Quantile Hedging of Defaultable Securities

by

Bohdan Horak

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Department of Mathematical and Statistical Sciences
University of Alberta

Abstract

This thesis investigates the hedging of equity-linked life insurance contracts with default time. In such a case the market is no longer complete and as such we consider imperfect hedging technique called Quantile Hedging. This hedging technique maximizes the probability of a successful hedge while allowing for a possibility of shortfall. This allows for a smaller amount of initial capital to be required for hedging.

First, we present a multi-dimensional market with default and then extending on previous results derive a general formula in the framework of a defaultable Black-Scholes model. We then formulate the hedging problem as a Neyman-Pearson problem with composite hypothesis against a simple alternative. We apply a convex duality approach to derive a solution to the quantile hedging problem of general derivative contract within a Black Scholes market with default.

We then introduce mortality of the client to the model, and using previously derived results provide closed form solutions to this problem in the case of one and two risky assets for an option to exchange one asset for another. We use these formulas to provide illustrative examples for both one and two risky asset cases and examine the relationships between shortfall probability, initial capital available for hedging, survival probability and default probability.

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Chapter 1

Introduction

A financial derivative is a contract whose value depends on the value of other financial assets. These financial assets can range from simple securities such as stocks or bonds to very complicated and esoteric ones such as weather futures and contracts on the perceived volatility of the market. The prototypical example of a derivative is the call option which gives the holder the right but not the obligation to purchase an asset at a pre-agreed upon price called the strike price. There are several variations of stock options with the most common type being the European option in which the holder can only exercise at a single point in time. This is contrasted with the American type in which the holder exercise the option at any point before the maturity date. These options and other derivatives are used for risk management as well as speculation purposes. Since the 1980's derivative such as these have become a cornerstone of the financial industry and such having ways of calculating the value of such options and ways of managing their risk is of paramount importance.

The major breakthrough in the field of option pricing came in 1973 with the Black-Scholes-Merton model with the first paper by Black and Schole's paper [3] as well as Merton's [14]. Their work showed that under several assumptions about the dynamics of the underlying assets it was possible to find a closed form solution to price of a call option. In addition, they and others showed that it was possible to replicate the price of the payoff of the call option. With this, it has became possible for an option seller to completely nullify the risk associated with this liability using a portfolio consisting of the asset and a some sort of risk free interest bearing instrument. This is first type of hedging technique.

Financial hedging and speculation are not the only applications of such instruments as they been becoming more and more popular in insurance. The combination of financial derivative instruments and actuarial contracts is called Equity-Linked life insurance and is quickly becoming a big area of research in Mathematical Finance. Unlike regular life insurance contracts, which provide a fixed payout that is contingent on the survival or death of the client, equity-linked life insurance contracts can have a payoff dependent on the performance of the financial markets. A large variety of such contracts have been developed, see [9] for an overview.

In this thesis we will examine the contract known as pure endowment with a guarantee. These contracts have a payoff that is contingent on the survival of the client to some maturity T. If the client survives he or she will receive a payoff that is based on an option to exchange on asset for another. We consider two cases: one the second asset is a fixed number K and two, the second is asset is another security. Thus, in addition to the financial component we introduce the uncertainty of mortality of the client.

To complicate matters further and to make the model more realistic we consider the default probability of the insurance company itself. Once this element has been introduced to the model we see the the market is no longer complete and a unique price for the option is no longer possible to find. If we employ a technique called Superhedging which attempts to hedge the worst case scenario, the initial cost of hedging become too great, see the paper by Karoui and Quenez [11]. Therefore we have to consider other types of hedging methodologies. We consider Quantile Hedging developed by Fôllmer and Leurkert [8]. Under this scheme, we accept some shortfall risk ϵ that the hedging portfolio will fall short of the terminal payoff of the option H.

We now come to the main goal of this thesis and that is: to develop a quantile hedging methodology for a financial market with three sources of uncertainty.

1. The dynamics of the risky assets.

2. The mortality of the client.

3. The default time of the insurer.

1.1 Summary of Thesis

This project contains 7 chapters including this introduction. The second chapter contains the foundational Mathematical Tools which will be used throughout the rest of the project.

Chapter 3 will extend the results of Nakano [15] and [16], together with results from [2] and [6], to extend his work on Quantile Hedging in the defaultable market to the multi-dimensional case. We will present a step-by-step derivation of the core result.

Chapter 4 will apply the original Nakano's results to the equity linked insurance setting which in addition to default time will account for the mortality of the client. We will use the standard "Brennnan and Scwartz" approach see [5], which uses the law of large numbers and some assumptions to manage this uncertainty. Formulas for initial capital cost of hedging will be presented. In addition, we will present methodologies which will allow practitioners to calculate the initial cost of capital as well as an acceptable shortfall risk along the minimum ages for which such a policy is suitable. Chapter 5 will present the very same results but in the case of two-risky assets.

Chapter 6 will present two numerical examples. Both assets will use real financial and actuarial data to estimate the parameters. One example will be using the results of Chapter 4 and one risky asset. The second example will present the case of two risky asset case. For both examples we present the relationships of the initial capital, shortfall probability, default probability and the mortality of the client.

Chapter 2

Mathematical Preliminaries

This chapter will go over some tools that will be used in the rest of the thesis. We will cover the basics of Stochastic Analysis such as the stochastic basis, martingales and stochastic integration. We will also present the Black-Scholes and Margrabe formulas. We will also present some results in the area of statistical testing such as Neyman Pearson Lemma and Composite Random tests.

2.1 Stochastic Basis

Stochastic analysis starts with a stochastic basis. This is a complete probability space

$$(\Omega, \mathcal{F}, P)$$

endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$, which is a non-decreasing continuous family of σ - algebras. We assume that all stochastic processes $X = (X_t(\omega))_{t\geq 0}$ that will be examined in this thesis are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ are $\mathcal{F}$ - adapted. That is, for each t, the random variable X_t is measurable with respect to $\mathcal{F}_t$.

Definition 2.1.1 *For a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, a stopping time is any non-negative random variable τ satisfying*

$$\{\tau \leq t\} \in \mathcal{F}_t$$

for any $t \geq 0$.

2.2 Martingales and Wiener Process

Definition 2.2.1 *A continuous process* W *is called Wiener process or Brownian motion, if the following conditions are satisfied*

1. $W_0 = 0$
2. $W_t - W_s$ *does not depend on* $\mathcal{F}_s$, $s \leq t$
3. $W_t - W_s$ *is normally distributed with zero mean and variance* $t - s$

An n-dimensional Wiener Process is an R^n valued process

$$W = (W^1, W^2, ..., W^d)'$$

with the the components W^i being independent one dimensional Wiener Processes as defined above.

Definition 2.2.2 *Consider stochastic process* $X = (X_t)_{t \geq 0}$ *and* $\mathbb{F}$ *adapted process satisfying* $E[X_t] < \infty$ *for all* $t \geq 0$.

1. X *is a supermartingale if*

$$E[X_t|\mathcal{F}_s] \leq X_s, \qquad 0 \leq s \leq t$$

2. X *is a submartingale if*

$$E[X_t|\mathcal{F}_s] \geq X_s, \qquad 0 \leq s \leq t$$

3. X *is a martingale if*

$$E[X_t|\mathcal{F}_s] = X_s, \qquad 0 \leq s \leq t$$

Theorem 2.2.1 *A one-dimenional Wiener Process* $W = \{W_t\}_{t \geq 0}$ *is a martingale.*

Definition 2.2.3 *A Wiener Process with drift* μ *and volatility* σ *is the process* $X = (X_t)_{t \geq 0}$ *and is given by*

$$X_t := \mu t + \sigma W_t, \quad t \geq 0$$

Corollary 2.2.1 *The Brownian motion with drift* μ *is a martingale if and only if* $\mu = 0$*, a submartingale if* $\mu \leq 0$*, and supermartingale if* $\mu \geq 0$*.*

2.3 Stochastic Integrals

Definition 2.3.1 *A martingale* $(M_t)_{t \geq 0}$ *defined on stochastic basis* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *is said to be square integrable if*

$$E(M_t^2) < +\infty, \quad \textit{for all } t \geq 0$$

Any stochastic process X is a function of two variables: the elementary event $\omega \in \Omega$ and time $t \leq T$.

Definition 2.3.2 *For a fixed elementary event* ω*, the function* $X(\omega, \cdot)$ *is called a trajectory.*

Definition 2.3.3 *We then divide the interval* $[0, T]$ *into* n *parts:* $0 = t_0 < t_1 < ... < t_n = T$*, then define the following*

$$\varphi(t, \omega) = \sum_{k=1}^{n} \varphi_{k-1}(\omega) \mathbf{1}_{(t_{k-1}, t_k]}(t)$$

where φ_{k-1} *are square-integrable random variables that are measurable with respect to* $\mathcal{F}_{t_{k-1}}$*. Then the stochastic integral of the random function with respect to the Wiener process* W *is defined as follows*

$$\int_0^t \varphi(s, \omega) dW_s := \sum_{k=1}^{n} \varphi_{k-1}(\omega)(W_{t_k \wedge t} - W_{t_{k-1} \wedge t})$$

Then we can consider stochastic processes like the following

$$X_t = X_0 = \int_0^t b(s,\omega)ds + \int_0^t a(s,\omega)dW_s, \tag{2.1}$$

where $\int_0^t b(s,\omega)ds$ is the standard Lebesque-type integral for each fixed ω and $\int_0^t a(s,\omega)dW_s$ is a stochastic integral. The equation 2.1 is often written in the form

$$dX_t = b_t dt + a_t dW_t$$

Let $F(t, X_t)$ be a real-valued function that is continuously differentiable in t and twice continuously differentiable in x. The then process $Y_t := F(t, X_t)$ can also be written of 2.1, which follows from the one dimensional Itô's formula

$$\begin{aligned} F(t,X_t) =& F(0,X_0) \\ &+ \int_0^t \left[\frac{\partial F}{\partial s}(s,X_s) + b_s\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}a_s^2\frac{\partial^2 F}{\partial s^2}(s,X_s)\right] ds \\ &+ \int_0^t a_s \frac{\partial F}{\partial x}(s,X_s)dW_s \end{aligned}$$

We can also consider the multi-dimensional case. Consider the n-dimensional process

$$X_i(t) = X_i(0) + \int_0^t K_i(s,\omega)ds + \sum_{j=1}^m \int_0^t H_{ij}(s,\omega)dW^j(s), \quad i = 1, 2, ...n$$

where $K_i(t), H_{ij}(t)$ are progressively measurable and square integrable random variables.

Then if F is a continuous function that differential with respect to the first variable and twice differentiable with respect to the last n variables. Then for $t \geq 0$, the following is true

$$
\begin{aligned}
F(t, X_{i,t}) = F(0, X_1(t), ..., X_n(t)) \\
&+ \int_0^t \frac{\partial F}{\partial s}(s, X_1(s), ..., X_n(s))ds + \sum_{i=1}^{n} \int_0^t \frac{\partial F}{\partial X_i}(s, X_1(s), ..., X_n(s))dX_i(s) \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^t \frac{\partial F}{\partial X_i}\frac{\partial F}{\partial X_j}(s, X_1(s), ..., X_n(s))d\langle X_i, X_j\rangle_s
\end{aligned}
$$

2.4 Black-Scholes Model and Formula

The Black-Scholes market consists of one risky asset modeled by process S_t and one risk-free asset denoted by B_t. Say B_t and S_t are processes defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and have the following form

$$
\begin{aligned}
B_t =& e^{rt} \\
S_t =& S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}, \quad S_0 > 0
\end{aligned}
\tag{2.2}
$$

Then using the Itô formula from above we can write rewrite them as

$$
\begin{aligned}
dB_t =& rB_t dt \\
dS_t =& S_t(\mu dt + \sigma dW_t), \quad S_0 > 0
\end{aligned}
\tag{2.3}
$$

We are looking to price a European Call Option with payoff $H = (S_T - K)^+$ where K is the strike price. The initial price, C_0 of such an is given by the Black-Scholes formula

$$
C_0 = S_0 \mathbf{\Phi}\left(\frac{log\frac{S_0}{K} + T(r + \sigma^2)}{\sigma\sqrt{T}}\right) - Ke^{-rt}\mathbf{\Phi}\left(\frac{log\frac{S_0}{K} + T(r - \sigma^2)}{\sigma\sqrt{T}}\right) \tag{2.4}
$$

where $y^{\pm} = \frac{ln(S_0/K) + T(r \pm \sigma^2)}{\sigma\sqrt{T}}$ and $\mathbf{\Phi}(\cdot)$ is the standard normal cumulative distribution function.

2.5 Two Risky Asset Model and Formula

The Two Risky Asset Market Model is the same as the Black Scholes market with the exception that there are two risky assets, S_t^1 and S_t^2 as well as the risk free account B_t. As in the one risky asset case each S_t^1 and S_t^2 have an expected return and a volatility parameter associated with them. In addition to this, we say that the underlying Brownian motions, W_t^1 and W_t^2 are related by saying that $\text{cov}(W_t^1, W_t^2) = \rho t$. Then if we assume these processes have the following dynamics

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t^i &= S_t^i(\mu_i dt + \sigma_i dW_t^i), \quad S_0 > 0, i = 1, 2 \end{aligned} \tag{2.5}$$

the the option with the payoff $H = (S_T^1 - S_T^2)^+$ will have the initial price, C_0 given by the Margrabe formula

$$C_0 = S_0^1 \mathbf{\Phi}\left(\frac{log\frac{S_0^1}{S_0^2} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - S_0^2 \mathbf{\Phi}\left(\frac{log\frac{S_0^1}{S_0^2} - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) \tag{2.6}$$

where $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$.

Note that we use different notation in the one and two dimensional markets compared to the d-dimensional market.

2.6 Neyman Pearson and Convex Duality

2.6.1 Simple Hypothesis

Say we are trying to distinguish between two probability measures Q, which corresponds to our null hypothesis and some alternative P. One way to do this is using a simple test, by considering the random variable $X : \Omega \to \{0, 1\}$ which rejects the null hypothesis Q when $\{X = 1\}$. Then, then the probability $Q(X = 1)$ is the probability of rejecting the null hypothesis when it is true, termed the Type I error. While the probability $P(X = 0)$ is the probability of accepting Q when it is false i.e. the Type II error. It is impossible to minimize both probabilities at once, so the standard procedure is to fix the Type I error to some low value, called the significance level and minimize the Type II error. This is the approach of the classical Neyman-Pearson approach, see the original paper by Neyman and Pearson [17].

Theorem 2.6.1 *Define third probability measure μ with*

$$P \ll \mu, \qquad Q \ll \mu,$$

and set

$$G := \frac{dP}{d\mu}, \qquad H := \frac{dQ}{d\mu}.$$

Then the optimal test has a solution

$$\hat{X} = \mathbf{1}_{\{\hat{z}H < G\}}$$

where

$$Q(\hat{z}H < G) = \alpha \text{ for some } 0 < \hat{z} < \infty$$

2.6.2 Composite Hypothesis

Say that on measurable space $(\Omega, \mathcal{F})$, we have an entire family $\mathbf{Q}$ of probability measures, i.e. a composite hypothesis. We are trying to discrimination against an alternative hypothesis family of measures $\mathbf{P}$.

