University of Alberta

K-Theory: The Brauer Group And Algebraic Cycles

by

Robert M. Deary ©

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(Signed) R Deary

Robert M. Deary
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
T6G 2G1
Canada

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UNIVERSITY OF ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **K-theory: The Brauer Group and Algebraic Cycles** submitted by **Robert M. Deary** in partial fulfilment of the requirements for the degree of Master of Science in Mathematics.

Dr. J.D. Lewis (Supervisor)

A. Weiss

Dr. A. Weiss (Chair and Examiner)

Dr. Barry Joe (External)

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An updated electronic version of this document is available by E-mailing a request to **rdeary@math.ubc.ca** or **fruitbat@beast.ualberta.ca**.

Abstract

We attempt to construct an alternate proof of Merkurjev and Suslin's result that if n is prime to the characteristic K of a field then we have an isomorphism

$$\alpha_K : H^2(K, \mu^n) \to \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right) \otimes K_2(K)$$

The key result needed is a generalization of Hilbert's 90 theorem for K_2 which, although it can be stated in terms of explicit generators and relations, requires surprisingly deep properties of higher K-theory.

In the first chapter we explore the relations between the Severi-Brauer varieties over K, the skew fields finite dimensional and central over K, and the cohomology group $H^2(\mathrm{Gal}(K^{sep}))$. In the second chapter we develop the formalism of higher K-theory as well as a Riemann-Roch type theorem. In the third chapter we construct the BQG spectral sequence and prove Hilbert's 90 for K_2. In the last chapter we collect our results and prove the isomorphism above.

Contents

CHAPTER I

The Brauer Group

1. Introduction

The study of skew fields and the Brauer group of equivalence classes of central simple algebras has a long and distinguished history tracing back nearly a century and a half to Hamilton's quaternions. The study reached its classical peak in the 1930's with the proof of the Brauer-Hasse-Noether theorem that showed that all skew fields finite dimensional and central over the rationals were products of the cyclic $\mathbb{Q}$-algebras, and that a similar result held for global fields. The natural generalization of these results asked whether all skew fields finite dimensional over their center K were similar to the product of cyclic algebras over K. Milnor conjectured that the norm residue homomorphism connecting K_2 of a field with its Brauer group was an isomorphism, which, along with other applications to algebraic cycles, implied the result. The conjecture resisted all attempts until the 1982 publication of [27] which required heavy use of higher K-theory as well as intimate cohomological knowledge of Severi-Brauer varieties. The proof has been simplified somewhat in recent years, particularly in [26] and [47]. The goal of this paper is to provide a simplified relatively self contained approach to the theorem of Merkurjev and Suslin.

In this document K will always represent a field and L some field extension of K. By a variety we mean a reduced and irreducible scheme of finite type over Spec K, and for any scheme $\mathfrak{X}$ over K we denote the base extension $\operatorname{Spec} L \times_{\operatorname{Spec} K} \mathfrak{X}$ by $\mathfrak{X}_L$.

2. The Brauer Group

Given a field K, recall that a *algebra* over K is a ring, not necessarily commutative, that contains K as a subring. We call a K-algebra A *central* if the center of the ring is K, and we call A *simple* if it contains no nontrivial two-sided ideals.

If we regard A simply as a module then A is just a vector space over K. The *dimension* of A is rank of this vector space and A is *finitely generated* if the rank is finite.

DEFINITION 1. We define a *division algebra* over K to be a finitely generated central K-algebra that is also a skew-field, and we call a finitely generated central simple algebra over K an *Azumaya algebra*.

EXAMPLE 2. Classically, we have Frobenius' theorem: The only division algebras over $\mathbb{R}$ are the real numbers, the complex numbers, and the quaternions. Thus the only central division algebras over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{H}$. See proposition 54 for a proof.

EXAMPLE 3. A general class of examples are the cyclic algebras. Let L/K be a Galois extension of degree n with a cyclic Galois group generated by an element σ. For each element $a \in K$ we let $(f, L/K, \sigma)$ be the n-dimensional vector space over L with basis vectors $(x_0, \dots x_{n-1})$ and define a multiplication on $(a, L/K, \sigma)$ by

$$(e x_i)(e' x_j) = \begin{cases} e\sigma(e')x_{i+j} & \text{if } i + j < n \\ e\sigma(e')f x_{i+j-n} & \text{if } i + j \geq n \end{cases}$$

and extending linearly. A little computation[1] shows that this forms an Azumaya algebra over K.

Since all nonzero elements of a skew field are invertible, all its ideals must be trivial. Thus division rings are Azumaya algebras. Some quick algebra shows that a matrix ring of a division algebra over K is also an Azumaya algebra over K. In fact, we have the following classical result.

THEOREM 4 (WEDDERBURN). All Azumaya algebras over K are isomorphic to $M_n(D)$, the matrix ring of a division algebra over K.

PROOF. See almost any text on non-commutative ring theory, such as chapter three of [8] or section IX.1 of [48].

We will need to understand how Azumaya algebras react to tensor products.

THEOREM 5. Let A and B be Azumaya algebras over K and let L/K be a field extension.

(1) $A \otimes_K B$ is an Azumaya algebra over K.
(2) If A^0 is the opposite algebra to A then $A \otimes_K A^0 = M_n(K)$ where n is the dimension of A over K.

[1] Modify the argument used in lemma 39

(3) $L \otimes_K A$ is an Azumaya algebra over L.

PROOF. See [48], IX.1 prop 3.
We can use Wedderburn's theorem to define an equivalence class on the set of Azumaya algebras over K by identifying algebras with the same underlying division ring. We call the resulting set the *Brauer group* $\mathrm{Br}(K)$.

PROPOSITION 6. $\mathrm{Br}(K)$ forms an abelian group with the group operation given by the tensor product of algebras over K.

PROOF. The tensor product is closed by theorem 5 and, as usual, forms a commutative and associative operation. Thus the set of Azumaya algebras form a commutative monoid. We compute that $M_n(K) \otimes M_m(K) = M_{mn}(K)$ and that $M_n(D) = M_n(K) \otimes_K D$, so the set of algebras equivalent to K form a multiplicative subgroup. We can thus take the quotient monoid, which forms a group with the inverse of an algebra given by the opposite algebra.

PROPOSITION 7. For any extension L/K we have the *transfer homomorphism*

$$\mathrm{res}_{L/K} : \mathrm{Br}(K) \longrightarrow \mathrm{Br}(L)$$

given by the map $A \longmapsto L \otimes_K A$.

PROOF. This is clear by the associativity of the tensor product.
This formalism allows some quick results.

PROPOSITION 8. There are no division rings over an algebraically closed field except for the field itself, so the Brauer group of an algebraically closed field is trivial.

PROOF. Let K be algebraically closed and let D be a central division algebra over K. If we could choose $x \in D$ with $x \notin K$, we would have a finite extension $K[x]$ of K. $K[x]$ would be a field as it would be a commutative subring of a skew field, but this would make it an algebraic field extension of an algebraically closed field. Thus K is the only division ring over K and the Brauer group is trivial.

COROLLARY 9. The dimension of an Azumaya algebra is a perfect square. We call the square root of the dimension the *index*.

PROOF. Given an Azumaya K-algebra A we let $\bar{K}$ be the algebraic closure of K and compute

$$\dim_K A = \dim_{\bar{K}} \bar{K} \otimes_K A = \dim_{\bar{K}} M_n(\bar{K}) = n^2$$

DEFINITION 10. Given an extension L/K and an element $[A]$ of $\mathrm{Br}(K)$, we say that L is a *splitting field* for $[A]$ or that L *splits* $[A]$ if the image of $[A]$ in the restriction map $\mathrm{res}_{L/K}$ is trivial. We define $\mathrm{br}(L/K)$ to be the subgroup of elements split by L.

Proposition 8 shows that all Azumaya algebras split over some algebraic extension. We, however, will require more. We would like to prove that all Azumaya algebras split over some finite dimensional Galois extension. We need the following two technical lemmas. By a *subfield* of a ring we mean a commutative subring containing the center that is also a field. By a *maximal subfield* of a ring we mean a subfield that is not a subset of any other subfield, and by a *maximal separable subfield* we mean a separable subfield that is not a subset of any other separable subfield.

LEMMA 11. If D is a non-commutative division algebra with center K then D contains a subfield that is a nontrivial separable extension of K.

PROOF. Let p be the characteristic of K and choose $d \in D - K$. If d is not purely inseparable over K then $K(D)$ contains a nontrivial subfield that is separable over K. If d is purely inseparable then we can assume wlog that $d^p \in K$ and construct an endomorphism τ of D by setting $\tau(x) = dxd^{-1}$. We compute that $(\tau - 1)^p = 0$, so we can find n maximal with $(\tau - 1)^n D \neq 0$. Thus we can pick $x \in (\tau - 1)^{n-1} D$ such that $y = \tau(x) - x \neq 0$ and $\tau(y) = y$. We compute

$$\tau\left(\frac{x}{y}\right) = \frac{y + x}{\tau y} = 1 + \frac{x}{y},$$

so τ induces a nontrivial automorphism of $K(x/y)$, which implies that x/y is not purely inseparable.

LEMMA 12. If D is a division algebra over K then every maximal separable subfield of D is a maximal subfield.

PROOF. We let L be a maximal separable extension of K in D, let L^s be a maximal subfield of D containing L, and set E to be the subring of D of all elements that commute with L. E is a division ring with center L, and we know that $L \subset L^s \subset E$. If E is not commutative, then lemma 11 would give us a separable extension L'/L contained in E. But then L' would be separable over K, which would contradict the maximality. Thus E is commutative and $L = L^s$.

LEMMA 13. Let D be a division algebra over K and let L be a maximal subfield. Then L is a splitting field for D and the index of D is $\dim_K L$.

PROOF. We regard D as a right $L \otimes D$ module by defining the action $x \cdot (l \otimes d) = lxd$, where $x \in D$ and $l \otimes d \in L \otimes D$. We can now consider $\mathrm{End}_{L \otimes D}\, D$, the ring of module endomorphisms of D. For any $y \in D$ and $\phi \in \mathrm{End}_{L \otimes D}\, D$ we compute

$$\phi(y) = \phi\left(1 \cdot (1 \otimes y)\right) = \phi(1) \cdot (1 \otimes y) = \phi(1)y,$$

so every endomorphism is just a left scalar multiplication. Letting $\phi(y) = xy$ for some $x \in D$, ϕ is an endomorphism precisely when

$$lxyd = \phi(y) \cdot (l \otimes d) = \phi(y \cdot (l \otimes d)) = xlyd,$$

so $\mathrm{End}_{L \otimes D}\, D$ is the subring of D of all elements that commute with the elements of L. We claim that $\mathrm{End}_{L \otimes D}\, D$ is exactly L. If not, we could choose an element x of the endomorphism ring not in L and consider the ring $L(x)$. As above, $L(x)$ would be a field, which contradicts the maximality of L. The right $L \otimes D$ module structure on D induces a right $L \otimes D$ module structure on the K algebra $L \otimes_K D$, so we have a K-algebra isomorphism

$$L \otimes D \cong \mathrm{End}_{L \otimes D}\,(L \otimes D)\,,$$

where the endomorphism ring is over $L \otimes D$ regarded only as an $L \otimes D$-module. Let $n = \dim_K L$ and note that a basis of L over K induces a surjective right $L \otimes D$ module homomorphism

$$f : D^n \to L \otimes D$$

which is an isomorphism by dimension considerations. Thus we have K-algebra isomorphisms

$$L \otimes_K D = \mathrm{End}_{L \otimes D}(L \otimes D) = \mathrm{End}_{L \otimes D}(D^n) = M_n\left(\mathrm{End}_{L \otimes D}(D)\right) = M_n(L).$$

Thus D splits over L and the index of D is n.

COROLLARY 14. Every Azumaya algebra over K splits over some separable extension L/K with $\dim_K L$ equal to the index of the underlying division algebra. Every element splits over some finite Galois extension and

$$\mathrm{Br}(K) = \bigcup_L \mathrm{Br}(L/K)$$

where the union ranges over all the finite Galois extensions.

PROOF. Any Azumaya algebra splits over a maximal separable subfield of its underlying division algebra, and hence splits over that field's normal closure. We now consider an extension L/K with automorphism group Γ. An Azumaya algebra A over L is given by the ring A and an inclusion $i : L \hookrightarrow A$. Given an element σ of Γ, we can induce a new L-algebra structure on the ring A by considering the inclusion $i \circ \sigma : L \hookrightarrow L$. We denote this new algebra by ${}^{\sigma}A$. This map lifts to give a left-Γ action on $\mathrm{Br}(L)$. An element $[A]$ of $\mathrm{Br}(L)$ is thus invariant under the action of σ when $A \cong^{\sigma} A$ as L-algebras. We denote the subgroup of elements fixed by all $\sigma \in \Gamma$ by $\mathrm{Br}^{\Gamma}(L)$.

PROPOSITION 15. We have an exact sequence

$$0 \longrightarrow \mathrm{Br}(L/K) \longrightarrow \mathrm{Br}(K) \xrightarrow{\mathrm{res}_{L/K}} \mathrm{Br}^{\Gamma}(L)$$

PROOF. The restriction maps $\mathrm{Br}(K)$ to $\mathrm{Br}(L)$, but for $[A] \in \mathrm{Br}(K)$ we compute

$${}^{\sigma}(L \otimes_K A) = {}^{\sigma} L \otimes_K A \cong L \otimes_K A.$$

So $\mathrm{res}[A]$ is invariant under the action.

PROPOSITION 16. Given an n-dimensional separable extension L/K of we can define a homomorphism $\mathrm{cor}_{L/K}\ \mathrm{Br}(L) \longrightarrow \mathrm{Br}(K)$ such that

$$\mathrm{cor}_{L/K} \circ \mathrm{res}_{L/K}[A] = n[A]$$

Moreover, if L/K is Galois, we have

$$\mathrm{res}_{L/K} \circ \mathrm{cor}_{L/K}[A] = N_{K/L}[A] \quad \text{where } N = \textstyle\sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma$$

PROOF. We set N equal to the normal closure of L, Γ to the Galois group of N/K and let Γ_L be the subgroup of Γ that fix L. We choose coset representatives $\{r_i\}$ so that $\Gamma = \cup r_i \Gamma_L$. For each $[A] \in \mathrm{Br}(L)$ we set

$$B = \bigotimes_i {}^{r_i}A$$

Clearly Γ acts trivially on A, so B is independent of the choice of r_i and Γ simply permutes the factors of B. Thus $[B]$ lies in $\mathrm{Br}(L)^{\Gamma}$ and we can show

$$B = \mathrm{res}_{L/K}\ {}^{\Gamma}B$$

We now compute

$$\mathrm{cor}_{L/K} \circ \mathrm{res}_{L/K}[A] = \left[{}^{\Gamma}\left(\bigotimes_{i=1}^{[L:K]} L \otimes_K A \right) \right] = n[A]$$

If L/K is Galois, Γ_L is trivial and $\{r_i\} = \Gamma$ and

$$\mathrm{res}_{L/K} \circ \mathrm{cor}_{L/K}[A] = L \otimes_K \left(\sum_{g \in \Gamma}^{\Gamma} [{}^g A] \right) = N_{L/K}[A]$$

COROLLARY 17. The Brauer group of a field K is torsion, with the order of an element dividing its index.

PROOF. Given a division algebra D we set L to be a maximal separable subfield. $\mathrm{res}_{L/K}[D]$ vanishes, so $\mathrm{cor}_{L/K} \circ \mathrm{res}\, L/K[D] = [L : K][D]$ vanishes.

3. Skolem-Noether

We require one lemma concerning automorphisms of simple rings.

LEMMA 18 (SKOLEM-NOETHER). Let B be an Azumaya algebra over K with a K-algebra subring A. If A is simple and $\phi : A \rightarrow B$ is an injective K-algebra homomorphism then there exists $b \in B^*$ such that $\phi(a) = bab^{-1}$.

PROOF. For any homomorphism ϕ we induce an $A \otimes_K B^{\mathrm{op}}$-module structure on the group B by defining $(a \otimes b)x = \phi(a)xb$ for $a \otimes b$ in $A \otimes_K B^{\mathrm{op}}$ and x in B. We denote this module by B_ϕ. We can show that any two such modules are isomorphic[2], so we choose such an isomorphism $f : B_\phi - B_i$, where $i : B - A$ is the inclusion map. We compute

$$f(x) = f((1 \otimes x) \cdot 1) = (1 \otimes x) \cdot f(1) = f(1)x$$
$$f(1)\phi(x) = f(\phi(x)) = f((x \otimes 1) \cdot 1) = (x \otimes 1) \cdot f(1) = xf(1)$$

Thus ϕ is given by conjugation by $b = f(1)$ in B^*.

COROLLARY 19. All automorphisms of a simple algebra fixing the center are inner. In particular, all K-automorphisms of $M_n(K)$ are inner.

PROOF. Choose $A = B$

4. Group cohomology

In order to get a cohomological interpretation of Brauer groups, we need to introduce Galois cohomology. We will assume the derived functor formalism in ([18] section III.1). For a complete treatment of group cohomology see ([42] Chapter VII)

We will need to develop an interpretation of Brauer groups in terms of Galois cohomology. For a complete treatment of regular group cohomology see ([42]

[2] $A \otimes_K B^{\mathrm{op}}$ is simple by theorem 5, and finitely generated modules over simple rings are classified by their dimension.

Chapter VII) or the excellent treatment in ([10]). Non-abelian group cohomology is discussed in ([42] Chapter VII appendix) and ([41] section 2). We will assume the derived functor formalism used in ([18] section III.1).

DEFINITION 20. Given a group G, not necessarily abelian, we define the category $\mathcal{M}(G)$ of G-modules to be the abelian category $\mathcal{M}(\mathbb{Z}G)$, where $\mathbb{Z}G$ is the integral group ring over G.

Note that the G-module structure of a G-modules M is completely given by an action[3] of G on M. Thus we can associate to each G-module M the abelian group M^G, the fixed points under the induced action. We can show that this gives a covariant left exact functor $\mathcal{F}$ from $\mathcal{M}(G)$ to $\mathfrak{Ab}$. Alternatively, we can define $\mathcal{F}(M)$ to be the subgroup of M whose elements are annihilated by the ideal I in $\mathbb{Z}G$ generated by elements of the form $1 - g$.

Since $\mathcal{M}(G)$ contains enough injectives ([18] III.2), we can construct a cohomology.

DEFINITION 21. We define the cohomology groups $H^*(G, -) : \mathcal{M}(G) \to \mathfrak{Ab}$ as the derived functors induced from $H^0(G, -) = \mathcal{F}$.

There exists explicit injective resolutions which allow us to compute these groups. $H^1(G, M)$ is given by the group of maps ρ from G into M that satisfy

$$\rho(gh) = g\rho(h) + \rho(g)$$

quotiented by the subgroup of elements of the form $\rho(g) = gm - m$ for some $m \in M$. $H^2(G, M)$ is given by

$$\frac{\{\rho : G \times G \to M \mid f\rho(g, h) + \rho(f, gh) = \rho(fg, h) + \rho(f, g)\}}{\{\rho(g, h) = g\tau(h) - \tau(gh) + \tau(g)\}}$$

Note that the above representations make sense even when M is not abelian but still has a G-action. The cocycle conditions, however, are no longer linear, so the resulting sets are no longer groups but are pointed sets containing a zero element. We can a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of pointed sets is *exact* if $f(A) = g^{-1}(0)$.

PROPOSITION 22. If we are given an exact sequence of groups $1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$ with A contained in the center of B then we have an exact sequence of pointed sets

$$1 \to H^0(G, A) \xrightarrow{H^0(G,f)} H^0(G, B) \xrightarrow{H^0(G,g)} H^0(G, C) \xrightarrow{\delta} H^1(G, A) \xrightarrow{H^1(G,f)} H^1(G, B) \xrightarrow{H^1(G,g)} H^1(G, C) \xrightarrow{\delta} H^2(G, A)$$

PROOF. See [42] proposition VII.A.2

[3] namely a group homomorphism from G to $\mathrm{Aut}(M)$

5. Severi-Brauer Varieties

We will now define a geometric analogue of the Brauer group whose elements are varieties.

DEFINITION 23. We call a variety $\mathfrak{X}$ over K a *Severi-Brauer* variety if we can choose a Galois extension L/K such that the base extension $\mathfrak{X}_L$ is isomorphic, as a variety over L, to the projective space $\mathbb{P}^n_L$. We call any such L a *splitting field* for $\mathfrak{X}$. We denote by $\mathfrak{Sb}(K)$ and $\mathfrak{Sb}(L/K)$ the set of Severi-Brauer varieties over K and the subset of those that split over L, respectively.

PROPOSITION 24. A Severi-Brauer variety over K either splits over K or it contains no points.

PROOF. [7]

The geometry of these varieties connect nicely to our constructions. There is a link [3], for example, between the Picard group of a Severi-Brauer variety over K and the Brauer group over K. The function fields, in particular, have a crucial property.

PROPOSITION 25. The function field $k(\mathfrak{X})$ is a splitting field for a Severi-Brauer variety $\mathfrak{X}$.

We represent $\mathfrak{X}$ as a subvariety of $\mathbb{P}^n_K$, so we choose elements $f_i \in K[x_0, \ldots x_n]$ such that

$$\mathfrak{X} \cong \text{Proj}\left(\frac{K[x_0, \ldots x_n]}{(f_i(x_0, \ldots x_n))}\right)$$

$\mathfrak{X}$ has an affine subset Spec R with

$$R = \frac{K[t_1, \ldots t_n]}{(f_i(1, t_1, \ldots t_n))}$$

R is a domain, so the field of functions $k(\mathfrak{X})$ is the field of fractions $R_{(0)}$. We now compute

$$\mathfrak{X}_{k(\mathfrak{X})} = \text{Proj}\left(\frac{R_{(0)}[x_0, \ldots x_n]}{(f(x_0, \ldots x_n))}\right)$$

Thus $k(\mathfrak{X})$ splits $\mathfrak{X}$ as it contains the rational point

$$\mathfrak{p} = (x_0 - 1, x_1 - t_1, \ldots x_n - t_n)$$

This splitting field was studied extensively by Châtelet [7] and Amitsur [1] and is called the *generic* splitting field. It enjoys many universal properties (see [1] for a detailed list), including

PROPOSITION 26. $\mathfrak{X}$ splits over L if and only if there is a place $p : k(\mathfrak{X}) \to L$
For a discussion on places we refer the reader to ([22], XII.4).
Recall that if we are given a field F we let $\{F, \infty\}$ be the union of F and the element ∞. We can extend the field structure of F by defining for all $f \in F$

$$f \pm \infty = \infty \quad f\infty = \infty \ (f \neq 0)$$
$$\infty\infty = \infty \qquad 1/0 = \infty \qquad\qquad 1/\infty = 0,$$

and by leaving $\infty \pm \infty$, 0∞, $0/0$, and ∞/∞ undefined. Given fields F and K we define a *place* $\phi : K \to F$ to be a map from K to $\{F, \infty\}$ such that

$$\phi(fg) = \phi(f)\phi(g) \quad \phi(f+g) = \phi(f) + \phi(g)$$

when both sides are defined. Note, for example, that $\mathfrak{o} = \phi^{-1}(K)$ is a valuation ring with maximal ideal $\mathfrak{m} = \phi^{-1}(0)$. We say that two places $\phi, \phi' : K \to F$ are equivalent if $\phi' = \phi \circ \tau$ for some automorphism τ of K.

