := 1$ and

$$c_j := \frac{1}{d^j - 1} \sum_{k=0}^{j-1} \binom{j}{k} [A^{(j-k)}(0) - d^k B^{(j-k)}(0)] c_k, \qquad j \in \mathbb{N}.$$

Obviously, there is a 2π-periodic trigonometric polynomial $\theta(\xi)$ such that $\theta^{(j)}(0) = c_j$ for all $j = 0, \ldots, n-1$. By the Leibniz differentiation formula, it is easy to verify that (1.3.30) holds for such a θ. ■

We now demonstrate that one can construct a pair of dual d-wavelet frames having certain vanishing moments derived from any two d-refinable function vectors. For simplicity of presentation, in the rest of this chapter, we assume that the dilation factor $d > 1$ and for any mask a, we assume that 1 is a simple eigenvalue of $a(0)$ and $d^k (k \in \mathbb{N})$ are not eigenvalues of $a(0)$.

Theorem 1.7. *Let ϕ and $\widetilde{\phi}$ be two $r \times 1$ d-refinable function vectors in $(L_2(\mathbb{R}))^r$ with dilation factor d and finitely supported masks a and b, respectively. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbb{Z}$ for some positive integers m and n, respectively. Then there are $r \times r$ matrices $\Theta, a^1, \ldots, a^d, b^1, \ldots, b^d$ of 2π-periodic trigonometric polynomials such that all the conditions in Theorem 1.6 are satisfied. Consequently, $\{\psi^1, \ldots, \psi^d\}$ and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$, which are defined in (1.3.27) for the special case $L = d$, generate a pair of dual d-wavelet frames in $L_2(\mathbb{R})$. Moreover, $\{\psi^1, \ldots, \psi^d\}$ has vanishing moments of order n and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ has vanishing moments of order m.*

Proof: Since a and b satisfy the sum rules of orders m and n respectively, by Lemma 1.1 and Theorem 1.2, without loss of generality, we can assume that

$$a(\xi) = \begin{bmatrix} a_{1,1}(\xi) & a_{1,2}(\xi) \\ a_{2,1}(\xi) & a_{2,2}(\xi) \end{bmatrix} \quad \text{and} \quad b(\xi) = \begin{bmatrix} b_{1,1}(\xi) & b_{1,2}(\xi) \\ b_{2,1}(\xi) & b_{2,2}(\xi) \end{bmatrix}$$

where $a_{2,2}(\xi)$ and $b_{2,2}(\xi)$ are $(r-1)\times(r-1)$ matrices of 2π-periodic trigonometric polynomials such that $a_{1,1}$, $a_{1,2}$, $a_{2,1}$, $a_{2,2}$, $b_{1,1}$, $b_{1,2}$, $b_{2,1}$ and $b_{2,2}$ satisfy $a_{1,1}(0) = b_{1,1}(0) = 1$ and

(1.3.31)

$$\begin{gathered}\left(1+\cdots+e^{-i(d-1)\xi}\right)^m \mid a_{1,1}(\xi), \quad (1-e^{-id\xi})^m \mid a_{1,2}(\xi), \quad (1-e^{-i\xi})^n \mid a_{2,1}(\xi), \\ \left(1+\cdots+e^{-i(d-1)\xi}\right)^n \mid b_{1,1}(\xi), \quad (1-e^{-id\xi})^n \mid b_{1,2}(\xi), \quad (1-e^{-i\xi})^m \mid b_{2,1}(\xi).\end{gathered}$$

Since $a_{1,1}(0) = b_{1,1}(0) = 1$, by Lemma 1.4, there exists a 2π-periodic trigonometric polynomial $\theta(\xi)$ such that

$$\theta(0) = 1 \quad \text{and} \quad (1-e^{-i\xi})^{m+n} \mid [\theta(\xi) - \overline{a_{1,1}(\xi)}\theta(d\xi)b_{1,1}(\xi)]. \tag{1.3.32}$$

Now we choose $\Theta(\xi) = \operatorname{diag}(\theta(\xi), 0, \ldots, 0)$. (In fact, in the following proof, one can choose $\Theta := \operatorname{diag}(\theta, *)$ where $*$ denotes some $(r-1)\times(r-1)$ matrix of 2π-periodic trigonometric polynomials.) Let $M(\xi)$ be defined in (1.3.26). Define

$$\widetilde{M}(\xi) := \left[\operatorname{diag}\left(D(\xi)^*, \ldots, D(\xi + 2\pi(d-1)/d)^*\right)\right]^{-n} M(\xi) D(\xi)^{-m}, \tag{1.3.33}$$

where $D(\xi) := \operatorname{diag}\left((1-e^{-i\xi}), 1, \ldots, 1\right)$. We now demonstrate that $\widetilde{M}(\xi)$ is an $rd \times r$ matrix of 2π-periodic trigonometric polynomials. From (1.3.33), by calculation, we have

$$\widetilde{M}(\xi) = \begin{bmatrix} \widetilde{M}_1(\xi) \\ \widetilde{M}_2(\xi) \\ \vdots \\ \widetilde{M}_d(\xi) \end{bmatrix} = \begin{bmatrix} [D(\xi)^*]^{-n}[\Theta(\xi) - a(\xi)^*\Theta(d\xi)b(\xi)]D(\xi)^{-m} \\ -[D(\xi+\frac{2\pi}{d})^*]^{-n} a(\xi+\frac{2\pi}{d})^*\Theta(d\xi)b(\xi)D(\xi)^{-m} \\ \vdots \\ -[D(\xi+\frac{2\pi(d-1)}{d})^*]^{-n} a(\xi+\frac{2\pi(d-1)}{d})^*\Theta(d\xi)b(\xi)D(\xi)^{-m} \end{bmatrix}.$$

Since $\Theta(\xi) = \mathrm{diag}(\theta(\xi), 0, \ldots, 0)$, by calculation, we have

(1.3.34)

$$a\Big(\xi + \frac{2\pi j}{d}\Big)^* \Theta(d\xi) b(\xi) = \theta(d\xi) \begin{bmatrix} \overline{a_{1,1}(\xi + \frac{2\pi j}{d})} b_{1,1}(\xi) & \overline{a_{1,1}(\xi + \frac{2\pi j}{d})} b_{1,2}(\xi) \\ a_{1,2}(\xi + \frac{2\pi j}{d})^* b_{1,1}(\xi) & a_{1,2}(\xi + \frac{2\pi j}{d})^* b_{1,2}(\xi) \end{bmatrix}.$$

Note that $D(\xi) = \mathrm{diag}\big((1 - e^{-i\xi}), I_{r-1}\big)$. For any $j = 1, \ldots, d-1$, it follows from (1.3.31) and (1.3.34) that

(1.3.35)

$$\begin{aligned} \widetilde{M}_{j+1}(\xi) &= -\Big[D\Big(\xi + \frac{2\pi j}{d}\Big)^*\Big]^{-n} a\Big(\xi + \frac{2\pi j}{d}\Big)^* \Theta(d\xi) b(\xi) D(\xi)^{-m} \\ &= -\begin{bmatrix} (1 - e^{i(\xi + \frac{2\pi j}{d})})^{-n} & \\ & I_{r-1} \end{bmatrix} \Big[a\Big(\xi + \frac{2\pi j}{d}\Big)^* \Theta(d\xi) b(\xi)\Big] \begin{bmatrix} (1 - e^{-i\xi})^{-m} & \\ & I_{r-1} \end{bmatrix} \\ &= \theta(d\xi) \begin{bmatrix} \frac{(1-e^{id\xi})^m (1-e^{-id\xi})^n}{\left[(1-e^{-i\xi})\left(1-e^{i(\xi+2\pi j/d)}\right)\right]^{m+n}} * & \frac{(1-e^{id\xi})^m (1-e^{-id\xi})^n}{\left(1-e^{i(\xi+2\pi j/d)}\right)^{m+n}} * \\ \frac{(1-e^{id\xi})^m (1-e^{-id\xi})^n}{(1-e^{-i\xi})^{m+n}} * & * \end{bmatrix}, \end{aligned}$$

where $*$ denotes some matrix of 2π-periodic trigonometric polynomials. Observe that

$$\frac{(1 - e^{id\xi})^m (1 - e^{-id\xi})^n}{[(1 - e^{-i\xi})(1 - e^{i(\xi + 2\pi j/d)})]^{m+n}} = (-1)^m e^{-idn\xi} \left[\frac{e^{i\xi}(1 - e^{id\xi})}{(1 - e^{i\xi})(1 - e^{i(\xi + 2\pi j/d)})}\right]^{m+n}$$

is a 2π-periodic trigonometric polynomial since $(1 - e^{i\xi})\big(1 - e^{i(\xi + 2\pi j/d)}\big) \mid (1 - e^{id\xi})$ for all $j = 1, \ldots, d-1$. We conclude that $\widetilde{M}_j(\xi), j = 2, \ldots, d$ are matrices of 2π-periodic trigonometric polynomials.

Similarly, by (1.3.31) and (1.3.34), we have

(1.3.36)

$$\begin{aligned} \widetilde{M}_1(\xi) &= [D(\xi)^*]^{-n} [\Theta(\xi) - a(\xi)^* \Theta(d\xi) b(\xi)] D(\xi)^{-m} \\ &= \begin{bmatrix} (1 - e^{i\xi})^{-n} & \\ & I_{r-1} \end{bmatrix} \Big[\Theta(\xi) - a(\xi)^* \Theta(d\xi) b(\xi)\Big] \begin{bmatrix} (1 - e^{-i\xi})^{-m} & \\ & I_{r-1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\theta(\xi) - \overline{a_{1,1}(\xi)} \theta(d\xi) b_{1,1}(\xi)}{(1-e^{i\xi})^n (1-e^{-i\xi})^m} & \frac{(1-e^{id\xi})^m (1-e^{-id\xi})^n}{(1-e^{i\xi})^{m+n}} \theta(d\xi) * \\ \frac{(1-e^{id\xi})^m (1-e^{-id\xi})^n}{(1-e^{-i\xi})^{m+n}} \theta(d\xi) * & \theta(d\xi) * \end{bmatrix}. \end{aligned}$$

By (1.3.32), $(1 - e^{i\xi})^n (1 - e^{-i\xi})^m \mid [\theta(\xi) - \overline{a_{1,1}(\xi)} \theta(d\xi) b_{1,1}(\xi)]$. Therefore, $\widetilde{M}_1(\xi)$ is a matrix of 2π-periodic trigonometric polynomials. In conclusion,

$\widetilde{M}(\xi)$ is an $rd \times r$ matrix of 2π-periodic trigonometric polynomials. Define $F^\ell(\xi) := e^{-i(\ell-1)\xi} I_r, \ell = 1, \ldots, d$ and

(1.3.37)
$$E(\xi) := \begin{bmatrix} F^1(\xi)^* & F^2(\xi)^* & \cdots & F^d(\xi)^* \\ F^1(\xi + \frac{2\pi}{d})^* & F^2(\xi + \frac{2\pi}{d})^* & \cdots & F^d\big(\xi + \frac{2\pi}{d}\big)^* \\ \vdots & \vdots & \ddots & \vdots \\ F^1\big(\xi + \frac{2\pi(d-1)}{d}\big)^* & F^2\big(\xi + \frac{2\pi(d-1)}{d}\big)^* & \cdots & F^d\big(\xi + \frac{2\pi(d-1)}{d}\big)^* \end{bmatrix}.$$

Denote

$$E_0(\xi) := \begin{bmatrix} 1 & e^{i\xi} & \cdots & e^{i(d-1)\xi} \\ 1 & e^{i(\xi+2\pi/d)} & \cdots & e^{i(d-1)(\xi+2\pi/d)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{i\left(\xi+2\pi(d-1)/d\right)} & \cdots & e^{i(d-1)\left(\xi+2\pi(d-1)/d\right)} \end{bmatrix}. \tag{1.3.38}$$

It is well known that $E_0(\xi)E_0(\xi)^* = dI_d$. Observe that $E(\xi) = E_0(\xi) \otimes I_r$, where $\otimes$ denotes the right Kronecker product. Consequently, we have $E(\xi)E(\xi)^* = dI_{rd}$ and $E(\xi)$ is invertible. Define the $r \times r$ matrices $\widetilde{F}^1(\xi), \ldots, \widetilde{F}^d(\xi)$ of 2π-periodic trigonometric polynomials by

$$\begin{bmatrix} \widetilde{F}^1(\xi) \\ \vdots \\ \widetilde{F}^d(\xi) \end{bmatrix} = E(\xi)^{-1}\widetilde{M}(\xi) = d^{-1}E(\xi)^*\widetilde{M}(\xi), \tag{1.3.39}$$

since $E(\xi)E(\xi)^* = dI_{rd}$. Now for $\ell = 1, \ldots, d$, define

$$\begin{aligned} a^\ell(\xi) &= F^\ell(\xi)\mathrm{diag}\big((1-e^{-i\xi})^n, 1\ldots, 1\big), \\ b^\ell(\xi) &= \widetilde{F}^\ell(\xi)\mathrm{diag}\big((1-e^{-i\xi})^m, 1\ldots, 1\big). \end{aligned} \tag{1.3.40}$$

Due to (1.3.31), we must have $\widehat{\phi}(0) = \widehat{\widetilde{\phi}}(0) = [1, 0, \ldots, 0]^T$. It can be easily verified that all the conditions in Theorem 1.6 are satisfied. In fact, by calculation, we have

$$\begin{bmatrix} a^1(\xi)^* & a^2(\xi)^* & \cdots & a^d(\xi)^* \\ a^1(\xi+\frac{2\pi}{d})^* & a^2(\xi+\frac{2\pi}{d})^* & \cdots & a^d(\xi+\frac{2\pi}{d})^* \\ \vdots & \vdots & \ddots & \vdots \\ a^1\big(\xi+\frac{2\pi(d-1)}{d}\big)^* & a^2\big(\xi+\frac{2\pi(d-1)}{d}\big)^* & \cdots & a^d\big(\xi+\frac{2\pi(d-1)}{d}\big)^* \end{bmatrix} \begin{bmatrix} b^1(\xi) \\ b^2(\xi) \\ \vdots \\ b^d(\xi) \end{bmatrix}$$

$$= \Big[\mathrm{diag}\Big(D(\xi)^*, \ldots, D\big(\xi+2\pi(d-1)/d\big)^*\Big)\Big]^n E(\xi) \begin{bmatrix} \widetilde{F}^1(\xi) \\ \vdots \\ \widetilde{F}^d(\xi) \end{bmatrix} D(\xi)^m$$

$$= \Big[\mathrm{diag}\Big(D(\xi)^*, \ldots, D\big(\xi+2\pi(d-1)/d\big)^*\Big)\Big]^n E(\xi)E(\xi)^{-1}\widetilde{M}(\xi)D(\xi)^m$$

$$= \Big[\mathrm{diag}\Big(D(\xi)^*, \ldots, D\big(\xi+2\pi(d-1)/d\big)^*\Big)\Big]^n \widetilde{M}(\xi)D(\xi)^m$$

$$= M(\xi).$$

So (1.3.25) holds. Define the wavelet function vectors $\psi^1, \ldots, \psi^d, \widetilde{\psi}^1, \ldots, \widetilde{\psi}^d$ as in (1.3.27). Then by Theorem 1.6, $\{\psi^1, \ldots, \psi^d\}$ and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbb{R})$. By Corollary 1.2, it follows from (1.3.31) and (1.3.40) that $\{\psi^1, \ldots, \psi^d\}$ has vanishing moments of order n and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ has vanishing moments of order m. ∎

A stronger version of Theorem 1.7 will be presented in Theorem 1.9 in Section 1.3.4, where we shall give a more general construction of pairs of dual d-wavelet frames derived from any two d-refinable function vectors.

1.3.3 Construction of pairs of symmetric dual wavelet frames from two symmetric refinable function vectors

Symmetry is a very important property of wavelet frames. Given two symmetric d-refinable function vectors, it is of interest to construct pairs of symmetric dual d-wavelet frames. In this subsection, we discuss how to obtain pairs of real-valued and symmetric dual d-wavelet frames when ϕ and $\widetilde{\phi}$ are real-valued and symmetric d-refinable functions.

Suppose that both ϕ and $\widetilde{\phi}$ are real-valued d-refinable function vectors in $L_2(\mathbb{R})$. Then their associating mask a and b must be real-valued as well. Let $\Theta, a^1, \ldots, a^L$, $b^1, \ldots, b^L$ be matrices of 2π-periodic trigonometric polynomials such that all the conditions in Theorem 1.6 are satisfied. When all the coefficients in $\Theta, a^1, \ldots, a^L$ are real-valued, define sequences $\widetilde{b}^\ell$ by $\widetilde{b}^\ell_k := \mathrm{Re}(b^\ell_k), k \in \mathbb{Z}$, where $\mathrm{Re}(b^\ell_k)$ denotes the real part of the complex matrix b^ℓ_k. Then it is easy to check that all the conditions in Theorem 1.6 still hold with b^ℓ being replaced by $\widetilde{b}^\ell$. Consequently, define $\widehat{\psi}^\ell(d\xi) = a^\ell(\xi)\widehat{\phi}(\xi)$ and $\widehat{\eta}^\ell(d\xi) = \widetilde{b}^\ell(\xi)\widehat{\widetilde{\phi}}(\xi)$ for $\ell = 1, \ldots, L$. Then $\psi^1, \ldots, \psi^L, \eta^1, \ldots, \eta^L$ are real-valued wavelet function vectors. By Theorem 1.6, $\{\psi^1, \ldots, \psi^L\}$ and $\{\eta^1, \ldots, \eta^L\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbb{R})$.

In the following, we show that when ϕ and $\widetilde{\phi}$ are real-valued and symmetric d-refinable function vectors such that the symmetry centers of all the components in ϕ and $\widetilde{\phi}$ differ by half integers, then we can derive pairs of real-valued and symmetric dual d-wavelet frames.

The following lemma plays a very important role in constructing real-valued and symmetric dual d-wavelet frames from real-valued and symmetric d-refinable function vectors.

Lemma 1.5. *Let d be a positive dilation factor and k be any integer. When k is odd, define*

$$f^j_{[d,k]}(\xi) := e^{-i\frac{k\xi}{2}}\left[e^{i(j-\frac{1}{2})\xi} + e^{-i(j-\frac{1}{2})\xi}\right],\ \ S_{d,k;j} := 1,\ \ j = 1, \ldots, \lfloor\frac{d+1}{2}\rfloor,$$
$$f^{j+\lfloor\frac{d+1}{2}\rfloor}_{[d,k]}(\xi) := e^{-i\frac{k\xi}{2}}\left[e^{i(j-\frac{1}{2})\xi} - e^{-i(j-\frac{1}{2})\xi}\right],\ \ S_{d,k;j+\lfloor\frac{d+1}{2}\rfloor} := -1,\ \ j = 1, \ldots, \lfloor\frac{d}{2}\rfloor.$$

When k is even, define

$$f^j_{[d,k]}(\xi) := e^{-i\frac{k\xi}{2}}\left[e^{i(j-1)\xi} + e^{-i(j-1)\xi}\right],\ \ S_{d,k;j} := 1,\ \ j = 1, \ldots, \lfloor\frac{d}{2}\rfloor + 1,$$
$$f^{j+\lfloor\frac{d}{2}\rfloor+1}_{[d,k]}(\xi) := e^{-i\frac{k\xi}{2}}\left[e^{ij\xi} - e^{-ij\xi}\right],\ \ S_{d,k;j+\lfloor\frac{d}{2}\rfloor+1} := -1,\ \ j = 1, \ldots, \lfloor\frac{d+1}{2}\rfloor - 1,$$

where $\lfloor \cdot \rfloor$ *is the floor function. Define a* $d \times d$ *matrix* $f_{[d,k]}(\xi)$ *as follows*

(1.3.41)

$$f_{[d,k]}(\xi) := \begin{bmatrix} f^1_{[d,k]}(\xi) & f^2_{[d,k]}(\xi) & \cdots & f^d_{[d,k]}(\xi) \\ f^1_{[d,k]}(\xi + \frac{2\pi}{d}) & f^2_{[d,k]}(\xi + \frac{2\pi}{d}) & \cdots & f^d_{[d,k]}(\xi + \frac{2\pi}{d}) \\ \vdots & \vdots & \ddots & \vdots \\ f^1_{[d,k]}(\xi + \frac{2\pi(d-1)}{d}) & f^2_{[d,k]}(\xi + \frac{2\pi(d-1)}{d}) & \cdots & f^d_{[d,k]}(\xi + \frac{2\pi(d-1)}{d}) \end{bmatrix}.$$

Then $\det[f_{[d,k]}(0)] \neq 0$ *and* $f^j_{[d,k]}(\xi), j = 1, \ldots, d$ *are* 2π*-periodic trigonometric polynomials such that*

$$\overline{f^j_{[d,k]}(\xi)} = S_{d,k;j} e^{ik\xi} f^j_{[d,k]}(\xi), \qquad j = 1, \ldots, d. \tag{1.3.42}$$

Proof: By definition, it is easy to see that $f^j_{[d,k]}(\xi), j = 1, \ldots, d$ are 2π-periodic trigonometric polynomials and (1.3.42) holds.

Consider four cases: k is even or odd and d is even or odd. After performing several elementary matrix transforms on $f_{[d,k]}(0)$, by a direct computation, it is not difficult to verify that the matrix $f_{[d,k]}(0)$ becomes the matrix $E_0(0)$, where the matrix $E_0(\xi)$ is defined in (1.3.38). Since it is well known that $E_0(0)E_0(0)^* = dI_d$, we conclude that $\det[f_{[d,k]}(0)] \neq 0$. ∎

As in [12], for any nonnegative integer N, we can even construct 2π-periodic trigonometric polynomials $f^j_{[d,k]}, j = 1, \ldots, d$ such that $\det[f_{[d,k]}(0)] \neq 0$, (1.3.42) holds and $[f^j_{[d,k]}(\cdot)]^{(\ell)}(0) = 0$ for all $j = 2, \ldots, d$ and $\ell = 0, \ldots, 2N$.

A similar result to Lemma 1.5 has been used in the construction of symmetric/antisymmetric semi-orthogonal d-band wavelets in Sun [47].

Now we can derive pairs of real-valued and symmetric dual d-wavelet frames from real-valued and symmetric d-refinable function vectors as follow.

Theorem 1.8. *Let* $\phi = (\phi_1, \ldots, \phi_r)^T$ *and* $\widetilde{\phi} = (\widetilde{\phi}_1, \ldots, \widetilde{\phi}_r)^T$ *be two* $r \times 1$ *d-refinable function vectors in* $(L_2(\mathbb{R}))^r$ *with finitely supported masks* a *and* b*, respectively. Suppose that* a *and* b *satisfy the sum rules of orders* m *and* n *with respect to the lattice* $d\mathbb{Z}$ *for some positive integers* m *and* n*, respectively.*

Assume that the masks a and b are real-valued and symmetric; that is, $\overline{a(\xi)} = a(-\xi), \overline{b(\xi)} = b(-\xi)$, *and*

$$\overline{a(\xi)} = S(d\xi)a(\xi)S(\xi)^{-1} \quad \textit{and} \quad \overline{b(\xi)} = \widetilde{S}(d\xi)b(\xi)\widetilde{S}(\xi)^{-1} \tag{1.3.43}$$

with

$$S(\xi) := diag(\varepsilon_1 e^{ic_1\xi}, \ldots, \varepsilon_r e^{ic_r\xi}) \quad \textit{and} \quad \widetilde{S}(\xi) := diag(\widetilde{\varepsilon}_1 e^{i\widetilde{c}_1\xi}, \ldots, \widetilde{\varepsilon}_r e^{i\widetilde{c}_r\xi}), \tag{1.3.44}$$

where $\varepsilon_1, \ldots, \varepsilon_r, \widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_r \in \{-1, 1\}$ *and the numbers* $c_1, \ldots, c_r, \widetilde{c}_1, \ldots, \widetilde{c}_r$ *satisfy*

$$dc_j - c_k \in \mathbb{Z}, \quad d\widetilde{c}_j - \widetilde{c}_k \in \mathbb{Z}, \qquad j, k = 1, \ldots, r \quad \textit{and} \quad \widetilde{c}_1 - c_1 \in \mathbb{Z}. \tag{1.3.45}$$

(In other words, the conditions in (1.3.43) *and* (1.3.45) *are equivalent to saying that all the symmetry centers of* $\phi_1, \ldots, \phi_r, \widetilde{\phi}_1, \ldots, \widetilde{\phi}_r$ *differ by half integers.) Then we can derive real-valued and symmetric wavelet function vectors* $\psi^1, \ldots, \psi^d$, $\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d$ *as in* (1.3.27) *such that* $\{\psi^1, \ldots, \psi^d\}$ *and* $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ *generate a pair of dual d-wavelet frames in* $L_2(\mathbb{R})$. *Moreover,* $\{\psi^1, \ldots, \psi^d\}$ *has vanishing moments of order n and* $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ *has vanishing moments of order m. In fact, each component in all the wavelet function vectors* $\psi^1, \ldots, \psi^d$, $\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d$ *is either symmetric or antisymmetric about the same point.*

Proof: By Theorem 1.5, without loss of generality, we can assume that (1.3.31) holds. Since $dc_1 - c_j \in \mathbb{Z}$ for all $j = 1, \ldots, r$, we can define $r \times r$ matrices

$$F^j(\xi) := \mathrm{diag}\Big(f^j_{[d,dc_1-c_1-n]}(\xi), f^j_{[d,dc_1-c_2]}(\xi), \ldots, f^j_{[d,dc_1-c_r]}(\xi)\Big), \qquad j = 1, \ldots, d,$$

where $f^j_{[d,k]}(\xi)(k \in \mathbb{Z})$ are defined in Lemma 1.5. By Lemma 1.5, $F^j(\xi), j = 1, \ldots, d$ are $r \times r$ matrices of 2π-periodic trigonometric polynomials with real coefficients. By (1.3.42), it can be easily verified that for $j = 1, \ldots, d$,

$$\overline{F^j(\xi)} = F^j(-\xi), \;\; F^j(\xi) = e^{-idc_1\xi} S_j \overline{F^j(\xi)} S(\xi) \mathrm{diag}\big((-1)^n e^{in\xi}, 1, \ldots, 1\big) \tag{1.3.46}$$

where

$$\text{(1.3.47)} \qquad S_j := \operatorname{diag}\big(\varepsilon_1(-1)^n S_{d,dc_1-c_1-n;j}, \varepsilon_2 S_{d,dc_1-c_2;j}, \ldots, \varepsilon_r S_{d,dc_1-c_r;j}\big)$$

with the numbers $S_{d,k;j} \in \{-1, 1\}$ being defined in Lemma 1.5. Let $E(\xi)$ be defined in (1.3.37). By Lemma 1.5, we observe that

$$|\det E(0)| = |\det[f_{[d,dc_1-c_1-n]}(0)]| \times |\det[f_{[d,dc_1-c_2]}(0)]| \cdots |\det[f_{[d,dc_1-c_r]}(0)]| \neq 0,$$

where the matrices $f_{[d,k]}(\xi)(k \in \mathbb{Z})$ are defined in (1.3.41).

Since $\det E\big(\xi + \frac{2\pi}{d}\big) = (-1)^{(d-1)r}\det E(\xi)$, it follows that $e^{-i\xi(d-1)r/2}\det E\big(\frac{\xi}{d}\big)$ is a 2π-periodic trigonometric polynomial by

$$e^{-i(\xi+2\pi)(d-1)r/2}\det E\Big(\frac{\xi+2\pi}{d}\Big) = e^{-i\xi(d-1)r/2}\det E\Big(\frac{\xi}{d}\Big).$$

Let

$$f(\xi) = \operatorname{lcm}(\det E(\xi), e^{-i\xi(d-1)r/2}\det E(\xi/d)) \quad \text{and} \quad g(\xi) := |f(\xi)|^2,$$

where lcm stands for least common multiple. (We can also choose $g(\xi) := \operatorname{lcm}(f(\xi), \overline{f(\xi)})$ such that $\overline{g(\xi)} = g(\xi)$.) Then $g(\xi)$ is a 2π-periodic trigonometric polynomial such that

(1.3.48)

$$\det E(\xi) \mid g(\xi), \;\; \det E(\xi) \mid g(d\xi) \quad \text{and} \quad \det E(\xi) \mid \overline{g(\xi)}, \;\; \det E(\xi) \mid \overline{g(d\xi)}.$$

Since $\det E(0) \neq 0$, we have $g(0) \neq 0$. Without loss of generality, we can assume $g(0) = 1$. Since $a_{1,1}(0) = b_{1,1}(0) = g(0) = 1$, by Lemma 1.4, there exists a 2π-periodic trigonometric polynomial θ_1 such that

$$\text{(1.3.49)} \quad \theta_1(0) = 1, \quad (1 - e^{-i\xi})^{m+n} \mid [\theta_1(\xi)g(\xi) - \theta_1(d\xi)\overline{a_{1,1}(\xi)}b_{1,1}(\xi)g(d\xi)].$$

Define

$$\theta(\xi) := [g(\xi)\theta_1(\xi) + e^{i(\tilde{c}_1 - c_1)\xi}\overline{g(\xi)\theta_1(\xi)}]/2.$$

Clearly, $\theta(0) = 1$ and $(1 - e^{-i\xi})^{m+n} \mid [\theta(\xi) - \theta(d\xi)\overline{a_{1,1}(\xi)}b_{1,1}(\xi)]$ since

$$\begin{aligned}
&\theta(\xi) - \theta(d\xi)\overline{a_{1,1}(\xi)}b_{1,1}(\xi) \\
&= [\theta_1(\xi)g(\xi) - \theta_1(\xi)\overline{a_{1,1}(\xi)}b_{1,1}(\xi)g(d\xi)]/2 \\
&\quad + \frac{1}{2}e^{i(\tilde{c}_1 - c_1)\xi}\left[\overline{\theta_1(\xi)g(\xi) - \theta_1(d\xi)e^{i(d-1)(c_1-\tilde{c}_1)\xi}a_{1,1}(\xi)\overline{b_{1,1}(\xi)}g(d\xi)}\right] \\
&= [\theta_1(\xi)g(\xi) - \theta_1(\xi)\overline{a_{1,1}(\xi)}b_{1,1}(\xi)g(d\xi)]/2 \\
&\quad + \frac{1}{2}e^{i(\tilde{c}_1 - c_1)\xi}\left[\overline{\theta_1(\xi)g(\xi) - \theta_1(d\xi)\overline{a_{1,1}(\xi)}b_{1,1}(\xi)g(d\xi)}\right]
\end{aligned}$$

where we used the fact that

$$\overline{a_{1,1}(\xi)} = e^{i(d-1)c_1\xi}a_{1,1}(\xi) \qquad \text{and} \qquad b_{1,1}(\xi) = e^{-i(d-1)\tilde{c}_1\xi}\overline{b_{1,1}(\xi)}.$$

By (1.3.48), $\det E(\xi) \mid \theta(\xi)$ and $\det E(\xi) \mid \theta(d\xi)$.