We assume,

$$\mathbf{P} \cap \mathbf{Q} = \emptyset$$

$$P \ll \mu, \qquad Q \ll \mu, \qquad \forall P \in \mathbf{P}, \quad \forall Q \in \mathbf{Q}$$

for some probability measure μ, we set

$$G_P := \frac{dP}{d\mu} \quad (P \in \mathbf{P}), \qquad H_Q := \frac{dQ}{d\mu} \quad (Q \in \mathbf{Q}).$$

Then looking at the set of randomized tests $X : \Omega \to [0,1]$,

$$\mathscr{X}_\alpha := \{X : \Omega \to [0,1]; E^Q(X) \le \alpha, \forall Q \in \mathbf{Q}\}$$

Since the alternative is an entire family $\mathbf{P}$ the optimization now becomes

$$V(\alpha) := \sup_{X \in \mathscr{X}_\alpha} (\inf_{P \in \mathbf{P}} E^P(X)).$$

Using the paper of Cvitanic and Karatzas [6] we have the following result. Under appropriate conditions on the family P of alternatives the optimal test $\hat{X}$ has the following form

$$\hat{X} = \mathbf{1}_{\{\hat{Z}\hat{H} < \hat{G}\}} + B \cdot \mathbf{1}_{\{\hat{Z}\hat{H} = \hat{G}\}}$$

where B is a random variable in the interval $[0,1]$. The random variable $\hat{G}$ is of the form $G_P = dP/d\mu$ for some $P \in \mathbf{P}$. The random variable $\hat{H}$ is chosen from a suitable family that contains the convex hull

$$C_0(H; \mathbf{Q}) := \{\lambda H_{Q_1} + (1 - \lambda H_{Q_2}) : Q_1 \in \mathbf{Q}, Q_2 \in \mathbf{Q}, 0 \le \lambda \le 1\}$$

of $\{H_Q\}_{Q \in \mathbf{Q}}$; and where $\hat{Z}$ is a positive number.

Chapter 3

Multi-Dimensional Quantile Hedging with Default

This chapter will implement the results of Nakano see [15] and [16], in the multi-dimensional case. There are a lot of similarities as well as some key differences.

3.1 The Default Free Black-Scholes Model

We consider the framework of a complete multi-dimensional financial market. This market consists of one risk-less asset, usually called a bank account or a money market account and several risky assets usually referred to as stocks, but can also be bonds, equity indices or even commodities. The price processes of the bank account $B(\cdot)$ and risky assets $S_1(\cdot), ..., S_d(\cdot)$ have the following dynamics

$$dB(t) = B(t)r(t)dt, \qquad t \in [0,T], \qquad B(0) = 1 \tag{3.1}$$

$$dS_i(t) = S_i(t)\big[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij} dW^j(t)\big], \qquad t \in [0,T], \qquad S_i(0) > 0, i = 1, ..., d.$$

The Brownian motion $W(\cdot) = (W^1(\cdot), ..., W^d(\cdot))'$ in $\mathbb{R}^d$ is defined on complete probability space $(\Omega, \mathbb{F} = (\mathcal{F}_{\mathcal{T}})_{0 \le t \le T}, P)$. The market coefficients: interest rate $r(\cdot)$, vector of the risky asset returns $b(\cdot) = (b_1(\cdot), ..., b_d(\cdot))'$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}\}_{1 \le i,j \le d}$ are progressively measurable with respect to $\mathbb{F}$. We also assume that the volatility matrix $\sigma(\cdot)$ is invertible and all processes $r(\cdot), b(\cdot), \sigma(\cdot)$ and $\sigma^{-1}(\cdot)$ are all uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$.

We define the risk premium process

$$\theta(t) := \sigma^{-1}(t) \left[b(t) - r(t)\tilde{\mathbf{1}} \right], \qquad t \in [0, T] \tag{3.2}$$

where $\tilde{\mathbf{1}} = (1, ..., 1)' \in \mathbb{R}^d$. We then define the $\mathbf{P}$-martingale,

$$Z^*(t) := exp \left[- \int_0^t \theta'(s) dW(s) - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds \right], \qquad t \in [0, T] \tag{3.3}$$

and

$$P^*(A) := E[Z^*(T)\mathbf{1}_A], \qquad A \in \mathbb{F} \tag{3.4}$$

is a probability measure equivalent to P. We then introduce the discount process

$$\gamma(t) := \frac{1}{B(t)} = exp \left(- \int_0^t r(s) ds \right), \qquad t \in [0, T] \tag{3.5}$$

The discounted risky assets $\gamma(\cdot)S_1(\cdot), ..., \gamma(\cdot)S_d(\cdot)$ are martingales under the equivalent martingale measure P^* and the process

$$W^*(t) := W(t) + \int_0^t \theta(s) ds, \qquad t \in [0, T] \tag{3.6}$$

is a P^* Brownian motion by the Girsanov theorem.

3.2 Default Time

The default time is denoted by $\tau \ge 0$ is a random time with $P(\tau = 0) \ge 0$ for all t. We can consider the filtrations for τ as follows.

$$\mathcal{G}_t = \mathcal{F}_t \vee \hat{\mathcal{H}}_t$$

Where $\hat{\mathcal{H}}_t := \sigma(\tau \wedge s, 0 \leq s \leq t$ for $t \in [0, T]$. This (B, S, τ) market is defined on $(\Omega, \mathbb{G} = (\mathbb{G}_t)_{0 \leq t \leq T} \subseteq \mathcal{G}, P)$.

From [15], with more a more detailed construction provided by [2], define the survival process

$$G_t = P(\tau > t | \mathcal{F}_t), \quad 0 \leq t \leq T$$

of τ with respect to $\hat{\mathbb{F}}$. We assume the $G_t > 0$ for $t \geq 0$.

Now consider the hazard process $\{\Gamma_t\}_{t \geq 1}$ of τ with respect to $\hat{\mathbb{F}}$ defined by $G_t = e^{-\Gamma_t}$ or $\Gamma_T = -logG_t$ for every $t \geq 0$. In addition, we assume that $\Gamma_t = \int_0^t \mu_s ds, t \geq 0$ for some non-negative process $(\mu_t)_{0 \leq t \leq T}$ that is $\hat{\mathbb{F}}$ predictable. Then $(\mu_t)_{0 \leq t \leq T}$ is called $\hat{\mathbb{F}}$ intensity of of the random time τ.

From [**?**] the process

$$M_t := N_t - \int_0^t \mu_s (1 - N_{s-}) ds = N_t - \int_0^{t \wedge \tau} \mu_s ds, \quad t \geq 0$$

follows a $\mathbb{G}$ martingale.

We make the assumption that $W_1(\cdot), ..., W_d(\cdot)$ are $(\mathbb{G}, \hat{P})$ martingales. This assumption is satisfied if τ is independent of $W_1(\cdot), ..., W_d(\cdot)$.

As we are dealing with option pricing it is natural to consider risk neutral probabilities and therefore risk neutral densities that we considered before in the form of the stochastic exponent Z_t^* we have to extend this to work with the model with default. In particular we consider the process

$$Z_t^k = (1 + k_\tau 1_{\tau \leq t}) exp \left(- \int_0^{\tau \wedge t} k_s \mu_s ds \right)$$

where $\{k_t\}_{0 \leq t \leq T}$ is taken from the class

$$\mathcal{D} = \{\{k_t\}_{0 \leq t \leq T} : bounded, \ \ \mathbb{G}predictable, \ \ k_t > -1 dt \times dP - a.e.\}$$

Then $\{Z_t^k\}$, $k \in \mathcal{D}$, satisfies

$$Z_t^k = 1 + \int_0^t k_s Z_{s-}^k dM_s, \;\; 0 \leq t \leq T$$

and follows a $(\mathbb{G}, \hat{P})$ martingale. Since the quadratic variation of $\left[Z^*, Z^k\right]$ is identically zero. We can use Itô's formula on both Z_t^* and Z_t^k and use product rule to find

Theorem 3.2.1 *$\{Z^*(t)Z^k(t)\}$ is a positive martingale.*

Proof. $Z^*(t)$ and $Z^k(t)$ are positive therefore their product is as well. In order to prove that $\{Z^*(t)Z^k(t)\}$ is a martingale we use the standard Itô Product Rule of the following form to calculate the derivative $dZ^*(t)Z^k(t)$

$$dZ^*(t)Z^k(t) = Z^*(t-)dZ^k(t) + Z^k(t-)dZ^*(t) + d[Z^*(t), Z^k(t)]$$

Now since we assume that the default time τ is independent of $W_1(\cdot), ..., W_d(\cdot)$ the quadratic covariation $d[Z^*(t), Z^k(t)]$ is equal to zero. With this all, that is left is to calculate the derivatives $dZ^*(t)$ and $dZ^k(t)$. The case of $dZ^k(t)$ is rather obvious it is simply

$$dZ^k(t) = k(t)Z^k(t-)dM(t), \qquad t \in [0, T]$$

In the case of $dZ^*(t)$ it is a slightly more involved application of Itô's Formula.

We rewrite the $Z^*(t)$ without vector notation

$$Z^*(t) = exp\left[-\sum_{i=1}^{d}\int_0^t \theta_i(s)dW_i(s) - \frac{1}{2}\left(\int_0^t \sum_{i=1}^{d}\theta_i^2(s)ds\right)\right]$$

First we find the partial derivatives,

$$\frac{\partial Z^*(t)}{\partial t} = -\frac{1}{2}Z^*(t)\sum_{i=1}^{d}\theta_i^2(t)$$
$$\frac{\partial Z^*(t)}{\partial W_i(t)} = -Z_t^*\theta_i(t)$$
$$\frac{\partial^2 Z^*(t)}{\partial W_i(t)W_j(t)} = Z^*(t)\theta_i^2 dW_i(t)dW_i(t) = Z^*(t)\sum_{i=1}^{d}\theta_i^2.$$

Then we can find its Ito representation using the Ito formula as

$$\begin{aligned} dZ^*(t) &= \frac{\partial Z^*(t)}{\partial t} + \sum_{i=1}^{d}\frac{\partial Z^*(t)}{\partial W_i(t)} + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 Z^*(t)}{\partial W_i(t)W_j(t)} \\ &= -\frac{1}{2}Z^*(t)\sum_{i=1}^{d}\theta_i^2(t)dt - \sum_{i=1}^{d}Z_t^*\theta_i(t)dW_i(t) + \frac{1}{2}Z^*(t)\sum_{i=1}^{d}\theta_i^2(t)dt \\ &= -\sum_{i=1}^{d}Z_t^*\theta_i(t)dW_i(t) \end{aligned}$$

Then noting that $Z^*(t)$ is continuous therefore $Z^*(t-) = Z^*(t)$, and using the Ito product formula above we have

$$\begin{aligned} dZ^*(t)Z^k(t) &= Z^*(t-)dZ^k(t) + Z^k(t-)dZ^*(t) + d[Z^*(t), Z^k(t)] \\ &= Z^*(t)k(t)Z^k(t-)dM(t) + Z^k(t-1)(-\sum_{i=1}^{d}Z_t^*\theta_i(t)dW_i(t)) \\ &= Z^*(t)Z^k(t-)\left(k(t)dM(t) - \sum_{i=1}^{d}\theta_i(t)dW_i(t)\right), \end{aligned}$$

As the above derivative has no drift term $\{Z^*(t)Z^k(t)\}$ is a positive martingale for $k \in \mathcal{D}$. □

From the [2] and [18] we can show that $\{Z_t^k\}$ is orthogonal to $(\hat{\mathbb{F}}, \hat{P})$ martingales, therefore we can show that $\{W_1(\cdot), ..., W_d(\cdot)\}$ are also standard Brownian

Motions under $\mathbb{Q}^k$ defined by density $d\mathbb{Q}^k/dP = Z_T^* Z_T^k$. Thus, $\{\mathbb{Q}^k : k \in \mathcal{D}\}$ defines the class of equivalent martingale measures.

3.3 Hedging Portfolio

The σ-algebra $\mathbb{G}$ is considered to be the available information to the market participants. The portfolio process $\{\pi_t\}_{0\le t\le T}$ is a $\mathbb{G}$ predictable process satisfying

$$\int_0^T |\pi_t|^2 dt < \infty, \quad a.s.$$

The self-financing wealth process $\{X_t^{x,\pi}\}_{0\le t\le T}$ for an initial wealth $x \ge 0$ and a portfolio process $\{\pi_t\}_{0\le t\le T}$ is described by

$$dX(t) = \left[X(t) - \sum_{i=1}^d \pi_i(t)\right] r(t)dt + \sum_{i=1}^d \pi_i(t)\left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW^j(t)\right], \quad X(0) = x.$$

The solution to the above equation is given by

$$\begin{aligned} X^{x,\pi}(t) = B(u)\Bigg[& x + \int_0^t B(u)^{-1} \sum_{i=1}^d \pi_i(u)(b_i(u) - r(u))du \\ & + \int_0^t B(u)^{-1} \sum_{i=1}^d \pi_i(u) \sum_{j=1}^d \sigma_{ij}(u)dW^j(u)\Bigg], \\ & t \in [0,T]. \end{aligned}$$

We denote $\mathcal{A}(x)$ as the set of all portfolio processes $\{\pi_t\}_{0\le t\le T}$ such that $X^{x,\pi}(t) \ge 0, 0 \le t \le T$ almost surely.

Next, we define

$$L^k(t) := B^{-1}(t)Z^*(t)Z^k(t), \quad k \in \mathcal{D} \tag{3.7}$$

and Itô's product rule we get for $\pi \in \mathcal{A}(x)$,

$$dL^k(t)X^{x,\pi}(t) = L^k_{t-}\left[\left[\sum_i^d \pi_i(t)\sum_j^d \sigma_{ij}(t)dW^j(t) - X^{x,\pi}_t\sum_j^d \theta_j(t)dW^j(t)\right] + X^{x,\pi}(t)k(t)dM(t)\right]$$

By assumption we assume the wealth process is non negative which combined with the above implies that $L^k(t)X^{x,\pi}(t)$ is a supermartingale for each $\pi \in \mathcal{A}(x)$. Denote $\mathcal{L}$ the set of all random variables $L^k_t, k \in \mathcal{D}$.

We consider the hedging problem for the defaultable claim G defined by

$$G = H\mathbf{1}_{\{\tau>T\}} + \delta H\mathbf{1}_{\{\tau<T\}} \tag{3.8}$$

Where $\delta \in [0,1]$ is the recovery rate. This is the percentage of the payoff that the client receives when the insurer defaults. If $\delta = 1$ the client receives the entire payoff he or she was due, this case makes the default irrelevant as far as the client is concerned. If $\delta = 0$ upon default the client receives nothing. While the next few intermediate results have a non-zero δ the main results in this paper are shown for $\delta = 0$.

The costs of superhedging this contract, denoted by $\Pi(G)$ is defined by

$$\Pi(G) = inf\{x \geq 0 : X^{x,\pi}_t \geq H \text{ a.s. for some } \pi \in \mathcal{A}(x)\}.$$

Proposition 3.3.1 *Let G be such that $E^*[H] < \infty$. Then we have*

$$\Pi(G) = E^*[B_T^{-1}H]$$

Also, the replicating portfolio for H becomes a superhedging portfolio for G.

