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The weighted compactification of a locally compact group and topological centres

by

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Abstract

Let G be a locally compact group. It is well-known that G^{LUC}, the spectrum of the algebra of left uniformly continuous functions on G, the so-called LUC-compactification of G, is a semigroup with product restricted from the Arens product on $LUC(G)^*$. Now consider the algebra of weighted left uniformly continuous functions on G, $LUC(G, \omega^{-1})$. The spectrum G_ω^{LUC} is a compactification of G homeomorphic to G^{LUC}, but is not a semigroup unless the weight is a homomorphism (in which case $G_\omega^{LUC} = G^{LUC}$). We study the algebraic and topological properties of G_ω^{LUC} and the semigroup it generates in $[0, 1]G_\omega^{LUC}$, including characterizing when it is dense, and use the results to attempt to extend some topological centre and determination results for G^{LUC} of Budak, Işık, and Pym [6] to G_ω^{LUC} and present some partial results. We also partially characterize the isometric isomorphisms of Beurling (weighted group) algebras. Finally, we show that the topological centre of the Fourier algebra of the Fell group is strongly Arens irregular.

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Chapter 1

Introduction

Let $\mathfrak{A}$ be a Banach algebra. Then the bidual $\mathfrak{A}^{**}$ is also a Banach algebra with the Arens product and the topological centre of $\mathfrak{A}^{**}$ is the set $\mathcal{Z}_t(\mathfrak{A})$ of elements of $\mathfrak{A}^{**}$ for which left multiplication is w^*-continuous. The topological centre contains $\mathfrak{A}$ and we call $\mathfrak{A}$ strongly Arens irregular if it is equal to $\mathfrak{A}$ and Arens regular if it is the whole bidual. C*-algebras are Arens regular [7] but the group algebra of any locally compact group is strongly Arens irregular [30]. A set $\mathfrak{D} \subseteq \mathfrak{A}^{**}$ is called a dtc set if the continuity of left multiplication by $m \in \mathfrak{A}^{**}$ at the points in $\mathfrak{D}$ implies that $m \in \mathcal{Z}_t(\mathfrak{A}^{**})$. These definitions generalize to right topological semigroups.

Abstract results for the topological centre(s) of Banach algebras are given in [10] and [26]. A very useful factorization technique for demonstrating the minimality of topological centres is given by Neufang in [40]. The first definition and examples of sets which determine the topological centre appeared in the memoir of Dales, Lau, and Strauss in 2010 in [11] (although their definition is slightly different), however many older topological centre results actually yielded dtc sets without explicitly stating them. Small dtc sets have also been found by Hu, Neufang, and Ruan [27], Filali and Salmi [20], and Mazowita and Neufang [38], [39].

Consider $LUC(G)$, the algebra of left uniformly continuous functions on G. The dual space $LUC(G)^*$ is a right continuous semigroup (even a Banach algebra) for a product which we also call the Arens product, and the spectrum of $LUC(G)$, denoted G^{LUC}, is a subsemigroup. The topological centres of $LUC(G)^*$ and G^{LUC} are the measure algebra $M(G)$ [29] and G [33], respectively, and G^{LUC} has a 2-point dtc set [11]. The topological centre of $LUC(G)^*$ for non-locally compact groups was studied in [17].

Now let ω be a *weight* on G. There are weighted analogues of the group algebra, the algebra of left uniformly continuous functions $LUC(G)$, the measure algebra $M(G)$, and the semigroup compactification G^{LUC} which we denote by $L^1(G,\omega)$, $LUC(G,\omega^{-1})$, $M(G,\omega)$, and G_ω^{LUC}, respectively. The weighted group algebras are called *Beurling algebras*. The LUC-compactification G^{LUC} is homeomorphic to its weighted analogue (so they are topologically identical), but the latter fails to have the algebraic properties enjoyed by the former. Crucially, unless the weight is a homomorphism, the weighted "compactification" G_ω^{LUC} is not a semigroup and does not even contain contain the group as a subsemigroup, so cannot be called a semigroup compactification (Theorem 4.1.2) . Nevertheless, we call it the *weighted compactification* and study it and the closed semigroup it generates. Moreso than the weight itself, it is the extent to which the weight fails to be a homomorphism (measured by the function Ω) that determines properties of G_ω^{LUC}.

The weight can drastically alter the topological centre. For example, Varopoulos showed in 1972 in [45] that there are weights ω on the integers such that the topological centre of $L^1(\mathbb{Z},\omega)^{**} = LUC(\mathbb{Z},\omega^{-1})^*$ is the whole space, that is, $L^1(\mathbb{Z},\omega)$ is Arens regular (more examples are given in [10]). Soon after, in 1974, Craw and Young characterized the Arens regular Beurling algebras in [8] and showed that in particular, the underlying group must be countable and discrete. In 1989 Baker and Rejali extended the characterization to Beurling algebras on semigroups in [5]. A systematic study of the Arens regularity of Beurling algebras was undertaken by Dales and Lau in 2005 in [10] and Dales and Dedania studied Beurling algebras on subsemigroups of $\mathbb{R}$ in 2009 in [9].

If the weight satisfies a mild boundedness property, that of *diagonal boundedness* (on a suitably large subset of G), then the Beurling algebra is strongly Arens irregular, as in the unweighted case [42]. It is also calculated in [42] that for such weights $\mathcal{Z}_t(LUC(G,\omega^{-1})^*) = M(G,\omega)$. Filali and Salmi extend these results in [18], building on other interesting work in [19] and [20].

This thesis is organized as follows. The next chapter gives the basic definitions and preliminary results for the classical versions of the objects we are mainly interested it. Chapter 3 introduces our primary focus, weights and weighted objects, and in Chapter 4 we study the weighted compactification and calculate the product of two elements of G_ω^{LUC} (an element of $LUC(G,\omega^{-1})^*$) and characterize when it lies again in G_ω^{LUC} and when G_ω^{LUC} is a semigroup. We also relate the diagonal boundedness property to algebraic and topological properties of G_ω^{LUC} and study the algebraic and topological properties of the semigroup generated by G_ω^{LUC} (which is contained in $[0,1]G_\omega^{LUC}$) and characterize when this containment is dense as well as presenting some examples. We wish to leverage these results to extend some recent results of Budak, Işık, and Pym [6] on the existence of small sets which determine the topological centres to the weighted case and present some partial results in Chapter 5. Chapter 6 concerns the isomorphism problem for Beurling algebras and includes an almost complete description of the isometric isomorphisms of Beurling algebras in terms of a stronger notion of equivalence of weights.

Finally, Chapter 7 presents a novel, unrelated result on the topological centre of the Fourier algebra of an interesting group, called the Fell group. We show that it is strongly Arens irregular using a criterion of Hu [25].

Chapter 2

Preliminaries

2.1 Topology and functional analysis

We do not provide an introduction to topology here, only the definitions we will use frequently and which are not standard. See Kelley or Willard's *General Topology* for a thorough introduction.

Definition 2.1.1. *Let Ξ be a topological space and S be a subset of Ξ.*

i. A covering of S is a collection (X_α) of subsets of Ξ such that $S \subseteq \bigcup_\alpha X_\alpha$.

ii. The compact covering number of S is the least cardinality $\kappa(S)$ of a covering of S by compact sets.

iii. A subset $S \subseteq \Xi$ is called dispersed if $\kappa(S) = \kappa(\Xi)$, that is, if S cannot be covered by fewer compacta than all of Ξ.

The compact covering number of Ξ can only be 1 (iff Ξ is compact) or infinite. If S is discrete then $\kappa(S) = \begin{cases} 1 & \text{if } |S| < \infty \\ |S| & \text{if } |S| = \infty \end{cases}$ so $S \subseteq \Xi$ is dispersed iff $|S| = |\Xi|$. The space $\mathbb{R}$ is σ-compact i.e. has $\kappa(\mathbb{R}) = \aleph_0$ and a subset is dispersed iff it is unbounded.

