DEFLATORS, LOG-OPTIMAL PORTFOLIO AND NUMÉRAIRE PORTFOLIO FOR MARKETS UNDER RANDOM HORIZON

by

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Abstract

This thesis addresses two important topics of deflators and log-utility-related optimal portfolios for markets stopped at a random time τ. This random time can model the death time of an agent in life insurance or the default time of a firm in credit risk. For the topic of deflators, the thesis elaborates extensively an explicit parametrization of the set of all deflators, which constitutes the dual set of all "admissible" wealth processes. We describe explicitly both cases of local martingale deflators and supermartingale delators as well. These results are essentially based on the martingales classification and representation introduced and developed recently by Choulli et al. [33] for progressive enlarged filtration.

Concerning the second topic of optimal portfolios, we focus on quantifying the impact of random time on these portfolios. In fact, we consider log-utility maximization problem, whose solution relies and is intimately related to the optimal deflators. Thus, we start by describing the optimal deflator for stopped models at random time τ, and then elaborate the duality which lead to the log-optimal portfolio. In meantime, as an important intermediate result, we characterize the log-optimal deflator and log-optimal portfolio for general semimartingale market models without the no-free-lunch-with-vanishing-risk assumption. Finally, the numéraire portfolio for model stopped at τ is also detailed and fully described in different manners.

For both topics, the thesis elaborates results for general semimartingales models and illustrates those results on several practical models. Among theses, we cite the exponential Lévy models (such as Jump-diffusion model and Black-Scholes market model), the volatility models (such as Corrected Stein and Stein Model and Barndorff-Nielsen Shephard Model), complete market model, and discrete time market models.

Preface

Chapter 3 of this thesis has been submitted as Choulli, T. and Yansori, S.,"Deflators and log-optimal portfolios under random horizon: Explicit description and optimization", submitted to Finance and Stochastics, 2018.

Chapter 4 of this thesis has been submitted as Choulli, T. and Yansori, S.,"Notes on the log-optimal portfolio: Necessary and sufficient conditions for its existence and duality", submitted to Applied Probability Trust, 2018.

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

– Carl Friedrich Gauss, 1808.

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Chapter 1

Introduction

About 2600 years ago, philosopher Thales came with an idea of a contract for pressing of olives in the forthcoming season which is known nowadays by "call option". He also had some "additional information" at the beginning, thanks to his expertise in astrology, which allowed him to anticipate a great harvest season (see [20]). We can find several examples of such interplay between finance and mathematics since then. However, modern finance and mathematical finance were born in 1900 only. In fact, on March 29th, 1900, Louis Bachelier defended his PhD thesis "Theory of speculation" at Sorbonne University in Paris. On that date, via Bachelier's thesis, modern finance, mathematical finance, and stochastic calculus as we know them today, were born. Since then, financial mathematics has been growing enormously in academia and industry.

1.1 Random horizon in finance and economy

The economic problem of how a random horizon will impact an investment is old and can be traced back to Fisher [70]. Fisher considered a time-horizon that is related to the death of a life. He argued that even at the time loan seems to be "risk-free" and the financial risks related to the loan doesn't exist, it is still affected by "uncertainty of human life" and as a consequence has an impact "on every business transaction into which time enters". In [70], he wrote

"*Even when there is no risk (humanly speaking) in the loan itself, the rate realized on it is affected by risk in other connections. The uncertainty of life itself casts a shadow on every business transaction into which time enters. The uncertainty of human life increases the rate of preference for present over future income for many people, although for those with loved dependents it may decrease impatience. Consequently, the rate of interest, even on the safest loans, will, in general, be raised by the existence of such life risks. The sailor or soldier who looks forward to a short or precarious existence will be less likely to make permanent investments, or, if he should make them, is less likely to pay a high price for them*".

Since Fisher's work, there were many debates and discussions in the economics literature about the impact of a random horizon on the market. In 1965, for instance, Yaari [124] investigated the optimal consumption by considering the uncertain lifetime in a deterministic market. Optimal investment and optimal consumption studied by [115] for a first time with respect to death time with known distribution. Later on in [78], the authors considered the optimal portfolio selection with two independent exit times, exogenous and endogenous exit time. Exogenous exit time is independent of the portfolio performance and can be considered as death time and can be modelled as the jump of a Poisson process. The waiting time for first Poisson jump is again the exponential distribution same as [29].

At the mathematical aspect, this problem is very difficult and only recently there were some advances (see [40, 74] for details). This problem of random horizon in finance can be viewed as a general setting for many other financial and economic frameworks. Among these, we cite the example of credit risk theory where the random horizon is the default time of a firm, and life insurance with its challenging mortality and/or longevity risks where the random time is the death time of an insured.

Since random time contains additional information, it is clear that the random horizon issue can be seen as a component of the important area of informational markets. In these markets, there are groups of financial agents with a different flow of information. A different and important component of this area is in the "insider trading problem". Herein, the insider group possesses extra valuable information from the very beginning of the investment period. Both the financial and mathematical finance literature on this component is very extensive and rich. We refer the reader to [75, 13, 52, 22, 23, 103] and the references therein. The key mathematical tool, in the analysis of insider problem, is the initial enlargement of filtration (see the works of Jeulin [86], Meyer and Dellacherie [60] and Jacod [82]. for more details). A random time cannot be seen before it occurs. For instance, no one knows ahead of time the death time or the time of financial crisis, and there is no single financial literature that models the information in the default of a firm as fully seen from the beginning. Hence the initial enlargement of the filtration is not adequate for the random horizon problem at all. The appropriate mathematical techniques, for modelling the discrepancy in the flow of information in the case of a random horizon, resides in the progressive enlargement of filtration, see Bielecki and Rutkowski [28], Jeulin [86], and Jeanblanc and Rutkowski [85] and the references therein. This information modelling allows us to apply our obtained results to credit risk theory and life insurance (mortality and/or longevity risk), where the progressive enlargement of filtration sounds tailor-fit, see [67, 15, 24].

1.2 Deflator and optimal portfolio

The theory of utility maximization and optimal portfolio are among the important and fundamental topics in Finance, Economics, and Mathematical Finance. The most fundamental and essential works on optimal portfolios started in on the paper of Markowitz [108] and the two seminal papers of Merton [110, 111]. On the one hand, in [108], Markowitz tends to work on single-period mean-variance portfolio selection (which is called *efficient frontier*) where he provided a platform for optimal portfolio theory. On the other hand, Merton's seminal papers developed the optimal portfolio problem by using the utility (i.e. a function that models the agent's preference) to address the risk and portfolio selection issues. He addressed the *capital asset pricing model* (CAPM). Afterwards, the theory of utility maximization and optimal portfolio has been developed successfully in many directions and in different frameworks since these seminal works, see [97, 102, 118, 37] and the reference therein. In this rich literature, the authors studied different utility functions for several popular market models as well as general semimartingale models with a fixed investment horizon. They practically neglected the impact of a random horizon on the optimal selection portfolio and/or investor's behaviour. For further details on this topic, we refer to the works of Karatzas et al. [97], Kramkov and Schachermayer [102], Cvitanic, Schachermayer, and Wang [53], Karatzas and Zitkovic [98], and the references therein to cite few.

The logarithm utility case attracts tremendous attention, even in the general semimartingale models, due to the nice feature of the log-utility. From the rich literature on the subject, it can be concluded that there are two types of optimum portfolio linked to the logarithm utility due to the discrepancy between the assumption of "admissibility" and the criterion of optimization. The first type is known as growth optimal portfolio (GOP hereafter), whose origins can be traced back to the Kelly's paper [100]. The growth optimal portfolio is the strategy which maximizes the growth rate of wealth up to some time horizon T. Many papers have examined the GOP since then, we refer to [71, 49] and the references therein. Mathematically, the growth optimal portfolio is defined as the portfolio θ^* solution to

$$\max E[\ln(W_T^\theta)],$$

over all θ with $W^\theta > 0$ and $E[\ln(W_T^\theta)^-] < +\infty$. In the thesis, we call it the log-optimal portfolio. For the GOP, the growth is a measure relative to a zero growth rate, and hence the GOP depends on the numéraire chosen.

The second type of log-utility related portfolio is known nowadays as numéraire portfolio. It is definitely a kind of GOP that does not depend on the numéraire chosen, it is a numéraire-independent-GOP. It can be defined as the portfolio θ^* having a positive wealth process W^{θ^*} such that

$$E\ln(\frac{W_T^\theta}{W_T^{\theta^*}}) \leq 0,$$

for any other portfolio θ with $W^\theta > 0$, where T is a fixed finite investment horizon. For details and related discussions, see [49], it is clear that this numéraire-independent-GOP coincides with the numéraire portfolio in the sense of [27], (i.e. the process W^θ/W^{θ^*} is a positive supermartingale). The portfolio numéraire was introduced in a paper of Long [106], where he examines the relationship between GOP and the numéraire portfolio. Since then, many authors extended and extensively investigated it differently (especially its relationship to market's viability and arbitrage), see [36, 49, 79, 91, 94] and the references therein. Today, it is known that GOP coincides with the numéraire portfolio whenever it exists. However, in general, the numéraire portfolio can exist while the GOP may fail to exit, see [36] for an example and related discussions. Since the time of Arrow and Debreu, there has been a growing interest in understanding the relationship between non-arbitrage, the viability of the market and the problem of utility maximization. They showed in [21] that the optimal portfolio, no-arbitrage and equivalent martingale measure coincide. Recently, Karatzas and Kardaras [94] showed that the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition and the existence of the NUPBR portfolio are equivalent. Under the No-Free-Lunch-with-Vanish-Risk (called NFLVR hereafter) assumption, the most advanced literature on the log-optimal portfolio, [71, 104, 13], characterizes explicitly this portfolio for the general semimartingale market models. However, see [36, 105, 117], there are many financial models that violates NFLVR, while they might admit the log-optimal portfolio. For market models under progressive enlargement of filtration, which incorporate the two important settings of credit risk and life insurance, NFLVR remains an open issue, and hence [71] is not applicable to these cases. However, for these latter models, see [6, 7], the no-unbounded-profit-with-bounded-risk (NUPBR) is fully analyzed, as it is the minimal no-arbitrage condition for a model to be financial “worthy”. Furthermore, see [31] for instance, recently there has been an interest in extending the existing results on utility maximization to models satisfying NUPBR, while they might violate NFLVR. One of the main contributions of this thesis is to establish the duality for the log-utility maximization and describe its solution without the No-Free-Lunch-with-Vanishing-Risk assumption on the market model.

In [30], the authers studied the optimal investment strategy problem with random horizon time which is not a stopping time with respect to the flow generated by the stock, but instead of knowing the conditional distribution function of the time horizon. In [1], Aase studied the optimal investment/consumption problem by considering the remaining lifetime as random time T_x and defined the survival probability function as an exponential density function. Many literature considers stopping times as death time so far, or the distribution function of the death time is known, contrary to our study. Here, in this thesis, the random horizon (death time) is as general as it can be, and it might not be a stopping time with with respect to the public flow of information.

The majority of the financial and the mathematical literature on the interplay of information and optimal portfolio are devoted to the optimal investment/portfolio problem and arbitrage for an insider who has a private information at the beginning of the investment period. This requires the study of an initial enlargement of filtration (see e.g. [13, 72, 96]). They have recently studied an insider's optimal portfolio problem for the Hara utility function and suppose that the insider has access to more information. You can find their achievements in the works of [62, 63, 75, 76, 77]. In a more general enlargement of filtration setting, the investment problem was discussed in [101]. The case of a progressive enlargement of filtration as in credit risk modelling or life insurance (mortality or longevity risk) is less investigated. Recently, in [88], they study the optimal investment problem for logarithmic utility under default risk where the information of default time is considered as an exogenous risk and consider the mixture of both enlargement methods. Recently, in [84, 125, 126], the optimal strategy under random horizon τ which is external to market was considered. Their method is based on the BSDE approach.

1.3 Summary of the Thesis

The thesis contains six chapters including the current one and one chapter for preliminaries. This thesis is based on several research papers co-authored by the candidate during his PhD studies under the supervision of Prof. Tahir Choulli, see[45, 46, 47]. The thesis includes four innovative chapters and a preliminary chapter. We keep each chapter of this thesis as independent and self-contained as possible. The organization of these six chapters is further detailed, below.

Chapter 2 recalls different financial concepts and some stochastic tools and theorems that will be very useful throughout the thesis. In particular, this chapter includes a review of deflators and no-arbitrage concepts, local martingale representations and the predictable characteristics of semimartingales. In the last two sections, we explain a mathematical model for capturing the additional information borne by the random time. Furthermore, we recall stochastic structures of the additional information and slightly extend a martingale representation result of Choulli et al. [33], which plays a vital role in our analysis and it is used frequently in this thesis.

Chapter 3 deals with the explicit parametrization of deflators for the models stopped at the random time in terms of deflators of their original counterpart models and the survival processes associated with the random time. We begin with local martingale deflators, and then deal with supermartingale deflators, as the largest class of deflators. The explicit parametrization of deflators is achieved without any assumption for the most general semimartingale model, whereas few sections of this chapter illustrate our results on various popular and practical market models. In addition, we will show how our parametrization is collective and certainly includes all restricted models.

Chapter 4 addresses the log-optimal portfolio for a general semimartingale model. We characterize a complete log-optimal portfolio and its associated optimal deflator, we provide necessary and sufficient conditions for their existence, and we elaborate their duality as well without NFLVR. There are many financial models that violate NFLVR, while they might admit the log-optimal portfolio. We also elaborate on the main result and discuss its relationship to the already existed literature. Chapter 4 closes the existing gaps in this research direction. Herein, it also contains some new intermediate useful results.

Chapter 5 addresses the dual problem of the log-utility maximization problem for the models stopped at the random time, and completely characterizes the log-optimal deflators for a quasi-left-continuous semimartingale model. Our main innovative contribution in this chapter is the description of an optimum deflator for a log-utility maximization in different manners and as explicit as possible. Thanks to the deep result of martingale classification and explicit deflator parametrization in Chapter 3, we assess the dual problem of the utility maximization problem. In the last two sections, we illustrate our results on the many important market models such as the case of the exponential Lévy market models, volatility models, and complete market model.

Chapter 6 focuses on the log-utility maximization problem itself, this chapter gives a complete characterization of the log-optimal portfolio under the random horizon, its relationship to the corresponding log-optimal deflator of Chapter 5, and beyond that. In particular, we discuss the impact of the random horizon on the numéraire portfolio in different manners, and how the random horizon induces randomness in an agent's utility. It is worth mentioning that random utilities appeared first in economics within the *random utility model theory* due to the psychometric literature that provided empirical evidence about stochastic choice behaviour. For details about this theme, we refer the reader to [121, 109, 51, 50] and the references therein to cite few. A random field utility represents the preference of an agent (or the agent's impatience as called in Fisher [70]), which is updated at each instant using the available aggregate flow of public information about the market. While, several popular and particular models are treated and discussed in this chapter.

Chapter 2

Notations and Preliminaries

In this chapter, we review some concepts (mathematical, statistical or financial/economic), and properties on stochastic processes, and their preliminary analysis. For all mathematical terminologies and techniques that are not included in this chapter, we refer to [57, 61, 81, 83, 59, 73].

This chapter contains six sections. Section 2.1 defines stochastic elements and processes that we used in this thesis. Section 2.2 introduces deflators and no-arbitrage concepts and some of their preliminary properties. Section 2.3 recalls Merton's optimal portfolio problem. In the fourth section, we present the predictable characteristics of a semimartingale. Section 2.5 presents the mathematical models for additional information. Section 2.6 introduces the $\mathbb{G}$-local martingale representation theorem and other related properties.

Throughout the rest of the thesis, our mathematical and economic model start with a filtered probability space $(\Omega, \mathcal{H}, \mathbb{H} := (\mathcal{H}_t)_{t\geq 0}, P)$, where filtration $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$ satisfies the usual conditions of completeness and right-continuity. The filtered probability space is also called a stochastic basis in the literature.

2.1 Universal notations and properties

This section recalls the stochastic elements, and related definitions that will be used through out the thesis. Most of the definitions and results of this section can be founded in Jacod and Shiryaev [83].

Definition 2.1: A process X is called càdlàg, or RCLL, if all its paths are right-continuous and admit left-hand limits.

Definition 2.2: **(i)** The *predictable σ-field* is the σ-field $\mathcal{P}(\mathbb{H})$ on $(\Omega \times \mathbb{R}_+)$ that is generated by all càg (left continuous) $\mathbb{H}$-adapted processes. Furthermore, a process X that is $\mathcal{P}(\mathbb{H})$-measurable is called predictable and it will be denoted by $X \in \mathcal{P}(\mathbb{H})$.

(ii) The *optional σ-field* is the σ-field $\mathcal{O}(\mathbb{H})$ on $(\Omega \times \mathbb{R}_+)$ that is generated by all $\mathbb{H}$-adapted and RCLL processes. Furthermore, a process X that is $\mathcal{O}(\mathbb{H})$-measurable is called optional and it will be denoted by $X \in \mathcal{O}(\mathbb{H})$.

Definition 2.3: If Y is a stochastic process and θ is a random time (i.e. a nonnegative random variable), then Y^θ is called stopped process and satisfies

$$Y^\theta_t := Y_{t\wedge\theta}, \qquad t \geq 0.$$

Here, we define the following stochastic intervals as

$$[\![\sigma, \theta]\!] := \{(\omega, t) : t \in \mathbb{R}_+, \sigma(\omega) \leq t \leq \theta(\omega)\},$$

$$[\![\sigma, \theta[\![:= \{(\omega, t) : t \in \mathbb{R}_+, \sigma(\omega) \leq t < \theta(\omega)\},$$

$$]\!]\sigma, \theta]\!] := \{(\omega, t) : t \in \mathbb{R}_+, \sigma(\omega) < t \leq \theta(\omega)\},$$

$$]\!]\sigma, \theta[\![:= \{(\omega, t) : t \in \mathbb{R}_+, \sigma(\omega) < t < \theta(\omega)\},$$

where σ and θ are random times.

Definition 2.4: [83] θ is called an $\mathbb{H}$-stopping time, if θ is a random time such that for all $t \geq 0$, the set $(\theta \leq t) := \{\omega \in \Omega : \theta(\omega) \leq t\}$ belongs to $\mathcal{H}_t$.

Lemma 2.1: *[112] Let θ be a random time. θ is an $\mathbb{H}$-stopping time if and only if $[\![0, \theta[\![$, or equivalently $[\![\theta \ +\infty[\![$, is an $\mathbb{H}$-optional set.*

If Y is an $\mathcal{H}_\sigma$-measurable random variable, then the following processes are optional:

$$YI_{]\!]\sigma,\theta[\![},\ YI_{[\![\sigma,\theta[\![},\ YI_{]\!]\sigma,\theta]\!]},\ YI_{[\![\sigma,\theta]\!]}.$$

Definition 2.5: An $\mathbb{H}$-predictable stopping time is an $\mathbb{H}$-stopping time such that the stochastic interval $[\![0, \theta[\![$ is $\mathbb{H}$-predictable.

Proposition 2.1: *[59, 112] Let T be an $\mathbb{H}$-stopping time, which is the debut $T(\omega) = \inf\{t : (\omega, t) \in A\}$ of an $\mathbb{H}$-predictable set A. If $[\![T]\!] \subset A$, then T is an $\mathbb{H}$-predictable time.*

The following concept could be a kind of partner for predictable stopping times.

Definition 2.6: Let T be an $\mathbb{H}$-stopping time.

(i) T is called $\mathbb{H}$-totally inaccessible if $P(T = \theta < +\infty) = 0$ for all $\mathbb{H}$-predictable times θ.

(ii) T is called $\mathbb{H}$-accessible if there exists a sequence of $\mathbb{H}$-predictable stopping times σ_n, such that $[\![T]\!] \subseteq \bigcup_{n\geq 1}^{\infty} [\![\sigma_n]\!]$.

Any stopping time can be presented by a totally inaccessible part and accessible part. This idea is presented by the following theorem and we refer to [83, Theorem 2.22] for its proof.

Theorem 2.1: *Let T be an $\mathbb{H}$-stopping time. There exist a sequence of $\mathbb{H}$-predictable stopping times $(S_n)_{n\geq 1}$ and a unique (up to $\mathbb{P}$-null set) $\mathcal{H}_T$-measurable subset A on $\{T < +\infty\}$, such that the stopping time T_A is totally inaccessible, and the stopping time T_{A^c} satisfies $[\![T_{A^c}]\!] \subset \bigcup [\![S_n]\!]$. T_A is called the $\mathbb{H}$-totally inaccessible part of T, and T_{A^c} its $\mathbb{H}$-accessible part. They are unique, up to a $\mathbb{P}$-null set.*

Definition 2.7: A càdlàg $\mathbb{H}$-adapted process X is called quasi-left-continuous if $\Delta X_T = 0$ a.s. on the set $\{T < +\infty\}$ for every $\mathbb{H}$-predictable stopping time T.

Here, we define martingale, sub-martingale, and super-martingale. We refer the reader to [64, 83] for more details and related properties.

Definition 2.8: Let X be an $\mathbb{H}$-adapted càdlàg process on the stochastic basis $(\Omega, \mathcal{H}, \mathbb{H}, P)$.

(i) X is called an $\mathbb{H}$-*martingale* (resp. *sub-martingale*, resp. *super-martingale*) if $E|X_t| < +\infty$ and for all $s \leq t$,

$$E[X_t|\mathcal{H}_s] = X_s, \text{ (resp. } E[X_t|\mathcal{H}_s] \geq X_s, \text{ resp. } E[X_t|\mathcal{H}_s] \leq X_s). \tag{2.1}$$

(ii) X is called an $\mathbb{H}$-*local martingale* if there is an increasing sequence of $\mathbb{H}$-stopping times $(\theta_n)_{n\geq 1} \uparrow +\infty$, such that each stopped process X^{θ_n} is an $\mathbb{H}$ martingale.

The set of all $\mathbb{H}$-martingales is denoted by $\mathcal{M}(\mathbb{H})$, and also the set of nondecreasing, right-continuous, $\mathbb{H}$-adapted and integrable processes will be denoted by $\mathcal{A}^+(\mathbb{H})$.

Definition 2.9: [99] Let M be a uniformly integrable martingale with $M_0 = 0$, and for $1 \leq p < +\infty$ we set

$$||M||_{BMO_p} = \sup_{\theta} ||\mathbb{E}[|M_\infty - M_{\theta_-}|^p|\mathcal{H}_\theta]^{1/p}||_\infty,$$

where the supremum is taken over all stopping time θ. Then we call

$$BMO_p := \{M : ||M||_{BMO_p} < \infty\},$$

is denoted by BMO_p. M is called an BMO_1-*martingale* , if $M \in BMO$ (i.e. p=1).

Definition 2.10: $\mathcal{C}_0(\mathbb{H})$ denotes a class of processes for the filtration $\mathbb{H}$, with $X_0 = 0$.

$\mathcal{C}_{0,loc}(\mathbb{H})$ denotes a class for the family of processes X, if there exists a sequence of $\mathbb{H}$-stopping times $(\theta_n)_{n\geq 1}$, such that the family of processes $X^{\theta_n} \in \mathcal{C}(\mathbb{H})$

Definition 2.11: **(i)** Two $\mathbb{H}$-local martingales M and N are called *orthogonal* if their product MN is an $\mathbb{H}$-local martingale.

(ii) An $\mathbb{H}$-local martingale X is called a *purely discontinuous* $\mathbb{H}$*-local martingale* (or a *pure jump* $\mathbb{H}$*-local martingale*) if $X_0 = 0$ and if it is orthogonal to all continuous $\mathbb{H}$-local martingales.

Below, the following corollary characterizing purely discontinuous $\mathbb{H}$-local martingales.

Corollary 2.1.1: [83, Corollary I.4.55] Let $M \in \mathcal{M}_{0,loc}(\mathbb{H})$. M is a pure jump $\mathbb{H}$-local martingale if for any continuous process $N \in \mathcal{M}_{0,loc}(\mathbb{H})$, we have $[M, N] = 0$.

Here we state the decomposition theorem for local martingales.

Theorem 2.2: *Any* $\mathbb{H}$*-local martingale* M *admits a unique (up to indistinguishability) decomposition*

$$M = M_0 + M^c + M^d,$$

where M^c *is a continuous* $\mathbb{H}$*-local martingale, and* M^d *is a pure jump* $\mathbb{H}$*-local martingale.*

For its proof, we refer to [83, Theorem I.4.18].

Definition 2.12: An $\mathbb{H}$-*semimartingale* is a càdlàg $\mathbb{H}$-adapted process X of the form $X = X_0 + M + A$, where X_0 is a finite-valued and $\mathcal{H}_0$-measurable random variable, M is an $\mathbb{H}$-local martingale and A is a finite variation process.

If A is predictable, we call X a *special semimartingale* and the decomposition $X = X_0+M+A$ is called the *canonical decomposition* of X. Furthermore, $L(X, \mathbb{H})$ denotes the set of $\mathbb{H}$-predictable processes φ, integrable with respect to X in the sense of semimartingale.

Below, the following theorem explaining the optional and predictable projection of a measurable process equipped with some integrability properties. For its proof, we refer to [59].

Theorem 2.3: *Let X be a positive or bounded $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{H}$-measurable process. There exist an $\mathbb{H}$-optional process ${}^{o,\mathbb{H}}(X)$ and an $\mathbb{H}$-predictable process ${}^{p,\mathbb{H}}(X)$ such that*

$$E\left[X_T I_{\{T<+\infty\}}|\mathcal{H}_T\right] = {}^{o,\mathbb{H}}(X)_T I_{\{T<+\infty\}} \text{ a.s. for any } \mathbb{H}\text{-stopping time } T,$$
$$E\left[X_T I_{\{T<+\infty\}}|\mathcal{H}_{T-}\right] = {}^{p,\mathbb{H}}(X)_T I_{\{T<+\infty\}} \text{ a.s. for any } \mathbb{H}\text{-predictable time } T.$$

The two processes ${}^{o,\mathbb{H}}(X)$ and ${}^{p,\mathbb{H}}(X)$ are unique up to evanescent set; and they are called the $\mathbb{H}$-optional projection and $\mathbb{H}$-predictable projection of X respectively.

Here, we state dual predictable projection theorem for an increasing, right-continuous and $\mathbb{H}$-adapted process.

Theorem 2.4: *Let A be a process in $\mathcal{A}^+_{loc}(\mathbb{H})$. There exists a process $A^{p,\mathbb{H}}$, which is unique up to an evanescent set, and is an $\mathbb{H}$-predictable process in $\mathcal{A}^+_{loc}(\mathbb{H})$ satisfying one of the following three equivalent properties:*

(a) $A - A^{p,\mathbb{H}}$ *is an $\mathbb{H}$-local martingale.*

(b) $E(A^{p,\mathbb{H}}_T) = E(A_T)$ *for all $\mathbb{H}$-stopping times T.*

(c) $E[H \centerdot A^{p,\mathbb{H}}_\infty] = E[H \centerdot A_\infty]$ *for all nonnegative $\mathbb{H}$-predictable process H.*

The process $A^{p,\mathbb{H}}$ is called the dual $\mathbb{H}$-predictable projection or compensator of A.

Proposition 2.2: *For any $\mathbb{H}$-semimartinagle L, we denote by $\mathcal{E}(L)$ the Doleans-Dade (stochastic) exponential, which is the unique solution to the stochastic differential equation*

$$dX = X_- dL, \quad X_0 = 1,$$

and is given by,

$$\mathcal{E}_t(L) = \exp(L_t - \frac{1}{2}\langle L^c\rangle_t) \prod_{0<s\leq t} (1 + \Delta L_s)e^{-\Delta L_s}\ ,\ t \geq 0.$$

Here is another simple but useful result for the nondecreasing and predictable processes.

Lemma 2.2: *Suppose V be a càdlàg (i.e. RCLL), nondecreasing and $\mathbb{H}$-predictable process such that $\Delta V < 1$. Then $1/(1-\Delta V)$ is $\mathbb{H}$-locally bounded.*

For its proof, we refer the reader to [8, Lemma A.2].

2.2 No-arbitrage and Deflators

In this section, we recall the definitions of no-arbitrage and deflators and we refer to [57, 61] for this topic. We start with no-arbitrage concepts and recall the definitions of No-Free-Lunch-with-Vanish-Risk (hereafter we called it NFLVR) and No-Unbounded-Profit-with-Bounded-Risk (NUPBR hereafter).

2.2.1 No-arbitrage concepts

Definition 2.13: Let a be a positive real number and X be an $\mathbb{H}$-semimartingale. An X-integrable $\mathbb{H}$-predictable process H is called a-admissible if $H_0 = 0$ and $H \centerdot X_t \geq -a$ for all $t \geq 0$. H is called admissible if it is admissible for some $a \in \mathbb{R}_+$.

Definition 2.14: Let X be an $\mathcal{H}$-semimartingale, and consider $\mathbb{C}_0$, as the cone of functions dominated by components of the set $\mathbb{K}_0$, where

$$\mathbb{K}_0 := \left\{ H \centerdot X_\infty \;\middle|\; H \text{ is } X-\text{admissible and } \lim_{t\to+\infty} H \centerdot X_t \text{ exists} \right\},$$

$$\mathbb{K} := \mathbb{K}_0 \cap L^\infty, \quad \text{and} \quad \mathbb{C} := \mathbb{C}_0 \cap L^\infty.$$

We say that the $\mathcal{H}$-semimartingale X satisfies the condition of

(a) *No Arbitrage* (NA) if $\mathbb{C} \cap L^\infty_+ = \{0\}$.

(b) *No Free Lunch with Vanishing Risk* (NFLVR) if $\overline{\mathbb{C}} \cap L^\infty_+ = \{0\}$.

Remark 2.1: (1) It is clear that (b) implies (a). The no-arbitrage property (NA) is equivalent to $\mathbb{K}_0 \cap L^0_+ = \{0\}$ and has an obvious interpretation: there should be no possibility of obtaining a positive profit by trading alone according to an admissible strategy.

(2) The condition of NFLVR has the following economic interpretation: there should be no sequence of final payoffs of admissible integrands, $f_n := H^n \bullet X_T$ such that the negative parts f_n^- tends to 0 uniformly and such that f_n tends almost surely to a $[0,\infty]$-valued function f_0 satisfying $P[f_0 > 0] > 0$. If (NFLVR) is not satisfied then there is a f_0 in L^∞_+, not identically 0, as well as a sequence $(f_n)_{n\geq 1}$ of elements in $\mathbb{C}$, tending almost surely to f_0, such that for all n, we have that $f_n \geq f_0 - \frac{1}{n}$. For more details we refer to [57].

Definition 2.15: The $\mathbb{H}$-semimartingale X is said to satisfy the *No-Unbounded-Profit-with-Bounded-Risk* (called NUPBR$(P,\mathcal{H})$) condition if the set

$$\mathbb{K}_1 := \left\{ H \bullet X_\infty \;\middle|\; H \bullet X \geq -1 \;\text{ and }\; \lim_{t\to+\infty} H \bullet X_t \;\text{ exists} \right\}, \qquad (2.2)$$

is bounded in $L^0(P)$ (i.e. bounded in probability under P).

Remark 2.2: The terminology of NUPBR is also articulated as *The First Kind of No Arbitrage* in Kardaras [93] or *(BK)* in Kabanov [90].

In the following lemma, we state relationships between (NA), (NUPBR) and (NFLVR). For its proof, we refer to [90].

Lemma 2.3: *The semimartingale X satisfies (NFLVR) if and only if (NA) and (NUPBR) are satisfied, i.e., NFLVR = NA + NUPBR.*

In the following theorem, we state one of the most important and fundamental concepts of asset pricing. We refer the reader to [55, 56].

Theorem 2.5: *Let X be an $(\mathbb{H}, P)$-semimartingale. Then X satisfies NFLVR if and only if there exists a probability measure $Q \sim P$ such that X is a σ-martingale* *with respect to Q (i.e. there exists $\mathbb{H}$-predictable process φ such that $0 < \varphi \leq 1$ and $\varphi \bullet X$ is a Q-martingale).*

2.2.2 Deflators

In this subsection, we recall the different class of deflators, which we use it frequently in this thesis. Namely, we define local martingale deflators, are also called in the literature by local σ-martingale density, and supermartingale deflators.

Definition 2.16: Let X be an $\mathbb{H}$-semimartingale and Z be a process.

(a) We call Z is an $\mathbb{H}$-local martingale deflator for X (or a local martingale deflator for $(X, \mathbb{H})$) if $Z > 0$ and there exists a real-valued and $\mathbb{H}$-predictable process φ such that $0 < \varphi \leq 1$ and both processes $Z > 0$ and $Z(\varphi \bullet X)$ are $\mathbb{H}$-local martingales. Throughout the thesis, the set of all local martingale deflators for $(X, \mathbb{H})$ will be denoted by $\mathcal{Z}_{loc}(X, \mathbb{H})$.

(b) We call Z is a $\mathbb{H}$-deflator for X (or a deflator for $(X, \mathbb{H})$) if $Z > 0$ and $Z\mathcal{E}(\varphi \bullet X)$ is an $\mathbb{H}$-supermartingale, for any $\varphi \in L(X, \mathbb{H})$ such that $\varphi \Delta X \geq -1$. Throughout the thesis, the set of all deflators for $(X, \mathbb{H})$ will be denoted by $\mathcal{D}(X, \mathbb{H})$.

Definition 2.17: Consider an $\mathbb{H}$-semimartingale X and an $\mathbb{H}$-positive local martingale $L > 0$. We call L is the local martingale density for X if the product LX is an $\mathbb{H}$-local martingale.

We end this subsection by the simple but important lemma.

Lemma 2.4: *Let σ be an $\mathbb{H}$-stopping time. Z is a deflator for $(X^\sigma, \mathbb{H})$ if and only if there exists unique pair of processes (K_1, K_2) such that $K_1 = (K_1)^\sigma$, $\mathcal{E}(K_1)$ is also a deflator for $(X^\sigma, \mathbb{H})$, K_2 is any $\mathbb{H}$-local supermartingale satisfying $(K_2)^\sigma \equiv 0$, $\Delta K_2 > -1$, and $Z = \mathcal{E}(K_1 + K_2) = \mathcal{E}(K_1)\mathcal{E}(K_2)$.*

Proof. Suppose Z is a deflator for $(X^\sigma, \mathbb{H})$. Hence Z is a positive $\mathbb{H}$-supermartingale. Then there exists an $\mathbb{H}$-local supermartingale K, such that $Z = \mathcal{E}(K)$. Put $K_1 := K^\sigma$

and $K_2 := K - K^\sigma$. Let φ be an $\mathbb{H}$-predictable process and X-integrable such that $\varphi\Delta X > -1$. Then $Z\mathcal{E}(\varphi \bullet X^\sigma) = \mathcal{E}(K + \varphi \bullet X^\sigma + \varphi \bullet [K, X^\sigma])$ is an $\mathbb{H}$-supermartingale if and only if $Y := K + \varphi \bullet X^\sigma + \varphi \bullet [K, X]^\sigma$ is an $\mathbb{H}$-local supermartingale. This is equivalent to $Y - Y^\sigma$ and $\mathcal{E}(Y^\sigma) = \mathcal{E}(K_1)\mathcal{E}(\varphi \bullet X^\sigma)$ are $\mathbb{H}$-local supermartingales. This ends the proof of the lemma. $\square$

This lemma shows, in a way or another, that when dealing with the stopped model $(X^\sigma, \mathbb{H})$, there is no loss of generality in focusing on the part up-to-σ of deflators, and assume that the deflator is flat after σ.

2.3 Log-optimal portfolio and numéraire portfolio

In this subsection, we provide the mathematical definitions of the utility and the corresponding Merton's optimal portfolio problem afterwards, we refer to [110, 111].

Definition 2.18: A utility function is a function U satisfying the following:

(a) U is continuously differentiable, strictly increasing, and strictly concave on its effective domain $\mathrm{dom}(U)$.

(b) There exists $u_0 \in [-\infty, 0]$ such that $\mathrm{dom}(U) \subset (u_0, +\infty)$.

The effective domain $\mathrm{dom}(U)$ is the set of $r \in \mathbb{R}$ satisfying $U(r) > -\infty$.

Given a utility function U, an $\mathbb{H}$-semimartingale X, and a probability P, we define the set of admissible portfolios as follows

$$\mathcal{A}(X_0) := \Big\{\psi \mid \ \psi \in L(X),\ \psi \bullet X \geq -X_0 \ \ \& \ \ E^P\Big[U^-(X_0 + (\psi \bullet X)_T)\Big] < +\infty\Big\}. \tag{2.3}$$

When $X = S$, and U is fixed, we simply denote $\mathcal{A}(X_0, S)$.

Utility function captures the agent's preferences.

Definition 2.19: The Merton's optimal portfolio problem is given by

$$u(x) = \max_{\psi \in \mathcal{A}(X_0)} \mathbb{E}[U(X_T^{\psi})], \qquad \text{subject to} \quad X^{(\psi, X_0)} > 0. \tag{2.4}$$

Here ψ which is X-integrable (i.e. $\psi \centerdot S$ exists) and belongs to $\mathcal{A}(X_0)$, which is the set of all admissible strategies. Then, the dual problem is given by

$$\min_{Z \in \mathcal{D}(S, \mathbb{H})} \quad \mathbb{E}[V(Z_T)], \tag{2.5}$$

where $V(y) = \max_{x>0}[U(x) - xy]$ is the conjugate function of U, and $\mathcal{D}(S, \mathbb{H})$ is the set of all deflators for the model $(S, \mathbb{H})$.

The solution to the Merton problem for a given initial wealth X_0 is the optimal strategy to determine our investment plan up to the horizon time T. The optimal portfolio ψ, maximizes the expected utility function of the terminal wealth X_T.

Below, we provide the mathematical definition of numéraire portfolio concept.

Definition 2.20: Let $(X, \mathbb{H}, P)$ be a model and Q be a probability measure such that $Q \ll P$. We call the numéraire portfolio, for the model $(X, \mathbb{H}, Q)$ when it exists, the unique $\mathbb{H}$-predictable process $\widetilde{\phi} \in L(X, \mathbb{H})$ such that $\mathcal{E}(\widetilde{\phi} \cdot X) > 0$, and $\mathcal{E}(\phi \cdot X)/\mathcal{E}(\widetilde{\phi} \cdot X)$ is a supermartingale under Q, for any $\phi \in L(X, \mathbb{H})$ satisfying $\mathcal{E}(\phi \cdot X) \geq 0$. When $Q = P$, we simply say numéraire portfolio for $(X, \mathbb{H})$.

By comparing Definitions 2.16 and 2.20, it is clear that if the numéraire portfolio $\widetilde{\phi}$ for $(X, \mathbb{H})$ exists, then $Z := 1/\mathcal{E}(\widetilde{\phi} \cdot X)$ belongs to $\mathcal{D}(X, \mathbb{H})$.

It is known that this numéraire portfolio, that was initially introduced in [106], is intimately related to the notion of deflator (or local martingale deflator) in a way or another. The connection of the existence of numéraire portfolio to deflators was first established by [94], see also [27, 36, 91] and the references therein for different proofs and/or related topics.

By taking into account a possible change of probability and or even a density, a natural extension of the above definition will be as follows.

Definition 2.21: Consider $(X, \mathbb{H}, P)$, and let Z be a positive $\mathbb{H}$-local martingale. We call numéraire portfolio for $(X, \mathbb{H}, Z)$, when it exists, is the unique $\widetilde{\psi} \in L(X, \mathbb{H})$ such that $\mathcal{E}(\widetilde{\psi} \bullet X) > 0$, and the process $Z\mathcal{E}(\phi \bullet X)/\mathcal{E}(\widetilde{\psi} \bullet X)$ is a supermartingale, for any $\phi \in L(X, \mathbb{H})$ satisfying $\mathcal{E}(\phi \bullet X) \geq 0$.

Remark 2.3: In the definition above, it is enough to consider the test processes $\phi \in L(X, \mathbb{H})$ such that $\mathcal{E}(\phi \bullet X) > 0$. In fact, we consider $\phi_0 \in L(X, \mathbb{H})$ such that $\mathcal{E}(\phi_0 \bullet X) > 0$. Then for any $\phi \in L(X, \mathbb{H})$ satisfying $\mathcal{E}(\phi \bullet X) \geq 0$ and any $\epsilon \in (0, 1]$ we have $\phi_\epsilon := \epsilon\phi_0 + (1 - \epsilon)\phi$ belongs to $L(X, \mathbb{H})$ satisfying $\mathcal{E}(\phi_\epsilon \bullet X) > 0$ and $\mathcal{E}(\phi_\epsilon \bullet X)$ converges to $\mathcal{E}(\phi \bullet X)$ when ϵ goes to zero.

2.4 Predictable characteristics of a semimartingale

In this section, we will recall the theory of semimartingale. The use of semimartingale characteristics in mathematical finance can be traced back to Yuri Kabanov in [90]. Most of the results presented in this section can be founded in [81], Jacod and Shiryaev [83], He et al [73], and Choulli and Schweizer [38]. For detailed proofs, we refer the reader to the aforementioned works of literature.

An auxiliary measurable space $(E, \mathcal{E})$ is *Blackwell space* if it is a separable space and for any $(E, \mathcal{E})$ measurable random variable ξ admits a regular condition distribution. Throughout this thesis, we consider $(E, \mathcal{E})$ is Blackwell space and it is presented by $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$.

Definition 2.22: A random measure on $\mathbb{R}_+ \times E$ is a family $\mu = \mu(\omega, dt, dx), \omega \in \Omega$ of nonnegative measure on $(\mathbb{R}_+ \times E,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E})$ satisfying $\mu(\omega, \{0\} \times E) = 0$

identically.

Throughout the thesis, on the space $\left(\widetilde{\Omega} := \Omega \times \mathbb{R}_+ \times E, \widetilde{\mathcal{H}} := \mathcal{H} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)\right)$, we will consider two σ-fields

$$\widetilde{\mathcal{O}}(\mathbb{H}) = \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(E) \qquad \text{and} \qquad \widetilde{\mathcal{P}}(\mathbb{H}) = \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(E). \tag{2.6}$$

For an $\mathbb{H}$-adapted càdlàg process X, we denote the jump random measure associate with X by μ ($\mu_X^{\mathbb{H}}$ if confusion may arise), which is given by

$$\mu(\omega, dt, dx) := \sum_{s>0} I_{\{\Delta X_s(\omega)\neq 0\}} \delta_{(s,\Delta X_s(\omega))}(dt, dx), \tag{2.7}$$

where δ_a is the Dirac measure at the point a.

For an $\mathcal{H}$-measurable function, we define the integral process $W \star \mu$ by

$$W \star \mu_t(\omega) := \begin{cases} \int_{[0,t]\times E} W(\omega, s, x)\mu(\omega, ds, dx), \text{ if } \int_{[0,t]\times E} |W(\omega, s, x)|\mu(\omega, ds, dx) < +\infty \\ +\infty, \qquad\qquad \text{otherwise.} \end{cases}$$

Another important and useful measure on $(\widetilde{\Omega}, \widetilde{\mathcal{H}})$ is given by

$$M_\mu^P(\widetilde{B}) := E^P\left[\int_{\mathbb{R}_+\times E} I_{\widetilde{B}}(\omega, t, x)\ \mu(\omega, dt, dx)\right], \text{ for all } \widetilde{B} \in \widetilde{\mathcal{H}}. \tag{2.8}$$

Thus, by $M_\mu^P[g|\widetilde{\mathcal{P}}(\mathbb{H})]$, we denote the unique $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable function, providing it exists, such that for any bounded $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable function W,

$$M_\mu^P(Wg) := E\left(\int_{\mathbb{R}_+}\int_E W(s,x)g(s,x)\ \mu(ds,dx)\right) = M_\mu^P\left(W M_\mu^P\left[g\middle|\widetilde{\mathcal{P}}(\mathbb{H})\right]\right).$$

Remark 2.4: In this thesis, the notation "$\star$" presents integrals with respect to random measures.

Definition 2.23: A random measure μ is called $\widetilde{\mathcal{O}}(\mathbb{H})$-*optional* (resp. $\mathbb{H}$-*predictable*) if the process $W\star\mu$ is $\mathbb{H}$-optional (resp. $\mathbb{H}$-*predictable*) for every $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable (resp. $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable) function W.

For any μ, a jump random measure of a process X, we associate a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable random measure ν satisfying $W\star\mu - W\star\nu$ is an $\mathbb{H}$-local martingale. Moreover, there exists a predictable process $A \in \mathcal{A}^+(\mathbb{H})$ and a kernel $F(\omega, t, dx)$ from $(\Omega\times\mathbb{R}_+, \mathcal{P}(\mathbb{H}))$ into $(E, \mathcal{E})$ such that

$$\nu(\omega, dt, dx) = dA_t(\omega)\ F(\omega, t, dx). \tag{2.9}$$

For any $\widetilde{\mathcal{H}}$-measurable function W, we define the following processes

$$\begin{aligned}\widehat{W}_t(\omega) &:= \int_E W(\omega, t, x)\nu(\omega, \{t\}\times dx),\\ \widetilde{W}_t(\omega) &:= W(\omega, t, \Delta X_t(\omega)) I_{\{\Delta X\neq 0\}}(\omega, t) - \widehat{W}_t(\omega).\end{aligned}$$

Here, we evaluate two types of integrals corresponding to the pair of random measures (μ, ν). Their integrals are denoted by $W\star(\mu-\nu)$ and $g\star\mu$, if W belongs to the set of integrands $\mathcal{G}^1_{loc}(\mu, \mathbb{H})$ and g belongs to the set of integrands $\mathcal{H}^1_{loc}(\mu, \mathbb{H})$ respectively. The two sets of integrands are defined by

$$\begin{aligned}\mathcal{G}^1_{loc}(\mu, \mathbb{H}) &:= \left\{ W \in \widetilde{\mathcal{P}}(\mathbb{H}) :\quad \sqrt{\sum_{s\leq \cdot} \widetilde{W}_s^2} \in \mathcal{A}^+_{loc}(\mathbb{H}) \right\} \quad \text{and}\\ \mathcal{H}^1_{loc}(\mathbb{H}, \mu) &:= \left\{ g : g \in \widetilde{\mathcal{O}}(\mathbb{H}), M^P_\mu[g|\mathcal{P}(\mathbb{H})] = 0, \sqrt{g^2\star\mu} \in \mathcal{A}^+_{loc}(\mathbb{H}) \right\}.\end{aligned} \tag{2.10}$$

Here, we state an important theorem of this subsection, which is the *canonical predictable representation* for a semimartingale. We refer reader to [83, Theorem 2.34] for more details.

Theorem 2.6: *Let X be an $\mathbb{H}$-semimartingale. Then X has the canonical representation:*

$$X = X_0 + X^c + h \star (\mu - \nu) + (x - h(x)) \star \mu + B, \tag{2.11}$$

where X^c is the continuous martingale part of X, B is a predictable finite variation process and h is a truncation function with the form of $h(x) = xI_{\{|x| \leq 1\}}$.

For the matrix C with entries $C^{ij} := [X^{c,i}, X^{c,j}]$, the triple (B, C, ν) is called *predictable characteristics* of X. Furthermore, we can find a version of the *characteristics triplet* satisfying

$$B = b \centerdot A, \;\; C = c \centerdot A, \;\text{ and }\;\; \nu(\omega, dt, dx) = dA_t(\omega)F_t(\omega, dx). \tag{2.12}$$

Here, A is an increasing and predictable process, b and c are predictable processes and $F_t(\omega, dx)$ is a predictable kernel such that

- $F_t(\omega, \{0\}) = 0, \qquad\qquad \int \left(|x|^2 \wedge 1\right) F_t(\omega, dx) \leq 1,$
- $\Delta B_t = \int h(x)\nu(\{t\}, dx), \quad c = 0$ on $\{\Delta A \neq 0\}$,
- $a_t := \nu(\{t\}, \mathbb{R}^d) = \Delta A_t F_t(\mathbb{R}^d) \leq 1.$

Below, the following corollary characterizing $\mathbb{H}$-special semimartingales.

Corollary 2.6.1: Let X be an $\mathbb{H}$-special semimartingale. Then X has the following decomposition:

$$X = X_0 + X^c + x \star (\mu - \nu) + B, \tag{2.13}$$

where B is a predictable process with finite variation.

In the next theorem, we present the important *Jacod's representation* of local martingale with respect to a semimartingale X. We refer the reader to [81, Theorem 3.75] for more details.

Theorem 2.7: *[Jacod's representation] Suppose that X is quasi-left-continuous, and let $N \in \mathcal{M}_{0,loc}(\mathbb{H})$. Then, there exist $\beta \in L(X^c, \mathbb{H})$, $N' \in \mathcal{M}_{0,loc}(\mathbb{H})$ with $[N', X] = 0$ and functionals $f \in \widetilde{\mathcal{P}}(\mathbb{H})$ and $g \in \widetilde{\mathcal{O}}(\mathbb{F})$ such that the following hold.*
(a) $\left(\sum_{s=0}^{t}(f(s,\Delta S_s)-1)^2 I_{\{\Delta S_s \neq 0\}}\right)^{1/2}$ *and* $\left(\sum_{s=0}^{t} g(s,\Delta S_s)^2 I_{\{\Delta S_s \neq 0\}}\right)^{1/2}$ *belong to* $\mathcal{A}^+_{loc}$.
(b) $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{H})) = 0$, $P \otimes \mu$*-.a.e., and the process N is given by*

$$N = \beta \cdot X^c + (f-1) \star (\mu - \nu) + g \star \mu + N'. \tag{2.14}$$

Moreover,

$$\Delta N = (f(\Delta X) + g(\Delta X))\, I_{\{\Delta X \neq 0\}} + \Delta N'. \tag{2.15}$$

Remark 2.5: The Jacod representation is used frequently in this thesis. From now on, we call (β, f, g, N') the *Jacod's parameters* of N with respect to X.

The following lemma can be found in [38, Proposition 2.2].

Lemma 2.5: *Let $\mathcal{E}(N)$ be a positive local martingale and (β, f, g, N') be the Jacod's parameters of N. Then, $\mathcal{E}(N) > 0$ (or equivalently $1 + \Delta N > 0$) implies that*

$$f > 0, \quad M^P_\mu - a.e.$$

Here we recall simple but an important definition for the relationship between a local martingale M and a semimartingale X with characteristics (B, C, ν).

Definition 2.24: [83] Let M be an $\mathbb{H}$-local martingale, then it has following representation property related to semimartingale X if it has the form

$$M = M_0 + H \bullet X^c + W \star (\mu - \nu),$$

where $H = (H^i)_{i \leq d} \in L^2_{loc}(X^c)$ and $W \in \mathcal{G}^1_{loc}(\mu, \mathbb{H})$.

For more details about *predictable characteristics* and related issues, we refer to [83, Section II.2]. For the sake of simplicity, we consider σ-*special* models, defined as follows.

Definition 2.25: The model $(X, \mathbb{H})$ is called σ-special if

$$\sum \varphi|\Delta X| I_{\{|\Delta X|>1\}} \in \mathcal{A}^+_{loc} \quad \text{for some real-valued and predictable } \varphi \text{ s.t. } 0<\varphi\le 1. \quad (2.16)$$

It is clear that (2.16) is equivalent to $\int_{(|x|>1)} |x| F(dx) < +\infty$ $P\otimes A$-a.e.. Throughout Thesis we consider the following set

$$\mathcal{L}(X, \mathbb{H}) := \left\{ \varphi \ d\text{-dimensional and predictable} \ \middle| \ \varphi^{tr}\Delta X > -1 \right\}. \qquad (2.17)$$

It is clear that φ belongs to $\mathcal{L}(X, \mathbb{H})$ if and only if $\varphi \in \widetilde{\mathcal{P}}(\mathbb{H})$ and $\varphi^{tr}_t(\omega)x > -1$ $P(d\omega) \otimes dA_t(\omega) F_t(\omega, dx)$-a.e. .

Lemma 2.6: *Let $Z = \mathcal{E}(N)$ be a nonnegative local martingale such that*

$$N = \beta \centerdot S^c + (f-1) \star (\mu - \nu) + g \star \mu + N'.$$

Then the following assertions hold.

(1) *ZS is a local martingale if and only if $(x - h(x) + x(f(x) - 1 + g(x))) \star \mu$ is a process with locally integrable variation and*

$$b \cdot A + c\beta \cdot A + x - h(x) + x(f(x) - 1) \star \nu = 0.$$

(2) *Consider $N_1 := \beta \cdot S^c + (f-1) \star (\mu - \nu)$. If ZS is a local martingale, then $\mathcal{E}(N_1)S$ also is a local martingale.*

For its proof, we refer reader to [43, lemma 2.4].

2.5 Enlargement of the flow of information

In this thesis, the additional information comes from a random time τ (a positive random variable) that would represent different concepts in finance, namely, a default or bankruptcy time in credit risk, retirement or death time in insurance, etc. To capture the additional information from τ, two main methods have been investigated. Precisely, these two methods are called initial enlargement of filtration, and progressive enlargement of filtration. In this thesis, in order to study the additional element and concept to the usual setup, in our case random horizon, we need to find a suitable framework. The most adequate method is progressive enlargement of filtration, as the random time cannot be seen before it occurs. By using this method we can capture the additional information carried by the random horizon. Below, we recall preliminary properties of progressive enlargement of filtration and we refer the reader to Jeulin [86] for their proofs and other related topics.

For the rest of the thesis, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions. Here $\mathbb{F}$ is the public flow of information. To this initial model, we consider a random time $\tau : \Omega \to \overline{\mathbb{R}}_+$, that represents the random horizon, which might not be an $\mathbb{F}$-stopping time. Throughout the thesis, we will be using the following associated process D and the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$,

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathcal{G}_t := \mathcal{G}^0_{t+} \text{ where } \mathcal{G}^0_t := \mathcal{F}_t \vee \sigma\left(D_s,\ s \leq t\right). \tag{2.18}$$

Thus, the agent who has access to $\mathbb{F}$, can only get information about τ through the following survival probabilities, called in the literature by Azéma supermartingales

$$G_t := P(\tau > t|\mathcal{F}_t) \quad \text{and} \quad \widetilde{G}_t := P(\tau \geq t|\mathcal{F}_t).$$

Both G and $\widetilde{G}$ are supermartingales, where G is RCLL, but $\widetilde{G}$ has right and left limits only. The process

$$m := G + D^{o,\mathbb{F}}, \tag{2.19}$$

is an $\mathbb{F}$-martingale. One has to note that the above process m is not the Doob-Meyer decomposition of G in general. Since the second term above is $\mathbb{H}$-dual optional projection of process D and in the Doob-Meyer decomposition of G, its $\mathbb{H}$-dual predictable projection appears. The following important theorem from Jeulin [86], provides a classic relationship between $\mathbb{F}$-local martingales and $\mathbb{G}$-local martingales on $[\![0,\tau]\!]$, which has been developed recently in [10, 33] and references therein.

Proposition 2.3: *Let τ be a random time. Then the following hold:*

(a) *If X is an $\mathbb{F}$-semimartingale, X^τ (i.e. X stopped at τ) is a $\mathbb{G}$-semimartingale.*

(b) *If X is an $\mathbb{F}$-local martingale, then*

$$\overline{X}_t = X_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{1}{G_{s-}} d\langle X, m\rangle_s^{\mathbb{F}}, \qquad t \geq 0. \tag{2.20}$$

is a $\mathbb{G}$-local martingale.

The following is an useful result, from [6, Lemma B.1], for relationships between functional measurability in $\mathbb{F}$ and $\mathbb{G}$.

Lemma 2.7: *Suppose that τ is a random time. Let $H^{\mathbb{G}}$ be an $\widetilde{\mathbb{P}}(\mathbb{G})$-measurable functional and $0 \leq H^{\mathbb{G}} \leq 1$. Then, there exists a $\widetilde{\mathbb{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$ and $0 \leq H^{\mathbb{F}} \leq 1$, such that*

$$H^{\mathbb{G}}(\omega, t, x) I_{[\![0,\tau]\!]} = H^{\mathbb{F}}(\omega, t, x) I_{[\![0,\tau]\!]}.$$

2.6 $\mathbb{G}$-stochastic structures versus those of $\mathbb{F}$

Next, we shall study about $\mathbb{G}$-local martingale representations theorems.

Theorem 2.8: *For any $\mathbb{F}$-local martingale, M, the process*

$$\mathcal{T}(M) := M^\tau - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet [M,m] + I_{[\![0,\tau]\!]} \bullet \Big(\sum \Delta M I_{\{\widetilde{G}=0<G_-\}} \Big)^{p,\mathbb{F}}, \qquad (2.21)$$

is a $\mathbb{G}$-local martingale.

For this claim we refer the reader to [10]. Thus, our first class of $\mathbb{G}$-martingales is given by

$$\mathcal{M}^{(1)}(\mathbb{G}) := \left\{ \mathcal{T}(M) \text{ defined in (2.21)} \;\middle|\; \mathcal{T}(M)_\infty \in L^1(P),\ M \in \mathcal{M}_{0,loc}(\mathbb{F}) \right\}. \qquad (2.22)$$

We recall from Choulli et al. [33] that the following process is a $\mathbb{G}$-martingale. In the following, they propose a new $\mathbb{G}$-martingale, and show that it has nice features such as a larger set of integrands than the usual $\mathbb{G}$-predictable integrands. This new martingale is the core generator of a second class of $\mathbb{G}$-(local)martingales.

Theorem 2.9: *Consider the following process*

$$N^{\mathbb{G}} := D - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet D^{o,\mathbb{F}}. \qquad (2.23)$$

Then, the following assertions hold.
(i) $N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with integrable variation.
(ii) Let K be an $\mathbb{F}$-optional process, which is Lebesgue-Stieltjes integrable with respect to $N^{\mathbb{G}}$. Then,

$$K \bullet N^{\mathbb{G}} \in \mathcal{A}(\mathbb{G}) \quad \textit{iff} \quad [K \bullet N^{\mathbb{G}}] \in \mathcal{A}^+(\mathbb{G}) \quad \textit{iff} \quad K \in \mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G}), \qquad (2.24)$$

where

$$\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) := \Big\{ K \in \mathcal{O}(\mathbb{F}) \;\; \Big| \;\; \mathbb{E}\Big[|K| G \widetilde{G}^{-1} I_{\widetilde{G}>0} \bullet D_\infty \Big] < +\infty \Big\}. \qquad (2.25)$$

Furthermore, in the case where K belongs to $\mathcal{I}^o(N^{\mathbb{G}},\mathbb{G})$ (resp. to $\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})$), the process $K \centerdot N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with integrable variation (resp. is a $\mathbb{G}$-local martingale with local integrable variation).

(iii) The elements of

$$\mathcal{M}^{(2)}(\mathbb{G}) := \Big\{ K \centerdot N^{\mathbb{G}} \ \big| \ K \in \mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G}) \ i.e.\ K \in \mathcal{O}(\mathbb{F}) \ and \ |K| \bullet Var(N^{\mathbb{G}}) \in \mathcal{A}^+_{loc}(\mathbb{G}) \Big\},$$

are $\mathbb{G}$-martingales orthogonal to locally bounded elements of $\mathcal{M}^{(1)}_{loc}(\mathbb{G})$ defined in (2.22).

(iv) For $K \in \mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})$ and $\mathbb{F}$-stopping time σ, the process $\left(K \centerdot N^{\mathbb{G}}\right)^{\sigma-}$ is also a $\mathbb{G}$-local martingale.