Proof. Set $\tilde{x} = E^*[B_T^{-1}H]$ and let $\tilde{\pi}$ be the replicating portfolio for H. Thus, $\tilde{\pi} \in \mathcal{A}(x)$ and $X^{\tilde{x},\tilde{\pi}}_t = H \geq G$, then $\tilde{x} \geq \Pi(G)$.

Now assume that $X^{x,\pi}_t \geq H$ for some $\pi \in \mathcal{A}$. Then as $\{L^k_t X^{x,\pi}_t\}$ is a supermartingale,

$$E[L_T^k G] \leq E[L_T^k X_T^{x,\pi}] \leq x, \quad k \in \mathcal{D}.$$

Then from $G = \delta H + (1-\delta)H\mathbf{1}_{\{\tau>T\}}$ the above can be rewritten as

$$E[L_T^k G] = E[B_T^{-1} Z_T^* Z_T^k \delta H] + E[L_T^k (1-\delta)H\mathbf{1}_{\{\tau>T\}}].$$

As the quadratic co-variation of $\{Z_t^k\}$ and an $\mathbb{F}$ martingale is equal to zero, the process

$$Z_t^k E[B_T^{-1} Z_T^* \delta Y]$$

is a local martingale. Then if H is bounded then this process is a martingale. We can then approximate G with $H \wedge n$, with the monotone convergence theorem the first term on the right hand side becomes $E[B_T^{-1} Z_T^* \delta H]$. With this we have

$$E[B_T^{-1} Z_T^* \delta H] + \sup_{k \in \mathcal{D}} E[L_T^k (1-\delta) H_{\{\tau>T\}}] \leq \Pi(G)$$

Then considering constant $k > -1$ and the definition of the survival probability

$$\begin{aligned} E[L_T^k H \mathbf{1}_{\{\tau>T\}}] &= E[B_T^{-1} Z_T^* (1 + k\mathbf{1}_{\{\tau \leq T\}}) e^{-k\int_0^{t\wedge T} \mu_t dt} \mathbf{1}_{\{\tau>T\}}] \\ &= E[L_T^k Z_T^* H \mathbf{1}_{\{\tau>T\}} e^{-k\int_0^{t\wedge T} \mu_t dt}] \\ &= E[L_T^k Z_T^* H e^{-(k+1)\int_0^T \mu_t dt}] \end{aligned}$$

Then

$$E[L_T^k H \mathbf{1}_{\tau>T}] \to E[B_T^{-1} Z_T^* H] \text{ as } k \searrow -1.$$

This concludes the proof.

□

3.4 Maximizing the Probability of the Super-Hedge

The above proposition implies that the insurer of the claim wants to hedge the contract then he or she needs to perfectly hedge contract as if it were non-defaultable. However, the price of such a contract should take into account the possibility of default and the resulting reduced payoff. This is because the buyer of such a contract should not be willing to purchase this contract when he or she could go into the default-free market and purchase this contract for the same price. Therefore, we need to construct a hedging portfolio whose initial price is less than the initial cost of the perfect hedging portfolio. Thus, we consider imperfect hedging. In this paper we consider a particular class of imperfect hedging called quantile hedging. In this type of hedging we accept that there will some probability that we will not successfully hedge the underlying contract. Our objective is to solve the following problem.

$$\max_{\pi \in \mathcal{A}(x)} P(X_T^{x,\pi} \geq G) \tag{3.9}$$

Given that,

$$x \leq E^*[B_T^{-1} G]$$

where x is the initial cost of the superhedging portfolio.

To solve this problem we consider the Neyman-Pearson type problem defined by

$$\max_{\phi \in \mathcal{R}} E[\phi]$$

where

$$\mathcal{R} = \{\phi : 0 \leq \phi \leq 1 \text{ a.s.}, \sup_{L \in \mathcal{L}} E[LG\phi] \leq x\}.$$

With the following proposition the problem above can be reduced to the Neyman-Pearson type problem with the following proposition.

Proposition 3.4.1 *Suppose there exists $A \in \mathcal{G}_t$ (recall that $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$) and $\{\hat{\pi}_t \in \mathcal{A}\}$ such that $\mathbf{1}_A$ solves the Neyman-Pearson type problem above and $X_T^{X,\hat{\pi}} \geq G\mathbf{1}_A$ a.s. Then such a $\hat{\pi}$ is optimal for the quantile hedging problem above.*

Proof. For $\pi \in \mathcal{A}(x)$,

$$E\left[L_T^k \mathbf{1}_{\{X_T^{u,\pi}\geq G\}}\right] \leq E\left[L_T^k \mathbf{1}_{\{X_T^{u,\pi}\geq G\}} X_T^{u,\pi}\right] \\ \leq \left[L_T^k X_T^{u,\pi}\right] \leq x, \quad k \in \mathcal{D}$$

Let $L \in \mathcal{L}$. Then there exists a sequence, $L_n \in \mathcal{L}$ such that $L = \lim_{n\to\infty} L_n$. Thereby Fatou's lemma $\mathbf{1}_{X_T^{x,\pi}} \in \mathcal{R}$ Therefore,

$$\max_{\pi\in\mathcal{A}(x)} P(X_T^{x,\pi} \geq G) \leq \max_{\phi\in\mathcal{R}} E[\phi]. \tag{3.10}$$

On the other hand $\hat{X} = X_T^{x,\hat{\pi}}$, we have

$$P[\hat{X} \geq G] \geq P[\hat{X} \geq G, A] = P[\hat{X} \geq G\mathbf{1}_A, A] = P[A] = \max_{\phi\in\mathcal{R}} E[\phi]$$

The above and 3.10 we have the result. □

We now consider the reduced problem in the default free case or where the recovery rate $\delta = 1$. In which case

$$G = H$$

In this case for every $L \in \mathcal{L}$ and $\phi \in \mathcal{R}$ we have

$$E[LH\phi] = E^*[B_T^{-1} H E[\phi|\mathcal{F}_T]].$$

Therefore the partial hedging problem is equivalent to the following

$$\max_{\phi\in\mathcal{R}_0} E[\phi]$$

where

$$\mathcal{R}_0 = \{\phi : 0 \le t \le T \text{ a.s.}, \mathcal{F}_T - \text{measurable}, E^*[B_T^{-1} H\phi] \le x\}$$

Define probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G})$ by

$$\frac{d\mathbb{Q}^*}{dP^*} = \frac{H}{B_T E^*[B_T^{-1} H]}$$

By the Neyman-Pearson Lemma the $[0,1]$ valued $\mathcal{F}_T$ - measurable random variable

$$\phi = \mathbf{1}_{\left\{\frac{dP}{dP^*} > z_0 \frac{d\mathbb{Q}^*}{dP^*}\right\}} + k\mathbf{1}_{\left\{\frac{dP}{dP^*} = z_0 \frac{d\mathbb{Q}^*}{dP^*}\right\}}$$

and

$$k = \begin{cases} 0, & \text{if } \mathbb{Q}^*\left(\frac{dP}{dP^*} = z_0 \frac{d\mathbb{Q}^*}{dP^*}\right) = 0, \\ \frac{\alpha - \frac{dP}{dP^*} > z_0 \frac{d\mathbb{Q}^*}{dP^*}}{\mathbb{Q}^*\left(\frac{dP}{dP^*} = z_0 \frac{d\mathbb{Q}^*}{dP^*}\right)}, & \text{if } \mathbb{Q}^*\left(\frac{dP}{dP^*} = z_0 \frac{d\mathbb{Q}^*}{dP^*}\right) > 0. \end{cases}$$

Define

$$y_0 = \frac{z_0}{E^*[B_T^{-1} Y],} \quad \xi_0 = y_0 B_T^{-1} Z_T^* Y.$$

With proposition 3.4.1 we obtain the following the following theorem.

Theorem 3.4.1 *Let G be a random variable satisfying the condition*

$$G = H$$

and $0 < E^*[B_T^{-1} H] < \infty$. Assume that $P(\xi_0 = 1) = 0$. Then the perfect hedging portfolio for $H\mathbf{1}_{\{\xi_0 < 1\}}$ solves the quantile hedging problem.

We have derived the solution for the quantile hedging problem in the default free case. Next consider the defaultable case with zero recovery rate. That is, in the case of default the client will receive a payout of 0. In this case G is of the form

$$G = H\mathbf{1}_{\{\tau > T\}}.$$

We need to adapt the convex duality approach first introduced by Cvitanic and Karatzas in [6]. Note that $\phi \in \mathcal{R}, y \geq 0, L \in \mathcal{L}$,

$$E[\phi] = E[\phi(1 - LG)] + E[LG\phi] \leq E[(1 - yLH)_+] + yx \tag{3.11}$$

The following dual problem arises:

$$\inf_{y \geq 0, l \in \mathcal{L}} \{E[(1 - yLG)_+] + yx\}$$

In order for this to work we have to consider $\bar{\mathcal{L}}$ instead of $\mathcal{L}$ defined by

$$\begin{aligned}\bar{\mathcal{L}} =& \{L \in L^1 : E[B_T L] \leq 1, E[LG] \\ &\leq \sup_{L^1 \in \mathcal{L}} E[L^{'}G], E[LH\phi] \leq x \quad (\phi \in \mathcal{R})\}\end{aligned}$$

Where $L^1 = L^1(\Omega, \mathcal{G}, P)$. Then the set $\bar{\mathcal{L}}$ includes $\mathcal{L}$ and is convex and closed under almost sure convergence. Therefore the existence of a solution $(\hat{\mathcal{L}}, \hat{y})$ to the dual problem

$$\inf_{y \geq 0, L \in \mathcal{L}} \{E[(1 - yLG)_+] + yx\}$$

in the class $\bar{\mathcal{L}} \times \mathbb{R}_+$ is guaranteed. There is also no duality gap for some $C : \Omega \to [0, 1]$ the random variable

$$\hat{\phi} := \mathbf{1}_{\{\hat{y}\hat{L}G < 1\}} + C\mathbf{1}_{\{\hat{y}\hat{L}G = 1\}}$$

is a solution to the randomized version of the Neyman-Pearson problem. Also, if $P(\hat{y}\hat{L}G = 1) = 0$ then by a previous proposition is a super-hedging portfolio of for $G\mathbf{1}_{\hat{y}\hat{L}G<1}$ becomes a solution to the quantile hedging problem.Since the abstract class $\bar{\mathcal{L}}$ is difficult to work with. We need to choose a more convenient class to work with. Then we need to find $\bar{\mathcal{L}}_1 \supset \mathcal{L}$ such that there is an explicit solution to the dual problem $(\hat{L}, \hat{y}) \in \bar{\mathcal{L}}_1 \times \mathbb{R}_+$ To do this, introduce the class $\bar{\mathcal{L}}$ defined by the closed hull of $\mathcal{L}$ with respect to the L^1 norm. Since $\mathcal{L}$ is convex, then so is $\bar{\mathcal{L}}_1$. Therefore, $\bar{\mathcal{L}}_1$ is a closed set in L^1. Then the Neyman-Pearson problem before is turns into

$$\max_{\phi \in \mathcal{R}_1} E[\phi]$$

where

$$\mathcal{R}_1 = \{\phi : 0 \leq \phi \leq 1 \text{ a.s, } sup_{L \in \bar{\mathcal{L}}_1} E[LG\phi] \leq x\}$$

Similar to the previous proposition we have the following result.

Proposition 3.4.2 *Suppose that there exists $A \in \mathcal{G}_T$ and $\hat{\pi} \in \mathcal{A}(x)$ such that $\mathbf{1}_A$ solves the Neyman-Pearson problem and $X_T^{x,\hat{\pi}} \geq G1_A$ a.s. Then $\hat{\pi}$ is optimal for the quantile hedging problem.*

Then the dual problem is modified to be

$$\inf_{y \geq 0, L \in \bar{\mathcal{L}}_1} \{E[(1 - yLG)_+] + yx\}.$$

Define

$$\hat{L} = B_T^{-1} Z_T^* \mathbf{1}_{\{\tau > T\}} e^{\int_0^T \mu_t dt}$$

Then we can state the following.

Theorem 3.4.2 *Suppose that $G = H\mathbf{1}_{\{\tau > T\}}$ with $E^*[B_T^- 1H] < \infty$. Then $\hat{L}$ defined above solves*

$$\inf_{L \in \bar{\mathcal{L}}_1} E[(1 - yLG)_+].$$

In addition, there exists $\hat{y} > 0$ that minimizes

$$h(y) := E[(1 - y\hat{L}G)_+] + yx$$

over $y \geq 0$. The pair $(\hat{y}, \hat{L})$ is optimal for the minimization problem above.

Proof. Then since $k_t > -1$ and $\hat{L} = \lim_{k \searrow -1} L_T^k$ a.s and in L^1. For $k \in \mathcal{D}$

$$\begin{aligned} E[1 \wedge (yL_T^k H)] =& E[1 \wedge (yZ_T^* Z_T^k H B_T^{-1}) \mathbf{1}_{\tau > T}] \\ =& E[1 \wedge (yZ_T^* e^{-\int_0^T k_s \mu_s ds} H B_T^{-1}) \mathbf{1}_{\tau > T}] \\ \leq& E[1 \wedge (yZ_T^* H B_T^{-1}) \mathbf{1}_{\tau > T}] \\ =& E[1 \wedge (y\hat{L}G)]. \end{aligned}$$

Therefore by Fatou's Lemma and the fact that $(1-z)_+ = 1 - 1 \wedge z$ we find that

$$E[1 - yLG]_+ = 1 - E[1 \wedge yLG]$$

Next, we need to prove the existence of y. We claim that there exists $y_0 > 0$ such that $h(y_0) < 1$. We assume that it does not exist and we find that $E[1 \wedge (y\hat{L}G) \leq yx$ for every $y > 0$. We then divide by y and then let $y \searrow 0$ we obtain that $E[\hat{L}G] \leq x$. This however, contradicts the assumption $x < E^*[B_T^{-1} H]$ as

$$E[\hat{L}G] = E[\hat{y}_1 B_t^{-1} Z_T^* e^{\int_0^T \mu_t dt} H P(\tau > T|\mathcal{F}_t)] = E^*[B_T^{-1} H].$$

The existence of the minimizer $\hat{y} > 0$ now follows from the convexity of h and the facts that $h(0) = 1$ and $h(+\infty) = +\infty$.

The optimality of the pair $(\hat{y}, \hat{L})$ is clear and so the proof omitted. □

Let $\hat{y} > 0$ be the as in the previous theorem and consider the $\mathcal{F}_T$ measurable random variable ξ_1 defined by

$$\xi_1 = \hat{y}_1 B_t^{-1} Z_T^* e^{\int_0^T \mu_t dt} H$$

Then we introduce the following theorem that summarizes the results for the quantile hedging problem.

Theorem 3.4.3 *Suppose that $G = H\mathbf{1}_{\{\tau>T\}}$ with $E^*[B_T^- 1H] < \infty$. and that $P(\xi_1 = 1) = 0$. Then the perfect hedging portfolio for $H\mathbf{1}_{\xi_1<1}$ is optimal for the quantile hedging problem*

Proof. Here we follow the derivations from [6] and [15].

