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# On The Gracefulness of Girders and Big Wheels

by

David Edwin Adams ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science.

in

Mathematics.

Department of Mathematical Sciences

Edmonton, Alberta

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled *On the Gracefulness of Girders and Big Wheels* submitted by David E. Adams in partial fulfilment of the requirements for the degree of Masters of Science in Mathematics.

Bruce N. Allison
Dr. Bruce Allison

L Stewart
Dr. Lorna Stewart

Dr. Andy Liu

December 4, 1998.

**Abstract:**

The first section (consisting of the first three chapters) of this thesis is a survey of results on graph labelling exercises, specifically that of graceful graphs. Some of basic examples of graceful graphs are sampled (complete, 2-regular, wheels, prisms, etc.), along with the origins of the theory (Rosa's $\alpha$-, $\beta$-valuations). Cordial graphs and harmonious graphs are also introduced, showing some of the variations associated with graph labelling exercises.

The fourth and fifth chapters are a representation of my own research in this field. I introduce a variety of graphs which are both planar and self-dual: the girder graphs, the high-diameter wheel graphs of both square-mesh and diamond-mesh varieties, and two unusual graphs which have neither symmetries nor vertices of degree > 4. The gracefulness of these graphs is investigated: all girders but one are graceful, and partial results on the high-diameter wheels were obtained. The thesis concludes with a brief discourse on a computer-implemented algorithm that was developed to compute the gracefulness of a particular graph of small size.

**Acknowledgments:**

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To Adrienne Grove and Karoline Coleman, two participants in the W.I.S.E.S.T. program (1997 - mathematics) at the University of Alberta, for their bibliography on Graceful Graphs. Although the bibliography missed a few key references and had a lot of red herrings, it did save me a lot of effort in filling in key holes.

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To my supervisor, Dr. Andy Liu, who gave me the freedom and idea to explore this subject, and inadvertently taught me that there was more to mathematics than just "Theorem - Proof".

And lastly, to my parents Clifford and Evelyn Adams, to whom I owe my life, my conscience, and probably several thousand dollars in unpaid debts. Thank-you.

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# Chapter 1: Introduction.

## 1.1 Background.

The act of labelling a graph is a simple one. With little motivation except that which is given by whimsey, we can arbitrarily assign numbers, symbols, and even colours, to the vertices and the edges of a graph, and with no difficulty whatsoever. It is only when we try to apply restrictions to the labelling, or attach a meaning or a purpose to the labels we choose that this problem becomes complicated.

In this thesis, we shall discuss a few varieties of the problems that are found in graph labelling exercises. In particular we shall discuss the problem of a "graceful" labelling: what is one, and which graphs have them? The first chapter shall introduce the basic terminology and notion involved in defining graceful graphs, as well as introduce a few key examples. The second chapter is a cornucopia of examples sampled from the literature which shows the many varieties of graceful graphs. Chapter three shows some of the major variations to graceful labellings: specifically cordial and harmonious labellings. And chapter four is a reflection upon my own research into this area, attempting to determine the gracefulness of certain families of planar, self-dual graphs.

At the simplest level, the problem of labelling a graph with specific properties, is a recreational one: can you label this diagram in such a way etc.? For a particular graph, singularly focused, finding a graceful labelling is akin to solving this puzzle. Sometimes we succeed; the level of difficulty varies on the size and complexity of a particular example. At other times, we discover a graph which has no graceful labelling; such proofs are usually difficult. But when we discover a pattern, that one particular labelling scheme will work for another, we begin to discover the rich combinatorial design associated with graph labelling, and find joy when we finally succeed.

## 1.2 Terminology and Notation

The following section is a brief guide and index to the notation and terminology used in this thesis. Unless otherwise stated, a graph G = (V(G), E(G)) is an ordered pair of sets, where V(G) represents the vertices of the graph, and E(G) the edges of the graph. The set of edges is a set of unordered pairs of vertices of the graph. A graph is said to be simple if there are no loops (edges of the form (u,u) or uu where u is a vertex), or multiple edges between two vertices. Throughout this thesis it will be assumed that all graphs are of the simple variety.

- The size or cardinality of a finite set X will be denoted by both |X| and by #X.

- The null graph, denoted by $N_k$ (where k is a natural number) is the graph which consists of k vertices and no edges.

- The path, denoted by $P_k$, is the graph consisting of k vertices $v_1, v_2, ..., v_k$ and the k-1 edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n$.

- The cyclic graph, denoted by $C_n$, is the graph consisting of k vertices $v_1, ..., v_n$ and the edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1$. The cyclic graph is merely the path $P_n$ of length n save that the edge $v_nv_1$ is added.

- The (simplicial) join of two graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, is the graph whose vertices are $V(G_1) \cup V(G_2)$ and edges are $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in (G_2)\}$.

- The complete graph, denoted $K_p$ is the graph $N_1 + N_1 + ... + N_1$ (p times) or, recursively: $K_p = K_{p-1} + N_1$, $K_1 = N_1$, the complete graph $K_n$ having n vertices and n(n-1)/2 edges - any two vertices share an edge between them in such a graph.

- The product graph $G_1 \times G_2$ is the graph whose vertices are the ordered pairs $\{(u,v): u\in V(G_1), v\in V(G_2)\}$ and whose edges are of the form $\{ (u_1, v_1)(u_2, v_2): u_1=u_2$ and $v_1v_2 \in E(G_2)$, or $v_1=v_2$ and $u_1u_2\in E(G_1)\}$. The product graph has $\#V(G_1)\#V(G_2)$ vertices and $\#V(G_1)\#E(G_2)+\#V(G_2)\#E(G_1)$ edges.

- The graph $G_1(G_2)$ is the graph $G_1$, only each vertex of $G_1$ is the graph $G_2$ (including edges), with edges in $G_1$ associated with the join of $G_2$. The graph $G_1(G_2)$ will have $\#V(G_1)\ \#V(G_2)$ vertices and $\#V(G_1)\#E(G_2) + \#E(G_1)\ (\#V(G_2))^2$ edges.

### 1.3 Rosa's α- ,β-, γ-, δ-valuations

The following section shall trace the beginning of the theory of graceful graphs as found in the first paper written on the subject: a paper written by Alexander Rosa entitled "On certain valuations of the vertices of a graph" and published in 1967 [35].

Definition 1.1

a.) *A (vertex-)valuation $\psi$ is an injective map from the set of vertices $V(G)$ of a graph $G$ to the set of integers. An edge-valuation $\sigma$ is a map from the set of edges $E(G)$ of a graph $G$ to the set of integers. An edge-valuation is said to be* induced *by $\psi$ if $\sigma(uv)$ is a function of $\psi(u)$ and $\psi(v)$, wherever $uv$ is an edge between vertices $u$ and $v$, i.e. $\sigma(uv) = f(\psi(u), \psi(v))$, where $f$ is independent of $u$ and $v$. In this instance we often abbreviate $f(\psi(u), \psi(v))$ by just $\psi(uv)$.*

b.) *A binary valuation is a valuation mapping the set of vertices $V(G)$ to the integers mod 2. This induces the map $\sigma(uv) = \psi(u) + \psi(v) \pmod 2$.*

Unless there is more than one valuation in the discourse of a particular example, we shall omit the reference to a particular valuation: we will simply refer the clause "$v = j$" to mean "$\psi(v) = j$", or "$uv = j$" to mean "$\sigma(uv) = j$". Throughout the text, the word "valuation" shall be used interchangeably with the word "labelling": their meaning is interpreted as being identical with respect to every incarnation.

We denote $V_\psi$ as being the set of integers which are images of the vertex-valuation $\psi$. As is often the case, if an edge-valuation $\sigma$ is induced by $\psi$, then we denote $E_\psi$ to be the set of integers mapped to by the induced vertex-valuation $\sigma(\psi)$.

We now induce the following edge-valuation: given an edge uv we set $\sigma(uv) := |\psi(u)- \psi(v)|$ . That is, the edge is labelled by the absolute difference of the labels of the vertices which make up that edge. With this in mind we define the following:

Definition 1.2 *Given a simple graph G=(V,E) where #E = e, we define the following properties:*

(a): $V_\psi \subset \{0,1,\ldots,e\}$;

(b): $V_\psi \subset \{0,1,\ldots,2e\}$;

(c): $E_\psi \subset \{1,2,\ldots,e\}$;

(d): $E_\psi \subset \{x_1,x_2,\ldots,x_e\}$, where $x_i = i$ or $x_i = 2e + 1 - i$,

(e): There exists an $x \in \{0,1,\ldots,e\}$, such that for any given edge $(v_i,v_j) \in E(G)$, either $\psi(v_i) \leq x$ and $\psi(v_j) > x$, or $\psi(v_i) > x$ and $\psi(v_j) \leq x$. (This condition necessarily implies that the graph G is bipartite.)

Definition 1.3: *Given a graph G=(V,E); if there exists a valuation $\psi$ satisfying the properties from definition 2.2, specifically:*

(i). (a), (c), (e): *$\psi$ is an $\alpha$ - valuation*

(ii). (a), (c): *$\psi$ is a $\beta$ (graceful) - valuation*

(iii). (b), (c): *$\psi$ is a $\gamma$ - valuation*

(iv). (b), (d): *$\psi$ is a $\delta$ - valuation.*

Definition 1.4: *A graph is said to be graceful if the graph has a $\beta$ - valuation .*

The term "graceful graph", used to describe those graphs having a $\beta$ - valuation, did not appear anywhere in [35] - it appeared later in a paper by Golomb in [21] and has been used since.

There are two distinct possibilities for a graceful graph in how the edge label $e-1$ may be formed: either by using the vertex labels $\{1, e\}$ to bracket the edge, or by using the vertex labels $\{0, e-1\}$, where n is the number of edges of a graph). These two possibilities are related to each other by means of the arithmetic complement. In some contradiction arguments, we often restrict ourselves to the use of one or the other, usually the labelling such that vertex labels e and 1 are adjacent to one another.

Definition 1.5: *The arithmetic complement of a $\beta$ - valuation $\psi$ is the valuation $\psi'(v) = e - \psi(v)$, where e is the number of edges of a particular graph. (Figure 1.1)*

Since the edge labels induced by the arithmetic complement $\psi'$ to $\psi$ are identical to those produced by $\psi$, it follows immediately that $\psi'$ is a $\beta$ - valuation whenever $\psi$ is.

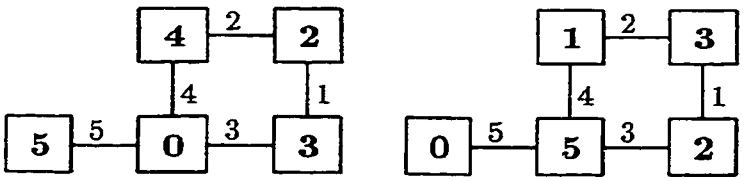


Figure 1.1 - A graceful labelling of a graph (left), with arithmetic complement (right).

With the exception of the path $P_2$ and the trivial graph $N_1$, the arithmetic complement of a ß-valuation will produce a valuation different than that provided by any graph symmetry. This is due to the fact that the vertex labels used to produce edge label {e-1} change from {0, e-1} to {e,1} and vice versa. If the action on the vertex labels caused by taking the arithmetic complement was also a graph automorphism, then either the edge label {1} is repeated, or e < 2. It immediately follows that the number of unique labellings of any graceful graph is divisible by 2 times the size of its automorphism group. This fact is useful when attempting to search for a graceful labelling, as it eliminates many configurations from consideration.

A graph is said to be Eulerian if there exists an unbroken cycle which traverses all the edges of a graph exactly once. (A graph is Eulerian if and only if it is connected and the degree of all the vertices is even.) The following theorem is quite useful, as it is one of the few principal negation-type results in graceful graph labelling:

Theorem 1.6 (Parity condition for graceful Eulerian graphs): *Let G be a graph whose vertices are all of even degree (if G is connected, then G is Eulerian). If G is graceful then* $e = |E(G)| \equiv 0$ *or 3 mod 4.*

Proof: Each component of a graph G must necessarily be Eulerian, so we may represent each component by a sequence of vertices $u_0, u_1, u_2, \ldots, u_k = u_0, v_0, v_1, v_2, \ldots v_j = v_0, \ldots$ whose disjoint edge paths traverse all the edges in G. Given a β- valuation, the edges $|u_0 - u_1|, |u_1 - u_2|, \ldots, |u_{k-1} - u_k|, | v_1 - v_0 |, \ldots, |v_j - v_{j-1}|, \ldots$ must necessarily be a permutation of the set $\{ 1, 2, \ldots, e\}$. Reducing all labels modulo 2, it follows that each individual cycle must have a sum congruent to 0 mod 2 as $| u_1 - u_0 | + \ldots + |u_k - u_{k-1}| \equiv | u_k - u_0 | \equiv 0$. So the binary sum of all the cycles is 0 modulo 2. However, $1 + 2 + \ldots + e \equiv 0 \bmod 2$ if and only if $e \equiv 3$ or $e \equiv 0$ modulo 4.

This parity criterion allows us to immediately exclude certain families of graphs from being graceful. The first example is given below:

Theorem 1.7: *The cycle graph* $C_n$ *is graceful if and only if* $n \equiv 3 \bmod 4$ *or* $n \equiv 0 \bmod 4$.

Proof: The cycle graph is regular of degree 2, so theorem 1.6 applies for these cases. An explicit valuation (and this is by no means a unique valuation) for the cases of when $n \equiv 0 \bmod 4$ or $n \equiv 3 \bmod 4$ are given below:

Case 1: $n = 4k$. Label the vertices $v_1,\ldots,v_n$ by the following:

| | | |
|---|---|---|
| $v_i =$ | $(i-1)/2$ | $i$ odd |
| | $n+1-i/2$ | $i$ even, $i \le n/2$ |
| | $n-i/2$ | $i$ even, $i > n/2$ |

*The valuation used in case 1 is an α-valuation. The missing vertex label out of the set {0, 1, ..., 4n} is 3n — this is not a trivial coincidence (See section 2.1).*

Case 2: $n = 4k+3$. Label the vertices $v_1,\ldots,v_n$ by the following:

| | | |
|---|---|---|
| $v_i =$ | $n+1 - i/2$ | $i$ even |
| | $(i-1)/2$ | $i$ odd, $i \le (n-1)/2$ |
| | $(i+1)/2$ | $i$ odd, $i > (n-1)/2$ |

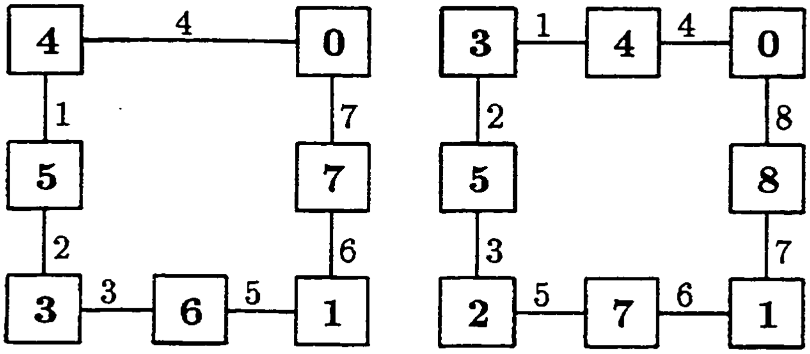


Figure 1.2: Graceful labelling of cycle graphs $C_7$ and $C_8$.