Let $\Theta(\xi) = \operatorname{diag}(\theta(\xi), 0, \ldots, 0)$. Since $S(\xi)^{-1} = \overline{S(\xi)}$ and $\theta(\xi)$ is a 2π-periodic trigonometric polynomial with real coefficients satisfying that $\theta(\xi) = e^{i(\tilde{c}_1 - c_1)\xi}\overline{\theta(\xi)}$, it is easy to verify that

$$\overline{\Theta(\xi)} = \Theta(-\xi) \qquad \text{and} \qquad \Theta(\xi) = S(\xi)^{-1}\overline{\Theta(\xi)}\widetilde{S}(\xi). \tag{1.3.50}$$

Let $M(\xi)$ be defined in (1.3.26) and $\widetilde{M}(\xi)$ be defined in (1.3.33). Denote

$$\begin{bmatrix} M_1(\xi) \\ \vdots \\ M_d(\xi) \end{bmatrix} = M(\xi) \qquad \text{and} \qquad \begin{bmatrix} \widetilde{M}_1(\xi) \\ \vdots \\ \widetilde{M}_d(\xi) \end{bmatrix} = \widetilde{M}(\xi).$$

Then

$$M_1(\xi) := \Theta(\xi) - a(\xi)^*\Theta(d\xi)b(\xi)$$

and

$$M_j(\xi) := -a\left(\xi + \frac{2\pi(j-1)}{d}\right)^*\Theta(d\xi)b(\xi), \qquad j = 2, \ldots, d.$$

Clearly, $M_j(\xi), j = 1, \ldots, d$ are matrices of 2π-periodic trigonometric polynomials with real coefficients. Since (1.3.45) implies $c_j - c_1 \in \mathbb{Z}$ for all

$j = 1, \ldots, r$, we deduce that $S(d\xi + 2\pi k) = e^{ic_1 2\pi k} S(d\xi)$ for all $k \in \mathbb{Z}$. By (1.3.43) and (1.3.50), for $j = 2, \ldots, d$, we have

$$
\begin{aligned}
M_j(\xi) &= -a(\xi + 2\pi(j-1)/d)^* \Theta(d\xi) b(\xi) \\
&= -S(\xi + 2\pi(j-1)/d)^{-1} a(\xi + 2\pi(j-1)/d)^T S(d\xi + 2\pi(j-1)) S(d\xi)^{-1} \times \\
&\quad \overline{\Theta(d\xi)} \widetilde{S}(d\xi) b(\xi) \\
&= -S(\xi + 2\pi(j-1)/d)^{-1} a(\xi + 2\pi(j-1)/d)^T e^{ic_1 2\pi(j-1)} S(d\xi) S(d\xi)^{-1} \times \\
&\quad \overline{\Theta(d\xi)}\, \overline{b(\xi)} \widetilde{S}(\xi) \\
&= -e^{ic_1 2\pi(j-1)} S(\xi + 2\pi(j-1)/d)^{-1} \overline{a(\xi + 2\pi(j-1)/d)^* \Theta(d\xi) b(\xi)} \widetilde{S}(\xi) \\
&= e^{ic_1 2\pi(j-1)} S(\xi + 2\pi(j-1)/d)^{-1} \overline{M_j(\xi)} \widetilde{S}(\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
M_1(\xi) &= \Theta(\xi) - a(\xi)^* \Theta(d\xi) b(\xi) \\
&= S(\xi)^{-1} \overline{\Theta(\xi)} \widetilde{S}(\xi) - S(\xi)^{-1} a(\xi)^T S(d\xi) S(d\xi)^{-1} \overline{\Theta(d\xi)} \widetilde{S}(d\xi) b(\xi) \\
&= S(\xi)^{-1} \overline{\Theta(\xi)} \widetilde{S}(\xi) - S(\xi)^{-1} a(\xi)^T \overline{\Theta(d\xi)}\, \overline{b(\xi)} \widetilde{S}(\xi) \\
&= S(\xi)^{-1} \left[\overline{\Theta(\xi) - a(\xi)^* \Theta(d\xi) b(\xi)} \right] \widetilde{S}(\xi) \\
&= S(\xi)^{-1} \overline{M_1(\xi)} \widetilde{S}(\xi).
\end{aligned}
$$

Therefore, for every $j = 1, \ldots, d$, we have

(1.3.51)

$$
\overline{M_j(\xi)} = M_j(-\xi) \quad \text{and} \quad M_j(\xi) = e^{ic_1 2\pi(j-1)} S(\xi + 2\pi(j-1)/d)^{-1} \overline{M_j(\xi)} \widetilde{S}(\xi).
$$

Since $\Theta(\xi) = \mathrm{diag}(\theta(\xi), 0, \ldots, 0)$, as in the proof of Theorem 1.7, for $j = 1, \ldots, d-1$, it follows from (1.3.35) that $\widetilde{M_j}(\xi), j = 2, \ldots, d$ are matrices of 2π-periodic trigonometric polynomials and $\theta(d\xi) \mid \widetilde{M_j}(\xi)$. Consequently, $\det E(\xi) \mid \widetilde{M_j}(\xi)$ for all $j = 2, \ldots, d$ since $\det E(\xi) \mid \theta(d\xi)$. Similarly, it follows from (1.3.36) that $\widetilde{M_1}(\xi)$ is a matrix of 2π-periodic trigonometric polynomial. By (1.3.48) and (1.3.49), it follows from $\det E(0) \neq 0$ that

$$
(1 - e^{i\xi})^n (1 - e^{-i\xi})^m \det E(\xi) \mid [\theta(\xi) - \overline{a_{1,1}(\xi)} \theta(d\xi) b_{1,1}(\xi)].
$$

Hence, by $\det E(\xi) \mid \theta(d\xi)$ and (1.3.36), we have $\det E(\xi) \mid \widetilde{M}_1(\xi)$. In conclusion, $\widetilde{M}(\xi)$ is a matrix of 2π-periodic trigonometric polynomials and

$$\det E(\xi) \mid \widetilde{M}(\xi).$$

Define

$$\begin{bmatrix} \widetilde{F}^1(\xi) \\ \vdots \\ \widetilde{F}^d(\xi) \end{bmatrix} = E(\xi)^{-1}\widetilde{M}(\xi) = [\mathrm{adj}E(\xi)][\det E(\xi)]^{-1}\widetilde{M}(\xi) \tag{1.3.52}$$

where $\mathrm{adj}E(\xi)$ is the adjacent matrix of $E(\xi)$ such that $E(\xi)\mathrm{adj}E(\xi) = I_{dr} \times [\det E(\xi)]$. Since $\det E(\xi) \mid \widetilde{M}(\xi)$, we have that $\widetilde{F}^1(\xi), \ldots, \widetilde{F}^d(\xi)$ are matrices of 2π-periodic trigonometric polynomials. Define the matrices $a^1, \ldots, a^d$, $b^1, \ldots, b^d$ as in (1.3.40). It follows from (1.3.40) and (1.3.46) that

$$\overline{a^\ell(\xi)} = a^\ell(-\xi) \quad \text{and} \quad a^\ell(\xi) = e^{-idc_1\xi} S_\ell \overline{a^\ell(\xi)} S(\xi), \qquad j = 1, \ldots, d, \tag{1.3.53}$$

where S_ℓ is defined in (1.3.47). Now as in the proof of Theorem 1.7, it is easy to check that $\Theta, a^1, \ldots, a^d, b^1, \ldots, b^d$ satisfy all the conditions in Theorem 1.6. By the remark before Lemma 1.5, we can assume that $b^\ell, \ell = 1, \ldots, d$ are matrices of 2π-periodic trigonometric polynomials with real coefficients (otherwise, we can replace $b^\ell(\xi)$ by $[b^\ell(\xi) + \overline{b^\ell(-\xi)}]/2$).

Define $\widetilde{b}^\ell(\xi) := [b^\ell(\xi) + e^{-idc_1\xi} S_\ell \overline{b^\ell(\xi)} \widetilde{S}(\xi)]/2$. By the definition of $\widetilde{b}^\ell$, it is easy to verify that

$$\overline{\widetilde{b}^\ell(\xi)} = \widetilde{b}^\ell(-\xi) \quad \text{and} \quad \widetilde{b}^\ell(\xi) = e^{-idc_1\xi} S_\ell \overline{\widetilde{b}^\ell(\xi)} \widetilde{S}(\xi), \qquad \ell = 1, \ldots, d. \tag{1.3.54}$$

We now demonstrate that Θ, a^1, ..., a^d, $\widetilde{b}^1$, ..., $\widetilde{b}^d$ also satisfy (1.3.25). Since (1.3.25) holds for Θ, a^1, ..., a^d, b^1, ..., b^d, we have

$$\sum_{\ell=1}^{d} a^\ell(\xi + 2\pi j/d)^* b^\ell(\xi) = M_{j+1}(\xi), \qquad j = 0, \ldots, d-1.$$

By (1.3.51), (1.3.53), (1.3.54) and the above identity, for $j = 0, \ldots, d-1$, we have

$$\sum_{\ell=1}^{d} a^\ell(\xi+2\pi j/d)^* \widetilde{b}^\ell(\xi) = M_{j+1}(\xi)/2 + \sum_{\ell=1}^{d} a^\ell(\xi+2\pi j/d)^* e^{-idc_1\xi} S_\ell \overline{b^\ell(\xi)} \widetilde{S}(\xi)/2$$

$$= M_{j+1}(\xi)/2+$$

$$\sum_{\ell=1}^{d} e^{idc_1(\xi+\frac{2\pi j}{d})} S(\xi+\frac{2\pi j}{d})^{-1} a^\ell(\xi+\frac{2\pi j}{d})^T S_\ell e^{-idc_1\xi} S_\ell \overline{b^\ell(\xi)} \widetilde{S}(\xi)/2$$

$$= M_{j+1}(\xi)/2 + e^{ic_1 2\pi j} S(\xi+\frac{2\pi j}{d})^{-1} \overline{\sum_{\ell=1}^{d} a^\ell(\xi+\frac{2\pi j}{d})^* b^\ell(\xi)} \widetilde{S}(\xi)/2$$

$$= M_{j+1}(\xi)/2 + e^{ic_1 2\pi j} S(\xi+2\pi j/d)^{-1} \overline{M_{j+1}(\xi)} \widetilde{S}(\xi)/2$$

$$= M_{j+1}(\xi)$$

where we used the fact that $S_\ell S_\ell = I_r$. Therefore, (1.3.25) holds for $\Theta, a^1, \ldots, a^d, \widetilde{b}^1, \ldots, \widetilde{b}^d$.

Define $\widehat{\psi^\ell}(d\xi) := a^\ell(\xi)\widehat{\phi}(\xi)$ and $\widehat{\widetilde{\psi}^\ell}(d\xi) := \widetilde{b}^\ell(\xi)\widehat{\widetilde{\phi}}(\xi)$ for $\ell = 1, \ldots, d$. By Lemma 1.3, it follows from (1.3.53) and (1.3.54) that for $\ell = 1, \ldots, d$,

$$\overline{\widehat{\psi^\ell}(\xi)} = \widehat{\psi^\ell}(-\xi),\ \overline{\widehat{\psi^\ell}(\xi)} = e^{ic_1\xi} S_\ell \widehat{\psi^\ell}(\xi),\ \overline{\widehat{\widetilde{\psi}^\ell}(\xi)} = \widehat{\widetilde{\psi}^\ell}(-\xi),\ \overline{\widehat{\widetilde{\psi}^\ell}(\xi)} = e^{ic_1\xi} S_\ell \widehat{\widetilde{\psi}^\ell}(\xi).$$

So all the wavelet functions in $\psi^1, \ldots, \psi^d, \widetilde{\psi}^1, \ldots, \widetilde{\psi}^d$ are real-valued functions and are either symmetric or antisymmetric about the point $c_1/2$. By Theorem 1.6, $\{\psi^1, \ldots, \psi^d\}$ and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbb{R})$. Moreover, $\{\psi^1, \ldots, \psi^d\}$ has vanishing moments of order n and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$ has vanishing moments of order m. ■

In order to assure that $S(d\xi)a(\xi)S(\xi)^{-1}$ and $\widetilde{S}(d\xi)b(\xi)\widetilde{S}(\xi)^{-1}$ in (1.3.43) are matrices of 2π-periodic trigonometric polynomials, it is natural and almost necessary to require that $dc_j - c_k \in \mathbb{Z}$ and $d\widetilde{c}_j - \widetilde{c}_k \in \mathbb{Z}$ for all $j, k = 1, \ldots, r$. The extra condition $\widetilde{c}_1 - c_1 \in \mathbb{Z}$ in (1.3.45) is automatically satisfied when $d = 2$ (since $c_j, \widetilde{c}_j \in \mathbb{Z}$) and is needed to guarantee the existence of a symmetric θ in (1.3.49). In other words, the condition in (1.3.45) seems necessary in order to obtain pairs of symmetric dual d-wavelet frames derived from symmetric d-refinable function vectors. By the remark after Lemma 1.5, as in Daubechies and Han [12], we see that in both Theorems 1.7 and 1.8, for any nonnegative integer N, we can even require that ψ^1, $\{\psi^2,\dots,\psi^d\}$, $\widetilde{\psi}^1$ and $\{\widetilde{\psi}^2,\dots,\widetilde{\psi}^d\}$ have vanishing moments of orders $n, n+2N, m+2N$ and m, respectively.

1.3.4 Wavelet frames from any refinable function vector

In this subsection, let us discuss how to derive wavelet frames from a single refinable function vector.

The following generalizes [12, Corollary 3.2] on d-wavelet frames to the multiwavelet case.

Theorem 1.9. *Let ϕ be an $r\times 1$ d-refinable function vector in $(L_2(\mathbb{R}))^r$ with a finitely supported mask a. Suppose that a satisfies the sum rules of order m with respect to the lattice $d\mathbb{Z}$. For any positive integer n, let $U_n(\xi)$ be an $r\times r$ invertible matrix of 2π-periodic trigonometric polynomials in Theorem 1.2 such that $U_n(d\xi)a(\xi)U_n(\xi)^{-1}$ takes the form of* (1.2.15). *For any nonnegative integer m_0 such that $0\leqslant m_0<m$, define matrices of 2π-periodic trigonometric polynomials $a^\ell(\xi)$, $\ell=1,\dots,d$ by*

$$a^\ell(\xi)=F^\ell(\xi)\mathrm{diag}\big((1-e^{-id\xi})^{m_0}(1-e^{-i\xi})^{n-m_0},1,\dots,1\big)U_n(\xi),\qquad \ell=1,\cdots,d,$$

where $F^1,\dots,F^d$ are some $r\times r$ matrices of 2π-periodic trigonometric polynomials such that $\det E(0)\neq 0$, where the matrix $E(\xi)$ is defined in (1.3.37). *Then $\{\psi^1,\dots,\psi^d\}$, which are defined in* (1.3.27), *generates a d-wavelet frame in $L_2(\mathbb{R})$ and has vanishing moments of order n. Moreover, for any d-refinable function vector $\widetilde{\phi}$ in $(L_2(\mathbb{R}))^r$ whose mask satisfies the sum rules of order n, there are matrices $b^1(\xi),\dots,b^d(\xi)$ of 2π-periodic trigonometric polynomials such that $\{\widetilde{\psi}^1,\dots,\widetilde{\psi}^d\}$, which are defined in* (1.3.27), *and $\{\psi^1,\dots,\psi^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbb{R})$.*

Proof: By observation, it suffices to prove the claim for the case $U_n(\xi)=I_r$. Let $\widetilde{\phi}$ be any d-refinable function vector in $(L_2(\mathbb{R}))^r$ with a finitely supported mask b such that b satisfies the sum rules of order n and by Theorem 1.2 we assume that (1.3.31) holds. For example, take $\widetilde{\phi} = (\widetilde{\phi}_1, 0, \ldots, 0)^T$ and $\widetilde{\phi}_1$ is a scalar d-refinable function in $L_2(\mathbb{R})$ whose mask satisfies the sum rules of order n. So such a d-refinable function vector $\widetilde{\phi}$ exists and in fact we can take $\widetilde{\phi}_1$ to be the B-spline function of order n.

Since $\det E(\xi + 2\pi/d) = (-1)^{(d-1)r} \det E(\xi)$, it follows that $e^{-i\xi(d-1)r/2} \times \det E(\frac{\xi}{d})$ is a 2π-periodic trigonometric polynomial. Let $f(\xi) := \operatorname{lcm}\big(\det E(\xi), e^{-i\xi(d-1)r/2} \det E(\xi/d)\big)$ and $g(\xi) := f(\xi)\big(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi}\big)^{m_0}$. Then $g(\xi)$ is a 2π-periodic trigonometric polynomial such that $\det E(\xi) \mid g(\xi)$ and $\det E(\xi) \mid g(d\xi)$.

By assumption $\det E(0) \neq 0$, so $f(0) \neq 0$ and therefore, $g(0) \neq 0$. Without loss of generality, we can assume $g(0) = 1$. Since $a_{1,1}(0) = b_{1,1}(0) = g(0) = 1$, by Lemma 1.4, there exists a 2π-periodic trigonometric polynomial θ_1 such that (1.3.49) holds. Take $\theta(\xi) = g(\xi)\theta_1(\xi)$ and $\Theta(\xi) = \operatorname{diag}(\theta(\xi), 0, \ldots, 0)$. Let M be defined in (1.3.26). Denote $D(\xi) := \operatorname{diag}(1 - e^{-i\xi}, 1, \ldots, 1)$ and $G(\xi) = \operatorname{diag}\big((1 - e^{i\xi})^{m_0 - n}(1 - e^{id\xi})^{-m_0}, 1, \ldots, 1\big)$. Define $\widetilde{M}(\xi)$ by

$$\widetilde{M}(\xi) := \operatorname{diag}\big(G(\xi), G(\xi + 2\pi/d), \ldots, G(\xi + 2\pi(d-1)/d)\big) M(\xi) D(\xi)^{m_0 - m}.$$

By (1.3.34) and $\det E(0) \neq 0$, as in the proof of Theorem 1.7, one can verify that $\widetilde{M}(\xi)$ is an $rd \times r$ matrix of 2π-periodic trigonometric polynomials and $\det E(\xi) \mid \widetilde{M}(\xi)$ since $\det E(\xi) \mid \theta(\xi)$ and $\det E(\xi) \mid \theta(d\xi)$. Define $\widetilde{F}^1, \ldots, \widetilde{F}^d$ as in (1.3.52). Then $\widetilde{F}^1(\xi), \ldots, \widetilde{F}^d(\xi)$ are matrices of 2π-periodic trigonometric polynomials. Let $b^\ell := \widetilde{F}^\ell(\xi) D(\xi)^{m - m_0}$ for $\ell = 1, \ldots, d$. Then all the conditions in Theorem 1.6 are satisfied and therefore, $\{\psi^1, \ldots, \psi^d\}$ and $\{\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d\}$, which are defined in (1.3.27), generate a pair of dual d-wavelet frames. ∎

Finally, using the same technique as in the proof of Theorem 1.9, we have the following result which generalizes [12, Corollary 3.3].

Corollary 1.6. Let ϕ be an $r \times 1$ d-refinable function vector in $\big(L_2(\mathbb{R})\big)^r$ with a finitely supported mask a which satisfies the sum rules of order m with respect to the lattice $d\mathbb{Z}$. Let $U(\xi)$ be an $r \times r$ invertible matrix of 2π-periodic trigonometric polynomials in Theorem 1.2 such that $U(d\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.15) with $n = 1$. Let $p(\xi)$ be an $r \times r$ matrix of 2π-periodic trigonometric polynomials such that

1) $\det P(\xi) \not\equiv 0$ and $P(0)\widehat{\phi}(0) = 0$;

2) $\lim_{\xi \to 0} e_j^T (1 - e^{-i\xi})^m U(0) P(\xi)^{-1} = 0$ for all $j = 2, \ldots, r$;

3) $\lim_{\xi \to 2\pi k/d} e_1^T U(0) a(\xi) P(\xi)^{-1} = 0$ for all $k = 1, \ldots, d-1$

where e_j denotes the jth unit coordinate vector in $\mathbb{R}^r$. Define a function vector ψ by $\widehat{\psi}(\xi) = P(\xi)\widehat{\phi}(\xi)$. Then $\{\psi\}$ generates a d-wavelet frame in $L_2(\mathbb{R})$. Moreover, there exist compactly supported function vectors $\widetilde{\psi}^1, \ldots, \widetilde{\psi}^d$ with arbitrary smoothness such that $\{\psi(d\cdot), \psi(d\cdot -1), \ldots, \psi(d\cdot -d+1)\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2, \ldots, \widetilde{\psi}^d\}$ generate a pair of dual d-wavelet frames.

Proof: It suffices to prove the claim for the case $U(\xi) = I_r$. Let $f_1(\xi) = \det P(\xi)$ and write $f_1(\xi) = f_2(\xi) \prod_{k=0}^{d-1} (1 - e^{-i(\xi + 2\pi k/d)})^{n_k}$ such that $n_k, k = 0, \ldots, d-1$ are nonnegative integers and $f_2(\xi)$ is a 2π-periodic trigonometric polynomial satisfying $f_2(2\pi k/d) \neq 0$ for all $k = 0, \ldots, d-1$. Let $n = n_1 + \cdots + n_d$ and define $f(\xi) = f_2(\xi)\big(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi}\big)^n \prod_{k=0}^{d-1} f_2(\frac{\xi + 2\pi k}{d})$. Clearly, $f(0) \neq 0$ and therefore, we can define $g(\xi) = f(\xi)/f(0)$.

Let $\widetilde{\phi} = (\widetilde{\phi}_1, 0, \ldots, 0)^T$ with mask $b(\xi) = \operatorname{diag}\Big(\big(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi}\big)^n, 0, \ldots, 0\Big)$, where $\widetilde{\phi}_1$ is the B-spline function of order n. Let θ be a 2π-periodic trigonometric polynomial such that (1.3.49) holds and $\Theta(\xi) = \operatorname{diag}\big(\theta(\xi)g(\xi), 0, \ldots, 0\big)$. Define $a^\ell(\xi) = e^{-i(\ell-1)\xi} P(\xi)$ for $\ell = 1, \ldots, d$. Define M in (1.3.26) and let

$$\widetilde{M}(\xi) = \operatorname{diag}\big((P(\xi)^*)^{-1}, (P(\xi + 2\pi/d)^*)^{-1}, \ldots, (P(\xi + 2\pi(d-1)/d)^*)^{-1}\big) M(\xi).$$

As in the proof of Theorem 1.7, by our assumption on $P(\xi)$, $\widetilde{M}$ must be a matrix of 2π-periodic trigonometric polynomials with the first column of $\widetilde{M}(0)$ being zeros. Let $b^1, \ldots, b^d$ be the solution of (1.3.25). Then $b^\ell(\xi), \ell = 1, \ldots, d$

must be matrices of 2π-periodic trigonometric polynomials such that the first column of $b^\ell(0)$ is zero for all $\ell = 1, \ldots, d$. The rest of the claim can be proved similarly as in the proof of Theorem 1.7 and [12, Corollary 3.3]. ∎

1.4 Examples of pairs of symmetric dual wavelet frames

I shall present a few examples of dual wavelet frames to illustrate the general procedure for constructing pairs of real-valued symmetric dual wavelet frames from real-valued symmetric refinable function vectors. The general procedure described in the proofs in Section 1.3 can be easily applied in practice. For simplicity, throughout this section, the dilation factor $d = 2$, the multiplicity $r = 2$ and we always denote $z := e^{-i\xi}$, $\xi \in \mathbb{R}$.

Example 1.1. Let us recall the mask of the well known piecewise Hermite cubics ϕ (it was discussed in Section 1.2) is given by

$$a(\xi) := \begin{bmatrix} (e^{i\xi} + 2 + e^{-i\xi})/4 & 3(e^{i\xi} - e^{-i\xi})/8 \\ (-e^{i\xi} + e^{-i\xi})/16 & (-e^{i\xi} + 4 - e^{-i\xi})/16 \end{bmatrix}. \tag{1.4.55}$$

The refinable function vector ϕ is known as a Hermite interpolant with a Hermite interpolatory mask a. For a general construction of Hermite interpolatory masks with multiplicity r and a general dilation factor d, see Han [20]. In Section 1.2, we already mentioned that a satisfies the sum rules of order 4 with a row vector $y(\xi) = [1, e^{i\xi}/3 + 1/2 - e^{-i\xi} + e^{-2i\xi}/6]$. Let $\tilde{\phi} = \phi$ and $b = a$. Take $m = n = 2$. By computation as in Theorem 1.2, let

$$U(\xi) := \begin{bmatrix} \frac{1}{60}(z^2 + z^{-2} + 58) & \frac{1}{2}(z - z^{-1}) \\ \frac{1}{30}(z - z^{-1}) & 1 \end{bmatrix}.$$

Then $U(2\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.15) with $m = n = 2$. Define

$$\Theta(\xi) := \begin{bmatrix} 1 & -(z - 1/z)/2 \\ (z - 1/z)/2 & 0 \end{bmatrix}.$$

Define a^1, a^2, b^1, b^2 by

$$a^1(\xi) := \begin{bmatrix} 0 & 1/z - z \\ 0 & 2 - z - 1/z \end{bmatrix}, \quad a^2(\xi) := \begin{bmatrix} 1/2 - z/4 - 1/(4z) & 0 \\ z/10 - 1/(10z) & -3 \end{bmatrix},$$

and

$$b^1(\xi) := \frac{1}{256}\begin{bmatrix} p_1(\xi) & p_2(\xi) \\ p_3(\xi) & p_4(\xi) \end{bmatrix}, \quad b^2(\xi) := \frac{1}{128}\begin{bmatrix} p_5(\xi) & p_6(\xi) \\ p_7(\xi) & p_8(\xi) \end{bmatrix},$$

where

$$\begin{aligned}
p_1(\xi) &= -89(z^3 + z^{-3}) - 64(z^2 + z^{-2}) + 209(z + 1/z) - 112,\\
p_2(\xi) &= 105(z^3 - z^{-3}) - 228(z^2 - z^{-2}) + 141(z - 1/z),\\
p_3(\xi) &= -94(z^3 - z^{-3}) - 68(z^2 - z^{-2}) + 418(z - 1/z),\\
p_4(\xi) &= 111(z^3 + z^{-3}) - 240(z^2 + z^{-2}) - 111(z + 1/z) + 480,\\
p_5(\xi) &= 4(z^3 + z^{-3}) - 36(z + 1/z) + 64,\\
p_6(\xi) &= -4(z^3 - z^{-3}) + 16(z^2 - z^{-2}) - 20(z - 1/z),\\
p_7(\xi) &= -30(z^3 - z^{-3}) - 20(z^2 - z^{-2}) + 130(z - 1/z),\\
p_8(\xi) &= 35(z^3 + z^{-3}) - 80(z^2 + z^{-2}) - 35(z + 1/z) + 160.
\end{aligned}$$

By a direct computation, one can verify that Θ, a^1, a^2, b^1, b^2 satisfy all the conditions in Theorem 1.6 with both the masks a and b in Theorem 1.6 being the mask in (1.4.55). Define function vectors ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ as in (1.3.27). Then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames. $\psi^1, \psi^2, \widetilde{\psi}^1, \widetilde{\psi}^2$ are real-valued and symmetric, and all of them have vanishing moments of order 2. For their graphs, see Figure 1.2.

Example 1.2. Let us use the same mask a and refinable function vector ϕ as in the previous example. Take $\widetilde{\phi} = \phi$ and $b = a$. Let $m = n = 4$. By computation as in Theorem 1.2, let

$$U(\xi) := \begin{bmatrix} u_{1,1}(\xi) & u_{1,2}(\xi) \\ u_{2,1}(\xi) & 1 \end{bmatrix},$$

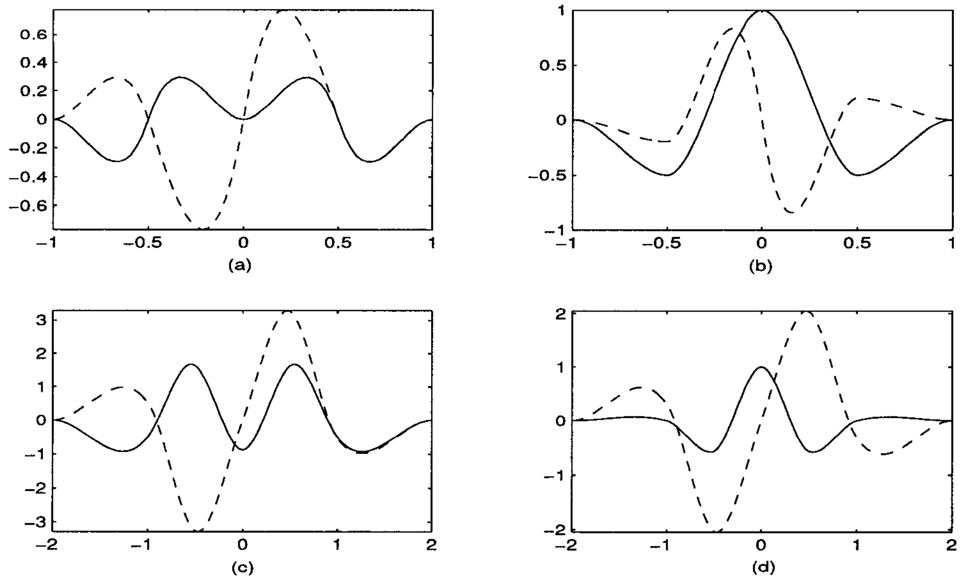

Figure 1.2: Generators for the pair of dual 2-wavelet frames in Example 1.1: (a) ψ^1 (b) ψ^2 (c) $\widetilde{\psi}^1$ (d) $\widetilde{\psi}^2$. All the components in the wavelet function vectors $\psi^1, \psi^2, \widetilde{\psi}^1, \widetilde{\psi}^2$ are either symmetric or antisymmetric about the origin and have vanishing moments of order 2.

where

$$\begin{aligned} u_{1,1}(\xi) &= \frac{1}{15120}\left[5(z^4+z^{-4}) - 92(z^3+z^{-3}) + 416(z^2+z^{-2})\right] + \\ &\qquad\qquad \frac{1}{15120}\left[92(z+z^{-1}) + 14278\right], \\ u_{1,2}(\xi) &= -\frac{1}{12}\left[(z^2-z^{-2}) - 8(z-z^{-1})\right], \\ u_{2,1}(\xi) &= -\frac{1}{1260}\left[5(z^2-z^{-2}) - 52(z-z^{-1})\right]. \end{aligned}$$

Then $\widetilde{a}(\xi) := U(2\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.15) with $m = n = 4$. Define $\Theta(\xi)$ to be the following matrix

$$\begin{bmatrix} -\frac{1}{162}\left(35(z+z^{-1}) - 232\right) & \frac{121}{108}(z-z^{-1}) \\ -\frac{121}{108}(z-z^{-1}) & \frac{1}{252}\left(12(z^2+z^{-2}) + 1345(z+z^{-1}) + 9536\right) \end{bmatrix}.$$

Define a^1, b^1, a^2, b^2 as

$$a^1(\xi) := \begin{bmatrix} 2 - z - z^{-1} & \frac{15}{2}(z - z^{-1}) \\ 0 & (2 - z - z^{-1})^2 \end{bmatrix}, \quad b^1 := \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix},$$

$$a^2(\xi) := \begin{bmatrix} 0 & (z - z^{-1})(2 - z - z^{-1}) \\ z - z^{-1} & -\frac{1}{7}\big(39(z + z^{-1}) + 132\big) \end{bmatrix}, \quad b^2 := \begin{bmatrix} p_5 & p_6 \\ p_7 & p_8 \end{bmatrix}$$

where

$$\begin{aligned}
p_1(\xi) :=& -\frac{1}{20736}\left[83(z^3 + z^{-3}) - 560(z^2 + z^{-2}) + 2333(z + z^{-1}) - 3712\right],\\
p_2(\xi) :=& -\frac{1}{6912}\left[19(z^3 - z^{-3}) + 484(z^2 - z^{-2}) - 3125(z - z^{-1})\right],\\
p_3(\xi) :=& \frac{1}{225792}\left[96(z^5 - z^{-5}) + 2005(z^3 - z^{-3}) - 11536(z^2 - z^{-2})\right] +\\
& \frac{1}{225792}\left[+23437(z - z^{-1})\right],\\
p_4(\xi) :=& -\frac{1}{75264}\left[32(z^5 + z^{-5}) - 128(z^4 + z^{-4}) - 229(z^3 + z^{-3})\right] -\\
& \frac{1}{75264}\left[-10492(z^2 + z^{-2}) + 42057(z + z^{-1}) + 6120\right],\\
p_5(\xi) :=& -\frac{1}{225792}\left[96(z^5 + z^{-5}) + 3541(z^3 + z^{-3}) - 11536(z^2 + z^{-2})\right] -\\
& \frac{1}{225792}\left[17867(z + z^{-1}) - 19936\right],\\
p_6(\xi) :=& \frac{1}{75264}\left[32(z^5 - z^{-5}) - 128(z^4 - z^{-4}) + 283(z^3 - z^{-3})\right] +\\
& \frac{1}{75264}\left[-12540(z^2 - z^{-2}) + 39563(z - z^{-1})\right],\\
p_7(\xi) :=& -\frac{1}{580608}\left[108(z^5 - z^{-5}) - 2815(z^3 - z^{-3})\right] -\\
& \frac{1}{580608}\left[11032(z^2 - z^{-2}) - 45029(z - z^{-1})\right],\\
p_8(\xi) :=& \frac{1}{193536}\left[36(z^5 + z^{-5}) - 144(z^4 + z^{-4}) + 53(z^3 + z^{-3})\right] +\\
& \frac{1}{193536}\left[10964(z^2 + z^{-2}) - 80939(z + z^{-1}) - 168640\right].
\end{aligned}$$

By a direct computation, one can verify that Θ, a^1, a^2, b^1, b^2 satisfy all the conditions in Theorem 1.6 with both the masks a and b in Theorem 1.6 being the mask in (1.4.55). Define function vectors ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ as in

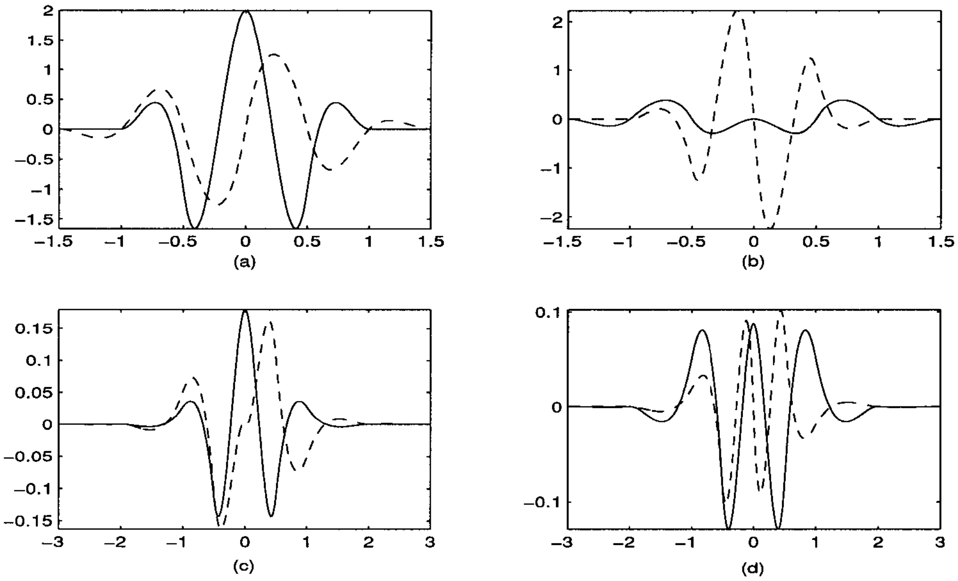

Figure 1.3: Generators for the pair of dual 2-wavelet frames in Example 1.2: (a) ψ^1 (b) ψ^2 (c) $\widetilde{\psi}^1$ (d) $\widetilde{\psi}^2$. All the components in the wavelet function vectors $\psi^1, \psi^2, \widetilde{\psi}^1, \widetilde{\psi}^2$ are (anti)symmetric and have vanishing moments of order 4.

(1.3.27). Then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames. $\psi^1, \psi^2, \widetilde{\psi}^1, \widetilde{\psi}^2$ are real-valued and symmetric, and all of them have vanishing moments of order 4. For their graphs, see Figure 1.3.

Chapter 2

Tight multiwavelet frames from refinable function vectors

2.1 Introduction

In Chapter 1, we discussed how to construct pairs of dual multiwavelet frames from refinable function vectors and the motivation of the construction. As a special kind of multiwavelet frames, tight multiwavelet frames are a generalization of orthogonal multiwavelet bases. It can carry an "orthogonal" property which is very interesting in some cases. In this chapter, I shall apply the results in Chapter 1 to the tight multiwavelet frame case and prove the existence of tight multiwavelet frames with the highest possible vanishing moments. It is proved in [4] and [6] the existence of tight multiwavelet frames for the scalar case, i.e., the case $r = 1$, where r is the multiplicity. The work of this chapter is to extend their result to the general multiwavelet case. Due to the complexity of matrix operation, the extension is not trivial.