The set

$$\mathcal{M} := \{(L, y) \in L^1 \times R : L \in \bar{\mathcal{L}}, y \geq 0\}$$

is closed and convex in the Banach space $L^1 \times R$ with the norm $\|(L, y)\| := E[L] + y$. In addition the functional

$$L^1 \times \mathbb{R} \ni (L, y) \mapsto U(L, y) := yx + E(1 - L)_+$$

is proper, convex and lower semi-continuous on $L^1 \times \mathbb{R}$ and

$$\inf_{(L,y)\in\mathcal{M}} U(yLH, y) = U(\hat{y}\hat{L}H, \hat{y}).$$

Denote $L^\infty(\Omega, \mathcal{G}, P)$ by L^∞. Let us consider the set $\mathcal{M}^* = \{(yLH, y) : (L, y) \in \mathcal{M}\}$, the normal cone

$$\mathcal{N}(\hat{y}\hat{L}G, \hat{y}) := \Big\{(\phi, u) \in L^\infty \times R : E[\hat{y}\hat{L}G\phi] + \hat{y}u \geq E[yLH\phi] + yu, (L, y) \in \mathcal{M}\Big\}$$

to the set $\mathcal{M}$ as $(\hat{y}\hat{L}G)$ and the sub-differential

$$\begin{aligned}\partial U(\hat{y}\hat{L}G, \hat{y}) := \big\{&(\phi, u) \in L^\infty \times R : U(\hat{y}\hat{L}G, \hat{y}) - U(L, y)\\ &\leq \hat{y}E[\phi(\hat{y}\hat{L} - L)] + u(\hat{y} - y), (L, y) \in L^1 \times R\big\}\end{aligned}$$

at this point. Then, from Corollary 4.6.3. of Aubing and Eckeland [1], we have

$$(0, 0) \in \partial U(\hat{y}\hat{L}G, \hat{y}) + \mathcal{N}(\hat{y}\hat{L}G, \hat{y}).$$

This implies that there exists $(\hat{\phi}, \hat{u}) \in L^\infty \times R$ such that $(\hat{\phi}, \hat{u}) \in \mathcal{N}(\hat{y}\hat{L}G, \hat{y})$

and $(\hat{\phi}, -\hat{u}) \in \partial U(\hat{y}\hat{L}G, \hat{y})$. Hence

$$E[G\hat{\phi}(\hat{y}\hat{L} - yL)] + (\hat{y} - y)\hat{u} \geq 0, \quad (L, y) \in \mathcal{M} \tag{3.12}$$

$$(x+\hat{u})(\hat{y}-y) \leq E[\hat{\phi}(L-\hat{y}\hat{L}G)] + E(1-L)_+ - E(1-\hat{y}\hat{L}G)_+, \quad (L, y) \in L^1 \times R. \tag{3.13}$$

By letting $y \to \pm\infty$, we see that 3.13 hold only if $\hat{u} = -x$. From 3.12 $\hat{u} = -x, y = \hat{y} \pm \delta(\delta > 0)$, and $L = \hat{L}$, we have

$$E[G\hat{\phi}\hat{L}] = x \tag{3.14}$$

Therefore, 3.12 with $y = \hat{y}$, we get

$$E[H\hat{\phi}L] \leq x, \quad L \in \bar{\mathcal{L}}. \tag{3.15}$$

Then 3.13 becomes

$$E[\hat{\phi}(L - \hat{y}\hat{L}G)] + E(1 - L)_+ - E(1 - \hat{y}\hat{L}G)_+ \geq 0, \quad L \in L^1 \tag{3.16}$$

Then 3.16 for $L = \hat{y}\hat{L}G + 1_A$ for some $A \in \mathcal{G}$, we see that $0 \leq E[\hat{\phi}1_A]$. Therefore, $\hat{\phi} \geq 0$ a.s. In the same note, looking at 3.16 for $L = \hat{y}\hat{L}G - 1_A$ for some $A \in \mathcal{G}$ and using $(x + y)_+ \leq (x)_+(y)_+$ for $x, y \in R$, we see that $0 \leq E(1 - \hat{\phi})1_A$. Thus $\hat{\phi} \leq 1$ a.s. This combined with 3.15 we have that $\hat{\phi} \in \mathcal{R}$

Then 3.16 implies that $E[\hat{\phi}(1 - \hat{y}\hat{L}G)] \geq E(1 - \hat{y}\hat{L}G)_+$. Then ,

$$\hat{\phi}(1 - \hat{y}\hat{L}H) = (1 - \hat{y}\hat{L}G) \quad \text{ a.s.}$$

From this and $\hat{\phi} \in \mathcal{R}$ we find that $\hat{\phi} = 1$ on $\{\hat{y}\hat{L}G < 1\}$ and $\hat{\phi} = 0$ on $\{\hat{y}\hat{L}G > 1\}$. Therefore there must some $[0, 1]$ - valued random variable C such that the representation

$$\hat{\phi} = \mathbf{1}_{\{\hat{y}\hat{L}G<1\}} + C\mathbf{1}_{\{\hat{y}\hat{L}G<1\}}$$

holds. In addition, we have that $E[\hat{\phi}] = E[(1 - \hat{y}\hat{L}G)_+] + \hat{y}x$. This and

3.11 imply that $\hat{\phi}$ is optimal for the Neyman-Pearson type problem and that there is no duality gap due to the assumption of the theorem. Moreover, since $G\hat{\phi} = Y\mathbf{1}_{\{\xi<1\}}\mathbf{1}_{\{\tau>T\}}$, we can apply Proposition 3.4.1 we see that that a superhedging portfolio for $G\hat{\phi}$ is given by a perfect hedging portfolio for $H\mathbf{1}_{\{\xi<1\}}$. With this and 3.3.1 we see that the theorem has been proven. □

Chapter 4

Case of One Risky Asset

In this chapter we use the results previously derived results in the case of a single risky asset. We study a defaultable insurance market model with three sources of uncertainty.

1. Standard market dynamics of the Black-Scholes market.
2. Default time of the insurer.
3. Mortality of the client.

4.1 Model

4.1.1 Default Free Black Scholes Model

Consider a financial market consisting of two assets B and S, defined by their prices $(B_t)_{0\leq t\leq T}$ and $(S_t)_{0\leq t\leq T}$. Asset B is typically referred to a Bank account or some other risk free investment such as a money market portfolio which is assumed to have a constant interest rate r and S is a risky asset. Here we call it the (B, S)-market and assume its price evolution is as follows:

$$dS_t = S_t(mdt + \sigma dW_t), \quad t \in [0, T], \quad S_0 \in (0, \infty) \tag{4.1}$$

$$dB_t = rB_tdt, \quad t \in [0, T], \quad B_0 = 1 \tag{4.2}$$

Where $(W_t)_{0 \leq t \leq T}$ is a standard Brownian motion on the complete probability space $(\Omega, \mathbb{F} = (\mathcal{F})_{0 \leq t \leq T} \subseteq \mathcal{G}, P)$ and m and σ are constant. We also assume that $\sigma > 0$. In the interest of simplicity we assume that the interest rate $r = 0$.

4.1.2 Default Time

The default tome construction is the same as in the previous chapter. The default time is denoted by τ, is a positive $\mathcal{G}$ random time with $P(\tau = 0) = 0$ and $P(\tau > 0) > 0$ for all $t \in [0, T]$. We again use progressive enlargements of the filtrations and define

$$\mathcal{G}_t = \mathcal{F}_t \vee H_t$$

where $\mathcal{H}_t := \sigma(\tau \wedge s, 0 \leq s \leq t)$ for $t \in [0, T]$. We call this market the (B, S, τ)-market. Therefore the defaultable $(\Omega, \mathbb{G} = (\mathcal{G})_{0 \leq t \leq T} \subseteq \mathcal{G}, P)$

As before we make the assumption that the default time τ is independent of $(W_t)_{0 \leq t \leq T}$. Levy's theorem together with this assumption imply that $(W_t)_{0 \leq t \leq T}$ is a $(\mathbb{G}, P)$ standard Brownian motion. This along with the martingale representation theorem imply that every $\mathbb{F}$- martingale is a $\mathbb{G}$ martingale. Therefore, the no arbitrage condition is satisfied for the defaultable (B, S, τ) market as on the (B, S) market.

As $(W_t)_{0 \leq t \leq T}$ is a (G, P) standard Brownian motion, by Girsanov's theorem

$$W_t := W_t + \frac{m}{\sigma} t, \quad t \in [0, T]$$

is a $(\mathbb{G}, P)$- standard Brownian motion, where

$$\frac{dP^*}{dP} := Z_T^*$$

and

$$Z_t^* = exp(-\frac{m}{\sigma} W_t - \frac{1}{2}(\frac{m}{\sigma})^2 t), \quad t \in [0, T] \tag{4.3}$$

By [15] this defaultable market is incomplete on $(\Omega, \mathbb{G}, P)$ with the following class of of equivalent martingale measures:

$$\mathcal{Q} := \{Q^k : k \in \mathcal{D}\}$$

where

$$\frac{dQ^k}{dP} := Z_T^* Z_T^k, \quad k \in \mathcal{D}$$

with

$$Z_t^k = 1 + \int_0^t k_s Z_{s-}^k dMs, \quad t \in [0, T]$$

and

$$\mathcal{D} = \{(k_t)_{0 \leq t \leq T} : \text{bounded}, \mathbb{G} - \text{predictable }, k_t > -1 dt \times dP \text{ a.e.}\}$$

As in the previous chapter we can also express Z_t^k in the following form

$$Z_t^k = (1 + k_\tau \mathbf{1}_{\{\tau \leq t\}}) exp(-\int_0^{\tau \wedge t} k_s \mu_s ds)$$

4.1.3 Mortality

The final source of uncertainty in our model is death risk of the client. We assume that is the client passes away before option maturity T none of his or her relatives will be able to exercise the option on their behalf. Therefore, the more likely the client to survive to time T the higher should be the premium charged. As in actuarial science, we denote the death time of the client as $T(x)$. This is a positive random variable defined on probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{P})$. Then the following

$$_tp_x = P(T(X) > T) \tag{4.4}$$

is called the survival probability of the client. This is the probability of a client whose age is x to survive to time T. These types of quantities are estimated by actuarial scientists and are listed in "Life Tables". Therefore when trying to account for type of uncertainty we refer to actuarial science literature. However, we do make the following assumptions.

Assumption 1: We assume that the death time of client $T(x)$ is independent of the market. That is the stochastic process S_1 is independent of the random variable $T(x)$. This is quite a natural assumption as the underlying developments in the market typically have little to do with the physical well being of market participants.

Assumption 2: Similar to the first assumption, we assume that $T(x)$ is independent of default time τ. This is a reasonable assumption as a single insurance company going bankrupt and defaulting on their liabilities will typically not affect the health of the clients.

4.2 The Problem

There are many types of equity-linked life insurance contract but in this thesis we are dealing with a class of contracts called pure endowment contracts. Such a contract has the following payoff

$$H\mathbf{1}_{\{T(x)>T\}}$$

where H is an $\hat{\mathcal{F}}_T$ measurable random variable which corresponds to a payment that the client receives if he or she survives till time T.

Next, we extend this model the case when the insurer has a possibility of default. This contract has the following payoff function

$$H\mathbf{1}_{\{T(x)>T\}}\mathbf{1}_{\{\tau>T\}}$$

Therefore, for the client to receive a payment at time T. He/she will have to survive to time T and the insurer and the insurer should default up to this time.

1. What is the price that the insurance company should charge the client

to sell this contract to him/her. In other words, what is the price of such a contract?

2. How would the insurance company go about hedging such a contract?

4.3 Hedging Mortality Risk

By far the easiest of the three types of uncertainties to deal with is that of mortality risk of a given client. While the actuarial science literature is rich with different methods of modelling and mitigating this type of risk here we focus on the Brennan-Schwartz approach. With this approach we simply assume that the number of contracts the insurer holds is large enough to use the strong law of large numbers. More concretely, we suppose

$$\sum_{i=1}^{n} \mathbf{1}_{\{T_i(x)>T\}} \approx n_T p_x$$

We also would like to know how should the insurance company hedge such exposure if it has many of such contracts. The first will be quantile hedging which looks like the following

$$\max P(X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}).$$

Then the problem to hedge these contracts simplifies to $n_T p_x H\mathbf{1}_{\{\tau>T\}}$. To hedge the credit risk of $H.\mathbf{1}_{\{T(x)>T\}}\mathbf{1}_{\{\tau>T\}}$ we apply superhedging techniques to $H\mathbf{1}_{\{\tau>T\}}$ in the incomplete market (B, S_1, S_2, τ) on probability space $(\Omega, \mathbb{G}, P)$. Then the initial value of the portfolio and the hedging portfolio are

$$U_0^{'} := n_T p_x U_0$$

and

$$\pi^{'} := n_T p_x \pi$$

where U_0 and π are corresponding to the initial value and super hedging portfolio for $H.\mathbf{1}_{\tau>T}$. Then the insurance company should superhedge $n_T p_x$

short positions of $H.\mathbf{1}_{\tau>T}$ in the defaultable market.

4.4 Hedging Portfolio

The hedging portfolio follows the standard definition as in [10].

1. An $\mathbb{G}$-portfolio process is an G - predictable process $\pi_t := (\pi_t)_{0\leq t\leq T}$ satisfying

$$\int_0^T |\pi_t|^2 dt < \infty, \quad P-\text{a.s}$$

2. $(\pi_t)_{0\leq t\leq T}$ is called self-financing if its corresponding wealth process $(X_t^{x_0,\pi})_{0\leq t\leq T}$, with initial wealth x_0, with initial wealth x_0, is defined by

$$X_t^{x_0,\pi} = x_0 + m\int_0^t \pi_s ds + \sigma\int_0^t \pi_s dW_s, \quad 0\leq t\leq T$$

3. A self-financing $\mathbb{G}$ - portfolio $(\pi)_{0\leq t\leq T}$ is called $\mathbb{F}$-admissible for the initial wealth $x_0 > 0$, if the corresponding wealth process $(X_t^{x_0,\pi})_{0\leq t\leq T}$ satisfies

$$X_t^{x_o,\pi} \geq 0, \quad P-\text{a.s.}$$

The set of all such portfolios with initial wealth x_0 is denoted by $\mathcal{A}(x_0)$

4.5 Final Formula

The premium the insurance company receives is ${}_Tp_xU_0$ for a single contract $H\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$. Thus we need to calculate U_0 the initial cost of hedging $H\mathbf{1}_{\{\tau>T\}}$. As mentioned earlier the superhedging cost of this contract is equivalent to the cost of hedging H without the risk of default. We need to do better than this as the market participants will take the probability of default into account when entering into new contracts. Thus we apply the results of Chapter 2 and use quantile hedging. Therefore our goal is

$$\max P(X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}) \tag{4.5}$$

under the constraint

$$u \leq \tilde{U}_0 \text{ and } \pi \in \mathcal{A}(u) \tag{4.6}$$

where $\tilde{U}_0 <_T p_x E^*[H]$, and U_0 is limit of how much capital the insurer has available to hedge a single contract.

To solve the problem 4.5 under the constraint 4.6 notice that for any $\pi \in \mathcal{A}(u)$ and $u \leq U_0$:

$$\begin{aligned}&\left\{X_t \geq H\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}\right\}\\ &= \left(\left\{X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}\right\} \times \{T(x) > T\}\right) \cup (\Omega \times \{T(x) \leq T\})\end{aligned}$$

with this and the independence of $\{S, \tau, T(x)\}$ imply that 4.5 and 4.6 are equivalent to the following for $\pi \in \mathcal{A}(u)$

$$n_T p_x \max_{u<U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}})) + (1 - n_T p_x) \tag{4.7}$$

Therefore we treat the survival probability independently and thus we can only consider the following

$$\max_{u<U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}})) \tag{4.8}$$

for $\pi \in \mathcal{A}(u)$. Which is the same problem considered in Chapter 2. In this case we present a solution to the problem in the case of $H = (S_T - K)^+$, for constant $K > 0$.