The converse of the parity condition given in theorem 1.7 is not true; we shall soon see that the complete graphs $K_{1-8t}$ and $K_{3+8t}$ (where $t > 0$ ) are both Eulerian, with the number of edges being congruent to 4 and 3 mod 4 respectively, and yet not graceful.

### 1.4 Complete (and almost complete) Graphs

One of the first families of graphs which were decided upon is the family of complete graphs $K_n$ and the complete bipartite graphs $K_{m,n}$.

Theorem 1.8: *The complete graph $K_n$ is graceful if and only if $n \leq 4$.*

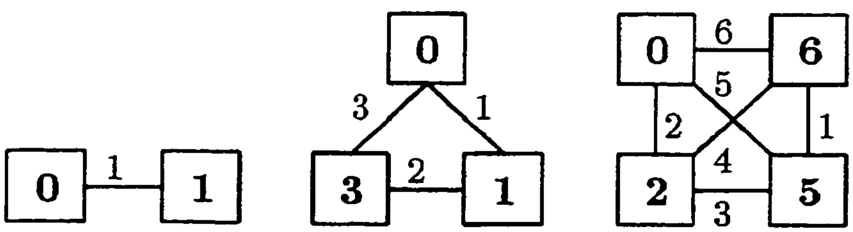


Figure 1.3: Graceful labelling of the complete graphs $K_2$, $K_3$, and $K_4$.

Proof: Graceful valuations of $K_2$, $K_3$, and $K_4$ are easily exhibited. Let e be the number of edges in the complete graph $K_n$ with vertices $v_1, v_2, ..., v_n$ where n>4. Suppose there is a graceful valuation $\psi$ on $K_n$. Without loss of generality (due to both symmetry and arithmetic complements) we have $\psi(v_1)=0$, $\psi(v_2)=e$, and $\psi(v_3)=1$. The only way to have $\psi(uv)=e-2$, is to either use the labels {0,e-2}, {1,e-1}, or {2,e}. Using $\psi(v_4)=2$ or $\psi(v_4)=e-1$ repeats the edge label "1", so $\psi(v_4)=e-2$. (In the case n=4, e=6, this would otherwise provide a graceful labelling of $K_4$.)

A contradiction is obtained when we try to pick a vertex label for $v_5$ so that we obtain the edge labels e-4: using {e-4,0} or {e-1,3} repeats edge label 2, using {e-3,1} or {e-2,2} repeats edge label 1, hence $v_5 = 4$. But in the case n=5, this repeats edge label 4. In the case n > 5, a similar analysis shows that this will prevent the formation of edge label e-5.

(A weaker version of this theorem, whose proof involves exploiting the solution to a diophantine equation, will be provided in section 3.4.)

The problem with the larger complete graphs is in that they seem to have too many edges, so one could ask the following question: if we could delete some edges from the complete graph $K_n$, how many (and which ones) would we have to delete in order to obtain a subgraph that is graceful? We have the following definition from a paper by Bloom and Golomb [9]:

Definition 1.9: *Given a positive integer n, an increasing sequence of positive integers* $0=v_1 < v_2 < v_3 < ... < v_n$ *is said to be a* restricted difference basis *for n, if the set formed by the elements* $v_j - v_i$ $(j > i)$ *— possibly with repetitions — is equivalent to the set* $\{1, 2, ..., v_n\}$. *Denote* $\Gamma(K_n)$ *to be the largest possible value k such that n has a restricted difference basis with* $v_n = k$.

The definition of the restricted difference basis (attributed to Leech [28]) allows us to answer question 1 above in the following manner: Given a restricted difference basis $\{v_i\}$ ($i = 1...n$) for n, we label the vertices of the complete graph $K_n$ by $v_i$. It is clear from theorem 1.8 that for $n > 4$ this will not induce a graceful labelling; the edge labels, in any case, shall possess inclusively all the integers from 1 to $v_n$. By deleting those edges which repeat induced labels, we obtain a subgraph of the complete graph which is graceful. In a superfluous manner, all graceful graphs can be obtained in the manner — although it may be the case that several edges would have to be deleted. Indeed, consider the graph $C_{11}$ with 11 edges: considering it as a subgraph of the complete graph $K_{11}$ with 55 edges, we would have to trim 80% of the edges — this is by no means an efficient method!

It then follows that the value of $\Gamma(K_n)$ in definition 1.9 gives the maximum number of edges in a graph with n nodes so that the graph has a graceful labelling. Consequently, the number of edges which we need to delete from $K_n$ to obtain a graceful graph is given by $n(n-1)/2-\Gamma(K_n)=\alpha(n)$.

There are many partial results for given values of n (found by computer search), specifically the following: (Figure 1.4 shows an example of graceful labelling of $K_5-\{e\}$ and $K_6-\{e_1, e_2\}$, two edge-deleted subgraphs of $K_5$ and $K_6$ respectively).

| n: | $\Gamma(K_n)$: | basis $\{v_1, v_2, ..., v_n\}$ |
|---|---|---|
| 5 | 9 | {0, 1, 4, 7, 9} |
| 6 | 13 | {0, 1, 2, 6, 10, 13} |
| 7 | 17 | {0, 1, 4, 10, 12, 15, 17} |
| 8 | 23 | {0, 1, 4, 10, 16, 18, 21, 23} |

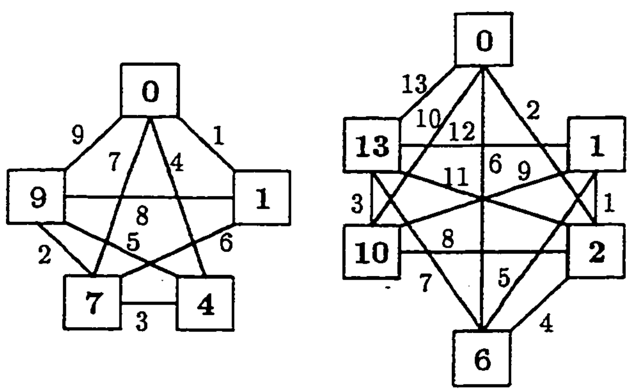


Figure 1.4: Largest subgraphs of $K_5$, $K_6$ respectively which has a graceful labelling.

An unpublished result of Erdös, stated in [Golomb, 21] claims that $\Gamma(K_n) \sim n^2$ and $\alpha(n) \sim cn^2$ where c is approximately 1/4. The definition of $\Gamma(K_n)$ can be extended for any graph G: $\Gamma(G)=k$ is the highest vertex label possible in G such that the edges labelled by $|v_j - v_i|$ (where $v_iv_j$ is an edge) produces the set 1, 2, ..., k. It is entirely clear that if A is a subgraph of B then $\Gamma(A) \leq \Gamma(B)$, since for any subgraph A one can't do worse than that for B. And, of course, a graph G is graceful if and only if $\Gamma(G) = \#E(G)$.

The bipartite complete graph $K_{m,n}$ ($= N_m + N_n$) is a graph consisting of m+n vertices, and mn edges which lie exclusively between two sets of vertices (of size n and m). Unlike their counterparts $K_n$, the family of bipartite complete graphs are quite graceful. Indeed, we have the following:

Theorem 1.10: *The bipartite complete graph $K_{m,n}$ has an α-valuation for all positive integers m , n.*

Proof: Let $u_1, u_2, ..., u_m$ and $v_1, v_2, ..., v_n$ be the vertices which make up the bipartitions of the complete graph $K_{m,n}$. Then set $\psi(u_i) = i-1$ for $i = 1, ..., m$, and $\psi(v_j) = jm$ for $j=1, ..., n$). The resultant edge labels are then of the form $jm-i$ where $i < m$, $j \leq n$; with each of $jm - i$ being distinct for each choice of (i,j) we obtain the edges 1, 2, ..., mn. ψ is then an α-valuation. An alternate labelling scheme is taken by reversing the parameters m and n - i.e. labelling the graph $K_{n,m}$. (See figure 1.5).

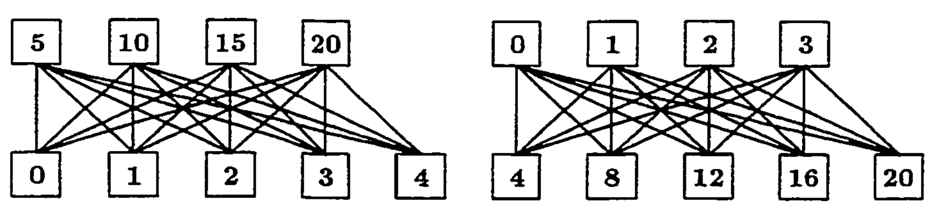


Figure 1.5: Two different ways to gracefully label the complete bipartite graph $K_{4,5}$.

### 1.5 Graceful Trees and the Ringel-Kotzig Conjecture.

A tree is a simply connected graph consisting of precisely n vertices and n−1 edges. The tree is thus a bipartite graph without any cycles whatsoever. The term *interlaced* was first used for some trees by Koh, Tan, and Rogers in 1978 [27]: a graceful tree is said to be interlaced if and only if the tree has an α-valuation.

The following question was posed in the 1960's:

Conjecture (Ringel-Kotzig): *There is a decomposition of the complete graph $K_{2n+1}$ into 2n+1 disjoint, isomorphic copies of an arbitrary tree T consisting of n edges.*

Rosa [35] had showed that this conjecture was equivalent to the following:

Conjecture: *All trees are graceful.*

This simple question is unsolved to this day, and has produced the largest volume of literature on any subject on graceful graphs. A few of the simpler cases are given below:

Theorem 1.11: *All paths ($P_n$) are graceful for all $n > 0$.*

Proof: Let $v_1, ..., v_n$ make up the path of length n. Set $v_i = i - 1$ for i odd and $v_i = 2n + 1 - i$ for i even. (This is also an α-valuation.)

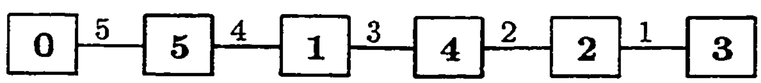


Figure 1.6: Graceful labelling of the path $P_6$.

An even stronger result, although not proved here (see [33]), is the following:

Theorem 1.12: *All paths, with but one exception, have an α-valuation such that an arbitrary vertex is assigned 0 — the exception being the path $P_5$ where the vertex given the label 0 is in the middle of the path.*

A caterpillar is a tree consisting of two sets of vertices $\{v_1, v_2, ... v_i\}$, $\{u_1, u_2, ..., u_j\}$, and edges consisting of the path $u_1u_2...u_j$ and edges of the form $v_xu_y$ (where $1 \leq x \leq i$, $1 \leq y \leq j$). Alternately, a caterpillar is a tree whose subgraph consisting of those vertices of degree $\geq 1$ is a path. (Figure 1.7 shows a typical example of a caterpillar, with associated graceful labelling).

Theorem 1.13: *All caterpillars are graceful.*

Proof: We will inductively construct a graceful labelling of a caterpillar, with the induction based on the number of vertices in the longest path in the tree.. If the length of the longest path is equal to 3, then the tree is also a bipartite complete graph: the middle vertex is labelled q (where q is the number of edges in the tree), the other vertices (being incident to the middle vertex) are to be labelled 0, 1, 2, ..., q-1.

Let T be a caterpillar with n+r edges and longest vertex path $u_0, u_1, u_2, u_3, ..., u_{k}.$, where $k \geq 3$. Here we have $u_0$ and $u_k$ as end-vertices, with $u_{k-1}$ being adjacent to $u_k$ and $u_{k-1}$ of degree r+1 (the remaining vertices incident to $u_{k-1}$ being $w_1, w_2, ..., w_{r-1}$.) Consider the subgraph $T'$ formed by deleting vertex $u_k$, and vertices $w_1, w_2, ..., w_{r-1}$. The subgraph $T'$ is a caterpillar and would have a graceful labelling $\psi'$ by induction hypothesis in a manner such that $u_{k-1} = 0$. We then form $\psi$ by labelling $w_j = n + j$ (for $j = 1, 2, ..., r-1$) and $u_k = n + r$. Then the arithmetical compliment of $\psi$ gives us the desired labelling. (Note that this algorithm will provide an $\alpha$-valuation.)

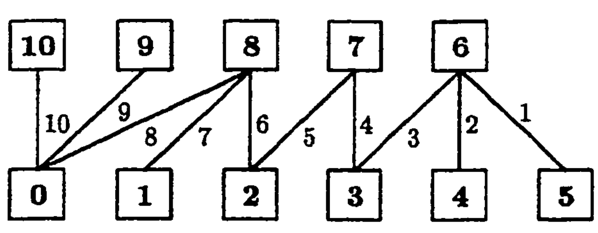


Figure 1.7: A caterpillar with graceful labelling.

A crab is a tree whose subgraph consisting of those vertices with degree > 1 is a caterpillar. It is still an open problem today as to whether even all crabs are graceful. Figure 1.8 shows an example of a crab tree with a graceful labelling. It is certainly untrue that all crab trees possess α-valuations: the graph shown in figure 1.9 has no α-valuation. However, there is one published result, attributed to Zhao [41], which states that:

Theorem 1.14: *All trees of diameter ≤ 4 are graceful. That is, every tree whose longest path consists of 5 vertices and 4 edges is graceful.*

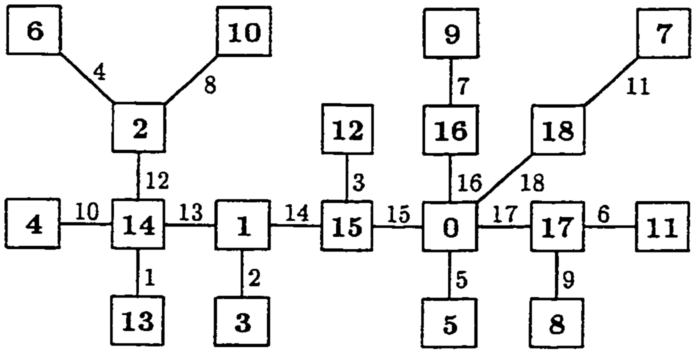

Figure 1.8: A large crab tree with graceful labelling.

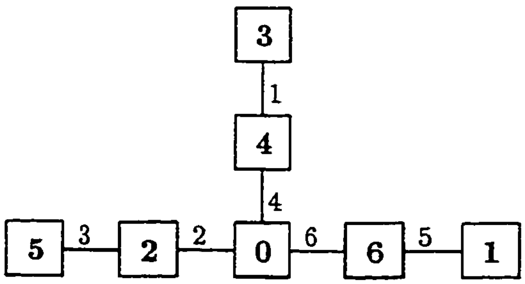

Figure 1.9: Smallest crab-tree without an α-valuation.