LEMMA 27. Let $\mathfrak{X}$ be a projective variety over K. There is a bijection between the closed points of $\mathfrak{X}$ and places $\phi : k(\mathfrak{X}) \to K$.

PROOF. The inclusion of a closed point Spec K into an affine open subset Spec R of $\mathfrak{X}$ corresponds to a homomorphism $f : R \to K$. We now define a place by

$$\phi : k(\mathfrak{X}) \cong R_0 \to \{K, \infty\}$$

by $\phi(a/b) = f(a)/f(b)$. Conversely, suppose Spec R is an open affine subset of $\mathfrak{X}$ with $R = k[x_1, \dots x_n]/\mathfrak{a}$ and a place

$$\phi : k(\mathfrak{X}) = R_0 \to K.$$

By composing with a suitable automorphisms of R_0 that map $x_i \mapsto x_i^{-1}$, we can assume that $\phi(x_i) \neq \infty$. Thus $\phi^{-1}(K) = \mathfrak{o} = R$ and $\phi^{-1}(0) = \mathfrak{m}$ forms a maximal ideal of R. $\mathfrak{m}$ is clearly generated by $(x_i - \phi(x_i))$, so it forms a closed point in Spec R and hence in $\mathfrak{X}$.
To prove the proposition, note that L is a splitting field for $\mathfrak{X}$ if and only if we have a place $\phi' : L \otimes_K k(\mathfrak{X}) = k(\mathfrak{X}_L) \to L$. But this exists if and only if we have a place $\phi : k(\mathfrak{X}) \to L$.
We will also need

PROPOSITION 28. Every Severi-Brauer variety has a splitting field finite dimensional and Galois over the base field.

PROOF. For any $\mathfrak{X}$ over K we choose an isomorphism as in proposition 25. For any splitting field L we have

$$\mathfrak{X}_L = \mathrm{Proj}\left(\frac{L[x_0, \dots x_n]}{(f(x_0, \dots x_n))}\right) \cong \mathbb{P}^m_L.$$

Closed points on $\mathfrak{X}_L$ are of the form $\mathfrak{p} = (x_0 - a_0, \dots x_n - a_n)$ with $a_j \in L$ and $f_i(a_0, \dots a_n) = 0$ for all I.

We claim that for any L/K and any sequence of polynomials $f_i \in K[x_0, \dots x_n]$ that have a common non-zero solution we can choose a solution $(a_0, \dots a_n)$ with a_i algebraic over K. If not, choose a counterexample with n minimal and a solution $(a_0, \dots a_n)$ over L where we assume wlog that a_0 is non-zero. Since $f_i(x, a_1, \dots a_n) \in K'[X]$ with $K' = K(a_1, \dots a_n)$ we can assume a_0 is nonzero and algebraic over K'. We let

$$f_i'(x_1, \dots x_n) = \prod_{\sigma \in \mathrm{Gal}(K'[a_0]/K')} f_i(\sigma(a_0), x_1, \dots x_n)$$

with $f_i' \in K'[x_1, \dots x_n]$. These polynomials lift through the projection

$$K[x_1, \dots x_n] \to K[a_1, \dots a_n]$$

to give polynomials $f'' \in K[x_1, \dots x_n]$. These polynomials have $(a_1, \dots a_n)$ as a common solution, and thus have a solution $(b_1, \dots b_n)$ algebraic over K by the minimality of n. Tracing back, f_i has a solution $(b_0, \dots b_n)$ where b_0 is algebraic over $K(b_0, \dots b_n)$. Contradiction.

To complete the proof, note that if we let L' be the splitting field of the algebraic elements a_i then $\mathfrak{X}_{L'}$ contains a point $\mathfrak{p}$, so L' is a splitting field.

PROPOSITION 29. Given a p-cyclic extension L/K, let $\mathcal{K}(L/K)$ be the composite of the function fields of the Severi-Brauer varieties associated to all the elements of $\mathrm{Br}(L/K)$. Then $\mathcal{K}(L/K)$ is a field extension of k such that $\mathcal{K}(L/K) \otimes L$ is a p-cyclic extension of $\mathcal{K}(L/K)$ and k is in the image of the resulting norm map.

PROPOSITION 30. A p-cyclic extension L/K has an extension $\mathcal{K}^\infty(L/K)$ of K such that

(1) $\mathcal{K}^\infty(L/K)$ is the direct limit of a sequence of k field extensions $k_{\mathfrak{X}}/F$ where $\mathfrak{X}$ is a Severi-Brauer variety over F that splits over some p-cyclic extension of F.

(2) $(\mathcal{K}^\infty(L/K) \otimes L)/\mathcal{K}^\infty(L/K)$ is a p-cyclic extension of fields.

(3) The norm map $N_{\mathcal{K}^\infty(L/K)\otimes L/\mathcal{K}^\infty(L/K)}$ is surjective.

PROOF. We define $\mathcal{K}^i$ inductively by

$$\mathcal{K}^0 = k$$
$$\mathcal{K}^i = \mathcal{K}\left((\mathcal{K}^{i-1} \otimes L)/\mathcal{K}^{i-1}\right)$$

and let $K^{\infty}(L/K)$ be the union.

6. Galois actions

Since Severi-Brauer varieties are étale open subsets of projective spaces, we can regard then as fixed points of some action. We fix a Galois extension L/K with Galois group Γ, choose $\mathfrak{X}$ in $\mathfrak{Sb}(L/K)$, and fix an isomorphism

$$\operatorname{Spec} L \times_{\operatorname{Spec} K} \mathfrak{X} = \mathfrak{X}_L \cong \mathbb{P}^n_L = \operatorname{Spec} L \times_{\operatorname{Spec} K} \mathbb{P}^n_K$$

Since Γ acts on Spec L on the left we can use this equation to define two left Γ actions on $\mathbb{P}^n_L$. If Γ acts on the RHS we get the usual group action, but if it acts on the LHS we get a second action that depends on $\mathfrak{X}$. The fixed points of this action are

$$(\mathbb{P}^n_L)^\Gamma = (\operatorname{Spec} L \times_{\operatorname{Spec} K} \mathfrak{X})^\Gamma = \operatorname{Spec} L^\Gamma \times_{\operatorname{Spec} K} \mathfrak{X} = \mathfrak{X}.$$

We can use this to define an additive structure. Choosing $\mathfrak{X}$ and $\mathfrak{Y}$ of dimensions n and m, we have the induced Γ actions on $\mathbb{P}^n_L$ and $\mathbb{P}^m_L$. We have the Segre embedding

$$\mathbb{P}^n_L \times_{\operatorname{Spec} L} \mathbb{P}^m_L \longrightarrow \mathbb{P}^{mn+m+n}_L,$$

so we can give $\mathbb{P}^{mn+m+n}_L$ a $\Gamma \oplus \Gamma$ action and we define $\mathfrak{X} + \mathfrak{Y}$ to be the fixed points of this action. This is Severi-Brauer as

$$(\mathbb{P}^{mn+m+n}_L)^{\Gamma\oplus\Gamma}_L = (\mathbb{P}^n_L)^\Gamma_L \times_{\operatorname{Spec} K} (\mathbb{P}^m_l)^\Gamma_L = (\mathbb{P}^{mn+m+n}_L)^\Gamma$$

PROPOSITION 31. This makes $\mathfrak{Sb}(L/K)$ into a commutative monoid.

PROOF. Associativity follows from the associativity of the tensor product. The identity is Spec K.

7. $\mathfrak{Sb}$ and the Projective Linear Group

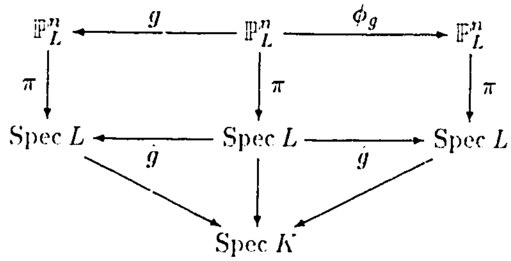
We fix a Galois extension L/K with Galois group Γ, let $\mathrm{PGL}_m(L)$ be the projective linear group of $m \times m$ matrices over L, and let Γ act on $\mathrm{PGL}_m(L)$ on the left.

PROPOSITION 32. We have an injective map Ω_1 from the n-dimensional elements of $\mathfrak{Sb}(L/K)$ into $H^1(\Gamma, \mathrm{PGL}_{n+1}(L))$

PROOF. We choose $\mathfrak{X} \in \mathfrak{Sb}(L/K)$ and let $\mathfrak{X}_L \cong \mathbb{P}^n_L$. We will use the usual group action notation for the natural left action by Γ on $\mathbb{P}^n_L$. To represent the Γ-action induced by $\mathfrak{X}$ we define an automorphism

$$\phi_g : \mathbb{P}^n_L \to \mathbb{P}^n_L$$

induced by the action of $g \in \Gamma$.
Since $\mathbb{P}^n_L$ is a variety over L, we have a projection map $\pi : \mathbb{P}^n_L \to \operatorname{Spec} L$. The morphisms g and ϕ_g factor through π to give a commuting diagram.



where g : Spec L — Spec L is given by the natural action of g. Note that neither actions yield automorphisms of $\mathbb{P}^n_L$ that are morphisms of varieties over L as they do not fix the base scheme. However $\tau_g = \phi_g \circ g^{-1}$ does fix Spec L, so it forms an automorphism of varieties over L. We know[4] that $\operatorname{Aut}(\mathbb{P}^n_L) = \mathrm{PGL}_{n+1}(L)$, so the group action is given by a map $\tau : \Gamma \to \mathrm{PGL}_{n+1}(L)$ satisfying $\tau_{gh} = \tau_g \cdot g(\tau_h)$.
Note that τ depends on a choice of an isomorphism $\mathfrak{X}_L \cong \mathbb{P}^n_L$. Since any two isomorphisms would differ only by an element of $\mathrm{PGL}_{n+1}(L)$, we identify τ and τ' if we can find an element $x \in \mathrm{PGL}_{n+1}(L)$ such that $\tau'_g = x\tau_g x^{-1}$.
Thus the induced Γ action is given by an element of $H^1(\Gamma, \mathrm{PGL}_{n+1}(L))$. Conversely, each cocycle in the image represents the conjugacy class of a Γ-action whose fixed points are the original scheme, so the map is injective.

8. Cohomology of the Brauer Group

We can construct a similar cohomological interpretation of Azumaya algebras.

PROPOSITION 33. For any Galois extension L/K with Galois group Γ we have an injective map Ω_2 from index n representatives of $\mathrm{Br}(L/K)$ to $H^1\Gamma, \mathrm{PGL}_n(L))$.
Choose such an Azumaya algebra A. We first fix an L-linear isomorphism

$$f : L \otimes_K A \longrightarrow M_n(L)$$

We use the usual group action notation for the element wise action of Γ on $M_n(L)$ and define a second action of Γ on $M_n(L)$ by having it act on the RHS of the equation. We will represent this action by the K-linear automorphisms

[4]Look at the projection $L^{n+1} \to \mathbb{P}^n_L$. Any automorphism of the bottom lifts to an automorphism of L^{n+1} which is an element of $\mathrm{GL}_{n+1}(L)$. Pushing down, we get an element of $\mathrm{PGL}_{n+1}(L)$.

$$\phi_g : M_n(L) \to M_n(L)$$

We note that the center of $M_n(L)$ is L so we compute for $l \in L \subset M_n(L)$ that

$$(\phi_g \circ g^{-1})(l) = f \circ g \circ f^{-1} \circ g(l) = f \circ g\,(g^{-1}l \otimes 1) = f(l \otimes 1) = l$$

Thus the Γ-action is given by the set of maps $\rho_g = \phi_g \circ g^{-1}$ satisfying

$$\rho_g : \Gamma \to \mathrm{Aut}_L\, M_n(L)$$
$$\rho_{gh} = \rho_g \cdot g(\rho_h)$$

Since all automorphisms of $M_n(L)$ are inner by corollary 19, the group of automorphisms is equal to $\mathrm{PGL}_n(L)$ and the above construction gives a representative of a cocycle in $H^1(\Gamma, \mathrm{PGL}_n(L))$. The choice of an isomorphism is unique up to an automorphism, so the cocycle induced is unique up to coboundary and the map is well defined. If ρ is in the image of this map then ρ induces a group action on $M_n(L)$ whose fixed points form a K-algebra. These maps are inverse to each other, so we have an injection.

Note that we have not shown whether this map is surjective.

Since both $n-1$ dimensional Severi-Brauer varieties over K and Azumaya algebras over K of index n are both represented by elements in the same cohomology group, we have a natural association between the algebras and the varieties. We can make this mapping explicit.

PROPOSITION 34. We have a map Ω_3 mapping index n Azumaya algebras to dimension $n-1$ Severi-Brauer varieties such that $\Omega_1 = \Omega_2 \circ \Omega_3$.

PROOF. Let A be an Azumaya algebra of index n that splits over L and view A as an n^2-dimensional vector space over K. We consider the Grasmannian $G_n(A)$ of n-dimensional subspaces of A which has the structure of an algebraic variety, and let W be the subset of $G_n(A)$ of subspaces $\mathfrak{a} \subset A$ that are also right ideals when A considered as a ring. For any element $a \in A$ the condition $\mathfrak{a}a \subset a$ is clearly algebraic on $G_n(A)$, so W is an algebraic subset of $G_n(A)$ and is thus an algebraic variety which we denote by $\mathfrak{X}(A)$

LEMMA 35. $\mathfrak{X}(M_n(K)) = \mathbb{P}_K^{n-1}$

PROOF. For any element $A \in M_n(K)$ right multiplication on A by $M_n(K)$ consists of column operations, so if A is contained in some ideal $\mathfrak{a}$ in W then $\mathfrak{a}$ contains a matrix B that vanishes off of the first column (that is, $B_{ij} = 0$ if $j > 1$). A dimension count shows that any such matrix B generates an ideal in

W and that B must be unique up to scalar multiplication. Thus any ideal in W is given uniquely by $b_{11}, \ldots b_{n1}$, so

$$\mathfrak{X}(M_n(K)) = \operatorname{Proj}(K[b_{11}, \ldots b_{n1}]) = \mathbb{P}_K^{n-1}$$

LEMMA 36. For any extension F/K, $\mathfrak{X}(F \otimes_K A) = \mathfrak{X}(A)_F$

PROOF. The Grassmanian $G_n(F \otimes_K A)$ is a variety over F and is by construction equal to $(G_n(A))_F$. The equations that cut out $\mathfrak{X}(A)$, when extended to F, are precisely the equations that cut out $\mathfrak{X}(F \otimes_K A)$.
The proof of the proposition is now clear. We compute

$$\operatorname{Spec} L \times_{\operatorname{Spec} K} \mathfrak{X}(A) = \mathfrak{X}(L \otimes_K A) \cong \mathfrak{X}(M_n(L)) = \mathbb{P}_L^{n-1} = \operatorname{Spec} L \times_{\operatorname{Spec} K} \mathbb{P}_K^{n-1}$$

so, in particular, $\mathfrak{X}(A)$ is a Severi-Brauer variety of the correct dimension. To show that both the algebra and the variety have the same cohomological description, we need to show that the action of $A \in \mathrm{PGL}_n(L)$ on $M_n(L)$ induces the same action on $\mathbb{P}_L^{n-1}$ as the usual action of A on $\mathbb{P}_L^{n-1}$. The action of A on $M_n(L)$ maps any n-dimensional right ideal $\mathfrak{a}$ of $M_n(L)$ to $A\mathfrak{a}A^{-1} = A\mathfrak{a}$, which matches the action of A on $\mathbb{P}_L^{n-1}$.

9. The Brauer Group and $H^2(\Gamma, L^*)$

We need to construct a final cohomological interpretation. We fix a finite Galois extension L/K of dimension n with Galois group Γ.

DEFINITION 37. Let $\tilde{\mathrm{Br}}(L/K)$ be the set of representatives A of classes $[A]$ in $\mathrm{Br}(L/K)$ of index n such that L is a subfield of A.
Note that there is at most one such representative for each class.

LEMMA 38. Given $A \in \tilde{\mathrm{Br}}(L/K)$ we can choose elements e_g for all $g \in \Gamma$ with $e_g x = g(x) e_g$ and

$$A = \bigoplus_{g \in \Gamma} L e_g$$

Moreover, if we let $e_g e_h = x(g,h) e_{gh}$ then $x(g,h) \in L^*$ gives us an element $\Omega(A)$ of $H^2(\Gamma, L^*)$.

PROOF. We apply lemma 18, the Skolem-Noether lemma, for the subfield L of A. Letting $i : L \hookrightarrow A$ be the inclusion map, we have for any $g \in \Gamma$ the inclusion $i \circ g : L \hookrightarrow A$. Applying lemma 18, we have that $i \circ g(l) = e_g l e_g^{-1}$ for some elements $e_g \in A$. Thus $e_g l = g(l) e_g$ as required.
We claim that the elements e_g are linearly independent over L. Suppose first that elements e_h are linearly independent for all h in some subset H of G. Suppose

further that for some $g \notin H$ we can write $e_g = \sum a_h e_h$ for $h \in H$, and compute for any $x \in L$

$$0 = e_g x - g(x)e_g = \sum_{h\in H} a_h e_h x - g(x)a_h e_h = \sum_{h\in H} a_h \left(h(x) - g(x)\right) e_g$$

By choosing x correctly, we can show that $a_h = 0$ for all h, contradicting the supposition. The claim now follows by induction. The collection e_g now forms a basis for A by a dimension count.

Suppose that $\sum a_g e_g$ commuted with all elements of L. We compute for $l \in L$ that

$$0 = \left(\sum_g a_g e_g\right) l - l \sum_g a_g e_g = \sum_g \left(g(l) - l\right) a_g e_g.$$

By choosing l correctly we can conclude that $a_g = 0$ for $g \neq 1$. Thus $\sum a_g e_g \in L$ and L is a maximal subfield. We now compute for all $x \in L$ and $g, h \in \Gamma$

$$e_{gh} x e_{gh}^{-1} = g\left(h(x)\right) = e_g e_h x e_h^{-1} e_g^{-1}.$$

Thus $e_g e_h e_{gh}^{-1} = x(g,h)$ commutes will all elements in L, and so must be an element of L by maximality. We use the associativity of A to compute

$$x(g,h)x(fg,h)e_{fgh} = x(g,h)e_{fg}e_h = e_f e_g e_h = e_f x(g,h) e_{gh}$$
$$= f\left(x(g,h)\right) e_f e_{gh} = f\left(x(g,h)\right) x(f,gh) e_{fgh}$$

Which is just the cocycle relation in multiplicative form. Suppose e_g and f_g are two choices of the basis elements. We compute that $y_g = e_g f_g^{-1}$ commutes with all elements of L, so it must lie in L by maximality. Substituting this back into the definition of $x(f,g)$ gives us the coboundary condition.

LEMMA 39. Given a finite Galois extension L/K with Galois group K we can associate to every element ζ of $H^2(\Gamma, L^*)$ an Azumaya algebra A in $\mathrm{Br}(L/K)$ such that $\Omega(A) = \zeta$.

PROOF. Choose symbols e_g for all $g \in \Gamma$ and define the vector space

$$A = \bigoplus_{g\in\Gamma} L e_g$$

We choose a representative $x(g,h)$ of a cocycle and make A into an ring by defining $xe_g \cdot ye_h = xg(y)x(g,h)e_{gh}$ and extending linearly[5] Associativity follows

[5]The identity is $x(1,1)^{-1}e_1$. Note that we can assume via a correct choice of cocycle that $x(1,1) = 1$.

directly from the cocycle condition. We embed L into A by the mapping $l \mapsto lx(1,1)^{-1}e_1$, and compute that for any $x \in L$

$$x\left(\sum_{g\in\Gamma} a_g e_g\right) - \left(\sum_{g\in\Gamma} a_g e_g\right) x = \sum_{g\in\Gamma} (x - g(x)) a_g e_g .$$

Since $x - g(x) \neq 0$ for some x if $g \neq 1$, the only elements that commute with L are those in L. Thus L is a maximal subfield. Since $e_g l = g(l) e_g$ for all $l \in L$ we conclude that $L^\Gamma = K$ is the center of A. We need now only show that A is simple.

Let $\mathfrak{a}$ be a two sided ideal of A containing an element a represented by $\sum a_g e_g$, and define $|a|$ to be the number of nonzero coefficients a_g. If $|a| = 1$ then a is clearly invertible and $\mathfrak{a} = A$. If $|a| > 1$ then we choose elements g_1 and g_2 such that $a_{g_i} \neq 0$. We pick $l \in L$ such that $g_1(l) \neq g_2(l)$ and compute

$$a - g_1(l) a l^{-1} = \sum_{g\in\Gamma} a_g \left[1 - g_1(l) g(l^{-1})\right] = \sum_{g\in\Gamma} b_i e_i$$

Since $b_{g_1} = 0$ and $b_{g_2} \neq 0$ we have $|b| > 0$ and $|b| < |a|$. By induction we have that $\mathfrak{a}$ must contain an element a with $|a| = 1$, so $\mathfrak{a} = A$.

The lemma follows as the two constructions are obviously inverses. We now have

PROPOSITION 40. We have a bijection $\Omega : \tilde{\mathrm{Br}}(L/K) \to H^2(\Gamma, L^*)$.

We have now for any element $A \in \tilde{\mathrm{Br}}(L/K)$ two cohomological representatives – one in $H^1(\Gamma, \mathrm{PGL}_n(L))$ and one in $H^2(\Gamma, L^*)$. To compare them, we consider the exact sequence of non-abelian Γ-modules

$$1 \to L^* \to \mathrm{GL}_n(L) \to \mathrm{PGL}_n(L) \to 1$$

By proposition 22, the analog of the long exact sequence gives us a Bockstein operator

$$\delta : H^1(\Gamma, \mathrm{PGL}_n(L)) \to H^2(\Gamma, L^*)$$

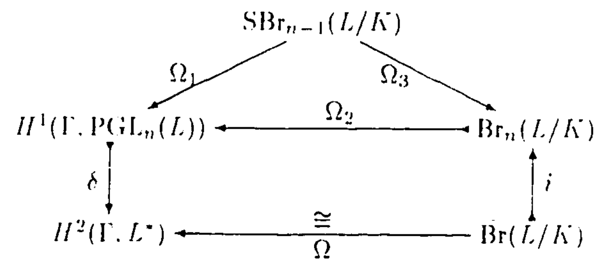
PROPOSITION 41. We have a commutative triangle

$$\begin{array}{ccc} & \tilde{\mathrm{Br}}(L/K) & \\ {\scriptstyle \omega_2} \swarrow & & \searrow {\scriptstyle \Omega_1} \\ H^1(\Gamma, \mathrm{PGL}_n(L)) & \xrightarrow[\delta]{} & H^2(\Gamma, L^*) \end{array}$$

PROOF. Computation. We have to construct the Bockstein explicitly via a diagram chase.