A Banach space X is a complete normed vector space. A Banach algebra is a Banach space $\mathfrak{A}$ on which there is a associative bilinear product $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ which satisfies $\|ab\| \leq \|a\|\|b\|$ for every $a, b \in \mathfrak{A}$, guaranteeing that the product is jointly continuous. See Palmer's *Algebras and Banach Algebras* or Dales' *Banach Algebras and Automatic Continuity* for the basic theory.

2.2 Arens products and topological centres

Let $\mathfrak{A}$ be a Banach algebra. We extend the product on $\mathfrak{A}$ to the bidual $\mathfrak{A}^{**}$ via the following definitions: for $m, n \in \mathfrak{A}^{**}$, $f \in \mathfrak{A}^*$, and $a, b \in \mathfrak{A}$,

$$\langle mn, f\rangle := \langle m, n \cdot f\rangle$$

$$\langle n \cdot f, a\rangle := \langle n, f \cdot a\rangle$$

$$\langle f \cdot a, b\rangle := \langle f, ab\rangle.$$

This is called the *Arens product*, after [1] and [2], and makes $\mathfrak{A}^{**}$ into a Banach algebra. The topological centre of $\mathfrak{A}^{**}$ is

$$\mathcal{Z}_t(\mathfrak{A}^{**}) = \mathcal{Z}_t^1(\mathfrak{A}^{**}) := \{m \in \mathfrak{A}^{**} \mid \mathfrak{A}^{**} \ni n \mapsto mn \text{ is } w^* \text{ continuous}\}.$$

It easily seen that $\mathfrak{A} \subseteq \mathcal{Z}_t(\mathfrak{A}^{**})$, and if $\mathcal{Z}_t(\mathfrak{A}^{**}) = \mathfrak{A}$, the algebra is called *(left) strongly Arens irregular*. One the other hand, if $\mathcal{Z}_t(\mathfrak{A}^{**}) = \mathfrak{A}^{**}$ then the algebra is called *Arens regular*.

There is also a right Arens product, defined in a similarly formal way by moving the variables on the outside of the duality bracket, and a right topological centre defined in the obvious way. If either the the left or right topological centre of $\mathfrak{A}^{**}$ is all of $\mathfrak{A}^{**}$ then they both are, which is why we do not distinguish between left and right Arens regularity. The same is not true of of strong Arens irregularity, see [41] for an example. We will restrict ourselves to the left topological centre in this thesis.

Definition 2.2.1. *Let $\mathfrak{A}$ be a Banach algebra. A set $\mathfrak{D} \subseteq \mathfrak{A}^{**}$ is called a dtc (determining for the topological centre) set if continuity of left multiplication by $m \in \mathfrak{A}^{**}$ at the points of $\mathfrak{D}$ implies $m \in \mathcal{Z}_t(\mathfrak{A}^{**})$.*

Note that (unless $\mathfrak{A}$ is strongly Arens irregular) our definition differs with that of Dales, Lau, and Strauss in [11] since they would require that the conclusion be that $m \in \mathfrak{A}$ rather than $m \in \mathcal{Z}_t(\mathfrak{A}^{**})$.

Many results concerning such sets (e.g. [6], [38]) require even less than continuity at a handful of points, asking only that multiplication is continuous against a single net to the determining points. So we propose the following definition.

Definition 2.2.2. *Let $\mathfrak{A}$ be a Banach algebra. Convergent nets $\mathfrak{N} = \{(\eta_\alpha)\}$ in $\mathfrak{A}^{**}$ are called dtc (determining for the topological centre) nets if for $m \in \mathfrak{A}^{**}$*

$$m \lim_\alpha \eta_\alpha = \lim_\alpha m\eta_\alpha \quad \forall (\eta_\alpha) \in \mathfrak{N} \quad \Rightarrow \quad m \in \mathcal{Z}_t(\mathfrak{A}^{**}).$$

Then the collection of limits of the dtc nets is a dtc set. The conclusion that $m \in \mathcal{Z}_t(A^{**})$ guarantees that multiplication is continuous against any nets to the same limits anyway, but only having to check the individual nets is a much easier criterion to verify.

More generally, if S is a right topological semigroup (i.e. right multiplication is continuous), we can define the topological centre of S to be

$$\mathcal{Z}_t(S) := \{r \in S \mid S \ni s \mapsto rs \text{ is continuous}\}$$

and call $\mathfrak{D} \subseteq S$ a dtc set if continuity of left multiplication by $r \in S$ at the points of $\mathfrak{D}$ implies that $r \in \mathcal{Z}_t(S)$. These definitions agree with the Banach algebra definitions (with the w^*-topology). We again call S Arens regular if $\mathcal{Z}_t(S) = S$, but we may have $\mathcal{Z}_t(S) = \varnothing$ (an example is given in the following section). However, there is often a subset of S which is "obviously" contained in the centre, and we still say that S is strongly Arens irregular if the centre is minimal, that is, it consists only of these elements. This is not a precise definition and depends on context, so it will be made especially clear what is meant by strong Arens irregularity of semigroups not arising from the second duals of Banach algebras with an Arens product.

The definitions of dtc sets and nets have obvious extensions to this more general setting.

2.3 Abstract harmonic analysis

A topological group is a group and a topological space in which the inversion and group operation are continuous. A topological group is *locally compact* if it is a Hausdorff space and every point has a neighbourhood which is precompact. Compact and discrete groups are locally compact. Locally compact groups are sometimes thought of as being finite-dimensional, since the additive group of a normed vector space is locally compact iff the space is finite-dimensional.

Abstract harmonic analysis is the study of locally compact groups since they are exactly those which have a nice measure theory, as the following theorem demonstrates.

Theorem 2.3.1. *Let G be a locally compact group. Then there are unique (up to scalar multiple) left and right Haar measures λ and ρ (respectively) on G. That is, there are regular Borel measures λ and ρ on G which are (respectively) left- and right-invariant, in the sense that*

$$\lambda(xA) = \lambda(A) \qquad \textit{and} \qquad \rho(Ax) = \rho(A) \qquad \forall x \in G, A \subseteq G$$

Moreover, the left and right Haar measures behave well under the opposite translations, as described by the modular function, a homomorphism $\Delta : G \to (0, \infty)$ with *the property that*

$$\lambda(Ax) = \Delta(x)\lambda(A) \qquad \text{and} \qquad \rho(xA) = \Delta(x)^{-1}\rho(A) \qquad \forall x \in G, A \subseteq G$$

and the left and right Haar measures are related via $d\lambda(x) = \Delta(x^{-1})d\rho(x)$.

Conversely, any topological group G *with a Haar measure must be locally compact.*

By convention, if G discrete then both the left and right Haar measures are taken to be the counting measure $\#$ defined by $\#(A) = |A|$ for $A \subseteq G$, and if G is compact then we normalize the Haar measures so that $\lambda(G) = \rho(G) = 1$. Note that these conventions are inconsistent if the group is finite (and hence both compact and discrete).

The group G is called *unimodular* if the modular function is constantly 1. Abelian, compact, and discrete groups are unimodular.

Then there is a (unique up to scalar multiple) left Haar measure d on G which we use to integrate over G and we define

$$L^1(G) := \left\{ f = [f] : G \to \mathbb{C} \,\middle|\, \|f\|_1 := \int_G |f(t)|dt < \infty \right\}$$

where the elements are actually equivalence classes of functions which disagree only on a set of measure zero, but it is standard to always (carefully) work with representatives and eschew discussing the values of the functions. Then $L^1(G)$ is a Banach algebra, called the *group algebra*, with norm $\|\cdot\|_1$ and *convolution* product given by, for $f, h \in L^1(G)$,

$$f * h(t) := \int_G f(s)h(s^{-1}t)ds \tag{2.1}$$

This algebra is commutative if and only if the group is abelian. See [14] for a study of operators defined via convolution.