The usual method of deriving new families of trees being graceful is to take certain families of trees which are known to have α-valuations (such as paths and complete k-ary trees) and "glue" them onto other trees with at least β-valuations in order to produce larger families of trees which are graceful. It is hoped that most of the families of trees can be shown to be graceful in this method in order to solve the Ringel-Kotzig Conjecture. One of the more successful examples to date of this is proved in a paper by Chen, Lu, and Yeh [13]. The following is an example of this method:

Theorem 1.15: *Let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be trees with vertex bipartitions $(X_1 \cup Y_1)$ and $(X_2 \cup Y_2)$ respectively. Let $\psi_1$ be an α-valuation of $T_1$ and $\psi_2$ be a β-valuation of $T_2$. A valuation $\psi$ on $T_1 \cup T_2$ is given by $\psi(v)=$*

$$\begin{array}{ll} \psi_1(v) & \text{if } v \in X_1 \\ \psi_2(v) + |X_1| & \text{if } v \in X_2 \cup Y_2 \\ \psi_1(v) + |X_2 \cup Y_2| & \text{if } v \in Y_1 \end{array}$$

*The tree T given by $T_1 \cup T_2 + (uv)$, where $u \in V_1$, $v \in V_2$, and $|\psi(u) - \psi(v)| = |X_2 \cup Y_2|$, is then graceful by means of $\psi$.*

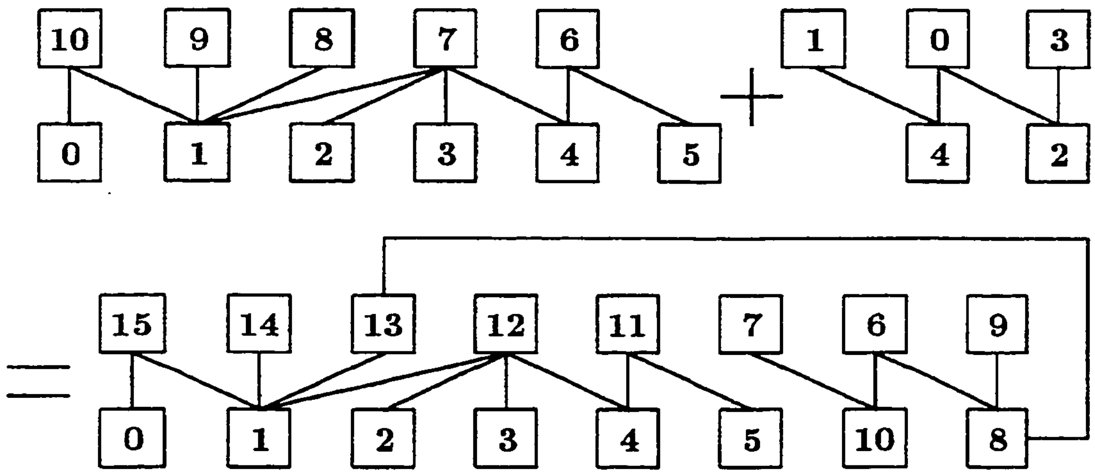


Figure 1.10: An example of Theorem 1.15 in action. The upper trees are $T_1$ and $T_2$. The edge connecting the vertices labelled 13 and 8, can be replaced by any edge whose vertices from different subtrees have labels differing by 5.

## Chapter 2: Potpourri of Results

This chapter is a selective survey on results gathered on graceful graphs, beyond those simple graphs discussed in Chapter 2.

### 2.1 2-regular graphs:

A graph is said to be 2-regular if all the vertices of the (not necessarily connected) graph all have degree 2. As we are only considering simple, finite graphs, 2-regular graphs then consist of disjoint unions of cycles. In section 1.1, it was shown that any graph whose vertices are all of even degree must have an edge count of the form 4k+3 or 4k, in order for that graph to be graceful. This was shown by using an Eulerian parity condition on each component to force the proper parity on the sum of the edges.

In a paper by Abraham and Kotzig [1] it was shown that in the case of the graph having precisely two components and being 2-regular, that the edge count condition is also sufficient. That is, every 2-regular graph with two components and 4k+3 or 4k edges is graceful. In addition, Abraham and Kotzig had also shown in [1] that the graph $C_p \cup C_q$ has an α-valuation if and only if both p and q are even and $p+q \equiv 0 \bmod 4$. *(Figure 2.1 shows a graceful labelling of the graph $C_6 \cup C_6$.)*

However the restriction that there be only two components is a serious one. Also stated in [1], is the result that for all positive integers $p \geq 11$ there exists a 2-regular graph with $|E| \equiv 3$ or 0 mod 4, and yet is not graceful, the first counter-example being $C_4 \cup C_4 \cup C_3$. However, certain families of 2-regular graphs with 3 components are not only graceful, but have α-valuations:

a.) $p, q > 1, p+q \leq m$: $C_{4p} \cup C_{4q} \cup C_{4m}$

b.) $p \geq 1, q \geq 2, p+q \leq m$: $C_{4p-2} \cup C_{4q-2} \cup C_m$.

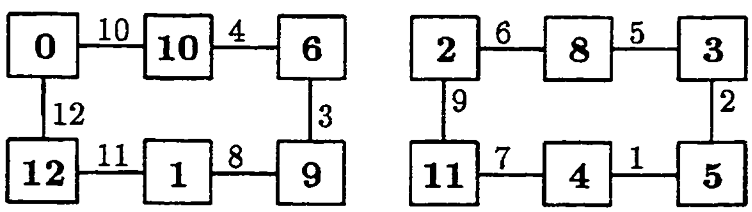


Figure 2.1: Graceful labelling of the graph $C_6 \cup C_6$. This is not an α-valuation, although one exists.

Another family of 2-regular graphs that was considered by Abraham and Kotzig in [3] is the family $A_k$, which consists of k disjoint copies of the cycle $C_4$. It is manually verifiable that $A_k$ has an $\alpha$-valuation for $k \leq 10$ excepting the graph $A_3$. Also in the above paper is the following result proved:

Theorem 2.1: *If $A_k$ has an $\alpha$-valuation, then $A_{4k-1}$, $A_{5k-1}$ and $A_{9k-2}$ also has an $\alpha$ valuation.*

On a curious note, it was shown in [2] that if G is a 2-regular, graceful graph with 4r vertices, then the vertex label not used (x, say) in the valuation, must satisfy the relation $r \leq x \leq 3r$. Furthermore, x=r or x=3r if and only the valuation is in fact an $\alpha$-valuation of G.

## 2.2 Triangular Snakes:

A triangular snake is a connected graph whose blocks are triangles arranged in a path (linked at the corners). The triangular snake $T_n$ then consists of 2n+1 vertices and 3n edges, where the vertices are all of even degree, so the parity condition applies: the triangular snakes $T_n$ cannot be graceful if $n \equiv 2$ or 3 mod 4. The other cases are resolved by the following theorem (found in [31]):

Theorem 2.2: *Every triangular snake $T_n$, when $n \equiv 0$ or 1 mod 4, is graceful.*

The proof of this theorem relies on random guesswork for the cases $n \leq 7$. For $n \geq 8$, the labelling is formed by means of a "Steiner Triple System", which was first used in [34] for this purpose.

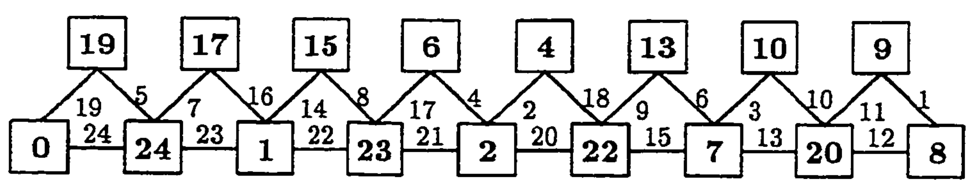


Figure 2.2: The triangular snake $T_8$ with graceful labelling.

## 2.3 θ-graphs

A θ-graph is a graph consisting of two vertices of degree 3 which are joined by three disjoint paths. (A θ-graph literally looks like the Greek character θ).

A conjecture, which was attributed to Bodendiek, Schumacher and Wegner in 1977 [10], is that all graphs consisting of a cycle and a chord are graceful. This cycle plus chord combination is an immature form of the θ-graph, where the smallest path is of length 1.

This first conjecture was verified in a paper published in 1980 by Delorme, Maheo, et. al. in [15], and so the more general question of all θ-graphs were considered. Their result was later extended in 1985 that all cycles with the chord $P_3$ were also graceful [26]. The family of θ-graphs was finally found to be graceful in 1986 by N. Pabhapote and N. Punnim using a severe case analysis [32], albeit a partial proof using matrix theory can be found in [12]. As the proof by Punnim and Pabhapote requires analysis of 64 separate cases, it shall not be included in its entirety, although two of these cases will be covered in appendix A as examples.

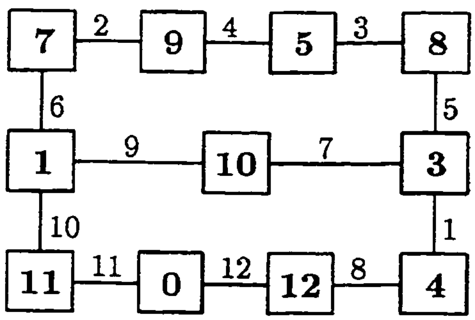


Figure 2.3: The θ-graph with cycle $C_{10}$ and chord $P_3$, with graceful labelling.

## 2.4 Graphs with dihedral symmetry: wheels, prisms, and some of their relatives

(Note: there are at least two families of graphs with dihedral symmetry, the high order wheel graphs and the girder graphs, each of which are detailed in sections 4.3—4.4. They are not included here as they possess a much more interesting property than just radial symmetry.)

The wheel graph $W_n = C_n + K_1$ was first conjectured to be graceful by Hebbare, and was later proved by Frucht in 1979 [18].

Theorem 2.3: *The wheel graph $W_n$ is graceful for all $n \geq 3$.*

The gear graph, which is taken from the wheel graph by subdividing each and every one of the edges, was shown to be graceful by Ma and Feng in [16]. *(Figure 2.4 shows the wheel graph $W_7$ with $\beta$-valuation, and the accompanying gear graph with 7 spokes is shown in figure 2.5 with $\beta$-valuation.)*

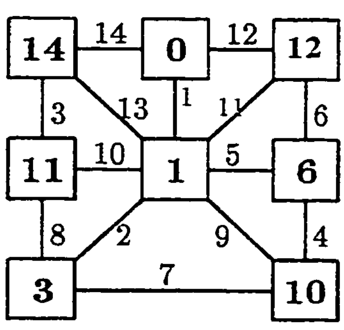


Figure 2.4: The wheel graph $W_7$.

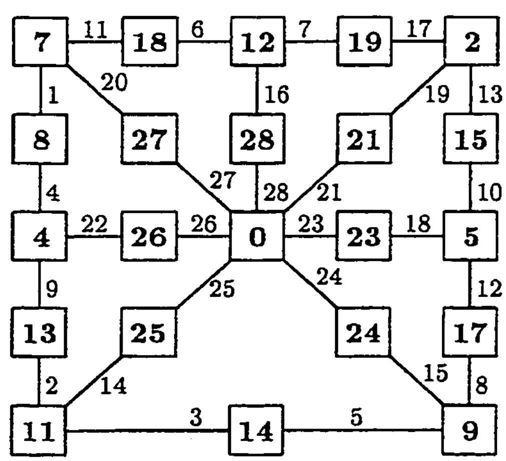


Figure 2.5: The gear graph with 7 spokes.

A graph which is similar to the wheel graph is the fan graph $P_n + K_1$, consisting of n+1 vertices and 2n−1 edges.

Theorem 2.4: *The fan graph $P_n + K_1$ is graceful for all $n \geq 3$. In fact, the graph $G + K_1$ is graceful for any graceful tree G.*

Proof: Let ψ be any graceful labelling of the tree $P_n$ (which exists by theorem 1.11). Set ψ′ on $P_n + K_1$ with $\psi'(v_i) = \psi(v_i)$ where $v_i$ is the path $(v_1, v_2, ..., v_n)$ in $P_n + K_1$, and let $\psi'(u) = 2n-1$ (where u is the single vertex of large degree). *(Figure 3.5 shows the fan graph $P_8 + K_1$ with β-valuation.)*

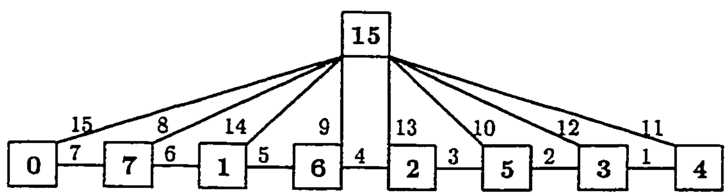


Figure 2.6: The fan graph $P_8 + K_1$, with graceful labelling.

In addition to the wheel graphs $C_n + K_1$, we have the prism graphs $C_n \times P_2$. This family has been studied in detail, including both edge-deleted and vertex-deleted variants.

Theorem 2.5

*a) The prism graphs $C_n \times P_2$ are graceful for all positive integers $n \geq 1$.*

*b) The vertex-deleted subgraph $C_n \times P_2 - \{u\}$ is graceful for all $n \geq 2$.*

*c) The edge-deleted subgraph $C_n \times P_2 - \{uv\}$ is graceful for all $n \geq 2$, where $\{uv\}$ is any edge in the graph.*

(This result is proven in [17,25].)

Gallian and Frucht [17] notes that the prism graph $C_n \times P_2$ has an $\alpha$-valuation precisely when n is even. *(Figure 2.7 gives an $\alpha$-valuation for the prism graph $C_8 \times P_2$).* Although it was stated in [19] that the cylindrical grid $C_n \times P_m$ has a graceful labelling for whenever both m and n are even, no such proof was published. However, successful results has been obtained for the following three cases: $C_{2m} \times P_n$ (for $n \geq 2$) [24], $C_{2m+1} \times P_n$ (for $3 \leq n \leq 12$) [ibid.], and $C_{4m+2} \times P_{4n+3}$ [39].

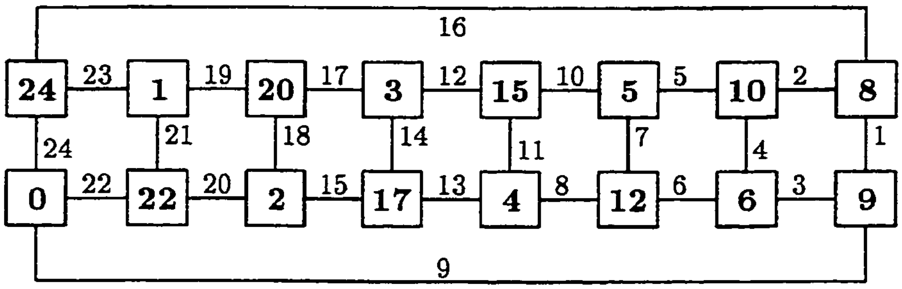


Figure 2.7: The prism graph $C_8 \times P_2$, with $\alpha$-valuation.

## 2.5 Miscellaneous examples

The following is a list (by no means an exhaustive one) of assorted families which are also graceful.

A crown $CR_n$ (although the notation $C_n^{(P_2)}$ is sometimes used) is a graph consisting of the cycle $\{u_1, u_2, ..., u_n\}$ and the vertices $\{v_1, v_2, ..., v_n\}$ with the edges $u_iv_i$ for $1 \leq i \leq n$ pendant to the cycle. ($CR_n$ has 2n vertices and 2n edges).

Theorem 2.6: *All crowns are graceful.*

Proof: Also proven by Frucht in [18]. Figures 2.8 and 2.9 (on the following page) show a graceful labelling for the cases n=7 and n=8.

