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SPACES OF LOCALLY CONSTANT CONNECTION

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

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FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SPACES OF LOCALLY CONSTANT CONNECTION submitted by EVA RUHNAU in partial fulfilment of the requirements for the degree of Master of Science.

J. G. Timourian
Supervisor

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ABSTRACT

Differentiable manifolds which admit a linear connection such that the connection coefficients are constant in a neighborhood of each point are considered. These manifolds are called spaces of locally constant connection. It is shown that they can be characterized in terms of an n-dimensional transitive, abelian group of local affine transformations.

The spaces of locally constant connection are investigated in the case when the connection is the Levi-Civita connection of a metric on these spaces. The relation between the Levi-Civita connection of a metric and the metric is given by a system of first order partial differential equations. The integrability conditions of this system are discussed and sufficient conditions in terms of the connection coefficients are displayed in order to ensure integrability. It is shown that the system of differential equations has a unique solution - given an initial value. An example of a metric of this type is discussed.

The spaces of locally constant Levi-Civita connection turn out to be flat in two dimensions. Furthermore, it is proved that the non-flat spaces of this type - in arbitrary dimensions - are not compact, not of constant curvature and reducible.

Finally, the conjecture is disproved that the linear connection of spaces of locally constant connection is locally symmetric.

PREFACE

The starting point of this thesis goes back to mathematical biology. The growth of a population is described by a differential equation, the Volterra equation. The idea is to interpret the coefficients of this equation as the coefficients of a linear metric connection on a differentiable manifold. In the simplest case, the Volterra equation has constant coefficients. This leads to the following problem: Which differentiable manifolds admit a linear metric connection such that the connection coefficients are constant in a neighborhood of each point? These manifolds will be called spaces of locally constant connection. It seems, that this problem has not yet been treated in differential geometry.

The program of the thesis is the following: The first chapter provides the background for a treatment of the problem just described. The linear connection is defined here in terms of bundles. The main sources for this chapter are [9] and [15].

Section 1 of Chapter 1 introduces Lie transformation groups and the frame bundle of a differentiable manifold. The basic material needed for these concepts is collected in short form in Appendix I.

Section 2 contains the definition of a linear connection and its description on the frame bundle.

The motivation for the introduction of a linear connection is to define parallel transport on a differentiable manifold. This geometric point of view is outlined in section 3.

Section 4 investigates the two basic geometrical invariants of a linear connection, the curvature tensor and the torsion tensor.

The second geometrical structure which can be defined on a differentiable manifold, a metric, is discussed in section 5. The relation between the connection and the metric is pointed out.

Chapter II is an investigation of spaces of locally constant linear connection.

In section 1 an exact definition is given. The ideas behind the introduction of these spaces as referred to above lead to a coordinate dependent description. Thus the aim of this section is to find a coordinate independent, and therefore geometric characterization. For this purpose, transformations on a differentiable manifold which leave the connection invariant are considered. It turns out that a space of locally constant linear connection admits an n-dimensional transitive, abelian group of affine transformations.

The formal relation between a connection and a metric is given by a system of first order partial differential equations considered in section 2. The integrability conditions of this system are investigated. They are conditions on the connection to be a metric connection. Therefore they have to have a geometrical meaning which is not seen in this way.

The geometric treatment of the integrability conditions is implemented in section 3 by means of the holonomy group of a linear connection. A sufficient condition for a connection to be a Levi-Civita connection is found for spaces of locally constant connection. A conjecture is that these restrictions on the connection are also necessary.

In section 4 it is shown that the system of differential equations which relate the connection and the metric has - given an initial value - a unique solution. Then the general form of the metric is displayed and an example of a metric for a space of locally constant connection is given. This metric is conformally flat and the scalar curvature of the connection is negative. If the conjecture of section 3 would turn out to be true, this would be the only possible metric for a space of locally constant connection.

Section 5 contains the proof that all spaces of locally constant connection are flat in two dimensions.

The characterization of the spaces of locally constant connection in terms of an n-dimensional transitive, abelian group of affine transformations is used in section 6. It is shown there that the spaces of locally constant connection cannot be compact, of constant curvature or irreducible, except for the flat case. Furthermore a counterexample to the conjecture that these spaces have a locally symmetric connection is provided in this section.

Although a lot of information is gained about the spaces of locally constant connection, it was not possible to classify them in this work. It should be possible to find out the additional restrictions on the connection to be locally symmetric. Further properties may be derived from the reducibility. But the most important open question is the necessity of the given sufficient integrability conditions. A solution of this problem would give a classification of the spaces of locally constant connection in a satisfactory way.

ACKNOWLEDGEMENT

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CHAPTER I

In order to perform analysis in spaces more general than Euclidean n-space $\mathbf{R}^n$ one introduces the notion of a differentiable manifold. This is a set which is locally in a 1-1 correspondence with open subsets of $\mathbf{R}^n$. Once one has this basic space, one is able to define differentiability of functions, tensor fields and Lie derivatives. These structures are intrinsic to the concept of a differentiable manifold.

But, if one wishes to do geometry on these spaces the analogue of a straight line in $\mathbf{R}^n$ is needed. A straight line in $\mathbf{R}^n$ has the property that the tangent vector remains parallel along the line. It turns out that there is no intrinsic way to define parallel transport on a differentiable manifold; hence an additional structure has to be imposed. This leads to the introduction of a linear connection.

Classically, a connection was defined by Christoffel, 1869 [2] to be a set of n^3 functions on the manifold. The motivation was the following observation: The partial derivatives of a function f on a differentiable manifold are the components of a tensor, the 1-form df. But the partial derivatives $\xi^i{}_{,k}$ of the components ξ^i of a vector field do not form the components of a tensor. Adding a linear combination of ξ^i to $\xi^i{}_{,k}$ — the coefficients are just the functions introduced by Christoffel — one can construct the components of a tensor. The relation to parallel propagation was established by Levi-Civita in 1917.

It was not until 1950 that Koszul [11] arrived at a more rigorous approach. He defined a linear connection as an assignment of a vector field $\nabla_X Y$ with certain properties to each pair X, Y of vector fields.

Also, in 1950 Ehresmann [4] gave the most geometric description of a connection in terms of principal fiber bundles.

In this these only linear connection, defined on the frame bundle of a differentiable manifold are introduced.

The basic ideas are as follows: Take a curve $\sigma: (a,b) \to \mathbb{R}^n$ in $\mathbb{R}^n$ and let t be the parameter along σ. Then the tangent vector to $\sigma(t)$ is denoted by $\frac{d\sigma(t)}{dt}$. In order to define a straight line the change of the tangent vector along the curve has to be zero,

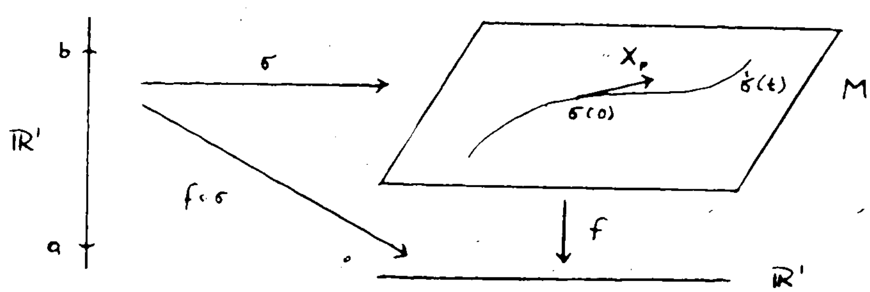
$$\frac{d^2\sigma(t)}{dt^2} = 0,$$

i.e. the tangent vector has to be parallel along the curve.
In analogy, on a differentiable manifold M, one has to describe the change of the tangent vector along a curve.

One knows already how to differentiate a function

$$f: M \to \mathbb{R}$$

along a curve $\sigma: (a,b) \to M$



Let X_p be the tangent vector to $\sigma(t)$ at $p = \sigma(0)$.
The derivative of f in the direction of X_p is given by

$$X_p f = [\frac{d}{dt} (f \circ \sigma)]_{t=0} ,$$

where $f \circ \sigma$ is a map of (a,b) into $\mathbb{R}^1$. In general, one would like to differentiate a tensor field with respect to a vector field defined by a curve. Thus it is a natural question to ask: can one describe tensor fields on M as functions on a differentiable manifold N? If this were possible, the next step would be to transform vector fields on M to vector fields on N and then differentiate the tensor fields of M in N. The last step would be to map the results back to M and compare with the usual definitions of a linear connection.

This program is made precise in this chapter.

1. The frame bundle of a differentiable manifold.

Definition 1.1. A *Lie group* G is a group which is also a differentiable manifold such that the maps

$$G \times G \to G$$

defined by $(a,b) \to a \cdot b$, and

$$G \to G$$

defined by $a \to a^{-1}$ are differentiable.

Example 1.1. Take the general linear group

$$GL(n, \mathbb{R}) := \{(a^i_j) : a^i_j \in \mathbb{R}^{n^2},\ \det a^i_j \neq 0\}.$$

The map $\det: \mathbb{R}^{n^2} \to \mathbb{R}$ is a polynomial map, hence det is continuous. $GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$ therefore $GL(n, \mathbb{R})$ is open in $\mathbb{R}^{n^2}$. This implies that $GL(n, \mathbb{R})$ is a differentiable manifold. Consider

$$g: GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$$
$$(a^i_j, b^k_\ell) \to a^i_j (b^{-1})^j{}_k .$$

Note that g is a polynomial map.

Thus g is differentiable, i.e. the group operations are differentiable functions with respect to the manifold structure. Therefore, $GL(n, \mathbb{R})$ is a Lie group.

<u>Definition 1.2</u>. A *frame* $u = (p, X_i)$ at a point $p \in M$ is an ordered basis $(X_i)_{i=1,\cdots,n}$ of the tangent space $T_p(M)$.

<u>Example 1.2</u>. Take a coordinate system x^i around $p \in M$. The coordinate vectors $\left(\frac{\partial}{\partial x^i}\right)_{i=1,\cdots,n}$ form a frame $u = \left(p, \frac{\partial}{\partial x^i}\right)$ at p.

Denote by $L(M)$ the set of all frames of M:

$$L(M) = \left\{(p, X_i): p \in M, \ X_i \text{ ordered basis of } T_p(M)\right\}.$$

Because there are many different frames at a point $p \in M$, the question arises: How can one transform one frame at a point to another frame at the same point? This leads to the following concept:

<u>Definition 1.3</u>. Let M be a differentiable manifold. A Lie group G acts as a *Lie transformation group* on the right on M if there exists a differentiable map

$$\Phi: M \times G \to M$$

such that

(i) the map

$$R_a: M \to M$$

$$p \mapsto \Phi(p, a) = pa$$

is a diffeomorphism and

(ii) $R_a \circ R_b = R_{ba} \quad \forall\, a,b \in G.$

<u>Note</u>: The action of a Lie group G on a manifold M as a Lie transformation group defines an equivalence relation on M: $p \sim q \iff q = pa$, for some $a \in G$. This equivalence relation identifies points which lie on the same orbit,

where the orbit through $p \in M$ is given by the set $\{pa \in M: \forall a \in G\}$.

Definition 1.4. A vector field X on M such that

$$(R_a)_* X_p = X_{pa} \quad \forall p \in M \quad \forall a \in G$$

is called *right invariant.*

Definition 1.5. The action of G on M is called *free* if

$$R_a p = p \quad \text{for some} \quad p \in M \quad \text{implies} \quad a = e,$$

where e denotes the identity of G.

A Lie transformation group defines a finite-dimensional Lie algebra of vector fields on M. Denote by $\mathfrak{g}$ the Lie algebra of G (see [9], p. 42).

Lemma 1.1. Let G act as a Lie transformation group on the right on a manifold M.

(i) The map

$$\sigma_p : G \to M \qquad \text{defined by}$$
$$a \to \Phi(p , a)$$

is differentiable.

(ii) For any $A \in \mathfrak{g}$, A^* defined by

$$(\sigma_p)_* A_a = A^*_{pa} \qquad A_a \in T_a(G)$$

is a vector field on M, called the fundamental vector field corresponding to A.

(iii) If G acts freely on M, then, for each nonzero $A \in \mathfrak{g}$, A^* never vanishes on M.

Proof: (i) The differentiability of Φ implies the differentiability of σ_p.

(ii) σ_p induces the linear map

$$(\sigma_p)_* : T_a(G) \to T_{pa}(M)$$
$$A_a \to A^*_{..} \quad \forall\, a \in G.$$

To show that $(\sigma_p)_* A_a = A^*_{pa}$ defines a vector field one has to show that A^* is well-defined. Using the left invariance of A one gets

$$(\sigma_p)_* A_a = (\sigma_p)_* (L_a)_* A_e = (\sigma_p \circ L_a)_* A_e = (\sigma_{pa})_* A_e .$$

Hence

$$pa = qb \text{ implies } (\sigma_p)_* A_a = (\sigma_q)_* A_b ,$$
$$q \in M, \quad b \in G$$

(iii) Let a_t be the integral curve of A in G. Because $(\sigma_{pa})_* A_e = (\sigma_p)_* A_a$, Ra_t is the local 1-parameter group generated by A^*.

Suppose $A^*_p = 0$ at some $p \in M$, then

$$R_{a_t}(p) = p \qquad \forall\, t.$$

If G acts freely on M this implies

$$a_t = e \qquad \forall\, t$$

and hence $A = 0$. □

<u>Note</u>: Furthermore, one can show that

$$[A,B]^* = [A^*,B^*] \qquad A,B \in \mathfrak{g} ,$$

hence σ_* defines a Lie algebra homomorphism.

The general linear group $GL(n, \mathbb{R})$ acts freely on the right on $L(M)$, transforming a frame at a point $p \in M$ to another frame at the same point. One can define a map

$$L(M) \times GL(n, \mathbb{R}) \to L(M) \quad \text{by}$$
$$\left((p,X_i),\, a^k_\ell\right) \to (p,\, X_k a^k_i)$$

where

$$R_a : L(M) \to L(M)$$

is given by

$$(p, X_i) \to (p, X_k a^k_i).$$

The action is free because

$$X_k a^k_i = X_i \Rightarrow a^k_i = \delta^k_i \quad .$$

Actually $GL(n, \mathbb{R})$ acts as a Lie transformation group on $L(M)$ because one can make $L(M)$ into a differentiable manifold:

Let $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ be an atlas of M. Every frame can be written uniquely as

$$\left(p,\ X_i = X^k_i \frac{\partial}{\partial x^k}\right)$$

where $X^k_i \in GL(n, \mathbb{R})$ and x^k is a coordinate system around $p \in M$. Define a projection π by

$$\pi : L(M) \to M$$

$$(p, X_i) \mapsto p.$$

Then $\left(\pi^{-1}(U_\alpha), \psi_\alpha\right)_{\alpha \in A}$, where

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times GL(n, \mathbb{R}) \subset \mathbb{R}^n \times \mathbb{R}^{n^2}$$

is given by $(p, X_i) \to (x^i(p), X^k_\ell)$, defines charts of $L(M)$ and

$$\bigcup_{\alpha \in A} \left(\pi^{-1}(U_\alpha)\right) = L(M).$$

It remains to show that these charts are compatible:

On the intersection of two charts $\left(\pi^{-1}(U_\alpha),\Psi_\alpha\right)$ with coordinates (x^i,X^k_ℓ) and $\left(\pi^{-1}(U_\beta),\Psi_\beta\right)$ with coordinates (z^i,Z^k_ℓ) one has

$$X_i = X^k_i \frac{\partial}{\partial x^k} = X^k_i \frac{\partial z^\ell}{\partial x^k}\frac{\partial}{\partial z^\ell} = Z^\ell_i \frac{\partial}{\partial z^\ell} .$$

But $\dfrac{\partial z^\ell}{\partial x^k}$, the Jacobian of the coordinate transformation in M, is a differentiable function because M is a differentiable manifold; hence the two charts are compatible.

Therefore one has established the following Lemma:

<u>Lemma 1.2</u>. If $(U_\alpha,\psi_\alpha)_{\alpha\in A}$ is an atlas of M then $\left(\pi^{-1}(U_\alpha),\Psi_\alpha\right)_{\alpha\in A}$ defines an atlas of L(M) of the same class of differentiability.

If one endows L(M) with the so defined differentiable structure one gets:

<u>Lemma 1.3</u>. The projection map

$$\pi : L(M) \to M$$
$$(p,X_i) \to p$$

is differentiable.

Proof: The map

$$\phi \circ \pi \circ \psi^{-1} : \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$$

defined by

$$(x^i,\, X^k_\ell) \to (x^i)$$

is differentiable.

Hence by definition of differentiability of a map between manifolds, π is differentiable. □

<u>Lemma 1.4</u>. $GL(n, \mathbb{R})$ acts as a Lie transformation group on the differentiable manifold $L(M)$.

Proof: The action of $GL(n, \mathbb{R})$ on $L(M)$ is given by

$$\Phi : L(M) \times GL(n, \mathbb{R}) \to L(M)$$
$$\big((p, X_i), a^k_\ell\big) \to (p, X_j a^j_i).$$

The Cartesian product of two manifolds is again a manifold, so let $(V_\alpha, \gamma_\alpha)_{\alpha\in B}$ be an atlas of $L(M) \times GL(n, \mathbb{R})$.

The map

$$\Psi_\beta \circ \Phi \circ \gamma_\alpha^{-1} : \mathbb{R}^{n(n+1)} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)}$$

defined by

$$\big((x^i, X^k_\ell), a^s_r\big) \to (x^i, X^k_\ell a^\ell_r)$$

is differentiable because $GL(n, \mathbb{R})$ is a Lie group. Hence Φ is differentiable.

Furthermore, one has to show that

$$R_a : L(M) \to L(M)$$

given by

$$\Big(p, X^i_k \frac{\partial}{\partial x^i}\Big) \to \Big(p, X^i_k a^k_\ell \frac{\partial}{\partial x^i}\Big)$$

is a diffeomorphism.