We summarize our results in a large diagram.

Proposition 42. Let L/K be a finite Galois extension with Galois group Γ, let $\mathrm{Br}_n(L/K)$ be the set of index n representatives of $\mathrm{Br}(L/K)$, and let $\mathrm{SBr}_{n-1}(L/K)$ be the subset of $n-1$ dimensional varieties in $\mathrm{SBr}_{n-1}(L/K)$. We have the following commutative diagram of injective maps.



Corollary 43. For any Galois extension L/K, $H^1(\Gamma, \mathrm{PGL}_n(L))$ classifies Azumaya algebras of index n that split over L

Proof. A diagram chase proves the result for finite extensions, and we can take the direct limit by corollary 14.

Corollary 44. For any Galois extension L/K, $H^1(\Gamma, \mathrm{PGL}_n(L))$ classifies n-dimensional Severi-Brauer varieties over K that split over L.

Proof. The finite case is, again, by a diagram chase and we can take the direct limit by proposition 28

Corollary 45. For a finite extension L/K, $\mathrm{Br}(L/K) = \mathrm{Br}(L/K)$. Thus L is a splitting field for A if and only if A is equivalent to some A' of index n containing L as a subfield.

Proof. A diagram chase shows that i is a bijection.

Corollary 46. There is a one to one correspondence between index n Azumaya algebras over K that split over L and n-dimensional Severi-Brauer varieties that split over L. This correspondence preserves the additive structure

Proof. A diagram chase shows that Ω_3 is surjective and the structure on $\mathrm{SBr}_{n-1}(L/K)$ was is compatible by construction.

Corollary 47. If we quotient $\mathfrak{Sb}(L/K)$ by the projective spaces over K we obtain a group isomorphic to $\mathrm{Br}(L/K)$.

Proof. Diagram chase.

Corollary 48. We have an isomorphism $\Omega : H^2(\Gamma, L^*) \cong \mathrm{Br}(L/K)$

PROOF. We need to construct an isomorphism

$$M_n(\omega(\zeta_1 \cdot \zeta_2)) \cong \omega(\zeta_1) \otimes_K (\zeta_2).$$

This is done via an explicit computation in ([8] Thm II.12.3).

COROLLARY 49. We have an isomorphism $\Omega : H^2(\mathrm{Gal}(K^{\mathrm{sep}}), K^{\mathrm{sep}}) \cong \mathrm{Br}(K)$

10. Brauer Groups of Cyclic Extensions

To conclude this chapter, we will explicitly compute some Brauer groups. Let L/K be a *cyclic*: an extension with a cyclic Galois group. We also choose a generator σ of $\mathrm{Gal}(L/K)$.
Recall that in example 3 we defined the cyclic algebras $(f, L/K, \sigma)$.

PROPOSITION 50. The product of $(f, L/K, \sigma)$ and $(fg, L/K, \sigma)$ in $\mathrm{Br}(L/K)$ is $(fg, L/K, \sigma)$

PROOF. ([8] 10.4).

PROPOSITION 51. $(f, L/K, \sigma)$ splits if and only if $f \in NL$, where $N = \sum_i \sigma^i$ is the norm element

PROOF. Compute the associated Severi-Brauer variety.

PROPOSITION 52. If L/K is cyclic then $\mathrm{Br}(L/K)$ consists entirely of cyclic algebras.

PROOF. Let $n = \dim_K L$. Using corollary 45 we can choose for any element of $\mathrm{Br}(L/K)$ a representative A of index n containing L as a maximal subfield. We choose elements e_τ for all $\tau \in <\sigma>$ as in proposition 33 and compute

$$e_\sigma^n l (e_\sigma^n)^{-1} = \sigma^n(l)$$

so we can assume by a correct chose of cocycle that $e_{\sigma^i} = e_\sigma^i$ for $0 \leq 1 < n$. Identifying $f = e_\sigma^n \in K$ and $x_i = e_\sigma^i$, A is equivalent to $(f, L/K, \sigma)$.

COROLLARY 53. $\mathrm{Br}(L/K) \cong K/NL$

COROLLARY 54. $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$

PROOF. The only algebraic extension of $\mathbb{R}$ is $\mathbb{C}$, and $\mathbb{R}^*/N\mathbb{C}^* \cong \mathbb{Z}/2\mathbb{Z}$.
The proposition raises the following conjecture.

CONJECTURE 55. Are all Azumaya algebras the tensor products of cyclic algebras?
A counterexample was discovered by Amitsur, Rowen, and Tignol who produced an algebra of index eight that was not the product of index two algebras. We can weaken the conjecture if we only require that every Azumaya algebra is equivalence to some such product.

CONJECTURE 56. Are all elements of the Brauer group over K the tensor products of cyclic algebras?
Classically, this was known for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{Q}_p$. If K contains the n'th roots of unity the conjecture for elements of order n is equivalent to Merkurjev-Suslin.
To make the conjecture clearer, we construct a second set of cyclic algebras.

DEFINITION 57. Suppose L/K is cyclic, K contains the n'th roots of unity μ_n, and the characteristic of K does not divide n. We fix an element $\omega \in \mu_n$ and define for $a, b \in K^*$ the *cyclic algebra* $A_\omega(a, b)$ to be the algebra generated over K by the symbols x and y with the relations $xy = \omega yx$, $x^n = a$ and $y^n = b$.

PROPOSITION 58. $A_\omega(a, b)$ forms an Azumaya algebra of index n over K.

PROOF. The index is clear by the construction. To show that it is an Azumaya algebra, modify lemma 39 as required.
We fix an element $l \in L - K$ and note that $\sigma l = \omega l$ for some root of unity ω. Thus l^n is invariant under the action of σ, so $l = \sqrt[n]{a}$ for some $a \in K$ and $L = K[\sqrt[n]{a}]$.

PROPOSITION 59. Choosing L/K, σ, and ω as above then $(b, L/K, \sigma) = A_\omega(a, b)$.

PROOF. Construct a map taking $x \mapsto x$ and $y \mapsto \sqrt[n]{a}$. Verification is trivial.

COROLLARY 60. If $L = K[\sqrt[n]{a}]$ then $\mathrm{Br}(L/K)$ consists entirely of algebras of the form $A_\omega(a, b)$ for all $b \in K^*$.
Note that all our cyclic algebras are of index n, so they have order dividing n by corollary 17. If we let $\mathrm{Br}_n(K)$ be the subgroup of $\mathrm{Br}(K)$ of elements of order n then we can assign to each pair $a, b \in K^*$ and element of $\mathrm{Br}_n(K)$.

PROPOSITION 61. We have a homomorphism $\alpha_K : K^* \otimes K^* \to \mathrm{Br}_n(K)$ given by mapping $a \otimes b \mapsto A_\omega(a, b)$. The map satisfies

(1) $\alpha(a \otimes b)\alpha(c \otimes b) = \alpha(ac \otimes b)$
(2) $\alpha(a \otimes b) = \alpha(b \otimes a)^{-1}$
(3) $\alpha(a^n \otimes b) = 1$
(4) If $a \neq 0, 1$ then $\alpha(a \otimes (1 - a)) = 1$

CHAPTER II

Category Theory

1. Exact Categories

We will be studying the properties classes of categories that have a well defined notion of an exact sequence. See [19], [18], [36], [46], or any standard text on category theory for more details.

DEFINITION 62. We call an additive category $\mathcal{C}$ *exact* if we can regard it as a full additive subcategory[1] of some abelian category $\mathcal{C}^0$ that is closed under extensions in the following sense. If we are given an exact sequence in $\mathcal{C}^0$

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

with both A' and A'' in $\mathcal{C}$ then the entire sequence is in $\mathcal{C}$. We call a sequence in $\mathcal{C}$ *exact* if it is an exact in $\mathcal{C}^0$, and we call a morphism in $\mathcal{C}$ *admissible* if it occurs as either the epimorphism or the monomorphism of some such exact sequence. We call a functor between exact categories *exact* if it maps all exact categories to exact categories.

Thus we have an interesting new category, $\mathfrak{Ex}$: the category of exact categories whose morphisms are exact functors. If $\mathcal{C}$ and $\mathcal{D}$ are exact categories, we also define the category $\mathfrak{Ex}(\mathcal{C}, \mathcal{D})$, the category of exact functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ together with their natural transformations.

PROPOSITION 63. $\mathfrak{Ex}(\mathcal{C}, \mathcal{D})$ is an exact category.

PROOF. (sketch) $\mathfrak{Ex}(\mathcal{C}, \mathcal{D})$ has a natural zero object given by $0(M) = 0$. Given a sequence of natural transformations

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

[1] That is, $\mathrm{Obj}\,\mathcal{C}$ is a subset of $\mathrm{Obj}\,\mathcal{C}^0$ with $\hom_{\mathcal{C}}(A, B)$ equal to $\hom_{\mathcal{C}^0}(A, B)$ as abelian groups.

We say that this sequence is exact if

$$0 \to \mathcal{F}'(M) \to \mathcal{F}(M) \to \mathcal{F}''(M) \to 0(M)$$

is exact for all elements M of $\mathcal{C}$. We can show that this satisfies an intrinsic criterion for exactness without reference to an embedding category (see [36] for details).

PROPOSITION 64. The following categories are abelian.

- $\mathfrak{Ab}$, the category of abelian groups.
- $\mathcal{V}(K)$, the category of vector spaces over a field K.
- $\mathcal{M}(R)$, the category of finitely generated R-modules for a commutative ring R
- $\mathfrak{Qco}(\mathfrak{X})$, the category of quasi-coherent sheaves over a scheme $\mathfrak{X}$
- $\mathcal{M}(\mathfrak{X})$, the category of coherent sheaves over a Noetherian scheme $\mathfrak{X}$

PROOF. See ([18], II 2.2-2.4)

PROPOSITION 65. The following categories are exact.

- $\mathcal{P}(R)$, the category of finitely generated projective modules over a ring R
- $\mathcal{P}(\mathfrak{X})$, the category of locally free sheaves of finite rank over a scheme $\mathfrak{X}$
- $\mathcal{M}_f(R)$, the category of finite length modules over a a local ring R.

PROOF. Since $\mathcal{P}(R)$ and $\mathcal{P}(\mathfrak{X})$ are full additive subcategories of $\mathcal{M}(R)$ and $\mathfrak{Qco}(\mathfrak{X})$ respectively, we need only check the extension properties. These are standard properties of the categories.
One of the more important categories that we will require is the category of codimension p supported modules on a Noetherian scheme.

DEFINITION 66. Given a module M in $\mathcal{M}(\mathfrak{X})$ we define the *support* of M, $\sup M$, by

$$\sup M = \{x \in \mathfrak{X} | M_x \neq 0\}$$

PROPOSITION 67. The support has the following properties:

- $\sup M =$ only if $M = 0$.
- $\sup(M \oplus N) = \sup M \cap \sup N$.
- Given $f : \mathfrak{X} \to \mathfrak{Y}$ and $m \in \mathcal{M}(\mathfrak{Y})$ then $\sup(f^* M) = f^{-1} \sup M$.
- $\sup(M \otimes N) = \sup M \cap \sup N$.
- $\sup M$ is a closed in $\mathfrak{X}$.

- Given any $x \in \mathfrak{X}$ we define $I_x \in \mathcal{M}(\mathfrak{X})$ by
$$I_x = i^* \left(\tilde{R/\mathfrak{p}}\right)$$
Where $i : \operatorname{Spec} R \longrightarrow \mathfrak{X}$ is the inclusion map of an affine cover with $x \in \operatorname{Spec} R$ corresponding to the prime ideal $\mathfrak{p}$ and $\tilde{R/\mathfrak{p}}$ is the $\mathcal{O}_{\operatorname{Spec} R}$-module corresponding to $R/\mathcal{P}$. For all such I_x we have
$$\sup(I_x) = \bar{x}$$

PROOF. These results are all local so we can reduce to the affine case. Since the support of Spec R corresponds to the usual support on R, the proposition is just a restatement of ([4] Ex. 3.19).

DEFINITION 68. Given $M \in \mathcal{M}(\mathfrak{X})$ we define the *codimension of support* of M to be the supremum of the codimension of the points x contained in the support of M.

DEFINITION 69. For $\mathfrak{X}$ Noetherian we define $\mathcal{M}^l(\mathfrak{X})$ to be the full additive subcategory of $\mathcal{M}(\mathfrak{X})$ of sheaves whose codimension of support is at least l. This gives a filtration
$$\cdots \subset \mathcal{M}^2(\mathfrak{X}) \subset \mathcal{M}^1(\mathfrak{X}) \subset \mathcal{M}^0(\mathfrak{X}) = \mathcal{M}(\mathfrak{X})$$
called the *filtration by codimension of support* or the *filtration by Coniveux.*

PROPOSITION 70. The category $\mathcal{M}^l(\mathfrak{X})$ is exact

PROOF. Suppose we are given an exact sequence of sheaves
$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''' \longrightarrow 0$$
with both $\mathcal{F}'$ and $\mathcal{F}''$ in $\mathcal{M}^l(\mathfrak{X})$. For any point x with codimension less than l we localize to get an exact sequence
$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$
with $\mathcal{F}'_x$ and $\mathcal{F}''_x$ both vanishing. Thus $\mathcal{F}_x$ vanishes and $\mathcal{M}^l(\mathfrak{X})$ is closed under extensions.

2. The Grothendieck group $K_0\mathcal{C}$

DEFINITION 71. If $\mathcal{C}$ is an exact category, we define the *Grothendieck group* $K_0(\mathcal{C})$ to be the free group generated by the objects in $\mathcal{C}$ modulo the ideal generated by $[A'] - [A] + [A'']$ for all exact sequences
$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

This map is functorial.

PROPOSITION 72. K_0 forms a covariant functor from the category of exact categories to the category of abelian groups.

PROOF. An exact functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ of exact categories induces a natural homomorphism from $\mathbb{Z}\mathcal{C}$ to $\mathbb{Z}\mathcal{D}$. Since $\mathcal{F}$ takes exact sequences to exact sequences, this lifts to define a homomorphism $K_0(\mathcal{F}) : K_0(\mathcal{C}) \to K_0(\mathcal{D})$.
If a category has additional structure then the Grothendieck group inherits further properties. The existence of a tensor product, for example, turns $K_0(\mathcal{C})$ into a ring.

PROPOSITION 73. If $\mathfrak{X}$ is a 'heme then $K_0(\mathcal{P}(\mathfrak{X}))$ is a commutative ring. This map is also functorial.

PROOF. $\mathbb{Z}\mathcal{P}(\mathfrak{X})$, the free group generated by the objects of $\mathcal{P}(\mathfrak{X})$ forms a commutative ring with multiplication given by extending the tensor product linearly. An exact sequence of locally free $\mathcal{O}_{\mathfrak{X}}$ modules splits *locally*, so the functor $A \otimes -$ maps maps is exact. The the subgroup of elements in $\mathbb{Z}\mathcal{P}(\mathfrak{X})$ equivalent to zero in $K_0(\mathcal{P}(\mathfrak{X}))$ forms an ideal, so the quotient $K_0(\mathfrak{X})$ is a commutative ring.

3. The Classifying space of a category.

We will need to associate to each category a topological space with similar properties.
We first define a category Δ with objects $\underline{n} = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are non-decreasing functions.

DEFINITION 74. A simplicial set is a contravariant functor from Δ to the category of sets.
Recall that a n-simplex is the topological space

$$\Delta_n = \{(x_0, \ldots x_n \in \mathbb{R}^{n+1} | x_i \geq 0, \sum_i x_i = 1\}$$

Given an element of $\hom_\Delta(\underline{n}, \underline{m})$ we define a continuous map $\tilde{f} : \Delta_n \to \Delta_m$ by

$$\tilde{f}(x_0, \ldots x_n) = \left(\sum_{i \in f^{-1}(0)} x_i, \sum_{i \in f^{-1}(1)} x_i, \ldots \sum_{i \in f^{-1}(m)} x_i \right)$$

For a simplicial set $\mathcal{F}$ we regard the set $\mathcal{F}(\underline{n})$ as a discrete topological space, so $\mathcal{F}(\underline{n}) \times \Delta_n$ is the disjoint union of $|\mathcal{F}(\underline{n})|$ copies of Δ_n. We then take the disjoint union

$$X = \coprod_{n\geq 0} \mathcal{F}(\underline{n}) \times \Delta_n$$

For every element f of $\hom_\Delta(\underline{m}, \underline{n})$, every δ in $\mathcal{F}(\underline{n})$, and every $x \in \Delta_n$ we identify the point $(\delta, \tilde{f}(x))$ of X with $(\mathcal{F}(f)\delta, x)$

DEFINITION 75. We call the resulting topological space the *geometric realization* $|\mathcal{F}|$ of $\mathcal{F}$.

Note that the geometric realizations is a CW complex with the compact open topology. Note also that the construction is functorial, so that a natural transformation of simplicial sets yields a continuous map on their geometric realizations. Finally, to each small category $\mathcal{C}$ we define a simplicial sets $\mathcal{NC}$ called the *nerve of the category*. Let $\mathcal{NC}(\underline{n})$ be the set of all diagrams

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} A_n$$

and for each $f \in \hom(\underline{m}, \underline{n})$ let $\mathcal{NC}(f)$ be the map sending the above sequence to

$$A_{f(0)} \xrightarrow{\varphi_{f(0),f(1)}} A_{f(1)} \xrightarrow{\varphi_{f(1),f(2)}} \cdots \xrightarrow{\varphi_{f(n-1),f(n)}} A_{f(m)}$$

where $\varphi_{i,j} : A_i \to A_j$ is defined as $\phi_{f(j)} \circ \phi_{f(j-1)} \circ \cdots \phi_{f(i)+1}$.

DEFINITION 76. We call the geometric realization of the nerve of a small category $\mathcal{C}$ the *geometric realization* of the category, and write $B\mathcal{C}$ for $|\mathcal{NC}$.

Note that this construction is also functorial. If we are given a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ of small categories then this induces a natural transformation on the nerves which gives a continuous function $B\mathcal{F} : B\mathcal{C} \to B\mathcal{D}$.

The properties of this space depend closely on the base space. $|\mathcal{C}|$ is an infinite dimensional CW complex with zero simplicies given by the objects of $\mathcal{C}$. The one simplicies are morphisms $\phi : A \to B$ with end points A and B. The two simplicies are diagrams $A \xrightarrow{f} B \xrightarrow{g} C$ with boundaries $B \xrightarrow{g} C$, $A \xrightarrow{g\circ f} C$ and $A \xrightarrow{f} B$. We can even compute the homology.

We define a homomorphism $d_n : \mathbb{Z}\mathcal{NC}(\underline{n}) \to \mathbb{Z}\mathcal{NC}(\underline{n-1})$ by

$$d_n\left[A_0 \to A_1 \to \cdots A_n\right] = \sum_{i=0}^{n} \left[A_0 \to A_1 \to \cdots \tilde{A}_i \cdots A_n\right]$$

Where $\tilde{A}_i$ indicates that this object is deleted. This gives us a complex

$$\cdots \xrightarrow{d_{n+1}} \mathbb{Z}\mathcal{NC}(\underline{n}) \xrightarrow{d_n} \mathbb{Z}\mathcal{NC}(\underline{n-1}) \xrightarrow{d_{n-1}} \cdots \mathbb{Z}\mathcal{NC}(\underline{0}) \to 0$$

PROPOSITION 77. The cohomology of the above complex is $H^1(|\mathcal{C}|, \mathbb{Z})$.

PROOF. Clack through the constructions or see [46] and [36].

One is tempted to study the geometric realizations of exact categories, but unfortunately these spaces contain little information. Every exact category contains a zero object which is both initial[2] and final.

PROPOSITION 78. If $\mathcal{C}$ has either an initial or a final object then $H^n(|\mathcal{C}|, \mathbb{Z})$ vanishes for $n > 0$.

PROOF. We represent a cocycle in $\mathbb{Z}\mathcal{N}\mathcal{C}(\underline{n})$ by the sum

$$x = \sum_i n_i \left[A_0^i \longrightarrow A_1^i \longrightarrow \cdots A_n^i\right]$$

and assume the category has an initial object I. We compute

$$d_{n+1}\left(\sum_i n_i \left[0 \longrightarrow A_0^i \longrightarrow \cdots A_n^i\right]\right) = x + \sum_i \sum_{j=1}^{n} n_i \left[0 \longrightarrow A_0^i \longrightarrow \cdots \hat{A}_j^i \longrightarrow \cdots A_n^i\right]$$

This second term vanishes as

$$0 = d_n x = \sum_i \sum_{j=1}^{n} n_i \left[A_0^i \longrightarrow \cdots \hat{A}_j^i \longrightarrow \cdots A_n^i\right]$$

We can even compute the homotopy groups.

Note that the fundamental group of a simplicial complex depends only on the zero, one, and two simplicies. We fix a base element 0 in a small category $\mathcal{C}$. Loops in $\pi_1(|\mathcal{C}|, 0)$ correspond to diagrams

$$A_0 = 0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots 0 = A_n$$

where $a_i : A_i \longrightarrow A_{i+1}$ is either a member of $\mathcal{C}$ or the opposite algebra $\mathcal{C}^o$. The composition of two loops are given by the concatation of the two diagrams, and the reverse loop is given by the dual of the diagram. The contractable loops are given by diagrams of the form

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{(j \circ i)^o} A$$

Where $(j \circ i)^o$ is the dual of $j \circ i$

4. Quillen Categories

We noted in the previous section that the geometric realization of an exact category is contractable and thus is not a useful invariant. To alleviate this problem, we need to develop a functor from $\mathfrak{Exto}$ $\mathfrak{Cat}$that will 'fiber' the category

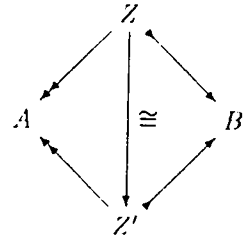
[2]An object I is *initial* if $\hom(I, A)$ contains precisely one morphism for all A. An object is *final* if $\hom(A, I)$ contains precisely one morphism for all A.

so that the resulting geometric realizations are non-trivial and take into account the exact structure. The simplest and most useful is Quillen's original Q-functor.

