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On a Generalization of the Gelfand Transform to Non-Commutative Banach Algebras

by

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To my parents

Abstract

A Gelfand theory for an arbitrary Banach algebra A is a pair $(\mathcal{G}, \mathfrak{A})$, such that: $\mathfrak{A}$ is a C^*-algebra and $\mathcal{G} : A \to \mathfrak{A}$ is an algebra homomorphism; $\mathcal{G}$ induces a bijection between the set of maximal modular left ideals of A and the set of maximal modular left ideals of $\mathfrak{A}$; and for every maximal modular left ideal L of $\mathfrak{A}$, the map $\mathcal{G}_L : A/\mathcal{G}^{-1}(L) \to \mathfrak{A}/L$ induced by $\mathcal{G}$ has dense range. We prove that if A is a postliminal C^*-algebra with Gelfand theory $(\mathcal{G}, \mathfrak{A})$, then no proper C^*-subalgebra of $\mathfrak{A}$ contains $\mathcal{G}A$. We also show that if J is an ideal of a Banach algebra A such that A/J and J both have Gelfand theories, then A also has a Gelfand theory if we impose some conditions on J and on its Gelfand theory.

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Introduction

The study of Banach algebras and their representation theory is one the main topics of functional analysis. The application of techniques from analysis, algebra and topology has allowed researchers in this field to obtain deep and sometimes surprising results.

A Banach algebra A is a Banach space that is also an algebra (i.e., it is a ring in which the multiplication is compatible with the vector space operations) such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. If the multiplication is commutative, then we say that A is commutative.

The idea of the representation theory for Banach algebras is to characterize an algebra as a subset of another algebra with more structure and whose properties we know fairly well and for which we have more tools available (e.g., the algebra of continuous functions on a compact Hausdorff space or the algebra of bounded operators on a Hilbert space).

One of the reasons commutative Banach algebras are important in analysis is because they have a very nice representation theory. The Gelfand transform establishes an algebra homomorphism between any commutative Banach algebra and $\mathcal{C}_0(K)$, the set of continuous complex-valued functions on a locally compact Hausdorff space that vanish at infinity. This set K is just the set of multiplicative linear functionals on A, i.e., the set of algebra homomorphisms from A to $\mathbb{C}$, which is a subset of the dual space of A.

For several reasons, the concept of Gelfand transform makes sense when the Banach algebra is commutative. The possibility of defining a notion of Gelfand transform in a non-commutative setting is studied in a paper by R. Choukri, E. H. Illoussamen and V. Runde [CIR02].

In that paper, the authors give an axiomatic definition of Gelfand theory that, in the commutative case, is not only consistent with the classical one, but also characterizes it. Furthermore, several properties of the Gelfand theory are established, as well as some existence and uniqueness results.

In this work, we review some of the main results of [CIR02]. In addition we examine several of the open problems in the paper. We answer one of the questions posted in [CIR02] and also find a solution for two other related problems under particular conditions.

Chapter 1

Introduction to Banach Algebras

In this first chapter we cover some basic definitions and classical results about Banach algebras and their representation theory.

Recall that a **Banach space** is a normed linear space that is complete, i.e., in which every Cauchy sequence converges.

Definition 1.1 An **algebra** A is a ring that is also a vector space under the addition of the ring and such that

$$\alpha(ab) = a(\alpha b) = (\alpha a)b \quad \text{for all}\, a, b \in A\, \text{and any scalar } \alpha.$$

If the multiplication of the algebra is associative, we say that A is associative. Throughout this work all algebras are assumed to be associative and to be complex vector spaces.

An algebra A is **commutative** if the multiplication is commutative.

Definition 1.2 An algebra A is **unital** if there exists a (necessarily unique) element $I \in A$ such that $Ia = aI = a$, for all $a \in A$. In that case we say that I is the identity of A. An algebra without a unit is **non-unital**. If A is a unital algebra, an element $a \in A$ is called **invertible** if there exists an element $b \in A$ such that $ab = ba = I$; in this case we say that b is the inverse of a.

When it is not clear by the context, we will indicate by a subindex the algebra or subalgebra on which I acts as the identity, e.g., I_A.

Given two unital algebras A and B, an algebra homomorphism $\varphi : A \to B$ is said to be **unital** if $\varphi(I_A) = I_B$.

Definition 1.3 We say that a subset B of an algebra A is a **subalgebra** if it is closed under all the operations (addition, multiplication, and multiplication by a scalar). If an algebra A has a unit, a given subalgebra is **unital** if it contains the unit of A.

Definition 1.4 Let A be an algebra. A subalgebra J of A is a **left ideal** if $aJ \in J$, for all $a \in A$. Similarly we say that J is a **right ideal** if $Ja \in J$, for all $a \in A$. If J is both a left and right ideal, then we say that J is an **ideal** of A. An ideal is **maximal** if it is not contained in any other proper ideal.

If an algebra A is also a normed space such that the norm is sub-multiplicative, i.e., $\|ab\| \leq \|a\|\|b\|$, for all $a, b \in A$, then we say that A is a **normed algebra**.

Definition 1.5 A **Banach algebra** is a normed algebra that considered as linear space is a Banach space over $\mathbb{C}$.

Example 1.6 The first obvious example of a Banach algebra is the set of complex numbers, $\mathbb{C}$, with the usual addition and multiplication, and with norm given by the modulus.

Example 1.7 Let K be a compact Hausdorff topological space. Let $\mathcal{C}(K)$ denote the set of complex-valued continuous functions on K. Endow this space with the ℓ^∞-norm:

$$\|f\| = \sup_{x \in K} |f(x)|, \text{ for all } f \in \mathcal{C}(K) \tag{1.1}$$

then $\mathcal{C}(K)$ becomes a commutative Banach algebra. This algebra is unital.

Example 1.8 Let K be a locally compact (but not necessarily compact)

Hausdorff toplogical space. Let $\mathcal{C}_0(K)$ denote the set of complex-valued functions that *vanish at infinity*, that is, $f \in \mathcal{C}_0(K)$ if given any $\epsilon > 0$ there exists a compact set $A \subset K$ such that $|f(x)| < \epsilon$, $x \in K \setminus A$. This space with the norm and operations defined in the previous example is also a Banach algebra.

The following example is very relevant for this work:

Example 1.9 Let E be a Banach space and let $\mathcal{B}(E)$ denote the set of bounded linear operators from E to E. Then $\mathcal{B}(E)$ is an algebra with pointwise addition, and multiplication given by composition of operators: $TQ = T \circ Q$, for $T, Q \in \mathcal{B}(E)$.

Furthermore, if we endow $\mathcal{B}(E)$ with the operator norm:

$$\|T\| = \sup_{\substack{x \in E \\ \|x\| \leq 1}} \|Tx\|, \text{ for all } T \in \mathcal{B}(E) \tag{1.2}$$

then $\mathcal{B}(E)$ becomes a (unital) Banach algebra. This Banach algebra is commutative if and only if E has dimension 1.

An immediate consequence of the sub-multiplicativity of the norm of Banach algebras is that multiplication is jointly continuous:

Theorem 1.10 *For a Banach algebra A, multiplication is jointly continuous.*

Proof. Let $a, b \in A$, let $(a_n)_{n=1}^{\infty}$ and $(b_m)_{m=1}^{\infty}$ be sequences in A such that $a_n \to a$ and $b_m \to b$. Then

$$\|a_n b_m - ab\| = \|a_n(b_m - b) + (a_n - a)b\| \leq \|a_n\|\|b_m - b\| + \|a_n - a\|\|b\|$$

Since $(a_n)_{n=1}^{\infty}$ and $(b_m)_{m=1}^{\infty}$ converge, they are bounded, so we have that

$$\lim_{n,m \to \infty} \|a_n b_m - ab\| = 0$$

holds. ∎

The proof of the next theorem, as well as most of the classical results regarding Banach algebras in this chapter, follows the treatment given in [Pal94].

Theorem 1.11 *If J is a closed ideal of a normed algebra A, then the quotient A/J (endowed with the usual quotient norm) is a normed algebra too. If A is a Banach algebra, then the quotient is a Banach algebra, too.*

Proof. The first part of the theorem is straightforward. Since J is a two sided ideal, clearly addition and multiplication are well-defined in the quotient space and the latter is closed under these operations.

For the second part, if A is a Banach space and J is closed, A/J with the quotient norm is a Banach space. So all we need to verify is that this norm is sub-multiplicative. Let $a, b \in A$:

$$\begin{aligned}\|(a+J)(b+J)\| = \|ab+J\| &= \inf_{c\in J}\|ab+c\| \\ &\leq \inf_{x,y\in J}\|ab+ax+yb+yx\| = \inf_{x,y\in J}\|(a+y)(b+x)\| \\ &\leq \inf_{x,y\in J}\|(a+y)\|\|(b+x)\| = \|a+J\|\|b+J\|. \qquad \blacksquare\end{aligned}$$

A non-unital Banach algebra can be embedded as an ideal into a unital Banach algebra:

Theorem 1.12 *Let A be a Banach algebra without unit. Define $A^{\#} = A \oplus \mathbb{C}$, with addition given by $a_1 \oplus \lambda_1 + a_2 \oplus \lambda_2 := (a_1 + a_2) \oplus (\lambda_1 + \lambda_2)$ and multiplication defined by $(a_1 \oplus \lambda_1)(a_2 \oplus \lambda_2) = (a_1a_2 + \lambda_1 a_2 + \lambda_2 a_1) \oplus \lambda_1\lambda_2$, for all $a_1, a_2 \in A$, $\lambda_1, \lambda_2 \in \mathbb{C}$. Then $A^{\#}$ with norm $\|a \oplus \lambda\| := \|a\| + |\lambda|$ is a unital Banach algebra with unit $0 \oplus 1$.*

Proof. The fact that $A^{\#}$ with the operations defined is an algebra with unit $0 \oplus 1$ is straightforward. Also it is clearly a Banach space (being the direct sum of two Banach spaces), so all that remains to be proven is that the norm is sub-multiplicative. Thus let $a, b \in A$, $\alpha, \beta \in \mathbb{C}$. Then

$$\|(a \oplus \alpha)(b \oplus \beta)\| = \|(ab + \beta a + \alpha b) \oplus \alpha\beta\| \leq \|ab\| + |\beta|\|a\| + |\alpha|\|b\| + |\alpha\beta|$$
$$= (\|a\| + |\alpha|)(\|b\| + |\beta|) = \|a \oplus \alpha\|\|b \oplus \beta\|.$$

Whence, $A^{\#}$ is a Banach algebra. Moreover, it is easy to see that $A \oplus 0$ is an ideal of $A^{\#}$ of codimension 1. ■

Definition 1.13 Let A be an algebra. The **quasi-product** is the map $(a, b) \mapsto a \circ b$ from $A \times A$ onto A given by $a \circ b = a + b - ab$.

Definition 1.14 Let A be an algebra and let a be an element of A. We say that $b \in A$ is a right (left) quasi-inverse for a if $a \circ b = 0$ (resp. $b \circ a = 0$). If $a \circ b = b \circ a = 0$ we say that a is quasi-invertible with quasi-inverse b. We will denote by $\mathcal{Q}(A)$ the set of quasi-invertible elements of A.

Definition 1.15 Let A be a non-unital algebra and let a be an element of A. The **spectrum** of a is defined as:

$$\sigma(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}a \notin \mathcal{Q}(A)\} \cup \{0\}. \tag{1.3}$$

When the given algebra A is unital the spectrum of an element $a \in A$ is given by

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda I \text{ is not invertible}\}. \tag{1.4}$$

Remark 1.16 If A is a non-unital algebra, the definition of spectrum based on quasi-invertibility (1.3) is consistent with the definition based on invertibility (1.4) when the calculations are made in $A^{\#}$. To see it, just observe that if $\lambda \in \mathbb{C}$ is not zero and $\lambda^{-1}a$ has a quasi-inverse b, then

$$(a - \lambda I)(\lambda^{-1}b - \lambda^{-1}I) = (\lambda^{-1}b - \lambda^{-1}I)(a - \lambda I)$$
$$= \lambda^{-1}ba - b - \lambda^{-1}a + I = I - b \circ \lambda^{-1}a = I.$$

On the other hand, if c is an inverse for $(a-\lambda I)$ we have

$$\begin{aligned} ac\circ\lambda^{-1}a=\lambda^{-1}a\circ ac &= \lambda^{-1}a+ac-\lambda^{-1}a^2c=\lambda^{-1}a+a(I-\lambda^{-1}a)c \\ &= \lambda^{-1}a-\lambda^{-1}a(a-\lambda I)c=0 \end{aligned}$$

and $\lambda^{-1}a$ is quasi-invertible. Therefore, for simplicity, when dealing with the spectrum sometimes we will assume that the algebra is unital.

Theorem 1.17 *Let A be an algebra. Then:*

(a) For any $a, b \in A$, we have

$$\sigma(ab)\cup 0=\sigma(ba)\cup 0.$$

(b) Let B be an algebra. If A is non-unital, then for any homomorphism $\varphi : A \to B$ we have:

$$\sigma_B(\varphi(a))\subseteq\sigma_A(a), \quad \textit{for all } a\in A.$$

Proof. (a) Assume first that A is non-unital. Let $\lambda\in\mathbb{C}\setminus\{0\}$. Suppose that c is a quasi-inverse for $\lambda^{-1}ab$, then

$$\lambda^{-1}ab\circ c=0=c\circ\lambda^{-1}ab,$$

therefore $\lambda^{-1}b(ca-a)$ is a quasi-inverse for $\lambda^{-1}ba$. Indeed:

$$\begin{aligned} \lambda^{-1}ba\circ(\lambda^{-1}b(ca-a)) &= \lambda^{-1}ba+\lambda^{-1}b(ca-a)-\lambda^{-2}b(abc)a+\lambda^{-2}b(ab)a \\ &= \lambda^{-1}ba+\lambda^{-1}bca-\lambda^{-1}ba+\lambda^{-2}b(ab-abc)a \\ &= \lambda^{-1}b(c+\lambda^{-1}ab-\lambda^{-1}abc)a \\ &= \lambda^{-1}b(\lambda^{-1}ab\circ c)a=0. \end{aligned}$$

Similarly, $\lambda^{-1}b(ca-a)\circ\lambda^{-1}ba=0$. If A is unital, $\lambda\in\mathbb{C}\setminus\{0\}$, and $(ab-\lambda I)$

has an inverse c, then

$$\begin{aligned}(ba-\lambda I)(\lambda^{-1}bca-\lambda^{-1}I) &= \lambda^{-1}babca-\lambda^{-1}ba-bca+I\\ &= b(ab-\lambda I)\lambda^{-1}ca+I-\lambda^{-1}ba\\ &= \lambda^{-1}ba+I-\lambda^{-1}ba=I.\end{aligned}$$

Therefore, $(ba-\lambda I)$ is invertible too. Thus in any case we have $\sigma(ab)\cup 0=\sigma(ba)\cup 0$.

(b) It is clear that if $\varphi(\lambda^{-1}a)$ is not quasi-invertible in B for $\lambda\neq 0$, then it is not quasi-invertible in A, and the claim holds. ■

It is not difficult to show that the inclusion in part b) of the previous theorem is true if we assume that A and B are unital and φ is a unital homomorphism.

Definition 1.18 Let A be a Banach algebra and let $a\in A$. The **spectral radius** of a is defined as

$$\rho(a)=\sup_{\lambda\in\sigma(a)}|\lambda|. \tag{1.5}$$

Since the spectrum of any element of a Banach algebra is a non-empty and compact set in $\mathbb{C}$ [Kan03, Theorem 7.6], the spectral radius is always well-defined.