Because $GL(n, \mathbb{R})$ is a Lie group and the action on $L(M)$ is free, R_a is injective, surjective and differentiable, i.e. it is in particular a homeomorphism.

The same is true for

$$R_a^{-1} : L(M) \to L(M)$$
$$\Big(p, X^i_k \frac{\partial}{\partial x^i}\Big) \to \Big(p, X^i_k (a^{-1})^k_\ell \frac{\partial}{\partial x^i}\Big)$$

which implies R_a is a diffeomorphism.

At last

$$R_a \circ R_b = R_{ba} \quad \text{holds} \quad \forall\, a,b \in GL(n, \mathbb{R})$$

because

$$X_k^i \in GL(n, \mathbb{R}). \; \square$$

<u>Definition 1.6</u>. The set $L(M)$ together with the defined differentiable structure is called the *frame bundle* of the differentiable manifold M.

<u>Note</u>: The frame bundle is a principal fiber bundle with structure group $GL(n, \mathbb{R})$. The orbit through a point $u \in L(M)$ is the fiber $\pi^{-1}(p)$, $p = \pi(u)$. The fiber $\pi^{-1}(p)$ is diffeomorphic to $GL(n, \mathbb{R})$.

The original idea was to describe tensor fields on M as functions on another manifold. With the frame bundle $L(M)$ one has found this manifold.

<u>Lemma 1.5</u>. Any tensor field S on M corresponds to a collection of real valued functions on $L(M)$. These functions transform according to a representation of GL determined by the type of S.

Proof: Consider a vector field X on M, i.e. a tensor field of type $\binom{1}{0}$.

Let (U,ϕ) [illegible] chart around $p \in M$ and x^i the coordinate functions. Then the comp[illegible] of X in the coordinate basis are given by

$$[illegible]\,p \quad [illegible] \quad \frac{\partial}{\partial x^i}$$

and in an arbitrary [illegible] defined by

$$X_p \quad [illegible]$$

where $u = (p, X_i)$.

Using the chart $(\pi^{-1}(U),\Psi)$ one calculates $\tilde{\xi}^i(x^k,X^\ell_s)$:

$$X_p = \xi^i(x^k)\frac{\partial}{\partial x^i} = \xi^i(x^k)Y^\ell_i X^s_\ell \frac{\partial}{\partial x^s}$$
$$= \tilde{\xi}^i(u)X_i = \tilde{\xi}^i(u)X^k_i \frac{\partial}{\partial x^k}$$

where

$$Y^\ell_i = (X^\ell_i)^{-1}$$
$$\Rightarrow \quad \tilde{\xi}^i(x^k,X^\ell_s) = \xi^r(x^k)Y^i_r(X^\ell_s).$$

Because of the group action the functions $\tilde{\xi}^i$ fulfil the following transformation property:

$$\tilde{\xi}^i(ua) = (a^{-1})^i_k\,\tilde{\xi}^k(u) .$$

If the values of the ξ^i's are known at one point $u \in L(M)$ they are determined on the fiber $\pi^{-1}(p)$, $p = \pi(u)$.

The generalization for arbitrary tensor fields is now straightforward: If $S \in T^r_s(M)$ is given in the coordinate bases by

$$S = S_{i_1\cdots i_s}{}^{k_1\cdots k_r}\, dx^{i_1}\otimes\cdots\otimes dx^{i_s}\otimes\frac{\partial}{\partial x^{k_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{k_r}}\, ,\quad \text{then}$$

one has

$$\tilde{S}_{\ell_1\cdots\ell_s}{}^{q_1\cdots q_r} = S_{i_1\cdots i_s}{}^{k_1\cdots k_r}\, X^{i_1}_{\ell_1}\cdots X^{i_s}_{\ell_s}\, Y^{q_1}_{k_1}\cdots Y^{q_r}_{k_r} .$$

The action of the group implies

$$\tilde{S}_{\ell_1\cdots\ell_s}{}^{q_1\cdots q_r}(ua) = (a^{-1})^{q_1}_{k_1}\cdots(a^{-1})^{q_r}_{k_r}\, S_{i_1\cdots i_s}{}^{k_1\cdots k_r}(u)\,a^{i_1}_{\ell_1}\cdots a^{i_s}_{\ell_s}. \quad \square$$

Having characterized tensor fields on M as functions on L(M) one is able to differentiate these functions with respect to a vector field on L(M).

Theorem 1.1. Let $\hat{X}$ be a vector field on $L(M)$ invariant under the action of $GL(n, \mathbb{R})$. If S is any tensor field on M, there exists a unique tensor field T of the same type such that

$$\tilde{T} = \hat{X}\,\tilde{S}$$

where $\tilde{S} := \tilde{S}_{i_1 \cdots i_s}{}^{k_1 \cdots k_r}$.

Proof: Given $\hat{X}$ and S, the functions $\hat{X}\,\tilde{S}$ are uniquely determined. To show that T is of the same type as S, one has to check the transformation property. Given $f : L(M) \to \mathbb{R}$, $(R_a)_* \hat{X} = \hat{X}$ $\forall\, a \in GL(n, \mathbb{R})$ implies

$$(\hat{X} f)_{ua} = \big(\hat{X} f(ua)\big)_u .$$

In particular

$$\tilde{T}(ua) = (\hat{X}\,\tilde{S})_{ua} = \big(\hat{X}\,\tilde{S}(ua)\big)_u .$$

With the transformation property of $\tilde{S}$ one gets:

$$\begin{aligned}
\tilde{T}_{\ell_1 \cdots \ell_s}{}^{q_1 \cdots q_r}(ua) &= \Big(\hat{X}\big((a^{-1})_{k_1}{}^{q_1} \cdots (a^{-1})_{k_r}{}^{q_r}\, \tilde{S}_{i_1 \cdots i_s}{}^{k_1 \cdots k_r}(u)\, a_{\ell_1}{}^{i_1} \cdots a_{\ell_s}{}^{i_s}\big)\Big)_u \\
&= (a^{-1})_{k_1}{}^{q_1} \cdots (a^{-1})_{k_r}{}^{q_r} \big(\hat{X}\tilde{S}_{i_1 \cdots i_s}{}^{k_1 \cdots k_r}(u)\big)_u\, a_{\ell_1}{}^{i_1} \cdots a_{\ell_s}{}^{i_s} \\
&= (a^{-1})_{k_1}{}^{q_1} \cdots (a^{-1})_{k_r}{}^{q_r}\, \tilde{T}_{i_1 \cdots i_s}{}^{k_1 \cdots k_r}(u)\, a_{\ell_1}{}^{i_1} \cdots a_{\ell_s}{}^{i_s}
\end{aligned}$$

hence $\tilde{T}$ has the correct transformation behavior. □

Note: If one takes a vector field $\hat{X}$ on $L(M)$, then $\pi_*(\hat{X})$ does - in general - not define a vector field on M. But if $\hat{X}$ is invariant under $(R_a)_*$, then $\pi_*(\hat{X})$ defines a vector field on M.

2. Linear connection.

The next step is to take a vector field X on M and transfer it to the frame bundle $L(M)$. Because one wants the derivative of a tensor field with respect to X depending only on X_p, $p \in M$ and not on X in a neighborhood of p, the vector field should be lifted in such a way that $\hat{X}_{\pi^{-1}(p)}$ is uniquely determined by X_p. So one needs an instruction to lift X in this way.

This instruction is gained by introducing a linear connection on M:

Definition 2.1. A *linear connection* Γ on M is an assignment of a subspace $H_u \subset T_u(L(M))$ to each $u \in L(M)$ such that

(i) $\pi_* H_u = T_{\pi(u)}(M)$

(ii) $(R_a)_* H_u = H_{ua}$

(iii) H_u depends differentiably on u.

The space H_u is called the *horizontal* subspace of $T_u(L(M))$.

Note: A linear connection is a distribution

$$u \to H_u \quad \text{on} \quad L(M).$$

Condition (ii) means that the distribution is invariant by $GL(n, \mathbb{R})$, condition (iii) means that the distribution is differentiable.

When does a linear connection on a differentiable manifold M exist? This question i swered by

Theorem 2.1. If a differentiable manifold M is paracompact, then the frame bundle $L(M)$ admits a linear connection.

Proof: See [; p. 67].

Note: Because the manifolds which are considered here are assumed to be paracompact they all admit a linear connection. Actually it is also necessary to assume that the manifold is paracompact. There exists a theorem due to Geroch [7] that if a Hausdorff C^3-manifold M admits a C^1-connection, then M is paracompact.

Now one can lift a vector field on M in the desired way:

Lemma 2.1. Let X, Y be vector fields on M and Γ a linear connection on M. There exists a unique horizontal lift $\hat{X}$ of X such that

$$(R_a)_* \hat{X} = \hat{X}$$

and

$$\hat{X} + \hat{Y} = \widehat{X + Y}.$$

Proof: Given the connection Γ, define $\hat{X}_u$, $u \in L(M)$ by $\hat{X}_u \in H_u$ and $\pi_* \hat{X}_u = X_{\pi(u)}$. The projection $\pi : L(M) \to M$ induces the linear map

$$\pi_* : T_u\big(L(M)\big) \to T_{\pi(u)}(M) \quad \forall\, u \in L(M)$$

and

$$\pi_*|H_u : H_u \to T_{\pi(u)}(M)$$

is an isomorphism.

This implies that $\hat{X}_u$ is uniquely defined and that

$$\hat{X} + \hat{Y} = \widehat{X + Y}.$$

Since the horizontal subspaces are invariant under $GL(n, \mathbb{R})$, it follows that $(R_a)_* \hat{X} = \hat{X}$. □

One is now able to define the directional derivative of a tensor field with respect to a vector field on a differentiable manifold M:

Definition 2.2. Let S be a tensor field defined around $p \in M$ and

$X_p \in T_p(M)$. The *covariant derivative* of S in the direction of X at p is a tensor $\nabla_{X_p} S$ defined by

$$\widetilde{\nabla_{X_p} S} = (\hat{X}_p \, \tilde{S})_{\pi^{-1}(p)} .$$

<u>Note</u>: Because of Theorem 1.1, $\nabla_{X_p} S$ is uniquely defined and is a tensor of the same type as S.

The covariant derivative satisfies the following properties:

<u>Theorem 2.2</u>. Let X, Y be vector fields and S, T tensor fields defined on M. Then

(1) $\nabla_X S + \nabla_Y S = \nabla_{X+Y} S$

(2) $\nabla_{fX} S = f \nabla_X S$

(3) $\nabla_X (S + T) = \nabla_X S + \nabla_X T$

(4) $\nabla_X (fS) = f \nabla_X S + (Xf) S$

(5) $\nabla_X (S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$

where $f : M \to \mathbb{R}$.

Proof: (1) is an immediate consequence of Lemma 1.5.

(2) is obvious.

For proofs of (3)-(5) see [9, p. 116]. □

<u>Note</u>: Properties (1)-(4) (for $S, T \in T^1_0(M)$) were the conditions used by Koszul to define a linear connection.

<u>Definition 2.3</u>. Let S be a tensor field, $S \in T^r_s(M)$. The tensor field $\nabla S \in T^r_{s+1}(M)$ is called the *covariant derivative* and is defined by

$$(\nabla S)(X, X_1, \cdots, X_s, \omega^1, \cdots, \omega^r) = (\nabla_X S)(X_1, \cdots, X_s, \omega^1, \cdots, \omega^r)$$

where $X_i \in T^1_0(M)$ and $\omega^i \in T^0_1(M)$ $\forall\, i$.

Once the linear connection is defined one has to ask how to describe this assignment of a horizontal vector space to each point of the frame bundle in more detail.

There can be defined n 1-forms on the frame bundle:

<u>Definition 2.4</u>. Let $\theta^i\colon T_u\big(L(M)\big) \to \mathbb{R}$ $(i = 1,\cdots,n)$ be defined by $\pi_*(X_u) = \theta^i(X_u)X_i$ for $u = (p, X_i)$. The n 1-forms θ^i are called the *canonical 1-forms* on the frame bundle.

Given a linear connection Γ on M, any $X \in T_u\big(L(M)\big)$ can be uniquely written as

$$X = hX + vX$$

where

$hX \in H_u$ and is called the horizontal component of X

and

vX is tangent to the fibers and called the vertical component of X.

One has an additional set of n^2 1-forms on $L(M)$:

<u>Definition 2.5</u>. Let E_k^i be a basis of $g\ell(n, \mathbb{R})$, $\overset{*}{E}{}_k^i$ the corresponding vector fields on $L(M)$ and X a vector field on M. The vertical component of X can be written

$$vX = \omega_k^i(X)\overset{*}{E}{}_i^k \,.$$

The 1-forms ω_k^i are called the *connection forms* of Γ on the frame bundle.

The canonical forms and the connection forms together form a basis of $T_u^*\big(L(M)\big)$ at any $u \in L(M)$. Now the question arises: What are the vector fields dual to these 1-forms?

Definition 2.6. Take $X \in T_u(L(M))$. It is called *horizontal* if $X \in H_u$.

Lemma 2.2. Take $\xi^i \in \mathbb{R}^n$. Then there exists a unique horizontal vector field $B(\xi^i)$ on $L(M)$ such that

$$\pi_* B_u(\xi^i) = \xi^i X_i \quad \text{if} \quad u = (p, X_i).$$

Proof: Because

$$\pi_*|H_u : H_u \to T_{\pi(u)}(M)$$

is a vector space isomorphism, the unique existence follows immediately. □

Denote by B_i $(i = 1, \cdots, n)$ the horizontal vector fields corresponding to the natural basis of $\mathbb{R}^n$, i.e. for example

$$\xi^1 = (1,0,\cdots,0) \quad \text{and} \quad \pi_* B_1 = X_1.$$

The vector fields B_i are called the *standard horizontal vector fields* and form a basis of the horizontal subspaces of $T_u(L(M))$ for each $u \in L(M)$.

Lemma 2.3. The vector fields $\overset{*}{E}{}^i_k$, B_ℓ and the 1-forms ω^i_k, θ^ℓ are dual and satisfy

$$\omega^i_k(\overset{*}{E}{}^\ell_s) = \delta^i_s \delta^\ell_k \qquad \omega^i_k(B_\ell) = 0$$

$$\theta^i(B_k) = \delta^i_k \qquad \theta^i(\overset{*}{E}{}^k_\ell) = 0.$$

Proof: These properties are a direct consequence of the definitions. The vector fields B_ℓ are horizontal, so $\omega^i_k(B_\ell) = 0$. The vector fields $\overset{*}{E}{}^k_\ell$ are tangent to the fibers, so $\theta^i(\overset{*}{E}{}^k_\ell) = 0$ $\omega^i_k(\overset{*}{E}{}^\ell_s)\overset{*}{E}{}^k_i = \overset{*}{E}{}^\ell_s$ which implies $\omega^i_k(\overset{*}{E}{}^\ell_s) = \delta^i_s \delta^\ell_k$. If $n = (p, X_i)$, then

$\pi_*(B_k)_u = \theta^i(B_k) X_i = X_k$ which implies $\theta^i(B_k) = \delta^i_k$. □

The next theorem shows that the connection is unique if one knows ω^i_k or B_i at one point of each fiber:

<u>Theorem 2.3</u>. For any $a^i_k \in GL(n, \mathbb{R})$ the vector fields B_i, E^{*k}_ℓ and the 1-forms θ^i, ω^k_ℓ transform in the following way:

(1) $(R_a)_* B_i = (a^{-1})^k_i B_k$

(2) $(R_a)_* E^{*i}_k = a^i_\ell E^{*\ell}_s (a^{-1})^s_k$

(3) $(R_a)^* \theta^i = (a^{-1})^i_k \theta^k$

(4) $(R_a)^* \omega^i_k = a^i_\ell \omega^\ell_s (a^{-1})^s_k$.

Proof: The transformation properties are easy to calculate using the fact that the horizontal subspaces H_u are invariant under $GL(n, \mathbb{R})$ and that $GL(n, \mathbb{R})$ acts freely on $L(M)$.
For details see [15; p. 312]. □

The description of the fields in a coordinate system is given by

<u>Theorem 2.4</u>. Let (U,ϕ) be a chart on M and (x^i, X^k_ℓ) the coordinates in $\pi^{-1}(U)$. If a connection Γ is given on M, there exist functions

$\Gamma^i_{k\ell}: U \to \mathbb{R}$, such that

(1) $$\frac{\hat{\partial}}{\partial x^i} = \frac{\partial}{\partial x^i} - \Gamma^k_{i\ell} X^\ell_s \frac{\partial}{\partial X^k_s}$$

(2) $$B_i = X^k_i \left(\frac{\partial}{\partial x^k} - \Gamma^\ell_{ks} X^s_r \frac{\partial}{\partial X^\ell_s}\right).$$

(3) $$\omega^i_k = Y^i_\ell dX^\ell_k + Y^i_r \Gamma^r_{\ell s} X^s_k dx^\ell$$

(4) $$\theta^i = Y^i_k dx^k$$

(5) $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = \Gamma^{\ell}_{ik} \frac{\partial}{\partial x^{\ell}}$.

Proof:

(1) Assume $\frac{\hat{\partial}}{\partial x^i} \in H_u$ where $u = (x^i, \delta^k_{\ell})$. Because $\pi_* \frac{\hat{\partial}}{\partial x^i} = \frac{\partial}{\partial x^i}$, there exist functions $\Gamma^i_{k\ell}$ such that

$$\frac{\hat{\partial}}{\partial x^i} = \frac{\partial}{\partial x^i} - \Gamma^k_{i\ell} \frac{\partial}{\partial X^k_{\ell}} .$$

Now

$$(R_a)_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} \qquad \text{and}$$

$$(R_a)_* \frac{\partial}{\partial X^k_{\ell}} = a^{\ell}_s \frac{\partial}{\partial X^k_s} \qquad \text{imply (1).}$$

(2) The standard horizontal vector fields B_i can be written as a linear combination of $\frac{\hat{\partial}}{\partial x^i}$:

$$(B_i)_u = C^k_i \left(\frac{\partial}{\partial x^k} - \Gamma^{\ell}_{ks} X^s_r \frac{\partial}{\partial X^{\ell}_r} \right) .$$

If $\quad u = (p, X_i) = \left(p, X^k_i \frac{\partial}{\partial x^k}\right)$ then

$$\pi_*(B_i)_u = X^k_i \frac{\partial}{\partial x^k} = C^k_i \frac{\partial}{\partial x^k}$$

$$\Longrightarrow C^k_i = X^k_i \qquad \Longrightarrow (2).$$

(3),(4) $E^{*i}_k = X^{\ell}_k \frac{\partial}{\partial X^{\ell}_i}$ and ω^i_k and θ^i are dual to E^{*i}_k and B_i. This, together with (2), gives (3) and (4).