Lemma 4.5.1 *Let B, S and τ be defined as before. If we consider the quantile hedging of $(S_t - K)^+\mathbf{1}_{\{\tau>T\}}$:*

$$\max_{u<U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}})) \tag{4.9}$$

for $\pi \in \mathcal{A}(u)$. Where,

$$U_0 <_T p_x E^*[(S_t - K)^+] < E^*[(S_T - K)^+] = \sup_{Q \in \mathcal{Q}} E^Q[(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}]$$

This problem has the following solution

1. *If* $\frac{m}{\sigma^2} \leq 1$:

   $$\max_{u \leq U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}) = \Phi(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}}) + \left(1 - \phi(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}})\right) P(\tau \leq T)$$

   where b_1 *is defined by*

   $$b_1 := \frac{1}{\sigma} ln(\frac{c_1}{S_0}) + \frac{1}{2}\sigma T$$

   and $c_1 > K$ *is the solution to the following equation:*

   $$(x - K)^+ = cP(\tau > T)x^{\frac{m}{\sigma^2}} \tag{4.10}$$

   where c *is a constant described later in the proof.*

2. *If* $\frac{m}{\sigma^2} > 1$:

   $$\max_{u<U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}))$$
   $$= (1 - P(c_1 \leq S_T \leq c_3)) + P(c_2 < S_T \leq c_3)P(\tau \leq T)$$

   where $c_3 > c_2 > K$ *are solutions to 4.10, and* b_2 *and* b_3 *defined similarly to* b_1

Proof. Using Nakano's original results without the need for the generalized version of Chapter 2. We see that the problem 5.9 has the following solution:

$$P((S_T - K)^+(\mathbf{1}_{\{\xi<1\}} - \mathbf{1}_{\{\tau>T\}}) \geq 1) \tag{4.11}$$

with

$$U_0 = E^*[(S_T - K)^+ \mathbf{1}_{\{\xi<1\}}] \tag{4.12}$$

Since we assumed that interest rate, $r = 0$, ξ looks like the following

$$\{\xi < 1\} = \{yZ_T^* e^{\int_0^T \mu_t dt}(S_T - K)^+ < 1\} \tag{4.13}$$

Then using our definition of the default of probability and the fact that we can rewrite the risk neutral density Z_T^* as a function of S_T.

$$\begin{aligned} S_T =& S_0 exp\big(\sigma W_T + (m - \frac{1}{2}\sigma^2)T\big) \\ Z_T^* =& c * S_T^{-\frac{m}{\sigma^2}} \end{aligned}$$

where c_1 is some constant of parameters of our model that are known at $T = 0$

Then 4.13 can be rewritten as the following

$$\Big\{(S_T - K)^+ < cS_T^{\frac{m}{\sigma^2}} P(\tau > T)\Big\} \tag{4.14}$$

1. $\frac{m}{\sigma^2} \leq 1$: In this case 4.10 has only one solution, c_1. From 4.14,

$$\{\xi < 1\} = \{S_T < c_1\} = \{W_T^* < b_1\} = \{W_T^* < b_1\} = \{W_T < b_1 - \frac{m}{\sigma}T\} \tag{4.15}$$

In order to use Nakano's results as we are we have to show that $E^*\big[H\big] < +\infty$ and $P(\xi = 1) = 0$. The first condition is satisfied as

$$E^*\big[(S_T - K)^+\big] < +\infty.$$

The second condition is also satisfied as

$$P(\xi = 1) = P(W_T = b_1 - \frac{m}{\sigma}T) = 0$$

thus the assumptions are satisfied. We can then perform the following decomposition

$$
\begin{aligned}
(S_T - K)^+ \mathbf{1}_{\{\xi<1\}} &= (S_T - K)^+ \mathbf{1}_{\{S_T<c_1\}} \\
&= (S_T - K)^+ - (S_T - c_1)^+ - (c_1 - K)\mathbf{1}_{\{S_T>c_1\}}
\end{aligned} \tag{4.16}
$$

Using 4.12 and 4.16 we can apply the Black Scholes formula and determine c_1:

$$
\begin{aligned}
U_0 =& E * [(S_T - K)^+ \mathbf{1}_{\{\xi<1\}}] \\
=& S_0 \Phi \left(\frac{ln \frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) - K\Phi \left(\frac{ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\
& - S_0 \Phi \left(\frac{ln \frac{S_0}{c_1} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) + K\Phi \left(\frac{ln \frac{S_0}{c_1} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right)
\end{aligned} \tag{4.17}
$$

Where ϕ is the standard normal distribution function defined by

$$
\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \quad x \in [-\infty, \infty]
$$

We have the result that the perfect hedging portfolio for $(S_T - K)\mathbf{1}_{\{\xi<1\}} = (S_T - K)^+ \mathbf{1}_{\{\xi<1\}}$ solves the hedging problem defined by 4.5. Now to find the probability of the successful hedge we can do the following

$$
\begin{aligned}
& \max_{u<U_0} P(X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}) \\
&= P\big(S_T - K)^+ (\mathbf{1}_{\{S_T<c_1\}} - \mathbf{1}_{\{\tau>T\}}) \geq 0\big) \\
&= 1 - P((S_T \geq 0) \cap (\tau > T)) \\
&= P(S_T < c_1) + (1 - P(S_T < c_1))\, P(\tau < T)
\end{aligned}
$$

Thus proof of part (i) is complete.

2. $\frac{m}{\sigma^2} > 1$: these case there are two distinct solutions c_2 and c_3. Then,

$$\{\xi < 1\} = \{S_T < c_2\} \cup \{S_T > c_3\} = \{W_T^* < b_2\} \cup \{W_T^* > b_3\} \quad (4.18)$$

In this case we again see that the two assumptions are satisfied. Namely,

$$E^*[H] = E^*[(S_T - K)^+] < +\infty$$

and also,

$$P(\xi = 1) = P(W_T = b_2 - \frac{m}{\sigma}T) + P(W_T = b_3 - \frac{m}{\sigma}T) = 0$$

and thus Nakano's results can be applied.

Similar to the first case 5.33 implies:

$$\begin{aligned}(S_T - K)^+ \mathbf{1}_{\{\xi<1\}} =& (S_T - K)^+ \mathbf{1}_{\{S_T<c_2\}} + (S_T - K)^+ \mathbf{1}_{\{S_T>c_3\}} \\ =& (S_T - K)^+ - (S_T - c_2)^+ + (S_T - c_3)^+ \\ & -(c_2 - K)\mathbf{1}_{\{S_T>c_2\}} + (c2 - K)\mathbf{1}_{\{S_T>c_3\}}\end{aligned} \quad (4.19)$$

Then c_2 and c_3 can be determined by

$$\begin{aligned}U_0 =& E*[(S_T - K)^+ \mathbf{1}_{\{\xi<1\}}] \\ =& S_0\Phi\left(\frac{ln\frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ & -S_0\Phi\left(\frac{ln\frac{S_0}{c_2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + K\Phi\left(\frac{ln\frac{S_0}{c_2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ & +S_0\Phi\left(\frac{ln\frac{S_0}{c_3} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{ln\frac{S_0}{c_3} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\end{aligned} \quad (4.20)$$

Once we have the solutions to c_2 and c_3 we can determine b_2 and b_3 then, as before, we can find the probability of a successful hedge with the following

$$\begin{aligned}
&\max_{u<U_0} P(X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}) \\
&= P\big((S_T - K)^+(\mathbf{1}_{\{S_T<c_2\}\cup\{S_T>c_3\}} - \mathbf{1}_{\{\tau>T\}}) \geq 0\big) \\
&= 1 - P((c_2 \leq S_T \leq c_3) \cap (\tau > T)) \\
&= 1 - P((c_2 \leq S_T \leq c_3))P(\tau > T) \\
&= \big(1 - P(c_2 \leq S_T \leq c_3)\big) + P(c_2 \leq S_T \leq c_3)P(\tau \leq T)
\end{aligned}$$

With this, the proof is concluded.

□

WE now having everything we need to price the original contract of interest. Namely, $H = \max(S_T, K)$. This contract is called a pure endowment contract with guarantee K.

We can rewrite the contract H in the following way

$$\begin{aligned}
H(\tau, T(x)) :=& \max(S_T, K)\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}} \\
=&(K + (S_T - K)^+)\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}} \qquad (4.21)
\end{aligned}$$

The value of the contract 4.21 at time zero is

$$\begin{aligned}
{}_{(\tau,U)}U_x :=& {}_T p_x E^*[K + (S_T - K)^+] \\
=& {}_T p_x (K + S_0\phi(d_+) - K\phi(d_-)) \qquad (4.22)
\end{aligned}$$

where

$$d_\pm = \frac{1}{\sigma\sqrt{T}} ln(\frac{S_0}{K}) \pm \frac{1}{2}\sigma\sqrt{T}$$

$$ {}_T p_x E^*[(S_T - K)^+] =_{(\tau,U)} U_x -_T p_x K \tag{4.23} $$

This implies we can superhedge $K\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$ by the initial cost of ${}_T p_x K$ and we can focus on the embedded call option $(S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$. Then the quantile hedging problem is

$$ \max P(X_T^{u,\pi} \geq (S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}) \tag{4.24} $$

given that

$$ u \leq U_0, \quad \text{and} \quad \pi \in \mathcal{A}(u) \tag{4.25} $$

where $U_0 <_T p_x E^*[(S_T - K)^+] =_{\tau,T} U_x -_T p_x K$ is fixed upper bound on the initial capital available to hedge the option. Thus the final premium that will be charged by the insurance company is $U_0 +_T p_x K$. With the risk of the contract being given by 4.14.

By substituting $H = (S_T - K)^+$ and using Lemma 4.5.1 we arrive to the following theorem.

Theorem 4.5.1 *The quantile hedging problem 4.24 under the constraints 4.25 has the following solution:*

1. *If* $\frac{m}{\sigma^2} \leq 1$:

$$ \max_{u \leq U_0} P\big(X_T^{u,\pi} \geq (S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}\big) $$
$$ =_T p_x \left[\phi\Big(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}}\Big) + \Big(1 - \phi\Big(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}}\Big)\Big)P(\tau \leq T\right] + (1 -_T p_x) $$

where b_1 is a constant determined by 4.10.

2. *if* $\frac{m}{\sigma^2} > 1$:

$$ \max_{u \leq U_0} P\big(X_T^{u,\pi} \geq (S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}\big) $$
$$ =_T p_x \left[(1 - P(c_2 \leq S_T \leq c_3)) + P(c_2 \leq S_T \leq c_3)P(\tau \leq T)\right] + (1 -_T p_x) $$

where c_2 and c_3 are constants determined by 4.10.

We extend the practicality of this theory by allowing the insurance company to set a lower bound age of client given a shortfall risk ϵ and initial capital U_0.

Given shortfall risk $\epsilon \in [0,1]$ and initial capital $U_0 \in (0, E^*[(S_T - K)^+])$ then we can find an acceptable survival probability ${}_Tp_x$:

1. If $\frac{m}{\sigma^2} \leq 1$: We can find c_1 and using part (i) of theorem 5.3.1 and given shortfall risk. We can find the following

$$1 - \epsilon =_T p_x \left[\phi(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}}) + \big(1 - \phi(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}})\big)P(\tau \leq T\right] + (1 -_T p_x)$$

   which we can then solve for ${}_Tp_x$ and find the acceptable ${}_TP_x$ of clients.

2. If $\frac{m}{\sigma^2} \leq 1$: We can do the same with

$$1-\epsilon =_T p_x \left[(1 - P(c_2 \leq S_T \leq c_3)) + P(c_2 \leq S_T \leq c_3)P(\tau \leq T)\right]+(1-{}_Tp_x)$$

Thus, results presented thus far can be used in a multitude of ways depending on which of ${}_TP_x, U_0$ or ϵ are known. Once we know 2 of the three we can use the results provided thus far and find a lower bound for the others.

Chapter 5

Case of Two Risky Assets

This Chapter deals with a similar problem to the previous with the exception that we have 2 risky assets instead of one. As before we are considering three distinct sources of risk: the market risk of the Black-Scholes Market, the default risk of the insurer and the mortality associated with the client.

5.1 Market Model

The framework of the market model is that of a financial Market that consists of one risk less asset, a bank account and several risky assets

$$\begin{aligned} dS_1(t) =& S_1(t)(\mu_1 dt + \sigma_1 dW_t^1), \\ dS_2(t) =& S_2(t)(\mu_2 dt + \sigma_2 dW_t^2) \end{aligned}$$

We assume that our market is driven by two different but correlated Wiener Processes W^1 and W^2. The correlation of the two Wiener Processes is W_t^1 and W_t^2 is ρt where $\rho^2 < 1$.

All processes are given on a standard Stochastic Basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ and are adapted to the filtration $\mathbb{F}$ generated by W_t^i. We also assume that the market coefficients μ_1, μ_2, σ_1 and σ_2 are constant.

In general, there are two ways to set up a multidimensional Black-Scholes market. One, in which we consider independent Wiener processes as we did in Chapter 2 before and one where we consider correlated Wiener processes. The reason we switch from one to the other depends on which form of the risk neutral density is more convenient to use. We note that from Theorem 7 in [7] we can go from one market type to another by transforming the covariance matrix.

As with the usual case with option pricing we need to consider risk neutral probability measures denoted by P^*. Under these risk neutral measures the expected returns of assets S_1 and S_2 are equal to the risk free interest rate r. The prices calculated with respect to P^* are also arbitrage free if such a P^* exists and are unique if such P^* is unique. By Girsanov theorem we can find a risk neutral density or deflator $Z_t^* = \dfrac{dP_t^*}{dP_t}$ of the martingale measure P^* using methodology given in [10]. We can then express Z^* as a stochastic exponent of martingale process N:

$$Z_t^* = \varepsilon(N_t)$$

We are dealing with two Brownian motion so the martingale takes the form $N_t = \phi_1 W_t^1 + \phi_2 W_t^2$. In order be able solve for ϕ_1 and ϕ_2 we can write the processes B_t, S_t^1 and S_t^2 as stochastic exponents of processes h, H_t^1, H_t^2 respectively

$$h = rt, \quad H_t^i = \mu_i t + \sigma_i W_t^i.$$

Using the method of finding martingale measures which states that the process

$$\psi(h, H, N) = H_t^i - h_t + N_t + \left\langle (h - H^i)^c, (h - N)^c \right\rangle_t$$

is a martingale with respect to P. Using this we can find constants ϕ_1 and ϕ_2.