Case 1: $n \equiv 0 \bmod 2$.

| | | |
|---|---|---|
| $v_i =$ | $2n - (i-1)$ | $i$ odd |
| | $i-1$ | $i$ even, $i \leq n/2$. |
| | $i$ | $i$ even, $i > n/2$. |
| $u_i =$ | $i-1$ | $i$ odd, $i < n/2$ |
| | $i$ | $i$ odd, $i > n/2$ |
| | $2n + 1 - i$ | $i$ even |

Case 2: $n \equiv 1 \bmod 2$.

| | | |
|---|---|---|
| $v_i =$ | $2n + 1 - i$ | $i$ odd, $i \leq n+1/2$ |
| | $2n - i$ | $i$ odd, $i > n+1/2$ |
| | $i - 1$ | $i$ even |
| $u_i =$ | $i - 1$ | $i$ odd |
| | $2n + 1 - i$ | $i$ even, $i \leq n+1/2$ |
| | $2n - i$ | $i$ even, $i > n+1/2$. |

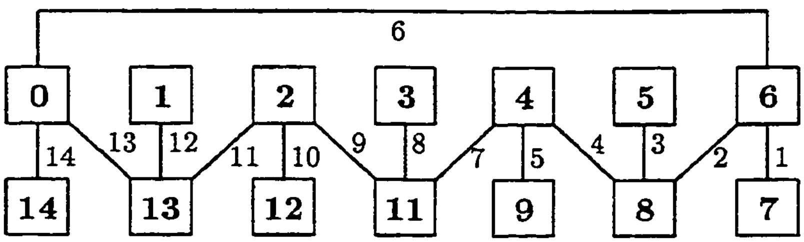


Figure 2.8: The crown graph $CR_7$

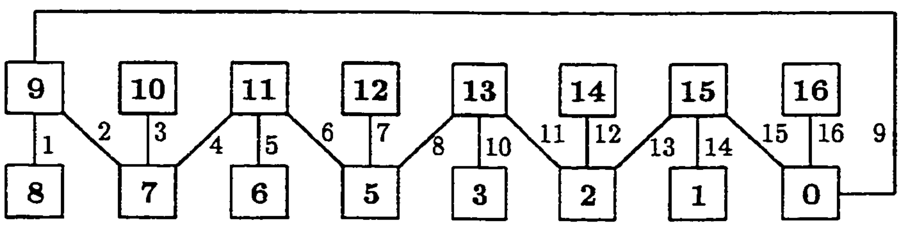


Figure 2.9: The crown graph $CR_8$.

Theorem 2.7: *The grid graph $P_m \times P_n$ is graceful for all choices of $m,n \geq 1$.*

Proof: Let $\{ s(i,j) : 1 \leq i \leq m , 1 \leq j \leq n \}$ denote the vertex labels for the product graph $P_m \times P_n$. The graph then consists of mn vertices and $e = 2mn - m - n$ edges. Set

| | | |
|---|---|---|
| $s(i, j) =$ | $(i-1)(m+n-2)/2 + (j-1)/2$ | i odd, j odd |
| | $e + 1 - (i-1)(m+n-2)/2 - j/2$ | i odd, j even |
| | $e + 1 - n - (i-2)(m+n-2)/2 - (j-1)/2$ | i even, j odd |
| | $n + (m+n-2)(i)/2 + (j)/2 - 2$ | i even, j even |

The edges are then labelled in decreasing order, going column by column from vertex s(1,1). This labelling scheme will actually produce an α-valuation. (Figure 2.10 shows the graceful labelling of the graph $P_5 \times P_4$, using the above proof.)

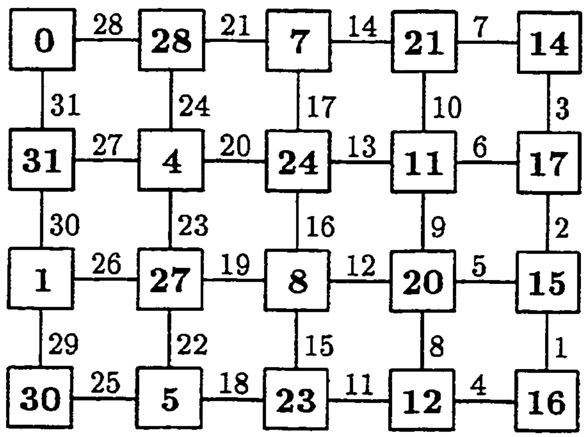


Figure 2.10: Graceful labelling of $P_5 \times P_4$.

One family of grids, the torus grids $C_m \times C_n$, has remained unsolved. $C_m \times C_n$ cannot be graceful if both m and n are odd, as this would otherwise violate the parity condition for Eulerian graphs (theorem 1.6) — the graph would be regular of degree 4 and have edge count $2mn \equiv 2 \bmod 4$. The only other case which has been decided is that of when n is even and $m \equiv 0 \bmod 4$ [42]. The first otherwise undecided case (m=4, n=3) is given on the following page:

Conjecture 2.8: *The torus grid $C_m \times C_n$ is graceful whenever one of {m, n} is even.*

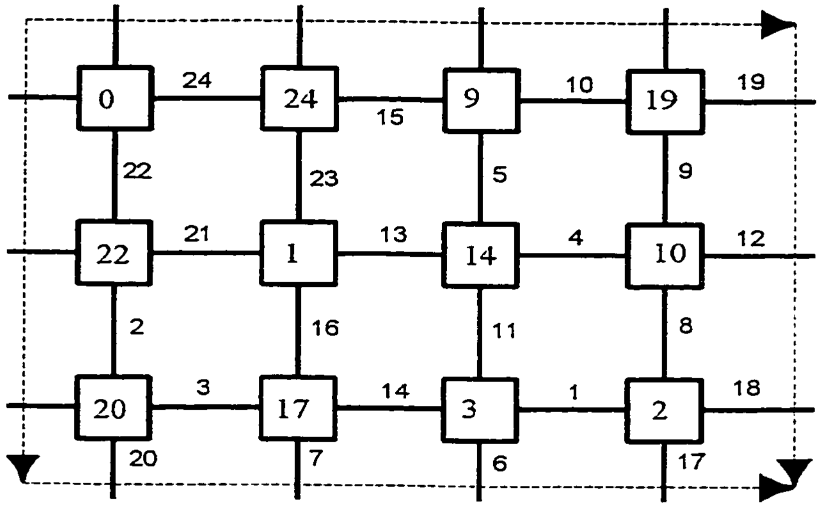


Figure 2.11: Torus grid $C_4 \times C_3$, with graceful labelling.

The windmill $K_m^n$ is the graph consisting of n copies of $K_m$ with one vertex in common. The case of when m = 3 are referred to as either Dutch n-windmills or friendship graphs, and the case of m=4 as French n-windmills.

Theorem 2.9: *The Dutch n-windmills are graceful precisely whenever n ≡ 0 or 1 mod 4.*
Proof: The proof follows from a solution to a problem by Skölem: for what values of m is it possible to partition the integers {1, 2, 3, ..., 2m} into pairs $(a_i, b_i)$, $1 \leq i \leq m$, such that $b_i - a_i = i$? (A solution is given in [37]). Given that the triangles are given by $(r, u_i, v_i)$ with vertex r being common, and given the sequence $\{(a_i, b_i)\}$ above, a graceful labelling scheme is derived by setting $\psi(r) = 0$, $\psi(u_i) = 2m + a_i$, and $\psi(v_i) = 2m+b_i$.

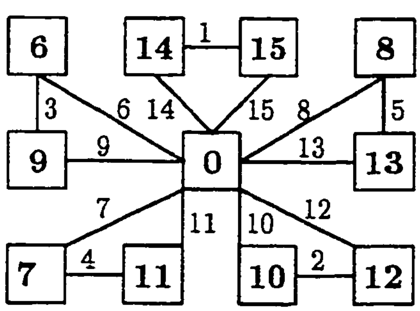


Figure 2.12: The Dutch 5-windmill.

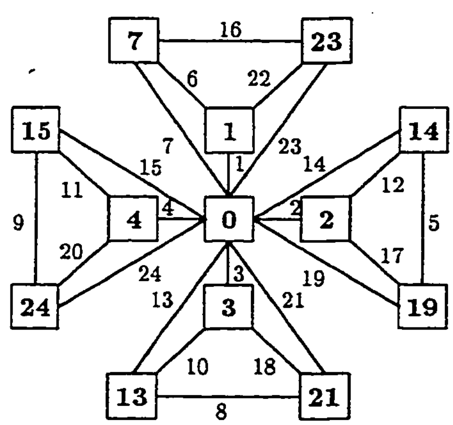


Figure 2.13: The French 4-windmill.

Conjecture 2.10: *The French n-windmills are graceful for n=1 or n ≥ 4.*

(The French 1-windmill is just $K_4$. The French 2-windmill and 3-windmill are not graceful [7]. The windmills are part of a larger question concerning difference bases and when they occur. A generalization of these graphs is given in section 4.1.)

Theorem 2.11: *If the windmill $K_m^n$ is graceful, then $m < 6$.*

A book consists of n copies of the cycle graph $C_4$ with one edge in common; alternately, a book is the product graph $K_{1,n}\times P_2$ — so called because of it's similarity to a book. (The book graph consists of 2n+2 vertices and 3n+1 edges.)

Theorem 2.12: *The books $K_{1,n}\times P_2$ are graceful if and only if $n \neq 3 \mod 4$.*

Proof: Found in [14, 30]. Necessity arises from the parity condition on Eulerian graphs whenever n is odd.

Also in [14] is the following:

Theorem 2.13: *The graphs consisting of n copies of $K_4$ with exactly one edge in common, $(N_n(P_2) + P_2)$, also known as $K_4$-books, is graceful.*

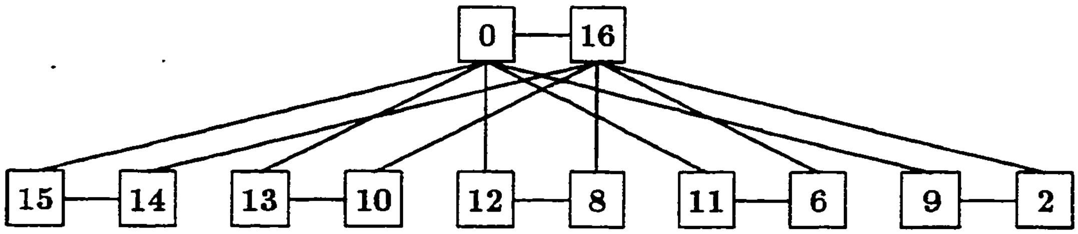


Figure 2.14: The book $K_{1,5} \times P_2$, with 5 pages

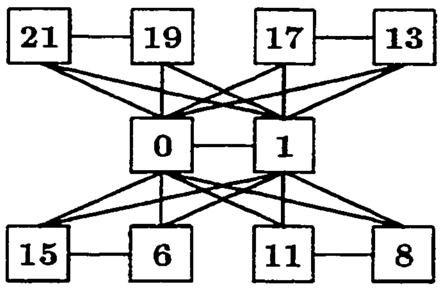


Figure 2.15: The $K_4$-book with 4 pages.

## 3. Extensions, Generalizations, and Modifications to the Basic Theory of Graceful Graphs

Most modifications to the basic theory are the result of the simple question: "What if ?" The realization that some certain graphs just won't lie amicably (pun intended) down to a graceful labelling has resulted in the tinkering of the notion of what comprises a ß-valuation in order to say that these abhorrent graphs at least have some sort of unusual property. Alternately, one could impose an even stronger condition than having a ß-valuation — i.e., the α-valuation which was discussed in section 1.3 — and see which graphs satisfy the new criterion. Or, one can take an even more bizarre tangent by changing the labelling scheme which is induced by vertex labelling. We shall examine at least a few of these "what-ifs" which comprise a more enriched theory.

### 3.1 K-Gracefulness

One possible generalization of "gracefulness" is to that of "k-gracefulness". This notion was arrived at independently by Slater [38], and by Maheo and Thuiller[29]. The term "k-graceful" is used for graphs whose edges are enumerated by successive integers starting with the positive integer k. The term "arbitrarily graceful" is used to denote those graphs which are k-graceful for any choice of k — this definition was first coined by Acharya [4].

Definition 3.1: *A graph is said to be k-graceful (where k is a positive integer), if there exists an injective mapping* $\psi: V \rightarrow \{0,1,2, \ldots, e+k-1\}$, *where e is the number of edges, such that the induced labelling* $\psi(v_i v_j) = |\psi(v_i) - \psi(v_j)|$ *produces the set* $\{k, k+1, \ldots, e+k-1\}$.

Definition 3.2: *A graph is said to be arbitrarily graceful if and only if it is k-graceful for all positive integers k.*

The following establishes a clear connection between k-graceful graphs and graphs with α-valuations:

Theorem 3.3: *All graphs with α-valuations (see section 2.1) are arbitrarily graceful.*

Proof: Given a graph G with α-valuation ψ, let q be the positive integer such that all edges are between vertex labels ≥ q and ≤ q. Then, to obtain a k-graceful labelling we use: $\psi'(v) = \psi(v) + k - 1$ if $\psi(v) \geq q$ and $\psi'(v) = \psi(v)$ otherwise.

The following results can be found in [29]:

Theorem 3.4: *The cyclic graph $C_n$ is k-graceful if and only if one of the following conditions hold:*

*a) $n \equiv 0 \bmod 4$, (the cycle graph $C_n$ has an α-valuation in this case)*

*b) $n \equiv 1 \bmod 4$, k is even and $2k \leq n - 1$,*

*c) $n \equiv 3 \bmod 4$, k is odd and $2k \leq n - 1$.*

Theorem 3.5: *The wheel graph $W_{2k-1}$ is k-graceful for any $k \geq 1$.*

Theorem 3.6: *If there exists a k-graceful labelling of the m-windmill (see section 2.5), then either:*

*a) $n=5$, m even and $m \geq 4k - 2$,*

*b) $n = 4$, $m \geq 2k - 1$, or*

*c) $n = 3$, $m \geq 2k - 1$ and if k is odd: $m \equiv 0,1 \bmod 4$,*

*or if k is even: $m \equiv 0,3 \bmod 4$.*

Proof: Found in [7, 8]. The case of when n=3 also happens to be a sufficiency condition. This question was not originally posed in terms of d-gracefulness, but as a combinatorial problem arising from radio-astronomy [7]. Maheo and Thuiller had also conjectured that the wheel graph $W_{2k}$ is k-graceful for $k \geq 5$, but this is known to be false when $k = 3$ or $k=4$.

### 3.2 "Nearly" Graceful

The property of being "nearly graceful" is a weakened variant of gracefulness which will allow at least some families of graphs — to wit, those graphs which are Eulerian but yet fail the parity condition in section 1.3 — to obtain some measure of gracefulness.

Definition 3.7: *A graph is said to be nearly graceful if there exists a vertex-valuation $\psi$, being an injection into the set $\{0, 1, ..., e+1\}$, such that the induced edge-labels (equal to the absolute difference in vertex labels, just as for graceful valuations) form an e-element subset of $\{1, 2, ..., e+1\}$.* (Definition 3.7 can be found in both [31] and [34].)

Another form of "nearly" graceful can be found in [31] — called "almost" graceful. An almost graceful labelling is a nearly graceful labelling such that the omitted edge label is either $\{e\}$ or $\{e+1\}$. (One example is the triangular snakes as discussed in section 2.2). This is largely a matter of hair-splitting: "almost" graceful implies "nearly" graceful, and there seems to be no advantage to differentiating between the two criteria, so given the choice between the two, both "nearly" and "almost" should apply to the former.