In Banach algebras, the analytic and the algebraic structures are connected by the spectral radius, as can be seen in the following result:

Theorem 1.19 *Let A be a Banach algebra and let $a\in A$. Then $\rho(a)\leq\|a\|$.*

Proof. For simplicity, assume that A is unital. Let $\lambda\in\mathbb{C}$ be such that $|\lambda|>\|a\|$. Observe that $I-\lambda^{-1}a$ is invertible with inverse $b:=\sum_{n=0}^{\infty}(\lambda^{-1}a)^n$. First, b is well-defined since the series is absolutely convergent, hence convergent. Furthermore,

$$-\lambda^{-1}b(a-\lambda I)=b(I-\lambda^{-1}a)=(I-\lambda^{-1}a)b=I.$$

Therefore, $-\lambda^{-1}b$ is the inverse of $a - \lambda I$ and λ is not in the spectrum of a. ■

In fact, there is an explicit formula for the spectral radius of an element of a Banach algebra, the so called **Beurling-Gelfand spectral radius formula** [Kan03, Theorem 7.9]:

Theorem 1.20 *Let A be a Banach algebra and let $a \in A$. Then*

$$\lim_n \|a^n\|^{\frac{1}{n}} = \rho(a). \tag{1.6}$$

One of the most useful features of algebras is functional calculus in its many forms. Next we will see one of the most modest results on functional calculus:

Theorem 1.21 *Let A be a unital algebra and let $a \in A$ be such that $\sigma(a)$ is not empty. If f is any rational continuous function on $\sigma(a)$, then $f(a)$ is well-defined in A, and the map*

$$\varphi : \mathcal{C}(\sigma(a)) \to A, \quad f \mapsto f(a)$$

is a unital homomorphism that sends the identity function on $\sigma(a)$ to a. Moreover this homomorphism is unique and the following holds:

$$\sigma(f(a)) = f(\sigma(a)).$$

Proof. Assume $\sigma(a) \neq \emptyset$; if f is constant, say $f(z) = \lambda_0$, then observe that $\varphi(f) = \lambda_0 I$, thus $\sigma(f(a)) = \{\lambda_0\} = f(\sigma(a))$. Therefore we can assume that f is a non-constant rational function such that $f = \dfrac{p}{q}$, where p, q are two polynomials with no common factors.

Observe that $p(a)$ and $q(a)$ are well-defined polynomials in the commutative algebra generated by a. Now for any given $\lambda_0 \in \mathbb{C}$, let

$$q(\lambda) = \alpha(\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Also let

$$\lambda_0 q(\lambda) - p(\lambda) = \beta(\mu_1 - \lambda)(\mu_2 - \lambda)\cdots(\mu_m - \lambda).$$

Note that $\alpha \neq 0$ ($q \neq 0$) and $\beta \neq 0$ ($f = \dfrac{p}{q}$ is not constant). Moreover, since $q \neq 0$ on $\sigma(a)$, then $q(a) = \alpha(\lambda_1 - a) \cdots (\lambda_n - a)$ is invertible, and we can write $f(a) := p(a)q(a)^{-1}$, which is well-defined.

As p and q have no common factor, we have:

$$\lambda_0 I - f(a) = \beta(\mu_1 - \lambda)...(\mu_m - \lambda)(\alpha(\lambda_1 - \lambda)...(\lambda_n - \lambda))^{-1},$$

which is invertible on $\sigma(a)$, unless some $\mu_i \in \sigma(a)$. But since the μ_i are the roots of the polynomial $\lambda_0 q(\lambda) - p(\lambda)$, λ_0 is in $\sigma(f(a))$ if and only if an element of $\sigma(a)$ satisfies the equation $\lambda_0 q(\lambda) - p(\lambda) = 0$, i.e., $\lambda_0 \in f(\sigma(a))$. ■

1.1 Gelfand Theory for Commutative Banach Algebras

It is well-known that any normed space can be identified with a subspace of the space of continuous functions on a compact Hausdorff space (this is a consequence of the Banach-Alaoglu theorem). The Gelfand transform is a very elegant construction that provides an identification somehow parallel to the one just mentioned. It allows us to represent a commutative Banach algebra as a subalgebra of the algebra of continuous functions on a compact Hausdorff space if the algebra is unital, or as a subalgebra of the algebra of continuous functions that *vanish at infinity* on a locally compact Hausdorff space if the algebra is not unital.

Definition 1.22 An ideal J of an algebra A is called **modular** if there exists an element $e \in A$ such that $ae - a, ea - a \in J$, for all $a \in A$. In this case, e is a unit for A/J, and is called a modular unit for J.

A left ideal J is called **modular left ideal** if there exists an element $e \in A$ such that $ae - a \in J$, for all $a \in A$; in this case we say that e is a right modular unit for J. Similarly, we define the notions of a **modular right ideal** and of a **left modular unit**.

Definition 1.23 Let A be a Banach algebra, a **character** is a non-zero algebra homomorphism from A to $\mathbb{C}$, i.e., is a multiplicative linear functional. It follows from multiplicativity that if A is unital, then $\varphi(I) = 1$ for every character φ of A.

Remark 1.24 Given a commutative Banach algebra A, its characters are automatically continuous. In fact, we will see that characters have norm at most one and lie in the closed unit ball of the dual space of A (denoted by $B_1(A^*)$).

Also in the case of commutative Banach algebras, there is a relation between the set of characters and the set of maximal modular ideals: a subset J of a commutative Banach algebra A is a maximal modular ideal if and only if it is the kernel of a character.

Definition 1.25 Let A be a commutative Banach algebra. If we endow the set of characters with the topology inherited from the weak-* topology, we obtain a Hausdorff topological space. In the sequel, we will refer to this topological space as the **character space** of A and will denote it by $\Phi(A)$.

Theorem 1.26 *Let A be a commutative Banach algebra, and let $a \in A$. The map*

$$\tau : A \to \mathcal{C}(\Phi(A)), \quad a \mapsto \widehat{a},$$

*where $\widehat{a}$ is the evaluation function $\varphi \mapsto \varphi(a)$, $\varphi \in \Phi(A)$, is a homomorphism from A into $\mathcal{C}(\Phi(A))$. This homomorphism is called the **Gelfand transform** of A.*

Proof. Since $\Phi(A)$ has the topology induced by the weak-* topology on A^*, it is clear that the functions $\widehat{a}$ are continuous on $\Phi(A)$. Furthermore, since characters preserve addition and multiplication, we have:

$$\widehat{a+b}(\varphi) = \varphi(a+b) = \varphi(a) + \varphi(b), \quad a, b \in A,\ \varphi \in \Phi(A)$$
$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in A,\ \varphi \in \Phi(A).$$

Thus τ is a homomorphism from A into $\mathcal{C}(\Phi(A))$. ■

Moreover, for any given element $a \in A$, the spectrum $\sigma(a)$ is just the range of the evaluation map $\widehat{a}$ over $\Phi(A)$.

In fact, if A is a unital Banach algebra, φ is character of A, and $a \in A$, then $\varphi(a) \in \sigma(a)$, otherwise $0 = \varphi(a - \varphi(a)I)\varphi((a - \varphi(a)I)^{-1}) = 1$, a contradiction. Conversely, if $\lambda \in \sigma(a)$, consider the ideal $M := A(a - \lambda I)$, which is contained in a maximal modular ideal M', by Zorn's lemma. The correspondence between maximal modular ideals and characters asserts that there exists a character ϕ of A such that $\ker(\phi) = M'$, thus $(a - \lambda I) \in \ker(\phi)$, implying that $\phi(a) = \lambda$. We have established that $\sigma(a) = \widehat{a}(\Phi(A))$.

From the previous argument, and using the fact that the spectral radius of an element of a Banach algebra is bounded by its norm, we can see that characters are continuous with norm at most 1.

Theorem 1.27 *Let A be a unital commutative Banach algebra, then $\Phi(A)$ is a compact Hausdorff topological space. If A is non-unital, then $\Phi(A)$ is a locally compact Hausdorff space such that the one-point compactification is given by $\Phi(A) \cup \{0\}$.*

Proof. First note that since $\varphi(a) \in \sigma(a), \forall a \in A$, then $|\varphi(a)| \leq \|a\|$, so $\|\varphi\| = 1$, for all $\varphi \in \Phi(A)$. Hence $\Phi(A) \subseteq B_1(A^*)$.

Let $(\phi_\alpha)_{\alpha \in \Delta}$ be a net in $\Phi(A)$ such that $\phi_\alpha \to \phi$. For $a, b \in A$, we have:

$$\phi(ab) = \lim_\alpha \phi_\alpha(ab) = \lim_\alpha \phi_\alpha(a) \lim_\alpha \phi_\alpha(b) = \phi(a)\phi(b).$$

Thus $\phi \in \Phi(A)$ or $\phi = 0$, implying that $\Phi(A) \cup \{0\}$ is a closed subset of the closed unit ball of A^*, and is compact and Hausdorff. If A is unital, then necessarily $\phi(I) = \lim_{\alpha} \phi_\alpha(I) = 1$, hence $\phi \in \Phi(A)$ and $\Phi(A)$ is compact. If A is non-unital then $\Phi(A) \cup \{0\}$ is compact, and so $\Phi(A)$ is locally compact. ■

Now we are ready to state the main properties of the Gelfand transform:

Theorem 1.28 *Let A be a commutative Banach algebra and let τ be the Gelfand transform for A. Then:*

(a) If A is unital, then $\tau(A) \subseteq \mathcal{C}(\Phi(A))$. If A is non-unital, then $\tau(A) \subseteq \mathcal{C}_0(\Phi(A))$;

(b) τ is contractive;

(c) $\tau(A)$ strongly separates points of $\Phi(A)$, i.e., for every $\phi, \psi \in \Phi(A)$ there is some $a \in A$ such that $\widehat{a}(\phi) \neq \widehat{a}(\psi)$ and $\tau(A)$ does not vanish at any point of $\Phi(A)$.

Proof. (a) If A is unital, $\Phi(A)$ is a closed subset of the closed unit ball of A^*, and so is compact, whence $\tau(A) \subseteq \mathcal{C}(\Phi(A))$. If A is non-unital, then $\Phi(A) \cup \{0\}$ provides the one-point compactification of $\Phi(A)$. Thus if (φ_α) is a net in $\Phi(A)$ such that $\varphi_\alpha \to 0$, then for any $a \in A$, $\varphi_\alpha(a) \to 0$ and $\tau(a) \in \mathcal{C}_0(\Phi(A))$.

(b) Since $\|\varphi\| \leq 1$ for all $\varphi \in \Phi(A)$, $\|\tau(a)\| = \sup_{\varphi \in \Phi(A)} |\varphi(a)| \leq \|a\|$, and τ is contractive.

(c) Assume that $\phi, \psi \in \Phi(A)$ with $\phi \neq \psi$, so there exists some $a \in A$ such that $\phi(a) \neq \psi(a)$, thus $\widehat{a}(\phi) \neq \widehat{a}(\psi)$. Finally, if there is a character $\varphi \in \Phi(A)$ such that $\tau(A)$ vanishes at φ, then $\varphi(a) = 0$ for all $a \in A$, which is not possible by the definition of character. ■

1.2 Representation of Banach Algebras

In this section we present some basic concepts and results that we will need about the representation theory of Banach algebras.

Definition 1.29 Let A be a Banach algebra, and let E be a linear space. A **representation** of A on E is a homomorphism $\pi : A \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ denotes the set of linear operators on E.

Definition 1.30 Let A be an algebra, let E, F be linear spaces, and let π_1, π_2 be representations of A on E and F, respectively. We say that π_1 and π_2 are **equivalent** if there exists an isomorphism $\varphi : E \to F$ such that $\varphi^{-1} \circ \pi_2(a) \circ \varphi = \pi_1(a)$, for all $a \in A$.

Definition 1.31 Let A be an algebra, let E be a linear space, and let π be a representation of A on E. A subspace F of E is called **invariant** under $\pi(A)$ if $\pi(A)F \subseteq F$.

A representation π of an algebra A on a linear space E is said to be **irreducible** if the only subspaces of E invariant under $\pi(A)$ are $\{0\}$ and E.

Definition 1.32 Let A be an algebra. A representation π of A on a normed space E is said to be **non-degenerate** if $\pi(A)E$ is dense in E.

Definition 1.33 Let A be an algebra. A representation π of A on a normed space E is said to be **normed** if for every $a \in A$, $\pi(a) \in \mathcal{B}(E)$.

Definition 1.34 Let A be an algebra, let E be a linear space, and let π be a representation of A on E. A vector $x \in E$ is called **cyclic** if $\pi(A)x = E$. A representation π of an algebra A on a linear space E is **cyclic** if there exists a cyclic vector for π on E.

Clearly, every cyclic representation is non-degenerate. Not so obvious is the fact that every irreducible representation is cyclic. Furthermore, in an irreducible representation every non-zero vector is cyclic.

Indeed, let A be an algebra and let π be an irreducible representation of A on a linear space E. Then the subspace $F := \{x \in A : \pi(A)x = 0\}$ is invariant under the action of $\pi(A)$. Since π is irreducible (and not trivial), F has to be 0. Therefore, for any $x \in E \setminus \{0\}$ the set $\pi(A)x$ is not 0 and is invariant, so must be all of E.

The converse is also true: if A is an algebra and π is a representation of A on a linear space E such that every non-zero vector is cyclic, then π is irreducible [Pal94, Theorem 4.1.3].

Definition 1.35 A representation π of an algebra A on a linear space E is **faithful** if the kernel of π is trivial.

Irreducible representations of algebras can be completely characterized by the set of maximal modular left ideals: given an algebra A and a maximal modular left ideal L, the representation π of A on $\mathcal{L}(A/L)$, $a \mapsto \pi(a)$, where $\pi(a)\,(x+L) = ax + L$, is an irreducible representation, and any irreducible representation of A is equivalent to a representation of this type [Pal94, Theorem 4.1.3].

The relation between irreducible representations and maximal modular left ideals has another expression. Let A be a Banach algebra, let L be a maximal modular left ideal of A, and let π_L be the irreducible representation of A on A/L; the kernel of π_L is an ideal. We call such an ideal a **primitive ideal**. We denote by Π_A the set of primitive ideals of A.

The set of primitive ideals of a Banach algebra A is related to the set of its maximal modular left ideals in the sense that any primitive ideal M is the kernel of an irreducible representation of A on A/L for some maximal modular left ideal L. Moreover, M is the largest ideal of A contained in L. Also, it can be shown that every primitive ideal is of the form [Pal94, Theorem 4.1.8]

$$M := \{a \in A : aA \subseteq L\} \tag{1.7}$$

and we will refer to M as the primitive ideal of A corresponding to the maximal modular left ideal L.

Chapter 2

C^*-Algebras

In this chapter we review some of the most fundamental results regarding the most important class of Banach algebras: C^*-algebras. First, we will introduce some preliminary definitions.

The material presented in this chapter can be found in any textbook on C^*-algebras; we follow in most of the proofs the arguments in [Mur90].

Definition 2.1 Let A be an algebra. An **involution** is a function from A to A denoted by * that is conjugate linear and anti-multiplicative, i.e.

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^* \tag{2.1}$$

for all $a, b \in A$, $\alpha, \beta \in \mathbb{C}$, and such that $a^{**} = a$.

Definition 2.2 A **Banach** ***-algebra** is a Banach algebra endowed with an involution.

When we put an extra condition on a Banach *-algebra we obtain a C^*-algebra:

Definition 2.3 A C^***-algebra** is a Banach *-algebra such that the norm satisfies the so-called C^*-condition:

$$\|a^* a\| = \|a\|^2, \text{ for all } a \in A. \tag{2.2}$$

This condition, which at first sight may not seem strong, endows C^*-algebras with a lot of structure. For example, as we well see, for commutative C^*-algebras the Gelfand transform is not only a contractive homomorphism, but also an isometric isomorphism from A onto $\mathcal{C}_0(\Phi_A)$. On the other hand, if A is a non-commutative C^*-algebra we can represent it isometrically on a subalgebra of bounded operators on some Hilbert space [Bla06]. This is very useful, as the algebra of bounded operators on a Hilbert space has great relevance in pure and applied mathematics.

Actually, the C^*-condition is equivalent to an apparently less restrictive condition:

Lemma 2.4 *Let A be a Banach *-algebra such that for all $a \in A$ we have $\|a\|^2 \leq \|a^*a\|$. Then A is a C^*-algebra.*

Proof. From $\|a\|^2 \leq \|a^*a\|$ it follows that $\|a\|^2 \leq \|a^*\|\|a\|$ and $\|a\| \leq \|a^*\|$, similarly $\|a^*\| \leq \|a\|$, so $\|a\| = \|a^*\|$. Whence

$$\|a\|^2 \leq \|a^*a\| \leq \|a\|\|a^*\| = \|a\|^2$$

and A is a C^*-algebra. ∎

Example 2.5 Again, a first trivial example of a C^*-algebra is $\mathbb{C}$, with involution given by complex conjugation.