Theorem 2.10: *[33] The following assertions hold.*

(i) For any $k \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P \otimes D\right)$, there exists a unique (up to a $\mu := P \otimes D$-negligible set) $\mathbb{F}$-optional process, h, satisfying

$$\mathbb{E}\left[k_\tau \ \big| \ \mathcal{F}_\tau\right] = h_\tau \qquad P\text{-}a.s. \quad on \ \ \{\tau < +\infty\}. \tag{2.26}$$

(ii) The elements of the set

$$\mathcal{M}^{(3)}(\mathbb{G}) := \left\{ k \centerdot D \ \Bigg| \ k \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P \otimes D\right) \quad and \quad \mathbb{E}[k_\tau | \mathcal{F}_\tau] = 0 \ P\text{-}a.s \right\} \tag{2.27}$$

are $\mathbb{G}$-martingales that are orthogonal to locally bounded elements of $\mathcal{M}^{(1)}_{loc}(\mathbb{G})$ and $\mathcal{M}^{(2)}_{loc}(\mathbb{G})$.

Now for parametrizing the class of $\mathbb{G}$-deflators for the model $(S^\tau, \mathbb{G})$, we need to see the decomposition of any $\mathbb{G}$-local martingales stopping at random time τ. For the sake of this goal, we slightly extend the representation theorem [33, Theorem 2.8] to the case of $\mathbb{G}$-local martingales when the process G never vanishes.

Theorem 2.11: *Suppose that $G > 0$. Then for any $\mathbb{G}$-local martingale $M^{\mathbb{G}}$, there* *exists a unique triplet* $(M^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)})$ *that belongs to*

$$\mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^{o}_{loc}\left(N^{\mathbb{G}}, \mathbb{G}\right) \times L^1_{loc}\left(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D\right)$$

and satisfies

$$\mathbb{E}\left[\varphi^{(pr)}_{\tau} \;\middle|\; \mathcal{F}_{\tau}\right] I_{\{\tau<+\infty\}} = 0, \quad P\text{-}a.s., \tag{2.28}$$

and

$$\left(M^{\mathbb{G}}\right)^{\tau} = M^{\mathbb{G}}_0 + G^{-2}_{-} I_{[\![0,\tau]\!]} \bullet \mathcal{T}(M^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{2.29}$$

Proof. Let $M^{\mathbb{G}} \in \mathcal{M}_{0,loc}(\mathbb{G})$, then there exists a sequence of $\mathbb{G}$-stopping times, $T_{n\{n\geq 0\}}$, that increases to infinity such that $(M^{\mathbb{G}})^{T_n}$ is a $\mathbb{G}$-martingale. On the one hand, thanks to [6, Proposition B.2-(b)], we deduce the existence of a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_n$ that increases to infinity and

$$T_n \wedge \tau = \sigma_n \wedge \tau, \qquad\qquad n \geq 1.$$

On the other hand, due to the assumption $G > 0$ and by applying [33, Theorem 2.20] to each $(M^{\mathbb{G}})^{T_n} - (M^{\mathbb{G}})^{T_{n-1}}$, we obtain the existence of sequence of triplet $(M^{\mathbb{F},n}, \varphi^{(o,n)}, \varphi^{(pr,n)})$ that belongs to

$$\mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^{o}_{loc}\left(N^{\mathbb{G}}, \mathbb{G}\right) \times L^1_{loc}\left(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D\right)$$

and satisfies

$$\mathbb{E}\left[\varphi^{(pr,n)}_{\tau} \;\middle|\; \mathcal{F}_{\tau}\right] I_{\{\tau<+\infty\}} = 0, \quad P\text{-a.s.},$$

and

$$(M^{\mathbb{G}})^{\tau\wedge T_n} - (M^{\mathbb{G}})^{\tau\wedge T_{n-1}} = G^{-2}_{-} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(M^{\mathbb{F},n}) + \varphi^{(o,n)} \bullet N^{\mathbb{G}} + \varphi^{(pr,n)} \bullet D.$$

Notice that $(M^{\mathbb{G}})^{\tau} - M^{\mathbb{G}}_0 = \sum_{n\geq 1}(M^{\mathbb{G}})^{\tau\wedge T_n} - (M^{\mathbb{G}})^{\tau\wedge T_{n-1}})$, and put

$$\sigma_0 = 0, \quad \varphi^{(o)} := \sum_{n\geq 1} I_{]\!]\sigma_{n-1},\sigma_n]\!]}\varphi^{(o,n)}, \quad \varphi^{(pr)} := \sum_{n\geq 1} I_{]\!]\sigma_{n-1},\sigma_n]\!]}\varphi^{(pr,n)},$$

$$M^{\mathbb{F}} := \sum_{n\geq 1} I_{]\!]\sigma_{n-1},\sigma_n]\!]} \bullet M^{\mathbb{F},n}.$$

This ends the proof of the theorem. □

Here, we recall important results about computing the compensator of $\mu^{\mathbb{G}}$ -which is the jump measure X^{τ}- and the canonical representation of X^{τ} in $\mathbb{G}$. For detailed proofs of the next two theorems, we refer to [6, 8, 10].

Proposition 2.4: *Let μ be the jump measure of X and ν be its $\mathbb{F}$-compensator. Then, on $[\![0,\tau]\!]$, the $\mathbb{G}$-compensator of $I_{[\![0,\tau]\!]} \bullet \mu$ is given by*

$$\nu^{\mathbb{G}} := \left(I_{[\![0,\tau]\!]} \bullet \mu\right)^{p,\mathbb{G}} = \frac{M^P_{\mu}[\widetilde{G}|\widetilde{\mathcal{P}}(\mathbb{H})]}{G_-} I_{[\![0,\tau]\!]} \bullet \nu. \tag{2.30}$$

Theorem 2.12: *Let X be an $\mathbb{F}$-semimartingale with the canonical representation*

$$X = X_0 + X^c + (x-h)\star\mu + h\star(\mu-\nu) + B.$$

Then the canonical representation of X^{τ} is given by

$$X^{\tau} = X_0 + \widehat{X^c} + h\star(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + (x-h)\star\mu^{\mathbb{G}} + \widetilde{B}, \tag{2.31}$$

where $\widehat{X^c}$ is defined via (2.20) and,

$$\widetilde{B} := B^{\tau} + \frac{I_{[\![0,\tau]\!]}}{G_-} \bullet \langle X^c, m\rangle^{\mathbb{H}} + h\frac{M^P_{\mu}[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{G_-} I_{[\![0,\tau]\!]} \star \ \nu.$$

The following lemma recalls the $\mathbb{G}$-compensator of any $\mathbb{F}$-optional process stopped at a random time τ.

Lemma 2.8: *[6] For any $\mathbb{F}$-adapted process V with locally integrable variation, we have* $(V^\tau)^{P,\mathbb{G}} = G_-^{-1} I_{]\!]0,\tau]\!]} \bullet (\widetilde{G} \bullet V)^{P,\mathbb{F}}$.

Chapter 3

Explicit description of all deflators for models stopped at random time

Throughout this chapter, we consider an initial market model, specified by its underlying assets S and its flow of information $\mathbb{F}$, and an arbitrary random time τ which might not be an $\mathbb{F}$-stopping time. In this setting, our principal goal resides in describing as explicit as possible the set of all deflators, which constitutes the dual set of all "admissible" wealth processes, for the stopped model S^τ. Since the death time and the default time (that τ might represent) can be seen when they occur only, the progressive enlargement of $\mathbb{F}$ with τ sounds tailor-fit for modelling the new flow of information that incorporates both $\mathbb{F}$ and τ. Thanks to the deep results of Choulli et al. [33], on martingales classification and representation for progressive enlarged filtration, our aim is fully achieved for both cases of local martingale deflators and supermartingale delators. The results are illustrated on several particular models for $(\tau, S, \mathbb{F})$ such as the discrete-time and the jump-diffusion settings for $(S, \mathbb{F})$, and the case when τ avoids $\mathbb{F}$-stopping times.

This chapter contains six sections. The first section presents some useful intermediate results that are interesting in themselves beyond their role in proving the main theorems of this chapter. The second section states the explicit parametrization of local martingale deflators for $(S^\tau, \mathbb{G})$. We illustrate the main theorems on particular models and their related discussions in the third and fourth sections. Section 5 addresses the case of supermartingale deflators. In the last section, we present particular models for the supermartingale deflators.

3.1 Preliminary results

This section contains some useful technical (new and existing) results with detailed proofs. The results might have applications beyond this chapter.

3.1.1 Some $\mathbb{G}$-properties versus those in $\mathbb{F}$

In the following proposition, we explain briefly some $\mathbb{G}$-properties versus those in $\mathbb{F}$. These results appear naturally in the proofs of our important theorems of this chapter. Besides this, they sound important in themselves.

Proposition 3.1: *The following assertions hold.*

(a) *For any $\mathbb{G}$-predictable process $\varphi^{\mathbb{G}}$, there exists an $\mathbb{F}$-predictable process $\varphi^{\mathbb{F}}$ such that*

$$\varphi^{\mathbb{G}} = \varphi^{\mathbb{F}} \quad \text{on} \quad]\!]0, \tau]\!].$$

If, furthermore, $\varphi^{\mathbb{G}}$ is bounded, then $\varphi^{\mathbb{F}}$ can be chosen to be bounded with the same constants.

(b) *Suppose $G > 0$. Then for a bounded $\theta \in \mathcal{L}(S^\tau, \mathbb{G})$ (i.e. θ is $\mathbb{G}$-predictable such that $\theta^{tr}\Delta S^\tau > -1$), there exists a bounded $\varphi \in \mathcal{L}(S, \mathbb{F})$ that coincides with θ on $[\![0, \tau]\!]$.*

(c) *Suppose $G > 0$. Let $V^{\mathbb{G}}$ be a $\mathbb{G}$-predictable, RCLL and nondecreasing process* *with finite values such that $(V^{\mathbb{G}})^\tau = V^{\mathbb{G}}$. Then there exists a unique nondecreasing with finite values, RCLL, and $\mathbb{F}$-predictable process, V, such that $V^{\mathbb{G}} = V^\tau$.*

If furthermore $\Delta V^{\mathbb{G}} < 1$, then $\Delta V < 1$ holds also.

Proof. Remark that the boundedness condition for $\varphi^{\mathbb{G}}$ can be reduced to the condition $0 \leq \varphi^{\mathbb{G}} \leq 1$. Thus, assertion (a) is a particular case of the general case treated in [6, Lemma B.1] (see also [86, Lemma 4.4 (b)]). Hence, its proof will be omitted, and we refer the reader to [6] and [86]. Thus, the remaining part of this proof focuses on proving assertions (b) and (c) in two parts.

Part 1. Here we prove assertion (b). Consider a bounded $\theta \in \mathcal{L}(S^\tau, \mathbb{G})$. Then θ is a bounded and $\mathbb{G}$-predictable process satisfying $\theta^{tr}\Delta S^\tau > -1$. Thus, in virtue of assertion (a), there exists a bounded and $\mathbb{F}$-predictable process φ such that

$$\theta I_{[\![0,\tau]\!]} = \varphi I_{[\![0,\tau]\!]}.$$

Then by inserting this equality in $\theta^{tr}\Delta S^\tau > -1$, we deduce that

$$\varphi^{tr}\Delta S I_{]\!]0,\tau]\!]} > -1,$$

which is equivalent to $I_{]\!]0,\tau]\!]} \leq I_{\{\varphi^{tr}\Delta S > -1\}}$. By taking the $\mathbb{F}$-optional projection on both sides of this inequality, we get $0 < G \leq I_{\{\varphi^{tr}\Delta S > -1\}}$ on $]\!]0, +\infty[\![$, or equivalently $\varphi^{tr}\Delta S > -1$. Hence φ belongs to $\mathcal{L}(S, \mathbb{F})$, and the proof of assertion (b) is complete.

Part 2. This part proves assertion (c). Consider a $\mathbb{G}$-predictable, RCLL, and nondecreasing process with finite values $V^{\mathbb{G}}$ such that $(V^{\mathbb{G}})^\tau = V^{\mathbb{G}}$. It is clear that there is no loss of generality in assuming that $V^{\mathbb{G}}$ is bounded. Then due to [86, Lemma 4.4 (b)] (see also [6, Lemma B.1]), there exists an $\mathbb{F}$-predictable process V such that

$$V^{\mathbb{G}} I_{[\![0,\tau]\!]} = V I_{[\![0,\tau]\!]}. \tag{3.1}$$

By writing $V^{\mathbb{G}} I_{[\![0,\tau[\![} = V^{\mathbb{G}} - V^{\mathbb{G}}_{\tau} I_{[\![\tau,+\infty[\![}$ –which is obviously a RCLL bounded $\mathbb{G}$-semimartingale– and by taking the $\mathbb{F}$-optional projection on both sides of (3.1), we get $V =^{o,\mathbb{F}} (V^{\mathbb{G}} I_{[\![0,\tau[\![})/G$. Hence V is a RCLL $\mathbb{F}$-semimartingale that is predictable. As a result, the Doob-Meyer decomposition guarantees the existence of a continuous $\mathbb{F}$-martingale L with $L_0 = 0$ and an $\mathbb{F}$-predictable process with finite variation B such that $V = L + B$. Since $V^{\mathbb{G}}$ is predictable with finite variation and

$$V^{\mathbb{G}} = V^{\tau} = L^{\tau} + B^{\tau} = \left(L^{\tau} - G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \langle L, m\rangle^{\mathbb{F}}\right) + G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \langle L, m\rangle^{\mathbb{F}} + B^{\tau},$$

then we conclude that the $\mathbb{G}$-local martingale $L^{\tau} - G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \langle L, m\rangle^{\mathbb{F}}$ is null. This implies that $[L, L]^{\tau}$ is also a null process since L is continuous, or equivalently $L \equiv 0$ due to the assumption $G > 0$. This proves that $V = B$ has a finite variation. To prove that V is nondecreasing, it is enough to remark that $(V^{\mathbb{G}})^{p,\mathbb{F}}$ is nondecreasing and $V = G_-^{-1} \bullet (V^{\mathbb{G}})^{p,\mathbb{F}}$. This proves the first statement of assertion (c), while the proof of the last statement of assertion (c) follows the same foot steps of part 1). Indeed $\Delta V^{\mathbb{G}} = \Delta V I_{]\!]0,\tau]\!]} < 1$ holds if and only if $I_{]\!]0,\tau]\!]} \leq I_{\{\Delta V < 1\}}$ holds, and this implies –after taking the $\mathbb{F}$-predictable projection on both sides of this inequality– that $0 < G_- \leq I_{\{\Delta V < 1\}}$ on $]\!]0, +\infty[\![$. This is equivalent to $\Delta V < 1$, due to $G_- > 0$. This property, ($i.e. G_- > 0$), follows from the assumption $G > 0$ and the fact that both sets $\{G_- > 0\}$ and $\{G > 0\}$ have the same début. For this last fact, we refer the reader to [86, Lemme (4.3)]. This ends the proof of the proposition. □

The following lemma recalls the $\mathbb{G}$-compensator of $\mathbb{F}$-optional processes stopped at a random time τ.

Lemma 3.1: *Let $V \in \mathcal{A}_{loc}(\mathbb{F})$, then we have*

$$(V^{\tau})^{p,\mathbb{G}} = I_{]\!]0,\tau]\!]} G_-^{-1} \bullet (\widetilde{G} \bullet V)^{p,\mathbb{F}}.$$

For the proof of this lemma and other related results, we refer to [6, 7, 8].

3.1.2 Useful integration properties

Lemma 3.2: *Let φ be a real-valued and $\mathbb{F}$-predictable process, $N^{\mathbb{F}} \in \mathcal{M}_{0,loc}(\mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, and $\varphi^{(pr)} \in L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$ such that*

$$\mathbb{E}[\varphi^{(pr)}_\tau \bigm| \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0 \quad P\text{-a.s., and}$$

$$\left(1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}\right) I_{]\!]0,\tau]\!]} + \varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D > 0, \quad 0 < \varphi \le 1. \tag{3.2}$$

Then the process

$$W := \sum \varphi \Delta S I_{\{|\Delta S|>1\}} \left[\left(1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}\right) I_{]\!]0,\tau]\!]} + \varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D\right]$$

has a $\mathbb{G}$-locally integrable variation if and only if both processes

$$\begin{aligned} W^{(1)} &:= \sum \varphi \Delta S I_{\{|\Delta S|>1\}} (1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}) I_{]\!]0,\tau]\!]} \quad \text{and} \\ W^{(2)} &:= \sum \varphi \Delta S I_{\{|\Delta S|>1\}} [\varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D] \end{aligned}$$

belong to $\mathcal{A}_{loc}(\mathbb{G})$.

Proof. Due to the first condition in (3.2), it is clear that $W \in \mathcal{A}_{loc}(\mathbb{G})$ iff

$$W^+ := \sum |\varphi \Delta S| I_{\{|\Delta S|>1\}} [(1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}) I_{]\!]0,\tau]\!]} + \varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D] \in \mathcal{A}^+_{loc}(\mathbb{G}).$$

By stopping, there is no loss of generality in assuming that $E[W^+_\infty] < +\infty$. Thus, since both processes $\sum |\varphi \Delta S| I_{\{k \ge |\Delta S|>1\}} \varphi^{(o)} \Delta N^{\mathbb{G}} = |\varphi \Delta S| I_{\{k \ge |\Delta S|>1\}} \varphi^{(o)} \bullet N^{\mathbb{G}}$ and $\sum |\varphi \Delta S| I_{\{k \ge |\Delta S|>1\}} \varphi^{(pr)} \Delta D = |\varphi \Delta S| I_{\{k \ge |\Delta S|>1\}} \varphi^{(pr)} \bullet D$ are $\mathbb{G}$-local martingale, we derive

$$E\left[\sum|\varphi\Delta S|I_{\{|\Delta S|>1\}}[(1+\frac{\Delta N^{\mathbb{F}}}{G_{-}\widetilde{G}})I_{]\!]0,\tau]\!]}\right]=\lim_{k\longrightarrow+\infty}E\left[\sum|\varphi\Delta S|I_{\{1<|\Delta S|\leq k\}}(1+\frac{\Delta N^{\mathbb{F}}}{G_{-}\widetilde{G}})I_{]\!]0,\tau]\!]}\right]$$
$$=\lim_{k\longrightarrow+\infty}E[(I_{\{|\Delta S|\leq k\}}\cdot W^{+})_{\infty}]\leq E[W^{+}_{\infty}]<+\infty.$$

This proves that $W^{(1)}\in\mathcal{A}(\mathbb{G})$, and hence $W^{(2)}=W-W^{(1)}\in\mathcal{A}(\mathbb{G})$. Thus, the proof of the lemma is complete. □

Lemma 3.3: *Suppose $G>0$. Let M be an $\mathbb{F}$-local martingale with bounded jumps. Then the following assertions hold.*

(a) *Let A be an $\mathbb{F}$-predictable process with finite variation. Then $[\mathcal{T}(M),A^{\tau}]$ is a $\mathbb{G}$-local martingale.*

(b) *Let $A^{(o)}\in I^{o}_{loc}\left(N^{\mathbb{G}},\mathbb{G}\right)$. Then $A^{(o)}\centerdot[N^{\mathbb{G}},M^{\tau}]$ is a $\mathbb{G}$-local martingale process.*

(c) *Let $A^{(pr)}\in L^{1}_{loc}\left(\mathcal{P}_{rog}(\mathbb{F}),P\otimes D\right)$ such that $\mathbb{E}\left[A^{(pr)}_{\tau}\mid F_{\tau}\right]I_{\{\tau<+\infty\}}=0$ P-a.s.. Then $A^{(pr)}\centerdot[D,M^{\tau}]$ is a $\mathbb{G}$-local martingale process.*

Proof. (a) Let A be an $\mathbb{F}$-predictable process with finite variation. Then by applying [83, lemma 4.49], we get $[\mathcal{T}(M),A^{\tau}]=\Delta A^{\tau}\centerdot\mathcal{T}(M)$ which is a $\mathbb{G}$-local martingale, since $\mathcal{T}(M)$ is a $\mathbb{G}$-local martingale and ΔA is locally bounded.

(b) By stopping, there is no loss of generality in assuming $A^{(o)}\in I^{o}\left(N^{\mathbb{G}},\mathbb{G}\right)$. Then, $A^{(o)}\centerdot N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale. Since $N^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale with finite variation and M^{τ} is a $\mathbb{G}$-semimartingale, we derive

$$A^{(o)}\centerdot[N^{\mathbb{G}},M^{\tau}]=[A^{(o)}\centerdot N^{\mathbb{G}},M^{\tau}]=\Sigma\Delta MA^{(o)}I_{]\!]0,\tau]\!]}\Delta N^{\mathbb{G}}=\Delta MA^{(o)}\centerdot N^{\mathbb{G}}.\quad(3.3)$$

Then $\tilde{A}^{(o)}:=\Delta MA^{(o)}$ is an $\mathbb{F}$-optional process as it is a product of ΔM and $A^{(o)}$ that are both $\mathbb{F}$-optional processes. Therefore, since $\mid\Delta M\mid$ is bounded and $A^{(o)}\in I^{o}\left(N^{\mathbb{G}},\mathbb{G}\right)$, we obtain

$$\mathbb{E}\left[|\tilde{A}^{(o)}|G\widetilde{G}^{-1}I_{\{\widetilde{G}>0\}}\centerdot D_{\infty}\right]=\mathbb{E}\left[|\Delta MA^{(o)}|G\widetilde{G}^{-1}I_{\{\widetilde{G}>0\}}\centerdot D_{\infty}\right]<+\infty\qquad(3.4)$$

This proves that $\tilde{A}^{(o)} \in I^{o}\left(N^{\mathbb{G}}, \mathbb{G}\right)$. Hence, in virtue of Theorem 2.9 and (3.3), we deduce that $A^{(o)} \centerdot [N^{\mathbb{G}}, M^{\tau}]$ is a $\mathbb{G}$-martingale process.

(c) Again, by stopping, we assume that $A^{(pr)} \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P \otimes D\right)$. Then, it is clear that $A^{(pr)} \centerdot D$ is a $\mathbb{G}$-martingale. Since both D and M^{τ} are $\mathbb{G}$-semimartingales, we obtain

$$A^{(pr)} \centerdot [D, M^{\tau}] = [A^{(pr)} \centerdot D, M^{\tau}] = \Sigma \Delta M A^{(pr)} I_{]\!]0,\tau]\!]} \Delta D = \Delta M A^{(pr)} \centerdot D. \quad (3.5)$$

Here, $\tilde{A}^{(pr)} := \Delta M A^{(pr)}$ is an $\mathbb{F}$-progressive process, since $A^{(pr)}$ is an $\mathbb{F}$-progressive process and ΔM is a bounded $\mathbb{F}$-optional process. Furthermore, these imply that $\tilde{A}^{(pr)} \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P \otimes D\right)$, and

$$\mathbb{E}\left[\tilde{A}^{(pr)}_{\tau} \mid F_{\tau}\right] I_{\{\tau<+\infty\}} = \Delta M_{\tau} \cdot \mathbb{E}\left[A^{(pr)}_{\tau} \mid F_{\tau}\right] I_{\{\tau<+\infty\}} = 0.$$

Hence, by combining (3.5) and Theorem 2.10, one can conclude that $A^{(pr)} \centerdot [D, M^{\tau}]$ is a $\mathbb{G}$-martingale. This completes the proof of the lemma. □

Lemma 3.4: *Suppose that X is an $\mathbb{F}$-local martingale, $Z := \mathcal{E}(X) > 0$ and Y is an $\mathbb{F}$-semimartingale. Then $\mathcal{E}(X)Y$ is an $\mathbb{F}$-local martingale if and only if $Y + [X, Y]$ is an $\mathbb{F}$-local martingale.*

Proof. First of all, since $Z > 0$, we conclude that both processes Z_- and $1/Z_-$ are $\mathbb{F}$-predictable and locally bounded. As a result, $Z_- \centerdot M$ is a $\mathbb{F}$-local martingale for any $\mathbb{F}$-local martingale M. By combining Ito's formula and $Z = \mathcal{E}(X) = 1 + Z_- \centerdot X$, we get the following

$$ZY = \mathcal{E}(X)Y = Y_- \centerdot Z + Z_- \centerdot Y + Z_- \centerdot [X, Y] = Y_- \centerdot Z + Z_- \centerdot (Y + [Y, X]).$$

Thus, since $Y_- \centerdot Z$ is an $\mathbb{F}$-local martingale and $1/Z_-$ is locally bounded, we deduce that $\mathcal{E}(X)Y$ is an $\mathbb{F}$-local martingale if and only if $(Y + [Y, X])$ is an $\mathbb{F}$-local martingale. □

The next section addresses the set of all local martingale deflators (also called in the literature by the set of all local martingale densities) for $(S^\tau, \mathbb{G})$.

3.2 Local martingale deflators: The general setting

This section focuses on describing completely the set of all local martingale deflators, defined in Definition 2.16-(a), for the model $(S^\tau, \mathbb{G})$ in terms of those of $(S, \mathbb{F})$. Throughout the rest of this section, we adopt the convention $1/0^+ = +\infty$.

Below, we state the main result of this section. Recall that the set of all local martingale deflators of $(X, \mathbb{H})$ is donated by $\mathcal{Z}_{loc}(X, \mathbb{H})$

Theorem 3.1: *Suppose $G > 0$, and let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ *is a local martingale deflator for* $(S^\tau, \mathbb{G})$.

(b) *There exists* $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ *such that* $(K^{\mathbb{F}}, \varphi^{(o)}) \in \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)} \in L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D)$ *and* $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ *P-a.s., and the following three conditions hold:*

(b.1) $\mathcal{E}\left(K^{\mathbb{F}}\right)$ *is a local martingale deflator for* $(S, \mathbb{F})$ *(i.e.* $Z^{\mathbb{F}} := \mathcal{E}\left(K^{\mathbb{F}}\right) \in \mathcal{Z}_{loc}(S, \mathbb{F})$*).*

(b.2) *The following inequalities hold*

$$\varphi^{(pr)} > -\left[G_-(1 + \Delta K^{\mathbb{F}}) + \varphi^{(o)} G\right]/\widetilde{G}, \qquad P \otimes D - a.e., \tag{3.6}$$

$$-\frac{G_-}{G}(1+\Delta K^{\mathbb{F}}) < \varphi^{(o)}, \quad P \otimes D^{o,\mathbb{F}}\text{-a.e. and} \quad \varphi^{(o)} \Delta D^{o,\mathbb{F}} < (1+\Delta K^{\mathbb{F}}) G_-. \tag{3.7}$$

(b.3) *The following decomposition holds.*

$$K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.8}$$

Proof. The proof will be achieved in two steps. The first step proves that $\mathcal{E}(K^{\mathbb{G}})$

is a local martingale for which there exists a $\mathbb{G}$-predictable process $\varphi^{\mathbb{G}}$ satisfying $0 < \varphi^{\mathbb{G}} \leq 1$ and $\mathcal{E}(K^{\mathbb{G}})(\varphi^{\mathbb{G}} \bullet S^{\tau})$ is a $\mathbb{G}$-local martingale if and only if there exists a quadruplet $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}, \varphi^{\mathbb{F}}\right)$ element of

$$\mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^{o}_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^{1}_{loc}(\widetilde{\Omega}, \text{Prog}(\mathbb{F}), P \otimes D) \times L(S, \mathbb{F})$$

such that (3.8) holds and $\mathcal{E}(K^{\mathbb{F}})(\varphi^{\mathbb{F}} \bullet S)$ is an $\mathbb{F}$-local martingale. The second step proves that $\mathcal{E}(K^{\mathbb{G}}) > 0$ if and only if the triplet $(K^{\mathbb{G}}, \varphi^{(o)}, \varphi^{(pr)})$, found in the first step satisfying (3.8), should fulfills (3.6)-(3.7) and $1 + \Delta K^{\mathbb{F}} > 0$.

Step 1) Suppose that $Z^{\mathbb{G}} = \mathcal{E}(K^{\mathbb{G}})$ is a local martingale deflator for $(S^{\tau}, \mathbb{G})$. On the one hand, thanks to [33, Theorem 2.20], there exists a triplet $(N^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)})$ such that $N^{\mathbb{F}}$ is an $\mathbb{F}$-local martingale, $\varphi^{(o)} \in \mathcal{I}^{o}_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)} \in L^{1}_{loc}(\text{Prog}(\mathbb{F}), P \otimes D)$, and $K^{\mathbb{G}}$ has the following unique decomposition in $\mathbb{G}$;

$$K^{\mathbb{G}} = K^{\mathbb{G}}_{0} + G_{-}^{-2} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(N^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.9}$$

On the other hand, thanks to a combination of Definition 2.16 and Lemma 3.1-(a), we deduce the existence of an $\mathbb{F}$-predictable process φ such that $0 < \varphi \leq 1$ and $\varphi^{\mathbb{G}} I_{]\!]0,\tau]\!]} = \varphi I_{]\!]0,\tau]\!]}$. Then $Z^{\mathbb{G}}(\varphi \bullet S^{\tau})$ is a $\mathbb{G}$-local martingale, or equivalently

$$\varphi \bullet S^{\tau} + [\varphi \bullet S^{\tau}, K^{\mathbb{G}}] \quad \text{is a } \mathbb{G}\text{-local martingale.} \tag{3.10}$$

Thus, using the decomposition $S = S_0 + M + A + \sum \Delta S I_{\{|\Delta|>1\}}$ where M is an $\mathbb{F}$-local martingale with bounded jumps and A is an $\mathbb{F}$-predictable process with finite variation, we derive

$$\begin{aligned}
\varphi \bullet S^\tau + \varphi \bullet [K^{\mathbb{G}}, S^\tau] &= \varphi \bullet M^\tau + \varphi \bullet A^\tau + \sum \varphi \Delta S I_{\{|\Delta S|>1\}} I_{]\!]0,\tau]\!]} + \varphi \bullet [K^{\mathbb{G}}, S^\tau], \\
&= \mathbb{G}\text{-local martingale} + \frac{\varphi}{\widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [M, m] + \varphi \bullet A^\tau \\
&+ \sum \varphi \Delta S I_{\{|\Delta S|>1\}} (1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}) I_{]\!]0,\tau]\!]} + \frac{\varphi}{G_- \widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [N^{\mathbb{F}}, M] \\
&+ \varphi \Delta S I_{\{|\Delta S|>1\}} \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi \Delta S I_{\{|\Delta S|>1\}} \varphi^{(pr)} \bullet D \\
&= \mathbb{G}\text{-LM} + \frac{\varphi}{\widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [M, m] + \varphi \bullet A^\tau + \frac{\varphi}{G_- \widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [N^{\mathbb{F}}, M] \\
&+ \sum \varphi \Delta S I_{\{|\Delta S|>1\}} \left[(1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}) I_{]\!]0,\tau]\!]} + \varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D \right].
\end{aligned}$$

Then $\varphi \bullet S^\tau + \varphi \bullet [K^{\mathbb{G}}, S^\tau]$ is a $\mathbb{G}$-local martingale if and only if

$$W := \sum \varphi \Delta S I_{\{|\Delta S|>1\}} [(1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}}) I_{]\!]0,\tau]\!]} + \varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D], \tag{3.11}$$

has a $\mathbb{G}$-locally integrable variation (i.e. $W \in \mathcal{A}_{loc}(\mathbb{G})$) and (due to Lemma 3.1)

$$0 \equiv \frac{\varphi}{G_-} I_{]\!]0,\tau]\!]} \bullet \langle M, m \rangle^{\mathbb{F}} + \varphi \bullet A^\tau + \frac{\varphi}{G_-^2} I_{]\!]0,\tau]\!]} \bullet \langle N^{\mathbb{F}}, M \rangle^{\mathbb{F}} + W^{p,\mathbb{G}}. \tag{3.12}$$

In virtue of Lemma 3.2, we conclude that $W \in \mathcal{A}_{loc}(\mathbb{G})$ iff both processes

$$W^{(1)} := \sum \varphi \Delta S I_{\{|\Delta S|>1\}} [(1 + \frac{\Delta N^{\mathbb{F}}}{G_- \widetilde{G}})] I_{]\!]0,\tau]\!]}$$

and $W^{(2)} := \sum \varphi \Delta S I_{\{|\Delta S|>1\}} [\varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D]$ belong to $\mathcal{A}_{loc}(\mathbb{G})$.
It is clear that $W^{(2)}$ belongs to $\mathcal{A}_{loc}(\mathbb{G})$ if and only if it is a $\mathbb{G}$-local martingale, and hence in this case we get

$$W^{p,\mathbb{G}} = (W^{(1)})^{p,\mathbb{G}} = G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \left(\sum \varphi \Delta S I_{\{|\Delta S|>1\}} \frac{G_- \widetilde{G} + \Delta N^{\mathbb{F}}}{G_-} \right)^{p,\mathbb{F}}.$$

As a result, by inserting these remarks in (3.12), we obtain

$$0 \equiv \varphi \bullet \langle M, m\rangle^{\mathbb{F}} + \varphi G_- \bullet A + \frac{\varphi}{G_-} \bullet \langle N^{\mathbb{F}}, M\rangle^{\mathbb{F}} + \\ + G_- \bullet \left(\sum \varphi \Delta S I_{\{|\Delta S|>1\}}[1 + \frac{\Delta m}{G_-} + \frac{\Delta N^{\mathbb{F}}}{G_-^2}]\right)^{p,\mathbb{F}},$$

or equivalently

$$0 = \varphi \bullet \langle M, \frac{1}{G_-} \bullet m + \frac{1}{G_-^2} \bullet N^{\mathbb{F}}\rangle^{\mathbb{F}} + \varphi \bullet A + \left(\sum \varphi \Delta S I_{\{|\Delta S|>1\}}[1 + \frac{\Delta m}{G_-} + \frac{\Delta N^{\mathbb{F}}}{G_-^2}]\right)^{p,\mathbb{F}}.$$

Thanks to Itô's formula, this is equivalent to $\mathcal{E}(K^{\mathbb{F}})(\varphi \bullet S)$ is an $\mathbb{F}$-local martingale with $K^{\mathbb{F}} := G_-^{-1} \bullet m + G_-^{-2} \bullet N^{\mathbb{F}}$, and the first step is completed.

Step 2). Herein, we assume that (3.8) holds, and prove that $\mathcal{E}(K^{\mathbb{G}}) > 0$ if and only if (3.6)-(3.7) and $1 + \Delta K^{\mathbb{F}} > 0$ hold. To this end, due to

$$\Delta \mathcal{T}(K^{\mathbb{F}}) = \frac{G_- \Delta K^{\mathbb{F}}}{\widetilde{G}} I_{]\!]0,\tau]\!]}, \ \Delta N^{\mathbb{G}} = \Delta D - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \Delta D^{o,\mathbb{F}}, \ \Delta m = \widetilde{G} - G_-$$

and $\Delta D^{o,\mathbb{F}} = \widetilde{G} - G$, we drive,

$$\begin{aligned}
\Delta K^{\mathbb{G}} &= \Delta \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \Delta \mathcal{T}(m) + \varphi^{(o)} \Delta N^{\mathbb{G}} + \varphi^{(pr)} \Delta D \\
&= \frac{G_- \Delta K^{\mathbb{F}} - \Delta m}{\widetilde{G}} I_{]\!]0,\tau]\!]} - \varphi^{(o)} \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}} I_{]\!]0,\tau]\!]} + (\varphi^{(pr)} + \varphi^{(o)}) I_{[\![\tau]\!]} \\
&= \frac{G_- \Delta K^{\mathbb{F}} - \Delta m}{\widetilde{G}} I_{]\!]0,\tau]\!]} - \varphi^{(o)} \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}} I_{]\!]0,\tau[\![} + (\varphi^{(pr)} + \varphi^{(o)} \frac{G}{\widetilde{G}}) I_{[\![\tau]\!]}, \\
&= \left[\frac{G_-}{\widetilde{G}}(1 + \Delta K^{\mathbb{F}}) - 1 - \varphi^{(o)} \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\right] I_{]\!]0,\tau[\![} + \left[\frac{G_-(1 + \Delta K^{\mathbb{F}})}{\widetilde{G}} - 1 + \varphi^{(pr)} + \varphi^{(o)} \frac{G}{\widetilde{G}}\right] I_{[\![\tau]\!]}.
\end{aligned}$$

Therefore, $\mathcal{E}(K^{\mathbb{G}}) > 0$ if and only if $1 + \Delta K^{\mathbb{G}} > 0$ which is equivalent to

$$\begin{aligned}
&]\!]0,\tau[\![\subset \left\{\frac{G_-}{\widetilde{G}}(1 + \Delta K^{\mathbb{F}}) - \varphi^{(o)} \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}} > 0\right\}, \\
\text{and} \quad & [\![\tau]\!] \subset \left\{\frac{G_-}{\widetilde{G}}(1 + \Delta K^{\mathbb{F}}) + \varphi^{(o)} \frac{G}{\widetilde{G}} + \varphi^{(pr)} > 0\right\}. \qquad (3.13)
\end{aligned}$$

Thus, by putting $\Sigma_1 := \left\{\frac{G_-}{\widetilde{G}}(1+\Delta K^{\mathbb{F}}) - \varphi^{(o)}\frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}} > 0\right\} \cap]\!]0,+\infty[\![$, the first inclusion in (3.13) is equivalent to $I_{]\!]0,\tau[\![} \leq I_{\Sigma_1}$. Hence, by taking the $\mathbb{F}$-optional projection on both sides of this inequality, we get $0 < G \leq I_{\Sigma_1}$ on $]\!]0,+\infty[\![$. This proves the right inequality in (3.7) since it means that $\Sigma_1 > 0$ holds. Notice that the second inclusion in (3.13) is equivalent to

$$\frac{G_-}{\widetilde{G}}(1+\Delta K^{\mathbb{F}}) + \varphi^{(o)}\frac{G}{\widetilde{G}} + \varphi^{(pr)} > 0, \quad P\otimes D - a.e.,$$

and (3.6) is proved. Now, we focus on proving the left inequality of (3.7). Thanks to $\mathbb{E}_{P\otimes D}[\varphi^{(pr)}|\mathcal{O}(\mathbb{F})] = 0$, $P\otimes D - a.e.$, by taking conditional expectation under $P\otimes D$ with respect to $\mathcal{O}(\mathbb{F})$ on the both sides of the above inequality, we get

$$\Sigma := G_-(1+\Delta K^{\mathbb{F}}) + \varphi^{(o)}G > 0, \quad P\otimes D - a.e., \tag{3.14}$$

or equivalently $I_{[\![\tau]\!]} \leq I_{\{\Sigma>0\}}$. Remark that the above inequality is equivalent to the left inequality in (3.7), and hence (3.7) is completely proved. By taking the $\mathbb{F}$-optional projection in both sides of $I_{[\![\tau]\!]} \leq I_{\{\Sigma>0\}}$, we get $\Delta D^{o,\mathbb{F}} \leq I_{\{\Sigma>0\}}$. Therefore, we derive

$$\{\Delta D^{o,\mathbb{F}} > 0\} \subseteq \{G_-(1+\Delta K^{\mathbb{F}}) > -\varphi^{(o)}G\}. \tag{3.15}$$

On the one hand, due to (3.7), we deduce that on set $\{\Delta D^{o,\mathbb{F}} = 0\}$, we have $1+\Delta K^{\mathbb{F}} > 0$. On the other hand, using (3.15) and (3.7) afterwards, we get

$$\{1+\Delta K^{\mathbb{F}} \leq 0\} \cap \{\Delta D^{o,\mathbb{F}} > 0\} \subseteq \{\varphi^{(o)} > 0, 1+\Delta K^{\mathbb{F}} \leq 0, \Delta D^{o,\mathbb{F}} > 0\} = \emptyset,$$

or equivalently $\{\Delta D^{o,\mathbb{F}} > 0\} \subseteq \{1+\Delta K^{\mathbb{F}} > 0\}$. Thus, $1+\Delta K^{\mathbb{F}} > 0$, or equivalently, $\mathcal{E}\left(K^{\mathbb{F}}\right) > 0$. Thus, the second step will be completed as soon as we prove the reverse sense (i.e. it means (3.6)-(3.7) implies that $\mathcal{E}(K^{\mathbb{G}}) > 0$), we assume that (3.6)-(3.7) hold. Then (3.13) follows immediately, and hence $\mathcal{E}(K^{\mathbb{G}}) > 0$. This ends the proof of the theorem. □

In the following, we derive a multiplicative version of Theorem 3.1.

Theorem 3.2: *Suppose $G > 0$, and let $Z^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following are equivalent.*

(a) $Z^{\mathbb{G}}$ *is a local martingale deflator for* $(S^\tau, \mathbb{G})$.

(b) *There exists a unique triplet* $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ *such that* $Z^{\mathbb{F}}$ *is a local martingale deflator for* $(S, \mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ *belongs to* $L^1_{loc}(\widetilde{\Omega}, \text{Prog}(\mathbb{F}), P \otimes D)$, *and satisfying* $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ *P-a.s.,*

$$\varphi^{(pr)} > -1, \quad P \otimes D - a.e., \; -\frac{\widetilde{G}}{G} < \varphi^{(o)} < \frac{\widetilde{G}}{\widetilde{G} - G}, \quad P \otimes D^{o,\mathbb{F}}\text{-}a.e.. \tag{3.16}$$

and

$$Z^{\mathbb{G}} = \frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D). \tag{3.17}$$

Proof. Thanks to Theorem 3.1, we conclude that $Z^{\mathbb{G}}$ is a local martingale deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a triplet $\left(Z^{\mathbb{F}}, \overline{\varphi}^{(o)}, \overline{\varphi}^{(pr)}\right)$ such that $Z^{\mathbb{F}} := \mathcal{E}(K^{\mathbb{F}})$ belongs to $\mathcal{Z}_{loc}(S, \mathbb{F})$, $\overline{\varphi}^{(o)}$ belongs to $\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\overline{\varphi}^{(pr)} \in L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D)$ such that $\mathbb{E}[\overline{\varphi}^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s., and (3.6), (3.7) and (3.8) hold. Thus, we put

$$\begin{aligned} &Y := \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), \quad X := Y + \overline{\varphi}^{(o)} \bullet N^{\mathbb{G}} \\ &\varphi^{(o)} := \frac{\widetilde{G}\overline{\varphi}^{(o)}}{G_-(1 + \Delta K^{\mathbb{F}})}, \quad \varphi^{(pr)} := \frac{\widetilde{G}\overline{\varphi}^{(pr)}}{G_-(1 + \Delta K^{\mathbb{F}}) + \overline{\varphi}^{(o)} G}. \end{aligned} \tag{3.18}$$

Since the pair $(\overline{\varphi}^{(o)}, \overline{\varphi}^{(pr)})$ satisfies (3.6)-(3.7), we conclude immediately that the pair $(\varphi^{(o)}, \varphi^{(pr)})$ satisfies (3.16). Furthermore, put

$$\Gamma := G_- \widetilde{G}^{-1}(1 + \Delta K^{\mathbb{F}}) - 1, \quad \widetilde{\Omega} := \Omega \times [0, +\infty),$$

and calculate

$$1+\Delta X \ = \left[\Gamma+1-\Delta D^{o,\mathbb{F}}\frac{\overline{\varphi}^{(o)}}{\widetilde{G}}\right] I_{]\!]0,\tau[\![} + \left[\Gamma+1+\overline{\varphi}^{(o)}\frac{G}{\widetilde{G}}\right] I_{[\![\tau]\!]} + I_{\widetilde{\Omega}\setminus]\!]0,\tau]\!]} > 0.$$

$$1+\Delta Y \ = \frac{G_-}{\widetilde{G}}(1+\Delta K^{\mathbb{F}})I_{]\!]0,\tau]\!]} + I_{]\!]-\infty,0]\!]\cup]\!]\tau,+\infty[\![} > 0.$$

Thanks to Yor's formula (i.e. $\mathcal{E}(X_1)\mathcal{E}(X_2)=\mathcal{E}(X_1+X_2+[X_1,X_2])$) we derive

$$\mathcal{E}(X_1+X_2)=\mathcal{E}(X_1)\mathcal{E}(X_2-\frac{1}{1+\Delta X_1}\bullet[X_1,X_2]),$$

for any semimartingales X_1, X_2 with $1+\Delta X_1>0$. By applying this formula repeatedly, and using $\varphi^{(o)}=\overline{\varphi}^{(o)}/(1+\Delta Y)$ and $\varphi^{(pr)}=\overline{\varphi}^{(pr)}/(1+\Delta X)$ $P\otimes D$-a.e. which follow directly from (3.18), we obtain

$$\mathcal{E}(K^{\mathbb{G}})=\mathcal{E}(X+\overline{\varphi}^{(pr)}\bullet D)=\mathcal{E}(X)\mathcal{E}(\frac{\overline{\varphi}^{(pr)}}{1+\Delta X}\bullet D)=\mathcal{E}(X)\mathcal{E}(\varphi^{(pr)}\bullet D)$$
$$=\mathcal{E}(Y)\mathcal{E}(\frac{\overline{\varphi}^{(o)}}{1+\Delta Y}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D)=\mathcal{E}(Y)\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D).$$

Therefore, the equality (3.17) follows immediately from combining this equality with $\mathcal{E}(Y)=\mathcal{E}(K^{\mathbb{F}})^{\tau}/\mathcal{E}(G_-^{-1}\bullet m)^{\tau}$. This latter equality is a direct consequence of $1/\mathcal{E}(G_-^{-1}\bullet m)^{\tau}=\mathcal{E}(-G_-^{-1}\bullet\mathcal{T}(m))$ and $\mathcal{E}(K)^{\tau}\mathcal{E}(-G_-^{-1}\bullet\mathcal{T}(m))=\mathcal{E}(\mathcal{T}(K)-G_-^{-1}\bullet\mathcal{T}(m))$, which both follow directly from Yor's formula (easy to check), as follows.

$$\mathcal{E}(-G_-^{-1}\bullet\mathcal{T}(m))\mathcal{E}(G_-^{-1}\bullet m)^{\tau}=\mathcal{E}(-\frac{1}{G_-}\bullet m^{\tau}+\frac{1}{\widetilde{G}G_-}\bullet[m]^{\tau})\mathcal{E}(G_-^{-1}\bullet m)^{\tau}=1,$$

and

$$
\begin{aligned}
\mathcal{E}(K)^\tau \mathcal{E}(-G_-^{-1} \bullet \mathcal{T}(m)) &= \mathcal{E}((K)^\tau - G_-^{-1} \bullet \mathcal{T}(m) - [(K)^\tau, G_-^{-1} \bullet \mathcal{T}(m)]) \\
&= \mathcal{E}(\mathcal{T}(K) + \frac{1}{\widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [K, m] - G_-^{-1} \bullet \mathcal{T}(m) - [(K)^\tau, G_-^{-1} \bullet \mathcal{T}(m)]) \\
&= \mathcal{E}(\mathcal{T}(K) + \frac{1}{\widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [K, m] - G_-^{-1} \bullet \mathcal{T}(m) - \frac{1}{\widetilde{G}} I_{]\!]0,\tau]\!]} \bullet [K, m]) \\
&= \mathcal{E}(\mathcal{T}(K) - G_-^{-1} \bullet \mathcal{T}(m)).
\end{aligned}
$$

This ends the proof of the theorem. □

As a direct consequence of Theorem 3.1, or equivalently Theorem 3.2, we describe below a family of local martingale deflators for $(S^\tau, \mathbb{G})$, that will play important role in Chapter 5, where we address the optimal deflator problem.

Corollary 3.2.1: For any $\mathbb{F}$-local martingale (respectively an element of $\mathcal{Z}_{loc}(S, \mathbb{F})$) $Z^{\mathbb{F}} := \mathcal{E}(K^{\mathbb{F}})$, the process $Z^{\mathbb{G}}$ given by

$$
Z^{\mathbb{G}} := \mathcal{E}\Big(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m)\Big) = \frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau}, \tag{3.19}
$$

is a $\mathbb{G}$-local martingale (respectively an element of $\mathcal{Z}_{loc}(S^\tau, \mathbb{G})$).

Proof. On the one hand, by choosing $\big(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\big) = \big(Z^{\mathbb{F}}, 0, 0\big)$ in Theorem 3.2, the decomposition (3.17) reduces to (3.19). On the other hand, $\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m)$ is the sum of two $\mathbb{G}$-local martingales, therefore the stochastic exponential of it, $Z^{\mathbb{G}}$, is also a $\mathbb{G}$-local martingale. Furthermore, if $Z^{\mathbb{F}} = \mathcal{E}(K^{\mathbb{F}})$ is a local martingale deflator for $(S, \mathbb{F})$, then $Z^{\mathbb{G}}$ is a local martingale deflator for $(S^\tau, \mathbb{G})$. □

3.3 Particular cases for τ

This section illustrates the results of Section 3.2 on several frequently studied models for the random time τ. We will consider the case of pseudo-stopping times, the case when either τ avoids all $\mathbb{F}$-stopping times, and the case when all $\mathbb{F}$-martingales are continuous.

3.3.1 Pseudo-stopping times

The family of pseudo-stopping times was introduced and studied in [113]. Below, we recall its definition.

Definition 3.1: τ is said a pseudo-stopping time if, for every bounded $\mathbb{F}$-martingale M, we have

$$E[M_\tau] = E[M_0].$$

The following lemma recalls some characterizations of pseudo-stopping times.

Lemma 3.5: *The following three properties are equivalent.*

(a) *τ is a pseudo-stopping time.*

(b) *$m = m_0$ (this implies that $\mathcal{E}\left(G_-^{-1} \bullet m\right) \equiv 1$ and $\Delta m \equiv 0$.)*

(c) *For any bounded $\mathbb{F}$-local martingale M, we have $\mathcal{T}(M) = M^\tau$.*

Proof. Thanks to [113, Theorem 1], τ is an $\mathbb{F}$-pseudo stopping time, if and only if $m \equiv m_0$. This leads to $\mathcal{T}(m) = m_0$ and $\widetilde{G} = G_-$. Hence, we get $\mathcal{E}\left(G_-^{-1} \bullet m\right) \equiv 1$ and $\Delta m = 0$. This proves (a) $\Longrightarrow$ (b). Furthermore, when τ is an $\mathbb{F}$-pseudo stopping time, $\{\widetilde{G} = 0 < G_-\} = \emptyset$, and $[M, m] = 0$ for any $\mathbb{F}$-local martingale M. Then, we derive

$$\begin{aligned}\mathcal{T}(M) &= M^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [M, m] + I_{]\!]0,\tau]\!]} \bullet \left(\sum \Delta M I_{\{\widetilde{G}=0<G_-\}}\right)^{p,\mathbb{F}} \\ &= M^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [M, m] + 0 = M^\tau.\end{aligned}$$

Hence, the proof of (a) $\Longrightarrow$ (c) is complete, while (c) $\Longrightarrow$ (a) is obvious. This is due to $\mathcal{T}(M)$ is $\mathbb{G}$-martingale for any bounded $\mathbb{F}$-martingale M, and hence

$$E[M_\tau] = E[\mathcal{T}_\tau(M)] = E[\mathcal{T}_0(M)] = E[M_0].$$

This ends the proof of the lemma. □

Theorem 3.3: *Suppose that τ is a pseudo-stopping time, and $G > 0$. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following are equivalent.*

(a) $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ *is a local martingale deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique triplet $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $\mathcal{E}(K^{\mathbb{F}}) \in \mathcal{Z}_{loc}(S, \mathbb{F})$, $\varphi^{(o)}$ belongs to $\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D)$, and satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s.,*

$$1 + \Delta K^{\mathbb{F}} > \varphi^{(o)}(1 - \frac{G}{G_-}), \quad \text{and} \quad -(1 + \Delta K^{\mathbb{F}}) < \frac{\varphi^{(o)} G}{G_-} \quad P \otimes D^{o,\mathbb{F}}\text{-a.e.}$$

$$\varphi^{(pr)} > -\left[1 + \Delta K^{\mathbb{F}} + \varphi^{(o)} \frac{G}{G_-}\right], \quad P \otimes D - a.e. \tag{3.20}$$

$$\text{and} \quad K^{\mathbb{G}} = \left(K^{\mathbb{F}}\right)^\tau + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.21}$$

(b) $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ *is a local martingale deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique triplet $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ that belongs to*

$$\mathcal{Z}_{loc}(S, \mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D),$$

and satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s.,

$$\varphi^{(pr)} > -1, \quad P \otimes D - a.e., \ -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \quad P \otimes D^{o,\mathbb{F}}\text{-a.e.} \tag{3.22}$$

$$Z^{\mathbb{G}} = (Z^{\mathbb{F}})^\tau \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D). \tag{3.23}$$

Proof. **(a)** Thanks to Theorem 3.1, we have the following decomposition

$$K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.24}$$

By Lemma 3.5, we deduce that $\mathcal{T}(K^{\mathbb{F}})$ reduces to $\left(K^{\mathbb{F}}\right)^{\tau}$ and $G_{-}^{-1}\bullet\mathcal{T}(m)$ vanishes. As a result, the processes in (3.24) and in (3.21) coincide. Both inequalities in (3.20) also follow from Theorem 3.1. This ends the proof of assertion (a).

(b) By combining Theorem 3.2, assertion (a), and the fact that $\mathcal{E}\left(G_{-}^{-1}\bullet m\right)\equiv 1$, the proof of assertion (b) follows immediately. $\square$

3.3.2 τ avoids all $\mathbb{F}$-stopping times

If a random time τ avoids all $\mathbb{F}$-stopping times, then for any $\mathbb{F}$-stopping time σ, we have $P(\sigma=\tau<+\infty)=0$. Throughout this section, put

$$\overline{N}^{\mathbb{G}} := D - G_{-}^{-1} I_{]\!]0,\tau]\!]}\bullet D^{p,\mathbb{F}}. \tag{3.25}$$

It is clear that $\overline{N}^{\mathbb{G}}$ is the $\mathbb{G}$-local martingale parts in the Doob-Meyer decomposition under $\mathbb{G}$ of D respectively (see [86, Remark 4.5] for details). The following lemma discusses the relationship between $\overline{N}^{\mathbb{G}}$ and $N^{\mathbb{G}}$, which is defined in (2.23).

Lemma 3.6: *Suppose that τ avoids all $\mathbb{F}$-stopping times. Then the following hold:*

(a) *It holds that $N^{\mathbb{G}}=\overline{N}^{\mathbb{G}}$.*

(b) *For any $\varphi^{(o)}\in\mathcal{I}^{o}_{loc}(N^{\mathbb{G}},\mathbb{G})$, there exists an $\mathbb{F}$-predictable process φ that is $N^{\mathbb{G}}$-integrable such that $(\varphi^{(o)}-\varphi)\bullet N^{\mathbb{G}}\equiv 0$.*

(c) $[\overline{N}^{\mathbb{G}},X]\equiv 0$ *for any $\mathbb{F}$-semimartingale X.*

Proof. **(a)** If τ avoids all $\mathbb{F}$-stopping times, then $\widetilde{G}=G$, or equivalently $\Delta D^{o,\mathbb{F}}=0$. This means that $D^{o,\mathbb{F}}$ is continuous and hence $D^{o,\mathbb{F}}=D^{p,\mathbb{F}}$. This implies that

$$\widetilde{G}^{-1}I_{]\!]0,\tau]\!]}\bullet D^{o,\mathbb{F}}=\widetilde{G}^{-1}I_{]\!]0,\tau]\!]}\bullet D^{p,\mathbb{F}}=G^{-1}I_{]\!]0,\tau]\!]}\bullet D^{p,\mathbb{F}}.$$

Thus, the $\mathbb{G}$-local martingale $N^{\mathbb{G}}$ given by (2.23) coincides with $\overline{N}^{\mathbb{G}}$.

(b) Since $\{\varphi^{(o)}\neq\varphi\}=\bigcup_{n\geq 0}[\![\sigma_n]\!]$ (see [59]), where $(\sigma_n)_{n\geq 0}$ is a sequence of $\mathbb{F}$-stopping times. Therefore, on one hand,

$$I_{\{\varphi^{(o)}\neq\varphi\}}\bullet D=\sum_{n\geq 0} I_{\{\sigma_n=\tau\}} I_{[\![\sigma_n,+\infty[\![}=0,$$

since τ avoids all $\mathbb{F}$-stopping times. On the other hand, $I_{\{\varphi^{(o)}\neq\varphi\}}\bullet D^{o,\mathbb{F}}$ is null as it is the $\mathbb{F}$-optional dual projection of $I_{\{\varphi^{(o)}\neq\varphi\}}\bullet D=0$. Thus, by definition of $N^{\mathbb{G}}$, we get,

$$(\varphi^{(o)}-\varphi)\bullet N^{\mathbb{G}}\equiv 0.$$

(c) Since $\overline{N}^{\mathbb{G}}$ has a finite variation, we get $[\overline{N}^{\mathbb{G}},X]=\sum\Delta\overline{N}^{\mathbb{G}}\Delta X=\Delta X\bullet\overline{N}^{\mathbb{G}}$. Furthermore, $\{\Delta X\neq 0\}=\bigcup_{n\geq 0}[\![\sigma_n]\!]$, where $(\sigma_n)_{n\geq 0}$ is a sequence of $\mathbb{F}$-stopping times. Thus,

$$\Delta\overline{N}^{\mathbb{G}}\Delta X=\sum_{n\geq 1}^{+\infty}\Delta X_{\sigma_n} I_{\{\sigma_n=\tau<+\infty\}} I_{[\![\sigma_n]\!]}=0,$$

due to the fact that τ avoids all $\mathbb{F}$-stopping times. This completes the proof. □

Theorem 3.4: *Suppose that $G>0$, and τ avoids all $\mathbb{F}$-stopping times. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following are equivalent.*

(a) $Z^{\mathbb{G}}:=\mathcal{E}\left(K^{\mathbb{G}}\right)$ *is a local martingale deflator for $(S^\tau,\mathbb{G})$ if and only if there exists a unique triplet $\left(K^{\mathbb{F}},\varphi^{(p)},\varphi^{(pr)}\right)$ such that $\mathcal{E}(K^{\mathbb{F}})\in\mathcal{Z}_{loc}(S,\mathbb{F})$, $\varphi^{(p)}$ belongs to $L^1_{loc}(\overline{N}^{\mathbb{G}},\mathbb{G})$, $\varphi^{(pr)}\in L^1_{loc}(\mathrm{Prog}(\mathbb{F}),P\otimes D)$ and $\mathbb{E}[\varphi^{(pr)}_\tau\mid\mathcal{F}_\tau]I_{\{\tau<+\infty\}}=0$ P-a.s., satisfying*

$$-1<\varphi^{(p)},\quad P\otimes D^{p,\mathbb{F}}-a.e.\qquad -\varphi^{(p)}-1<\varphi^{(pr)},\quad P\otimes D-a.e.$$
$$\text{and}\qquad K^{\mathbb{G}}\;=\;\mathcal{T}(K^{\mathbb{F}})-G_-^{-1}\bullet\mathcal{T}(m)+\varphi^{(p)}\bullet\overline{N}^{\mathbb{G}}+\varphi^{(pr)}\bullet D.\qquad(3.26)$$

(b) $Z^{\mathbb{G}}$ *is a local martingale deflator for $(S^\tau,\mathbb{G})$ if and only if there exists a unique triplet $\left(Z^{\mathbb{F}},\varphi^{(p)},\varphi^{(pr)}\right)$ belongs to $\mathcal{Z}(S,\mathbb{F})\times L^1_{loc}(\overline{N}^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\widetilde{\Omega},\mathrm{Prog}(\mathbb{F}),P\otimes D)$,* *and* $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau]I_{\{\tau<+\infty\}} = 0$ *P-a.s., such that* $\varphi^{(pr)} > -1, \quad P\otimes D - a.e.,$

$$-1 < \varphi^{(p)}, \quad P\otimes D^{p,\mathbb{F}} - a.e. \text{ and } Z^{\mathbb{G}} = \frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1}\bullet m)^\tau}\mathcal{E}(\varphi^{(p)}\bullet \overline{N}^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D). \quad (3.27)$$

Proof. **(a).** The proof of assertion (a) follows from combining Theorem 3.1 with the three assertions of Lemma 3.6. In fact, on the one hand, thanks to Theorem 3.1, we have

$$K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1}\bullet\mathcal{T}(m) + \varphi^{(o)}\bullet N^{\mathbb{G}} + \varphi^{(pr)}\bullet D. \quad (3.28)$$

On the other hand, since τ avoids all $\mathbb{F}$-stopping times, Lemma 3.6 implies that $N^{\mathbb{G}} = \overline{N}$. Thus by plugging these in (3.26),(3.28) follows and the proof of assertion (a) is completed.