Theorem 3.8: *All cyclic graphs $C_n$ are "nearly" graceful.*

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of the cyclic graph $C_n$ (with edges $v_i v_{i+1}$ mod n). The cyclic graphs $C_n$ are already graceful for $n \equiv 3$ or 0 mod 4. An explicit labelling scheme is given below for the cases $n \equiv 1$ or 2 mod 4:

| | | | |
|---|---|---|---|
| $n = 4k+1$: | $v_i =$ | $(i-1)/2$ : | $i$ odd, $i \neq n$ |
| | | $(n+3)/2$: | $i = n$ |
| | | $n+2 - i/2$: | $i$ even, $i \leq (n-1)/2$ |
| | | $n+1 - i/2$: | $i$ even, $i > (n-1)/2$. |
| $n = 4k+2$: | $v_i =$ | $(i-1)/2$: | $i$ odd, $i \neq n-1$ |
| | | $n/2$: | $i = n-1$ |
| | | $n+2-i/2$ | $i$ even, $i < (n-1)/2$ |
| | | $n+1-i/2$ | $i$ even, $(n-1)/2 < i < n$ |
| | | $n/2 + 2$ | $i = n$. |

(Of course, 2-graceful implies "nearly" graceful.)

### 3.3 Harmonious Graphs

Instead of maintaining the scheme of labelling the edges by subtracting the absolute value of the incident vertex labels, we adopt a different scheme. This time we add the vertex labels and take their sum modulo e (where e is the number of edges), and produce an entirely different problem: harmonious graphs. Much of the groundwork on harmonious graphs was laid out in [22] by Graham and Sloane in 1980.

Definition 3.9: *Suppose there is a vertex valuation $\psi$ on $V(G)$ of a graph G such that $V_\psi \subseteq \{1, ..., e\}$. The graph G is said to be harmonious if the valuation $\sigma(uv) = \psi(u) + \psi(v) \pmod e$ induced by $\psi(V)$ is such that $E_\sigma \equiv \{0, 1, 2, ..., e-1\}$. The valuation (or labelling) $\psi$ is said to be harmonious in this case. In the case of when the number of vertices is one greater than the number of edges (i.e. for trees), then exactly one of the vertex labels modulo e may be repeated.*

It is not the case that the set of harmonious graphs and the set of graceful graphs are either disjoint or coincident, as this next result shows:

Theorem 3.10: *The cycle graphs $C_n$ are harmonious if and only if n is an odd number.*

Proof: The sum of the edge labels is given by $(v_1 + v_2) + (v_2 + v_3) + ... + (v_n + v_1) = 2(v_1 + v_2 + ... + v_n) = n(n-1)$, since each of the numbers 1, 2, 3, ..., n must be used exactly once as vertex labels. This sum is then equal to zero modulo n. On the other hand, the sum of the edge labels is given by $1 + 2 + ... + n = n(n-1)/2$. If n is an even number, then $n(n-1)/2$ is not zero modulo n. This shows necessity. For sufficiency, we have the following harmonious labelling for $C_n$, when n is odd: $v_i = (i - 1) / 2$ for i odd, $v_i = (n+i-1) / 2$ for i even.

A survey of graphs which are known to be harmonious can be found in [19, 22, 36]. There is a labelling scheme related to harmonious labelling — called a felicitous labelling — which is analogous to nearly graceful graphs in comparison to graceful graphs. In a felicitous labelling, the vertices are to be labelled by distinct integers {0, 1, ..., n} mod n+1, with edge labels induced by the sum mod n. A survey of results on felicitous graphs can be found in [36], noting that "felicitous" and "harmonious" have identical meaning in the case of trees.

Unlike that for graceful graphs (excepting those possessing α-valuations), harmonious graphs have the following cyclic structure:

Theorem 3.11: *Let $\psi$ be a harmonious labelling for a graph G with n edges, and let a, b be elements of $Z_n$ such that a is invertible. Then the valuation $\psi(v)' := a\psi(v)+b \pmod n$ is also an harmonious labelling.* (See [22].)

As immediate corollaries: if a graph is harmonious, then there is a harmonious labelling such that any arbitrary vertex is given the label 0, and that in any harmonious tree, any particular label can repeat itself (mod n).

Harmonious graphs have the following parity condition which is analogous to that for graceful graphs (Theorem 1.6).

Theorem 3.12: *Let G be a harmonious graph with an even number of edges e. If the degree of every vertex is divisible by $2^k$, for some $k > 0$, then e is divisible by $2^{k-1}$.*

Proof: Let d(v) denote the degree of a vertex v, $\psi(v)$ the vertex label of a harmonious labelling $\psi$. Then the sum of the edge labels induced by $\psi$ is given by the sum $0+1+\ldots+e-1=e(e-1)/2$. The sum of these labels is also given by $\sum_{v\in V} d(v)\psi(v)$. Since $2^k$ divides d(v), $2^k$ must also divide $e(e-1)/2$, so $2^{k+1}$ divides $e(e-1)$. That is, $2^{k+1}$ divides e.

As an example, the book $P_2 \times K_{1,7}$, (from section 2.5) a graph consisting of 22 edges, with every vertex being of even degree, is not harmonious.

As for graceful graphs, the complete graph $K_n$ is harmonious if and only if $n < 5$ [22]. For complete bipartite graphs, the result is far different:

Theorem 3.13: *The complete bipartite graph $K_{m,n}$ is harmonious if and only if $m=1$ or $n=1$.*

Proof: The graph $K_{1,n}$ is harmonious: set $\psi(u)=0$ and label the vertices in the opposite bipartition with the numbers $\{0, 1, ..., n-1\}$. If $m > 1$ and $n > 1$, then the sets $A=\{\psi(u_i) \mid u_i \in U\}$, $B=\{\psi(v_i) \mid v_i \in V\}$, (where U, V are the bipartitions of the vertex set) form a disjoint, direct-sum decomposition of the abelian group $Z_{mn}$ under addition. This means that the elements $\{a+b \mid a \in A, b \in B\}$ are distinct elements of $Z_{mn}$. But then $\{a-b \mid a \in A, b \in B\}$ would also be distinct elements of $Z_{mn}$. That means $a-b=0$ and hence $a=b$ for some $a \in A$, $b \in B$ — contradiction.

## 3.4 Binary gracefulness

The following definition and idea were used to attempt to solve a particular problem (detailed in chapter 4), but were scrapped in favour of something else. However, the idea of what I call "binary gracefulness" was at least a little useful:

Definition 3.14: *Consider a labelling $\psi$ on the vertices of a graph such that $V_\psi = \{0, 1\}$, inducing the edge labelling $\sigma(uv) \equiv \psi(u) + \psi(v) \mod 2$. The graph is said to be binary graceful or well balanced if $|\{v \in V: \psi(v)=0\}| \leq 1+e/2$, and $1 \geq |\{e \in E: \psi(e)=1\}| - |\{e \in E: \psi(e)=0\}| \geq 0$. Any valuation of V(G) which shows G to be well balanced is said to be a binary $\beta$-valuation (or binary graceful labelling).*

The preceding definition was called a binary labelling in Golomb [21]. On the surface such a definition could be considered fanciful. However, finding vertex valuations which are binary β-valuations for a graph allows one to search more efficiently for a β-valuation when using brute force. Hence the usefulness in such a definition being formulated is in the application of a computer search.

The following result is quite trivial:

Theorem 3.15: *All graceful graphs are binary graceful. All harmonious graphs with an even number of edges are binary graceful.*

Proof: Reduce all labels in a graceful or harmonious labelling modulo 2.

We can use this to prove a slightly weaker result than theorem 1.8:

Theorem 3.16: *The complete graph $K_n$ is graceful only if n or n-2 is a square number.*

Proof: Suppose $K_n$ has a graceful labelling $\psi$, hence is well-balanced. Let $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$ ($p+q = n$) be the vertices of $K_n$ such that $\psi(u_i) \equiv 0 \bmod 2$ and $\psi(v_i) \equiv 1 \bmod 2$. (Unless n = 1, both these sets are non-empty.) There are two cases, assuming that n > 4 (the cases n = 2, 3, 4 are already dealt with):

Case 1: The number of edges $\#E = (n)(n-1)/2$ is even (i.e. $n \equiv 0$ or 1 mod 4). It then follows that there are $p(p-1)2 + q(q-1)/2 = (n)(n-1)/4$ edges with even parity. That is:

$$n(n-1) = 2p^2 - 2p + 2q^2 - 2q.$$

which, when simplified using the identity $p+q = n$, yields:

$$4q^2 - 4nq + (n^2 - n) = 0.$$

Solving this equation as a quadratic polynomial in terms of q yields the solution for both p and q:

$$q = ½ ( n \pm \sqrt{n} ), p = ½ ( n \mp \sqrt{n} ).$$

Since both p and q are positive integers, it immediately follows that n is a square number.

Case 2: The number of edges $\#E = (n)(n-1)/2$ is odd (i.e. $n \equiv 2$ or 3 mod 4). In a similar vein, there are $p(p-1)/2 + q(q-1)/2 = (n)(n-1)/4 - 1$ edges with even parity. That is:

$$n(n-1) - 2 = 2p^2 - 2p + 2q^2 - 2q,$$

which provides, when simplified yet again using the identity $p+q = n$:

$$4q^2 - 4nq + (n^2 - n + 2) = 0.$$

Solving for p, q yields:

$$q = ½ ( n \pm \sqrt{n-2} ) , p = ½ ( n \mp \sqrt{n-2} ).$$

Hence under this circumstance, $n - 2$ must also be the square of an integer.

(However, this is also a sufficient condition for $K_n$ to be binary graceful: set $v_i = 1$ for $i = 1, 2, ..., q$ and $v_i = 0$ for $i = q+1, ..., q+p=n$.)

Note: Obviously this theorem and proof is supplanted by that found in section 1.4 on complete graphs, but it seemed unusual to be able to express a labelling exercise in terms of the solution to a set of diophantine equations, and so was included for interest only. A slightly stronger result, which was obtained in [40] using complex polynomials, asserted that for $2 \le n \le 100$, n must equal either 2, 3, 4, 27, 36, 38, 49, 64 or 81. This is merely a subset of those values permissible in theorem 3.15. Both the theorem and its proof was found in [21], but the latter proof required more detail.

**3.5 Cordial Graphs:**

Cordial graphs are quite similar in nature to that of binary graceful graphs. Indeed, the definitions are almost identical, although cordial graph theory shares a common weakness with that discussed in section 3.4 above in that few results could be used to make statements about graceful or harmonious graphs. Cordial graphs were first defined in a paper by Cahit in 1987 [11].

Definition 3.17: *Let $\psi$ be a vertex labelling of a graph G=(V,E) which maps the set V to the set {0,1}, $\psi$ inducing the edge labelling $\psi(uv) = \psi(u) + \psi(v) \pmod 2$. The valuation $\psi$ is then a cordial valuation (or cordial labelling) if $|\#\{\psi(v)=0\} - \#\{\psi(v)=1\}| \le 1$ and $|\#\{\psi(e) = 0\} - \#\{\psi(e) = 1\}| \le 1$. The graph is said to be cordial if it admits a cordial valuation.*

As can be seen, the formation of cordial valuations differ from that of binary graceful valuations. The restriction that $|\#\{\psi(v) = 0\} - \#\{\psi(v) = 1\}| \leq 1$ is not present in binary graceful labellings, hence binary gracefulness is a weaker condition than cordiality as a property.

We have the following observations:

Theorem 3.18: *All graceful trees are cordial.*

Proof: Reduce a graceful labelling modulo 2 of any tree to obtain a cordial labelling.

Theorem 3.19: *All harmonious trees are cordial.*

Proof: There is no restriction on the repeated label in a harmonious labelling for a tree (see the note to theorem 3.11), so we may as well assume that one of {e,e-1} is repeated, and that the repeated label is odd. Hence the reduction modulo 2 of the harmonious labelling will produce a cordial labelling.

To show that all trees are cordial, one could prove the Ringel/Kotzig conjecture and then use theorem 3.17. However, there turns out to be two separate proofs (both in [11]) that all trees are cordial, one using induction on the number of vertices, the other algorithmic, and neither providing any elucidation on the Ringel/Kotzig conjecture.

Theorem 3.20: *All trees are cordial.*

Theorem 3.21: *The complete graph $K_n$ is cordial if and only if $n \leq 3$.*

Proof: Using the same notation as that of theorem 3.16, we have p = q if n is even, or $|q-p|=1$ if n is odd. If n is even and n > 4, then we have p = q, or $(n - \sqrt{n})/2 = (n + \sqrt{n})/2$ i.e. $\sqrt{n}=0$. If n > 4 is odd, then we have $|p-q| = 1$, or $\left|\frac{n-\sqrt{n+2}}{2} + \frac{n+\sqrt{n+2}}{2}\right| = \left|\sqrt{n+2}\right| = 1$, i.e. n+2 = 1 (Impossible). The cases n=1, n=2 and n=3 are trivial enough.

As for graceful graphs, there exists a parity condition for an Eulerian graph to be cordial. Unlike for graceful graphs, this parity condition can be extended to "odd" graphs (graphs whose vertices are all of odd degree):

Theorem 3.22: (Parity condition for Eulerian and odd graphs)
*a.) Let G be a (not necessarily connected) Eulerian graph with 4k+2 edges for some non-negative integer k. Then G is not cordial.*
*b.) Let G be an odd graph such that* $|V| + |E| = 4k+2$ *for some integer k, then G is not cordial.*

Proof: Part b. is proved from part a by noting that if G is an odd graph with #V+#E = 4k+2, then G + $\{K_1\}$ is an Eulerian graph with 4k+2 edges, and that any cordial labelling for G would induce a cordial labelling for G + $\{K_0\}$. Part a is proven in a similar manner as for the parity condition for graceful Eulerian graphs — the sum of the edges must be an even number in any component of an Eulerian graph, yet the sum of the edges in a cordial labelling of a graph with 4k+2 edges is odd.

As an immediate corollary we have the following:

Corollary 3.22: *Every cubic graph with 8k+4 edges is not cordial.*

The two parity conditions for cordial graphs immediately allows us to discount a large variety of graphs from being cordial, specifically the following:

- The ladder graphs $L_n = P_2 \cup P_2 \cup P_2 \cup \ldots \cup P_2$, whenever n = 4k+2
- The wheel graphs $W_n$, whenever n = 4k+3
- The cycle graph $C_n$ and the triangular snake $T_n$ (see section 2.2) whenever n = 4k+2
- The prism graph $C_n \times P_2$, and Möbius strip ladder $M_n$, whenever n = 4k+2.

Those families listed above are otherwise cordial whenever they do not satisfy the parity condition.

## 4. Graceful Planar Graphs with Self-Duality

*(It will be assumed that the reader has a passing knowledge on the definition of planar graphs and some (very) basic theory on dualism. A beginner's explanation can be found in [20].)*

### 4.1 A Curious Planar Property

This section is an attempt to correlate graphs with two seemingly unrelated properties of planar graphs: whether a graph has a ß-valuation, and whether it has the property of being self-dual. Although the property of a graph being planar has nothing to do with a graph being graceful (not all planar graphs are graceful — to wit, the cyclic graphs of order 1 or 2 modulo 4), it is hoped that self-duality imposes the existence of a ß-valuation. One simple family of self-dual graphs are already known to be graceful: the wheel graphs $C_n + K_1$ as discussed in chapter 2, are both self-dual and graceful. In this section we will consider a few families of self-dual planar graphs for investigation of a ß-valuation, but quickly realize the question to be in the negative. We especially pay attention to a family of graphs with one additional, restrictive, property.