Before proceeding further, let us review some definitions. Define A^{adj} to be the adjoint matrix of a matrix A. Define $\mathbb{T} = \mathbb{R}/2\pi\mathbb{R}$. Denote $\mathcal{P}(\mathbb{T})^{r\times r}$ the space of $r \times r$ matrices of 2π-periodic trigonometric polynomials, $C^{\infty}(\mathbb{T})^{r\times r}$

the space of $r \times r$ matrices of 2π-periodic C^∞-functions and $C(\mathbb{T})^{r\times r}$ the space of $r \times r$ matrices of 2π-periodic continuous functions. For any $A \in C(\mathbb{T})^{r\times r}$, we say $A > 0$ (or $A \geqslant 0$) if for all $\xi \in \mathbb{T}$, $A(\xi) > 0$ which means $A(\xi)$ is positive definite (or $A(\xi) \geqslant 0$ which means $A(\xi)$ is positive semi-definite). Let $\phi = (\phi_1, \ldots, \phi_r)^T \in (L_2(\mathbb{R}))^r$ be a refinable function vector and let a be the matrix mask of ϕ. The **transition operator** T_a is very important in wavelet theory. It is defined by

(2.1.1)

$$(T_a F)(d\xi) := \sum_{j=0}^{|d|-1} a(\xi + 2\pi j/d) F(\xi + 2\pi j/d) a(\xi + 2\pi j/d)^* \quad \forall F \in C(\mathbb{T})^{r\times r}.$$

By the definition, we know that T_a is well defined and T_a maps $C(\mathbb{T})^{r\times r}$ and $\mathcal{P}(\mathbb{T})^{r\times r}$ into $C(\mathbb{T})^{r\times r}$ and $\mathcal{P}(\mathbb{T})^{r\times r}$, respectively. A special eigenfunction of T_a is Φ, the bracket product of $\widehat{\phi}$ and $\widehat{\phi}$. For $f, g \in (L_2(\mathbb{R}))^r$, define the **bracket product** (see [33]) as

$$[f, g](\xi) := \sum_{j\in\mathbb{Z}} f(\xi + 2\pi j) g(\xi + 2\pi j)^* \qquad \forall \xi \in \mathbb{T}.$$

Since $f, g \in \left(L_2(\mathbb{R})\right)^r$, $[f, g]$ is an $r \times r$ matrix of functions in $L_1(\mathbb{T})$. Define

$$\Phi := \left[\widehat{\phi}, \widehat{\phi}\right].$$

Then we can verify that Φ is an eigenfunction of T_a as follows.

$$\begin{aligned}
T_a\Phi(d\xi) &= \sum_{j=0}^{|d|-1} a(\xi + 2\pi j/d)\Phi(\xi + 2\pi j/d) a(\xi + 2\pi j/d)^* \\
&= \sum_{j=0}^{|d|-1} \sum_{k\in\mathbb{Z}} a\left(\xi + \frac{2\pi j}{d}\right) \widehat{\phi}\left(\xi + \frac{2\pi j}{d} + 2\pi k\right) \widehat{\phi}\left(\xi + \frac{2\pi j}{d} + 2\pi k\right)^* a\left(\xi + \frac{2\pi j}{d}\right)^* \\
&= \sum_{j=0}^{|d|-1} \sum_{k\in\mathbb{Z}} \widehat{\phi}(d\xi + 2\pi j + 2\pi dk) \widehat{\phi}(d\xi + 2\pi j + 2\pi dk)^* \\
&= \Phi(d\xi).
\end{aligned}$$

Thus we have $T_a\Phi = \Phi$.

The remaining part of this chapter is as follows. In Section 2.2, I shall build some auxiliary lemmas for matrix inequalities. Based on these lemmas, combined with an interesting property of the transition operator T_a, we can prove the existence of tight multiwavelet frames with highest possible vanishing moments. Although Section 2.2 just proved the existence of tight multiwavelet frames, in Section 2.3, I shall show an example of tight multiwavelet frames for the case $r = 2$ and $d = 2$.

2.2 Existence of tight multiwavelet frames

First let us continue our discussion on Φ.

Lemma 2.1. *Let $\phi = (\phi_1, \dots, \phi_r)^T \in \left(L_2(\mathbb{R})\right)^r$ be a refinable function vector with its matrix mask a. Define $\Phi := \left[\widehat{\phi}, \widehat{\phi}\right]$. It is evident that $\Phi^* = \Phi$. Assume the matrix mask a takes the canonical form (1.2.15) with $n = m$, then we have*

$$(1 - e^{-i\xi})^m \mid \Phi(\xi)_{1,j} = \overline{\Phi(\xi)}_{j,1}, \qquad j = 2, \dots, r$$

and

$$(1 - e^{-i\xi})^m \mid \Phi^{\mathrm{adj}}(\xi)_{1,j} = \overline{\Phi^{\mathrm{adj}}(\xi)}_{j,1}, \qquad j = 2, \dots, r.$$

Moreover, if $\Phi > 0$, then we have

$$(1 - e^{-i\xi})^m \mid \Phi^{-1}(\xi)_{1,j} = \overline{\Phi^{-1}(\xi)}_{j,1}, \qquad j = 2, \dots, r$$

and

$$(1 - e^{-i\xi})^{2m} \mid \left[\Phi^{-1}(\xi) - a(\xi)^*\Phi^{-1}(d\xi)a(\xi)\right]_{1,1}.$$

Proof: By assumption, we have

$$a(\xi) = \begin{bmatrix} (1 - e^{-id\xi})^m/(1 - e^{-i\xi})^m * & (1 - e^{-id\xi})^m * \\ (1 - e^{-i\xi})^m * & * \end{bmatrix} \tag{2.2.2}$$

where $*$ denotes some 1×1, $1 \times (r-1)$, $(r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of trigonometric polynomials. Since ϕ is refinable,

$$\widehat{\phi}(d\xi) = a(\xi)\widehat{\phi}(\xi).$$

Taking 0th, ..., $(m-1)$th derivatives at $\xi = 0$ from both sides of the equation above, since $a(\xi)$ has the canonical form (2.2.2), we have

$$\widehat{\phi}_j^{(\ell)}(0) = 0, \qquad j = 2, \ldots, r;\ \ell = 0, 1, \ldots, m-1.$$

Notice that ϕ_1 is a superfunction, we have for $\ell = 0, 1, \ldots, m-1$,

$$\sum_{k \in \mathbb{Z}} k^\ell \phi_1(x+k) = p_\ell(x)$$

where p_ℓ, $\ell = 0, 1, \ldots, m-1$, is a polynomial with degree at most ℓ. Notice

$$\Phi(\xi)_{1,j} = \sum_{k \in \mathbb{Z}} \left[\int \phi_1(x+k)\overline{\phi_j}(x)dx \right] e^{-ik\xi} =: \sum_{k \in \mathbb{Z}} b_{k,j} e^{-ik\xi}$$

and for $\ell = 0, \ldots, m-1$, $j = 2, \ldots, r$,

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \overline{k^\ell b_{k,j}} &= \int \sum_{k \in \mathbb{Z}} \overline{[k^\ell \phi_1(x+k)]} \phi_j(x) dx \\
&= \int \overline{p_\ell(x)} \phi_j(x) dx \\
&= \left(\overline{p_\ell(-iD)} \widehat{\phi}_j \right)(0) = 0.
\end{aligned}$$

Thus for $j = 2, \ldots, r$,

$$(1 - e^{-i\xi})^m \mid \Phi(\xi)_{1,j} = \overline{\Phi(\xi)}_{j,1}. \tag{2.2.3}$$

By the definition of adjoint matrices, $(-1)^{j+1}\Phi^{\mathrm{adj}}_{1,j}$, $j = 2, \ldots, r$, is the determinant of a sub-matrix of Φ. Since the first row of this sub-matrix has a common factor $(1 - e^{-i\xi})^m$, we have

$$(1 - e^{-i\xi})^m \mid \Phi^{\mathrm{adj}}(\xi)_{1,j} = \overline{\Phi^{\mathrm{adj}}(\xi)}_{j,1}. \tag{2.2.4}$$

Moreover, if $\Phi > 0$, then $\det\Phi(\xi) > 0$ for all $\xi \in \mathbb{T}$. Therefore, $\det\Phi(0) \neq 0$. By the matrix identity $A^{-1} = A^{\mathrm{adj}}/\det A$ and (2.2.4), for $j = 2, \ldots, r$, we have

$$(1 - e^{-i\xi})^m \mid \Phi^{-1}(\xi)_{1,j} = \overline{\Phi^{-1}(\xi)}_{j,1}. \tag{2.2.5}$$

As we proved in Section 2.1, $T_a\Phi = \Phi$, i.e.,

$$\Phi(d\xi) = \sum_{j=0}^{|d|-1} a(\xi + 2\pi j/d)\Phi(\xi + 2\pi j/d)a(\xi + 2\pi j/d)^*.$$

By (2.2.2), we have

$$(1-e^{-i\xi})^{2m} \mid \left[a(\xi+2\pi j/d)\Phi(\xi+2\pi j/d)a(\xi+2\pi j/d)^*\right]_{1,1}, \qquad j=2,\ldots,r.$$

Therefore,

$$\Phi(d\xi)_{1,1} = \left[a(\xi)\Phi(\xi)a(\xi)^*\right]_{1,1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.$$

Combining the equality above with (2.2.2) and (2.2.3), we have

$$\Phi(d\xi)_{1,1} = |a(\xi)_{1,1}|^2 \Phi(\xi)_{1,1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.$$

Hence

$$\text{(2.2.6)} \qquad \left(\Phi(\xi)_{1,1}\right)^{-1} = |a(\xi)_{1,1}|^2 \left(\Phi(d\xi)_{1,1}\right)^{-1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.$$

By (2.2.3), (2.2.5) and $\Phi(\xi)\Phi^{-1}(\xi) = I_r$, we have

$$1 = \Phi(\xi)_{1,1}\Phi^{-1}(\xi)_{1,1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.$$

Therefore,

$$\text{(2.2.7)} \qquad \left(\Phi(\xi)_{1,1}\right)^{-1} = \Phi^{-1}(\xi)_{1,1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.$$

By (2.2.2), (2.2.6) and (2.2.7), we have

$$\begin{aligned}
\Phi^{-1}(\xi)_{1,1} &= \left(\Phi(\xi)_{1,1}\right)^{-1} + O(|\xi|^{2m}) \\
&= |a(\xi)_{1,1}|^2 \left(\Phi(d\xi)_{1,1}\right)^{-1} + O(|\xi|^{2m}) \\
&= |a(\xi)_{1,1}|^2 \Phi^{-1}(d\xi)_{1,1} + O(|\xi|^{2m}) \\
&= \left[a(\xi)^*\Phi^{-1}(d\xi)a(\xi)\right]_{1,1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.
\end{aligned}$$

Therefore

$$(1-e^{-i\xi})^{2m} \mid \left[\Phi^{-1}(\xi) - a(\xi)^*\Phi^{-1}(d\xi)a(\xi)\right]_{1,1}.$$

∎

Now let us consider the eigenvectors of T_a. As we showed in Section 2.1, $T_a\Phi = \Phi$. Moreover, if the integer shifts of ϕ are stable, then the cascade algorithm associated with the mask a converges in L_2. Hence, we have that T_a has a simple eigenvalue 1 and all other eigenvalues of T_a are less than 1 in modulus (see [45]). Thus under the condition the integer shifts of ϕ are stable, we have that Φ is the unique 1-eigenfunction of T_a and for all $\xi \in \mathbb{T}$, $\Phi(\xi) > 0$. Define $\widetilde{a}$ as

$$\widetilde{a}(\xi) := \begin{bmatrix} (1-e^{-id\xi})^m & 0 \\ 0 & I_{r-1} \end{bmatrix}^{-1} a(\xi) \begin{bmatrix} (1-e^{-i\xi})^m & 0 \\ 0 & I_{r-1} \end{bmatrix}.$$

Notice we assume that a takes the form (2.2.2), so $\widetilde{a}$ is an $r \times r$ matrix of trigonometric polynomials. Define a new operator $T_{\widetilde{a}}$ mapping $C(\mathbb{T})^{r\times r}$ into $C(\mathbb{T})^{r\times r}$ by

$$(2.2.8) \qquad T_{\widetilde{a}}F(d\xi) := \sum_{j=0}^{|d|-1} \widetilde{a}(\xi + 2j\pi/d)F(\xi + 2j\pi/d)\widetilde{a}(\xi + 2j\pi/d)^*.$$

Since $\widetilde{a}$ is an $r \times r$ matrix of trigonometric polynomials, we denote $\mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}$ a subspace containing all $r \times r$ matrices of trigonometric polynomials up to a finite degree determined by $\widetilde{a}$ such that it is an invariant subspace of $T_{\widetilde{a}}$ and $\rho(T_{\widetilde{a}}) = \rho(T_{\widetilde{a}}|_{\mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}})$ (see [22]). Similarly, since a is an $r \times r$ matrix of trigonometric polynomials, denote $\mathcal{P}_a(\mathbb{T})^{r\times r}$ a subspace containing all $r \times r$ matrices of trigonometric polynomials up to a finite degree determined by a such that it is an invariant subspace of T_a and $\rho(T_a) = \rho(T_a|_{\mathcal{P}_a(\mathbb{T})^{r\times r}})$. It is obvious that

$$T_{\widetilde{a}}\widetilde{F} = \lambda\widetilde{F} \implies T_a F = \lambda F$$

where F is derived from $\widetilde{F}$ by

$$(2.2.9) \quad F(\xi) = \begin{bmatrix} (1-e^{-i\xi})^m & 0 \\ 0 & I_{r-1} \end{bmatrix} \widetilde{F}(\xi) \begin{bmatrix} (1-e^{-i\xi})^m & 0 \\ 0 & I_{r-1} \end{bmatrix}^* \qquad \forall \xi \in \mathbb{T}.$$

Therefore, we have

$$(2.2.10) \qquad \rho(T_{\widetilde{a}}) \leqslant \rho(T_a).$$

Now we are in the position to prove the following Theorem.

Theorem 2.1. *Let ϕ and a be defined as in Lemma 2.1. If ϕ has stable integer shifts, then there exist a positive number $\rho < 1$ and some $\widetilde{F} \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that $\widetilde{F} > 0$ and $T_a F \leqslant \rho F$ where F is derived from $\widetilde{F}$ by (2.2.9).*

Proof: Since $T_{\widetilde{a}}$ is a linear operator acting on $\mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}$ which is a finite dimensional space, by the definition of spectrum, we know that there exists $0 \neq \widetilde{G}(\xi) \in \mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}$ such that $T_{\widetilde{a}}\widetilde{G} = \lambda_0 \widetilde{G}$ and $|\lambda_0| = \rho(T_{\widetilde{a}}|_{\mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}})$. Define

$$G(\xi) = \begin{bmatrix} (1-e^{-i\xi})^m & 0 \\ 0 & I_{r-1} \end{bmatrix} \widetilde{G}(\xi) \begin{bmatrix} (1-e^{-i\xi})^m & 0 \\ 0 & I_{r-1} \end{bmatrix}^*,$$

then it is evident to see that we have $T_a G = \lambda_0 G$.

Since $\Phi(0)_{1,1} \geqslant |\widehat{\phi}_1(0)|^2 \neq 0$, $\Phi(0)_{1,1} \neq 0$ and $G(0)_{1,1} = 0$ by the definition of G. Thus for all $\lambda \in \mathbb{C}$, we have $\Phi \neq \lambda G$. Notice that Φ is the unique 1-eigenfunction of T_a, we have $|\lambda_0| = \rho(T_{\widetilde{a}}) \neq 1$. Since ϕ has stable integer shifts, T_a has a simple eigenvalue 1 and all other eigenvalues of T_a are less than 1 in modulus (see [45]). By (2.2.10), we have $\rho(T_{\widetilde{a}}) \leqslant 1$. Combining the fact that $\rho(T_{\widetilde{a}}) \neq 1$, we have $|\lambda_0| = \rho(T_{\widetilde{a}}) < 1$.

Choose $\rho_1 := [1 + \rho(T_{\widetilde{a}})]/2$. Then we have $\rho(T_{\widetilde{a}}) < \rho_1 < 1$. Borrowing the idea from the proof of [38, Theorem 3], since $\rho(T_{\widetilde{a}}/\rho_1) < 1$, $(Id - T_{\widetilde{a}}/\rho_1)^{-1}$ is a well defined operator acting on $\mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}$ and

$$(Id - T_{\widetilde{a}}/\rho_1)^{-1} = Id + T_{\widetilde{a}}/\rho_1 + T_{\widetilde{a}}^2/\rho_1^2 + \cdots,$$

where Id denotes the identity operator. Define $\widetilde{F} = (Id - T_{\widetilde{a}}/\rho_1)^{-1} I_r$, then $\widetilde{F} = I_r + T_{\widetilde{a}} I_r/\rho_1 + \cdots$. Thus $\widetilde{F} \in \mathcal{P}_{\widetilde{a}}(\mathbb{T})^{r\times r}$ and $\widetilde{F} > 0$. By the definition of $\widetilde{F}$, we have $(Id - T_{\widetilde{a}}/\rho_1)\widetilde{F} > 0$. Therefore, $T_{\widetilde{a}}\widetilde{F} < \rho_1 \widetilde{F}$. Let F be derived from $\widetilde{F}$ by (2.2.9), then we have $T_a F \leqslant \rho_1 F$. ∎

Remark: Theorem 2.1 is inspired by [4, Lemma 5]. When I was trying to generalize [4, Lemma 5], I met some difficulty to apply Perron-Frobenius theory. Later I realized that the conclusion of [4, Lemma 5] is somehow stronger than its role in [4], i.e., we can use "$T_a \leqslant \rho F$" instead of "$T_a F = \rho F$". That is how Theorem 2.1 comes out.

After Theorem 2.1, our next goal is to prove that there exist a positive number ϵ and some $\Theta \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that for all $\xi \in \mathbb{T}$,

$$\big(\Phi(\xi) + \epsilon F(\xi)\big)^{-1} \leqslant \Theta(\xi) \leqslant \big(\Phi(\xi) + \epsilon\rho F(\xi)\big)^{-1}.$$

Let us prove the following lemmas first.

Lemma 2.2. *If $A, B \in C(\mathbb{T})^{r\times r}$ such that $A^* = A$, $B^* = B$ and $A < B$. Then there exists some $P \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that $A < P < B$.*

Proof: For every $\xi \in \mathbb{T}$, suppose $\lambda_1(\xi)$, ..., $\lambda_r(\xi)$ are all the eigenvalues of $(B-A)(\xi)$ and we have $\lambda_1(\xi) \geqslant \cdots \geqslant \lambda_r(\xi)$. Thus $B(\xi) - A(\xi) \geqslant \lambda_r(\xi) I_r$. Since $B(\xi) - A(\xi) > 0$, we have

$$\lambda_1(\xi) \geqslant \cdots \geqslant \lambda_r(\xi) > 0.$$

Hence we have, for all $\xi \in \mathbb{T}$,

$$\lambda_r(\xi) = \frac{\det\big(B(\xi) - A(\xi)\big)}{\lambda_1(\xi)\cdots\lambda_{r-1}(\xi)} \geqslant \frac{\det\big(B(\xi) - A(\xi)\big)}{\big[\mathrm{trace}\big(B(\xi) - A(\xi)\big)\big]^{r-1}} \geqslant c_1 I_r.$$

where c_1 is a suitable positive constant number we choose to satisfy the above inequality. Therefore, for all $u \in \mathbb{C}^r$, $u^*(B-A)u \geqslant c_1 u^* u$. Then we can get a trigonometric polynomial $P_1 \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that for $1 \leqslant i,j \leqslant r$ and for all $\xi \in \mathbb{T}$,

$$|[P_1 - (A+B)/2]_{i,j}(\xi)| < c_1/(4r^2).$$

Define $P := (P_1 + P_1^*)/2$, then we have $P^* = P$ and for all $u \in \mathbb{C}^r$,

$$\begin{aligned} u^*(P-A)u &= u^*[(A+B)/2 - A]u + u^*[P_1 - (A+B)/2]u/2 \\ &\quad + u^*[P_1^* - (A+B)^*/2]u/2 \\ &\geqslant c_1 u^* u/2 - c_1 u^* u/4/2 - c_1 u^* u/4/2 \\ &= c_1 u^* u/4. \end{aligned}$$

Thus $P - A \geqslant \frac{c_1}{4} I_r > 0$. Similarly, we have $B - P \geqslant \frac{c_1}{4} I_r > 0$. Thus $B < P < A$. ■

Lemma 2.3. *If $A, B \in C^\infty(\mathbb{T})^{r\times r}$ such that $A^* = A$, $B^* = B$ and $B - A = P_1 F P_1^*$, where $P_1 \in \mathcal{P}(\mathbb{T})^{r\times r}$, $\det P_1 \not\equiv 0$ and $F \in C(\mathbb{T})^{r\times r}$, $F > 0$, then there exists some $P \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that $A \leqslant P \leqslant B$.*

Proof: For all $\xi \in \mathbb{T}$, define $p_1(\xi) := \det P_1(\xi)$. Since $P_1 \in \mathcal{P}(\mathbb{T})^{r\times r}$, and $\det P_1 \not\equiv 0$, p_1 is a non-zero 2π-periodic trigonometric polynomial. Therefore, $|p_1|^2$ has finitely many roots in $\mathbb{T}$. Suppose all the roots of $|p_1|^2$ in $\mathbb{T}$ are

$$-\pi \leqslant \xi_1 < \ldots < \xi_N < \pi$$

with multiplicities $\alpha_1, \ldots, \alpha_N$, respectively. Then we have $|p_1|^2 = p_2 p_3$, where p_2 and p_3 are two 2π-periodic trigonometric polynomials such that $p_2(\xi) \neq 0$ for all $\xi \in \mathbb{T}$ and

$$p_3(\xi) = \Pi_{k=1}^N (e^{-i\xi} - e^{-i\xi_k})^{\alpha_k}.$$

By the Lagrange Interpolation Theorem, for $1 \leqslant i, j \leqslant r$, there exist unique $p_{i,j} \in \mathcal{P}(\mathbb{T})^{1\times 1}$ such that $\deg p_{i,j} < \deg p_3$ and

$$p_{i,j}^{(\ell)}(\xi_k) = A_{i,j}^{(\ell)}(\xi_k), \qquad \ell = 0, \ldots, \alpha_k - 1;\ k = 1, \ldots, N.$$

Define $\widetilde{A}_{i,j} := |p_1|^{-2}(A_{i,j} - p_{i,j})$, $P_0 := [p_{i,j}]_{1\leqslant i,j\leqslant r}$ and $\widetilde{A} = [\widetilde{A}_{i,j}]_{1\leqslant i,j\leqslant r}$. It is obvious that $P_0 \in \mathcal{P}(\mathbb{T})^{r\times r}$ and $A = P_0 + p_1 \widetilde{A} p_1^*$. Also, by the definition of $p_{i,j}$ and the fact that $A_{i,j}$ is a C^∞-function, it is evident to see that $\widetilde{A}_{i,j} = p_2^{-1}\frac{A_{i,j} - p_{i,j}}{p_3}$ is a continuous function on $\mathbb{T}$. Since $A^* = A$ and P_0 is uniquely determined by A, we have $P_0^* = P_0$ and $\widetilde{A}^* = \widetilde{A}$. By $F > 0$, $\widetilde{A} \in C(\mathbb{T})^{r\times r}$ and Lemma 2.2, there exists some $P_2 \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that

$$P_1^{\text{adj}} \widetilde{A} (P_1^{\text{adj}})^* \leqslant P_2 \leqslant P_1^{\text{adj}} \widetilde{A} (P_1^{\text{adj}})^* + F.$$

Let $P := P_0 + P_1 P_2 P_1^*$. We have

$$P - A = P_1 [P_2 - P_1^{\text{adj}} \widetilde{A} (P_1^{\text{adj}})^*] P_1^* \geqslant 0.$$

Similarly $B - P \geqslant 0$. Thus $A \leqslant P \leqslant B$. ∎

Recall $\Phi := \left[\widehat{\phi}, \widehat{\phi}\right]$, now we can prove the following proposition.

Proposition 2.4. Suppose $0 < \rho < 1$, $\Phi > 0$, $\widetilde{F} \in C(\mathbb{T})^{r\times r}$, $\widetilde{F} > 0$, F is derived from $\widetilde{F}$ by (2.2.9), then there exist $\epsilon > 0$ and $\Theta \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that

$$(\Phi + \epsilon F)^{-1} \leqslant \Theta \leqslant (\Phi + \epsilon\rho F)^{-1}.$$

Proof: For a constant $r \times r$ matrix $A \geqslant 0$, we have

$$(I + A)^{-1} = I - A + A^2(I + A)^{-1} = I - A + A(I + A)^{-1}A.$$

Hence we have that for all $A \geqslant 0$,

$$(I + A)^{-1} \geqslant I - A.$$

Also for a given positive number λ, if $A \leqslant \lambda I_r$, we have

$$(I + A)(I - A + \lambda A) = I - A^2 + \lambda A + \lambda A^2 \geqslant I + A(\lambda I_r - A) \geqslant I.$$

Hence

(2.2.11) $$(I + A)^{-1} \leqslant I - A + \lambda A \qquad \text{when } A \leqslant \lambda I_r.$$

Since $\Phi > 0$, we have $\Phi_1 := \Phi^{1/2} > 0$. Thus we have

$$\begin{aligned}(\Phi + \epsilon\rho F)^{-1} &= \Phi_1^{-1}[I + \epsilon\rho\Phi_1^{-1}F\Phi_1^{-1}]^{-1}\Phi_1^{-1} \\ &\geqslant \Phi_1^{-1}[I - \epsilon\rho\Phi_1^{-1}F\Phi_1^{-1}]\Phi_1^{-1} = \Phi^{-1} - \epsilon\rho\Phi^{-1}F\Phi^{-1},\end{aligned}$$

i.e.,

(2.2.12) $$(\Phi + \epsilon\rho F)^{-1} \geqslant \Phi^{-1} - \epsilon\rho\Phi^{-1}F\Phi^{-1}.$$

Choose $\epsilon > 0$ small enough such that $\epsilon F \leqslant (1-\rho)\Phi/2$ since $\Phi \geqslant cI_r$ for some positive number c. By inequality (2.2.11), choosing $\lambda = (1-\rho)/2$, similarly to the proof of inequality (2.2.12), we have

(2.2.13) $$(\Phi + \epsilon F)^{-1} \leqslant \Phi^{-1} - \epsilon(1+\rho)\Phi^{-1}F\Phi^{-1}/2.$$

Define

(2.2.14) $$A_0(\xi) := \operatorname{diag}[(1 - e^{-i\xi}), 1, \ldots, 1]$$

and

$$\widetilde{\Phi} := A_0^{-m}\Phi^{-1}A_0^m.$$

By $\Phi > 0$ we know that the determinant of $\widetilde{\Phi}$ is positive. By Lemma 2.1 we know that $\widetilde{\Phi} \in C^\infty(\mathbb{T})^{r\times r}$. By the definition of $\widetilde{\Phi}$, we have

$$\begin{aligned}
&[\Phi^{-1} - \epsilon\rho\Phi^{-1}F\Phi^{-1}] - [\Phi^{-1} - \epsilon(1+\rho)\Phi^{-1}F\Phi^{-1}/2] \\
&= \epsilon(1-\rho)\Phi^{-1}F\Phi^{-1}/2 \\
&= \epsilon(1-\rho)A_0^m[\widetilde{\Phi}A_0^{-m}F(A_0^{-m})^*\widetilde{\Phi}^*](A_0^m)^*/2 \\
&= \epsilon(1-\rho)A_0^m[\widetilde{\Phi}\widetilde{F}\widetilde{\Phi}^*](A_0^m)^*/2.
\end{aligned} \tag{2.2.15}$$

It is obvious that $\widetilde{\Phi}\widetilde{F}\widetilde{\Phi}^* > 0$. By Lemma 2.3, there exists some $P \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that

$$\Phi^{-1} - \epsilon(1+\rho)\Phi^{-1}F\Phi^{-1}/2 \leqslant P \leqslant \Phi^{-1} - \epsilon\rho\Phi^{-1}F\Phi^{-1}. \tag{2.2.16}$$

Plus inequalities (2.2.12) and (2.2.13), we have

$$(\Phi + \epsilon F)^{-1} \leqslant P \leqslant (\Phi + \epsilon\rho F)^{-1}.$$

■

Now we can prove the following proposition.

Proposition 2.5. Let ϕ and a be defined as in Theorem 2.1. If ϕ has stable integer shifts, then there exists $\Theta \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that $\Theta(0)_{1,1} = 1$, $\Theta > 0$ and

$$\Theta^{-1} - T_a(\Theta^{-1}) \geqslant 0. \tag{2.2.17}$$

Proof: By Theorem 2.1, there exists some positive number $\rho < 1$ and $\widetilde{F} > 0$ such that $T_aF < \rho F$ where F is defined as in (2.2.9). As we discussed before, Φ satisfies $T_a\Phi = \Phi > 0$. Hence by Proposition 2.4, there exist $\epsilon > 0$ and some $P \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that

$$0 < (\Phi + \epsilon F)^{-1} \leqslant P \leqslant (\Phi + \epsilon\rho F)^{-1},$$

i.e.,

$$\Phi + \rho\epsilon F \leqslant P^{-1} \leqslant \Phi + \epsilon F. \tag{2.2.18}$$

Let $\Theta := P > 0$, then by inequality (2.2.18) we have $\Theta(0)_{1,1} = 1$ and

$$\Theta^{-1} - T_a(\Theta^{-1}) \geqslant (\Phi + \rho\epsilon F) - T_a(\Theta^{-1}) \geqslant (\Phi + \rho\epsilon F) - T_a(\Phi + \epsilon F) \geqslant 0.$$

■

Now let us go through the following well-known lemmas.

Lemma 2.6. *Suppose A is an $m \times n$ matrix and B is an $n \times m$ matrix. Then we have*

$$\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA).$$

Proof: We have the following identities.

$$\begin{bmatrix} I_m & A \\ B & \lambda I_n \end{bmatrix} \begin{bmatrix} \lambda I_m & 0 \\ -B & I_n \end{bmatrix} = \begin{bmatrix} \lambda I_m - AB & A \\ 0 & \lambda I_n \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & A \\ B & \lambda I_n \end{bmatrix} \begin{bmatrix} \lambda I_m & -A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} \lambda I_m & 0 \\ \lambda B & \lambda I_n - BA \end{bmatrix}.$$

Taking the determinants of the matrices in the above two equations, we have

$$\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA).$$

■

Lemma 2.7. *Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. If $(AB)^* = AB$ and $(BA)^* = BA$, then we have*

$$I_m - AB \geqslant 0 \Longleftrightarrow I_n - BA \geqslant 0.$$

Proof: Suppose $I_m - AB \geqslant 0$. Then all the eigenvalues of $(I_m - AB)$ are nonnegative. Hence

$$\det[\lambda I_m - (I_m - AB)] = \prod_{j=1}^{m} (\lambda - \lambda_j),$$

where for $j = 1, \dots, m$, λ_j is nonnegative. By Lemma 2.6, we have

$$\begin{aligned}(\lambda-1)^{m-n}\det[\lambda I_n - (I_n - BA)] &= (\lambda-1)^{m-n}\det[(\lambda-1)I_n + BA)] \\ &= \det[(\lambda-1)I_m + AB] = \prod_{j=1}^{m}(\lambda - \lambda_j).\end{aligned}$$

Thus all the eigenvalues of $(I_n - BA)$ are nonnegative. Combining that $(BA)^* = BA$, we know that $I_n - BA \geqslant 0$. Therefore, $I_m - AB \geqslant 0$ implies $I_n - BA \geqslant 0$. Similarly, $I_n - BA \geqslant 0$ implies $I_m - AB \geqslant 0$. ■

Finally, we are in the position to state our main theorem in this chapter:

Theorem 2.2. *Let $\phi \in \big(L_2(\mathbb{R})\big)^r$ be a refinable function vector. If the integer shifts of ϕ are stable and the matrix mask a of ϕ satisfies sum rules of order m, then there exists a tight multiwavelet frame derived from ϕ and it has vanishing moments of order m.*

Proof: By Theorem 1.2, we can assume that the matrix mask a takes the canonical form (2.2.2). By Proposition 2.5, there exist $\Theta \in \mathcal{P}(\mathbb{T})^{r\times r}$ such that $\Theta(0)_{1,1} = 1$, $\Theta > 0$ and $\Theta^{-1} - T_a(\Theta^{-1}) \geqslant 0$.

First we want to prove that $M \geqslant 0$, where M is defined by

$$\begin{aligned}M(\xi) := {} & \operatorname{diag}\big[\Theta(\xi), \Theta(\xi + 2\pi/d), \dots, \Theta(\xi + 2(|d|-1)\pi/d)\big] - \\ & \big[a(\xi), a(\xi + 2\pi/d), \dots, a(\xi + 2(|d|-1)\pi/d)\big]^* \,\Theta(d\xi) \times \\ & \big[a(\xi), a(\xi + 2\pi/d), \dots, a(\xi + 2(|d|-1)\pi/d)\big] \qquad \forall \xi \in \mathbb{T}.\end{aligned} \tag{2.2.19}$$

By the fact $\Theta > 0$ we have $\Theta_1 := \Theta^{1/2} > 0$. Therefore, by the definition of M in (2.2.19), we have

$$\begin{aligned}M(\xi) = {} & \operatorname{diag}[\Theta_1(\xi), \Theta_1(\xi + 2\pi/d), \dots, \Theta_1(\xi + 2(|d|-1)\pi/d)] \\ & \big(I_{|d|r} - [\Theta_1(d\xi)a(\xi)\Theta_1(\xi)^{-1}, \Theta_1(d\xi)a(\xi + 2\pi/d)\Theta_1(\xi + 2\pi/d)^{-1}, \dots, \\ & \quad \Theta_1(d\xi)a(\xi + 2(|d|-1)\pi/d)\Theta_1(\xi + 2(|d|-1)\pi/d)^{-1}]^* \times \\ & \quad [\Theta_1(d\xi)a(\xi)\Theta_1(\xi)^{-1}, \Theta_1(d\xi)a(\xi + 2\pi/d)\Theta_1(\xi + 2\pi/d)^{-1}, \dots, \\ & \quad \Theta_1(d\xi)a(\xi + 2(|d|-1)\pi/d)\Theta_1(\xi + 2(|d|-1)\pi/d)^{-1}]\big) \\ & \operatorname{diag}[\Theta_1(\xi), \Theta_1(\xi + 2\pi/d), \dots, \Theta_1(\xi + 2(|d|-1)\pi/d)].\end{aligned}$$

Hence,

$$
\begin{aligned}
&M(\xi) \geqslant 0\\
&\Longleftrightarrow I_{|d|r} - [\Theta_1(d\xi)a(\xi)\Theta_1(\xi)^{-1}, \Theta_1(d\xi)a(\xi+2\pi/d)\Theta_1(\xi+2\pi/d)^{-1}, \dots,\\
&\quad \Theta_1(d\xi)a(\xi+2(|d|-1)\pi/d)\Theta_1(\xi+2(|d|-1)\pi/d)^{-1}]^* \times\\
&\quad [\Theta_1(d\xi)a(\xi)\Theta_1(\xi)^{-1}, \Theta_1(d\xi)a(\xi+2\pi/d)\Theta_1(\xi+2\pi/d)^{-1}, \dots,\\
&\quad \Theta_1(d\xi)a(\xi+2(|d|-1)\pi/d)\Theta_1(\xi+2(|d|-1)\pi/d)^{-1}] \geqslant 0.
\end{aligned}
$$

By Lemma 2.7, for all $\xi \in \mathbb{T}$, we have that

$$
\begin{aligned}
&M(\xi) \geqslant 0\\
&\Longleftrightarrow I_r - \big[\Theta_1(d\xi)a(\xi)\Theta_1(\xi)^{-1}, \Theta_1(d\xi)a(\xi+2\pi/d)\Theta_1(\xi+2\pi/d)^{-1}, \dots,\\
&\quad \Theta_1(d\xi)a(\xi+2(|d|-1)\pi/d)\Theta_1(\xi+2(|d|-1)\pi/d)^{-1}\big] \times\\
&\quad \big[\Theta_1(d\xi)a(\xi)\Theta_1(\xi)^{-1}, \Theta_1(d\xi)a(\xi+2\pi/d)\Theta_1(\xi+2\pi/d)^{-1}, \dots,\\
&\quad \Theta_1(d\xi)a(\xi+2(|d|-1)\pi/d)\Theta_1(\xi+2(|d|-1)\pi/d)^{-1}\big]^* \geqslant 0.
\end{aligned}
$$

By $\Theta = \Theta_1^2 = \Theta_1^*\Theta_1$, we have

$$M(\xi) \geqslant 0 \Leftrightarrow I_r - \Theta_1(d\xi)T_a(\Theta^{-1})(d\xi)\Theta_1^*(d\xi) \geqslant 0 \Leftrightarrow \Theta^{-1}(d\xi) \geqslant T_a(\Theta^{-1})(d\xi),$$

i.e.,

$$M \geqslant 0 \Longleftrightarrow \Theta^{-1} \geqslant T_a(\Theta^{-1}).$$

Thus by inequality (2.2.17), we know $M \geqslant 0$.