For ψ_1 and ψ_2 we have the following expressions:

$$\psi_1 = \mu_1 t + \sigma_1 W_1(t) - rt + \phi_1 W_1(t) + \phi_2 W_2(t) + \sigma_1\phi_1 t + \sigma_1\phi_2\rho t$$
$$\psi_2 = \mu_2 t + \sigma_2 W_2(t) - rt + \phi_1 W_1(t) + \phi_2 W_2(t) + \sigma_1\phi_1 t + \sigma_1\phi_2\rho t$$

For these to be martingales, the terms with "t" must be equal to zero. In other words

$$\mu_1 t - rt + \sigma_1\phi_1 t + \sigma_1\phi_2\rho t = 0$$
$$\mu_2 t - rt + \sigma_2\phi_2 t + \sigma_2\phi_1\rho t = 0$$

After solving this system of equations we have

$$\phi_1 = \frac{r(\sigma_2 - \sigma_1\rho) + \rho\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1\sigma_2(1-\rho^2)}$$
$$\phi_2 = \frac{r(\sigma_1 - \sigma_2\rho) + \rho\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_1\sigma_2(1-\rho^2)}$$

As the interest rate, r is assumed to be zero.

$$\phi_1 = \frac{\rho\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1\sigma_2(1-\rho^2)}$$
$$\phi_2 = \frac{\rho\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_1\sigma_2(1-\rho^2)}$$

Therefore in the Stochastic Exponent for Z_t^* becomes

$$Z_t^* = \frac{dP_t^*}{dP_t} = \varepsilon(N_t) = \varepsilon(\phi_1 W_t^1 + \phi_2 W_t^2)$$
$$= exp\left\{\phi_1 W_t^1 + \phi_2 W_t^2 - \frac{1}{2}(\phi_1^2 + \phi_1^2 + 2\rho\phi_1\phi_2)t\right\} \tag{5.1}$$

5.2 Default Time and Mortality

As before we have default time of the insurer as well as the mortality of the client as a source of uncertainty in our model. Please refer to Chapter 2 and 3 for more details on how these are defined.

5.3 Main Formula and Results

The contract we will be attempting to hedge is called the Pure Endowment with a stochastic guarantee. It allows the client to gain exposure to two types of assets simultaneously. The first asset is meant to be seen as the riskier of the two but with a higher return and higher standard deviation than the second. Such a contract has the following payoff

$$H := \max\{S_T^1, S_T^2\}\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{T(x)>T} \tag{5.2}$$

where we assume that

$$\mu_1 > \mu_2 \quad \text{and} \quad \sigma_1 > \sigma_2.$$

Note that

$$\max\{S_T^1, S_T^2\} = S_T^2 + (S_T^1 - S_T^2)^+ \tag{5.3}$$

Which implies that we can write 5.2 as the following

$$H = \max\{S_T^1, S_T^2\}\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{T(x)>T} = S_T^1\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{T(x)>T} + (S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{T(x)>T}$$

Hence, once we take the expectation with respect to the risk neutral density the first term becomes

$$E(S_T^1\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{T(x)>T}) = S_0^1 P(\tau > T)P(T(x) > T)$$

which can be be estimate at time $t = 0$ and as such we can focus on the problem of hedging of

$$H' = (S_T^1 - S_T^2)^+ \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{T(x) > T} \tag{5.4}$$

We introduce the ratio

$$Y_T = \frac{S_T^1}{S_T^2}$$

which follows the following dynamics

$$Y_T = \frac{S_T^1}{S_T^2} = \frac{S_0^1}{S_0^2} exp\left\{ -\frac{\sigma_1^2 - \sigma_2^2}{2} T + \sigma_1 W_T^{1*} + \sigma_2 W_T^{2*} \right\} \tag{5.5}$$

where

$$W_T^* = \frac{\sigma_1 W_T^{1*} - \sigma_2 W_T^{2*}}{\sigma} \tag{5.6}$$

where $\sigma = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho$. Then W_T^* is a new Brownian motion with respect to P^* with the following covariances

$$\begin{aligned} cov(W_t^*, W_t^{1*}) =& E^*[W_t^*, W_t^{1*}] = \frac{(\sigma_1 - \sigma_2\rho)t}{\sigma} \\ cov(W_t^*, W_t^{2*}) =& E^*[W_t^*, W_t^{2*}] = \frac{(\sigma_1\rho - \sigma_2)t}{\sigma} \end{aligned}$$

the we can rewrite 5.5 as the following

$$Y_T = \frac{S_0^1}{S_0^2} exp\left\{ -\frac{(\sigma_1^2 - \sigma_2^2)}{2} T + \sigma W_T^* \right\} \tag{5.7}$$

$$\max_{u < U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau > T\}})) \tag{5.8}$$

for $\pi \in \mathcal{A}(u)$. Which is the same problem considered in Chapter 2. In this case we present a solution to the problem in the case of $H = (S_T - K)^+$, for constant $K > 0$.

Lemma 5.3.1 *Let B, S^1, S^2 and τ be defined as before. If we consider the quantile hedging of $(S_T^1 - S_T^2)^+ \mathbf{1}_{\{\tau > T\}}$:*

$$\max_{u<U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}})) \tag{5.9}$$

for $\pi \in \mathcal{A}(u)$. *Where,*

$$U_0 <_T p_x E^*[(S_T^1 - S_T^2)^+] < E^*[(S_T^1 - S_T^2)^+] = \sup_{Q\in\mathcal{Q}} E^Q[(S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}]$$

This problem has the following solution

1. *If* $\frac{\phi_1}{\sigma_1} \geq -1$:

   The hedging price is

$$\begin{aligned} U_0 =& E * [(S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}] \\ =& S_0^1\Phi\left(\frac{ln\frac{S_0^1}{S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - S_0^2\Phi\left(\frac{ln\frac{S_0^1}{S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ -& S_0^1\Phi\left(\frac{ln\frac{S_0^1}{c_1 S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + S_0^2\Phi\left(\frac{ln\frac{S_0^1}{c_1 S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{5.10}$$

   with the probability of a successful hedge being

$$\max_{u\leq U_0} P(X_t^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}) = \left(1 - P\left(\frac{S_T^1}{S_T^2} \leq c_1\right)\right) + P\left(\frac{S_T^1}{S_T^2} \leq c_1\right) P(\tau \leq T)$$

   and $c_1 > K$ *is the solution to the following equation:*

$$x^{-\alpha} = g \cdot (x-1)^+. \tag{5.11}$$

   where g *is a constant described later in the proof*

2. *If* $\frac{\phi_1}{\sigma_1} < -1$: *The hedging price is*

$$\begin{aligned}U_0 =&E*[(S_T^1-S_T^2)^+\mathbf{1}_{\{\tau>T\}}]\\=&S_0^1\Phi\left(\frac{ln\frac{S_0^1}{S_0^2}+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)-S_0^2\Phi\left(\frac{ln\frac{S_0^1}{S_0^2}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)\\-&S_0^1\Phi\left(\frac{ln\frac{S_0^1}{c_2S_0^2}+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)+S_0^2\Phi\left(\frac{ln\frac{S_0^1}{c_2S_0^2}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)\\+&S_0^1\Phi\left(\frac{ln\frac{S_0^1}{c_3S_0^2}+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)-S_0^2\Phi\left(\frac{ln\frac{S_0^1}{c_3S_0^2}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)\end{aligned} \tag{5.12}$$

with the probability being

$$\begin{aligned}&\max_{u<U_0}P(X_t^{u,\pi}\geq H\mathbf{1}_{\{\tau>T\}}))\\&=\left(1-P(c_1\leq\frac{S_T^1}{S_T^2}\leq c_3)\right)+P\left(c_2\leq\frac{S_T^1}{S_T^2}\leq c_3\right)P(\tau\leq T)\end{aligned}$$

where $c_3>c_2>1$ *are solutions to 5.11*

Proof. Here we the results of Chapter 2. Like in the 1 dimensional case they state that the maximal probability is

$$P((S_T-K)^+(\mathbf{1}_{\{\xi<1\}}-\mathbf{1}_{\{\tau>T\}})\geq 1) \tag{5.13}$$

where

$$U_0=E^*[(S_T^1-S_T^2)^+\mathbf{1}_{\{\xi<1\}}] \tag{5.14}$$

Since we assumed that interest rate, $r=0$, ξ looks like the following

$$\{\xi<1\}=\{\hat{y}(S_T^1-S_T^2)^+Z_T^*e^{\int_0^T\mu_t dt}<1\} \tag{5.15}$$

Note that

$$(S_T^1 - S_T^2)^+ = S_T^2 \left(\frac{S_T^1}{S_T^2} - 1\right)^+ = S_T^2 \left(Y_T - 1\right)^+$$

Then we can rewrite 5.15 as

$$\begin{aligned}
\{\xi < 1\} =& \{\hat{y} S_T^2 (Y_T - 1)^+ Z_T^* e^{\int_0^T \mu_t dt} < 1\} && (5.16)\\
=& \left\{(Z_T^* S_T^2)^{-1} \geq \hat{y}^{-1} P(\tau > T)(Y_T - 1)^+\right\} && (5.17)
\end{aligned}$$

The goal now is to simplify this set in terms of Y_T. Our approach, as in [13] is to find $Z_T S_T^2$ in terms of Y_T multiplied by some constant.

To find the $Z_T S_T^2$ as a function of Y_T, we rewrite 5.1 as follows

$$\begin{aligned}
Z_T =& exp\left\{\phi_1 W_t^1 + \phi_2 W_t^2 - \frac{1}{2}(\phi_1^2 + \phi_1^2 + 2\rho\phi_1\phi_2)t\right\}\\
=& exp\left\{\phi_1\left(W_T^{1*} - T\frac{\mu_1}{\sigma_1}\right) + \phi_2\left(W_T^{2*} - T\frac{\mu_2}{\sigma_2}\right)\right\}\\
\times& exp\left\{-\frac{1}{2}(\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2)T\right\}\\
=& exp\{\phi_1 W_T^{1*} + \phi_2 W_T^{2*}\}\\
\times& exp\left\{\left(-\frac{\phi_1\mu_1}{\sigma_1} + \frac{\phi_2\mu_2}{\sigma_2} + \frac{1}{2}(\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2)\right)T\right\} && (5.18)
\end{aligned}$$

We then use 5.6 and 5.7 to rewrite the term $Z_T S_T^2$ in the form gY_T^α where g and α are functions of the model parameters.

$$
\begin{aligned}
Z_T S_T^2 = exp&\left\{\phi_1 W_T^{1*} - T\frac{\phi_1\mu_1}{\sigma_1} + \phi_2 W_T^{2*} - T\frac{\phi_2\mu_2}{\sigma_2}\right\} \\
&\times exp\left\{-\frac{1}{2}(\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2)T\right\} \\
&\times S_0^2 exp\left\{rT - \frac{\sigma_2^2}{2}T + \sigma W_T^{2*}\right\} \\
&= S_0^2 exp\{\phi_1 W_T^{1*} + (\phi_2 + \sigma_2 W_t^{2*})\} \\
&\times exp\left\{rT - \frac{\sigma^2}{2}T - \frac{\phi_1\mu_1}{\sigma_1}T - \frac{\phi_2\mu_2}{\sigma_2}T\right\} \\
&\times exp\left\{-\frac{1}{2}(\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2)T\right\} \\
= \left(\frac{S_0^1}{S_0^2}\right)^\alpha exp&\left\{\sigma_1\alpha W_T^{1*} - \sigma_2\alpha W_T^2 - T\alpha\frac{\sigma_1^2 - \sigma_2^2}{\sigma_2}\right\} \qquad (5.19) \\
&\times \left(\frac{S_0^1}{S_0^2}\right)^\alpha exp\left\{T\alpha\frac{\sigma_1^2 - \sigma_2^2}{2}\right\} \\
&\times S_0^2 exp\left\{rT - \frac{\sigma^2}{2}T - \frac{\phi_1\mu_1}{\sigma_1}T - \frac{\phi_2\mu_2}{\sigma_2}T\right\} \\
&\times exp\left\{-\frac{1}{2}(\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2)T\right\} \\
&= gY_T^\alpha
\end{aligned}
$$

Where g is the constant of the following form

$$
g = \frac{(S_0^2)^{\alpha+1}}{(S_0^1)^\alpha} exp\left\{T\alpha\left(\frac{\sigma_1^2 - \sigma_2^2}{2}\right) + rT - \frac{\sigma_2^2}{2}T\right\} \qquad (5.20)
$$

$$
\times exp\left\{-\left(\frac{\phi_1\mu_1}{\sigma_1}T + \frac{\phi_2\mu_2}{\sigma_2}T + \frac{1}{2}(\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2)\right)\right\} \qquad (5.21)
$$

In 5.19 we have that

$$
\sigma_1 \cdot \alpha = \phi_1 \quad -\sigma_2 \cdot \alpha = \phi_2 + \sigma_2 \qquad (5.22)
$$

which leads to the the condition

$$\alpha = \frac{\phi_1}{\sigma_1} = -1 - \frac{\phi_2}{\sigma_2}. \tag{5.23}$$

We can define a new constant g' as

$$g' = \hat{y}^{-1} \cdot P(\tau > T) \cdot g. \tag{5.24}$$

With this we can now represent the set 5.15 as

$$\{\xi < 1\} = \left\{ \frac{1}{Y_T^\alpha} \geq g' \cdot (Y_T - 1)^+ \right\} \tag{5.25}$$

which has the following characteristic equation

$$x^{-\alpha} = g' \cdot (x-1)^+. \tag{5.26}$$

1. $\frac{\phi_1}{\sigma} \geq -1$: In this case 5.26 has only one solution, c_1. From 5.25,

$$\{\xi < 1\} = \{S_T < c_1\} = \{W_T^* < b_1\} \tag{5.27}$$

In order to use Nakano's results as we are we have to show that $E^*\big[H\big] < +\infty$ and $P(\xi = 1) = 0$. The first condition is satisfied as

$$E^*\big[(S_T^1 - S_T^2)^+\big] < +\infty.$$

The second condition is also satisfied as

$$P(\xi = 1) = P(W_T^* = b_1) = 0,$$

thus the assumptions are satisfied. We can then perform the following decomposition

$$\begin{aligned}(S_T^1 - S_T^2)^+ \mathbf{1}_{\{\xi<1\}} &= (S_T^1 - S_T^2)^+ \mathbf{1}_{\{Y_T < c_1\}} \\ &= (S_T^1 - S_T^2)^+ - S_T^1 + S_T^2 + (S_T^1 - S_T^2)\mathbf{1}_{\{Y_T<c_1\}}. \end{aligned} \tag{5.28}$$

We then take the expectations under risk neutral probabilities and obtain

$$U_0 = E^*(S_T^1 - S_T^2)^+ - E^* S_T^1 + E^* S_T^2 + E^*(S_T^1 - S_T^2)\mathbf{1}_{\{Y_T < c_1\}}. \quad (5.29)$$

The first term of 5.29, $E^*(S_T^1 - S_T^2)^+$ can be calculated using Margrabe's Formula. The second and third terms, $E^* S_T^1$ and $E^* S_T^2$ are martingales with respect to the risk neutral density and therefore they are known at $t = 0$ as $E^* S_T^1 = S_0^1$ and $E^* S_T^2 = S_0^2$. The last term is more complicated and requires the use of Lemma 2.3 in [13] . Using which, we obtain

$$\begin{aligned} E^*(S_T^1 - S_T^2)\mathbf{1}_{\{Y_T < c_1\}} &= E^*(S_T^1)\mathbf{1}_{\{Y_T < c_1\}} - E^*(S_T^2)\mathbf{1}_{\{Y_T < c_1\}} \\ = -S_0^1 &\left(\frac{-\left[ln\frac{S_0^1}{c_1 S_0^2} - \frac{\sigma^2}{2}T \right]}{\sigma\sqrt{T}} \right) + S_0^2 \left(\frac{-\left[ln\frac{S_0^1}{c_1 S_0^2} + \frac{\sigma^2}{2}T \right]}{\sigma\sqrt{T}} \right) \end{aligned} \quad (5.30)$$