Definition 4.1: *A planar graph is said to have property "Q" if the graph is*

*a). Self-dual, (i.e. isomorphic to its topological / combinatorial dual )*

*b). The maximum degree of a vertex is at most 4.*

Conjecture 4.2: *For any natural number n > 3, there exists a planar graph of order n with property "Q".*

As is immediately apparent, the above definition and conjecture have no relationship to the theory of graceful graphs. They were derived independently by the author a full year before work on this thesis began.

Property a in definition 4.1 forces the number of faces and the number of vertices of a graph to be equal in number. So then by Euler's formula, a graph with n vertices must also have n faces and 2n−2 edges. Property b in definition 4.1 is even more restrictive: all the vertices must now be of degree 4 or 3, and there must be exactly 4 triangles (i.e. 4 vertices of degree 3) — all other faces must be quadrangles. This is under the assumption that both G and G* would be simple graphs: if in a simple planar graph there is a vertex of degree one, then the dual has a looped edge, whereas a vertex with degree two will give rise to multiple edges in the dual.)

The veracity of this conjecture is so far unknown to the author. However, one can find graphs with property "Q" for certain natural numbers. When $n \geq 4$ is an even number, one can find a planar graph with property "Q": the girder graphs in section 4.3. When n is of the form 4k+1 or 3k+1, there is also a planar graph with property "Q" — the higher-diameter wheel graphs W(3,k), W(4,k), X(3,k) and X(4,k) of 3 or 4 spokes will be discussed in section 4.4. The key values for which the conjecture is undecided are those integers of the form 12k−1 and 12k+3, although (as it will be shown in section 5.4) there are graphs with property "Q" for n=11 and n=15.

A complete classification of all spherical polyhedra (read: planar graphs) with self-duality can be found in [5]. This paper, written by Archdeacon and Richter, had described all the constructions that were sufficient to generate all spherical polyhedra, but had not made it clear that polyhedra with max. degree 4 and 12k−1 or 12k+3 vertices ($k \geq 2$) could be constructed, so the problem seems to remain open.

### 4.2: A Potential Correlation Between Conservative and Self-dual Graceful Graphs

The following is a possible correlation between the notions of a graph being "conservative" and being graceful. The notion of "conservative" graphs was first seen in [6] by Slater, Bange, and Barkauskas, and is defined below:

Definition 4.3: *A directed numbering is a numbering of the edges (using the integers 1, 2, 3, ..., e where e=|E(G)|), with orientation, such that the flow into a vertex is equal to the flow out of that vertex.*

Definition 4.4: *A graph is conservative if and only if there is a direct numbering of the edges using the integers 1,2, ..., e, where e = #{edges in G}.*

Theorem 4.5: *A graph G has a directed numbering if and only if it is 3-edge connected (hence our restriction that G has minimum degree 3.).*

A relationship between conservative graphs and graceful graphs is provided by the following theorem, also found in [6]:

Theorem 4.6: *Let G be a planar, graceful graph. Then the dual graph G* is conservative.*

Proof: Give G a graceful labelling. Number the edges in G* so that an edge in G* is given that number attached to the edges in G which this edge crosses. Direct this edge outward (or inward) from a face in G if the larger vertex number in the graceful labelling of G is to the right (or left, respectively) as the edge is crossed. If a vertex of the dual G* corresponds to a triangular face, it is easy to see that the flow is conserved at a particular vertex using orientation and numbering. The theorem then follows by induction on the number of edges in a face of G.

The converse of theorem 4.6 does not happen to be true. In figure 4.1 is shown a planar graph which has a conservative labelling, yet whose dual (itself — the graph is self-dual) is not graceful. Figure 4.2, showing a graceful labelling of a wheel-type graph, provides an example of how theorem 4.6 works to induce a conservative labelling of its planar dual.

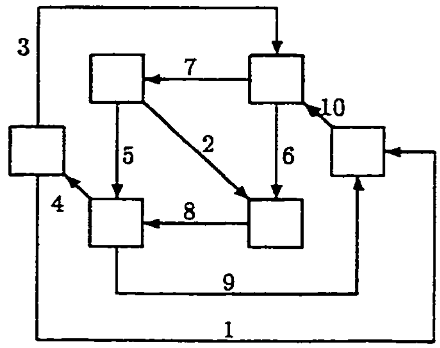


Figure 4.1: A conservative labelling of the girder graph $G_3$ - an example of a self-dual graph which is not graceful.

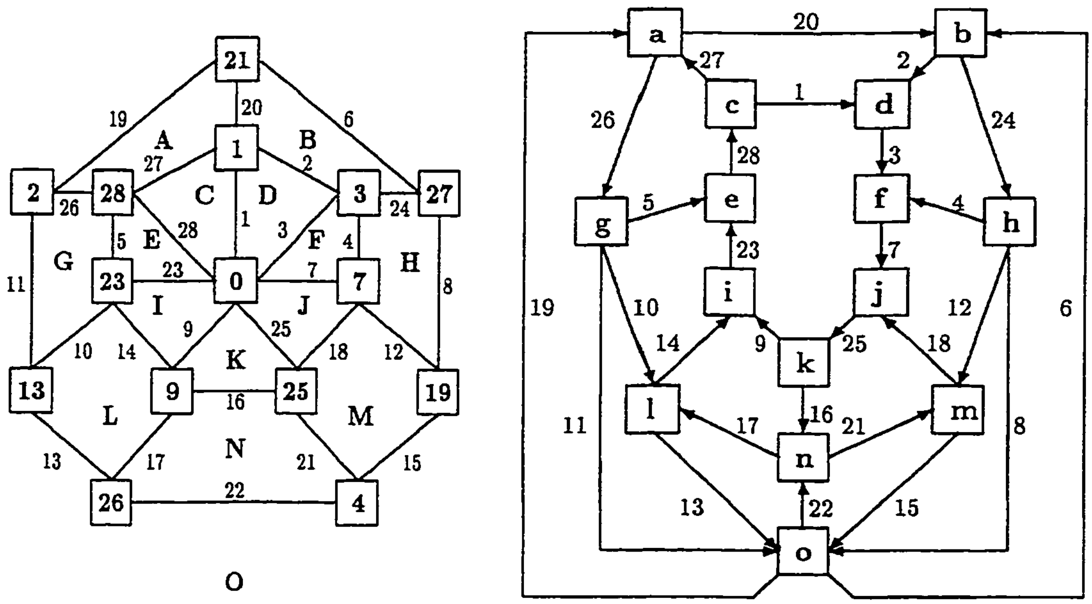


Figure 4.2: An example of a planar, graceful graph (left - the wheel graph W(7,2) ), which induces a conservative labelling of its dual graph (right). The left graph is a high-diameter wheel graph, which are defined in section 4.4.

### 4.3 **The "Girder Graphs":**

One particular family which exhibits property "Q" are the "girder graphs". Holistically, a girder graph is a long, straight sequence of X's tied to one another at the corners, bracketed by a long cycle on the exterior (see figure 4.3). Explicitly, the girder graphs $G_n$ (where n is a natural number) is a graph consisting of 2n+2 vertices (labelled $u_0$, $u_1$, ..., $u_n$ and $v_0$, $v_1$, ... $v_n$) and the following 4n+2 edges:

$(u_0, v_0)$ , $(u_n, v_n)$ ,

$(u_i, u_{i+1})$ , $(u_i, v_{i+1})$ , $(v_i, u_{i+1})$, $(v_i , v_{i+1})$.   (for i = 0,1,..., n−1).

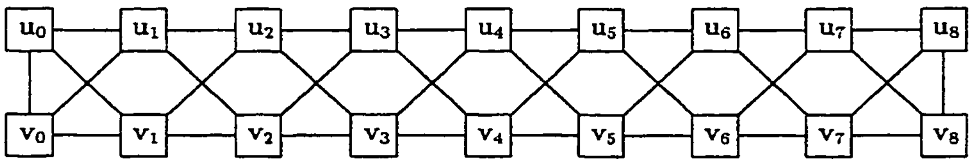


Figure 4.3: Vertex labelling of the girder graph $G_8$.

The girder graph $G_1$ is only the complete graph $K_4$. As shown in figure 4.4, the girder graphs $G_n$ are planar. Alternatively, the girder graphs are merely the graphs $P_{n+1}(N_2)$ with two additional edges at the "ends" to create the minimum four triangles necessary in self-dual graphs.

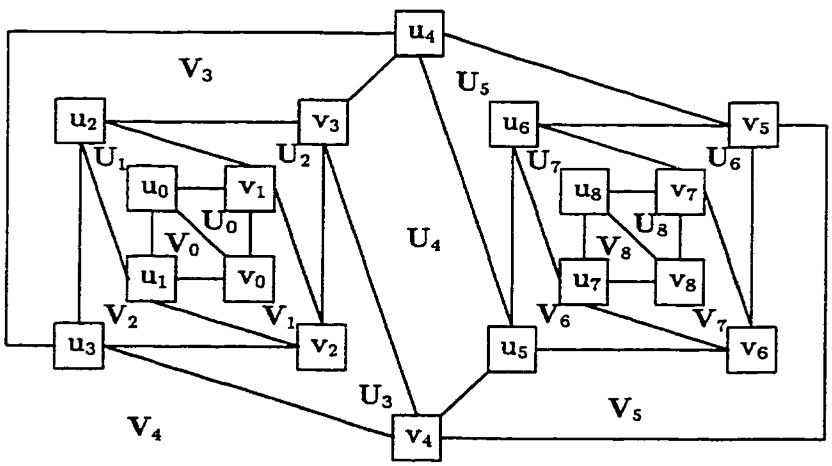


Figure 4.4: The planar, self-dual representation of the girder graph $G_8$. The small-case vertex labels $u_i$, $v_i$, correspond to the upper-case face labels $U_i$, $V_i$ respectively.

Theorem 4.7: *The girder graphs $G_n$ are self-dual. (The pattern shown in figure 4.4 can be extended or contracted as necessary.)*

We shall now attempt to construct a graceful labelling for the girder graphs $G_n$. Of course, this has already been done for the case n=1. To construct the labelling of the higher-ordered graphs we shall make use of an extension lemma and then use induction to construct the exact labellings we require.

Lemma 4.8: *Suppose the girder graph $G_n$ has a graceful labelling for some positive integer n. Furthermore, suppose that in this graceful labelling the vertex $v_0$ is given label "0", and $u_0$ is given either the label 1 or 2. Then there is a graceful labelling of the girder graph $G_{n-2}$ with $v_0 = 0$ and $u_0 = 1$ or 2.*

Proof: Given the girder graph $G_{n+2}$ with vertices $u_0, \ldots, u_{n+2}, v_0, \ldots, v_{n+2}$ and edges as given above, use a graceful valuation $\psi$ of $G_n$ such that $v_0 = 0$, and $u_0 = 1$ to construct $\psi'$ as follows:

$\psi'(u_{i+2}) = 4 + \psi(u_i)$, $(i = 0, 1, \ldots, n)$
$\psi'(v_{i+2}) = 4 + \psi(v_i)$, $(i = 0, 1, \ldots, n)$
$\psi'(u_0) = 1$,
$\psi'(v_0) = 0$,
$\psi'(u_1) = 4n+10$,
$\psi'(v_1) = 4n+8$.

Since $\psi$ is a graceful labelling of $G_n$, it follows that $\psi'$ will label those edges, consisting only of those vertices $u_2, v_2, \ldots$, with the numbers 2, 3, 4, 5, ..., 4n+2, with the vertices using only some of the integers from 4 to 4n+6. The edge labeled 1 is relocated to $u_0v_0$ and the remaining edges 4n+3,..., 4n+10 are inserted between the vertices $u_0, v_0, u_1, v_1, u_2$, and $v_2$. The proof is identical when $u_0 = 1$ is replaced with $u_0 = 2$, save only that $\psi'(v_1) = 4n+9$.

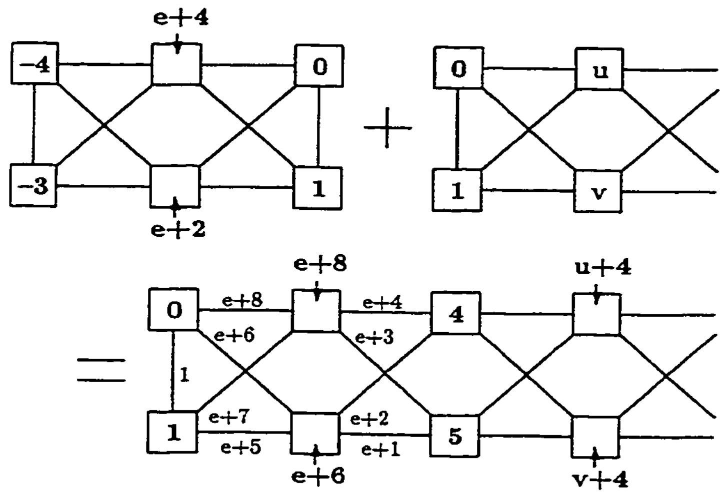


Figure 4.5 - extension lemma in action

Figure 4.5 provides a different look at the proof of lemma 4.8. Given a graceful girder $G_k$, a separate girder $G_2$ is glued to the end. The two overlapping edges are deleted. The added component provides the additional high-numbered edge labels, whereas the 1-edge is retained in the end edge $u_0v_0$ position. With a simple correction (adding 4 to all vertex labels), a graceful labelling of the girder $G_{k+2}$ is obtained.

Lemma 4.9:

a.) *There is no graceful labelling of the girder graph $G_2$.*

b.) *There is no graceful labelling for the girder graph $G_4$ such that $v_0 = 0$ and $u_0 = 1$ or $2$ — although the girder graph is indeed graceful.*

c.) *There is a graceful labelling of $G_6$ such that $v_0 = 0$ and $u_0 = 1$.*

Proof: There is no simple proof of part a. except either by computer search or by brute force. However, the graph was noted as not being graceful in [23]. The proof that there is no graceful labelling of $G_4$ where $v_0 = 0$ and $u_0 = 1$ or 2 is also by means of computer search. The following labelling for $G_4$ and $G_6$ are given on the next page:

$G_4$ : $u_0 = 1, u_1 = 7, \ u_2 = 0, u_3 = 14, u_4 = 3.$
$v_0 = 2, v_1 = 18, v_2 = 5, v_3 = 15, v_4 = 11.$

$G_6$ : $u_0 = 0 \ , u_1 = 24, u_2 = 4, u_3 = 9, u_4 = 16, u_5 = 7, u_6 = 23.$
$v_0 = 1, v_1 = 26, v_2 = 5, v_3 = 22, v_4 = 19, v_5 = 8, v_6 = 21.$

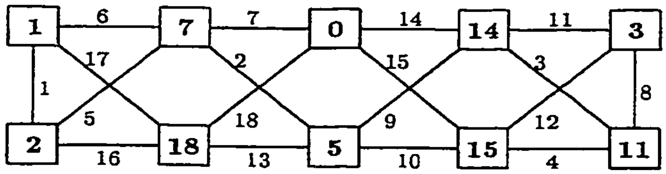


Figure 4.6: Graceful labelling of girder graph $G_4$.