Secondly, we want to prove that we can derive a tight wavelet frame from the given Θ. As a special case of Theorem 1.6, to derive a tight wavelet frame from the given Θ, we need to find $r \times r$ matrices $a^1(\xi)$,..., $a^L(\xi)$ of trigonometric polynomials such that

$$
\begin{bmatrix}
a^1(\xi)^* & \cdots & a^L(\xi)^*\\
a^1(\xi+\frac{2\pi}{d})^* & \cdots & a^L(\xi+\frac{2\pi}{d})^*\\
\vdots & \ddots & \vdots\\
a^1\Big(\xi+\frac{2\pi(|d|-1)}{d}\Big)^* & \cdots & a^L\Big(\xi+\frac{2\pi(|d|-1)}{d}\Big)^*
\end{bmatrix}
\begin{bmatrix}
a^1(\xi)\\ a^2(\xi)\\ \vdots\\ a^L(\xi)
\end{bmatrix}
= M_1(\xi), \tag{2.2.20}
$$

where

$$M_1(\xi) := \begin{bmatrix} \Theta(\xi) - a(\xi)^*\Theta(d\xi)a(\xi) \\ -a(\xi + \frac{2\pi}{d})^*\Theta(d\xi)a(\xi) \\ \vdots \\ -a\left(\xi + \frac{2\pi(|d|-1)}{d}\right)^*\Theta(d\xi)a(\xi) \end{bmatrix}.$$

Recall M is defined as

$$M(\xi) = \begin{bmatrix} \Theta(\xi) & & \\ & \ddots & \\ & & \Theta\left(\xi + \frac{2\pi(|d|-1)}{d}\right) \end{bmatrix} -$$

$$\begin{bmatrix} a(\xi)^* \\ a(\xi + \frac{2\pi}{d})^* \\ \vdots \\ a\left(\xi + \frac{2\pi(|d|-1)}{d}\right)^* \end{bmatrix} \Theta(d\xi) \begin{bmatrix} a(\xi)^* \\ a(\xi + \frac{2\pi}{d})^* \\ \vdots \\ a\left(\xi + \frac{2\pi(|d|-1)}{d}\right)^* \end{bmatrix}^*.$$

It is evident to see that (2.2.20) is equivalent to

$$A(\xi)^*A(\xi) = M(\xi), \tag{2.2.21}$$

where

$$A(\xi) := \begin{bmatrix} a^1(\xi) & \cdots & a^1\left(\xi + \frac{2\pi(|d|-1)}{d}\right) \\ \vdots & \ddots & \vdots \\ a^L(\xi) & \cdots & a^L\left(\xi + \frac{2\pi(|d|-1)}{d}\right) \end{bmatrix}.$$

Define

$$E(\xi) := \begin{bmatrix} I_r & e^{-i\xi}I_r & \cdots & e^{-i(|d|-1)\xi}I_r \\ I_r & e^{-i(\xi+2\pi/d)}I_r & \cdots & e^{-i(|d|-1)(\xi+2\pi/d)}I_r \\ \vdots & \vdots & \ddots & \vdots \\ I_r & e^{-i(\xi+2\pi(|d|-1)/d)}I_r & \cdots & e^{-i(|d|-1)(\xi+2\pi(|d|-1)/d)}I_r \end{bmatrix}.$$

To solve (2.2.21), using the polyphase decomposition, define

$$\widetilde{A}(d\xi) := A(\xi)E(\xi)$$

and

$$\widetilde{M}(d\xi) := E(\xi)^*M(\xi)E(\xi).$$

By direct calculation, we can verify that

$$M(\xi) \in \mathcal{P}(\mathbb{T})^{|d|r\times|d|r} \Longleftrightarrow \widetilde{M}(\xi) \in \mathcal{P}(\mathbb{T})^{|d|r\times|d|r}$$

and

$$a^1(\xi), \dots, a^L(\xi) \in \mathcal{P}(\mathbb{T})^{r\times r} \Longleftrightarrow \widetilde{A}(\xi) \in \mathcal{P}(\mathbb{T})^{Lr\times|d|r}.$$

It is evident to see that (2.2.21) is equivalent to

$$(2.2.22) \qquad \widetilde{A}(\xi)^*\widetilde{A}(\xi) = \widetilde{M}(\xi).$$

We are especially interested in the case $L = |d|$. In this case, $\widetilde{A}(\xi)$ is a $|d|r \times |d|r$ matrix. Since $M(\xi) \in \mathcal{P}(\mathbb{T})^{|d|r\times|d|r}$, we know that $\widetilde{M}(\xi) \in \mathcal{P}(\mathbb{T})^{|d|r\times|d|r}$. Moreover, by $M \geqslant 0$, we have $\widetilde{M} \geqslant 0$. Hence by the matrix-valued Fejér-Riesz Lemma([18], [28], [41]), there exists $\widetilde{A} \in \mathcal{P}(\mathbb{T})^{|d|r\times|d|r}$ such that $\widetilde{A}(\xi)^*\widetilde{A}(\xi) = \widetilde{M}(\xi)$. Therefore we can obtain trigonometric polynomial matrices $a^1(\xi)$, ..., $a^{|d|}(\xi)$ by the relation $A(\xi) = \widetilde{A}(d\xi)E(\xi)^{-1}$. Moreover, by the choice of a^1, ..., $a^{|d|}$, we know that $a^1(\xi)$, ..., $a^{|d|}(\xi)$ are $r \times r$ matrices of trigonometric polynomials and satisfy (2.2.20). Hence by Theorem 1.6 we can derive a tight wavelet frame with generators $\{\psi^1, ..., \psi^{|d|}\}$ such that ψ^1, ..., $\psi^{|d|}$ are $r \times 1$ function vectors and

$$\widehat{\psi^j}(d\xi) = a^j(\xi)\widehat{\phi}(\xi), \qquad j = 1, \dots, |d|.$$

Finally we want to prove that ψ^1, ..., $\psi^{|d|}$ all have vanishing moments of order m. By Corollary 1.2, we only need to prove that

$$(1 - e^{-i\xi})^m \mid a^j(\xi)_{k,1}, \qquad k = 1, \dots, r; j = 1, \dots, |d|,$$

where $a^j(\xi)_{k,1}$ denotes the (k, 1)-entry of $a^j(\xi)$. By (2.2.20), we have that

$$\begin{aligned}
\sum_{j=1}^{|d|}\sum_{k=1}^{r} |a^j(\xi)_{k,1}|^2 &= \left[\Theta(\xi) - a(\xi)^*\Theta(d\xi)a(\xi)\right]_{1,1} \\
&= \big[\Phi^{-1}(\xi) + A_0^m(\xi)G(\xi)A_0^m(\xi)^* - a(\xi)^*\Phi^{-1}(d\xi)a(\xi) \\
&\qquad - a(\xi)^*A_0^m(d\xi)G(d\xi)A_0^m(d\xi)^*a(\xi)\big]_{1,1} \\
&= \left[\Phi^{-1}(\xi) - a(\xi)^*\Phi^{-1}(d\xi)a(\xi)\right]_{1,1} + O(|\xi|^{2m}) \text{ as } \xi \to 0.
\end{aligned}$$

By Lemma 2.1, we have

$$(1-e^{-i\xi})^{2m} \mid \left[\Phi^{-1} - a(\xi)^*\Phi^{-1}(d\xi)a(\xi)\right]_{1,1}.$$

Hence

$$\sum_{j=1}^{|d|}\sum_{k=1}^{r} |a^j(\xi)_{k,1}|^2 = O(|\xi|^{2m}) \text{ as } \xi \to 0.$$

Therefore,

$$\sum_{j=1}^{|d|}\sum_{k=1}^{r} |a^j(\xi)_{k,1}/\xi^m|^2 < +\infty \text{ as } \xi \to 0.$$

Noticing the fact that the summation of nonnegative numbers is still nonnegative, we have

$$|a^j(\xi)_{k,1}/\xi^m| < +\infty \text{ as } \xi \to 0, \qquad k = 1,\dots,r; j = 1,\dots,|d|.$$

Thus,

$$(1-e^{-i\xi})^m \mid a^j(\xi)_{k,1}, \qquad k = 1,\dots,r; j = 1,\dots,|d|.$$

Hence ψ^1, ..., $\psi^{|d|}$ all have vanishing moments of order m. ■

2.3 Example

Example 2.1. Let us recall the mask of the well known piecewise Hermite cubics ϕ (it was discussed in Section 1.2) which is given by

$$(2.3.23) \qquad a(\xi) := \begin{bmatrix} (e^{i\xi}+2+e^{-i\xi})/4 & 3(e^{i\xi}-e^{-i\xi})/8 \\ (-e^{i\xi}+e^{-i\xi})/16 & (-e^{i\xi}+4-e^{-i\xi})/16 \end{bmatrix}.$$

In Section 1.2, we already proved that a satisfies the sum rules of order 4 with a row vector $y(\xi) = [1, e^{i\xi}/3 + 1/2 - e^{-i\xi} + e^{-2i\xi}/6]$. Take $m = n = 1$. Define $U(\xi) := I_2$. Then $U(2\xi)a(\xi)U(\xi)^{-1}$ takes the form of (1.2.15) with $m = n = 1$. Define

$$\Theta(\xi) := \begin{bmatrix} 1 & 0 \\ 0 & 15+9\sqrt{2} \end{bmatrix},$$

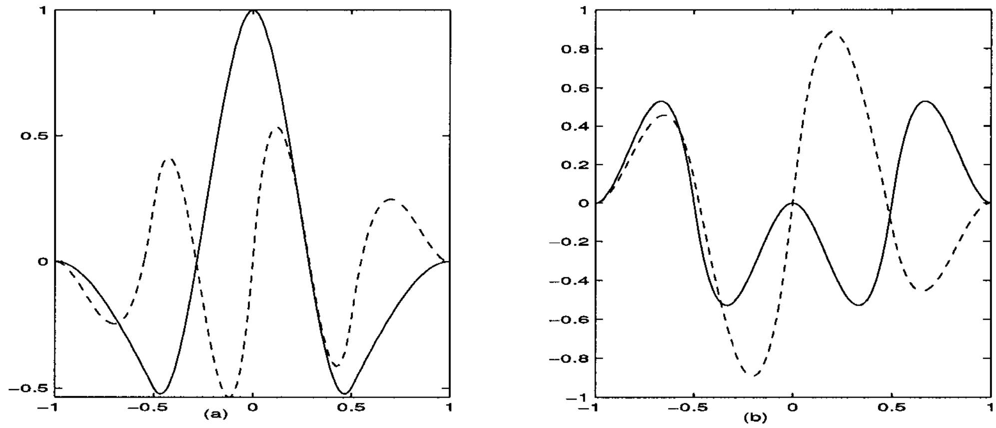

Figure 2.1: Generators for the tight 2-wavelet frame in Example 2.1: (a) ψ^1 (b) ψ^2. All the components in the wavelet function vectors ψ^1, ψ^2 are either symmetric or antisymmetric about the origin and have vanishing moments of order 1.

and

$$a^1(\xi) := d_1 \begin{bmatrix} (2-e^{-i\xi}-e^{i\xi})/4 & 3(e^{-i\xi}-e^{i\xi})/8 \\ (-29+16\sqrt{2})(e^{-i\xi}-e^{i\xi})/784 & (20+11e^{-i\xi}+11e^{i\xi})/112 \end{bmatrix},$$

$$a^2(\xi) := d_2 \begin{bmatrix} 0 & (3\sqrt{6}+4\sqrt{3})(e^{-i\xi}-e^{i\xi})/8 \\ (6-5\sqrt{2})(e^{-i\xi}-e^{i\xi})/196 & 3(2-e^{-i\xi}-e^{i\xi})/28 \end{bmatrix}$$

where

$$d_1 := \operatorname{diag}\left[1, \sqrt{105+63\sqrt{2}}\right] \text{ and } d_2 := \operatorname{diag}\left[1, \sqrt{70+42\sqrt{2}}\right].$$

Define function vectors ψ^1 and ψ^2 by

$$\widehat{\psi^1}(2\xi) := a^1(\xi)\widehat{\phi}(\xi), \quad \widehat{\psi^2}(2\xi) := a^2(\xi)\widehat{\phi}(\xi).$$

By a direct computation based on Theorem 1.6, one can verify that $\{\psi^1, \psi^2\}$ generates a tight 2-wavelet frame. Moreover, function vectors ψ^1, ψ^2 are real-valued and symmetric, and all of them have vanishing moments of order 1. For their graphs, see Figure 2.1.

Chapter 3

Tight wavelet frames with three symmetric generators having high vanishing moments

3.1 Introduction

In Chapters 1 and 2, we discussed how to construct multiwavelet frames from refinable function vectors and the existence of a tight multiwavelet frame derived from any given refinable function vector with stable shifts. These two chapters generalize the corresponding result in the classical wavelet theory, i.e., the case of multiplicity $r = 1$. However, even in the classical wavelet theory, some interesting questions have not been answered yet. In classical wavelet theory, tight wavelet frames generated by symmetric functions are very interesting. A question that had not been answered is: for a given positive integer n, can we find a tight wavelet frame with some symmetric generators having vanishing moments n and the generators belong to C^{n-2}? By numerical computation, it was verified in [13] (also in [4]) that the answer to the above question is positive for $n = 1, \ldots, 6$. In this chapter, we are going to discuss how to construct a family of tight wavelet frames with three symmetric generators having arbitrary smoothness and arbitrary vanishing moments. As discussed before, in many applications, symmetry and orthogonality are highly desirable properties. Since Daubechies proved that there does not exist smooth compactly supported symmetric orthogonal wavelet, we have to loosen some conditions. One way is to construct tight wavelet frames with some smooth symmetric generators. Then we have a choice on how many generators we are going to pick. In practice, it is highly desirable to use few generators. By Daubechies' proof, we know that using one generator is not possible and it is easy to see that using four generators is trivial. So the question will naturally be restricted to use two or three generators to construct wavelet frames. Using two generators is very difficult, we will discuss it in the next chapter. So it is suitable to discuss the case of three generators constructing wavelet frames in this chapter. After that, the next choice is how we can choose the refinable function to derive frame generators. One natural choice is to use B-splines. Let us recall that a B-spline of order n is defined by the following inductive way: $B_1 := \chi_{[0,1)}$ and $B_m := B_1 * B_{m-1}$ for $m = 2, 3, \ldots$. Here "$*$" denotes the standard convolution. In this chapter, we will prove the following main result.

Theorem 3.1. *Let m be a positive integer and let B_m denote the B-spline function of order m. Then there exist three finitely supported sequences b^1, b^2, b^3 on $\mathbb{Z}$, which can be easily constructed by a simple procedure, such that by defining*

$$\psi^\ell := \sum_{k\in\mathbb{Z}} b^\ell(k) B_m(2\cdot -k), \qquad \ell = 1, 2, 3,$$

one has

(1) *$\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$ and has the vanishing moments of order m;*

(2) *ψ^1, ψ^2, ψ^3 are real-valued, symmetric and compactly supported functions such that $\psi^1(1-m-t) = (-1)^m\psi^1(t)$, $\psi^2(m-t) = \psi^2(t)$, and $\psi^3(m-t) = -\psi^3(t)$ for all $t \in \mathbb{R}$.*

The rest of this chapter is organized by the following way. In Section 3.2, I shall prove an interesting inequality. Based on this inequality, in Section 3.3, I shall present a simple step by step procedure to construct the sequences b^1, b^2 and b^3 in Theorem 3.1. All the results in this chapter have been summarized in the paper [24] which has been published in *Proceedings of the American Mathematical Society.*

3.2 Auxiliary inequalities

In order to prove Theorem 3.1, let us introduce some auxiliary inequalities in this section.

Let us recall the following inequality. For $a \geqslant 0$, $c \geqslant 0$, $b > 0$, $d > 0$, it is very easy to prove that if $\frac{a}{b} \geqslant \frac{c}{d}$, then we have $\frac{a}{b} \geqslant \frac{a+c}{b+d} \geqslant \frac{c}{d}$. Repeat the above inequality n times, we have the following lemma.

Lemma 3.1. *If $a_j \geqslant 0$ and $b_j > 0$ for all $j = 1, \dots, n$ such that $\frac{a_1}{b_1} \geqslant \frac{a_2}{b_2} \geqslant \cdots \geqslant \frac{a_n}{b_n}$, then*

$$\frac{a_1}{b_1} \geqslant \frac{a_1 + a_2}{b_1 + b_2} \geqslant \cdots \geqslant \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}.$$

For any positive integer m, throughout this chapter, we define $c_0 = 1$ and

$$c_j := \frac{(2j-1)!!}{(2j)!!(2j+1)} = \frac{1}{2j+1} \prod_{k=1}^{j} \left(1 - \frac{1}{2k}\right), \qquad j \in \mathbb{N}, \tag{3.2.1}$$

where $0!! := 1$, $1!! := 1$ and $n!! := (n-2)!! \times n$ for $n = 2, 3, \dots$. Note that

$$\frac{\xi/2}{\sin(\xi/2)} = \frac{\arcsin(\sin(\xi/2))}{\sin(\xi/2)} = \sum_{j=0}^{\infty} c_j \sin^{2j} \frac{\xi}{2}, \qquad \xi \in [-\pi, \pi]. \tag{3.2.2}$$

So we have

$$\sum_{j=0}^{\infty} c_j = \pi/2. \tag{3.2.3}$$

The following estimate will be needed later.

Lemma 3.2. *Let $c_0 := 1$ and $c_j (j \in \mathbb{N})$ be defined in (3.2.1). For any positive integer m, for $x \in [0,1]$, define*

$$f_m(x) := 4m(1+x)^{2m} \sum_{j=0}^{\infty} c_{j+m}(1-x^2)^j.$$

Then for $m \geqslant 2$, f_m is an increasing function on the interval $[0,1]$ and for $m \geqslant 3$, we have

$$4m(1+x)^{2m} \sum_{j=0}^{\infty} c_{j+m}(1-x^2)^j \geqslant \frac{\pi}{1-2^{1/2-m}} \qquad \forall\, x \in [0,1]. \tag{3.2.4}$$

Proof: By equality (3.2.3), it is evident that f_m is a continuous function on the interval $[0,1]$. For $x \in (0,1)$, by the definition of f_m, we have

$$f_m'(x) = 8m^2(1+x)^{2m-1} \sum_{j=0}^{\infty} c_{j+m}(1-x^2)^j - 8mx(1+x)^{2m} \sum_{j=0}^{\infty} jc_{j+m}(1-x^2)^{j-1}.$$

Consequently, for $x \in (0,1)$,

$$\frac{f_m'(x)}{8mx(1+x)^{2m}} = m\frac{1/x-1}{1-x^2} \sum_{j=0}^{\infty} c_{j+m}(1-x^2)^j - \sum_{j=0}^{\infty} jc_{j+m}(1-x^2)^{j-1}.$$

Denote $y := 1-x^2$. Then

$$1/x = 1/\sqrt{1-y} = (1-y)^{-1/2} = 1 + \sum_{j=1}^{\infty}(2j+1)c_j y^j, \qquad x \in (0,1).$$

Therefore, for $x \in (0,1)$, we have

$$\begin{aligned} \frac{f_m'(x)}{8mx(1+x)^{2m}} &= m[\sum_{j=0}^{\infty}(2j+3)c_{j+1}y^j][\sum_{j=0}^{\infty} c_{j+m}y^j] - \sum_{j=0}^{\infty}(j+1)c_{j+m+1}y^j \\ &=: \sum_{j=0}^{\infty} g_{m,j}c_{j+m+1}y^j, \end{aligned}$$

where $y = 1-x^2$ and the numbers $g_{m,j}$ are defined by

$$g_{m,j} := m\sum_{k=0}^{j}(2k+3)c_{k+1}\frac{c_{j+m-k}}{c_{j+m+1}} - (j+1), \qquad j \in \mathbb{N} \cup \{0\}. \tag{3.2.5}$$

In the following several pages, we want to estimate $g_{m,j}$, $j \in \mathbb{N} \cup \{0\}$ to prove that $g_{m,j}$ are all positive numbers. By $c_0 := 1$ and the definition of the numbers c_j ($j \in \mathbb{N}$) in (3.2.1), we have

$$(3.2.6) \qquad (2k+3)c_{k+1}\frac{c_{j+m-k}}{c_{j+m+1}} = \frac{2j+2m+3}{2j+2m-2k+1}\frac{\prod_{\ell=1}^{k+1}[1-1/(2\ell)]}{\prod_{\ell=j+m-k+1}^{j+m+1}[1-1/(2\ell)]}.$$

Note that for any nonnegative integer k, we have

$$\prod_{\ell=1}^{k+1}[1-1/(2\ell)] = \frac{1}{2}\prod_{\ell=2}^{k+1}[1-1/(2\ell)] \geqslant \frac{1}{2}\prod_{\ell=2}^{k+1}\sqrt{1-\frac{1}{2\ell}}\sqrt{1-\frac{1}{2\ell-1}}$$
$$= \frac{1}{2}\prod_{\ell=2}^{k+1}\sqrt{\frac{\ell-1}{\ell}} = \frac{1}{2\sqrt{k+1}}$$

and for $0 \leqslant k \leqslant j$,

$$\prod_{\ell=j+m-k+1}^{j+m+1}[1-1/(2\ell)] \leqslant \prod_{\ell=j+m-k+1}^{j+m+1}\sqrt{1-\frac{1}{2\ell}}\sqrt{1-\frac{1}{2\ell+1}}$$
$$= \prod_{\ell=j+m-k+1}^{j+m+1}\sqrt{\frac{2\ell-1}{2\ell+1}} = \frac{\sqrt{2j+2m-2k+1}}{\sqrt{2j+2m+3}}.$$

It follows from (3.2.6) and the above two inequalities that

$$(2k+3)c_{k+1}\frac{c_{j+m-k}}{c_{j+m+1}} \geqslant \frac{2j+2m+3}{2j+2m-2k+1}\cdot\frac{1}{2\sqrt{k+1}}\cdot\frac{\sqrt{2j+2m+3}}{\sqrt{2j+2m-2k+1}}$$
$$= \frac{1}{2\sqrt{k+1}}\frac{(j+m+3/2)^{3/2}}{(j+m+1/2-k)^{3/2}}.$$

Hence,

$$g_{m,j} + (j+1) = m\sum_{k=0}^{j}(2k+3)c_{k+1}\frac{c_{j+m-k}}{c_{j+m+1}}$$
$$\geqslant m(j+m+3/2)^{3/2}\sum_{k=0}^{j}\frac{1}{2\sqrt{k+1}(j+m+1/2-k)^{3/2}}$$
$$= m(j+m+3/2)^{3/2}\sum_{k=1}^{j+1}\frac{1}{2\sqrt{k}(j+m+3/2-k)^{3/2}}.$$

Let $t^1_{m,j,k} = \sqrt{(k+1)(j+m+\frac{5}{2}-k)}$ and $t^2_{m,j,k} = \sqrt{k[j+m+\frac{5}{2}-(k+1)]}$. It is evident to see that $t^1_{j,k,m} \geqslant t^2_{j,k,m}$ and moreover, we have

$$\begin{aligned}
&\frac{\sqrt{k+1}}{\sqrt{j+m+5/2-(k+1)}} - \frac{\sqrt{k}}{\sqrt{j+m+5/2-k}} \\
&= \frac{t^1_{m,j,k} - t^2_{m,j,k}}{\sqrt{j+m+3/2-k}\sqrt{j+m+5/2-k}} \\
&= \frac{(k+1)(j+m+5/2-k) - k[j+m+5/2-(k+1)]}{\sqrt{j+m+3/2-k}\sqrt{j+m+5/2-k}(t^1_{m,j,k}+t^2_{m,j,k})} \\
&= \frac{j+m+5/2}{\sqrt{j+m+3/2-k}\sqrt{j+m+5/2-k}(t^1_{m,j,k}+t^2_{m,j,k})} \\
&\leqslant \frac{j+m+5/2}{2\sqrt{k}(j+m+3/2-k)^{3/2}}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&g_{m,j} + (j+1) \\
&\geqslant m\frac{(j+m+3/2)^{3/2}}{j+m+5/2}\sum_{k=1}^{j+1}\left[\frac{\sqrt{k+1}}{\sqrt{j+m+5/2-(k+1)}} - \frac{\sqrt{k}}{\sqrt{j+m+5/2-k}}\right] \\
&= \frac{m(j+m+3/2)^{3/2}}{j+m+5/2}\left[\frac{\sqrt{j+2}}{\sqrt{m+1/2}} - \frac{\sqrt{1}}{\sqrt{j+m+3/2}}\right] \\
&= \frac{m(j+m+3/2)^{3/2}}{j+m+5/2}\times \\
&\frac{(j+2)(j+m+3/2)-(m+1/2)}{\sqrt{m+1/2}\sqrt{j+m+3/2}[\sqrt{(j+2)(j+m+3/2)}+\sqrt{m+1/2}]} \\
&= (j+1)\frac{m(j+m+3/2)}{\sqrt{(m+1/2)(j+2)(j+m+3/2)}+(m+1/2)}.
\end{aligned}$$

By computation, for $m \geqslant 2$ and $j \geqslant 0$, we have

$$\begin{aligned}
&[m(j+m+3/2)-m-1/2]^2 - (m+1/2)(j+2)(j+m+3/2) \\
=&[m(m-1)-1/2]j^2 + [m(2m^2-5)-7/4]j + m(m+1)(m^2-4) \\
&+ 5m(m-1)/4 + (3m-5)/4 > 0.
\end{aligned}$$

Consequently, for all $m \geqslant 2$ and $j \geqslant 0$, we have

$$m(j+m+3/2) > \sqrt{(m+1/2)(j+2)(j+m+3/2)} + (m+1/2).$$

Hence, $g_{m,j} + (j+1) > j+1$ for all $m \geqslant 2$ and $j \geqslant 0$. That is

$$g_{m,j} > 0 \qquad \forall m \geqslant 2 \text{and} j \geqslant 0$$

which implies $f'_m(x) > 0$ for all $x \in (0,1)$ and $m \geqslant 2$.

So, for $m \geqslant 2$, f_m is an increasing function on the interval $[0,1]$ and $f_m \geqslant f_m(0)$. Note that $1 - 1/(2k) = (2k-1)/(2k) \geqslant \sqrt{(k-1)/k}$ for all $k \in \mathbb{N}$. We deduce that

$$\begin{aligned} c_j &= \frac{1}{2j+1}\prod_{k=1}^{j}[1-1/(2k)] \geqslant \frac{1}{2j+1}\frac{1}{2}\prod_{k=2}^{j}\sqrt{(k-1)/k} \\ &= \frac{1}{2(2j+1)\sqrt{j}} \geqslant \frac{1}{2\sqrt{j}} - \frac{1}{2\sqrt{j+1}} \end{aligned} \tag{3.2.7}$$

for all $j \in \mathbb{N}$. When $m \geqslant 4$, by inequality (3.2.7), we have

$$\begin{aligned} f_m(0) &= 4m\sum_{j=m}^{\infty} c_j \geqslant 4m\sum_{j=m}^{\infty}\left(\frac{1}{2\sqrt{j}} - \frac{1}{2\sqrt{j+1}}\right) = 2\sqrt{m} \\ &\geqslant 4 > \pi/(1-2^{1/2-4}) \geqslant \pi/(1-2^{1/2-m}). \end{aligned}$$

Note that $c_0 = 1$, $c_1 = 1/6$ and $c_2 = 3/40$. When $m = 3$, by (3.2.3), we have

$$f_3(0) = 12\sum_{j=3}^{\infty} c_j = 12(\pi/2 - c_0 - c_1 - c_2) = 6\pi - 149/10 \geqslant \pi/(1-2^{1/2-3})$$

which completes the proof. ■

Now the main result in this section, which plays a critical role in our proof of Theorem 3.1, is as follows:

Theorem 3.2. *For any positive integer m,*

$$\frac{\sum_{j=0}^{m-1} c_j \sin^{2j}\frac{\xi}{2}}{\sum_{j=0}^{m-1} c_j \sin^{2j}\xi} \geqslant \left(\cos^{2m}\frac{\xi}{2} + \sin^{2m}\frac{\xi}{2}\right)^{\frac{1}{2m}} \qquad \forall\, \xi \in \mathbb{R}, \tag{3.2.8}$$

where $c_0 := 1$ and the numbers $c_j (j \in \mathbb{N})$ are defined in (3.2.1).

Proof: It is easy to see that in order to show the inequality (3.2.8), it suffices to prove it for $\xi \in [0, \pi/2]$ since $\sin^2 \frac{\pi-\xi}{2} = \cos^2 \frac{\xi}{2} \geqslant \sin^2 \frac{\xi}{2}$ for all $\xi \in [0, \pi/2]$. Let $x = \cos(\xi)$. By Lemma 3.2, for $m \geqslant 3$, we have the following estimate

$$4m\Big(2\cos^2\frac{\xi}{2}\Big)^{2m}\sum_{j=0}^{\infty} c_{j+m}\sin^{2j}\xi \geqslant \frac{\pi}{1-2^{1/2-m}} \qquad \forall\, \xi \in [0,\pi/2]. \tag{3.2.9}$$

Define

$$A(\xi) := \sum_{j=0}^{m-1} c_j \sin^{2j}\frac{\xi}{2} \qquad \text{and} \qquad B(\xi) := \frac{\xi/2}{\sin(\xi/2)} - A(\xi).$$

By (3.2.2), $B(\xi) = \sum_{j=m}^{\infty} c_j \sin^{2j}\frac{\xi}{2}$. It follows from (3.2.9) that for $\xi \in [0, \pi/2]$,

$$\begin{aligned}
B(2\xi) &= \sum_{j=m}^{\infty} c_j \sin^{2j}\xi = \sin^{2m}\xi \sum_{j=0}^{\infty} c_{j+m}\sin^{2j}\xi \\
&= \frac{1}{4m}\cdot\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\cdot 4m\Big(2\cos^2\frac{\xi}{2}\Big)^{2m}\sum_{j=0}^{\infty} c_{j+m}\sin^{2j}\xi \\
&\geqslant \frac{\frac{\pi}{2}\cdot\frac{1}{2m}\cdot\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}}{1-2^{1/2-m}} \\
&\geqslant \frac{\xi}{\sin\xi}\cdot\frac{\frac{1}{2m}\cdot\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}}{1-\frac{1}{[4\cos^2(\xi/2)]^m}\cdot\frac{1}{\cos(\xi/2)}},
\end{aligned}$$

where we use the fact that $\cos(\xi/2) \geqslant 2^{-1/2}$ for all $\xi \in [0, \pi/2]$. Observe that $(1+x)^{\frac{1}{2m}} - 1 \leqslant \frac{x}{2m}$ for all $x \geqslant 0$ and $m \in \mathbb{N}$. It follows that

$$\Big(1+\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\Big)^{\frac{1}{2m}} - 1 \leqslant \frac{1}{2m}\cdot\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}, \qquad \xi \in [0,\pi/2].$$

Since $A(2\xi) + B(2\xi) = \frac{\xi}{\sin\xi}, \xi \in [0, \pi/2]$, from the above two inequalities, we have

$$\begin{aligned}
\frac{B(2\xi)}{A(2\xi)+B(2\xi)} = \frac{B(2\xi)}{\frac{\xi}{\sin\xi}} &\geqslant \frac{\Big(1+\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\Big)^{\frac{1}{2m}} - 1}{\Big(1+\frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\Big)^{\frac{1}{2m}} - \frac{1}{[4\cos^2(\xi/2)]^m}\cdot\frac{1}{\cos(\xi/2)}} \\
&= \frac{\Big(\cos^{2m}(\xi/2)+\sin^{2m}(\xi/2)\Big)^{\frac{1}{2m}} - \cos(\xi/2)}{\Big(\cos^{2m}(\xi/2)+\sin^{2m}(\xi/2)\Big)^{\frac{1}{2m}} - \frac{1}{[4\cos^2(\xi/2)]^m}}.
\end{aligned}$$

The above inequality is equivalent to

$$\text{(3.2.10)}\qquad \frac{\cos\frac{\xi}{2}-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}\cdot\frac{1}{[4\cos^2(\xi/2)]^m}}{1-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}}\geqslant\Big(\cos^{2m}(\xi/2)+\sin^{2m}(\xi/2)\Big)^{\frac{1}{2m}}.$$

Since $2\cos(\xi/2)>1$ for all $\xi\in[0,\pi/2]$ and

$$\frac{c_j\sin^{2j}\frac{\xi}{2}}{c_j\cos^{2j}\xi}=\frac{1}{\left(2\cos\frac{\xi}{2}\right)^{2j}}\geqslant\frac{1}{\left(2\cos\frac{\xi}{2}\right)^{2j+2}}=\frac{c_{j+1}\sin^{2j+2}\frac{\xi}{2}}{c_{j+1}\cos^{2j+2}\xi},\qquad \xi\in[0,\pi/2],$$

by Lemma 3.1, we have

$$\frac{B(\xi)}{B(2\xi)}=\frac{\sum_{j=m}^{\infty}c_j\sin^{2j}\frac{\xi}{2}}{\sum_{j=m}^{\infty}c_j\sin^{2j}\xi}\leqslant\frac{\sin^{2m}\frac{\xi}{2}}{\sin^{2m}\xi}=\frac{1}{\left[4\cos^2(\xi/2)\right]^m},\qquad \xi\in[0,\pi/2].$$

Hence, $B(\xi)\leqslant\frac{B(2\xi)}{[4\cos^2(\xi/2)]^m}$ for all $\xi\in[0,\pi/2]$ and

$$\begin{aligned}\frac{A(\xi)}{A(2\xi)}&=\frac{[A(\xi)+B(\xi)]-B(\xi)}{[A(2\xi)+B(2\xi)]-B(2\xi)}\geqslant\frac{[A(\xi)+B(\xi)]-\frac{B(2\xi)}{[4\cos^2(\xi/2)]^m}}{[A(2\xi)+B(2\xi)]-B(2\xi)}\\&=\frac{\frac{A(\xi)+B(\xi)}{A(2\xi)+B(2\xi)}-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}\cdot\frac{1}{[4\cos^2(\xi/2)]^m}}{1-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}}\\&=\frac{\cos\frac{\xi}{2}-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}\cdot\frac{1}{[4\cos^2(\xi/2)]^m}}{1-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}}\end{aligned}$$

since $A(\xi)+B(\xi)=\frac{\xi/2}{\sin(\xi/2)}$ for $\xi\in[-\pi,\pi]$. It follows from the above inequality and (3.2.10) that for $\xi\in[0,\pi/2]$,

$$\frac{A(\xi)}{A(2\xi)}\geqslant\frac{\cos\frac{\xi}{2}-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}\cdot\frac{1}{[4\cos^2(\xi/2)]^m}}{1-\frac{B(2\xi)}{A(2\xi)+B(2\xi)}}\geqslant\Big(\cos^{2m}(\xi/2)+\sin^{2m}(\xi/2)\Big)^{\frac{1}{2m}}.$$

Therefore, (3.2.8) holds when $m\geqslant 3$. It is evident that (3.2.8) holds for $m=1$. In the following, let us check the case $m=2$. Let $x=\sin^2(\xi/2)$. Then $\sin^2\xi=4x(1-x)$. When $m=2$, to prove (3.2.8), it suffices to prove

$$\text{(3.2.11)}\qquad \frac{1+\frac{x}{6}}{1+\frac{2}{3}x(1-x)}\geqslant[1-2x(1-x)]^{1/4}\qquad\forall x\in[0,1].$$

By computation, for $x \in [0,1]$, we have

$$\begin{aligned}&[1+\frac{2}{3}x(1-x)][1-2x(1-x)]^{1/4} \leqslant [1+\frac{2}{3}x(1-x)][1-\frac{1}{4}2x(1-x)]\\&=1+(\frac{2}{3}-\frac{1}{2})x(1-x)-\frac{1}{3}x^2(1-x)^2\\&\leqslant 1+\frac{1}{6}x(1-x)\leqslant 1+\frac{1}{6}x\end{aligned}$$

which verifies inequality (3.2.11). Therefore, the proof is completed. ■

3.3 Construction of tight wavelet frames with three symmetric generators

In this section, using the auxiliary inequalities in Section 3.2, we shall prove Theorem 3.1. In particular, we shall give a step by step procedure for constructing the sequences b^1, b^2, b^3 in Theorem 3.1.