(b). The proof of this part follows immediately from combining assertion (a), Yor's formula, and Lemma 3.6-(c). This ends the proof of the theorem. □

3.3.3 The case when all $\mathbb{F}$-martingales are continuous

Throughout this section, we consider the following notations

$$\begin{aligned}\overline{M} &:= M^\tau - G_-^{-1}I_{]\!]0,\tau]\!]}\bullet\langle M,m\rangle^{\mathbb{F}}, \quad \text{for any } M\in\mathcal{M}_{loc}(\mathbb{F}),\\ \overline{N}^{\mathbb{G}} &:= D - G_-^{-1}I_{]\!]0,\tau]\!]}\bullet D^{p,\mathbb{F}}.\end{aligned} \quad (3.29)$$

Here, we compare the processes $\overline{M}$ and $\overline{N}^{\mathbb{G}}$ to $\mathcal{T}(M)$ and $N^{\mathbb{G}}$ respectively, in the following Lemma.

Lemma 3.7: *Suppose that all* $\mathbb{F}$*-martingales are continuous. Then* $N^{\mathbb{G}} = \overline{N}^{\mathbb{G}}$*, and for any* $\mathbb{F}$*-local martingale* M*, we have* $\mathcal{T}(M) = \overline{M}$*.*

Proof. If we assume that all $\mathbb{F}$-martingales are continuous then we have $\Delta m = 0$, or equivalently, $\widetilde{G} = G_-$, $D^{o,\mathbb{F}} - D^{p,\mathbb{F}} = 0$, and $[M,m] = \langle M,m\rangle$ for any $\mathbb{F}$-local martingale M. As a result, we deduce that $N^{\mathbb{G}}$ given by (2.23) coincides with $\overline{N}^{\mathbb{G}}$

immediately. Furthermore, due to $\Delta M = 0$ for any $M \in \mathcal{M}_{loc}(\mathbb{F})$, we get

$$\begin{aligned}\mathcal{T}(M) &= M^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [M, m] + I_{]\!]0,\tau]\!]} \bullet \Big(\sum \Delta M I_{\{\widetilde{G}=0<G_-\}} \Big)^{p,\mathbb{F}} \\ &= M^\tau - G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \langle M, m\rangle^{\mathbb{F}} = \overline{M}.\end{aligned}$$

This ends the proof of the lemma. □

Theorem 3.5: *Suppose that $G > 0$, and all $\mathbb{F}$-martingales are continuous. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following are equivalent.*

(a) *$Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ is a local martingale deflator for $(S^\tau, \mathbb{G})$ if and only if there exists $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $\mathcal{E}(K^{\mathbb{F}}) \in \mathcal{Z}_{loc}(S, \mathbb{F})$, $\varphi^{(o)}$ belongs to $L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)} \in L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$ and $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s., satisfying*

$$-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \ P\otimes D^{o,\mathbb{F}}-a.e., \ \ \varphi^{(pr)} > -\varphi^{(o)}\frac{G}{G_-} - 1, \qquad P\otimes D-a.e.,$$

$$\textit{and} \qquad K^{\mathbb{G}} = \overline{K^{\mathbb{F}}} - G_-^{-1} \bullet \overline{m} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.30}$$

(b) *$Z^{\mathbb{G}}$ is a local martingale deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique triplet $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $Z^{\mathbb{F}}$ is a local martingale deflator for $(S, \mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D)$, satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s.,*

$$\begin{aligned}-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, &\quad P \otimes D^{o,\mathbb{F}} - a.e. \qquad \varphi^{(pr)} > -1, \quad P \otimes D - a.e., \\ Z^{\mathbb{G}} \ = \ &\frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet \overline{N}^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D). \end{aligned} \tag{3.31}$$

Proof. **(a)** The proof of assertion (a) follows from combining Theorem 3.1 with the three assertions of Lemma 3.7. Thanks to Theorem 3.1, we obtain

$$K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.32}$$

By Lemma 3.7, we deduce that $N^{\mathbb{G}} = \overline{N}$, and $\mathcal{T}(M) = \overline{M}$, respectively. As a result, the processes in (3.32) and (3.30) coincide. This ends the proof of assertion (a).

(b) The proof is fairly similar to Theorem 3.4-(b), and hence we omit it. □

3.4 Particular cases for $(S, \mathbb{F})$

This section illustrates Theorems 3.1-3.2 on several regularly used models, such as the case of Lévy market model, jump-diffusion for S, two important volatility market frameworks and few other cases.

3.4.1 The exponential Lévy market models

In this subsection, we focus on the case of Lévy market model. Precisely, we assume that S is given by the following stochastic differential equation

$$\begin{aligned} X_t &= \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|\leq 1} x\widetilde{N}^{\mathbb{F}}(dt, dx) + \int_0^t \int_{|x|>1} xN(dt, dx) + \int_0^t \mu_s ds, \\ S_t &= S_0\mathcal{E}(X)_t, \qquad \widetilde{N}^{\mathbb{F}}(dt, dx) = N(dt, dx) - F^X(dx)dt, \end{aligned} \tag{3.33}$$

where $\sigma > 0$ and μ are bounded adapted processes, W is a one-dimensional Brownian motion, $N(dt, dx)$ and is a Poisson random measure on $[0, T] \times \mathbb{R} \setminus \{0\}$ and $\widetilde{N}^{\mathbb{F}}$ is a compensated Poisson measure with Lévy measure (intensity) $F^X(dx)dt$. Stock price process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F}$ is the completed and right continuous filtration generated by W and $\widetilde{N}^{\mathbb{F}}$. For more details about the application of Lévy market model in finance, we refer the reader to [25], [65], [66] and the references therein. Lévy model is a quasi-left-continuous process and has some extra nice features comparing to general semimartingale. We parametrize local martingale deflators for general Lévy market models in this section, then in Chapters 5 and 6, we continue to study their characterization for optimal deflator and Log-optimal portfolio. Once the general Lévy model is treated in the next paragraph, the remaining part of this section is divided into two subsections where we discuss two popular cases of Lévy market model.

For the parametrization of local martingale deflator for $(S^\tau, \mathbb{G})$, we need to know its parametrization for model $(S, \mathbb{F})$. This is the aim of the following Lemma.

Lemma 3.8: *Suppose S is given by (3.33). Then $\mathcal{E}\left(K^{\mathbb{F}}\right)$ is a local martingale deflator for $(S, \mathbb{F})$ if and only if there exists a unique quadruple (β, f, g, K') that belongs to $(L^1_{loc}(W, \mathbb{F}) \times \mathcal{G}^1_{loc}(\mu, \mathbb{F}) \times \mathcal{H}^1_{loc}(\mu, \mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F})$, $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0$, $P \otimes \mu$-.a.e., $\Delta K^{\mathbb{F}} > -1$, and $[K', S] = 0$, satisfying*

$$\begin{aligned} &\text{(i)} \int |x(f(x) - I_{|x|\leq 1})F^X(dx) < +\infty \quad P \otimes dt - a.e. \ , \\ &\text{(ii)}\ \mu + \sigma\beta + \int x(f(x) - I_{|x|\leq 1})F^X(dx) \equiv 0, \quad P \otimes dt - a.e. \ , \end{aligned} \tag{3.34}$$

and then the process $K^{\mathbb{F}}$ has following decomposition

$$K^{\mathbb{F}} := \beta \bullet W + (f - 1) \star (\mu - \nu) + g \star \mu + K'. \tag{3.35}$$

Proof. For any $K^{\mathbb{F}} \in \mathcal{M}_{loc}(\mathbb{F})$, the Jacod's representation with respect to S, is given by (3.35). For more details about the decomposition of $K^{\mathbb{F}}$, we refer the reader to the Theorem 2.7. Therefore, by applying Lemma 3.4, $S + [S, K^{\mathbb{F}}]$ is $\mathbb{F}$-local martingale if and only if the process

$$X + [X, K^{\mathbb{F}}] = \mathbb{F} - \text{local martingale} + \int_0^t \left[\mu + \sigma_s\beta_s + \int x(f(x) - I_{|x|\leq 1})F^X(dx)\right] ds$$

is an $\mathbb{F}$-local martingale. This is equivalent to (3.34), and the proof of the lemma is complete. □

The main result in this section is given in the following.

Theorem 3.6: *Suppose S given by (3.33) and $G > 0$. Let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) $Z^{\mathbb{G}}$ *is a local martingale deflator for* $(S^\tau, \mathbb{G})$,

(b) *There exists a unique* $(\beta, f, g, K', \varphi^{(o)}, \varphi^{(pr)})$ *that belongs to*

$$L^1_{loc}(W,\mathbb{F})\times\mathcal{G}^1_{loc}(\mu,\mathbb{F})\times\mathcal{H}^1_{loc}(\mu,\mathbb{F})\times\mathcal{M}_{0,loc}(\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\widetilde{\Omega}, Prog(\mathbb{F}), P\otimes D),$$

satisfying $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0 \;\; P\otimes\mu$*-.a.e.*, $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau]I_{\{\tau<+\infty\}} = 0$ *P-a.s.*,

$$\begin{aligned}
&K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1}\bullet\mathcal{T}(m) + \varphi^{(o)}\bullet N^{\mathbb{G}} + \varphi^{(pr)}\bullet D,\\
&K^{\mathbb{F}} = \beta\bullet W + (f-1)\star(\mu-\nu) + g\star\mu + K',\\
&\varphi^{(pr)} > -\big[G_-\Big(f(\Delta X)+g(\Delta X))I_{\{\Delta X\neq 0\}} + (1+\Delta K')I_{\{\Delta X=0\}}\Big) + \varphi^{(o)}G\big]/\tilde{G},\;\; P\otimes D\text{-a.e.},\\
&-\frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{G} < \varphi^{(o)} < \frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{\Delta D^{o,\mathbb{F}}} \quad P\otimes D^{o,\mathbb{F}}\text{-a.e. on } \{\Delta X\neq 0\},\\
&-\frac{(1+\Delta K')G_-}{G} < \varphi^{(o)} < \frac{(1+\Delta K')G_-}{\Delta D^{o,\mathbb{F}}} \qquad P\otimes D^{o,\mathbb{F}}\text{-a.e. on } \{\Delta X = 0\},\\
&\int |x(f(x) - I_{|x|\leq 1})F^X(dx) < +\infty \;\; P\otimes dt - a.e.\;,\\
&\text{and}\quad \mu + \sigma\beta + \int x(f(x) - I_{|x|\leq 1})F^X(dx) \equiv 0,\; P\otimes dt - a.e.,\; \Delta K^{\mathbb{F}} > -1,\; [K', S] = 0.
\end{aligned} \tag{3.36}$$

(c) *There exists a unique* $(\beta, f, g, K', \varphi^{(o)}, \varphi^{(pr)})$ *belongs to*

$$L^1_{loc}(W,\mathbb{F})\times\mathcal{G}^1_{loc}(\mu,\mathbb{F})\times\mathcal{H}^1_{loc}(\mu,\mathbb{F})\times\mathcal{M}_{0,loc}(\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\widetilde{\Omega}, Prog(\mathbb{F}), P\otimes D),$$

satisfying $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0 \;\; P\otimes\mu$*-.a.e.*, $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau]I_{\{\tau<+\infty\}} = 0$ *P-a.s.*,

$$\begin{aligned}
&Z^{\mathbb{G}} = \frac{\mathcal{E}(K^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1}\bullet m)^\tau}\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D).\\
&K^{\mathbb{F}} = \beta\bullet W + (f-1)\star(\mu-\nu) + g\star\mu + K',\\
&\varphi^{(pr)} > -1,\; P\otimes D\text{-a.e.},\; -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{\Delta D^{o,\mathbb{F}}}\; P\otimes D^{o,\mathbb{F}}\text{-a.e.}
\end{aligned} \tag{3.37}$$

$$\int |x(f(x) - I_{|x|\leq 1})F^X(dx) < +\infty \quad P\otimes dt - a.e. \ ,$$

$$\text{and} \quad \mu + \sigma\beta + \int x(f(x) - I_{|x|\leq 1})F^X(dx) \equiv 0,\ P\otimes dt - a.e.,\ \Delta K^{\mathbb{F}} > -1,\ [K', S] = 0.$$

Proof. The proof follows from combining Theorems 3.1 and 3.2 with Theorem 2.7.

□

3.4.1.1 Jump-diffusion market model

This subsection focuses on the important case of jump-diffusion framework for the market model $(S, \mathbb{F}, P)$ defined as follows. Herein, we suppose that a standard Brownian motion W and a Poisson process N with intensity $\lambda > 0$ are defined on the probability space $(\Omega, \mathcal{F}, P)$, and the filtration $\mathbb{F}$ is the complete and right continuous filtration generated by W and N. Then the stock price process is given by the following dynamics.

$$S_t = S_0\mathcal{E}(X)_t, \quad X_t = \int_0^t \sigma_s dW_s + \int_0^t \zeta_s dN_s^{\mathbb{F}} + \int_0^t \mu_s ds, \qquad N_t^{\mathbb{F}} := N_t - \lambda t. \tag{3.38}$$

We suppose that there exists a constant $\delta \in (0, +\infty)$ such that

$$\mu,\ \sigma,\ \text{and } \zeta \text{ are bounded } \mathbb{F}\text{-adapted processes,} \tag{3.39}$$

$$\text{and} \quad \sigma > 0, \quad \zeta > -1, \quad \sigma + |\zeta| \geq \delta,\ P\otimes dt\text{-a.e.}.$$

Lemma 3.9: *Suppose S is given by (3.38). Then $\mathcal{E}\left(K^{\mathbb{F}}\right)$ is an $\mathbb{F}$-local martingale deflator for $(S, \mathbb{F})$ if and only if there exists a unique pair (ψ_1, ψ_2) that belongs to $L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F})$ and satisfies*

$$K^{\mathbb{F}} = \psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}}, \tag{3.40}$$

$$\int_0^t \left((\psi_{1_s})^2 + |\psi_{2_s}|\right) ds < +\infty \quad P\text{-}a.s.\ for\ all\ t \geq 0$$
$$and \quad \mu + \psi_1\sigma + (\psi_2 - 1)\zeta\lambda \equiv 0, \quad \psi_2 > 0\ P \otimes dt - a.e.\ . \tag{3.41}$$

Proof. Define $(\beta, f, g, K') = (\psi_1, \psi_2, 0, 0)$, where $f(x) = f(1)$, in Lemma 3.8, then (3.41)-(3.40) coincide with (3.34)-(3.35) respectively. This ends the proof. □

Lemma 3.10: *There exists two $\mathbb{F}$-predictable processes $\varphi^{(m)}$ and $\psi^{(m)}$ such that*

$$\int_0^t \left((\varphi_s^{(m)})^2 + |\psi_s^{(m)}|\right) ds < +\infty \quad P\text{-}a.s.\ for\ all\ t \geq 0$$
$$G_-^{-1} \bullet m = \varphi^{(m)} \bullet W + (\psi^{(m)} - 1) \bullet N^{\mathbb{F}}.$$

Below, we state our main result of this subsection.

Theorem 3.7: *Suppose S given by (3.38) and $G > 0$. Let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a local martingale deflator for $(S^\tau, \mathbb{G})$,*

(b) *There exist $(\psi_1, \psi_2) \in L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$ and $\varphi^{(pr)}) \in L^1_{loc}(\mathrm{Prog}(\mathbb{F}), \mathrm{P} \otimes \mathrm{D})$ satisfying $\mathbb{E}[\varphi_\tau^{(pr)} \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s.,*

$$K^{\mathbb{G}} = (\psi_1 - \varphi^{(m)}) \bullet \mathcal{T}(W) + (\psi_2 - \psi^{(m)}) \bullet \mathcal{T}(N^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$
$$\varphi^{(pr)} > -[\frac{\psi_2}{\psi^{(m)}} + \frac{\varphi^{(o)} G}{\psi^{(m)} G_-}], \ and\ -\frac{\psi_2 G_-}{G} < \varphi^{(o)} < \frac{\psi_2 G_-}{\Delta D^{o,\mathbb{F}}} \quad P \otimes D^{o,\mathbb{F}}\text{-}a.e.\ on\ \{\Delta N \neq 0\},$$
$$\varphi^{(pr)} > -\left[1 + \frac{\varphi^{(o)} G}{G_-}\right], \ and \quad -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G} \quad P \otimes D^{o,\mathbb{F}}\text{-}a.e.\ on\ \{\Delta N = 0\},$$
$$\int_0^t \left((\psi_1(s))^2 + |\psi_2(s)|\right) ds < +\infty \quad P\text{-}a.s.\ for\ all\ t \geq 0,$$
$$and \quad \mu + \psi_1\sigma + (\psi_2 - 1)\zeta\lambda \equiv 0, \quad \psi_2 > 0\ P \otimes dt - a.e.\ .$$

(c) *There exists unique quadruplet $(\psi_1, \psi_2, \varphi^{(o)}, \varphi^{(pr)})$ that belongs to the following set $L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), \mathrm{P} \otimes \mathrm{D})$, and satisfies*

$\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0 \ P\text{-}a.s.,$

$$Z^{\mathbb{G}} = \frac{\mathcal{E}(\psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D),$$

$$\varphi^{(pr)} > -1, \quad P \otimes D\text{-}a.e., \quad -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{\Delta D^{o,\mathbb{F}}} \quad P \otimes D^{o,\mathbb{F}}\text{-}a.e.,$$

$$\int_0^t \left((\psi_1(s))^2 + |\psi_2(s)| \right) ds < +\infty \quad P\text{-}a.s. \ for \ all \ t \geq 0,$$

$$and \quad \mu + \psi_1\sigma + (\psi_2 - 1)\zeta\lambda \equiv 0, \quad \psi_2 > 0 \ P \otimes dt - a.e. \ .$$

Proof. The proof follows immediately from Theorems 3.1-3.2 and the fact that for any $M \in \mathcal{M}_{loc}(\mathbb{F})$, there exists a unique $(\psi_1, \psi_2) \in L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F})$ such that $M = M_0 + \psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}}$, for more details about the representation of M, we refer the reader to the Definition 2.24 and Theorem 2.7. □

3.4.1.2 Black-Scholes market model

In this subsection, we focus on the case of Black-Scholes market model, where S is given by the following stochastic differential equation

$$S_t = S_0 \mathcal{E}(X)_t, \quad X_t := \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds. \tag{3.42}$$

Here W is a one-dimensional Brownian motion, $\sigma > 0$ and (μ, σ) are bounded adapted processes. The filtration $\mathbb{F}$ is the right continuous and complete filtration that is generated by Brownian motion W. Then, by Theorem 2.7, any $\mathbb{F}$-local martingale M, can be represented as

$$M = M_0 + h \bullet W,$$

where h is progressively measurable process satisfying $\int_0^T h_t^2 dt < \infty \ P - a.s.$.

Lemma 3.11: *Suppose that S is given by (3.42). Then $\mathcal{E}\left(K^{\mathbb{F}}\right)$ is local martingale deflator for the model $(S, \mathbb{F})$ if and only if $K^{\mathbb{F}} = \frac{-\mu}{\sigma} \bullet W$.*

Proof. Thanks to Theorem 2.7, $K^{\mathbb{F}}$ is a continuous local martingale and hence (ψ_1, ψ_2) of Lemma 3.9 takes the form of $(\beta, 0)$ and the proof of the lemma follows immediately. □

By combining the parametrization of local martingale deflator for Jump-diffusion model given by Theorem 3.7, with the case that all $\mathbb{F}$-local martingales are continuous, we derive the following result.

Theorem 3.8: *Suppose S given by (3.42) and $G > 0$. Let $Z^{\mathbb{G}} := \mathcal{E}\big(K^{\mathbb{G}}\big)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) $Z^{\mathbb{G}}$ *is a local martingale deflator for* $(S^{\tau}, \mathbb{G})$,

(b) *There exists a pair* $(\varphi^{(o)}, \varphi^{(pr)}) \in (L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D))$ *such that* $\mathbb{E}[\varphi^{(pr)}_{\tau} \mid \mathcal{F}_{\tau}] I_{\{\tau < +\infty\}} = 0$ *P-a.s.,*

$$K^{\mathbb{G}} = -(\frac{\mu}{\sigma} + \varphi^{(m)}) \cdot \overline{W} + \varphi^{(o)} \cdot \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \cdot D, \qquad \textit{and} \tag{3.43}$$
$$-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - \ G}, \ P \otimes D^{o,\mathbb{F}} - a.e., \ \varphi^{(pr)} > -\varphi^{(o)} \frac{G}{G_-} - 1, \ P \otimes D - a.e. \ .$$

(c) *There exists a unique pair* $(\varphi^{(o)}, \varphi^{(pr)}) \in (L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D))$ *such that* $\mathbb{E}[\varphi^{(pr)}_{\tau} \mid \mathcal{F}_{\tau}] I_{\{\tau < +\infty\}} = 0$ *P-a.s.,*

$$Z^{\mathbb{G}} = \mathcal{E}(-(\frac{\mu}{\sigma} + \varphi^{(m)}) \cdot \overline{W}) \mathcal{E}(\varphi^{(o)} \cdot \overline{N}^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \cdot D). \tag{3.44}$$
$$-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - \ G}, \quad P \otimes D^{o,\mathbb{F}} - a.e. \ \textit{and} \ \varphi^{(pr)} > -1, \ P \otimes D - a.e. \ .$$

Proof. In Lemma 3.10 and Theorem 3.7, put $\psi^{(m)} = 1, \ \psi^{(m)} = \psi_2 = 1$ respectively. Then the Poisson process vanishes, and we have the following decomposition

$$K^{\mathbb{G}} = -(\frac{\mu}{\sigma} + \varphi^{(m)}) \cdot \mathcal{T}(W) + \varphi^{(o)} \cdot N^{\mathbb{G}} + \varphi^{(pr)} \cdot D. \tag{3.45}$$

Since all $\mathbb{F}$-local martingale are continuous, thanks to Theorem 3.5, the decomposition (3.45) coincides with

$$K^{\mathbb{G}} = -(\frac{\mu}{\sigma} + \varphi^{(m)}) \bullet \overline{W} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$

To prove the multiplicative representation in assertion (c), the same calculation using Yor's formula as in the previous sections applies, and we will omit it here. □

When the Black-Scholes market model is stopped at random time τ, it is not complete market anymore, and the random time τ causes this incompleteness. Therefore the class of deflators for this model is characterized by $(\varphi^{(o)}, \varphi^{(pr)})$.

3.4.2 Jump-diffusion for $(S, \mathbb{F})$ and a particular random time

In the subsection 3.4.2, we discuss a particular model for τ that was introduced in [6, Example 2.12] and [9, Subsection 5.2.2, page 108]. Consider the same model for $(S, \mathbb{F})$ as in Theorem 3.7, and let $\tau := (aT_2) \wedge T_1$, where $a \in (0,1)$ and T_1 and T_2 are the first and the second jump times of the Poisson process N (i.e. $N := \sum_{n=1}^{+\infty} I_{[\![T_n,+\infty[\![}$). Since $\mathbb{F}$ is generated by (W, N) and W is independent of τ, the same calculations for the three processes $(G, G_-, \widetilde{G})$ as in [6, 9] remain valid. Thus, we get

$$\widetilde{G}_t = e^{-\beta t}(\beta t + 1) I_{[\![0,T_1[\![}(t) + e^{-\beta t} I_{[\![T_1]\!]}(t), \; G_t = e^{-\beta t}(\beta t + 1) I_{[\![0,T_1[\![}(t)$$
$$G_{t-} = \widetilde{G}_{t-} = e^{-\beta t}(\beta t + 1) I_{[\![0,T_1]\!]}(t).$$

However the arguments for the calculations of m and $D^{o,\mathbb{F}}$ differ slightly from that of [6, 9]. Let m^c be the continuous local martingale part of m, and hence $m - m^c$ is a pure jump local martingale with jumps equal to

$$\Delta m = \widetilde{G} - G_- = \phi \Delta N^{\mathbb{F}} \text{ where } \phi_t := -\beta t e^{-\beta t}, \; \beta := \lambda(a^{-1} - 1).$$

Hence $m = m^c + \phi \bullet N^{\mathbb{F}}$ on the one hand. On the other hand, by writing

$$G_t = e^{-\beta t}(\beta t + 1)(1 - H_t^{(1)}),\ H^{(1)} := I_{[\![T_1,+\infty[\![},\ M^{(1)} := H^{(1)} - \lambda(t \wedge T_1) = (N^{\mathbb{F}})^{T_1},$$

and by applying Itô's formula to the process G and using $G = m + D^{o,\mathbb{F}}$ (see (2.19)), we deduce that $m^c \equiv 0$. Hence, we get

$$m = m_0 + \phi \bullet N^{\mathbb{F}} \quad \text{and} \quad \mathrm{D}_{\mathrm{t}}^{\mathrm{o},\mathbb{F}} = \int_0^{\mathrm{t}} \mathrm{e}^{-\beta \mathrm{s}} \mathrm{dH}_{\mathrm{s}}^{(1)} + (\beta + \lambda)\beta \int_0^{\mathrm{t}\wedge \mathrm{T}_1} \mathrm{s}\mathrm{e}^{-\beta \mathrm{s}} \mathrm{ds}. \tag{3.46}$$

Since in the current case we have $\{\widetilde{G} = 0 < G_-\} = \emptyset$, we derive

$$\mathcal{T}(W) = W^\tau,\ \mathcal{T}(N^{\mathbb{F}}) = (N^{\mathbb{F}})^\tau + \frac{\beta t}{1+\beta t} \bullet (N)^\tau,\ \frac{1}{G_-} \bullet \mathcal{T}(m) = -\frac{\beta t}{1+\beta t} \bullet \mathcal{T}(N^{\mathbb{F}}),$$
$$N^{\mathbb{G}} = I_{\{aT_2 < T_1\}} I_{[\![aT_2,+\infty[\![} - (\lambda + \beta) \int_0^{t\wedge\tau} \frac{\beta s}{1+\beta s} ds. \tag{3.47}$$

Furthermore, we have $]\!]0,\tau]\!] \subset \{G_- > 0\} =]\!]0, T_1]\!]$, and hence the condition $G > 0$ is redundant in the current case, as one can work with S^{T_1} instead. This confirms our claim that the condition $G > 0$ is technical and can be relaxed at the expenses of technicalities (in both the statements of the results and the proofs) that we tried to avoid here. A combination of this analysis with Theorem 3.7 leads to the following.

Corollary 3.8.1: $Z^{\mathbb{G}} \in \mathcal{Z}(S^\tau, \mathbb{G})$ is equivalent to each of the following:

(a) There exist $(\psi_1, \psi_2) \in L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F})$, $\varphi^{(o)} = \varphi^{(o)} I_{]\!]0,T_1]\!]}$ belongs to $\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$ and $\varphi^{(pr)} \in L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$ satisfying the following

$$K^{\mathbb{G}} = \psi_1 \bullet W^\tau + (\psi_2 - \frac{1}{1+\beta t}) \bullet \mathcal{T}(N^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D,$$
$$\mu + \psi_1 \sigma + (\psi_2 - 1)\zeta\lambda \equiv 0, \quad \psi_2 > 0,\ -\psi_2 < \varphi^{(o)} I_{]\!]0,T_1[\![}, \quad P \otimes dt - a.e.$$

and the following inequalities hold P-a.s.

$$\varphi^{(o)}(T_1) < \psi_2(T_1)(1+\beta T_1),$$
$$\varphi^{(pr)}(aT_2 \wedge T_1) > -[\psi_2(aT_2) + \varphi^{(o)}(aT_2)]I_{\{aT_2<T_1\}} - \psi_2(T_1)[1+\beta T_1]I_{\{aT_2\geq T_1\}},$$

(b) There exists a unique quadruplet $(\psi_1, \psi_2, \varphi^{(o)}, \varphi^{(pr)})$ belonging to the following set $L^1_{loc}(W,\mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}},\mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), \mathrm{P}\otimes \mathrm{D})$, satisfying

$$\begin{aligned}
&Z^{\mathbb{G}} = \mathcal{E}(L)^{\tau}\mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)} \bullet D),\\
&L := \psi_1 \bullet W + ((1+\beta t)\psi_2 - 1) \bullet N^{\mathbb{F}} + \int_0^{\cdot} \frac{\lambda\beta t}{1+\beta t}[(1+\beta t)\psi_2(t) - 1]dt,\\
&\varphi^{(pr)}(aT_2 \wedge T_1) > -1, \quad P-a.s., \quad \varphi^{(o)}(T_1) < \psi_2(T_1)(1+\beta T_1) \quad P\text{-a.s.},\\
&\text{and} \quad \mu + \psi_1\sigma + (\psi_2 - 1)\zeta\lambda \equiv 0, \;\; \psi_2 > 0, \; -1 < \varphi^{(o)} I_{]\!]0,T_1[\![} \;\; P\otimes dt - a.e.
\end{aligned}$$

Proof. By Theorem 3.7, we have the following decomposition for any local martingale deflators $Z^{\mathbb{G}} \in \mathcal{Z}_{loc}(S^{\tau},\mathbb{G})$, where S is given by 3.38,

$$K^{\mathbb{G}} = \psi_1 \bullet \mathcal{T}(W) + (\psi_2 - 1)\bullet \mathcal{T}(N^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$

Since in the current particular case of stopping time τ, we have $\{\widetilde{G} = 0 < G_-\} = \emptyset$ and by inserting (3.46) and (3.47) in $K^{\mathbb{G}}$, we get

$$K^{\mathbb{G}} = \psi_1 \bullet W^{\tau} + (\psi_2 - \frac{1}{1+\beta t})\bullet \mathcal{T}(N^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$

The proof for the multiplicative representation, stated in assertion (b), follows immediately from assertion (a), and will be omitted here. □

3.4.3 Volatility market models

In this section, we discuss two popular volatility models. Precisely, we address the corrected Stein and Stein Model and the Barndorff-Nielsen Shephard Model. For more details about these models and their role in the financial markets we refer to [26] and [114]. We evaluate local martingale deflators parametrization for these two models in the next two subsections.

3.4.3.1 Corrected Stein and Stein Model

The corrected Stein and Stein financial market is given by the following stochastic differential equations and it has no jumps,

$$S_t = S_0\mathcal{E}(X)_t \quad , \quad X_t = X_0 + \int_0^t V_s\sigma_s dW_s^{(1)} + \int_0^t \mu V_s^2 dt,$$
$$\text{where,} \quad V_t = V_0 + \int_0^t (m - aV_s)dt + \int_0^t \alpha_s dW_s^{(2)}, \ \text{ and } t \in [0,T]. \tag{3.48}$$

$W^{(i)}, i = 1,2$, are two one-dimensional Brownian motions with the correlation coefficient $\rho \in (-1,+1)$ and the filtration $\mathbb{F}$ is generated by the $W^{(i)}, i = 1,2$, where $\mathbb{F} = (\mathcal{F}_t^{W^1,W^2})_{t\in[0,T]}$. All the coefficients σ, α, m, a and μ are positive constants.
Thanks to Theorem 2.7, we deduce that any $\mathbb{F}$-local martingale M can be represented as follows

$$M = M_0 + h^1 \bullet W^{(1)} + h^2 \bullet W^{(2)}. \tag{3.49}$$

Here $h^{(i)}, i = 1,2$, are progressively measurable processes satisfying the following fact that $\int_0^T (h_t^1)^2 + (h_t^2)^2 dt < \infty \ \ P-a.s.$.

Lemma 3.12: *Suppose S is given by (3.48). Then $\mathcal{E}\left(K^{\mathbb{F}}\right)$ is an $\mathbb{F}$-local martingale deflator for $(S,\mathbb{F})$ if and only if there exists a unique pair $(\beta^{(1)},\beta^{(2)})$ of $\mathbb{F}$-progressively measurable process such that $\int_0^T [(\beta_t^1)^2 + (\beta_t^2)^2]dt < \infty. \ \ P-a.s.$,*

$$K^{\mathbb{F}} = \beta^{(1)} \bullet W^{(1)} + \beta^{(2)} \bullet W^{(2)}, \tag{3.50}$$
$$\textit{and} \qquad V\sigma\beta^{(1)} + V\sigma\beta^{(2)}\rho + \mu V^2 \equiv 0, \quad P\otimes dt\text{-}a.e. \ . \tag{3.51}$$

Proof. The representation (3.50) for the process $K^{\mathbb{F}}$ follows from the equation (3.49). Therefore, by applying Lemma 3.4, we conclude that $S+[S,K^{\mathbb{F}}]$ is $\mathbb{F}$-local martingale if and only if $X + [X, K^{\mathbb{F}}]$ is $\mathbb{F}$-local martingale on one hand. On the other hand, we have

$$X + [X, K^{\mathbb{F}}] = \mathbb{F} - \text{local martingale} + \int_0^t V_s \sigma_s \beta_s^{(1)} + V_s \sigma_s \beta_s^{(2)} \rho + \mu V_s^2 ds.$$

Thus, the condition (3.51) follows immediately. This ends the proof. □

For the corrected Stein and Stein financial market model, we have the following parametrization for local martingale deflators.

Theorem 3.9: *Suppose S given by (3.48) and $G > 0$. Let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a local martingale deflator for $(S^\tau, \mathbb{G})$.*

(b) *There exists a unique triplet $(\beta^{(2)}, \varphi^{(o)}, \varphi^{(pr)})$ belongs to*

$$L^1_{loc}(W^{(2)}, \mathbb{F}) \times L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D),$$

satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ *P-a.s.,*

$$K^{\mathbb{G}} = -(\frac{\mu V}{\sigma} + \rho \beta^{(2)}) \bullet \overline{W}^{(1)} + \beta^{(2)} \bullet \overline{W}^{(2)} - G_-^{-1} \bullet \overline{m} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D,$$

$$\textit{and} - \frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \ P \otimes D^{o,\mathbb{F}} - a.e., \ \varphi^{(pr)} > -\varphi^{(o)} \frac{G}{G_-} - 1, \ P \otimes D - a.e. \ .$$

(c) *There exists a unique triple $(\beta^{(2)}, \varphi^{(o)}, \varphi^{(pr)})$ belongs to*

$$L^1_{loc}(W^{(2)}, \mathbb{F}) \times L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D),$$

satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ *P-a.s.,*

$$Z^{\mathbb{G}} = \frac{\mathcal{E}(-(\mu V \sigma^{-1} + \rho \beta^{(2)}) \bullet W^{(1)} + \beta^{(2)} \bullet W^{(2)})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet \overline{N}^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D), \quad (3.52)$$

$$-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \ P \otimes D^{o,\mathbb{F}} - a.e. \ \ \varphi^{(pr)} > -1, \ P \otimes D - a.e. \ . \quad (3.53)$$

Proof. From the dynamics of S in (3.48), we deduce that all $\mathbb{F}$-local martingale are continuous. On one hand, thanks to Theorem 3.5, there exists a unique triplet $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that the process $K^{\mathbb{G}}$ gets the following decomposition

$$K^{\mathbb{G}} = \overline{K^{\mathbb{F}}} - G_{-}^{-1} \bullet \overline{m} + \varphi^{(0)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.54}$$

On the other hand, by Lemma 3.12, we get

$$\overline{K^{\mathbb{F}}} = -(\frac{\mu V}{\sigma} + \rho\beta^{(2)}) \bullet \overline{W}^{(1)} + \beta^{(2)} \bullet \overline{W}^{(2)}.$$

By inserting it in (3.54), this ends the proof of assertion (b).

The proof of assertion (c) follows immediately from combining assertion (b), equation (3.77) and Lemma 3.12, and we will omit it here. □

Remark When $\rho = 0$, it's clear that the decomposition (3.52) becomes

$$K^{\mathbb{G}} = \frac{-\mu V}{\sigma} \bullet \overline{W}^{(1)} + \beta^{(2)} \bullet \overline{W}^{(2)} - G_{-}^{-1} \bullet \overline{m} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}},$$

This fact follows from the equation (3.51) in Lemma 3.12.

3.4.3.2 Barndorff-Nielsen Shephard Model

The Barndorff-Nielsen Shephard financial market is presented by the filtration generated by a one dimensional Lévy process Y, where $Y := Y^c + \widetilde{Y}^d$. The process Y^c is the continuous part of Leѵ́y process and $\widetilde{Y}^d$ is driven by random measure of Y, donated by $\widetilde{\mu}(dt \times dx)$ with componsator measure $\widetilde{\nu}(dt \times dx) = \widetilde{F}(dx)dA_t$, for more details about the model see [26] and [114].

By considering $A_s = s$, we have $< Y^c >_s = s$. Then Barndorff-Nielsen Shephard process, S_t, follows stochastic differential equations below

$$S_t = S_0 e^{X_t}, \quad dX_s = (\mu + \xi\sigma_s^2)ds + \sigma_s dY_s^c + d(\rho x \star \widetilde{\mu}_Y)_s,$$
$$\text{and} \quad d\sigma_s = -\lambda\sigma_s^2 dt + d(x \star \widetilde{\mu}_Y)_s, \tag{3.55}$$

where all the coefficients ξ, ρ, λ and μ are real constants with $\rho < 0$ and $\lambda > 0$. Therefore by Ito's formula, we calculate the dynamics of S,

$$\frac{dS_t}{S_{t_-}} = \alpha_t dt + \sigma_t dY_t^c + d(e^{\rho x} - 1) \star (\widetilde{\mu} - \widetilde{\nu})_t, \tag{3.56}$$

where $\alpha := \mu + \sigma^2(\xi + \frac{1}{2}) + \int(e^{\rho x} - 1)\widetilde{F}(dx)$. The predictable characteristic of Section 2.4 for Barndorff-Nielsen Shephard process is

$$S_t^c = S_{t_-}\sigma_t \bullet Y_t^c, \quad \Delta S_t = S_{t_-}(e^{\rho\Delta\sigma_t^2} - 1), \quad b_t = S_{t_-}\alpha,$$
$$c_t = S_{t_-}^2\sigma_t^2, \quad \nu^S(dt \times dx) = F_t^S(dx)dt, \quad f(x)F_t^S(dx) = f(S_{t_-}(e^{\rho x} - 1))\widetilde{F}_t(dx).$$

Here, we don't compute the truncation function $h(x)$, since we don't have big jumps. In the following lemma, we parametrize deflators for the model $(S, \mathbb{F})$, given by (3.55).

Lemma 3.13: *Suppose S is given by (3.55). Then $\mathcal{E}\left(K^{\mathbb{F}}\right)$ is an $\mathbb{F}$-local martingale deflator for S if and only if there exists a unique quadruple (β, f, g, K') that belongs to $(L^1_{loc}(Y^c, \mathbb{F}) \times \mathcal{G}^1_{loc}(\widetilde{\mu}, \mathbb{F}) \times \mathcal{H}^1_{loc}(\widetilde{\mu}, \mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F})$, satisfying $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0$, $P \otimes \mu$-.a.e., $\Delta K^{\mathbb{F}} > -1$ and $[K', Y] = 0$, satisfiying*

(i) $\displaystyle\int |(e^{\rho x} - 1)f(S_{t_-}(e^{\rho x} - 1))|\widetilde{F}(dx) < +\infty \quad P \otimes dt - a.e.$,

(ii) $\displaystyle\mu + \sigma(\sigma\xi + \frac{\sigma}{2} + \beta) + \int(e^{\rho x} - 1)f(S_{t_-}(e^{\rho x} - 1))\widetilde{F}(dx) \equiv 0, \ P \otimes dt\text{-}a.e.$ (3.57) *then the process $K^{\mathbb{F}}$ has following decomposition*

$$K^{\mathbb{F}} := \beta \bullet Y^c + (f-1) \star (\widetilde{\mu} - \widetilde{\nu}) + g \star \widetilde{\mu} + K'. \tag{3.58}$$

Proof. For any $K^{\mathbb{F}} \in \mathcal{M}_{loc}(\mathbb{F})$, by applying the Jacod's representation with respect to S, we get (3.58). For more details about the decomposition of $K^{\mathbb{F}}$, we refer the reader to the Theorem 2.7. Therefore by applying Lemma 3.4, $S + [S, K^{\mathbb{F}}]$ is $\mathbb{F}$-local martingale if and only if

$$Y + [Y, K^{\mathbb{F}}] = \mathbb{F} - \text{local martingale} + \int_0^t \mu + \sigma_s(\sigma_s \xi + \frac{\sigma_s}{2} + \beta_s) ds$$
$$+ \int_0^t \int (e^{\rho x} - 1) f_s (e^{\rho x} - 1) \widetilde{F}(dx) ds$$

is an $\mathbb{F}$-local martingale if and only if the right side of above equation vanishes. Thus, the condition (3.57) follows immediately. This ends the proof. □

Our main result in this subsection is presented by the following Theorem.

Theorem 3.10: *Suppose S is given by (3.55) and $G > 0$. Let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a local martingale deflator for $(S^\tau, \mathbb{G})$.*

(b) *There exists a unique $(\beta, f, g, K', \varphi^{(o)}, \varphi^{(pr)})$ that belongs to*

$$L^1_{loc}(Y^c, \mathbb{F}) \times \mathcal{G}^1_{loc}(\widetilde{\mu}, \mathbb{F}) \times \mathcal{H}^1_{loc}(\widetilde{\mu}, \mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\widetilde{\Omega}, \mathit{Prog}(\mathbb{F}), P \otimes D),$$

satisfying $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0$, $P \otimes \mu$-.a.e., $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s.,

$$K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D,$$

$$
\begin{aligned}
&\varphi^{(pr)} > -\big[G_-\Big(f(\Delta X)+g(\Delta X))I_{\{\Delta X\neq 0\}}+(1+\Delta K')I_{\{\Delta X=0\}}\Big)+\varphi^{(o)}G\big]/\tilde{G},\ P\otimes D\text{-}a.e.,\\
&-\frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{G} < \varphi^{(o)} < \frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{\Delta D^{o,\mathbb{F}}}\quad P\otimes D^{o,\mathbb{F}}\text{-}a.e.\ on\ \{\Delta X\neq 0\},\\
&-\frac{(1+\Delta K')G_-}{G} < \varphi^{(o)} < \frac{(1+\Delta K')G_-}{\Delta D^{o,\mathbb{F}}}\qquad P\otimes D^{o,\mathbb{F}}\text{-}a.e.\ on\ \{\Delta X=0\},\\
&\int |(e^{\rho x}-1)f(S_{t_-}(e^{\rho x}-1))|\widetilde{F}(dx) < +\infty\quad P\otimes dt-a.e.\ ,\\
&and\quad \mu+\sigma(\sigma\xi+\frac{\sigma}{2}+\beta)+\int(e^{\rho x}-1)f(S_-(e^{\rho x}-1))\widetilde{F}(dx)\equiv 0,\quad P\otimes dt-a.e.\ ,\\
&where\quad K^{\mathbb{F}}=\beta\bullet Y^c+(f-1)\star(\widetilde{\mu}-\widetilde{\nu})+g\star\widetilde{\mu}+K',\quad \Delta K^{\mathbb{F}}>-1,\ [K',Y]=0.
\end{aligned}
\tag{3.59}
$$

(c) *There exists a unique* $(\beta, f, g, K', \varphi^{(o)}, \varphi^{(pr)})$ *that belongs to*

$$
L^1_{loc}(Y^c,\mathbb{F})\times\mathcal{G}^1_{loc}(\widetilde{\mu},\mathbb{F})\times\mathcal{H}^1_{loc}(\widetilde{\mu},\mathbb{F})\times\mathcal{M}_{0,loc}(\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\widetilde{\Omega}, Prog(\mathbb{F}), P\otimes D),
$$

satisfying $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F}))=0,\ P\otimes\mu\text{-}.a.e.,\ \mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau]I_{\{\tau<+\infty\}}=0\ P\text{-}a.s.,$

$$
Z^{\mathbb{G}}=\frac{\mathcal{E}(K^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1}\bullet m)^\tau}\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D). \tag{3.60}
$$

$$
\begin{aligned}
&and\ \ \varphi^{(pr)}>-1,\ P\otimes D\text{-}a.e.,\ -\frac{G_-}{G}<\varphi^{(o)}<\frac{G_-}{\Delta D^{o,\mathbb{F}}}\quad P\otimes D^{o,\mathbb{F}}\text{-}a.e.,\\
&\int |(e^{\rho x}-1)f(S_{t_-}(e^{\rho x}-1))|\widetilde{F}(dx) < +\infty\quad P\otimes dt-a.e.\ ,\\
&\mu+\sigma(\sigma\xi+\frac{\sigma}{2}+\beta)+\int(e^{\rho x}-1)f(S_-(e^{\rho x}-1))\widetilde{F}(dx)\equiv 0,\quad P\otimes dt-a.e.\ ,\\
&where\quad K^{\mathbb{F}}=\beta\bullet Y^c+(f-1)\star(\widetilde{\mu}-\widetilde{\nu})+g\star\widetilde{\mu}+K',\ \Delta K^{\mathbb{F}}>-1,\ [K',Y]=0.
\end{aligned}
$$

Proof. The proof follows immediately from Theorems 3.1-3.2. By Theorem 3.1, we have the following decomposition

$$
K^{\mathbb{G}}=\mathcal{T}(K^{\mathbb{F}})-G_-^{-1}\bullet\mathcal{T}(m)+\varphi^{(o)}\bullet N^{\mathbb{G}}+\varphi^{(pr)}\bullet D. \tag{3.61}
$$

Thanks to Lemma 3.13, define $K^{\mathbb{F}}$ as follows $K^{\mathbb{F}}:=\beta\bullet Y^c+(f-1)\star(\widetilde{\mu}-\widetilde{\nu})+g\star\widetilde{\mu}+K'$, then decomposition (3.61) coincides with (3.59). Similarly, one can see the prove of the multiplicative form. This ends the proof of theorem. □

3.4.4 The case of the complete market model $(S, \mathbb{F})$

In this subsection, we illustrate our results of Section 3.2 in the case where the initial market model, $(S, \mathbb{F}, P)$, is complete.

Theorem 3.11: *Suppose $G > 0$ and S be a $\mathbb{F}$-semimartingale such that the model $(S, \mathbb{F})$ is complete and $Z^{(1)} := \mathcal{E}(K^{(1)})$ is its unique local martingale deflator (i.e. $Z^{\mathbb{F}} \in \mathcal{Z}_{loc}(S, \mathbb{F})$). Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale, then the following assertions are equivalent.*

(a) $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ *is a local martingale deflator for $(S^\tau, \mathbb{G})$ (i.e. $Z^{\mathbb{G}} \in \mathcal{Z}(S^\tau, \mathbb{G})$).*

(b) *There exists a unique pair $\left(\varphi^{(o)}, \varphi^{(pr)}\right)$, such that $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D)$, satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s.,*

$$\varphi^{(pr)} > -\big[G_-(1 + \Delta K^{(1)}) + \varphi^{(o)} G\big]/\widetilde{G}, \qquad P \otimes D - a.e., \tag{3.62}$$

$$-\frac{G_-}{G}(1 + \Delta K^{(1)}) < \varphi^{(o)}, \quad P \otimes D^{o,\mathbb{F}}\text{-a.e. and} \quad \varphi^{(o)} \Delta D^{o,\mathbb{F}} < (1 + \Delta K^{(1)}) G_-, \tag{3.63}$$

and

$$K^{\mathbb{G}} = \mathcal{T}(K^{(1)}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.64}$$

(c) *There exists a unique pair $\left(\varphi^{(o)}, \varphi^{(pr)}\right)$, such that $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D)$, and satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s.,*

$$\varphi^{(pr)} > -1, \quad P \otimes D - a.e., \; -\frac{\widetilde{G}}{G} < \varphi^{(o)} < \frac{\widetilde{G}}{\widetilde{G} - G}, \quad P \otimes D^{o,\mathbb{F}}\text{-a.e.} \; , \tag{3.65}$$

$$\text{and} \quad Z^{\mathbb{G}} = \frac{(Z^{(1)})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D). \tag{3.66}$$

Proof. Since the model $(S, \mathbb{F})$ is complete, there exists a unique local martingale deflator for this model that we denote by $Z^{(1)} = \mathcal{E}(K^{(1)})$. Thus, by combining this fact with Theorem 3.1, we deduce the existence of $\left(\varphi^{(o)}, \varphi^{(pr)}\right)$ and the decomposition

$K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D$. By substituting $(K^{\mathbb{F}} = K^{(1)}$, the proofs of assertions (a) and (b) follow.

The proof for the multiplicative representation, stated in assertion (c), follows immediately from assertion (b) and Yor's formula, and it will be omitted here. □

3.5 Super-martingale deflator: The general setting

Throughout this section, for any $\mathbb{H}$-semimartingale X, we denote by $X^{p,\mathbb{H}}$ and defined in Section 2.1, when it exists (i.e. when X is a special semimartingale), the $\mathbb{H}$-predictable with finite variation part in its Doob-Meyer decomposition. Recall that all the set of deflators for the model $(X, \mathbb{H})$, defined in Section 2.2.2, will be denoted by $\mathcal{D}(X, \mathbb{H})$ throughout the rest of the thesis. Furthermore, stochastic processes will be compared to each other in the following sense.

Definition 3.2: Let X and Y be two processes with $X_0 = Y_0$. Then

$$X \succeq Y \quad \text{if } X - Y \text{ is an increasing process.}$$

We start this section by parametrizing deflators as follows.

Lemma 3.14: *Let X be an $\mathbb{H}$-semimartingale, and Z be a positive $\mathbb{H}$-supermartingale. Then the following assertions are equivalent.*

(a) *Z is a deflator for $(X, \mathbb{H})$.*

(b) *There exists a unique pair (N, V) of $N \in \mathcal{M}_{0,loc}(\mathbb{H})$, and a nondecreasing, RCLL and $\mathbb{H}$-predictable process V, such*

$$Z := Z_0 \mathcal{E}(N)\mathcal{E}(-V), \quad N_0 = V_0 = 0, \quad \Delta N > -1, \quad \Delta V < 1 \qquad (3.67)$$

$$\sup_{0<s\leq \cdot} |\Delta Y_s^{(\varphi)}| \in \mathcal{A}_{loc}^+(\mathbb{H}) \text{ and } \frac{1}{1-\Delta V} \bullet V \succeq (Y^{(\varphi)})^{p,\mathbb{H}}. \qquad (3.68)$$

Here $(Y^{(\varphi)})^{p,\mathbb{H}}$ *is the predictable with finite variation process such that*

$$Y^{(\varphi)} - (Y^{(\varphi)})^{p,\mathbb{H}} \in \mathcal{M}_{loc}(\mathbb{H}) \quad \text{and} \quad \mathrm{Y}^{(\varphi)} := \varphi \bullet \mathrm{X} + [\varphi \bullet \mathrm{X}, \mathrm{N}],$$

for any bounded φ *that belongs to* $\mathcal{L}(X, \mathbb{H})$ *given by*

$$\mathcal{L}(X, \mathbb{H}) := \left\{ \varphi \textit{ is } \mathbb{H}\textit{-predictable} \;\middle|\; \varphi \Delta X > -1 \right\}. \tag{3.69}$$

Proof. The proof of this lemma will be achieved in two steps. The first step proves that there exists a unique pair (N, V) satisfying (3.67) as soon as Z is a deflator for $(X, \mathbb{H})$. The second step shows that for a process Z, for which there exists a pair (N, V) satisfying (3.67), there is equivalence between $Z\mathcal{E}(\varphi \bullet X)$ is supermartingale and (3.68), for any bounded $\varphi \in \mathcal{L}(X, \mathbb{H})$.

Step 1. Suppose that Z is a deflator. This implies that Z is a positive supermartingale (since $\varphi = 0 \in \mathcal{L}(X, \mathbb{H})$), and hence $X := Z_-^{-1} \bullet Z$ is a local supermartingale having the unique Doom-Meyer decomposition $X := K - V$, where $K \in \mathcal{M}_{0,loc}(\mathbb{H})$ and V is nondecreasing and predictable with $\Delta V < 1$ (since $Z > 0$). It is clear that the predictable process $(1 - \Delta V)^{-1}$ is well defined and is locally bounded. Hence

$$N := \frac{1}{1 - \Delta V} \bullet K \in \mathcal{M}_{0,loc}(\mathbb{H}), \quad \Delta N > -1 \quad \text{and} \quad \mathrm{Z} = \mathrm{Z}_0 \mathcal{E}(\mathrm{N}) \mathcal{E}(-\mathrm{V}).$$

This ends the first step.

Step 2. Suppose that there exists a pair (N, V) such that $Z = Z_0 \mathcal{E}(N) \mathcal{E}(-V)$ and (3.67) holds. Let φ be a bounded element of $\mathcal{L}(X, \mathbb{H})$. Then by applying Yor's formula to $Z\mathcal{E}(\varphi \bullet X) = Z_0 \mathcal{E}(N) \mathcal{E}(-V) \mathcal{E}(\varphi \bullet X)$, one get

$$\begin{aligned} Z\mathcal{E}(\varphi \bullet X) &= Z_0 \mathcal{E}(N) \mathcal{E}(\varphi \bullet X) \mathcal{E}(-V) = Z_0 \mathcal{E}\Big(N + \varphi \bullet X + \varphi \bullet [X, N]\Big) \mathcal{E}(-V) \\ &= Z_0 \mathcal{E}(Y^{(\varphi)}) \mathcal{E}(-V) = Z_0 \mathcal{E}\left(Y^{(\varphi)} - V - [Y^{(\varphi)}, V]\right) \\ &= Z_0 \mathcal{E}\left((1 - \Delta V) \bullet Y^{(\varphi)} - V\right), \end{aligned}$$

where $Y^{(\varphi)} := N + \varphi \bullet X + \varphi \bullet [X, N]$. Since Z is positive and $\varphi \in \mathcal{L}(X, \mathbb{H})$, then the process $Z\mathcal{E}(\varphi \bullet X)$ is an $\mathbb{H}$-supermartingale if and only if $(1 - \Delta V) \bullet Y^{(\varphi)} - V$ is a local $\mathbb{H}$-supermartingale, or equivalently $Y^{(\varphi)}$ is a special semimartingale (which is equivalent to the first condition of (3.68)) and its predictable with finite variation part, $(Y^{(\varphi)})^{p,\mathbb{H}}$, satisfies $(Y^{(\varphi)})^{p,\mathbb{H}} \preceq (1 - \Delta V)^{-1} \bullet V$. This finishes the second step, and the proof of the lemma as well. □

Below, we state our last main theorem of this chapter.

Theorem 3.12: *Suppose $G > 0$, and let $Z^{\mathbb{G}}$ be a $\mathbb{G}$-semimartingale. Then the following assertions are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$ (i.e. $Z^{\mathbb{G}} \in \mathcal{D}(S^\tau, \mathbb{G})$).*

(b) *There exists a unique $\left(K^{\mathbb{F}}, V^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $K^{\mathbb{F}} \in \mathcal{M}_{0,loc}(\mathbb{F})$, $V^{\mathbb{F}}$ is an $\mathbb{F}$-predictable and nondecreasing process, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$ such that $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s., the process $\mathcal{E}(K^{\mathbb{F}})\mathcal{E}(-V^{\mathbb{F}})$ belongs to $\mathcal{D}(S, \mathbb{F})$,*

$$\varphi^{(pr)} > -\left[G_-(1 + \Delta K^{\mathbb{F}}) + \varphi^{(o)} G\right]/\widetilde{G}, \qquad P \otimes D - a.e., \tag{3.70}$$

$$-\frac{G_-}{G}(1 + \Delta K^{\mathbb{F}}) < \varphi^{(o)} < \frac{(1 + \Delta K^{\mathbb{F}})G_-}{\Delta D^{o,\mathbb{F}}}, \quad P \otimes D^{o,\mathbb{F}}\text{-}a.e.. \tag{3.71}$$

$$Z^{\mathbb{G}} = Z^{\mathbb{G}}_0 \mathcal{E}(K^{\mathbb{G}})\mathcal{E}(-V^{\mathbb{F}})^\tau, \quad K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.72}$$

(c) *There exists unique $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $Z^{\mathbb{F}} \in \mathcal{D}(S, \mathbb{F})$, $(\varphi^{(o)}, \varphi^{(pr)})$ belongs to $\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$, $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s.,*

$$\varphi^{(pr)} > -1, \ P \otimes D - a.e., \quad -\frac{\widetilde{G}}{G} < \varphi^{(o)} < \frac{\widetilde{G}}{\widetilde{G} - G}, \ P \otimes D^{o,\mathbb{F}}\text{-}a.e., \tag{3.73}$$

and

$$Z^{\mathbb{G}} = \frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D). \tag{3.74}$$

Proof. The proof will be achieved in three steps, where we prove the implications (a)$\Longrightarrow$(b), (b)$\Longrightarrow$(c), and (c)$\Longrightarrow$(a) respectively.