The group algebra is (left and right) strongly Arens irregular for any locally compact group [30]. The dual space of the group algebra is the space

$$L^\infty(G) := \left\{ f = [f] : G \to \mathbb{C} \;\middle|\; \|f\|_\infty := \operatorname*{ess\,sup}_{t \in G} |f(t)| < \infty \right\}$$

again consisting actually of equivalence classes of functions which agree almost everywhere. This space is actually a commutative C*-algebra with norm $\|\cdot\|_\infty$, involution $f \mapsto f^*$, $f^*(x) := \overline{f(x)}$, and pointwise product. We will also need the *measure algebra* of G,

$$M(G) := \{\text{regular Borel measures } \mu \text{ on } G \mid \|\mu\| := |\mu|(G) < \infty\}$$

with the convolution product $*$ defined implicitly via

$$\int_G f(t) d(\mu * \nu)(t) := \int_G \int_G f(st) d\mu(s) d\nu(t) \qquad (\mu, \nu \in M(G)).$$

The measure algebra contains the group algebra as absolutely continuous measures. Very recently Losert, Neufang, Pachl, and Steprāns have shown that the measure algebra is (left and right) strongly Arens irregular for any locally compact group [36], answering a conjecture of Ghahramani and Lau [22].

Let $C_b(G)$ be the bounded continuous functions on G. This algebra sits in $L^\infty(G)$ but continuity guarantees that the almost-everywhere-agreement equivalence classes are all singletons, so for continuous functions we can actually discuss the values of the functions and we have point evaluations. Also on $C_b(G)$ the $\|\cdot\|_\infty$-norm is just the supremum norm $\|f\|_{\sup} := \sup_{t \in G} |f(t)|$.

For $g \in G$ let ℓ_g be the left translation given by $\ell_g f(x) = f(gx)$ for $f \in C_b(G)$. Also let

$$LUC(G) = \{f \in C_b(G) : G \ni g \mapsto \ell_g f \text{ is } \|\cdot\|_\infty\text{-continuous}\}$$

be the *left uniformly continuous functions* on G. Then we have

$$LUC(G) = L^\infty(G) \cdot L^1(G)$$

where $\cdot$ is the action of $L^\infty(G) = L^1(G)^*$ on $L^1(G)$ from the definition of the Arens product, and $LUC(G)$ is a unital commutative C*-algebra (with pointwise product), so $LUC(G) = C(G^{LUC})$ where G^{LUC} is the (Gelfand) spectrum of $LUC(G)$.

The dual space $LUC(G)^*$ acts on $LUC(G)$ via the action

$$(m \cdot f)(t) := \langle m, \ell_t f \rangle \quad (f \in LUC(G), m \in LUC(G)^*)$$

that this $m \cdot f$ lies in $LUC(G)$ is not obvious, but follows from the the fact that $LUC(G)$ enjoys the property of being an introverted subspace of $L^\infty(G)$. Then the product (which we denote by juxtaposition)

$$\langle mn, f \rangle := \langle m, n \cdot f \rangle \quad (m, n \in LUC(G)^*, f \in LUC(G))$$

makes $LUC(G)^*$ a Banach algebra and a right topological semigroup (with the w^* topology) such that $\mathcal{Z}_t(LUC(G)^*) = M(G)$ [29].

With this product on $LUC(G)^*$, G^{LUC} is actually a compact right topological semigroup which densely contains G, so it is a semigroup compactification called the LUC-compactification of G. In fact it is the largest such compactification in the sense of the following universal property: any semigroup compactification of G which is a right topological semigroup is a quotient of G^{LUC}. We have $\mathcal{Z}_t(G^{LUC}) = G$ and $\mathcal{Z}_t(G^{LUC} \setminus G) = \varnothing$ [33].

One of the most important properties a locally compact group can possess is the following:

Definition 2.3.2. *A locally compact group is amenable if there is a left-invariant mean m on $L^\infty(G)$, that is, a linear functional $m \in L^\infty(G)^*$ with $\|m\| = 1$ which is* *i. positive: if* $L^\infty(G) \ni f \geq 0$ *then* $m(f) \geq 0$*, and*

ii. left-invariant: for any $f \in L^\infty(G)$ *and* $g \in G$*,* $m(\ell_{g^{-1}} f) = m(f)$

The definition is equivalent to the existence of a left-invariant mean on $LUC(G)$, which can be a better definition to work with since $LUC(G)^*$ has point evaluations, which we can use, for example, to rephrase the left-invariance condition.

Abelian and compact groups are amenable, but any group containing a closed free subgroup is not. An excellent reference is [43], see e.g. [13] for further results.

There is also a notion of amenability for Banach algebras which we do not discuss here, we only remark that the definition (a cohomology condition) is chosen so that the group algebra $L^1(G)$ is amenable (as a Banach algebra) if and only if G is amenable (as a locally compact group). See [28] and [44] for pioneering work and a recent reference, respectively.

Chapter 3

Weighted objects

3.1 Weights

The main objects of study in this thesis are generalizations of classical Banach algebras on locally compact groups in which the group elements are weighted. We must restrict our attention to weightings with a few properties to ensure that the analogues are still Banach algebras and are not otherwise too wild.

Definition 3.1.1. *A weight ω on a locally compact group G with identity $e \in G$ is a continuous function $w : G \to (0, \infty)$ with $\omega(e) = 1$ and $\omega(gh) \leq \omega(g)\omega(h) \ \ \forall g, h \in G$.*

We could of course define weights on non-locally compact groups (with the same definition), but we are only interested in locally compact groups anyway.

Here are several examples:

i. If ω is constantly 1 then ω is a weight called the *trivial weight*.

ii. Any continuous homomorphism $\omega : G \to (0, \infty)$ is a weight, these weights are nearly trivial.

iii. For any $\alpha \geq 0$, $\omega_\alpha(k) := (1 + |k|)^\alpha$ is an important weight on $\mathbb{Z}$.

iv. If $\eta : G \to \mathbb{R}$ is a subadditive function with $\eta(e) = 0$ then $\omega := e^\eta$ is a weight on G. This construction yields several weights on the free group $\mathbb{F}_2$ with unusual properties, see [10, Chapter 10].

v. If ω_1 and ω_2 are weights on G then so is the pointwise product $\omega_1\omega_2$.

vi. If ω_G and ω_H are weights on groups G and H respectively, then $\omega_G \otimes \omega_H$ defined by $(\omega_G \otimes \omega_H)(g, h) = \omega_G(g)\omega_H(h)$ is a weight on $G \times H$.

See [9] for several other interesting weights.

To extend results to classical unweighted objects to their weighted analogues, some boundedness condition on the weights is usually required. Actual boundedness is too strong an assumption, boundedness of kind of diagonal product is enough.

Definition 3.1.2. A weight on G is said to be *diagonally bounded* (by C) on $S \subseteq G$ if

$$\sup_{g \in S} \omega(g)\omega(g^{-1}) = C < \infty.$$

We say that the weight is diagonally bounded *if it is diagonally bounded on G.*

If ω is diagonally bounded on $S \subseteq G$ and $g \in S$ then for any $h \in G$

$$\omega(g)\omega(h) = \omega(g)\omega(g^{-1}gh) \leq \omega(g)\omega(g^{-1})\omega(gh) \leq C\omega(gh)$$

and if we instead take $g \in G$ and $h \in S$ then we still have

$$\omega(g)\omega(h) = \omega(ghh^{-1})\omega(h) \leq \omega(gh)\omega(h^{-1})\omega(h) \leq C\omega(gh)$$

so diagonal boundedness by C on $S \subseteq G$ is in fact equivalent to

$$\omega(g)\omega(h) \leq C\omega(gh) \quad \forall g, h \in G \text{ with } g \text{ or } h \in S.$$

That we can have this inequality with only one of the two elements in S is vitally important.

Any bounded weight (such as the trivial weight $\omega \equiv 1$ or any weight on a compact group) is obviously diagonally bounded. The weight $\omega : \mathbb{R} \to (0, \infty)$, $x \mapsto e^x$ is diagonally bounded but not bounded, as is any other unbounded homomorphism of a group. See [10, Example 10.1] for a more interesting example of a weight on $\mathbb{F}_2$ which is diagonally bounded but not bounded.