(5) In correspondence with the vector fields $\frac{\partial}{\partial x^i}$ are the real valued functions

$$\frac{\tilde{\partial}}{\partial x^i} = Y^q_i \quad \text{on} \quad L(M). \qquad \text{(See Lemma 1.5.)}$$

One gets

$$\frac{\hat{\partial}}{\partial x^k}\left(\frac{\tilde{\partial}}{\partial x^i}\right) = \left(\frac{\partial}{\partial x^k} - \Gamma^{\ell}_{ks} X^s_r \frac{\partial}{\partial X^{\ell}_r}\right) Y^q_i$$

$$= \Gamma^{\ell}_{ks} X^s_r Y^q_{\ell} Y^r_i$$

$$= \Gamma^{\ell}_{ki} Y^q_{\ell} = \Gamma^{\ell}_{ki} \widetilde{\frac{\partial}{\partial x^{\ell}}}$$

because $\dfrac{\partial Y^q_i}{\partial X^{\ell}_r} = -Y^q_{\ell} Y^r_i$. This implies (5). □

<u>Note</u>: As seen from the last theorem, the n^3 functions $\Gamma^i_{k\ell}$ determine the connection Γ uniquely.

These functions $\Gamma^i_{k\ell}$ are the same as the ones used to introduce a classical connection since they fulfil the transformation property required:

<u>Lemma 2.4</u>. Let (U,ϕ) and (U',ϕ') be two charts around $p \in M$ with coordinate functions x^i and y^i.
Given a connection Γ on M, the connection coefficients $\Gamma^i_{k\ell}$ transform in the following way:

$$\underset{\phi'}{\Gamma}{}^r_{st} = \underset{\phi}{\Gamma}{}^i_{k\ell} \frac{\partial x^k}{\partial y^s} \frac{\partial x^{\ell}}{\partial y^t} \frac{\partial y^r}{\partial x^i} + \frac{\partial^2 x^{\ell}}{\partial x^s \partial x^t} \frac{\partial y^r}{\partial x^{\ell}} .$$

Proof: $\dfrac{\partial}{\partial y^k} = \dfrac{\partial x^i}{\partial y^k} \dfrac{\partial}{\partial x^i}$ and the properties of the covariant derivative (Theorem 2.2) together with Theorem 2.4(5) imply this transformation rule. □

<u>Note</u>: As seen from the transformation behaviour the $\Gamma^i_{k\ell}$ do not define a tensor. But on the frame bundle a linear connection corresponds to tensors (the connection forms ω^i_k).

A consequence of Theorem 2.4 and Lemma 2.4 is the following:

<u>Theorem 2.5</u>. A linear connection on M can be defined by a map which assigns to any pair of vector fields X, Y a vector field $\nabla_X Y$ on M such that

(1) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$

(2) $\nabla_{fX} Y = f\nabla_X Y$

(3) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

(4) $\nabla_X (fY) = f\nabla_X Y + (Xf)Y$,

where $f: M \to \mathbb{R}$.

This gives the equivalence of a linear connection defined on the frame bundle and a Koszul connection.

3. Parallelism.

Given a connection on a differentiable manifold M, one can define the concept of parallel displacement of a frame along a curve in M. This leads to the problem of lifting curves in M horizontally to the frame bundle of M.

Definition 3.1. Let

$$\tau : (a,b) \to L(M)$$
$$s \to \tau(s)$$

be a differentiable curve in $L(M)$. The curve $\tau(s)$ is called *horizontal* if

$$\dot{\tau}(s) \in H_{\tau(s)} \quad \forall\, s \in (a,b)$$

where $\dot{\tau}(s)$ is tangent to τ at s.

It is possible to get a unique horizontal lift of a curve in M:

Theorem 3.1. Let

$$\sigma : [0,1] \to M$$
$$t \to \sigma(t)$$

be a differentiable curve in M. Take $p = \sigma(0) \in M$ and $u_0 \in \pi^{-1}(p)$. Then there exists a unique curve $\hat{\sigma} : [0,1] \to L(M)$ such that

$$\hat{\sigma}(0) = u_0$$
$$\pi\left(\hat{\sigma}(t)\right) = \sigma(t)$$
$$\dot{\hat{\sigma}}(t) \in H_{\hat{\sigma}(t)} .$$

Proof: Because there exists the unique lift of a vector field, the problem of the unique lift of a curve is already solved piecewise. One has to show that one can join these lifted pieces to get a lift of the whole curve. For details see [15; p. 311]. □

The relation between horizontal curves and the covariant derivative is given by:

Lemma 3.1. Let $\sigma(t)$ be a curve on M and $\hat{\sigma}(t)$ its horizontal lift. For any tensor field S on M one has

$$\widetilde{(\nabla_{\dot\sigma(0)} S)}_{\hat\sigma(0)} = [\frac{d}{dt}\left(\tilde{S}(\hat\sigma(t))\right)]_{t=0} .$$

Proof: Since $\hat{\sigma}(t)$ is a horizontal curve $\Rightarrow \dot{\hat{\sigma}}(0) \in H_{\hat\sigma(0)}$. Thus

$$\widetilde{(\nabla_{\dot\sigma(0)} S)}_{\hat\sigma(0)} = \left(\dot{\hat\sigma}(0)\tilde{S}\right)_{\hat\sigma(0)}$$

$$= [\frac{d}{dt}\left(\tilde{S}(\hat\sigma(t))\right)]_{t=0} . \quad \square$$

This leads to the following definition:

Definition 3.2. Let X be a vector field and $\sigma(t)$ a curve in M. Then X is said to be *parallel* along $\sigma(t)$ if

$$\nabla_{\dot\sigma(t)} X = 0 \quad \forall\ t .$$

Note: If $\tau(t)$ in $L(M)$ is described by $\tau(t) = \left(\sigma(t), X_i(t)\right)$ then the frame X_i is parallel along $\sigma(t)$ when (t) is a horizontal curve.

Because the horizontal subspaces H_u are univariant under $GL(n, \mathbb{R})$, every horizontal curve is mapped into a horizontal curve by the action of $GL(n, \mathbb{R})$. Hence the parallel propagation along any differentiable curve commutes with the action of $GL(n, \mathbb{R})$ on $L(M)$ and defines a vector space isomorphism between the tangent spaces along this curve.

Remember that the tangent vector of a straight line in $\mathbb{R}^n$ remains parallel along the line. Thus one is led quite naturally to the following definition which gives the analogy of straight lines in a differentiable manifold M:

Definition 3.4. Let $\sigma : (a,b) \to M$ be a differentiable curve in M.

Then $\sigma(t)$ is called a *geodesic* if its tangent vector $\dot{\sigma}(t)$ is parallel along $\sigma(t)$, i.e.

$$\nabla_{\dot{\sigma}(t)}\dot{\sigma}(t) = 0 \quad \forall\ t \in (a,b).$$

The relation between a geodesic and the description of a linear connection developed in section 2 is established by the following theorem:

Theorem 3.2. Let B be a horizontal vector field on $L(M)$ and $\tau(t)$ the integral curve of B. Then $\pi\left(\tau(t)\right)$ is a geodesic and every geodesic is obtained in this way.

Proof: See [9; p. 139].

4. Curvature and torsion of a linear connection.

All information about a linear connection Γ on M is contained in the standard horizontal vector fields. These vector fields are used to construct two geometrical objects, the curvature and the torsion tensor fields. They are the basic invariants of a given linear connection.

Theorem 4.1. Let B_i be the standard horizontal vector fields of the connection Γ on M. The functions $\tilde{R}^\ell{}_{sik}$ and $\tilde{T}^\ell{}_{ik}$ on $L(M)$ defined by

$$(*) \qquad [B_i, B_k] = -\tilde{R}^\ell{}_{sik} \, E^{*s} - \tilde{T}^\ell{}_{ik} \, B_\ell$$

are the components of two tensor fields $R \in T^1_3(M)$, the *curvature tensor* and $T \in T^1_2(M)$, the *torsion tensor*.

Proof: One has to show that $\tilde{R}^\ell{}_{iks}$ and $\tilde{T}^\ell{}_{ik}$ have the right transformation properties, i.e.

$$\tilde{R}^\ell{}_{iks}(ua) = (a^{-1})^\ell_r \, \tilde{R}^r{}_{qvt}(u) a^q_i \, a^v_k \, a^t_s$$

$$\tilde{T}^\ell{}_{ik}(ua) = (a^{-1})^\ell_r \, \tilde{T}^r{}_{qv}(u) a^q_i \, a^v_k \, .$$

Apply $(R_a)_*$ to both sides of equation (*). Then the transformation properties are an immediate consequence of the transformation properties of the standard horizontal vector fields. □

Note: If one introduces a linear connection as a Koszul connection, it can be shown that the curvature and torsion tensors are given by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, where X, Y, Z are vector fields on M. The first equation expresses the fact that the second covariant derivatives do not generally commute.

Theorem 4.2. Let Γ be a linear connection on M and the connection forms ω^i_k be defined with respect to the natural basis in $GL(n, \mathbb{R})$. Then

$$d\omega^i_k = -\omega^i_\ell \wedge \omega^\ell_k + \frac{1}{2}\tilde{R}^i{}_{k\ell s}\ \theta^\ell \wedge \theta^s$$

$$d\theta^i = -\omega^i_k \wedge \theta^k + \frac{1}{2}\tilde{T}^i{}_{k\ell}\ \theta^k \wedge \theta^\ell.$$

These equations are called the *structure equations* on $L(M)$.

Proof: Let e_a $(a = 1, \cdots, n(n+1))$ be a basis of $T_u(L(M))$ for all $u \in L(M)$ and let e^a be the dual 1-forms, i.e. $e^a(e_b) = \delta^a_b$. If $[e_a, e_b] = C^c{}_{ab}\ e_c$ then

$$de^a = -\frac{1}{2}C^a{}_{bc}\ e^b \wedge e^c\ ,$$

because $de^a(e_b, e_c) = \frac{1}{2}\left(e_b e^a(e_c) - e_c e^a(e_b) - e^a([e_b, e_c])\right)$. Using the local expressions for B_i given in Theorem 2.4, one calculates

$$[E^{*i}_k, B_\ell] = \delta^i_\ell\ B_k \qquad \text{and}$$

$$[E^{*i}_k, E^{*\ell}_s] = \delta^i_s\ E^{*\ell}_k - \delta^\ell_k\ E^{*i}_s\ .$$

Replacing e_a and e^a by B_i, E^{*k}_ℓ and ω^i_k, θ^ℓ respectively, these equations together with Theorem 4.1 imply the structure equations. □

What meaning do the structure equations on $L(M)$ have for M? They can be expressed on M in the following way:

Definition 4.1. Let (U, ϕ) be a chart on M. A differentiable map

$$\alpha : U \to L(M) \qquad \text{such that}$$

$$\pi \circ \alpha = id_U$$

is called a *local cross section* of $L(M)$.

Note: A local cross section is just a differentiably varying frame at each point of U.

Any coordinate system x^i induces a local cross section

$$\alpha : U \to L(M)$$
$$(x^i) \to \left(x^i, \frac{\partial}{\partial x^i}\right) .$$

Using the induced linear map

$$\alpha^* : T^r_s\big(L(M)\big)_u \to T^r_s(M)_{\pi(u)}$$

one gets the pulled back equations on M.

<u>Theorem 4.3</u>. Let α be a local cross section. The structure equations on M are

$$d(\alpha^* \omega^i_k) = -\alpha^* \omega^i_\ell \wedge \alpha^* \omega^\ell_k + \frac{1}{2} R^i{}_{k\ell s}\, \alpha^* \theta^\ell \wedge \alpha^* \theta^s$$
$$d(\alpha^* \theta^i) = -\alpha^* \omega^i_k \wedge \alpha^* \theta^k + \frac{1}{2} T^i{}_{k\ell}\, \alpha^* \theta^k \wedge \alpha^* \theta^\ell$$

where

$$R^i{}_{k\ell s} := \alpha^* \tilde{R}^i{}_{k\ell s}$$
$$T^i{}_{k\ell} := \alpha^* \tilde{T}^i{}_{k\ell} .$$

Proof: The structure equations on M are just a consequence of the structure equations on $L(M)$ using the properties of the induced linear map α^*.
For details see [15; p. 317]. □

The structure equations on M will be used later on to calculate the Riemann tensor.

In a coordinate system, the forms $\alpha^* \omega^i_k$ and $\alpha^* \theta^i$ are described in the following way:

<u>Lemma 4.1</u>. Let x^i be a coordinate system on M and α the local cross

section

$$\alpha : U \to L(M)$$

$$(x^i) \to \left(x^i, \frac{\partial}{\partial x^i}\right) .$$

Then

$$\alpha^* \omega^i_k = \Gamma^i{}_{\ell k}\, dx^\ell$$

$$\alpha^* \theta^i = dx^i .$$

Proof: This is an immediate consequence of Theorem 2.4. □

5. Riemannian Connection.

One can add to a differentiable paracompact manifold M a structure which makes M into a Riemannian metric space.

Definition 5.1. A metric on a differentiable manifold M is an assignment of a tensor $g \in T_2^0(M)$ to each $p \in M$ such that

$$g(X,Y) = g(Y,X) \quad \text{for all } X, Y ,$$

where X, Y are vector fields on M.

In other words, g assigns an inner product in each tangent space $T_p(M)$.

Example 5.1. The Euclidean metric on $\mathbf{R}^n$ is defined by

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = \delta_{ik}$$

where x^i denotes the natural coordinate system.

The metric is called *positive definite* if

$$g(X,X) \geq 0 \qquad \forall X$$

and

$$g(X,X) = 0 \iff X = 0 ,$$

and *non-degenerate* if

$$g(X,Y) = 0 \quad \forall X \Rightarrow Y = 0.$$

If the metric is non-degenerate and positive definite it is called a Riemannian metric.

Theorem 5.1. Every paracompact differentiable manifold admits a Riemannian metric.

Proof: See [9]. The proof is based on the fact that the frame bundle

L(M) can be reduced to a bundle of orthonormal frames in the paracompact case. □

<u>Definition 5.2</u>. A connection Γ on M is called a *metric connection* if

$$\nabla g = 0.$$

It is called the *Levi-Civita connection* of g if, in addition, the torsion of Γ vanishes.

<u>Lemma 5.1</u>. $\nabla g = 0 \iff$ parallel propagation preserves the inner product.

Proof: $\nabla g = 0 \iff (\nabla_X g)(Y,Z) = 0$

$$\iff \hat{X}\,\tilde{g} = 0$$

$$\forall\, X,Y,Z \in T^1_0(M). \quad \square$$

Is there a relation between two geometric structures, the metrics and the connections on a differentiable manifold? The choice of a metric connection for a given metric is by no means unique. But the restriction to torsion free connections gives the following important result:

<u>Theorem 5.2</u>. Every non-degenerate metric g on M admits a unique connection with vanishing torsion.

Proof:

$$(\nabla_X g)(Y,Z) = X\,g(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z).$$

Suppose $\nabla g = 0 \Rightarrow Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) = 0.$ Cyclically permutating X, Y, Z gives

$$Z\,g(X,Y) - g(\nabla_Z X,Y) - g(X,\nabla_Z Y) = 0$$

$$Y\,g(Z,X) - g(\nabla_Y Z,X) - g(Z,\nabla_Y X) = 0.$$

Adding the first and third equation and subtracting the second leads to

$$Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) - g(\nabla_Y Z,X) -$$

$$- g(Z,\nabla_Y X) + g(\nabla_Z X,Y) + g(X,\nabla_Z Y) = 0.$$

Assume further that Γ is torsion free

$$\Rightarrow \quad \nabla_X Y - \nabla_Y X = [X,Y].$$

Inserting this in the last equation on gets:

$$(*) \qquad 2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) +$$

$$+ g([Z,X],Y) - g([Y,Z],X).$$

Hence from the assumption that Γ is metric and torsion free one can derive an expression which determines Γ uniquely in terms of g. Defining Γ by the last equation one can show that $\nabla_X Y$ satisfies the condition of Theorem 2.5 and therefore defines a linear connection. □

<u>Note</u>: A proof of this theorem in the frame bundle can be found in [9; p. 159].

<u>Proposition 5.1</u>. Let x^i be a coordinate system on M. Let g be a non-degenerate metric on M and Γ the unique Levi-Civita connection determined by g. Then

$$\Gamma^i_{k\ell}\, g_{is} = g_{s(\ell,k)} - \frac{1}{2}\, g_{k\ell,s}$$

holds.

Proof: For the coordinate basis $\frac{\partial}{\partial x^i}$ one has

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right] = 0 \quad \text{and}$$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = \Gamma^\ell_{ik} \frac{\partial}{\partial x^\ell}.$$

Thus the assertion follows from (*). □

CHAPTER II

1. Spaces of locally constant linear connection.

It turned out in Chap I that a linear connection defined a geometrical structure on a differentiable manifold M because it provided M with the concept of parallel propagation. Moreover, every paracompact differentiable manifold admits a linear connection as well as a Riemannian metric. If one wants to consider spaces with a specified geometry, for example spaces in which the parallel propagation of tensors between two points does not depend on the curve joining these two points, one has to impose restrictions on the linear connection.

The question is then: What are the underlying spaces allowing such a specified connection?

In case that parallel transport does not depend on the curve, the curvature tensor of the connection has to vanish. These spaces are called *flat*. The most elementary example of a flat differentiable manifold is Euclidean n-space $\mathbb{R}^n$. It has a global chart such that the connection coefficients with respect to this chart are zero.