With this, the initial price U_0 equals

$$\begin{aligned} U_0 =& S_0^1\Phi(b_+(1)) - S_0^2(b_-(1)) - S_0^1 + S_0^2 + S_0^1\Phi(-b_+(c_1)) - S_0^2(-b_-(c_1)) \\ =& S_0^1\Phi(b_+(1)) - S_0^2(b_-(1)) - S_0^1 + S_0^2 + S_0^1(1 - \Phi(b_+(c_1))) - S_0^2(1 - \Phi(b_-(c_1)) \end{aligned} \quad (5.31)$$

$$\begin{aligned} U_0 =& E * [(S_T - K)^+ \mathbf{1}_{\{\xi < 1\}}] \\ =& S_0^1\Phi\left(\frac{ln\frac{S_0^1}{S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) - S_0^2\Phi\left(\frac{ln\frac{S_0^1}{S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\ & - S_0^1\Phi\left(\frac{ln\frac{S_0^1}{c_1 S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) + S_0^2\Phi\left(\frac{ln\frac{S_0^1}{c_1 S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \end{aligned} \quad (5.32)$$

Where ϕ is the standard normal distribution function defined by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \quad x \in [-\infty, \infty]$$

Now to find the probability of the successful hedge we can do the following

$$\begin{aligned}
&\max_{u<U_0} P(X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}} \\
&= P\left(\left(\frac{S_T^1}{S_T^2} - K\right)^+ \left(\mathbf{1}_{\{\frac{S_T^1}{S_T^2}<c_1\}} - \mathbf{1}_{\{\tau>T\}}\right) \geq 0\right) \\
&= 1 - P\left(\left(\frac{S_T^1}{S_T^2} \geq 0\right) \cap (\tau > T)\right) \\
&= \left(1 - P\left(\frac{S_T^1}{S_T^2} \leq c_1\right)\right) + P\left(\frac{S_T^1}{S_T^2} \leq c_1\right) P(\tau \leq T)
\end{aligned}$$

Thus proof of part (i) is complete.

2. $\frac{\phi_1}{\sigma} < -1$: these case there are two distinct solutions c_2 and c_3. Then,

$$\{\xi < 1\} = \left\{\frac{S_T^1}{S_T^2} < c_2\right\} \cup \left\{\frac{S_T^1}{S_T^2} > c_3\right\} = \{W_T^* < b_2\} \cup \{W_T^* > b_3\} \quad (5.33)$$

In this case we again see that the two assumptions are satisfied. Namely,

$$E^*[H] = E^*\left[\left(\frac{S_T^1}{S_T^2} - K\right)^+\right] < +\infty$$

and also,

$$P(\xi = 1) = P(W_T^* = b_2) + P(W_T^* = b_3) = 0$$

and thus Nakano's results can be applied.

Similar to the first case 5.33 implies:

$$\begin{aligned}(S_T^1 - S_T^2)^+\mathbf{1}_{\{\xi<1\}} &= (S_T^1 - S_T^2)^+\mathbf{1}_{\{Y_T\le c_2\}} + (S_T^1 - S_T^2)^+\mathbf{1}_{\{S_T\ge c_3\}}\\ = (S_T^1 - S_T^2)^+ - S_T^1 + S_T^2 &+ (S_T^1 - S_T^2)\mathbf{1}_{\{Y_T\le c_2\}} + (S_T^1 - S_T^2)\mathbf{1}_{\{Y_T\ge c_3\}}\\ &= (S_T^1 - S_T^2)^+ + (S_T^1 - S_T^2)\mathbf{1}_{\{Y_T\le c_2\}} - (S_T^1 - S_T^2)\mathbf{1}_{\{Y_T\le c_3\}}\end{aligned} \tag{5.34}$$

as

$$\mathbf{1}_{\{Y_T\ge c_2\}} = 1 - \mathbf{1}_{\{Y_T\le c_2\}}.$$

Then c_2 and c_3 can be determined by

$$\begin{aligned}U_0 =& E*[(S_T^1 - S_T^2)^+\mathbf{1}_{\{\xi<1\}}]\\ =& S_0^1\Phi\left(\frac{ln\frac{S_0^1}{S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - S_0^2\Phi\left(\frac{ln\frac{S_0^1}{S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\\ &- S_0^1\Phi\left(\frac{ln\frac{S_0^1}{c_2 S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + S_0^2\Phi\left(\frac{ln\frac{S_0^1}{c_2 S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\\ &+ S_0^1\Phi\left(\frac{ln\frac{S_0^1}{c_3 S_0^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - S_0^2\Phi\left(\frac{ln\frac{S_0^1}{c_3 S_0^2} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\end{aligned} \tag{5.35}$$

Once we have the solutions to c_2 and c_3 we can determine b_2 and b_3 then, as before, we can find the probability of a successful hedge with the following

$$
\begin{aligned}
&\max_{u<U_0} P(X_T^{u,\pi} \geq H\mathbf{1}_{\{\tau>T\}}) \\
&= P\big((S_T^1 - S_T^2)^+(\mathbf{1}_{\{Y_T<c_2\}\cup\{Y_T>c_3\}} - \mathbf{1}_{\{\tau>T\}}) \geq 0\big) \\
&= 1 - P((c_2 \leq Y_T \leq c_3) \cap (\tau > T)) \\
&= 1 - P((c_2 \leq Y_T \leq c_3))P(\tau > T) \\
&= \left(1 - P(c_1 \leq \frac{S_T^1}{S_T^2} \leq c_3)\right) + P\left(c_2 \leq \frac{S_T^1}{S_T^2} \leq c_3\right) P(\tau \leq T)
\end{aligned}
$$

With this, the proof is concluded.

□

Theorem 5.3.1 *The quantile hedging problem 4.24 under the constraints 4.25 has the following solution:*

1. *If* $\frac{\phi_1}{\sigma^2} \geq -1$:

$$
\begin{aligned}
&\max_{u\leq U_0} P\big(X_T^{u,\pi} \geq (S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}\big) \\
&=_T p_x \left[\left(1 - P\left(\frac{S_T^1}{S_T^2} \leq c_1\right)\right) + P\left(\frac{S_T^1}{S_T^2} \leq c_1\right) P(\tau \leq T)\right] + (1 -_T p_x)
\end{aligned}
$$

where b_1 *is a constant determined by 4.10.*

2. *if* $\frac{\phi_1}{\sigma^2} < -1$:

$$
\begin{aligned}
&\max_{u\leq U_0} P\big(X_T^{u,\pi} \geq (S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}\big) \\
&=_T p_x \left[\left(1 - P\left(c_1 \leq \frac{S_T^1}{S_T^2} \leq c_3\right)\right) + P\left(c_2 \leq \frac{S_T^1}{S_T^2} \leq c_3\right) P(\tau \leq T)\right] + (1 -_T p_x)
\end{aligned}
$$

where c_2 *and* c_3 *are constants determined by 4.10.*

We extend the practicality of this theory by allowing the insurance company to set a lower bound age of client given a shortfall risk ϵ and initial capital U_0.

Corollary 5.3.1 *Given shortfall risk $\epsilon \in [0,1]$ and initial capital $U_0 \in (0, E^*[(S_T - K)^+])$ then we can find an acceptable survival probability ${}_Tp_x$:*

1. *If $\frac{\phi_1}{\sigma^2} \geq -1$: We can find c_1 and using part (i) of theorem 5.3.1 and given shortfall risk. We can find the following*

$$1-\epsilon = {}_T p_x \left[\left(1 - P\left(\frac{S_T^1}{S_T^2} \leq c_1 \right) \right) + P\left(\frac{S_T^1}{S_T^2} \leq c_1 \right) P(\tau \leq T) \right] + (1 - {}_Tp_x)$$

   which we can then solve for ${}_Tp_x$ and find the acceptable ${}_Tp_x$ of clients.

2. *If $\frac{\phi_1}{\sigma^2} < -1$: We can do the same with*

$$1-\epsilon = {}_T p_x \left[\left(1 - P\left(c_1 \leq \frac{S_T^1}{S_T^2} \leq c_3 \right) \right) + P\left(c_2 \leq \frac{S_T^1}{S_T^2} \leq c_3 \right) P(\tau \leq T) \right] + (1 - {}_Tp_x)$$

Chapter 6

Numerical Results

The chapter will demonstrate the use cases of the formulas presented in the preceding two chapters with real financial and actuarial data. We will present two examples corresponding to the one and two risky asset cases and confirm that the resulting prices and success ratios are as one would expect from examining the formulas.

Example 1 will present a simple case when an equity index is used to calculate the Black-Scholes model parameters. If these results were to be used in the financial industry we expect this scenario to be the most common. The second example will present the use case for the two risky asset model with an equity index being the first asset and and a bond index being the flexible guarantee. This is a more nuanced and more financially involved contract and therefore will not see widespread use in the industry but it has some advantages over the one risky asset case due to the ability of picking assets which perform better in different economic scenarios.

6.1 One Risky Asset Example

The parameters of the Black-Scholes model are estimated from daily returns of the MSCI Emerging Market Index for the period of Nov 10, 2017 to October 19, 2018. The calculated parameters are as below

We assume the interest rate, $r = 0$. The real level of the MSCI Emerging

	MSCI Emerging Market Index
appreciation rate, m_i	0.0097
volatility, σ_i	0.1075
Initial Price, S_0^i	100

Table 6.1: Model parameters estimated using daily price returns of the MSCI Emerging Market Index for the period of Nov 10, 2017 to October 19,2018.

Market index was not $100 on Nov 10, 2017 but to make the to make comparisons to the following example we normalized the price. Also, note that the reason why the mean appreciation rate is so small is due to the generally poor performance of the emerging market securities over the last year.

The call option has a strike price of K=80 and the maturity time T=15 and the contract is written for a client at age $x = 30$. To calculate the needed probability of survival we refer to the life tables provided by [4] to get the survival probability ${}_Tp_x = 0.970$ and $x = 30$ and $T = 15$. As for the default probability we assume $\mu = 0.01$ which given our assumption will produce a survival probability, (i.e. the probability that the insurer will not default) of $P(\tau > T) = 0.8607$. Which implies the probability of default at $P(\tau \leq T) = 0.1393$

As seen from figure 6.1 the results are as expected. The initial capital required for each type of hedge is relatively close. However, each time we add a source of randomness the probability of a successful hedge increases for a given amount of initial capital. Which is inline with the intuition that if either the insurer goes bankrupt or the client passes away before maturity the payoff of the liability will not need to be paid and therefore is considered successful. More concretely,

$$(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \leq (S_T - K)^+$$

which implies a smaller required hedge is needed in the case of default. Similarly, once we add the mortality to the model

$$(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{\{T(x) > T\}} \leq (S_T - K)^+ \mathbf{1}_{\{\tau > T\}}.$$

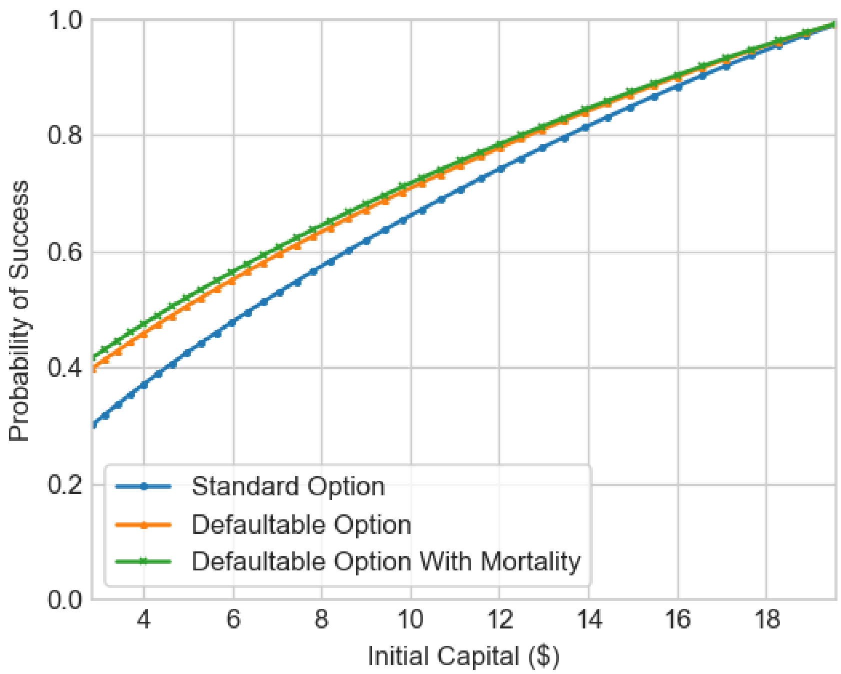

Figure 6.1: Probability of Success vs. Initial Capital for $(S_T - K)^+$, $(S_T - K)^+\mathbf{1}_{\{\tau>T\}}$ and $(S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$

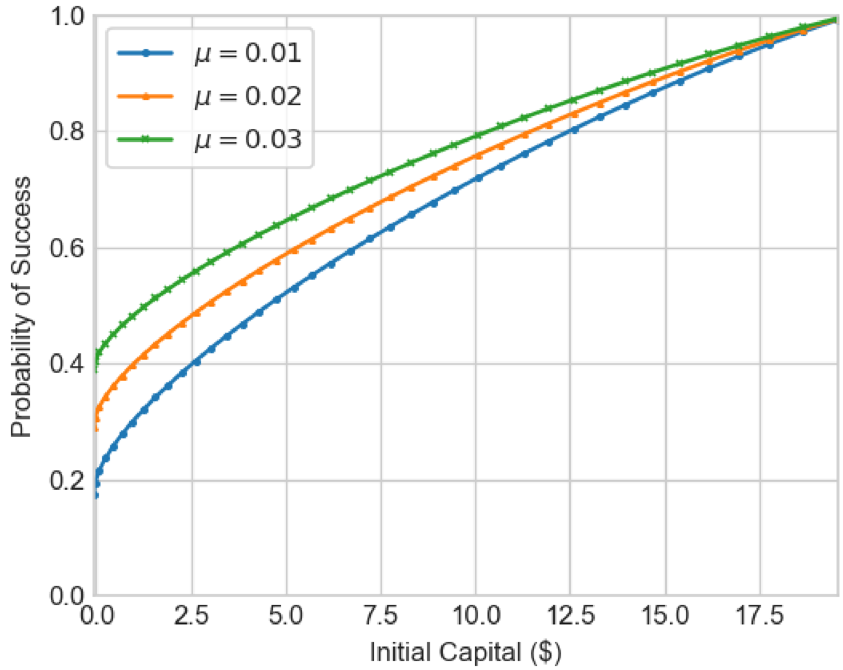

Figure 6.2: Probability of Success vs. Initial Capital for $(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{\{T(x) > T\}}$ with $\mu = 0.01, 0.02, 0.03$, or $P(\tau < T) = 0.1393, 0.2592, 0.3624$

Figure 6.2 demonstrates the relationship between the parameter μ and probability of successful hedge. For default intensities $\mu = 0.01$, $\mu = 0.02$ and $\mu = 0.02$ yield the following default probabilities $P(\tau \leq T) = 0.1393, 0.2592, 3624$ respectively. Using the same logic a above we can argue that as μ increases the probability of default $P(\tau < T)$ also increases which implies a higher probability of a successful hedge.