For $g, h \in G$ let

$$\omega_h(g) := \frac{\omega(gh)}{\omega(h)} \leq \omega(g)$$

and note that if ω is diagonally bounded by C on $S \subseteq G$ then

$$\omega(g) \leq C\omega_h(g) \quad \forall g, h \in G \text{ with } g \text{ or } h \in S.$$

Technical results for weighted objects usually only require that the weight is diagonally bounded on a sufficiently large (e.g. dispersed, see Definition 2.1.1) subset of the group. This can be a strictly weaker assumption, for example, if G is an infinite discrete group and ω_1 and ω_2 are weights on G such that ω_1 is diagonally bounded but ω_2 is not, then $\omega_1 \otimes \omega_2$ on $G \times G$ cannot be diagonally bounded but is on the dispersed subset $\{e\} \times G$.

However, sometimes full diagonal boundedness is required. For example (and also to justify the importance of the definition), see Theorem 3.2.1 in the following section.

We now introduce the most important function in this work. If ω is a weight on a locally compact group G we define $\Omega : G \times G \to (0, 1]$ by

$$\Omega(g, h) = \frac{\omega(gh)}{\omega(g)\omega(h)}.$$

When it is important to identify the underlying weight, we write Ω_ω. The function Ω measures to what extent ω fails to be a homomorphism: $0 < \Omega \leq 1$ and $\Omega \equiv 1$

exactly when ω is a homomorphism. Note that ω is diagonally bounded by C on $S \subseteq G$ if and only if $\Omega \geq \frac{1}{C}$ on $(G \times S) \cup (S \times G)$. Also define, following [10, (8.7)] for $k \geq 1$, $\Omega^k : G^k \to (0,1]$ by, for $g_1, \ldots, g_k \in G$,

$$\Omega^k(g_1, \ldots, g_k) = \frac{\omega(g_1 \cdots g_k)}{\omega(g_1) \cdots \omega(g_k)}$$

and let $\Omega^1 \equiv 1$. Then $\Omega^2 = \Omega$ and ω is diagonally bounded by C if and only if $\Omega^k \geq C^{1-k}$ for any $k > 1$. The motivation for this definition will be seen in Section 4.2. We will see that it is often actually the function Ω, rather that the weight itself, which determines the properties of weighted objects.

We end this section with a simple but important inequality. For any $g, h \in G$ we have

$$\Omega(g, g^{-1}) = \frac{1}{\omega(g)\omega(g^{-1})} = \frac{\omega(gh)}{\omega(g)\omega(h)} \frac{\omega(h)}{\omega(g^{-1})\omega(gh)} = \Omega(g,h)\Omega(g^{-1}, gh) \leq \Omega(g,h)$$

and thus for any $g \in G$

$$\Omega(g, g^{-1}) \leq \min\{\Omega(g,h),\, \Omega(h,g) : h \in H\}. \tag{3.1}$$

We paraphrase this result by saying that the diagonal values $\Omega(g, g^{-1})$ are the smallest values of Ω.

3.2 Weighted algebras

Given a fixed weight ω on a locally compact group G, we have weighted L^1, L^∞, and LUC algebras, given by

$$L^1(G,\omega) := \{f : G \to \mathbb{C} \mid \|f\|_{1,\omega} := \int_G |f(g)|\omega(g) < \infty\}$$

$$L^\infty(G,\omega^{-1}) := \{f : G \to C \mid \|f\|_{\infty,\omega} := \operatorname*{ess\,sup}_{g\in G} |f(g)|\omega(g)^{-1} < \infty\}$$

$$LUC(G,\omega^{-1}) := \{f : G \to C \mid f\omega \in LUC(G)\}$$

where the product on $L^1(G,\omega)$ is convolution (the same as defined in 2.1), but the product on $L^\infty(G,\omega^{-1})$ and, by restriction, $LUC(G,\omega^{-1})$, is denoted by $\cdot_\omega$ and is given by, for $f, g \in L^\infty(G,\omega^{-1})$ or $LUC(G,\omega^{-1})$,

$$f \cdot_\omega g := \frac{fg}{\omega},$$

so $L^\infty(G,\omega^{-1})$ and $LUC(G,\omega^{-1})$ are unital with identity ω. The algebras $L^1(G,\omega)$ are called *Beurling* (or *weighted convolution*) algebras. Again we have

$$L^1(G,\omega)^* = L^\infty(G,\omega^{-1}) \quad \text{and} \quad L^\infty(G,\omega^{-1}) \cdot L^1(G,\omega) = LUC(G,\omega^{-1}).$$

These algebras are linearly isomorphic to to their classical, unweighted counterparts by simple maps which just multiply or divide by the weight, but they they can be very different considered as algebras. In fact as Banach spaces, the Beurling algebra is just the L^1 space of integrable functions against the measure whose Radon-Nikodym derivative with respect to the Haar measure is the weight, but one should not be lulled into a false sense that Beurling algebras are mild generalizations of group algebras.

Often the diagonal boundedness condition is necessary to extend results from the group algebra to Beurling algebras. For example, consider the following.

Theorem 3.2.1 (Grønbæk [24]). *Let G be a locally compact group and ω be a weight on G. Then the Beurling algebra $L^1(G, \omega)$ is amenable (as a Banach algebra) if and only if G is amenable (as a locally compact group) and ω is diagonally bounded.*

3.3 Topological centres of weighted objects

In the classical, unweighted case, the group algebra $L^1(G)$ of any locally compact group G is strongly Arens irregular, but adding a weight can drastically affect the topological centre. For example, consider the following result.

Theorem 3.3.1 (Craw and Young [8]). *The Beurling algebra $\ell^1(\mathbb{Z}, \omega_\alpha)$ with $\omega_\alpha(k) = (1 + |k|)^\alpha$ is Arens regular if and only if $\alpha > 0$.*

They also prove a much more powerful result, still the best general result in this theory.

Theorem 3.3.2 (Theorem 2 in [8]). *Let G be a locally compact group. There is a weight on G such that the Beurling algebra is Arens regular if and only if the group is discrete and countable.*

Theorem 8.11 in [10] gives conditions for weights on discrete groups which are equivalent to Arens regularity of the associated Beurling algebra. The situation for non-discrete groups is not as well understood. The best general result is the following.

Theorem 3.3.3 (Dales and Lau, 12.3 in [10]). *Let ω be a weight on a locally compact group G such that the compact covering number $\kappa(G)$ is non-measurable. Then if ω is diagonally bounded on a dispersed subset of G then the Beurling algebra $L_1(G, \omega)$ is strongly Arens irregular.*

The nonexistence of measurable cardinals is consistent with ZFC and the existence of measurable cardinals cannot be proved consistent with ZFC. We resist the urge to expound on foundational and philosophical issues here, instead we remark that this assumption on $\kappa(G)$ is considered "mild" and direct the reader to [16] for a discussion of these matters.

Recall that for discrete groups the condition that a subset is dispersed is equivalent to having the same cardinality as the group (as long as the group is infinite).

We have a similar result for $LUC(G, \omega^{-1})^*$ under the same assumptions.

Theorem 3.3.4 (Neufang, 1.2 in [42])**.** *Let ω be a weight on a locally compact group G which is diagonally bounded on a dispersed subset. Then the topological centre of $LUC(G, \omega^{-1})^*$ is $M(G, \omega)$, that is, $LUC(G, \omega^{-1})^*$ is strongly Arens irregular.*

However, nondiscrete Beurling algebras are not always strongly Arens irregular. For example, consider the following result (note that the weight is similar to that in Theorem 3.3.1).