Now we present the two most important examples of commutative C^*-algebra; in fact, as was already mentioned, every commutative C^*-algebra can be identified with one of these:

Example 2.6 Let K be a compact Hausdorff topological space. Then $\mathcal{C}(K)$ endowed with the ℓ^∞-norm becomes a commutative C^*-algebra, with pointwise addition and multiplication and involution given by $f^*(x) := \overline{f(x)}$.

Example 2.7 Let K be a locally compact Hausdorff toplogical space. Then $\mathcal{C}_0(K)$ with the norm and operations defined in the previous example is also a

C^*-algebra.

Next, we see an example of a non-commutative C^*-algebra:

Example 2.8 Let $\mathcal{H}$ be a Hilbert space. We already know that $\mathcal{B}(\mathcal{H})$ is a Banach algebra. Now, for any operator $T \in \mathcal{B}(\mathcal{H})$, let T^* be the adjoint operator, i.e., T^* is the unique operator such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x, y \in \mathcal{H}. \tag{2.3}$$

Using the properties of the inner product it is not difficult to verify that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. In fact, later we will see that every non-commutative C^*-algebra can be identified with a subalgebra of an algebra of this class.

The following consequence of the C^*-condition is already included in the argument of the proof of Lemma 2.4.

Theorem 2.9 *In a C^*-algebra the involution is isometric.*

Proof. Let A be a C^*-algebra and let $a \in A$, then by the C^*-condition:

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|, \text{ which implies } \|a\| \leq \|a^*\|.$$

Similarly,

$$\|a^*\|^2 = \|aa^*\| \leq \|a\|\|a^*\|, \text{ thus } \|a^*\| \leq \|a\|.$$

Hence, $\|a\| = \|a^*\|$ holds. ■

Definition 2.10 Let A be a C^*-algebra and let $a \in A$. We say that a is **normal** if $aa^* = a^*a$; we say that a is **self-adjoint** if $a^* = a$. The set of self-adjoint elements of A will be denoted by A_{sa}.

Normal and self-adjoint elements are very important in the theory of C^*-algebras; for now let us point out the following:

Remark 2.11 In a C^*-algebra any element can be written as a linear combination of two self-adjoint elements. Indeed, let A be a C^*-algebra and let $a \in A$, then $x := \frac{1}{2}(a + a^*)$ and $y := \frac{1}{2}i(a - a^*)$ are self-adjoint elements of A and $a = \frac{1}{2}x - \frac{1}{2}iy$.

It is also a very useful fact that self-adjoint elements in a C^*-algebra have real spectrum [Mur90, Theorem 2.1.8].

Theorem 2.12 *Let A be a C^*-algebra and let $a \in A$ be normal. Then $\|a\| = \rho(a)$ (the spectral radius of a).*

Proof. Let $a \in A$ be normal. We have:

$$\|a^2\|^2 = \|(a^2)^*(a^2)\| = \|a^*a^*aa\| = \|(a^*a)(a^*a)\| = \|a^*a\|^2 = \|a\|^4$$

and by induction

$$\|a^{2^n}\| = \|a\|^{2^n}.$$

Thus $\|a\| = \|a^{2^n}\|^{\frac{1}{2^n}}$ for all $n \in \mathbb{N}$, whence

$$\rho(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|. \qquad \blacksquare$$

Definition 2.13 Let A and B be C^*-algebras. An algebra homomorphism $\varphi : A \to B$ is a *-homomorphism if it preserves the involution, i.e., $\varphi(a^*) = \varphi(a)^*$.

Theorem 2.14 *Any *-homomorphism from a C^*-algebra A to a C^*-algebra B is contractive.*

Proof. Let A and B be C^*-algebras and let $\varphi : A \to B$ be a *-homomorphism. If A is not unital, we can extend φ to a homomorphism from $A^\#$ to $B^\#$ by letting $\widehat{\varphi}(0 \oplus 1) = 0 \oplus 1$. Since $\widehat{\varphi}(0 \oplus 1)$ is a unit for $\widehat{\varphi}(A^\#)$, we can assume without loss of generality that A and B are unital and that φ is a unital *-homomorphism.

In this setting, we have that $\sigma_B(\varphi(a)) \subseteq \sigma_A(a)$ and

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = \rho(\varphi(a^*a)) \leq \rho(a^*a) = \|a^*a\| = \|a\|^2$$

so that $\|\varphi(a)\| \leq \|a\|$. Thus any $^*-$homomorphism is contractive. ■

Definition 2.15 Let A be a Banach *-algebra. A representation π of A on a Hilbert space E is called a ***-representation** if $\pi(a^*) = \pi(a)^*$, where $\pi(a)^*$ denotes the adjoint of the operator $\pi(a)$.

As a straightforward consequence of the previous theorem, every *-representation of a C^*-algebra on a Hilbert space is automatically continuous (and contractive).

Another immediate consequence is the following:

Corollary 2.16 *Let A, B be C^*-algebras. If $\varphi : A \to B$ is a *-isomorphism, then φ is also an isometry.*

Proof. Since φ^{-1} is also a *-isomorphism, we have:

$$\|\varphi(a)\| \leq \|a\| = \|\varphi^{-1}(\varphi(a))\| \leq \|\varphi(a)\|.$$ ■

From the previous corollary, a C^*-algebra has a *unique* norm that satisfies the C^*-condition.

Now recall that for an arbitrary Banach algebra A, the unitization $A^\#$ is also a Banach algebra. It turns out that this unitization can be endowed with a (necessarily unique) norm that makes it a C^*-algebra:

Theorem 2.17 *Let A be a non-unital C^*-algebra and let $A^\#$ be the unitization of A. Endow $A^\#$ with the following norm:*

$$\|a \oplus \lambda\| := \sup\{\|ab + \lambda b\| : \|b\| \leq 1, b \in A\} \tag{2.4}$$

for all $a \in A, \lambda \in \mathbb{C}$. Then $A^\#$ with this norm is a C^*-algebra.

The proof of the previous theorem can be found in [DF88, Proposition VI.3.10].

In the setting of C^*-algebras we can say a lot more about the Gelfand transform:

Theorem 2.18 *Let A be a commutative C^*-algebra. The Gelfand homomorphism is an isometric *-isomorphism from A onto $\mathcal{C}_0(\Phi(A))$.*

Proof. First observe that the Gelfand homomorphism preserves the involution. Indeed, let $a \in A$, $\varphi \in \Phi(A)$ and write $a = x + iy$, with x, y self-adjoint. Then $a^* = x - iy$ so that

$$\widehat{a^*}(\varphi) = \varphi(a^*) = \varphi(x) - i\varphi(y) = \overline{\varphi(a)} = \overline{\widehat{a}(\varphi)}.$$

To verify that the homomorphism is isometric, note that since A is commutative, every element is normal, hence

$$\|a\| = \rho(a) = \sup_{\varphi \in \Phi(A)} = \varphi(a) = \|\widehat{a}\|.$$

Finally, recall that $\tau(A)$ is a subalgebra of $\mathcal{C}_0(\Phi(A))$ that does not vanish at any point and is closed under complex conjugation. Moreover, if A is unital, $\tau(A)$ is a unital subalgebra of $\mathcal{C}(\Phi(A))$ that is closed under conjugation. In either case, we have by the Stone-Weierstrass theorem that $\tau(A)$ is dense in $\mathcal{C}_0(\Phi(A))$ (or $\mathcal{C}(\Phi(A))$ if A is unital). But since τ is isometric, its image is closed, therefore the map is onto. ■

2.1 Positive Elements and Positive Linear Functionals

Positive elements and positive linear functionals play an important part in the representation theory of C^*-algebras. Next, we review some definitions and basic facts on this topic that will be of use later.

Definition 2.19 Let A be a C^*-algebra. An element $a \in A$ is **positive** if it is self-adjoint and has non-negative spectrum. We will denote by A_+ the set of positive elements of A.

Remark 2.20 In a C^*-algebra every self-adjoint element can be written as a linear combination of two positive elements; therefore, every element can be written as a linear combination of four positive elements [Bla06, Proposition II.3.1.2].

Definition 2.21 A **bounded approximate identity** for a Banach algebra A is a net of elements $(e_\alpha)_\alpha$ of A such that, for some $M > 0$,

1. $\sup_\alpha \|e_\alpha\| \leq M$ and
2. $\lim_\alpha e_\alpha a = \lim_\alpha a e_\alpha = a$,

for all $a \in A$.

Lemma 2.22 *Let A be a non-unital C^*-algebra, let $a \in A$ be normal, and let $f \in \mathcal{C}(\sigma(a))$ be such that $f(0) = 0$. Then $f(a) \in A^\#$ lies in A.*

Proof. Let $f \in \mathcal{C}(\sigma(a))$, by the Stone-Weierstrass theorem there exists a sequence (p_n) of polynomials in $(\lambda, \overline{\lambda})$ such that $f = \lim_{n\to\infty} p_n$, and by taking $p_n - p_n(0)$ if necessary, we can assume that $p_n(0) = 0$, for all $n \in \mathbb{N}$. Each $p_n(a)$ is in the non-unital commutative algebra generated by a, a^* and so is $f(a) = \lim_{n\to\infty} p_n(a)$. ∎

Lemma 2.23 *Let A be a unital C^*-algebra and let $a \in A$ be self-adjoint. Given any constant $m \geq \|a\|$, a is positive if and only if $\|a - mI\| \leq m$.*

Proof. Since a is self-adjoint, consider $C^*(a)$, the commutative C^*-algebra generated by a and the identity of A. Since this algebra is isometrically isomorphic to $\mathcal{C}(\sigma(a))$ and the statement is clearly true in the latter algebra, then it is true in general. ∎

Theorem 2.24 *Let A be a C^*-algebra and let A_+ be the set of positive elements in A:*

(a) If $a, -a \in A_+$ then $a = 0$.

(b) If $a, b \in A_+$ commute, then $ab \in A_+$.

Proof. Since positivity is not affected by passing to the unitization, assume that A is unital. For the first claim observe that $\sigma(a) = -\sigma(-a)$ and that $\sigma(a), \sigma(-a) \subset \mathbb{R}_+$, hence $\sigma(a) = 0 = \|0\|$.

Now assume that a, b commute, so that $(ab)^* = b^*a^* = ba = ab$ and ab is self-adjoint. Also, since $\sigma(ab) \subseteq \{\alpha\beta : \alpha \in \sigma(a), \beta \in \sigma(b)\}$, ab has a non-negative spectrum. ■

We can endow the set of self-adjoint elements of a C^*-algebra with the following order relation: $a \geq b \Leftrightarrow a - b \in A_+$. It is not difficult to show that A_{sa} thus becomes a partially ordered set and A_+ becomes a closed cone. Furthermore we have the following [Mur90, Theorem 2.2.5]:

Theorem 2.25 *Let A be a C^*-algebra and let $a, b \in A$ be self-adjoint. Then:*

(a) If $-b \leq a \leq b$, then $\|a\| \leq \|b\|$.

(b) Let $c \in A$. If $a \leq b$ then $c^*ac \leq c^*bc$.

(c) If $0 \leq a \leq b$ and a is invertible, then so is b with $0 \leq b^{-1} \leq a^{-1}$.

(d) An element $a \in A$ is positive if and only if $a = x^*x$ for some $x \in A$.

Now let A be a C^*-algebra, we will denote by Δ_A the intersection of A_+ and the open unit ball of A. We will show that every C^*-algebra has a bounded approximate identity, but first we need the following lemma [Mur90]:

Lemma 2.26 *Let A be a C^*-algebra and let $a, b \in A$ be positive such that*

$a \leq b$, then $a(I+a)^{-1}, b(I+b)^{-1} \in \Delta_A$ with $a(I+a)^{-1} \leq b(I+b)^{-1}$.

Theorem 2.27 *Let A be a C^*-algebra, then Δ_A with the order induced by A_{sa} is a bounded approximate identity for A.*

Proof. First we show that Δ_A is upward directed. For that, let $a, b \in \Delta_A$, and set set $a' = a(I-a)^{-1}$, $b' = b(I-b)^{-1}$, $c = (a'+b')(I+a'+b')^{-1}$. From the previous lemma we have $a = a'(I+a')^{-1}$, $b = b'(I+b')^{-1}$, and $c \in \Delta_A$. Also $a, b \leq c$, showing that Δ_A is upward directed.

With the foregoing, we have that $(e_\alpha)_{\alpha \in \Delta_A}$ is an increasing net. Now let $a \in \Delta_A$ and let $\tau : C^*(a) \to \mathcal{C}_0(\Phi(C^*(a)))$ be the Gelfand transform, and set $f := \tau(a)$. Fix $\epsilon > 0$ and set $K := \{x \in \Phi(C^*(a)) : |f(x)| \geq \epsilon\}$. By Uryshon's lemma, there exists a continuous $g \in \mathcal{C}(\Phi(C^*(a)))$ with support on some compact set that contains K such that $g\,|_K = 1 = \|g\|$. Now choose $\delta > 0$ such that $\delta < 1$ and $1 - \delta < \epsilon$.

Since $\|f\| = 1$, we have $\|f - \delta g f\| \leq \epsilon$. Next, let $e_{\alpha 0} = \tau^{-1}(\delta g)$, so that $e_{\alpha 0} \in \Delta_A$ and $\|a - e_{\alpha 0} a\| \leq \epsilon$. Now for any $e_\alpha \in \Delta_A$ such that $e_{\alpha 0} \leq e_\alpha$, we have $I - e_\alpha \leq I - e_{\alpha 0}$, consequently $a(I-e_\alpha)a \leq a(I - e_{\alpha 0})a$ and

$$\begin{aligned} \|a - e_\alpha a\|^2 &= \|(I-e_\alpha)^{\frac{1}{2}}(I-e_\alpha)^{\frac{1}{2}} a\|^2 \leq \|(I-e_\alpha)^{\frac{1}{2}} a\|^2 \\ &= \|a(I-e_\alpha)a\| \leq \|a(I-e_{\alpha 0})\| \leq \epsilon. \end{aligned}$$

Therefore, $\lim_\alpha a - a e_\alpha = 0$. Similarly, we can show that $\lim_\alpha a - e_\alpha a = 0$. Since this is true for any $a \in \Delta_A$ and Δ_A spans A, (e_α) is a bounded approximate identity for A. ∎

Theorem 2.28 *Let A be a C^*-algebra and let $L \subseteq A$ be a closed left ideal. Then L has a right bounded approximate identity (e_α). Moreover (e_α) can be chosen to be increasing with all the the elements positive and with norm less than or equal to one.*

Proof. Let $\mathcal{B} := L \cap L^*$, so that $\mathcal{B}$ is self-adjoint, hence a C^*-algebra. Let

(e_α) be a bounded approximate identity for $\mathcal{B}$. For any $a \in L$ we have $\lim_\alpha (a^*a - a^*ae_\alpha) = 0$. By taking the unitization of $\mathcal{B}$, we get:

$$\|a - ae_\alpha\|^2 = \|a(I - e_\alpha)\|^2 = \|(I - e_\alpha)a^*a(I - e_\alpha)\| \leq \|a^*a(I - e_\alpha)\| \to 0.$$

This completes the proof. ■

The existence of bounded approximate identities for any C^*-algebra has the following important consequence:

Corollary 2.29 *Let A be a C^*-algebra and let $J \subseteq A$ be a closed ideal. Then J is self-adjoint.*

Proof. Let $(e_\alpha)_\alpha$ be a right bounded approximate identity for J; then $a = \lim_\alpha ae_\alpha$, where $a \in J$. Thus $a^* = \lim_\alpha e_\alpha a^* \in J$ and hence J is closed under the involution. ■

Theorem 2.30 *Let A be a C^*-algebra and let $J \subseteq A$ be a closed ideal. Then A/J is a C^*-algebra.*

Proof. We already know that A/J is a Banach algebra. It is in fact a Banach *-algebra, just let $(a + J)^* := a^* + J$, for all $a \in A$. Now let us show that the quotient norm is, in fact, a C^*-norm. For that, let $a \in A$ and let $(e_\alpha)_\alpha$ be a bounded approximate identity for J. Clearly $\|a - ae_\alpha\| \geq \|a + J\|$. Moreover, since for any $\epsilon > 0$ there exists an element $b \in J$ such that $\|a+b\| \leq \|a+J\|+\epsilon$, we have (again making the calculations in $A^{\#}$):

$$\|a - ae_\alpha\| = \|(a - b)(I - e_\alpha) + (b - be_\alpha)\| \leq \|a - b\|\|I - e_\alpha\| + \|b - be_\alpha\|.$$

Since $\|b - be_\alpha\| \to 0$, we conclude that $\limsup_\alpha \|a - ae_\alpha\| \leq \|a + J\|$. Thus $\lim_\alpha \|a - ae_\alpha\| = \|a + J\|$. Now

$$\begin{aligned}\|a + J\|^2 = \lim_\alpha \|a - ae_\alpha\|^2 &= \lim_\alpha \|(I - e_\alpha)a^*a(I - e_\alpha)\| \\ &\leq \lim_\alpha \|a^*a(I - e_\alpha)\| = \|a^*a + J\|\end{aligned}$$

and so $\|a+J\|^2 = \|a^*a+J\|$ holds. ■

As we mentioned before, C^*-algebras have a very nice representation theory and positive linear functionals are of central importance in the development of that theory.