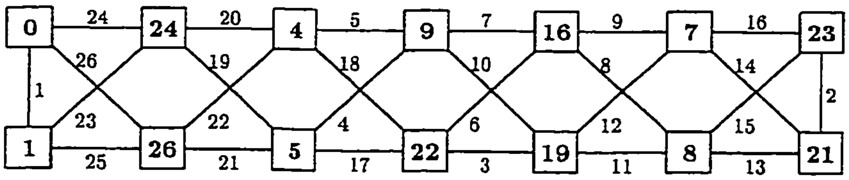


Figure 4.7: Graceful labelling of girder graph $G_6$.

Note that the labelling for $G_6$ cannot be contracted (by "chopping" $u_0$, $v_0$, $u_1$, and $v_1$, and subtracting 4 from the labels of the remaining vertices) to provide a labelling for $G_4$ such that $v_0 = 0$ and $u_0 = 1$ — the label for $u_6$ would then lie beyond the accepted range.

Initially, I had thought of using definition 3.13 in an attempt to show that the girder graphs $G_n$ are not graceful for n even. This naïve idea was based upon my apparent lack of success when n=2. The planned method was to characterize all the possible binary configurations of all the girder graphs $G_n$ that made them well-balanced, and then show that no such configuration could exist when n was even. Unfortunately, there were plenty of configurations for when n was even, and that particular method was scrapped. However, knowing all the possible binary configurations in advance allowed for a rapid search of $G_6$ to find β-valuations, thereby completing the theorem:

Theorem 4.10: *The girder graphs $G_n$ are graceful for all $n > 2$.*

Proof: The extension lemma 4.8 allows us to inductively construct a graceful labelling for $G_{n+2}$, given a graceful labelling of $G_n$ with certain properties. For odd numbers, the labelling is constructed from the base case of n=1 ( $u_0 = 1$, $v_0 = 0$, $u_1 = 6$, $v_1 = 4$). For even numbers, the base case is given by the labelling given for $G_6$ (lemma 4.9). The case of $G_4$ was handled separately.

An explicit labelling scheme for the girder graphs will be given in an appendix. It should be pointed out that if we delete either the edge $u_0v_0$ or the edge $u_nv_n$ (or both!) from $G_n$, then the resultant edge-deleted girders are graceful, especially in the case n=2.

In the case of one "end-"edge being removed, the extension lemma 4.8 can be used for a graceful labelling given $v_0$=0 and $u_0$=1 for $G_n - u_nv_n$, to obtain a graceful labelling for $G_{n+2} - u_{n+2}v_{n+2}$. Such a graceful labelling exists for the troublesome case n=2. In the even simpler case of $G_n - u_0v_0 - u_nv_n$, (which is just $P_n(N_2)$), it is easier to simply provide an exact labelling, also given in appendix A.

The edge-contracted girders $G_n \setminus u_0v_0$ are also graceful for all $n > 1$. (The edge-contracted girder $G_1 \setminus u_0v_0$ being $K_3$ is obviously graceful.) Again, the proof is by direct construction, provided in appendix A.

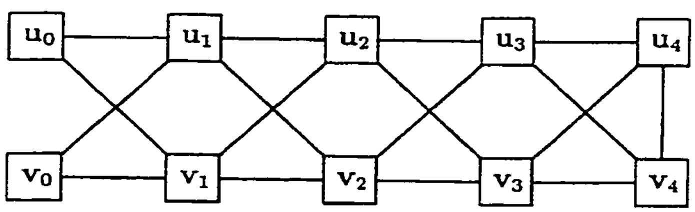


Figure 4.8: The edge-deleted girder $G_4$ - $\{u_0v_0\}$

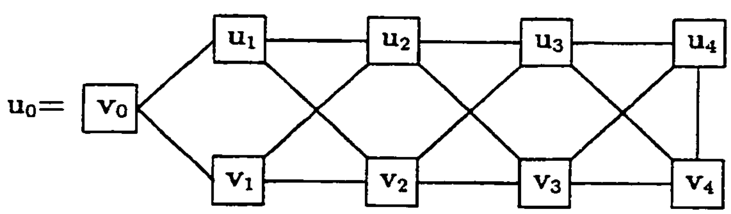


Figure 4.9: The edge-contracted girder $G_4 \setminus \{u_0,v_0\}$

**4.4 High-diameter wheels: Two Families of dihedral-symmetric graphs with property Q.**

Let W(m,n) (and X(m,n)) denote the square-meshed wheel (respectively diamond-mesh) wheel of diameter 2n, with m radial "spokes" (where m>2). The graphs W(m,n) and X(m,n) are then graphs with 1+mn vertices and 2mn edges.

Let r and s(i,j) ($1 \le i \le m$, $1 \le j \le n$) be the 1+mn vertices of W(m,n) and X(m,n). The edges for these graphs are given by the formulae:

| | | |
|---|---|---|
| W(m,n): | ( r, s(i,1) ): | i = 1..m |
| | ( s(i, j−1), s(i, j) ): | i = 1..m, j=2..n |
| | ( s(i, j), s( i+1, j) ): | i = 1..m−1, j=1..n |
| | ( s(m,j), s(0,j) ) | j = 1..n |
| X(m,n): | ( r, s(i,1) ): | i = 1..m |
| | ( s( i , j), s( i, j+1) ); | i= 1..m , j=1..n−1 |
| | ( s( i, j), s( i+1 (mod m) , j+1 ): | i=1..m, j=1..n−1 |
| | ( s( i, n), s(i+1 (mod m) , n ): | i=1..m. |

The simple wheel graphs $W_m = C_m + K_1$ are merely the high-diameter wheels for the case n=1. Figures 4.9 through 4.13 show the structural differences between the two families. Like their smaller cousins, the high-diameter wheels W(m,n) and X(m,n) are planar and are easily seen to be self-dual. So the high-diameter wheels W(m,n) and X(m,n) will also exhibit property "Q" when m=3 or m=4.

Conjecture 4.11: *The graphs W(m,n) and X(m,n) are graceful for all numbers m $\ge$ 3 and n $\ge$ 1.*

It is still an open problem as to whether the diamond-mesh wheel graphs X(m,n), or whether the square-mesh wheels W(m,n) are graceful or not. A few particular valuations have been found for various high-diameter wheels — they are to be found in the appendix.

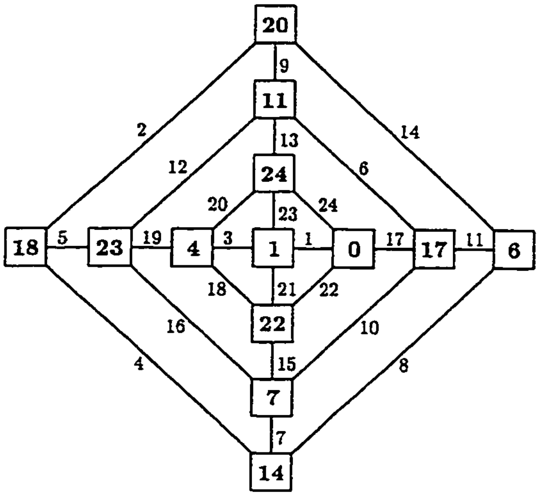

Figure 4.10: The wheel graph W(4,3).

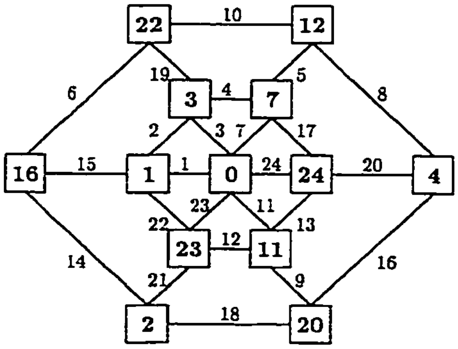

Figure 4.11: The wheel graph W(6,2)

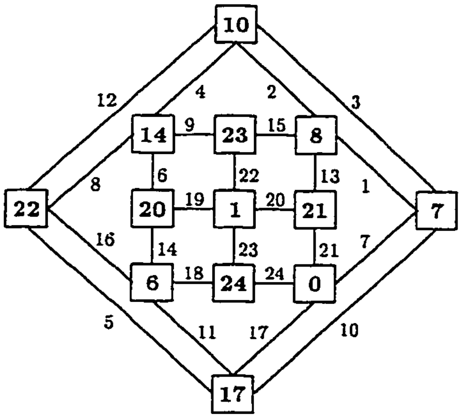

Figure 4.11: The wheel graph X(4,3).

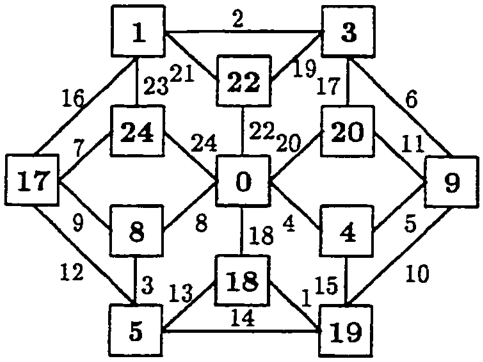

Figure 4.13: The wheel graph X(6,2).

While searching for β-valuations for these graphs, I had attempted to exploit certain binary graceful valuations, and had met with some success for certain families. The following conjectures hope to exploit this:

Conjecture 4.12: *There exists a graceful labelling of X(m,2) for all m ≥3 such that the middle vertex is zero, its adjacent vertices are all even, and all other vertices with odd labels. (Shown to be true for n ≤ 7).*

Conjecture 4.13: *There exist a graceful labelling of W(m,n), for all $m \geq 3$, $n \geq 2$, in which vertex s(i,j) = j mod 2 (and middle vertex zero).*

One possible method which could be used to show that these conjectures are true, is to find a way of inductively extending a graceful valuation for a particular sub-family of wheel graphs, and then use a computer to search for a base case. This was the exact method used in proving that all girders are graceful. Using binary graceful valuations — or cordial valuations — and noting how we could extend those patterns, we may assay an attempt to construct a valid extension lemma. I was able to prove that the wheel graphs W(m,n) and X(m,n) are *cordial* for all n>1 and m>2 — the proof is omitted here due to space limitations.

If we allow m=1 or m=2 in the definition of W(m,n) and X(m,n), we obtain classes of non-simple, planar graphs. By removing multiple edges between pairs of vertices (which will certainly be the case for W(2,n) ), we obtain three families of simple, planar (but non-self-dual) graphs. W(1,n) and X(1,n) become, in fact, a path consisting of n edges: its gracefulness is already evident. The family X(2,n) degenerates into the edge-contracted girder graph $G_n \setminus u_0v_0$, shown earlier to be graceful for all n > 1 (X(2,1) is just $K_3$). The family W(2,n) becomes the edge-contracted polyomino grid $P_2 \times P_{n-1} \setminus u_0v_0$: these graphs, which look like a string of squares with a triangle on top, is also graceful (see the labelling scheme for this family in appendix A).

### 4.5 Two Graphs with Unusual Characteristics

We now present two planar graphs with property "Q". The two graphs, one of order 11, the other of order 15, are given in figures 4.14 and 4.15 respectively, with labelling for both the vertices and the faces. Self-duality is obtained by pairing the vertices labeled by a, b, c, ..., m, n, o, with faces A, B, C, D, ..., N, O respectively.

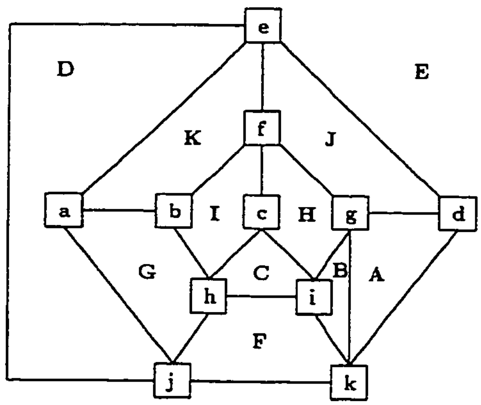


Figure 4.14: The unusual graph $U_{11}$.

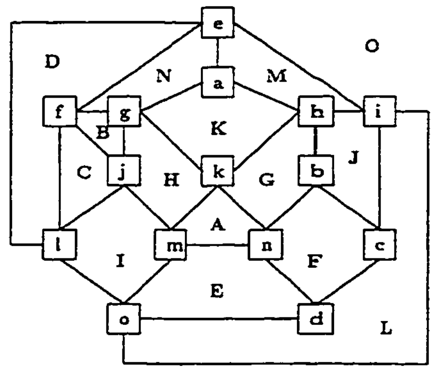


Figure 4.15: The unusual graph $U_{15}$.

In addition, these graphs assume one further, unusual property:

Theorem 4.14: *The unusual graphs $U_{11}$ and $U_{15}$ have trivial automorphism groups. That is, the graphs possess no symmetries.*

Proof: Let $\Phi$ be an automorphism of the graph $U_{11}$. We shall show that $\Phi(v) = v$ for any vertex v in $U_{11}$.

Since $\Phi$ preserves the degree of a vertex, we must have $\Phi\{a,b,c,d\} = \{a,b,c,d\}$, i.e. the vertices of degree 3 are mapped to themselves. Since $\Phi$ preserves adjacencies between vertices, we must also have $\Phi(a) = a$, $\Phi(b) = b$, or $\Phi(a)=b$, $\Phi(b)=a$. Since vertices b and c are the only pair of cubic vertices sharing two neighbours, we must have $\Phi(b) = b$ or c. Hence, $\Phi(b) = b$, $\Phi(c) = c$, $\Phi(a) = a$, and consequently $\Phi(d)=d$.

Vertex e is the only vertex adjacent to both a and d, so $\Phi(e) = e$. $\Phi(f) = f$, as vertex f is the only vertex adjacent to b and e, and is of degree 4. Vertex g is the only one in a cycle involving vertices e, f, and d; $\Phi(g) = g$. $\Phi(h) = h$, due to the cycle (b,f,c,h). $\Phi(i)=i$: cycle (c,h,i). $\Phi(j)=j$: cycle (a,b,h,j). Lastly, $\Phi(k)=k$ — the other vertices have been assigned.

The proof for $U_{15}$ having the trivial automorphism group, is similar in nature. There is only one way with which to map the vertices of degree 3 ( a, b, c, d), and this in turn forces the mapping of vertex h by automorphism to itself, as vertex h is the only vertex adjacent to both vertices a and b. Inductively, the other cycles are mapped individually to themselves in a manner similar to $U_{11}$.

This differs from the other graphs previously mentioned: the high-diameter wheel graphs in section 4.4 have dihedral symmetric groups, and the girder graphs possess an automorphism group of order $2^k$ for some integer k.

This lack of symmetry becomes an unfortunate burden when it comes to a brute-force search for ß-valuations. The burden comes in the fact that we gain no advantage computationally in forcing a particular edge to be labelled with the number e (where e is the highest-labelled edge). As there is no symmetry to exploit, the number of computations cannot be easily decreased — brute force becomes even more brutish. However, with a little bit of luck, we are rewarded:

Theorem 4.15: *The unusual graphs $U_{11}$ and $U_{15}$ are graceful.*

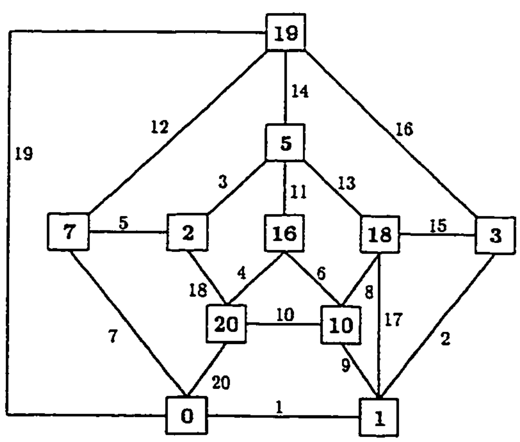


Figure 4.16: Graceful labelling of $U_{11}$.