Theorem 3.3.5 (Dales and Lau, Theorem 12.6 in [10])**.** *Let w_α be the weight on $\mathbb{R}$ given by $\omega_\alpha(x) = (1 + |x|)^\alpha$ with $\alpha > 0$. Then the Beurling algebra $L^1(\mathbb{R}, \omega_\alpha)$ is neither Arens regular nor strongly Arens irregular.*

Chapter 4

The weighted compactification

4.1 Algebra in the compactification

Following Filali and Salmi, we denote by G_ω^{LUC} the spectrum of $LUC(G, \omega^{-1})$, which we call the *weighted (LUC-)compactification* of G. In the unweighted ($\omega \equiv 1$) case, $G_\omega^{LUC} = G^{LUC}$ is a semigroup which densely contains G. But if ω is a nontrivial weight, then G_ω^{LUC} does not even contain G as point evaluations, since $\|\delta_g\| = \omega(g)$. So we identify G with $\widetilde{G} \subset G_\omega^{LUC}$ via

$$G \ni g \mapsto \tilde{g} = \delta_g \omega(g)^{-1}.$$

Then $\widetilde{G}$ is dense in the compact space G_ω^{LUC}, which is homeomorphic to G^{LUC} via the map Φ given by, for $m \in G_\omega^{LUC}$ and $f \in LUC(G, \omega^{-1})$,

$$\langle \Phi(m), f \rangle = \langle m, f\omega^{-1} \rangle$$

but this embedding of $\widetilde{G}$ in G_ω^{LUC} does not respect products.

In this section we study G_ω^{LUC} and products of its elements. By restricting the

Arens product on $LUC(G,\omega^{-1})^*$, we obtain a map

$$G_\omega^{LUC} \times G_\omega^{LUC} \to LUC(G,\omega^{-1})^*$$

which we denote simply by juxtaposition. However, unlike the unweighted case, the product may not land back in G_ω^{LUC}. Let $g,h \in G$. Then

$$\tilde{g}\tilde{h} = \omega(g)^{-1}\omega(h)^{-1}\delta_g\delta_h = \Omega(g,h)w(gh)^{-1}\delta_{gh} = \Omega(g,h)\,\widetilde{gh} \tag{4.1}$$

which immediately gives the following.

Proposition 4.1.1. *Let ω be a weight on a locally compact group G and $g,h \in G$. Then the following are equivalent*

i. $\omega(gh) = \omega(g)\omega(h)$

ii. $\Omega(g,h) = 1$

iii. $\tilde{g}\tilde{h} = \widetilde{gh}$

iv. $\tilde{g}\tilde{h} \in G_\omega^{LUC}$.

The obvious extension of this result to all of G_ω^{LUC} holds.

Theorem 4.1.2. *Let ω be a weight on a locally compact group G. Then G_ω^{LUC} is a semigroup if and only if ω is a homomorphism.*

Proof. If ω is not a homomorphism then there are $g,h \in G$ with $\omega(gh) \neq \omega(g)\omega(h)$ and then $\tilde{g}\tilde{h} \notin G_\omega^{LUC}$ by the preceding Proposition, which establishes necessity. For sufficiency, suppose that ω is a homomorphism and let $m,n \in G_\omega^{LUC}$, $g \in G$, and

$x, y \in LUC(G, \omega^{-1})$. Then

$$\begin{aligned}\ell_g(x \cdot_\omega y) &= \ell_g(xy\omega^{-1}) \\ &= (\ell_g x)(\ell_g y)(\ell_g \omega^{-1}) \\ &= (\ell_g x)(\ell_g y)\, \omega(g)^{-1}\omega^{-1} \\ &= (\ell_g x \ \cdot_\omega \ \ell_g y)\, \omega(g)^{-1}\end{aligned}$$

so

$$\begin{aligned}n \cdot (x \cdot_\omega y)(g) &= \langle n, \ell_g(x \cdot_\omega y)\rangle \\ &= \langle n, \ell_g x \ \cdot_\omega \ \ell_g y\rangle \omega(g)^{-1} \\ &= \langle n, \ell_g x\rangle \langle n, \ell_g y\rangle \omega(g)^{-1} \\ &= n \cdot x(g)\, n \cdot y(g)\, \omega(g)^{-1}\end{aligned}$$

hence

$$n \cdot (x \cdot_\omega y) = (n \cdot x)(n \cdot y)\omega^{-1} = (n \cdot x) \cdot_\omega (n \cdot y)$$

and then

$$\begin{aligned}\langle mn, x \cdot_\omega y\rangle &= \langle m, n \cdot (x \cdot_\omega y)\rangle \\ &= \langle m, (n \cdot x) \cdot_\omega (n \cdot y)\rangle \\ &= \langle m, n \cdot x\rangle \langle m, n \cdot y\rangle \\ &= \langle mn, x\rangle \langle mn, y\rangle\end{aligned}$$

thus mn is multiplicative. It remains to show that $mn \neq 0$. Note that

$$n \cdot \omega(g) = \langle n, \ell_g \omega\rangle = \langle n, \omega(g)\omega\rangle = \langle n, \omega\rangle \omega(g) = \omega(g)$$

so $n \cdot \omega = \omega$, and then

$$\langle mn, \omega\rangle = \langle m, n \cdot w\rangle = \langle m, \omega\rangle = 1$$

so $mn \neq 0$ and hence $mn \in G_\omega^{LUC}$. □

We wish to calculate the product of two elements of G_ω^{LUC}. First we need an extension of Ω to G_ω^{LUC} and the following identity. Let $m, n \in G_\omega^{LUC}$ and $x \in LUC(G, \omega^{-1})$. Write $m = \lim_\alpha \widetilde{g_\alpha}$ and $n = \lim_\beta \widetilde{h_\beta}$ with the g_α, h_β in $\widetilde{G}$. Then

$$\begin{aligned}
\langle mn, x\rangle &= \langle m, n \cdot x\rangle \\
&= \left\langle \lim_\alpha \widetilde{g_\alpha}, \left(\lim_\beta \widetilde{h_\beta}\right) \cdot x \right\rangle \\
&= \lim_\alpha \left\langle \widetilde{g_\alpha}, \left(\lim_\beta \widetilde{h_\beta}\right) \cdot x \right\rangle \\
&= \lim_\alpha \left(\lim_\beta \widetilde{h_\beta}\right) \cdot x\,(g_\alpha)\,\omega(g_\alpha)^{-1} \qquad (4.2)\\
&= \lim_\alpha \left\langle \lim_\beta \widetilde{h_\beta}, \ell_{g_\alpha} x \right\rangle \omega(g_\alpha)^{-1} \\
&= \lim_\alpha \lim_\beta \langle \widetilde{h_\beta}, \ell_{g_\alpha} x\rangle \omega(g_\alpha)^{-1} \\
&= \lim_\alpha \lim_\beta x(g_\alpha h_\beta)\omega(g_\alpha)^{-1}\omega(h_\beta)^{-1}.
\end{aligned}$$

which, in the special case $x = \omega$, yields

$$\langle mn, \omega\rangle = \lim_\alpha \lim_\beta \Omega(g_\alpha, h_\beta). \qquad (4.3)$$

We now extend Ω to G_ω^{LUC}. Define $\widetilde{\Omega} : G_\omega^{LUC} \times G_\omega^{LUC} \to [0, 1]$ by, for $m, n \in G_\omega^{LUC}$,

$$\widetilde{\Omega}(m, n) = \langle mn, \omega\rangle. \qquad (4.4)$$

Note that $\widetilde{\Omega}$ agrees with Ω on G (identified with $\widetilde{G}$) and that if ω is diagonally bounded by K, then $\widetilde{\Omega} \geq \frac{1}{K}$. We will see that the function $\widetilde{\Omega}$ is intimately related to the semigroup generated by G_ω^{LUC}.

The definition of $\widetilde{\Omega}$ is anticipated by that of $\Omega_\Box$ in [9, Equation 4.5]. Indeed the definition there looks identical to the right side of Equation 4.3, but is actually defined on the Stone-Čech compactifcation of G, which is homeomorphic to G_ω^{LUC} if an only if G is discrete, so the definitions agree in the discrete case.

Following [9, Definition 5.3], we call a weight weakly diagonally bounded by $K > 0$ if $\widetilde{\Omega}(m, n) \geq \frac{1}{K}$ for every $m, n \in G_\omega^{LUC} \backslash \widetilde{G}$. This is equivalent to the condition that for any $\varepsilon > 0$, there is a compact set $C \subset G$ such that $\Omega(g, h) \geq \frac{1}{K}(1 - \varepsilon)$ for every $g, h \in G \setminus C$.

Clearly a diagonally bounded weight is weakly diagonally bounded by the same bound, but the conditions are distinct. [10, Example 9.7] is an example (due to J.F. Feinstein) of a weight on $\mathbb{Z}$ which is diagonally bounded by e^2 but weakly diagonally bounded by 1.