Definition 2.31 Let A be a C^*-algebra and let φ be a linear functional on A. We say that φ is **positive** if $\varphi(a) \geq 0$, for all $a \in A_+$. A **state** is a positive linear functional of norm 1. We denote by $S(A)$ the set of states of A.

We begin by stating a very basic fact that has strong consequences. First, recall that a semi-inner product on a linear space is a positive-semidefinite sesquilinear functional.

Proposition 2.32 *Let A be a C^*-algebra, let φ be a positive linear functional on A and let $a, b \in A$. Then $|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b)$.*

Proof. Observe that the map $(a, b) \mapsto \varphi(b^*a)$ is a semi-inner product, hence the statement follows from the Cauchy-Schwarz inequality. ■

Positive linear functionals are automatically continuous:

Theorem 2.33 *Let A be a C^*-algebra. If φ is a positive linear functional on A, then φ is bounded.*

Proof. Let $M := \sup\{\varphi(a) : a \in A_+ \text{ and } \|a\| \leq 1\}$. We claim that M is finite. Otherwise we can find a sequence (a_n) of positive elements in the unit ball of A such that $\varphi(a_n) \geq 2^n$, $n \in \mathbb{N}$; implying that the element $a := \sum_{n=0}^{\infty} \frac{1}{2^n} a_n$ is well-defined and positive with norm at most 1. Now, for any fixed $N \in \mathbb{N}$ we have that $\varphi(a) \geq \sum_{n=0}^{N} \frac{1}{2^n}\varphi(a_n) \geq N$, which is not possible.

Since any element in a C^*-algebra can be written as a linear combination of 4 positive elements, it follows that φ is bounded. ■

Remark 2.34 One can say even more. If $(e_\alpha)_\alpha$ is a bounded approximate identity for A and φ is a positive linear functional on A, then $\|\varphi\| = \lim_\alpha \varphi(e_\alpha)$. In particular, if A is unital, a bounded linear functional φ on A is positive if and only if $\|\varphi\| = \varphi(I)$ [Mur90, Theorem 3.3.3].

A very useful consequence of the previous fact is the following:

Corollary 2.35 *Let A be a C^*-algebra, let B be a C^*-subalgebra of A, and let φ be a positive linear functional on B. Then φ can be extended to a positive linear functional $\tilde{\varphi}$ on A that preserves the norm.*

Proof. If B is not unital, extend φ to a linear functional $\widetilde{\varphi}$ on $B^{\#}$ by letting $\widetilde{\varphi}(a \oplus \lambda) = \varphi(a) + \lambda\|\varphi\|$, for $a \in B$, $\lambda \in \mathbb{C}$. Note that $\widetilde{\varphi}$ preserves the norm of φ. To see this, let $(e_\alpha)_\alpha$ be a bounded approximate identity for B, then

$$\begin{aligned} |\widetilde{\varphi}(a \oplus \lambda)| &= |\varphi(a) + \lambda\|\varphi\|| = |\varphi(a) + \lambda \lim_\alpha \varphi(e_\alpha)| \\ &= \lim_\alpha |\varphi(ae_\alpha) + \lambda\varphi(e_\alpha)| = \lim_\alpha |\varphi((a+\lambda)e_\alpha)| \leq \|\varphi\|\|a \oplus \lambda\| \end{aligned}$$

Therefore the norm of $\widetilde{\varphi}$ is just $\widetilde{\varphi}(0 \oplus 1) = \|\varphi\| = 1$ and by the previous remark $\widetilde{\varphi}$ is a state of $B^{\#}$. Now by the Hahn-Banach theorem we can extend $\widetilde{\varphi}$ to a linear functional on $A^{\#}$ preserving its norm, and applying again the remark this extension is a state on $A^{\#}$ and its restriction to A is a state too. ∎

Of course, the aforementioned extension does not have to be unique. As we shall see later, the existence of unique extensions for positive linear functionals is very significant in the context of this work. There exists a particular kind of C^*-subalgebra for which we can guarantee that this extension is unique.

Definition 2.36 Let A be a C^*-algebra, let B be a C^*-subalgebra of A. We say that B is a **hereditary** subalgebra if for any $a \in A_+$ and $b \in B_+$ we have that $a \leq b$ implies $a \in B$.

For hereditary C^*-subalgebras every positive linear functional has a unique norm-preserving extension [Mur90, Section 3.3]:

Theorem 2.37 *Let A be a C^*-algebra, let B be a hereditary C^*-subalgebra of A, and let φ be a positive linear functional on B. Then there exists a unique norm-preserving positive linear functional $\tilde{\varphi}$ on A that extends φ.*

We finish this section by noting that there exists a large quantity of positive linear functionals. As we will see, this makes it possible to faithfully represent any C^*-algebra on a subalgebra of $\mathcal{B}(H)$, for some Hilbert space H.

Theorem 2.38 *Let A be a C^*-algebra and let $a \in A$ be normal. Then there exists a state $\tilde{\varphi}$ on A such that $|\tilde{\varphi}(a)| = \|a\|$.*

Proof. First, assume that A is unital and that $a \neq 0$. Let $C^*(a)$ be the C^*-algebra generated by a and the identity. Note that this is a commutative C^*-algebra; so there exists a character φ on it such that $|\varphi(a)| = \|a\|$.

By Corollary 2.35, characters from this commutative subalgebra can be extended to states, so if we let $\widetilde{\varphi}$ be such an extension, the claim follows in the case A is unital. If A is not unital, take its unitization $A^{\#}$ and apply the same reasoning to get the desired character and its corresponding extension $\widetilde{\varphi}$. Of course, the restriction of $\widetilde{\varphi}$ to A will be a state, too. Since the statement holds trivially if $a = 0$, this finishes the proof. ■

2.2 The GNS Construction

As we saw before, every positive linear functional on a C^*-algebra defines a semi-inner product. The Gelfand-Naimark-Segal (GNS) construction shows how to get a *-representation of the given C^*-algebra on the completion of the pre-Hilbert space obtained from the semi-inner product induced by a positive linear functional.

Specifically, let φ be a positive linear functional on a C^*-algebra A. Let $L_\varphi := \{a \in A : \varphi(a^*a) = 0\}$. It is not difficult to see that L_φ is a left ideal of A, and that it is closed. Indeed, for any $a \in A$ the left multiplication map L_a, $b \mapsto ab$ is continuous in the semi-norm defined by φ:

$$\varphi((ab)^*(ab)) = \varphi(b^*a^*ab) \leq \|a^*a\|\varphi(b^*b).$$

Now we can define a representation π_φ on A/L_φ by letting $a \in A$ act on A/L_φ via left multiplication, i.e., $\pi_\varphi(a)(b + L_\varphi) = ab + L_\varphi$, $a, b \in A$. By the previous argument, this map is well-defined. Moreover, $\|\pi_\varphi\| \leq \|a\|$. From Theorem 2.38, for any $a \in A$ there exists a state τ such that $\|\pi_\tau(a)\| = \|a\|$. Let $\mathcal{H}_\varphi$ denote the completion of A/L_φ. Then we can extend π_φ to a representation on $\mathcal{H}_\varphi$. Putting all this together we get:

Theorem 2.39 *Every C^*-algebra has a faithful *-representation on some Hilbert space. If A is unital, this representation can be chosen to be unital, too.*

Proof. For any given state φ of A, let π_φ be the GNS representation of A on $\mathcal{H}_\varphi$. Now set $\mathcal{H} := \bigoplus_{\varphi \in S(A)} \mathcal{H}_\varphi$ (the Hilbert space direct sum). By the previous argument $\pi := \ell^\infty\text{-}\bigoplus_{\varphi \in S(A)} \pi_\varphi$ is a faithful representation. Indeed, for any non-zero $a \in A$, there exists a state φ of A such that $\varphi(a^*a) = \|a\|^2$, so $\pi_\varphi(a) \neq 0$. Thus π is faithful (and even isometric). ■

Thus every C^*-algebra has a faithful representation on some Hilbert space; but from the way in which this space was constructed we can see it is *very large*. Therefore a natural question that arises is if given a C^*-algebra A there exists a smaller Hilbert space such that we can still faithfully represent A on it.

The answer to the previous question is positive and is related to set of *pure* *states* of the given C^*-algebra. Indeed, the GNS construction uses more states than is strictly needed to obtain a faithful representation.

As we know, $S(A)$ is a closed subset of the closed unit ball of A^*; from Remark 2.34 this set is also convex. The extreme points of $S(A)$ are called **pure states** and we will denote this set by $PS(A)$. By the Krein-Milman theorem $S(A)$ has extreme points, but we can say even more:

Theorem 2.40 *Let A be a C^*-algebra and let $a \in A$ be self-adjoint. Then there is a pure state φ of A such that $|\varphi(a)| = \|a\|$.*

The detailed proof of this theorem can be found in many textbooks, such as [Bla06], but the basic idea is that $C^*(a)$ is a commutative C^*-algebra and so by its Gelfand representation there is a character ϕ such that $\|a\| = \rho(a) = |\phi(a)|$. Observe that characters are pure states of $C^*(a)$ and thus can be extended to states of A; since the set of all such extensions is closed (in the weak-* topology) and convex, by the Krein-Milman theorem it has extreme points and any of them will be a pure state of A.

We saw in Theorem 2.37 that states of a C^*-subalgebra B of a given C^*-algebra A can be extended to elements of $S(A)$. The same is true for pure states:

Theorem 2.41 *Let A be a C^*-algebra, let B be a C^*-subalgebra of A, and let φ be a pure state of B. Then there exists a pure state $\widetilde{\varphi}$ of A that extends φ.*

Again, the idea of the proof is to note that the set of all the extensions of φ from B to A as states, is weak-* closed and convex, so has extreme points, which are pure states of A.

Just as we mentioned before when we talked about the extension of states, the extension of a pure state does not have to be unique. Again, when the given subalgebra is hereditary, we can guarantee uniqueness of extensions:

Theorem 2.42 *Let A be a C^*-algebra, let B be a hereditary C^*-subalgebra of A, and let φ be a pure state of B, then there exists a* unique *a pure state $\widetilde{\varphi}$ of A that extends φ.*

Proof. Just note that the set of extensions of φ to A is weak-* closed and convex, so it has extreme points. Since B is hereditary, by Theorem 2.37 the set of all such extensions only has one element, say $\widetilde{\varphi}$, which of course is an extreme point of $S(A)$, whence has to be the unique pure state that extends φ. ■

Remark 2.43 For a C^*-algebra A with a hereditary C^*-subalgebra B, there is also a relation between the restriction to B of pure states of A, and pure states of B. Specifically, given a pure state ϕ of A, there exists a $t \in [0, 1]$ and a pure state φ' of B such that the restriction of φ to B is just $t\varphi'$ [Mur90, Corollary 5.5.3].

One of the main reasons for the importance of pure states is the following [Mur90, Theorem 5.1.6]:

Theorem 2.44 *Let A be a C^*-algebra and let φ be a state of A. Then the representation π_φ of A on A/L_φ is irreducible if and only if φ is a pure state.*

As a corollary, we have that for a given state φ of a C^*-algebra A, L_φ is a maximal modular left ideal if and only if φ is pure. Furthermore, there exists a one-to-one correspondence between pure states of a C^*-algebra and the set of all its maximal modular left ideals, given by $\varphi \mapsto L_\varphi$, for $\varphi \in PS(A)$ [DF88, Theorem VI.25.12].

Moreover, if φ is a pure state, then A/L_φ is complete in the norm induced by φ, therefore is a Hilbert space [DF88, Proposition VI.25.9].

Another corollary is that we can obtain a faithful representation of A if, in the proof of Theorem 2.39, we only take the ℓ^∞-direct sum over all the irreducible representations of A, i.e., over all the pure states of A.

2.3 Liminal and Postliminal C^*-Algebras

We finish this chapter by introducing two classes of C^*-algebras that will be very significant in the sequel, because their properties are very well-known and there are very strong results about their representation theory.

Recall that if we have two normed spaces E, F and we let $B_1(E)$ denote the closed unit ball of E, then a linear operator $T : E \to F$ is called **compact** if the closure of $T(B_1(E))$ is compact. We will denote by $\mathcal{K}(E)$ the set of compact operators on E.

Definition 2.45 A C^*-algebra A is called **liminal** if $\pi(A) = \mathcal{K}(\mathcal{H})$ for each irreducible *-representation π of A on a Hilbert space $\mathcal{H}$.

Remark 2.46 The previous condition is equivalent to saying that for each irreducible *-representation π of A on a Hilbert space $\mathcal{H}$, $\pi(A) \subseteq \mathcal{K}(\mathcal{H})$. This formulation is sometimes more useful [Mur90, Theorem 2.4.9].

In the literature, liminal C^*-algebras are sometimes referred to as *CCR* C^*-algebras, where CCR stands for *Completely Continuous Representation.* A standard reference for liminal C^*-algebras is [Dix77, Chapter 4].

Example 2.47 Any finite dimensional C^*-algebra is liminal. To see this, recall that given a C^*-algebra A with an irreducible representation π on a Hilbert space $\mathcal{H}$, any non-zero vector $x \in \mathcal{H}$ is cyclic for π. Thus $\mathcal{H} = \pi(A)x$ and $\mathcal{H}$ is finite dimensional. Since in finite dimensional normed spaces every linear operator is compact, $\pi(A) \subseteq \mathcal{K}(\mathcal{H})$.

Example 2.48 Every commutative C^*-algebra is liminal. In fact, let A be a

C^*-algebra and let π be an irreducible *-representation of A on a Hilbert space $\mathcal{H}$. Then the commutant of $\pi(A)$—denoted by $\pi(A)'$—is just $\mathbb{C}I$, [Mur90, Theorem 4.1.12]; and since A is commutative, $\pi(A) \in \pi(A)'$. From the irreducibility of π, there is a vector $x \in \mathcal{H}$ such that $\mathcal{H} = \pi(A)x = \mathbb{C}x$ and $\mathcal{H}$ has dimension one. Thus $\pi(A) = \mathcal{K}(\mathcal{H})$.

Definition 2.49 A C^*-algebra A is called **postliminal** if $\pi(A) \supseteq \mathcal{K}(\mathcal{H})$, for each irreducible *-representation π of A on a Hilbert space $\mathcal{H}$.

Of course, every liminal C^*-algebra is postliminal.

Postliminal C^*-algebras are also called *GCR* C^*-algebras, which stands for *Generalized Continuous Representation*. Again, for the main results and examples about postliminal C^*-algebras we refer the reader to [Dix77].

As we will see in Chapter 4, for postliminal C^*-algebras there is a version of the Stone-Weierstrass theorem, which will be very useful in the context of this thesis.

Chapter 3

Gelfand Theory for Non-Commutative Banach Algebras

If we try to generalize the notion of Gelfand transform to the setting of non-commutative Banach algebras we find several inconveniences. For example, it is not difficult to find a Banach algebra such that its character space is empty.