Another interesting case to consider could be a linear connection which does not vary in a neighborhood of each point of the manifold.

Differentiable manifolds admitting a linear connection of this type are treated in this chapter.

Definition 1.1. Let M be a differentiable manifold. If M admits a linear connection Γ such that for every $p \in M$ there exists a chart containing p in which the connection coefficients with respect to this chart are constant, M is called a *space of locally constant linear connection.*

As defined, the property of a differentiable manifold being a space of locally constant connection is described in terms of local coordinate systems. The aim of this section is to give a more geometrical and coordinate independent characterization of these spaces by means of transformations and groups of transformations.

Definition 1.2. Let $\Phi : M \to M$ be a map. If Φ is a diffeomorphism, Φ is called a *transformation* of M.

Looking at the frame bundle $L(M)$ one gets:

Any transformation Φ of M induces an automorphism

$$\tilde{\Phi} : L(M) \to L(M) \text{ of the frame bundle}$$

$$(p, X_i) \to \big(\Phi(p), \Phi_* X_i\big).$$

Note: Since $\tilde{\Phi}$ is an automorphism of the bundle $L(M)$ it leaves the fibers invariant.

Certain tensor fields such as the canonical forms or the connection forms are already defined on the frame bundle. What are the effects of a transformation on M on these tensor fields?

Using the canonical $\mathbb{R}$-valued 1-forms θ^i one can introduce an $\mathbb{R}^n$-valued 1-form on $L(M)$:

Definition 1.3. The $\mathbb{R}^n$-valued 1-form $\theta = (\theta^1, \cdots, \theta^n)$ defined by

$$\theta : T_u\big(L(M)\big) \to \mathbb{R}^n$$

$$\hat{X}_u \to \theta^i(\hat{X}_u) e_i ,$$

where e_i is the natural basis in $\mathbb{R}^n$, is called the *canonical form* on $L(M)$.

This gives another description of a frame:
A frame $u = (p, X_i)$ can be considered as a map

$$u : \mathbb{R}^n \to T_p(M)$$
$$e_i a_k^i \to X_i a_k^i \ .$$

Then θ can be defined by

$$\theta : T_u\big(L(M)\big) \to \mathbb{R}^n$$
$$\hat{X}_u \to u^{-1}(\pi_* \hat{X}_u)$$

for all $u \in L(M)$.

<u>Lemma 1.1</u>. Let $\Phi : M \to M$ be a transformation on a differentiable manifold M. The induced automorphism

$$\tilde{\Phi} : L(M) \to L(M)$$

leaves the canonical form invariant, i.e.

$$\tilde{\Phi}^* \theta_{\tilde{\Phi}(u)} = \theta_u \quad \text{for all} \quad u \in L(M).$$

Conversely, every fiber preserving transformation of $L(M)$ leaving θ invariant is induced by a transformation on M.

Proof: Let $\hat{X}_u \in T_u\big(L(M)\big)$, $\pi_*(\hat{X}_u) = X_p \in T_p(M)$ where $p = \pi(u)$.
Then $\Theta(\hat{X}_u) = u^{-1}(X_p)$
and $\Theta(\tilde{\Phi}_* \hat{X}_u) = \big(\tilde{\Phi}(u)\big)^{-1}(\Phi_* X_p)$.
The diagram

$$\begin{array}{ccc} & \mathbb{R}^n & \\ {\scriptstyle u} \swarrow & & \searrow {\scriptstyle \tilde{\Phi}(u)} \\ T_p(M) & \xrightarrow[\Phi_*]{} & T_{\Phi(p)}(M) \end{array}$$

commutes by definition of $\tilde{\Phi}$.

This implies $u^{-1}(X_p) = (\tilde{\Phi}(u))^{-1}(\phi_* X_p)$, hence θ is invariant by $\tilde{\Phi}$. For the proof of the converse see [9; p. 227]. □

Lemma 1.2. Let M be a differentiable manifold with linear connection Γ. Let ϕ be a transformation on M and $\tilde{\Phi}$ the induced automorphism on L(M). There exists a unique linear connection $\tilde{\Phi}(\Gamma)$ on M such that the horizontal subspaces of Γ are mapped into horizontal subspaces of $\tilde{\Phi}(\Gamma)$ by the induced linear map

$$\tilde{\Phi}_* : T(L(M)) \to T(L(M)).$$

Proof: Take $u, v \in L(M)$, $a \in GL(n, \mathbb{R})$ such that $v = \tilde{\Phi}(u)a$. Define the horizontal subspace H_v of $T_v(L(M))$ by

$$H_v = (R_a)_* \circ \tilde{\Phi}_*(H_u) ,$$

where H_u is the horizontal subspace of $T_u(L(M))$ with respect to the linear connection Γ.

One has to show that H_v is independent of the choice of u and a; the uniqueness is already clear from the definition. Take

$$u' \in L(M), \quad a' \in GL(n, \mathbb{R})$$

such that

$$v = \tilde{\Phi}(u')a'.$$

Since $\tilde{\Phi}$ maps fibers into fibers,

$$u' = ub \quad \text{for some} \quad b \in GL(n, \mathbb{R}).$$

Let $\phi' : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ be the map induced by the automorphism $\tilde{\Phi}$. Take

$b' = \phi'(b)$. Then

$v = \tilde{\Phi}(u')a' = \tilde{\Phi}(ub)a' = \tilde{\Phi}(u)b'a'$ and thus

$a = b'a'$.

Now
$$\begin{aligned}(R_{a'})_* \circ \tilde{\Phi}_*(H_{u'}) &= (R_{a'})_* \circ \tilde{\Phi}_*(H_{ub}) \\ &= (R_{a'})_* \circ \tilde{\Phi}_* \circ (R_b)_*(H_u)\end{aligned}$$

$$= (R_{a'})_* \circ (R_{b'})_* \circ \tilde{\Phi}_*(H_u)$$

$$= (R_a)_* \circ \tilde{\Phi}_*(H_u),$$

hence H_v is well defined.

It remains to show that the assignment $v \to H_v$ defines a linear connection on M. Since

$$v = \tilde{\Phi}(u)a , \qquad vb = \tilde{\Phi}(u)ab \quad b \in GL(n, \mathbb{R}).$$

Hence

$$H_{vb} = (R_{ab})_* \circ \tilde{\Phi}_*(H_u) = (R_b)_* \circ (R_a)_* \circ \tilde{\Phi}_*(H_u)$$
$$= (R_b)_*(H_v).$$

Now one may assume that $v = \Phi(u)$, because H_v is invariant under $(R_a)_*$. The diagram

$$\begin{array}{ccc} H_u & \xrightarrow{\tilde{\Phi}_*} & H_v \\ \downarrow{\pi_*} & & \downarrow{\pi_*} \\ T_{\pi(u)}(M) & \xrightarrow[\Phi_*]{} & T_{\pi(v)}(M) \end{array}$$

commutes. The maps

$$\pi_* : H_u \to T_{\pi(u)}(M)$$

and $$\Phi_* : T_{\pi(u)}(M) \to T_{\pi(v)}(M)$$

are linear isomorphisms which imply that

$$\pi_* : H_v \to T_{\pi(v)}(M)$$

is a linear isomorphism. □

Note: Furthermore, it can be shown that $\tilde{\Phi}^* \omega^i_k$ are the connection forms of $\tilde{\Phi}(\Gamma)$.

It was shown earlier that every transformation Φ on M induces a transformation $\tilde{\Phi}$ on L(M). Taking a vector field X on M and $p \in M$, then the local 1-parameter group of local transformations ϕ_t

generated by X in a neighborhood U of p induces for each t a transformation $\tilde{\psi}_t$. Therefore starting with a vector field on M a corresponding vector field on $L(M)$ can be introduced in a natural way:

Lemma 1.3. Let X be a vector field on M. Then there exists a unique vector field $\overline{X}$ on $L(M)$ called the *natural lift* of X such that

(1) $(R_a)_*\overline{X} = \overline{X}$ for all $a \in G(n, \mathbb{R})$

(2) $L_{\overline{X}}\theta = 0$, where $L_{\overline{X}}$ denotes the Lie derivative with respect to $\overline{X}$,

(3) $\pi_*(\overline{X}_u) = X_{\pi(u)}$ for all $u \in L(M)$.

Proof: (1) Let ψ_t be a local 1-parameter group of local transformations generated by X in a neighborhood U of $p \in M$. Then ψ_t induces a transformation

$$\tilde{\psi}_t : \pi^{-1}(U) \to \pi^{-1}\big(\psi_t(U)\big)$$

for all t, i.e. a local 1-parameter group of local transformations on $L(M)$. Denote by $\overline{X}$ the vector field on $L(M)$ induced by $\tilde{\psi}_t$. For all $a \in GL(n, \mathbb{R})$, $\tilde{\psi}_t$ commutes with R_a. This implies (1).

(2) ψ_t preserves θ (Lemma 1.1) hence $L_{\overline{X}}\theta = 0$.

(3) Since $\pi \circ \tilde{\psi}_t = \psi_t \circ \pi$, one has (3).

Now assume that $\overline{Y}$ is another vector field on $L(M)$ satisfying (1)-(3). Let $\tilde{\phi}_t$ be the local 1-parameter group of local transformations generated by $\overline{Y}$.

(1) implies that $\tilde{\phi}_t$ commutes with R_a and

(2) implies that $\tilde{\phi}_t$ preserves the canonical form θ.

Hence one knows from Lemma 1.1 that $\tilde{\phi}_t$ is induced by a local 1-parameter group of local transformations ϕ_t on M.
Now $\pi_*(\overline{Y}_u) = X_{\pi(u)}$, so that ϕ_t induces the vector field X on M. Therefore $\phi_t = \psi_t$ and $\tilde{\phi}_t = \tilde{\psi}_t$. This proves the uniqueness of $\overline{X}$. □

Note: One can use the natural lift of a vector field X to introduce the Lie derivative in a similar way as the covariant derivative is introduced (see Chapter I). When S denotes a tensor field on M the Lie derivative of S with respect to X is defined by

$$L_X S = \overline{X}\tilde{S} .$$

Now one can define transformations on M which leave the linear connection invariant:

Definition 1.4. Let $\Phi : M \to M$ be a transformation. It is called an *affine transformation* if

$$\nabla_{\dot{\sigma}(t)} X = 0 \quad \text{implies} \quad \nabla_{\Phi_*(\dot{\sigma}(t))} \Phi_* X = 0,$$

where $\sigma(t)$ is a curve in M, X a vector field.

Note: This means that the induced map Φ_* maps each parallel vector field X along each curve $\sigma(t)$ of M into a parallel vector field along the curve $\Phi \circ \sigma(t)$. In other words the induced map Φ_* maps every horizontal curve into a horizontal curve, i.e. affine transformations preserve geodesics.

Now one can look at the effect of an affine transformation on the connection forms on $L(M)$. As in the case of canonical forms one can use the connection forms ω^i_k to introduce a 1-form on $L(M)$ with values in the Lie algebra $g\ell(n, \mathbb{R})$.
Remember that a Lie algebra homomorphism

$$\sigma_* : g\ell(n, \mathbb{R}) \to T(L(M))$$
$$A \to A^*$$

where A^*, the fundamental vector field corresponding to A, has already been defined (Lemma 1.1). One gets

<u>Definition 1.5</u>. The 1-form $\omega = (\omega_1^1, \cdots, \omega_n^n)$ defined by

$$\omega : T(L(M)) \to g\ell(n, \mathbb{R})$$
$$X \to A$$

such that $A^* = vX$ where vX denotes the vertical component of X, is called the *connection form* on $L(M)$.

Since $vX = \omega_k^i(X)E_i^{*k}$, $\omega(X) = \omega_k^i(X)E_i^k$.

<u>Lemma 1.4</u>. Let M be a differentiable manifold with connection Γ and let Φ be an affine transformation. The induced automorphism $\tilde{\Phi}$ leaves both the canonical form θ and the connection form ω, invariant. Conversely, every fiber preserving transformation on $L(M)$ leaving both θ and ω invariant is induced by an affine transformation on M.

Proof: The first statement is a direct consequence of Lemmas 1.1 and 1.2, and the definition of an affine transformation. For the converse see [9; p. 226].

Furthermore, one has

<u>Lemma 1.5</u>. Let Γ be a linear connection on M. For a transformation Φ of M the following conditions are equivalent:

(1) Φ is an affine transformation,

(2) $\tilde{\Phi}^*\omega = \omega$,

(3) $\tilde{\Phi}_{*}B(\xi) = B(\xi)$, where $B(\xi)$ is a standard horizontal vector field

(4) $\Phi_{*}(\nabla_X Y) = \nabla_{\Phi_{*}X}\Phi_{*}Y$, where X and Y are vector fields on M.

Proof: The equivalence of (1) and (2) is already given by Lemma 1.4. For further proofs see [9; p. 228]. □

Again the affine transformation can be induced by a vector field on M.

Definition 1.6. A vector field X on M is called an *infinitesimal affine transformation* of M if for all $p \in M$ the local 1-parameter groups of local transformations $\psi_t : U \to M$, $p \in U$ is an affine transformation for all t.
Here U is provided with the connection Γ/U which is the restric of Γ to U.

To relate the concept of affine transformations and the problem of characterizing spaces of locally constant connection in a coordinate independent way, the description of the affine transformations in local coordinates is necessary.

Lemma 1.6. Let X be a vector field on M and $\bar{X}$ the natural lift of X to $L(M)$. If x^i is a coordinate system, then $\bar{X}$ is described in the chart (x^i, X^k_ℓ) on $L(M)$ by

$$\bar{X} = \xi^i(x^k)\frac{\partial}{\partial x^i} + X^\ell_s\, \xi^r_{,\ell}\,(x^t)\frac{\partial}{\partial X^r_s},$$

where $X = \xi^i(x^k)\dfrac{\partial}{\partial x^i}$.

Proof: Take $u \in L(M)$. In the coordinate basis one has

$$\overline{X}_u = a^i(u)\frac{\partial}{\partial x^i} + A^i_k(u)\frac{\partial}{\partial X^i_k}.$$

Now $\pi_*(\overline{X}_u) = X_{\pi(u)}$ implies $a^i(u) = \xi^i(x^k)$ and $\theta = \theta^i e_i$, $L_{\overline{X}}\theta = 0$ implies

$$\overline{X}\left(\theta^k\left(\frac{\partial}{\partial x^i}\right)\right) - \theta^k\left([\overline{X}, \frac{\partial}{\partial x^i}]\right) = 0.$$

Inserting the quantities θ^k and $\overline{X}$ one gets

$$A^i_k = X^\ell_k\, \xi^i_{,\ell}$$

and this implies the statement. □

<u>Lemma 1.7</u>. Let Γ be a linear connection on M. For a vector field X on M the following conditions are equivalent:

(1) X is an infinitesimal affine transformation on M,

(2) $L_{\overline{X}}\omega = 0$,

(3) $[\overline{X}, B(\xi)] = 0$ for every $\xi \in \mathbb{R}^n$, where $B(\xi)$ is the horizontal vector field corresponding to ξ,

(4) $L_X \circ \nabla_Y - \nabla_Y \circ L_X = \nabla_{[X,Y]}$ for every vector field Y on M.

Proof: Denote by ψ_t the local 1-parameter group of local transformations of M generated by X and by $\tilde{\psi}_t$ the induced transformation of L(M).

(1) → (2): If X is an infinitesimal affine transformation, $\tilde{\psi}_t$ preserves ω by Lemma 1.5.

(2) → (3): $\theta^i(B(\xi)) = \xi^i$, hence $\theta(B(\xi)) = \xi \in \mathbb{R}^n$. Thus

$$\overline{X}(\theta(B(\xi))) = 0 = (L_{\overline{X}}\theta)(B(\xi)) + \theta([\overline{X}, B(\xi)]).$$

Now $L_{\overline{X}}\theta = 0$, so this implies that

$[\overline{X}, B(\xi)]$ is vertical.

But $\omega(B(\xi)) = 0$, which leads to

$$\overline{X}(\omega(B(\xi))) = 0 = (L_{\overline{X}}\omega)(B(\xi)) + \omega([\overline{X}, B(\xi)]).$$

Thus $L_{\overline{X}}\omega = 0$ implies $[\overline{X}, B(\xi)]$ is horizontal. Therefore $[\overline{X}, B(\xi)] = 0$.

(3) → (1): $[\overline{X}, B(\xi)] = 0$ implies that $\tilde{\psi}_t$ leaves $B(\xi)$ invariant. Now the B_i span the horizontal subspaces, so $\tilde{\psi}_t$ preserves the connection.

For the equivalence of (4) see [9; p. 231]. □

Note: In general an infinitesimal affine transformation generates only a local 1-parameter group of local affine transformations. If the infinitesimal affine transformation X generates a global 1-parameter group of affine transformations, the vector field X is called *complete*.

One can show (see [10]) that the affine transformations form a Lie group which has as its Lie algebra the set of complete infinitesimal affine transformations.

Example 1.1. Denote by A^n the Euclidean n-space $\mathbb{R}^n$ regarded as an affine space.

The group of affine transformations of A^n is represented by the group of all matrices of the form

$$T = \begin{pmatrix} a & \xi \\ & \\ 0 & 1 \end{pmatrix},$$

where $a \in GL(n, \mathbb{R})$ and ξ is a column vector, $\xi \in \mathbb{R}^n$.