Proof of Theorem 3.1: Let B_m be the B-spline function of order m and let $a(\xi) := \left(\frac{1+e^{-i\xi}}{2}\right)^m$. Then it is known that $\widehat{B}_m(2\xi) = a(\xi)\widehat{B}_m(\xi)$. Let $c_0 := 1$ and $c_j (j \in \mathbb{N})$ be the numbers which are defined in (3.2.1). The numbers $d_{m,j}$ are uniquely determined by the following identity

$$(3.3.12) \qquad \Big(\sum_{j=0}^{m-1} c_j x^j\Big)^m = \sum_{j=0}^{\infty} d_{m,j} x^j.$$

Clearly, $d_{m,0} = [c_0]^m = 1$ and $d_{m,j} = 0$ for all $j > m(m-1)$. Define two 2π-periodic trigonometric polynomials θ_1 and θ as follows:

$$(3.3.13) \quad \theta_1(\xi) := 1 + \sum_{j=1}^{m-1} d_{m,j} \sin^{2j}\frac{\xi}{2} \qquad \text{and} \qquad \theta(\xi) := |\theta_1(\xi)|^2 = [\theta_1(\xi)]^2.$$

Since $\frac{\sin^2(\xi/2)}{\sin^2\xi} = \frac{1}{4\cos^2(\xi/2)} \leqslant \frac{1}{2}$ for $\xi \in [0,\pi/2]$, we have $\left(\frac{\sin^2(\frac{\xi}{2})}{\sin^2\xi}\right)^{j-1} \geqslant \left(\frac{\sin^2(\frac{\xi}{2})}{\sin^2\xi}\right)^j$ for all $j \in \mathbb{N}$ and for all $\xi \in [0,\pi/2]$. By Lemma 3.1, for $\xi \in [0,\pi/2]$, we have

$$\frac{\theta_1(\xi)}{\theta_1(2\xi)} = \frac{\sum_{j=0}^{m-1} d_{m,j}\sin^{2j}(\frac{\xi}{2})}{\sum_{j=0}^{m-1} d_{m,j}\sin^{2j}\xi} \geqslant \frac{\sum_{j=0}^{m(m-1)} d_{m,j}\sin^{2j}(\frac{\xi}{2})}{\sum_{j=0}^{m(m-1)} d_{m,j}\sin^{2j}\xi} = \left[\frac{\sum_{j=0}^{m-1} c_j\sin^{2j}(\frac{\xi}{2})}{\sum_{j=0}^{m-1} c_j\sin^{2j}\xi}\right]^m.$$

By Theorem 3.2, for $\xi \in [0, \pi/2]$, we have

$$\frac{\theta_1(\xi)}{\theta_1(2\xi)} \geqslant \left(\frac{\sum_{j=0}^{m-1} c_j \sin^{2j}(\xi/2)}{\sum_{j=0}^{m-1} c_j \sin^{2j} \xi}\right)^m \geqslant \left(\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)\right)^{1/2}$$
$$= \sqrt{|a(\xi)|^2 + |a(\xi+\pi)|^2}.$$

In other words, $\theta(\xi) - \theta(2\xi)(|a(\xi)|^2 + |a(\xi+\pi)|^2) \geqslant 0$ for all $\xi \in [0, \pi/2]$. By the definition of θ, we have $\theta(-\xi) = \theta(\xi)$ and $\theta(\xi) \leqslant \theta(\pi - \xi)$ for all $\xi \in [0, \pi/2]$. Consequently, we have

$$\theta(\xi) - \theta(2\xi)(|a(\xi)|^2 + |a(\xi+\pi)|^2) \geqslant 0 \qquad \forall\, \xi \in [-\pi, \pi], \tag{3.3.14}$$

which is the same conclusion in [13, Proposition 3.5] with different θ here. Inspired by the "wavelet mask construction" in [13] (at page 21), by the Fejer-Riesz lemma, there exists a 2π-periodic trigonometric polynomial θ_2 such that

$$|\theta_2(\xi)|^2 = \theta(\xi) - \theta(2\xi)(|a(\xi)|^2 + |a(\xi+\pi)|^2). \tag{3.3.15}$$

Now define

$$\begin{aligned}
b^1(\xi) &:= \overline{a(\xi+\pi)} e^{-i\xi}\theta_1(2\xi) = \left(\frac{1-e^{i\xi}}{2}\right)^m e^{-i\xi}\theta_1(2\xi),\\
b^2(\xi) &:= a(\xi)[\theta_2(2\xi) + \overline{\theta_2(2\xi)}]/2 = \left(\frac{1+e^{-i\xi}}{2}\right)^m [\theta_2(2\xi) + \overline{\theta_2(2\xi)}]/2,\\
b^3(\xi) &:= a(\xi)[\theta_2(2\xi) - \overline{\theta_2(2\xi)}]/2 = \left(\frac{1+e^{-i\xi}}{2}\right)^m [\theta_2(2\xi) - \overline{\theta_2(2\xi)}]/2.
\end{aligned} \tag{3.3.16}$$

It is evident that b^1, b^2, b^3 are real-valued finitely supported sequences on $\mathbb{Z}$ such that

$$\overline{b^1(\xi)} = (-1)^m e^{i(2-m)\xi} b^1(\xi), \quad \overline{b^2(\xi)} = e^{im\xi} b^2(\xi), \quad \overline{b^3(\xi)} = -e^{im\xi} b^3(\xi). \tag{3.3.17}$$

Let

$$\Theta(\xi) := \theta(2\xi)(|a(\xi)|^2 + |a(\xi+\pi)|^2). \tag{3.3.18}$$

Note that $|\theta_2(\xi)+\overline{\theta_2(\xi)}|^2+|\theta_2(\xi)-\overline{\theta_2(\xi)}|^2=4|\theta_2(\xi)|^2$. By calculation, we have

$$
\begin{aligned}
&|a(\xi)|^2\Theta(2\xi)+|b^1(\xi)|^2+|b^2(\xi)|^2+|b^3(\xi)|^2\\
=&|a(\xi)|^2\Theta(2\xi)+|\widehat{a}(\xi+\pi)|^2|\theta_1(2\xi)|^2+|a(\xi)|^2|\theta_2(2\xi)|^2\\
=&|a(\xi)|^2\theta(4\xi)(|a(2\xi)|^2+|a(2\xi+\pi)|^2)+|a(\xi+\pi)|^2\theta(2\xi)\\
&\quad+|\widehat{a}(\xi)|^2(\theta(2\xi)-\theta(4\xi)(|a(2\xi)|^2+|\widehat{a}(2\xi+\pi)|^2)\\
=&\theta(2\xi)(|a(\xi)|^2+|a(\xi+\pi)|^2)\\
=&\Theta(\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
&a(\xi)\overline{a(\xi+\pi)}\Theta(2\xi)+b^1(\xi)\overline{b^1(\xi+\pi)}+b^2(\xi)\overline{b^2(\xi+\pi)}+b^3(\xi)\overline{b^3(\xi+\pi)}\\
=&a(\xi)\overline{a(\xi+\pi)}\Theta(2\xi)-a(\xi)\overline{a(\xi+\pi)}|\theta_1(2\xi)|^2+a(\xi)\overline{a(\xi+\pi)}|\theta_2(2\xi)|^2\\
=&a(\xi)\overline{a(\xi+\pi)}\big[\Theta(2\xi)-\theta(2\xi)-\big(\theta(2\xi)-\theta(4\xi)(|\widehat{a}(2\xi)|^2+|a(2\xi+\pi)|^2)\big)\big]\\
=&0.
\end{aligned}
$$

Define $\widehat{\psi}^\ell(2\cdot)=b^\ell\widehat{B}_m, \ell=1,2,3$. Since $\Theta(0)=1$, by Theorem 1.6, $\{\psi^1\psi^2,\psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$.

It follows from (3.2.2) that $\theta(\xi)-\theta(2\xi)|a(\xi)|^2=O(|\xi|^{2m}), \xi\to 0$ (also see [13]). Thus, we deduce that $\Theta(\xi)-\Theta(2\xi)|\widehat{a}(\xi)|^2=O(|\xi|^{2m}), \xi\to 0$. Consequently, each wavelet function $\psi^\ell, \ell=1,2,3$ has the vanishing moments of order m. The symmetry of the wavelet functions ψ^1,ψ^2,ψ^3 follows directly from (3.3.17). ∎

Chapter 4

Tight wavelet frames with two symmetric generators having high vanishing moments

4.1 Introduction and motivation

Matrix theory plays an important role in wavelet analysis [11] and filter banks [46, 48, 49]. In this chapter, we are interested in splitting a 2×2 matrix of Laurent polynomials with real coefficients and symmetry into the form $U(z)U(1/z)^T$ for some 2×2 matrix U whose entries are Laurent polynomials with real coefficients and symmetry. Our investigation on this matrix splitting problem is greatly motivated by the recent development of symmetric tight wavelet frames and framelet filter banks which have been found to be useful and interesting in many applications [3, 4, 10, 11, 12, 13, 19, 26, 36, 37, 39, 44].

For simplicity, we use a Laurent polynomial $a(z)$ with $z = e^{-i\xi}$ to represent a mask or a finitely supported sequence a. Throughout this chapter, we assume that all Laurent polynomials have real coefficients. In other words, all the filters discussed in this chapter are of finite-impulse-response (FIR).

As an important family of refinable functions, B-spline functions are useful in applications. The B-spline function $B_n \in C^{n-2}(\mathbb{R})$ is a symmetric refinable function satisfying $\widehat{B}_n(2\xi) = 2^{-n}(1+e^{-i\xi})^n \widehat{B}_n(\xi)$ for $\xi \in \mathbb{R}$.

In order to obtain an orthonormal wavelet basis from a refinable function ϕ via the multiresolution analysis, the refinable function ϕ must satisfy the following condition ([11, 46, 49]):

$$\int_{\mathbb{R}} \phi(x+k)\overline{\phi(x)}\,dx = \delta_k \qquad \forall\, k \in \mathbb{Z}. \tag{4.1.1}$$

By a simple argument, (4.1.1) implies that its mask a must satisfy the condition ([11, 46, 49]):

$$|a(z)|^2 + |a(-z)|^2 = 1 \qquad \forall\, z \in \mathbb{T} := \{z \in \mathbb{C} \,:\, |z| = 1\}. \tag{4.1.2}$$

If (4.1.1) holds, one can define a wavelet function by $\widehat{\psi}(2\xi) = e^{-i\xi} a(-e^{i\xi})\widehat{\phi}(\xi)$. Then $\{\psi\}$ generates an orthonormal wavelet basis in $L_2(\mathbb{R})$ (see [11]). Note that the Haar wavelet, derived from the B-spline function B_1, is discontinuous.

The conditions in (4.1.1) and (4.1.2) impose a very restrict constraint on a refinable function and its low-pass filter, i.e., its mask. Many refinable functions such as the B-spline functions $B_n (n > 1)$ do not satisfy (4.1.1). In fact, up to an integer shift, B_1 is the only example of real-valued compactly supported refinable function that can have symmetry and satisfy (4.1.1) (see [11]).

As discussed above, an orthonormal wavelet basis has only one generator. By increasing the number of generators in a tight wavelet frame, recently it was found that one has a lot of freedom in the construction of tight wavelet frames derived from refinable functions which may not satisfy the condition in (4.1.1). For example, it was demonstrated in Ron and Shen [39] that from any B-spline function of order n, one can construct a symmetric tight wavelet frame with n generators. More recently, Chui and He [3] (also see Petukhov [36]) showed that if the mask a for a symmetric refinable function satisfies

$$|a(z)|^2 + |a(-z)|^2 \leqslant 1 \qquad \forall\, z \in \mathbb{T}, \tag{4.1.3}$$

then one can derive a symmetric tight wavelet frame with three generators. Recently, Daubechies *et al.* [13] and Chui *et al.* [4] obtained the following interesting procedure(also see Theorem 1.6 in Chapter 1) that yields all possible MRA tight wavelet frames derived from a refinable function.

Theorem 4.1. *Let ϕ be a refinable function in $L_2(\mathbb{R})$ such that $\widehat{\phi}(0) \neq 0$ and $\widehat{\phi}(2\xi) = a(e^{-i\xi})\widehat{\phi}(\xi)$ for a Laurent polynomial a with $a(1) = 1$. Suppose that there exist Laurent polynomials $\Theta, a^1, \dots, a^L$ such that $\Theta(1) = 1$ and*

$$\begin{bmatrix} a^1(z) & \dots & a^L(z) \\ a^1(-z) & \dots & a^L(-z) \end{bmatrix} \begin{bmatrix} a^1(1/z) & a^1(-1/z) \\ \vdots & \vdots \\ a^L(1/z) & a^L(-1/z) \end{bmatrix} = M_\Theta(z), \tag{4.1.4}$$

where for all $z \in \mathbb{C}\backslash\{0\}$,

(4.1.5)

$$M_\Theta(z) := \begin{bmatrix} \Theta(z) - \Theta(z^2)a(z)a(1/z) & -\Theta(z^2)a(z)a(-1/z) \\ -\Theta(z^2)a(-z)a(1/z) & \Theta(-z) - \Theta(z^2)a(-z)a(-1/z) \end{bmatrix}.$$

Define the wavelet functions $\psi^1, \dots, \psi^L$ by $\widehat{\psi^\ell}(2\xi) = a^\ell(e^{-i\xi})\widehat{\phi}(\xi)$, $\ell = 1, \dots, L$. Then $\{\psi^1, \dots, \psi^L\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$.

According to Theorem 4.1, a framelet filter bank consists of a low-pass filter a and L high-pass filters $a^1, \dots, a^L$. In order to design a framelet filter bank, one has to split the matrix M_Θ in (4.1.5) into the form of (4.1.4).

Using Theorem 4.1, it was demonstrated in [4] (also c.f. [13]) that for any refinable function $\phi \in L_2(\mathbb{R})$ whose integer shifts are stable, one can obtain an MRA tight wavelet frame with two generators. Unfortunately, when ϕ is symmetric, the construction in [4, 13] cannot guarantee the symmetry of the two constructed generators which do not have symmetry in most cases.

Though by increasing the number of generators in a tight wavelet frame one has a great deal of freedom to construct them from refinable functions, in many applications, for various purposes such as computational cost and storage concern, one prefers a symmetric tight wavelet frame with as small as possible number of generators (or equivalently, high-pass filters). Ideally, a tight wavelet frame with a single symmetric generator is desirable. However, as shown in [4, 13], it is impossible to have an MRA symmetric tight wavelet frame with one continuous generator. All the above discussions naturally motivate us to consider construction of symmetric MRA tight wavelet frames with two generators (that is, symmetric framelet filter banks with two high-pass filters) for the following possible advantages.

1. Such framelet filter banks have symmetry which is a much desired property in applications.
2. By using two high-pass filters, one still has much freedom to construct symmetric framelet filter banks from many low-pass filters without imposing strict conditions on them.
3. By limiting to two high-pass filters, the associated framelet transform for decomposition and reconstruction is efficient in terms of computational and storage costs.
4. Such symmetric framelet filter banks can have good vanishing moments, short support and many other desired properties.

In order to construct a symmetric framelet filter bank with two high-pass filters, according to Theorem 4.1, the core problem is to find two symmetric high-pass filters a^1 and a^2 such that (4.1.4) holds with $L = 2$. In other words, we have to split the 2×2 matrix M_Θ of Laurent polynomials into the desirable form in (4.1.4). This motivates us to investigate the problem of splitting a matrix of Laurent polynomials with symmetry which may be of interest in other applications such as construction of symmetric orthonormal multiwavelets and dual framelet filter banks [4, 12, 13].

The following is an outline of this chapter. In Section 2, we shall present a general result on splitting a matrix of Laurent polynomials with symmetry. As an application of this result to symmetric framelet filter banks, we shall present a necessary and sufficient condition for the construction of a symmetric tight wavelet frame with two generators derived from a given symmetric refinable function through Theorem 4.1. Once the necessary and sufficient condition is satisfied, we shall present a step-by-step algorithm (see Algorithm 4.3 in Section 2) to derive the two symmetric high-pass filters from a given low-pass filter. In Section 3, we shall present some examples of symmetric framelet filter banks with two high-pass filters which are derived from various low-pass filters including some B-spline filters. Our work in this chapter was also motivated by [37, 44] where symmetric tight wavelet frames with two generators were considered but using the unitary extension principle in [39], which is a special case of Theorem 4.1 by taking $\Theta = 1$. As discussed in [4, 13], a nonconstant Θ is very important in order to have a tight wavelet frame with good vanishing moments. Also, in order to use the unitary extension principle, the mask must satisfy (4.1.3) which excludes some interesting low-pass filters ([3, 4, 13, 36]). We shall see that by using the general construction in Theorem 4.1 the investigation of symmetric tight wavelet frames and symmetric framelet filter banks becomes much more complicated. In Section 3, by using Algorithm 4.3 and Theorem 4.1 we shall give examples to show that symmetric framelet filter banks with two high-pass filters having good vanishing moments can be constructed. For applications of framelet filter banks, see [44]. In order to prove the main results in this chapter, in Section 4, we shall provide some auxiliary results. In Section 5, we shall prove our main result on splitting a matrix of Laurent polynomials with symmetry. Though the whole proof of the main result is somewhat technical, we shall present a step-by-step algorithm (see Algorithm 4.7 in Section 5) to implement the main result on splitting a matrix of Laurent polynomials with symmetry which may be of interest in other applications.

4.2 Main results

In this section, we shall present the main results of this chapter. We shall obtain a general result on splitting a matrix of Laurent polynomials with symmetry. As an application of such a result, we shall give a necessary and sufficient condition for the construction of symmetric MRA tight wavelet frames with two compactly supported generators. A step-by-step algorithm (Algorithm 4.3) will be given for construction of symmetric framelet filter banks.

In order to state the results in this section, let us introduce some notation first. We remind the reader that all of the Laurent polynomials discussed in this chapter have real coefficients and we say that a Laurent polynomial p with real coefficients is symmetric(or antisymmetric) about $k/2$ for some $k \in \mathbb{Z}$ if $p(z) = z^k p(1/z)$(or $p(z) = -z^k p(1/z)$). Throughout this chapter, we say that a Laurent polynomial p is **(anti)symmetric** if p is either symmetric or antisymmetric. For a nonzero Laurent polynomial p, we define an operator S to be

$$(4.2.6) \qquad [Sp](z) := \frac{p(z)}{p(1/z)}, \qquad z \in \mathbb{C}\backslash\{0\}.$$

When $p \equiv 0$, by convention Sp is undefined and can be anything.

The following result can be easily verified.

Proposition 4.1. Let p and q be two Laurent polynomials with real coefficients. Then

(1) p is (anti)symmetric about $k/2$ for some $k \in \mathbb{Z}$ if and only if $[Sp](z) = \pm z^k$.

(2) $[S(p(1/\cdot))](z) = [Sp](1/z) = 1/[Sp](z)$.

(3) $[S(pq)](z) = [Sp](z)[Sq](z)$ and $[S((\cdot)^k)](z) = z^{2k}$ for $k \in \mathbb{Z}$.

(4) If p and q are (anti)symmetric such that $Sp = Sq$, then $p \pm q$ is (anti)symmetric and $S(p \pm q) = Sp = Sq$.

For a nonzero Laurent polynomial $p(z) = \sum_{k=\ell}^{h} p_k z^k$ such that $p_\ell \neq 0$ and $p_h \neq 0$, we denote the **degree** of p by $\deg(p) = h - \ell$. In other words, $\deg(p)$ measures the length of the filter p. By convention, $\deg(0) = -\infty$. For any two Laurent polynomials p and q, we say that $p \mid q$ if there is another Laurent polynomial h such that $q(z) = p(z)h(z)$ for all $z \in \mathbb{C}\backslash\{0\}$. We define $\gcd(p, q)$ to be a nonzero Laurent polynomial h with maximum degree such that $h \mid p$ and $h \mid q$. By convention, $\gcd(0, 0) = 0$. We say that a Laurent polynomial p is **trivial** if $p(z) = cz^k$ for some $c \in \mathbb{R}\backslash\{0\}$ and $k \in \mathbb{Z}$. Up to a factor of a trivial Laurent polynomial, $\gcd(p, q)$ is unique.

Proposition 4.2. Let $A(z) = A_0 + \sum_{k=1}^{N} A_k(z^{-k} + z^k)$ with $A_N \neq 0$ be a Laurent polynomial with real coefficients. Then $A(z) = d(z)d(1/z)$ for some (anti)symmetric Laurent polynomial d with real coefficients if and only if $A(z) = d_A(z)d_A(1/z)$ for the Laurent polynomial d_A which is uniquely determined by one of the following four cases:

Case 1: When $N = 2n$ and $A_N > 0$, define $d_A(z) = c_0 + \sum_{k=1}^{n} c_k(z^k + z^{-k})$ and $\operatorname{sgn}(A_N) = 1$.

Case 2: When $N = 2n$ and $A_N < 0$, define $d_A(z) = \sum_{k=1}^{n} c_k(z^k - z^{-k})$ and $\operatorname{sgn}(A_N) = -1$.

Case 3: When $N = 2n+1$ and $A_N > 0$, define $d_A(z) = \sum_{k=0}^{n} c_k(z^k + z^{-1-k})$and $\operatorname{sgn}(A_N) = 1$.

Case 4: When $N = 2n + 1$ and $A_N < 0$, define $d_A(z) = \sum_{k=0}^{n} c_k(z^k - z^{-1-k})$ and $\operatorname{sgn}(A_N) = -1$.

The coefficients $c_0, \ldots, c_n$ are uniquely determined by the following recursive formula: $c_n := \sqrt{|A_N|}$ and

$$(4.2.7) \quad c_{n-j} := \frac{1}{2c_n}\Big[\operatorname{sgn}(A_N)A_{N-j} - \sum_{k=n-j+1}^{n-1} c_k c_{2n-j-k}\Big], \qquad j = 1, 2, \ldots, n.$$

Moreover, if $A(z) = d(z)d(1/z)$ for an (anti)symmetric Laurent polynomial d with real coefficients, then we must have $d(z) = \pm z^k d_A(z)$ for some $k \in \mathbb{Z}$.

Proof: If a Laurent polynomial d is (anti)symmetric and satisfies $A(z) = d(z)d(1/z)$, then it is easy to see that $d(z) = \pm z^k d_A(z)$ for some $k \in \mathbb{Z}$. By comparing the coefficients of $A(z)$ and $d_A(z)d_A(1/z)$, all the claims can be easily verified. ■

For a matrix M, we denote by $M_{j,k}$ the (j,k)-entry of the matrix M. For a Laurent polynomial p, we denote by $Z(p, z_0)$ the multiplicity of zeros of p at $z = z_0$, that is,

$$Z(p, z_0) = \sup\{n \in \mathbb{N} \cup \{0\} \; : \; (z - z_0)^n \,|\, p(z)\}. \tag{4.2.8}$$

Now we are ready to state the main results in this chapter.

Theorem 4.2. *Let A, B and C be (anti)symmetric Laurent polynomials with real coefficients. Denote a 2×2 matrix M by*

$$M(z) = \begin{bmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{bmatrix}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{4.2.9}$$

Then there exist (anti)symmetric Laurent polynomials u_1, u_2, v_1, v_2 with real coefficients such that

(4.2.10)

$$U(z)U(1/z)^T = M(z) \qquad \forall\, z \in \mathbb{C}\backslash\{0\} \quad \textit{with} \qquad U(z) := \begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix}$$

and

$$[Su_1](z)[Sv_2](z) = [Sv_1](z)[Su_2](z), \qquad z \in \mathbb{C}\backslash\{0\} \tag{4.2.11}$$

if and only if all the following conditions are satisfied:

(a) *The matrix $M(z)$ is positive semi-definite (that is, $M(z) \geqslant 0$) for all $z \in \mathbb{T}$.*

(b) $\det M(z) = d(z)d(1/z)$ *for some (anti)symmetric Laurent polynomial d with real coefficients.*

(c) *Define* $g = gcd(A, B, C)$. *If* $Bd \equiv 0$, *then there is no condition on* g. *If both* B *and* d *are not identically zero, then the matrix* M *satisfies the following "gcd" condition, that is, one of the following conditions must be true:*

(1) *If* $[SB](z)[Sd](z) = z^{2n}$ *for some* $n \in \mathbb{Z}$, *then* $Z(g, x)$ *is an even number for every* $x \in (-1, 0) \cup (0, 1)$.

(2) *If* $[SB](z)[Sd](z) = z^{2n+1}$ *for some* $n \in \mathbb{Z}$, *then* $Z(g, x)$ *is an even number for every* $x \in (0, 1)$.

(3) *If* $[SB](z)[Sd](z) = -z^{2n}$ *for some* $n \in \mathbb{Z}$, *then there is no condition on* g.

(4) *If* $[SB](z)[Sd](z) = -z^{2n+1}$ *for some* $n \in \mathbb{Z}$, *then* $Z(g, x)$ *is an even number for every* $x \in (-1, 0)$.

We shall prove Theorem 4.2 in Section 5 in a constructive way and a step-by-step algorithm (see Algorithm 4.7) will be given to construct the desired filters u_1, u_2, v_1, v_2 from the matrix M. We shall also show that the "gcd" condition in Theorem 4.2 cannot be removed. Note that by Proposition 4.1 and (4.2.10), it is easy to see that when $B \not\equiv 0$, (4.2.11) is equivalent to

$$\frac{[Su_1](z)}{[Su_2](z)} = [SB](z) = \frac{[Sv_1](z)}{[Sv_2](z)}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{4.2.12}$$

As an application of Theorem 4.2 to symmetric framelet filter banks, we have the following result for constructing symmetric MRA tight wavelet frames with two generators.

Theorem 4.3. *Let* $\phi \in L_2(\mathbb{R})$ *be a refinable function satisfying* $\widehat{\phi}(2\xi) = a(e^{-i\xi})\widehat{\phi}(\xi)$ *for a symmetric Laurent polynomial* a *with real coefficients such that* $a(1) = 1$. *Let* Θ *be a Laurent polynomial with real coefficients such that* $\Theta(z) = \Theta(1/z)$ *and* $\Theta(1) = 1$. *Let* M_Θ *be defined in* (4.1.5). *Then there exist two (anti)symmetric Laurent polynomials* a^1 *and* a^2 *with real coefficients such that* (4.1.4) *in Theorem 4.1 holds with* $r = 2$ *if and only if the following conditions are satisfied:*

(a) $M_\Theta(z) \geqslant 0$ *for all* $z \in \mathbb{T}$. *(This condition can be replaced by* $\Theta(z) \geqslant 0$ *for all* $z \in \mathbb{T}$ *while conditions (b) and (c) keep unchanged.)*

(b) $\det M_\Theta(z) = d(z^2)d(z^{-2})$ *for an (anti)symmetric Laurent polynomial* d *with real coefficients.*

(c) *Define* $g(z^2) = gcd([M_\Theta]_{1,1}, [M_\Theta]_{1,2}, [M_\Theta]_{2,2})$. *If* $\det M_\theta \equiv 0$, *there is no condition on* g. *If* $\det M_\theta \not\equiv 0$, *then the matrix* M_Θ *satisfies the following "gcd" condition, that is, one of the following conditions must be true:*

(1) *If* $[Sa](-z)[Sd](z) = z^{2n+1}$ *for some* $n \in \mathbb{Z}$, *then* $Z(g, x)$ *is an even number for every* $x \in (-1, 0) \cup (0, 1)$.

(2) *If* $[Sa](-z)[Sd](z) = z^{2n}$ *for some* $n \in \mathbb{Z}$, *then* $Z(g, x)$ *is an even number for every* $x \in (0, 1)$.

(3) *If* $[Sa](-z)[Sd](z) = -z^{2n+1}$ *for some* $n \in \mathbb{Z}$, *then there is no condition on* g.

(4) *If* $[Sa](-z)[Sd](z) = -z^{2n}$ *for some* $n \in \mathbb{Z}$, *then* $Z(g, x)$ *is an even number for every* $x \in (-1, 0)$.

Proof: Let us make some connections to Theorem 4.2 first. With $r = 2$, (4.1.4) becomes

$$W(z)W(1/z)^T = M_\Theta(z) \quad \text{where} \quad W(z) = \begin{bmatrix} a^1(z) & a^2(z) \\ a^1(-z) & a^2(-z) \end{bmatrix}. \tag{4.2.13}$$

Since the mask a is symmetric, we have $[Sa](z) = z^k$ for some $k \in \mathbb{Z}$. Borrowing the idea of polyphase decomposition, define

(4.2.14)

$$P(z) := \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \text{ if } k \text{ is even;} \quad P(z) := \frac{1}{2}\begin{bmatrix} 1+z & 1-z \\ 1-z & 1+z \end{bmatrix} \text{ if } k \text{ is odd.}$$

Then $P(z)P(1/z)^T = I_2$ and $P(-z) = P(z)J_2$, where $J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Now (4.2.13) can be rewritten as

$$U(z)U(1/z)^T = M(z) \tag{4.2.15}$$

with

(4.2.16)
$$U(z^2) = \widetilde{W}(z) := P(z)W(z), \quad M(z^2) = \widetilde{M}(z) := P(z)M_\Theta(z)P(1/z)^T.$$

When k is even, by computation we have

$$\widetilde{W}(z) = \frac{\sqrt{2}}{2}\begin{bmatrix} a^1(z) + a^1(-z) & a^2(z) + a^2(-z) \\ za^1(z) - za^1(-z) & za^2(z) - za^2(-z) \end{bmatrix} \tag{4.2.17}$$

and

$$[\widetilde{M}(z)]_{1,2} = \frac{1}{2z}\big(\Theta(z) - \Theta(-z) - \Theta(z^2)[a(z) + a(-z)][a(1/z) - a(-1/z)]\big).$$

It is easy to see that $\widetilde{W}(-z) = \widetilde{W}(z)$ and

$$\widetilde{M}(-z) = P(z)J_2M_\Theta(-z)J_2P(1/z)^T = P(z)M_\Theta(z)P(1/z)^T = \widetilde{M}(z).$$

So, U and M are well-defined. Moreover, It is easy to see that $M_{1,2} \not\equiv 0$ and $[SM_{1,2}](z) = z^{-1}$.

When k is odd, by computation we have

(4.2.18)
$$\widetilde{W}(z) = \frac{1}{2}\begin{bmatrix} (1+z)a^1(z) + (1-z)a^1(-z) & (1+z)a^2(z) + (1-z)a^2(-z) \\ (1-z)a^1(z) + (1+z)a^1(-z) & (1-z)a^2(z) + (1+z)a^2(-z) \end{bmatrix}$$

and

$$\widetilde{M}_{1,2}(z) = \frac{1}{4}(z - 1/z)[\Theta(z) - \Theta(-z)] - \frac{1}{4}\Theta(z^2)[(1+z)a(z) + (1-z)a(-z)]\times \\ [(1-1/z)a(1/z) + (1+1/z)a(-1/z)].$$

It is clear that $\widetilde{W}(-z) = \widetilde{W}(z)$ and $\widetilde{M}(-z) = \widetilde{M}(z)$. So, U and M are well-defined. Moreover, It is easy to see that $M_{1,2} \not\equiv 0$ and $[SM_{1,2}](z) = -1$.