Let $\Phi : G_\omega^{LUC} \to G^{LUC}$ be the homeomorphism given by, for $m \in G_\omega^{LUC}$,

$$\langle \Phi(m), x \rangle = \langle m, x\omega \rangle \qquad (x \in LUC(G)) \tag{4.5}$$

and let $\pi : G^{LUC} \times G^{LUC} \to G^{LUC}$ be the product. Then consider the binary operation

$$\odot : G_\omega^{LUC} \times G_\omega^{LUC} \xrightarrow{\Phi \times \Phi} G^{LUC} \times G^{LUC} \xrightarrow{\pi} G^{LUC} \xrightarrow{\Phi^{-1}} G_\omega^{LUC} \tag{4.6}$$

It is easily verified that for $g, h \in G$,

$$\Phi(\tilde{g}) = g \tag{4.7}$$

here identifying g with the point evaluation $\delta_g \in G^{LUC}$, and that

$$\tilde{g} \odot \tilde{h} = \widetilde{gh}. \tag{4.8}$$

Using this operation makes one feel dirty, since it simply ignores the algebraic difficulties introduced by the weight (which it forgets entirely), but we need it to calculate the product of two arbitrary elements of G_ω^{LUC}, which we do now.

Proposition 4.1.3. *Let ω be a weight on a locally compact group G and*

$m, n \in G_\omega^{LUC}$. Then

$$mn = \widetilde{\Omega}(m,n)\, m \odot n. \tag{4.9}$$

and hence

$$\|mn\| = \widetilde{\Omega}(m,n). \tag{4.10}$$

Proof. Write $m = \lim_\alpha \widetilde{g_\alpha}$ and $n = \lim_\beta \widetilde{h_\beta}$ with the $\widetilde{g_\alpha}, \widetilde{h_\beta} \in \widetilde{G}$. Then

$$\begin{aligned}
m \odot n &= \Phi^{-1} \circ \pi \circ (\Phi \times \Phi)(m,n) \\
&= \Phi^{-1}(\Phi(m)\Phi(n)) \\
&= \Phi^{-1}\left(\Phi\left(\lim_\alpha \widetilde{g_\alpha}\right)\Phi\left(\lim_\beta \widetilde{h_\beta}\right)\right) \\
&= \lim_\alpha \lim_\beta \Phi^{-1}(\Phi(\widetilde{g_\alpha})\Phi(\widetilde{h_\beta})) \\
&= \lim_\alpha \lim_\beta \Phi^{-1}(g_\alpha h_\beta) \\
&= \lim_\alpha \lim_\beta \widetilde{g_\alpha h_\beta}
\end{aligned}$$

Now let $x \in LUC(G, \omega^{-1})$. Then

$$\begin{aligned}
\langle \widetilde{\Omega}(m,n)\, m \odot n, x\rangle &= \left\langle \lim_\alpha \lim_\beta \widetilde{g_\alpha h_\beta}, x \right\rangle \widetilde{\Omega}(m,n) \\
&= \lim_\alpha \lim_\beta \langle \widetilde{g_\alpha h_\beta}, x\rangle \widetilde{\Omega}(m,n) \\
&= \lim_\alpha \lim_\beta x(g_\alpha h_\beta)\omega(g_\alpha h_\beta)^{-1}\widetilde{\Omega}(m,n) \\
&= \lim_\alpha \lim_\beta x(g_\alpha h_\beta)\omega(g_\alpha h_\beta)^{-1}\langle mn, \omega\rangle \\
&= \lim_\alpha \lim_\beta x(g_\alpha h_\beta)\omega(g_\alpha h_\beta)^{-1} \lim_\alpha \lim_\beta \omega(g_\alpha h_\beta)\omega(g_\alpha)^{-1}\omega(h_\beta)^{-1} \\
&= \lim_\alpha \lim_\beta x(g_\alpha y_\beta)\omega(g_\alpha)^{-1}\omega(h_\beta)^{-1} \\
&= \langle mn, x\rangle. \quad \square
\end{aligned}$$

Expanding Equation 4.9 with limits yields the following. Let $m, n \in G_\omega^{LUC}$ and write $m = \lim_\alpha \widetilde{g_\alpha}$ and $n = \lim_\beta \widetilde{h_\beta}$ with the $\widetilde{g_\alpha}$, $\widetilde{h_\beta} \in \widetilde{G}$. Then

$$mn = \lim_\alpha \lim_\beta \Omega(g_\alpha, h_\beta)\widetilde{g_\alpha h_\beta}. \tag{4.11}$$

The Proposition immediately gives us the following.

Corollary 4.1.4. *The closed semigroup generated by G_ω^{LUC} is contained in $[0,1]G_\omega^{LUC}$, and hence is compact, being a closed subset of a continuous image of the compact space $[0,1] \times G^{LUC}$.*

We denote the semigroups generated by $\widetilde{G}$ and G_ω^{LUC} by $\langle\widetilde{G}\rangle$ and $\langle G_\omega^{LUC}\rangle$ and their closures by $\overline{\langle\widetilde{G}\rangle}$ and $\overline{\langle G_\omega^{LUC}\rangle}$, respectively. We study these semigroups in the following section.

Another result which follows immediately from the Proposition is the following, which is reminiscent of Proposition 4.1.1.

Corollary 4.1.5. *Let $m, n \in G_\omega^{LUC}$. Then $mn \in G_\omega^{LUC}$ if and only if*

$$\widetilde{\Omega}(m,n) = \|mn\| = 1.$$

We end this section with some results for G_ω^{LUC} related to diagonal boundedness.

Theorem 4.1.6. *Let ω be a weight on a topological group G. Then we have the following.*

i. If $0 \in \langle G_\omega^{LUC}\rangle$ then $0 \in G_\omega^{LUC} G_\omega^{LUC}$.

ii. If ω is diagonally bounded then $0 \notin \langle G_\omega^{LUC}\rangle$.

iii. ω is diagonally bounded if and only if $0 \notin \overline{\widetilde{G}\widetilde{G}}$.

Proof. i. Take the least $k \in \mathbb{N}$ such that $0 \in (G_\omega^{LUC})^n$. If $k > 2$ then take $m_1, \ldots, m_k \in G_\omega^{LUC}$ with $m_1 \cdots m_k = 0$. But then

$$\widetilde{\Omega}(m_1, m_2)(m_1 \odot m_2) m_3 \cdots m_k = 0$$

so $\Omega(m_1, m_2) = \|m_1 m_2\| = 0$ or $(G^{LUC})^{k-1} \ni (m_1 \odot m_2) m_3 \cdots m_k = 0$, both of which are impossible by minimality of k. So $k = 2$.

ii. Since ω is diagonally bounded there is a $K > 0$ such that $\widetilde{\Omega} \geq \frac{1}{K}$. Then for any $m, n \in G_\omega^{LUC}$ we have $\|mn\| = \widetilde{\Omega}(m, n) \geq \frac{1}{K}$ so $mn \neq 0$. Therefore $0 \notin G_\omega^{LUC} G_\omega^{LUC}$ and hence $0 \notin \langle G_\omega^{LUC} \rangle$ by part i.

iii. If ω is not diagonally bounded then there is a sequence (g_n) in G with $\omega(g_n)\omega(g_n^{-1}) \to \infty$ and thus $\Omega(g_n, g_n^{-1}) \to 0$. So

$$0 = \lim_n \Omega(g_n, g_n^{-1})\tilde{e} = \lim_n \tilde{g}_n \widetilde{g_n^{-1}} \in \overline{\tilde{G}\tilde{G}}.$$

Now suppose that $0 \in \overline{\tilde{G}\tilde{G}}$. Then there are nets (g_α) and (h_α) (indexed by the same set) in G such that $\lim_\alpha \Omega(g_\alpha, h_\alpha) = 0$. Then

$$\inf_\alpha \Omega(g_\alpha, g_\alpha^{-1}) \leq \inf_\alpha \Omega(g_\alpha, h_\alpha) = 0 \tag{4.12}$$

so ω cannot be diagonally bounded. □

Note that (carefully inspecting equation 4.12) it is moreover true that for any $S \subseteq \tilde{G}$ if $0 \in \overline{\tilde{G}S} \cup \overline{S\tilde{G}}$ then ω is not diagonally bounded on S.