For instance, let $A := \mathcal{M}_{2\times 2}(\mathbb{C})$, the algebra of 2×2 matrices with entries from $\mathbb{C}$. Assume that there exists a character φ of A, then $\ker(\varphi)$ is an ideal of A. Let $e_{ij} \in A$ be the matrix with 1 in the $ij-th$ entry and 0 everywhere else. Observe that $e_{1,2} \in \ker(\varphi)$, because $e_{1,2}e_{1,2} = 0$ implies that $\varphi(e_{1,2}e_{1,2}) = \varphi(e_{1,2})^2 = 0$. The same is true for $e_{2,1}$, but since $e_{11} = e_{1,2}e_{2,1}$ and $e_{22} = e_{2,1}e_{1,2}$, it follows that $e_{ij} \in \ker(\varphi)$, $i, j \in (1, 2)$. As the set $\{e_{ij},\, i, j \in (1, 2)\}$ spans A, it follows that $\ker(\varphi) = A$. Hence the character space of A has to be empty.

Another problem in extending the classical approach to the non-commutative setting is that, even if the character space is not empty, the kernel of the Gelfand transform could be very large. In particular, for any element in the commutant of A, that is, any element of the form $ab - ba$, $a, b \in A$, if τ is the classical Gelfand transform, then $\tau(ab-ba) = 0$, thus there is a significant loss of information and the usefulness of the construction could be very limited.

In order to overcome these shortcomings, [CIR02] makes a generalization of the Gelfand theory to non-commutative Banach algebras. To do that, the authors define a non-commutative Gelfand theory that captures some of the most important properties of the classical Gelfand transform; for example, the Gelfand transform provides an algebra homomorphism between a commutative Banach algebra A and the C^*-algebra $\mathcal{C}_0(\Phi_A)$ that induces a bijection between the sets of maximal modular ideals of A and $\mathcal{C}_0(\Phi_A)$.

Thus, the following definition is introduced by the authors in [CIR02]:

Definition 3.1 For a given Banach algebra A, let Λ_A be the set of maximal modular left ideals. A **Gelfand theory** for A is any pair $(\mathcal{G}, \mathfrak{A})$ that satisfies the following conditions:

(G1) $\mathcal{G} : A \to \mathfrak{A}$ is an algebra homomorphism from A into the C^*-algebra $\mathfrak{A}$. We refer to $\mathcal{G}$ as the **Gelfand transform** of A corresponding to $(\mathcal{G}, \mathfrak{A})$.

(G2) The map $L \mapsto \mathcal{G}^{-1}(L)$ is a bijection between $\Lambda_{\mathfrak{A}}$ and Λ_A.

(G3) For each $L \in \Lambda_{\mathfrak{A}}$, the map $\mathcal{G}_L : A/\mathcal{G}^{-1}(L) \to \mathfrak{A}/L$ induced by $\mathcal{G}$ has dense range.

From the previous remarks, we can see that if A is commutative, then $(\tau, C_0(\Phi_A))$ satisfies the aforementioned conditions. But these properties also characterize, up to isomorphism, the Gelfand transform for commutative Banach algebras[1].

First, recall that a Banach algebra is **semisimple** if the intersection of the kernels of all its irreducible representations is zero. C^*-algebras are semisimple; which implies that the intersection of all the maximal modular left ideals of a

[1]Unless otherwise explicitly stated, all the results regarding the Gelfand theory in this chapter and the next one are taken from [CIR02].

C^*-algebra is 0 [Bla06, Corollary II.1.6.4].

Proposition 3.2 *Let A be a commutative Banach algebra and let $(\mathcal{G}, \mathfrak{A})$ be a pair satisfying (G1),(G2), and (G3). Then there is an isomorphism $\theta : \mathfrak{A} \to C_0(\Phi_A)$ such that $\tau = \theta \circ \mathcal{G}$.*

Proof. First note that from (G2) and (G3) $\mathfrak{A}$ has to be commutative: for a given $L \in \Lambda_{\mathfrak{A}}$, and some $x, y \in \mathfrak{A}$ such that $xy - yx \neq 0$, consider $(xy - yx + L)$. Then for simplicity we can assume that there are $a, b \in A$ such that

$$\mathcal{G}_L(a + \mathcal{G}^{-1}(L)) = (x + L), \quad \mathcal{G}_L(b + \mathcal{G}^{-1}(L)) = (y + L),$$

$$\text{and hence } (xy - yx) + L = \mathcal{G}(a)\mathcal{G}(b) - \mathcal{G}(b)\mathcal{G}(a) + L = L.$$

Thus $xy - yx \in L$ for every $L \in \Lambda_{\mathfrak{A}}$, and since $\mathfrak{A}$ is semisimple, it follows that $xy - yx = 0$. Also, since $\mathcal{G}$ is a homomorphism from a Banach algebra to a commutative C^*-algebra, it is continuous: any homomorphism from a Banach algebra to a commutative semisimple Banach algebra is automatically continuous [Pal94, Theorem 3.1.11].

Thus, we have that $\mathcal{G}^*$ induces a homeomorphism between Φ_A and $\Phi_{\mathfrak{A}}$: just observe that if $\mathfrak{A}$ is unital, $\Phi_{\mathfrak{A}}$ is compact and $\mathcal{G}^*$ establishes a continuous bijection from a compact Hausdorff space onto a locally compact Hausdorff space, therefore a homeomorphism, and if $\mathfrak{A}$ is not unital, just take the one-point compactification of $\Phi_{\mathfrak{A}}$, i.e., $\Phi_{\mathfrak{A}} \cup \{0\}$, on which $\mathcal{G}^*$ induces a continuous bijection onto $\Phi_A \cup \{0\}$, with the restriction to $\Phi_{\mathfrak{A}}$ establishing a homeomorphism between $\Phi_{\mathfrak{A}}$ and Φ_A.

Since $\mathfrak{A}$ is isometrically isomorphic to $C_0(\Phi_{\mathfrak{A}})$, we can define

$$\theta : \mathfrak{A} \to C_0(\Phi_A), \quad f \to f \circ (\mathcal{G}^*)^{-1}.$$

It is not difficult to check that $\tau = \theta \circ \mathcal{G}$: note that

$$\theta \circ \mathcal{G} : A \to C_0(\Phi_A), \quad a \mapsto \mathcal{G}(a) \circ \mathcal{G}^{*-1}$$

so for any $\varphi \in \Phi_A$ we have

$$(\theta \circ \mathcal{G} \circ a)(\varphi) = \mathcal{G}(a) \circ \mathcal{G}^{*-1} \circ \varphi = (\mathcal{G}^{*-1} \circ \varphi) \circ (\mathcal{G}(a))$$
$$= \mathcal{G}^{*} \circ (\mathcal{G}^{*-1} \circ \varphi)(a) = \varphi(a).$$

Therefore $\tau = \theta \circ \mathcal{G}$ holds. ■

Thus, we adopt the following:

Definition 3.3 Let A be a Banach algebra, we say that two Gelfand theories $(\mathcal{G}_1, \mathfrak{A}_1)$ and $(\mathcal{G}_2, \mathfrak{A}_2)$ for A are equivalent if there exists an isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\mathcal{G}_2 = \theta \circ \mathcal{G}_1$.

3.1 Basic Properties of Gelfand Theories

The notion of Gelfand theory adopted in [CIR02] shares many of the properties of the classical Gelfand transform. For example, we know that the Gelfand transform of a commutative Banach algebra is a continuous algebra homomorphism. This is also true for the generalized Gelfand transform just defined.

Lemma 3.4 *Let A be a Banach algebra, let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A, let $L \in \Lambda_{\mathfrak{A}}$, and let $\pi : \mathfrak{A} \to \mathcal{B}(\mathfrak{A}/L)$ be the corresponding irreducible representation of $\mathfrak{A}$. Then $\pi_L \circ \mathcal{G}$ is continuous.*

Proof. Denote by E_L the image of $\mathcal{G}_L$ in $\mathfrak{A}/L$, then it is a pre-Hilbert space (recall that $\mathcal{G}_L$ has dense range and $\mathfrak{A}/L$ is a Hilbert space). By (G2), $\mathcal{G}^{-1}(L)$ is a maximal modular ideal of A, thus $A \to \mathcal{B}(E_L)$, $a \mapsto (\pi_L \circ \mathcal{G})(a)|_{E_L}$ is an irreducible representation of A.

Since irreducible normed representations of a Banach algebra on a normed space are continuous [Pal94, Theorem 4.2.15], there is a $C \geq 0$ such that

$\|(\pi_L \circ \mathcal{G})(a)\|_{\mathcal{B}(E_L)} \leq C\,\|a\|$, for $a \in A$. But E_L is dense in $\mathfrak{A}/L$, so this is just the operator norm of $(\pi_L \circ \mathcal{G})(a)$ on $\mathfrak{A}/L$. ■

Theorem 3.5 *Let A be a Banach algebra and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. Then $\mathcal{G}$ is continuous.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a sequence in A such that $a_n \to 0$ and $\mathcal{G}(a_n) \to b$, for some $b \in \mathfrak{A}$. By the closed graph theorem, if $b = 0$ then $\mathcal{G}$ is continuous. By the previous lemma we know that for $L \in \Lambda_{\mathfrak{A}}$ we have that $\pi_L \circ \mathcal{G}$ is continuous, thus $\lim\limits_{n\to\infty} (\pi_L \circ \mathcal{G})(a_n) = 0$. That is, $b \in \ker(\pi_L)$ for every $L \in \Lambda_{\mathfrak{A}}$. By semisimplicity of $\mathfrak{A}$, $b = 0$ and $\mathcal{G}$ is continuous. ■

We saw that for a commutative Banach algebra the Gelfand transform τ preserves the spectra, specifically

$$\sigma_A(a) \cup \{0\} = \sigma_{C_0(\Phi_A)}(\tau(a)) \cup \{0\}\,.$$

The generalized Gelfand theory preserves this relation. First, we need a couple of lemmas:

Lemma 3.6 *Let A be a unital Banach algebra, and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. Then $\mathfrak{A}$ is unital and $\mathcal{G}$ is a unital homomorphism.*

Proof. Let $L \in \Lambda_{\mathfrak{A}}$, let $\pi_L : \mathfrak{A} \to \mathcal{B}(\mathfrak{A}/L)$ be the corresponding irreducible representation of $\mathfrak{A}$ on $\mathfrak{A}/L$, and let E_L be the image of $\mathcal{G}_L$ in $\mathfrak{A}/L$. Denote by I_A the unit of A. Then $(\pi_L \circ \mathcal{G})(I_A)$ is an identity for E_L and by the density of E_L we have that $(\pi_L \circ \mathcal{G})(I_A) = I_{\mathfrak{A}/L}$.

Therefore, for any $L \in \Lambda_{\mathfrak{A}}$ and $a \in \mathfrak{A}$ we have $\mathcal{G}(I_A)a - a \in L$. By semisimplicity of $\mathfrak{A}$, $\mathcal{G}(I_A)a - a = 0$, which implies that $\mathfrak{A}$ is unital and that $\mathcal{G}$ is a unital homomorphism. ■

Lemma 3.7 *Let A be a Banach algebra and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. Then for $a \in A$ the element $\mathcal{G}a$ is quasi-invertible in $\mathcal{G}A$ if and only if it* *is quasi-invertible in* $\mathfrak{A}$.

Proof. Assume $\mathcal{G}a$ is quasi-invertible in $\mathfrak{A}$ but not in $\mathcal{G}A$, then a is not quasi-invertible in A, therefore it is not left quasi-invertible in A (otherwise $\mathcal{G}a$ would have a left quasi-inverse in $\mathcal{G}A$ which of course would be a quasi-inverse, given that $\mathcal{G}a$ is quasi-invertible). Since an element of A is a right modular identity for some maximal modular left ideal if and only if it is not left quasi-invertible [Pal94, Theorem 2.4.6], we can find $\widetilde{L} \in \Lambda_A$ such that a is a modular right identity for $\widetilde{L}$. Let $\tilde{L} = \mathcal{G}^{-1}(L)$, for $L \in \Lambda_{\mathfrak{A}}$. If we again let $\pi_L : \mathfrak{A} \to \mathcal{B}(\mathfrak{A}/L)$ be the corresponding irreducible representation of $\mathfrak{A}$ in $\mathcal{B}(\mathfrak{A}/L)$ and E_L be the image of $\mathcal{G}_L$ in $\mathfrak{A}/L$, and denote

$$x := \mathcal{G}_L(a + \tilde{L}) \in E_L \setminus \{0\},$$

then

$$\pi_L(\mathcal{G}a)\mathcal{G}_L(a + \tilde{L}) = \mathcal{G}_L(a^2 + \tilde{L}) = \mathcal{G}_L(a + \tilde{L}) = x.$$

Thus, if we let $b \in \mathfrak{A}$ be the quasi-inverse of $\mathcal{G}a$ in $\mathfrak{A}$, then

$$x = \pi_L(\mathcal{G}a)x = \pi_L(b\mathcal{G}a - b)x = \pi_L(b)\pi_L(\mathcal{G}a)x - \pi_L(b)x = \pi_L(b)x - \pi_L(b)x = 0,$$

a contradiction. ∎

Theorem 3.8 *Let A be a Banach algebra and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. If A is unital,*

$$\sigma_A(a) = \sigma_{\mathfrak{A}}(\mathcal{G}a),\ a \in A, \tag{3.1}$$

and if A is non-unital

$$\sigma_A(a) = \sigma_{\mathfrak{A}}(\mathcal{G}a) \cup \{0\}\,,\ a \in A. \tag{3.2}$$

Proof. From the previous lemma, we have that

$$\sigma_A(a) \cup \{0\} = \sigma_{\mathfrak{A}}(\mathcal{G}a) \cup \{0\}\,,\ a \in A,$$

whether A has an identity or not. If A is non-unital then 0 is in the spectrum of a and so

$$\sigma_A(a) = \sigma_{\mathfrak{A}}(\mathcal{G}a) \cup \{0\}\,,\ a \in A$$

holds. In the case when A is unital, assume first that a is invertible, so that $0 \notin \sigma(a)$. Since $\mathfrak{A}$ is also unital and $\mathcal{G}$ is a unital homomorphism, $\mathcal{G}a$ is invertible in $\mathfrak{A}$, hence $0 \notin \sigma(\mathcal{G}a)$.

Now assume that $0 \notin \sigma(\mathcal{G}a)$ but $0 \in \sigma(a)$, that is, a is not invertible in A. Suppose first that a does not have a left inverse. Then there exists a maximal modular left ideal $\widetilde{L} \in \Lambda_A$ such that $a \in \widetilde{L}$ and we can find a unique $L \in \Lambda_{\mathfrak{A}}$ such that $\tilde{L} = \mathcal{G}^{-}(L)$, a contradiction because $\mathcal{G}a \in L$ is invertible.

Suppose now that a has a left inverse b. Then $(\mathcal{G}a)^{-1} = \mathcal{G}b$, hence $0 \notin \sigma(\mathcal{G}b)$. Also note that, since a is a right inverse of b, it cannot have a left inverse in A (otherwise b would be invertible, with inverse a). Since we just showed this leads to a contradiction, the theorem is proven. ■

3.2 Existence of Gelfand Theories

The first result that we establish is about non-existence, which uses the following fact that can be found in [BP69, Corollary 6.15]:

Lemma 3.9 *Let X and Y be chosen among the spaces ℓ^p, $1 < p < \infty$, and c_0. If $X \neq Y$, the only Banach algebra homomorphism of $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ is the zero homomorphism.*

Proposition 3.10 *Let E be the Banach space c_0 or ℓ^p, $1 < p < \infty$. Then there is no Gelfand theory for $\mathcal{B}(E)$.*

Proof. Let $A := \mathcal{B}(E)$ and assume that there exists a C^*-algebra $\mathfrak{A}$ and a homomorphism $\mathcal{G} : A \to \mathfrak{A}$ such that $(\mathcal{G}, \mathfrak{A})$ is a Gelfand theory for A. Take $x \in A \setminus \{0\}$ and let $\widetilde{L} := \{T \in \mathcal{B}(E) : Tx = 0\}$, so that $\widetilde{L}$ is maximal modular left ideal of A. We have that $A/\widetilde{L} \cong E$ and we know that there exists a maximal modular left ideal, say L, in $\mathfrak{A}$ such that $\widetilde{L} = \mathcal{G}^{-1}(L)$. Since $\mathcal{G}$ is continuous, so is $\mathcal{G}_L : A/\widetilde{L} \to \mathfrak{A}/L$. We have that

$$\mathcal{G}_L : E \cong A/\widetilde{L} \to \mathfrak{A}/L.$$

But since E is separable, so is $A/\widetilde{L}$, and since $\mathcal{G}_L$ has dense range, $\mathfrak{A}/L$ is separable too. Thus the Hilbert space $\mathfrak{A}/L$ is isomorphic to ℓ^2 and $\mathcal{G}_L$ induces a non-zero homomorphism from $\mathcal{B}(E)$ into $\mathcal{B}(\ell^2)$. This contradicts the previous lemma.