By the definition of P and the definition (4.2.16), we have

$$\det M(z^2) = \det M_\Theta(z)$$

and

$$g(z) = \gcd(M_{1,1}, M_{1,2}, M_{2,2}),$$

where

$$g(z^2) = \gcd([M_\Theta]_{1,1}, [M_\Theta]_{1,2}, [M_\Theta]_{2,2}).$$

By the discussion above, we can clearly see the relation between conditions (a), (b) and (c) in Theorem 4.2 and conditions (a), (b) and (c) in this theorem, respectively. Based on the relation, we will prove the necessity and sufficiency respectively.

Necessity. Suppose that there exist two (anti)symmetric Laurent polynomials a^1 and a^2 with real coefficients such that (4.1.4) holds, by (4.2.13) we have $M_\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$ and therefore condition (a) holds. Note that $\det W(-z) = -\det W(z)$. Thus we can define a Laurent polynomial d by $d(z^2) = z\det W(z)$. Clearly,

$$\det M_\Theta(z) = \det W(z)\det W(1/z) = d(z^2)d(z^{-2}).$$

We now show that d is (anti)symmetric. Since a^1 and a^2 are (anti)symmetric, we have $[Sa^1](z) = \varepsilon_1 z^{k_1}$ and $[Sa^2](z) = \varepsilon_2 z^{k_2}$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and $k_1, k_2 \in \mathbb{Z}$. By (4.2.13) and (4.1.5), we have

$$a^1(z)a^1(-1/z)+a^2(z)a^2(-1/z) = [M_\Theta]_{1,2}(z) = -\Theta(z^2)a(z)a(-1/z). \tag{4.2.19}$$

Note that

$$S[a^1(z)a^1(-1/z)] = \varepsilon_1 z^{k_1}\varepsilon_1(-1/z)^{k_1} = (-1)^{k_1}$$

and similarly, $S[a^2(z)a^2(-1/z)] = (-1)^{k_2}$. Since

$$S[\Theta(z^2)a(z)a(-1/z)] = S[\Theta(z^2)]S[a(z)a(-1/z)] = (-1)^k,$$

by a simple argument, it follows from (4.2.19) that $(-1)^{k_1} = (-1)^{k_2} = (-1)^k$. (Note that there are at least two even (or odd) numbers among k_1, k_2 and k. Say, k_1 and k_2 are even. Then by item (4) in Proposition 4.1, we conclude that $(-1)^{k_1} = (-1)^{k_2} = (-1)^k$.) Note that $\det W(z) = a^1(z)a^2(-z) - a^1(-z)a^2(z)$. Since

$$\begin{aligned} &S[a^1(z)a^2(-z)] = \varepsilon_1\varepsilon_2(-1)^{k_2}z^{k_1+k_2} = \varepsilon_1\varepsilon_2(-1)^k z^{k_1+k_2} \\ =&\varepsilon_1\varepsilon_2(-1)^{k_1}z^{k_1+k_2} = S[a^1(-z)a^2(z)], \end{aligned}$$

by Proposition 4.1, we conclude that $S[\det W(z)] = \varepsilon_1\varepsilon_2(-1)^k z^{k_1+k_2}$. So, $\det W$ is (anti)symmetric and therefore, by $d(z^2) = z\det W(z)$, d is (anti)symmetric. Hence condition (b) holds.

Recall that $[Sa](z) = z^k$. When k is even, By Proposition 4.1 and the fact that $(-1)^{k_1} = (-1)^{k_2} = (-1)^k = 1$, it follows from (4.2.17) that

$$[S\widetilde{W}_{1,1}](z) = \varepsilon_1 z^{k_1}, \quad [S\widetilde{W}_{1,2}](z) = \varepsilon_2 z^{k_2}, \quad [S\widetilde{W}_{2,1}](z) = \varepsilon_1 z^{2+k_1},$$
$$[S\widetilde{W}_{2,2}](z) = \varepsilon_2 z^{2+k_2} \quad \text{and} \quad [S\widetilde{M}_{1,2}](z) = z^{-2}.$$

So, when k is even, (4.2.10) and (4.2.11) are satisfied. Since $P(z)P(1/z)^T = I_2$, we must have $g = \gcd(M_{1,1}, M_{1,2}, M_{2,2})$. Note that $[SM_{1,2}](z)[Sd](z) = z^{-1}[Sd](z) = z^{-1-k}[Sa](-z)[Sd](z)$ and k is an even integer. Therefore, by Theorem 4.2, condition (c) must be true.

When k is odd, by Proposition 4.1 and the fact that $(-1)^{k_1} = (-1)^{k_2} = (-1)^k = -1$, it follows from (4.2.18) that

$$[S\widetilde{W}_{1,1}](z) = \varepsilon_1 z^{1+k_1}, \quad [S\widetilde{W}_{1,2}](z) = \varepsilon_2 z^{1+k_2}, \quad [S\widetilde{W}_{2,1}](z) = -\varepsilon_1 z^{1+k_1},$$
$$[S\widetilde{W}_{2,2}](z) = -\varepsilon_2 z^{1+k_2} \quad \text{and} \quad [S\widetilde{M}_{1,2}](z) = -1.$$

Thus when k is odd, (4.2.10) and (4.2.11) are satisfied. Note that

$$\begin{aligned}&[SM_{1,2}](z)[Sd](z) = -[Sd](z) = -(-z)^{-k}[Sa](-z)[Sd](z)\\ =&z^{-k}[Sa](-z)[Sd](z)\end{aligned}$$

and k is an odd integer. Therefore, by Theorem 4.2, condition (c) must be true.

Sufficiency. Suppose that conditions (a), (b) and (c) in this theorem are satisfied. From the discussion before the necessity part, applying Theorem 4.2 on $M(z)$, we know that there exist (anti)symmetric Laurent polynomials u_1, u_2, v_1, v_2 with real coefficients such that (4.2.10) and (4.2.12) hold. Define

$$\begin{bmatrix} a^1(z) & a^2(z) \\ a^1(-z) & a^2(-z) \end{bmatrix} := P(1/z)^T U(z^2) = P(1/z)^T \begin{bmatrix} u_1(z^2) & v_1(z^2) \\ u_2(z^2) & v_2(z^2) \end{bmatrix}.$$

We show that a^1 and a^2 must be (anti)symmetric. Since $[Sa](z) = z^k$, when k is even, we have

$$a^1(z) = \frac{\sqrt{2}}{2}[u_1(z^2) + u_2(z^2)/z] \quad \text{and} \quad a^2(z) = \frac{\sqrt{2}}{2}[v_1(z^2) + v_2(z^2)/z].$$

By $[SM_{1,2}](z) = z^{-1}$, it follows from (4.2.12) that

$$S(u_1(z^2)) = S(M_{1,2}(z^2))S(u_2(z^2)) = z^{-2}S(u_2(z^2)) = S(u_2(z^2)/z)$$

and $S(v_1(z^2)) = S(v_2(z^2)/z)$. By Proposition 4.1, we have $[Sa^1](z) = [Su_1](z^2)$ and $[Sa^2](z) = [Sv_1](z^2)$. Since u_1 and v_1 are (anti)symmetric, so are the Laurent polynomials a^1 and a^2.

When k is odd, we have

$$\begin{aligned} a^1(z) &= [(1 + 1/z)u_1(z^2) + (1 - 1/z)u_2(z^2)]/2 \quad \text{and} \\ a^2(z) &= [(1 + 1/z)v_1(z^2) + (1 - 1/z)v_2(z^2)]/2. \end{aligned}$$

By $[SM_{1,2}](z) = -1$, it follows from (4.2.12) that

$$\begin{aligned} &S((1 + 1/z)u_1(z^2)) = z^{-1}S(u_1(z^2)) \\ =&z^{-1}S(M_{1,2}(z^2))S(u_2(z^2)) = S((1 - 1/z)u_2(z^2)) \end{aligned}$$

and $S((1 + 1/z)v_1(z^2)) = S((1 - 1/z)v_2(z^2))$. By Proposition 4.1, we deduce that $[Sa^1](z) = z^{-1}[Su_1](z^2)$ and $[Sa^2](z) = z^{-1}[Sv_1](z^2)$. Since u_1 and v_1 are (anti)symmetric, so are the Laurent polynomials a^1 and a^2. Now it is straightforward to verify that (4.2.13) holds. ∎

In order to construct symmetric framelet filter banks with two high-pass filters, by the proof of Theorem 4.3, we present the following algorithm.

Algorithm 4.3. Let a be a symmetric Laurent polynomial with real coefficients such that $a(1) = 1$ (that is, a is a low-pass filter). Suppose that we have a Laurent polynomial Θ such that all the conditions in Theorem 4.3 are satisfied.

1) Compute the symmetry center of the low-pass filter a: $[Sa](z) := \frac{a(z)}{a(1/z)} = z^k$ for some integer k. Define the 2×2 matrix P in (4.2.14) according to the integer k.

2) Calculate the 2×2 matrix $M(z^2) := P(z)M_\Theta(z)P(1/z)^T$, where M_Θ is defined in (4.1.5).

3) Using Algorithm 4.7 in Section 5 to split the matrix M into the desired form:

$$M(z) = \begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix} \begin{bmatrix} u_1(1/z) & u_2(1/z) \\ v_1(1/z) & v_2(1/z) \end{bmatrix} \quad \text{and}$$
$$[Su_1](z)[Sv_2](z) = [Su_2](z)[Sv_1](z).$$

In most cases $g(z^2) = \gcd([M_\Theta]_{1,1}, [M_\Theta]_{1,2}, [M_\Theta]_{2,2}) = 1$ and consequently, by solving a system of linear equations, we have all the symmetric filters u_1, u_2, v_1, v_2 by Algorithm 4.7.

4) Obtain the symmetric high-pass filters a^1 and a^2 by

$$a^1(z) := P_{1,1}(1/z)u_1(z^2) + P_{2,1}(1/z)u_2(z^2), \text{ and}$$
$$a^2(z) := P_{1,1}(1/z)v_1(z^2) + P_{2,1}(1/z)v_2(z^2).$$

Then (4.2.13) holds and we have a symmetric framelet filter bank consisting of a low-pass filter a and two high-pass filters a^1 and a^2.

In order to design a desired filter Θ such that all the conditions in Theorem 4.3 are satisfied, quite often one constructs a Θ such that $\Theta(1) = 1$, $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$ and

$$\begin{aligned}\det M(z^2) &= \Theta(z)\Theta(-z) - \Theta(z^2)[\Theta(z)a(-z)a(-1/z) + \Theta(-z)a(z)a(1/z)] \\ &= d(z^2)d(z^{-2})\end{aligned}$$

where d is determined in Proposition 4.2. In most cases the "gcd" condition in Theorem 4.3 is automatically satisfied.

4.3 Some examples of symmetric framelet filter banks

First, we illustrate that the "gcd" condition in Theorem 4.3 cannot be removed. Then by Algorithm 4.3 we provide several examples of symmetric framelet filter banks with two high-pass filters. In Theorem 4.3, the "gcd" condition seems unnatural. One may conjecture that the "gcd" condition would be automatically satisfied if $M_\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\det M_\Theta(z) = d(z^2)d(z^{-2})$ holds for some (anti)symmetric Laurent polynomial d, i.e., in Theorem 4.3, conditions (a) and (b) could imply condition (c). The following example shows that this conjecture is not true.

Example 4.1. Let the low-pass filter a be given by

$$a(z) := \frac{1}{4}(1+z)^2[1 + c_1(2 - z - z^{-1})/2]$$

where $c_1 \approx 0.07391$ is a root of $x^8 + 8x^7 + 35x^6 + 58x^5 - 10x^4 - 72x^3 - x^2 + 14x - 1 = 0$. By a simple calculation, it is easy to verify that the refinable function ϕ with the mask a lies in $L_2(\mathbb{R})$ and in fact is a continuous function. Define $b := c_1^2 + 2c_1 - 1$, $f(z) := 1 + b(2 - z - z^{-1})/4$ and $\Theta(z) := f(z^2)f(z)$. It is easy to verify that $M_\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\det M_\Theta(z) = d(z^2)d(z^{-2})$ for some antisymmetric Laurent polynomial d such that $[Sd](z) = -z^2$. Let $x_0 = 1 + 2(1 - \sqrt{b+1})/b \approx -0.43729 \in (-1, 0)$ which satisfies $f(x_0) = 0$. By a simple computation, we have $Z(g, x_0) = 1$, where $g(z^2) = f(z^2) = \gcd([M_\Theta]_{1,1}, [M_\Theta]_{1,2}, [M_\Theta]_{2,2})$. Since $[Sa](-z)[Sd](z) = z^2(-z^2) = -z^4$, the "gcd" condition fails while conditions (a) and (b) in Theorem 4.3 are satisfied. Therefore, the "gcd" condition in Theorem 4.3 cannot be removed.

In the following, let us apply Algorithm 4.3 to obtain several examples of symmetric framelet filter banks with two high-pass filters.

Example 4.2. Let $\phi = B_3$ be the B-Spline function of order 3. It is known that the low-pass filter for B_3 is $a(z) = (z+1)^3/8$. Define $\Theta(z) := 1 + w +$

$13/15\,w^2+c_1w^3+c_2w^4$ with $w=(2-z-z^{-1})/4$. In order to satisfy the condition $\det M_\Theta(z)=d(z^2)d(z^{-2})$ for some (anti)symmetric Laurent polynomial d, we find that c_2 must be one of the 6 real roots of a polynomial of degree 16 and c_1 can be expressed as a rational polynomial with variable c_2. For simplicity, we present them in decimal notation:

$$c_1 \approx -0.95151049593786685032042323190785445091649974062921$$

and

$$c_2 \approx 3.8031271585681554877256781103577526459109075403804.$$

It is easy to check that $g=1$ and all the conditions in Theorem 4.3 are satisfied. By Algorithms 4.3 and 4.7, solving a system of linear equations, we have the high-pass filters a^1 and a^2 as follows:

$$\begin{aligned}
a^1(z) := z(z-1)^3\Big[&0.01231796418812551(z^3+z^{-3})+\\
&0.07390778512875306(z^2+z^{-2})+\\
&0.1935907748598208(z+z^{-1})-0.01145080836662162\Big],\\
a^2(z) := (z-1)^3\Big[&0.01523563127546168(z^4+z^{-4})+\\
&0.09141378765277004(z^3+z^{-3})+\\
&0.2159429726473255(z^2+z^{-2})+\\
&0.2291636466016358(z+z^{-1})+0.06272019447988098\Big].
\end{aligned}$$

Therefore, $\{\psi^1,\psi^2\}$, which is defined in Theorem 4.1, generates a symmetric tight wavelet frame and has vanishing moments of order 3. See Figure 4.1 for their graphs.

Example 4.3. Let $\phi=B_4$ be the B-Spline function of order 4. The low-pass filter for B_4 is $a(z)=(z+1)^4/16$. Define $\Theta(z):=1+3/4\,w+62/45\,w^2+c_1w^3+c_2w^4+c_3w^5$ with $w=(2-z-z^{-1})/4$. In order to satisfy the condition $\det M_\Theta(z)=d(z^2)d(z^{-2})$ for some (anti)symmetric Laurent polynomial d, we

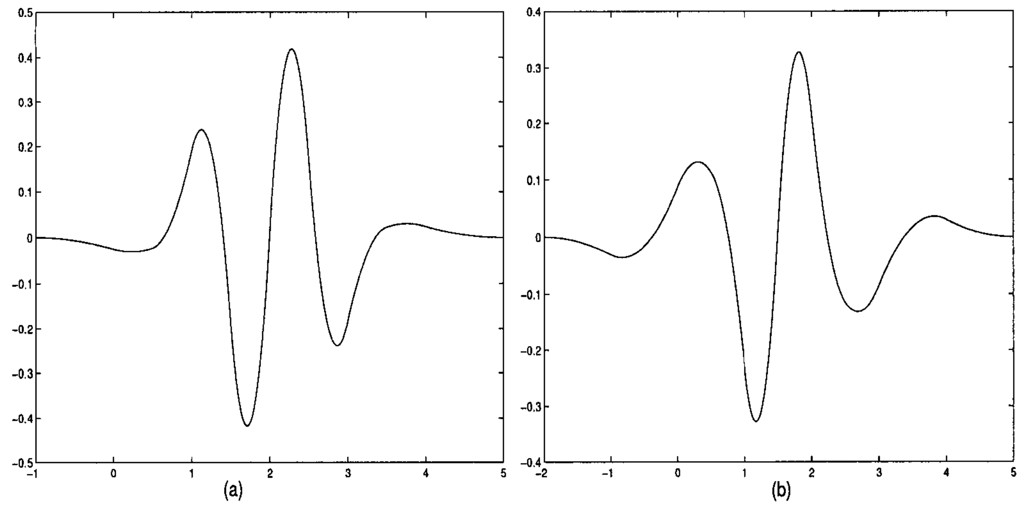

Figure 4.1: (a) is the graph of ψ^1. (b) is the graph of ψ^2. $\{\psi^1, \psi^2\}$ in Example 4.2 generates a symmetric tight wavelet frame with vanishing moments of order 3.

find a solution $\{c_1, c_2, c_3\}$ in decimal notation as follows

$$c_1 \approx -0.875585655474017949144276175704355556,$$

$$c_2 \approx -.0984256534670124429462445137273111111,$$

$$c_3 \approx .000969725649530081100446091981262222222.$$

Then $g = 1$ and all the conditions in Theorem 4.3 hold. By Algorithms 4.3 and 4.7, solving a system of linear equations, we have the high-pass filters a^1 and a^2 as follows:

$$\begin{aligned} a^1(z) :=& z(z+1)(z-1)^3 \Big[0.000021000455515458106(z^5 + z^{-5}) + \\ & 0.0001260027309274863(z^4 + z^{-4}) + 0.01944570184560223(z^3 + z^{-3}) + \\ & 0.1152041792127928(z^2 + z^{-2}) + 0.2275150394894326(z + z^{-1}) + \\ & 0.009838194257376166 \Big], \\ a^2(z) :=& z(z-1)^4 \Big[0.00006434461049978000(z^5 + z^{-5}) + \\ & 0.0005147568839982400(z^4 + z^{-4}) + 0.01966520045452812(z^3 + z^{-3}) + \\ & 0.1465117090722619(z^2 + z^{-2}) + 0.4466955026709126(z + z^{-1}) + \\ & 0.5353777065261440 \Big]. \end{aligned}$$

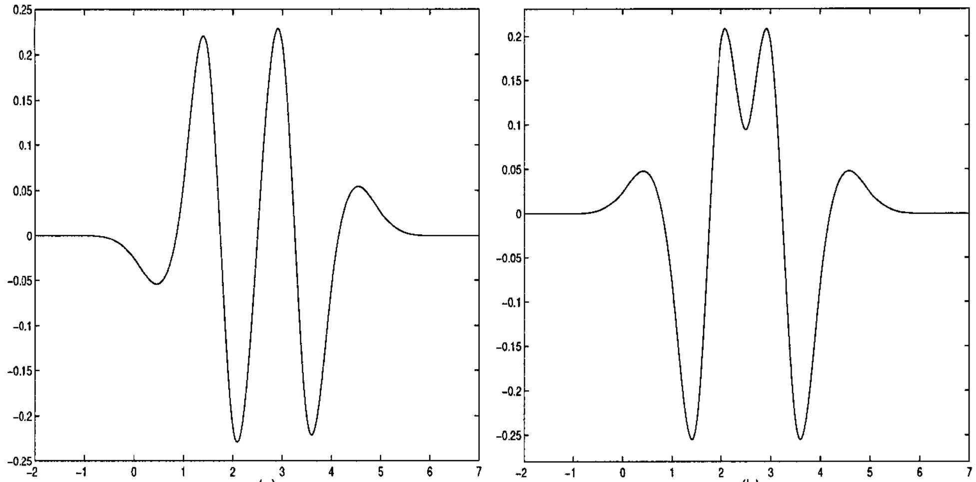

Figure 4.2: (a) is the graph of ψ^1. (b) is the graph of ψ^2. $\{\psi^1, \psi^2\}$ in Example 4.3 generates a symmetric tight wavelet frame with 3 vanishing moments.

Therefore, $\{\psi^1, \psi^2\}$, which is defined in Theorem 4.1, generates a symmetric tight wavelet frame and has 3 vanishing moments. See Figure 4.2 for their graphs.

Example 4.4. The low-pass filter a is given by

$$a(z) = (z+1)^4(4 - z - z^{-1})/16 = -(z^3 + z^{-3})/16 + 9(z + z^{-1})/32 + 1/2.$$

Define $\Theta(z) := 1 + 2/5\, w^2 + 44/315\, w^3 + c_1 w^4 + c_2 w^5 + c_3 w^6 + c_4 w^7 + c_5 w^8 + c_6 w^9$ with $w := (2 - z - z^{-1})/4$. In order to satisfy $\det M_\Theta(z) = d(z^2)d(z^{-2})$ for some (anti)symmetric Laurent polynomial d, we find a solution $\{c_1, c_2, c_3, c_4, c_5, c_6\}$ in decimal notation as follows

$$c_1 \approx -0.5391476369353669, \qquad c_2 \approx 0.03123065991448046,$$
$$c_3 \approx 0.1404437899699654, \qquad c_4 \approx -0.008183355709257437,$$
$$c_5 \approx -0.02305770106687993, \qquad c_6 \approx 0.005166592059270131.$$

It is easy to check that $g = 1$ and all the conditions in Theorem 4.3 hold. By Algorithms 4.3 and 4.7, solving a system of linear equations, we have the high-pass filters a^1 and a^2 as follows:

$$a^1(z) := (z-1)^4\Big[0.000009949295438893275(z^9+z^{-9})+$$
$$0.00003979718175557310(z^8+z^{-8})+0.00005349425360331152(z^7+z^{-7})-$$
$$0.0001441976213869118(z^6+z^{-6})-0.001526840787475249(z^5+z^{-5})-$$
$$0.005764716919552544(z^4+z^{-4})-0.01264520660352171(z^3+z^{-3})-$$
$$0.01724394516308753(z^2+z^{-2})+0.01039096409945511(z+z^{-1})+$$
$$0.1033772717787854\Big],$$
$$a^2(z) := (z-1)^4\Big[0.000004387146598246904(z^9+z^{-11})+$$
$$0.00001754858639298762(z^8+z^{-10})-0.000007220506808539094(z^7+z^{-9})-$$
$$0.0001868193047710449(z^6+z^{-8})-0.001033777502667078(z^5+z^{-7})-$$
$$0.002874902341160608(z^4+z^{-6})-0.002673048014126028(z^3+z^{-5})+$$
$$0.009978772639517269(z^2+z^{-4})+0.06388250373593019(z+z^{-3})+$$
$$0.2021738981781012(1+z^{-2})+0.3153550969816685z^{-1}\Big].$$

Therefore, $\{\psi^1,\psi^2\}$, which is defined in Theorem 4.1, generates a symmetric tight wavelet frame and has vanishing moments of order 4. See Figure 4.3 for their graphs.

4.4 Some auxiliary results

In order to prove Theorem 4.2, in this section we establish some auxiliary results. The following result generalizes [4, Theorem 4] by taking into account symmetry.

Theorem 4.4. *Let A, B and C be (anti)symmetric Laurent polynomials with real coefficients. Let M be defined in (4.2.9). Suppose that $M(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\det M(z) = d(z)d(1/z)$ for some (anti)symmetric Laurent polynomial d with real coefficients. If A and B have no common zeros in $\mathbb{C}\backslash\{0\}$*

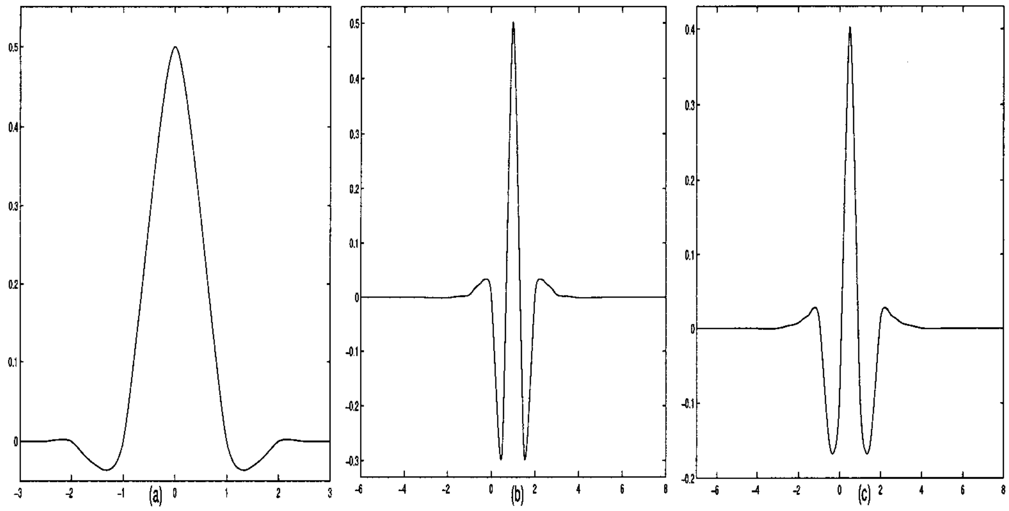

Figure 4.3: (a) is the graph of the interpolating refinable function ϕ. (b) is the graph of ψ^1. (c) is the graph of ψ^2. $\{\psi^1, \psi^2\}$ in Example 4.4 generates a symmetric tight wavelet frame with vanishing moments of order 4.

and $A(z) = A_0 + \sum_{k=1}^{N} A_k(z^k + z^{-k})$ *with* $A_N \neq 0$, *then there exist four (anti)symmetric Laurent polynomials* u_1, u_2, v_1, v_2 *with real coefficients such that* (4.2.10) *and* (4.2.12) *are satisfied with the degrees of* u_1 *and* v_1 *being at most* N. *In fact, if* u_1, u_2, v_1, v_2 *are (anti)symmetric Laurent polynomials with real coefficients such that the degrees of* u_1 *and* v_1 *are at most* N, (4.2.12) *holds, and* $\{u_1, u_2, v_1, v_2\}$ *is a solution to the following system of linear equations*

$$\begin{cases} B(1/z)u_1(z) - d(z)v_1(1/z) - A(z)u_2(z) = 0, \\ B(1/z)v_1(z) + d(z)u_1(1/z) - A(z)v_2(z) = 0, \end{cases} \tag{4.4.20}$$

with the following normalization condition

$$u_1(1)^2 + v_1(1)^2 = A(1), \tag{4.4.21}$$

then (4.2.10) *holds.*

Proof: If $B(z) \equiv 0$, by $\gcd(A, B) = 1$, then $A(z)$ must be a positive constant and all the claims can be easily verified by taking $u_1 = \sqrt{A}, u_2 = 0, v_1 = 0$ and $v_2 = d/\sqrt{A}$. So, we can assume that B is not identically zero.

If $d(z) \equiv 0$, then $A(z)C(z) = B(z)B(1/z)$. Since $\gcd(A, B) = 1$ and B is (anti)symmetric, it follows from $A(z)C(z) = B(z)B(1/z) = B(z)^2/[SB](z)$ that A must be a positive constant. All the claims hold by taking $u_1 = \sqrt{A}, u_2 = B(1/z)/\sqrt{A}, v_1 = 0$ and $v_2 = 0$. So, we can assume that d is not identically zero.

The following proof (about 2 pages) is borrowed from the proof of [4, Theorem 4]. Under the assumption that (4.2.12) holds and the degrees of u_1 and v_1 are at most N, we first show that (4.2.10) is equivalent to the system of linear equations in (4.4.20) with the condition in (4.4.21).

Since $M(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\gcd(A, B) = 1$, if we have $A(z_0) = 0$ for some $z_0 \in \mathbb{T}$, then by condition $M(z_0) \geqslant 0$, we have

$$0 \leqslant \det M(z_0) = A(z_0)C(z_0) - B(z_0)B(1/z_0) = -B(z_0)\overline{B(z_0)} = -|B(z_0)|^2.$$

Hence, $B(z_0) = 0$. Therefore, $(z - z_0) \mid A(z)$ and $(z - z_0) \mid B(z)$. So, $(z - z_0) \mid \gcd(A, B)$ which is a contradiction to the assumption $\gcd(A, B) = 1$. So, $A(z) \neq 0$ for all $z \in \mathbb{T}$. Since $A(z) \geqslant 0$ for all $z \in \mathbb{T}$, we must have $A(z) > 0$ for all $z \in \mathbb{T}$. By Proposition 4.2, without loss of generality, we can assume that $d(z) = \det U(z)$. By $U(z)U(1/z)^T = M(z)$, we have $u_1(1)^2 + v_1(1)^2 = A(1)$ and therefore (4.4.21) holds. Since $d(z) \not\equiv 0$ and $d(z)U(z)^{-1} = \operatorname{adj} U(z)$, it follows from $U(z)U(1/z)^T = M(z)$ that

$$\begin{aligned} d(z)U(1/z)^T &= d(z)U(z)^{-1}M(z) = [\operatorname{adj} U(z)]M(z) \\ &= \begin{bmatrix} v_2(z) & -v_1(z) \\ -u_2(z) & u_1(z) \end{bmatrix} \begin{bmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{bmatrix}. \end{aligned}$$

Comparing the $(1, 1)$ and $(2, 1)$-entries of the above matrices, we see that (4.4.20) holds.

Conversely, let u_1, u_2, v_1, v_2 be (anti)symmetric Laurent polynomials with real coefficients such that (4.2.12) holds and the degrees of u_1 and v_1 are at most N. If $\{u_1, u_2, v_1, v_2\}$ is a solution to the system of linear equations in (4.4.20) and satisfies the normalization condition in (4.4.21), then we show that (4.2.10) must be true.

Multiplying $u_1(1/z)$ with the first equation and multiplying $v_1(1/z)$ with the second equation in (4.4.20), by adding them together we have
(4.4.22)

$$B(1/z)[u_1(z)u_1(1/z) + v_1(z)v_1(1/z)] = A(z)[u_1(1/z)u_2(z) + v_1(1/z)v_2(z)].$$

Since A and B have no common zeros in $\mathbb{C}\backslash\{0\}$, we must have that $A(z)$ divides $[u_1(z)u_1(1/z) + v_1(z)v_1(1/z)]$. That is, there is a Laurent polynomial p such that $u_1(z)u_1(1/z) + v_1(z)v_1(1/z) = p(z)A(z)$. Since the degrees of u_1 and v_1 are at most N and $A(z) = A_0 + \sum_{k=1}^{N} A_k(z^k + z^{-k})$ with $A_N \neq 0$, we conclude that p must be a constant. By (4.4.21) and $A(1) > 0$, we must further have $p \equiv 1$. Therefore,

$$u_1(z)u_1(1/z) + v_1(z)v_1(1/z) = A(z).$$

It follows from (4.4.22) that $B(1/z) = u_1(1/z)u_2(z) + v_1(1/z)v_2(z)$ and consequently, $B(z) = u_1(z)u_2(1/z) + v_1(z)v_2(1/z)$. In other words, $[U(z)U(\frac{1}{z})^T]_{j,k} = [M(z)]_{j,k}$ for all $1 \leqslant j, k \leqslant 2$ except for the case $j = k = 2$.

Multiplying $v_2(z)$ with the first equation and multiplying $u_2(z)$ with the second equation in (4.4.20), by subtracting the second one from the first one, we have

$$B(\frac{1}{z})[u_1(z)v_2(z) - u_2(z)v_1(z)] = d(z)[u_1(1/z)u_2(z) + v_1(1/z)v_2(z)] = d(z)B(\frac{1}{z}).$$

So, by $B \not\equiv 0$, $d(z) = u_1(z)v_2(z) - u_2(z)v_1(z) = \det U(z)$. Consequently, $\det[U(z)U(1/z)^T] = d(z)d(1/z) = \det M(z)$. Now it is easy to deduce that $[U(z)U(1/z)^T]_{2,2} = [M(z)]_{2,2}$ from the fact that $\det[U(z)U(1/z)^T] = \det M(z)$ and $[U(z)U(1/z)^T]_{j,k} = [M(z)]_{j,k}$ for all $1 \leqslant j, k \leqslant 2$ except for $j = k = 2$. So (4.2.10) holds.

Now we will add some new discussion by taking symmetry into account.

In the second part of the proof, let us show the existence of a desirable solution $\{u_1, u_2, v_1, v_2\}$ to the system of linear equations in (4.4.20) with the normalization condition in (4.4.21).

First, we demonstrate that there are desirable Laurent polynomials u_1 and v_1 satisfying

$$A(z) \mid [B(1/z)u_1(z) - d(z)v_1(1/z)] \tag{4.4.23}$$

and

$$[Su_1](z)[Sv_1](z) = [SB](z)[Sd](z). \tag{4.4.24}$$

Let u_0 and v_0 be two symmetric Laurent polynomials in the following parametric forms:

$$u_0(z) = b_0 + \sum_{j=1}^{h_b} b_j(z^j + z^{-j}) \quad \text{and} \quad v_0(z) = c_0 + \sum_{k=1}^{h_c} c_k(z^k + z^{-k})$$

where h_b, h_c are nonnegative integers and $b_j, c_k, j = 0, \ldots, h_b, k = 0, \ldots, h_c$ are real numbers which are to be determined later. Let us consider the following four cases.

Case 1: $[SB](z)[Sd](z) = z^{2n}$ for some $n \in \mathbb{Z}$. We choose $u_1(z) = z^n u_0(z)$ and $v_1(z) = v_0(z)$. When N is even, set $h_b = h_c = N/2$; when N is odd, set $h_b = h_c = (N-1)/2$.

Case 2: $[SB](z)[Sd](z) = z^{2n+1}$ for some $n \in \mathbb{Z}$. We choose $u_1(z) = z^n(1 + z)u_0(z)$ and $v_1(z) = v_0(z)$. When N is even, set $h_b = N/2 - 1$ and $h_c = N/2$; when N is odd, set $h_b = h_c = (N-1)/2$.

Case 3: $[SB](z)[Sd](z) = -z^{2n}$ for some $n \in \mathbb{Z}$. When N is even, we choose $u_1(z) = z^n(z - 1/z)u_0(z)$, $v_1(z) = v_0(z)$ and set $h_b = N/2 - 1, h_c = N/2$; when N is odd, we choose $u_1(z) = z^n(1 - z)u_0(z)$, $v_1(z) = (1 + 1/z)v_0(z)$ and set $h_b = h_c = (N-1)/2$.