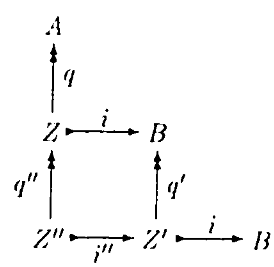
DEFINITION 79. We define a functor $Q : \mathfrak{Ex} \longrightarrow \mathfrak{Cat}$ as follows. If we fix an exact category $\mathcal{C}$, we let the objects of $Q\mathcal{C}$ be the objects of $\mathcal{C}$. Given $A, B \in \mathcal{C}$, we define $\hom_{Q\mathcal{C}}(A, B)$ to be the set of triplets (q, i, Z) where

- Z is an object in $\mathcal{C}$
- q is an admissible epimorphism in $\hom_{\mathcal{C}}(Z, A)$
- q is an admissible monomorphism in $\hom_{\mathcal{C}}(Z, B)$

We identify (q, i, Z) with (q', i', Z') if Z is isomorphic to Z' and we have a commuting diagram:



Given morphisms $(q, i, Z) \in \hom_{Q\mathcal{C}}(A, B)$ and $(q', i', Z') \in \hom_{Q\mathcal{C}}(B, C)$, we define the composition of the morphisms by the triplet $(q \circ q'', i' \circ i'', Z'')$ if Z'' is given by the following push-out diagram



If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is an exact functor of exact categories, we can define $Q\mathcal{F}(A) = \mathcal{F}(A)$ for all objects in $\mathcal{C}$. Given a morphism $(q, i, Z) \in \hom_{Q\mathcal{C}}(A, B)$, we define[3]. $\mathcal{F}(q, i, Z) = (\mathcal{F}q, \mathcal{F}i, \mathcal{F}Z)$.

The Q construction enjoys several nice properties. Any admissible monomorphism $i : A \rightarrowtail B$ induces an element of $\hom_{Q\mathcal{C}}(A, B)$ given by the triplet $(1 : A \rightarrowtail A, A, i : A \rightarrowtail B)$ which we will denote by $i_!$. Dually, for any admissible epimorphism $q : A \twoheadrightarrow B$ we construct the element $q^! \in \hom_{Q\mathcal{C}}(B, A)$ given by $(q : A \rightarrowtail B, A, 1 : A \rightarrowtail A)$.

[3]This is well defined as an exact functor preserves exact sequences, and thus preserves admissibility

Proposition 80. QC has the following properties

(1) Any morphism in QC is of the form $i_! \circ q^!$ for some admissible morphisms i and j which are unique up to a unique isomorphism.

(2) If i and j are admissible monomorphisms that can be composed then $(i \circ j)_! = i_! \circ j_!$.

(3) If p and q are admissible epimorphisms that can be composed then $(i \circ j)^! = j^! \circ I^!$.

(4) If $1 : A \to A$ is the the identity map then $1^! = 1_!$ is the identity map in $\hom(A, A)$.

(5) If we have a bicartesian square

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\ j\ ]{} & A' \end{array}$$

with i, j admissible monomorphisms and p, q admissible epimorphisms then $i_! \circ p^! = q^! \circ j_!$.

Proof. Any morphism $(q : Z \twoheadrightarrow A, Z, i : Z \rightarrowtail B)$ in $\hom_{QC}(A, B)$ is clearly equal to $i_! \circ q^!$. The assertions all follow by direct computation.

Proposition 81. Suppose we have an exact category C equipped with a category C' such that $\mathrm{Obj}(C') = \mathrm{Obj}(C)$. If we can associate to each admissible map in C the morphisms $*^!$ and $*_!$ as above such that they satisfy proposition 80 then there is a unique functor from C' to QC compatible with these constructions.

Proof. See [36] section 2.2.

5. Quotient Categories

For any integral domain R with field of fractions K we let $\mathcal{M}_{\text{tors}}$ be the full subcategory of $\mathcal{M}(R)$ of torsion modules[4] and let $\mathcal{M}_{\text{free}}$ be the subcategory of of free R modules. The tensor product $K \otimes_R -$ induces a functor from $\mathcal{M}(R)$ to $\mathcal{M}(K)$ which, since $\mathcal{M}(K)$ is equivalent to $\mathcal{M}_{\text{free}}(R)$, gives us a functor

$$T : \mathcal{M}(R) \to \mathcal{M}_{\text{free}}(R)$$

. Note that T is just the identity when restricted to $\mathcal{M}_{\text{free}}(R)$ and that $\mathcal{M}_{\text{tors}}(R)$ is the subcategory killed by T. Thus, in some sense, $\mathcal{M}_{\text{free}}(R)$ is the 'quotient' of $\mathcal{M}(R)$ by $\mathcal{M}_{\text{tors}}(R)$.

We will need to make this notion more explicit.

[4]A module in *torsion* if $rM = 0$ for some nonzero r in R.

DEFINITION 82. We call a subcategory $\mathcal{C}$ of an abelian category $\mathcal{D}$ a *Serre* subcategory if it satisfies the following.

(1) $\mathcal{C}$ is a full subcategory of $\mathcal{D}$ such that $\mathcal{C}$ inherits the structure of an abelian category.
(2) All the zero objects of $\mathcal{D}$ are in $\mathcal{C}$.
(3) Given any exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathcal{D}$ then $A \in \mathcal{C}$ if and only if both A' and A'' are in $\mathcal{C}$.

DEFINITION 83. If $\mathcal{C}$ is a Serre subcategory of $\mathcal{D}$ we define the ***quotient subcategory*** $\mathcal{D}/\mathcal{C}$ to be the category with $\operatorname{Obj}\mathcal{D}/\mathcal{C} = \operatorname{Obj}\mathcal{C}$ with $\hom_{\mathcal{D}/\mathcal{C}}(A, B)$ given by the direct limit of the system of groups $\hom_{\mathcal{D}}(K_k, C_k)$ under the natural maps

$$\hom_{\mathcal{D}}(K_k, C_k) \longrightarrow \hom_{\mathcal{D}}(K_j, C_j),$$

where $i_k : K_k \longrightarrow A_k$ is a monomorphism with cokernel in $\mathcal{C}$, $q_k : B_k \longrightarrow C_k$ is an epimorphism with kernel in $\mathcal{C}$, K_j is a subobject of K_i and C_j is a quotient object of C_i. We let $T : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ be the natural functor.
The basic result is

PROPOSITION 84. If $\mathcal{C}$ is a Serre subcategory of $\mathcal{D}$ then $\mathcal{D}/\mathcal{C}$ is an abelian category such that T is an exact functor with $Tx = 0$ if and only if $x \in \mathcal{C}$.

PROOF. ([46] B.7)

PROPOSITION 85. $\mathcal{M}_{\mathrm{tors}}(R)$ is a Serre subcategory of $\mathcal{M}(R)$ with $\mathcal{M}_{\mathrm{free}}(R)$ naturally equivalent to $\mathcal{M}(R)/\mathcal{M}_{\mathrm{tors}}(R)$.

PROOF. We let T be the functor to the quotient as above. For any M in $\mathcal{M}(R)$ we choose n such that $K \otimes_R M \cong K^K$. If we choose a basis then this lifts to a morphism $f : R^n \longrightarrow M$ such that both the kernel and the cokernel are torsion elements. Thus $Tf : T(R^n) \longrightarrow T(M)$ and we can define $S : \mathcal{M}(R) \longrightarrow \mathcal{M}_{\mathrm{free}}(R)$ by $S(M) = R^n$. Thus we have $T : \mathcal{M}_{\mathrm{free}}(R) \longrightarrow \mathcal{M}(R)/\mathcal{M}_{\mathrm{free}}(R)$ such that $S \circ T$ is the identity and $T \circ S$ is isomorphism to the identity. The result follows from the computation

$$\hom_{\mathcal{M}_{\mathrm{free}}(R)}(M, N) \cong \hom_{\frac{\mathcal{M}(R)}{\mathcal{M}_{\mathrm{tors}}(R)}}(TM, TN)$$

We can generalize this to any localization.

PROPOSITION 86. If C is a prime in R we let $\mathcal{M}_{\mathfrak{p}}(R)$ be the full subcategory of $\mathcal{M}(R)$ such that $xM = 0$ for some $x \notin \mathfrak{p}$. $\mathcal{M}_{\mathfrak{p}}(R)$ is a Serre subcategory of $\mathcal{M}(R)$ and $\mathcal{M}(R)/\mathcal{M}_{\mathfrak{p}}(R)$ is naturally equivalent to $\mathcal{M}(R_{\mathfrak{p}})$.

PROOF. ([46] B.7).
Similarly,

PROPOSITION 87. Let $\mathfrak{X}$ be a Noetherian scheme with a closed subscheme $\mathfrak{Z}$. and let $\mathcal{M}_{\mathfrak{Z}}(\mathfrak{X})$ be the full subcategory of sheaves supported on $\mathfrak{X}$. $\mathcal{M}(\mathfrak{X} - \mathfrak{Z})$ is naturally equivalent to $\mathcal{M}(\mathfrak{X})/\mathcal{M}_{\mathfrak{Z}}(\mathfrak{X})$.

PROOF. ([46] B.8).
We will be concerning ourselves chiefly with the following quotient.

PROPOSITION 88. If $\mathfrak{X}$ is a Noetherian scheme then we have an equivalence of categories between

$$\frac{\mathcal{M}^p(\mathfrak{X})}{\mathcal{M}^{p+1}(\mathfrak{X})} \cong \bigoplus_{x \in \mathfrak{X}^p} \mathcal{M}_{\ell}(\mathcal{O}_{\mathfrak{X},x}).$$

where $\mathcal{M}^p(\mathfrak{X})$ the category of codimension p modules and $\mathcal{M}_{\ell}(R)$ is the category of finite length modules over a local ring.

PROOF. We claim that if $M \in \mathcal{M}^p(\mathfrak{X})$ then the set of codimension p points in the support of M is finite. If we set $M_0 = M$ we define a sequence of modules M_i inductively by choosing (if possible) a codimension p element x_i in the support of M_i and defining M_{i+1} via the sequence

$$0 \to M_i \otimes_{\mathcal{O}_{\mathfrak{X}}} I_{x_i} \to M_i \to M_{i+1} \to 0$$

where $I_x \in \mathcal{M}(\mathfrak{X})$ is as in proposition 67. Locallizing, we see that

$$\sup(M_{i+1}) \cap \mathfrak{X}^p = \sup(M_i) cap \mathfrak{X}^p - \{x\}.$$

Thus we have constructed a descending chain of nested closed subschemes of a Noetherian space, so we know the chain terminates and

$$\sup M \cap \mathfrak{X}^p = \bigcup_{i=1}^{n} \{x_i\}$$

which proves our claim.
Note that if M is a coherent $\mathcal{O}_{\mathfrak{X}}$-module then the localization M_x is a $\mathcal{O}_{\mathfrak{X},x}$-module, so we can define a functor

$$s : \mathcal{M}^p(\mathfrak{X}) \to \bigoplus_{x \in \mathfrak{X}^p} \mathcal{M}_{\ell}(\mathcal{O}_{\mathfrak{X},x})$$

by $S(M) = \oplus_{x \in \mathfrak{X}^p} M_x$. Note that this is an exact functor that kills $\mathcal{M}^{p+1}(\mathfrak{X})$, so it lifts to give

$$S : \frac{\mathcal{M}^p(\mathfrak{X})}{\mathcal{M}^{p+1}(\mathfrak{X})} \to \bigoplus_{x \in \mathfrak{X}^p} \mathcal{M}_\ell(\mathcal{O}_{\mathfrak{X},x})$$

which we can show is an equivalence.

6. Higher K-groups

Combining the geometric realization and the Quillen functor we get a map that takes an exact category $\mathcal{C}$ to the topological space $BQ\mathcal{C}$. We let M be the point in $BQ\mathcal{C}$ represented by the object M and let 0 be the zero object.

DEFINITION 89. We define the *higher K-groups* of a small exact category $\mathcal{C}$ by

$$K_i(\mathcal{C}) = \pi(BQ\mathcal{C}, 0)$$

PROPOSITION 90. The higher K-group $K_0(\mathcal{C})$ is equal to the Grothendieck group $K_0(\mathcal{C})$.

PROOF. (sketch)
For every object A in $Q\mathcal{C}$ we have the canonical monomorphism $i^A : 0 \to A$ and the canonical epimorphism $q_A : A \to 0$. Together, these maps give us morphisms $i_!^A, j_A^!$ in $\hom_{Q\mathcal{C}}(0, A)$, which give us paths from the point 0 to the point A in $|Q\mathcal{C}$. Thus we can associate to A the closed loop $l(A) = (j_A^!)^{-1} \circ i_!^A$, where $(j_A^!)^{-1}$ denotes the inverse path of $j_A^!$ from A to 0. This gives us a map from $\mathbb{Z}\mathcal{C}$ to $\pi_1(|Q\mathcal{C}, 0)$.
Since every morphism in $Q\mathcal{C}$ is of the form $q^! \circ i_!$, we can factor the diagrams in example so that the fundamental group is given entirely by morphisms of this form. We can then use the formal properties in proposition 80 to show that every diagram is constructed from loops of the for $l(A)$, so the homomorphism is surjective. If we are given a short exact sequence $A' \to A \to A''$ then $l(A') \circ l(A'') \circ l(A)^{-1}$ turns out to be the boundary of a collection of two two-simplicies. Moreover, if we are given any two-simplex we can show that the boundary is the composition of loops of this form. Thus the kernel of the map from $\mathbb{Z}\mathcal{C}$ to $\pi_1(|Q\mathcal{C}, 0)$ is generated by elements of the form $[A] = [A'] + [A'']$ for each exact sequence above, so the map lifts to give the desired isomorphism.
This method is computational, but can be done using the universal properties of $Q\mathcal{C}$. The original method is slicker, but requires developing some covering space theory for $|Q\mathcal{C}|$. See [36] for details of this argument.

7. Properties of K-groups

We will state several useful properties of the higher K groups omitting all the proofs. There are many useful surveys of algebraic K-theory, including [14] and [15]. The best starting points are Quillen's original paper [36] or [46].

Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an exact functor of exact categories. This induces a functor $Q\mathcal{F} : Q\mathcal{C} \to Q\mathcal{D}$, which in turn lifts to give a continuous map $BQ\mathcal{F} : BQ\mathcal{C} \to BQ\mathcal{D}$. This map induces a map on the homotopy groups $\pi_i(BQ\mathcal{F}) : \pi_i(BQ\mathcal{C}, 0) \to \pi_1(BQ\mathcal{D}, 0)$, which we write as $\mathcal{F}_* : K_i(\mathcal{C}) \to K_i(\mathcal{D})$. So the K-groups are covariant functors.

We now recall the category $\mathfrak{Ex}(\mathcal{C}, \mathcal{D})$ of exact functors from $\mathcal{C}$ to $\mathcal{D}$. Each element $\mathcal{F}$ of $\mathfrak{Ex}(\mathcal{C}, \mathcal{D})$ now gives us a group homomorphism $\mathcal{F}_* : K_i(\mathcal{C}) \to K_i(\mathcal{D})$, so we can construct a pairing

$$(1) \qquad \tau : \mathbb{Z}\mathfrak{Ex}(\mathcal{C}, \mathcal{D}) \otimes K_i(\mathcal{C}) \to K_i(\mathcal{D})$$

Where $\tau(\mathcal{F}, x) = \mathcal{F}_*(x)$.

PROPOSITION 91. If we are given an exact sequence[5] of exact functors

$$(2) \qquad 0 \to \mathcal{F}' \xrightarrow{t} \mathcal{F} \xrightarrow{t'} \mathcal{F}'' \to 0$$

then the homomorphism $\mathcal{F}_* - \mathcal{F}'_* - \mathcal{F}''_* : K_i(\mathcal{C}) \to K_i(\mathcal{D})$ is the zero map. Thus Equation 1 lifts to a homomorphism

$$\tau_* : K_0(\mathfrak{Ex}(\mathcal{C}, \mathcal{D})) \otimes K_i(\mathcal{C}) \to K_i(\mathcal{D})$$

PROOF. [36]

PROPOSITION 92. If we have an admissible long exact sequence

$$0 \to \mathcal{F}_0 \xrightarrow{f_0} \mathcal{F}_1 \xrightarrow{f_1} \cdots \mathcal{F}_n \to 0$$

between exact categories $\mathcal{C}$ and $\mathcal{D}$ then

$$\sum_{k=0}^{n} (-1)^k (\mathcal{F}_k)_* : K_i(\mathcal{C}) \to K_i(\mathcal{D})$$

[5]We say that sequence 2 is *exact* if t and t' are natural transformations of functors such that for all objects M the sequence

$$0 \to \mathcal{F}'(M) \xrightarrow{t(M)} \mathcal{F}(M) \xrightarrow{t'(M)} \mathcal{F}''(M) \to 0$$

is exact

is the zero homomorphism.

PROOF. We say the sequence is *admissible* if for each natural transformation f_k we have an exact sequence of exact functors

$$0 \to (\ker f_k) \xrightarrow{i_k} \mathcal{F}_k \xrightarrow{f_k} \mathcal{F}_{k+1}.$$

The result is immediate if we apply proposition 91 to the short exact sequences

$$0 \to (\ker f_k) \xrightarrow{i_k} \mathcal{F}_k \xrightarrow{f_k} (\ker f_{k+1}) \to 0$$

DEFINITION 93. Let $\mathcal{C}$ be a full additive subcategory of an exact category $\mathcal{D}$ such that $\mathcal{D}$ induces the structure of an exact category on $\mathcal{C}$ [6]. Suppose also that if we have an exact sequence in $\mathcal{D}$

$$0 \to M \to C' \to C'' \to 0$$

such that C' and C'' are objects in $\mathcal{C}$ then M is also an object in $\mathcal{C}$ and that every object M in $\mathcal{D}$ has a finite resolution

$$0 \to C_n \to C_{n-1} \to \cdots C_0 \to M \to 0$$

with C_k in $\mathcal{C}$. We call $\mathcal{C}$ a *resolution subcategory* of $\mathcal{D}$.

EXAMPLE 94. If R is a Noetherian ring then every finitely generated module has a projective resolution if and only if R is a regular ring. Thus for Noetherian rings the notion of regularity is equivalent to having $\mathcal{P}(R)$ a resolution subcategory of $\mathcal{M}(R)$.

EXAMPLE 95. Similarly, $\mathcal{P}(\mathfrak{X})$ is a resolution subcategory of $\mathcal{D}$ if $\mathfrak{X}$ is regular and Noetherian.

THEOREM 96 (RESOLUTION). If $\mathcal{C}$ is a resolution subcategory of $\mathcal{D}$ then $K_i(\mathcal{C}) = K_i(\mathcal{D})$.

PROOF. See([46] Thm 6.11)

An immediate consequence of this explores the relations between the cohomology of a category and its K-groups.

THEOREM 97. Let H^* be any cohomology functor on an exact category $\mathcal{C}$ such that every objects in $\mathcal{C}$ has a finite cohomological dimension and every element is the image of an admissible epimorphism from an acyclic element. The full subcategory $\mathcal{A}$ of acyclic elements is a resolution subcategory of $\mathcal{C}$ and thus $K_i(\mathcal{A}) = K_i(\mathcal{C})$.

[6] To be precise, we require that $\mathcal{C}$ is closed under extensions in the sense of definition 62.

PROOF. See ([46] Theorem 4.7).

PROPOSITION 98. If $\mathcal{C}_i$ is an exact category then

$$K_i\left(\bigoplus_i \mathcal{C}_i\right) = \bigoplus_i K_i(\mathcal{C}_i)$$

PROOF. ([46] lemma 5.9).
Lastly, we will need the following.

DEFINITION 99. We call a full abelian subcategory $\mathcal{C}$ of an abelian category $\mathcal{D}$ *devissage* if $\mathcal{C}$ is close under subobjects, quotient objects and direct products. We also require that every element M of $\mathcal{D}$ has a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots M_n = M$$

such that M_k/M_{k-1} is an object in $\mathcal{C}$.

THEOREM 100 (DEVISSAGE). If $\mathcal{C}$ is a devissage subcategory of $\mathcal{D}$ then $K_i(\mathcal{C}) = K_i(\mathcal{D})$.

PROOF. See ([46] Theorem 4.8).

8. K-theory of rings

For the convenience of the reader we will summarize some of what is known about the K-theory of rings and fields. A recent survey of the subject can be found in [14].

DEFINITION 101. For a ring R let $K_i(R) = K_i(\mathcal{P}(R))$ and $K_i'(R) = K_i(\mathcal{M}(R))$.

PROPOSITION 102. $K_i(R) = K_i'(R)$ when R is a regular Noetherian ring.

PROOF. Combine theorem 97 and example 94.
The classical[7] K-theory of rings was developed as an analog of topological K-theory and involved explicitly defining $K_1(R)$ as the quotient of the infinite general linear group $GL(R)$ by its commutator subgroup $[GL(R), GL(R)]$ and setting $K_2(R)$ to be the homology group $H_2([GL(R), GL(R)], \mathbb{Z})$. In the early seventies several different definitions of the higher K-groups were developed for

[7] \begin{irony} We define the *classical era* to be before the 1973 publication of [36]. \end{irony}

rings extending the above definitions, such as Quillen's 'plus' construction[8]. Unfortunately, although the Q construction is defined in the most general setting possible, it does not lend itself for explicit computations.

PROPOSITION 103. If R is a domain then we have natural decompositions $K_0(R) = \mathbb{Z} \oplus \tilde{K}_0(R)$ and $K_0'(R) = \mathbb{Z} \oplus \tilde{K}_0'(R)$.

PROOF. Every R-module M has a length given by $\ell(M) = \dim_{R_0} M_{(0)}$. The length is additive on exact sequences, so it lifts naturally to a map $\ell : K_0(R) \to \mathbb{Z}$ or $\ell : K_0'(R) \to \mathbb{Z}$ such that $\ell(R^n)$ is n.
$\tilde{K}_0(R)$ measures to what extent projective modules fail to be free.

PROPOSITION 104. For any Dedekind domain R (ie. a one dimensional Noetherian domain) $\tilde{K}_0(R)$ is the ideal class group of R.

PROOF. See [31].
The basis theorem of basic linear algebra states that all projective modules over a field are free, and one can also show that this also holds for local rings. Thus

PROPOSITION 105. $K_0(R) = \mathbb{Z}$ for all local rings R.
For most other rings the Grothendieck group is difficult to compute. Using the classical definition and some computation we can show

PROPOSITION 106. $K_1(R) = R^*$ for all local rings R

PROPOSITION 107. We can construct a product $* : K_i(R) \otimes_{\mathbb{Z}} K_j(R) \to K_{i+j}(R)$ such that $x * y = (-1)^{ij} y * x$. This product makes $K_*(R)$ into a graded skew ring.

PROOF. See ([46] section 2) or ([31]). The proof involves constructing a homomorphism $GL(R) \otimes GL(R) \to GL(R)$ which induces a continuous map on the 'plus' spaces as used in the plus construction. The product is then induced by the smash product of spheres.