(The isomorphism above is given by $T \mapsto \widetilde{T}$, where $\widetilde{T}(\mathcal{G}_L a) := \mathcal{G}_L(Ta)$ is defined on a dense subset of $\mathfrak{A}/L$ and extended by continuity). ■

Now let us see what we can say about the Gelfand theory of a Banach *-algebra. For that, first we need to review a couple of facts:

Definition 3.11 Given a Banach *-algebra A, there exists a largest C^*-semi-norm γ_A. The completion of $A/\ker(\gamma_A)$ in this semi-norm is a C^*-algebra, which is called the **enveloping C^*-algebra** of A and is denoted $C^*(A)$ [Pal01, Chapter 11]. Let us denote by ι_A the *-homomorphism from A into $C^*(A)$ given by $a \mapsto a + \ker(\gamma_A)$.

A *-algebra is called **hermitian** is every self-adjoint element has real spectrum. Note that C^*-algebras are always hermitian [Mur90, Theorem 2.1.8].

Remark 3.12 We showed that Gelfand theories preserves spectrum, implying that $(\iota_A, C^*(A))$ is a Gelfand theory for A only if A is hermitian.

Remark 3.13 Every irreducible *-representation of a hermitian Banach *-algebra A has a unique extension to an irreducible *-representation on $C^*(A)$ [Pal72].

Proposition 3.14 *Let A be a Banach *-algebra. Then $(\iota_A, C^*(A))$ is a Gelfand* *theory for A if and only if A is hermitian.*

Proof. Assume that A is hermitian. Let $L \in \Lambda_{C^*(A)}$, so that there exists a pure state ϕ on $C^*(A)$ such that

$$L = \{a \in C^*(A) : \phi(a^*a) = 0\}. \tag{3.3}$$

By the previous remark, L is the unique extension of a maximal modular left ideal $\widetilde{L} \in \Lambda_A$, i.e., $\widetilde{L} = \iota_A^{-1}(L)$. Moreover, since A is dense in $C^*(A)$, any pure state of $C^*(A)$ is determined by its restriction to A, so for each maximal modular left ideal in $C^*(A)$ there exists a corresponding one in A. Thus there is a bijection between $\Lambda_{C^*(A)}$ and Λ_A. From the density of A in $C^*(A)$, it follows that $\mathcal{G}_L : A/\iota_A^{-1}(L) \to C^*(A)/L$ has a dense range, for every $L \in \Lambda_{C^*(A)}$. ■

3.3 Hereditary Properties

Lemma 3.15 *Let A be an algebra, let J be an ideal of A, and let π be an irreducible representation of J on a linear space E. Then π extends uniquely to an irreducible representation of A on E. Moreover, if π is an irreducible representation of A on a linear space E such that $\pi|_J \neq \{0\}$, then $\pi|_J$ is an irreducible representation of J on E.*

Proof. Let $a \in A$ and let $x \in E$. Since π is irreducible, there exists $y \in J$, $b \in E$, such that $\pi(y)b = x$. Now define $\pi(a)x := \pi(ay)b$. Note that this does not depend on the selection of y and b, because, if $\pi(y)b = \pi(z)c$ with $z \in J, c \in E$, then $\pi(y)b - \pi(z)c = 0$. Since b is cyclic vector, we can find $k \in J$ such that $\pi(k)b = c$, hence

$$\pi(y)b - \pi(z)c = 0 \quad \text{and so} \quad \pi(y)b - \pi(z)\pi(k)b = 0.$$

Therefore, for any $d \in J$,

$$\pi(y)b - \pi(zk)b = 0, \quad \text{hence} \quad \pi(d)\pi(ay)b - \pi(d)\pi(azk)b = 0, \text{ and}$$

$$\pi(day)b - \pi(dazk)b = \pi(da)\pi(y)b - \pi(da)\pi(zk)b = 0,$$

so for any $d \in J$ we have

$$\pi(d)\pi(ay - azk)b = 0.$$

Thus $\pi(ay - azk)b = 0$ for every $a \in A$ and the operator $\pi(a)$ is well-defined.

Moreover, this extension is unique because if we assume that there are two such extensions, say π_1, π_2, then

$$\pi_1(a)x = \pi_1(a)\pi(b)y = \pi(ab)y = \pi_2(ab)y = \pi_2(a)\pi(b)y = \pi_2(a)x,$$

hence $\pi_1 = \pi_2$.

Conversely, if π is an irreducible representation of A on some linear space E, set $Z := \{x \in E : \pi(J)x = 0\}$, then Z is an invariant subspace for $\pi(A)$, and Z is either E or $\{0\}$. In the first case, the restriction is trivial. In the second case, we have that $\pi(J)x \neq 0$ for any $x \in E \setminus \{0\}$, and since this will be also an invariant subspace for $\pi(A)$, it will be all of E. As every non-zero vector is cyclic for $\pi(J)$, the representation is irreducible. ■

Since there exists a correspondence between maximal modular left ideals of an algebra and equivalence classes of irreducible representations, from the previous lemma we have the following:

Corollary 3.16 *Let A be a Banach algebra and let J be a closed ideal of A. Then*

$$\{L \in \Lambda_A : J \nsubseteq L\} \to \Lambda_J, \quad L \mapsto L \cap J \tag{3.4}$$

is a bijection.

Proposition 3.17 *Let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. Then $A^{\#}$ has a Gelfand theory too.*

Proof. Let $\mathfrak{A}^{\#}$ denote the unconditional unitization of $\mathfrak{A}$, that is, if $\mathfrak{A}$ has an identity, we adjoin another one. Define a homomorphism

$$\mathcal{G}^{\#} : A^{\#} \to \mathfrak{A}^{\#}, \quad a + \lambda I_{A^{\#}} \mapsto \mathcal{G}a + \lambda I_{\mathfrak{A}^{\#}}. \tag{3.5}$$

By the previous corollary, $(\mathcal{G}^{\#}, \mathfrak{A}^{\#})$ satisfies (G2). If $\mathfrak{A}$ is non-unital, one can define a C^*-norm on $\mathfrak{A}^{\#}$ (Theorem 2.17). If $\mathfrak{A}$ is unital, we have a *-isomorphism

$$\mathfrak{A}^{\#} \to \mathfrak{A} \oplus C, \quad a + \lambda I_{\mathfrak{A}^{\#}} \mapsto a + \lambda I_{\mathfrak{A}} \oplus \lambda. \tag{3.6}$$

In this case we also get a C^*-norm on $\mathfrak{A}^{\#}$.

Clearly (G3) also holds, i.e., the map $\mathcal{G}_L$ has dense range, so $(\mathcal{G}^{\#}, \mathfrak{A}^{\#})$ is a Gelfand theory for $A^{\#}$. ■

Proposition 3.18 *Let A be a Banach algebra with a Gelfand theory $(\mathcal{G}, \mathfrak{A})$ and let J be a closed ideal of A. Then J has a Gelfand theory.*

Proof. Let

$$\mathfrak{J} := \bigcap \{L \in \Lambda_{\mathfrak{A}} : \mathcal{G}J \subseteq L\} \tag{3.7}$$

It is clear that $\mathfrak{J}$ is a left ideal, being the intersection of maximal modular left ideals. We claim it is in fact an ideal. Let $L \in \Lambda_{\mathfrak{A}}$ be such that $\mathcal{G}J \subseteq L$ and let

$$P := \{a \in \mathfrak{A} : a\mathfrak{A} \subseteq L\} \tag{3.8}$$

(i.e., P is the primitive ideal of $\mathfrak{A}$ associated to L.)

Of course,

$$\mathcal{G}^{-1}(P) \subseteq \{a \in A : aA \in \mathcal{G}^{-1}(L)\} =: Q. \tag{3.9}$$

Now let $a \in Q$ and let π_L be the irreducible representation of $\mathfrak{A}$ on $\mathfrak{A}/L$. As usual, denote by E_L the image of $\mathcal{G}_L$ in $\mathfrak{A}/L$. Observe that $\pi_L(\mathcal{G}a)E_L = 0$ and, since E_L is dense in $\mathfrak{A}/L$, we have that $\pi_L(\mathcal{G}a) = 0$. Thus $\mathcal{G}a \in P$, whence $a \in \mathcal{G}^{-1}(P)$ and the previous inclusion is actually an equality. Furthermore, from $J \subseteq \mathcal{G}^{-1}(L)$ we have that $J \subseteq \mathcal{G}^{-1}(P)$ (note that $\mathcal{G}^{-1}(P)$ is the primitive ideal of A associated with $\mathcal{G}^{-1}(L)$, hence the largest ideal contained in the latter) and $\mathcal{G}J \subseteq P$. Consequently

$$\mathfrak{J} := \bigcap\{P \in \Pi_{\mathfrak{A}} : \mathcal{G}J \subseteq P\} \tag{3.10}$$

is a two-sided ideal.

Now, by the previous corollary we can see that $(\mathcal{G}|_J, \mathfrak{J})$ verifies condition (G2). In addition, for each $L \in \Lambda_{\mathfrak{A}}$ such that $\mathcal{G}J \nsubseteq L$ we have the following isomorphisms

$$\mathfrak{J}/\mathfrak{J} \cap L \cong \mathfrak{A}/L \quad \text{and} \quad J/J \cap \mathcal{G}^{-1}(L) \cong A/\mathcal{G}^{-1}(L), \tag{3.11}$$

implying that $(\mathcal{G}|_J, \mathfrak{J})$ also satisfies condition (G3) and so is a Gelfand theory for J. ■

Corollary 3.19 *Let A be a Banach algebra. Then $A^\#$ has a Gelfand theory if and only if A has one.*

Proof. One direction was already established. For the other, just note that A is an ideal of $A^\#$, thus if the latter has a Gelfand theory, by the previous proposition the former has one too. ■

Proposition 3.20 *Let A be a Banach algebra with a Gelfand theory and let J be a closed ideal of A. Then A/J has a Gelfand theory.*

Proof. Let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. Just as in Proposition 3.17, let

$$\mathfrak{J} := \bigcap\{L \in \Lambda_{\mathfrak{A}} : \mathcal{G}J \subseteq L\}.$$

Then $\mathfrak{A}/\mathfrak{J}$ is a C^*-algebra. Let $\pi : \mathfrak{A} \mapsto \mathfrak{A}/\mathfrak{J}$ be the quotient map. From the way in which $\mathfrak{J}$ was defined, it is clear that $\tilde{\mathcal{G}} := \pi \circ \mathcal{G}$ vanishes on J, thus is a well-defined algebra homomorphism from A/J to $\mathfrak{A}/\mathfrak{J}$.

Observe that there is a bijection between maximal modular left ideals of A that contain J and maximal modular left ideals of A/J. Also, by definition there is a bijection between maximal modular left ideals of $\mathfrak{A}$ that contain $\mathfrak{J}$

and maximal modular left ideals of A that contain J. Thus $(\tilde{\mathcal{G}}, \mathfrak{A}/\mathfrak{J})$ satisfies condition (G2). It is straightforward to check that it also satisfies condition (G3). Thus A/J has a Gelfand theory. ■

In the study of Banach algebras (or Banach spaces, making the obvious substitutions), a property that is satisfied by an algebra if and only if it is satisfied by a closed ideal and the quotient algebra induced by it is called a *three-space property.*

So far we have seen that given a Banach algebra A with a Gelfand theory $(\mathcal{G}, \mathfrak{A})$, if J is an ideal of A, then A/J and J have Gelfand theories. Thus, a natural question was raised at the end of [CIR02]: if the opposite implication holds, specifically, we would like to know if, given a Banach algebra A with an ideal J (closed, of course) such that both J and A/J have Gelfand theories, can we conclude that A has a Gelfand theory too?

This is one of the problems we focused on in this thesis. As we will see, we obtained a positive answer if we impose some rather strong restrictions on J and its Gelfand theory.

As a remark, let us note that if we have a Banach algebra A with an ideal J that possesses a unit I_J and a Gelfand theory $(\mathcal{G}_2, \mathfrak{A}_2)$, then we can extend $\mathcal{G}_2$ from J to A by setting

$$\widetilde{\mathcal{G}_2} : A \to \mathfrak{A}_2, \quad a \mapsto \mathcal{G}_2(aI_J). \tag{3.12}$$

In this situation, if A/J has a Gelfand theory $(\mathcal{G}_1, \mathfrak{A}_1)$, define

$$\iota : A \to A/J \quad a \mapsto a + J, \tag{3.13}$$

then we have the homomorphism

$$\Gamma : A \to \mathfrak{A}_1 \oplus \mathfrak{A}_2,\ a \mapsto \mathcal{G}_1 \circ \iota(a) \oplus \widetilde{\mathcal{G}_2}(a). \tag{3.14}$$

Since there is a bijection between maximal modular left ideals of A and maximal modular left ideals of $J \oplus A/J$, $(\Gamma, \mathfrak{A}_1 \oplus \mathfrak{A}_2)$ satisfies (G2) for A. It is also not too difficult to see that for each $L \in \Lambda_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$ the map $\Gamma_L : A/\Gamma^{-1}(L) \to (\mathfrak{A}_1 \oplus \mathfrak{A}_2)/L$ has dense range, therefore (G3) is also satisfied. In conclusion, if J and A/J have Gelfand theories and J has a unit, A has a Gelfand theory.

Now we show that if instead of letting J have a unit, we just assume it has a bounded approximate identity, we still obtain a Gelfand theory for A, but we have to impose further restrictions on the Gelfand theory of J.

As preparation, we need the following lemma. But first recall that given two normed spaces E and F, the **strong operator topology** on $\mathcal{B}(E, F)$—the set of bounded operators from E to F—is just the topology of pointwise convergence, i.e., $(T_\alpha)_\alpha \in \mathcal{B}(E, F) \to T$ if and only if $\|(T_\alpha - T)x\| \to 0$ for all $x \in E$.