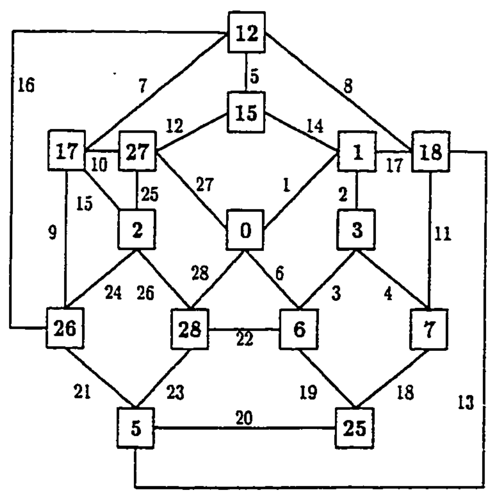


Figure 4.17: Graceful labelling of $U_{15}$.

## Chapter 5: Conclusion and Further Remarks

Perhaps the most valuable tool I had developed in the study of this subject, is designing an implementation of a computer algorithm for the search and cataloguing of graceful valuations. The resultant program allowed me to determine, at least for graphs of small size (< 30 edges), all possible valuations which are graceful. (A change of but two lines of code converts the program to finding any harmonious labelling.) The usefulness of such a program is this: it allows me to not only verify patterns suspected of producing graceful valuations, but to determine the initial case of these patterns.

This was done in the case of the girders described in section 4.3. Although I had the idea of the extension lemma beforehand, I had resorted to the computer to determine upon which cases this extension lemma could be used. The complete graph $K_4 = G_1$ was an easy example, but the girder graph $G_2$ was already known to be disgraceful (the computer program verified this quickly). A search by hand succeeded in finding a graceful labelling of $G_4$, but it did not have the desired placement of particular labels — and the computer search confirmed this. The computer finally found the appropriate base case of $G_6$ (in fact, it found several), and so the general theorem (Theorem 4.6) was proven. Given an extension lemma, the computer can aid in the proof of a family being graceful.

The algorithm goes as follows:

The boolean function *isgraceful*, accepts the following parameters:

| | |
|---|---|
| G=(V,E) — | the graph G with vertex sets V and E. The vertices are labelled $v_1, v_2, \ldots, v_n$, where n is the number of vertices. |
| d — | the current depth of the graph |
| A — | the current set of vertex labels already assigned to vertices in G, |
| B — | the current set of edge labels as yet assigned to edges in G, |

The function returns TRUE if the current vertex labels induce a graceful graph, and FALSE if the current set of vertex labels cannot produce a graceful labelling.

```
boolean isgraceful ( G, d, A, B):

        if d = n+1, return TRUE, exit function. /* Current vertex labels are graceful. */
        int i = 0.

        Repeat
                /* If we can safely add label a to vertex v_d without causing repetition...*/

                if ( i ∉ A) and (∀c):{(v_d,v_c) ∈ E(G), c < d, and |v_d - v_c| ∉ B}

                        v_d = i;
                        isgraceful( G, d + 1, A + {i}, B + { |v_d - v_c| : c < d, (v_d,v_c) ∈ E(G)})

                i = i + 1;          /* try the next possible label */
        until (i > e).

        return FALSE.   /* All possible valuations using vertices x_1, ..., x_d are exhausted*/
End (of boolean function isgraceful).
```

The disadvantage in the "graceful graph" algorithm I had implemented is in its lack of speed — it employs a brute-force search by trying all the possible permutations in order and evaluating each permutation to see if it is graceful. The algorithm is, therefore, exponential in complexity: in the worst-case scenario, an attempt to label a star-tree ($N_1 \times N_k$) by labelling the outer vertices first, would examine all (k+1)! permutations. Having even a modest number of edges (say, 20) means that the program would take an excessive amount of time to execute. Even with these modifications, however, only graphs of a modest size ( 30 or less edges) could be examined.

The algorithm can use the following modifications:

a.) Implementing a recursive search. Instead of applying all the labels in a particular permutation simultaneously, the program would attempt to insert a label one at a time, providing that the insertion would not already violate one of the conditions of gracefulness (repeating edge-labels or vertex labels). This allowed for entire branches of permutations to be omitted.

b.) Forcing the parity of particular vertices. The number of permutations to check decreases drastically once the parity of all the vertex labels is known. Determining the parity is merely an exercise in finding binary valuations — also exponential, but far easier. (In a sample case of 20 edges and 10 vertices: knowing the parity of the vertices results in a total of $({}_{10}P_5)^2 =$ 914,457,600 permutations for a worst-case cordial labelling, whereas not specifying the parity gives a possible ${}_{20}P_{10}$ = 670,442,572,800 permutations — a one-thousand-fold improvement!)

c.) Utilizing graph symmetry and selecting a particular edge to have the maximum weight (i.e. selecting the labels 0 and "max".) This avoids repeats and eliminates branches of permutations to check. Using the above example of 20 edges and 10 vertices (assuming no symmetry), knowing the location of 0 and 20: ${}_{18}P_8$ = 158,789,030,400. Knowing the parity of the other vertices, assuming a cordial labelling: ${}_{10}P_5 \cdot {}_8P_5 = 1.0\times10^7$. (It is the difference between seconds and hours.)

The most obvious direction to take in terms of future research is to find further families of graphs which are graceful, harmonious, or cordial. There are always more families to discover, although the more complex the graph is, the harder it is to find a labelling without the means of brute-force computer checking; of course, the larger the graph, the harder it is to even use brute force. Another direction is to find different labelling problems and solve those in turn; most new problems however do not provide as simple a solution for particular cases as that of graceful graphs. Of a certainty there is also the Ringel-Kotzig conjecture to consider: current methods have done little to solve this problem except to attack particular sub-cases in a piece-meal fashion, and it seems unlikely that this problem will be solved even in the next half-century.

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**Appendices: Algorithmic Graceful labelling for certain families of graphs**

A.1: Girder Graphs (Section 4.3), and sub-variants.

The graph $G_n$, where $n \geq 1$, is a planar, self-dual graph with $2n+2$ vertices and $4n$ edges. The vertices are labelled $u_0, u_1, ..., u_n$ and $v_0, v_1, ..., v_n$, and the edges are of the following form: $u_0v_0$, $u_nv_n$, and $u_iv_{i+1}$, $u_iu_{i+1}$, $v_iv_{i+1}$, $v_iu_{i+1}$, for $i=0, 1, ... n-1$. (Alternately, the girder graph $G_n$ is the graph $P_n(N_2) + u_0v_0 + u_nv_n$.)

$n > 1$, n odd: $u_i = 2(n+i)$, $v_i = 2(n+i)+2$ (i odd)
$u_i = 2(n-i)-2$, $v_i = 2(n-i)-1$ (i even)

$n > 6$, n even:

| | | |
|---|---|---|
| $i < n-3$, i even: | $u_i = 0 + 2i$, | $v_i = 1 + 2i$. |
| $i < n-3$, i odd: | $u_i = 4n - 2i$, | $v_i = 4n - 2i + 2$. |
| $i = n-3$: | $u_i = 2n - 3$, | $v_i = 2n + 10$ |
| $i = n-2$: | $u_i = 2n + 4$, | $v_i = 2n + 7$ |
| $i = n-1$: | $u_i = 2n - 5$, | $v_i = 2n - 4$ |
| $i = n$: | $u_i = 2n + 9$, | $v_i = 2n+11$. |

The vertex labelling of $G_4$ and $G_6$ were already given in section 4.3. Remember that $G_2$ was not found to be graceful.

The edge-deleted girders $G_n - u_nv_n$: $(n > 0, 0 \leq i \leq n)$

n even:

i even: $u_i = 2(n-i) - 1$, $v_i = 2(n-i) - 3$;
i odd: $u_i = 2(n+i) + 1$, $v_i = 2(n+i) - 1$;

n odd:

i even: $u_i = 2(n-i) + 1$, $v_i = 2(n-i)$;
i odd: $u_i = 2(n+i) + 3$, $v_i = 2(n+i) + 1$;

The edge-contracted girders $G_n \setminus u_0v_0$: The degenerate wheel graph X(2,n)

n even: $(i > 1)$

| | | |
|---|---|---|
| i even: | $u_i = 2(n-i+1)$, $v_i = 2(n-i)$; | |
| i odd: | $u_i = 2(n+i) + 1$, | $v_i = 2(n+i)$; |
| $u_0 = 2n-3$ , $u_1 = 2n+2$ , $v_1 = 2n$. | | |

n odd: $(i > 0)$

| | | |
|---|---|---|
| i even: | $u_i = 2(n+i) + 1$, | $v_i = 2(n+i)-1$; |
| i odd: | $u_i = 2(n-i) + 1$, | $v_i = 2(n-i)$; |
| $u_0$: | $2n+1$. | |

The degenerate wheel graph W(2,n):

The graph consists of 2n+1 vertices and 3n edges. The vertices are labelled $u_0$, $u_1$, ..., $u_n$, $v_1$, $v_2$, ..., $v_n$, and the edges are given by $u_0u_i$, $u_0v_i$, $u_iv_i$ for i=1, 2, ..., n, $u_iu_{i+1}$ and $v_iv_{i+1}$ for i = 1, 2, ..., n−1. A graceful labelling exists for all n > 0 (with $W(2,1) = K_3$), and is given by (for all values of n > 0)

| | | |
|---|---|---|
| i even $(i \neq 0)$: | $u_i = 2(n-i)$ | $v_i = 3n - (n - i)$ |
| i odd: | $u_i = 3n - (n - i)$ | $v_i = 2(n-i)$ |
| | $u_0 = 2(n-i) + 1$ | |

A.2 θ-graphs:

What follows is a graceful labelling for some families of θ-graphs (see section 2.3) as found in the proof that all such graphs are graceful [32].

Let $G=C_n(l,k)$ be the θ-graph with vertex set $\{v_1, v_2, ..., v_n, ..., v_{n-k-2}\}$ and edge sets consisting of the cycle of length n: $\{v_1v_2, v_2v_3, ..., v_nv_1\}$ and chord of length l: $\{v_nv_{n-1}, v_{n+1}v_{n+2}, ..., v_{n+k-2}, v_{l-1}\}$.

Case 1: $n = 4t$, $l=4a$, $k=4b$, where $t \geq 2$, $a \geq 1$, $b \geq 1$:

| | | | |
|---|---|---|---|
| $\psi(v_i)=$ | $j$ | $: i = 4t + 4b - 3 - 2j$, | $j = 0, 1, ..., t-a+b-1$ |
| | $t-a+b+1+j$ | $: i = 2t + 2a + 2b - 3 - 2j$ | $j = 0, 1, ..., a+b-2$ |
| | $t+2b+1+j$ | $: i = 2t + 1 - 2j$ | $j = 0, 1, ..., t-1$ |
| | $2t+2b+1+j$ | $: i = 2 + 2j$ | $j = 0, 1, ..., 2a-2$ |
| | $2t+2a+2b$ | $: i = 4t + 4b - 2$ | |
| | $2t+2a+2b+1+j$ | $: i = 4a + 2j$ | $j = 0, 1, ..., 2t-2a+2b-2$ |

Case 2: $n = 4t + 1$, $l = 4a$, $k = 4b+3$; $t \geq 4$, $a \geq 2$ $b \geq 1$:

| | | | |
|---|---|---|---|
| $\psi(v_i)=$ | $j$ | $: i = 4t + 4b + 2-2j$ | $j = 0, 1, ..., t+2b$ |
| | $t+2b+2+j$ | $: i = 2t - 2j$ | $j = 0, 1, ..., t-1$ |
| | $2t+2b+2+j$ | $: i = 1 + 2j$ | $j = 0, 1, ..., t+a+b-1$ |
| | $3t+a+3b+3+j$ | $: i = 2t + 2a + 2b + 1 + 2j$ | $j = 0, 1, ..., t - a + b$. |

A.3: Current results on high-diameter wheels

This section contains graceful valuations for those high-diameter wheels which are known to have them. (Labelling schemes: using notation as in section 4.4, first row is given by vertex r, each row thereafter is given by s(i,j) for a fixed row j.)

Square-mesh Wheels W(m,n):

W(3,2)

| | | |
|---|---|---|
| 1 | | |
| 0 | 9 | 12 |
| 10 | 3 | 8 |

W(4,2)

| | | | |
|---|---|---|---|
| 0 | | | |
| 2 | 9 | 15 | 16 |
| 14 | 4 | 7 | 3 |

W(5,2)

| | | | | |
|---|---|---|---|---|
| 0 | | | | |
| 1 | 3 | 16 | 5 | 20 |
| 9 | 13 | 7 | 19 | 2 |

W(6,2)

| | | | | | |
|---|---|---|---|---|---|
| 0 | | | | | |
| 1 | 3 | 7 | 24 | 11 | 23 |
| 16 | 22 | 12 | 4 | 20 | 2. |

W(7,2)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 1 | 3 | 7 | 25 | 9 | 23 | 28 |
| 21 | 27 | 19 | 4 | 26 | 13 | 2 |

W(3,3)

| | | |
|---|---|---|
| 0, | | |
| 18 | 15 | 1 |
| 10 | 4 | 17 |
| 12 | 16 | 7 |

W(4,3)

| | | | |
|---|---|---|---|
| 1 | | | |
| 24 | 0 | 22 | 4 |
| 11 | 17 | 7 | 23 |
| 20 | 6 | 14 | 18 |

W(5,3)

0

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 7 | 12 | 30 |
| 27 | 14 | 23 | 29 | 2 |
| 6 | 28 | 13 | 5 | 25 |

W(3,4)

1

| | | |
|---|---|---|
| 22 | 12 | 4 |
| 5 | 21 | 17 |
| 0 | 2 | 24 |
| 23 | 3 | 9 |

Diamond-mesh Wheels X(m,n):

X(3,2)

0

| | | |
|---|---|---|
| 4 | 10 | 12 |
| 3 | 1 | 9 |

X(4,2)

0

| | | | |
|---|---|---|---|
| 6 | 8 | 16 | 14 |
| 11 | 1 | 3 | 15 |

X(5,2)

0

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 18 | 19 | 20 |
| 16 | 5 | 9 | 3 | 8 |

X(6,2)

0

| | | | | | |
|---|---|---|---|---|---|
| 24 | 22 | 20 | 4 | 18 | 8 |
| 1 | 3 | 9 | 19 | 5 | 17 |

X(7,2)

0

| | | | | | | |
|---|---|---|---|---|---|---|
| 28 | 26 | 24 | 22 | 14 | 12 | 4 |
| 1 | 3 | 21 | 5 | 25 | 19 | 9 |

X(3,3)

0

| | | |
|---|---|---|
| 18 | 17 | 6 |
| 2 | 7 | 11 |
| 16 | 14 | 3 |

X(4,3)

1

| | | | |
|---|---|---|---|
| 23 | 21 | 24 | 20 |
| 8 | 0 | 6 | 14 |
| 7 | 17 | 22 | 10 |