Our next result is that (except for the trivial case) we always have $0 \in \overline{\langle G_\omega^{LUC} \rangle}$.

4.2 The semigroup generated by G_ω^{LUC}

Here we study the semigroups (in $LUC(G, \omega^{-1})^*$) generated $\widetilde{G}$ and G_ω^{LUC}. We begin with $\langle \widetilde{G} \rangle$, which can easily describe in terms of Ω_k. First we make the promised observation that we always have $0 \in \overline{\langle G_\omega^{LUC} \rangle}$ unless G_ω^{LUC} is already a semigroup.

Proposition 4.2.1. *Let ω be a non-homomorphic weight on a locally compact group*

G (so that G_ω^{LUC} is not a semigroup). Then

$$0 \in \delta\langle\widetilde{G}\rangle \subset \overline{\langle G_\omega^{LUC}\rangle}.$$

Proof. Since ω is not a homomorphism, $\exists g, h \in G$ such that $\omega(gh) \neq \omega(g)\omega(h)$ and hence $\Omega(g,h) < 1$. Then

$$\left\|(\tilde{g}\tilde{h})^n\right\| = \left\|\left(\Omega(g,h)\widetilde{gh}\right)^n\right\| = \Omega(g,h)^n \left\|\widetilde{gh}^n\right\| \leq \Omega(g,h)^n \to 0$$

as $n \to \infty$, so $0 \in \overline{\langle\widetilde{G}\rangle}$. □

Proposition 4.2.2.

$$\langle\widetilde{G}\rangle = \bigcup_{k\in\mathbb{N}} \{\Omega^k(g_1, \ldots, g_k)\tilde{g} : g = g_1 \cdots g_k\}.$$

Recall that if ω is diagonally bounded on a dispersed set, then the topological centre of G_ω^{LUC} is $\widetilde{G}$ [42]. From this we immediately obtain the following.

Proposition 4.2.3. *Let G be a locally compact group with a weight ω which is diagonally bounded on a dispersed set. Then the topological centre of $\overline{\langle G_\omega^{LUC}\rangle}$ is $\overline{\langle G_\omega^{LUC}\rangle} \cap [0,1]\widetilde{G}$.*

Note that $\widetilde{G}$ is contained in the topological centre.

The condition that the weight be diagonally bounded is necessary: for $\alpha > 0$ and $\omega_alpha(k) = (1+|k|)^\alpha$ on the integers has $\ell^1(\mathbb{Z}, \omega_\alpha)$ is Arens regular [8]. But since $\mathbb{Z}$ is discrete we have

$$LUC(G,\omega^{-1})^* = \ell^\infty(G,\omega^{-1})^* = \ell^1(G,\omega)^{**}$$
$$\mathcal{Z}_t(LUC(G,\omega^{-1})^*) = LUC(G,\omega^{-1})^*$$
$$\mathcal{Z}_t(G_\omega^{LUC}) = G_\omega^{LUC}$$
$$\mathcal{Z}_t(\overline{\langle G_\omega^{LUC}\rangle}) = \overline{\langle G_\omega^{LUC}\rangle}$$

In the unweighted case, it is known that G^{LUC} determines the group in the sense that for locally compact groups G and H, if G^{LUC} is topologically isomorphic to H^{LUC} then G is topologically isomorphic to H. We can generalize this result to the following.

Theorem 4.2.4. *Let G and H be locally compact groups with weights ω_G and ω_H respectively such that $\overline{\langle G_{\omega_G}^{LUC}\rangle}$ and $\overline{\langle H_{\omega_H}^{LUC}\rangle}$ are strongly Arens irregular (for example, this is the case if the weights are diagonally bounded on dispersed subsets of the groups). Let φ be a topological isomorphism between $\overline{\langle G_{\omega_G}^{LUC}\rangle}$ and $\overline{\langle H_{\omega_H}^{LUC}\rangle}$. Then φ preserves the norm if and only if there is a topological (group) isomorphism $\overline{\varphi}: G \cong H$ such that*

$$\Omega_G = \Omega_H \circ (\overline{\varphi} \times \overline{\varphi}).$$

Proof. Let φ be a topological isomorphism between $\overline{\langle G_{\omega_G}^{LUC}\rangle}$ and $\overline{\langle H_{\omega_H}^{LUC}\rangle}$. Then φ restricts to a topological isomorphism $\overline{\langle G_{\omega_G}^{LUC}\rangle} \cap [0,1]\widetilde{G} \cong \overline{\langle H_{\omega_H}^{LUC}\rangle} \cap [0,1]\widetilde{H}$ by Proposition 4.2.3, and since φ is multiplicative and preserves the norm, $\overline{\varphi} := \frac{\varphi|_G}{\|\varphi|_G\|}$ is a topological isomorphism $G \to H$. Also note that for $m, n \in G_{\omega_G}^{LUC}$,

$$\|mn\| = \|\Omega(m,n)\, m \odot n\| = \Omega(m,n),$$

so

$$\widetilde{\Omega}_G(m,n) = \|mn\| = \|\varphi(mn)\| = \|\varphi(m)\varphi(n)\| = \widetilde{\Omega}_H(\varphi(m), \varphi(n)).$$

The converse is clear. □

Since the topological centre condition is only used to identify the groups, if we have two weights on the same group then we do not need this assumption.

Theorem 4.2.5. *Let ω_1 and ω_2 be two weights on a locally compact group G. Then $\overline{\langle G_{\omega_1}^{LUC}\rangle} \cong \overline{\langle G_{\omega_2}^{LUC}\rangle}$ preserving the norm if only if $\Omega_1 = \Omega_2$.*

Note that since G_{ω}^{LUC} depends only on Ω, it cannot recover the weight itself since it cannot (in particular) distinguish between homomorphic weights.

The assumption that the map preserves norms is necessary to guarantee that the Ω functions agree. For example, consider $\mathbb{Z}_2$ with a weight given by $\omega_\alpha(0) = 1$ and $\omega_\alpha(1) = \alpha > 1$. Then $\Omega(1,1) = \alpha^{-2}$ and

$$\langle\widetilde{\mathbb{Z}_2}\rangle = \{\alpha^{-2n} : n \geq 0\}\mathbb{Z}_2$$

so

$$\overline{\langle\widetilde{\mathbb{Z}_2}\rangle} = \{\alpha^{-2n} : n \geq 0\}\mathbb{Z}_2 \cup \{0\} \cong (\mathbb{Z}_2 \times \mathbb{N}) \cup \{0\}$$

with operation

$$(x, m)(y, n) = (x + y, m + n + xy)$$

So if $\alpha \neq \beta$ then $\Omega_\alpha \neq \Omega_\beta$ but $\overline{\langle\widetilde{\mathbb{Z}_2}\rangle_\alpha} \cong \overline{\langle\widetilde{\mathbb{Z}_2}\rangle_\beta}$. Note also that although this is the simplest non-trivial example of a weighted compactification, already the algebra of the semigroup is not straightforward.

The final result of this section is a characterization of when $\langle\widetilde{G}\rangle$ is dense in $[0,1]G_\omega^{LUC}$. First we need the following basic Lemma on density in $\mathbb{R}$.

Lemma 4.2.6. *i. If* 0 *is an accumulation point of* $A \subseteq [0,\infty)$ *and* A *is closed under (whole) multiples then* A *is dense in* $[0,\infty)$.

ii. If 1 *is an accumulation point of* $B \subseteq [0,1]$ *and* B *is closed under (positive) powers then* B *is dense in* $[0,1]$.

Proof. i. $0 \in \overline{A}$ so let $x \in (0,\infty)$ and $0 < \varepsilon < x$. Take $y \in A$ such that $y < x - \varepsilon$. Then some multiple of y is in $A \cap (x-\varepsilon, x+\varepsilon)$ and hence A is dense in$[0,\infty)$.

ii. This follows from (i) and the homeomorphism $[0,\infty) \to (0,1]$, $x \mapsto \log -x$.