Lemma 3.21 *Let A be a Banach algebra and let J be a closed ideal of A with a bounded approximate identity $(e_\alpha)_{\alpha \in \Delta}$. Then any continuous normed irreducible representation of J on a normed space E can be extended to A, and moreover this extension is also normed.*

Proof. Denote by $\rho : J \to \mathcal{B}(E)$ the irreducible representation of J in E. By [Pal94, Theorem 4.1.3], we know that $\rho(J)E = E$. Also, for any $\xi \neq 0$ in E, every $\eta \in E$ can be represented as $\eta = \rho(x)\xi$, for some x in J. Hence, for any a in A, just as in [DF88, V.2.2], we can define $\tilde{\rho}(a)$ (an extension of ρ to A) by:

$$\tilde{\rho}(a)\eta := \rho(ax)\xi = \lim_{\alpha \in \Delta} \rho(ae_\alpha x)\xi = \lim_{\alpha \in \Delta} \rho(ae_\alpha)\rho(x)\xi \tag{3.15}$$

$$= \lim_{\alpha \in \Delta} \rho(ae_\alpha)\eta, \text{ for all } \eta \in E. \tag{3.16}$$

Note that $\tilde{\rho}(a)$ is in the strong operator closure of $\rho(J)$. We will show that this is a well-defined extension of ρ from J to A and that it gives a normed representation. Indeed, given $a, b \in A$, $\eta \in E$, we have:

$$\tilde{\rho}(ab)\eta = \rho(abx)\xi = \lim_{\alpha\in\Delta} \rho(ae_\alpha bx)\xi = \lim_{\alpha\in\Delta} \rho(ae_\alpha)\rho(bx)\xi$$
$$= \lim_{\alpha\in\Delta} \rho(ae_\alpha) \lim_{\alpha\in\Delta} \rho(be_\alpha)\rho(x)\xi = \tilde{\rho}(a)\tilde{\rho}(b)\eta.$$

Also, if we let $K := \sup\{\|e_\alpha\|, \alpha \in \Delta\}$, then for any $a \in A$, $\eta \in E$:

$$\|\tilde{\rho}(a)\eta\| = \|\lim_{\alpha\in\Delta} \rho(ae_\alpha)\rho(x)\xi\| \leq \limsup_{\alpha\in\Delta} \|\rho\|\|ae_\alpha\|\|\eta\| \leq K\|\rho\|\|a\|\|\eta\|$$

and $\tilde{\rho}$ is a normed representation of A in E. ■

Now assume again that we have a C^*-algebra A with an ideal J that has a bounded approximate identity and such that A/J and J have Gelfand theories $(\mathcal{G}_1, \mathfrak{A}_1)$ and $(\mathcal{G}_2, \mathfrak{A}_2)$, respectively. If we identify the C^*-algebra $\mathfrak{A}_2$ with a subalgebra of $\mathcal{B}(\mathfrak{H})$, for some Hilbert space $\mathfrak{H}$, using the GNS construction, then $\mathcal{G}_2 J$ will be a subalgebra of $\mathcal{B}(\mathfrak{H})$, too. Therefore, $\mathcal{G}_2$ induces a normed representation of J on $\mathfrak{H}$. The construction of Lemma 3.21 allows us to extend that representation from J to A. Consequently, we can define a homomorphism from A to the C^*-algebra $\mathfrak{A}_1 \oplus \mathcal{B}(\mathfrak{H})$. The problem in this case is that we are representing A on a subalgebra of $\mathcal{B}(\mathfrak{H})$ that may be larger than $\mathfrak{A}_2$, hence we cannot guarantee that condition (G2) is satisfied.

We could say more if, for example, $\mathfrak{A}_2$ is a unital hereditary subalgebra of $\mathcal{B}(\mathfrak{H})$. In that case, from Theorem 2.42 and the subsequent remark, there would be a bijection between pure states of $\mathfrak{A}_2$ and pure states of $\mathcal{B}(\mathfrak{H})$. By the correspondence between pure states and maximal modular left ideals, there would be a one-to-one assignment between Λ_J and $\Lambda_{\mathcal{B}(\mathfrak{H})}$. In this setting there would be a bijection between Λ_A and $\Lambda_{\mathfrak{A}_1\oplus\mathcal{B}(\mathfrak{H})}$. But we still would not be able to say much about the fulfilment of condition (G3).

Nevertheless, we can use the construction of Lemma 3.21 to get a Gelfand theory for A if we impose another condition on the Gelfand theory of J, as we show in the next theorem.

Theorem 3.22 *Let A be a Banach algebra and let J be an ideal of A with a bounded approximate identity. If A/J and J have Gelfand theories $(\mathcal{G}_1, \mathfrak{A}_1)$, $(\mathcal{G}_2, \mathfrak{A}_2)$, respectively, such that $\mathfrak{A}_2$ is closed in the strong operator topology (given by the faithful representation of $\mathfrak{A}_2$ on the Hilbert space $\ell^\infty\text{-}\bigoplus_{L \in \Lambda_{\mathfrak{A}_2}} \mathfrak{A}_2/L$ obtained via the GNS construction), then A has a Gelfand theory too.*

Proof. Let $L \in \Lambda_{\mathfrak{A}_2}$, let $\mathfrak{H}_L$ denote the Hilbert space $\mathfrak{A}_2/L$, and let $\pi_L : \mathfrak{A}_2 \to \mathcal{B}(\mathfrak{H}_L)$ be the corresponding irreducible representation. Let $E_L \cong J/\mathcal{G}_2^{-1}(L)$ be the image of $\mathcal{G}_{2L}$ in $\mathfrak{H}_L$, so that E_L becomes a pre-Hilbert space. Let $\rho_L : J \to \mathcal{B}(J/\mathcal{G}_2^{-1}(L))$ be the irreducible representation of J given by

$$\rho_L(a)(x + \mathcal{G}_2^{-1}(L)) := ax + \mathcal{G}_2^{-1}(L) \ (a, x \in J). \tag{3.17}$$

Of course, as $E_L \cong J/\mathcal{G}_2^{-1}(L)$ holds algebraically, we can also view ρ_L as an irreducible representation of J on E_L. Since

$$\rho_L(a) = (\pi_L \circ \mathcal{G}_2)(a)\,|_{E_L} \ \ (a \in J) \tag{3.18}$$

it follows that $\rho_L(J) \subset \mathcal{B}(E_L)$, i.e., ρ_L is a normed representation of J on E_L. Since J has a bounded approximate identity, by the previous lemma we can extend ρ_L to a representation $\tilde{\rho_L}$ of A on E_L such that the image of this extension lies in the strong operator closure of the image of ρ.

Furthermore, by (G3), $\rho_L(J) \subseteq \mathcal{B}(\mathfrak{A}_2/L)$. Thus $\tilde{\rho_L}$ goes from A into $\mathcal{B}(\mathfrak{A}_2/L)$. We can identify $\mathfrak{A}_2$ with a subalgebra of $\ell^\infty\text{-}\bigoplus_{L \in \Lambda_{\mathfrak{A}_2}} \mathcal{B}(\mathfrak{A}_2/L)$; and by assumption this subalgebra is closed in the strong operator topology. Thus the map

$$\Gamma_2 := A \to \ell^\infty\text{-}\bigoplus_{L \in \Lambda_{\mathfrak{A}_2}} \mathcal{B}(\mathfrak{A}_2/L) \quad a \mapsto \tilde{\rho_L}(a),\, L \in \Lambda_{\mathfrak{A}_2} \tag{3.19}$$

is a homomorphism from A into $\mathfrak{A}_2$.

If we denote by ι the canonical map from A onto A/J and define

$$\Gamma : A \to \mathfrak{A}_1 \oplus \mathfrak{A}_2, \ \ a \mapsto \mathcal{G}_1 \circ \iota(a) \oplus \Gamma_2(a) \tag{3.20}$$

then by the argument following Proposition 3.20 we have that $(\Gamma, \mathfrak{A}_1 \oplus \mathfrak{A}_2)$ is a Gelfand theory for A. ■

Chapter 4

Gelfand Theory for C^*-Algebras

As we show in Chapter 2, for commutative C^*-algebras the classical Gelfand transform provides far more information than in the case of Banach algebras. We would expect something similar to happen with the generalized Gelfand theory. In this chapter we see that this is in fact the case and we can obtain stronger results in this setting.

It is obvious that if A is a C^*-algebra, then (Id, A) is a Gelfand theory for A. Now some questions arise, for example, when does a C^*-algebra A have a unique Gelfand theory, that is, when is any Gelfand theory of A equivalent to (Id, A).

In this chapter we will explore these issues. In particular, we will see that the Gelfand theory behaves very nicely in the case of liminal and postliminal C^*-algebras.

Let A be a C^*-algebra, and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. We know that for $L \in \Lambda_{\mathfrak{A}}$, $\mathfrak{A}/L$ and $A/\mathcal{G}^{-1}(L)$ are Hilbert spaces. For any Hilbert space $\mathfrak{H}$ and ξ, η in $\mathfrak{H}$ define the rank-one operator $\xi \odot \eta$ by:

$$x \mapsto \xi \odot \eta(x) = \langle x, \eta \rangle \xi \quad \text{for all } x \in \mathfrak{H}. \tag{4.1}$$

By the Riesz Representation theorem, we have that, since $\mathcal{G}_L$ is a linear map from the Hilbert space $A/\mathcal{G}^{-1}(L)$ to the Hilbert space $\mathfrak{A}/L$, the adjoint map $\mathcal{G}_L^*$ induces a map from $\mathfrak{A}/L$ to $A/\mathcal{G}^{-1}(L)$. With this preparation we are ready for the following lemma, which will be used frequently in the rest of this chapter.

Lemma 4.1 *Let A be a C^*-algebra, let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A, and let $a \in A$ be such that $\pi_{\mathcal{G}^{-1}(L)}(a) = \xi \odot \mathcal{G}_L^* \eta$, with $\xi \in A/\mathcal{G}^{-1}(L)$ and $\eta \in \mathfrak{A}/L$. Then $\pi_L(\mathcal{G}a) = \mathcal{G}_L \xi \odot \eta$ holds.*

Proof. First note that

$$\mathcal{G}_L \circ \pi_{\mathcal{G}^{-1}(L)}(a) = \pi_L(\mathcal{G}a) \circ \mathcal{G}_L \tag{4.2}$$

as operators from $A/\mathcal{G}^{-1}(L)$ to $\mathfrak{A}/L$. Thus by (G3), $\pi_L(\mathcal{G}a)$ is a rank-one operator such that its image is spanned by $\mathcal{G}_L\xi$. So there is a unique vector $\tilde{\eta} \in \mathfrak{A}$ such that $\pi_L(\mathcal{G}a) = \mathcal{G}_L \xi \odot \tilde{\eta}$. By 4.2 we have

$$\langle x, \mathcal{G}_L^* \eta \rangle = \langle \mathcal{G}_L x, \tilde{\eta} \rangle \quad (x \in A/\mathcal{G}^{-1}(L)) \tag{4.3}$$

therefore $\eta = \tilde{\eta}$ and $\pi_L(\mathcal{G}a) = \mathcal{G}_L \xi \odot \eta$. ■

For the next result, we need the following theorem, whose proof can be found in [Dix77, Corollary 4.1.10].

Theorem 4.2 *Let A be a C^*-algebra and let π be an irreducible representation of A on a Hilbert space $\mathfrak{H}$, then if $\pi(A) \cap \mathcal{K}(\mathfrak{H}) \neq \emptyset$ we have that $\pi(A) \supseteq \mathcal{K}(\mathfrak{H})$.*

Corollary 4.3 *Let A be a C^*-algebra and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. If A is liminal, so is $\mathfrak{A}$ and if A is postliminal, so is $\mathfrak{A}$.*

Proof. First, let us assume that A is postliminal. Let $L \in \Lambda_{\mathfrak{A}}$. From Lemma 4.1 we know that $\pi_L(\mathfrak{A}) \cap \mathcal{K}(\mathfrak{A}/L) \neq \emptyset$. Since π_L is irreducible for every $L \in \Lambda_{\mathfrak{A}}$, $\mathfrak{A}$ is postliminal, by Theorem 4.2.

Now assume that A is liminal. By the previous sentence $\mathfrak{A}$ is postliminal. Let $\mathfrak{J}$ be the largest closed liminal ideal of $\mathfrak{A}$. Assume that $\mathfrak{J} \subsetneq \mathfrak{A}$, so there exists $L \in \Lambda_{\mathfrak{A}}$ such that $\mathfrak{J} \subseteq L$. Since $\mathcal{G}_L$ is injective, so is $\mathcal{G}_L^*$. Thus all the rank-one operators on $A/\mathcal{G}^{-1}(L)$ can be written as $\xi \odot \mathcal{G}_L^* \eta$, $\xi \in A, \eta \in \mathfrak{A}/L$, so the closed linear span of vectors of this form is $\mathcal{K}(A/\mathcal{G}^{-1}(L))$. Then, using

Lemma 4.1 one more time, we have that $\mathcal{G}(A) \subseteq \mathfrak{J}$ and thus $\mathcal{G}^{-1}(L) = A \notin \Lambda_A$, a contradiction. ■

Recall that given two unital C^*-algebras A and B and a unital, contractive algebra homomorphism $\mathcal{G} : A \to B$, then $\mathcal{G}$ is a *-homomorphism and $\mathcal{G}A$ is a closed C^*-subalgebra of B [Pau02, Proposition 2.11].

In the following theorems we rely heavily on the next Stone-Weierstrass type theorem for postliminal C^*-algebras, whose proof can be found in [Dix77, Theorem 11.1.8].

Theorem 4.4 *Let A be a postliminal C^*-algebra and B a C^*-subalgebra of A. If B separates* 0 *and the set of pure states of A, then $B = A$.*

Proposition 4.5 *Let A be a unital, postliminal C^*-algebra, and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A such that $\|\mathcal{G}\| \leq 1$. Then $(\mathcal{G}, \mathfrak{A})$ and (Id, A) are equivalent.*

Proof. Since A is unital, we already know that $\mathfrak{A}$ is also unital and that $\mathcal{G}$ is a unital homomorphism. Thus $\mathcal{G}A$ is a C^*-subalgebra of $\mathfrak{A}$. Note that, by Corollary 4.3, we have that $\mathfrak{A}$ is also postliminal. Since $(\mathcal{G}, \mathcal{G}A)$ is also a Gelfand theory for A, by the correspondence between pure states and maximal modular left ideals, we have that $\mathcal{G}A$ separates 0 and the pure states of $\mathfrak{A}$, and so $\mathfrak{A} = \mathcal{G}A$, by Theorem 4.4. ■

For the next result we need a very strong theorem, whose proof is beyond the scope of this work and can be found in [Pis01, Theorem 7.5]. Observe that the notion of cyclic vector used in the next theorem is less strong than the one we defined in Chapter 1.

Theorem 4.6 *Let A be a unital C^*-algebra, let $\mathcal{H}$ be a Hilbert space, and let $\pi : A \to B(\mathcal{H})$ be a bounded unital homomorphism. Assume that π has a* *cyclic vector ξ, i.e. ξ is such that*

$$\overline{\pi(A)\xi} = \mathcal{H}.$$

Then there is an isomorphism

$$S : \mathcal{H} \to \mathcal{H} \quad \textit{with} \quad \|S\|\|S^{-1}\| \leq \|\pi\|^4, \tag{4.4}$$

such that

$$a \mapsto S^{-1}\pi(a)S \tag{4.5}$$

is a $^*-$representation.

Lemma 4.7 *Let A be a C^*-algebra and let π be a bounded representation of A on a Hilbert space $\mathfrak{H}$ such that there is a cyclic vector $\xi \in \mathfrak{H}$ for π. Then there is an invertible operator $S \in \mathcal{B}(\mathfrak{H})$ with $\|S\|\|S^{-1}\| \leq (1 + 2\|\pi\|)^4$ such that*

$$A \to \mathcal{B}(\mathfrak{H}), \quad a \mapsto S\pi(a)S^{-1} \tag{4.6}$$

is a $^* -$ representation.

Proof. Since there is a cyclic vector for π, if A is unital, π is a unital representation (observe that $\pi(I_A)$ will be the identity operator on a dense subset of $\mathfrak{H}$). In this case the claim is a consequence of Theorem 4.6, including the norm estimate.

In the case when A is not unital, let $A^{\#}$ be the unitization of A, and let

$$\pi^{\#} : A^{\#} \to \mathcal{B}(\mathfrak{H}), \;\; a \oplus \lambda \mapsto \pi(a) + \lambda I, \; a \in A, \lambda \in \mathbb{C}. \tag{4.7}$$

Since A is a maximal ideal of $A^{\#}$ such that $A^{\#}/A \cong \mathbb{C}$, it has a corresponding character (with norm less than 1, of course). Let us denote this character by ϕ, so then

$$\|\pi^{\#}(a)\| = \|\langle a, \phi\rangle I + \pi(a - \langle a, \phi\rangle)\| \leq (1 + 2\|\pi\|)\|a\| \quad a \in A^{\#}. \tag{4.8}$$

In this situation we can again apply Theorem 4.6. ■

Lemma 4.8 *Let A be a liminal C^*-algebra with Gelfand theory $(\mathcal{G}, \mathfrak{A})$. Then* *for each* $L \in \Lambda_{\mathfrak{A}}$*, the Gelfand transform induces an isomorphism between* $\mathcal{K}(A/\mathcal{G}^{-1}(L))$ *and* $\mathcal{K}(\mathfrak{A}/L)$*. The inverse of this isomorphism has norm at most* $(1+2\|\mathcal{G}\|)^4$.