The affine transformation is described by

$$T : A^n \to A^n$$
$$\eta \to a\eta + \xi.$$

The geometric invariants of a linear connection, the curvature and torsion tensors are given in local coordinates by

Lemma 1.8. Let M be a differentiable manifold with connection Γ and x^i a coordinate system. The curvature and the torsion tensor components with respect to the coordinate basis are

$$R^i_{\ k\ell s} = \Gamma^i_{\ sk,\ell} - \Gamma^i_{\ \ell k,s} + \Gamma^i_{\ \ell r}\Gamma^r_{\ sk} - \Gamma^i_{\ sr}\Gamma^r_{\ \ell k}$$

$$T^i_{\ k\ell} = \Gamma^i_{\ k\ell} - \Gamma^i_{\ \ell k}\ .$$

Proof: These equations are a direct consequence of the structure equations on M using the fact that the definitions of R and T (Theorem 4.1, Chapter I) imply $R^i_{\ k(\ell s)} = 0$ and $T^i_{\ (k\ell)} = 0$. □

Lemma 1.9. A vector field X on M with components ξ^i with respect to a local coordinate system is an infinitesimal affine transformation if and only if it is a solution of the differential equation:

$$\xi^i_{\ ;k\ell} - (T^i_{\ sk}\xi^s)_{;\ell} + R^i_{\ ks\ell}\xi^s = 0.$$

Proof: The differential equation is calculated from $L_{\overline{X}}\omega = 0$, i.e.

$$\overline{X}\left(\omega^i_k\left(\frac{\partial}{\partial x^\ell}\right)\right) - \omega^i_k\left([\overline{X}, \frac{\partial}{\partial x^\ell}]\right) = 0. \quad \square$$

Note: A vector field X is an infinitesimal affine transformation iff $L_X\Gamma^i_{k\ell} = 0$.

Theorem 1.1. Let M be a differentiable n-dimensional manifold with linear connection .

Then M is a space of locally constant linear connection if and only if M admits an n-dimensional transitive, abelian group of local affine transformations.

Proof:

"$\Rightarrow$": Let (U,ϕ) be a chart around $p \in M$ such that

$\Gamma^i_{k\ell}(x^s)$ = constant.

A vector field $X = \xi^i \frac{\partial}{\partial x^i}$ defined on U is an infinitesimal affine transformation if and only if it is a solution of

$$\xi^i_{;\ell k} - (T^i_{ks}\xi^s)_{;\ell} + R^i_{\ell sk}\xi^s = 0.$$

Using the local expressions for R and T one obtains

$$\xi^i_{,k\ell} + \Gamma^i_{s\ell}\xi^s_{,k} - \Gamma^s_{k\ell}\xi^i_{,s} + \Gamma^i_{ks}\xi^s_{,\ell} + \Gamma^i_{k\ell,s}\xi^s = 0.$$

Hence a coordinate vector field

$$\frac{\partial}{\partial x^r} = \delta^s_r \frac{\partial}{\partial x^s}$$

is an infinitesimal affine transformation if and only if $\Gamma^i_{k\ell,r} = 0$. Therefore the n coordinate vector fields $\frac{\partial}{\partial x^i}$ are infinitesimal affine transformat[illegible]s. They generate translations and satisfy $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right] = 0$ for all $i,k \in \{1,\cdots,n\}$. Thus they form an n-dimensional transitive, abelian group of local affine transformations.

"$\Leftarrow$": Let G be an n-dimensional transitive abelian group of affine transformations defined on a neighborhood U of $p \in M$

Take a basis e_a in the Lie algebra of vector fields on U which is isomorphic to the Lie algebra of G.

The fact that G is abelian implies $[e_a,e_b] = 0$ for all $a,b \in \{1,\cdots,n\}$. Thus the vector fields e_a can be chosen as coordinate vector fields. They are infinitesimal affine transformations, hence one obtains $\Gamma^i_{k\ell,r} = 0$ for $r = 1,\cdots,n$. Thus the connection coefficients are constant with respect to this coordinate system, and therefore M is a space of locally constant linear connection. □

Corollary 1.1. Let M be a differentiable manifold with linear connection Γ. In order that M admit an infinitesimal affine transformation it is necessary and sufficient that there exists a coordinate system such that the connection coefficients are independent of one of the coordinates.

Proof: This is just a consequence of the proof of Theorem 1.1. □

2. Levi-Civita connection.

It was pointed out in Chapter I that on a differentiable manifold another important structure can exist, a metric g. A first relation between the connection and the metric was shown by Theorem 5.2, (Chapter I): Every non-degenerate metric g determines a unique torsion free linear connection, the so-called Levi-Civita connection. Now the opposite question arises: When is the linear connection Γ on M the Levi-Civita connection of a metric g? This section and the next is a discussion of the problem in the case of spaces of locally constant connection.

It is clear from Theorem 5.2 that a first restriction on the linear connection Γ has to be that the connection is torsion free. This implies that the connection coefficients are symmetric, i.e.

$$\Gamma^i{}_{[k\ell]} = 0.$$

From now on, a linear connection means always a torsion free connection.

With respect to a coordinate system the coefficients of a Levi-Civita connection were related to the components of the metric in the following way (Proposition 5.1, Chapter I):

$$g_{is}\Gamma^s_{k\ell} = \frac{1}{2}(g_{i\ell,k} + g_{ki,\ell} - g_{k\ell,i}).$$

Changing the indices i and k gives

$$g_{ks}\Gamma^s_{i\ell} = \frac{1}{2}(g_{k\ell,i} + g_{ik,\ell} - g_{i\ell,k}).$$

Adding both equations leads to

$$g_{ik,\ell} = g_{is}\Gamma^s_{k\ell} + g_{ks}\Gamma^s_{i\ell}\ .$$

Therefore the metric (if it exists) for which the given connection is a Levi-Civita connection has to satisfy this set of differential equations. This implies that the question when a given connection is a Levi-Civita connection is equivalent to the problem of finding the integrability conditions of this set of first order partial differential equations.

In order to handle the system of differential equations more easily it is rewritten:

There exists a 1-1 correspondence between each pair of indices of the components g_{ik} of the metric tensor and an element of the set $\{1,\cdots,\frac{1}{2}n(n+1)\}$, where n is the dimension of the manifold. This correspondence can be described by a map

$$\phi : \{1,\cdots,n\} \times \{1,\cdots,n\} \to \{1,\cdots,\tfrac{1}{2}n(n+1)\}$$

$$(a,b) \to \phi(a,b) .$$

Note: This correspondence can be given by

$$\phi(a,b) = n(a-1) + b - \tfrac{1}{2}a(a-1).$$

Define a 3-index xymbol

$$\sigma^{\alpha}_{ab} \quad a,b \in \{1,\cdots,n\}, \quad \alpha \quad \{1,\cdots,\tfrac{1}{2}n(n+1)\}$$

by

$$\sigma^{\alpha}_{ab} = \delta^{\alpha}_{\phi(a,b)} = \begin{cases} 1 & \text{if } \alpha = \phi(a,b) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sigma^{\alpha}_{[ab]} = 0.$$

[illegible]at

$$\sigma^{\alpha}_{ab}\ \sigma^{ab}_{\beta} \quad [illegible] \quad \beta$$

$$\sigma^{ab}_{\alpha}\ \sigma^{\alpha}_{cd} = \delta [illegible] ,d) .$$

Then one can define

$$g_\alpha := \sigma_\alpha^{ab} g_{ab} ,$$

which implies

$$g_{ab} = \sigma_{ab}^\alpha g_\alpha .$$

The set of differential equations of interest is

$$g_{ik,\ell} = g_{is}\Gamma^s_{k\ell} + g_{ks}\Gamma^s_{i\ell}$$

or

$$g_{ik,\ell} = g_{rs}(\delta^r_i\Gamma^s_{k\ell} + \delta^r_k\Gamma^s_{i\ell}) .$$

It can be rewritten as

$$g_{\alpha,i} = \underset{i}{A}{}^\beta_\alpha g_\beta ,$$

where $\underset{i}{A}{}^\beta_\alpha = \sigma_\alpha^{k\ell}\sigma^\beta_{ks}\Gamma^s_{\ell i} + \sigma_\alpha^{k\ell}\sigma^\beta_{\ell s}\Gamma^s_{ki}$.

Hence if the connection coefficients $\Gamma^i_{k\ell}$ are constant, $\underset{i}{A}{}^\beta_\alpha$ is a $r \times r$ matrix $\left(r = \frac{1}{2}n(n+1)\right)$ with constant coefficients.

From now on all considerations are restricted to the case of constant connection coefficients.

Now the integrability of the above system of differential equations is investigated.

<u>Definition 2.1</u>. Let x^i be a coordinate system on M. The system

$$g_{\alpha,i} = \underset{i}{A}{}^\beta_\alpha g_\beta$$

of partial differential equations of first order defined in a domain G is *completely integrable* if to each $(x^i, \underset{\circ}{g}_\alpha) \in G$ there exists at least one solution $g_\alpha(x^i)$ such that the initial condition $g_\alpha(\underset{\circ}{x}^i) = \underset{\circ}{g}_\alpha$ is satisfied.

Proposition 2.1. Let M be a space of locally constant linear connection. The system of differential equations

$$g_{ik,\ell} = g_{is}\Gamma^{s}_{k\ell} + g_{ks}\Gamma^{s}_{i\ell}$$

is completely integrable if and only if M is flat.

Proof: Write the system as

$$g_{\alpha,i} = \underset{i}{A}{}^{\beta}_{\alpha}\, g_{\beta}.$$

It is completely integrable if and only if the conditions for complete integrability which are

$$\underset{i}{A}{}^{\beta}_{\alpha}\, g_{\beta,k} + \sum_{\gamma} \underset{i}{A}{}^{\beta}_{\alpha}\, g_{\beta,\gamma}\, \underset{k}{A}{}^{\delta}_{\gamma}\, g_{\delta} =$$

$$\underset{k}{A}{}^{\beta}_{\alpha}\, g_{\beta,i} + \sum_{\gamma} \underset{k}{A}{}^{\beta}_{\alpha}\, g_{\beta,\gamma}\, \underset{i}{A}{}^{\delta}_{\gamma}\, g_{\delta}$$

are satisfies (see [12; p. 176].

Putting $g_{\alpha,i} = \underset{i}{A}{}^{\beta}_{\alpha}\, g_{\beta}$ back into these equations gives

$$(*) \qquad g_{\gamma}(\underset{i}{A}{}^{\beta}_{\alpha}\, \underset{k}{A}{}^{\gamma}_{\beta} - \underset{k}{A}{}^{\beta}_{\alpha}\, \underset{i}{A}{}^{\gamma}_{\beta}) = 0.$$

Using

$$\underset{i}{A}{}^{\beta}_{\alpha} = \sigma_{\alpha}^{\ k\ell}\sigma^{\beta}_{ks}\Gamma^{s}_{\ell i} + \sigma_{\alpha}^{\ k\ell}\, \sigma^{\beta}_{\ell s}\Gamma^{s}_{ki}$$

and the fact that the components of the curvature tensor are expressed by

$$R^{i}_{\ k\ell s} = \Gamma^{i}_{\ell r}\Gamma^{r}_{sk} - \Gamma^{i}_{sr}\Gamma^{r}_{\ell k}$$

one can see (after a lengthy calculation) that the condition (*) is equivalent to

$$g_{ir}R^{r}_{\ k\ell s} + g_{kr}\, R^{r}_{\ i\ell s} = 0$$

or

$$R_{(ik)\ell s} = 0.$$

Now complete integrability means that the initial value $\underset{o}{g}_{ik}$ can be chosen arbitrarily. Hence the system is completely integrable iff

$$R^{i}{}_{k\ell s} = 0. \quad \square$$

In general, the condition of complete integrability is much too strong. A necessary and sufficient condition that the system admit is gained in the following way:

Denote the set of equations

$$R_{(ik)\ell s} = 0 \qquad \text{by } F_o$$

and

$$R_{(ik)\ell s, q_1 \cdots q_\alpha} = 0 \quad \text{by } F_\alpha, \quad \alpha \in \mathbb{N}.$$

Proposition 2.2. Let M be an n-dimensional space of locally constant connection Γ with curvature tensor R. In order that the system of differential equations

$$g_{ik,\ell} = g_{is}\Gamma^{s}_{k\ell} + g_{ks}\Gamma^{s}_{i\ell}$$

admit a solution it is necessary and sufficient that there exists a positive integer $N \leq n$ such that the equations of the sets $F_o, \cdots, F_N$ are compatible and that the equations of the set F_{N+1} are satisfied on account of the former sets.

Proof: The result is a direct consequence of a theorem about integrability of such a system of first order partial differential equations as stated in [5 , p. 3] . According to the general theory the set F_o is the set of equations for complete integrability. By Proposition 2.1, F_o turned out to be

$$R_{(ik)\ell s} = 0. \quad \square$$

The above formulation has a crucial disadvantage. The restrictions on a linear connection to be a Levi-Civita connection should have a geometrical meaning since the concepts involved (metric, connection) are geometrical. That this is so cannot be seen in this formal way. More insight is gained into the geometrical meaning of the integrability conditions if they are described in terms of the holonomy group of a linear connection.

3. Holonomy group and integrability conditions.

A linear connection on M provides the manifold with the concept of parallel propagation. Let $\sigma(t)$ be a differentiable curve on M. A frame $u = (p, X_i)$ at $p \in M$ is parallel propagated along $\sigma(t)$ if

$$\nabla_{\dot\sigma(t)} X_i = 0 \quad \text{for all } X_i .$$

There exists a special class of curves on M:

<u>Definition 3.1</u>. The *loop space* $C(p)$ at $p \in M$ is the set of all closed differentiable curves starting and ending at p.

Parallel propagation of a frame along a loop $\sigma \in C(p)$ defines a linear transformation

$$L_\sigma : T_p(M) \to T_p(M).$$

If σ is given by

$$\sigma : (a,b) \to M$$

define the inverse by $\sigma^{-1}(t) := \sigma(a+b-t)$. Now it can be shown:

<u>Lemma 3.1</u>. Let $\sigma, \tau \in C(p)$, $p \in M$.

(i) The parallel propagation along σ^{-1} is the inverse of the parallel propagation along σ.

(ii) The parallel propagation along the composite curve $\tau \cdot \sigma$ is the composite of the parallel propagation along σ and τ..

Proof: See [9 , p. 71]. □

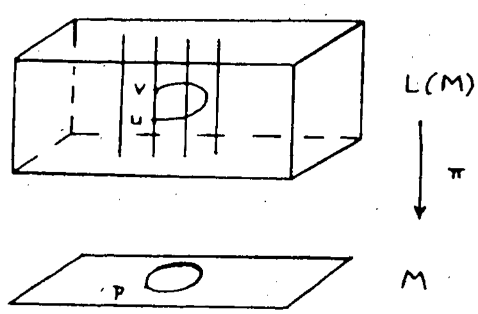
This Lemma implies that the linear transformations just considered form a group.

<u>Definition 3.2</u>. Let M be a differentiable manifold with connection Γ.

The *holonomy group* $\Psi(p)$ of Γ with reference point p is the group of linear transformations in the tangent space $T_p(M)$ of $p \in M$ defined by parallel propagation of frames at p along loops starting at p.

Note: Another way to realize the holonomy group as a subgroup of $GL(n, \mathbb{R})$ is the following:

Let $u = (p, X_i) \in L(M)$ be an arbitrary but fixed point of the fiber $\pi^{-1}(p)$. Under parallel propagation along $\sigma(t) \in C(p)$ the frame $u = (p, X_i)$ is mapped into the frame $v = (p, L_\sigma(X_i))$.



Hence parallel propagation along loops defines a map (an isomorphism):

$$\hat{L}_\sigma : \pi^{-1}(p) \to \pi^{-1}(p)$$
$$(p, X_i) \to (p, L_\sigma(X_i))$$

of the fiber $\pi^{-1}(p)$ onto itself.

Thus each $\sigma \in C(p)$ determines an element $a \in GL(n, \mathbb{R})$. The set of elements $a \in GL(n, \mathbb{R})$ determined by all $\sigma \in C(p)$ form a subgroup of $GL(n, \mathbb{R})$. This subgroup is called the holonomy group of Γ with reference point u and is denoted by $\Psi(u)$.

It can be shown that the holonomy groups are Lie groups:

Theorem 3.1. Let M be a paracompact, connected differentiable manifold with linear connection Γ. The holonomy group $\Psi(u)$ of Γ with reference point $u \in L(M)$ is a Lie subgroup of $GL(n, \mathbb{R})$. The holonomy group $\Psi(u)$ is isomorphic to $\Psi(v)$ for all $v \in L(M)$.

Proof: See [9 , p. 73]. □

The Lie algebra of the holonomy group is totally determined by the curvature tensor and its covariant derivatives if Γ is analytic.

Theorem 3.2. Let M be a paracompact, connected differentiable manifold with analytic linear connection Γ. The Lie algebra $\mathcal{g}(p)$ of the holonomy group $\Psi(p)$, $p \in M$ is spanned by all linear endomorphisms of $T_p(M)$ of the form

$$\nabla^k R(X,Y;\ V_1,\cdots,V_k) \qquad 0 \leq k < \infty .$$

where

$$X,Y,V_I \in T_p(M).$$

Proof: See [9 ; p. 152]. □

The starting point of these considerations was to determine when a linear connection is also a Levi-Civita connection. Suppose Γ is the Levi-Civita connection of a metric g, which implies g has to be invariant under parallel propagation along each curve

$$\nabla g = 0.$$

Therefore if the connection Γ is a Levi-Civita connection with respect to g the holonomy group of Γ has to be a subgroup of the generalized orthogonal group corresponding to the signature of g.
It turns out that this necessary condition is also sufficient.

Theorem 3.3. Let M be a connected differentiable manifold with torsion free connection Γ. Then Γ is the Levi-Civita connection of a metric g if the holonomy group of Γ keeps a non-degenerate quadratic form g_o invariant. The signature of g is the same as that of g_o.

Proof: See [16]. □

Thus the procedure is: Look for non-degenerate quadratic forms

$$g : T_p(M) \times T_p(M) \to \mathbb{R}$$

which are invariant under the holonomy group $\Psi(p)$, i.e.

$$(*) \qquad g(aX,aY) = g(X,Y) \qquad \forall\, a \in \Psi(p).$$

Now the holonomy group $\Psi(p)$ is uniquely determined by its Lie algebra $\mathcal{g}(p)$. Take $A \in \mathcal{g}(p)$ such that $\exp A = a$ (exp denotes the exponential map). Then (*) is equivalent to

$$g(AX,Y) = -g(X,AY).$$

Thus A has to be self-adjoint. If the non-degenerate quadratic form is

$$g_{ab} = \delta_{ab}$$

then this implies

$$A = -A^T$$

for the Lie algebra of the holonomy group.