Case 4: $[SB](z)[Sd](z) = -z^{2n+1}$ for some $n \in \mathbb{Z}$. We choose $u_1(z) = z^n(1 - z)u_0(z)$ and $v_1(z) = v_0(z)$. When N is even, set $h_b = N/2 - 1$ and $h_c = N/2$; when N is odd, set $h_b = h_c = (N-1)/2$.

It is easy to see that both u_1 and v_1 are (anti)symmetric and (4.4.24) holds. Moreover, the degrees of u_1 and v_1 are at most N and it is easy to verify that $h_b+h_c+2>N$. Since $A(z)>0$ for all $z\in\mathbb{T}$, by Fejér-Riesz Lemma, we have $A(z)=\widetilde{A}(z)\widetilde{A}(1/z)$ for some Laurent polynomial $\widetilde{A}$ with real coefficients such that all of the roots of $\widetilde{A}$ are contained in $\{z\in\mathbb{C}\,:\,|z|<1\}$. Therefore, $\widetilde{A}(z)$ and $\widetilde{A}(1/z)$ have no common zeros in $\mathbb{C}\backslash\{0\}$. Since $A(z)=A_0+\sum_{k=1}^{N}A_k(z^k+z^{-k})$, $\widetilde{A}(z)$ can have at most N zeros in $\mathbb{C}\backslash\{0\}$, say, $\{z_1,\ldots,z_{N'}\}$ which are all of the distinct roots of the Laurent polynomial $\widetilde{A}(z)$ in $\mathbb{C}\backslash\{0\}$ such that $Z(\widetilde{A},z_1)+\cdots+Z(\widetilde{A},z_{N'})=N$. Define $F(z):=B(1/z)u_1(z)-d(z)v_1(1/z)$. Now we have the following system of homogeneous linear equations:

$$F^{(j)}(z_k)=0,\qquad k=0,\ldots,N',\ j=0,\ldots,Z(\widetilde{A},z_k)-1. \tag{4.4.25}$$

Since the number of free parameters in $\{c_j,d_k\,:\,j=0,\ldots,h_b,k=0,\ldots,h_c\}$ is $h_b+h_c+2>N$ and we have N homogeneous linear equations, there must be a nonzero solution $\{c_j,d_k\,:\,j=0,\ldots,h_b,k=0,\ldots,h_c\}$ to the system of homogeneous linear equations in (4.4.25). So there exist u_1 and v_1 satisfying (4.4.25) with at least one of them nonzero. In other words, we deduce from (4.4.25) that

$$\widetilde{A}(z)\mid[B(1/z)u_1(z)-d(z)v_1(1/z)]. \tag{4.4.26}$$

Since $z_1,\ldots,z_{N'}$ are complex numbers, a solution $\{c_j,d_k\,:\,j=0,\ldots,h_b,k=0,\ldots,h_c\}$ may be complex numbers too. However, since $\widetilde{A},B$ and C are Laurent polynomials with real coefficients, we can simply replace the numbers c_j,d_k by either their real parts or their imaginary part so that (4.4.26) is still true and at least one of u_1 and v_1 is nonzero.

On the other hand, by (4.4.24) and Proposition 4.1, we deduce that $B(1/z)u_1(z)-d(z)v_1(1/z)$ is (anti)symmetric. So,

$$B(z)u_1(1/z)-d(1/z)v_1(z)=p(z)[B(1/z)u_1(z)-d(z)v_1(1/z)] \tag{4.4.27}$$

for some nonzero trivial Laurent polynomial p. Consequently, it follows from (4.4.26) and (4.4.27) that

$$\widetilde{A}(1/z)\mid[B(1/z)u_1(z)-d(z)v_1(1/z)].$$

Since $\widetilde{A}(z)$ and $\widetilde{A}(\frac{1}{z})$ have no common zeros in $\mathbb{C}\backslash\{0\}$ and $A(z) = \widetilde{A}(z)\widetilde{A}(\frac{1}{z})$, we conclude that (4.4.23) holds. Later on we shall show that $u_1(1)^2 + v_1(1)^2 \neq 0$. If $u_1(1)^2 + v_1(1)^2 \neq 0$, then we can properly scale u_1 and v_1 such that $u_1(1)^2 + v_1(1)^2 = A(1)$ holds. Note that without factorizing A we can solve the system of linear equations given by $[B(1/z)u_1(z) - d(z)v_1(1/z)] \equiv 0($ mod $A(z))$ to obtain the desired u_1 and v_1.

Since $A(z) \not\equiv 0$, we can define

(4.4.28)
$$u_2(z) := \frac{B(\frac{1}{z})u_1(z) - d(z)v_1(\frac{1}{z})}{A(z)} \quad \text{and} \quad v_2(z) := \frac{d(z)u_1(\frac{1}{z}) + B(\frac{1}{z})v_1(z)}{A(z)}.$$

By (4.4.23) we see that u_2 is an (anti)symmetric Laurent polynomial with real coefficients. Now we show that v_2 is also an (anti)symmetric Laurent polynomial. By definition of u_2 and the fact that $d(z)d(1/z) = \det M(z) = A(z)C(z) - B(z)B(1/z)$, we have

$$\begin{aligned} A(z)d(1/z)u_2(z) &= B(1/z)d(1/z)u_1(z) - \det M(z)v_1(1/z) \\ &= B(1/z)d(1/z)u_1(z) - A(z)C(z)v_1(1/z) + B(1/z)B(z)v_1(1/z). \end{aligned}$$

From the above identity, we have

$$A(z)[d(1/z)u_2(z) + C(z)v_1(1/z)] = B(1/z)[d(1/z)u_1(z) + B(z)v_1(1/z)].$$

Since $A(z) = A(1/z)$ and $\gcd(A, B) = 1$, we conclude that $A(z)$ divides $[d(1/z)u_1(z) + B(z)v_1(1/z)]$ and therefore, by $A(1/z) = A(z)$, $A(z)$ divides $[d(z)u_1(1/z) + B(1/z)v_1(z)]$. So v_2 is a Laurent polynomial with real coefficients. By (4.4.24) and Proposition 4.1, we see that v_2 is (anti)symmetric.

By (4.4.28) and Proposition 4.1, we see that (4.2.12) and the system of linear equations in (4.4.20) must hold. In the following, let us show that $u_1(1)^2 + v_1(1)^2 \neq 0$. Since both (4.2.12) and (4.4.20) are satisfied, as we demonstrated in the first part of the proof, we must have $u_1(z)u_1(1/z) + v_1(z)v_1(1/z) = pA(z)$ for some constant p. If $u_1(1) = v_1(1) = 0$, by $A(1) > 0$, then we must have $p = 0$. That is, $|u_1(z)|^2 + |v_1(z)|^2 = u_1(z)u_1(1/z) +$

$v_1(z)v_1(1/z) = 0$ for all $z \in \mathbb{T}$. So, u_1 and v_1 must be identically zero which is a contradiction to our choice of u_1 and v_1 since one of them must be nonzero. So $u_1(1)^2 + v_1(1)^2 \neq 0$. Now replacing u_1 and v_1 by cu_1 and cv_1 with $c = \sqrt{A(1)/(u_1(1)^2 + v_1(1)^2)}$ in the above proof, we see that (4.4.20) and (4.2.12) still hold. Moreover, we have $u_1(1)^2 + v_1(1)^2 = A(1)$ which completes the proof. ∎

Let $\mathbb{R}[z, z^{-1}]$ denote the set of all Laurent polynomials with real coefficients. For a Laurent polynomial $p \in \mathbb{R}[z, z^{-1}]$, we say that p is **irreducible** in $\mathbb{R}[z, z^{-1}]$ if $q \mid p$ for some $q \in \mathbb{R}[z, z^{-1}]$ implies that $q = p_0$ or $q = p_0 p$ for some trivial Laurent polynomial $p_0 \in \mathbb{R}[z, z^{-1}]$ (that is, $p_0 = cz^k$ for some $c \in \mathbb{R} \backslash \{0\}$ and $k \in \mathbb{Z}$).

Inspired by [4, Lemma 4], we have a stronger version of Theorem 4.4.

Corollary 4.4. Let A, B and C be (anti)symmetric Laurent polynomials with real coefficients. Let M be defined in (4.2.9). Suppose that $M(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\det M(z) = d(z)d(1/z)$ for some (anti)symmetric Laurent polynomial d with real coefficients. If $\gcd(A, B, C) = 1$, then there exist four (anti)symmetric Laurent polynomials u_1, u_2, v_1, v_2 with real coefficients such that (4.2.10) and (4.2.12) are satisfied.

Proof: If $C(z) \equiv 0$, then $\gcd(A, B) = \gcd(A, B, C) = 1$ and all the claims follow from Theorem 4.4. So, we can assume that C is not identically zero.

Define $h(z) = \gcd(A(z), B(z)B(1/z))$. By the symmetry of A and B, we see that h must be (anti)symmetric. Now, we show that $\gcd(h, C) = 1$. Suppose not. Then there is a nontrivial irreducible $p \in \mathbb{R}[z, z^{-1}]$ such that $p \mid \gcd(h, C)$. So, $p \mid h$ and $p \mid C$. Consequently, $p \mid A$ and $p \mid B(z)B(1/z)$. Note that $B(1/z) = B(z)/[SB](z)$ and SB is trivial. So $p \mid B^2$. Since p is irreducible, we must have $p \mid B$. So, $p \mid \gcd(A, B, C)$ which is a contradiction since p is nontrivial but by assumption $\gcd(A, B, C) = 1$.

Next, we show that for a nontrivial irreducible $p \in \mathbb{R}[z, z^{-1}]$, if $p^{2n-1} \mid h$ for some $n \in \mathbb{N}$, then we must have $p^{2n} \mid h$. Since $p^{2n-1} \mid h$, we have $p^{2n-1} \mid$

$B(z)B(1/z)$ and therefore, $p^{2n-1} \mid B^2$. Since p is irreducible, we must have $p^n \mid B$ and consequently $p^{2n} \mid B(z)B(1/z)$.

On the other hand, by $p^{2n-1} \mid h$ and $h = \gcd(A(z), B(z)B(1/z))$, we have

$$p^{2n-1} \mid [A(z)C(z) - B(z)B(1/z)].$$

Since $A(z)C(z) - B(z)B(1/z) = \det M(z) = d(z)d(1/z)$, we have $p^{2n-1} \mid d(z)d(1/z)$. Since $d(1/z) = d(z)/[Sd](z)$ and Sd is trivial, it follows from $p^{2n-1} \mid d^2$ that $p^{2n} \mid d(z)d(1/z)$. Since $C \not\equiv 0$, by $d(z)d(1/z) = \det M(z) = A(z)C(z) - B(z)B(1/z)$, we have

$$A(z) = \frac{d(z)d(1/z) + B(z)B(1/z)}{C(z)}.$$

By $\gcd(h, C) = 1$ and $p \mid h$, we must have $p \nmid C$ since p is not trivial. Hence, we must have $p^{2n} \mid A$. So, $p^{2n} \mid h$. As a consequence of the fact that $p^{2n-1} \mid h$ implies $p^{2n} \mid h$, factorize h as

$$h(z) = p_0(z) \prod_{j=1}^{m} p_j^{2n_j}(z),$$

where p_0 is a trivial Laurent polynomial and $p_1, \dots, p_m$ are essentially different nontrivial irreducible Laurent polynomials in $\mathbb{R}[z, z^{-1}]$. Now define

$$d_h(z) := \prod_{j=1}^{m} p_j^{n_j}(z).$$

Then $h(z) = p_0(z)d_h(z)d_h(z)$. Note that by Proposition 4.2 we can directly obtain d_h from h without factorizing h. Since $([Sd_h](z))^2 = [Sh](z)/[Sp_0](z)$ is a trivial Laurent polynomial, Sd_h must be trivial and therefore d_h is (anti)-symmetric. So, $\gcd(A(z), B(z)B(1/z)) = d_h(z)d_h(1/z)$. Since both d_h and B are (anti)symmetric, it follows from $d_h(z)d_h(1/z) \mid B(z)B(1/z)$ that $d_h^2 \mid B^2$ and consequently $d_h \mid B$. Define

$$\widetilde{A}(z) := \frac{A(z)}{d_h(z)d_h(1/z)}, \quad \widetilde{B}(z) := \frac{B(z)}{d_h(z)} \quad \text{and} \quad \widetilde{M}(z) = \begin{bmatrix} \widetilde{A}(z) & \widetilde{B}(z) \\ \widetilde{B}(1/z) & C(z) \end{bmatrix}.$$

Clearly, $\widetilde{A}$, $\widetilde{B}$ and C are (anti)symmetric Laurent polynomials and $\gcd(\widetilde{A}, \widetilde{B}) = 1$. By Theorem 4.4, there exist four (anti)symmetric Laurent polynomials $\widetilde{u}_1, \widetilde{u}_2, \widetilde{v}_1, \widetilde{v}_2$ with real coefficients such that

$$\widetilde{M}(z) = \begin{bmatrix} \widetilde{u}_1(z) & \widetilde{v}_1(z) \\ \widetilde{u}_2(z) & \widetilde{v}_2(z) \end{bmatrix} \begin{bmatrix} \widetilde{u}_1(1/z) & \widetilde{u}_2(1/z) \\ \widetilde{v}_1(1/z) & \widetilde{v}_2(1/z) \end{bmatrix} \tag{4.4.29}$$

and

$$\frac{[S\widetilde{u}_1](z)}{[S\widetilde{u}_2](z)} = [S\widetilde{M}_{1,2}](z) = \frac{[S\widetilde{v}_1](z)}{[S\widetilde{v}_2](z)}. \tag{4.4.30}$$

Note that

$$M(z) = \begin{bmatrix} d_h(z) & 0 \\ 0 & 1 \end{bmatrix} \widetilde{M}(z) \begin{bmatrix} d_h(1/z) & 0 \\ 0 & 1 \end{bmatrix}.$$

Define

$$u_1(z) = \widetilde{u}_1(z) d_h(z), \quad v_1(z) = \widetilde{v}_1(z) d_h(z), \quad u_2(z) = \widetilde{u}_2(z), \quad v_2(z) = \widetilde{v}_2(z).$$

Then it follows directly from (4.4.29) and (4.4.30) that (4.2.10) and (4.2.12) are satisfied. ■

Lemma 4.5. *Let p be a nonzero (anti)symmetric Laurent polynomial with real coefficients. Then there exist $c \in \{-1, 1\}$ and $k \in \mathbb{Z}$ such that $cz^k p(z) \geqslant 0$ for all $z \in \mathbb{T}$ if and only if $Z(p, z_0)$ is an even integer for every $z_0 \in \mathbb{T}$.*

Proof: If $cz^k p(z) \geqslant 0$ for all $z \in \mathbb{T}$, then by Fejér-Riesz Lemma, $cz^k p(z) = q(z)q(1/z)$ for some Laurent polynomial q with real coefficients. Hence for all $z_0 \in \mathbb{T}$, we have

$$Z(p, z_0) = Z(cz^k p(z), z_0) = Z(q(z), z_0) + Z(q(1/z), z_0) = 2Z(q(z), z_0)$$

where we used the fact that $Z(q(1/z), z_0) = Z(\overline{q(z)}, z_0) = Z(q, z_0)$ for all $z_0 \in \mathbb{T}$ since q is a Laurent polynomial with real coefficients. So $Z(p, z_0)$ must be an even integer for every $z_0 \in \mathbb{T}$.

Conversely, write $p(z) = q(z)h(z)$ such that $q(z) \neq 0$ for all $z \in \mathbb{T}$ and all of the zeros of h lie on $\mathbb{T}$. Since p is (anti)symmetric and $Z(p, z_0) = Z(h, z_0)$

is an even integer for all $z_0 \in \mathbb{T}$, there exist $c_1 \in \{-1, 1\}$ and $k_1 \in \mathbb{Z}$ such that $c_1 z^{k_1} h(z) \geqslant 0$ for all $z \in \mathbb{T}$. Since $[Sp](z) = [Sq](z)[Sh](z)$, q must be symmetric. Since $q(z) \neq 0$ for all $z \in \mathbb{T}$, we must have $[Sq](z) = z^{2k_2}$ for some $k_2 \in \mathbb{Z}$. So $z^{-k_2} q(z) \neq 0$ and is real-valued for all $z \in \mathbb{T}$. Consequently, there exists $c_2 \in \{-1, 1\}$ such that $c_2 z^{-k_2} q(z) > 0$ for all $z \in \mathbb{T}$. So, $c_1 c_2 z^{k_1 - k_2} p(z) = c_1 z^{k_1} h(z) c_2 z^{-k_2} q(z) \geqslant 0$ for all $z \in \mathbb{T}$. ∎

Remark: When p is antisymmetric, it is evident that both conditions in Lemma 4.5 cannot be satisfied.

Lemma 4.6. *Let g be a nonzero Laurent polynomial with real coefficients. Then there exist two (anti)symmetric Laurent polynomials q_1 and q_2 with real coefficients such that*

$$q_1(z)q_1(1/z) + q_2(z)q_2(1/z) = g(z) \tag{4.4.31}$$

and

$$[Sq_1](z)/[Sq_2](z) = z^{2k}, -z^{2k}, z^{2k+1}, \text{ or } -z^{2k+1} \qquad \text{for some integer } k$$

if and only if $g(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $Z(g, x)$ is an even integer for every $x \in (-1, 0) \cup (0, 1)$, $x \in \emptyset$, $x \in (0, 1)$, or $x \in (-1, 0)$, respectively.

Proof: Necessity. If (4.4.31) holds, then it is evident that $g(z) \geqslant 0$ for all $z \in \mathbb{T}$. Since $q_1(1/z) = q_1(z)/[Sq_1](z)$ and $q_2(1/z) = q_2(z)/[Sq_2](z)$, we can rewrite (4.4.31) as follows:

$$q_1^2(z) + q_2^2(z)[Sq_1](z)/[Sq_2](z) = g(z)[Sq_1](z).$$

If $[Sq_1](z)/[Sq_2](z) = z^{2k}$, then we have $q_1^2(x) + x^{2k} q_2^2(x) = g(x)[Sq_1](x)$ for all $x \in \mathbb{R}\backslash\{0\}$ and consequently, it is easy to see that for every $x \in (-1, 0) \cup (0, 1)$, we have

$$Z(g, x) = Z(g[Sq_1], x) = \min(Z(q_1^2, x), Z(q_2^2, x)) = 2\min(Z(q_1, x), Z(q_2, x)).$$

So, when $[Sq_1](z)/[Sq_2](z) = z^{2k}$, $Z(g, x)$ must be an even integer for all $x \in (-1, 0) \cup (0, 1)$.

If $[Sq_1](z)/[Sq_2](z) = z^{2k+1}$, then we have $q_1^2(x)+x^{2k+1}q_2^2(x) = g(x)[Sq_1](x)$ for all $x \in \mathbb{R}\backslash\{0\}$. Similarly, it is easy to prove that for every $x \in (0,1)$, $Z(g,x) = 2\min\bigl(Z(q_1,x),Z(q_2,x)\bigr)$ must be an even integer.

If $[Sq_1](z)/[Sq_2](z) = -z^{2k+1}$, then we have $q_1^2(x) + (-x)^{2k+1}q_2^2(x) = g(x)[Sq_1](x)$ for all $x \in \mathbb{R}\backslash\{0\}$. Similarly, it is easy to prove that for every $x \in (-1,0)$, $Z(g,x) = 2\min(Z(q_1,x), Z(q_2,x))$ must be an even integer.

Sufficiency. Since $g(z) \geqslant 0$ for all $z \in \mathbb{T}$, by Fejér-Riesz lemma, we can write $g(z) = h(z)h(1/z)$ for some Laurent polynomial h with real coefficients such that all of the roots of h are contained in $\{z : |z| \leqslant 1\}$. Set $q_1(z) = z^k[h(z)+h(1/z)]/2$ and $q_2(z) = [h(z)-h(1/z)]/2$. Then it is easy to verify that (4.4.31) holds and $[Sq_1](z)/[Sq_2](z) = -z^{2k}$. In the following, let us consider the other three cases. Factorize h as

$$h(z) = p_0(z)(z-1)^{Z(h,1)}(z+1)^{Z(h,-1)}\prod_{j=1}^{m} p_j^{n_j}(z),$$

where p_0 is a trivial Laurent polynomial and all $p_j, j = 1,\dots,m$ are essentially different nontrivial irreducible Laurent polynomials in $\mathbb{R}[z,z^{-1}]$. Since there are only two types of nontrivial irreducible Laurent polynomials in $\mathbb{R}[z,z^{-1}]$, without loss of generality we can assume that either $p_j = z - a_j$ for some $a_j \in (-1,1)\backslash\{0\}$ or $p_j(z) = z^2 + b_jz + c_j$ for some $b_j, c_j \in \mathbb{R}$ satisfying $b_j^2 - 4c_j < 0$. Let us consider the following two cases.

If $p_j(z) = z^2 + b_jz + c_j$ for some $b_j, c_j \in \mathbb{R}$ satisfying $4c_j > b_j^2$, since $c_j \geqslant 0$ and $-2\sqrt{c_j} \leqslant b_j \leqslant 2\sqrt{c_j}$, then we have

$$\begin{aligned} p_j(z) &= [z + b_j/2]^2 + z^{2k}\left[\sqrt{c_j - b_j^2/4}\, z^{-k}\right]^2 \\ &= [z - \sqrt{c_j}]^2 + z^{2k+1}\left[\sqrt{2\sqrt{c_j} + b_j}\, z^{-k}\right]^2 \\ &= [z + \sqrt{c_j}]^2 - z^{2k+1}\left[\sqrt{2\sqrt{c_j} - b_j}\, z^{-k}\right]^2. \end{aligned}$$

If $p_j(z) = z - a_j$ for some $a_j \in (-1,1)\backslash\{0\}$, then by assumption, we have the following cases:

Case 1: If $Z(g,x)$ is an even integer for all $x \in (-1,0) \cup (0,1)$, then n_j must be an even integer and therefore, $p_j^{n_j}(z) = [(z-a_j)^{n_j/2}]^2 + z^{2k} \times 0$.

Case 2: If $Z(g,x)$ is an even integer for all $x \in (0,1)$, then n_j must be an even integer when $a_j \in (0,1)$. Therefore, when $a_j \in (0,1)$, we have $p_j^{n_j}(z) = [(z-a_j)^{n_j/2}]^2 + z^{2k+1} \times 0$. When $a_j \in (-1,0)$, we also have $p_j(z) = z - a_j = [\sqrt{-a_j}]^2 + z^{2k+1}[z^{-k}]^2$.

Case 3: If $Z(g,x)$ is an even integer for all $x \in (-1,0)$, then n_j must be an even integer if $a_j \in (-1,0)$. When $a_j \in (-1,0)$, we have $p_j^{n_j}(z) = [(z-a_j)^{n_j/2}]^2 - z^{2k+1} \times 0$. When $a_j \in (0,1)$, we also have $p_j(z) = z - a_j = -\big([\sqrt{a_j}]^2 - z^{2k+1}[z^{-k}]^2\big)$.

By a direct computation, it is easy to verify the following identify

$$(f_1^2 + wf_2^2)(f_3^2 + wf_4^2) = (f_1f_3 - wf_2f_4)^2 + w(f_1f_4 + f_2f_3)^2. \tag{4.4.32}$$

By the above argument, using the identity in (4.4.32) we have

$$h(z) = \widetilde{q}_0(z)(z-1)^{Z(h,1)}(z+1)^{Z(h,-1)}\big(\widetilde{q}_1^2(z) + w(z)\widetilde{q}_2^2(z)\big),$$

where $\widetilde{q}_0$ is a trivial Laurent polynomial, $w(z) = z^{2k}, z^{2k+1}$ or $-z^{2k+1}$ according to the assumption, and $\widetilde{q}_1$ and $\widetilde{q}_2$ are Laurent polynomials with real coefficients. Observing that $w(1/z) = w(z)^{-1}$, we have

$$h(1/z) = \widetilde{q}_0(1/z)(1/z-1)^{Z(h,1)}(1/z+1)^{Z(h,-1)}\big(\widetilde{q}_1^2(1/z) + w(z)[\widetilde{q}_2(1/z)/w(z)]^2\big).$$

Note that $\widetilde{q}_0(z)\widetilde{q}_0(1/z)$ is a positive constant since $\widetilde{q}_0$ is trivial. By a simple computation, we deduce that

$$g(z) = h(z)h(1/z) = q_1(z)q_1(1/z) + q_2(z)q_2(1/z)$$

where

$$q_1(z) := \sqrt{\widetilde{q}_0(z)\widetilde{q}_0(\tfrac{1}{z})}(z-1)^{Z(h,1)}(z+1)^{Z(h,-1)}[\widetilde{q}_1(z)\widetilde{q}_1(\tfrac{1}{z}) - \widetilde{q}_2(z)\widetilde{q}_2(\tfrac{1}{z})],$$

$$q_2(z) :=$$

$$\sqrt{\widetilde{q}_0(z)\widetilde{q}_0(\tfrac{1}{z})}(z-1)^{Z(h,1)}(z+1)^{Z(h,-1)}[\widetilde{q}_1(z)\widetilde{q}_2(\tfrac{1}{z})w(z)^{-1} + \widetilde{q}_2(z)\widetilde{q}_1(\tfrac{1}{z})].$$

Since $w(z)^{-1} = w(1/z)$, by a simple computation, we have

$$q_1(1/z) = (-1)^{Z(h,1)} z^{-Z(h,1)-Z(h,-1)} q_1(z),$$
$$q_2(1/z) = (-1)^{Z(h,1)} z^{-Z(h,1)-Z(h,-1)} w(z) q_2(z).$$

Therefore, both q_1 and q_2 are (anti)symmetric and $\frac{[Sq_1](z)}{[Sq_2](z)} = w(z)$. ∎

4.5 Proof of Theorem 4.2 and its associated algorithm

In this section, we shall prove Theorem 4.2 and give a step-by-step algorithm to implement it.

Proof of Theorem 4.2: If $g = \gcd(A, B, C) \equiv 0$, then $A = B = C = 0$ and all the claims are obviously true by taking $u_1 = u_2 = v_1 = v_2 = 0$. So, we will assume $g \not\equiv 0$. Since $g = \gcd(A, B, C)$, by the symmetry of A, B and C, g is (anti)symmetric. Since $\det M(z) \geqslant 0$ for all $z \in \mathbb{T}$, we see that

$$0 \leqslant B(z)B(1/z) \leqslant A(z)C(z) \qquad \forall\, z \in \mathbb{T}.$$

Since $B(1/z) = B(z)/[SB](z)$, it yields that $2Z(B, z) \geqslant Z(A, z) + Z(C, z)$ for all $z \in \mathbb{T}$. So, by the definition of g, we have for every $z \in \mathbb{T}$,

$$Z(g, z) = \min(Z(A, z), Z(B, z), Z(C, z)) = \min(Z(A, z), Z(C, z)).$$

Since $A(z) \geqslant 0$ and $C(z) \geqslant 0$ for all $z \in \mathbb{T}$, by Lemma 4.5, $Z(A, z)$ and $Z(C, z)$ are even integers. Consequently, $Z(g, z) = \min(Z(A, z), Z(C, z))$ is an even integer for all $z \in \mathbb{T}$. Since g is (anti)symmetric, by Lemma 4.5, there exist $c \in \{-1, 1\}$ and $k \in \mathbb{Z}$ such that $cz^k g(z) \geqslant 0$ for all $z \in \mathbb{T}$. Since $g = \gcd(A, B, C)$, without loss of generality, we can assume that $g(z) \geqslant 0$ for all $z \in \mathbb{T}$ by replacing g by $cz^k g(z)$. Now define $\widetilde{M}(z) = M(z)/g(z)$ by

(4.5.33)
$$\widetilde{M}(z) = \begin{bmatrix} \widetilde{A}(z) & \widetilde{B}(z) \\ \widetilde{B}(1/z) & \widetilde{C}(z) \end{bmatrix} \text{ with } \widetilde{A}(z) = \frac{A(z)}{g(z)},\ \widetilde{B}(z) = \frac{B(z)}{g(z)},\ \widetilde{C}(z) = \frac{C(z)}{g(z)}.$$

Since $g(z) \geqslant 0$ for all $z \in \mathbb{T}$, it is easy to see that all $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are (anti)symmetric Laurent polynomials and $\widetilde{M}(z) \geqslant 0$ for all $z \in \mathbb{T}$.

Sufficiency. Since $d(z)d(1/z) = \det M(z) = g(z)^2 \det\widetilde{M}(z)$, we have $g^2 \mid d(z)d(1/z)$. Since $d(\frac{1}{z}) = d(z)/[Sd](z)$, $g^2 \mid d^2$ and therefore, $g \mid d$. So define $d_1(z) = d(z)/g(z)$. Then d_1 is an (anti)symmetric Laurent polynomial and $\det\widetilde{M}(z) = d_1(z)d_1(1/z)$. Note that $\gcd(\widetilde{A}, \widetilde{B}, \widetilde{C}) = 1$. By Corollary 4.4, there exist four (anti)symmetric Laurent polynomials $\widetilde{u}_1, \widetilde{u}_2, \widetilde{v}_1, \widetilde{v}_2$ with real coefficients such that (4.4.29) and (4.4.30) are satisfied. Define $\widetilde{d}(z) := \widetilde{u}_1(z)\widetilde{v}_2(z) - \widetilde{u}_2(z)\widetilde{v}_1(z)$. By (4.4.30), $\widetilde{d}$ is (anti)symmetric and by Proposition 4.1 $[S\widetilde{d}](z) = [S\widetilde{u}_1](z)[S\widetilde{v}_2](z)$.

By Proposition 4.2, it follows from $\widetilde{d}(z)\widetilde{d}(1/z) = \det\widetilde{M}(z) = d_1(z)d_1(1/z)$ that we must have $\widetilde{d}(z) = \pm z^k d_1(z) = \pm z^k d(z)/g(z)$ for some $k \in \mathbb{Z}$. So, $[S\widetilde{d}](z) = z^{2k}[Sd](z)$. Rewrite (4.4.30) as

$$\frac{[S\widetilde{u}_1](z)}{[S\widetilde{u}_2](z)} = [S\widetilde{B}](z) = [SB](z) = \frac{[S\widetilde{v}_1](z)}{[S\widetilde{v}_2](z)}.$$

So, we have

$$\frac{[S\widetilde{v}_1](z)}{[S\widetilde{u}_1](z)} = \frac{[S\widetilde{v}_1](z)}{[S\widetilde{v}_2](z)} \frac{[S\widetilde{u}_1](z)[S\widetilde{v}_2](z)}{([S\widetilde{u}_1](z))^2} = [SB](z)\frac{[S\widetilde{d}](z)}{([S\widetilde{u}_1](z))^2}$$
$$= \left(\frac{z^k}{[S\widetilde{u}_1](z)}\right)^2 [SB](z)[Sd](z)$$

and

$$\frac{[S\widetilde{v}_2](z)}{[S\widetilde{u}_2](z)} = \frac{[S\widetilde{u}_1](z)}{[S\widetilde{u}_2](z)} \frac{[S\widetilde{u}_1](z)[S\widetilde{v}_2](z)}{(S\widetilde{u}_1](z))^2} = [SB](z)\frac{[S\widetilde{d}](z)}{([S\widetilde{u}_1](z))^2} = \frac{[S\widetilde{v}_1](z)}{[S\widetilde{u}_1](z)}.$$

By assumption in (c) and Lemma 4.6, there exist two (anti)symmetric Laurent polynomials q_1 and q_2 such that

$$\frac{[Sq_1](z)}{[Sq_2](z)} = \frac{[S\widetilde{v}_1](z)}{[S\widetilde{u}_1](z)} = \frac{[S\widetilde{v}_2](z)}{[S\widetilde{u}_2](z)} = \left(\frac{z^k}{[S\widetilde{u}_1](z)}\right)^2 [SB](z)[Sd](z) \tag{4.5.34}$$

and $g(z) = q_1(z)q_1(1/z) + q_2(z)q_2(1/z)$. Define

$$\begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix} = \begin{bmatrix} \widetilde{u}_1(z) & \widetilde{v}_1(z) \\ \widetilde{u}_2(z) & \widetilde{v}_2(z) \end{bmatrix} \begin{bmatrix} q_1(z) & -q_2(1/z) \\ q_2(z) & q_1(1/z) \end{bmatrix}.$$

Now by (4.5.34) and Proposition 4.1, it is easy to check that all u_1, u_2, v_1, v_2 are (anti)symmetric Laurent polynomials. By a direct computation, it is easy to see that (4.2.10) and (4.2.12) are satisfied.