Step 1. Herein, we prove (a)$\Longrightarrow$(b). To this end, we suppose that $Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$. Thus, due to Lemma 3.14, we deduce the existence of $K^{\mathbb{G}} \in \mathcal{M}_{0,loc}(\mathbb{G})$ and $V^{\mathbb{G}}$ a $\mathbb{G}$-predictable and nondecreasing process such that

$$\begin{array}{l} Z^{\mathbb{G}} = Z_0 \mathcal{E}(K^{\mathbb{G}})\mathcal{E}(-V^{\mathbb{G}}), \quad \sup\limits_{0<s\leq \cdot} |\varphi_s \Delta S^\tau_s|(1+\Delta K^{\mathbb{G}}_s) \in \mathcal{A}^+_{loc}(\mathbb{G}) \\ \Delta K^{\mathbb{G}} > -1, \quad \Delta V^{\mathbb{G}} < 1, \quad (1-\Delta V^{\mathbb{G}})^{-1} \bullet V^{\mathbb{G}} \succeq (\varphi \bullet S^\tau + [\varphi \bullet S^\tau, K^{\mathbb{G}}])^{p,\mathbb{G}}, \end{array}$$

for any bounded $\varphi \in \mathcal{L}(S^\tau, \mathbb{G})$. Then a direct application of Theorem 2.11 to $K^{\mathbb{G}}$ and Lemma 3.1 to $V^{\mathbb{G}}$, leads to the existence of the triplet $(N^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)})$ that belongs to $\mathcal{M}_{loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$ and an $\mathbb{F}$-predictable and nondecreasing process V with finite values such that

$$K^{\mathbb{G}} = K^{\mathbb{G}}_0 + \frac{1}{G^2_-} I_{\rrbracket 0,\tau\rrbracket} \bullet \mathcal{T}(N^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D, \quad V^{\mathbb{G}} = V^\tau.$$

Consider a bounded $\varphi \in \mathcal{L}(S, \mathbb{F})$, and remark that $\varphi \bullet S^\tau + [\varphi \bullet S^\tau, K^{\mathbb{G}}] \in \mathcal{A}_{loc}(\mathbb{G})$ is equivalent to $W := \sum (\varphi \Delta S^\tau + \varphi \Delta [S^\tau, K^{\mathbb{G}}]) I_{\{|\Delta S|>1\}} \in \mathcal{A}_{loc}(\mathbb{G})$. As a result,

$$W^{p,\mathbb{G}} = \lim_{n \longrightarrow +\infty} \Big(I_{\{n \geq |\Delta S| > 1\}} \bullet W \Big)^{p,\mathbb{G}}.$$

By combining

$$\begin{aligned} \varphi \bullet S^\tau + [\varphi \bullet S^\tau, K^{\mathbb{G}}] &= \varphi \bullet S^\tau + \frac{I_{\rrbracket 0,\tau\rrbracket}}{G^2_-} \bullet [\mathcal{T}(N^{\mathbb{F}}), \varphi \bullet S^\tau] \\ &\quad + \varphi \Delta S \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi \Delta S \varphi^{(pr)} \bullet D, \end{aligned}$$

the fact that $\varphi \Delta S \varphi^{(o)} I_{\{|\Delta S| \leq n\}} \bullet N^{\mathbb{G}}$ and $\varphi \Delta S \varphi^{(pr)} I_{\{|\Delta S| \leq n\}} \bullet D$ are $\mathbb{G}$-local martingales for any $n \geq 1$, and $[\mathcal{T}(N^{\mathbb{F}}), \varphi \bullet S^\tau] = G_- \widetilde{G}^{-1} I_{\rrbracket 0,\tau\rrbracket} \bullet [N^{\mathbb{F}}, \varphi \bullet S^\tau]$, we deduce that

$$
\begin{aligned}
\left(\varphi\bullet S^\tau + [\varphi\bullet S^\tau, K^{\mathbb{G}}]\right)^{p,\mathbb{G}} &= \left(\varphi\bullet S^\tau + \frac{1}{G_-^2} I_{]\!]0,\tau]\!]}\bullet[\mathcal{T}(N^{\mathbb{F}}),\varphi\bullet S]\right)^{p,\mathbb{G}}\\
&= \left(\varphi\bullet S^\tau + \frac{1}{G_-\widetilde{G}} I_{]\!]0,\tau]\!]}\bullet[N^{\mathbb{F}},\varphi\bullet S]\right)^{p,\mathbb{G}}.
\end{aligned}
$$

By inserting in this equation the following decomposition of S,

$$
S = S_0 + M + A + \sum \Delta S I_{\{|\Delta S|>1\}},
$$

where M is a locally bounded $\mathbb{F}$-local martingale and A is an $\mathbb{F}$-predictable process with finite variation, we obtain

$$
\begin{aligned}
\left(\varphi\bullet S^\tau + [\varphi\bullet S^\tau, K^{\mathbb{G}}]\right)^{p,\mathbb{G}} &= \varphi\bullet A^\tau + \frac{I_{]\!]0,\tau]\!]}}{G_-}\bullet\langle m,\varphi\bullet M\rangle^{\mathbb{F}} + \frac{I_{]\!]0,\tau]\!]}}{G_-^2}\bullet\langle N^{\mathbb{F}},\varphi\bullet M\rangle^{\mathbb{F}} +\\
&+\left(\sum\varphi\Delta S\Big(1+\frac{\Delta N^{\mathbb{F}}}{G_-\widetilde{G}}\Big)I_{]\!]0,\tau]\!]}I_{\{|\Delta S|>1\}}\right)^{p,\mathbb{G}}\\
&= \varphi\bullet A^\tau + \frac{I_{]\!]0,\tau]\!]}}{G_-}\bullet\langle m,\varphi\bullet M\rangle^{\mathbb{F}} + \frac{I_{]\!]0,\tau]\!]}}{G_-^2}\bullet\langle N^{\mathbb{F}},\varphi\bullet M\rangle^{\mathbb{F}} +\\
&+G_-^{-1}I_{]\!]0,\tau]\!]}\bullet\left(\sum\varphi\Delta S\Big(\widetilde{G}+\frac{\Delta N^{\mathbb{F}}}{G_-}\Big)I_{\{|\Delta S|>1\}}\right)^{p,\mathbb{F}}.
\end{aligned}
$$

As a result, for any bounded $\varphi\in\mathcal{L}(S,\mathbb{F})$, $V(\varphi) := \varphi\bullet S + [\varphi\bullet S, \frac{1}{G_-}\bullet m + \frac{1}{G_-^2}\bullet N^{\mathbb{F}}]$ has an $\mathbb{F}$-compensator, and

$$
\begin{aligned}
\frac{1}{1-\Delta V^{\mathbb{F}}}\bullet V^{\mathbb{F}} \succeq\; & \varphi\bullet A + \langle\frac{1}{G_-}\bullet m + \frac{1}{G_-^2}\bullet N^{\mathbb{F}},\varphi\bullet M\rangle^{\mathbb{F}} + \left(\sum\Delta V(\varphi)I_{\{|\Delta S|>1\}}\right)^{p,\mathbb{F}}\\
=\;& \left(\varphi\bullet S + [\varphi\bullet S,\frac{1}{G_-}\bullet m + \frac{1}{G_-^2}\bullet N^{\mathbb{F}}]\right)^{p,\mathbb{F}}.
\end{aligned}
$$

On the one hand, this is equivalent to $\mathcal{E}(K^{\mathbb{F}})\mathcal{E}(-V)\mathcal{E}(\varphi\bullet S)$ is an $\mathbb{F}$-supermartingale for any bounded $\varphi\in\mathcal{L}(S,\mathbb{F})$, where $K^{\mathbb{F}} := G_-^{-1}\bullet m + G_-^{-2}\bullet N^{\mathbb{F}}$. On the other hand, as in the proof of Theorem 3.1 (see step 2 of that proof), it is clear that $\mathcal{E}(K^{\mathbb{G}})>0$ if and only if $1+\Delta K^{\mathbb{G}}>0$ if and only if the triplet $(K^{\mathbb{F}},\varphi^{(o)},\varphi^{(pr)})$

satisfies $1+\Delta K^{\mathbb{F}}>0$ and (3.70)-(3.71). This proves assertion (b), and the first step is completed.

Step 2. This step proves (b)$\Longrightarrow$(c). Hence, we suppose that assertion (b) holds. Then there exists a unique $\left(K^{\mathbb{F}}, V^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $\mathbb{E}[\varphi^{(pr)}_{\tau} \mid \mathcal{F}_{\tau}]I_{\{\tau<+\infty\}}=0$ P-a.s., and

$$Z^{\mathbb{G}}=\mathcal{E}\Big(\mathcal{T}(K^{\mathbb{F}})-G_{-}^{-1}\bullet\mathcal{T}(m)+\varphi^{(o)}\bullet N^{\mathbb{G}}+\varphi^{(pr)}\bullet D\Big)\mathcal{E}(-V^{\mathbb{F}})^{\tau}.$$

Then by mimicking the analysis (calculations) that starts from (3.18), we derive

$$\begin{aligned}Z^{\mathbb{G}} &= \frac{\mathcal{E}\big(K^{\mathbb{F}}\big)^{\tau}\mathcal{E}(-V^{\mathbb{F}})^{\tau}}{\mathcal{E}(G_{-}^{-1}\bullet m)^{\tau}}\mathcal{E}(\overline{\varphi}^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\overline{\varphi}^{(pr)}\bullet D)\\ &= \frac{(Z^{\mathbb{F}})^{\tau}}{\mathcal{E}(G_{-}^{-1}\bullet m)^{\tau}}\mathcal{E}(\overline{\varphi}^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\overline{\varphi}^{(pr)}\bullet D).\end{aligned}$$

Here $\overline{\varphi}^{(o)}:=\widetilde{G}\varphi^{(o)}/G_{-}(1+\Delta K^{\mathbb{F}})$ and $\overline{\varphi}^{(pr)}:=\widetilde{G}\varphi^{(pr)}[G_{-}(1+\Delta K^{\mathbb{F}})+G\varphi^{(o)}]^{-1}$ are the $\mathbb{F}$-optional and $\mathbb{F}$-progressive processes respectively that satisfy (3.73) as a direct consequence of the conditions (3.70)-(3.71) fulfilled by the pair $(\varphi^{(o)}, \varphi^{(pr)})$. This ends the proof of (b)$\Longrightarrow$(c).

Step 3. Herein, we deal with (c)$\Longrightarrow$(a). Thus, we suppose that assertion (c) holds, and deduce the existence of a triplet $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ that belongs to $\mathcal{D}(S,\mathbb{F})\times\mathcal{I}^{o}_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^{1}_{loc}(\mathrm{Prog}(\mathbb{F}), P\otimes D)$ satisfying (3.73) and

$$Z^{\mathbb{G}}=\frac{(Z^{\mathbb{F}})^{\tau}}{\mathcal{E}(G_{-}^{-1}\bullet m)^{\tau}}\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D). \tag{3.75}$$

Then for any bounded $\varphi\in\mathcal{L}(S,\mathbb{F})$, $Z^{\mathbb{F}}\mathcal{E}(\varphi\bullet S)$ is an $\mathbb{F}$-supermartingale, and hence there exist $N\in\mathcal{M}_{0,loc}(\mathbb{F})$ and V is an $\mathbb{F}$-predictable and non decreasing process such that

$$Z^{\mathbb{F}}\mathcal{E}(\varphi\bullet S)=\mathcal{E}(N)\mathcal{E}(-V).$$

Therefore, by combining the above product with (3.75), we deduce that, for any bounded $\varphi \in \mathcal{L}(S, \mathbb{F})$,

$$Z^{\mathbb{G}}\mathcal{E}(\varphi \bullet S)^{\tau} \; = \frac{\mathcal{E}(N)^{\tau}}{\mathcal{E}(G_{-}^{-1} \bullet m)^{\tau}}\mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)} \bullet D)\mathcal{E}(-V)^{\tau}.$$

Thus, thanks to Corollary 3.2.1 which allows us to conclude that the process

$$(\mathcal{E}(N)^{\tau}/\mathcal{E}(G_{-}^{-1} \bullet m)^{\tau})\mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)} \bullet D)$$

is in fact a $\mathbb{G}$-local martingale, we deduce that $Z^{\mathbb{G}}\mathcal{E}(\varphi \bullet S)^{\tau}$ is a $\mathbb{G}$-supermartingale, for any bounded $\varphi \in \mathcal{L}(S, \mathbb{F})$. Then assertion (a) follows immediately from combining this with the fact that, for any bounded $\varphi^{\mathbb{G}}$ that belongs to $\mathcal{L}(S^{\tau}, \mathbb{G})$, there exists a bounded $\varphi^{\mathbb{F}} \in \mathcal{L}(S, \mathbb{F})$ such that $\varphi^{\mathbb{G}} = \varphi^{\mathbb{F}}$ on $]\!]0, \tau]\!]$ (see Lemma 3.1-(a)). This ends the proof of the theorem. □

3.6 Particular cases for supermartingale deflators

This section illustrates the results of Theorem 3.12 on several frequently studied cases. We will discuss the general Lévy market model, two popular cases of Lévy market model such as the case of jump-diffusion and Black-Scholes model, different cases for random times τ, two important volatility market frameworks and few more cases.

3.6.1 The case when all $\mathbb{F}$-martingales are continuous

In this subsection, we consider the following notations, defines in (3.29), that we recall here for the reader's convenience.

$$\begin{aligned} \overline{M} &= M^{\tau} - G_{-}^{-1} I_{]\!]0,\tau]\!]} \bullet \langle M, m \rangle^{\mathbb{F}}, \quad \text{for any } M \in \mathcal{M}_{loc}(\mathbb{F}), \\ \overline{N}^{\mathbb{G}} &= D - G_{-}^{-1} I_{]\!]0,\tau]\!]} \bullet D^{p,\mathbb{F}}. \end{aligned}$$

It is clear that both $\overline{M}$ and $\overline{N}^{\mathbb{G}}$ are the $\mathbb{G}$-local martingale parts in the Doob-Meyer decomposition, of M^τ and D respectively, under $\mathbb{G}$. In Lemma 3.7, we discuss the relations between $\overline{N}^{\mathbb{G}}$ and $N^{\mathbb{G}}$, $\overline{M}$ and $\mathcal{T}(M)$, and their properties.

Theorem 3.13: *Suppose that $G > 0$, all $\mathbb{F}$-martingales are continuous. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale and $V^{\mathbb{G}}$ be a nondecreasing and $\mathbb{G}$-predictable. Then the following assertions are equivalent.*

(a) $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)\mathcal{E}(-V^{\mathbb{G}})$ *is a deflator for* $(S^\tau, \mathbb{G})$ *if and only if there exists a unique* $\left(K^{\mathbb{F}}, V^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ *such that* $(K^{\mathbb{F}}, \varphi^{(o)}) \in \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ *belongs to* $L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$ *and* $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ *P-a.s.,* $V^{\mathbb{F}}$ *is nondecreasing with finite values and $\mathbb{F}$-predictable,* $\mathcal{E}(K^{\mathbb{F}})\mathcal{E}(-V^{\mathbb{F}}) \in \mathcal{D}(S, \mathbb{F})$,

$$-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \ P \otimes D^{o,\mathbb{F}} - a.e., \quad \varphi^{(pr)} > -\varphi^{(o)}\frac{G}{G_-} - 1, \ P \otimes D - a.e.,$$
$$V^{\mathbb{G}} := (V^{\mathbb{F}})^\tau, \ \textit{and}\ K^{\mathbb{G}} := \overline{K^{\mathbb{F}}} - G_-^{-1} \bullet \overline{m} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{3.76}$$

(b) $Z^{\mathbb{G}}$ *is a deflator for* $(S^\tau, \mathbb{G})$ *if and only if there exists a unique unique triplet* $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ *such that* $Z^{\mathbb{F}}$ *is a supermartingale deflator for* $(S, \mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)}$ *belongs to* $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D)$, *and satisfying the following* $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ *P-a.s.,*

$$-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \quad P \otimes D^{o,\mathbb{F}} - a.e. \quad \varphi^{(pr)} > -1, \quad P \otimes D - a.e.,$$
$$\textit{and}\ Z^{\mathbb{G}} = (Z^{\mathbb{F}})^\tau \frac{\mathcal{E}(\varphi^{(o)} \bullet \overline{N}^{\mathbb{G}})}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(pr)} \bullet D). \tag{3.77}$$

Proof. (a) The proof of assertion (a) follows from combining Theorem 3.12-(a) with the following three facts. Under the assumption that either τ avoids $\mathbb{F}$-stopping times or all $\mathbb{F}$-martingales are continuous, then, by Lemma 3.7, we can get these two equalities: $N^{\mathbb{G}} = \overline{N}$, and $\mathcal{T}(M) = \overline{M}$ for any $\mathbb{F}$-local martingale M. This proves assertion (a).

(b) The proof of assertion (b) follows immediately from combining assertion (a) and the fact that, under the assumption that when all $\mathbb{F}$-martingales are continuous, we have $[\overline{N}^{\mathbb{G}}, \overline{M}] = [\overline{N}^{\mathbb{G}}, M] \equiv 0$ for any $\mathbb{F}$-local martingale M. This fact, indeed, implies that

$$\mathcal{E}(\overline{M} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}}) = \mathcal{E}(\overline{M})\mathcal{E}(\varphi^{(o)} \bullet \overline{N}^{\mathbb{G}}),$$

for $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ with $1 + \Delta\overline{M} > 0$. This ends the proof of the theorem. □

Theorem 3.13 states results that work for both cases of τ avoids $\mathbb{F}$-stopping times and when all $\mathbb{F}$-martingales are continuous. The main difference between the two cases lies in the condition on the parameter $\varphi^{(p)}$. Indeed, for the case when τ avoids $\mathbb{F}$-stopping times, the condition (3.76) (or equivalently the inequalities in (3.77)) becomes $-1 < \varphi^{(p)}$ $P \otimes D^{p,\mathbb{F}}$-a.e. instead. This is due to $\widetilde{G} = G$ which follows from the avoidance property of τ. However, when all $\mathbb{F}$-martingales are continuous, (3.76) takes the form of $-\widetilde{G}/G < \varphi^{(p)} < \widetilde{G}/(\widetilde{G} - G)$ $P \otimes D^{p,\mathbb{F}}$-a.e., since in this case $\widetilde{G} = G_-$ and ${}^{p,\mathbb{F}}(G) = G$.

Corollary 3.13.1: Suppose τ is a pseudo-stopping time, and $G > 0$. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale and $V^{\mathbb{G}}$ be a nondecreasing and $\mathbb{G}$-predictable. Then $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)\mathcal{E}(-V^{\mathbb{G}})$ is a deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique triple $\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ that belongs to $\mathcal{D}(S, \mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\widetilde{\Omega}, \text{Prog}(\mathbb{F}), P \otimes D)$ and satisfying $\varphi^{(pr)} > -1, \quad P \otimes D - a.e., \; -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \quad P \otimes D^{o,\mathbb{F}}$-a.e.

$$\text{and} \quad Z^{\mathbb{G}} \;=\; (Z^{\mathbb{F}})^\tau \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)} \bullet D). \tag{3.78}$$

3.6.2 The exponential Lévy market model

In this section, we focus on the case of Lévy market model. Consider the Lévy market models given in Section 3.4.1. The stock price process, S, presented by $S_t = S_0\mathcal{E}(X)_t$, is a locally bounded Lévy process and set up by the stochastic differential equation (3.33). All elements are all the same way as in Section 3.4.1, where $\sigma > 0$ and μ are bounded adapted processes, W is a one-dimensional Brownian Motion,

$N(dt, dx)$ and is a Poisson random measure on $[0, T] \times \mathbb{R}/\{0\}$, $\widetilde{N}^{\mathbb{F}}$ is a compensated Poisson measure with Lévy measure (intensity) $F^X(dx)dt$ and A^X is an increasing and predictable process. Filtration $\mathbb{F}$ is generated by W and $\widetilde{N}^{\mathbb{F}}$. For more details about the application of Lévy market model in finance, we refer the reader to [25], [65], [66] and the references therein. Lévy model is a quasi-left-continuous process, so it has some extra nice features comparing to general semimartingale. For instance, For locally bounded Lévy model, the truncation function $h(x)$ in characteristics of semimartingales, vanishes.

For the parametrization of general deflators for $(S^\tau, \mathbb{G})$, we need to know its parametrization for model $(S, \mathbb{F})$ in advance. We present it in the following lemma.

Lemma 3.15: *Suppose S is given by (3.33) and Z is a positive $\mathbb{F}$-supermartingale. Then Z is a deflator for $(S, \mathbb{F})$ if and only if there exists a unique (β, f, g, K', V) such that (β, f, g, K') belongs to $(L^1_{loc}(W, \mathbb{F}) \times \mathcal{G}^1_{loc}(\mu, \mathbb{F}) \times \mathcal{H}^1_{loc}(\mu, \mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F})$, $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0$, $P \otimes \mu$-.a.e., $[K', S] = 0$, and a nondecreasing, RCLL and $\mathbb{F}$-predictable process V, such that*

$$Z := Z_0 \mathcal{E}(N) \mathcal{E}(-V), \quad \Delta V < 1, \quad \Delta N > -1, \quad N_0 = V_0 = 0, \tag{3.79}$$

where $N := \beta \bullet W + (f - 1) \star (\mu - \nu) + g \star \mu + K'$,

and the triple (β, f, v) satisfying the following inequality

$$\text{(i)} \int |x(f(x) - I_{|x|\le 1}) F^X(dx) < +\infty \quad P \otimes dt - a.e. \ ,$$

$$\text{(ii)} \ \big(\mu + \sigma\beta + \int x(f(x) - I_{|x|\le 1}) F^X(dx)\big) \bullet A^X \preceq \frac{1}{1 - \Delta V} \bullet V. \tag{3.80}$$

Proof. In the same spirit as Theorem 2.7, we have the following decomposition for any local martingale N with respect to a quasi-left-continuous process, X,

$$N = N_0 + \beta \bullet W + (f - 1) \star (\mu - \nu) + g \star \mu + K'.$$

Assume that Z is a positive $\mathbb{F}$-supermartingale. Therefore, by applying Lemma 3.14, Z is a deflator if and only if $(1-\Delta V)\bullet Y^{(\varphi)} - V$ is an $\mathbb{F}$-supermartingale, or equivalently, $\frac{1}{1-\Delta V}\bullet V \succeq (Y^{(\varphi)})^{p,\mathbb{F}}$, where $Y^{(\varphi)} := N + \varphi\bullet X + \varphi\bullet [X,N]$, and $\varphi \in \mathcal{L}(X,\mathbb{F})$. Thus, by computing the dual predictable projection of $Y^{(\varphi)}$, we get

$$(Y^{(\varphi)})^{p,\mathbb{F}} = \varphi\bullet (X+[X,N])^{P,\mathbb{F}} = \int_0^t \varphi_s\Big(\mu_s + \sigma_s\beta_s + \int x f_s(x) F^X(dx)\Big) dA_s^X.$$

Therefore, we obtain the following inequality

$$\begin{aligned}(Y^{(\varphi)})^{p,\mathbb{F}} - \frac{1}{1-\Delta V}\bullet V \ &= \int_0^t \varphi_s\Big(\mu_s + \sigma_s\beta_s + \int x f_s(x) F^X(dx)\Big) dA_s^X \\ &\quad - \frac{1}{1-\Delta V}\bullet V \preceq 0,\end{aligned}$$

if and only if the condition (3.80) holds. This ends the proof. □

Since the Lévy model is a quasi-left-continuous and the process A_t is continuous, there is no loss of generality in defining $A_t = t$. The main result in this section is given in the following theorem.

Theorem 3.14: *Suppose S is given by (3.33), $G > 0$, and let $Z^{\mathbb{G}}$ be a $\mathbb{G}$-semimartingale. Then the following are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$ (i.e. $Z^{\mathbb{G}} \in \mathcal{D}(S^\tau,\mathbb{G})$).*

(b) *There exists a unique $\big(\beta, f, g, K', V^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\big)$ such that(β, f, g, K') belongs to $(L^1_{loc}(W,\mathbb{F}) \times \mathcal{G}^1_{loc}(\mu,\mathbb{F}) \times \mathcal{H}^1_{loc}(\mu,\mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P\otimes D)$, and a nondecreasing, RCLL and $\mathbb{F}$-predictable process $V^{\mathbb{F}}$, satisfying*

$$M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0,\ P\otimes\mu,\ \text{-a.e.},\ \mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0,\ P\text{-a.s.}\ ,$$

$$\varphi^{(pr)} > -\big[G_-\Big(f(\Delta X)+g(\Delta X))I_{\{\Delta X\neq 0\}} + (1+\Delta K')I_{\{\Delta X=0\}}\Big) + \varphi^{(o)}G\big]/\tilde{G},\ \ P\otimes D\text{-}a.e.,$$

$$-\frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{G} < \varphi^{(o)} < \frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{\Delta D^{o,\mathbb{F}}} \quad P\otimes D^{o,\mathbb{F}}\text{-}a.e.\ \ on\ \{\Delta X\neq 0\},$$

$$-\frac{(1+\Delta K')G_-}{G} < \varphi^{(o)} < \frac{(1+\Delta K')G_-}{\Delta D^{o,\mathbb{F}}} \qquad P\otimes D^{o,\mathbb{F}}\text{-}a.e.\ \ on\ \{\Delta X = 0\},$$

$$\int |x(f(x)-I_{|x|\leq 1})F^X(dx) < +\infty \quad P\otimes dt - a.e.\ ,$$

$$and\ \ \big(\mu+\sigma\beta+\int x(f(x)-I_{|x|\leq 1})F^X(dx)\big)\bullet A^X \preceq \frac{1}{1-\Delta V^{\mathbb{F}}}\bullet V^{\mathbb{F}},\ \ \Delta V^{\mathbb{F}}<1, \quad f+g>0,$$

$$Z^{\mathbb{G}} = Z_0^{\mathbb{G}}\mathcal{E}(K^{\mathbb{G}})\mathcal{E}(-V^{\mathbb{F}})^{\tau}, \tag{3.81}$$

$$K^{\mathbb{G}} = \mathcal{T}(\beta\bullet W + (f-1)\star(\mu-\nu) + g\star\mu + K') - G_-^{-1}\bullet\mathcal{T}(m) + \varphi^{(o)}\bullet N^{\mathbb{G}} + \varphi^{(pr)}\bullet D.$$

(c) *There exists a unique* $\big(\beta, f, g, K', V^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\big)$ *such that*(β, f, g, K') *belongs to* $(L^1_{loc}(W,\mathbb{F}) \times \mathcal{G}^1_{loc}(\mu,\mathbb{F}) \times \mathcal{H}^1_{loc}(\mu,\mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F}),\ \varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G}),\ \varphi^{(pr)}$ *belongs to* $L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P\otimes D)$*,* $V^{\mathbb{F}}$ *is an* $\mathbb{F}$*-predictable and nondecreasing process, satisfying* $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0$*,* $P\otimes\mu$*-.a.e.,* $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau]I_{\{\tau<+\infty\}} = 0$ P*-a.s.,*

$$Z^{\mathbb{G}} = \frac{\mathcal{E}(\beta\bullet W + (f-1)\star(\mu-\nu) + g\star\mu + K')^{\tau}\mathcal{E}(-V^{\mathbb{F}})^{\tau}}{\mathcal{E}(G_-^{-1}\bullet m)^{\tau}}\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D).$$

$$and\ \ \varphi^{(pr)} > -1, \quad P\otimes D\text{-}a.e., \quad -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{\Delta D^{o,\mathbb{F}}} \quad P\otimes D^{o,\mathbb{F}}\text{-}a.e.,$$

$$\int |x(f(x)-I_{|x|\leq 1})F^X(dx) < +\infty \quad P\otimes dt - a.e.\ ,$$

$$\big(\mu+\sigma\beta+\int x(f(x)-I_{|x|\leq 1})F^X(dx)\big)\bullet A^X \preceq \frac{1}{1-\Delta V^{\mathbb{F}}}\bullet V^{\mathbb{F}}, \Delta V^{\mathbb{F}}<1, \quad f+g>0.$$

Proof. The proof is immediately follows from combing Theorem 3.12, Theorem 3.6 and lemma 3.15 with the similar discussion of next subsection (i.e. the Jump-diffusion model) and it is omitted here. □

3.6.2.1 Jump-diffusion framework

This subsection focuses on the important case of jump-diffusion framework for the market model $(S, \mathbb{F}, P)$ defined in Section 3.4.1.1. Herein, we suppose that a onedimentional Brownian motion W and a Poisson process N with intensity $\lambda > 0$ are defined on the probability space $(\Omega, \mathcal{F}, P)$, and the filtration $\mathbb{F}$ is the completed and right continuous filtration generated by W and $N^{\mathbb{F}}$. Then the stock price process is given by the following dynamics

$$S_t = S_0\mathcal{E}(X)_t, \quad X_t = \int_0^t \sigma_s dW_s + \int_0^t \zeta_s dN_s^{\mathbb{F}} + \int_0^t \mu_s ds, \qquad N_t^{\mathbb{F}} := N_t - \lambda t. \quad (3.82)$$

All elements are all the same way as in Section 3.4.1.1, where a constant $\delta \in (0, +\infty)$ such that μ, σ, and ζ are bounded $\mathbb{F}$-adapted processes and we also assume that $\sigma > 0,\ \zeta > -1,\ \sigma + |\zeta| \geq \delta,\ P \otimes dt$-a.e. . $\mathbb{F}$-martingale, m, defines in Lemma 3.10 and it has following decomposition $G_-^{-1} \bullet m = \varphi^{(m)} \bullet W + (\psi^{(m)} - 1) \bullet N^{\mathbb{F}}$.

Theorem 3.15: *Suppose S given by (3.82) and $G > 0$. Let $Z^{\mathbb{G}}$ be a $\mathbb{G}$-semimartingale. Then the following are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$ (i.e. $Z^{\mathbb{G}} \in \mathcal{D}(S^\tau, \mathbb{G})$).*

(b) *There exists a unique $\left(\psi_1, \psi_2, V^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $(\psi_1, \psi_2) \in L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F})$, $V^{\mathbb{F}}$ is a nondecreasing, RCLL and $\mathbb{F}$-predictable process, $\varphi^{(o)} \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$, $\varphi^{(pr)} \in L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D)$ such that $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s., satisfying the following*

$$\mathcal{E}(\psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}}) \exp(-V^{\mathbb{F}}) \in \mathcal{D}(S, \mathbb{F}), \quad (3.83)$$

$$\varphi^{(pr)} > -\Big[\frac{\psi_2}{\psi^{(m)}} + \frac{\varphi^{(o)} G}{\psi^{(m)} G_-}\Big],\ \textit{and}\ -\frac{\psi_2 G_-}{G} < \varphi^{(o)} < \frac{\psi_2 G_-}{\Delta D^{o,\mathbb{F}}} \quad P \otimes D^{o,\mathbb{F}}\textit{-a.e. on } \{\Delta N \neq 0\},$$

$$\varphi^{(pr)} > -\left[1 + \frac{\varphi^{(o)} G}{G_-}\right],\ \textit{and}\ -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G} \quad P \otimes D^{o,\mathbb{F}}\textit{-a.e. on } \{\Delta N = 0\}, \quad (3.84)$$

$$\int_0^t \left((\psi_1(s))^2 + |\psi_{2(s)}|\right) ds < +\infty \quad P\textit{-a.s. for all } t \geq 0$$

$$\textit{and} \int_0^\cdot (\mu + \psi_1\sigma + (\psi_2 - 1)\zeta\lambda) ds \preceq \frac{1}{1 - \Delta V^{\mathbb{F}}} \bullet V^{\mathbb{F}},\ \Delta V^{\mathbb{F}} < 1,\ \psi_2 > 0,\ P \otimes dt - a.e. \quad (3.85)$$

$$\textit{such that } Z^{\mathbb{G}} = Z_0^{\mathbb{G}} \mathcal{E}(K^{\mathbb{G}}) \exp(-V^{\mathbb{F}})^\tau, \quad (3.86)$$

$$K^{\mathbb{G}} = \psi_1 \bullet \mathcal{T}(W) + (\psi_2 - 1) \bullet \mathcal{T}(N^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$

(c) *There exists* $(\psi_1, \psi_2, \varphi^{(o)}, \varphi^{(pr)})$ *that belongs to*

$$L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^{\mathbb{F}}, \mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\widetilde{\Omega}, Prog(\mathbb{F}), P \otimes D),$$

and $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ *P-a.s., and a nondecreasing, RCLL and* $\mathbb{H}$*-predictable process* $V^{\mathbb{F}}$*, satisfying the following*

$$Z^{\mathbb{G}} = \frac{\mathcal{E}(\psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}} - V^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet N^{\mathbb{G}}) \mathcal{E}(\varphi^{(pr)} \bullet D) \tag{3.87}$$

$$\varphi^{(pr)} > -1 \quad P \otimes D\text{-a.e. }, \quad -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{\Delta D^{o,\mathbb{F}}} \quad P \otimes D^{o,\mathbb{F}}\text{-a.e. }, \tag{3.88}$$

$$\int_0^t \left((\psi_1(s))^2 + |\psi_{2(s)}|\right) ds < +\infty \quad P\text{-a.s. for all } t \geq 0$$

$$\text{and} \quad \int_0^{\cdot} (\mu + \psi_1 \sigma + (\psi_2 - 1)\zeta\lambda) ds \preceq \frac{1}{1 - \Delta V^{\mathbb{F}}} \bullet V^{\mathbb{F}}, \ \Delta V^{\mathbb{F}} < 1, \quad \psi_2 > 0, \quad P \otimes dt - a.e.$$

Proof. In Lemma 3.15, define $(\beta, f) = (\psi_1, \psi_2)$, where $f(x) = f(1)$. Therefore, $\mathcal{E}(N)\mathcal{E}(V^{\mathbb{F}}) \in \mathcal{D}(S, \mathbb{F})$ if and only if $N := \psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}}$, and $V^{\mathbb{F}} = V$,

$$\int_0^{\cdot} (\mu + \psi_1 \sigma + (\psi_2 - 1)\zeta\lambda) ds \preceq \frac{1}{1 - \Delta V} \bullet V, \Delta V < 1, \quad \psi_2 > 0, \quad P \otimes dt\text{-a.e. .}$$

Then by Theorem 3.12 and above explanation, there exists a unique $\left(\psi_1, \psi_2, \varphi^{(o)}, \varphi^{(pr)}\right)$ such that $K^{\mathbb{G}}$ has the following form

$$K^{\mathbb{G}} = \psi_1 \bullet \mathcal{T}(W) + (\psi_2 - 1) \bullet \mathcal{T}(N^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$

Moreover, the inequalities (3.85)-(3.88) are fulfilled. It ends the proof of assertion (b). For the remaining parts of the proof, in Theorem 3.12 assertion (c), define $Z^{\mathbb{F}} := \mathcal{E}(\psi_1 \bullet W + (\psi_2 - 1) \bullet N^{\mathbb{F}}) \exp(-V^{\mathbb{F}})$. Then, by using the Yor's formula, we get (3.87). This complete the proof of Theorem. □

3.6.2.2 Black-Scholes market model

In this subsection we focus on the case of Black-Scholes market model, as it defines in subsection 3.4.1.2. The financial market is presented by following stochastic differential equation

$$S_t = S_0\mathcal{E}(X)_t, \quad X_t := \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds, \tag{3.89}$$

where W is a one-dimensional Brownian motion. For this model, we have the following parametrization of supermartingale deflators.

Theorem 3.16: *Suppose S is given by (3.89) and $G > 0$. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique $(\beta, \varphi^{(o)}, \varphi^{(pr)}) \in (L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D))$ such that $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s., and a nondecreasing, RCLL and $\mathbb{F}$-predictable process $V^{\mathbb{F}}$,*

$$\begin{aligned}
&\mathcal{E}(\beta \bullet W)\exp(-V^{\mathbb{F}}) \in \mathcal{D}(S, \mathbb{F}), \ \textit{and} \int \mu + \sigma\beta dt \preceq (1 - \Delta V^{\mathbb{F}})^{-1} \bullet V^{\mathbb{F}}, \ \Delta V^{\mathbb{F}} < 1 \qquad (3.90)\\
&-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - \ G}, \ P \otimes D^{o,\mathbb{F}} - a.e., \ \varphi^{(pr)} > -\varphi^{(o)}\frac{G}{G_-} - 1, \ P \otimes D - a.e.,\\
&Z^{\mathbb{G}} = Z^{\mathbb{G}}_0 \mathcal{E}(K^{\mathbb{G}})\exp(-V^{\mathbb{F}})^\tau, \ \textit{and } K^{\mathbb{G}} = -(\frac{\mu}{\sigma} + \varphi^{(m)}) \bullet \overline{W} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D.
\end{aligned}$$

(b) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique triplet*

$$\left(Z^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right) \ \textit{that belongs to } \mathcal{D}(S, \mathbb{F}) \times L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P \otimes D)$$

such that $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0$ P-a.s.,

$$\begin{aligned}
-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - \ G}, \quad & P \otimes D^{o,\mathbb{F}} - a.e. \qquad \varphi^{(pr)} > -1, \quad P \otimes D - a.e.,\\
Z^{\mathbb{G}} \ = \ & \frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\varphi^{(o)} \bullet \overline{N}^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)} \bullet D).
\end{aligned} \tag{3.91}$$

Proof. In Lemma 3.15, define $(\beta, f) = (\beta, 0)$. Therefore, $\mathcal{E}(N)\mathcal{E}(V^{\mathbb{F}}) \in \mathcal{D}(S, \mathbb{F})$ if and only if $N := \beta \bullet W$, $V^{\mathbb{F}} := V$, and $\int \mu + \sigma\beta dt \preceq (1 - \Delta V)^{-1} \bullet V$, $\Delta V < 1$. The decompositions (3.90)-(3.91) for $Z^{\mathbb{G}}$ and inequalities (3.90)-(3.91) follow from Theorem 3.13 due to the fact that all $\mathbb{F}$-local martingale are continuous. □

Remark. It clear that $\mathbb{F}$-deflator in the multiplicative representation can be replace by $Z^{\mathbb{F}} = \mathcal{E}(\beta \bullet W - V^{\mathbb{F}})$, where the condition (3.90) holds.

3.6.3 Volatility market models

In this subsection we focus on the case of Volatility Models model, which has huge application in financial industry. Corrected Stein and Stein Model and Barndorff-Nielsen Shephard Model are well know and most interested in Finance. We evaluate parametrization of these two models in this subsection.

3.6.3.1 Corrected Stein and Stein Model

In this subsection we focus on the case of corrected Stein and Stein financial market. Consider the corrected Stein and Stein market model given in Section 3.4.3.1. The stock price process, S, presented by $S_t = S_0\mathcal{E}(X)_t$, is represented by the following stochastic differential equations

$$\begin{aligned} X_t &= X_0 + \int_0^t V_s\sigma_s dW_s^{(1)} + \int_0^t \mu V_s^2 dt \\ V_t &= V_0 + \int_0^t (m - aV_s)dt + \alpha_s dW_s^{(2)} \end{aligned} \tag{3.92}$$

$W^{(i)}, i = 1, 2$, is a one-dimensional Brownian motion with the filtration is generated by the processes $W^{(i)}, i = 1, 2$, and all coefficients are all the same way as in Section 3.4.3.1. Any $\mathbb{F}$-local martingale M represents same as equation (3.49) (i.e. $M = M_0 + h^1 \bullet W^1 + h^2 \bullet W^2$). For this model, we have the following decomposition of $\mathbb{F}$-deflator.

Lemma 3.16: *Suppose S is given by (3.92) and Z be a positive $\mathbb{F}$-supermartingale. Then Z is a deflator for $(S,\mathbb{F})$ if and only if there exists a unique triple $(\beta^{(1)},\beta^{(2)},v)$ such that $(\beta^{(1)},\beta^{(2)})$ are $\mathbb{F}$-progressively measurable processes such that*

$$\int_0^T [(\beta_t^1)^2+(\beta_t^2)^2]dt<\infty. \;\; P-a.s. \;,$$

and a nondecreasing, RCLL and $\mathbb{H}$-predictable process $V^{\mathbb{F}}$, such that

$$Z:=Z_0\mathcal{E}(N)\mathcal{E}(-V^{\mathbb{F}}),\;\; N:=\beta^{(1)}\bullet W^{(1)}+\beta^{(2)}\bullet W^{(2)}, \tag{3.93}$$

and the following inequality holds

$$\int_0^t \varphi_s\Big(\sigma_s\beta_s^{(1)}+\sigma_s\beta_s^{(2)}\rho+\mu V_s^2\Big)dt\preceq(1-\Delta V^{\mathbb{F}})^{-1}\bullet V^{\mathbb{F}},\quad \Delta V^{\mathbb{F}}<1. \tag{3.94}$$

Proof. The representation (3.93) for process N, follows from the equation (3.49). Therefore by applying Lemma 3.14, $Z\mathcal{E}(\varphi\bullet X)$ is an $\mathbb{F}$-supermartingale, if and only if, $\frac{1}{1-\Delta V^{\mathbb{F}}}\bullet V^{\mathbb{F}}\succeq(Y^{(\varphi)})^{p,\mathbb{F}}$ is an $\mathbb{F}$-supermartingale, where $Y^{(\varphi)}:=N+\varphi\bullet X+\varphi\bullet[X,N]$, and $\varphi\in\mathcal{L}(X,\mathbb{F})$. Therefore,

$$(Y^{(\varphi)})^{p,\mathbb{F}}-\frac{1}{1-\Delta V}\bullet V\;=\int_0^t\varphi_s\Big(\sigma_s\beta_s^{(1)}+\sigma_s\beta_s^{(2)}\rho+\mu V_s^2\Big)dt-(1-\Delta V^{\mathbb{F}})^{-1}\bullet V^{\mathbb{F}}\preceq 0,$$

which is equivalent to the condition (3.94). This is the end of the proof. □

For the corrected Stein and Stein financial market model, we have the following parametrization for deflators in $(S^\tau,\mathbb{G})$.

Theorem 3.17: *Suppose S given by (3.92) and $G>0$. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau,\mathbb{G})$ if and only if there exists a unique there exists a unique $(\beta^{(1)},\beta^{(2)},V^{\mathbb{F}},\varphi^{(o)},\varphi^{(pr)})$ and a nondecreasing, RCLL and $\mathbb{F}$-predictable process $V^{\mathbb{F}}$, such that $(\beta^{(1)},\beta^{(2)},\varphi^{(o)})$ belongs to $L^1_{loc}(W^1,\mathbb{F})\times L^1_{loc}(W^2,\mathbb{F})\times$*

$L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G})$ and $\varphi^{(pr)}$ belongs to $L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P\otimes D)$, satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s.,

$$\begin{aligned}
&\mathcal{E}(\beta^{(1)} \bullet W^{(1)} + \beta^{(2)} \bullet W^{(2)})\mathcal{E}(-V^{\mathbb{F}}) \in \mathcal{D}(S, \mathbb{F}), && (3.95)\\
&-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \ P\otimes D^{o,\mathbb{F}} - a.e., \quad \varphi^{(pr)} > -\varphi^{(o)}\frac{G}{G_-} - 1, \ P\otimes D - a.e., &&\\
&Z^{\mathbb{G}} = Z^{\mathbb{G}}_0 \mathcal{E}(K^{\mathbb{G}}) \exp{(-V^{\mathbb{F}})}^{\tau}, && (3.96)\\
&\int_0^t \varphi_s\Big(\sigma_s\beta^{(1)}_s + \sigma_s\beta^{(2)}_s\rho + \mu V_s^2\Big)dt \preceq (1-\Delta V^{\mathbb{F}})^{-1} \bullet V^{\mathbb{F}}, \quad \Delta V^{\mathbb{F}} < 1, &&\\
&\text{and } K^{\mathbb{G}} = \beta^{(1)} \bullet (\overline{W}^{(1)}) + \beta^{(2)} \bullet (\overline{W}^{(2)}) - G_-^{-1} \bullet \overline{m} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D. && (3.97)
\end{aligned}$$

(b) *$Z^{\mathbb{G}}$ is a deflator for $(S^\tau, \mathbb{G})$ if and only if there exists a unique $(\beta^{(1)}, \beta^{(2)}, \varphi^{(o)}, \varphi^{(pr)})$ belongs to $L^1_{loc}(W^1, \mathbb{F}) \times L^1_{loc}(W^1, \mathbb{F}) \times L^1_{loc}(\overline{N}^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}(\mathrm{Prog}(\mathbb{F}), P\otimes D)$ and a nondecreasing, RCLL and $\mathbb{F}$-predictable process $V^{\mathbb{F}}$, satisfying $\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau<+\infty\}} = 0$ P-a.s.,*

$$\begin{aligned}
&-\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{G_- - G}, \quad P\otimes D^{o,\mathbb{F}} - a.e. \qquad \varphi^{(pr)} > -1, \quad P\otimes D - a.e., &&\\
&\int_0^t \varphi_s\Big(\sigma_s\beta^{(1)}_s + \sigma_s\beta^{(2)}_s\rho + \mu V_s^2\Big)dt \preceq (1-\Delta V^{\mathbb{F}})^{-1} \bullet V^{\mathbb{F}}, \quad \Delta V^{\mathbb{F}} < 1, &&\\
&\text{and } Z^{\mathbb{G}} = (\beta^{(1)} \bullet W^{(1)} + \beta^{(2)} \bullet W^{(2)} - V^{\mathbb{F}})^{\tau} \frac{\mathcal{E}(\varphi^{(o)} \bullet \overline{N}^{\mathbb{G}})}{\mathcal{E}(G_-^{-1} \bullet m)^{\tau}} \mathcal{E}(\varphi^{(pr)} \bullet D). && (3.98)
\end{aligned}$$

Proof. On one hand, by Lemma 3.16, we have

$$N = N_0 + \int_0^t \beta^{(1)}_s dW^{(1)}_s + \int_0^t \beta^{(2)}_s dW^{(2)}_s, \ V^{\mathbb{F}} = V$$

On the other hand, since all $\mathbb{F}$-martingales are continuous, thanks to Theorem 3.13, there exists a unique parametrization for general deflator $Z^{\mathbb{G}}$ such that $Z^{\mathbb{G}} = Z^{\mathbb{G}}_0 \mathcal{E}(K^{\mathbb{G}})\mathcal{E}(-(V)^\tau)$, and $K^{\mathbb{G}} = \overline{K^{\mathbb{F}}} - G_-^{-1} \bullet \overline{m} + \varphi^{(o)} \bullet \overline{N}^{\mathbb{G}} + \varphi^{(pr)} \bullet D$. Therefore, by putting the pair $(K^{\mathbb{F}}, V^{\mathbb{F}})$ in (N, V) respectively, and also considering the following

$$\overline{(\beta^{(1)} \bullet W^{(1)} + \beta^{(2)} \bullet W^{(2)})} = \beta^{(1)} \bullet (\overline{W}^{(1)}) + \beta^{(2)} \bullet (\overline{W}^{(2)}),$$

the proof of assertion (a) follows immediately. The remaining parts of proof is easy to see with same explanation. □

3.6.3.2 Barndorff-Nielsen Shephard Model

Here, we turn to the Barndorff-Nielsen Shephard financial market which is presented in Section 3.4.3.2. The filtration is generated by a one dimensional Lévy process Y, where we have $Y = Y^c + \widetilde{Y}^d$. The stock price process is an exponential of a Lévy process such that $S_t = S_0 e^{(X)_t}$ and follows these stochastic differential equations

$$\begin{aligned} dX_s &= (\mu + \xi\sigma_s^2)ds + \sigma_s dY_s^c + d(\rho x \star \widetilde{\mu}_Y)_s \\ d\sigma_s^2 &= -\lambda\sigma_s^2 dt + d(x \star \widetilde{\mu}_Y)_s \qquad (3.99) \\ \frac{dS_t}{S_{t_-}} &= \alpha_t dt + \sigma_t dY_t^c + d(e^{\rho x} - 1) \star (\widetilde{\mu} - \widetilde{\nu})_t, \qquad (3.100) \end{aligned}$$

where all the coefficients B, ρ, λ and μ are real constants with $\rho < 0$ and $\lambda > 0$, and

$$\alpha = \mu + \sigma_t^2(\xi + \frac{1}{2}) + \int (e^{\rho x} - 1)\widetilde{F}(dx), \quad S_t^c = S_{t_-}\sigma_t \bullet Y_t^c, \quad \Delta S_t = S_{t_-}(e^{\rho\Delta\sigma_t^2} - 1),$$

$$b_t = S_{t_-}\alpha, \quad c_t = S_{t_-}^2\sigma_t^2, \quad \nu^S(dt \times dx) = F_t^S(dx)dt.$$

By the above-mentioned decomposition for Barndorff-Nielsen Shephard process, we parametrize supermartingale deflators for model $(S, \mathbb{F})$ in the following lemma. The model is a quasi-left-continuous and the process A_t is continuous. Thus, there is no loss of generality on defining $A_t = t$.

Lemma 3.17: *Suppose S is given by (3.99) and Z be a positive $\mathbb{F}$-supermartingale. Then Z is a deflator for $(S, \mathbb{F})$ if and only if there exists a unique (β, f, g, K', V) such that (β, f, g, K') belongs to $(L_{loc}^1(Y^c, \mathbb{F}) \times \mathcal{G}_{loc}^1(\mu, \mathbb{F}) \times \mathcal{H}_{loc}^1(\mu, \mathbb{F}) \times \mathcal{M}_{0,loc}(\mathbb{F})$, $[K', Y] = 0$, and a nondecreasing, RCLL and $\mathbb{F}$-predictable process V, such that*

$M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0,\ P\otimes\mu\text{-}.a.e.,$

$$\begin{aligned} Z &:= Z_0\mathcal{E}(N)\mathcal{E}(-V), \quad \Delta V < 1\ \Delta N > -1,\\ N &:= N_0^{\mathbb{F}} + \beta_t \bullet Y^c + (f-1)\star(\widetilde{\mu}-\widetilde{\nu})_t + g\star\widetilde{\mu} + K'\ , \end{aligned} \tag{3.101}$$

and the following inequalities hold

$$\text{(i)} \int |(e^{\rho x}-1)f(S_-(e^{\rho x}-1))|\widetilde{F}(dx) < +\infty \quad P\otimes dt - a.e.\ , \tag{3.102}$$

$$\text{(ii)} \int_0^{\cdot} \varphi_s\Big(\mu_s + \sigma_s(\sigma_s\xi_s + \frac{\sigma_s}{2} + \beta_s)\Big)ds + \int (e^{\rho x}-1)f(S_-(e^{\rho x}-1))\widetilde{F}(dx) \preceq (1-\Delta V)^{-1}\bullet V.$$

Proof. Thanks to Lemma 3.13, we have the representation (3.101) for process N. Therefore by applying Lemma 3.14, Z is a positive $\mathbb{F}$-supermartingale and $Z\mathcal{E}(\varphi\bullet X)$ is an $\mathbb{F}$-supermartingale, if and only if

$$\begin{aligned} (Y^{(\varphi)})^{p,\mathbb{F}} - \frac{1}{1-\Delta V}\bullet V \ &= \int_0^{\cdot} \varphi_s\Big(\mu_s + \sigma_s(\sigma_s\xi_s + \frac{\sigma_s}{2} + \beta_s)\Big)ds \\ &+ \int_0^{\cdot} \varphi_s\Big(\int (e^{\rho x}-1)f_s(e^{\rho x}-1)\widetilde{F}(dx)\Big) - (1-\Delta V)^{-1}\bullet V \preceq 0, \end{aligned}$$

which is equivalent to the condition (3.102). This is the end of proof. □

The main result in this subsection is given in the following theorem.

Theorem 3.18: *Suppose S given by (3.99) and $G > 0$. Let $K^{\mathbb{G}}$ be a $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) $Z^{\mathbb{G}}$ *is a deflator for* $(S^\tau, \mathbb{G})$ *if and only if there exists a unique* $\big(\beta, f, g, K', v, \varphi^{(o)}, \varphi^{(pr)}\big)$ *and a nondecreasing, RCLL and* $\mathbb{F}$*-predictable process* $V^{\mathbb{F}}$*, such that,* $(\beta, f, g, K', \varphi^{(o)}, \varphi^{(pr)})$ *belongs to*

$$L^1_{loc}(Y^c,\mathbb{F})\times\mathcal{G}^1_{loc}(\mu,\mathbb{F})\times\mathcal{H}^1_{loc}(\mu,\mathbb{F})\times\mathcal{M}_{0,loc}(\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\widetilde{\Omega}, Prog(\mathbb{F}), P\otimes D),$$

satisfying the following $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{F})) = 0,\ P\otimes\mu\text{-}.a.e.,\ \mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau]I_{\{\tau<+\infty\}} = 0$

P-a.s., $\Delta V^{\mathbb{F}} < 1,\ f+g>0$,

$$\varphi^{(pr)} > -\big[G_-\Big(f(\Delta X)+g(\Delta X))I_{\{\Delta X\neq 0\}}+(1+\Delta K')I_{\{\Delta X=0\}}\Big)+\varphi^{(o)}G\big]/\tilde{G},\ P\otimes D\text{-a.e.},$$
$$-\frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{G} < \varphi^{(o)} < \frac{\Big(f(\Delta X)+g(\Delta X)\Big)G_-}{\Delta D^{o,\mathbb{F}}}\quad P\otimes D^{o,\mathbb{F}}\text{-a.e. on } \{\Delta X\neq 0\},$$
$$-\frac{(1+\Delta K')G_-}{G} < \varphi^{(o)} < \frac{(1+\Delta K')G_-}{\Delta D^{o,\mathbb{F}}}\qquad P\otimes D^{o,\mathbb{F}}\text{-a.e. on } \{\Delta X = 0\},$$
$$\int |(e^{\rho x}-1)f(S_-(e^{\rho x}-1))|\widetilde{F}(dx) < +\infty\quad P\otimes dt-a.e.\ , \tag{3.103}$$
$$\int_0^{\cdot}\varphi_s\Big(\mu_s+\sigma_s(\sigma_s\xi_s+\frac{\sigma_s}{2}+\beta_s)\Big)ds+\int(e^{\rho x}-1)f(e^{\rho x}-1)\widetilde{F}(dx)\preceq(1-\Delta V^{\mathbb{F}})^{-1}\bullet V^{\mathbb{F}},$$
$$Z^{\mathbb{G}} = Z_0^{\mathbb{G}}\mathcal{E}(K^{\mathbb{G}})\exp(V^{\mathbb{F}})^{\tau}, \tag{3.104}$$
$$K^{\mathbb{G}} = \mathcal{T}(\beta\bullet Y^c+(f-1)\star(\widetilde{\mu}-\widetilde{\nu})+g\star\widetilde{\mu}+K')-G_-^{-1}\bullet\mathcal{T}(m)+\varphi^{(o)}\bullet N^{\mathbb{G}}+\varphi^{(pr)}\bullet D.$$

(b) $Z^{\mathbb{G}}$ *is a deflator for* $(S^{\tau},\mathbb{G})$ *if and only if there exists a unique* $\big(\beta,f,g,K',V^{\mathbb{F}},\varphi^{(o)},\varphi^{(pr)}\big)$ *such that* $V^{\mathbb{F}}$ *is an* $\mathbb{F}$*-predictable and nondecreasing process,* $(\beta,f,g,K',\varphi^{(o)},\varphi^{(pr)})\in L^1_{loc}(Y^c,\mathbb{F})\times\mathcal{G}^1_{loc}(\mu,\mathbb{F})\times\mathcal{H}^1_{loc}(\mu,\mathbb{F})\times\mathcal{M}_{0,loc}(\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\widetilde{\Omega},Prog(\mathbb{F}),P\otimes D)$*, such that* $M^P_{\mu}(g\mid\widetilde{\mathcal{P}}(\mathbb{F}))=0,\ P\otimes\mu\text{-.a.e.},\ \mathbb{E}[\varphi^{(pr)}_{\tau}\mid\mathcal{F}_{\tau}]I_{\{\tau<+\infty\}}=0\ P\text{-a.s.},$ $\Delta V^{\mathbb{F}}<1,\ f+g>0,$

$$Z^{\mathbb{G}} = \frac{\mathcal{E}(\beta\bullet Y^c+(f-1)\star(\widetilde{\mu}-\widetilde{\nu})+g\star\widetilde{\mu}+K'-V^{\mathbb{F}})^{\tau}}{\mathcal{E}(G_-^{-1}\bullet m)^{\tau}}\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D),$$
$$\varphi^{(pr)} > -1,\quad P\otimes D\text{-a.e.},\quad -\frac{G_-}{G} < \varphi^{(o)} < \frac{G_-}{\Delta D^{o,\mathbb{F}}}\quad P\otimes D^{o,\mathbb{F}}\text{-a.e.}, \tag{3.105}$$
$$\int |(e^{\rho x}-1)f(S_-(e^{\rho x}-1))|\widetilde{F}(dx) < +\infty\quad P\otimes dt-a.e.\ , \tag{3.106}$$
$$\text{and}\int_0^{\cdot}\varphi_s\Big(\mu_s+\sigma_s(\sigma_s\xi_s+\frac{\sigma_s}{2}+\beta_s)\Big)ds+\int(e^{\rho x}-1)f(e^{\rho x}-1)\widetilde{F}(dx)\preceq(1-\Delta V^{\mathbb{F}})^{-1}\bullet V^{\mathbb{F}}.$$

Proof. On the one hand, in Lemma 3.17, we calculate the corresponding $\mathbb{F}$-supermartingale deflator $Z=Z_0\mathcal{E}(N)\mathcal{E}(-V)$ for Barndorff-Nielsen Shephard financial market model, where $N:=N_0^{\mathbb{F}}+\beta\bullet Y^c+(f-1)\star(\widetilde{\mu}-\widetilde{\nu})+g\star\widetilde{\mu}+K'$, satisfying condition (3.102). On the other hand, thanks to Theorem 3.12, there exists a unique $\big(\beta,f,V^{\mathbb{F}},\varphi^{(o)},\varphi^{(pr)}\big)$

such that $K^{\mathbb{G}}$ has the following decomposition

$$Z^{\mathbb{G}} = Z_0^{\mathbb{G}}\mathcal{E}(K^{\mathbb{G}})\mathcal{E}(-(V^{\mathbb{F}})^{\tau}), \quad K^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_{-}^{-1}\bullet\mathcal{T}(m) + \varphi^{(o)}\bullet N^{\mathbb{G}} + \varphi^{(pr)}\bullet D.$$

Therefore, by substituting pair $(K^{\mathbb{F}}, V^{\mathbb{F}})$ with N and V respectively, and the fact that

$$\mathcal{E}(-V^{\mathbb{F}}) = \exp(-V^{\mathbb{F}}),$$

the proof follows immediately. □

3.6.4 The discrete-time market models

This subsection considers the setting where the time is $n = 0, 1, 2, ..., T$. Our intial market model takes the form $\big(S = (S_n), \mathbb{F} := (\mathcal{F}_n)_{n\geq 0}\big)$. To this initial model, we consider random variable τ that takes values in $\{0, 1, 2, ..., T\}$. In this setting, our principal goals reside in describing as explicit as possible the set of all deflators (i.e. the dual set of all wealth processes) for the stopped model S^{τ} and the discrete-time version of the representation of $\mathbb{G}$-martingales.

Here, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_n)_{n\geq 0}, P)$. The trading time is finite and countable. Thus, the agent who has access to $\mathbb{F}$, can only get information about τ through the following survival probabilities,

$$G_n := \sum_{k=n+1}^{T} P[\tau = k|\mathcal{F}_n],\ \widetilde{G}_n := \sum_{k=n}^{T} P[\tau = k|\mathcal{F}_n],\ n = 0, ..., T..$$

To capture the additional information of random time τ, we adopt the progressive enlargement of filtration in discrete time. Thus, we consider $\mathbb{G} := (\mathcal{G}_n)_{n\geq 0}$ given by

$$\mathcal{G}_n = \mathcal{F}_n \vee \sigma\left(\tau\wedge 1, ..., \tau\wedge n\right),\ S = (S_n)_{n=0,1,...,T},$$

Besides the processes G and $\widetilde{G}$, the process m given by

$$m_n := G_n + \sum_{j=0}^{n} P[\tau = j|\mathcal{F}_j], \tag{3.107}$$

is an $\mathbb{F}$-martingale and plays crucial role in our analysis. In the following lemma, we prove that indeed m is $\mathbb{F}$-martingale.

Lemma 3.18: *The process m is an $\mathbb{F}$-martingale.*

Proof. For $n = 0, 1, 2, ..., (T-1)$, we drive

$$\begin{aligned} E[\Delta m_{(n+1)}|\mathcal{F}_n] &= E\big[\Delta G_{(n+1)} + E[I_{\{\tau=n+1\}}|\mathcal{F}_{n+1}]|\mathcal{F}_n\big] \\ &= E\big[-E[I_{\{\tau=n+1\}}|\mathcal{F}_{n+1}] + E[I_{\{\tau=n+1\}}|\mathcal{F}_{n+1}]|\mathcal{F}_n\big] \\ &= 0. \end{aligned}$$

□

Here, we recall an important and critical Lemma. It demonstrates the relationship between $\mathbb{F}$-predictable projection and $\mathbb{G}$-predictable projection of X.