The integrability conditions for spaces of locally constant linear connection are now given by

<u>Proposition 3.1</u>. Let M be a space of locally constant connection Γ. Then is a Levi-Civita connection with respect to a Riemannian metric g if and only if the equations

$$\delta_{is}\nabla_{p_k}\cdots\nabla_{p_1}R^s{}_{\ell xy} + \delta_{\ell s}\nabla_{p_k}\cdots\nabla_{p_1}R^s{}_{ixy} = 0 \qquad 0 \le k < \infty$$

are satisfied.

Here $R^s{}_{ixy}$ are the components of the curvature tensor with respect to a coordinate chart (U,ϕ) in which the connection coefficients are constant.

Proof: If M is a space of locally constant connection, there exists a coordinate chart (V,ψ) such that the components of the curvature tensor ${}_{\psi}R^{i}{}_{k\ell s}$ with respect to this chart are constant:

$$ {}_{\psi}R^{i} \text{[illegible]} \Gamma^{r}{}_{sk} - {}_{\psi}\Gamma^{i}{}_{sr}\, {}_{\psi}\Gamma^{r}{}_{\ell k} . $$

Now it was seen [illegible] 3) that the ${}_{\psi}\Gamma^{i}{}_{k\ell}$ are the connection coefficients of a Levi-Civita connection if and only if the holonomy group leaves a non-degenerate quadratic form g invariant. Because of Theorem [illegible].2 this statement is e[illegible]ivalent to: The components ${}_{\psi}R^{i}{}_{k\ell s}$ of the curvature tensor have to satisfy the following set of equations

$$ (*) \quad g_{is}\nabla_{p_k}\cdots\nabla_{p_1}R^{s}{}_{\ell xy}\xi^{x}\eta^{y}v_1^{p_1}\cdots v_k^{p_k} + $$

$$ + g_{\ell s}\nabla_{p_k}\cdots\nabla_{p_1}R^{s}{}_{ixy}\xi^{x}\eta^{y}v_1^{p_1}\cdots v_k^{p_k} = 0, \qquad 0 \le k < \infty, $$

where $\xi,\eta,v_i \in T_p(M)$.

Now any non-degenerate positive definite quadratic form g_{ik} can be transformed at one point to the form δ_{ik} by a linear transformation. As seen from the transformation equation for the connection coefficients (Lemma 2.4, Chapter I), the constancy of the connection coefficients is not changed by a linear transformation. Choose

$$ \xi = \delta^{i}_{x}\frac{\partial}{\partial x^{i}}, \qquad \eta = \delta^{i}_{y}\frac{\partial}{\partial x^{i}}, \ldots . $$

Then the set of equations (*) is fulfilled if and only if the set

$$ \delta_{is}\nabla_{p_k}\cdots\nabla_{p_1}R^{s}{}_{\ell xy} + \delta_{\ell s}\nabla_{p_k}\cdots\nabla_{p_1}R^{s}{}_{ixy} = 0 $$

is satisfied, where $R^{i}_{\ k\ell s}$ is gained by the linear transformation from $_{\psi}R^{i}_{\ k\ell s}$. □

These equations establish the integrability conditions for the constant connection coefficients. Which connections solve these equations? A sufficient condition that a space of locally constant connection admits a constant Levi-Civita connection is given now:

Proposition 3.2. Let M be a space of locally constant connection Γ. If there exists a chart such that the connection coefficients with respect to this chart are constant and satisfy the equation

$$\delta_{is}\Gamma^{s}_{k\ell} + \delta_{\ell s}\Gamma^{s}_{ki} = \gamma_k\delta_{i\ell}$$

(γ_k denotes a constant vector), then the linear connection is a Levi-Civita connection with respect to a Riemannian metric.

Proof: One wants to show

$$\delta_{is}\nabla_{p_k}\cdots\nabla_{p_1}R^{s}_{\ \ell xy} + \delta_{\ell s}\nabla_{p_k}\cdots\nabla_{p_1}R^{s}_{\ ixy} = 0 \qquad 0 \leq k < \infty$$

is satisfied by

$$\Lambda_{\ell ik} = \gamma_\ell\delta_{ik}$$

with

$$\Lambda_{\ell ik} = \delta_{is}\Gamma^{s}_{\ell k} + \delta_{ks}\Gamma^{s}_{\ell i} .$$

The proof is carried out by induction:

k = 1:

$$\delta_{is}\nabla_{p_1}R^{s}_{\ \ell xy} + \delta_{\ell s}\nabla_{p_1}R^{s}_{\ ixy} = 0$$

Writing out the covariant derivative and adding terms which cancel each other one obtains:

$$(\delta_{is}\Gamma^s_{p_1 r} + \delta_{rs}\Gamma^s_{p_1 i})R^r{}_{\ell xy} + (\delta_{\ell s}\Gamma^s_{p_1 r} + \delta_{rs}\Gamma^s_{p_1 \ell})R^r{}_{ixy}$$

$$- (\delta_{is}R^s{}_{rxy} + \delta_{rs}R^s{}_{ixy})\Gamma^r_{p_1 \ell} - (\delta_{\ell s}R^s{}_{rxy} + \delta_{rs}R^s{}_{\ell xy})\Gamma^r_{p_1 i}$$

$$- (\delta_{is}R^s{}_{\ell ry} + \delta_{\ell s}R^s{}_{iry})\Gamma^r_{p_1 x} - (\delta_{is}R^s{}_{\ell xr} + \delta_{\ell s}R^s{}_{ixr})\Gamma^r_{p_1 y}$$

$$= \Lambda_{p_1 ir}R^r{}_{\ell xy} + \Lambda_{p_1 \ell r}R^r{}_{ixy} - \cdots \qquad = 0$$

Inserting $\Lambda_{\ell ik} = \gamma_\ell \delta_{ik}$ leads to

$$\gamma_{p_1}(\delta_{ir}R^r{}_{\ell xy} + \delta_{\ell r}R^r{}_{ixy}) - (\delta_{is}R^s{}_{rxy} + \delta_{rs}R^s{}_{ixy})\Gamma^r_{p_1 \ell} -$$

$$-(\delta_{\ell s}R^s{}_{rxy} + \delta_{rs}R^s{}_{\ell xy})\Gamma^r_{p_1 i} - (\delta_{is}R^s{}_{\ell ry} + \delta_{\ell s}R^s{}_{iry})\Gamma^r_{p_1 x} -$$

$$- (\delta_{is}R^s{}_{\ell xr} + \delta_{\ell s}R^s{}_{ixr})\Gamma^r_{p_1 y} = 0 .$$

Thus every parenthesis is of the form

$$\delta_{is}R^s{}_{\ell xy} + \delta_{\ell s}R^s{}_{ixy}.$$

It can be shown that this term actually vanishes with the above choice of $\Lambda_{\ell ik}$. First of all,

$$(*) \qquad \delta_{is}R^s{}_{\ell xy} + \delta_{\ell s}R^s{}_{ixy} \equiv \Lambda_{x\ell s}\Gamma^s_{iy} - \Lambda_{y\ell s}\Gamma^s_{ix} + \Lambda_{xis}\Gamma^s_{\ell y} - \Lambda_{yis}\Gamma^s_{\ell x} .$$

This can be seen by plugging the definition of Λ into the right hand side. Now insert $\Lambda_{\ell ik} = \gamma_\ell \delta_{ik}$ into the right hand side of (*)

$$\delta_{is}R^s{}_{\ell xy} + \delta_{\ell s}R^s{}_{ixy} = \gamma_x(\delta_{\ell s}\Gamma^s_{iy} + \delta_{is}\Gamma^s_{\ell y}) -$$

$$- \gamma_y(\delta_{\ell s}\Gamma^s_{ix} + \delta_{is}\Gamma^s_{\ell x})$$

$$= \gamma_x\Lambda_{yi\ell} - \gamma_y\Lambda_{xi\ell} = 0.$$

Hence the above choice of Λ satisfies the equations for $k = 1$.

Suppose $\Lambda_{\ell ik} = \gamma_\ell \delta_{ik}$ fulfils the equation for $k = n$. Writing out the equations for the $(n+1)$-set one has

$$\delta_{is}\nabla_{p_{n+1}}\cdots\nabla_{p_1}R^s{}_{\ell xy} + \delta_{\ell s}\nabla_{p_{n+1}}\cdots\nabla_{p_1}R^s{}_{ixy} =$$

$$\delta_{is}\Gamma^s_{p_{n+1}r}\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{\ell xy} + \delta_{\ell s}\Gamma^s_{p_{n+1}r}\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{ixy} -$$

$$-\Gamma^r_{p_{n+1}p_n}(\delta_{is}\nabla_r\nabla_{p_{n-1}}\cdots\nabla_{p_1}R^s{}_{\ell xy} + \delta_{\ell s}\nabla_r\nabla_{p_{n-1}}\cdots\nabla_{p_1}R^s{}_{ixy}) - \cdots$$

$$-\Gamma^r_{p_{n+1}p_1}(\delta_{is}\nabla_{p_n}\cdots\nabla_{p_2}\nabla_r R^s{}_{\ell xy} + \delta_{\ell s}\nabla_{p_n}\cdots\nabla_{p_2}\nabla_r R^s{}_{ixy}) -$$

$$-\delta_{is}\Gamma^r_{p_{n+1}\ell}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{rxy} + \delta_{\ell s}\Gamma^r_{p_{n+1}i}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{rxy} -$$

$$-\Gamma_{p_{n+1}x}(\delta_{is}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{\ell ry} + \delta_{\ell s}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{iry}) -$$

$$-\Gamma^r_{p_{n+1}y}(\delta_{is}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{\ell xr} + \delta_{\ell s}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{ixr}).$$

By assumption, all parentheses which have a Γ-factor in front vanish. Adding terms which cancel each other, one is left with

$$= (\delta_{is}\Gamma^s_{p_{n+1}r} + \delta_{rs}\Gamma^s_{p_{n+1}i})\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{\ell xy} +$$

$$+ (\delta_{\ell s}\Gamma^s_{p_{n+1}r} + \delta_{rs}\Gamma^s_{p_{n+1}\ell})\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{ixy} -$$

$$- (\delta_{is}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{rxy} + \delta_{rs}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{ixy})\Gamma^r_{p_{n+1}\ell} -$$

$$- (\delta_{\ell s}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{rxy} + \delta_{rs}\nabla_{p_n}\cdots\nabla_{p_1}R^s{}_{\ell xy})\Gamma^r_{p_{n+1}i}\ .$$

But the last two parentheses vanish again by assumption. Therefore

$$= \Lambda_{p_{n+1}ir}\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{\ell xy} + \Lambda_{p_{n+1}\ell r}\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{ixy}$$

$$= \gamma_{p_{n+1}}(\delta_{ir}\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{\ell xy} + \delta_{\ell r}\nabla_{p_n}\cdots\nabla_{p_1}R^r{}_{ixy}) = 0$$

by the assumption.

Thus the conclusion from n to $n+1$ is possible. □

It may happen that these conditions in Proposition 3.2 are also necessary.

4. General form of the metric and an example.

In this section it is assumed that the linear connection Γ of a space of locally constant connection is a Levi-Civita connection with respect to a metric g. If one knew these metrics g, it would be possible to get further information about the spaces of locally constant connection. The general functional form of g is discussed here and an explicit example of such a metric is given.

Let A be an $r \times r$ constant matrix $(r = \frac{1}{2}n(n+1))$, n the dimension of the manifold.

Define the matrix exponential function by means of the infinite series

$$e^{At} = I + At + \cdots + \frac{A^n t^n}{n!} + \cdots .$$

This series exists for all A and for any fixed value of t and for all t for any fixed A. It converges uniformly (see [1 ; p. 166]). Denote by ρ a constant vector, $\rho \in \mathbb{R}^r$ and $g = \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}$.

<u>Proposition 4.1</u>. Let M be a space of locally constant Levi-Civita connection with respect to a metric g which is at least C^3. Then the metric has to be of the form

$$g = e^{\underset{1}{A} x^1} \cdots e^{\underset{n}{A} x^n} \rho ,$$

where $\underset{i}{A}$ are the constant matrices

$$\underset{i\alpha}{A}^{\beta} = \sigma^{k\ell}_{\alpha}\sigma^{\beta}_{ks}\Gamma^{s}_{\ell i} + \sigma^{k\ell}_{\alpha}\sigma^{\beta}_{\ell s}\Gamma^{s}_{ki} .$$

Proof: The metric g is a solution of the system

$$g_{\alpha,i} = \underset{i\alpha}{A}^{\beta} g_{\beta} .$$

This system of partial differential equations of first order splits into systems of ordinary differential equations, each system depending on (n-1) parameters:

$$g_{\alpha,1} = A_{1\alpha}^{\beta}\, g_{\beta}$$
$$\vdots$$
$$g_{\alpha,n} = A_{n\alpha}^{\beta}\, g_{\beta}\ .$$

Assume the metric is C^3. Then, given an initial value, each of the systems has a unique solution. [illegible]side,

$$g_{\alpha,1} = A_{1\alpha}^{\beta}\, g_{\beta}\ .$$

The solution is $g_{\alpha} = e^{A_1 x^1}\, \gamma(x^2,\cdots,x^n)$, where γ has to satisfy the initial value equation.

Hence, given an initial value, one gets n unique solutions for the n systems of ordinary differential equations

It is possible to combine together to give the solution of the whole system:

$$g = e^{A_1 x^1} \cdots e^{A_n x^n}\, \rho\ ,$$

where ρ has to fulfil the initial value equation.

If g is assumed to be analytic, then the uniqueness of the solution (for a fixed initial value) is given by a theorem of Holmgren about systems of partial differential equations ([3 ; p. 237]).
The assumption of analyticity is in this case not very strong because the connection coefficients are analytic (constant). □

Knowing this general form of the metric it is – in principle – possible to construct the solution explicitely for some given sets of connection coefficients which satisfy the integrability conditions. In practice, it is nearly impossible to generate an explicite solution in this way. (In three dimensions, the matrices A_i are already 6×6 matrices.)

Nevertheless, an explicite solution can be constructed in the following way:

Definition 4.1. Let g and $\hat{g}$ be metrics on a differentiable manifold M. They are called *conformal* if $g = e^{2\sigma}\hat{g}$ for some suitable differentiable function σ.
If the components of $\hat{g}$ are $\hat{g}_{ik} = \delta_{ik}$ in some coordinate system, i.e. $\hat{g}$ is the flat Riemannian metric, then g is said to be *conformally flat*.

Note: Angles and ratios of magnitudes are preserved under a conformal change of the metric.

One can ask now:

Given a space of locally constant Levi-Civita connection with metric g. Can g be of the form

$$g_{ik} = e^{2\sigma}\delta_{ik}$$

in a coordinate system in which the connection coefficients are constant. Note that g has to satisfy $g_{\alpha,i} = A^{\beta}_{i\alpha}\, g_{\beta}$.
Taking this ansatz for g one obtains

$$g_{ik,\ell} = 2\sigma_{,\ell}\, g_{ik} = g_{is}\Gamma^{s}_{k\ell} + g_{ks}\Gamma^{s}_{i\ell} .$$

Contracting with g^{ik} leads to

$$\sigma_{,\ell} = \frac{1}{n}\,\Gamma^{i}_{i\ell} ,$$

where n is the dimension of the manifold. The integrability conditions for this system are satisfied identically. Hence

$$\sigma = \frac{1}{n}\,\Gamma^{i}_{i\ell}\, x^{\ell} \quad \text{for} \quad \mathop{g}_{\circ}{}_{ik} = \delta_{ik}.$$

This means if σ is this linear function of the coordinates x^i, then $g_{ik} = e^{2\sigma}\delta_{ik}$ is a solution of $g_{\alpha,i} = A^{\beta}_{i\alpha}\, g_{\beta}$.

Now all connection coefficients can be calculated through

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{is}(g_{sk,\ell} + g_{s\ell,k} - g_{k\ell,s}).$$

This leads to

$$\Gamma^i_{k\ell} = 2\delta^i_{(k}\sigma_{,\ell)} - \delta^{is}\delta_{k\ell}\sigma_{,s} .$$

Thus the connection coefficients with respect to

$$g_{ik} = e^{2\sigma}\,\delta_{ik}$$

are

$$\Gamma^i_{k\ell} = 0 \qquad i \neq k \neq \ell$$

$$\Gamma^i_{ii} = \sigma_{,i}$$

$$\Gamma^i_{ik} = \Gamma^i_{ki} = \sigma_{,k} \qquad i \neq k \qquad \text{no summation}$$

$$\Gamma^i_{kk} = -\delta^{is}\,\sigma_{,s} \qquad i \neq k .$$

In the last section (Proposition 3.2) sufficient integrability conditions were derived:

$$\delta_{is}\Gamma^s_{k\ell} + \delta_{\ell s}\Gamma^s_{ki} = \gamma_k\delta_{i\ell} \qquad \gamma_k \in \mathbb{R}^n.$$

It is easily seen that the just discovered set of constant connection coefficients satisfy exactly these integrability conditions for

$$\gamma_k = 2\sigma_{,n} = \frac{2}{n}\,\Gamma^i_{ik} .$$

Hence one has the following Proposition:

Proposition 4.2. Let M be a space of locally constant Levi-Civita connection Γ. If the connection coefficients $\Gamma^i_{k\ell}$ with respect to a chart in which they are constant satisfy

$$\delta_{is}\Gamma^s_{k\ell} + \delta_{\ell s}\Gamma^s_{ki} = \gamma_k\delta_{i\ell},$$

then the unique solution of the system of partial differential equations

$$g_{ik,\ell} = g_{is}\Gamma^s_{k\ell} + g_{ks}\Gamma^s_{i\ell}$$

for the initial value $\underset{\circ}{g}_{ik} = \delta_{ik}$ is given by

$$g_{ik} = e^{2\sigma}\delta_{ik} ,$$

where $\sigma = \frac{1}{n}\,\Gamma^i_{i\ell}\,x^\ell .$

Proof: This is a direct consequence of Proposition 4.1. The solution of the system is unique for a given set of constant $\Gamma^i_{k\ell}$'s and a fixed initial value $\underset{\circ}{g}_{ik}$. It is always possible to choose the initial value $\underset{\circ}{g}_{ik} = \delta_{ik}$ because every non-degenerate positive definite quadratic form can be changed to the form δ_{ik} by a linear transformation. The connection coefficients remain constant under this transformation. □

Note: If the sufficient conditions for integrability introduced in Section 3 turn out to be necessary, then the only Riemannian metric on a space of locally constant connection is the just described conformally flat metric.