Necessity. Obviously, (a) and (b) must be true. As we proved in the part of sufficiency, we can assume that $g(z) \geqslant 0$ for all $z \in \mathbb{T}$. Let $\widetilde{M}$ be defined in (4.5.33). We have $g(z)^2 \det\widetilde{M}(z) = \det M(z) = d(z)d(1/z)$. So, $g^2 \mid d(z)d(1/z)$. Since $d(1/z) = d(z)/[Sd](z)$, we deduce that $g^2 \mid d^2$ and therefore, $g \mid d$. Define $\widetilde{d}(z) = d(z)/g(z)$. Then $\widetilde{d}$ is (anti)symmetric and $\det\widetilde{M}(z) = \widetilde{d}(z)\widetilde{d}(1/z)$. Since $M(z) \geqslant 0$ and $g(z) \geqslant 0$ for all $z \in \mathbb{T}$, it is easy to see that $\widetilde{M}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Since $\gcd(\widetilde{A}, \widetilde{B}, \widetilde{C}) = 1$, by Corollary 4.4, (4.4.29) and (4.4.30) are satisfied. So,

$$\begin{aligned}
&\begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix} \begin{bmatrix} u_1(1/z) & u_2(1/z) \\ v_1(1/z) & v_2(1/z) \end{bmatrix} = g(z)\widetilde{M}(z) \\
=&g(z) \begin{bmatrix} \widetilde{u}_1(z) & \widetilde{v}_1(z) \\ \widetilde{u}_2(z) & \widetilde{v}_2(z) \end{bmatrix} \begin{bmatrix} \widetilde{u}_1(1/z) & \widetilde{u}_2(1/z) \\ \widetilde{v}_1(1/z) & \widetilde{v}_2(1/z) \end{bmatrix}
\end{aligned}$$

Define

$$Q(z) := \begin{bmatrix} q_1(z) & q_2(z) \\ q_3(z) & q_4(z) \end{bmatrix} := \begin{bmatrix} \widetilde{v}_2(z) & -\widetilde{v}_1(z) \\ -\widetilde{u}_2(z) & \widetilde{u}_1(z) \end{bmatrix} \begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix}.$$

Then $Q(z)Q(1/z)^T = g(z)\widetilde{d}(z)\widetilde{d}(1/z)I_2$. In particular, we have

$$q_1(z)q_1(1/z) + q_2(z)q_2(1/z) = g(z)\widetilde{d}(z)\widetilde{d}(1/z).$$

By (4.2.12) and (4.4.30), we have

$$\frac{[Su_1](z)}{[Su_2](z)} = \frac{[Sv_1](z)}{[Sv_2](z)} = [SB](z) = [S\widetilde{B}](z) = \frac{[S\widetilde{u}_1](z)}{[S\widetilde{u}_2](z)} = \frac{[S\widetilde{v}_1](z)}{[S\widetilde{v}_2](z)}.$$

By Proposition 4.1, q_1 and q_2 are (anti)symmetric. By Proposition 4.2, $d(z) = \pm z^k[u_1(z)v_2(z) - u_2(z)v_1(z)]$. So

$$[Sd](z) = z^{2k}[Su_1](z)[Sv_2](z) = z^{2k}[Su_2](z)[Sv_1](z).$$

Observing that $[Sq_1](z) = [S\widetilde{v}_2](z)[Su_1](z)$ and $[Sq_2](z) = [S\widetilde{v}_2](z)[Sv_1](z)$, we have

$$\frac{[Sq_1](z)}{[Sq_2](z)} = \frac{[S\widetilde{v}_2](z)[Su_1](z)}{[S\widetilde{v}_2](z)[Sv_1](z)} = \frac{[Su_1](z)}{[Su_2](z)} \frac{z^{2k}[Su_2](z)[Sv_1](z)}{(z^k[Sv_1](z))^2} = \frac{[SB](z)[Sd](z)}{(z^k[Sv_1](z))^2}.$$

By Lemma 4.6, $Z(g(z)\widetilde{d}(z)\widetilde{d}(1/z), x)$ must be an even integer for the corresponding cases. Note that $\widetilde{d}(1/z) = \widetilde{d}(z)/[S\widetilde{d}](z)$. So, $Z(\widetilde{d}(z)\widetilde{d}(1/z), x)$ is always an even integer for all $x \in \mathbb{R}$. So, $Z(g, x) = Z(g(z)\widetilde{d}(z)\widetilde{d}(1/z), x) - Z(\widetilde{d}(z)\widetilde{d}(1/z), x)$ must be an even integer for the corresponding cases. Therefore, (c) must be true. ∎

Finally, by the proof of Theorem 4.2 and all the auxiliary results in Section 4, let us present the following algorithm on splitting a matrix of Laurent polynomials with symmetry.

Algorithm 4.7. Let A, B and C be (anti)symmetric Laurent polynomials with real coefficients. Let M be the 2×2 matrix defined in (4.2.9) such that all the conditions in Theorem 4.2 are satisfied.

1) Compute $g = \gcd(A, B, C)$. By the proof of Theorem 4.2, without loss of generality, we can assume that $g \not\equiv 0$ and $g(z) \geqslant 0$ for all $z \in \mathbb{T}$.

2) Compute $h(z) = \gcd(A(z)/g(z), B(z)B(1/z)/g(z)^2)$. By the proof of Corollary 4.4, we can assume that $h(z) \geqslant 0$ for all $z \in \mathbb{T}$ and we can calculate d_h such that $h(z) = d_h(z)d_h(1/z)$ by Proposition 4.2.

3) Define a 2×2 matrix $\widetilde{M}$ of Laurent polynomials with real coefficients by

$$\widetilde{M}(z) = \begin{bmatrix} \widetilde{A}(z) & \widetilde{B}(z) \\ \widetilde{B}(1/z) & \widetilde{C}(z) \end{bmatrix}$$

with $\widetilde{A}(z) = \dfrac{A(z)}{g(z)h(z)}$, $\widetilde{B}(z) = \dfrac{B(z)}{g(z)d_h(z)}$, $\widetilde{C}(z) = \dfrac{C(z)}{g(z)}$.

By Proposition 4.2, we can calculate $\widetilde{d}$ such that $\det\widetilde{M}(z) = \widetilde{d}(z)\widetilde{d}(1/z)$. (If we have an (anti)symmetric Laurent polynomial d satisfying that $\det M(z) = d(z)d(1/z)$, then we can take $\widetilde{d}(z) = d(z)/(g(z)d_h(z))$).

4) Assume $\widetilde{A}(z) = \widetilde{A}_0 + \sum_{k=1}^{N} \widetilde{A}_k(z^k + z^{-k})$ with $\widetilde{A}_N \neq 0$. Parameterize the (anti)symmetric Laurent polynomials $\widetilde{u}_1$ and $\widetilde{v}_1$ with $[S\widetilde{u}_1](z)[S\widetilde{v}_1](z) = [S\widetilde{B}](z)[S\widetilde{d}](z)$ and the degrees of $\widetilde{u}_1$ and $\widetilde{v}_1$ are at most N (see the paragraph after the formula (4.4.24) about how to parameterize $\widetilde{u}_1$ and $\widetilde{v}_1$). Then according to Theorem 4.4 there must be a nonzero solution $\{\widetilde{u}_1, \widetilde{v}_1\}$ to the system of linear homogeneous equations derived from

$$\widetilde{B}(1/z)\widetilde{u}_1(z) - \widetilde{d}(z)\widetilde{v}_1(z) \equiv 0 \qquad \text{mod} \quad \widetilde{A}(z).$$

By the proof of Theorem 4.4, we must have $\widetilde{u}_1(1)^2 + \widetilde{v}_1(1)^2 \neq 0$. Multiplying $\widetilde{u}_1$ and $\widetilde{v}_1$ by a constant, we can require that the solution $\{\widetilde{u}_1, \widetilde{v}_1\}$ satisfy $\widetilde{u}_1(1)^2 + \widetilde{v}_1(1)^2 = \widetilde{A}(1)$.

5) Define the symmetric filters $\widetilde{u}_2$ and $\widetilde{v}_2$ by

$$\widetilde{u}_2(z) := \frac{\widetilde{B}(\frac{1}{z})\widetilde{u}_1(z) - \widetilde{d}(z)\widetilde{v}_1(\frac{1}{z})}{\widetilde{A}(z)} \quad \text{and} \quad \widetilde{v}_2(z) := \frac{\widetilde{d}(z)\widetilde{u}_1(\frac{1}{z}) + \widetilde{B}(\frac{1}{z})\widetilde{v}_1(z)}{\widetilde{A}(z)}.$$

6) By Lemma 4.6, write $g(z) = q_1(z)q_1(1/z) + q_2(z)q_2(1/z)$ for some (anti)-symmetric Laurent polynomials q_1 and q_2 such that $[Sq_1](z)/[Sq_2](z) = [S\widetilde{v}_1](z)/[S\widetilde{u}_1](z)$. (In most cases, $g = 1$ and we can simply choose $q_1 = 1$ and $q_2 = 0$.)

7) Obtain the (anti)symmetric Laurent polynomials (or symmetric FIR filters) u_1, u_2, v_1, v_2 by

$$U(z) := \begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix} = \begin{bmatrix} d_h(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{u}_1(z) & \widetilde{v}_1(z) \\ \widetilde{u}_2(z) & \widetilde{v}_2(z) \end{bmatrix} \begin{bmatrix} q_1(z) & -q_2(\frac{1}{z}) \\ q_2(z) & q_1(\frac{1}{z}) \end{bmatrix}.$$

Then $U(z)U(1/z)^T = M(z)$ and $[Su_1](z)[Sv_2](z) = [Su_2](z)[Sv_1](z)$.

It is not necessary to check all the conditions in Theorem 4.2 in advance. If at some step one cannot carry out Algorithm 4.7, then the conditions in Theorem 4.2 cannot be satisfied.

Chapter 5

An algorithm for constructing pairs of dual wavelet frames with two symmetric generators

5.1 Introduction

As we mentioned before, symmetry is a highly desirable property of a wavelet system. In Chapters 3 and 4, we discussed symmetric tight wavelet frames derived from symmetric refinable functions.

For simplicity, as we did in Chapter 4, we still use a Laurent polynomial $a(z)$ with $z = e^{-i\xi}$ to represent a mask or a finitely supported sequence a. When we were constructing examples of symmetric tight wavelet frames in Chapter 4, for a given symmetric refinable function ϕ with its mask a, we found that it is difficult to construct a Laurent polynomial Θ such that

$$\Theta(z)\Theta(-z) - \Theta(z^2)[\Theta(z)a(-z)a(-1/z) + \Theta(-z)a(z)a(1/z)] = d(z^2)d(1/z^2)$$

for some (anti)symmetric Laurent polynomial d. Even if we have constructed such a Θ, the coefficients of Θ, in many cases, are irrational. In practice, a function with rational coefficients is desired.

These two disadvantages of constructing a symmetric tight wavelet frame motivate us to study pairs of dual wavelet frames with two symmetric generators.

By doing it, we will lose the "orthogonality" that a tight wavelet frame carries, but other properties can be still kept. It is much easier to construct a pair of dual symmetric wavelet frames and the coefficients of the high-pass filters will be rational numbers in most cases.

So in this chapter, for a given pair of symmetric 2-refinable functions ϕ and $\tilde{\phi}$ with its masks a and b, respectively, we are interested in constructing a pair of dual wavelet frames with two symmetric generators.

In Section 5.2, I shall give an algorithm to construct pairs of dual wavelet frames with two symmetric generators. As a consequence, pairs of dual wavelet frames with two symmetric generators of balanced length can be easily constructed via our algorithm for constructing pairs of symmetric dual wavelet frames. In Section 5.3, several examples are provided to demonstrate the algorithm.

5.2 Algorithm

Let $\phi \in L_2(\mathbb{R})$ and $\tilde{\phi} \in L_2(\mathbb{R})$ be two refinable functions with masks a and b respectively such that $\widehat{\phi}(2\xi) = a(e^{-i\xi})\widehat{\phi}(\xi)$ and $\widehat{\tilde{\phi}}(2\xi) = b(e^{-i\xi})\widehat{\tilde{\phi}}(\xi)$. Then we have that $a(1) = b(1) = 1$. By [12, Theorem 2.2], if we can find Laurent polynomials Θ, a^1, a^2, b^1 and b^2 satisfying $\Theta(1) = 1$, $a^1(1) = a^2(1) = b^1(1) = b^2(1) = 0$, and

$$\begin{bmatrix} a^1(z) & a^2(z) \\ a^1(-z) & a^2(-z) \end{bmatrix} \begin{bmatrix} b^1(1/z) & b^1(-1/z) \\ b^2(1/z) & b^2(-1/z) \end{bmatrix} = M_\Theta(z) \tag{5.2.1}$$

where

$$M_\Theta(z) := \begin{bmatrix} \Theta(z) - \Theta(z^2)a(z)b(1/z) & -\Theta(z^2)a(z)b(-1/z) \\ -\Theta(z^2)a(-z)b(1/z) & \Theta(-z) - \Theta(z^2)a(-z)b(-1/z) \end{bmatrix},$$

then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual wavelet frames where

$$\widehat{\psi^1}(2\xi) = a^1(e^{-i\xi})\widehat{\phi}(\xi), \qquad \widehat{\psi^2}(2\xi) = a^2(e^{-i\xi})\widehat{\phi}(\xi),$$
$$\widehat{\widetilde{\psi}^1}(2\xi) = b^1(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi), \qquad \widehat{\widetilde{\psi}^2}(2\xi) = b^2(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi).$$

Since we are interested in constructing symmetric wavelet frames, we assume that a, b and Θ are symmetric and

$$\frac{[S\Theta](z)}{[S\Theta](z^2)} = \frac{[Sa](z)}{[Sb](z)}, \tag{5.2.2}$$

where for a nonzero Laurent polynomial p, $[Sp](z) := p(z)/p(1/z)$ which is defined in Chapter 4. Moreover, we assume that

$$\frac{[Sa^1](z)}{[Sa^1](-z)} = \frac{[Sa^2](z)}{[Sa^2](-z)} = \frac{[Sa](z)}{[Sa](-z)} \tag{5.2.3}$$

and the following facts: The symmetric masks a and b are given and they have sum rules of orders m_1 and m_2, respectively; The symmetric Laurent polynomial Θ is given, $\Theta(1) = 1$, and $(1-z)^{n_1+n_2} \mid [M_\Theta(z)]_{1,1}$ with some integers n_1 and n_2 such that $1 \leqslant n_1 \leqslant m_2$ and $1 \leqslant n_2 \leqslant m_1$.

Define a Laurent polynomial $d(z)$ such that $d(z^2) := \det M_\Theta(z)/(1 - z^2)^{n_1+n_2}$. It is easy to see that d is a well defined Laurent polynomial and d is symmetric.

Now let us state our algorithm.

Algorithm 5.1. If $d \equiv 0$, then we can construct a pair of symmetric dual wavelet frames with one (anti)symmetric generator by the following steps.

1. Define

$$\widetilde{M_\Theta}(z) := \begin{bmatrix} \frac{\Theta(z)-\Theta(z^2)a(z)b(1/z)}{(1-z)^{n_1}(1-1/z)^{n_2}} & -\frac{\Theta(z^2)a(z)b(-1/z)}{(1-z)^{n_1}(1+1/z)^{n_2}} \\ -\frac{\Theta(z^2)a(-z)b(1/z)}{(1+z)^{n_1}(1-1/z)^{n_2}} & \frac{\Theta(-z)-\Theta(z^2)a(-z)b(-1/z)}{(1+z)^{n_1}(1+1/z)^{n_2}} \end{bmatrix}.$$

Compute

$$
(5.2.4)\qquad
\begin{aligned}
&g_0(z^2) := \gcd([\widetilde{M_\Theta}(z)]_{1,1}, [\widetilde{M_\Theta}(z)]_{1,2}, [\widetilde{M_\Theta}(z)]_{2,1}, [\widetilde{M_\Theta}(z)]_{2,2}),\\
&g_1(z) := \gcd([\widetilde{M_\Theta}(z)]_{1,1}, [\widetilde{M_\Theta}(z)]_{1,2})/g_0(z^2),\\
&g_2(z) := [\widetilde{M_\Theta}(1/z)]_{1,1}/\big(g_1(1/z)g_0(1/z^2)\big),\\
&\widetilde{g}_2(z) := [\widetilde{M_\Theta}(1/z)]_{2,1}/\big(g_1(1/z)g_0(1/z^2)\big),.
\end{aligned}
$$

2. It is easy to see that g_0, g_1, g_2, $\widetilde{g}_2$ are well defined symmetric Laurent polynomials. We will have $g_2(z) = \pm\widetilde{g}_2(-z)$. If $g_2(z) = \widetilde{g}_2(-z)$, then define $w := 1$; If $g_2(z) = -\widetilde{g}_2(-z)$, then define $w := e^{-i\xi}$. Chose a symmetric Laurent polynomial g_3 such that g_3 divides g_0 and define $g_4 := g_0/g_3$.

3. Define ψ and $\widetilde{\psi}$ by

$$
\begin{aligned}
\widehat{\psi}(2\xi) &:= w(1-e^{-i\xi})^{n_1} g_3(e^{-i2\xi}) g_1(e^{-i\xi})\widehat{\phi}(\xi),\\
\widehat{\widetilde{\psi}}(2\xi) &:= w(1-e^{-i\xi})^{n_2} g_4(e^{-i2\xi}) g_2(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi).
\end{aligned}
$$

Then $\{\psi\}$ and $\{\widetilde{\psi}\}$ generate a pair of symmetric dual wavelet frames with vanishing moments of orders n_1 and n_2, respectively.

If $d \not\equiv 0$, then we use the following steps to construct a pair of symmetric dual wavelet frames with two (anti)symmetric generators having vanishing moments of orders n_1 and n_2, respectively.

1. Find a symmetric Laurent polynomial d_1 such that d_1 divides d.

2. Find a symmetric Laurent polynomial a^2 which satisfies the following conditions:

$$
(5.2.5)\qquad
\begin{aligned}
&(1-z)^{n_1} \mid a^2(z), \qquad [Sa^2](z)/[Sa^2](-z) = [Sa](z)/[Sa](-z),\\
&d_1(z^2)(1-z^2)^{n_1+n_2} \mid a^2(-z)[M_\Theta(z)]_{1,1} - a^2(z)[M_\Theta(z)]_{2,1},\\
&\gcd\big(a^2(-z)/(1-z)^{n_1}, a^2(z)/(1+z)^{n_1}\big) \mid d_1(z^2)
\end{aligned}
$$

and

$$
(5.2.6)\qquad d_1(z^2)(1-z^2)^{n_1+n_2} \mid \big(a^1(-z)[M_\Theta(z)]_{1,1} - a^1(z)[M_\Theta(z)]_{2,1}\big)
$$

where a^1 is a symmetric Laurent polynomial satisfying the following equations.

$$(5.2.7)\qquad \begin{aligned} &(1-z)^{n_1} \mid a^1(z), \qquad [Sa^1](z)/[Sa^1](-z) = [Sa](z)/[Sa](-z),\\ &a^2(-z)a^1(z) - a^2(z)a^1(-z) = z(1-z^2)^{n_1} d_1(z^2). \end{aligned}$$

3. Define

$$b^1(z) := z\Big(a^2(-\frac{1}{z})[M_\Theta(\frac{1}{z})]_{1,1} - a^2(\frac{1}{z})[M_\Theta(\frac{1}{z})]_{2,1}\Big)/\big(d_1(\frac{1}{z^2})(1-z^2)^{n_1}\big),$$
$$b^2(z) := z\Big(a^1(-\frac{1}{z})[M_\Theta(\frac{1}{z})]_{1,1} - a^1(\frac{1}{z})[M_\Theta(\frac{1}{z})]_{2,1}\Big)/\big(d_1(\frac{1}{z^2})(1-z^2)^{n_1}\big).$$

4. Define ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ by

$$\widehat{\psi^1}(2\xi) = a^1(e^{-i\xi})\widehat{\phi}(\xi), \qquad \widehat{\psi^2}(2\xi) = a^2(e^{-i\xi})\widehat{\phi}(\xi),$$
$$\widehat{\widetilde{\psi}^1}(2\xi) = b^1(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi), \qquad \widehat{\widetilde{\psi}^2}(2\xi) = b^2(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi).$$

Then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual wavelet frames with vanishing moments of orders n_1 and n_2, respectively.

Proof of Algorithm 5.1: First let us consider the case $d \equiv 0$. In this case, it is easy to see that

$$(5.2.8)\qquad \widetilde{M}_\Theta(z) = \begin{bmatrix} k_0(z)k_2(z) & k_1(z)k_2(z) \\ k_0(z)k_3(z) & k_1(z)k_3(z) \end{bmatrix}$$

with some nonzero Laurent polynomials k_0, k_1, k_2 and k_3. Moreover, we can assume that $\gcd(k_2(z), k_3(z)) = 1$. Notice $[\widetilde{M}_\Theta(z)]_{1,1} = [\widetilde{M}_\Theta(-z)]_{2,2}$. So we have

$$(5.2.9)\qquad k_0(z)k_2(z) = k_1(-z)k_3(-z).$$

Similarly,

$$(5.2.10)\qquad k_0(z)k_3(z) = k_1(-z)k_2(-z).$$

Divide (5.2.9) by (5.2.10), we have

$$\frac{k_2(z)}{k_3(z)} = \frac{k_3(-z)}{k_2(-z)}.$$

Therefore, by $\gcd(k_2(z), k_3(z)) = 1$, we have $k_2(z) = \pm k_3(-z)$. If $k_2(z) = -k_3(-z)$, then we replace k_0, k_1, k_2 and k_3 by $zk_0(z)$, $zk_1(z)$, $k_2(z)/z$ and $k_3(z)/z$, respectively. So we can assume that $k_2(z) = k_3(-z)$. Therefore, by condition (5.2.9), we have $k_0(z) = k_1(-z)$ and $k_2(z) = k_3(-z)$. So

$$\begin{aligned}\widetilde{M}_\Theta(z) &= [k_2(z) \;\; k_2(-z)]^T[k_1(-z) \;\; k_1(z)] \\ &= g_0(z^2)[k_2(z) \;\; k_2(-z)]^T[k_1(-z)/g_0(z^2) \;\; k_1(z)/g_0(z^2)]\end{aligned}$$

where

$$\begin{aligned}g_0(z^2) &:= \gcd(k_1(z), k_1(-z)) \\ &= \gcd([\widetilde{M}_\Theta(z)]_{1,1}, [\widetilde{M}_\Theta(z)]_{2,2}, [\widetilde{M}_\Theta(z)]_{1,2}, [\widetilde{M}_\Theta(z)]_{2,1}).\end{aligned}$$

Therefore, we proved the algorithm for the case $d \equiv 0$ and it covers all the possible pairs of symmetric wavelet frames with one (anti)symmetric generator derived from ϕ and $\widetilde{\phi}$.

Secondly, let us consider the case $d \not\equiv 0$. As discussed before, we want to find some symmetric polynomials a^1, a^2, b^1 and b^2 such that

$$(5.2.11) \quad (1-z)^{n_1} \mid a^1, \qquad (1-z)^{n_1} \mid a^2, \qquad (1-z)^{n_2} \mid b^1, \qquad (1-z)^{n_2} \mid b^2$$

and (5.2.1) are satisfied. So we have $\det \begin{bmatrix} a^1(z) & a^2(z) \\ a^1(-z) & a^2(-z) \end{bmatrix}$ is symmetric. Define

$$(5.2.12) \qquad d_1(z^2) = \frac{1}{z} \det \begin{bmatrix} a^1(z) & a^2(z) \\ a^1(-z) & a^2(-z) \end{bmatrix}.$$

It is easy to see that d_1 is well defined and d_1 is symmetric by assumption (5.2.3). Under the condition $d_1 \not\equiv 0$ since $d \not\equiv 0$, it is evident to see that (5.2.1) is equivalent to

$$(5.2.13) \qquad \begin{bmatrix} b^1(1/z) & b^1(-1/z) \\ b^2(1/z) & b^2(-1/z) \end{bmatrix} = \frac{1}{zd_1(z^2)} \begin{bmatrix} a^2(-z) & -a^2(z) \\ -a^1(-z) & a^1(z) \end{bmatrix} M_\Theta(z).$$

Also, it is easy to see that (5.2.13) is equivalent to

$$(5.2.14) \qquad \begin{bmatrix} b^1(1/z) \\ b^2(1/z) \end{bmatrix} = \frac{1}{zd_1(z^2)} \begin{bmatrix} a^2(-z)[M_\Theta(z)]_{1,1} - a^2(z)[M_\Theta(z)]_{2,1} \\ -a^1(-z)[M_\Theta(z)]_{1,1} + a^1(z)[M_\Theta(z)]_{2,1} \end{bmatrix}.$$

So the combination of conditions (5.2.1) and (5.2.12) is equivalent to the combination of conditions (5.2.12) and (5.2.14). By this reason and condition (5.2.3), checking the symmetry pattern, it is evident to see that our algorithm for the case $d \not\equiv 0$ is correct and in fact it covers all the possible pairs of symmetric dual wavelet frames derived from ϕ and $\widetilde{\phi}$. Moreover, since it covers all the possible pairs of symmetric dual wavelet frames, we can search some optimal ones according to our algorithm. For instance, in practice, a^1, a^2, b^1 and b^2 can be used as high-pass filters. In terms of efficiency, the maximal length of $\{a^1, a^2, b^1, b^2\}$ is desired to be as short as possible. In the next section, we shall refine our algorithm to search for the "shortest" pair of symmetric dual wavelet frames and several examples are provided to demonstrate our algorithm. ■

5.3 Examples

Using Maple program and based on the algorithm in last section, we calculated many examples. Especially, we would like to show the following examples:

Example 5.1. Let ϕ be a refinable function with its mask

$$a(z) = z^{-2}[18 - 5(z + z^{-1})](1 + z)^5/256.$$

Set $\widetilde{\phi} = \phi$ and $b = a$. We have $m_1 = m_2 = 4$. Define $\Theta = 1$, $n_1 = n_2 = 2$,

$$\begin{aligned}
a^1(z) &= 15z^{-1}(1 - z)^3[526 + 242(z + z^{-1}) + 55(z^2 + z^{-2})]/15808,\\
a^2(z) &= -z^{-1}(1 + z)(1 - z)^2[106 + 36(z + z^{-1}) + 15(z^2 + z^{-2})]/240,\\
b^1(z) &= z^{-1}(1 - z)^3[86 + 22(z + z^{-1}) + 5(z^2 + z^{-2})]/480,\\
b^2(z) &= -525z^{-1}(1 + z)(1 - z)^2[270 + 12(z + z^{-1}) + 5(z^2 + z^{-2})]/1011712
\end{aligned}$$

and define ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ by

$$\begin{aligned}
&\widehat{\psi^1}(2\xi) = a^1(e^{-i\xi})\widehat{\phi}(\xi), \qquad \widehat{\psi^2}(2\xi) = a^2(e^{-i\xi})\widehat{\phi}(\xi),\\
&\widehat{\widetilde{\psi}^1}(2\xi) = b^1(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi), \qquad \widehat{\widetilde{\psi}^2}(2\xi) = b^2(e^{-i\xi})\widehat{\widetilde{\phi}}(\xi).
\end{aligned} \tag{5.3.15}$$

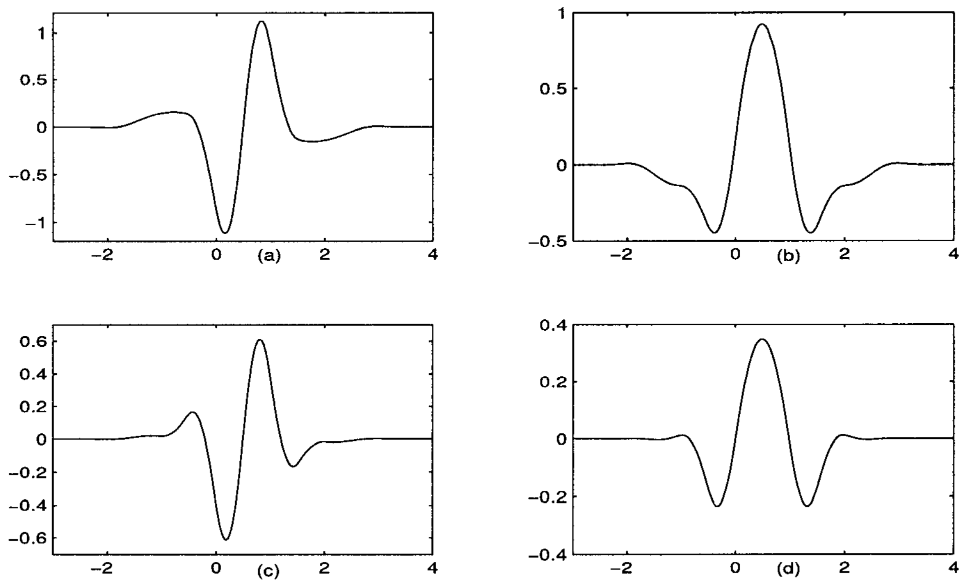

Figure 5.1: Generators for the pair of dual 2-wavelet frames in Example 5.1: (a) ψ^1 (b) ψ^2 (c) $\widetilde{\psi}^1$ (d) $\widetilde{\psi}^2$. Functions ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ are symmetric or antisymmetric and have vanishing moments of order 2.

Then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual wavelet frames with vanishing moments of orders 2. See Figure 5.1 for their graphs.

Example 5.2. Let ϕ be the cubic B-spline with its mask $a(z) = (1+z)^4/16$. Set $\widetilde{\phi} = \phi$ and $b = a$. We have $m_1 = m_2 = 4$. Define

$$\Theta = 1 + (2 - z - z^{-1})/3 + 31(2 - z - z^{-1})^2/360,$$

$n_1 = 2$, $n_2 = 4$,

$$\begin{aligned}
a^1(z) &= -z^{-4}(1-z)^2[38 + 18(z + z^{-1}) + 3(z^2 + z^{-2})]/32,\\
a^2(z) &= -z^{-6}(1-z)^2(1+z)^6/16,\\
b^1(z) &= z^{-5}(1-z)^4[786 - 244(z + z^{-1}) + 31(z^2 + z^{-2})]/5760,\\
b^2(e^{-i\xi}) &= z^{-4}(1-z)^4[2352 + 403(z + z^{-1}) + 62(z^2 + z^{-2})]/11520
\end{aligned}$$

and define ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ by (5.3.15). Then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual wavelet frames with vanishing moments of orders 2 and 4, respectively. See Figure 5.2 for their graphs. Similarly, we can construct a

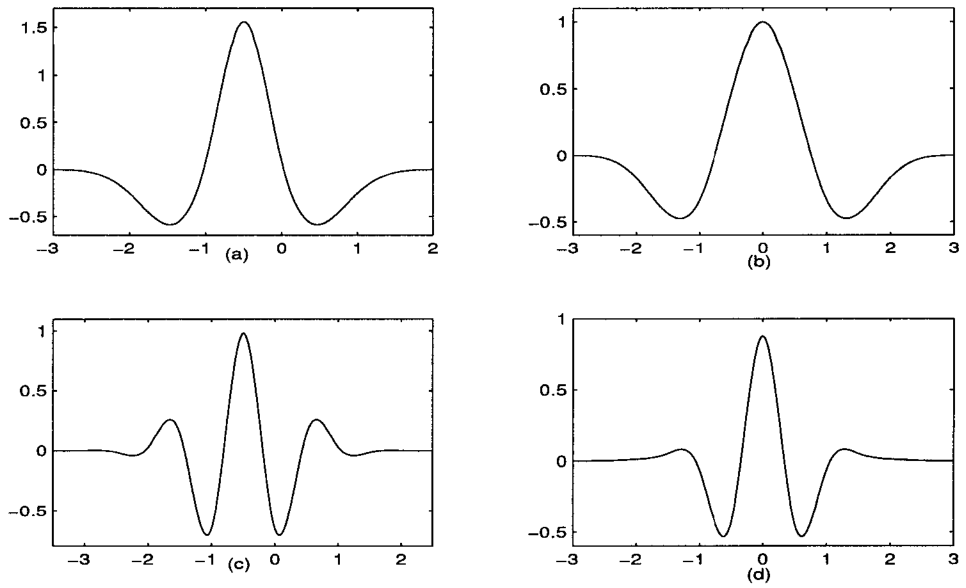

Figure 5.2: Generators for the pair of dual 2-wavelet frames in Example 5.2: (a) ψ^1 (b) ψ^2 (c) $\widetilde{\psi}^1$ (d) $\widetilde{\psi}^2$. Functions $\psi^1, \psi^2, \widetilde{\psi}^1, \widetilde{\psi}^2$ are symmetric and have vanishing moments of order 2 or 4.

"shortest" pair of dual wavelet frames with vanishing moments of order 3 which is derived from the same mask a. A similar example has been given in [4, Example 1.(iii)].

Example 5.3. Let ϕ be the cubic B-spline with its mask $a(z) = (1+z)^4/16$. Set $\widetilde{\phi} = \phi$ and $b = a$. We have $m_1 = m_2 = 4$. Define

$$\Theta = 1 + (2 - z - z^{-1})/3 + 31(2 - z - z^{-1})^2/360 + 311(2 - z - z^{-1})^3/15210,$$

$n_1 = 4$, $n_2 = 4$,

$$\begin{aligned}
a^1(z) &= (1-z)^4[22 + 8(z + z^{-1}) + (z^2 + z^{-2})]/64, \\
a^2(z) &= z(1-z)^4[208 + 131(z + z^{-1}) + 40(z^2 + z^{-2}) + \\
&\qquad 5(z^3 + z^{-3})]/320, \\
b^1(z) &= (1-z)^4[28602 + 3424(z + z^{-1}) + 933(z^2 + z^{-2})]/172800, \\
b^2(z) &= z(1-z)^4[61024 + 33045(z + z^{-1}) + 9952(z^2 + z^{-2}) + \\
&\qquad 1244(z^3 + z^{-3})]/241920
\end{aligned}$$

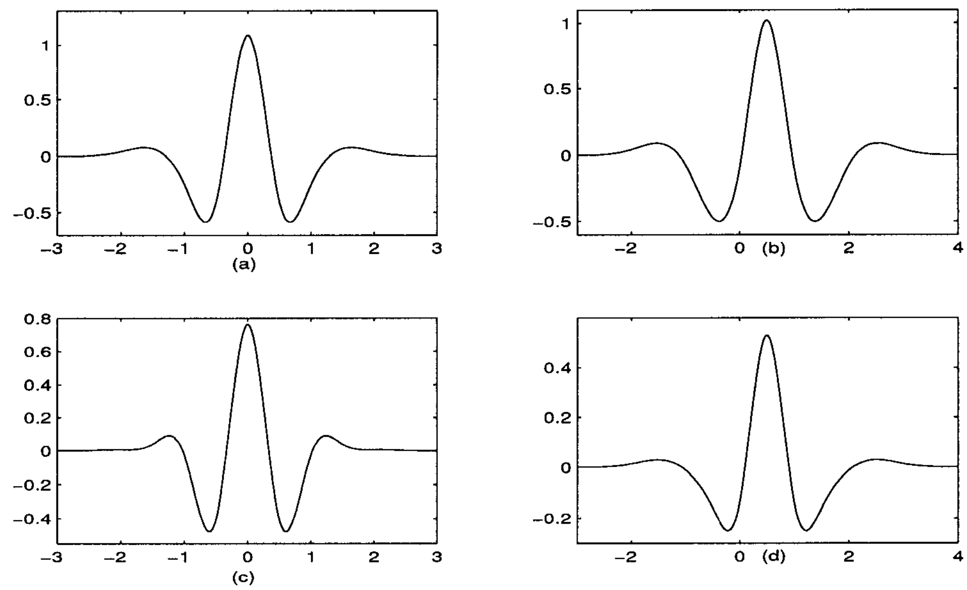

Figure 5.3: Generators for the pair of dual 2-wavelet frames in Example 5.3: (a) ψ^1 (b) ψ^2 (c) $\widetilde{\psi}^1$ (d) $\widetilde{\psi}^2$. Functions $\psi^1, \psi^2, \widetilde{\psi}^1, \widetilde{\psi}^2$ are symmetric and have vanishing moments of order 4.

and define ψ^1, ψ^2, $\widetilde{\psi}^1$, $\widetilde{\psi}^2$ by (5.3.15). Then $\{\psi^1, \psi^2\}$ and $\{\widetilde{\psi}^1, \widetilde{\psi}^2\}$ generate a pair of dual wavelet frames with vanishing moments of order 4. See Figure 5.3 for their graphs. Similar examples have been given in [4, 13].

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