Lemma 3.19: *Let X be an integrable and $\mathbb{F}$-measurable random variable. Then,*

$$(\mathbf{a})\ \mathbb{E}[X|\mathcal{G}_n]I_{\{\tau>n\}} = E[XI_{\{\tau>n\}}|\mathbb{F}_n]\frac{1}{G_n}I_{\{\tau>n\}}, \quad on\ \ \{\tau > n\}.P-a.s. \tag{3.108}$$

(**b**) *If X is an $\mathbb{F}_n$-measurable on $\{n = 0,1,2,...,T\}$, then*

$$E[X|\mathcal{G}_{n-1}]I_{\{\tau\geq n\}} = E[X\widetilde{G}_n|\mathbb{F}_{n-1}]\frac{1}{G_{n-1}}I_{\{\tau\geq n\}}, \quad on\ \ \{\tau \geq n\},\ P-a.s.. \tag{3.109}$$

Proof. We refer readers to Jeulin [86, Theorem 2.3] for its proof. □

For the reader's convenience, we recall the discrete version of the Doleans-Dade (stochastic) exponential. For any Y, we denote by $\mathcal{E}(Y)$ the Doleans-Dade (stochastic) exponential, it is the unique solution to the stochastic differential equation

$$\Delta X_n = X_{n-1}\Delta Y_n, \quad X_0 = 1, \quad \text{and is given by} \quad \mathcal{E}_n(Y) = \prod_{s=0}^{n}(1+\Delta Y_s).$$

In the following lemma, we state the discrete-time version of the operator $\mathcal{T}(.)$ defined in Theorem 2.21.

Lemma 3.20: *For any $\mathbb{F}$-martingale, M, and the random time τ, the process*

$$\mathcal{T}_n(M) = \sum_{k=1}^{n\wedge\tau} \frac{P(\tau \geq k|\mathcal{F}_{k-1})}{P(\tau \geq k|\mathcal{F}_k)}\Delta M_k + \sum_{k=1}^{n\wedge\tau} E(\Delta M_k I_{\{P(\tau\geq k|\mathcal{F}_k)=0\}}|\mathcal{F}_{k-1}), \tag{3.110}$$

is a $\mathbb{G}$ martingale, where $\Delta M_n := M_n - M_{n-1}$, for all $n = 1, ..., T$.

Proof. Let M be an $\mathbb{F}$ martingale and $n = 1, ..., (T-1)$. We start our proof with a simple but useful remark: for any processes $Y = (Y_n)_{n=1,...,T}$, we have

$$Y_{(n+1)\wedge\tau} - Y_{n\wedge\tau} = (Y_{(n+1)} - Y_n)I_{\{\tau\geq n+1\}},$$

Since one can write $\mathcal{T}_n(M) = Y_{n\wedge\tau}$, where

$$Y_n := \sum_{k=1}^{n} \frac{P(\tau \geq k|\mathcal{F}_{k-1})}{P(\tau \geq k|\mathcal{F}_k)}\Delta M_k + \sum_{k=1}^{n} E(\Delta M_k I_{\{P(\tau\geq k|\mathcal{F}_k)=0\}}|\mathcal{F}_{k-1}),$$

we deduce that

$$\begin{aligned}\mathcal{T}_{n+1}(M) - \mathcal{T}_n(M) &= (Y_{n+1} - Y_n)I_{\{\tau\geq n+1\}} \\ &= \Big(\frac{P(\tau \geq n+1|\mathcal{F}_n)}{P(\tau \geq n+1|\mathcal{F}_{n+1})}\Delta M_{n+1} + E(\Delta M_{n+1} I_{\{P(\tau\geq n+1|\mathcal{F}_{n+1})=0\}}|\mathcal{F}_n)\Big)I_{\{\tau\geq n+1\}}.\end{aligned}$$

Hence by taking the expectation with respect to $\mathcal{G}_n$, in both sides of the above equation, we deduce that

$$\mathbb{E}[\mathcal{T}_{n+1}(M)|\mathcal{G}_n] = \mathcal{T}_n(M) + \mathbb{E}[\frac{P(\tau \geq n+1|\mathcal{F}_n)}{P(\tau \geq n+1|\mathcal{F}_{n+1})}\Delta M_{n+1}|\mathcal{G}_n]I_{\{\tau\geq n+1\}}$$
$$+\mathbb{E}[\mathbb{E}(\Delta M_{n+1}I_{\{P(\tau\geq n+1|\mathcal{F}_{n+1})=0\}}|\mathcal{F}_n)I_{\{\tau\geq n+1\}}|\mathcal{G}_n]I_{\{\tau\geq n+1\}}$$

Thus, as a direct application of Lemma 3.19-(b), we obtain that

$$\mathbb{E}[\mathcal{T}_{n+1}(M)|\mathcal{G}_n] = \mathcal{T}_n(M).$$

□

Here, we state a $\mathbb{G}$-martingale representation in discrete time. Basically, it is a discrete-version of Theorem 2.11 (see here [33, Theorem 2.8] for more details). To this end, we start by providing the discrete-time version of the process $N^{\mathbb{G}}$ defined in (2.23)

Theorem 3.19: *Consider the following process,*

$$N_n^{\mathbb{G}} := I_{\{n\geq\tau\}} - \sum_{k=1}^{n\wedge\tau}\frac{P(\tau = k|\mathcal{F}_k)}{P(\tau \geq k|\mathcal{F}_k)}. \tag{3.111}$$

Then the following assertions hold:

(a) $N^{\mathbb{G}}$ *is a* $\mathbb{G}$*-martingale.*

(b) *For any integrable* $\mathbb{F}$*-adapted process* k*, the process* $\sum\varphi_n\Delta N^{\mathbb{G}}$ *is a* $\mathbb{G}$*-martingale.*

Proof. For $n = 1, ..., (T-1)$, we drive

$$N_{n+1}^{\mathbb{G}} - N_n^{\mathbb{G}} = I_{\{\tau=n+1\}} - \frac{p(\tau = n+1|\mathcal{F}_{n+1})}{P(\tau \geq n+1|\mathcal{F}_{n+1})}I_{\{\tau\geq n+1\}}.$$

Therefore, by taking the expectation with respect to $\mathcal{G}_n$, in both sides of the above equation and applying Lemma 3.19-(b) afterwards, the proof follows immediately.

□

τ is a $\mathbb{G}$-stopping time and $\mathcal{G}_\tau$ is defined as usual, while $\mathcal{F}_\tau$ is given by

$$\mathcal{F}_\tau := \sigma\Big(X_\tau,\ X\ \mathbb{F}\text{-adapted and bounded}\Big).$$

Below, we discuss the relationship between $\mathcal{G}_\tau$ and $\mathcal{F}_\tau$, as this role is very important in our analysis.

Lemma 3.21: *Consider the discrete-time setting of (3.107). Then the σ-fields $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ coincide, and hence for any $\mathcal{G}_\tau$-measurable random variable X, there exists an $\mathbb{F}$-adapted process ξ such that $X = \xi_\tau$ P-a.s.*

Proof. Since τ is a $\mathbb{G}$-stopping time and due to [58, Theorem 64, Chapter IV], we conclude that for any $\mathcal{G}_\tau$-measurable random variable X, there exists a $\mathbb{G}$-adapted process, $\xi^{\mathbb{G}} = (\xi^{\mathbb{G}}_n)_{n=0,1,...,T}$ such that $X = \xi^{\mathbb{G}}_\tau$. Thus, the lemma follows immediately if we prove that for any $n \in \{0, ..., T\}$, and any $\mathcal{G}_n$-measurable random variable $X^{\mathbb{G}}_n$, there exists an $\mathcal{F}_n$-measurable random variable $X^{\mathbb{F}}_n$ such that

$$X^{\mathbb{G}}_n = X^{\mathbb{F}}_n \quad \text{on} \quad (\tau = n). \tag{3.112}$$

Thus, on the one hand, it is clear that (3.112) holds for random variables having the form of $X^{\mathbb{G}}_n = \xi^{\mathbb{F}}_n f(\tau \wedge 1, ..., \tau \wedge n)$, where $\xi^{\mathbb{F}}_n$ is a bounded and $\mathcal{F}_n$-measurable random variable and f is a bounded and Borel-measurable real-valued function on $\mathbb{R}^n$. On the other hand, these random variables generate the vector space of bounded and $\mathcal{G}_n$-measurable random variables. Hence, the fulfillment of (3.112) for general random variables, follows from this remark and the class monotone theorem (see [58, Theorem 21, Chapter I]). This proves the lemma. □

The impact of Lemma 3.21 can be noticed immediately. Here, we state our last result of this section. Since we study the decomposition for any $\mathbb{G}$-martingale stopped at random time τ.

Theorem 3.20: *Let assume that* $H_n^{\mathbb{G}} := \sum_{k=1}^{n} \Delta H_k^{\mathbb{G}}$ *is any* $\mathbb{G}$*-martingale, then there exists a unique pair* $\left(M^{\mathbb{F}}, \varphi\right)$ *such that* $M^{\mathbb{F}} \in \mathcal{M}_{0,loc}(\mathbb{F})$, φ *is* $\mathbb{F}$*-adopted and* $\mathbb{E}|\varphi_\tau| < +\infty$, *and*

$$H_{n\wedge\tau} = H_0 + \sum_{k=1}^{n} \frac{\Delta \mathcal{T}_k(M^{\mathbb{F}})}{P(\tau \geq k|\mathcal{F}_{k-1})^2} + \sum_{k=1}^{n} \varphi_k \Delta N_k^{\mathbb{G}}. \tag{3.113}$$

Proof. The proof follows the same process as in [33, Theorem 2.8] and it is omitted here. □

In the following theorem, we describe the class of all deflators for the model $(S^\tau, \mathbb{G})$.

Theorem 3.21: *Let* $Z^{\mathbb{G}}$ *be a* $\mathbb{G}$*-adapted process and* $\widetilde{Q}$ *be a probability given by*

$$\widetilde{Q} := \overline{Z}_T \cdot P \text{ and } \overline{Z}_n := \prod_{k=1}^{n} \left(\frac{\widetilde{G}_k}{G_{k-1}} I_{\{G_{k-1}>0\}} + I_{\{G_{k-1}=0\}} \right). \tag{3.114}$$

Then the following assertions are equivalent.

(a) $Z^{\mathbb{G}}$ *is a deflator for* $(S^\tau, \mathbb{G})$ *(i.e.* $Z^{\mathbb{G}} \in \mathcal{D}(S^\tau, \mathbb{G})$*).*

(b) *There exists a unique pair* $\left(Z^{(\widetilde{Q},\mathbb{F})}, \varphi\right)$ *such that* $Z^{(\widetilde{Q},\mathbb{F})} \in \mathcal{D}(S, \widetilde{Q}, \mathbb{F})$, φ *is an* $\mathbb{F}$*-adapted process satisfying for all* $n = 0, ..., T$ P*-a.s.*

$$-\frac{P(\tau \geq n|\mathcal{F}_n)}{P(\tau > n|\mathcal{F}_n)} < \varphi_n < \frac{P(\tau \geq n|\mathcal{F}_n)}{P(\tau = n|\mathcal{F}_n)}, \quad \text{and} \quad Z^{\mathbb{G}} = (Z^{(\widetilde{Q},\mathbb{F})})^\tau Z^{(\varphi)}. \tag{3.115}$$

Here $Z^{(\varphi)}$ *is given by*

$$Z_t^{(\varphi)} := \prod_{n=1}^{t} \left(1 + \varphi_n \frac{P(\tau > n|\mathcal{F}_n)}{P(\tau \geq n|\mathcal{F}_n)} I_{\{\tau=n\}} - \varphi_n \frac{P(\tau = n|\mathcal{F}_n)}{P(\tau \geq n|\mathcal{F}_n)} I_{\{\tau>n\}} \right). \tag{3.116}$$

Proof. We start this proof by making the following three remarks:

1) It is easy to check that (see also [34] for details and related results) the process $\overline{Z}$ is a martingale and hence $\widetilde{Q}$ is a well defined probability. Furthermore, the process $\overline{Z}$ is the discrete-time version of the process $\mathcal{E}(G_-^{-1} I_{\{G_->0\}} \cdot m)$ (which is well defined even in the case where G might vanish) see [92, Subsection 2.3].

2) It is clear that X is a supermartingale under $\widetilde{Q}$ if and only if $Y := \overline{Z}X$ is a supermartingale .

3) Thanks to (3.111), the discrete-time version of $\mathcal{E}(\varphi \bullet N^{\mathbb{G}})$ coincides with $Z^{(\varphi)}$ given in (3.116), for any $\mathbb{F}$-optional process φ.

Then by combining these remarks and Theorem 3.12, the proof of the theorem follows immediately. □

As a direct consequence of Theorem 3.21, we describe below a family of $\mathbb{G}$-local martingales.

Corollary 3.21.1: For any $\mathbb{F}$-local martingale $Z^{(\widetilde{Q},\mathbb{F})} \in \mathcal{D}(S, \widetilde{Q}, \mathbb{F})$, the process $Z^{\mathbb{G}}$ is given by

$$Z_n^{\mathbb{G}} := (Z^{(\widetilde{Q},\mathbb{F})})^{\tau}, \tag{3.117}$$

is a $\mathbb{G}$-local martingale (respectively belongs to $\mathcal{Z}_{loc}(S^{\tau}, \mathbb{G})$).

Chapter 4

log-optimal portfolio without NFLVR

This chapter addresses the log-optimal portfolio for a general semimartingale model. The most advanced literature on the topic elaborates existence and characterization of this portfolio under no-free-lunch-with-vanishing-risk assumption (NFLVR). In [71], the authors addressed the log-optimal portfolio problem, but their necessary condition for the existence of optimal portfolio is elaborated under NFLVR. Recently, see [31], there has been an interest in extending the existing results on utility maximization to models satisfying NUPBR and having finite primal and dual value functions, while they might violate NFLVR. There are many financial models violating NFLVR, see [36, 105, 117], while admitting the log-optimal portfolio on the one hand. On the other hand, for financial markets under progressive enlargement of filtration, NFLVR remains completely an open issue, and hence the literature on the log-optimal portfolio cannot be applied to these models. Herein, we provide a complete characterization of log-optimal portfolio and its associated optimal deflator, necessary and sufficient conditions for their existence, and we elaborate their duality as well without NFLVR.

This chapter contains three sections. The first section presents some useful intermediate results that are interesting in themselves beyond their role in the proof of the main result of this chapter. This result and its related discussions are presented in the second section. The third (and last) section deals with the proof of the main result.

4.1 Intermediate results

This section prepares the ground for the main result and its proof (Section 2 and Section 3), and it has three subsections. The first subsection discusses some general integrability properties that we will use later on. In the second subsection, we recall (see Section 2.4) the important martingale representation using the predictable characteristics and extends slightly a result on the characterization of deflators using the predictable characteristics. The last subsection proves a measurability selection result that we will encounter in the proof of the main result.

We start by recalling the following definition, that was introduced in [43].

Definition 4.1: Let N be an $\mathbb{H}$-local martingale such that $1+\Delta N > 0$. We call a Hellinger process of order zero for N, denoted by $h^{(0)}(N,\mathbb{H})$, the process $h^{(0)}(N,\mathbb{H}) := \left(H^{(0)}(N,\mathbb{H})\right)^{p,\mathbb{H}}$ when this projection exists, where

$$H^{(0)}(N,\mathbb{H}) := \frac{1}{2}\langle N^c\rangle^{\mathbb{H}} + \sum\left(\Delta N - \ln(1+\Delta N)\right). \tag{4.1}$$

4.1.1 Some useful integrability properties

The results of this section are new and are general, not technical at all, and very useful, especially the first lemma and the proposition.

Lemma 4.1: *Consider $K \in \mathcal{M}_{0,loc}(\mathbb{H})$ with $1+\Delta K > 0$. If*

$$E[\langle K^c\rangle_T + \sum_{0<s\leq T}(\Delta K_s - \ln(1+\Delta K_s))] < +\infty, \tag{4.2}$$

then $E[\sqrt{[K,K]_T}] < +\infty$ *or equivalently* $E[\sup_{0\leq t\leq T}|K_t|] < +\infty$.

Proof. Let $K \in \mathcal{M}_{0,loc}(\mathbb{H})$ such that $1+\Delta K > 0$ and (4.2) holds. Then it is enough to remark that for $\delta \in (0,1)$, we have

$$\Delta K - \ln(1+\Delta K) \geq \frac{\delta|\Delta K|}{\max(2(1-\delta), 1+\delta^2)} I_{\{|\Delta K|>\delta\}} + \frac{(\Delta K)^2}{1+\delta} I_{\{|\Delta K|\leq\delta\}}.$$

By using this inequality and (4.2), on the one hand, we deduce that

$$E\left[\langle K^c\rangle_T + \sum_{0<t\leq T}|\Delta K_t| I_{\{|\Delta K_t|>\delta\}} + \sum_{0<t\leq T}(\Delta K)^2 I_{\{|\Delta K|\leq\delta\}}\right]$$
$$\leq C_\delta E\left[\langle K^c\rangle_T + \sum_{0<s\leq T}(\Delta K_s - \ln(1+\Delta K_s))\right] < +\infty,$$

where $C_\delta := 1+\delta+\max(2(1-\delta), 1+\delta^2)/\delta$. On the other hand, it is clear that

$$[K,K]_T^{1/2} \leq \sqrt{\langle K^c\rangle} + \sum_{0<t\leq T}|\Delta K_t| I_{\{|\Delta K_t|>\delta\}} + \sqrt{\sum_{0<t\leq T}(\Delta K)^2 I_{\{|\Delta K|\leq\delta\}}}.$$

This ends the proof of the lemma. □

The following lemma uses the set $\mathcal{L}(X,\mathbb{H})$, defined in (2.17), that we recall here for the reader's convenience.

$$\mathcal{L}(X,\mathbb{H}) := \left\{\varphi \ d\text{-dimensional and } \mathbb{H}\text{-predictable} \ \middle| \ \varphi^{tr}\Delta X > -1\right\}. \tag{4.3}$$

Lemma 4.2: *Let* $\lambda \in \mathcal{L}(X,\mathbb{H})$, *and* $\delta \in (0,1)$ *such that*

$$\frac{|\lambda^{tr}x|}{1+\lambda^{tr}x} I_{\{|\lambda^{tr}x|>\delta\}} \star \mu + \left(\frac{\lambda^{tr}x}{1+\lambda^{tr}x}\right)^2 I_{\{|\lambda^{tr}x|\leq\delta\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{4.4}$$

Then $\sqrt{((1+\lambda^{tr}x)^{-1}-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H})$.

Proof. By using $\sqrt{\sum_i x_i^2} \leq \sum_i |x_i|$, we derive

$$\sqrt{((1+\lambda^{tr}x)^{-1}-1)^2 \star \mu} = \sqrt{\sum \left(\frac{\lambda^{tr}\Delta X}{1+\lambda^{tr}\Delta X}\right)^2}$$
$$\leq \sqrt{\sum \frac{(\lambda^{tr}\Delta X)^2}{(1+\lambda^{tr}\Delta X)^2} I_{\{|\lambda^{tr}\Delta X|\leq\delta\}}} + \sum \frac{|\lambda^{tr}\Delta X|}{1+\lambda^{tr}\Delta X} I_{\{|\lambda^{tr}\Delta X|>\delta\}}.$$

Thus, the lemma follows immediately from the latter inequality. □

Proposition 4.1: *Let Z be a positive supermartingale such that $Z_0 = 1$. Then the following assertions hold.*

(a) $-\ln(Z)$ *is a uniformly integrable submartingale if and only if there exists a local martingale N and a nondecreasing and predictable process V such that $\Delta N > -1,\ Z = \mathcal{E}(N)\exp(-V)$ and*

$$E\left[V_T + H_T^{(0)}(N, \mathbb{H})\right] < +\infty, \tag{4.5}$$

where $H_\cdot^{(0)}(N, \mathbb{H})$ is defined in (4.1).

(b) *Suppose that there exists a finite sequence of positive supermartingale $(Z^{(i)})_{i=1,...,n}$ such that the product $Z := \prod_{i=1}^{n} Z^{(i)}$. Then $-\ln(Z)$ is uniformly integrable submartingale if and only if all $-\ln(Z^{(i)})$, $i = 1, ..., n$, are uniformly integrable submartingales.*

Proof. The proof of this proposition is achieved in three parts. The first and second parts prove assertion (a), while part three proves assertion (b).

Part 1. This parts proves that Z is a positive super martingale if and only if there exist unique local martingale N and a nondecreasing and predictable process V such that $N_0 = V_0 = 0$,

$$\Delta N > -1, \qquad Z = \mathcal{E}(N)\exp(-V).$$

Since Z is a positive supermartingale ($Z > 0$), then $1/Z_-$ is locally bounded and hence $(Z_-)^{-1} \bullet Z$ is a local supermartingale. Thanks to Doob-Meyer decomposition, there exists unique $M \in \mathcal{M}_{0,loc}(\mathbb{H})$ and a nondecreasing and predictable process A, such that

$$\frac{1}{Z_-} \bullet Z = M - A \text{ or equivalently, } \quad Z = \mathcal{E}(M - A).$$

Furthermore, the condition $Z > 0$ is equivalent to $1 + \Delta M - \Delta A > 0$, and this implies that

$$1 - \Delta A =^{P,\mathbb{F}} (1 + \Delta M - \Delta A) > 0, \text{ and } \quad 1 + \frac{\Delta M}{1 - \Delta A} > 0.$$

Therefore, thanks to Lemma 2.2, $1/(1 - \Delta A)$ is predictable and locally bounded, and using Yor's formula, we get

$$Z = \mathcal{E}\big(\frac{1}{1 - \Delta A} \bullet M\big)\mathcal{E}\big(- A\big).$$

Then, by putting

$$N := (1 - \Delta A)^{-1} \bullet M \quad \text{and} \quad V := A + \sum[-\Delta A - \ln(1 - \Delta A)],$$

the first part of assertion (a) is proved.

Part 2. Here we prove that $-\ln(Z)$ is a uniformly integrable submartingale if and only if (4.5) holds. Let $Z = \mathcal{E}(N)\exp(-V)$, where N is a local martingale, and V is a nondecreasing and predictable process such that $\Delta N > -1$. To this end, we derive

$$-\ln(Z) \ = -\ln(\mathcal{E}(N)) + V = -N + H^{(0)}(N, \mathbb{H}) + V, \tag{4.6}$$

where both processes V and $H^{(0)}(N, \mathbb{H})$ are nondecreasing.

Suppose that $-\ln(Z)$ is a uniformly integrable submartingale, and let $(\tau_n)_n$ be a sequence of stopping times that increases to infinity and N^{τ_n} is a martingale. Then, on the one hand, by stopping (4.6) with τ_n, and taking expectation afterwards we get

$$E[-\ln(Z_{\tau_n\wedge T})] = E\left[V_{\tau_n\wedge T} + H^{(0)}_{\tau_n\wedge T}(N,\mathbb{H})\right].$$

On the other hand, since $\{-\ln(Z_{\tau_n\wedge T}),\ n \geq 0\}$ is uniformly integrable and the RHS term of the above equality is increasing, by letting n goes to infinity in this equality, (4.5) follows immediately. Now suppose that (4.5) holds. As a consequence $E[H^{(0)}_T(N,\mathbb{H})] < +\infty$, and by combining this with Lemma 4.1 and (4.6), we deduce that $-\ln(Z)$ is a uniformly integrable submartingale.

Part 3. Here we prove assertion (b). From the proof of assertion (a), it is clear that there exists a sequence of local martingale $N^{(i)}$ and a sequence of nondecreasing and predictable processes $V^{(i)}$ such that for all $i = 1, ..., n$,

$$\Delta N^{(i)} > -1, \qquad Z^{(i)} = \mathcal{E}(N^{(i)})\exp(-V^{(i)}).$$

Furthermore, we derive

$$-\ln(Z)\ = -\sum_{i=1}^{n} N^{(i)} + \sum_{i=1}^{n} H^{(0)}(N^{(i)},\mathbb{H}) + \sum_{i=1}^{n} V^{(i)}.$$

If $-\ln(Z)$ is a uniformly integrable submartingale, then thanks to assertion (a), we deduce that

$$\sum_{i=1}^{n} H^{(0)}_T(N^{(i)},\mathbb{H}) + \sum_{i=1}^{n} V^{(i)}_T \tag{4.7}$$

is integrable. Hence, thanks again to assertion (a), we deduce that $(-\ln(Z^{(i)}))_{i=1,...,n}$ are uniformly integrable submartingales. Now suppose that $(-\ln(Z^{(i)}))_{i=1,...,n}$ are uniformly integrable submartingales, then in virtue of assertion (a) and Lemma 4.1, $(N^{(i)})_{i=1,...,n}$ are uniformly integrable martingales, and hence $-\ln(Z)$ is a uniformly integrable submartingale. This ends the proof of the proposition. □

4.1.2 Deflators via predictable characteristics

The following theorem describes how general deflators can be characterized using the predictable characteristics. A version of this theorem can be found in [107]. The following theorem uses the set $\mathcal{D}_{log}(X, \mathbb{H})$, that is defined as follows

$$\mathcal{D}_{log}(X, \mathbb{H}) := \left\{ Z \in \mathcal{D}(X, \mathbb{H}) \quad \middle| \quad E[-\ln(Z_T)] < +\infty \right\}. \tag{4.8}$$

Theorem 4.1: *Suppose X is quasi-left-continuous. If $Z \in \mathcal{D}_{log}(X, \mathbb{H})$, then there exists a triplet (β, f, V) such that $\beta \in L(X^c, \mathbb{H})$, f is $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable, positive and $\sqrt{(f-1)^2 \star \mu}$ belongs to $\mathcal{A}^+_{loc}(\mathbb{H})$, V is an $\mathbb{H}$-predictable and nondecreasing process, and the following hold for any bounded process $\theta \in \mathcal{L}(X, \mathbb{H})$.*

$$E\left[V_T + \left(\frac{1}{2}\beta^{tr} c\beta + \int (f(x) - 1 - \ln(f(x)))F(dx)\right) \cdot A_T\right] \leq E[-\ln(Z_T)], \tag{4.9}$$

$$\left(\int |f(x)\theta^{tr} x - \theta^{tr} h(x)| F(dx)\right) \cdot A_T < +\infty \quad P\text{-a.s.} \tag{4.10}$$

$$\left(\theta^{tr} b + \theta^{tr} c\beta + \int [f(x)\theta^{tr} x - \theta^{tr} h(x)] F(dx)\right) \cdot A \preceq V. \tag{4.11}$$

Proof. Let $Z \in \mathcal{D}_{log}(X, \mathbb{H})$, then $Z_-^{-1} \cdot Z$ a local supermartingale (which follows from $Z \in \mathcal{D}(X, \mathbb{H})$ only). Hence, there exists a local martingale N and a nondecreasing and predictable process V such that $Z = \mathcal{E}(N)\exp(-V)$. Then we derive

$$-\ln(Z) = -N + V + \frac{1}{2}\langle N^c \rangle + \sum (\Delta N - \ln(1 + \Delta N)).$$

Thus $Z \in \mathcal{D}_{log}(X, \mathbb{H})$ if and only if $V + \frac{1}{2}\langle N^c \rangle + \sum(\Delta N - \ln(1 + \Delta N))$ is integrable. Thanks to Theorem 2.7, we deduce the existence of a positive and $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional f such that $\sqrt{(f-1)^2 \star \mu}$ is locally integrable, $\beta \in L(X^c, \mathbb{H})$, $g \in \mathcal{H}^1_{loc}(\mu, \mathbb{F})$, and $N' \in \mathcal{M}_{0,loc}(\mathbb{H})$ such that $M^P_\mu(g \mid \widetilde{\mathcal{P}}(\mathbb{H})) = 0$, $[N', X] = 0$, and

$$N = \beta \cdot X^c + (f - 1) \star (\mu - \nu) + g \star \mu + N'. \tag{4.12}$$

Then $Z \in \mathcal{D}_{log}(X, \mathbb{H})$ if and only if the three processes V, $\frac{1}{2}\langle N^c \rangle = \frac{1}{2}\beta^{tr} c \beta \cdot A$ and $\sum(\Delta N - \ln(1 + \Delta N))$ are integrable. Since,

$$\sum(\Delta N - \ln(1 + \Delta N)) = [f + g - 1 - \ln(f + g)] \star \mu + \sum(\Delta N' - \ln(1 + \Delta N)'),$$

which is due to $[N', X] = 0$. There is equivalence between $\sum(\Delta N - \ln(1 + \Delta N))$ belongs to $\mathcal{A}^+(\mathbb{H})$ and both nondecreasing processes $[f + g - 1 - \ln(f + g)] \star \mu$ and $\sum(\Delta N' - \ln(1 + \Delta N)')$ belong to $\mathcal{A}^+(\mathbb{H})$. Furthermore, by writing

$$[f + g - 1 - \ln(f + g)] \star \mu = [f - 1 - \ln(f)] \star \mu + g(1 - \frac{1}{f}) \star \mu + \left[\frac{g}{f} - \ln(1 + \frac{g}{f})\right] \star \mu,$$

we conclude that $[f + g - 1 - \ln(f + g)] \star \mu \in \mathcal{A}^+(\mathbb{H})$ if and only if both nondecreasing processes $[f - 1 - \ln(f)] \star \mu$ and $\left[\frac{g}{f} - \ln(1 + \frac{g}{f})\right] \star \mu$ belongs to $\mathcal{A}^+(\mathbb{H})$ and $g(1 - \frac{1}{f}) \star \mu$ belongs to $\mathcal{M}_0(\mathbb{H})$. Therefore, by combining all the above remarks, we get

$$\begin{aligned}\mathbb{E}[-\ln(Z_T)] = \mathbb{E}\Big[V_T + \frac{1}{2}\beta^{tr} c \beta \cdot A + \int \big(f - 1 - \ln(f)\big) F(dx) \cdot A_T\Big] \\ + \mathbb{E}\Big[(\frac{g}{f} - \ln(1 + \frac{g}{f})) \star \mu\Big] + \mathbb{E}\Big[\sum(\Delta N' - \ln(1 + \Delta N)')\Big],\end{aligned}$$

which implies (4.9) immediately.

Since $Z \in \mathcal{D}_{log}(X, \mathbb{H})$, then for any bounded $\theta \in \mathcal{L}(X, \mathbb{H})$, the process $Z\mathcal{E}(\theta \bullet X)$ is a supermartingale. This implies that

$$\mathcal{E}\big(-V + \beta \cdot X^c + (f - 1) \star (\mu - \nu)\big)\mathcal{E}(\theta \bullet X),$$

is also a supermartingale (the processes β and f are defined above), or equivalently

$$Y := -V + \theta \cdot X + \beta^{tr} c \theta \cdot A + \theta^{tr} x (f - 1) \star \mu,$$

is a local supermartingale. By simplifying, we get

$$Y = \text{local martingale } - V + \theta^{tr} b \cdot A + \beta^{tr} c\theta \cdot A + (\theta^{tr} x f(x) - \theta^{tr} h(x)) \star \mu.$$

Then, Y is a local supermartingale if and only if $(\theta^{tr} x f(x) - \theta^{tr} h(x)) \star \mu$ belongs to $\mathcal{A}_{loc}(\mathbb{H})$ which is equivalent to (4.10), and

$$-V + \theta^{tr} b \cdot A + \beta^{tr} c\theta \cdot A + (\theta^{tr} x f(x) - \theta^{tr} h(x)) \star \nu \preceq 0.$$

This latter inequality is obviously (4.11), and the proof of the theorem is complete.

□

4.1.3 A measurability result

Lemma 4.3: *Consider the triplet* $(\Omega \times [0, +\infty), \mathcal{P}(\mathbb{H}), P \otimes A)$*, and the functional* $L(\omega, t, \lambda) := L_{(\omega,t)}(\lambda)$ *given by*

$$L_{(\omega,t)}(\lambda) := -\lambda^{tr} b(\omega, t) + \frac{1}{2}\lambda^{tr} c(\omega, t)\lambda + \int \left(\lambda^{tr} h(x) - \ln((1 + \lambda^{tr} x)^{+})\right) F_{(\omega,t)}(dx), \tag{4.13}$$

for any $\lambda \in \mathbb{R}^d$ *and any* $(\omega, t) \in \Omega \times [0, +\infty)$*. Then the functional* L*, as a map on* $\Omega \times [0, +\infty) \times \mathbb{R}^d$ *(i.e.* $(\omega, t, \lambda) \longrightarrow L(\omega, t, \lambda)$*), is* $\mathcal{P}(\mathbb{H}) \times \mathcal{B}(\mathbb{R}^d)$*-measurable.*

Proof. The proof of the lemma will be achieved in two steps. The first step defines a family of functionals $\{L_\delta(\omega, t, \cdot),\ \delta \in (0, 1)\}$ for $(\omega, t) \in \Omega \times [0, +\infty)$, and proves that these functionals are indeed $\mathcal{P}(\mathbb{H}) \times \mathcal{B}(\mathbb{R}^d)$-measurable (i.e. jointly measurable in (ω, t) and λ). Then the second step proves that $L_\delta(\omega, t, \lambda)$ converges to $L(\omega, t, \lambda)$ when δ goes to one for any $(\omega, t, \lambda) \in \Omega \times [0, +\infty) \times \mathbb{R}^d$.

Step 1: Let $(\omega, t) \in \Omega \times [0, +\infty)$ and $\delta \in (0, 1)$. Then for all $\lambda \in \mathbb{R}^d$, put

$$L_\delta(\omega, t, \lambda) := -\lambda^{tr} b(\omega, t) + \frac{1}{2}\lambda^{tr} c(\omega, t)\lambda + \int_{\mathbb{R}^d} f_\delta(\lambda, x) F_{(\omega,t)}(dx),$$
$$f_\delta(\lambda, x) := \delta\lambda^{tr} h(x) - \ln(1 - \delta + \delta(1 + \lambda^{tr} x)^{+}).$$

It is clear that for any $\lambda \in \mathbb{R}^d$, $L_\delta(\omega, t, \lambda)$ is predictable. Thus, in order to prove that L_δ is jointly measurable (i.e. $\mathcal{P}(\mathbb{H}) \times \mathcal{B}(\mathbb{R}^d)$-measurable), it is enough to prove that this functional is continuous in λ (in this case our functional L_δ falls into the class of Carathéodory functions), and hence one can conclude immediately that it is jointly measurable due to [11, Lemma 4.51]. Thus, the rest of this step focuses on proving that L_δ is continuous in λ. To this end, we first remark that $-\lambda^{tr} b(\omega,t) + \frac{1}{2}\lambda^{tr} c(\omega,t)\lambda$ is continuous, and we derive

$$-\delta|\lambda|^2|x|^2 \leq f_\delta(\lambda, x) \leq \max(\frac{1}{2(1-\delta)^2}, -\delta - \ln(1-\delta))|\lambda|^2|x|^2 \text{ on } \{|x| \leq 1\}$$
$$-\delta|\lambda||x| \leq f_\delta(\lambda, x) \leq -\ln(1-\delta) \quad \text{on } \{|x| > 1\}.$$

Therefore, thanks to the dominated convergence theorem and these inequalities, we deduce that in fact L_δ is continuous in λ, and the first step is complete.

Step 2: Herein, we prove that for any $(\omega, t) \in \Omega \times [0, +\infty)$ and any $\lambda \in \mathbb{R}^d$, $L_\delta(\omega, t, \lambda)$ converges to $L(\omega, t, \lambda)$ when δ goes to one. To this end, we first write

$$\begin{aligned}
&L_\delta(\omega, t, \lambda) + \lambda^{tr} b(\omega, t) - \frac{1}{2}\lambda^{tr} c(\omega, t)\lambda = \\
&= \delta \int_{\{\lambda^{tr} x \leq -1\}} \lambda^{tr} h(x) F(dx) - \delta \int_{\{\lambda^{tr} x \leq -1\}} \lambda^{tr} x I_{\{|x|>1\}} F(dx) \\
&- \ln(1-\delta) F\left(\{\lambda^{tr} x \leq -1\}\right) + \int I_{\{\lambda^{tr} x > -1\}} \left(\delta \lambda^{tr} x - \ln(1 + \delta \lambda^{tr} x)\right) F(dx).
\end{aligned}$$

Remark that $\int_{\{\lambda^{tr} x \leq -1\}} \lambda^{tr} h(x) F(dx)$ and $\int_{\{\lambda^{tr} x \leq -1\}} \lambda^{tr} x I_{\{|x|>1\}} F(dx)$ are well defined and take finite values, while $I_{\{\lambda^{tr} x > -1\}} \left(\delta \lambda^{tr} x - \ln(1 + \delta \lambda^{tr} x)\right)$ is nonnegative and increasing in δ. By distinguishing the cases whether $F\left(\{\lambda^{tr} x \leq -1\}\right)$ is null or not, thanks to the convergence monotone theorem, we conclude that $L_\delta(\omega, t, \lambda)$ converges to $L(\omega, t, \lambda)$. This ends the second step and the proof of the lemma. $\square$

4.2 Log-optimal portfolio: Duality and description

This section addresses the main contribution of this chapter that lies in characterizing, in different manners and as explicit as possible, the log-optimal portfolio, as well as the associated optimal deflator, for the general case of semimartingale without varying the flow of information. This section focuses on the following maximization problem for the economic model $(U, X, \mathbb{H}, Q)$, where $\mathbb{H}$ is a filtration satisfying the usual conditions, X is an $\mathbb{H}$-semimartingale, U is a utility function, and Q is a probability measure on $(\Omega, \mathcal{H}_T)$.

$$\max_{\theta\in\Theta(X,\mathbb{H})}\ \mathbb{E}_Q[U\left(1+\theta\bullet X\right)_T)], \tag{4.14}$$
$$\Theta(U,X,\mathbb{H}) := \left\{\theta\in L(X,Q,\mathbb{H})\big|\ E_Q\left[\max(0,-U(1+\theta\bullet X)_T)\right] < +\infty\right\}.$$

To this end, we recall the set of admissible portfolios $\Theta(X, \mathbb{H})$ given by

$$\Theta(X,\mathbb{H}) := \Big\{\theta\in L(X,\mathbb{H})\Big|\quad E\left[\max(0,-\ln(1+(\theta\cdot X)_T)\right] < +\infty\Big\}. \tag{4.15}$$

Since this section focuses on the logarithm utility maximization problem, for the sake of simplifying notation, we simply write $\Theta(X, \mathbb{H}) := \Theta(\ln, X, \mathbb{H})$. Our result in this section extends deeply the existing literature on the log-optimal portfolio and establish the duality for the log utility without the No-Free-Lunch-with-Vanishing-Risk assumption for general model $(\ln, X, \mathbb{H})$.

Theses results require the powerful techniques of semimartingale characteristics that we recalled in Section 2.4. For the reader's convenience, we recall *the canonical decomposition* of X (for more related details, we refer the reader to Section 2.4 or [83, Theorem 2.34, Section II.2])

$$X = X_0 + X^c + h\star(\mu_X - \nu_X) + b\bullet A^X + (x-h)\star\mu_X,$$

where h, defined as $h(x) := xI_{\{|x|\leq 1\}}$, is the truncation function, and $h \star (\mu_X - \nu_X)$ is the unique pure jump $\mathbb{H}$-local martingale with jumps given by $h(\Delta X)I_{\{\Delta X\neq 0\}}$. The quadruplet

$$(b, c, F, A) \text{ are the predictable characteristics of } X.$$

For more details about these *predictable characteristics* and other related issues (such as the martingale representation of local martingales using Jacod's parameters, we refer the reader to [83, Section II.2]. For the sake of simplicity, we will consider models, that we call *σ-special*, defines in (2.25), that we recall here for the reader's convenience.

Definition 4.2: A model $(X, \mathbb{H})$ is called σ-special if there exists a real-valued and $\mathbb{H}$-predictable process φ such that

$$0 < \varphi \leq 1 \quad \text{and} \quad \sum \varphi|\Delta X| I_{\{|\Delta X|>1\}} \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{4.16}$$

This assumption is equivalent to the fact that $\varphi \bullet X$ is a special semimartingale (i.e. it can be written as the sum of a local martingale and a predictable process with finite variation), or equivalently $\int_{(|x|>1)} |x|F(dx) < +\infty$ $P\otimes A$-a.e.. This assumption is for the sake of simplifying the proofs only, and it can be dropped at the expenses of technical intermediate results. Now, we are in the stage of stating our main result of this section.

Theorem 4.2: *Suppose $(X, \mathbb{H})$ is σ-special and quasi-left-continuous with predictable characteristics (b, c, F, A). Then the following assertions are equivalent.*

(a) *The set $\mathcal{D}_{log}(X, \mathbb{H})$, given by*

$$\mathcal{D}_{log}(X, \mathbb{H}) = \left\{ Z \in \mathcal{D}(X, \mathbb{H}) \quad \Big| \quad E[-\ln(Z_T)] < +\infty \right\}, \tag{4.17}$$

is not empty (i.e. $\mathcal{D}_{log}(X,\mathbb{H}) \neq \emptyset$*).*

(b) *There exists an* $\mathbb{H}$*-predictable process* $\widetilde{\varphi} \in \mathcal{L}(X,\mathbb{H})$ *such that, for any* φ *belonging to* $\mathcal{L}(X,\mathbb{H})$*, the following hold*

$$E\left[\widetilde{V}_T + \frac{1}{2}(\widetilde{\varphi}^{tr} c\widetilde{\varphi} \cdot A)_T + (\int (\frac{-\widetilde{\varphi}^{tr} x}{1+\widetilde{\varphi}^{tr} x} + \ln(1+\widetilde{\varphi}^{tr} x))F(dx)\cdot A)_T\right] < +\infty, \tag{4.18}$$

$$\widetilde{V} := \left|\widetilde{\varphi}^{tr}(b - c\widetilde{\varphi}) + \int \left[\frac{\widetilde{\varphi}^{tr} x}{1+\widetilde{\varphi}^{tr} x} - \widetilde{\varphi}^{tr} h(x)\right] F(dx)\right| \cdot A, \tag{4.19}$$

$$(\varphi - \widetilde{\varphi})^{tr}(b - c\widetilde{\varphi}) + \int \left(\frac{(\varphi-\widetilde{\varphi})^{tr} x}{1+\widetilde{\varphi}^{tr} x} - (\varphi - \widetilde{\varphi})^{tr} h(x)\right) F(dx) \leq 0. \tag{4.20}$$

(c) *There exists a unique* $\widetilde{Z} \in \mathcal{D}(X,\mathbb{H})$ *such that*

$$\inf_{Z \in \mathcal{D}(X,\mathbb{H})} E[-\ln(Z_T)] = E[-\ln(\widetilde{Z}_T)] < +\infty. \tag{4.21}$$

(d) *There exists a unique* $\widetilde{\theta} \in \Theta(X,\mathbb{H})$ *such that*

$$\sup_{\theta \in \Theta(X,\mathbb{H})} E[\ln(1 + (\theta \cdot X)_T)] = E[\ln(1 + (\widetilde{\theta} \cdot X)_T)] < +\infty. \tag{4.22}$$

Furthermore, when these assertions hold, the following hold.

$$\widetilde{\varphi} \in L(X^c,\mathbb{H}) \cap \mathcal{L}(X,\mathbb{H}), \quad \sqrt{((1+\widetilde{\varphi}^{tr} x)^{-1} - 1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H}), \tag{4.23}$$

$$\frac{1}{\widetilde{Z}} = \mathcal{E}(\widetilde{\varphi} \cdot X), \ \widetilde{Z} := \mathcal{E}(K - \widetilde{V}), \ K := -\widetilde{\varphi} \cdot X^c + \frac{-\widetilde{\varphi}^{tr} x}{1+\widetilde{\varphi}^{tr} x} \star (\mu - \nu). \tag{4.24}$$

$$\widetilde{\varphi} = \widetilde{\theta}(1 + (\widetilde{\theta} \cdot X)_-)^{-1} \quad \text{and} \quad \widetilde{\theta} = \widetilde{\varphi}\mathcal{E}_-(\widetilde{\varphi} \cdot \mathrm{X}) \quad \mathrm{P} \otimes \mathrm{A}\text{-a.e.}. \tag{4.25}$$

It is important to notice that, for any $Z \in \mathcal{D}(X,\mathbb{H})$, we have $E\ln^+(Z_T) \leq \ln(2)$. Furthermore, one can easily prove that the following two assertions are equivalent:
(a) $Z \in \mathcal{D}_{log}(X,\mathbb{H})$ (i.e. $-\ln(Z_T)$ is integrable or equivalently $(\ln(Z_T))^-$ is integrable),
(b) $\{-\ln(Z_t),\ 0 \leq t \leq T\}$, or equivalently $\{(\ln(Z_t))^-,\ 0 \leq t \leq T\}$, is uniformly integrable submartingale.

Besides this, for a positive local martingale Z, the condition $E[-\ln(Z_T)] < +\infty$ does not guarantee that this Z is a martingale, while it implies that $K := Z_-^{-1} \cdot Z$ is a martingale satisfying $E[\sup_{0\leq t\leq T} |K_t|] < +\infty$ instead, see Lemma 4.1 for this latter fact. As a result of this discussion, we conclude that Theorem 4.2 extends deeply the existing literature on the log-optimal portfolio by dropping the no-free-lunch-with-vanishing-risk condition on the model. This assumption is really a vital assumption for the analysis of [71]. This achievement is due to our approach that differs fundamentally from that of [71], while it is inspired from the approach of [43] with a major difference. This difference lies in dropping all assumptions on the model $(X, \mathbb{H})$ considered in [43], which guarantee that the minimizer of a functional belongs to the interior of its effective domain. We recall our aforementioned claim that the "σ-special assumption" for $(X, \mathbb{H})$ is purely technical and is not related at all to the minimizer of this functional. In conclusion, our theorem establishes the duality, under basically no assumption, besides describing the optimal dual solution when it exists as explicit as possible. This, furthermore, proves that in general, this optimal deflator might not be a local martingale deflator.

Remark 4.1: It is clear that the process V is well defined. This is due to

$$\begin{aligned}\int [\frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x} - \widetilde{\varphi}^{tr}h(x)]F(dx) \ &= -\int_{(|x|\leq 1)} \frac{(\widetilde{\varphi}^{tr}x)^2}{1+\widetilde{\varphi}^{tr}x}F(dx) \\ &\quad -\int_{(|x|>1)} \frac{1}{1+\widetilde{\varphi}^{tr}x}F(dx) + F(|x|>1),\end{aligned}$$

which is a well defined integral with values in $[-\infty, +\infty)$.

Similarly for the LHS term of (4.20), the integral term is well defined for any $\varphi \in \mathcal{L}(X, \mathbb{H})$. Indeed, due to $\Omega \times [0, +\infty) = \cup_{n\geq 0}(|\varphi| \leq n)$, for any process $\varphi \in \mathcal{L}(X, \mathbb{H})$, it is enough to prove that the integral term is well defined for bounded $\varphi \in \mathcal{L}(X, \mathbb{H})$. To this end, on the one hand, we write

$$\int \left(\frac{\varphi^{tr} x}{1 + \widetilde{\varphi}^{tr} x} - \varphi^{tr} h(x) \right) F(dx) = - \int_{(|x| \leq 1)} \frac{(\varphi^{tr} x)(\widetilde{\varphi}^{tr} x)}{1 + \widetilde{\varphi}^{tr} x} F(dx) - F(|x| > 1) + \int_{(|x|>1)} \frac{\varphi^{tr} x + 1}{1 + \widetilde{\varphi}^{tr} x} F(dx) - \int_{(|x|>1)} \frac{\widetilde{\varphi}^{tr} x}{1 + \widetilde{\varphi}^{tr} x} F(dx).$$

On the other hand, since φ is X-integrable (as it is bounded), both processes $I_{\{|\Delta X| \leq 1\}} \cdot [K, \varphi \cdot X]$ and $[\sum I_{\{|\Delta X|>1\}}, K]$ have locally integrable variations and their compensators are

$$- \int_{(|x| \leq 1)} \frac{(\varphi^{tr} x)(\widetilde{\varphi}^{tr} x)}{1 + \widetilde{\varphi}^{tr} x} F(dx) \cdot A \quad \text{and} \quad - \int_{(|x|>1)} \frac{\widetilde{\varphi}^{tr} x}{1 + \widetilde{\varphi}^{tr} x} F(dx) \cdot A$$

respectively. This proves that the integral is well defined with values in $(-\infty, +\infty]$.

The remaining part of this section illustrates the result of Theorem 4.2 on the case of jump-diffusion model, where the uncertainties in the initial model $(S, \mathbb{H})$ is a one-dimensional process generated by Poisson process and a Brownian motion. Precisely, we suppose that a standard Brownian motion W and a Poisson process N (with intensity $\lambda > 0$) that are defined on the probability space $(\Omega, \mathcal{H}, P)$. The filtration $\mathbb{H}$ is the completed and right continuous filtration generated by W and N, and the stock price process is given by the following dynamics.

$$S_t := S_0 \mathcal{E}(X)_t, \; X_t := \int_0^t \sigma_s dW_s + \int_0^t \zeta_s dN_s^{\mathbb{H}} + \int_0^t \mu_s ds, \; N_t^{\mathbb{H}} := N_t - \lambda t, \quad (4.26)$$

and there exists a constant $\delta \in (0, +\infty)$ such that μ, σ and ζ are bounded adapted processes satisfying the following

$$\sigma > 0, \quad \zeta > -1, \quad \sigma + |\zeta| \geq \delta, \; P \otimes dt\text{-a.e.}. \quad (4.27)$$

Theorem 4.3: *Suppose S and X are given by (4.26)-(4.27). Then the following*

$\mathbb{H}$*-predictable process*

$$\widetilde{\theta} := \frac{\alpha + sign(\zeta)\sqrt{\alpha^2 + 4\lambda}}{2\sigma} - \frac{1}{\zeta}, \text{ where } \alpha := \frac{\mu - \lambda\zeta}{\sigma} + \frac{\sigma}{\zeta}, \tag{4.28}$$

is S-integrable satisfying $1 + \widetilde{\theta}\zeta > 0$ $P \otimes dt$-a.e., and the following hold.

(a) *The solution to*

$$\inf_{Z \in \mathcal{D}(X,\mathbb{H})} E[-\ln(Z_T)] = E[-\ln(\widetilde{Z}_T)] < +\infty. \tag{4.29}$$

exists and is given by $\widetilde{Z}^{\mathbb{H}} := \mathcal{E}(\widetilde{K}^{\mathbb{H}})$ where

$$\widetilde{K}^{\mathbb{H}} := -\sigma\widetilde{\theta} \bullet W - \frac{\zeta\widetilde{\theta}}{1 + \widetilde{\theta}\zeta} \bullet N^{\mathbb{H}}. \tag{4.30}$$

(b) *It holds that*

$$\sup_{\theta \in \Theta(X,\mathbb{H})} E[\ln(1 + (\theta \cdot X)_T)] = E[\ln(1 + (\widetilde{\theta} \cdot X)_T)] < +\infty, \tag{4.31}$$

Proof. For the jump-diffusion model with parameters given by (4.27), the predictable characteristics of Section 2.4 is as follows. Let $\delta_a(dx)$ be the Dirac mass at the point a. Then in this case we have $d = 1$ and

$$A_t = t\ , \quad c = (S_-\sigma)^2\ , \quad b = (\mu - \lambda\zeta I_{\{|\zeta|S_- > 1\}})S_-\ , \quad S^c = S_-\sigma \bullet W$$
$$\mu(dt, dx) = \delta_{\zeta_t S_{t-}}(dx)dN_t\ , \quad \nu(dt, dx) = \delta_{\zeta_t S_{t-}}(dx)\lambda dt\ , \quad F_t(dx) = \lambda\delta_{\zeta_t S_{t-}}(dx).$$

As a result, the set

$$\begin{aligned}\mathcal{L}_{(\omega,t)}(S,\mathbb{H}) &:= \{\varphi \in \mathbb{R} \mid \varphi x > -1\ F_{(\omega,t)}(dx) - a.e.\} = \{\varphi \in \mathbb{R} \mid \varphi S_-\zeta > -1\} \\ &= \left(-1/(S_-\zeta)^+, +\infty\right) \cap \left(-\infty, 1/(S_-\zeta)^-\right)\end{aligned}$$

is an open set in $\mathbb{R}$ (with the convention $1/0^+ = +\infty$). Then the inequality condition

(4.20) in Theorem 4.2 becomes the following equality

$$\begin{aligned} 0 &= \mu - \lambda\zeta I_{\{|\zeta|>1/S_-\}} - S_-\sigma^2\varphi + \frac{\lambda\zeta}{1+S_-\varphi\zeta} - \lambda\zeta I_{\{|\zeta|\le 1/S_-\}} \\ &= \mu - \lambda\zeta - S_-\sigma^2\varphi + \frac{\lambda\zeta}{1+S_-\varphi\zeta}. \end{aligned} \tag{4.32}$$

Put $\psi := 1+\varphi S_-\zeta > 0$, then the above equation is equivalent to

$$0 = -\frac{\sigma^2}{\zeta}\psi^2 + \sigma\alpha\psi + \lambda\zeta,$$

where $\alpha = \dfrac{\mu-\lambda\zeta}{\sigma} + \dfrac{\sigma}{\zeta}$.

This equation has always a unique positive solution

$$\widetilde{\psi} := \frac{\sigma\alpha\zeta + |\zeta|\sqrt{(\sigma\alpha)^2+4\sigma^2\lambda}}{2\sigma^2} = \frac{\alpha\zeta+|\zeta|\sqrt{\alpha^2+4\lambda}}{2\sigma}.$$

Therefore, $\widetilde{\varphi} := \frac{\widetilde{\theta}}{S_-}$, where $\widetilde{\theta}$ is given by (4.28), coincides with $(\widetilde{\psi}-1)/(S_-\zeta)$, satisfies $1+\zeta\widetilde{\theta} > 0$. Hence, we deduce that $\widetilde{\theta}$ is the unique solution to (4.32). $\widetilde{\theta}$ is also S-integrable due to the parameters that we assume in (4.27). As a result, the optimal wealth process is $\mathcal{E}(\widetilde{\theta}\bullet X)$ and hence $\widetilde{\theta}$ is the solution to (4.31) and the proof of assertion (b) is completed. The assertion (a) follows immediately using the decomposition of deflator (4.24) in Theorem 4.2, $\widetilde{\varphi} = \frac{\widetilde{\theta}}{S_-}$ and above predictable characteristics for S.

□

4.3 Proof of the main theorem

Proof of Theorem 4.2. It is clear that (c)$\Longrightarrow$(a) is obvious, and hence the proof of the theorem reduces to proving (a)$\Longrightarrow$(b)$\Longrightarrow$(c), (b)$\Longrightarrow$(d), (d)$\Longrightarrow$(a), and as long as assertion (b) holds the properties in (4.23)-(4.24) hold also. Thus, the rest of this proof is divided into three steps. The first step proves that assertion (b) implies both assertions (c) and (d) and (4.23)-(4.24). The second step deals with (d)$\Longrightarrow$ (a), while the third step addresses (a)$\Longrightarrow$ (b).

Step 1. Here, we assume that assertion (b) holds, and focus on proving assertions (c) and (d), and (4.23)-(4.24). Then due to (4.18) and

$$(1+y)\ln(1+y) - y \geq \frac{1-\delta}{2}\frac{y^2}{1+y}I_{\{|y|\leq\delta\}} + \frac{\delta}{2(1+\delta)}|y|I_{\{|y|>\delta\}}$$

for any $\delta \in (0,1)$ and any $y \geq -1$, we deduce that for $\delta \in (0,1)$ the following

$$\begin{aligned}
&\widetilde{\varphi}^{tr}c\widetilde{\varphi}\cdot A, \quad \int_{\mathbb{R}^d\setminus\{0\}}\left(\frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x}\right)^2 I_{\{|\widetilde{\varphi}^{tr}x|\leq\delta\}}F(dx)\cdot A,\\
&\text{and} \quad \int_{\mathbb{R}^d\setminus\{0\}}\frac{|\widetilde{\varphi}^{tr}x|}{1+\widetilde{\varphi}^{tr}x}I_{\{|\widetilde{\varphi}^{tr}x|>\delta\}}F(dx)\cdot A
\end{aligned}$$

are integrable processes, and hence $\widetilde{\varphi} \in L(X^c, \mathbb{H})$ and $\sqrt{((1+\widetilde{\varphi}^{tr}x)^{-1}-1)^2 \star \mu} \in \mathcal{A}^+_{loc}$ due to Lemma 4.2. Hence K, defined in (4.24), is a well defined local martingale satisfying $\Delta K + 1 = (1+\widetilde{\varphi}^{tr}\Delta X)^{-1} > 0$. Furthermore, thanks to Yor's formula and the continuity of A, on the one hand, we conclude that for any bounded $\varphi \in \mathcal{L}(X, \mathbb{H})$,

$$\mathcal{E}(\varphi\cdot X)\widetilde{Z} = \mathcal{E}\left(\varphi\cdot X + [\varphi\cdot X, K] + K - \widetilde{V}\right).$$

On the other hand, we drive,

$$\begin{aligned}
\varphi\bullet X + [\varphi\bullet X, K] + K - V &= \varphi\bullet X^c + \varphi h\star(\mu-\nu) + \varphi^{tr}b\bullet A + \varphi^{tr}(x-h(x))\star\mu + K - V\\
&- \varphi^{tr}c\widetilde{\varphi}\bullet A - \left(\frac{(\widetilde{\varphi}^{tr}x)(\varphi^{tr}x)}{1+\widetilde{\varphi}^{tr}x}\right)\star\mu\\
&= \text{ local martingale} + (\varphi^{tr}b - \varphi^{tr}c\widetilde{\varphi})\bullet A - V\\
&+ \left(\varphi^{tr}(x-h(x)) - \frac{(\widetilde{\varphi}^{tr}x)(\varphi^{tr}x)}{1+\widetilde{\varphi}^{tr}x}\right)\star\mu\\
&= \text{ local martingale} + (\varphi^{tr}b - \varphi^{tr}c\widetilde{\varphi})\bullet A - V + \left(\frac{\varphi^{tr}x}{1+\widetilde{\varphi}^{tr}x} - \varphi^{tr}h(x)\right)\star\mu,
\end{aligned}$$

By writing

$$\frac{\varphi^{tr}x}{1+\widetilde{\varphi}^{tr}x} - \varphi^{tr}h(x) = \frac{-\varphi^{tr}x\varphi^{tr}x}{1+\widetilde{\varphi}^{tr}x}I_{\{|X|\leq 1\}} + \frac{\varphi^{tr}x+1}{1+\widetilde{\varphi}^{tr}x}I_{\{|X|>1\}}$$
$$-I_{\{|X|>1\}} - \frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x}I_{\{|X|>1\}},$$

and using similar argument as in Remark 4.1, we deduce that (4.20) implies that $\varphi \bullet X + [\varphi \bullet X, K] + K - V$ is a special semimartingale and

$$(\varphi \bullet X + [\varphi \bullet X, K] + K - V)^{p,\mathbb{H}} \preceq 0.$$

This latter inequality is due to (4.19) and (4.20). This proves that the process $\varphi \cdot X + [\varphi \cdot X, K] + K - \widetilde{V}$ is a local supermartingale. As a consequence, $\mathcal{E}(\varphi \cdot X)\widetilde{Z}$ is a positive supermatingale, and hence $\widetilde{Z} \in \mathcal{D}(X, \mathbb{H})$ on the one hand. On the other hand, due to Itô, we derive

$$-\ln(\widetilde{Z}) = \text{local martingale} + \widetilde{V} + \frac{1}{2}\widetilde{\varphi}^{tr}c\widetilde{\varphi}\cdot A + \left[-\frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x} + \ln(1+\widetilde{\varphi}^{tr}x)\right]\star\mu.$$

By combining this with (4.18), we deduce that $\widetilde{Z} \in \mathcal{D}_{log}(X, \mathbb{H})$. This proves that assertion (a) holds. Furthermore, since $\widetilde{Z}$ is a positive supermartingale, $\widetilde{Z}^{-1}$ is a positive semimartingale, and

$$\widetilde{Z}_{-}I_{\{|\widetilde{\varphi}|\leq n\}} \cdot (\widetilde{Z})^{-1} = \widetilde{\varphi}I_{\{|\widetilde{\varphi}|\leq n\}} \cdot X.$$

Since the LHS term, of the above-mentioned equality converges (in probability at any time $t \in (0, T]$), we deduce that $\widetilde{\varphi} \in L(X, \mathbb{H})$ (i.e. it is X-integrable in the semimartingale sense), and $(\widetilde{Z})^{-1} = \mathcal{E}(\widetilde{\varphi} \cdot X)$. Therefore, on the one hand, this ends the proof for the properties (4.23)-(4.24). On the other hand, we notice that $\ln(\mathcal{E}(\widetilde{\varphi}\cdot X)_T) = -\ln(\widetilde{Z}_T)$ is an integrable random variable, and for any $\varphi \in \mathcal{L}(X, \mathbb{H})\cap$

$L(X,\mathbb{H})$ satisfying the condition $E\ln^-(\mathcal{E}(\varphi\cdot X)_T)<+\infty$, we get

$$E[\ln(\mathcal{E}(\varphi\cdot X)_T/\mathcal{E}(\widetilde{\varphi}\cdot X)_T)]=E[\ln(\mathcal{E}(\varphi\cdot X)_T)-E\ln(\mathcal{E}(\widetilde{\varphi}\cdot X)_T)]\leq 0.$$

This is due to a combining of Jensen's inequality and the fact $\mathcal{E}(\varphi\bullet X)\widetilde{Z}$ is a positive supermartingale. Thus, assertion (d), and (4.25) follow immediately. The rest of this step focuses on proving assertion (c).