Now the curvature tensor of the spaces which admit this metric is calculated. Its components for constant connection are

$$R^i_{k\ell s} = \Gamma^i_{\ell r}\Gamma^r_{sk} - \Gamma^i_{sr}\Gamma^r_{\ell k} .$$

Inserting

$$\Gamma^i_{k\ell} = 2\delta^i_{(k}\sigma_{,\ell)} - \delta^{is}\delta_{k\ell}\sigma_{,s}$$

leads to

$$(*) \quad R^i_{k\ell s} = 2\big(\sigma_{,k}\sigma_{,[s}\delta^i_{\ell]} + \delta^{ir}\sigma_{,r}\delta_{k[s}\sigma_{,\ell]} + \\ + \delta^{rt}\sigma_{,r}\sigma_{,t}\,\delta_{k[\ell}\delta^i_{s]}\big)\,.$$

Therefore the components of the curvature tensor are

$$R^i_{ki\ell} = \sigma_{,k}\sigma_{,\ell} \qquad i \neq k \neq \ell$$

$$R^i_{kik} = -\sum_{r\neq i,k} (\sigma_{,r})^2 \qquad i \neq k \,. \qquad \text{no summation}$$

$$R^i_{kk\ell} = -\delta^{ir}\sigma_{,r}\sigma_{,\ell} \qquad i \neq k \neq \ell,$$

while the other components are zero and $\sigma_{,i} = \frac{1}{n}\Gamma^s_{si}$

Definition 4.2. Let M be a Riemannian manifold. The *Ricci tensor* is a covariant tensor of degree two whose components are defined by

$$R_{ik} = R^s_{isk} \,.$$

The transvection of the Ricci-tensor is called the *scalar curvature* R and is given by

$$R = g^{ik} R_{ik} \,.$$

Proposition 4.3. Let M be an n-dimensional space of locally constant Levi-Civita connection with metric

$$g_{ik} = e^{2\sigma}\delta_{ik} \,.$$

The scalar curvature is negative or zero for $n > 2$ and zero for $n \leq 2$.

Proof: Construct the Ricci tensor from equation (*):

$$R_{ik} = (n-2)(\sigma_{,i}\sigma_{,k} - \delta_{ik}\delta^{rs}\sigma_{,r}\sigma_{,s}).$$

Contraction with $g^{ik} = e^{-2\sigma}\delta^{ik}$ leads to

$$R = -e^{-2\sigma}(n-2)(n-1)\delta^{ik}\sigma_{,i}\sigma_{,k} \,. \quad \square$$

Looking at the formula for the curvature tensor it can be seen that a space of locally constant Levi-Civita connection admitting a conformally flat metric is flat in two dimensions. The general two dimensional case will be considered in the next section.

5. The two-dimensional case.

The spaces of locally constant Levi-Civita connection in two dimensions can be classified now. For this purpose the following definition is needed:

Definition 5.1. Let M be a differentiable manifold with metric g. If g and the Ricci tensor are connected by

$$R_{ik} = \frac{R}{n} g_{ik} ,$$

where R denotes the scalar curvature, then M is called an *Einstein space*.

Lemma 5.1. All two dimensional differentiable manifolds with metric are Einstein spaces.

Proof: The proof is just a simple calculation which will not be carried out here.
For details see [6; p. 47]. □

Theorem 5.1. Let M be a two dimensional differentiable manifold. Then M is a space of locally constant Levi-Civita connection if and only if M is flat.

Proof: The proof is done in two different ways. The first proof uses the fact that all two dimensional manifolds are Einstein spaces to construct the general two dimensional metric for the considered spaces. The second proof is based on the integrability conditions on the connection for it to be a Levi-Civita connection.

First proof:

" $\Rightarrow$ " : M is an Einstein space, therefore

$$R_{ik} = \frac{1}{2} R g_{ik}.$$

Because M is a space of locally constant connection there exists a coordinate chart around each point such that the components of the Ricci tensor are constant.

Differentiate the last equation:

$$2\, R_{ik,\ell} = 0 = R_{,\ell}\, g_{ik} + R\, g_{ik,\ell}.$$

Now

$$g_{ik,\ell} = g_{is}\,\Gamma^{s}_{k\ell} + g_{ks}\,\Gamma^{s}_{i\ell}\;,$$

hence

$$R_{,\ell} g_{ik} = -R(g_{is}\Gamma^{s}_{k\ell} + g_{ks}\Gamma^{s}_{i\ell}).$$

Contraction with g^{ik} gives

$$R_{,\ell} = -R\;\Gamma^{i}_{i\ell}\;.$$

The unique solution of this system of differential equations is

$$R = c \cdot \exp\{-(\Gamma^{1}_{11}+\Gamma^{2}_{12})x^{1} - (\Gamma^{1}_{12}+\Gamma^{2}_{22})x^{2}\}$$

where c is constant, depending on the initialvalue, while the connection coefficients have to fulfill integrability conditions.

Using the first equation, this implies

$$g_{ik} = \exp\{(\Gamma^{1}_{11}+\Gamma^{2}_{12})x^{1} + (\Gamma^{1}_{12}+\Gamma^{2}_{22})x^{2}\}\lambda_{ik}$$

where

$$\lambda_{ik} = \frac{2}{c}\, R_{ik} = \text{constant}.$$

Now the curvature tensor is calculated for this metric (see appendix). All components vanish, thus $\underline{M}$ is flat.

<u>Second proof</u>;

" ⟹ " : To be a Levi-Civita connection the holonomy group of the connection has to satisfy the equations (see Proposition 3.1)

$$\delta_{is}\nabla_{p_k}\cdots\nabla_{p_1} R^s{}_{\ell xy} + \delta_{\ell s}\nabla_{p_k}\cdots\nabla_{p_1} R^s{}_{ixy} = 0, \quad 0 \le k < \infty.$$

Consider the first two sets of equations:

$$\delta_{is} R^s{}_{\ell xy} + \delta_{\ell s} R^s{}_{ixy} = 0$$

$$\delta_{is} \nabla_p R^s{}_{\ell xy} + \delta_{\ell s} \nabla_p R^s{}_{ixy} = 0.$$

From these two sets one obtains the following conditions on the connection coefficients:

$$\Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22}$$

$$\Gamma^2_{22} = \Gamma^1_{12} = -\Gamma^2_{11} .$$

The curvature tensor has only 3 independent components in two dimensions. Writing out these components in terms of the connection coefficients and using the just obtained conditions, the curvature tensor turns out to be zero, hence M is flat.

" $\Longrightarrow$ " : Let M be flat. Then it is a well known result that there exists around each point $p \in M$ a chart such that the connection coefficients are zero. Hence M is a space of locally constant connection. □

6. Further properties.

In section 1 of this chapter a coordinate independent characterization of spaces of locally constant connection was given. It was found that these spaces admit an n-dimensional abelian, transitive group of affine transformations. The effects of this property are investigated in this section.

In a manner analogous to that for a connection, one can introduce transformations which leave the metric invariant.

Definition 6.1. Let M be a Riemannian manifold. A transformation

$$\Phi : M \to M$$

is called an *isometry* if

$$\Phi^* g = g,$$

i.e.
$$g_p(X,Y) = g_{\Phi(p)}(\Phi_* X, \Phi_* Y)$$

for all $p \in M$ and $X,Y \in T_p(M)$.

Definition 6.2. A vector field X on M is called an *infinitesimal isometry* on M if for all $p \in M$ the local 1-parameter group of local transformations

$$\phi_t : U \to M \qquad p \in U$$

is an isometry for all t.

Note: It can be shown that the isometries on M form a Lie group.

The next two propositions are mentioned without proof. They demonstrate the properties of isometries in a similar way as they are shown for affine transformations.

Denote by $O(M)$ a bundle of orthonormal frames. This is a sub-bundle of the frame bundle $L(M)$.

Proposition 6.1. Let Φ be a transformation on M.

(i) It is an isometry if and only if the induced transformation $\tilde{\Phi}$ on $L(M)$ maps $O(M)$ into itself.

(ii) A fiber preserving transformation which leaves the canonical form θ on $O(M)$ invariant is induced by an isometry of M.

Proposition 6.2. For a vector field X on M, the following conditions are equivalent:

(1) X is an infinitesimal affine transformation.

(2) $\overline{X}\,\tilde{g} = 0$, where $\overline{X}$ is the natural lift of X and g the metric on M.

Proposition 6.3. Let X be an infinitesimal isometry on M. Then X satisfies

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for $Y, Z \in T(M)$. With respect to a chart (x^i) one has

$$\xi_{(i;k)} = 0$$

where $X = \xi^i \frac{\partial}{\partial x^i}$. This equation is called the *Killing equation*.

Proof: Since X is an infinitesimal isometry,

$$L_X g = 0.$$

Then $\nabla_X g = 0$ implies $(L_X - \nabla_X)g = 0$. But

$$\big((L_X - \nabla_X)g\big)(Y,Z) = (L_X - \nabla_X)g(Y,Z) - g\big((L_X - \nabla_X)Y, Z\big) - g\big(Y, (L_X - \nabla_X)Z\big),$$

therefore

$$g\big((L_x - \nabla_x)Y,Z\big) + g\big(Y,(L_x - \nabla_x)Z\big) = 0.$$

The torsion vanishes, thus

$$L_xY - \nabla_xY = -\nabla_yX$$

and this implies

$$g(\nabla_YX,Z) + g(Y,\nabla_XZ) = 0. \quad \square$$

Every isometry is also an affine transformation:

Lemma 6.1. Let Φ be an isometry on M. Then the induced linear map Φ_* commutes with parallel propagation.

Proof: This is a consequence of the uniqueness of the Levi-Civita connection (see Theorem 5.2, Chapter I).
The Levi-Civita connection Γ on M is mapped by the induced automorphism $\tilde{\Phi}$ on L(M) to a unique linear connection $\tilde{\Phi}(\Gamma)$ (see Lemma 1.2, Chapter II). If Φ is an isometry, this connection t out to be torsion free and metric. Thus

$$\Phi_*(\nabla_XY) = \nabla_{\Phi_*X}\Phi_*Y\ , \qquad X,Y \in T(M).$$

For more details see [9; p. 161]. $\square$

Now in the case of spaces of locally constant connection one has an n-dimensional abelian, transitive group of affine transformations. The following Lemma is easily seen:

Lemma 6.2. Let M be a Riemannian manifold admitting an n-dimensional transitive, abelian group G of affine transformations. If G is also a group of isometries, then M is flat.

Proof: Let G be defined on a neighborhood U of $p \in M$.

Take a basis e_a in the Lie algebra of vector fields on U which is isomorphic to the Lie algebra of G. The vector fields e_a can be chosen as coordinate vector fields, because G is abelian. Each

$$e_a = \xi_a^i \frac{\partial}{\partial x^i} = \xi_a^i \frac{\partial}{\partial x^i}$$

satisfies the Killing equation

$$\xi_{(i;k)} = 0.$$

Using

$$g_{ik,\ell} = g_{is}\Gamma^s_{k\ell} + g_{ks}\Gamma^s_{i\ell}$$

leads to $g_{ik,a} = 0$.

But there exist n vector fields e_a, hence g_{ik} is constant. Thus the curvature tensor vanishes, M is flat. □

Because of the last lemma, the interesting questic e investigation of spaces of locally constant Levi-Civita connection is: When is an affine transformation an isometry?

First compact manifolds are considered. The following proposition is found:

Proposition 6.4. Let M be a compact Riemannian manifold. Then every infinitesimal affine transformation is an infinitesimal i metry.

Proof: This result is due to Yano [18]. The proof involves several Lemmas. It can be found also in [10; p. 45]. □

Now an immediate consequence is

Proposition 6.5. Let M be a compact Riemannian manifold. Then M is a space of locally constant Levi-Civita connection if and only if M is flat.

Proof: " $\Rightarrow$ " : M admits an n-dimensional transitive, abelian group affine transformations which is also a group of isometries. Hence Proposition 6.2 implies that M is flat.

" $\Leftarrow$ " : This is already known, see proof of Theorem 5.1, Chapter II. □

Note. All spheres $S^{n-1} = \{(x^i) \in \mathbb{R}^n : {x^1}^2 + \cdots + {x^n}^2 = 1\}$ are compact and not flat $(n \geq 2)$. Therefore no sphere $(n \geq 2)$ is a space of locally constant Levi-Civita connection.

Next, spaces of constant curvature are considered.

Definition 6.3. Let M be a Riemannian manifold. For each plane S in the tangent space $T_p(M)$, $p \in M$ the *sectional curvature* $K(s)$ of S is defined by $K(S) = g(R(X_1, X_2)X_2, X_1)$, where X_1, X_2 is an orthonormal basis for S.

Note: The sectional curvature $K(S)$ is independent of the choice of an orthonormal basis X_1, X_2.

Definition 6.4. Let M be a Riemannian manifold. If the sectional curvature $K(S)$ is a constant for all planes S in $T_p(M)$ and for all $p \in M$, then M is called a space of *constant curvature*.

In a space of constant curvature the curvature tensor and the metric are related by

Lemma 6.3. Let M be a space of constant curvature k. Then

$$R(X,Y)Z = k\big(g(Z,Y)X - g(Z,X)Y\big),$$

where X, Y, Z are vector fields on M.

Proof: See [9; p. 203]. □

Note: If the sectional curvature depends only on $p \in M$, and not on the plane S, then $K(p)$ and the scalar curvature are related by

$$K(p) = \frac{R}{n(n-1)} .$$

If $\dim M \geq 3$, then M is already a space of constant curvature.

For example, the n-sphere S^n, is a space of constant curvature. But S^n is also compact and hence (Proposition 6.5) not a space of locally constant Levi-Civita connection. Therefore the question arises: Are there spaces of locally constant connection which have constant curvature? The answer is given by

Proposition 6.6. Let M be a space of constant curvature. Then M is a space of locally constant Levi-Civita connection if and only if M is flat.

Proof: Since M is a space of constant curvature,

$$R(X,Y)Z = k\big(g(Z,Y)X - g(Z,X)Y\big).$$

In local coordinates one gets

$$R^i_{\ k\ell s} = k(\delta^i_\ell \, g_{ks} - \delta^i_s \, g_{k\ell}).$$

If M is a space of locally constant connection then it admits around each $p \in M$ a chart such that the components of the curvature tensor are constant. Now

$$R^i_{\ k(\ell s)} = 0$$

thus choose $s \neq \ell = i$.

This implies

$$R^i_{kis} = k \cdot g_{ks} \qquad \text{(no summation)}.$$

Therefore the metric components are constant, hence M is flat. The reverse direction was already outlined in the proof of Theorem 5.1, Chapter II.

This result can be used to draw a conclusion about the Lie group of isometries for spaces of constant curvature.
It is possible to determine the maximal dimension of the affine group as well as the isometry group.

<u>Theorem 6.1</u>. Let M be an n-dimensional connected manifold with connection Γ. The Lie algebra $A(M)$ of infinitesimal affine transformations of M is of dimension at most $n^2 + n$. If $\dim A(M) = n^2 + n$, then M is flat.

<u>Theorem 6.2</u>. Let M be a connected, simply connected n-dimensional Riemannian manifold. The Lie algebra $T(M)$ of infinitesimal isometries is of dimension at most $\frac{1}{2}n(n+1)$. If $\dim T(M) = \frac{1}{2}n(n+1)$, then M is a space of constant curvature of the following type:

(a) n-dimensional Euclidean space $\mathbb{R}^n$,

(b) n-dimensional sphere S^n,

(c) n-dimensional projective space $\mathbb{P}_n(\mathbb{R})$,

(d) n-dimensional simply connected hyperbolic space.

For proofs of both Theorems see [9 ; p. 234,238].

It was shown (Proposition 6.6) that a space of constant curvature is a space of locally constant Levi-Civita connection iff M is flat. This implies

Corollary 6.1. Let M be an n-dimensional manifold of type (b)-(d). Then the isometry group of M does not contain an n-dimensional transitive, abelian subgroup.

Proof: Assume the isometry group of M contains an n-dimensional transitive, abelian subgroup. All isometries are in particular affine transformations, thus M is a space of locally constant Levi-Civita connection. But M is also a space of constant curvature, which implies by Proposition 6.6 that M is flat.
Hence one has a contradiction to the fact that the spaces (b)-(d) are not flat. □

Example 6.1. The isometry groups of the n-spheres S^n are the special orthogonal groups SO(n+1). Consider the two sphere S^2 with isometry group SO(3). This group does not admit a two dimensional subgroup of translations.

Another class of spaces whose affine transformations are also isometries is the class of irreducible Riemannian manifolds.

Definition 6.5. Let M be a connected Riemannian manifold and $\Psi(p)$, $p \in M$, the holonomy group. The holonomy group is *reducible* if $T_p(M)$ contains a subspace P which is invariant under all transformations $a \in \Psi(p)$. Hereby P is neither the whole space $T_p(M)$ nor the zero space.

Definition 6.6. Let M be a connected Riemannian manifold. Then M is *reducible* (irreducible) according as the holonomy group $\Psi(p)$, $p \in M$, is reducible (irreducible) as a linear group acting on $T_p(M)$.

For an irreducible Riemannian manifold the following holds:

Proposition 6.7. Let M be an irreducible Riemannian manifold. Any infinitesimal affine transformation on M is also an isometry.