[8] The +-construction involves constructing for any ring R the Eilenberg-Maclane space $BGL(R)$ of the infinite general linear group $GL(R)$. By definition this is a connected space with trivial higher homotopy groups with $\pi_1(BGL(R), 1) = GL(R)$ and can be constructed by taking the classifying space of the one point category with $GL(R)$ being the group of morphisms. It can be shown that by adding a two-cell and a three-cell to $BGL(R)$ we get a space $BGL(R)^+$ such that

$$\pi_1(BGL(R)^+, 1) = \frac{GL(R)}{[GL(R), GL(R)]}$$

and $\pi_2(BGL(R)^+, 1) = H_2([GL(R), GL(R)], \mathbb{Z})$. Quillen defined $K_i(R) = \pi_i(BGL(R)^+, 1)$ for $i > 0$.

For a local ring R the isomorphism $K_1(R) = R^*$ and the product induces a map $R^* \otimes R^* \to K_2(R)$. The image and kernel of this map was first computed by Matsumoto in his thesis.

Proposition 108 (Matsumoto). If R is a field or a local ring with more than twelve elements then the map $R^* \otimes R^* \to K_2(R)$ is surjective with kernel generated by the elements $a \otimes (1-a)$ with $a \in R^*$ and $a \neq 1$.

Proof. See [31].

Definition 109. If K is a field we define the *Milnor K-groups* $K_i^M(K)$ to be the i'th graded component of the tensor algebra of R^* modulo the *Steinberg* relation $a \otimes (1-a) = 0$. We define a map $m : K_k^M \to K_k$ to be the map induced by the product on K_i.

Clearly $K_i(K) = K_i^M(K)$ if $i \leq 2$. Example 1.6 and Cor 16.2 in [14] shows that m is, in general, neither injective nor surjective.

To conclude this section, we will list some computations of some higher K-groups.

Proposition 110. If K is a finite field on n elements then

$$K_i(K) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \text{ is even} \\ \dfrac{\mathbb{Z}}{\left(n^{\frac{i+1}{2}} - 1\right)\mathbb{Z}} & \text{is odd} \end{cases}$$

Proof. See [32].

Proposition 111.

$$K_i(\mathbb{Q}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Q}^* & i = 1 \\ \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \bigoplus_{p \text{ prime}} \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^* & i = 2 \\ \frac{\mathbb{Z}}{48\mathbb{Z}} & i = 3 \end{cases}$$

Moreover, for $i > 3$ the rank of $K_i(\mathbb{Q})$ is one when $i \equiv 1 \mod 4$ and is zero otherwise.

Proof. See [42] and [14].

There is a wonderful conjecture for the values of $K_i(\mathbb{Z})$ in [15] which has been verified for $i \leq 4$. See [23] and [40]. The higher K-groups for $\mathbb{R}$ and $\mathbb{C}$ are also known, and are computed in [14].

9. K-theory of schemes

DEFINITION 112. If $\mathfrak{X}$ is a scheme we define $K_i(\mathfrak{X}) = K_i(\mathcal{P}(\mathfrak{X}))$ and $K'_i(\mathfrak{X}) = K_i(\mathcal{M}(\mathfrak{X}))$.
These two groups are equal if $\mathfrak{X}$ is Noetherian and regular by theorem 97 and example 95. Note that we have natural equivalences of categories between $\mathcal{M}(R)$ and $\mathcal{M}(\mathrm{Spec}\ R)$, and $\mathcal{P}(R)$ and $\mathcal{P}(\mathrm{Spec}\ R)$, so $K_i(R) = K_i(\mathrm{Spec}\ R)$ and $K'_i(R) = K'_i(\mathrm{Spec}\ R)$.

PROPOSITION 113. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of schemes we have a natural functor $f^* : K_i(\mathfrak{Y}) \to K_i(\mathfrak{X})$. Moreover, if f is flat, we can also define $f^* : K'_i(\mathfrak{Y}) \to K'_i(\mathfrak{X})$.

PROOF. See [46] or [31]. The functors are just the lifting of the usual $f^* : \mathcal{M}(\mathfrak{Y}) \to \mathcal{M}(\mathfrak{X})$ and $f^* : \mathcal{P}(\mathfrak{Y}) \to \mathcal{P}(\mathfrak{X})$.
If the morphism f is sufficiently nice we can construct a covariant functors $f_* : K_i(\mathfrak{X}) \to K_i(\mathfrak{Y})$ and $f_* . K'_i(\mathfrak{X}) \to K'_i(\mathfrak{Y})$ from lifting the push-forward of $\mathcal{O}_{\mathfrak{X}}$ modules. One can also show that the usual projection formula lifts to operations on the K-groups. See ([46] proposition 5.12) for details.

10. Torsion in $K_2(K)$

Recall that if we are given a field K then both $K_0(K)$ and $K_1(K)$ are trivial to compute. The higher K-groups, however, are far more difficult to compute. Even $K_2(K)$, although given by an explicit set of generators and relations, is still highly nontrivial. To demonstrate this complexity we will develop some technical tools that we will need later concerning the torsion of $K_2(K)$.

DEFINITION 114. If p is a prime different to the characteristic of K, we define ${}_pK_2(K) = K_2(K)/pK_2(K)$.
We will also write $\tilde{K}$ for K^*/pK^*.

PROPOSITION 115. Let G be the subgroup of $\tilde{K} \otimes \tilde{K}$ generated by elements of the form $x \otimes y$ where $y = N_{K(\sqrt[p]{x})/K}\, a$ for some $a \in K(\sqrt[p]{x})$. Then

$${}_pK_2(K) \cong \frac{\tilde{K} \otimes \tilde{K}}{G}$$

PROOF. The natural projection map $K^* \otimes K^* \to \tilde{K} \otimes \tilde{K}$ lefts to give a surjection

$$\pi : K_2(K) \to \frac{\tilde{K} \otimes \tilde{K}}{G}$$

as

$$(x, 1-x) = (x, N_{K(\sqrt[p]{x})/K}(1 - \sqrt[p]{x})).$$

The kernel of π is generated by elements $(x, Na) \in K_2(K)$ with $a \in \sqrt[p]{x}$. We compute

$$(x, Na) = \mathrm{cor}_{K(\sqrt[p]{x})/K}(x, a) = p\,\mathrm{cor}(\sqrt[p]{x}, a) \in pK_2(L)$$

As the kernel of π is precisely $pK_2(L)$, the result follows.

We will require some notation. We will write $\vec{t}$ for the set $(t_1, \ldots, t_m)$ and write $\vec{t}^{\vec{s}}$ for the product $t_1^{s_1} t_2^{s_2} \cdots t_m^{s_m}$. We need to develop some method to determine if an element of $2k2$ vanishes.

PROPOSITION 116. Suppose we have elements $x_i, y_i \in K^*$ such that

$$\sum_{i=1}^{n} \{x_i, y_i\} = 0 \in {}_pK_2(K)$$

with the set x_i forming a linearly independent set[9] in $\tilde{K}$, where $\tilde{K}$ is regarded as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Then we can choose an integers l and m, elements $x_i \in K^*$ $(n < i \leq m)$, a collection $\vec{r}_i \in \mathbb{Z}^m$ $(1 \leq i \leq l)$, and elements $r_i \in K(\sqrt[p]{\vec{x}^{\vec{r}_i}})$ $(1 \leq i \leq l)$ such that (if we set $y_{n+1} = y_{n+2} \cdots y_m = 1$)

$$y_j^{-1} \prod_{i=1}^{l} N_{K(\sqrt[p]{\vec{x}^{\vec{r}_i}})/K}\, r_i^{(\vec{r}_i)_j} \tag{3}$$

is a p'th power in K for all j with $1 \leq j \leq m$.

PROOF. If we let π be the map in proposition 115, we can choose q_i and r_i such that

$$0 = \pi\left(\sum_{i=1}^{n} \{x_i, y_i\}\right) = \sum_{i=1}^{n} x_i \cdot y_i = \sum_{i=1}^{l} q_i \cdot N_{K(\sqrt[p]{\vec{x}^{\vec{r}_i}})/K}\, r_i.$$

The set $\{x_1, \ldots x_n, q_1 \ldots q_l\}$ span some subspace in $\tilde{K}$ of rank m, so we choose a basis $\{x_1, \ldots x_m\}$ of this space. We now use this basis to select $\vec{r}_i$ such that $q_i = \vec{x}^{\vec{r}_i}$. We now compute

[9] Since the symbols are linear, this condition can always be satisfied.

$$\sum_{i=1}^{l} q_i \odot N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i$$
$$= \sum_{i=1}^{l} \vec{x}^{\vec{v}_i} \odot N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i$$
$$= \sum_{i=1}^{l} x_1^{(v_i)_1} \cdots x_m^{(v_i)_m} \odot N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i$$
$$= \sum_{i=1}^{l} \sum_{j=1}^{m} x_j \odot N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} (r_i)^{(v_i)_j}$$
$$= \sum_{j=1}^{m} x_j \odot \left(\prod_{i=1}^{l} N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i^{(v_i)_j}\right)$$
$$= \sum_{j=1}^{m} x_j \odot y_j$$

Since the x_j form a basis of $\tilde{K}$, we have that for all j

$$y_j = \prod_{i=1}^{l} N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i^{(v_i)_j}$$

as elements of $\tilde{K}$. Hence they must differ only by a p'th power.

PROPOSITION 117. The converse of proposition 116 also holds: Given elements $\vec{v}_i$, r_i, x_i, and y_i as above and if equation 3 holds then $\sum_{i=1}^{n} \{x_i, y_i\}$ vanishes in ${}_pK_2(K)$.

PROOF. We compute

$$\pi\left(\sum_{j=1}^{n} \{x_j, y_j\}\right)$$
$$= \sum_{j=1}^{m} x_j \odot y_j$$
$$= \sum_{j=1}^{m} x_j \odot \prod_{i=1}^{l} N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i^{(v_i)_j}$$
$$= \sum_{j=1}^{m} \sum_{i=1}^{l} x_j^{(v_i)_j} \odot N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i$$
$$= \sum_{i=1}^{l} \vec{x}^{\vec{v}_i} \odot N_{K(\sqrt[p]{\vec{x}^{\vec{v}_i}})/K} r_i$$
$$= 0$$

We now come to an overly complicated corollary.

COROLLARY 118. Let $L = K(\sqrt[p]{a})$. Every element of $\ker(\mathrm{res} : {}_pK_2(K) \to {}_pK_2(L))$ is of the form

$$\sum_{j=1}^{n} \{x_j, y_j\}$$

with x_i linearly independent over K^*/pK^* such that we can choose

(1) Positive integers l, m
(2) $v_{ij} \in \mathbb{Z}$ for $1 \leq i \leq l$ and $1 \leq j \leq m$.
(3) $x_{ij} \in K$ for $n < i \leq m$ and $0 \leq j < p$.
(4) $r_{ijk} \in K$ for $1 \leq i \leq m$, $0 \leq j < p$, and $0 \leq k < p$.
(5) $z_{ij} \in K$ for $1 \leq i \leq m$ and $0 \leq j < p$.

such that if we set

$$(4) \qquad \tau_i = \sqrt[p]{\prod_{j=1}^{n} x_j^{v_{ij}} \prod_{j=n+1}^{m} \left(\sum_{k=0}^{p-1} x_{jk} \left(\sqrt[p]{a} \right)^k \right)^{v_{ij}}} .$$

then

$$(5) \quad y_j^{-1} \prod_{i=1}^{l} N_{K(\tau_i, \sqrt[p]{a})/K(\sqrt[p]{a})} \left(\sum_{f,g=0}^{p-1} r_{ifg} \tau_i^f \left(\sqrt[p]{a} \right)^g \right)^{t_{ij}} = \left(\sum_{i=0}^{p-1} z_{ji} \left(\sqrt[p]{a} \right)^i \right)^p$$

PROOF. This is clear if we substitute into the previous proposition the following:

$$\begin{aligned} x_i &= \textstyle\sum_{j=0}^{p-1} x_{ij} \left(\sqrt[p]{a} \right)^j \\ \vec{v}_i &= (v_{i1}, \ldots, v_{im}) \\ r_i &= \textstyle\sum_{j,k=0}^{p-1} r_{ijk} \left(\sqrt[p]{\vec{x}^{\vec{v}_i}} \right)^j \left(\sqrt[p]{a} \right)^k \\ z_i &= \textstyle\sum_{j=0}^{p-1} z_{ij} \left(\sqrt[p]{a} \right)^j \\ \tau_i &= \sqrt[p]{\vec{x}^{\vec{v}_i}} \end{aligned}$$

11. λ-rings

In this section we will let $\mathfrak{X}$ be a scheme over a field K and exploit the properties of $\mathcal{P}(\mathfrak{X})$ to develop some additional structures on the ring $K_0(\mathfrak{X})$. We will need to develop the λ-ring formalism in order to construct a slightly stronger version of the Grothendieck-Riemann-Roch theorem. First, we need to define some universal polynomials.

For any $n > 0$ we choose indeterminates $x_1, \ldots x_n$ and write

$$\begin{aligned} \textstyle\prod_{i=1}^{n} (1 + x_i) &= 1 + \textstyle\sum_{i=1}^{n} x_i + \ldots \sum_{1 \le i_1 < \ldots i_k \le n} x_{i_1} \ldots x_{i_k} + \ldots x_1 \cdots x_n \\ &= \sigma_0^n + \sigma_1^n + \ldots \sigma_k^n + \ldots \sigma_n^n \end{aligned}$$

where $\sigma_k^n = \sigma_k^n(x_1, \ldots x_n)$ is the sum of all the degree k monomials in the expansion. These polynomials are independent of n since, if $m > n$ then

$$\sigma_k^n(x_1, \ldots x_n) = \sigma_k^m(x_1, \ldots x_n, 0, \ldots 0)$$

DEFINITION 119. We call the polynomials $\sigma_k = \sigma_k^n$ the *elementary symmetric polynomials*.

We can show that every symmetric polynomial in in $x_1 \ldots x_n$ can be written uniquely as a function of the elementary symmetric polynomials. We also define for any n, m the polynomials $P_k(a_1, \ldots a_{2k})$ and $P_{ik}(a_1, \ldots a_{ik})$ by

$$\prod_{i=1}^{n}\prod_{j=1}^{m} 1 + x_i y_j t = \sum_{k=0}^{nm} P_k\left(\sigma_1(x_1, \dots x_n), \dots, \sigma_1(y_1, \dots y_n), \dots \sigma_k(y_1, \dots y_n)\right) t^k$$

$$\prod_{1 \le i_1 < \dots i_k \le n} x_{i_1} \dots x_{i_k} 1 + x_{i_1} \cdots x_{i_k} t = \sum_{i=0} \left(\frac{k}{n}\right) P_{ik}\left(\sigma_1(x_1, \dots), \dots \sigma_{ik}(x_1, \dots)\right) t^i$$

Suppose we have a natural functor from linear algebra, say $\mathcal{F} : \mathcal{V}(K) \to \mathcal{V}(K)$. Given any $P \in \mathcal{P}(\mathfrak{X})$ we can, by definition, choose an open cover U_α of $\mathfrak{X}$ such that

$$P|_{U_i} \cong \mathcal{O}_{U_i}^r = \mathcal{O}_{U_i} \otimes_K K^n.$$

We construct the sheaf $\mathcal{F}P$ on $\mathfrak{X}$ given by defining locally

$$\mathcal{F}P|_{U_i} = \mathcal{O}_{U_i} \otimes_K \mathcal{F}(K^n).$$

This patches together nicely by the naturality assumption, so we have an induced functor $\mathcal{F} : \mathcal{P}(\mathfrak{X}) \to \mathcal{P}(\mathfrak{X})$.
Now suppose that $\Lambda^i : \mathcal{V}(K) \to \mathcal{V}(K)$ is the exterior product, and note that these functors are exact and respect tensor products. Thus Λ^i induces an exact functor from $\mathcal{P}(\mathfrak{X})$ to itself that induces an endomorphism of $K_0(\mathfrak{X})$.

DEFINITION 120. We define $\lambda^i : K_0(\mathfrak{X}) \to K_0(\mathfrak{X})$ to be the endomorphism induced by the exterior product Λ^i. We also define a map $\lambda_t : K_0(\mathfrak{X}) \to K_0(\mathfrak{X})[[t]]$ by

$$\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x) t^i$$

PROPOSITION 121. The operations λ^i have the following properties on $K_0(\mathfrak{X})$ for any scheme $\mathfrak{X}$.

(1) $\lambda^0(x) = 1$
(2) $\lambda^0(x) = x$
(3) $\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x) \cdot \lambda^{k-i}(y)$, so

$$\lambda_t(x+y) = \lambda_t(x) \cdot \lambda_t(y)$$

(4) If $x = [P]$ for some $P \in \mathcal{P}(\mathfrak{X})$ then we say x is a *positive* element and denote the rank of x by $\epsilon(x)$. ϵ extends to a nontrivial homomorphism from the group $K_0(\mathfrak{X})$ to $\mathbb{Z}$, and we call all positive elements x with $\epsilon(x) = 1$ *line* elements. If x is positive then $\lambda^{\epsilon(x)}(x) \neq 0$ and $\lambda^k(x) = 0$ if $k > \epsilon(x)$.

(5) Every element is the difference of positive elements, so λ_t is a h... rom the group $K_0(\mathfrak{X})$ to the multiplicative group $K(t)^*$. We say a positive element x *splits* if it can be written as the sum of line elements.

(6) If x and y split then $\lambda^k(xy) = P_k(\lambda^1(x), \ldots \lambda^k(x), \lambda^1(y), \ldots \lambda^k(y))$

(7) If x splits then $\lambda^n \circ \lambda^m(x) = P_{n,m}(\lambda^1(x), \ldots \lambda^{nm}(x))$.

(8) We will also assume that we have an involution $x \mapsto \bar{x}$. By this we mean a ring endomorphism that maps the line elements to their inverses and the composition of the involution with itself is the identity.

PROOF. (1) This follows as $\Lambda^0(K^n) = K$ and $\Lambda^0(f) = 1$ for all $f : K^n \to K^n$, so the patching operation builds the rank one trivial module.

(2) Λ^1 is the identity functor.

(3) This follows from the identity

$$\Lambda^n(A \oplus B) = \bigoplus_{i=0}^{n} (\Lambda^i A) \otimes (\Lambda^{n-i} B)$$

(4) $\Lambda^{\epsilon(x)}(x)$ is a line bundle and thus is nonzero.

(5) The first part is clear from the definition of K_0. If x is positive then $\lambda_t(x)$ is just a polynomial, so property 3) shows that $\lambda_t(x-y) = \lambda_t(x)/\lambda_t(y)$ is a rational function.

(6) We write $x = x_1 + \ldots x_n$ and $y = y_1 + \ldots y_m$. We compute that

$$\lambda_t(x) = \prod_{i=1} n\lambda_t(x_i) = \prod_{i=1} n 1 + x_i t = \sum_{i=0}^{n} \sigma_i^n(x_1, \ldots x_n)t^i = \sigma_i^n \lambda^i(x)t^i.$$

Thus $\sigma_i^n(x_1, \ldots x_n) = \lambda^i(x)$, so λ ... given by the i'th symmetric polynomial. We now compute

$$\begin{aligned} \lambda_t(xy) &= \textstyle\prod_{i=1}^{n} \prod_{j=1}^{m} \lambda_t(x_i y_j) = \prod_{i=1}^{n} \prod_{j=1}^{m} 1 + x_i y_j t \\ &= \textstyle\sum_{k=0}^{nm} P_k(\sigma_1(x), \ldots \sigma_k(x), \sigma_1(y), \ldots \sigma_k(y))\, t^k \\ &= \textstyle\sum_{k=0}^{nm} P_k(\lambda^1(x), \ldots \lambda^k(x), \lambda^1(y), \ldots \lambda^k(y))\, t^k \end{aligned}$$

(7) This follows from expanding $\lambda_t \sigma_m(x)$.

We can now define a λ-ring

DEFINITION 122. We call a commutative ring K a λ-ring if it comes equipped with a set of operations $\lambda^i : K \to K$, a nontrivial ring homomorphism $\epsilon : K \to \mathbb{Z}$, and a subset E of *positive* elements such that:

(1) $\lambda^0(x) = 1$.

(2) $\lambda^1(x) = x$.

(3) $\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^k(x)\lambda^{k-i}(y)$.

(4) $\lambda^k(xy) = P_k(\lambda^1(x), \ldots \lambda^k(x), \lambda^1(y), \ldots \lambda^k(y))$ when x and y are positive.

(5) $\lambda^n \circ \lambda^m(x) = P_{n,m}(\lambda^1(x), \dots \lambda^{nm}(x))$ when x is positive.

(6) If x is positive then $\epsilon(x) > 0$ and $\lambda^{\epsilon(x)}(x)$ is a unit in K. Moreover, $\lambda^k(x) = 0$ if $n > \epsilon(x)$.

(7) Every element is the difference of finite elements. We say that a positive element x is a *line* element if $\epsilon(x) = 1$, and say that an element *splits* if it is the sum of line elements.

(8) K also comes equipped with an involution mapping line elements to their multiplicative inverses. Thus the line elements have the structure of a multiplicative group called the *Picard group*.

DEFINITION 123. A homomorphism $f : K \to K'$ between two lambda rings is a *lambda ring extension* if $\lambda^k(x) = \lambda^k(fx)$, $\epsilon(x) = \epsilon(fx)$, and f is injective.

EXAMPLE 124. If $\mathfrak{X}$ is a smooth compact manifold then we can regard $\mathfrak{X}$ as a scheme with the local rings being smooth functions. $\mathcal{P}(\mathfrak{X})$ is then equivalent to the category of smooth vector bundles over $\mathfrak{X}$. Since we have partitions of unity, any vector bundle P contains a line subbundle L, so we can construct the quotient bundle and write $P = L \oplus P/L$. Thus every vector bundle is the direct sum of line bundles and every positive element of $K_0(\mathfrak{X})$ splits, and we can show that $K_0(\mathfrak{X})$ is a λ-ring. It was shown by Grothendieck that all vector bundles over rational curves split, but this is not true for all algebraic varieties. See the excellent paper [4] by Atiyah on the splitting of vector bundles on elliptic curves.

EXAMPLE 125. We will show that the Grothendieck groups $K_0(R)$ for a Noetherian integral ring and $K_0(\mathfrak{X})$ for an algebraic variety are λ-rings. The positive elements correspond to the projective modules or $\mathcal{O}_{\mathfrak{X}}$-modules, and the line elements correspond to the rank one modules. Thus the Picard group of a ring or variety corresponds to the Picard group of the λ-ring.
The next proposition is crucial.

PROPOSITION 126. Let $\mathfrak{X}$ be an integral Noetherian separated scheme over a field. If x is a positive element of $K_0(\mathfrak{X})$ then there exists a scheme $\mathfrak{Y}$ of finite type over K such that the induced map $\pi^* : K_0(\mathfrak{X}) \to K_0(\mathfrak{Y})$ is an injective map preserving the λ structure such that $\pi^*(x)$ splits.