Suppose that $Z\in\mathcal{D}_{log}(X,\mathbb{H})$, then by applying Theorem 4.1, we deduce the existence of (β,f,V) such that

$$\begin{aligned}
&\varphi^{tr}xf(x)\geq-[f(x)-1-\ln(f(x))]+\ln(1+\varphi^{tr}x),\qquad\text{for any }\varphi\in\mathcal{L}(X,\mathbb{H}),\\
&V\succeq\left(\varphi^{tr}b+\varphi^{tr}c\beta+\int\left(\varphi^{tr}xf(x)-\varphi^{tr}h(x)\right)F(dx)\right)\cdot A,\\
&E[-\ln(Z_T)]\geq E\left[V_T+\frac{1}{2}\beta^{tr}c\beta\cdot A_T+\int[f(x)-1-\ln(f(x))]F(dx)\cdot A_T\right].
\end{aligned}$$

Then by combining these properties (take $\varphi=\widetilde{\varphi}$) with (4.18)-(4.19)-(4.20) and the fact that under (4.20) we have $\widetilde{\varphi}^{tr}(b-c\widetilde{\varphi})+\int[\frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x}-\widetilde{\varphi}^{tr}h(x)]F(dx)\geq 0$, we derive

$$\begin{aligned}
E[-\ln(\widetilde{Z}_T)]&=E\left[\widetilde{V}_T+\frac{1}{2}(\widetilde{\varphi}^{tr}c\widetilde{\varphi}\cdot A)_T+(-\frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x}+\ln(1+(\widetilde{\varphi}^{tr}x))\star\mu)_T\right]\\
&=E\left[\widetilde{V}_T+\left(\frac{1}{2}\widetilde{\varphi}^{tr}c\widetilde{\varphi}+\int(-\frac{\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x}+\ln(1+(\widetilde{\varphi}^{tr}x))F(dx)\right)\cdot A_T\right]\\
&=E\left[(\widetilde{\varphi}^{tr}b-\frac{1}{2}\widetilde{\varphi}^{tr}c\widetilde{\varphi})\cdot A_T+\int(\ln(1+\widetilde{\varphi}^{tr}x)-\widetilde{\varphi}^{tr}h(x))F(dx)\cdot A_T\right]\\
&\leq E\left[(\widetilde{\varphi}^{tr}b-\frac{1}{2}\widetilde{\varphi}^{tr}c\widetilde{\varphi})\cdot A_T+\int(\widetilde{\varphi}^{tr}xf(x)-\widetilde{\varphi}^{tr}h(x))F(dx)\cdot A_T\right]+\\
&\qquad\qquad+E\left[\left(\int[f(x)-1-\ln(f(x))]F(dx)\right)\cdot A_T\right]\\
&\leq E\left[\left(-\widetilde{\varphi}^{tr}c\beta-\frac{1}{2}\widetilde{\varphi}^{tr}c\widetilde{\varphi}+\int[f(x)-1-\ln(f(x))]F(dx)\right)\cdot A_T+V_T\right]\\
&\leq E\left[V_T+\frac{1}{2}\beta^{tr}c\beta\cdot A_T+\int[f(x)-1-\ln(f(x))]F(dx)\cdot A_T\right]\\
&\leq E[-\ln(Z_T)].
\end{aligned}$$

This proves assertion (c), and the first step is complete.

Step 2. This step proves (d)$\Longrightarrow$ (a). Thus, we suppose that assertion (d) holds. Then there exists a portfolio $\widetilde{\theta} \in \Theta(X,\mathbb{H})$ such that (4.22) holds. Thanks to [36, Theorem 2.8] (see also [49] and [79, Theorem 2.3]), we deduce that $\mathcal{D}(X,\mathbb{H}) \neq \emptyset$. By combining this with $1 + (\widetilde{\theta} \cdot X)_T > 0$, we conclude the positivity of both processes $1+\widetilde{\theta}\cdot X$ and $1+(\widetilde{\theta}\cdot X)_-$, and hence the existence of $\widetilde{\varphi} \in \mathcal{L}(X,\mathbb{H})\cap L(X,\mathbb{H})$ such that $1+\widetilde{\theta}\cdot X = \mathcal{E}(\widetilde{\varphi}\cdot X)$ on the one hand. On the other hand, the condition $\mathcal{D}(X,\mathbb{H}) \neq \emptyset$ is equivalent to the existence of the numéraire portfolio, that we denote by $\widehat{\varphi}$ (see [36, 91, 94] and the references therein to cite few). This means that there exists $\widehat{\varphi} \in L(X,\mathbb{H})$ such that $\mathcal{E}(\widehat{\varphi}\cdot X) > 0$ and $\mathcal{E}(\varphi\cdot X)/\mathcal{E}(\widehat{\varphi}\cdot X)$ is a supermartingale for any $\varphi \in L(X,\mathbb{H})$ with $1 + \varphi\Delta X \geq 0$. In particular, the process

$$M := \frac{\mathcal{E}(\widetilde{\varphi}\cdot X)}{\mathcal{E}(\widehat{\varphi}\cdot X)} - 1,$$

is a suprermartingale. Due to $\ln(x) \leq x - 1$, we get

$$-\ln(\mathcal{E}(\widehat{\varphi}\cdot X)) \leq -\ln(\mathcal{E}(\widetilde{\varphi}\cdot X)) + \frac{\mathcal{E}(\widetilde{\varphi}\cdot X)}{\mathcal{E}(\widehat{\varphi}\cdot X)} - 1, \tag{4.33}$$

and deduce that $\ln^-(\mathcal{E}_T(\widehat{\varphi}\cdot X))$ is integrable. As a result, $\widehat{\theta} := \widehat{\varphi}\mathcal{E}(\widehat{\varphi}\cdot X)_- \in \Theta(X,\mathbb{H})$, and the following hold

$$\begin{aligned} E[\ln(\mathcal{E}_T(\widehat{\varphi}\cdot X))] &= E[\ln(1 + (\widehat{\theta}\cdot X)_T)] \\ &\leq E[\ln(1 + (\widetilde{\theta}\cdot X)_T)] = E[\ln(\mathcal{E}_T(\widetilde{\varphi}\cdot X))]. \end{aligned} \tag{4.34}$$

This, in particular, implies that $\ln(\mathcal{E}_T(\widehat{\varphi}\cdot X))$ is an integrable random variable, or equivalently $\ln(\mathcal{E}_T(\widetilde{\varphi}\cdot X)/\mathcal{E}_T(\widehat{\varphi}\cdot X) = \ln(\mathcal{E}_T(\widetilde{\varphi}\cdot X)) - \ln(\mathcal{E}_T(\widehat{\varphi}\cdot X))$ is integrable. Then using Jensen's inequality, we deduce that

$$E[\ln(\mathcal{E}_T(\widetilde{\varphi}\cdot X)/\mathcal{E}_T(\widehat{\varphi}\cdot X)] \leq \ln(E[\mathcal{E}_T(\widetilde{\varphi}\cdot X)/\mathcal{E}_T(\widehat{\varphi}\cdot X)]) \leq 0.$$

This combined with (4.34) implies that

$$E[\ln(\mathcal{E}_T(\widehat{\varphi}\cdot X))] = E[\ln(\mathcal{E}_T(\widetilde{\varphi}\cdot X))]. \tag{4.35}$$

A combination of this with (4.33) leads to $E[\mathcal{E}_T(\widetilde{\varphi}\cdot X)/\mathcal{E}_T(\widehat{\varphi}\cdot X)] = 1$, and hence the process $M+1$ is in fact a martingale (a positive supermartingale with constant expectation is a martingale). It is clear that $f(x) := x - \ln(1+x),\ x > -1$, is a nonnegative and strictly convex function that vanishes at $x = 0$ only. Since $E[f(M_T)] < +\infty$, we conclude that $f(M)$ is a nonnegative submartingale satisfying

$$0 = E[f(M_0)] \leq E[f(M_t)] \leq E[f(M_T)] = E[M_T + \ln(\frac{\mathcal{E}(\widetilde{\varphi}\bullet X)}{\mathcal{E}(\widehat{\varphi}\bullet X)})] = \ln(1) = 0,$$

where the last equality follows from combining (4.35) with the fact that M is martingale. Thus, we conclude that $M \equiv 0$ and hence $\mathcal{E}(\widehat{\varphi}\cdot X) \equiv \mathcal{E}(\widetilde{\varphi}\cdot X)$. As a consequence the process $Z := 1/\mathcal{E}(\widetilde{\varphi}\cdot X)$ belongs to $\mathcal{D}(X,\mathbb{H})$. Therefore, assertion (a) follows immediately from this and

$$E[-\ln(Z_T)] = E[\ln(\mathcal{E}_T(\widetilde{\varphi}\cdot X))] = E[1 + (\widetilde{\theta}\cdot X)_T] < +\infty,$$

and the proof of (d)$\Longrightarrow$ (a) is complete.

Step 3. This step proves the implication (a) $\Longrightarrow$ (b). Hence, we assume that assertion (a) holds for the rest of this proof. In virtue of Theorem 4.1, which guarantees the existence of (β, f, V) such that $\beta \in L(X^c,\mathbb{H})$, f is $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable, positive and $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}$, V is a predictable and nondecreasing process, and the following hold for any bounded $\theta \in \mathcal{L}(X, \mathbb{H})$.

$$E\left[V_T + \frac{1}{2}(\beta^{tr} c\beta \cdot A)_T + \left(\int (f(x) - 1 - \ln(f(x)))F(dx)\right) \cdot A_T\right]$$
$$\leq E[-\ln(Z_T)] < +\infty, \tag{4.36}$$
$$\left(\int |f(x)\theta^{tr} x - \theta^{tr} h(x)|F(dx)\right) \cdot A_T < +\infty \ P\text{-a.s.,} \quad \text{and} \tag{4.37}$$
$$\left(\theta^{tr} b + \theta^{tr} c\beta + \int [f(x)\theta^{tr} x - \theta^{tr} h(x)]F(dx)\right) \cdot A \preceq V, \tag{4.38}$$

The rest of this proof is divided into two sub-steps, and uses these properties. The first sub-step proves that a functional L, that we will define below, attains its minimal value, while the second sub-step proves that this minimum fulfills (4.18)-(4.19)-(4.20).

Step 3.a. Throughout the rest of the proof, we denote by $L_{(\omega,t)}$ $-P \otimes A$-almost all $(\omega, t) \in \Omega \times [0, +\infty)-$ the function given by

$$L_{(\omega,t)}(\lambda) = -\lambda^{tr} b(\omega, t) + \frac{1}{2}\lambda^{tr} c(\omega, t)\lambda + \int \left(\lambda^{tr} h(x) - \ln((1 + \lambda^{tr} x)^+)\right) F_{(\omega,t)}(dx), \tag{4.39}$$

for any $\lambda \in \mathbb{R}^d$ with the convention $\ln(0^+) = -\infty$. This sub-step proves the existence of a predictable process $\widetilde{\varphi}$ such that $P \otimes A$-almost all $(\omega, t) \in \Omega \times [0, +\infty)$

$$\widetilde{\varphi}(\omega, t) \in \mathcal{L}_{(\omega,t)}(X, \mathbb{H}) \quad \text{and} \quad \mathrm{L}_{(\omega,t)}(\widetilde{\varphi}(\omega, \mathrm{t})) = \min\nolimits_{\lambda \in \mathcal{L}_{(\omega,t)}(\mathrm{X},\mathbb{H})} \mathrm{L}_{(\omega,t)}(\lambda). \tag{4.40}$$

To this end, we start by noticing that in virtue of a combination of Remark 4.1 (which implies that this functional takes values in $(-\infty, +\infty]$), Lemma 4.3, and [69, Proposition 1] (which guarantees the existence of a predictable selection for the minimizer when it exists), this proof boils down to prove that $L_{(\omega,t)}$ attains in fact its minimum for all $(\omega, t) \in \Omega \times [0, +\infty)$. This is the aim of the rest of this sub-step. For the sake of simplicity, we denote $L_{(\omega,t)}(\cdot)$ by L throughout the rest of this proof. In order to prove that L attains it minimum value, we start by proving that this function L is convex, proper and closed. Let first recall some definitions from convex analysis. Consider a convex function f. The effective domain of f, denoted by $\mathrm{dom}(f)$, is the set of all $x \in \mathbb{R}^d$ such that $f(x) < +\infty$. The function f is said to be proper if, for any $x \in \mathbb{R}^d$, $f(x) > -\infty$ and if its effective domain $\mathrm{dom}(f\ )$ is not empty. For all undefined or unexplained concepts from convex analysis, we refer the reader to Rockafellar [116].

Let θ be a bounded element of $\mathcal{L}(X, \mathbb{H})$, and due to $\ln(1+(\theta^{tr}x)^+) \leq (\theta^{tr}x)^+ \leq |\theta||x|$ and $\int_{(|x|>1)} |x| F(dx) < +\infty$ (since X is σ-special), we obtain $P \otimes A$-a.e.

$$\int_{(|x|>1)} \ln(1+(\theta^{tr}x)^+) F(dx) \leq \int_{(|x|>1)} (\theta^{tr}x)^+ F(dx) \leq |\theta| \int_{(|x|>1)} |x| F(dx) < +\infty. \tag{4.41}$$

Then by combining this with

$$\begin{aligned} \int \left(\lambda^{tr}h(x) - \ln(1+\lambda^{tr}x)\right) F(dx) &= \int_{(|x|\leq 1)} \left(\lambda^{tr}x - \ln(1+\lambda^{tr}x)\right) F(dx) \\ &\quad - \int_{(|x|>1)} \ln(1-(\lambda^{tr}x)^-) F(dx) \\ &\quad - \int_{(|x|>1)} \ln(1+(\lambda^{tr}x)^+) F(dx) \\ &\geq - \int_{(|x|>1)} \ln(1+(\lambda^{tr}x)^+) F(dx) \\ &> -\infty, \end{aligned}$$

and $L(0) = 0 < +\infty$ (i.e. $0 \in \mathrm{dom}(L) \subset \mathcal{L}(X, \mathbb{H})$), we deduce that L is a convex and proper function. Now we prove that L is closed or equivalently L is lower semi-continuous. Let θ_n be a sequence in $\mathbb{R}^d$ that converges to θ such that $L(\theta_n)$ converges. Then it is clear that $\theta_n^{tr}b + \theta_n^{tr}c\beta$ converges to $\theta^{tr}b + \theta^{tr}c\beta$ and $\int_{(|x|>1)} \ln(1+(\theta_n^{tr}x)^+) F(dx)$ converges to $\int_{(|x|>1)} \ln(1+(\theta^{tr}x)^+) F(dx)$. This latter is due to a combination of $\ln(1+(\theta_n^{tr}x)^+) \leq (\theta_n^{tr}x)^+ \leq (\sup_n |\theta_n|)|x|$, (4.41), and dominated convergence theorem. Now consider the assumption

$$\theta \in \mathcal{L}(X, \mathbb{H}) \quad \text{and there exists } n_0 \text{ such that for all } n \geq n_0 \quad \theta_n \in \mathcal{L}(X, \mathbb{H}). \tag{4.42}$$

Under (4.42), by combining Fatou's lemma and the above remarks, we get

$$\begin{aligned} L(\theta) &= -\theta^{tr}b + \frac{1}{2}\theta^{tr}c\theta + \int \left(\theta^{tr}h(x) - \ln(1+\theta^{tr}x)\right) F(dx) \\ &= -\theta^{tr}b + \frac{1}{2}\theta^{tr}c\theta - \int_{|x|>1} \ln(1+(\theta^{tr}x)^+)F(dx) \\ &- \int_{|x|>1} \ln(1-(\theta^{tr}x)^-)F(dx) + \int_{|x|\le 1} (\theta^{tr}x - \ln(1+\theta^{tr}x))F(dx) \\ &\le \lim_{n\longrightarrow+\infty} L(\theta_n). \end{aligned}$$

This proves that L is closed under (4.42) on the one hand. On the other hand, it is clear that, when (4.42) is violated, there exists a subsequence $(\theta_{k(n)})_n$ such that $\theta_{k(n)} \notin \mathcal{L}(X,\mathbb{H})$ for all $n \ge 1$. As a result, since $L(\theta_n)$ converges, we conclude that $L(\theta) \le \lim_{n\longrightarrow+\infty} L(\theta_n) = \lim_{n\longrightarrow+\infty} L(\theta_{k(n)}) = +\infty$. This proves that L is closed, convex and proper. Thus, we can apply [116, Theorem 27.1(b)] which states that, for L to attain its minimal value, it is sufficient to prove that the set of recession for L is contained in the set of directions in which L is constant. To check this last condition, we calculate the recession function for L. For $\lambda \in \mathrm{dom}(L)$ and $y \in \mathbb{R}^d$, the recession function for L is by definition

$$L0^+(y) := \lim_{\alpha\longrightarrow+\infty} \frac{L(\lambda+\alpha y) - L(\lambda)}{\alpha}.$$

Consider the following sets $\Gamma^+(\lambda) := \{x \in \mathbb{R}^d \mid \lambda^{tr}x > 0\}$, $\Gamma^-(\lambda) := \{x \in \mathbb{R}^d \mid \lambda^{tr}x < 0\}$, and remark that we have

$$\begin{aligned} &\frac{L(\lambda+\alpha y) - L(\lambda)}{\alpha} \\ &= -y^{tr}b + \frac{\alpha}{2}y^{tr}cy + y^{tr}c\lambda + \int \left(y^{tr}h(x) - \frac{1}{\alpha}\ln(1+\frac{\alpha y^{tr}x}{1+\lambda^{tr}x})\right) F(dx) \\ &= -y^{tr}b + \frac{\alpha}{2}y^{tr}cy + y^{tr}c\lambda + \int_{\Gamma^+(y)} \left(y^{tr}h(x) - \frac{1}{\alpha}\ln(1+\frac{\alpha y^{tr}x}{1+\lambda^{tr}x})\right) F(dx) \\ &+ \int_{\Gamma^-(y)} \left(y^{tr}h(x) - \frac{1}{\alpha}\ln(1+\frac{\alpha y^{tr}x}{1+\lambda^{tr}x})\right) F(dx). \end{aligned}$$

Then, on the one hand, we calculate the recession function $L0^+(y)$ as follows.

$$L0^+(y) = \begin{cases} +\infty & \text{if either } F(\Gamma^-(y)) > 0 \text{ or } y^{tr}cy > 0, \\ -y^{tr}b + \int_{\Gamma^+(y)} y^{tr}h(x)F(dx) & \text{otherwise} \end{cases}$$

On the other hand, we have

$$\alpha\{y \in \mathbb{R}^d \ : \ cy = 0 \text{ and } F(\Gamma^-(y)) = 0\} \subset \mathcal{L}(X, \mathbb{H}) \ \text{ for any } \alpha \in (0, +\infty),$$
$$-\alpha\{y \in \mathbb{R}^d \ : \ cy = 0 \text{ and } F(\Gamma^+(y)) = 0\} \subset \mathcal{L}(X, \mathbb{H}) \ \text{ for any } \alpha \in (0, +\infty).$$

Thus, by combining these with (4.38), we deduce that

$$-y^{tr}b + \int_{\Gamma^+(y)} y^{tr}h(x)F(dx) > 0 \quad \text{if } F(\Gamma^-(y)) = 0 < F(\Gamma^+(y)), \ cy = 0,$$
$$y^{tr}b - \int_{\Gamma^-(y)} y^{tr}h(x)F(dx) > 0 \quad \text{if } F(\Gamma^+(y)) = 0 < F(\Gamma^-(y)), \ cy = 0.$$

Thus, thanks to these remarks, the recession cone for L and the set of directions in which L is constant, that we denote RC and CD respectively, are defined and calculated as follows

$$\begin{aligned} RC &:= \{y \in \mathbb{R}^d \ \big| \ L0^+(y) \leq 0\} = \{y \in \mathbb{R}^d \ \big| \ cy = y^{tr}b = F(\Gamma^-(y)) = F(\Gamma^+(y)) = 0\} \\ CD &:= \{y \in \mathbb{R}^d \ \big| \ L0^+(y) \leq 0, \ L0^+(-y) \leq 0\} \\ &= \{y \in \mathbb{R}^d \ \big| \ y^{tr}b = cy = F(\Gamma^-(y)) = F(\Gamma^+(y)) = 0\}. \end{aligned}$$

This proves that both sets (RC and CD) are equal. Hence, thanks to [116, Theorem 27.1(b)], we conclude that $L_{(\omega,t)}$ attains its minimal value at $\varphi(\omega, t)$ which satisfies $1 + x^{tr}\varphi(\omega, t) > 0$ $F_{(\omega,t)}(dx)$-a.e. since $L_{(\omega,t)}(\varphi(\omega, t)) \leq L_{(\omega,t)}(0) = 0 < +\infty$. This ends the first part of the third step.

Step 3.b. This sub-step proves that $\widetilde{\varphi}$, a minimizer for L proved in the previous sub-step, fulfills in fact the conditions of assertion (b) (i.e. the properties (4.18)-

(4.19)-(4.20)).

Since $L(\widetilde{\varphi}) \leq L(\varphi)$ for any $\varphi \in \mathcal{L}(X, \mathbb{H})$. Let $\varphi \in \mathcal{L}(X, \mathbb{H})$ and $\alpha \in (0,1)$, then using similar calculations as above, we get

$$\frac{L(\widetilde{\varphi}) - L(\widetilde{\varphi} + \alpha(\varphi - \widetilde{\varphi}))}{\alpha} = (\varphi - \widetilde{\varphi})^{tr} b - \frac{\alpha}{2}(\varphi - \widetilde{\varphi})^{tr} c (\varphi - \widetilde{\varphi}) - (\varphi - \widetilde{\varphi})^{tr} c \widetilde{\varphi} + \\ + \int \Big(\frac{1}{\alpha} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x}) - (\varphi - \widetilde{\varphi})^{tr} h(x) \Big) F(dx).$$

Here we have,

$$\begin{aligned} \frac{1}{\alpha} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x}) &= \frac{\ln\Big(\alpha \varphi^{tr} x + 1 + (1-\alpha)\widetilde{\varphi}^{tr} x\Big) - \ln(1 + \widetilde{\varphi}^{tr} x)}{\alpha} \\ &= \frac{\ln\Big(\alpha(1 + \varphi^{tr} x) + (1-\alpha)(1 + \widetilde{\varphi}^{tr} x)\Big) - \ln(1 + \widetilde{\varphi}^{tr} x)}{\alpha} \\ &\geq \frac{\alpha \ln(1 + \varphi^{tr} x) + (1-\alpha)\ln(1 + \widetilde{\varphi}^{tr} x) - \ln(1 + \widetilde{\varphi}^{tr} x)}{\alpha} \\ &= \ln(1 + \varphi^{tr} x) - \ln(1 + \widetilde{\varphi}^{tr} x) \end{aligned} \tag{4.43}$$

Due to the concavity of $\ln(x)$, since $\ln(\alpha y + (1-\alpha)x) \geq \alpha \ln(y) + (1-\alpha)\ln(x)$. On the other hand, by computing its derivative,

$$\begin{aligned} \Big(\frac{1}{\alpha} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x})\Big)' &= \frac{-1}{\alpha^2} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x}) \\ &\quad + \frac{(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x} \Big(\frac{1}{\alpha(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x})} \Big) \\ &= \frac{1}{\alpha^2} \Big(-\ln(1 + \alpha x) + \frac{\alpha x}{1 + \alpha x} \Big) \\ &= \frac{1 + \alpha x}{\alpha^2} \Big(\alpha x - (1 + \alpha x) \ln(1 + \alpha x) \Big) \\ &\leq 0. \end{aligned} \tag{4.44}$$

It is clear that, as a function of α, $\alpha^{-1} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x})$ is decreasing and hence

$$\ln(1 + \varphi^{tr} x) - \ln(1 + \widetilde{\varphi}^{tr} x) \leq \frac{1}{\alpha} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x}) \leq \frac{(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x}.$$

As a result of this, combined with the convergence monotone theorem, we deduce that

$$\int \left(\frac{1}{\alpha} \ln(1 + \frac{\alpha(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x}) - (\varphi - \widetilde{\varphi})^{tr} h(x) \right) F(dx)$$

converges to $\int [\frac{(\varphi-\widetilde{\varphi})^{tr} x}{1+\widetilde{\varphi}^{tr} x} - (\varphi - \widetilde{\varphi})^{tr} h(x)] F(dx)$, when α goes to zero and hence (4.20) is proved. By using (4.20) for $\varphi = 0$, and $L(\widetilde{\varphi}) \leq L(0) = 0$, we get

$$0 \leq \widetilde{\varphi}^{tr} b - \widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int \left(-\widetilde{\varphi}^{tr} h(x) + \frac{\widetilde{\varphi}^{tr} x}{1 + \widetilde{\varphi}^{tr} x} \right) F(dx) \tag{4.45}$$

$$0 \leq \widetilde{\varphi}^{tr} b - \frac{1}{2} \widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int \left(\ln(1 + \widetilde{\varphi}^{tr} x) - \widetilde{\varphi}^{tr} h(x) \right) F(dx). \tag{4.46}$$

The second inequality is $-L(\widetilde{\varphi}) \geq 0$. We also know that $0 \leq ||\sigma(a + b)||^2$, hence

$$0 \leq (a + b)^{tr} c(a + b) = a^{tr} ca + b^{tr} cb + 2a^{tr} cb,$$

and the following inequality holds.

$$-\frac{1}{2} \widetilde{\varphi}^{tr} c \widetilde{\varphi} - \frac{1}{2} \beta^{tr} c \beta \leq \widetilde{\varphi}^{tr} c \beta.$$

Therefore, by combining (4.45) and (4.46), with $-\frac{1}{2} \widetilde{\varphi}^{tr} c \widetilde{\varphi} - \frac{1}{2} \beta^{tr} c \beta \leq \widetilde{\varphi}^{tr} c \beta$, and

$$f(x) - 1 - \ln(f(x)) \geq \ln(1 + \widetilde{\varphi}^{tr} x) - f(x) \widetilde{\varphi}^{tr} x \qquad \text{(Young's inequality)},$$

we derive

$$\begin{aligned} &\widetilde{\varphi}^{tr} b - \frac{1}{2} \widetilde{\varphi}^{tr} c \widetilde{\varphi} - \frac{1}{2} \beta^{tr} c \beta + \int [\ln(1 + \widetilde{\varphi}^{tr} x) - \widetilde{\varphi}^{tr} h(x)] F(dx) + \\ &\qquad - \int [f(x) - 1 - \ln(f(x))] F(dx) \\ &\leq \widetilde{\varphi}^{tr} b - \frac{1}{2} \widetilde{\varphi}^{tr} c \widetilde{\varphi} - \frac{1}{2} \beta^{tr} c \beta + \int [f(x) \widetilde{\varphi}^{tr} x - \widetilde{\varphi}^{tr} h(x)] F(dx) \\ &\leq \widetilde{\varphi}^{tr} b + \widetilde{\varphi}^{tr} c \beta + \int [f(x) \widetilde{\varphi}^{tr} x - \widetilde{\varphi}^{tr} h(x)] F(dx). \end{aligned}$$

Therefore, thanks to this latter inequality and (4.38), we deduce that

$$\begin{aligned}0 &\preceq \left(\widetilde{\varphi}^{tr} b - \frac{1}{2}\widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int [\ln(1+\widetilde{\varphi}^{tr} x) - \widetilde{\varphi}^{tr} h(x)] F(dx)\right) \cdot A \\ &\preceq \left(\int [f(x) - 1 - \ln(f(x))] F(dx) + \frac{1}{2}\beta^{tr} c \beta\right) \cdot A + V.\end{aligned}$$

By combining this, (4.36), the fact that

$$\begin{aligned}&\widetilde{\varphi}^{tr} b - \frac{1}{2}\widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int [\ln(1+\widetilde{\varphi}^{tr} x) - \widetilde{\varphi}^{tr} h(x)] F(dx) \\ &= \left(\frac{1}{2}\widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int \left(\ln(1+\widetilde{\varphi}^{tr} x) - \frac{\widetilde{\varphi}^{tr} x}{1+\widetilde{\varphi}^{tr} x}\right) F(dx)\right) \\ &\quad + \left(\widetilde{\varphi}^{tr} b - \widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int \left(-\widetilde{\varphi}^{tr} h(x) + \frac{\widetilde{\varphi}^{tr} x}{1+\widetilde{\varphi}^{tr} x}\right) F(dx)\right),\end{aligned}$$

where both terms of the RHS are nonnegative, and the second term of this RHS coincides with $\frac{d\widetilde{V}}{dA}$, we conclude that

$$E\left[\widetilde{V}_T + \left(\frac{1}{2}\widetilde{\varphi}^{tr} c \widetilde{\varphi} + \int \left(\ln(1+\widetilde{\varphi}^{tr} x) - \frac{\widetilde{\varphi}^{tr} x}{1+\widetilde{\varphi}^{tr} x}\right) F(dx)\right) \cdot A_T\right] < +\infty.$$

This proves (4.18), and assertion (b) follows. This ends the proof of the theorem.

Chapter 5

Log-optimal deflators under random horizon

This chapter addresses the dual problem of the log-utility maximization problem under the random horizon. Here, we consider an initial market model specified by the pair $(S, \mathbb{F})$, where S is its discounted assets' price process and $\mathbb{F}$ its flow of information, and an arbitrary random time τ. This random time can represent the death time and the default time, and in both cases, τ can be seen when it occurs only. Thus, the progressive enlargement of $\mathbb{F}$ with τ, that we denote by $\mathbb{G}$, sounds tailor-fit for modelling the new flow of information that incorporates both $\mathbb{F}$ and τ. In this setting, our main aim resides in describing as explicit as possible the log-optimal deflator for the stopped model $(S^\tau, \mathbb{G})$. The primal maximization of this latter problem is fully considered in Chapter 6. For more details, about utility optimization problem, we refer the reader to Section 2.3 where we describe the Merton's optimal portfolio problem briefly.

Since this chapter focuses on the logarithm utility maximization problem, for the sake of simplifying notation, we simply write $\Theta(X, \mathbb{H}) := \Theta(\ln, X, \mathbb{H})$, where the set is initially defined in (4.15). The main goal of this chapter is the model $(S^\tau, \mathbb{G})$, when S is an $\mathbb{F}$-semimartingale that is quasi-left-continuous and has some integrability condition (to avoid some technicalities).

In this chapter, we will use frequently the operator function $\mathcal{T}(.)$ defined in (2.21), that we recall below for the reader's convenience.

$$\mathcal{T}(M) := M^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [M,m] + I_{]\!]0,\tau]\!]} \bullet \Big(\sum \Delta M I_{\{\widetilde{G}=0<G_-\}} \Big)^{p,\mathbb{F}}, \ \forall\, M \in \mathcal{M}_{0,loc}(\mathbb{F}).$$

The following lemma illustrates some simple but useful properties on this operator function $\mathcal{T}(.)$.

Lemma 5.1: *Let K be an $\mathbb{F}$-local martingale, and α belongs to $L^1_{loc}(K,\mathbb{F})$, then*

(a) *The continous local martingale part of $\mathcal{T}(K)$ (i.e. $\mathcal{T}(K)^c$), is equal to $\mathcal{T}(K^c)$.*

(b) *$\mathcal{T}(\alpha \bullet K) = \alpha \bullet \mathcal{T}(K)$, and it is an $\mathbb{F}$-local martingale.*

Proof. Suppose that $K = K^c + K^d$, where K^c is continuous $\mathbb{F}$-local martingale, and K^d is a pure jumps $\mathbb{F}$-local martingale (see Theorem 2.2). Then, we have

$$\mathcal{T}(K) = \mathcal{T}(K^c) + \mathcal{T}(K^d). \tag{5.1}$$

Thus, $\mathcal{T}(K^c) = \mathcal{T}(K)^c$ if and only if $\mathcal{T}(K^d)$ is a pure jump $\mathbb{G}$-local martingale. Let $M^{\mathbb{G}}$ be continuous and bounded $\mathbb{G}$-martingale. Then we drive

$$\begin{aligned}
[M^{\mathbb{G}}, \mathcal{T}(K^d)] &= [M^{\mathbb{G}}, (K^d)^\tau] - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [M^{\mathbb{G}}, [K^d, m]] \\
&\quad + I_{]\!]0,\tau]\!]} \bullet [M^{\mathbb{G}}, \Big(\sum \Delta K^d I_{\{\widetilde{G}=0<G_-\}} \Big)^{p,\mathbb{F}}] \\
&= \sum \Delta M^{\mathbb{G}} \Delta (K^d)^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet \sum \Delta M^{\mathbb{G}} \Delta (K^d)^\tau \Delta m + 0 = 0,
\end{aligned}$$

which is due to continuity of $M^{\mathbb{G}}$. Therefore, by Corollary 2.1.1, $\mathcal{T}(K^d)$ is a pure jump $\mathbb{G}$-local martingale. This proves assertion (a).

For part (b), we write

$$\begin{aligned}\mathcal{T}(\alpha\bullet K) &= (\alpha\bullet K)^\tau - \widetilde{G}^{-1}I_{]\!]0,\tau]\!]}\bullet[(\alpha\bullet K),m] + I_{]\!]0,\tau]\!]}\bullet\Big(\sum\Delta(\alpha\bullet K)I_{\{\widetilde{G}=0<G_-\}}\Big)^{p,\mathbb{F}}\\ &= \alpha\bullet(K)^\tau - \widetilde{G}^{-1}\alpha I_{]\!]0,\tau]\!]}\bullet[K,m] + I_{]\!]0,\tau]\!]}\bullet\Big(\sum\alpha\Delta K I_{\{\widetilde{G}=0<G_-\}}\Big)^{p,\mathbb{F}}\\ &= \alpha\bullet\mathcal{T}(K).\end{aligned}$$

This ends the proof of lemma. □

The rest of this chapter is divided into three sections. The first section states the results in the general setting of semimartingales, and proves them. The second and third sections illustrate the main results, elaborated in the first section, on particular, practical, and popular models for random time τ and $(S,\mathbb{F})$ respectively.

5.1 Optimal deflator for $(\ln, S^\tau, \mathbb{G})$

This section focuses on the following minimization problem

$$\min_{Z^{\mathbb{G}}\in\mathcal{D}_{log}(S^\tau,\mathbb{G})}\ \mathbb{E}[-\ln(Z^{\mathbb{G}}_T)], \tag{5.2}$$

where $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ is defined by

$$\mathcal{D}_{log}(S^\tau,\mathbb{G}) := \left\{Z\in\mathcal{D}(S^\tau,\mathbb{G})\quad\Big|\quad E[-\ln(Z_\tau)]<+\infty\right\}.$$

This requires the powerful techniques of predictable characteristics of semimartingales (see Section 2.4). For the sake of simplicity, we will consider models, that we call σ*-special* and are defined in Definition (2.25).

Now, in this stage, for reader's convenience we recall the main theorem of Chapter 4 here, and for the sake of avoiding some big notations, we change the form slightly. As we explained in Chapter 4, this result establishes the duality for the log utility without the No-Free-Lunch-with-Vanishing-Risk assumption.

Theorem 5.1: *Let X be an $\mathbb{H}$-quasi-left-continuous semimartingale with predictable characteristics (b, c, F, A). We define*

$$\mathcal{K}_{log}(y) := \frac{-y}{1+y} + \ln(1+y) \quad \text{for any} \quad y > -1. \tag{5.3}$$

If $(X, \mathbb{H})$ is σ-special, then the following assertions are equivalent.

(a) *The set $\mathcal{D}_{log}(X, \mathbb{H})$, given by*

$$\mathcal{D}_{log}(X, \mathbb{H}) = \left\{ Z \in \mathcal{D}(X, \mathbb{H}) \quad \Big| \quad E[-\ln(Z_T)] < +\infty \right\}, \tag{5.4}$$

is not empty (i.e. $\mathcal{D}_{log}(X, \mathbb{H}) \neq \emptyset$).

(b) *There exists an $\mathbb{H}$-predictable process $\widetilde{\varphi} \in \mathcal{L}(X, \mathbb{H})$ such that, for any φ belonging to $\mathcal{L}(X, \mathbb{H})$, the following hold*

$$E\left[V_T^X + \frac{1}{2}(\widetilde{\varphi}^{tr} c \widetilde{\varphi} \bullet A)_T + (\int \mathcal{K}_{log}(\widetilde{\varphi}^{tr} x) F(dx) \bullet A)_T \right] < +\infty, \tag{5.5}$$

$$V^X := \left| \widetilde{\varphi}^{tr}(b - c\widetilde{\varphi}) + \int \left[\frac{\widetilde{\varphi}^{tr} x}{1 + \widetilde{\varphi}^{tr} x} - \widetilde{\varphi}^{tr} h(x) \right] F(dx) \right| \bullet A, \tag{5.6}$$

$$(\varphi - \widetilde{\varphi})^{tr}(b - c\widetilde{\varphi}) + \int \left(\frac{(\varphi - \widetilde{\varphi})^{tr} x}{1 + \widetilde{\varphi}^{tr} x} - (\varphi - \widetilde{\varphi})^{tr} h(x) \right) F(dx) \leq 0. \tag{5.7}$$

(c) *There exists a unique $\widetilde{Z} \in \mathcal{D}(X, \mathbb{H})$ such that*

$$\inf_{Z \in \mathcal{D}(X,\mathbb{H})} E[-\ln(Z_T)] = E[-\ln(\widetilde{Z}_T)] < +\infty. \tag{5.8}$$

(d) *There exists a unique $\widetilde{\theta} \in \Theta(X, \mathbb{H})$ such that*

$$\sup_{\theta \in \Theta(X,\mathbb{H})} E[\ln(1 + (\theta \bullet X)_T)] = E[\ln(1 + (\widetilde{\theta} \bullet X)_T)] < +\infty. \tag{5.9}$$

Furthermore, $\widetilde{\theta}(1+(\widetilde{\theta}\bullet X)_-)^{-1}$ *and* $\widetilde{\varphi}$ *coincide* $P\otimes A$*-a.e., and*

$$\widetilde{\varphi}\in L(X^c,\mathbb{H})\cap\mathcal{L}(X,\mathbb{H}),\quad \sqrt{((1+\widetilde{\varphi}^{tr}x)^{-1}-1)^2\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{H}),\tag{5.10}$$

$$\frac{1}{\widetilde{Z}}=\mathcal{E}(\widetilde{\varphi}\bullet X),\ \widetilde{Z}:=\mathcal{E}(K^X-V^X),\ K^X:=\widetilde{\varphi}\bullet X^c+\frac{-\widetilde{\varphi}^{tr}x}{1+\widetilde{\varphi}^{tr}x}\star(\mu-\nu).\tag{5.11}$$

In this section, we describe, in different manners and as explicit as possible using $\mathbb{F}$-adapted processes only, the optimal deflator for the model $(S^\tau,\mathbb{G})$ solutions to (5.2). The first result of this section allows us to simplify the optimization problem in order to apply Theorem 5.1 (i.e. Section 4.2). To this end, we recall the following notation and definition, that was initially introduced in [43].

Definition 5.1: Let N be an $\mathbb{H}$-local martingale such that $1+\Delta N>0$. We call an entropy-Hellinger process for N, denoted by $h^{(E)}(N,\mathbb{H})$, the process

$$h^{(E)}(N,\mathbb{H}):=\left(H^{(E)}(N,\mathbb{H})\right)^{p,\mathbb{H}},$$

when this projection exists, where

$$H^{(E)}(N,\mathbb{H}):=\frac{1}{2}\langle N^c\rangle^{\mathbb{H}}+\sum\left((1+\Delta N)\ln(1+\Delta N)-\Delta N\right).\tag{5.12}$$

For reader's convenience, we recall the definition of Hellinger process, below.

Definition 5.2: Let N be an $\mathbb{H}$-local martingale such that $1+\Delta N>0$. We call a Hellinger process of order zero for N, denoted by $h^{(0)}(N,\mathbb{H})$, the process $h^{(0)}(N,\mathbb{H}):=\left(H^{(0)}(N,\mathbb{H})\right)^{p,\mathbb{H}}$ when this projection exists, where

$$H^{(0)}(N,\mathbb{H}):=\frac{1}{2}\langle N^c\rangle^{\mathbb{H}}+\sum\left(\Delta N-\ln(1+\Delta N)\right).\tag{5.13}$$

In the following theorem, we elaborate our first result of this subsection.

Theorem 5.2: *The following equality holds.*

$$\inf_{Z^{\mathbb{G}}\in\mathcal{D}(S^\tau,\mathbb{G})} E\Big[-\ln(Z^{\mathbb{G}}_T)\Big] = \inf_{Z\in\mathcal{D}(S,\mathbb{F})} E\Big[-\ln(Z_{T\wedge\tau}/\mathcal{E}_{\tau\wedge T}(G_-^{-1}\bullet m))\Big]. \quad (5.14)$$

Proof. Thanks to Theorem 3.12 -(c), we deduce that $Z^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau$ belongs to $\mathcal{D}(S^\tau,\mathbb{G})$ for any $Z\in\mathcal{D}(S,\mathbb{F})$, and the following inequality holds

$$\inf_{Z^{\mathbb{G}}\in\mathcal{D}(S^\tau,\mathbb{G})} E\Big[-\ln(Z^{\mathbb{G}}_T)\Big] \leq \inf_{Z\in\mathcal{D}(S,\mathbb{F})} E\Big[-\ln(Z_{T\wedge\tau}/\mathcal{E}_{\tau\wedge T}(G_-^{-1}\bullet m))\Big].$$

To prove the reverse inequality, we consider $Z^{\mathbb{G}}\in\mathcal{D}(S^\tau,\mathbb{G})$, and apply Theorem 3.12. This implies the existence of a triplet $\big(Z^{\mathbb{F}},\varphi^{(o)},\varphi^{(pr)}\big)$ that belongs to

$$\mathcal{D}(S,\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\text{Prog}(\mathbb{F}),P\otimes D),$$

and satisfies

$$\varphi^{(pr)}>-1,\quad P\otimes D-a.e.,\quad -\frac{\widetilde{G}}{G}<\varphi^{(o)}<\frac{\widetilde{G}}{\widetilde{G}-G},\qquad P\otimes D^{o,\mathbb{F}}\text{-a.e..}$$

and

$$Z^{\mathbb{G}}=\frac{(Z^{\mathbb{F}})^\tau}{\mathcal{E}(G_-^{-1}\bullet m)^\tau}\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D).$$

As a result, we get

$$-\ln(Z^{\mathbb{G}})=-\ln\Big((Z^{\mathbb{F}})^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau\Big)-\ln(\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}}))-\ln(\mathcal{E}(\varphi^{(pr)}\bullet D)).$$

Thus, in virtue of Proposition 4.1, the process $-\ln(Z^{\mathbb{G}})$ is uniformly integrable if and only if $-\ln\big((Z^{\mathbb{F}})^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau\big)$, $-\ln(\mathcal{E}(\varphi^{(o)}\bullet N^{\mathbb{G}}))$ and $-\ln(\mathcal{E}(\varphi^{(pr)}\bullet D))$ are uniformly integrable, and hence

$$
\begin{aligned}
E[-\ln(Z_T^{\mathbb{G}})] &= E\left[-\ln\left(Z^{\mathbb{F}}_{\tau\wedge T}/\mathcal{E}_{\tau\wedge T}(G_-^{-1}\bullet m)\right)\right] + E[-\ln(\mathcal{E}_T(\varphi^{(o)}\bullet N^{\mathbb{G}}))] \\
&+E[-\ln(\mathcal{E}_T(\varphi^{(pr)}\bullet D))] \\
&\geq E\left[-\ln\left(Z^{\mathbb{F}}_{\tau\wedge T}/\mathcal{E}_{\tau\wedge T}(G_-^{-1}\bullet m)\right)\right] \\
&\geq \inf_{Z\in\mathcal{D}(S,\mathbb{F})} E\Big[-\ln(Z_{T\wedge\tau}/\mathcal{E}_{\tau\wedge T}(G_-^{-1}\bullet m)\Big].
\end{aligned}
$$

The first inequality is due to the fact that both quantifies $E[-\ln(\mathcal{E}_T(\varphi^{(o)}\bullet N^{\mathbb{G}}))]$ and $E[-\ln(\mathcal{E}_T(\varphi^{(pr)}\bullet D))]$ are nonnegative. Then, the proof of the theorem follows immediately. □

We end this section by describing explicitly the optimal deflator for the model $(\ln, S^\tau, \mathbb{G})$. This requires the predictable characteristics of $(S,\mathbb{F})$ and/or that of $(S^\tau,\mathbb{G})$. Thus, throughout the rest of the thesis, for the sake of simplicity, the random measure μ_S associated with the jumps of S will be denoted for simplicity by μ, while S^c denotes the continuous $\mathbb{F}$-local martingale part of S, and the quadruplet

$$(b,c,F,A)\ \text{ are the predictable characteristics of } (S,\mathbb{F}).$$

Or equivalently *the canonical decomposition* of S (see Section 2.4 and the reference herein for details) is given by

$$S = S_0 + S^c + h\star(\mu-\nu) + b\bullet A + (x-h)\star\mu, \quad h(x) := xI_{\{|x|\leq 1\}}. \tag{5.15}$$

Throughout the rest of this chapter, we consider Jacod's decomposition for the space $\mathcal{L}(S,\mathbb{F})$ and the $\mathbb{F}$-martingale $G_-^{-1}\bullet m$ given by

$$\mathcal{L}(S,\mathbb{F}) := \Big\{\theta\in\mathcal{P}(\mathbb{F})\ \big|\ 1+x^{tr}\theta_t(\omega)>0 \quad P\otimes F_t\otimes dA_t\text{-.a.e}\Big\}, \tag{5.16}$$

$$G_-^{-1}\bullet m = \beta_m\bullet S^c + (f_m-1)\star(\mu-\nu) + g_m\star\mu + m^{\perp}, \tag{5.17}$$

where β_m is an $\mathbb{F}$-predictable and also S^c-integrable process, $m^{\perp} \in \mathcal{M}_{0,loc}(\mathbb{F})$ with $[m^{\perp}, S] = 0$, $f_m \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$ and $g_m \in \mathcal{H}^1_{loc}(\mu, \mathbb{F})$.

Theorem 5.3: *Let $\mathcal{K}_{log}(.)$ be given by (5.3), and suppose S is quasi-left-continuous and σ-special, and $G > 0$. Then the following assertions are equivalent.*

(a) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S, \mathbb{F})$ and the nondecreasing process*

$$G_- \bullet V + G_- \bullet h^E(G_-^{-1} \bullet m, P) - \langle K, m\rangle^{\mathbb{F}} + \Big(\widetilde{G} \bullet H^{(0)}(K, P)\Big)^{p,\mathbb{F}}, \qquad (5.18)$$

is integrable.

(b) *The set $\mathcal{D}_{log}(S^{\tau}, \mathbb{G})$ is not empty.*

(c) *There exists a unique $\widetilde{Z}^{\mathbb{G}} \in \mathcal{D}_{log}(S^{\tau}, \mathbb{G})$ such that*

$$\min_{Z \in \mathcal{D}_{log}(S^{\tau},\mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\Big] \qquad (5.19)$$

(d) *There exists $\widetilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold*

$$E\left[(G_- \bullet \widetilde{V})_T + G_-\Big(\int f_m(x)\mathcal{K}_{log}(\widetilde{\lambda}^{tr}x)F(dx) + \widetilde{\lambda}^{tr}c\widetilde{\lambda}\Big) \bullet A_T\right] < +\infty, \qquad (5.20)$$

$$(\theta - \widetilde{\lambda})^{tr}\left[b + c(\beta_m - \widetilde{\lambda}) + \int_{\mathbb{R}^d\setminus\{0\}} \left(\frac{f_m(x)}{1 + \widetilde{\lambda}^{tr}x}x - h(x)\right)F(dx)\right] \leq 0, \qquad (5.21)$$

$$\widetilde{V} := \widetilde{\lambda}^{tr}\left[b + c(\beta_m - \widetilde{\lambda}) + \int_{\mathbb{R}^d\setminus\{0\}} \left(\frac{f_m(x)}{1 + \widetilde{\lambda}^{tr}x}x - h(x)\right)F(dx)\right] \bullet A. \qquad (5.22)$$

If furthermore one of the above assertions holds, then $\widetilde{Z}^{\mathbb{G}}$ solution to (5.19) and *the process $\widetilde{\lambda}$ of assertion (d) are related via*

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}})\exp(-\widetilde{V}^{\tau}), \quad \textit{where } \widetilde{K}^{\mathbb{G}} := \mathcal{T}(K^{\mathbb{F}}) - G_{-}^{-1}\bullet\mathcal{T}(m), \textit{ and} \tag{5.23}$$

$$K^{\mathbb{F}} := (\beta_m - \widetilde{\lambda})\bullet S^c + \frac{f_m - 1 - \widetilde{\lambda}^{tr}x}{1+\widetilde{\lambda}^{tr}x}\star(\mu-\nu) + \frac{g_m}{1+\widetilde{\lambda}^{tr}x}\star\mu + m^{\perp}. \tag{5.24}$$

Proof. The proof of (b)$\Longleftrightarrow$ (c) is a direct application of Theorem 5.1 for the model $(X,\mathbb{H}) = (S^{\tau},\mathbb{G})$. Thus, the remaining part of the proof will be achieved in three steps. The first step proves the equivalence (a)$\Longleftrightarrow$(b), the second step proves (b)$\Longleftrightarrow$(d) and the last step proves (5.23)-(5.24).

Step 1. Here we prove (a)$\Longleftrightarrow$(b). Let $Z^{\mathbb{G}} \in \mathcal{D}(S^{\tau},\mathbb{G})$. Thus, thanks to Theorem 5.1, there exist two $\mathbb{G}$-local martingales $\mathcal{E}(\varphi^{(0)}\bullet N^{\mathbb{G}})$ and $\mathcal{E}(\varphi^{(pr)}\bullet D)$ and $Z^{\mathbb{F}} := \mathcal{E}(K)\exp(-V) \in \mathcal{D}(S,\mathbb{F})$, where $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and V is an nondecreasing and $\mathbb{F}$-predictable process, such that

$$Z^{\mathbb{G}} = \frac{(Z^{\mathbb{F}})^{\tau}}{\mathcal{E}(G_{-}^{-1}\bullet m)^{\tau}}\mathcal{E}(\varphi^{(0)}\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)}\bullet D).$$

As a result, we obtain

$$-\ln(Z^{\mathbb{G}}) = -\ln((Z^{\mathbb{F}})^{\tau}/\mathcal{E}(G_{-}^{-1}\bullet m)^{\tau}) - \ln(\mathcal{E}(\varphi^{(0)}\bullet N^{\mathbb{G}})) - \ln(\mathcal{E}(\varphi^{(pr)}\bullet D)).$$

Thanks to Proposition 4.1, we deduce that $Z^{\mathbb{G}} \in \mathcal{D}_{log}(S^{\tau},\mathbb{G})$ if and only if the three $\mathbb{G}$-local martingale $(Z^{\mathbb{F}})^{\tau}/\mathcal{E}(G_{-}^{-1}\bullet m)^{\tau}$, $\mathcal{E}(\varphi^{(0)}\bullet N^{\mathbb{G}})$ and $\mathcal{E}(\varphi^{(pr)}\bullet D)$ belong to $\mathcal{D}_{log}(S^{\tau},\mathbb{G})$. Then by combining this with

$$
\begin{aligned}
-\ln((Z^{\mathbb{F}})^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau) &= -\ln((Z^{\mathbb{F}})^\tau) + \ln(\mathcal{E}(G_-^{-1}\bullet m)^\tau)\\
&= \mathbb{G}\text{-local mart.} - \frac{I_{]\!]0,\tau]\!]}}{G_-}\bullet \langle K, m\rangle^{\mathbb{F}} + H^{(0)}(K,P)^\tau\\
&+V^\tau + \frac{I_{]\!]0,\tau]\!]}}{G_-^2}\bullet\langle m\rangle^{\mathbb{F}} - H^{(0)}(G_-^{-1}\bullet m, P)^\tau,
\end{aligned}
$$

we conclude that the process in the RHS term is nondecreasing and $\mathbb{G}$-integrable, or equivalently its $\mathbb{F}$-predictable dual projection ($\mathbb{F}$-compensator) is a nondecreasing and integrable process. This resulting predictable process coincides with the process defined in (5.18) due to

$$
\begin{aligned}
&\left(\frac{I_{]\!]0,\tau]\!]}}{G_-^2}\bullet\langle m\rangle^{\mathbb{F}} - H^{(0)}(G_-^{-1}\bullet m, P)^\tau\right)^{p,\mathbb{F}} = \frac{1}{G_-}\bullet\langle m\rangle^{\mathbb{F}} - \left(\widetilde{G}\bullet H^{(0)}(G_-^{-1}\bullet m,P)\right)^{p,\mathbb{F}}\\
&= \frac{1}{2G_-}\bullet\langle m^c\rangle^{\mathbb{F}} + G_-\bullet\left(\sum(\frac{\Delta m}{G_-})^2\right)^{p,\mathbb{F}} - \left(\sum\widetilde{G}(\frac{\Delta m}{G_-} - \ln(1+\frac{\Delta m}{G_-}))\right)^{p,\mathbb{F}}\\
&= \frac{1}{2G_-}\bullet\langle m^c\rangle^{\mathbb{F}} + \left(\sum(\Delta m + G_-)\ln(1+\frac{\Delta m}{G_-})) - \Delta m)\right)^{p,\mathbb{F}}\\
&= G_-\bullet h^E(G_-^{-1}\bullet m, P).
\end{aligned}
$$

This ends the proof of the equivalence between assertions (a) and (b).

Step 2. Here we prove (b)$\Longleftrightarrow$(d) using Theorem 5.1. To this end, we start deriving the predictable characteristics of $(S^\tau,\mathbb{G})$, denoted by $(b^{\mathbb{G}}, c^{\mathbb{G}}, F^{\mathbb{G}}, A^{\mathbb{G}})$ are given by

$$
\begin{aligned}
&b^{\mathbb{G}} := b + c\beta_m + \int h(x)(f_m(x)-1)F(dx),\ \ \mu^{\mathbb{G}} := I_{]\!]0,\tau]\!]}\star\mu,\ \ c^{\mathbb{G}} := c\\
&\nu^{\mathbb{G}} := I_{]\!]0,\tau]\!]}f_m\star\nu, \quad F^{\mathbb{G}}(dx) := I_{]\!]0,\tau]\!]}f_m(x)F(dx), \quad A^{\mathbb{G}} := A^\tau. \qquad (5.25)
\end{aligned}
$$

Thus, by directly applying Theorem 5.1 to $(S^\tau,\mathbb{G})$, we deduce $\mathcal{D}_{log}(S^\tau,\mathbb{G})\neq\emptyset$ is equivalent to the existence of a $\mathbb{G}$-predictable process $\varphi \in \mathcal{L}(S^\tau, \mathbb{G})$ satisfying

$$E\left[V_T^{\mathbb{G}} + \frac{1}{2}(\varphi^{tr} c^{\mathbb{G}} \varphi \bullet A^{\mathbb{G}})_T + (\mathcal{K}_{log}(\varphi^{tr} x) \star \mu^{\mathbb{G}})_T\right] < +\infty, \tag{5.26}$$

$$V^{\mathbb{G}} := |\varphi^{tr} b^{\mathbb{G}} - \varphi^{tr} c^{\mathbb{G}} \varphi + \int [\frac{\varphi^{tr} x}{1 + \varphi^{tr} x} - \varphi^{tr} h(x)] F^{\mathbb{G}}(dx)| \bullet A^{\mathbb{G}}, \tag{5.27}$$

$$(\theta - \varphi)^{tr}(b^{\mathbb{G}} - c^{\mathbb{G}}\varphi) + \int \left(\frac{(\theta - \varphi)^{tr} x}{1 + \varphi^{tr} x} - (\theta - \varphi)^{tr} h(x) \right) F^{\mathbb{G}}(dx) \leq 0, \tag{5.28}$$

for any bounded $\theta \in \mathcal{L}(S^\tau, \mathbb{G})$. Thanks to Proposition 3.1, we deduce the existence of $\widetilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that $\varphi I_{]\!]0,\tau]\!]} = \widetilde{\lambda} I_{]\!]0,\tau]\!]}$ $P \otimes A$-a.e.. Thus, by inserting this in (5.26)-(5.27)-(5.28), we conclude that $V^{\mathbb{G}} = \widetilde{V}^\tau \in \mathcal{A}^+_{loc}(\mathbb{G})$, which is equivalent to (5.22) due to Lemma 3.1, and

$$E\left[(G_- \bullet \widetilde{V})_T + \frac{1}{2}(G_- \widetilde{\lambda}^{tr} c \widetilde{\lambda} \bullet A)_T + \int \mathcal{K}_{log}(\widetilde{\lambda}^{tr} x) f_m(x) F(dx) G_- \bullet A_T\right] < +\infty,$$

$$(\theta - \widetilde{\lambda})^{tr}(b + c(\beta_m - \widetilde{\lambda})) + \int \left(\frac{(\theta - \widetilde{\lambda})^{tr} x}{1 + \widetilde{\lambda}^{tr} x} f_m(x) - (\theta - \widetilde{\lambda})^{tr} h(x) \right) F(dx) \leq 0,$$

$P \otimes A$-a.e. on $]\!]0, \tau]\!]$ for any bounded $\theta \in \mathcal{L}(S, \mathbb{F})$. The above first inequality is obviously (5.20), while (5.21) follows immediately from combining the second inequality above and Lemma 3.1 again. This proves (b)$\Longrightarrow$(d), while the converse follows from the fact that assertion (d) implies (5.26)-(5.27)-(5.28) with $\varphi = \widetilde{\lambda} I_{]\!]0,\tau]\!]}$. This latter fact is obviously equivalent to assertion (b) due to Theorem 5.1 as stated above. This ends the second step, and the proof of the theorem is complete. □

5.2 Particular cases for τ

This section illustrates the results of this chapter on several frequently studied models for the random time τ. We will consider the case of pseudo-stopping times, and the case when all $\mathbb{F}$-martingales are continuous.