Proof: See [12; p. 128]. □

This result leads to

Proposition 6.8. Let M be a space of locally constant Levi-Civita connection whose curvature tensor does not vanish. Then M is reducible.

Proof: Assume M is irreducible and not flat. Then all infinitesimal affine transformations are isometries, hence there exists an n-dimensional abelian, transitive group of isometries. This implies that M is flat, which contradicts the assumption. □

The importance of reducibility lies in the fact that the manifolds can be split into the direct product of irreducible parts. This statement is shown by means of the decomposition theorem of de Rham.

Theorem 6.3. A connected, simply connected and complete Riemannian manifold M is isometric to the direct product

$$M_0 \times M_1 \times \cdots \times M_k,$$

where M_0 is a Euclidean space (possibly of dimension 0) and $\cdots, M_k$ are all simply connected, complete irreducible Riemannian manifolds. Such a decomposition is unique up to an order.

See [9; p. 192] for further details.

The last class of spaces which will be treated in context with the spaces of locally constant connection are the spaces with locally symmetric connection.

Definition 6.7. Let M be a differentiable manifold and let U be a neighborhood of $p \in M$. If the map

$$S_p : U \to U$$

$$\exp X \to \exp(-X) \qquad X \in T_p(M)$$

is a diffeomorphism, S_p is called the *symmetry* at $p \in M$.

Definition 6.8. Let M be a differentiable manifold with connection Γ. If the symmetry S_p at $p \in M$ is an affine transformation for all p, then Γ is called *locally symmetric.*

The spaces of locally constant connection possess a whole group of affine transformations, so a conjecture might be that their linear connection is locally symmetric. It will be shown that this is not true in general. For this purpose one needs the following characterization of locally symmetric connections.

Proposition 6.9. Let M be a differentiable manifold with connection Γ. Then Γ is locally symmetric if and only if

$$T = 0$$

$$\nabla R = 0,$$

where T and R are the torsion and the curvature tensor respectively.

Proof: See [9; p. 303]. □

It can be seen by a counterexample that the connection of spaces of locally constant connection is not locally symmetric in general. Consider the uniformally flat metrics discussed in section 4 of this chapter:

$$g_{ik} = e^{2\sigma}\delta_{ik} ,$$

where

$$\sigma = \frac{1}{n}\Gamma^{i}_{i\ell}\, x^{i} .$$

Choose a chart such that the components of the curvature tensor, with respect to this chart, are constant.

Then the components of ∇R are given by

$$R^{i}{}_{k\ell s;r} = \Gamma^{i}_{tr}\, R^{t}{}_{k\ell s} - \Gamma^{t}_{kr}\, R^{i}{}_{t\ell s} - \Gamma^{t}_{\ell r}\, R^{i}{}_{kts} - \Gamma^{t}_{sr}\, R^{i}{}_{k\ell t} .$$

It suffices to find one component which does not vanish.

Carrying out the calculation one obtains

$$R^{1}{}_{212;1} = 4\sigma_{,1}(\sigma_{,3})^{2} .$$

Obviously, a connection can be given such that this component does not vanish.

Therefore if a manifold M is a space of locally constant connection, this is not sufficient to ensure that the connection is locally symmetric.

NOTATION

Throughout the thesis the *Einstein summation convention* is adopted: For example,

$$X = \xi^i \frac{\partial}{\partial x^i}$$

means summation over all values of i.

The Euclidean n-space is denoted by $\mathbb{R}^n$.

The *natural basis* of $\mathbb{R}^n$ is given by

$$e_1 = (1,0,\cdots,0)$$
$$\vdots$$
$$e_n = (0,\cdots,0,1)$$

The differentiable manifolds M considered are assumed to be of class C^∞ (if not otherwise stated) and to be paracompact.

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APPENDIX I

Basic Concepts of Differential Geometry

Definition A.1. Let M be a set. An n-*chart* at a point $p \in M$ is a pair (U,ϕ), where

n integer ≥ 0

U open set in M, $p \in U$

$\phi: U \to$ open set in $\mathbb{R}^n$ is a one-to-one map.

Definition A.2. Let
$$pr^i : \mathbb{R}^n \to \mathbb{R}$$
$$(a^1, \cdots, a^n) \to a^i$$
be the projection maps. The maps
$$x^i = pr^i \circ \phi : U \to \mathbb{R}$$
$$q \to a^i$$
are called the *coordinates* of $q \in U$.

Definition A.3. Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are C^r-*compatible* if the maps
$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$
are r-times continuously differentiable.

Definition A.4. An *atlas* on M is a collection of compatible charts $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ (A index set), such that
$$\bigcup_{\alpha \in A} U_\alpha = M.$$

Definition A.5. A C^r-atlas is called *maximal* if it contains any chart which is C^r-compatible to its charts.

A maximal C^r-atlas on M is also called a *differentiable structure* of class C^r on M.

<u>Definition A.6</u>. The set M together with a maximal C^r-atlas is a C^r-n-*dimensional manifold*.

<u>Definition A.7</u>. Let f be a real-valued function defined on an open subset $V \subset M$

$$f : V \to \mathbb{R}.$$

f is called *differentiable* at $p \in V$ if

$$f \circ \phi^{-1} : \phi(V) \subset \mathbb{R}^n \to \mathbb{R}$$

is differentiable at $\phi(p)$ for all charts (U,ϕ), at p.
f is called differentiable on V if it is differentiable at every $p \in V$.

<u>Definition A.8</u>. A *differentiable curve* in M is a map

$$\sigma : (a,b) \to M$$

where (a,b) is an open interval in $\mathbb{R}$ such that the map

$$\phi \circ \sigma : (a,b) \to \mathbb{R}^n$$

is differentiable - wherever it is defined - for all charts (U,ϕ).

<u>Definition A.9</u>. Let $F_p(M)$ denote the set of all differentiable real valued functions defined around $p \in M$ and let $\sigma(t)_{-\varepsilon < t < \varepsilon}$ be a differentiable curve with $\sigma(0) = p$.
The *tangent vector* X_p to $\sigma(t)$ at p is a map

$$X_p : F_p(M) \to \mathbb{R}$$
$$f \to [\frac{d}{dt} (f \circ \sigma)]_{t=0} = X_{pf}.$$

<u>Lemma A.1</u>. Let M be a differentiable manifold, $p \in M$. The set $T_p(M) := \{X_p : X_p$ tangent vector of a curve through $p\}$ forms a real vector space whose dimension is that of the manifold. The space $T_p(M)$ is called the *tangent space* to M at p.

<u>Definition A.10</u>. The *dual* of the tangent space to M at p or the *cotangent space* is defined by

$$T_p^*(M) = \{\omega : T_p(M) \to \mathbb{R} \quad \omega \text{ linear}\}$$
$$X_p \to \omega(X_p).$$

<u>Definition A.11</u>. A *tensor* S on M at p is a map

$$S : \underbrace{T_p(M) \times \cdots \times T_p(M)}_{r\text{-times}} \times \underbrace{T_p^*(M) \times \cdots \times T_p^*(M)}_{s\text{-times}} \to \mathbb{R}$$

which is linear in each argument.

Denote by $T_p{}^r_s(M)$ the vector space of all tensors of covariant degree s and contravariant degree r.

<u>Definition A.12</u>. The *tensor product* is a map

$$T_p{}^r_s(M) \times T_p{}^k_\ell(M) \to T_p{}^{r+k}_{s+\ell}$$

defined by

$$(S \otimes R)(X_1,\cdots,X_s, Y_1,\cdots,Y_\ell, \omega^1,\cdots,\omega^r, \eta^1,\cdots,\eta^k) =$$
$$S(X_1,\cdots,\omega^r) \cdot R(Y_1,\cdots,\eta^k)$$

where X, Y denote vectors, ω, η 1-forms.

<u>Definition A.13</u>. Let e_a be a basis of $T_p(M)$, e^a the dual basis. The *components* of a tensor field $S \in T_p{}^r_s(M)$ in the basis e_a, e^a are functions

$$S_{i_1 \cdots i_s}^{k_1 \cdots k_r} : M \to \mathbb{R}$$

such that

$$S = S_{i_1 \cdots i_s}^{k_1 \cdots k_r} e^{i_1} \otimes e^{i_s} \otimes e_{k_1} \otimes \cdots \otimes e_{k_r},$$

i.e.

$$S_{i_1 \cdots i_s}^{k_1 \cdots k_r} = S(e_{i_1}, \cdots, e_{i_s}, e^{k_1}, \cdots e^{k_r}).$$

Definition A.14. A tensor $S \in T_p{}^o_s(M)$ is *symmetric* if for any permutation $\sigma = \begin{pmatrix} 1 & \cdots & s \\ \sigma(1) & \cdots & \sigma(s) \end{pmatrix}$

$$S(X_1, \cdots, X_s) = S(X_{\sigma(1)}, \cdots, X_{\sigma(s)})$$

holds.

Definition A.15. A *tensor field* on M is a map

$$S : M \to T^r_s(M) := \bigcup_{p \in M} T_p{}^r_s(M).$$

S is differentiable if its components $S\left(\left(\frac{\partial}{\partial x^{i_1}}\right), \cdots, dx^{k_r}\right)$ with respect to a chart are differentiable.

Definition A.16. Let $\omega \in T^o_s(M)$ and σ be a permutation. The *skew part* of ω is defined by

$$(A\omega)(X_1, \cdots, X_s) = \frac{1}{s!} \sum_\sigma (\text{sgn } \sigma)\omega(X_{\sigma(1)}, \cdots, X_{\sigma(s)}).$$

Definition A.17. Let ω, η be two forms. The *exterior product* $\wedge$ is defined by $\omega \wedge \eta = A(\omega \otimes \eta)$.

Theorem A.1. There exists unique map

$$d : T^o_s(M) \to T^o_{s+1}(M)$$

called *exterior differentiation* such that

(1) $(df)X = Xf \qquad f : M \to \mathbb{R}$,

(2) $d(d\omega) = 0 \qquad$ for any s-form ω,

(3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^s \omega \wedge d\eta$, where ω is s-form,

(4) $d(\omega + \eta) = d\omega + d\eta$.

Definition A.18. Let X be a vector field on M. Let $\sigma : (a,b) \to M$ be a curve with tangent vector $\dot{\sigma}(t) = \frac{d\sigma(t)}{dt}$. σ is an *integral curve* of X if $\dot{\sigma}(t) = X_{\sigma(t)} \quad \forall\, t \in (a,b)$.

Lemma A.2. Let X be a vector field on M, $p \in M$. There exists a unique maximal integral curve $\sigma(t)$ with $\sigma(0) = p$.

Lemma A.3. Let $\sigma_p(t)$ be the maximal integral curve of the vector field X with $\sigma_p(0) = p \in M$. Then there exists a neighborhood U of p and an $\varepsilon > 0$ such that the maps

$$\phi_t : U \to M \qquad \text{defined by}$$

$$q \to \sigma_q(t) \qquad t \in (-\varepsilon, \varepsilon)$$

are diffeomorphisms $U \to \phi_t(U)$ and satisfy

$$\phi_t \circ \phi_s = \phi_{t+s} \qquad s \in (-\varepsilon, \varepsilon).$$

The collection ϕ_t is called the *local one-parameter group of local transformations* generated by X.

Definition A.19. Let M^n, N^k be differentiable manifolds, (U, ϕ) be a chart at $p \in M^n$, (U', ϕ') be a chart at $\alpha(p) \in N^k$ where α is a map

$$\alpha : M^n \to N^k.$$

α is called *differentiable* if the map

$$\phi' \circ \alpha \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^k$$

is differentiable where it is defined.

Definition A.20. $\alpha : M \to N$ is called a *diffeomorphism* if α is a homeomorphism and α and α^{-1} are differentiable.

Theorem A.2. Let f be a function

$$f : N \to \mathbb{R}.$$

Any differentiable map

$$\alpha : M \to N$$

induces a linear map

$$(\alpha_*)_p : T_p(M) \to T_{\alpha(p)}(N) \qquad \forall\, p \in M$$

defined by

$$\big((\alpha_*)_p\, X\big) f = X_p(f \circ \alpha)$$

and a linear map

$$(\alpha^*)_p : T^*_{\alpha(p)}(N) \to T^*_p(M) \qquad \forall\, p \in M$$

defined by

$$\big((\alpha^*)_p\, \omega\big)(X) = \omega_{\alpha(p)}\big((\alpha_*)_p\, X\big),$$

where

$$\omega \in T^*_{\alpha(p)}(N) \quad \text{and} \quad X \in T_p(M).$$

Definition A.21. Let X, Y be vector fields on M. The Lie bracket (or commutator) of X and Y is defined by the map

$$\begin{aligned} \phi : F(M) &\to F(M) \\ f &\to [X,Y]f \\ &= X(Yf) - Y(Xf). \end{aligned}$$

<u>Definition A.22</u>. Let G be a Lie group and

$$L_a : G \to G$$
$$p \to ap.$$

A vector field X on G is called *left invariant* if

$$(L_a)_* X_p = X_{ap} \qquad p \in G.$$

<u>Theorem A.3</u>. Let G be a Lie group. The set of all left invariant vector fields on G form a Lie algebra g and $\dim g = \dim G$.

<u>Definition A.23</u>. Let $\sigma(t)$ be the geodesic with

$$\sigma(0) = p$$
$$\dot{\sigma}(0) = X \qquad X \in T_p(M)$$

The *exponential map* is defined by

$$\exp : T_p(M) \to M$$
$$tX \to \sigma(t)$$

for all $p \in M$.

APPENDIX II

Curvature Computed Using Exterior Differential Forms

Given a metric, it is possible to compute the curvature tensor components by calculating the connection coefficients and then using the relation between curvature components and connection components given in Lemma 1.8, Chapter II.

A more effective way is described here in a short form. This approach is taken to compute the curvature tensor for the two dimensional case (see Section 5, Chapter II).

The line element is given by

$$ds^2 = g_{ik}\, dx^i \otimes dx^k.$$

Introduce a frame

$$e_\alpha = e^i_\alpha \frac{\partial}{\partial x^i}$$

and the dual 1-forms

$$e^\alpha = e^\alpha_i\, dx^i$$

such that the components of the metric, with respect to this frame are

$$g_{\alpha\beta} = \delta_{\alpha\beta}.$$

Hence,

$$ds^2 = \delta_{\alpha\beta}\, e^\alpha \otimes e^\beta.$$

Now

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = \Gamma^\ell_{ik} \frac{\partial}{\partial x^\ell}.$$

Define $\Gamma^\alpha_{\beta\gamma}$ by

$$\nabla_{e_\alpha} e_\beta = -\Gamma^\gamma_{\alpha\beta} e_\gamma$$

which implies

$$\Gamma^\gamma_{\alpha\beta} = e^\gamma_{i;k}\, e^i_\alpha\, e^k_\beta .$$

Define 1-forms ω^α_β by

$$\omega^\alpha_\beta := \Gamma^\alpha_{\beta\gamma}\, e^\gamma$$

$$\omega_{(\alpha\beta)} = 0 ,$$

then

$$de^\alpha = -\omega^\alpha_\beta \wedge e^\beta .$$

The structure equations lead to

$$d\omega^\alpha_\beta = R^\alpha{}_{\beta\gamma\delta}\, e^\gamma \wedge e^\delta - \omega^\alpha_\gamma \wedge \omega^\gamma_\beta$$

where $R^\alpha{}_{\beta\gamma\delta}$ are the components of the curvature tensor in the frame $e_\alpha (e^\alpha)$. They are related to the components of the curvature tensor in the coordinate frame by

$$R^\alpha{}_{\beta\gamma\delta} = R^i{}_{k\ell s}\, e^\alpha_i\, e^k_\beta\, e^\ell_\gamma\, e^s_\delta .$$

The general two dimensional line element for spaces of locally constant Levi-Civita connection was

$$ds^2 = e^{ax^1+bx^2}(A\, dx^1 + B\, dx^1 dx^2 + C\, dx^2)$$

where a, b, A, B, C are constants.

Choose

$$e^1 = \sqrt{f}\, (\alpha\, dx^1 + \beta\, dx^2)$$

$$e^2 = \sqrt{f}\, \delta\, dx^2$$

with

$$f = e^{ax^1+bx^2}$$

$$\alpha = \sqrt{A}$$

$$\beta = \frac{B}{2\sqrt{A}}$$

$$\delta = \sqrt{C - \frac{B^2}{4A}} \quad .$$

Hence

$$ds^2 = e^1 \otimes e^1 + e^2 \otimes e^2$$

and

$$dx^1 = \frac{1}{\alpha\sqrt{f}} (e^1 - \frac{\beta}{\delta} e^2)$$

$$dx^2 = \frac{1}{\delta\sqrt{f}} e^2$$

$$de^1 = \frac{(a\beta - b\alpha)}{2\sqrt{f}\ \alpha\delta} (e^1 \wedge e^2)$$

$$de^2 = \frac{1}{2\sqrt{f}} \frac{a}{\alpha} e^1 \wedge e^2 .$$

Therefore

$$\omega^1_2 = - \frac{1}{2\sqrt{f}} \left(\frac{a\beta - b\alpha}{\alpha\delta} e^1 + \frac{a}{\alpha} e^2 \right) .$$

Also $\omega^\alpha_\gamma \wedge \omega^\gamma_\beta = 0$ in two dimensions because of $\omega_{(\alpha\beta)} = 0$.

Thus

$$d\omega^1_2 = R^1{}_{2\gamma\delta}\ e^\gamma \wedge e^\delta$$

gives all curvature components.

One gets

$$d\omega^1_2 = \frac{1}{4\sqrt{f}} (adx^1 + bdx^2) \wedge \left(\frac{a\beta - b\alpha}{\alpha\delta} e^1 + \frac{a}{\alpha} e^2 \right)$$

$$- \frac{1}{2\sqrt{f}} \left(\frac{a\beta - b\alpha}{\alpha\delta} de^1 + \frac{a}{\alpha} de^2 \right)$$

$$= 0, \quad \text{inserting } de^\alpha \text{ and } dx^i .$$

Therefore the curvature tensor vanishes.