Proof. We saw earlier that the linear span of the set $\{\xi \odot \mathcal{G}_L^*\eta, \, \xi \in A, \, \eta \in \mathfrak{A}/L\}$ is dense in $\mathcal{K}(A/\mathcal{G}^{-1}(L))$. Thus $\mathcal{G}$ induces a homomorphism $\tilde{\mathcal{G}} : \mathcal{K}(A/\mathcal{G}^{-1}(L)) \to \mathcal{K}(\mathfrak{A}/L)$ that has dense range (recall that the map $\mathcal{G}_L : A/\mathcal{G}^{-1}(L) \to \mathfrak{A}/L$ induced by $\mathcal{G}$ has dense range and that finite rank operators are dense in the set of compact operators).

Also, since the representation of A in $A/\mathcal{G}^{-1}(L)$ is irreducible, every non-zero vector is cyclic and so any non-zero vector in the image of $\mathcal{G}_L$ is cyclic for the representation of A in $\mathfrak{A}/L$. Since A and $\mathfrak{A}$ are liminal, the representation of A on $A/\mathcal{G}^{-1}(L)$ is $\mathcal{K}(A/\mathcal{G}^{-1}(L))$. Thus, by the foregoing, any vector in the image of $\mathcal{G}_L$ is cyclic for the representation $\tilde{\mathcal{G}}$.

By the previous lemma, there is an invertible operator $S \in \mathcal{B}(\mathfrak{A}/L)$ such that $\|S\|\|S^{-1}\| \leq (1+2\|\tilde{\mathcal{G}}\|)^4 \leq (1+2\|\mathcal{G}\|)^4$ and

$$\mathcal{K}(A/\mathcal{G}^{-1}(L)) \to \mathcal{K}(\mathfrak{A}/L), \;\; a \mapsto S(\tilde{\mathcal{G}}a)S^{-1} \tag{4.9}$$

is a *-homomorphism, hence it has closed dense range [Bla06, Corollary II.5.1.2]. Therefore, $\tilde{\mathcal{G}}$ is an isomorphism. Since (4.9) is a *-isomorphism, it has norm 1, which combined with the estimate for $\|S\|\|S^{-1}\|$ gives the desired bound for the norm of $\tilde{\mathcal{G}}^{-1}$. ■

Theorem 4.9 *Let* A *be a* C^**-algebra and let* $(\mathcal{G}, \mathfrak{A})$ *be a Gelfand theory for* A*. Then* $\mathcal{G}$ *has closed range in* $\mathfrak{A}$.

Proof. Again, for each $L \in \Lambda_{\mathfrak{A}}$, denote by π_L the irreducible representation of $\mathfrak{A}$ on $\mathfrak{A}/L$ and by E_L the image of $\mathcal{G}_L$ in $\mathfrak{A}/L$. Set $\rho_L := \pi_L \circ \mathcal{G}$. Note that ρ_L is a bounded representation with $\|\rho_L\| \leq \|\mathcal{G}\|$ and moreover, by (G3), every non-zero vector in E_L is cyclic for ρ_L. Applying the same argument used in the proof of Lemma 4.8, we can find a invertible operator $S \in \mathcal{B}(\mathfrak{A}/L)$ with $\|S\|\|S^{-1}\| \leq (1+2\|\mathcal{G}\|)^4$ and such that

$$A \to \mathcal{B}(\mathfrak{A}/L), \qquad a \mapsto S_L \rho_L(a) S_L^{-1} \tag{4.10}$$

is a *-homomorphism. Next, define

$$\theta : \mathfrak{A} \to \ell^\infty\text{-}\bigoplus_{L\in\Lambda_{\mathfrak{A}}} \mathcal{B}(\mathfrak{A}/L), \qquad a \mapsto (S_L \pi_L(a) S_L^{-1})_{L\in\Lambda_{\mathfrak{A}}}. \tag{4.11}$$

The image of $\theta \circ \mathcal{G}$ is a C^*-subalgebra of $\ell^\infty\text{-}\bigoplus_{L\in\Lambda_{\mathfrak{A}}} \mathcal{B}(\mathfrak{A}/L)$ (note that the image of θ is not necessarily self-adjoint). Therefore, A is *-isomorphic to the image of $\theta \circ \mathcal{G}$ and we can identify A with the aforementioned subalgebra of $\ell^\infty\text{-}\bigoplus_{L\in\Lambda_{\mathfrak{A}}} \mathcal{B}(\mathfrak{A}/L)$.

Now, let $(a_n)_{n=1}^\infty$ be a sequence in A such that $\mathcal{G}(a_n) \to b \in \mathfrak{A}$. Since $a_n = \theta \circ \mathcal{G}(a_n)$ for $n \in \mathbb{N}$, (a_n) is a Cauchy sequence. Denote by a the limit of (a_n) in A. It is straightforward to verify that $b = \mathcal{G}a$. ∎

From the previous result we have that, given a C^*-algebra A with a Gelfand theory $(\mathcal{G}, \mathfrak{A})$, $\mathcal{G}A$ is a closed subalgebra of $\mathfrak{A}$.

Next, we will investigate under what conditions this subalgebra generates $\mathfrak{A}$ as C^*-algebra, that is, when is $\mathfrak{A}$ the smallest C^*-subalgebra of $\mathfrak{A}$ that contains $\mathcal{G}A$. This is one of the open problems stated on [CIR02]. We are able to answer this question when A is postliminal. As we will see, the result is a consequence of Theorem 4.4.

In the following arguments we will denote by $C^*(\mathcal{G}A)$ the smallest C^*-subalgebra of $\mathfrak{A}$ that contains $\mathcal{G}A$.

Theorem 4.10 *Let A be a postliminal C^*-algebra and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. Then $\mathfrak{A}$ is the smallest C^*-subalgebra of $\mathfrak{A}$ that contains $\mathcal{G}A$.*

Proof. By Corollary 4.3, we already know that $\mathfrak{A}$ is postliminal. Let ϕ_1 and ϕ_2 be pure states of $\mathfrak{A}$ and denote by L_{ϕ_1} and L_{ϕ_2} the corresponding maximal modular left ideals of $\mathfrak{A}$. Assume that

$$\phi_1|_{C^*(\mathcal{G}A)} = \phi_2|_{C^*(\mathcal{G}A)}, \tag{4.12}$$

so that $L_{\phi_1} \cap \mathcal{G}A = L_{\phi_2} \cap \mathcal{G}A$. Otherwise, assume there exists $a \in L_{\phi_1} \cap \mathcal{G}A$ such that $a \notin L_{\phi_2} \cap \mathcal{G}A$, so $\phi_2(a^*a) \neq 0$, while $\phi_1(a^*a) = 0$, a contradiction, since $a^*a \in C^*(\mathcal{G}A)$.

By the properties of the Gelfand theory we know that there exists two unique maximal modular left ideals $\widetilde{L_1}, \widetilde{L_2}$ of A such that

$$\widetilde{L_1} = \mathcal{G}^{-1}(L_{\phi_1}), \widetilde{L_2} = \mathcal{G}^{-1}(L_{\phi_2}). \tag{4.13}$$

But we also have

$$\widetilde{L_1} = \mathcal{G}^{-1}(L_{\phi_1} \cap \mathcal{G}A) = \mathcal{G}^{-1}(L_{\phi_2} \cap \mathcal{G}A) = \widetilde{L_2}, \tag{4.14}$$

a contradiction. Hence $C^*(\mathcal{G}A)$ separates the pure states of $\mathfrak{A}$.

Now assume that there exists a pure state ψ on $\mathfrak{A}$ such that $\psi|_{C^*(\mathcal{G}A)} = 0$. Let L_ψ be the corresponding maximal modular left ideal of $\mathfrak{A}$.

Observe that $(\mathcal{G}|_{C^*(\mathcal{G}A)}, C^*(\mathcal{G}A))$ is also a Gelfand theory for A. Thus there exists a maximal modular left ideal in $C^*(\mathcal{G}A)$ corresponding to $\mathcal{G}^{-1}(L_\psi)$, and of course there is a pure state of $C^*(\mathcal{G}A)$ associated with this maximal modular ideal. This pure state can be extended to a pure state of $\mathfrak{A}$. The restriction to $C^*(\mathcal{G}A)$ of this extension is not zero, a contradiction. Thus $C^*(\mathcal{G}A)$ separates the pure states of $\mathfrak{A}$ and 0, and, since $\mathfrak{A}$ is postliminal, $\mathfrak{A} = C^*(\mathcal{G}A)$ holds, by Theorem 4.4. ■

Closely related to the previous result is the following:

Corollary 4.11 *Let A be a Banach algebra and let $(\mathcal{G}, \mathfrak{A})$ be a Gelfand theory for A. If $\mathfrak{A}$ is postliminal, then $C^*(\mathcal{G}A) = \mathfrak{A}$.*

Proof. Note that in the proof of the previous theorem we only used the assumption that A is a C^*-algebra (and postliminal) to establish that $\mathfrak{A}$ was postliminal. Hence the previous result clearly holds in the case when A is any Banach algebra, as long as $\mathfrak{A}$ is postliminal. ■

The foregoing suggests that when A is a postliminal C^*-algebra the range of a Gelfand theory is *very large*. This raises the question of the existence of a minimal Gelfand theory or if any two Gelfand theories for A are equivalent. We already saw that in case A is a unital postliminal C^*-algebra with a Gelfand theory $(\mathcal{G}, \mathfrak{A})$ such that $\|\mathcal{G}\| \leq 1$, $(\mathcal{G}, \mathfrak{A})$ is equivalent to (Id_A, A). Next, we will examine what we can say in a more general case.

Let A be a postliminal C^*-algebra and let $(\mathcal{G}_1, \mathfrak{A}_1)$ and $(\mathcal{G}_2, \mathfrak{A}_2)$ be two Gelfand theories for A. Define

$$\mathfrak{B} := C^*(\{(\mathcal{G}_1(a), \mathcal{G}_2(a)) : a \in A\}), \tag{4.15}$$

let

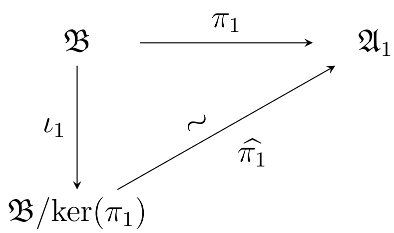
$$\pi_1 : \mathfrak{B} \to \mathfrak{A}_1, \quad (x_1, x_2) \mapsto x_1, \tag{4.16}$$

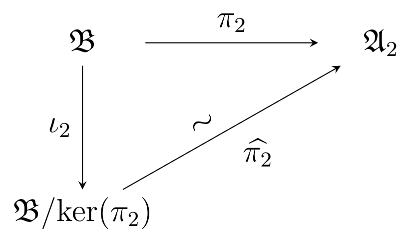
and define π_2 similarly.

Since A is postliminal, we know that $C^*(\mathcal{G}_i A) = \mathfrak{A}_i$ for $i = 1, 2$, so each of the maps π_i has dense range, and being *-homomorphisms also have closed range and thus are onto.

Set $\Gamma : A \to \mathfrak{B}, a \mapsto (\mathcal{G}_1 a, \mathcal{G}_2 a)$. We want to investigate when $(\Gamma, \mathfrak{B})$ is a Gelfand theory for A.

Let I_1, I_2 be the kernels of π_1, π_2, respectively. Then $\mathfrak{B}/I_1$ is *-isomorphic to $\mathfrak{A}_1$ and $\mathfrak{B}/I_2$ is *-isomorphic to $\mathfrak{A}_2$, so we have that the following diagrams commute:





Observe that for every $L \in \Lambda_{\mathfrak{B}}$ we have either $\pi_1(L) \in \Lambda_{\mathfrak{A}_1}$ or $\pi_2(L) \in \Lambda_{\mathfrak{A}_2}$, otherwise $\pi_i(L) = \mathfrak{A}_i, i = 1, 2$. If either $I_1 \in L$ or $I_2 \in L$, then by the previous diagrams one of the projections of L would be a maximal modular ideal. On the other hand, if there exists $(x, 0) \notin L$, then if we let ϕ be the pure state of $\mathfrak{B}$ that corresponds to L, since $\pi_1(L) = \mathfrak{A}_1$, there exists an element $(x, y) \in L$, and so $(x^*, 0)(x, y) = (x^*x, 0)$, hence $\phi(x^*x, 0) = 0$ and $(x, 0) \in L$, a contradiction. Thus if both maps are onto, then L contains the kernel of each of them, therefore $L/I_i \simeq \mathfrak{A}_i, i = 1, 2$, which is not possible.

So assume that $\pi_1(L) \in \Lambda_{\mathfrak{A}_1}$, then since $\widehat{\pi_1}^{-1} \circ \pi_1(L) \in \Lambda_{\mathfrak{B}/I_1}$, we have $I_1 \in L$. Now consider $\pi_2(L)$. If $I_2 \notin L$, then $\pi_2(L) = \mathfrak{A}_2$. On the other hand,

$$\mathcal{G}_1^{-1}(\pi_1(L)) = \mathcal{G}_2^{-1}(\overline{M}),\ \overline{M} \in \Lambda_{\mathfrak{A}_2}, \tag{4.17}$$

where $\overline{M}$ denotes the unique maximal modular left ideal of $\mathfrak{A}_2$ that contains $\mathcal{G}_2 \circ \mathcal{G}_1^{-1} \circ \pi_1(L)$.

If we denote by M the lifting of $\overline{M}$ to $\mathfrak{B}$ via $\mathfrak{B}/I_2$, then $I_2 \in M$ and

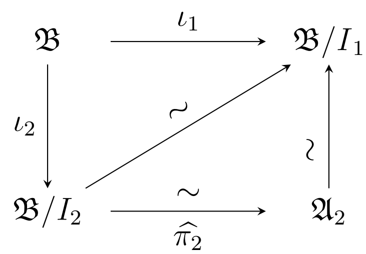
$$\Gamma^{-1}(L) = \Gamma^{-1}(M) \in \Lambda_A, \tag{4.18}$$

so in this case we have that the assignment $L \mapsto \Gamma^{-1}(L)$ is not bijective.

Thus, as long as we have an $L \in \Lambda_{\mathfrak{B}}$ such that either $I_1 \nsubseteq L$ or $I_2 \nsubseteq L$, then the assignment $L \mapsto \Gamma^{-1}(L)$ will not be bijective. And of course, if for each $L \in \Lambda_{\mathfrak{B}}$ we have that $I_1, I_2 \in L$, then the correspondence between $\Lambda_{\mathfrak{B}}$ and Λ_A will be bijective.

Hence, the bijection will exist if and only if each $L \in \Lambda_{\mathfrak{B}}$ contains both I_1 and I_2, but that would imply that $I_1 = I_2 = 0$, hence $(\Gamma, \mathfrak{B})$ will be a Gelfand theory for A if and only if $\mathfrak{B}$ is isomorphic to both $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Conversely, if we have that $(\mathcal{G}_1, \mathfrak{A}_1)$ and $(\mathcal{G}_2, \mathfrak{A}_2)$ are equivalent Gelfand theories, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ will be isomorphic and we will have the situation described in the following diagram:



Then, by the previous reasoning, we have that for any $L \in \Lambda_{\mathfrak{B}}$ either $I_1 \subseteq L$ or $I_2 \subseteq L$, but since $\mathfrak{B}/I_1$ is isomorphic to $\mathfrak{B}/I_2$, if $I_1 \subseteq L$ then $I_2 \subseteq L$. Of course, if $I_i \subseteq L$, $i = 1, 2$, then we can assign L to a unique maximal modular left ideal of A. This will be true for every maximal modular left ideal of $\mathfrak{B}$, thus $(\Gamma, \mathfrak{B})$ satisfies (G2). It is straightforward to check that (G3) is also verified, so $(\Gamma, \mathfrak{B})$ is a Gelfand theory for A too.

Thus, in this case we will again have that $I_1 = I_2 = 0$ and that $(\Gamma, \mathfrak{B})$ will be a Gelfand theory equivalent to $(\mathcal{G}_1, \mathfrak{A}_1)$ and to $(\mathcal{G}_2, \mathfrak{A}_2)$.

In conclusion, $(\mathcal{G}_1, \mathfrak{A}_1)$ and $(\mathcal{G}_2, \mathfrak{A}_2)$ will be equivalent Gelfand theories for A if and only if they are equivalent to $(\Gamma, \mathfrak{B})$.

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