PROOF. (sketch) Given $P \in \mathcal{P}(\mathfrak{X})$ we construct a new scheme Proj P of finite type over $\mathfrak{X}$ as follows: We choose an affine open trivialising cover $U_i = \operatorname{Spec} R_i$ such that $P|_{U_i} \cong \mathcal{O}_{U_i} \otimes_K K^n$ and record the patching information by choosing maps $f_{ij} : U_i \cap U_j \to \mathrm{GL}_n(K)$. We define Proj P by

$$\operatorname{Proj} P|_{U_i} = \operatorname{Proj}(R_i[x_1, \dots x_n])$$

Where construct the patchings by interpreting the maps f_{ij} as endomorphisms of $K[x_1, \dots x_n]$. A calculation shows that $\pi^*(x)$ has n line subbundles and that π^* is injective.

COROLLARY 127. $K_0(\mathfrak{X})$ is a λ-ring for all integral schemes of finite type.

PROOF. If x and y do not split, we construct by the proposition an inclusion $K_0(\mathfrak{X}) \to K_0(\mathfrak{Y})$ such that x and y split in the extension. We then verify properties 4) and 5) in $K_0(\mathfrak{Y})$ so they must also hold in $K_0(\mathfrak{X})$.
This result can be generalized

PROPOSITION 128. $K_0(\mathfrak{X})$ is a λ-ring for every separated scheme $\mathfrak{X}$. Moreover, there exists a λ-ring extension $f : K_0(\mathfrak{X}) \to K$ such that all positive elements of $K_0(\mathfrak{X})$ split in K.

PROOF. See ([44] I.5.3)
This splitting phenomenon is a general property of λ-rings.

PROPOSITION 129. If K is a λ-ring then for all positive elements there exists a λ ring extension such that the element splits in the extension.

PROOF. We consider the ring

$$K' = \frac{K[t]}{\sum_{i=0}^{\epsilon(x)+1} (-1)^i \lambda^i(x) t^{\epsilon(x)+1-i}}$$

and make it into a λ-ring by making t a line element and adjoining t and $x - t$ to the set of positive elements. This gives us a λ-ring extension of K with $x = t + (x - t)$. If we iterate this process then x is decomposed into the sum of line elements. See ([13] for details.
An immediate consequence of this is the *verification principle*: If we are given a relation between the λ classes that holds for all split elements then it holds for all positive elements. This is the argument we used in Corollary 127. We also prove, for example,

PROPOSITION 130. If K is a λ-ring with Picard group L then we have a homomorphism $\det : K \to L$ taking the additive group to the multiplicative group.

PROOF. We define $\det x = \lambda^{\epsilon(x)}$ if x is a positive element. We choose an extension such that x splits, so $x = x_1 + \cdots x_n$. Then

$$\lambda^n(x) = \sigma_n(x_1, \dots x_n) = x_1 \cdots x_n$$

Since this is the product of line elements, $\lambda^{\epsilon(x)}(x)$ is a line element. For any x and y positive we have the equation $\lambda_t(x)\lambda_t(y) = \lambda_t(x + y)$, so if we look at the

$t^{\epsilon(xy)}$ term we get the equation $\det x \det y = \det x + y$. Thus det is linear and we can define $\det : K \to L$ by $\det x - y = \det x / \det y$.

12. Adams Operations

We now need to define the Adams operations and the γ-filtration in order to build the Grothendieck-Riemann-Roch. We will fix a λ-ring K and and write for any $x \in K$ and $s \in K(t)$

$$\lambda_s(x) = \sum_{i=0}^{\infty} \lambda^i s^i.$$

Note that $\lambda_s(x + y) = \lambda_s(x) \cdot \lambda_s(y)$.

For any $k > 0$ we have an natural endomorphism of the Picard group given by mapping a line element x to x^k. In the case where the ring has characteristic k then this corresponds to the Frobenius endomorphism and is defined for the entire ring. We would like to extend the k'th power map to the entire ring for a general k.

DEFINITION 131. We define the *Adams character* $\varphi_t : K \to K(t)^*$ and the *Adams operations* $\varphi^k : K \to K$ by

$$\varphi_t = -t \frac{\frac{d}{dt}\lambda_{-t}(x)}{\lambda_{-t}(x)} = -t\frac{d}{dt}\log \lambda_{-t}(x) = \sum_{i=1}^{\infty} \varphi^i(x) t^i$$

PROPOSITION 132. $\varphi^k : K \to K$ is a ring endomorphism such that $\varphi^n \circ \varphi^m(x) = \varphi^{nm}$ and $\varphi^k(x) = x^k$ if x is a line element.

PROOF. If x is a line element then

$$\varphi_t(x) = -t \frac{\frac{d}{dt}}{1 - tx} = \frac{tx}{1 - tx} = sum_{i=0}^{\infty} x^i t^i$$

so $\varphi(x) = x^k$. For any positive x and y we also have

$$\begin{aligned} \varphi_t(x + y) &= -t\tfrac{d}{dt}\log \lambda_{-t}(x + y) = -t\tfrac{d}{dt}\log \lambda_{-t}(x)\lambda_{-t}(y) \\ &= -t\tfrac{d}{dt}\log \lambda_{-t}(x) - t\tfrac{d}{dt}\log \lambda_{-t}(y) \\ &= \varphi_t(x) + \varphi_t(y) \end{aligned}$$

so $\varphi^k(x + y) = \varphi^k(x) + \varphi^k(y)$. Assuming x and y split we also compute

$$\varphi^k(xy) = \sum_{i,j} \varphi^k(x_i y_j) = \sum_{i,j} \varphi^k(x_i)\varphi^k(y_j) = \varphi^k(x)\varphi^k(y)$$

$$\varphi^n \varphi^m(x) = \sum_i \varphi^n \varphi^m(x_i) = \sum_i \varphi^{nm}(x_i) = \varphi^{nm}(x)$$

and apply the verification principle.

Recall that, unfortunately, the lambda operations are not additive homomorphisms like the Adams operations. It can be shown that the exterior products induce a λ-ring like operation on the higher K-groups which turns out to be a group homomorphisms. We introduce the γ-operations in an attempt to repair this non additivity.

DEFINITION 133. We define the γ-class $\gamma_t : K \to K(t)^*$ and the γ operations $\gamma^k : K \to K$ by

$$\gamma_t(x) = \sum_{i=0}^{\infty} \gamma^i(x) = \lambda_{\frac{t}{1-t}}(x)$$

We say that that an element $x \in K$ is γ-*positive* if we can find y positive such that $x = y - \epsilon(y)$, and we say that x is a γ-*line* element if we can choose y to be a line element. Note that every element of rank zero is the difference of two γ-positive elements.

PROPOSITION 134. γ has the following properties:

1. (1) $\gamma^0(x) = 1$ and $\gamma^1(x) = x$.
2. (2) $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$
3. (3) $\gamma_t(x) = 1 + tx$ if x is a γ-line element.
4. (4) If x is γ-positive then $\gamma^k(x)$ vanishes for k sufficiently large. Moreover, if x splits as the sum of γ-line elements $x_1, \ldots, x_n$ then

$$\gamma^k(x) = \sigma^k(x_1, \ldots, x_n)$$

PROOF. (1)

$$\begin{aligned}\gamma_t(x) &= \lambda_0(x) + \lambda_1(x)\tfrac{t}{1-t} + \lambda_2(x)\left(\tfrac{t}{1-t}\right)^2 + \ldots \\ &= 1 + x(t + t^2 + t^3 \ldots) + \lambda_2(x)(t^2 + 2t^3 + \ldots) \\ &= 1 + xt + (x + \lambda_2(x))t^2 + \ldots\end{aligned}$$

(2) Immediate

(3) If x is a line element we compute

$$\gamma_t(x-1) = \frac{\gamma_t(x)}{\gamma_t(1)} = \frac{1 + x\frac{t}{1-t}}{1 + \frac{t}{1-t}} = 1 + (x-1)t.$$

(4) We use the γ-*splitting principle*: For every γ-positive element x we can find a λ-ring extension such that x splits as the sum of γ-line elements. If x is γ-positive and splits then we compute

$$\gamma_t(x) = \gamma_t\left(\sum_i^n x_i\right) = \prod_i^n \gamma_t(x_i) = \prod_i^n 1 + x_i t = sum_i^n \sigma^i(x_1 \ldots x_n)t^i$$

We now wish to define a filtration on our lambda ring of the form

$$K = K^{(0)} \supseteq K^{(1)} \supseteq K^{(2)} \dots$$

DEFINITION 135. For all λ-rings K we define the subring $K^{(1)} = \ker \epsilon : K \to \mathbb{Z}$, and for $\ell > 1$ we define

$$K^{(\ell)} = \bigcup_{\tilde{K}} K \cap \mathbb{Z}\left\{x_1 x_2 \cdot x_n | x_i \in \tilde{K}^{(1)}, n \geq \ell\right\}$$

where the summation runs over all λ-ring extensions $\tilde{K}$ of K.

PROPOSITION 136. For any λ-ring extension $\tilde{K}$ of K we have $K \cap \tilde{K}^{(\ell)} = K^{(\ell)}$

PROOF. Clear.

PROPOSITION 137. The sets $K^{(i)}$ are ideals and thus form a filtration called the *weight* filtration.

PROOF. $K^{(\ell)}$ is closed under addition and if $x \in K^{(\ell)}$ and $y \in K$ then $yx = (y - \epsilon(y))x + \epsilon(y)x$ with both terms in $K^{(\ell)}$.

PROPOSITION 138. $K^{(\ell)} = \mathbb{Z}\{\gamma^{a_1}(y_1)\gamma^{a_2}(y_2) \cdot \gamma^{a_n}(y_n) | y_i \in K^{(1)}, \sum a_i \geq \ell\}$

PROOF. Given a collection y_i and a_i as above, we choose $\tilde{K}$ such that the y_i's γ-split. Setting $y_i = \sum_j y_{ij}$ with y_{ij} a γ-line element in $\tilde{K}$,

$$\gamma^{a_1}(y_1) \cdot \gamma^{a_n}(y_n) = \prod_{i=1}^{n} \sigma_{a_i}(y_{i1}, y_{i2}, \dots)$$

Which is homogeneous polynomial with integral coefficients of degree $\sum a_i$, and thus is an element of $K^{(\ell)}$. Conversely, every generator of $K^{(\ell)}$ must have symmetric γ-roots and so can be placed into this form.

PROPOSITION 139. We have group homomorphisms $\epsilon : K^{(0)}/K^{(1)} \xrightarrow{\cong} \mathbb{Z}$ and $\det : K^{(1)}/K^{(2)} \to L$ where L is the Picard group of line bundles.
The first isomorphism is clear. A quick computation shows that the determinant map is surjective and vanishes on $K^{(2)}$, and we construct an inverse map $v : L \to K^{(0)}/K^{(1)}$ by $v(x) = x - 1$. We compute

$$v(xy) = v(x) + v(y) + v(x)v(y) = v(x) + v(y)$$

as $v(x)v(y) \in K^{(2)}$, so v is a homomorphism and

$$\det v(x) = \det(x - 1) = \frac{\lambda^1(x)}{\lambda^0(1)} = x$$

We now have a key relation between the Adams operations and this weight filtration

PROPOSITION 140. For any λ-ring K and $x \in K^{(\ell)}$ we have

$$\varphi^j(x) - j^\ell x \in K^{(\ell+1)}$$

PROOF. For the $\ell = 0$ case, we assume by the linearity of both sides and the splitting principle that x is a line element. We compute

$$\epsilon\left(\varphi^j(x) - j^\ell x\right) = \epsilon\left(x^j - x\right) = 0$$

For $\ell > 0$ the linearity and splitting principle indicate that we need test only for $x = (x_1 - 1) \cdot (x_\ell - 1)$ t with $x_i \in L$. We compute

$$\begin{aligned}
\varphi^j(x) \prod_{i=1}^{\ell} \varphi^j(x_i - 1) &= \prod_{i=1}^{\ell} x_i^j - 1 \\
&= \prod_{i=1}^{\ell} (x_i - 1) \prod_{i=1}^{l} \left(1 + x_i + x_i^2 + \cdots x_i^{j-1}\right) \\
&= \prod_{i=1}^{\ell} (x_i - 1) \prod_{i=1}^{l} \left(\overbrace{1 + x_i + x_i^2 + \cdots x_i^{j-1} - j}^{\in K^{(1)}} + j\right) \\
&\equiv j^l \prod_{i=1}^{\ell} x_i - 1 \mod K^{(\ell+1)}
\end{aligned}$$

CHAPTER III

Hilbert 90

1. Introduction

We now come to the technical heart of this exposition: The proof of Hilbert's 90 theorem for K_2.

DEFINITION 141. If K is a field and p is a prime different to the characteristic of K we say a field extension L/K is *p-cyclic* if it is a degree p Galois extension.

CONJECTURE 142 (HILBERT'S 90 FOR K_n). If L/K is p-cyclic with σ a generator of $\mathrm{Gal}(L/K)$ then we have an exact sequence

$$K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{\mathrm{cor}_{L/K}} K_n(K).$$

Note that in the case $n = 1$ this reduces to

$$L^* \xrightarrow{1-\sigma} L^* \xrightarrow{\mathrm{cor}_{L/K}} K^*,$$

which is the usual Hilbert's theorem 90.

2. The Codimension Quotient

We first need to complete our study of the categories $\mathcal{M}^p(\mathfrak{X})$.

PROPOSITION 143. If $\mathfrak{X}$ is a regular Noetherian scheme then we have a natural isomorphism

$$\begin{aligned} K_i\left(\frac{\mathcal{M}^p(\mathfrak{X})}{\mathcal{M}^{p-1}(\mathfrak{X})}\right) &= K_i\left(\bigoplus_{x\in\mathfrak{X}^p} \mathcal{M}_\ell(\mathcal{O}_{\mathfrak{X},x})\right) \\ &= K_i\left(\bigoplus_{x\in\mathfrak{X}^p} \mathcal{M}\left(k(x)\right)\right) \\ &= \bigoplus_{x\in\mathfrak{X}^p} K_i\left(k(x)\right) \end{aligned}$$

PROOF. Combine proposition 88, the devissage theorem, and proposition 98.

We let $\mathfrak{X}$ be a Noetherian integral regular scheme of dimension n and consider the filtration by codimension of support

$$\mathcal{M}(\mathfrak{X}) = \mathcal{M}^0(\mathfrak{X}) \supseteq \mathcal{M}^1(\mathfrak{X}) \supseteq \cdots \mathcal{M}^{n+1}(\mathfrak{X}) = 0.$$

If we apply the functor K_n to this equation we get a sequence of maps

$$f_p : K_n\left(\mathcal{M}^p(\mathfrak{X})\right) \to K_n\left(\mathcal{M}^{p-1}(\mathfrak{X})\right).$$

If we now apply the localization theorem and use proposition 143 we get an exact fragment

$$\bigoplus_{x \in \mathfrak{X}^p} K_{n+1}(k(x)) \to K_n\left(\mathcal{M}^p(\mathfrak{X})\right) \xrightarrow{f_p} K_n\left(\mathcal{M}^{p-}(\mathfrak{X})\right),$$

so we see that the map f_p is, in general, not injective.

DEFINITION 144. We define the *filtration by codimension of the support} on* K_n

$$0 = F^{n+1}K_n(\mathfrak{X}) \subseteq= F^n K_n(\mathfrak{X}) \subseteq \cdots = F^0 K_n(\mathfrak{X}) = K_n(\mathfrak{X})$$

by setting

$$F^p K_i(\mathfrak{X}) = \text{image}(f_1 \circ \cdots f_p : K_n\left(\mathcal{M}^p(\mathfrak{X})\right) \to K_n\left(\mathfrak{X}\right))$$

We will only be interested in studying the filtration on K_0 *(See* [34] *for a discussion of the higher Chow groups resulting from the higher weight filtrations). Applying the localization theorem again with* $n = 0$ *we have a second fragment*

$$K_0\left(\mathcal{M}^{p+1}(\mathfrak{X})\right) \xrightarrow{f_{p+1}} K_0\left(\mathcal{M}^p(\mathfrak{X})\right) \to \bigoplus_{x \in \mathfrak{X}^p} K_0(k(x)) \to 0$$

which gives

$$F^{p+1}K_0(\mathcal{M}(\mathfrak{X})) \to F^p K_0(\mathcal{M}(\mathfrak{X})) \to \bigoplus_{x \in \mathfrak{X}^p} K_0(k(x)) \to 0$$

Recall that $K_0(\mathfrak{X})$ *has the structure of a* λ*-ring and thus has an Adams character. It turns out that the filtration by codimension of support respects the* γ*-weight filtration*

PROPOSITION 145. Given $x \in F^p K_0(\mathcal{M}(\mathfrak{X}))$ then $\varphi_i(x) - i^p \in F^{p+1}K_0(\mathfrak{X})$.

LEMMA 146. If $\pi : \mathfrak{Y} \to \mathfrak{X}$ be the scheme constructed in proposition 126 then

$$\pi^* : F^p(\mathfrak{X}) \to F^p(\mathfrak{Y})$$

PROOF. We need to show that π^* preserves codimension. By construction, there exists an open cover of $\mathfrak{Y}$ such that locally π is of the form

$$\pi : \operatorname{Spec} R[x_1, \dots x_m] \to \operatorname{Spec} R,$$

which reduces to a simple affine computation.

PROOF. Since the codimension of support respects λ-ring extensions, we can assume the γ-splitting principle. The entire calculation reduces to showing that if x is a codimension one point then I_x represents an element of $K_0(\mathfrak{X})^{(1)}$. By definition, $I_x = \tilde{R}/\mathfrak{p}$ where $\mathfrak{p}$ is a minimal non-zero prime ideal, and thus is of the form $\mathfrak{p} = (f)$ for some irreducible element f of R. We have an exact sequence

$$0 \to fR \to R \to \frac{R}{(f)R} = \frac{R}{\mathfrak{p}} \to 0$$

so I_x is the form $1 - l$ where l is a line element. Thus I_x is a γ-line element and we are done.

Lemma 146 can be generalized

PROPOSITION 147. If $f : \mathfrak{Y} \to \mathfrak{X}$ is a flat map of Noetherian schemes then

$$f^* : F^p(\mathfrak{X}) \to F^p(\mathfrak{Y})$$

PROOF. ([46] 5.19).

3. BQG: Level One

THEOREM 148 (BROWN-QUILLEN-GERSTEN SPECTRAL SEQUENCE). Given $\mathfrak{X}$ of finite Krull dimension we have a convergent fourth quadrant spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in \mathfrak{X}^p} K_{-p-q}\left(k_{x,\mathfrak{X}}\right) \Longrightarrow G_{-p-q}(\mathfrak{X}),$$

where $\mathfrak{X}^p$is the set of points of codimension p. The induced filtration is by codimension of support, and the spectral sequence is contravariant under flat morphisms.

PROOF. We know that that $\mathcal{M}^{p+1}(\mathfrak{X})$ is a Serre subcategory of $\mathcal{M}^p(\mathfrak{X})$, so we can construct the quotient $\mathcal{M}^p(\mathfrak{X})/\mathcal{M}^{p+1}(\mathfrak{X})$ and apply the localization theorem to get an exact sequence

$$K_i\left(\mathcal{M}^{p+1}(\mathfrak{X})\right) \xrightarrow{a} K_i\left(\mathcal{M}^p(\mathfrak{X})\right) \xrightarrow{b} K_i\left(\frac{\mathcal{M}^p(\mathfrak{X})}{\mathcal{M}^{p+1}(\mathfrak{X})}\right) \xrightarrow{c} K_{i-1}\left(\mathcal{M}^{p+1}(\mathfrak{X})\right)$$

By proposition 143 we have a natural equivalence

$$K_i\left(\frac{\mathcal{M}^p(\mathfrak{X})}{\mathcal{M}^{p+1}(\mathfrak{X})}\right) = K_i\left(\bigoplus_{x\in\mathfrak{X}^p} \mathcal{M}_l(\mathcal{O}_{\mathfrak{X},x})\right) = \bigoplus_{x\in\mathfrak{X}^p} K_i(k_x)\,.$$

so we can define bi-graded algebras A_1 and E_1 by

$$\begin{aligned} A_1^{p,q} &= K_{-p-q}(\mathcal{M}^p(\mathfrak{X})) \\ E_1^{p,q} &= K_{-p-q}(\mathcal{M}^p(\mathfrak{X})/\mathcal{M}^{p+1}(\mathfrak{X})) \\ &= \bigoplus_{x\in\mathfrak{X}^p} K_{-p-q}(k_x) \end{aligned}\,.$$

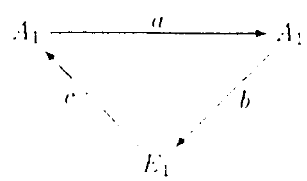
The maps a, b, and c can be interpreted as bi-graded morphisms with bi-degrees $(-1,1)$, $(0,0)$, and $(1,0)$ respectively. Since $\mathfrak{X}$ is finite dimensional we have a direct limit

$$A^n = \varinjlim A^{n-i,i} = G_{-n}(\mathfrak{X}),$$

where the direct limit is taken over the maps $a : A^{n-i+1,i} \longmapsto A^{n-i,i+1}$. We can induce a filtration on A^n by

$$F^pA^{p+q} = \text{Image}\,(a^p : A^{p,q} \longmapsto A^{p+q})\,.$$

This is exactly the filtration by codimension of support of G^{p+q}. We now set $d_1 = c \circ d$ and consider the spectral sequence with base $\{E_1^{p,q}, d_1\}$. Since $\mathfrak{X}$ has finite Krull dimension, $\mathcal{M}^p = \mathcal{M}^{p+1}$ for p sufficiently large. Thus the $E_2^{p,q}$ terms vanish except for a band of finite width, so the spectral sequence converges. We now consider the exact couple



This couple gives rise to the spectral sequence $\{E_1^{p,q}, d_1\}$ which, since it converges, converges to $A^n = G_{-n}(\mathfrak{X})$. To show the functoriality, note that a flat morphism $f : \mathfrak{X} \longmapsto \mathfrak{Y}$ induces a functor f^* from $\mathcal{M}(\mathfrak{Y})$ to $\mathcal{M}(\mathfrak{X})$. Proposition 147 tells us that codimension of support is compatible with flat maps, so f^* induces a natural contravariant functor from $\mathcal{M}^p(\mathfrak{Y})$ to $\mathcal{M}^p(\mathfrak{X})$ compatible with K^i and G^i for flat maps and the result follows.

Note that this implies that the spectral sequence commutes with direct limits.

4. BQG: Level Two

We wish to study the second level of the above spectral sequence. We first characterize the schemes where the second level is, in some sense, trivial.

DEFINITION 149. A Noetherian scheme is *Gersten* if it satisfies the following equivalent conditions.