5.2.1 Pseudo-stopping times

In this subsection, we consider the family of pseudo-stopping times, τ, defined in Definition 3.1. Here, we recall some of their important properties elaborated in Lemma 3.5. τ is a pseudo-stopping time if and only if $m \equiv m_0$. This implies that $\mathcal{E}\left(G_-^{-1} \bullet m\right) \equiv 1$, $\Delta m \equiv 0$ and for any bounded $\mathbb{F}$-local martingale M, we have $\mathcal{T}(M) = M^\tau$. The Jacod's components for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$ are simply given by the vector of processes $(0, 1, 0, 0)$ in equation (5.16), since $m \equiv m_0$.

Theorem 5.4: *Let $\mathcal{K}_{log}(.)$ be given by (5.3), and suppose S is quasi-left-continuous and σ-special, τ is a pseudo-stopping time, and $G > 0$. Then the following assertions are equivalent:*

(a) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S, \mathbb{F})$ and the nondecreasing process*

$$G_- \bullet V + G_- \bullet \left(H^{(0)}(K, P)\right)^{p,\mathbb{F}} \quad \text{is integrable.}$$

(b) *The set $\mathcal{D}_{log}(S^\tau, \mathbb{G})$ is not empty.*

(c) *There exists a unique $\widetilde{Z}^{\mathbb{G}} \in \mathcal{D}_{log}(S^\tau, \mathbb{G})$ such that*

$$\min_{Z \in \mathcal{D}_{log}(S^\tau, \mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\Big] < +\infty. \tag{5.29}$$

(d) *There exists $\widetilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold*

$$E\left[(G_- \bullet \widetilde{V})_T + G_-(\int \mathcal{K}_{log}(\widetilde{\lambda}^{tr}x)F(dx) + \widetilde{\lambda}^{tr}c\widetilde{\lambda}) \bullet A_T\right] < +\infty,$$

$$(\theta - \widetilde{\lambda})^{tr}\left[b - c\widetilde{\lambda} + \int_{\mathbb{R}^d\setminus\{0\}}\left(\frac{x}{1 + \widetilde{\lambda}^{tr}x} - h(x)\right)F(dx)\right] \leq 0, \tag{5.30}$$

$$\widetilde{V} := \widetilde{\lambda}^{tr}\left[b - c\widetilde{\lambda} + \int_{\mathbb{R}^d\setminus\{0\}}\left(\frac{x}{1 + \widetilde{\lambda}^{tr}x} - h(x)\right)F(dx)\right] \bullet A. \tag{5.31}$$

Furthermore, if one of the above assertions holds, then $\widetilde{Z}^{\mathbb{G}}$ solution to (5.29) and *the process $\widetilde{\lambda}$ of assertion (d) are related via*

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{F}})^{\tau} \exp(-\widetilde{V}^{\tau}), \quad \textit{where } \widetilde{K}^{\mathbb{F}} := -\widetilde{\lambda} \bullet S^c - \frac{\widetilde{\lambda}^{tr} x}{1 + \widetilde{\lambda}^{tr} x} \star (\mu - \nu). \tag{5.32}$$

Proof. Thanks to Theorem 5.3, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ solution to (5.29) exists and the proof of all assertions follows immediately. For the family of pseudo-stopping times, the inequality condition (5.21) reduces to the condition (5.30). □

Remark. By comparing both, the optimality condition (5.30) and the solution for process $\widetilde{V}$ in (5.31), with the inequality condition (4.20) and process $\widetilde{V}$ in (4.19) in Theorem 4.2, we deduce that the optimal deflator for the model $(S, \mathbb{F})$ and the model $(S^{\tau}, \mathbb{G})$ for the family of pseudo-stopping times coincides on $[\![0, \tau]\!]$ i.e.

$$\widetilde{Z}^{\mathbb{G}} = \widetilde{Z}^{\mathbb{F}} \quad \text{on} \quad [\![0, \tau]\!].$$

5.2.2 The case when all $\mathbb{F}$-martingales are continuous

Now, we address a random time τ when all $\mathbb{F}$-martingales are continuous. If all $\mathbb{F}$-martingales are continuous, simply we have $\Delta m = 0$. For the definition and properties of τ, we refer the reader to Section 3.3.3. If all $\mathbb{F}$-martingales are continuous, then processes $\overline{N}^{\mathbb{G}}$ and $\overline{M}$, defined in (3.29), coincide with $N^{\mathbb{G}}$ and $\mathcal{T}(M)$, respectively.

Throughout the rest of this subsection, we consider the following Jacod's decomposition for the $\mathbb{F}$-martingale $G_{-}^{-1} \bullet m$,

$$G_{-}^{-1} \bullet m = \beta_m \bullet S^c + m^{\perp}. \tag{5.33}$$

As a result, the Jacod's parameters of $G_{-}^{-1} \bullet m$ is $(\beta_m, 1, 0, m^{\perp})$.

Now, we parametrize optimal deflators for this model in the following theorem.

Theorem 5.5: *Let $\mathcal{K}_{log}(.)$ be given by (5.3), and suppose S is an $\mathbb{F}$-semimartingale* *that is quasi-left-continuous and all $\mathbb{F}$-martingales are continuous, and $G > 0$. Then the following assertions are equivalent.*

(a) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ such that $\mathcal{E}(K) \in \mathcal{Z}_{loc}(S,\mathbb{F})$ and the nondecreasing process*

$$\frac{1}{2G_-} \bullet \langle m \rangle^{\mathbb{F}} - \langle K, m\rangle^{\mathbb{F}} + \frac{G_-}{2} \bullet \langle K\rangle^{\mathbb{F}} \quad \textit{is integrable.}$$

(b) *The set $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ is not empty if and only of there exists a unique $\widetilde{Z}^{\mathbb{G}}$ belongs to $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ such that*

$$\min_{Z\in\mathcal{D}_{log}(S^\tau,\mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\Big] < +\infty. \tag{5.34}$$

(c) *There exists $\widetilde{\lambda} \in \mathcal{L}(S,\mathbb{F})$ such that the following hold*

$$E\left[G_- \int \widetilde{\lambda}^{tr} c \widetilde{\lambda} \bullet A_T\right] < +\infty, \quad \textit{and} \quad b + c(\beta_m - \widetilde{\lambda}) = 0. \tag{5.35}$$

If furthermore one of the above assertions holds, then $\widetilde{Z}^{\mathbb{G}}$ solution to (5.34) and the process $\widetilde{\lambda}$ are related via

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}}), \quad \textit{where} \quad K^{\mathbb{G}} := -\widetilde{\lambda} \bullet \overline{S}^c. \tag{5.36}$$

Proof. **(a)** The existence of the unique optimal deflator $\widetilde{Z}^{\mathbb{G}}$ follows from Theorem 5.3, immediately. The integrablility condition (5.18) is equal to

$$\frac{1}{2G_-} \bullet \langle m \rangle^{\mathbb{F}} - \langle K, m\rangle^{\mathbb{F}} + \frac{G_-}{2} \bullet \langle K\rangle^{\mathbb{F}} \quad \text{and it is integrable.}$$

Since all $\mathbb{F}$-martingales are continuous, we deduce that the measure jumps μ of S is $\mathbb{F}$-predictable that means that $\mu = \nu$ (equals to its $\mathbb{F}$-predictable dual). Hence $F \equiv 0$, and S is predictable as well as a special semimartingale. Since $\mathcal{Z}_{loc}(S,\mathbb{F}) \neq 0$, we deduce that S is continuous. Suppose $W = I_{\{|x|\geq k\}}$ and $0 < k < +\infty$. Then, since $W \star \mu$ belongs to $\mathcal{A}^+_{loc}(,\mathbb{F})$, we have

$$W \star \mu - (W \star \mu)^{p,\mathbb{F}} = W \star \mu - W \star \nu \in \mathcal{M}_{loc}(\mathbb{F}).$$

Hence, we have

$$W \star \mu = W \star \nu, \quad \text{or equivalently,} \quad I_{\{|x|\geq k\}} \star \mu = I_{\{|x|\geq k\}} \star \nu,$$

for any $k \geq 0$. Then, let k goes to zero, therefore we have $\mu = \nu$. It means that S is a predictable semimartingale, locally bounded, and $F \equiv 0$. Furthermore when $\mathcal{D}_{log}(S,\mathbb{F}) \neq 0$ one can prove that S is continuous and $\mathcal{L}(S,\mathbb{F}) = \mathbb{R}$. It implies that $V \equiv 0$. The rest of proof follows immediately by considering the above-mentioned Jacod components of $G_-^{-1} \bullet m$. □

5.3 Particular cases for $(S,\mathbb{F})$

In this section, we address the optimal deflator problem for several frequently studied cases of the general market model. More precisely, it contains two main subsections by considering two models with particular examples: The general Lévy market model and volatility market models.

5.3.1 The exponential Lévy market model

In this subsection we address the Lévy market model. Consider the Lévy market models given by Section 3.4.1. The stock price process, S, presented by $S_t = S_0\mathcal{E}(X)_t$, is a locally bounded Lévy process and setup by the stochastic differential equation (3.33). All elements are all the same way as it mentioned in Section 3.4.1, where $\sigma > 0$ and μ are bounded adapted processes. Filtration $\mathbb{F}$ is generated by W and $\widetilde{N}^{\mathbb{F}}$, where W is a standard Brownian Motion, $N(dt,dx)$ and is a Poisson random measure on $[0,T]\times\mathbb{R}/\{0\}$, $\widetilde{N}^{\mathbb{F}}$ is a compensated Poisson measure with Lévy measure (intensity) $F^X(dx)dt$ and A^X is an increasing and predictable process. For the model (3.33), the predictable characteristics of Section 5.1 or Section 2.4 can be derived as follows.

$$\mu(dt,dx) = N(dt,dx),\ \nu(dt,dx) = F^X(dx)dt,\ S^c = S_-\sigma \bullet W\ ,$$
$$A_t = A_t^X,\ c = (S_-\sigma)^2,\ b = S_-\mu. \tag{5.37}$$

As a result, throughout the rest of this subsection, we consider the following Jacod's decomposition for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$ and the space $\mathcal{L}(S,\mathbb{F})$,

$$\mathcal{L}(S,\mathbb{F}) := \Big\{\lambda \in \mathcal{P}(\mathbb{F}) \ \big|\ 1 + S_- x^{tr}\lambda_t(\omega) > 0 \quad P\otimes F_t^X \otimes dA_t^X\text{-.a.e}\Big\}, \tag{5.38}$$
$$G_-^{-1} \bullet m = \hat\beta_m \bullet W + (\hat f_m - 1)\star(\mu-\nu) + \hat g_m \star \mu + m^\perp. \tag{5.39}$$

Here, $(\hat\beta_m, \hat f_m, \hat g_m, \hat m^\perp)$ are the Jacod's components of $G_-^{-1}\bullet m$, where $\hat\beta_m(S_-\sigma)^{-1} := \beta_m$ and $(\hat f_m(S_-x), \hat g_m(S_-x), \hat m^\perp) = (f_m(x), g_m(x), m^\perp)$ in equation (5.16).

Theorem 5.6: *Suppose S given by (3.33) and $G > 0$. Then the following assertions are equivalent.*

(a) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S,\mathbb{F})$ and the nondecreasing process*

$$G_-\bullet V + G_-\bullet h^E(G_-^{-1}\bullet m, P) - \langle K, m\rangle^{\mathbb{F}} + \Big(\widetilde G \bullet H^{(0)}(K,P)\Big)^{p,\mathbb{F}}, \tag{5.40}$$

is integrable.

(b) *The set $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ is not empty.*

(c) *There exists a unique $\widetilde Z^{\mathbb{G}} \in \mathcal{D}_{log}(S^\tau,\mathbb{G})$ such that*

$$\min_{Z\in\mathcal{D}_{log}(S^\tau,\mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde Z_T^{\mathbb{G}})\Big] < +\infty. \tag{5.41}$$

(d) *There exists $\widetilde{\varphi} \in \mathcal{L}(S, \mathbb{F})$ such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold*

$$(\theta - \widetilde{\varphi})^{tr} \left[\mu + \sigma\hat{\beta}_m - \sigma^2\widetilde{\varphi} + \int (\frac{\hat{f}_m(S_- x)}{1 + \widetilde{\varphi}x} - I_{S_-|x|<1}) S_- x F^X(dx) \right] \leq 0, \; P \otimes dA^X\text{-}a.s. \quad (5.42)$$

$$\mathbb{E}\left[(G_- \cdot \widetilde{V})_T + G_-(\int \hat{f}_m(S_- x)\mathcal{K}_{log}(\widetilde{\varphi}^{tr}x)F(dx) + \widetilde{\varphi}^{tr}(\sigma)^2\widetilde{\varphi}) \cdot A^X_T \right] < +\infty,$$

$$\widetilde{V} := \widetilde{\varphi}^{tr} \left[\mu + \sigma\hat{\beta}_m - \sigma^2\widetilde{\varphi} + \int (\frac{\hat{f}_m(S_- x)}{1 + \widetilde{\varphi}x} - I_{S_-|x|<1}) S_- x F^X(dx) \right] \cdot A^X. \quad (5.43)$$

Furthermore, when one of the above assertions holds, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ solution to (5.41) and the process $\widetilde{\varphi}$ of assertion (d) are related via the following

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}}) \exp(-\widetilde{V}^{\tau}), \quad \textit{where} \quad \widetilde{K}^{\mathbb{G}} = \mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \cdot \mathcal{T}(m) \quad (5.44)$$

$$\textit{and} \quad K^{\mathbb{F}} := (\hat{\beta}_m - \sigma\widetilde{\varphi}) \cdot W + \frac{\hat{f}_m - 1 - \widetilde{\varphi}^{tr}x}{1 + \widetilde{\varphi}^{tr}x} \star (\mu - \nu) + \frac{\hat{g}_m}{1 + \widetilde{\varphi}^{tr}x} \star \mu + \hat{m}^{\perp}.$$

Proof. Thanks to Theorem 5.3, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ solution to (5.41) exists. For the Lévy model, consider the above-mentioned predictable characteristics (5.37). Then, the inequality condition (5.21), characterizing the optimal deflator $\widetilde{Z}^{\mathbb{G}}$, becomes as follows.

$$\begin{aligned} 0 &\geq \mu + S_-\sigma\hat{\beta}_m - (S_-\sigma)^2\widetilde{\lambda} + \int (\frac{\hat{f}_m(S_- x)}{1 + S_-\widetilde{\lambda}x} S_- x - S_- x I_{\{|x|S_- \leq 1\}}) F^X(dx) \\ &= \mu + \sigma\hat{\beta}_m - S_-\sigma^2\widetilde{\lambda} + \int (\frac{\hat{f}_m(S_- x)}{1 + S_-\widetilde{\lambda}x} x - x I_{\{|x|S_- \leq 1\}}) F^X(dx), \quad P \otimes dA^X\text{-a.s.,} \end{aligned}$$

and the solution for $\mathbb{F}$-predictable process $\widetilde{V}$ reduces to (5.43). Therefore, $\widetilde{\varphi} = \widetilde{\lambda} S_-$, where $\widetilde{\lambda}$ is given by (5.21). We deduce that $1 + xS_-\widetilde{\varphi} > 0$ and $\widetilde{\varphi}$ is the unique solution to (5.42). As a result, the optimal deflator follows immediately using the decomposition of deflator (5.24) in Theorem 5.3, putting $\widetilde{\lambda} = \frac{\widetilde{\varphi}}{S_-}$ and above predictable characteristics for the Lévy process S. □

5.3.1.1 Jump-diffusion framework

This subsection focuses on the important case of jump-diffusion model defined in Section 3.4.1.1. On this case, the uncertainties in the initial model $(S,\mathbb{F})$ is a one-dimensional process generated by a standard Brownian Motion W and a Poisson process N with intensity $\lambda > 0$, defined on the probability space $(\Omega, \mathcal{F}, P)$. Then the stock price process is given by the following dynamics

$$S_t = S_0\mathcal{E}(X)_t, \quad X_t = \int_0^t \sigma_s dW_s + \int_0^t \zeta_s dN_s^{\mathbb{F}} + \int_0^t \mu_s ds, \qquad N_t^{\mathbb{F}} := N_t - \lambda t. \tag{5.45}$$

All elements are all the same way as in Section 3.4.1.1, where a constant $\delta \in (0,+\infty)$ such that μ, σ, and ζ are bounded $\mathbb{F}$-adapted processes and we also assume that $\sigma > 0,\ \zeta > -1,\ \sigma + |\zeta| \geq \delta,\ P\otimes dt$-a.e.. Since m is $\mathbb{F}$-martingale, we consider Jacod's decomposition for the $\mathbb{F}$-martingale $G_-^{-1}\bullet m$ and the space $\mathcal{L}(S,\mathbb{F})$ given by

$$\begin{aligned}
&\int_0^{\cdot} \left((\varphi^{(m)})^2 + |\psi^{(m)}|\right) ds < +\infty \quad P\text{-a.s.}\\
&G_-^{-1}\bullet m = \varphi_s^{(m)}\bullet W + (\psi_s^{(m)} - 1)\bullet N^{\mathbb{F}}, \\
&\mathcal{L}(S,\mathbb{F}) = \Big\{\mathbb{F}\text{-predictable process } \theta \ \Big|\ 1 + x^{tr}\theta_t(\omega) > 0 \quad P(d\omega)\otimes F_t(dx)\otimes dA_t\text{-.a.e}\Big\}.
\end{aligned} \tag{5.46}$$

Theorem 5.7: *Suppose S given by (5.45), $G > 0$, and let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *There exists $\mathcal{E}(K) \in \mathcal{Z}(S,\mathbb{F})$, where $K \in \mathcal{M}_{0,loc}(\mathbb{F})$, such that the nondecreasing process*

$$G_-\bullet h^E(G_-^{-1}\bullet m, P) - \langle K, m\rangle^{\mathbb{F}} + \left(\widetilde{G}\bullet H^{(0)}(K,P)\right)^{p,\mathbb{F}}$$

is integrable.

(b) *The set $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ is not empty if and only if there exists a unique $\widetilde{Z}^{\mathbb{G}} \in$*

$\mathcal{D}_{log}(S^\tau, \mathbb{G})$ *such that*

$$\min_{Z\in\mathcal{D}_{log}(S^\tau,\mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\Big] < +\infty. \tag{5.47}$$

(c) *There exists a unique solution* $\widetilde{\varphi} \in \mathcal{L}(S,\mathbb{F})$ *for the following hold,*

$$E\left[G_-(\int \frac{-\widetilde{\varphi}\lambda\psi^{(m)}}{1+\widetilde{\varphi}\lambda} + \psi^{(m)}\ln(1+\widetilde{\varphi}\lambda) + (\widetilde{\varphi}S_-\sigma)^2 dt\right] < +\infty,$$

$$\mu - \lambda\zeta + \sigma\varphi^{(m)} - \sigma^2\varphi + \frac{\varphi^{(m)}\lambda\zeta}{1+\varphi\zeta} = 0.\ dt - a.e. \tag{5.48}$$

Furthermore, when one of the above assertions holds, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ *solution to (5.47) and the process* $\widetilde{\varphi}$ *are related via the following*

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}}) \quad \text{and} \quad \widetilde{K}^{\mathbb{G}} := -\sigma\widetilde{\varphi} \bullet \overline{W} + \left(\frac{\sigma^2}{\lambda\zeta}\widetilde{\varphi} - \frac{\mu}{\lambda\zeta} - \frac{\varphi^{(m)}\sigma}{\lambda\zeta} - \psi^{(m)}\right) \bullet \mathcal{T}(N^{\mathbb{F}}). \tag{5.49}$$

Proof. For the model (5.45), the predictable characteristics of Section 5.1 can be derived as follows. Let $\delta_a(dx)$ be the Dirac mass at the point a. Then in this case we have $d = 1$ and

$$\mu(dt,dx) = \delta_{\zeta_t S_{t-}}(dx)dN_t,\ \nu(dt,dx) = \delta_{\zeta_t S_{t-}}(dx)\lambda dt,\ F_t(dx) = \lambda\delta_{\zeta_t S_{t-}}(dx),$$
$$A_t = t,\ c = (S_-\sigma)^2,\ b = (\mu - \lambda\zeta I_{\{|\zeta|S_->1\}})S_-,\ (\beta_m, g_m, m^\perp) = (\frac{\varphi^{(m)}}{S_-\sigma}, 0, 0).$$

As a result, the set

$$\begin{aligned}\mathcal{L}_{(\omega,t)}(S,\mathbb{F}) &:= \{\varphi\in\mathbb{R} \mid \varphi x > -1\ F_{(\omega,t)}(dx) - a.e.\} = \{\varphi\in\mathbb{R} \mid \varphi S_-\zeta > -1\}\\ &= \big(-1/(S_-\zeta)^+, +\infty\big) \cap \big(-\infty, 1/(S_-\zeta)^-\big)\end{aligned}$$

is an open set in $\mathbb{R}$ (with the convention $1/0^+ = +\infty$). Then the condition (5.21), characterizing the optimal deflator, becomes an equation as follows.

$$\begin{aligned}0 &= \mu - \lambda\zeta I_{\{|\zeta|>1/S_-\}} + S_-\sigma^2(\frac{\varphi^{(m)}}{S_-\sigma} - \hat{\varphi}) + \lambda\frac{\psi^{(m)}\zeta}{1+S_-\hat{\varphi}\zeta} - \lambda\zeta I_{\{|\zeta|\leq 1/S_-\}}\\ &= \mu + \sigma\varphi^{(m)} - S_-\sigma^2\hat{\varphi} + \frac{\psi^{(m)}\lambda\zeta}{1+\hat{\varphi}\zeta} - \lambda\zeta I_{\{|\zeta|\leq 1/S_-\}} - \lambda\zeta I_{\{|\zeta|>1/S_-\}}\\ &= \mu - \lambda\zeta + \sigma\varphi^{(m)} - S_-\sigma^2\hat{\varphi} + \frac{\psi^{(m)}\lambda\zeta}{1+\hat{\varphi}S_-\zeta}. \qquad (5.50)\end{aligned}$$

By changing the variable $\varphi := \hat{\varphi}S_-$, the above equation is equivalent to (5.48). Hence, $\widetilde{\varphi}$ is the unique solution to (5.47). It is also clear that $\widetilde{\varphi}$ is S-integrable due to the assumption in (3.39).

For a model like Jump-Diffusion framework with nice features, one can prove the theorem directly as follows. The proof of (a)$\Longleftrightarrow$ (b) immediately follows from Proposition 4.1. Then thanks to Theorem 5.2, the proof is to determine the functional $H^{(0)}(Z^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau)$ and its compensator. Then, we minimize the compensator functional in the sense of Definition 3.2.

By the decomposition of $G_-^{-1}\bullet m$, we get $\widetilde{G} = G_- + \Delta m = G_-\psi^{(m)}(\Delta N)I_{\{\Delta N\neq 0\}}$. Therefore,

$$\begin{aligned}&\inf_{Z\in\mathcal{Z}_{loc}(S,\mathbb{F})}(H^{(0)}(Z^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau)^{p,\mathbb{F}}\\ &= \inf_{(\psi_1,\psi_2)\in L^1_{loc}(W)\times L_{loc}(N^{\mathbb{F}},\mathbb{F})}\int_0^T G_-(t)\frac{1}{2}(\psi_1(t)-\varphi^{(m)}(t))^2\\ &\quad + G_-\lambda\big[\psi_2(t)-\psi^{(m)}(t)\big] - \lambda\psi^{(m)}(t)G_-(t)\big(\ln(\psi_2(t)) + \ln\psi^{(m)}(t)\big)dt,\end{aligned}$$

where the minimization is over the set of (ψ_1,ψ_2) such that

$$\mu_t + \psi_1\sigma + (\psi_2-1)\zeta\lambda_t \equiv 0, \quad \psi_2 > 0 \quad dt - a.e.$$

By solving the above-mention constraint for ψ_1, we get $\psi_1 = -\dfrac{\gamma(\psi_2-1)+\mu}{\sigma}$. By putting it in above minimization, it reduce to the following problem

$$\inf_{\psi_2 \in L_{loc}(N^{\mathbb{F}},\mathbb{F})} G_-(t)\Big[\frac{1}{2}(\frac{\gamma(\psi_2-1)+\mu}{\sigma}+\varphi^{(m)})^2 + \lambda\big[\psi_2-\psi^{(m)}\big] - \lambda\psi^{(m)}G_-\big(\ln(\psi_2)+\ln\psi^{(m)}\big)\Big].$$

We define $\theta := \frac{1}{\sigma}(\dfrac{\gamma(\psi_2-1)+\mu}{\sigma}+\varphi^{(m)})$. Therefore by substituting it in minimization equation and computing the derivative, we obtain

$$\mu - \lambda\zeta + \sigma\varphi^{(m)} - \sigma^2\theta + \frac{\varphi^{(m)}\lambda\zeta}{1+\theta\zeta} = 0, \; P\otimes dt - a.e. \;,$$

which is equal to the equation (5.48). □

5.3.1.2 Black-Scholes market model

In this subsection we focus on the case of Black-Scholes market model. The financial market is setup by following stochastic differential equation

$$S_t = S_0\mathcal{E}(X)_t, \quad X_t := \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds, \tag{5.51}$$

where W is a one-dimensional Brownian motion. Let $\psi_{(m)} = 1$ in the decomposition (5.46). Then the Poisson process vanishes and we get $G_-^{-1}\bullet m := \varphi_{(m)}\bullet W$. The space $\mathcal{L}(S,\mathbb{F})$ is simply equal to $\mathbb{R}$. $\mathcal{L}(S,\mathbb{F})$ is an open set in $\mathbb{R}$ (with the convention $1/0^+ = +\infty$).

Theorem 5.8: *Suppose S given by (5.51) and $G>0$. Let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. There exists a unique $\widetilde{Z}^{\mathbb{G}} \in \mathcal{D}_{log}(S^\tau,\mathbb{G})$ such that*

$$\min_{Z\in\mathcal{D}_{log}(S^\tau,\mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\Big] < +\infty, \tag{5.52}$$

if and only if the nondecreasing process

$$\int_0^T \frac{1}{2G_-}(\varphi_s^{(m)})^2 - \frac{\mu\varphi_s^{(m)}}{\sigma_s} + \frac{G_-}{2}(\frac{\mu}{\sigma_s})^2 ds \quad \textit{is integrable.}$$

Furthermore, if the above assertion holds, the optimal deflator, $\widetilde{Z}^{\mathbb{G}}$*, is a unique solution to (5.52) and it is given by* $\widetilde{Z}^{\mathbb{G}} := \mathcal{E}\big(-(\varphi_{(m)} + \frac{\mu}{\sigma}) \bullet \overline{W}\big)$.

Proof. On one hand, Black-Scholes market model $(S, \mathbb{F})$ is complete. Therefore, by the Theorem 5.2, we have

$$\inf_{Z^{\mathbb{G}} \in \mathcal{D}(S^{\tau}, \mathbb{G})} E\Big[-\ln(Z_T^{\mathbb{G}})\Big] = \inf_{Z \in \mathcal{D}(S, \mathbb{F})} E\Big[-\ln(Z_{T\wedge\tau}/\mathcal{E}_{\tau\wedge T}(G_-^{-1} \bullet m))\Big].$$

It means we reduce the dual problem in an incomplete market to the second dual problem, where the market is complete. On the other hand, all $\mathbb{F}$-martingales are with respect to a Brownian motion and continuous. Hence, $\mathcal{T}(W) = \overline{W}$. This ends the proof. □

5.3.2 Volatility market models

In this subsection, we focus on the case of Volatility market models, which has great applications in the financial industry. For these models, the volatility is a stochastic process. This subsection is divided into two parts. First, we address the optimal deflator problem for the corrected Stein and Stein Model, then we discuss it for the Barndorff-Nielsen Shephard Model. The corrected Stein and Stein Model is continuous and the Barndorff-Nielsen Shephard Model might have Lévy jumps.

5.3.2.1 Corrected Stein and Stein Model

In the corrected Stein and Stein financial market, the price process $(S := S_0\mathcal{E}(X))$ follows the dynamic

$$\begin{aligned} X_t &= X_0 + \int_0^t V_s \sigma_s dW_s^{(1)} + \int_0^t \mu V_s^2 dt \\ V_t &= V_0 + \int_0^t (m - aV_s)dt + \alpha_s dW_s^{(2)} \end{aligned} \tag{5.53}$$

where $W^{(i)},\ i=1,2$ are one-dimensional Brownian motions with the correlation coefficient $\rho \in (-1,+1)$ and all the coefficients σ, α, m, a and μ are the same as in Section 3.4.3.1. Any $\mathbb{F}$-local martingale M represents as follows

$$M = M_0 + \int_0^t h_s^1 dW_s^{(1)} + \int_0^t h_s^2 dW_s^{(2)}, \quad t \in [0,T], \tag{5.54}$$

Therefore, the decomposition for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$ given by is

$$G_-^{-1} \bullet m = \varphi_{(m)} \bullet W^{(1)} + \psi_{(m)} \bullet W^{(2)}. \tag{5.55}$$

The space $\mathcal{L}(S,\mathbb{F})$ is simply equal to $\mathbb{R}$. $\mathcal{L}(S,\mathbb{F})$ is an open set in $\mathbb{R}$ (with the convention $1/0^+ = +\infty$). For the corrected Stein and Stein financial market model, we have the following parametrization for the optimal deflator.

Theorem 5.9: *Suppose S given by (5.53), $G>0$. Let $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.*

(a) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ such that $\mathcal{E}(K) \in \mathcal{Z}_{loc}(S,\mathbb{F})$ and the nondecreasing process*

$$\frac{1}{2G_-} \bullet \langle m \rangle^{\mathbb{F}} - \langle K, m\rangle^{\mathbb{F}} + \frac{G_-}{2} \bullet \langle K \rangle^{\mathbb{F}} \quad \textit{is integrable.}$$

(b) *The set $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ is not empty if and only of there exists a unique $\widetilde{Z}^{\mathbb{G}}$ belongs to $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ such that*

$$\min_{Z\in\mathcal{D}_{log}(S^\tau,\mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}_T^{\mathbb{G}})\Big] < +\infty. \tag{5.56}$$

(c) $E\left[(G_- \int_0^\cdot (\frac{\mu}{\sigma})^2 + \varphi_{(m)}^2 + \rho^2 \psi_{(m)}^2 dt\right] < +\infty.$

Furthermore, when one of the above assertions holds, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ solution to (5.56) is

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}\Big(-(\frac{\mu}{\sigma} + \varphi_{(m)} + \rho\psi_{(m)}) \bullet \overline{W}^{(1)}\Big). \tag{5.57}$$

Proof. By the model (5.53), the predictable characteristics and Jacod's parameters of Section 5.1 are

$$A_t = t,\ c = (S_t\sigma_t V_t)^2,\ b = \mu S_t V_t^2,\ (\beta_m, f_m, g_m, m^{\perp}) = (\frac{\varphi_{(m)}}{\sigma S_t} + \frac{\rho\psi_{(m)}}{\sigma S_t}, 1, 0, m^{\perp}).$$

As a result, the condition (5.21), characterizing the optimal deflator $\widetilde{\lambda}$, becomes an equation as follows.

$$\begin{aligned} 0 &= \mu S V^2 + (S_t\sigma V)^2(\frac{\varphi_{(m)}}{\sigma S} + \frac{\rho\psi_{(m)}}{\sigma S} - \lambda) \\ &= \mu + \sigma\varphi_{(m)} + \sigma_t\rho\psi_{(m)} - \lambda S\sigma^2. \end{aligned} \tag{5.58}$$

Hence, $\widetilde{\lambda} = \dfrac{\mu + \sigma\varphi_{(m)} + \sigma\rho\psi_{(m)}}{S\sigma^2}$. By putting $\widetilde{\lambda}$ in (5.36) and considering the fact that $S^c = S\sigma \cdot W$, we get

$$\widetilde{Z}^{\mathbb{G}} = \mathcal{E}(-\widetilde{\lambda} \cdot \overline{S}^c) = \mathcal{E}(-\frac{\mu + \sigma\varphi_{(m)} + \sigma\rho\psi_{(m)}}{\sigma} \cdot \overline{W}^{(1)}).$$

The last equality occurs from the continuity of all $\mathbb{F}$-martingales (i.e. $\mathcal{T}(W) = \overline{W}$). ☐

5.3.2.2 Barndorff-Nielsen Shephard Model

Now, we turn to the Barndorff-Nielsen Shephard financial market which is presented in Section 3.4.3.2. The filtration is generated by a one dimensional Lévy process Y, where we have $Y = Y^c + \widetilde{Y}^d$. The stock price process is an exponential of a Lévy process such that $S_t = S_0 e^{(X)_t}$ and follows these stochastic differential equations

$$\begin{aligned} dX_s &= (\mu + \xi\sigma_s^2)ds + \sigma_s dY_s^c + d(\rho x \star \widetilde{\mu}_Y)_s \\ d\sigma_s^2 &= -\lambda\sigma_s^2 dt + d(x \star \widetilde{\mu}_Y)_s & (5.59) \\ \frac{dS_t}{S_{t_-}} &= \alpha_t dt + \sigma_t dY_t^c + d(e^{\rho x} - 1) \star (\widetilde{\mu} - \widetilde{\nu})_t, & (5.60) \end{aligned}$$

where $\alpha = \mu + \sigma_t^2(\xi + \frac{1}{2}) + \int (e^{\rho x} - 1)\widetilde{F}(dx)$, and all the coefficients are the same as in Section 3.4.3.2. The model is a quasi-left-continuous and the process A_t is continuous. Thus, there is no loss of generality on defining $A_t = t$. Throughout the rest of this subsection, we consider the following Jacod's decomposition for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$ and the space $\mathcal{L}(S, \mathbb{F})$, given by

$$\mathcal{L}(S, \mathbb{F}) := \Big\{ \lambda \in \mathcal{P}(\mathbb{F}) \ \Big| \ 1 + S_{t-}\lambda^{tr}(e^{\rho x} - 1) > 0 \quad P \otimes \widetilde{F} \otimes dt\text{-.a.e} \Big\}, \quad (5.61)$$

$$G_-^{-1} \bullet m = \hat{\beta}_m \bullet Y^c + (\hat{f}_m - 1) \star (\widetilde{\mu} - \widetilde{\nu}) + \hat{g}_m \star \widetilde{\mu} + m^\perp. \quad (5.62)$$

For the Barndorff-Nielsen Shephard model, the predictable characteristics of Section 5.1 can be derived as follows.

$$b_t = S_{t-}\alpha, \quad S_t^c = S_{t-}\sigma_t \bullet Y_t^c, \quad \Delta S_t = S_{t-}(e^{\rho \Delta \sigma_t^2} - 1), \quad (5.63)$$

$$c_t = S_{t-}^2 \sigma_t^2, \quad \nu^S(dt \times dx) = F_t^S(dx)dt, \quad f(x)F_t^S(dx) = \ f(S_{t-}(e^{\rho x} - 1))\widetilde{F}_t(dx).$$

As a result, the Jacod's components of $G_-^{-1} \bullet m$ is $(\hat{\beta}_m, \hat{f}_m, \hat{g}_m, m^\perp)$, where $\hat{\beta}_m(S_-\sigma)^{-1} := \beta_m$, $f_m(x) = \hat{f}_m(S_{t-}(e^{\rho x} - 1))$ and $g_m = \hat{g}_m(S_{t-}(e^{\rho x} - 1))$ in equation (5.16).

Here, we parametrize optimal deflators for model $(S^\tau, \mathbb{G})$ given by (5.59), in the following theorem.

Theorem 5.10: *Suppose S given by (5.59) and might not be locally bounded and $G > 0$. Then the following assertions are equivalent.*

(a) *There exist $K \in \mathcal{M}_{loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that*

$$G_- \bullet V + G_- \bullet h^E(G_-^{-1} \bullet m, P) - \langle K, m \rangle^{\mathbb{F}} + \Big(\widetilde{G} \bullet H^{(0)}(K, P) \Big)^{p,\mathbb{F}}$$

is integrable and $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S, \mathbb{F})$.

(b) *The set* $\mathcal{D}_{log}(S^\tau,\mathbb{G})$ *is not empty.*

(c) *There exists a unique* $\widetilde{Z}^{\mathbb{G}}\in\mathcal{D}_{log}(S^\tau,\mathbb{G})$ *such that*

$$\min_{Z\in\mathcal{D}_{log}(S^\tau,\mathbb{G})} E\Big[-\ln(Z_T)\Big]=E\Big[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\Big]<+\infty. \tag{5.64}$$

(d) *There exists* $\widetilde{\varphi}\in\mathcal{L}(S,\mathbb{F})$ *such that, for any* $\theta\in\mathcal{L}(S,\mathbb{F})$*, the following hold*

$$\begin{aligned}
&E\left[(G_-\bullet\widetilde{V})_T+\int_0^{\cdot} G_-(\int \hat{f}_m\big(S_-(e^{\rho x}-1)\big)\mathcal{K}_{log}(\widetilde{\varphi}^{tr}(e^{\rho x}-1))\widetilde{F}(dx)+\widetilde{\varphi}^{tr}\sigma^2\widetilde{\varphi})dt\right]<+\infty,\\
&(\theta-\widetilde{\varphi})^{tr}\left[\mu+\sigma^2(\xi+\frac{1}{2})+\hat{\beta}_m\sigma-\sigma^2\widetilde{\varphi}+\int(\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\varphi}(e^{\rho x}-1)}+1)(e^{\rho x}-1)\widetilde{F}(dx)\right]\leq 0,\\
&\widetilde{V}:=\int_0^T\widetilde{\varphi}^{tr}[\mu+\sigma^2(\xi+\frac{1}{2})+\hat{\beta}_m\sigma-\sigma^2\widetilde{\varphi}+\int(\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\varphi}(e^{\rho x}-1)}+1)(e^{\rho x}-1)\widetilde{F}(dx)]dt.
\end{aligned} \tag{5.65}$$

Furthermore, when one of the above assertions holds, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ *solution to (5.19) and the process* $\widetilde{\varphi}$ *are related via the following*

$$\begin{aligned}
&\widetilde{Z}^{\mathbb{G}}:=\mathcal{E}(\widetilde{K}^{\mathbb{G}})\exp(-\widetilde{V}^\tau),\quad \textit{where } \widetilde{K}^{\mathbb{G}}:=\mathcal{T}(K^{\mathbb{F}})-G_-^{-1}\bullet\mathcal{T}(m),\ \textit{and}\\
&K^{\mathbb{F}}:=(\hat{\beta}_m-\sigma\widetilde{\varphi})\bullet Y^c+\frac{\hat{f}_m-1-\widetilde{\varphi}^{tr}(e^{\rho x}-1)}{1+\widetilde{\varphi}^{tr}(e^{\rho x}-1)}\star(\widetilde{\mu}-\widetilde{\nu})+\frac{\hat{g}_m}{1+\widetilde{\varphi}^{tr}(e^{\rho x}-1)}\star\widetilde{\mu}+m^{\perp}.
\end{aligned} \tag{5.66}$$

Proof. Thanks to Theorem 5.3, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ solution to (5.66), exists. Consider the above-mentioned predictable characteristics (6.51) for the model (5.59). Then, the inequality condition (5.21), characterizing the optimal deflator $\widetilde{Z}^{\mathbb{G}}$, becomes as follows.

$$\begin{aligned}
0\ &\geq S_-(\mu+\sigma_t^2(\xi+\frac{1}{2})+\int\big(S_-(e^{\rho x}-1)\big)\widetilde{F}(dx))+(S_{t-})^2\sigma^2(\frac{\hat{\beta}_m}{S_-\sigma}-\widetilde{\lambda})\\
&\quad+\int\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\lambda}S_{t-}(e^{\rho x}-1)}S_{t-}(e^{\rho x}-1)\widetilde{F}(dx)\\
&=S_{t-}\Big((\mu+\sigma_t^2(\xi+\frac{1}{2}))+\hat{\beta}_m-S_{t-}\sigma^2\widetilde{\lambda}+\int(\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\lambda}S_{t-}(e^{\rho x}-1)}+1)(e^{\rho x}-1)\widetilde{F}(dx)\Big),
\end{aligned}$$

and the solution for $\mathbb{F}$-predictable process V reduces to (5.65). Therefore, $\widetilde{\varphi} = \widetilde{\lambda} S_-$, where $\widetilde{\lambda}$ is given by (5.21). We deduce that $1 + xS_-\widetilde{\varphi} > 0$ and $\widetilde{\varphi}$ is the unique solution to (5.65). As a result, the optimal deflator follows immediately using the decomposition of deflator (5.24) in Theorem 5.3, putting $\widetilde{\lambda} = \widetilde{\varphi}/S_-$ and above predictable characteristics for the this model. ☐

5.3.3 Complete market model $(S, \mathbb{F})$

Theorem 5.11: *Suppose $G > 0$, S is quasi-left-continuous and σ-special, such that the model $(S, \mathbb{F})$ is complete with $Z = \mathcal{E}(K^{(1)}) \in \mathcal{Z}(S, \mathbb{F})$. Then the following are equivalent.*

(a) *The nondecreasing process*

$$G_- \bullet h^E(G_-^{-1} \bullet m, P) - \langle K^{(1)}, m\rangle^{\mathbb{F}} + \Big(\widetilde{G} \bullet H^{(0)}(K^{(1)}, P)\Big)^{p,\mathbb{F}}, \tag{5.67}$$

is integrable.

(b) *The set $\mathcal{D}_{log}(S^\tau, \mathbb{G})$ is not empty.*

(c) *There exists a unique $\widetilde{Z}^{\mathbb{G}} \in \mathcal{D}_{log}(S^\tau, \mathbb{G})$ such that*

$$\min_{Z \in \mathcal{D}_{log}(S^\tau, \mathbb{G})} E\Big[-\ln(Z_T)\Big] = E\Big[-\ln(\widetilde{Z}_T^{\mathbb{G}})\Big]. \tag{5.68}$$

Furthermore, when one of the above assertions holds, the optimal deflator $\widetilde{Z}^{\mathbb{G}}$ solution to (5.68) exists and has the following representation

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}}), \quad where \quad \widetilde{K}^{\mathbb{G}} = \mathcal{T}(K^{(1)}) - G_-^{-1} \bullet \mathcal{T}(m). \tag{5.69}$$

Proof. The proof immediately follows fromTtheorem 5.2, since we get

$$\inf_{Z^{\mathbb{G}} \in \mathcal{D}(S^\tau, \mathbb{G})} E\Big[-\ln(Z^{\mathbb{G}}_T)\Big] = \inf_{Z \in \mathcal{D}(S,\mathbb{F})} E\Big[-\ln\big(Z_{T\wedge\tau}/\mathcal{E}_{\tau\wedge T}(G_-^{-1} \bullet m)\big)\Big],$$
$$= \mathcal{E}\big(\mathcal{T}(K^{(1)}) - G_-^{-1} \bullet \mathcal{T}(m)\big),$$

and the set $\mathcal{D}(S, \mathbb{F})$ has only one unique member, since the market $(S, \mathbb{F})$ is complete.

□

Chapter 6

Log-optimal portfolio and numéraire portfolio

This chapter addresses the log-optimal portfolio and the numéraire portfolio for the stopped model $(S^\tau, \mathbb{G})$. Thus, we consider $\Theta(X, \mathbb{H}) := \Theta(\ln, X, \mathbb{H})$, where the set is initially defined in (4.15). Our aim of this chapter is the model $(S^\tau, \mathbb{G})$, when S is an $\mathbb{F}$-semimartingale that is quasi-left-continuous and has some integrability condition (to avoid some technicalities).

This chapter contains 3 sections. The first section addresses the optimal portfolio, solution to the primal problem, for $(S^\tau, \mathbb{G})$. While the second section illustrates our results on various popular models. The last section discusses the impact of the random time τ on the numéraire portfolio. For the reader's convenience, we recall the operator function $\mathcal{T}(.)$, the function $\mathcal{K}_{log}(y)$, and Jacod's decomposition for $G_-^{-1} \centerdot m$ defined in (2.21), (5.3) and (5.16) respectively.

$$
\begin{aligned}
&\mathcal{T}(M) := M^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \centerdot [M, m] + I_{]\!]0,\tau]\!]} \centerdot \Big(\sum \Delta M I_{\{\widetilde{G}=0<G_-\}} \Big)^{p,\mathbb{F}}, \\
&\mathcal{K}_{log}(y) := \frac{-y}{1+y} + \ln(1+y) \quad \text{for any} \quad y > -1, && (6.1) \\
&G_-^{-1} \centerdot m \, = \, \beta_m \centerdot S^c + (f_m - 1) \star (\mu - \nu) + g_m \star \mu + m^\perp, && (6.2)
\end{aligned}
$$

where β_m is an $\mathbb{F}$-predictable and also S^c-integrable process, $m^\perp \in \mathcal{M}_{0,loc}(\mathbb{F})$ with $[m^\perp, S] = 0$, $f_m \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$ and $g_m \in \mathcal{H}^1_{loc}(\mu, \mathbb{F})$.

6.1 Optimal portfolio for $(\ln, S^\tau, \mathbb{G})$: The general setting

This section addresses the optimal investment for the economic model $(\ln, S^\tau, \mathbb{G})$ when S is a general $\mathbb{F}$-semimartingale. Below, we elaborate the main result of this section that characterizes, in different manners and using the processes under $\mathbb{F}$ only, the existence of the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$. In particular, our theorems naturally connects the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$ with the optimal portfolio for $(\ln, S, \mathbb{F})$ under a specific random utility.

Theorem 6.1: *Let $\mathcal{K}_{log}(.)$ be given by (6.1). Suppose $G > 0$, S is quasi-left-continuous and σ-special, and $\mathcal{D}(S, \mathbb{F}) \neq \emptyset$. Then the following are equivalent.*

(a) *There exists $\widetilde{\theta}^{\mathbb{G}} \in \Theta(S^\tau, \mathbb{G})$ such that*

$$\max_{\theta \in \Theta(S^\tau, \mathbb{G})} E\Big[\ln\left(1 + (\theta \bullet S^\tau)_T\right)\Big] = E\Big[\ln\Big(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^\tau)_T\Big)\Big] < +\infty. \tag{6.3}$$

(b) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S, \mathbb{F})$, and*

$$E\left[(G_- \bullet V)_T + (\widetilde{G} \bullet H^{(0)}(K,P))^{p,\mathbb{F}}_T + (G_- \bullet h^E(G_-^{-1} \bullet m, P))_T - \langle K, m\rangle^{\mathbb{F}}_T\right] < +\infty.$$

(c) *There exists $\widetilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ (i.e. $\widetilde{\lambda}$ is $\mathbb{F}$-predictable and $\widetilde{\lambda}^{tr}\Delta S > -1$) such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold*

$$E\left[(G_- \bullet \widetilde{V})_T + (G_- \widetilde{\lambda}^{tr} c \widetilde{\lambda} \bullet A)_T + (G_- \mathcal{K}_{log}(\widetilde{\lambda}^{tr} x) \star \nu)_T\right] < +\infty, \tag{6.4}$$

$$(\theta - \widetilde{\lambda})^{tr}(b + c(\beta_m - \widetilde{\lambda})) + \int (\theta - \widetilde{\lambda})^{tr}\left[\frac{f_m(x)}{1 + \widetilde{\lambda}^{tr} x} x - h(x)\right] F(dx) \leq 0, \tag{6.5}$$

$$\widetilde{V} := \left[\widetilde{\lambda}^{tr} b + \widetilde{\lambda}^{tr} c(\beta_m - \widetilde{\lambda}) + \int \widetilde{\lambda}^{tr}\left(\frac{f_m(x)}{1 + \widetilde{\lambda}^{tr} x} x - h(x)\right) F(dx)\right] \bullet A. \tag{6.6}$$

Furthermore, on $]\!]0,\tau]\!]$, *we have*

$$\widetilde{\theta}^{\mathbb{G}}(1+(\widetilde{\theta}^{\mathbb{G}}\bullet S^{\tau})_{-})^{-1}=\widetilde{\lambda}\quad and\ \widetilde{\theta}^{\mathbb{G}}=\widetilde{\lambda}\mathcal{E}_{-}(\widetilde{\lambda}\bullet S).$$

Proof. The proof of this theorem follows immediately from combining Theorem 4.2 in Chapter 4 and Theorem 5.3 in Chapter 5. □

The following is a consequence of the above theorem, and naturally connects the optimal portfolio for $(\ln, S^{\tau}, \mathbb{G})$ with the optimal portfolio for $(S, \mathbb{F}, \widetilde{U})$ where $\widetilde{U}$ is a random field utility that will be specified.

Theorem 6.2: *Suppose* S *is quasi-left-continuous and* σ*-special,* $\mathcal{E}(G_{-}^{-1}\bullet m)$ *is a martingale and* $G>0$*. Define the measure* Q *as follows*

$$Q:=\mathcal{E}_T(G_{-}^{-1}\bullet m)\bullet P.$$

Then the following assertions are equivalent.

(a) *The optimal portfolio for* $(\ln, S^{\tau}, \mathbb{G})$ *exists*

(b) $(\widetilde{U}, S, \mathbb{F})$ *admits the optimal portfolio, where* $\widetilde{U}(t,x):=\mathcal{E}\left(G_{-}^{-1}\bullet m\right)\ln(x)$ *for any* $x>0$*.*

(c) *The model* $(\ln, S, Q, \mathbb{F})$ *admits the optimal portfolio.*

Furthermore, the three portfolios coincide on $]\!]0,\tau]\!]$ *when they exists.*

Proof. It is clear that (b) $\Longleftrightarrow$ (c) is obvious. Thus, the remaining part of this proof focuses on proving (a) $\Longleftrightarrow$ (c). This follows as a direct applications of Theorems 4.2-6.1 as follows. To this end, we start by noticing that $\nu^{Q}(dt,dx)=f_m(x)\nu(dt,dx)$, and we derive

$$\begin{aligned} S &= S_0+(S^c-\langle S^c, G_{-}^{-1}\bullet m\rangle^{\mathbb{F}})+c\beta_m\bullet A+(h\star(\mu-\nu)-\langle h\star(\mu-\nu), G_{-}^{-1}\bullet m\rangle^{\mathbb{F}}) \\ &\quad +(f_m-1)h\star\nu+b\bullet A+(x-h)\star\mu \\ &\quad = S_0+S^{c,Q}+h\star(\mu-\nu^{Q})+[b+\int(f_m(x)-1)h(x)F(dx)+c\beta_m]\bullet A+(x-h)\star\mu. \end{aligned}$$

Here $S^{c,Q}$ is the continuous local martingale part of S under measure Q. Then the predictable characteristics of $(S, \mathbb{F})$ under Q, denoted by (b^Q, c^Q, F^Q, A^Q), are given by

$$b^Q := b + \int (f_m(x) - 1)h(x)F(dx) + c\beta_m, \quad c^Q := c,$$

$$F^Q(dx) := f_m(x)F(dx), \quad A^Q := A.$$

Therefore, using these characteristics and applying Theorem 4.2, we deduce that $(\ln, S, Q, \mathbb{F})$ admits the optimal portfolio if and only if assertion (d) of Theorem 5.3 (or assertion (c) of Theorem 6.1) holds, which is equivalent to the existence of the optimal portfolio for $(\ln, S^\tau, \mathbb{G})$. This ends the proof of the theorem. □

The following theorem discusses the existence of the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$.

Theorem 6.3: *Suppose $G > 0$, S is quasi-left-continuous and σ-special, and $(\ln, S, \mathbb{F})$ admits the optimal portfolio. Then the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$ exists if and only if*

$$E\left[(G_- \bullet h^E(G_-^{-1} \bullet m, P))_T\right] < +\infty. \tag{6.7}$$

Here $h^E(N, P)$ is given by Definition 4.1, for any $N \in \mathcal{M}_{0,loc}(\mathbb{F})$ such that we have $1 + \Delta N \geq 0$.

Proof. Recall that the process $h^{(E)}(N, \mathbb{H}) := \left(H^{(E)}(N, \mathbb{H})\right)^{p,\mathbb{H}}$, when this projection exists, where $H^{(E)}(N, \mathbb{H}) := \frac{1}{2}\langle N^c\rangle^{\mathbb{H}} + \sum((1 + \Delta N)\ln(1 + \Delta N) - \Delta N)$. Due to Theorem 4.2, $(\ln, S, \mathbb{F})$ admits the optimal portfolio if and only if there exists a deflator, given by $Z := \mathcal{E}(K)\exp(-V)$, where $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and V is a nondecreasing and $\mathbb{F}$-predictable process, such that

$$E[-\ln(Z_T)] = E[V_T + H^{(0)}(K, P)_T] < +\infty.$$

Thus, due to Lemma 4.1, we conclude that $\sqrt{[K,K]}$ is an integrable process (or equivalently K is a martingale such that $\sup_{0\leq t\leq T}|K_t|\in L^1(P)$), and hence the process $\langle K,m\rangle^{\mathbb{F}}$ has integrable variation as m is a BMO martingale, defined in Definition 2.9. Therefore, by combining these facts with Theorem 6.1, we deduce that the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$ exists if and only if (6.7) holds. This ends the proof of the theorem. □

In the following simple but important corollary, we consider the existence of the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$ in a different manner.

Corollary 6.3.1: Suppose $G>0$, S is quasi-left-continuous and σ-special, and

$$E\left[(G_-\bullet h^E(G_-^{-1}\bullet m, P))_T\right]<+\infty.$$

Then the optimal portfolio for $(\ln, S^\tau, \mathbb{G})$ exists if and only if there exists $\mathcal{E}(K-V)$ belongs to $\mathcal{D}(S,\mathbb{F})$, where $K\in\mathcal{M}_{0,loc}(\mathbb{F})$ and V is nondecreasing and $\mathbb{F}$-predictable, such that $G_-\bullet V+\langle K,m\rangle^{\mathbb{F}}+\widetilde{G}\bullet H^{(0)}(K,P)$ has integrable variation.

Proof. The proof follows immediately from Theorem 6.1. □

6.2 Particular cases for the triple $(S,\mathbb{F},\tau)$

In this section, we illustrate our results of Section 6.1 in the various order of generality. This section has three subsections. The first subsection illustrates the result of Theorem 6.1 on several frequently studied models for the random time τ. We will consider the case of pseudo-stopping times and the case when all $\mathbb{F}$-martingales are continuous. In the second (and last) subsection, we address the optimal portfolio problem for the general Lévy market model, and the volatility market models, and the case when $(S,\mathbb{F},P)$ is a complete market model.

6.2.1 Particular cases for τ

The random time that we consider in this subsection is the family of pseudo-stopping times, where it is defined in Definition 3.1. Recall that τ is a pseudo-stopping time if and only if $m \equiv m_0$. Hence $\Delta m = 0$. The Jacod's parameters for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$ is simply given by the vector of processes $(0, 1, 0, 0)$ in equation (6.2). Furthermore, for any bounded $\mathbb{F}$-local martingale M, we have $\mathcal{T}(M) = M^\tau$ (for more details see Lemma 3.5).

Theorem 6.4: *Let $\mathcal{K}_{log}(.)$ be given by (6.1), and suppose S is quasi-left-continuous and σ-special, τ is a pseudo-stopping time, $G > 0$, and $\mathcal{D}(S, \mathbb{F}) \neq \emptyset$. Then the following assertions are equivalent:*

(a) *There exists $\widetilde{\theta}^{\mathbb{G}} \in \Theta(S^\tau, \mathbb{G})$ such that*

$$\max_{\theta \in \Theta(S^\tau, \mathbb{G})} E\Big[\ln\left(1 + (\theta \bullet S^\tau)_T\right)\Big] = E\Big[\ln\Big(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^\tau)_T\Big)\Big] < +\infty. \tag{6.8}$$

(b) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S, \mathbb{F})$ and*

$$E\Big((G_- \bullet V)_T + \Big(G_- \bullet H^{(0)}(K, P)\Big)^{p,\mathbb{F}}_T\Big] < +\infty.$$

(c) *There exists $\widetilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold*

$$E\left[(G_- \bullet \widetilde{V})_T + G_-(\int \mathcal{K}_{log}(\widetilde{\lambda}^{tr}x)F(dx) + \widetilde{\lambda}^{tr}c\widetilde{\lambda}) \bullet A_T\right] < +\infty,$$

$$(\theta - \widetilde{\lambda})^{tr}\left[b - c\widetilde{\lambda} + \int_{\mathbb{R}^d\setminus\{0\}}\left(\frac{x}{1 + \widetilde{\lambda}^{tr}x} - h(x)\right)F(dx)\right] \leq 0, \tag{6.9}$$

$$\widetilde{V} := \widetilde{\lambda}^{tr}\left[b - c\widetilde{\lambda} + \int_{\mathbb{R}^d\setminus\{0\}}\left(\frac{x}{1 + \widetilde{\lambda}^{tr}x} - h(x)\right)F(dx)\right] \bullet A. \tag{6.10}$$

Therefore, the optimal portfolio solution to (6.8) and the process $\widetilde{\lambda}$ are related via $\widetilde{\theta}^{\mathbb{G}}(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^\tau)_-)^{-1} = \widetilde{\lambda}$ on $]\!]0, \tau]\!]$.

Proof. By combing Theorem 5.4 and Theorem 6.1, the optimal portfolio $\widetilde{\theta}^{\mathbb{G}}$ solution to (5.29) exists and the proof of all assertions follows immediately. For the family of pseudo-stopping times, (6.5) and (6.6) become the conditions (6.9) and (6.10) respectively. □

In the following corollary, we combine the above result with another important class of random time (the case when immersion holds). We say that "immersion" holds if and only if τ is a random time such that any $\mathbb{F}$-martingale is an $\mathbb{G}$-local martingale.

Corollary 6.4.1: Suppose that $G > 0$. Then the following conditions are all sufficient for the fact that the optimal portfolios for $(S^\tau, \mathbb{G}, \ln)$ exists if and only if the optimal portfolio for $(S, \mathbb{F}, \ln)$ does also, and both portfolios coincide on $\rrbracket 0, \tau \rrbracket$ when they exist.

(a) τ is a pseudo-stopping time.

(b) τ is such that immersion holds.

Proof. For assertion (a), the proof follows immediately from combining Theorem 6.1 with the fact that τ is a pseudo-stopping time. By comparing both, the optimality condition (6.9) and the process $\widetilde{V}$ in (6.10), with the inequality condition (4.20) and process $\widetilde{V}$ in (4.19) in Theorem 4.2, we deduce that the optimal portfolio for the model $(S, \mathbb{F})$ and the model $(S^\tau, \mathbb{G})$ for the family of pseudo-stopping times coincides, i.e. $\widetilde{\theta}^{\mathbb{G}} = \widetilde{\theta}^{\mathbb{F}}$ on $\rrbracket 0, \tau \rrbracket$. For assertion (b), suppose that the immersion holds then we get $\mathcal{E}\left(G_-^{-1} \bullet m\right) \equiv 1$, this latter fact can be found in [112]. Therefore, the proof follows immediately the same discussion. □

The remaining part of this subsection deals with the case when all $\mathbb{F}$-martingales are continuous. For this case, we have $\Delta m = 0$, and processes $\overline{N}^{\mathbb{G}}$ and $\overline{M}$, defined in (3.29), coincide with $N^{\mathbb{G}}$ and $\mathcal{T}(M)$, respectively. For the definition and properties of τ, we refer the reader to Section 3.3.3. Here, we consider the following decomposition for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$, which we discuss it in Section 5.2.2,

$G_-^{-1} \bullet m = \beta_m \bullet S^c + m^{\perp}$. As a result, the vector of processes, $(\beta_m, 1, 0, m^{\perp})$, is the Jacod components of $G_-^{-1} \bullet m$. We parametrize optimal portfolio for this model in the following theorem.