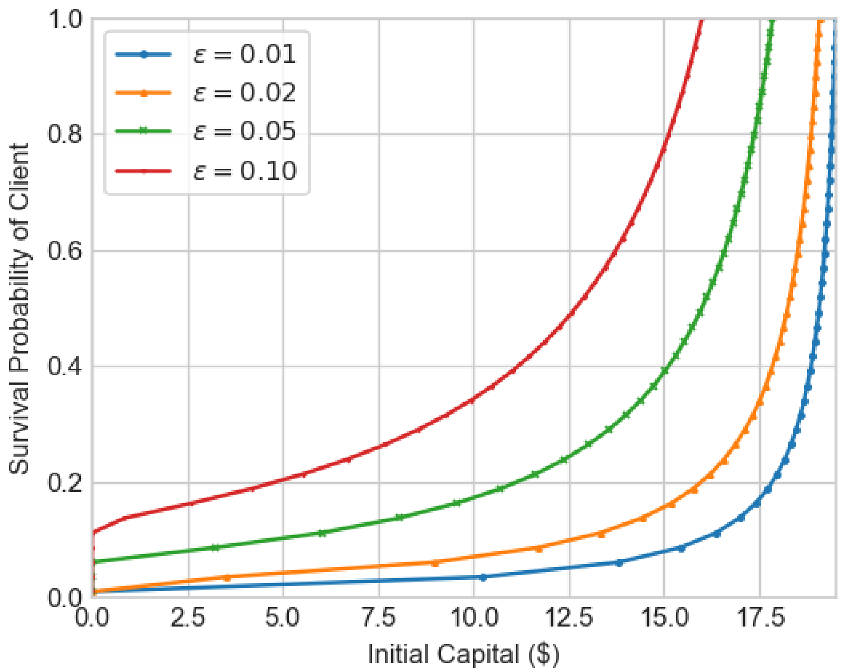

Figure 6.3: Probability of Success vs. Initial Capital for $(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{\{T(x) > T\}}$ with $\epsilon = 0.01, 0.02, 0.05, 0.10$, or $P(\tau < T) = 0.1393, 0.2592, 0.3624$

Figure 6.3 examines the relationship of the Survival Probability of a client and the required capital to hedge the option for a given amount of risk. We perform this analysis using the the result of Corollary 5.3.1. We examine this relationship by fixing the overall shortfall probabilities $\epsilon = 0.01, 0.02, 0.05$ and 0.01. We then calculate what the survival probability of the client would have to be for the given amount of risk for a given amount of Initial Capital. As the higher the probability of survival the higher the probability that the insurer will be liable for the the payoff $(S_T - K)^+$ leading to a more capital being required to successfully hedge a contract for a given amount of risk. Using these probabilities we can then use actuarial tables and estimate the minimum age of the client that fits the shortfall probability and initial capital that the insurer has available. These results are summarized for several points in the following table using the actuarial tables from [4]. We can note from the table above is that accepting a higher shortfall risk ϵ or having a larger pool of capital with which to hedge with widens the acceptable range of clients for which this contract can be offered.

Shortfall Risk	Initial Capital	Survival Probability	Age of Client (x)
1%	\$19.00	46.67%	≥ 70
2%	\$18.50	59.38%	≥ 66
5%	\$17.00	69.53%	≥ 63
10%	\$16.00	100%	Any Age

Table 6.2: Given an acceptable shortfall probability ϵ and available Initial Capital we find the maximal survival probability of the client and the minimum acceptable age.

6.2 Two Risky Asset Example

We repeat the analysis from before using the results of the two stock scenario. In this case we present a novel case where the client does not only want exposure equity markets but also to bond markets. As equity markets tend to be seen as a more conservative investment to equity, a bond market index can be seen as a suitable flexible guarantee.Therefore, the more risky asset in our model, S^1, is the equity index and the stochastic guarantee, S^2, is the bond index. In this example we focus on the Canadian markets for the equity index we use the S&P TSX 60 index and for the bond index we use the Universe Bond Index constructed by FTSE.

The parameters for the S&P Index are summarized in Table 6.3. In addition to the appreciation rate and volatility we need the correlation between the two indices. For period in question the correlation was, $\rho = -0.145$. With

	S&P TSX	FTSE Universe Bond Index
appreciation rate, m_i	0.035	0.018
volatility, σ_i	0.066	0.027
initial price, S_0^i	100	100

Table 6.3: Model Parameters Estimated for the two indices using period of November 06, 2017 to September 21, 2018.

rest of the model parameters are the same as in the previous example. Namely, $\mu = 0.01$ which gives a probability of default, $P(\tau < T) = 0.1393$, the survival probability, ${}_Tp_x = 0.970$ and interest rate, $r = 0$.

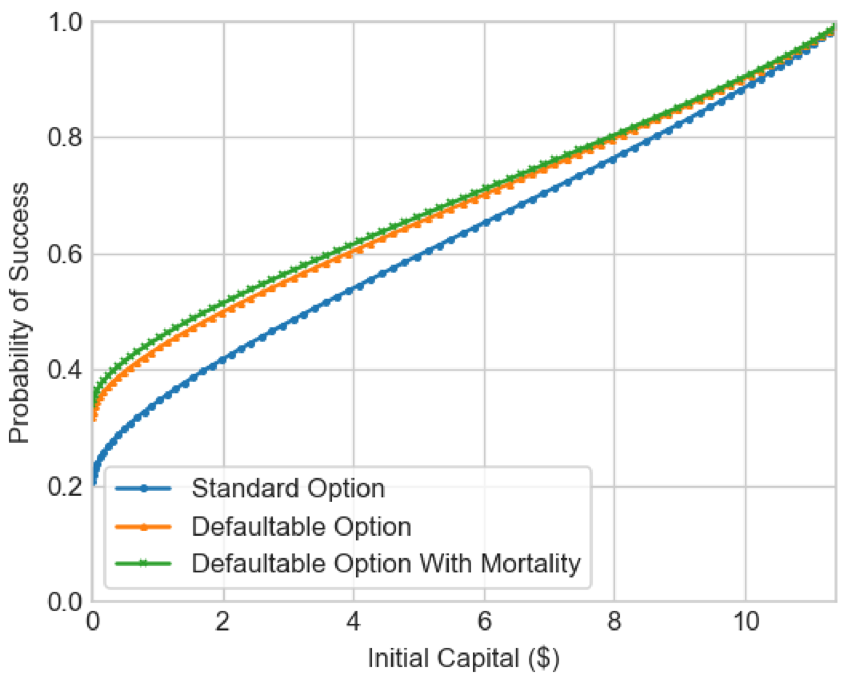

Figure 6.4: Probability of Success vs. Initial Capital for $(S_T^1 - S_T^2)^+$, $(S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}$ and $(S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$

In Figure 6.4 presents the results of the derived model for the two risky assets. The default and mortality free option with payoff $(S_T^1 - S_T^2)^+$ has the lowest probability of a successful hedge using quantile hedging. While as before, the model with both default and mortality has the highest success chance.

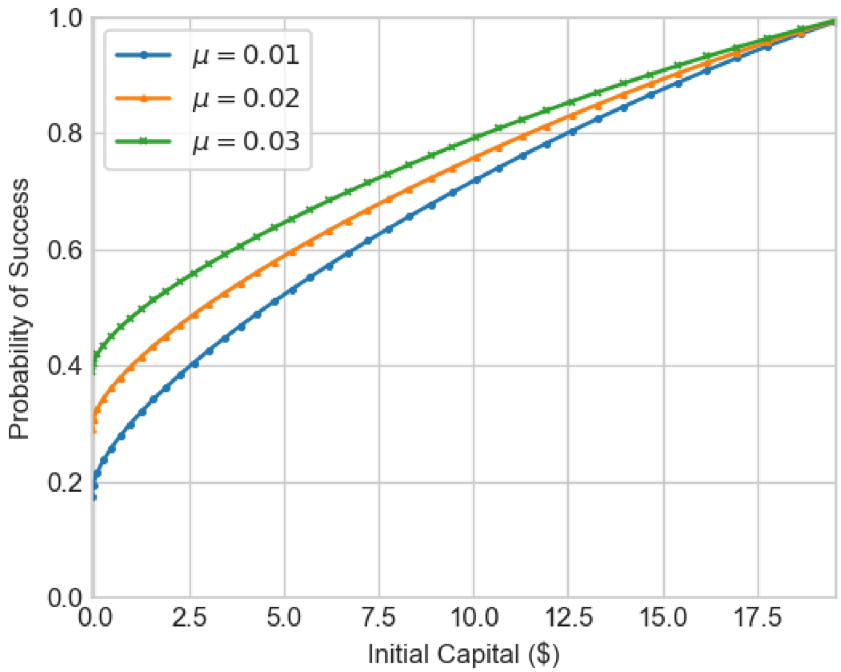

Figure 6.5: Probability of Success versus Initial Capital for $(S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$ with $\mu = 0.01,\ 0.02,$ and 0.03 or $P(\tau < T) = 0.1393, 0.2592, 0.3624$

Likewise Figure 6.5 contains result that is similar to Figure 6.2. This is as one would expect as all that has changed in the construction of the model is the payoff function of the financial contract when we replaced the strike price K with the value of the second risky asset S_T^2. Namely, we go from

$$(S_T - K)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$$

to

$$(S_T^1 - S_T^2)^+\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T(x)>T\}}$$

which results in few changes as far as the numerical results are concerned.

Using 4.4 we estimate the lower bound of survival probabilities for a given amount of shortfall risk ϵ and Initial Capital, see figure 6.6. We find that for a given amount of risk we see that the more risk we accept and the more Initial Capital that is available for hedging the more greater the lower bound survival probability.

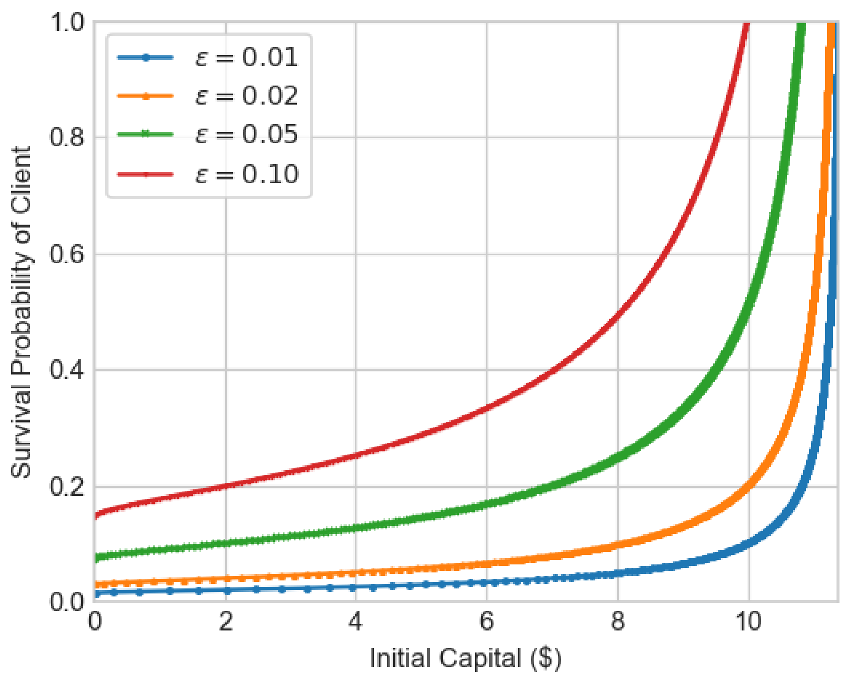

Figure 6.6: Probability of Survival vs. Initial Capital for $(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{\{T(x) > T\}}$ with $\epsilon = 0.01, 0.02, 0.05, 0.10$, or $P(\tau < T) = 0.1393, 0.2592, 0.3624$

We can also estimate the minimum age of a satisfactory customer as we did in the one risky asset example see Table 6.4. The results are very much similar with small differences due to the different dynamics of the assets. As expected the higher the amount of initial capital available the more freedom the insurance company has in accepting new clients of different ages.

Shortfall Risk	Initial Capital	Survival Probability	Age of Client (x)
1%	\$11.25	50.0%	≥ 70
2%	\$11.00	52.4%	≥ 68
5%	\$10.50	71.67%	≥ 61
10%	\$9.50	79.58%	≥ 57

Table 6.4: Given an acceptable shortfall probability ϵ and available Initial Capital we find the maximal survival probability of the client and the minimum acceptable age in the two risky asset scenario.

Chapter 7

Conclusion

7.1 Summary

Equity linked life insurance contracts have become a large part of mathematical finance research as well a growing part of the insurance industry. Thus, having an realistic model for these contracts and an accurate method of pricing and hedging is important.

In this thesis, we chose to augment a general class of contracts to take into account the default of the insurer within a multi-dimensional Black-Scholes market. The inclusion of a default time made our model incomplete. However, after applying the superhedging technique to our market we saw that the cost of hedging such an option is not realistic. Thus, we applied the Quantile Hedging technique to our problem to see if we could reduce the cost of hedging if we allowed a specified amount of shortfall risk. Re-framing the problem into terms of a Neyman Pearson type problem and using Convex Duality approach of finding optimal random tests we came up with a solution.

We then augmented this model to take into account the mortality of clients using the Brennan and Scwartz approach. We used the preceding results to find closed form solutions for the initial capital requirements as well as shortfall probability. Using these formulas we were able to isolate what the required survival probability should be for perspective client given some hedging capital constraint and a lower bound on the probability of a successful hedge.

In the last chapter we explored some numerical examples of the two sets of results. These results confirmed our intuitions as how the survival probability, default probability and acceptable shortfall probability effect the initial capital requirements. We saw that the higher the shortfall risk the company was willing to accept the lower the capital requirements. Similarly the higher the default probability, and the lower the survival probability led to lower initial capital requirement.

7.2 Future Work

We solved the problem with using a standard Black-Scholes market where the default of the insurer meant that the client receives no payout. Thus, we see two possible extensions to this thesis. One, consider a more general market framework such as the jump diffusion market considered in [12]. We believe this should be possible in the case one or two risky assets as this already been done without default. Second, we can consider the case of non-zero recovery rate. In this case the client will receive some reduced portion of the payoff instead of zero. This presents some complications as solving the dual problem becomes more tricky. In Nakano's paper, whose results heavily drew upon, presented a solution to this problem with a restricted class of portfolios and one risky asset. At this point we are not sure whether a solution is possible and how restrictive such a solution would be, but this problem warrants some exploration. Of the two, this is less important practically as depending on the size contract a client would receive pennies on the dollar in the case of a default event as most insurance companies have millions of such clients.

Another extension would be to a different type of efficient hedging methodology such minimizing the expected loss rather than the probability of a loss. In the former case we would consider the magnitude of a loss rather than just the existence of a loss. One could also consider a type of hedging which attempts to minimize some loss function associated with such a contract. An example would be mean-variance hedging which minimizes the squared difference between the terminal value of the hedging portfolio and the payoff of the contract.

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