- The natural map
$$K_i\left(\mathcal{M}^{p+1}(\mathfrak{X})\right) \mapsto K_i\left(\mathcal{M}^{p}(\mathfrak{X})\right)$$
vanishes for all nonnegative p.
- $E_2^{p,q}$ vanishes for all $p \neq 0$.
- The spectral sequence converges at the second level with the filtration on the abutment concentrated in codimension zero[1].
- We have an exact sequence
$$0 \longrightarrow G_n(\mathfrak{X}) \xrightarrow{d_1} \bigoplus_{x\in\mathfrak{X}^0} K_n(k_x) \xrightarrow{d_1} \bigoplus_{x\in\mathfrak{X}^1} K_{n-1}(k_x) \xrightarrow{d_1} \bigoplus_{x\in\mathfrak{X}^2} K_{n-2}(k_x)\cdots$$

PROPOSITION 150. Let R be a regular semi-local ring essentially of finite type over a field. Then R is Gersten

PROOF. [36]

One can prove that all regular semi-local rings essentially of finite type over a field are Gersten, and it is conjectured that all regular local rings are Gersten. We define a scheme to be locally Gersten if it satisfies Gersten's conjecture for all points on the scheme. If $\mathfrak{X}$ is such a scheme, we can sheafify the groups $E_1^{i,-n}$ by defining $\mathfrak{E}^{i,-n}$ to be the sheaf given by

$$U \mapsto \bigoplus_{x\in U\cap\mathfrak{X}^i} K_{n-i}(k_x) = E_1^{i,-n}(U)$$

Over a Gersten subscheme $\mathfrak{Y}$ of $\mathfrak{X}$ we have, by the definition, a natural exact sequence

$$0 \longrightarrow \mathfrak{G}_{n,\mathfrak{X}}\Big|_{\mathfrak{Y}} \xrightarrow{d_1} \mathfrak{E}^{0,n}|_{\mathfrak{Y}} \xrightarrow{d_1} \mathfrak{E}^{1,n}|_{\mathfrak{Y}} \xrightarrow{d_1} \mathfrak{E}^{2,n}|_{\mathfrak{Y}} \xrightarrow{d_1} \mathfrak{E}^{3,n}|_{\mathfrak{Y}}\cdots$$

Patching together, we have an exact sequence of sheaves giving a flasque resolution $\mathfrak{F}_{i,-n}$ of $\mathfrak{G}_{n,\mathfrak{X}}$. But the global sections $\Gamma(\mathfrak{X}, \mathfrak{E}_{i,-n})$ are just $E_1^{i,-n}$ with maps d_1, so the p^{th} cohomology group of $\mathfrak{G}_{n,\mathfrak{X}}$ is given by the p^{th} cohomology of $E_1^{i,-n}$. Since a scheme of finite type over a field is locally Gersten,

[1] In other words we have an isomorphism $K_j(\ \ ) \cong E_2^{0,-j}$

THEOREM 151. For a scheme of finite type over a field $\mathfrak{X}$ we have

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(\mathfrak{X}, \mathfrak{G}_{-q,\mathfrak{X}}).$$

Since we have explicit representations of the lower K-groups we can analyse the spectral sequence near the diagonal. Let $\mathfrak{X}$ be a field of finite type over a field, and note that d_1 has bi-degree $(1,0)$. This gives us a diagram

$$\begin{array}{ccccccc}
E^{p-2,-p} & \xrightarrow{d_1} & E^{p-1,-p} & \xrightarrow{d_1} & E^{p,-p} & \xrightarrow{d_1} & 0 \\
\| \downarrow & & \| \downarrow & & \downarrow \| & & \\
\bigoplus_{x\in\mathfrak{X}^{p-2}} K^2(k_x) & \xrightarrow[\partial_p]{} & \bigoplus_{x\in\mathfrak{X}^{p-1}} k_x^* & \longrightarrow & \bigoplus_{x\in\mathfrak{X}^{p}} \mathbb{Z} & \longrightarrow & 0 \\
 & & & \searrow^{\mathrm{ord}} & \downarrow \| & & \\
 & & & & \mathbb{Z}\mathfrak{X}^p & &
\end{array}$$

Where $\mathbb{Z}\mathfrak{X}^p$ is the free group generated by subschemes of codimension p and the image of ord is a subgroup of algebraic cycles. Examining ord locally, we find the image is precisely the cycles rationally equivalent to zero. Thus $E_2^{p,-p}$ is the Chow group of codimension p cycles. We can not obtain much more information from the spectral sequence as the higher K-groups are difficult to compute[2].

5. A Steinberg Relation

We require a small lemma which will allow us to define a symbol.

LEMMA 152. Given a p-cyclic extension L/K with σ a generator of the Galois group and any x in L^* with $N_{L/K}(x) \neq 0$, we have

$$\{x, 1 - N_{L/K}(x)\} \in (1-\sigma)K_2(E)$$

PROOF. Let $f(t)$ be any irreducible monic polynomial in $k[t]$ dividing $t^p - N_{L/K}(x)$ with a splitting field F and a root y. As usual, the automorphism group of $L \otimes_k F/F$ is generated by σ, so we can compute

$$N_{L\otimes F/F}(x) = N_{L/K}(x) = y^p = N_{L\otimes F/F}(y).$$

[2]This would be a good place to do the examples for $\mathbb{Q}[x,y]$, $(\mathbb{Z}/p\mathbb{Z})[x,y]$, and projective spaces. The first is interesting as the second and zeroth codimensions are related while the first codimension is composed of the K groups of curves. The second and third cases provide results as both the irreducible polynomials and the K groups are known. The last provides not much more than a computation of K_0, but is useful.

Thus $N_{L\otimes F/F}(x/y) = (1-\sigma)z$ for some z in F by Hilbert 90, and we have

$$\begin{aligned}\{x, f(1)\} &= \{x, N_{L\otimes F/F}(1-y)\}\\ &= \mathrm{cor}_{L\otimes F/F}\{x, 1-y\}\\ &= \mathrm{cor}_{L\otimes F/F}\{\tfrac{x}{1-y}, 1-y\}\\ &= (1-\sigma)\,\mathrm{cor}_{L\otimes F/F}\{z, 1-y\}\end{aligned}$$

If we decompose $t^p - N_{L/K}(x)$ as the product of irreducible monic polynomials f_i we have that

$$\{x, 1-N_{L/K}(x)\} = \prod_i \{x, f_i(1)\}$$

and the result follows.

6. Injectivity of K_1

Given a field k and a central simple algebra D of dimension p^2 over k, we can choose a cyclic extension L of order p over k such that D splits over L. If $\mathfrak{X}$ is the Severi-Brauer variety associated to D we have an étale inclusion $i : \mathfrak{X}_L \mapsto \mathfrak{X}$.

PROPOSITION 153. The map i^* induces a natural inclusion $K_1(\mathfrak{X}) \mapsto K_1(\mathfrak{X}_L)$ that respect the codimension of the support.

This comes from calculating

$$\begin{aligned}K_1(\mathfrak{X}_L) &= K_1\left(\mathbb{P}_L^{p-1}\right) = \textstyle\bigoplus_{i=0}^{p-1} K_1(L)\\ K_1(\mathfrak{X}) &= \textstyle\bigoplus_{i=0}^{p-1} K_1\left(D^{\otimes i}\right)\end{aligned}$$

and by explicitly computing i^*. The first is proved by viewing $K_1(\mathbb{P}_L^{p-1})$ as the quotient of the category of finite positively graded $L[t_0, \ldots t_{p-1}]$ modules by the Serre subcategory of those of finite length. We then apply devissage to get a special case of the projective homotopy theorem

$$K_i(\mathbb{P}_L^r) = K_0(\mathbb{P}_L^r) \otimes_{K_0(L)} K_i(L)$$

The computation of $K_1(\mathfrak{X})$ can be done étale locally followed by a patching argument.

7. The Diagonal $E_i^{p,-p}$ and the Limit $E_\infty^{1,-2}$

PROPOSITION 154. Let L/K be a field extension of order p and let $\mathfrak{X}$ be a Severi-Brauer variety associated to a central simple algebra D over k that is split by L. Then the spectral sequence of $\mathfrak{X}$ converges along the diagonal at the second level, that is, $E_2^{p,-p}(\mathfrak{X}) = E_\infty^{p,-p}(\mathfrak{X})$.

To prove this, note that the inclusion map $i : \mathfrak{X}_L \mapsto \mathfrak{X}$ induces a sequence

$$\mathrm{CH}^i(\mathfrak{X}) \xrightarrow{i^*} \mathrm{CH}^i(\mathfrak{X}_L) \xrightarrow{i_*} \mathrm{CH}^i(\mathfrak{X}).$$

For any $y \in \mathrm{CH}^i(\mathfrak{X})$ the projection formula yields

$$i_* i^*(y) = i_*(\mathfrak{X}_L \cdot i^* y) = y \cdot i_*(\mathfrak{X}_L) = [L:k]\mathfrak{X} \cdot y = py.$$

Since $\mathrm{CH}^i(\mathfrak{X}_L) = \mathbb{Z}$ is free, we can conclude that all the torsion of $\mathrm{CH}^i(\mathfrak{X})$ is p-torsion. But by a consequence of our Riemann-Roch theorem we know that the kernel of the map from $E_2^{p,-p}$ to $E_\infty^{p,-p}$ is annihilated by $(p-1)!$[45], so the kernel must be trivial.

COROLLARY 155. $\mathrm{CH}^i(\mathfrak{X})$ has rank at most one with all torsion p-torsion.

Our next result concerns the sheaf cohomology groups $H^1(\mathfrak{X}, \mathfrak{K})$. Note that, although we have an explicit representation of K_2 via Matsumoto's theorem, we still require the machinery of higher K-theory to obtain our results.

PROPOSITION 156. Let L/K be a p-cyclic extension and let $\mathfrak{X}$ be a Severi-Brauer variety over k split by L. If $i : \mathfrak{X}_L \longmapsto \mathfrak{X}$ then we have an injection

$$H^1(\mathfrak{X}, \mathfrak{K}_2) \hookrightarrow H^1(\mathfrak{X}_L, \mathfrak{K}_2)$$

We note that $E^{1-n,-3+n}(\mathfrak{X}) = 0$ for all $n \geq 2$ as the spectral sequence is fourth quadrant, and the image of $d_n^{1,-2} : E_n^{1,-2} \longmapsto E_n^{1+n,-n+1}$ vanishes by proposition 154. Thus we have that for all $n \geq 2$

$$E_{n+1}^{1,-2} = \frac{\ker\left(d_n : E_n^{1,-2} \longmapsto E_n^{1+n,-n-1}\right)}{\operatorname{image}\left(d_n : E_n^{1-n,-3+n} \longmapsto E_n^{1,-2}\right)} = \frac{E_n^{1,-2}}{0}$$

Thus $H^1(\mathfrak{X}, \mathfrak{K}_2) = K_3(\mathfrak{X})^{1/2}$ which maps injectively into $H^1(\mathfrak{X}_L, \mathfrak{K}_2) = K_3(\mathfrak{X}_L)^{1/2}$ by proposition 153.

8. Coprime lifting

As usual, we let L/K be a p-cyclic extension and fix a generator σ of the Galois group. For any F/k we note that $F \otimes L$ is a finite F-algebra with an automorphism given by σ acting on the right, so we can define

$$V_{L/k}(F) = \frac{\ker\left(K_2(L \otimes_k F) \xrightarrow{\text{cor}} K_2(F)\right)}{(1-\sigma)K_2(L \otimes_k F)}$$

The standard relations between the restriction and the corestriction lift to properties of $V_{L/k}(F)$. For any tower of fields $k \subset F \subset F'$ we can define maps

$$\operatorname{res} : V_{L/k}(F) \longrightarrow V_{L/k}(F')$$
$$\operatorname{cor} : V_{L/k}(F') \longrightarrow V_{L/k}(F)$$

such that cor ∘ res is multiplication by $[F' : F]$. We can also 'lift' the base extensions by noting that $V_{L/K}(F) = V_{F \otimes L/F}(F)$ if $F \otimes L$ is a field. Our eventual goal is to show that $V_{L/k}(k) = 0$, but we first try something less ambitious.

PROPOSITION 157. $V_{L/k}(L) = 0$

To see this, we fix $E \in L$ such that $k(e) = L$ and note that $L \otimes_k L$ is a regular zero dimensional ring with maximal ideals $\mathcal{M}_i = (e \otimes \sigma^i(e))$. We can compute (use Theorem 148) that

$$K_2(L \otimes_k L) = \bigoplus_{x \in (\mathrm{Spec}\, L \otimes_k)^0} K_2(k_x) = \oplus_{i=1}^{p} K_2(L)$$

and that the corestr[illegible] by

$$\mathrm{cor}_{L \otimes_k L/L} : \oplus [illegible] \to K_2(L) \quad \mathrm{cor}_{L \otimes_k L/L}(a_1, \ldots a_p) = \sum_{i=1}^{p}$$

For any element [illegible] the kernel of equation 157 we set $b_i = a_1 + \cdots a_i$ and, since the Galois g[illegible]acts with $\sigma \mathcal{M}_i = \mathcal{M}_{i+1}$, we can compute

$$(1 - \sigma)(b_1, \ldots b_n) = (a_1 + (a_1 + \ldots a_p), a_2, \ldots a_p) = (a_1, \ldots a_p)$$

COROLLARY 158. $pV_{L/k}(k) = 0$ and for any finite extension F/k with $[F : k]$ coprime to p we have an injective map res : $V_{L/k}(k) \hookrightarrow V_{L/k}(F')$

PROOF. The first follows as the following is multiplication by p and trivial.

$$\begin{array}{ccccc} V_{L/k}(k) & \xrightarrow{\text{res}} & V_{L/k}(L) & \xrightarrow{\text{cor}} & V_{L/k}(k) \\ & & \| & & \\ & & 0 & & \end{array}$$

The second is from noting that the composite $V_{L/k}(k) \xrightarrow{\text{res}} V_{L/k}(F) \xrightarrow{\text{cor}} V_{L/k}(k)$ has no kernel killed by p.

9. Severi-Brauer Lifting

The main goal of this section is to prove the following proposition.

PROPOSITION 159. Let L/K be a p-cyclic extension and let $\mathfrak{X}$ be a Severi-Brauer variety over k that is split by L. The map

$$\mathrm{res}_{k_{\mathfrak{X}}/k} : V_{L/k}(k) \hookrightarrow V_{L/k}(k_{\mathfrak{X}})$$

is injective.

For any $v \in K_2(k_{\mathfrak{X}_L})$ we will first construct an element η_v of $H^1(\mathfrak{X}, \mathfrak{K}_2)$. Recall that in diagram 150 we had a map

$$\partial_2 : K^2\left(k_{\mathfrak{X}_L}\right) \longrightarrow \bigoplus_{y \in \mathfrak{X}_L{}^1} k^*_{y.\mathfrak{X}_L}$$

So we can define a map

$$\partial_{2,y} : K^2\left(k_{\mathfrak{X}_L}\right) \longrightarrow k^*_{y.\mathfrak{X}_L}$$

by

$$\partial_2(v) = \bigoplus_{y \in \mathfrak{X}_L{}^1} \partial_{2,y}(v).$$

Note that for any point x in $\mathfrak{X}^1$ we have

$$L \otimes k_{x.\mathfrak{X}} = \prod_{y \in I_x} k_{y.\mathfrak{X}_L}$$

where the product is taken over the elements of $\mathfrak{X}_L$ that lie over x, so we can define an element $\partial_x(v)$ by

$$\partial_x(v) = \prod_{y \in I_x} \partial_{2,y}(v) \in \left(\prod_{y \in I_x} k_{y.\mathfrak{X}_L}\right)^* = \left(L \otimes k_{x.\mathfrak{X}}\right)^*.$$

Note, however, that the action of the Galois group simply permutes the order of the factors, so $\partial_x(v)$ is fixed and thus defines an element of $k_{x.\mathfrak{X}}$.

LEMMA 160. If we have $v \in K_2(k_{\mathfrak{X}_L})$ such that $\partial_x(v)$ is trivial for all x in $\mathfrak{X}^1$, then $v = \mathrm{res}_{\mathfrak{X}_L/L}(z)$ for some z in $K_2(L)$

To see this, note that by equation 159, $\partial_x(v)$ is trivial only if $\partial_{2,y}(v)$ is trivial for all y over x. Since the map from $\mathfrak{X}_L$ to $\mathfrak{X}$ is an étale inclusion, we must have $\partial_{2,y}(v)$ vanishing for all y of codimension one in $\mathfrak{X}_L$. Recall that we have an exact sequence

$$0 \longrightarrow K_2(L) \longrightarrow K_2\left(\mathbb{P}_L^{r-1}\right) \xrightarrow{\partial_2} \bigoplus_{x \in \left(\mathbb{P}_L^{r-1}\right)^1} K_1\left(k_{x.\mathbb{P}_L^{r-1}}\right)$$

which, since $\mathfrak{X}$ splits over L, gives us an exact sequence

$$0 \longrightarrow K_2(L) \longrightarrow K_2(\mathfrak{X}_L) \xrightarrow{\partial_2} \bigoplus_{x \in \mathfrak{X}^1} \left(k_{x.\mathfrak{X}_L}\right)^*$$

Thus $v = \mathrm{res}_{\mathfrak{X}_L/L}(z)$ for some z in $K_2(L)$.

We now examine the following commuting diagram.

$$\begin{array}{ccccc}
K^2(k_{\mathfrak{X}}) & \xrightarrow{\partial_2} & \bigoplus_{x\in\mathfrak{X}^1} k^*_{x,\mathfrak{X}} & \xrightarrow{\text{ord}} & \bigoplus_{y\in\mathfrak{X}^2}\mathbb{Z} \\
\mid \text{res}' & & \mid \text{res}'' & & \mid \text{res}''' \\
K^2\left(k_{\mathfrak{X}_L}\right) & \xrightarrow{\partial_2} & \bigoplus_{x\in\mathfrak{X}_L{}^1} k^*_{x,\mathfrak{X}_L} & \xrightarrow{\text{ord}} & \bigoplus_{y\in\mathfrak{X}_L{}^2}\mathbb{Z}
\end{array}$$

and let η be the element of the middle term in the top row given by $\bigoplus_x \partial_x(v)$.

LEMMA 161. $\eta = \partial_2(\tilde{v})$ for some element $\tilde{v}$ in $K^2(k_{\mathfrak{X}})$.

Note first that

$$\text{res}''(\eta) = \prod_{y\in I_x} \partial_2(v)$$

and that

$$\text{res}''' \circ \text{ord}(\eta) = \text{ord} \circ \text{res}''(\eta) = \prod_{y\in I_x} \text{ord} \circ \partial_2(v) = 1.$$

Since res''' is clearly injective, ord(η) vanishes. If we take the homology of both rows at the center term, we get a map

$$\text{res}'' : H^1(\mathfrak{X}, \mathfrak{K}_2) \hookrightarrow H^1(\mathfrak{X}_L, \mathfrak{K}_2)$$

with η representing an element of $H^1(\mathfrak{X}, \mathfrak{K}_2)$. We know the map is injective by proposition 153 and that res$''\eta$ vanishes by equation 159, so η represents the trivial element.

We can now prove the proposition. We choose a in the kernel of $\text{cor} : K_2() \to K_2(k)$ to represent an element of $V_{L/k}(k)$ and note that

$$\text{res}_{k_{\mathfrak{X}_L}/L}(a) = (1-\sigma)v$$

for some $v \in K_2(k_{\mathfrak{X}_L})$ when a is in the kernel of $\text{res} : V_{L/k}(k) \to V_{L/k}(k_{\mathfrak{X}})$. We choose $\tilde{v}$ as above, and compute that $\partial_2(v - \text{res}'(\tilde{v})) = 0$ for all x in $\mathfrak{X}_L{}^1$. Thus

$$\begin{aligned}
&\text{res}_{k_{\mathfrak{X}_L}/L}(a) = (1-\sigma)(v - \text{res}'(\tilde{v})) = (1-\sigma)\,\text{res}_{k_{\mathfrak{X}_L}/L}(z) \\
&a = (1-\sigma)z
\end{aligned}$$

10. Reduction to the Extension

PROPOSITION 162. Given a p-cyclic extension L/K we can find an extension $\mathcal{L}$ of k such that

(1) We have an injective map $\text{res} : V_{L/k}(k) \hookrightarrow V_{\mathcal{L}\otimes L/\mathcal{L}}(\mathcal{L})$
(2) $\mathcal{L} \otimes L/\mathcal{L}$ is a p-cyclic extension of fields.
(3) The norm map $N_{\mathcal{L}\otimes L/\mathcal{L}}$ is surjective.
(4) $\mathcal{L}$ has no finite field extension with degree prime to p

We define a sequence of extensions $\mathcal{L}^i$ of k inductively. Let $\mathcal{L}^1 = k$, and assume that $\mathcal{L}^n \otimes_k L$ is a field. We consider the class $\mathcal{C}$ of algebraic field extensions of $\mathcal{L}^n$ with the property that any finite index subextension has degree coprime to p, and use a Zorn argument to choose a maximal extension $\mathcal{L}^n_{\infty}$. Note that every element F in $\mathcal{C}$ has the property that $F \otimes L$ is a field, so for every finite extension F'/F in $\mathcal{C}$ we have an injection

$$V_{L\otimes\mathcal{L}^n/\mathcal{L}^n}(F) = V_{L\otimes F/F}(F) \hookrightarrow V_{L\otimes F/F}(F') = V_{L\otimes\mathcal{L}^n/\mathcal{L}^n}(F')$$

by corollary 158. Thus $\mathcal{L}^n_{\infty}$ is a field extension of $\mathcal{L}^n$ such that every finite field extension of $\mathcal{L}^n_{\infty}$ has order dividing p. $\mathcal{L}^n_{\infty} \otimes L$ is a field, and we have an injection

$$V_{L\otimes\mathcal{L}^n/\mathcal{L}^n}(\mathcal{L}^n) \hookrightarrow V_{L\otimes\mathcal{L}^n_{\infty}/\mathcal{L}^n_{\infty}}(\mathcal{L}^n_{\infty})$$

We now use proposition 30 and set $\mathcal{L}^{n+1} = K^{*}(L \otimes \mathcal{L}^n_{\infty}/\mathcal{L}^n_{\infty})$, which gives us an injection

$$V_{L\otimes\mathcal{L}^n/\mathcal{L}^n}(\mathcal{L}^n) \hookrightarrow V_{L\otimes\mathcal{L}^{n+1}/\mathcal{L}^{n+1}}(\mathcal{L}^{n+1})$$

We now define $\mathcal{L}$ to be the union of all $\mathcal{L}^n$.

11. Generators of $K_2(L)$

Given A p-cyclic extension L/K of fields, we know that $K_2(L)$ has a presentation given by Matsumoto's theorem so we can define the subgroup $K_2^{gn}(L)$of $K_2(L)$ generated by elements (x, a) with $x \in L$ and $a \in k$. We wish to find a condition under which these two groups are equal.