Theorem 6.5: *Suppose S is an $\mathbb{F}$-semimartingale that is quasi-left-continuous and all $\mathbb{F}$-martingales are continuous, $G > 0$, and $\mathcal{D}(S, \mathbb{F}) \neq \emptyset$. Then the following assertions are equivalent:*

(a) *There exists $\widetilde{\theta}^{\mathbb{G}} \in \Theta(S^{\tau}, \mathbb{G})$ such that*

$$\max_{\theta \in \Theta(S^{\tau}, \mathbb{G})} E\Big[\ln\left(1 + (\theta \bullet S^{\tau})_T\right)\Big] = E\Big[\ln\Big(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^{\tau})_T\Big)\Big] < +\infty. \quad (6.11)$$

(b) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ such that $\mathcal{E}(K) \in \mathcal{Z}_{loc}(S, \mathbb{F})$ and the nondecreasing process*

$$E\left[\frac{1}{2G_-} \bullet \langle m \rangle_T^{\mathbb{F}} - \langle K, m \rangle_T^{\mathbb{F}} + \frac{G_-}{2} \bullet \langle K \rangle_T^{\mathbb{F}}\right] < +\infty.$$

(c) *There exists $\widetilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold*

$$E\left[G_- \int \widetilde{\lambda}^{tr} c \widetilde{\lambda} \bullet A_T\right] < +\infty, \quad \textit{and} \quad b + c(\beta_m - \widetilde{\lambda}) = 0. \quad (6.12)$$

Furthermore, on $]\!]0, \tau]\!]$, we have $\widetilde{\theta}^{\mathbb{G}} = \widetilde{\lambda}\mathcal{E}_-(\widetilde{\lambda} \bullet S)$.

Proof. Under the assumption that all $\mathbb{F}$-martingales are continuous, any semimartingale is predictable. Therefore, it is locally bounded. Thus, the process S is locally bounded $\mathbb{F}$-semimartingale. Since $\mathcal{D}(S, \mathbb{F}) \neq \emptyset$, one can prove that even S is continuous. Hence $\mu \equiv 0 \equiv \nu$ and $F \equiv 0$, and $\mathcal{L}(S, \mathbb{F}) = \mathbb{R}^d$. Therefore, by combining these with Theorem 6.1, the proof of theorem follows immediately. □

6.2.2 Particular cases for $(S, \mathbb{F})$

The first model, that we discuss in this subsection, is the Lévy market model. Consider the Lévy market models given by Section 3.4.1. The stock price process, S, presented by $S_t = S_0 \mathcal{E}(X)_t$, is a locally bounded Lévy process and satisfies the stochastic differential equation (3.33). All elements are all the same as in Section 3.4.1. We recall the predictable characteristics of Section 5.1 for the Lévy here.

$$\begin{aligned} &\mu(dt,dx) = N(dt,dx),\ \nu(dt,dx) = F^X(dx)dt,\ S^c = S_-\sigma \bullet W\ , \\ &A_t = A_t^X,\ c = (S_-\sigma)^2,\ b = S_-\mu. \end{aligned} \tag{6.13}$$

Therefore, we have the following setting

$$\mathcal{L}(S,\mathbb{F}) := \Big\{\theta \in \mathcal{P}(\mathbb{F})\ \big|\ 1 + S_- x^{tr}\theta_t(\omega) > 0 \quad P \otimes F_t^X \otimes dA_t^X\text{-.a.e}\Big\}, \tag{6.14}$$

$$G_-^{-1} \bullet m = \hat{\beta}_m \bullet W + (f_m - 1) \star (\mu - \nu) + g_m \star \mu + m^{\perp}. \tag{6.15}$$

The Jacod's components of $G_-^{-1} \bullet m$ is $(\hat{\beta}_m, \hat{f}_m, \hat{g}_m, \hat{m}^{\perp})$, where $\hat{\beta}_m(S_-\sigma)^{-1} := \beta_m$ and $(\hat{f}_m(S_-x), \hat{g}_m(S_-x), \hat{m}^{\perp}) = (f_m(x), g_m(x), m^{\perp})$ are the same as in (6.2). We parametrize the optimal portfolio for the Lévy model in the following theorem.

Theorem 6.6: *Let $\mathcal{K}_{log}(.)$ be given by (6.1). Suppose $G > 0$, S is quasi-left-continuous and σ-special, and $\mathcal{D}(S,\mathbb{F}) \neq \emptyset$. Then the following are equivalent.*

(a) *There exists $\widetilde{\theta}^{\mathbb{G}} \in \Theta(S^\tau,\mathbb{G})$ such that*

$$\max_{\theta\in\Theta(S^\tau,\mathbb{G})} E\Big[\ln\left(1 + (\theta \bullet S^\tau)_T\right)\Big] = E\Big[\ln\Big(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^\tau)_T\Big)\Big] < +\infty. \tag{6.16}$$

(b) *There exist $K \in \mathcal{M}_{loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S,\mathbb{F})$, and*

$$E\left[(G_- \bullet V)_T + (\widetilde{G} \bullet H^{(0)}(K,P))_T^{p,\mathbb{F}} + (G_- \bullet h^E(\frac{1}{G_-} \bullet m, P))_T - \langle K, m\rangle_T^{\mathbb{F}}\right] < +\infty.$$

(c) *There exists* $\widetilde{\varphi} \in \mathcal{L}(S, \mathbb{F})$ *such that, for any* $\theta \in \mathcal{L}(S, \mathbb{F})$*, the following hold*

$$(\theta - \widetilde{\varphi})^{tr} \left[\mu + \sigma \hat{\beta}_m - \sigma^2 \widetilde{\varphi} + \int \left(\frac{\hat{f}_m(S_- x)}{1 + \widetilde{\varphi} x} - I_{S_- |x| < 1} \right) S_- x F^X(dx) \right] \leq 0, \ P \otimes dA^X\text{-}a.s. \quad (6.17)$$

$$\mathbb{E} \left[(G_- \bullet \widetilde{V})_T + G_- \left(\int \hat{f}_m(S_- x) \mathcal{K}_{log}(\widetilde{\varphi}^{tr} x) F(dx) + \widetilde{\varphi}^{tr} (\sigma)^2 \widetilde{\varphi} \right) \bullet A^X_T \right] < +\infty,$$

$$\widetilde{V} := \widetilde{\varphi}^{tr} \left[\mu + \sigma \hat{\beta}_m - \sigma^2 \widetilde{\varphi} + \int \left(\frac{\hat{f}_m(S_- x)}{1 + \widetilde{\varphi} x} - I_{S_- |x| < 1} \right) S_- x F^X(dx) \right] \bullet A^X. \quad (6.18)$$

Furthermore, on $]\!]0, \tau]\!]$*, we have*

$$\widetilde{\theta}^{\mathbb{G}} (1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^\tau)_-)^{-1} = \widetilde{\varphi} \quad \text{and} \quad \widetilde{\theta}^{\mathbb{G}} = \widetilde{\varphi} \mathcal{E}_-(\widetilde{\varphi} \bullet S).$$

Proof. Thanks to Theorem 6.1, the optimal portfolio $\widetilde{\theta}^{\mathbb{G}}$ solution to (6.16) exists. By considering the above-mentioned predictable characteristics (6.13) and the same discussion in the proof of Theorem 5.6, we deduce that conditions (6.4), (6.5), and (6.6) reduce to (6.17)-(6.18) immediately. □

Now, we consider one of the important example of Lévy model : The case when S follows a jump-diffusion model. The stock price process is given by the following dynamics

$$S_t = S_0 \mathcal{E}(X)_t, \quad X_t = \int_0^t \sigma_s dW_s + \int_0^t \zeta_s dN^{\mathbb{F}}_s + \int_0^t \mu_s ds, \quad N^{\mathbb{F}}_t := N_t - \lambda t, \quad (6.19)$$

and all elements are same as in Subsection 3.4.1.1 and we consider the following decomposition

$$G_-^{-1} \bullet m = \varphi^{(m)} \bullet W + (\psi^{(m)} - 1) \bullet N^{\mathbb{F}}. \quad (6.20)$$

Theorem 6.7: *Suppose* $G > 0$ *and* S *and* X *are given by (6.19). Then the following* $\mathbb{F}$*-predictable process*

$$\widetilde{\theta} := \frac{\xi + sign(\zeta) \sqrt{\xi^2 + 4 \lambda \psi^{(m)}}}{2\sigma} - \frac{1}{\zeta}, \quad \text{where } \xi := \frac{\mu - \lambda \zeta}{\sigma} + \varphi^{(m)} + \frac{\sigma}{\zeta}, \quad (6.21)$$

is S-integrable satisfying $1+\widetilde{\theta}\zeta>0$, and the following assertions hold.

(a) *The solution to*

$$\min_{Z\in\mathcal{D}(S^{\tau},\mathbb{G})} E\left[-\ln(Z_T)\right] = E\left[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\right] < +\infty, \tag{6.22}$$

is given by $\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}})$ where

$$\widetilde{K}^{\mathbb{G}} := -\sigma\widetilde{\theta}\bullet\mathcal{T}(W) - \frac{\psi^{(m)}\zeta\widetilde{\theta}}{1+\widetilde{\theta}\zeta}\bullet\mathcal{T}(N^{\mathbb{F}}). \tag{6.23}$$

(b) *It holds that*

$$\max_{\theta\in\Theta(S^{\tau},\mathbb{G})} \mathbb{E}[\ln(\mathcal{E}_T(\theta\bullet X^{\tau}))] = \mathbb{E}[\ln(\mathcal{E}_T(\widetilde{\theta}\bullet X^{\tau}))]. \tag{6.24}$$

(c) *It holds that*

$$\max_{\theta\in\Theta(\overline{S},\mathbb{F})} \mathbb{E}[\ln(\mathcal{E}_T(\theta\bullet \overline{X}))] = \mathbb{E}\left[\ln\left(\mathcal{E}_T\left(\frac{\widetilde{\theta}}{\psi^{(m)}}\bullet\overline{X}\right)\right)\right] < +\infty, \tag{6.25}$$

where $\overline{S} := S_0\mathcal{E}(\overline{X})$, $\overline{X}_0=0$, and

$$\begin{aligned} d\overline{X} \;&:= \sqrt{\psi^{(m)}}\sigma dW + \psi^{(m)}\zeta dN^{\mathbb{F}} + \\ &+\left[\lambda\zeta(\psi^{(m)}-1)+\mu+\sigma\varphi^{(m)}(1-\sqrt{\psi^{(m)}})\right]ds. \end{aligned} \tag{6.26}$$

(d) *There exists $\mathcal{E}(K)\in\mathcal{Z}(S,\mathbb{F})$, where $K\in\mathcal{M}_{0,loc}(\mathbb{F})$, such that the nondecreasing process*

$$G_-\bullet h^E(G_-^{-1}\bullet m, P) - \langle K, m\rangle^{\mathbb{F}} + \left(\widetilde{G}\bullet H^{(0)}(K,P)\right)^{p,\mathbb{F}}$$

is integrable.

Proof. This proof is achieved in three steps. The first step proves assertions (a) and (b), while step 2 proves assertion (c).

Step 1. For the model (6.19), the predictable characteristics can be derived as follows (for more details see here 2.4). Let $\delta_a(dx)$ be the Dirac mass at the point a. Then in this case we have $d = 1$ and

$$\begin{aligned}
&\mu(dt,dx) = \delta_{\zeta_t S_{t-}}(dx)dN_t,\ \nu(dt,dx) = \delta_{\zeta_t S_{t-}}(dx)\lambda dt,\ F_t(dx) = \lambda\delta_{\zeta_t S_{t-}}(dx),\\
&A_t = t,\ c = (S_-\sigma)^2,\ b = (\mu - \lambda\zeta I_{\{|\zeta|S_->1\}})S_-,\ (\beta_m, g_m, m^\perp) = (\frac{\varphi^{(m)}}{S_-\sigma}, 0, 0).
\end{aligned}$$

As a result, we define the set

$$\begin{aligned}
\mathcal{L}_{(\omega,t)}(S,\mathbb{F}) &:= \{\varphi \in \mathbb{R} \bigm| \varphi x > -1\ F_{(\omega,t)}(dx) - a.e.\} = \{\varphi \in \mathbb{R} \bigm| \varphi S_-\zeta > -1\}\\
&= \left(-1/(S_-\zeta)^+, +\infty\right) \cap \left(-\infty, 1/(S_-\zeta)^-\right).
\end{aligned}$$

Then the condition (6.5), characterizing the optimal portfolio $\widetilde{\varphi}$, becomes an equation as follows.

$$\begin{aligned}
0 &= \mu - \lambda\zeta I_{\{|\zeta|>1/S_-\}} + S_-\sigma^2(\frac{\varphi^{(m)}}{S_-\sigma} - \theta) + \lambda\frac{\psi^{(m)}\zeta}{1 + S_-\theta\zeta} - \lambda\zeta I_{\{|\zeta|\le 1/S_-\}}\\
&= \mu - \lambda\zeta + \sigma\varphi^{(m)} - S_-\sigma^2\theta + \frac{\psi^{(m)}\lambda\zeta}{1 + \theta S_-\zeta}. \qquad (6.27)
\end{aligned}$$

By changing the variable and putting $\varphi := 1 + \theta S_-\zeta > 0$, the above equation is equivalent to

$$0 = -\frac{\sigma^2}{\zeta}\varphi^2 + [\mu - \lambda\zeta + \sigma\varphi^{(m)} + \frac{\sigma^2}{\zeta}]\varphi + \psi^{(m)}\lambda\zeta.$$

This equation has always (since $\psi^{(m)} > 0$) a unique positive solution

$$\widetilde{\varphi} := \frac{\Gamma\zeta + |\zeta|\sqrt{\Gamma^2 + 4\sigma^2\lambda\psi^{(m)}}}{2\sigma^2}, \quad \Gamma := \mu - \lambda\zeta + \sigma\varphi^{(m)} + \frac{\sigma^2}{\zeta}.$$

Hence, we deduce that $\widetilde{\lambda} := \widetilde{\theta}/S_-$ ($\widetilde{\lambda}$ from Theorem 6.1, and it's not the Poisson jump intensity!), where $\widetilde{\theta}$ is given by (6.21), coincides with $(\widetilde{\varphi}-1)/(S_-\zeta)$, satisfies $1+\zeta\widetilde{\theta}>0$, and hence it is the unique solution to (6.27). It is also clear that $\widetilde{\theta}$ is S-integrable (or equivalently $\widetilde{\lambda}$ is S-integrable). As a result, the optimal wealth process is $\mathcal{E}(\widetilde{\lambda}\bullet S^\tau)=\mathcal{E}(\widetilde{\theta}\bullet X^\tau)$ and hence $\widetilde{\theta}$ is the solution to (6.22) and assertions (a) and (b) follow immediately using the above analysis and Theorems 6.1.

Step 2. Herein, we prove assertion (c) using Theorem 4.2. To this end, we calculate the random measure jumps $\overline{\mu}$ and its compensator $\overline{\nu}$, and the predictable characteristics $(\overline{b},\overline{c},\overline{F},\overline{A})$ for the model $(\overline{S},\mathbb{F})$ as follows.

$$\begin{aligned}
&\overline{\mu}(dt,dx):=\mu_{\overline{S}}(dt,dx)=\delta_{\zeta_t\psi_t^{(m)}\overline{S}_{t-}}(dx)dN_t,\ \overline{\nu}(dt,dx)=\delta_{\zeta_t\psi_t^{(m)}\overline{S}_{t-}}(dx)\lambda dt\\
&\overline{b}=\overline{S}_-(\mu-\lambda\zeta+\lambda\psi^{(m)}\zeta I_{\{|\zeta|\overline{S}_-\psi^{(m)}\leq 1/\overline{S}_-\}}+\sigma\varphi^{(m)}(1-\sqrt{\psi^{(m)}})),\ \overline{A}_t=t,\\
&\overline{F}_t(dx)=\lambda\delta_{\zeta_t\psi_t^{(m)}\overline{S}_{t-}}(dx),\ \overline{c}=\psi^{(m)}(\overline{S}_-\sigma)^2,\ \beta_m=\frac{\varphi^{(m)}}{\overline{S}_-\sigma\sqrt{\psi^{(m)}}},\ m^\perp\equiv 0.
\end{aligned}$$

Then similarly as in the first step, we deduce that the set $\mathcal{L}_{(\omega,t)}(\overline{S},\mathbb{F})$ is an open real set (since $\mathcal{L}_{(\omega,t)}(\overline{S},\mathbb{F})=(-(\overline{S}_-\psi^{(m)}\zeta^+)^{-1},+\infty)\cap(-\infty,(\overline{S}_-\psi^{(m)}\zeta^-)^{-1}))$ and hence the condition (6.5) becomes

$$0=\overline{b}+\overline{c}(\beta_m-\varphi)+\int(\frac{x}{1+\varphi x}-h(x))\overline{F}(dx).$$

This is equivalent to

$$0\ =\mu-\lambda\zeta+\sigma\varphi^{(m)}-\sigma^2\psi^{(m)}\overline{S}_-\varphi+\frac{\psi^{(m)}\lambda\zeta}{1+\zeta\psi^{(m)}\overline{S}_-\varphi}.$$

Thus, by comparing this equation to (6.27), we deduce that the optimal strategy for the problem (6.25), that we denote by $\overline{\theta}$ satisfies

$$\overline{\theta}=\overline{S}_-\overline{\varphi}=S_-\widetilde{\varphi}=\frac{\widetilde{\theta}}{\psi^{(m)}}$$

(where $\overline{\varphi}$ is the root of the above equation). This ends the proof of assertion (c), the proof of assertion (d) follows immediately by Theorem (5.7), and the proof of the theorem as well. □

Remark. Here, we can elaborate the duality relations between the optimal deflator and optimal portfolio for jump-diffusion model. The optimal deflator, solution for the dual problem, is given by the Theorem (6.7) and it is as follows

$$\widetilde{Z}^{\mathbb{G}} = \mathcal{E}\left(\widetilde{K}^{\mathbb{G}}\right) \quad , \quad \widetilde{K}^{\mathbb{G}} := -\sigma\theta^* \bullet \mathcal{T}(W) - \frac{\psi^{(m)}\zeta\theta^*}{1+\theta^*\zeta} \bullet \mathcal{T}(N^{\mathbb{F}})$$

or, equivalently,

$$\widetilde{K}^{\mathbb{G}} = (-\sigma\theta^*) \bullet \mathcal{T}(W) + (\frac{\sigma^2}{\gamma}\theta^* - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)}) \bullet \mathcal{T}(N^{\mathbb{F}}),$$

Therefore, we discuss the following duality relations: $X_\tau^{\widetilde{\theta}^{\mathbb{G}}} = -V'(\widetilde{Z}^{\mathbb{G}})$ - a.s. , where X is the optimal wealth process and V is the conjugate function of log-utility. Thus, $-V'(\widetilde{Z}^{\mathbb{G}}) = \frac{1}{(\widetilde{Z}^{\mathbb{G}})} = \frac{1}{\mathcal{E}(\widetilde{K}^{\mathbb{G}})}$. Thanks to Ito's formula, we get

$$d\frac{1}{\widetilde{Z}^{\mathbb{G}}} = -\frac{1}{(\widetilde{Z}^{\mathbb{G}})^2_-}d\widetilde{Z}^{\mathbb{G}} + \frac{1}{(\widetilde{Z}^{\mathbb{G}})^3_-}d < (\widetilde{Z}^{\mathbb{G}})^c >_. + \sum_{0<s<.} \Delta\frac{1}{(\widetilde{Z}^{\mathbb{G}})_s} + \frac{1}{(\widetilde{Z}^{\mathbb{G}})^2_-}\Delta(\widetilde{Z}^{\mathbb{G}})_s$$

$$= \frac{1}{(\widetilde{Z}^{\mathbb{G}})_-}(-d\widetilde{K}^{\mathbb{G}} + d < \widetilde{K}^{\mathbb{G},c} > + \sum_{0<s<.} \frac{(\Delta\widetilde{K}^{\mathbb{G}})^2}{1+\Delta\widetilde{K}^{\mathbb{G}}}) = \frac{1}{(\widetilde{Z}^{\mathbb{G}})_-}(d\ddot{K}^{\mathbb{G}}),$$

where $d\ddot{K}^{\mathbb{G}} := -d\widetilde{K}^{\mathbb{G}} + (1+\Delta\widetilde{K}^{\mathbb{G}})^{-1}d[\widetilde{K}^{\mathbb{G}}]$, or equivalently, $\frac{1}{\mathcal{E}(\widetilde{K}^{\mathbb{G}})} = \mathcal{E}(\ddot{K}^{\mathbb{G}})$. On one hand, by characterization of $\widetilde{K}^{\mathbb{G}}$, we obtain

$$\ddot{K}^{\mathbb{G}} = \sigma\theta^* \bullet \mathcal{T}(W) - (\frac{\sigma^2}{\gamma}\theta^* - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)}) \bullet \mathcal{T}(N^{\mathbb{F}}) + \int_0^T I_{]\!]0,\tau]\!]}(\sigma\theta^*)^2 dt$$

$$+ \int_0^T \left(\frac{(\frac{\sigma^2}{\gamma}\theta^* - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)})^2(\frac{1}{\psi^{(m)}})^2}{1+(\frac{\sigma^2}{\gamma}\theta^* - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)})(\frac{1}{\psi^{(m)}})}\right) \bullet N^\tau$$

$$= \sigma\theta^{*} \bullet \big[W^{\tau} - \int_0^T I_{]\!]0,\tau]\!]}\varphi^{(m)}dt\big] - (\frac{\sigma^2}{\gamma}\theta^{*} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)}) \bullet \big[\frac{1}{\psi^{(m)}} \bullet N^{\tau} - \int_0^T I_{]\!]0,\tau]\!]}\lambda dt\big] +$$

$$\int_0^T I_{]\!]0,\tau]\!]}(\sigma\theta^{*})^2 dt + \int_0^T \Big(\frac{(\frac{\sigma^2}{\gamma}\theta^{*} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)})^2(\frac{1}{\psi^{(m)}})^2}{1 + (\frac{\sigma^2}{\gamma}\theta^{*} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)})(\frac{1}{\psi^{(m)}})}\Big) \bullet N^{\tau}$$

$$= \sigma\theta^{*} \bullet W^{\tau} + [\psi^{(m)}(\frac{\sigma^2}{\gamma}\theta^{*} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma})^{(-1)} - 1] \bullet N^{\tau} +$$

$$\int_0^T I_{]\!]0,\tau]\!]}\Big[(\sigma\theta^{*})^2 - \sigma\theta^{*}\varphi^{(m)} + (\frac{\sigma^2}{\gamma}\theta^{*} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)})\lambda\Big]dt. \qquad (6.28)$$

On the other hand, the wealth process $X_{\tau}^{\widetilde{\theta}^{\mathbb{G}}} = X_0\mathcal{E}(\theta \bullet S^{\tau})$ with an initial endowment W_0 is given by

$$dX_{\tau}^{\widetilde{\theta}^{\mathbb{G}}} = I_{]\!]0,\tau]\!]}X_{-}(\sigma\widetilde{\theta}^{\mathbb{G}}dW_t + \widetilde{\theta}_t^{\mathbb{G}}\mu_t dt + \widetilde{\theta}_t^{\mathbb{G}}\zeta dN_t),$$

where $\widetilde{\theta}^{\mathbb{G}}$ is the self-financing trading strategy at time t.

$$\widetilde{\theta}^{\mathbb{G}} \bullet S^{\tau} = \int_0^T I_{]\!]0,\tau]\!]}\widetilde{\theta}^{\mathbb{G}}\mu_t dt + \sigma\widetilde{\theta}^{\mathbb{G}} \bullet W^{\tau} + \widetilde{\theta}^{\mathbb{G}}\zeta \bullet N^{\tau}. \qquad (6.29)$$

Both coefficient of Brownian motion in (6.28) and (6.29) coincide. For the coefficient of the Poisson process of optimal values, we have

$$\psi^{(m)}(\frac{\sigma^2}{\gamma}\widetilde{\theta}^{\mathbb{G}} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma})^{(-1)} - 1 = \zeta\Big(\frac{\gamma\psi^{(m)}}{\sigma^2\big(\zeta\theta^{*} - \zeta[\frac{\mu}{\sigma} + \varphi^{(m)} + \lambda\frac{\sigma}{\gamma} - \frac{1}{\zeta}]\big)} - \frac{1}{\zeta}\Big)$$

$$= \zeta\Big(\frac{\gamma\psi^{(m)}}{\sigma^2\big(\zeta\widetilde{\theta}^{\mathbb{G}} - \zeta[\widetilde{\theta}^{\mathbb{G}} - \frac{\gamma\psi^{(m)}}{\sigma^2(1+\zeta\widetilde{\theta}^{\mathbb{G}})}]\big)} - \frac{1}{\zeta}\Big) = \zeta\theta^{*}.$$

For last remaining part,

$$(\sigma\widetilde{\theta}^{\mathbb{G}})^2 - \sigma\widetilde{\theta}^{\mathbb{G}}\varphi^{(m)} + (\frac{\sigma^2}{\gamma}\widetilde{\theta}^{\mathbb{G}} - \frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma} - \psi^{(m)})\lambda$$

$$= \mu\widetilde{\theta}^{\mathbb{G}} + T(\widetilde{\theta}^{\mathbb{G}}) + \frac{\varphi^{(m)}\sigma}{\zeta} + \frac{1}{\zeta} + \lambda(-\frac{\mu}{\gamma} - \frac{\varphi^{(m)}\sigma}{\gamma}) = \mu\widetilde{\theta}^{\mathbb{G}},$$

where, $T(\widetilde{\theta}^{\mathbb{G}})$ is equivalent to the equation 6.27.

Here, we discuss the impact of τ on the optimal portfolio in this setting of the jump-diffusion model. Here, we compute the difference of both maximization utility function and we call it the *additional gain utility.* In Theorem 6.7 and Theorem 4.3, we compute the optimal deflator and optimal portfolio for both models, $(\ln, S, \mathbb{F})$ and $(\ln, S^{\tau}, \mathbb{G})$. In the following problem, we formulate all these results.

Theorem 6.8: *Suppose that S is given by (6.19). Then,*

(a) *The optimal portfolio for $(\ln, S, \mathbb{F})$, denoted by $\widetilde{\theta}^{\mathbb{F}}$, exists and is given by:*

$$\widetilde{\theta}^{\mathbb{F}} := \frac{\alpha + sign(\zeta)\sqrt{\alpha^2 + 4\lambda}}{2\sigma} - \frac{1}{\zeta}, \ \ where\ \alpha := \frac{\mu - \lambda\zeta}{\sigma} + \frac{\sigma}{\zeta}, \tag{6.30}$$

and the maximal expected utility up to the terminal time is

$$\mathbb{E}[\ln(X_s^{\widetilde{\theta}^{\mathbb{F}}})] = \ln X_0 + \mathbb{E}\left[\int_0^s \left(\widetilde{\theta}_t^{\mathbb{F}}\mu_t - \frac{1}{2}(\widetilde{\theta}_t^{\mathbb{F}})^2\sigma^2 + \ln(1 + \widetilde{\theta}_t^{\mathbb{F}}\zeta)\lambda\right) dt\right]. \tag{6.31}$$

(b) *The optimal portfolio up to the default time τ, $\widetilde{\theta}^{\mathbb{G}}$, is given by:*

$$\widetilde{\theta}^{\mathbb{G}} = \frac{\xi + sign(\zeta)\sqrt{\xi^2 + 4\lambda\psi^{(m)}}}{2\sigma} - \frac{1}{\zeta}, \ \ where\ \xi := \frac{\mu - \lambda\zeta}{\sigma} + \varphi^{(m)} + \frac{\sigma}{\zeta}, \tag{6.32}$$

and the maximal expected utility up to the default time is

$$\mathbb{E}[\ln(X_\tau^{\widetilde{\theta}^{\mathbb{G}}})] = \ln X_0 + \mathbb{E}[\int_0^T G_-(\sigma\widetilde{\theta}_t^{\mathbb{G}}\varphi_t^m + \widetilde{\theta}_t^{\mathbb{G}}\mu_t - \frac{1}{2}(\widetilde{\theta}_t^{\mathbb{G}})^2\sigma^2) + G_-\psi^{(m)}\ln(1 + \widetilde{\theta}_t^{\mathbb{G}}\zeta)\lambda dt]. \tag{6.33}$$

(c) *The optimal informational investment difference up-to-τ is*

$$D(\varphi_s^m, \psi_s^{(m)}) := \widetilde{\theta}^{\mathbb{G}} - \widetilde{\theta}^{\mathbb{F}} = \frac{\varphi_s^m}{\sigma_s} + \frac{(\varphi_s^m)^2 + (2\alpha_s - \Sigma_s)\varphi_s^m + 4\lambda_s(\psi_s^{(m)} - 1)}{2\sigma_s\Sigma_s}, \qquad (6.34)$$

where $\Sigma_s := \sqrt{\xi_s^2 + 4\lambda_s\psi_s^{(m)}} + \sqrt{\alpha_s^2 + 4\lambda_s}$.

(d) *The expected informational value up-to-τ, which we call it the additional gain utility, is given by*

$$\mathbb{E}[A(X_\tau)] := \mathbb{E}[\ln(X_\tau^{\widetilde{\theta}^{\mathbb{G}}}) - \ln(X_\tau^{\widetilde{\theta}^{\mathbb{F}}})] = \mathbb{E}[\ln(X_\tau^{D(\varphi^m, \psi^{(m)})})] + \mathbb{E}[R_\tau(\widetilde{\theta}^{\mathbb{F}}, \varphi^m, \psi^{(m)})], \quad (6.35)$$

where $$R_t(\widetilde{\theta}^{\mathbb{F}}, \varphi^m, \psi^{(m)}) := \int_0^t \sigma_s \widetilde{\theta}_s^{\mathbb{F}}(\varphi_s^m - \sigma_s D_s(\varphi^m, \psi^{(m)}))ds + \int_0^t \lambda_s[\psi_s^{(m)} \ln(\frac{1 + \zeta(D_s(\varphi^m, \psi^{(m)}) + \widetilde{\theta}_s^{\mathbb{F}})}{1 + \zeta_s D_s(\varphi^m, \psi^{(m)})}) - \ln(1 + \zeta_s\widetilde{\theta}_s^{\mathbb{F}})]ds.$$

(e) *Fix terminal time T. Then additional gain utility, $A^T(X)$, is given by*

$$\mathbb{E}[A^T(X)] := \mathbb{E}[\ln(X_{\tau\wedge T}^{\widetilde{\theta}^{\mathbb{G}}}) - \ln(X_T^{\widetilde{\theta}^{\mathbb{F}}})] = \mathbb{E}[G_- A(X_\tau)] - (1 - G_-)\mathbb{E}[\ln(X_T^{\widetilde{\theta}^{\mathbb{F}}})]. \quad (6.36)$$

Proof. Assertions (a) and (b) follow immediately from Theorem 4.3 and Theorem 6.7 respectively. Assertion (c) is trivial. For assertion (d), by using the equation (6.34), we get $\widetilde{\theta}_s^{\mathbb{G}} = D(\varphi_s^m, \psi_s^{(m)}) + \widetilde{\theta}_s^{\mathbb{F}}$. Therefore,

$$\mathbb{E}[\ln(X_\tau^{\widetilde{\theta}^{\mathbb{G}}})] = \mathbb{E}[\ln(X_\tau^{\widetilde{\theta}^{\mathbb{F}}})] + \mathbb{E}[\ln(X_\tau^{D(\varphi^m, \psi^{(m)})})] + \mathbb{E}[I_{]\!]0,\tau]\!]} R(\widetilde{\theta}^{\mathbb{F}}, \varphi^m, \psi^{(m)})]. \quad (6.37)$$

Thus, by putting (6.37) in additional gain utility function, we drive

$$\mathbb{E}[A(X_\tau)] = \mathbb{E}[\ln(X_\tau^{\widetilde{\theta}^{\mathbb{G}}})] - \mathbb{E}[\ln(X_\tau^{\widetilde{\theta}^{\mathbb{F}}})]$$

$$= \mathbb{E}[\ln(X_\tau^{D(\varphi^m,\psi^{(m)})})] + \mathbb{E}[R_\tau(\widetilde{\theta}^{\mathbb{F}}, \varphi^m, \psi^{(m)})].$$

Then both assertions (d) and (e) follow immediately. This ends the proof. □

Here, we consider another and the last example of Lévy model : The of Black-Scholes market model. The stock price process is given by the following dynamics

$$S_t = S_0\mathcal{E}(X)_t, \quad X_t := \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds, \tag{6.38}$$

where W is a one-dimensional Brownian motion. Let $\psi_{(m)} = 1$ in the decomposition (6.20).

Theorem 6.9: *Suppose S given by (6.38) and $G > 0$. Then there exists $\widetilde{\theta}^{\mathbb{G}} \in \Theta(S^\tau, \mathbb{G})$ such that*

$$\max_{\theta\in\Theta(S^\tau,\mathbb{G})} E\Big[\ln\left(1 + (\theta \bullet S^\tau)_T\right)\Big] = E\Big[\ln\Big(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^\tau)_T\Big)\Big] < +\infty. \tag{6.39}$$

if and only if the following hold

$$E\left[\int_0^T \frac{1}{2G_-}(\varphi_s^{(m)})^2 - \frac{\mu\varphi_s^{(m)}}{\sigma_s} + \frac{G_-}{2}(\frac{\mu}{\sigma_s})^2 ds\right] < +\infty.$$

$$\text{and} \qquad E\left[(G_- \int (\varphi_{(m)} + \frac{\mu}{\sigma})^2 dt\right] < +\infty, \tag{6.40}$$

Furthermore, on $]\!]0, \tau]\!]$, we have

$$\widetilde{\theta}^{\mathbb{G}} = \varphi_{(m)} + \frac{\mu}{\sigma}.$$

Proof. All $\mathbb{F}$-martingales are continuous for the Black-Scholes market model. Therefore, the proof of the theorem follows from Theorem 6.5 immediately. □

Corollary 6.9.1: Suppose that the $S_t = S_0\mathcal{E}(Y)_t$ is generated by a Brownian motion $\omega^{\mathbb{F}}$ only (ex. the Black-Scholes model). Then,

(a) The optimal informational investment difference up-to-τ is $D(\varphi^m, 1) := \varphi^m$.

(b) The expected informational value up-to-τ is given by $\mathbb{E}[A(X_\tau)] = \mathbb{E}[\ln(X_\tau^{D(\varphi^m,1)})]$

(i.e. for equation (6.35), we have $R_t(\widetilde{\theta}^{\mathbb{F}}, \varphi^m, 1) = 0$).

6.2.3 Volatility market model

In this subsection we discuss are result of this chaptor on the case of Volatility market models. Precisely, we address the optimal portfolio problem for the corrected Stein and Stein Model, then we discuss it for the Barndorff-Nielsen Shephard Model. The asset price processes in corrected Stein and Stein Model is continuous process and is given by

$$\begin{aligned} X_t &= X_0 + \int_0^t V_s \sigma_s dW_s^{(1)} + \int_0^t \mu V_s^2 dt \\ V_t &= V_0 + \int_0^t (m - aV_s)dt + \alpha_s dW_s^{(2)} \end{aligned} \tag{6.41}$$

where $W^{(i)}$, $i = 1, 2$ are one-dimensional Brownian motions with the correlation coefficient $\rho \in (-1, +1)$ and all the coefficients σ, α, m, a and μ are the same as in Section 3.4.3.1. For the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$, we have the following decomposition

$$G_-^{-1} \bullet m = \varphi_{(m)} \bullet W^{(1)} + \psi_{(m)} \bullet W^{(2)}.$$

In the next theorem, we formulate the optimal portfolio for the case of corrected Stein and Stein financial market model.

Theorem 6.10: *Suppose $G > 0$ and S is given by (6.41). Then the following $\mathbb{F}$-predictable process*

$$\widetilde{\theta} := \varphi^{(m)} + \rho\psi^{(m)} + \frac{\mu}{\sigma}, \tag{6.42}$$

is S-integrable belonging to $\mathcal{L}(S, \mathbb{F})$, and the following assertions hold.

(a) *The solution to*

$$\min_{Z\in\mathcal{D}(S^\tau,\mathbb{G})} E\left[-\ln(Z_T)\right] = E\left[-\ln(\widetilde{Z}^{\mathbb{G}}_T)\right] < +\infty, \tag{6.43}$$

is given by

$$\widetilde{Z}^{\mathbb{G}} := \mathcal{E}(\widetilde{K}^{\mathbb{G}}) \quad \textit{and} \quad \widetilde{K}^{\mathbb{G}} := \mathcal{E}(-\widetilde{\theta}_t \bullet \overline{W}^{(1)}). \tag{6.44}$$

(b) *It holds that*

$$\max_{\theta\in\Theta(S^\tau)} \mathbb{E}[\ln(\mathcal{E}_T(\theta\bullet S^\tau))] = \mathbb{E}[\ln(\mathcal{E}_T(\widetilde{\theta}\bullet S^\tau))]. \tag{6.45}$$

(c) $E\left[G_- \int (\frac{\mu}{\sigma})^2 + \varphi^2_{(m)} + \rho^2\psi^2_{(m)}dt\right] < +\infty.$

Proof. For the model (6.41), the predictable characteristics of Section 3 can be derived as follows

$$F_t(dx) = 0, \quad A_t = t, \quad c = \sigma^2 V_t^2 S_t^2, \quad \beta_m = (\varphi^{(m)} + \rho\psi^{(m)})(S_t\sigma)^{-1}, \quad b = \mu V_t^2 S_t.$$

Thus, the inequality (6.5) that characterizes the optimal solution $\widetilde{\varphi}$ becomes an equation as follows.

$$\begin{aligned} 0 &= \mu S V^2 + (S_t\sigma V)^2\left(\frac{\varphi_{(m)}}{\sigma S} + \frac{\rho\psi_{(m)}}{\sigma S} - \theta\right) \\ &= \mu + \sigma\varphi_{(m)} + \sigma_t\rho\psi_{(m)} - \theta S\sigma^2. \end{aligned}$$

It is clear that $\widetilde{\theta} := \theta S$, given by (6.42), is the unique solution to the above equation and $\widetilde{\theta}$ is S-integrable. The optimal mortality deflator, the solution for the dual problem, is given by the Theorem (5.9). We can simply formulate it with respect to $\widetilde{\theta}$

$$\widetilde{Z}^{\mathbb{G}} = \mathcal{E}\left(\widetilde{K}^{\mathbb{G}}\right) \quad , \quad \widetilde{K}^{\mathbb{G}} = -\frac{\mu + \sigma\varphi_{(m)} + \sigma\rho\psi_{(m)}}{\sigma} \bullet \overline{W}^{(1)} = \mathcal{E}(-\widetilde{\theta}_t \bullet \overline{W}^{(1)}).$$

□

The last model we discuss in this subsection, is Barndorff-Nielsen Shephard which is presented in Section 3.4.3.2. The asset price process, S_t, is given by the following stochastic differential equation

$$\begin{aligned} S_t &= S_0 e^{(X)_t}, \quad dX_s = (\mu + \xi\sigma_s^2)ds + \sigma_s dY_s^c + d(\rho x \star \widetilde{\mu}_Y)_s \\ d\sigma_s^2 &= -\lambda\sigma_s^2 dt + d(x \star \widetilde{\mu}_Y)_s, \end{aligned} \tag{6.46}$$

By Ito's formula, we calculate the dynamics of S,

$$\frac{dS_t}{S_{t_-}} = \alpha_t dt + \sigma_t dY_t^c + d(e^{\rho x} - 1) \star (\widetilde{\mu} - \widetilde{\nu})_t, \tag{6.47}$$

where $\alpha = \mu + \sigma_t^2(\xi + \frac{1}{2}) + \int (e^{\rho x} - 1)\widetilde{F}(dx)$, and all the coefficients are the same as in Section 3.4.3.2. The filtration generated by a one dimensional Lev́y process Y, $Y = Y^c + \widetilde{Y}^d$. Y^c is the continuous part of Lev́y process and $\widetilde{Y}^d$ is driven by random measure of Y, donated by $\widetilde{\mu}(dt \times dx)$ with compensator measure $\widetilde{\nu}(dt \times dx) = \widetilde{F}(dx)dA_t$. Throughout the rest of this subsection, we consider the following decomposition for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$ and the space $\mathcal{L}(S, \mathbb{F})$ given by

$$\begin{aligned} G_-^{-1} \bullet m \ &= \ \hat{\beta}_m \bullet Y^c + (\hat{f}_m - 1) \star (\widetilde{\mu} - \widetilde{\nu}) + \hat{g}_m \star \widetilde{\mu} + m^{\perp}. \\ \mathcal{L}(S, \mathbb{F}) &:= \Big\{ \theta \in \mathcal{P}(\mathbb{F}) \ \Big| \ 1 + S_{t\_}\theta^{tr}(e^{\rho x} - 1) > 0 \quad P \otimes \widetilde{F} \otimes dt\text{-.a.e} \Big\}. \end{aligned} \tag{6.48}$$

The model is a quasi-left-continuous and the process A_t is continuous. Thus, there is no loss of generality on defining $A_t = t$. Throughout the rest of this subsection, we consider the following Jacod's decomposition for the $\mathbb{F}$-martingale $G_-^{-1} \bullet m$

and the space $\mathcal{L}(S,\mathbb{F})$, given by

$$\mathcal{L}(S,\mathbb{F}) := \Big\{\lambda \in \mathcal{P}(\mathbb{F}) \;\Big|\; 1 + S_{t_-}\lambda^{tr}(e^{\rho x}-1) > 0 \quad P\otimes \widetilde{F}\otimes dt\text{-.a.e}\Big\}, \tag{6.49}$$

$$G_-^{-1} \centerdot m = \hat{\beta}_m \centerdot Y^c + (\hat{f}_m - 1)\star(\widetilde{\mu}-\widetilde{\nu}) + \hat{g}_m \star \widetilde{\mu} + m^{\perp}. \tag{6.50}$$

For the Barndorff-Nielsen Shephard model, the predictable characteristics of Section 5.1 can be derived as follows.

$$\begin{aligned} &b_t = S_{t_-}\alpha, \quad S_t^c = S_{t_-}\sigma_t \bullet Y_t^c, \quad \Delta S_t = S_{t_-}(e^{\rho\Delta\sigma_t^2}-1), \qquad (6.51)\\ &c_t = S_{t_-}^2\sigma_t^2, \quad \nu^S(dt\times dx) = F_t^S(dx)dt, \quad f(x)F_t^S(dx) = \; f(S_{t_-}(e^{\rho x}-1))\widetilde{F}_t(dx). \end{aligned}$$

As a result, the Jacod's components of $G_-^{-1} \bullet m$ is $(\hat{\beta}_m, \hat{f}_m, \hat{g}_m, m^{\perp})$, where $\hat{\beta}_m(S_-\sigma)^{-1} := \beta_m$, $f_m(x) = \hat{f}_m(S_{t_-}(e^{\rho x}-1))$ and $g_m = \hat{g}_m(S_{t_-}(e^{\rho x}-1))$ in equation (5.16).

Theorem 6.11: *Suppose that $G > 0$, S is given by (6.46), and $\mathcal{D}(S,\mathbb{F}) \neq \emptyset$. Then the following are equivalent.*

(a) *There exists $\widetilde{\theta}^{\mathbb{G}} \in \Theta(S^{\tau},\mathbb{G})$ such that*

$$\max_{\theta\in\Theta(S^{\tau},\mathbb{G})} E\Big[\ln\left(1+(\theta\bullet S^{\tau})_T\right)\Big] = E\Big[\ln\Big(1+(\widetilde{\theta}^{\mathbb{G}}\bullet S^{\tau})_T\Big)\Big] < +\infty. \tag{6.52}$$

(b) *There exist $K \in \mathcal{M}_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process V such that $\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S,\mathbb{F})$, and*

$$E\left[(G_-\bullet V)_T + (\widetilde{G}\bullet H^{(0)}(K,P))_T^{p,\mathbb{F}} + (G_-\bullet h^E(G_-^{-1}\bullet m, P))_T - \langle K, m\rangle_T^{\mathbb{F}}\right] < +\infty.$$

(c) *There exists* $\widetilde{\varphi} \in \mathcal{L}(S,\mathbb{F})$ *such that, for any* $\theta \in \mathcal{L}(S,\mathbb{F})$, *the following hold*

$$
\begin{aligned}
&E\left[(G_- \bullet \widetilde{V})_T + (G_-\widetilde{\varphi}^{tr}\sigma_t^2\widetilde{\varphi} \bullet A)_T + (G_-\mathcal{K}_{log}(\widetilde{\varphi}^{tr}(e^{\rho x}-1)) \star \nu)_T\right] < +\infty,\\
&(\theta - \widetilde{\varphi})^{tr}\left[\mu + \sigma^2(\xi + \frac{1}{2}) + \hat{\beta}_m\sigma - \sigma^2\widetilde{\varphi} + \int (\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\varphi}(e^{\rho x}-1)} + 1)(e^{\rho x}-1)\widetilde{F}(dx)\right] \leq 0,\\
&\widetilde{V} := \int_0^T \widetilde{\varphi}^{tr}[\mu + \sigma^2(\xi + \frac{1}{2}) + \hat{\beta}_m\sigma - \sigma^2\widetilde{\varphi} + \int (\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\varphi}(e^{\rho x}-1)} + 1)(e^{\rho x}-1)\widetilde{F}(dx)]dt. \quad (6.53)
\end{aligned}
$$

Furthermore, on $]\!]0,\tau]\!]$, *we have*

$$\widetilde{\theta}^{\mathbb{G}}(1 + (\widetilde{\theta}^{\mathbb{G}} \bullet S^{\tau})_-)^{-1} = \widetilde{\varphi} \quad \textit{and} \ \widetilde{\theta}^{\mathbb{G}} = \widetilde{\varphi}\mathcal{E}_-(\widetilde{\varphi} \bullet S).$$

Proof. For the model (6.47)-(6.48), the predictable characteristics can be derived as follows (for more details see here 2.4). Then, we get

$$
\begin{aligned}
&\alpha := \mu + \sigma_t^2(\xi + \frac{1}{2}) + \int (e^{\rho x}-1)\widetilde{F}(dx), \quad S_t^c = S_{t-}\sigma_t \bullet Y_t^c, \quad \Delta S_t = S_{t-}(e^{\rho\Delta\sigma_t^2}-1),\\
&b_t = S_{t-}\alpha, \quad c_t = S_{t-}^2\sigma_t^2, \quad \nu^S(dt \times dx) = F_t^S(dx)dt, \quad f(x)F_t^S(dx) = \ f(S_{t-}(e^{\rho x}-1))\widetilde{F}_t(dx).
\end{aligned}
$$

As a result, the Jacod's components of $G_-^{-1}\bullet m$ is $(\hat{\beta}_m, \hat{f}_m, \hat{g}_m, m^{\perp})$, where $\hat{\beta}_m(S_-\sigma)^{-1} := \beta_m$, $f_m(x) = \hat{f}_m(S_{t-}(e^{\rho x}-1))$ and $g_m = \hat{g}_m(S_{t-}(e^{\rho x}-1))$ in equation (6.2). Hence, the condition (6.5), characterizing the optimal portfolio $\widetilde{\theta}$, satisfying an inequality as follows.

$$
\begin{aligned}
0 \ &\geq S_-(\mu + \sigma_t^2(\xi + \frac{1}{2}) + \int \big(S_-(e^{\rho x}-1)\big)\widetilde{F}(dx)) + (S_{t-})^2\sigma^2(\frac{\hat{\beta}_m}{S_-\sigma} - \widetilde{\lambda})\\
&\quad + \int \frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\lambda}S_{t-}(e^{\rho x}-1)} S_{t-}(e^{\rho x}-1)\widetilde{F}(dx)\\
&= S_{t-}\Big(\mu + \sigma_t^2(\xi + \frac{1}{2}) + \hat{\beta}_m - S_{t-}\sigma_t^2\widetilde{\lambda} + \int (\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\lambda}S_{t-}(e^{\rho x}-1)} + 1)(e^{\rho x}-1)\widetilde{F}(dx)\Big),
\end{aligned}
$$

and for the equation (6.6) in Theorem 6.1, we have

$$\frac{d\widetilde{V}}{dt} = S_-\widetilde{\lambda}^{tr}\left[\mu + \sigma^2(\xi + \frac{1}{2}) + \hat{\beta}_m - S_-\sigma^2\widetilde{\lambda} + \int (\frac{\hat{f}_m\big(S_-(e^{\rho x}-1)\big)}{1+\widetilde{\lambda}S_-(e^{\rho x}-1)} + 1)(e^{\rho x}-1)\widetilde{F}(dx)\right].$$

Hence, we deduce that $S_-\widetilde{\lambda} = \widetilde{\varphi}$, and it is S-integrable. As a result, $\widetilde{\theta}$ is the solution to (6.52) and assertions (a) and (b) follow immediately by Theorem 6.1. □

6.3 Numéraire portfolio under random horizon

This section addresses the impact of τ on the numéraire portfolio, defined in Section (2.3). To this end, for the reader's convenience, we recall the mathematical definition of this financial concept.

Definition 6.1: Consider $(X, \mathbb{H}, P)$, and let Z be a positive $\mathbb{H}$-local martingale. We call numéraire portfolio for $(X, \mathbb{H}, Z)$, when it exists, the unique $\widetilde{\theta} \in L(X, \mathbb{H})$ such that $\mathcal{E}(\widetilde{\theta} \bullet X) > 0$, and the process $Z\mathcal{E}(\phi \bullet X)/\mathcal{E}(\widetilde{\theta} \bullet X)$ is a supermartingale, for any $\phi \in L(X, \mathbb{H})$ satisfying $\mathcal{E}(\phi \bullet X) \geq 0$.

Below, we elaborate on the principal result of this section.

Theorem 6.12: *Let $Z^{(m)} := \mathcal{E}(G_-^{-1} \bullet m)$. Then the numéraire portfolio for $(S^\tau, \mathbb{G}, P)$, denoted by $\widetilde{\theta}^{\mathbb{G}}$, exists if and only if the numéraire portfolio for $(S, \mathbb{F}, Z^{(m)})$, denoted by $\widetilde{\theta}^{(\mathbb{F})}$, does exist also. Furthermore,*

$$\widetilde{\theta}^{\mathbb{G}} = \widetilde{\theta}^{(\mathbb{F})} I_{]\!]0,\tau]\!]}. \tag{6.54}$$

Proof. The proof is achieved in two parts, where we prove (a) $\Longrightarrow$ (b) and its converse respectively.

Part 1. Suppose that the numéraire portfolio $\widetilde{\theta}^{(\mathbb{G})}$, for the model $(S^\tau, \mathbb{G}, P)$, exists. Then on the one hand, there exists an $\mathbb{F}$-predictable process $\theta^{\mathbb{F}}$ such that $\widetilde{\theta}^{(\mathbb{G})} I_{]\!]0,\tau]\!]} = \theta^{(\mathbb{F})} I_{]\!]0,\tau]\!]}$. On the other hand, for $\theta \in L(S, \mathbb{F})$ such that $\mathcal{E}(\theta \bullet S) > 0$, the process

$X := \mathcal{E}(\theta \bullet S^\tau)/\mathcal{E}(\widetilde{\theta} \bullet S^\tau)$ is a positive supermartingale. Or equivalently

$$\begin{aligned}\frac{\mathcal{E}(\theta \bullet S^\tau)}{\mathcal{E}(\widetilde{\theta} \bullet S^\tau)} &= \mathcal{E}(\theta \bullet S^\tau)\mathcal{E}(\widetilde{\theta} \bullet S^\tau)^{-1} = \mathcal{E}(\theta \bullet S^\tau)\mathcal{E}\left(-\widetilde{\theta} \bullet S^\tau + \frac{1}{1+\widetilde{\theta}\Delta S^\tau} \bullet [\widetilde{\theta} \bullet S^\tau]\right) \\ &= \mathcal{E}\left((\theta - \widetilde{\theta}) \bullet S^\tau + \frac{\widetilde{\theta} - \theta}{1+\widetilde{\theta}\Delta S^\tau} \bullet [S^\tau, \widetilde{\theta} \bullet S^\tau]\right) := \mathcal{E}(L),\end{aligned}$$

and L is a $\mathbb{G}$-local supermartingale. Consider the decomposition for S given by $S = S_0 + M + A + \sum \Delta S I_{\{|\Delta S|>1\}}$, where M is a $\mathbb{G}$-local martingale with bounded jumps and A is a finite variation and predictable process. Therefore, we derive

$$\begin{aligned}L &= (\theta - \widetilde{\theta}) \bullet (M^\tau + A^\tau) + \sum (\theta - \widetilde{\theta})^{tr} \Delta S^\tau I_{\{|\Delta S|>1\}} - \frac{\theta - \widetilde{\theta}}{1+\widetilde{\theta}\Delta S^\tau} \bullet [S^\tau, \widetilde{\theta} \bullet S^\tau] \\ &= \mathbb{G}\text{-local martingale} + \frac{\theta - \widetilde{\theta}}{G_-} I_{]\!]0,\tau]\!]} \bullet \langle M, m\rangle^{\mathbb{F}} + W^\theta,\end{aligned}$$

where W^θ is given by

$$W^\theta := (\theta - \widetilde{\theta}) \bullet \left\{A^\tau + \sum \Delta S^\tau I_{\{|\Delta S|>1\}} - \frac{1}{1+\widetilde{\theta}\Delta S^\tau} \bullet [S^\tau, \widetilde{\theta} \bullet S^\tau]\right\}.$$

Thus, the process L is a $\mathbb{G}$-local supermartingale if and only if $W^\theta \in \mathcal{A}_{loc}(\mathbb{G})$ and

$$\Big(\frac{(\theta - \widetilde{\theta})}{G_-} I_{]\!]0,\tau]\!]} \bullet \langle M, m\rangle^{\mathbb{F}} + W_\theta\Big)^{p,\mathbb{F}} \preceq 0.$$

This is equivalent to

$$\begin{aligned}&(\theta - \widetilde{\theta}) \bullet \Big\{\langle M, m\rangle^{\mathbb{F}} + G_- \bullet A\Big\} + \\ &+(\theta - \widetilde{\theta}) \bullet \left(\sum \widetilde{G}\Delta S I_{\{|\Delta S|>1\}} - \frac{\widetilde{G}}{1+\widetilde{\theta}\Delta S} \bullet [S, \widetilde{\theta} \bullet S]\right)^{p,\mathbb{F}} \preceq 0. \qquad (6.55)\end{aligned}$$

Now, we derive

$$
\begin{aligned}
X &:= \mathcal{E}(G_-^{-1}\bullet m)\frac{\mathcal{E}(\theta\bullet S)}{\mathcal{E}(\widetilde{\theta}\bullet S)} = \mathcal{E}(G_-^{-1}\bullet m)\mathcal{E}\left((\theta-\widetilde{\theta})\bullet S + \frac{\widetilde{\theta}-\theta}{1+\widetilde{\theta}\Delta S}\bullet[S,\widetilde{\theta}\bullet S]\right)\\
&= \mathcal{E}(L_1),
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &:= G_-^{-1}\bullet m + (\theta-\widetilde{\theta})\bullet S + \frac{\theta-\widetilde{\theta}}{G_-}\bullet[m,S] + \frac{\widetilde{G}}{G_-}\frac{\widetilde{\theta}-\theta}{1+\widetilde{\theta}\Delta S}\bullet[S,\widetilde{\theta}\bullet S]\\
&= \mathbb{G}\text{-local martingale} + \frac{(\theta-\widetilde{\theta})}{G_-}\bullet(\langle M,m\rangle^{\mathbb{F}} + G_-\bullet A)\\
&+\frac{(\theta-\widetilde{\theta})}{G_-}\bullet\left(\sum \widetilde{G}\Delta S I_{\{|\Delta S|>1\}} - \frac{\widetilde{G}}{1+\widetilde{\theta}\Delta S}\bullet[S,\widetilde{\theta}\bullet S]\right)^{p,\mathbb{F}}.
\end{aligned}
$$

Thanks to the inequality (6.55), we deduce that L_1 is an $\mathbb{F}$-local supermartingale, and hence X is a nonnegative $\mathbb{F}$-supermartingale. This proves assertion (b).

Part 2. Here we prove that assertion (b) implies assertion (a). Suppose that the numéraire portfolio for $(X,\mathbb{F},Z)$ exists that we denote by $\widetilde{\theta}^{(\mathbb{F})}$. Then for any $\phi\in L(X,\mathbb{F})$ satisfying $\mathcal{E}(\phi\bullet X)>0$, $X := Z\mathcal{E}(G_-^{-1}\bullet m)\mathcal{E}(\phi\bullet S)/\mathcal{E}(\widetilde{\theta}\bullet S)$ is a positive supermartingale. On the one hand, it is known that there exists a local martingale M and a nondecreasing and $\mathbb{F}$-predictable process V such that $X=\mathcal{E}(M)\exp(-V)$. On the other hand, we have

$$
\frac{\mathcal{E}(M)^\tau}{\mathcal{E}(G_-^{-1}\bullet m)^\tau} = \mathcal{E}(\mathcal{T}(M) - G_-^{-1}\bullet\mathcal{T}(m)) \quad \text{is a nonnegative } \mathbb{G}\text{-local martingale.}
$$

Therefore we easily conclude that $X^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau$ is a nonnegative $\mathbb{G}$-supermartingale, or equivalently the process

$$
\frac{\mathcal{E}(\phi\bullet S^\tau)}{\mathcal{E}(\widetilde{\theta}\bullet S^\tau)} = \frac{\mathcal{E}(\phi\bullet S)^\tau}{\mathcal{E}(\widetilde{\theta}\bullet S)^\tau} = X^\tau/\mathcal{E}(G_-^{-1}\bullet m)^\tau,
$$

is a nonnegative $\mathbb{G}$-supermartingale. This ends the proof of theorem. □

Corollary 6.12.1: The following assertions hold.

(a) Suppose that $\mathcal{E}(G_-^{-1} \bullet m)$ is a uniformly integrable martingale, and consider $Q := \mathcal{E}(G_-^{-1} \bullet m)_\infty \cdot P$. Then the numéraire portfolio for $(S^\tau, \mathbb{G}, P)$ exists if and only if the numéraire portfolio for $(S, \mathbb{F}, Q)$ does exists, and both portfolios coincide on $]\!]0, \tau]\!]$.

(b) Suppose that τ is a pseudo-stopping time. Then the numéraire portfolio for $(S^\tau, \mathbb{G}, P)$ exists if and only if the numéraire portfolio for $(S, \mathbb{F}, P)$ does exists, and both portfolios coincide on $]\!]0, \tau]\!]$.

Proof. It is clear that assertion (a) follows immediately from combining Theorem 6.12 with the fact that when $\mathcal{E}(G_-^{-1} \bullet m)$ is a uniformly integrable martingale, the numéraire portfolio for $\big(S, \mathbb{F}, \mathcal{E}(G_-^{-1} \bullet m)\big)$ coincides with the numéraire portfolio for $(S, \mathbb{F})$ under Q. The second assertion follows also from combining Theorem 6.12 with the fact that τ is a pseudo-stopping time if and only if $m \equiv m_0$, and hence $\mathcal{E}(G_-^{-1} \bullet m) \equiv 1$. This ends the proof of the corollary. $\square$

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