Choosing an element $t \in L$ such that $L = k(t)$ allows us, for each element $x \in L$, to choose a unique polynomial $f \in k[t]$ such that the degree of f is less than p and $f(t) = x$. We can then consider f as the product of an element of k by irreducible monic polynomials in $k[t]$, so for x and y in L we can compute

$$\begin{aligned}(x, y) &= \left(a\prod_i f_i(t), b\prod_j g_j(t)\right) \\ &= (\prod_i f_i(t), b)) - \left(b\prod_j g_j(t), a)\right) + \sum_{i,j}(f_i(t), g_j(t)))\end{aligned}$$

Thus we are reduced to studying elements $(f(t), g(t)$ where f and g are irreducible monic polynomials in $k[t]$. The most direct approach would be to assume that $\deg g \leq \deg f$ and use the Chinese remainder theorem to write $f = cg + d$ where $\deg d \leq \deg g$. We can now define an element of $K_2(L)$ by

$$\begin{aligned}\left(\frac{cg(t)}{f(t)}, \frac{d(t)}{f(t)}\right) = \quad &(c(t), d(t)) + (g(t), d(t)) + (f(t), f(t)) \\ &-(c(t), f(t)) + (f(t), g(t)) - (f(t), d(t)).\end{aligned}$$

The left side vanishes by the Steinberg relation, so we could hope to express

$(f(t), g(t))$ as a combination of elements of lower bi-degree and proceed by induction. Unfortunately, the induction process fails when $\deg f$ is less than half of $\deg g$. We can, however, prove the following.

LEMMA 163. If x and y are elements of L with $x - y \in k$ then $(x, y) \in K_2^{lin}(L)$.

PROOF. If $x = y$ then we note that $(x, x) = (x, -1) + (x, -x) = (x, -1)$ as $(x, -x)$ vanishes for all x. If $x \neq y$, we set $a = x - y$ and compute

$$0 = \left(\frac{a}{x}, \frac{y}{x}\right) = (a, y) + (x, x) - (a, x) - (x, y),$$

so (x, y) is in $K_2^{lin}(L)$.
Note that lemma 163 gives us that $(f_i(t), g_j(t))$ is an element of $K_2^{lin}(L)$if both f_i and f_j are of degree one. So we can conclude

PROPOSITION 164. Let L/K be a p-cyclic extension. $K_2(L) = K_2^{lin}(L)$ if all elements of $k[t]$ of degree less than p split completely.

12. Hilbert 90

We can now prove the main result of this chapter.

THEOREM 165 (HILBERT'S 90 FOR K_2). For any p-cyclic extension L/K with σ generating the Galois group we have $V_{L/k}(k) = 0$. Thus we have an exact sequence

$$K_2(L) \xrightarrow{1-\sigma} K_2(L) \xrightarrow{cor_{L/K}} K_2(L)$$

PROOF. By Proposition 152 we can assume that all finite extensions of k have degree coprime to p and that $N_{L/K}$ is surjective.
We define a bilinear map

$$\phi : k^* \times k^* \to K_2(L)/(1-\sigma)K_2(L)$$

by setting $\phi(a, b) = (x, b)$ where $N_{L/K}(x) = a$. Proposition 152 states that $\phi(a, 1-a)$ vanishes, so ϕ lifts to give us a map from $K_2(k)$ to $K_2(L)$. We now compute that the composite

$$K_2(k) \xrightarrow{\phi} K_2(L) \xrightarrow{cor_{L/K}} K_2(k)$$

is just the identity map, so we are reduced to proving that ϕ is surjective.
But since k has no extensions of degree coprime to p, any element of $k[t]$ with degree less than p must split completely. The image of ϕ is just $K_2^{lin}(L)$, so ϕ is surjective by proposition 164.

COROLLARY 166. If L/K is a p-cyclic extension with σ generating the Galois group then we have an exact sequence

$$K_2(L) \xrightarrow{1-\sigma} K_2(L) \xrightarrow{cor_{L/K}} K_2(K)$$

CHAPTER IV

Merkurjev-Suslin

1. Galois Lift

In this section we will study the commutative diagram

$$\begin{array}{ccc} {}_pK_2(K) & \xrightarrow{\alpha_K} & {}_p\operatorname{Br}(K) \\ \Big\downarrow {\scriptstyle \operatorname{res}_{L/K}} & & \Big\downarrow {\scriptstyle \operatorname{res}_{L/K}} \\ {}_pK_2(L) & \xrightarrow[\alpha_L]{} & {}_p\operatorname{Br}(L) \end{array} \tag{6}$$

where L/K is a p-cyclic extension. We will assume for this section that α_K is injective, and prove that this will imply that α_L is also injective in preparation for a Galois descent argument.

We choose $a \in K$ such that $L = K(\sqrt[p]{a})$ and a generator σ of the Galois group.

LEMMA 167. If α_K is injective then

$$\ker\left({}_pK_2(K) \xrightarrow{\text{res}} {}_pK_2(L)\right) = \{\{a, b\} | b \in K\}$$

PROOF. One direction is trivial, as

$$\operatorname{res}\{a, b\} = \{a, b\} = p\{\sqrt[p]{a}, b\} = 0.$$

Conversely, if we are given an element $x \in {}_pK_2(K)$ of the kernel then the commutivity of the diagram implies that $\operatorname{res} \circ \alpha_K(x) = 0$. Note that the kernel of $\operatorname{res} : {}_p\operatorname{Br}(K) \to {}_p\operatorname{Br}(L)$ is just$\operatorname{Br}(L/K)$, which was computed in corollary 60. Thus for some $b \in K$ we have

$$\alpha_K(x) = A_\omega(a, b) = \operatorname{res}\{a, b\}$$

so $x = \{a, b\}$ as α_K is injective.
Since we know the kernel of this restriction map, we would like to compute the image.

LEMMA 168. Suppose α_K is injective and we have an element $x \in (1-\sigma)_p K_2(L)$. Then $\alpha_L(x) = \sigma\alpha_L(x)$ if and only if $x \in \mathrm{image}({}_pK_2(K) \xrightarrow{\mathrm{res}} {}_pK_2(L))$

PROOF. The 'if' direction holds as σ acts trivially on the image of the restrictions. To prove the other direction, we claim that we can produce a sequence $x_1 \ldots x_{p-2}$ such that $(1-\sigma)x_1 = x$, $(1-\sigma)^{i+1}\alpha_K(x_i) = 0$ and

$$(1-\sigma)^i x_i - (1-\sigma)^{i+1} x_{i+1} \in \mathrm{image}\left({}_pK_2(K) \xrightarrow{\mathrm{res}} {}_pK_2(L)\right)$$

If $i < p-2$ then recall that

$$(1-\sigma)^{p-1} = 1 + \sigma + \sigma^2 + \ldots \sigma^{p-1} = N_{L/K}.$$

so we compute

$$\mathrm{res} \circ \mathrm{cor} \circ \alpha_L(x_i) = N_{L/K}\alpha_L(x_i) = (1-\sigma)^{p-2-i}(1-\sigma)^{i+1}\alpha_L(x_i) = 0$$

By corollary 60 and since the corestriction commutes with α we can find $b \in K$ such that

$$\alpha_K\{a, b\} = A_\omega(a, b) = \mathrm{cor} \circ \alpha_L(x_i) = \alpha_K \circ \mathrm{cor}(x_i)$$

and again by the injectivity of α_K we have $\{a, b\} = \mathrm{cor}(x_i)$. We now compute

$$\mathrm{cor}\left(\{\sqrt[p]{a}, b\} - x_i\right) = \{a, b\} - \mathrm{cor}(x_i) = 0$$

We now use corollary 166 of Hilbert's 90 for K_2 to choose elements $y \in {}_pK_2(K)$ and $x_{i+1} \in {}_pK_2(L)$ such that

$$x_i - \{\sqrt[p]{a}, b\} = \mathrm{res}(y) + (1-\sigma)x_{i+1}.$$

Thus

$$\begin{aligned}(1-\sigma)^i x_i \quad &= (1-\sigma)^i\{\sqrt[p]{a}, b\} + (1-\sigma)^i\,\mathrm{res}(y) + (1-\sigma)^{i+1}x_{i+1}\\ &\left\{\left(\frac{\sqrt[p]{a}}{\sigma\sqrt[p]{a}}\right)^i, b\right\} + (1-\sigma)^{i+1}x_{i+1}\\ &\{\tau, b\} + (1-\sigma)^{i+1}x_{i+1}\end{aligned}$$

Since τ is a p'th root of unity, it lies in K by assumption and $\{\tau, b\} \in \mathrm{res}\,{}_pK_2(K)$. Using the above equation we can also show that $(1-\sigma)^{i+1}\alpha_L(x_{i+1}) = 0$ and the claim follows.
We now note that $(1-\sigma)^{p-1}x_{p-1} = \mathrm{res} \circ \mathrm{cor}\, x_{p-1}$, so

$$x = (1-\sigma)x_1 - (1-\sigma)^{p-1}x_{p-1} + \sum_{i=1}^{p-3}\left[(1-\sigma)^{i+1}x_{i+1} - (1-\sigma)^i x_i\right] \in \operatorname{res}{}_p K_2(K)$$

Having proved the above, heavily computational, lemma, we can now prove

Proposition 169. If α_K is injective then Diagram 6 is a pushout square.

Proof. We need only show that is we are given $x \in {}_p \operatorname{Br} K$ and $y \in {}_p K_2(L)$ with $\alpha_L(y) = \operatorname{res}(x)$ then we can find a unique element $z \in {}_p K_2(K)$ making the diagram commute.
Since the corestriction commutes with the norm we have

$$\alpha_K \circ \operatorname{cor}(y) = \operatorname{cor} \circ \alpha_L(y) = \operatorname{cor} \circ \operatorname{res}(x) = 0$$

Thus, by the assumption on α_K, $\operatorname{cor}(y) = 0$. We now apply corollary 166 of Hilbert's 90 for K_2 to choose $u \in {}_p K_2(K)$ and $v \in {}_p K_2(L)$ such that $y = \operatorname{res}(u) + (1-\sigma)v$. If set $v' = (1-\sigma)v$ then

$$(1-\sigma)\alpha_L v' = (1-\sigma)\alpha_L y - (1-\sigma)\alpha_L \operatorname{res}(u) = (1-\sigma)\operatorname{res} x - (1-\sigma)\alpha_L \operatorname{res} u = 0$$

Thus v satisfies the conditions of lemma 168, so $(1-\sigma)v = \operatorname{res} v''$ for some $v'' \in {}_p K_2(K)$.
The sum $u + v''$ is almost what we are looking for, as

$$\operatorname{res}(u + v'') = \operatorname{res} u + (1-\sigma)v'' = y.$$

However,

$$\operatorname{res} \alpha_K(u + v'') = \alpha_L \circ \operatorname{res}(u + v'') = \alpha_L(y) = \operatorname{res}(x),$$

so, by lemma 167 one, last, time, we can choose $b \in K^*$ such that

$$x = \alpha_K(u + v'') + A_\omega(a, b) = \alpha_K(u + v'' + \{a, b\}) = \alpha_K(z)$$

and

$$\operatorname{res} z = \operatorname{res}(u + v'') + \operatorname{res}\{a, b\} = y + p\{\sqrt[p]{a}, b\} = y.$$

Since α_K is injective, z must be unique.

Corollary 170. If α_K is injective then α_L is also injective.

Proof. Setting $x = 0$ and y to be any element of $\ker(\alpha_L : {}_p K_2(L) \to {}_p \operatorname{Br} L)$, we select z as above. Since $\alpha_K(z) = 0$, z vanishes by the injectivity assumption

2. K_2 and Places

We need a relation between ${}_pK_2(K)$ and ${}_pK_2(L)$ if we have a place between the two fields.

PROPOSITION !71. Given a place $p : K \to L$ we can choose homomorphisms

$$\theta : \frac{K^\bullet}{pK^\bullet} \to \frac{L^\bullet}{pL^\bullet}$$
$$\tau : {}_pK_2(K) \to {}_pK_2(L)$$

Such that $\theta(x) = p(x)$ whenever $p(x) \neq \infty$ and

$$\tau\{x, y\} = \{\theta(x), \theta(y)\}$$

PROOF. **3. A Large Construction**

We now come to the main construction. We let $L = K(\sqrt[p]{a})$ as usual and construct the following fields.

Let K_0 be the smallest subfield of K containing the p'th roots of unity and a. Thus K_0 is either $\mathbb{F}_p(\omega, a)$ or $\mathbb{Q}(\omega, a)$ so it is either a global field or a purely transcendental extension of a global field.

For any positive integers l and m we choose integers r_{ij} where $1 \le i \le l$ and $1 \le j \le m$, and define the following variables:

(1) $\tilde{x}_i$ for $1 < i \le n$.
(2) $\tilde{y}_i$ for $1 < i \le n$.
(3) $\tilde{x}_{ij}$ for $n < i \le m$ and $0 \le j < p$.
(4) $\tilde{r}_{ijk}$ for $1 \le i \le m$, $0 \le j < p$, and $0 \le k <$
(5) $\tilde{z}_{ij}$ for $1 \le i \le m$ and $0 \le j < p$.

We now set $K_1 = K_0(\tilde{x}_i, \tilde{y}_i, \tilde{x}_{ij}, \tilde{r}_{ijk})$ to be the purely transcendental extension of K_0 generated by these variables. We now recall equations 4 and 5 and write them in terms of these new variables, which gives us m equations in terms of new variables $\tilde{z}_{ij}$.

$$\tilde{\tau}_i = \sqrt[p]{\prod_{j=1}^{n} \tilde{x}_j^{r_{ij}} \prod_{j=n+1}^{m} \left(\sum_{k=0}^{p-1} \tilde{x}_{jk} \, (\sqrt[p]{a})^k \right)^{r_{ij}}}$$

$$(7) \qquad \tilde{y}_j^{-1} \prod_{i=1}^{l} N_{K(\tau_i)/K} \left(\sum_{f,g=0}^{p-1} \tilde{r}_{ifg} \tilde{\tau}_i^f \, (\sqrt[p]{a})^g \right)^{r_{ij}} = \left(\sum_{i=0}^{p-1} \tilde{z}_{ji} \, (\sqrt[p]{a})^i \right)^p$$

We can expand out the above and collect the terms containing the same powers of $\sqrt[p]{a}$. This gives us a total of pm equations $f_1, \dots, f_{pm}$ in $K_1[\tilde{z}_{ij}]$. We now define $K_2 = K_1[\tilde{z}_{ij}]/(f_1, \dots, f_{pm})$, and define $L_i = K_i(\sqrt[p]{a})$.

PROPOSITION 172. L_i is a p-cyclic extension of K_i and $\dim_{K_1} K_2 = p^{mp}$.

PROOF. We choose a field $\tilde{K} \neq L_2$ such that $L_2 = \tilde{K}(\sqrt[p]{a})$, and let σ be the element of $\mathrm{Gal}(L_2/\tilde{K})$ that maps $\sqrt[p]{a}$ to $\omega\sqrt[p]{a}$. We now define $p_{ij} \in L_2$ by $(1 \leq i \leq m, 0 \leq j < p)$

$$p_{ji} = \sum_{i=0}^{p-1} z_{ji}\omega^{ij} \left(\sqrt[p]{a}\right)^i = \sigma^j \left(\sum_{i=0}^{p-1} z_{ji}\omega^i \left(\sqrt[p]{a}\right)^i\right)$$

Since the determinant of $A_{ij} = \omega^{ij}(\sqrt[p]{a})^i$ is nonzero, we know the system above is invertible and the elements p_{ij} generate L_2 over L_1. We now let LHS_j be the left hand side of equation 7 for a fixed value of j.

$$\sigma^k \left(\mathrm{LHS}_j\right) = \sigma^k \left(\sum_{i=0}^{p-1} z_{ji} \left(\sqrt[p]{a}\right)^i\right)^p = p_{kj}^p$$

Thus L_2 is constructed by adjoining to L_1 a total of pm p'th roots. The result follows as clearly $\sqrt[p]{a} \neq K_2$.

We now consider the diagram

$$\begin{array}{ccc}
{}_pK_2(K_0) & \xrightarrow{\alpha_{K_0}} & {}_p\mathrm{Br}(K_0) \\
\downarrow \mathrm{res} & & \downarrow \mathrm{res} \\
{}_pK_2(K_1) & \xrightarrow[\alpha_{K_1}]{\alpha_{K_1}} & {}_p\mathrm{Br}(K_1) \\
\downarrow \mathrm{res} & & \downarrow \mathrm{res} \\
{}_pK_2(K_2) & \xrightarrow[\alpha_{K_2}]{} & {}_p\mathrm{Br}(K_2)
\end{array}$$

PROPOSITION 173. α_{K_0}, α_{K_1}, and α_{K_2} are injective.

PROOF. We know by [6] that the norm residue homomorphism is an isomorphism for global fields and pure trancendental extensions of global fields. Since K_0 is either global or a purely transcendental extension of a global field and K_1 is a purely transcendental extension of K_0, we know α_{K_0} and α_{K_1} are injective. Since K_2 is formed from continuously adding p'th roots, we can apply proposition 170 pm times so conclude that α_{K_2} is also injective.

With these preliminaries out of the way, we come to the penultimate theorem before Mercury-Suslin.

THEOREM 174. If L/K is a p-cyclic extension with $L = K(\sqrt[p]{a})$ then the kernel of $\mathrm{res} : {}_pK_2(K) \to {}_pK_2(L)$ is precisely $\{\{a, b\} | b \in K^*\}$.

PROOF. We choose n, l, m, v_{ij}, x_i, y_i, x_{ij}, r_{ijk}, and z_{ij} as in corollary 118, so any element of the kernel is given by

$$\sum_{i=1}^{n}\{x_i, y_i\}.$$

We now consider the following diagram:

$$\begin{array}{ccc} {}_pK_2(K_2) & \xrightarrow{\alpha_{K_2}} & {}_p\mathrm{Br}(K_2) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\ {}_pK_2(L_2) & \xrightarrow{\alpha_{L_2}} & {}_p\mathrm{Br}(L_2) \end{array}$$

Note that $\sum\{\tilde{x}_i, \tilde{y}_i\} \in \ker(\mathrm{res} : {}_pK_2(K_2) \to {}_pK_2(L_2))$ as $\tilde{x}_i$ and $\tilde{y}_i$ satisfy, by construction, corollary 118. Since α_{K_2} is injective by proposition 173 we apply proposition 171 one, final, time to conclude that

$$\sum_{i=1}^{n}\{\tilde{x}_i, \tilde{y}_i\} = \{a, \tilde{y}\}$$

for some $\tilde{y} \in K_2$. We have a natural homomorphism

$$\phi : K_0[\tilde{x}_i, \tilde{y}_i, \tilde{x}_{ij}, \tilde{r}_{ijk}, z_{ij}] \to K$$

mapping $\tilde{x}_i \mapsto x_i$, $\tilde{y}_i \mapsto y_i$, etc., which is well defined by construction. This lifts to the field of fractions K_2 to give a place $p : K_2 \to K$. We now apply proposition 171 to choose maps

$$\theta : \frac{K_2^*}{pK_2^*} \to \frac{K^*}{pK^*}$$
$$\tau : {}_pK_2(K_2) \to {}_pK_2(K)$$

such that

$$\sum_{i=1}^{n}\{x_i, y_i\} = \sum_{i=1}^{n}\{\theta\tilde{x}_i, \tilde{y}_i\} = \tau\left(\sum_{i=1}^{n}\{\tilde{x}_i, \tilde{y}_i\}\right) = \tau(\{a, \tilde{y}\}) = \{a, \theta(\tilde{y})\}$$

4. More Lifting

We also have,

PROPOSITION 175. If L/K is a finite Galois extension with $[L : K]$ coprime to p and both α_K and α_L are injective then

$$\begin{array}{ccc} {}_pK_2(L_{i-1}) & \xrightarrow{\alpha_{L_{i-1}}} & {}_p\mathrm{Br}(L_{i-1}) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\ {}_pK_2(L_i) & \xrightarrow{\alpha_{L_i}} & {}_p\mathrm{Br}(L_i) \end{array}$$

is a pushout square.

PROOF. We consider the diagram

$$\begin{array}{ccc} {}_pK_2(K) & \xrightarrow{\alpha_K} & {}_p\operatorname{Br}(K) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ {}_pK_2(L) & \xrightarrow[\alpha_L]{\alpha_L} & {}_p\operatorname{Br}(L) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ {}_pK_2(K) & \xrightarrow[\alpha_K]{} & {}_p\operatorname{Br}(K). \end{array}$$

Since cor $\circ$ res is multiplication of an invertible element, the vertical maps are all isomorphisms.

5. Mercury-Suslin

We can now prove the full Mercury-Suslin theorem.

THEOREM 176. For all K, α_K is injective.

PROOF. We choose n minimal such that we can find a field K such that

$$0 \neq \sum_{i=1}^{n} \{x_i, y_i\} \in \ker \alpha_K .$$

If $n = 1$ then $A_\omega(x_1, y_1) = \alpha_K\{x_1, y_1\}$ is a trivial cyclic algebra. Proposition 53 shows that $y = N_{K(\sqrt[p]{x_1})/K}\, a$ for some $a \in K(\sqrt[p]{x_1})$, so $\{x_1, y_1\} = 0$ by proposition 115. Contradiction.
If $n > 1$ then we set $L = K(\sqrt[p]{x_1})$. We compute

$$0 = \operatorname{res} \alpha_K \left(\sum_{i=1}^{n} \{x_i, y_i\} \right) = \alpha_L \operatorname{res} \left(\sum_{i=1}^{n} \{x_i, y_i\} \right) = \alpha_L \left(0 + \operatorname{res} \sum_{i=2}^{n} \{x_i, y_i\} \right),$$

so res $\sum\{x_i, y_i\}$ vanishes by the minimality of n. Using theorem 174,

$$\sum_{i=1}^{n} \{x_i, y_i\} = \{x_1, a\}$$

which implies $n = 1$. Contradiction.

THEOREM 177. For all K, α_K is an isomorphism.

PROOF. Choose any element $u \in {}_p\operatorname{Br}(K)$ and let L/K be a finite dimensional Galois splitting field. We construct a filtration

$$K = L_0 \subset L_1 \subset \ldots L_n = L$$

such that $\dim_{L_i} L_{i+1}$ is either p or is not divisible by p. We claim we can choose $x_i \in {}_pK_2(L_n)$ such that $\alpha_L(x_i) = \mathrm{res}_{L_i/K}\, u$. We can choose $x_n = 0$ as L is a splitting field for u, and if x_i exists we consider the diagram

$$\begin{array}{ccc} {}_pK_2(L_{i-1}) & \xrightarrow{\alpha_{L_{i-1}}} & {}_p\mathrm{Br}(L_{i-1}) \\ \mathrm{res}\downarrow & & \downarrow\mathrm{res} \\ {}_pK_2(L_i) & \xrightarrow[\alpha_{L_i}]{} & {}_p\mathrm{Br}(L_i) \end{array}.$$

We have $x_i \in {}_pK_2(L_i)$ and $u_i = \mathrm{res}_{L_{i-1}/K}\, u$ such that $\alpha_{L_i}(x_i) = \mathrm{res}\, u_i$, so by either proposition or proposition 175 We can choose $x_{i-1} \in {}_pK_2(L_{i-1})$ with the right property.

Thus we can construct $x_0 \in {}_pK_2(K)$ with $\alpha_K x_0 = u$.

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