□

Theorem 4.2.7. *Let* ω *be a weight on a locally compact group* G*. Then the following are equivalent* *i.* 1 *is an accumulation point of* $\{\Omega(g, g^{-1}) : g \in G\}$.

ii. $\langle \widetilde{G} \rangle$ *is* $\|\cdot\|$*-dense in* $[0,1]\widetilde{G}$.

iii. $\langle \widetilde{G} \rangle$ *is* w^**-dense in* $[0,1]G_\omega^{LUC}$.

Proof. (i) $\Rightarrow$ (ii): Let

$$B = \{\gamma \in [0,1] : \gamma\tilde{e} \in \langle \widetilde{G} \rangle\}.$$

Since $\tilde{g}\widetilde{g^{-1}} = \Omega(g, g^{-1})\tilde{e}$, we have $\{\Omega(g, g^{-1}) : g \in G\} \subseteq B$, so 1 is an accumulation point of B. Also B is clearly closed under multiplication, so by the Lemma, B is dense in $[0,1]$. So $[0,1]\tilde{e} \subset \overline{\langle \widetilde{G} \rangle}$, then by multiplying by elements of $\widetilde{G}$, $\langle \widetilde{G} \rangle$ is dense in $[0,1]\widetilde{G}$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): It suffices to show that for every $n \in \mathbb{N}$ $\exists g_n \in G$ such that

$$1 - \frac{1}{n} < \Omega(g_n, g_n^{-1}) < 1.$$

For every $n \in \mathbb{N}$, $(1 - \frac{1}{2n})\tilde{e} \in \overline{\langle \widetilde{G} \rangle}$, so $\exists$ a net (m_α) in $\langle \widetilde{G} \rangle$ such that $m_\alpha \to (1 - \frac{1}{2n})\tilde{e}$. Write $m_\alpha = \gamma_\alpha \widetilde{g_\alpha}$ with $\gamma_\alpha \in (0,1]$ and $g_\alpha \in G$. Without loss of generality none of the $\gamma_\alpha = 1$. Then $g_\alpha \to e$ in G, so

$$\gamma_\alpha \Omega(g_\alpha, g_\alpha^{-1})\tilde{e} = \gamma_\alpha \widetilde{g_\alpha}\widetilde{g_\alpha^{-1}} = m_\alpha \widetilde{g_\alpha^{-1}} \to \left(1 - \frac{1}{2n}\right)\tilde{e}^2 = \left(1 - \frac{1}{2n}\right)\tilde{e}$$

and hence

$$\gamma_\alpha \geq \gamma_\alpha \Omega(g_\alpha, g_\alpha^{-1}) \to 1 - \frac{1}{2n}.$$

So $\exists \beta$ such that $\gamma_\beta \Omega(g_\beta, g_\beta^{-1}) > 1 - \frac{1}{n}$. Let $\gamma := \gamma_\beta$. If $\Omega(g_\beta, g_\beta^{-1}) < 1$ then we have

$$1 - \frac{1}{n} < \gamma\Omega(g_\beta, g_\beta^{-1}) \leq \Omega(g_\beta, g_\beta^{-1}) < 1$$

so we may take $g_n = g_\beta$ and we are done. Otherwise $\Omega(g_\beta, g_\beta^{-1}) = 1$ and then

$$\gamma\Omega(g_\beta, g_\beta^{-1})\tilde{e} = \gamma\tilde{e} \in \langle\widetilde{G}\rangle \setminus \widetilde{G}$$

since $\gamma < 1$ so we can write $\gamma\tilde{e} = \prod_{i=1}^k \widetilde{g_i}$ for some $k > 1$ and $\widetilde{g_1}, \ldots, \widetilde{g_k} \in \widetilde{G}$. Then by multiplying the expression $\widetilde{g_1} \cdots \widetilde{g_k}$ by pairing adjacent factors in different order, we obtain many different descriptions of γ as a product values of Ω. The Catalan numbers C_{n-1} count the number of (*a priori*) different ways that we can multiply the $\widetilde{g_1}, \ldots, \widetilde{g_n}$ to decompose γ, but we are only interested in $k-1$ decompositions.

We introduce now a shorthand notation: for $1 \le a \le b < c \le d \le k$, we write

$${}_a^b\Omega_c^d = \Omega(g_a \cdots g_b, g_c \cdots g_d)$$

and omit the b or d if $a = b$ or $c = d$, respectively.

Then the product of any row of the following table is equal to γ.

$$\begin{array}{cccccc}
{}_1\Omega_2^k & {}_2\Omega_3^k & {}_3\Omega_4^k & \cdots & {}_{k-2}\Omega_{k-1}^k & {}_{k-1}\Omega_k \\
{}_1\Omega_2 & {}_1^2\Omega_3^k & {}_3\Omega_4^k & & \cdots & {}_{k-1}\Omega_k \\
{}_1\Omega_2 & {}_1^2\Omega_3 & {}_1^3\Omega_4^k & {}_4\Omega_5^k & \cdots & {}_{k-1}\Omega_k \\
 & & \ddots & & & \\
{}_1\Omega_2 & {}_1^2\Omega_3 & \cdots & {}_1^i\Omega_{i+1}^k & \cdots & {}_{k-1}\Omega_k \\
 & & & & \ddots & \\
{}_1\Omega_2 & \cdots & \cdots & {}_1^{k-3}\Omega_{k-2} & {}_1^{k-2}\Omega_{k-1}^k & {}_{k-1}\Omega_k \\
{}_1\Omega_2 & \cdots & \cdots & \cdots & {}_1^{k-2}\Omega_{k-1} & {}_1^{k-1}\Omega_k
\end{array}$$

Note that the diagonal terms are of the form

$${}_1^i\Omega_{i+1}^k = \Omega(g_1 \cdots g_i, g_{i+1} \cdots g_k) = \Omega(g_1 \cdots g_i, (g_1 \cdots g_i)^{-1}).$$

We claim that at least one of these terms is strictly less than 1. If not, then by the inequality (3.1), for every $1 \le i \le k$, both ${}_1^{i-1}\Omega_i = \Omega(g_1 \cdots g_{i-1}, g_i)$ and

${}_{i-1}\Omega_i^k = \Omega(g_{i-1}, g_i \cdots g_k)$ are greater than ${}_1^{i-1}\Omega_i^k = \Omega(g_1 \cdots g_{i-1}, g_i \cdots g_k) = 1$. So in the table, the diagonal entries are less than or equal the entries above and below it. So if the diagonal entries are all 1, then so is every entry in the table, so $\gamma = 1$, which is a contradiction.

So fix i such that $\Omega(g_1 \cdots g_i, g_{i+1} \cdots g_k) < 1$ and let $g_n = g_1 \cdots g_i$. Then $g^{-1} = g_{i+1} \cdots g_k$ and

$$1 - \frac{1}{n} < \gamma \leq \Omega(g_1 \cdots g_i, g_{i+1} \cdots g_k) = \Omega(g_n, g_n^{-1}) < 1.$$

Now for every $n \in \mathbb{N}$ there is a $g_n \in G$ such that

$$1 - \frac{1}{n} < \Omega(g_n, g_n^{-1}) < 1$$

as desired. □

For example, the weight $\omega_\alpha(x) = (1 + |x|)^\alpha$, $\alpha > 0$ on $\mathbb{R}$ from Theorem 3.3.5 satisfies the conditions of the theorem: as $x, y \searrow 0$, $\Omega(x, y) \nearrow 1$.

Chapter 5

Sets determining the topological centre

5.1 The classical case

Budak, Işık, and Pym found small sets which determine the topological centres of $L^1(G)^{**}$, $LUC(G)^*$, and G^{LUC} in [6].

Theorem 5.1.1. *Let G be a locally compact, non-compact group. Then there exists a convergent net $\nu_j \to \nu$ in G^{LUC} such that*

i. If $\mu \in G^{LUC}$ and $\mu\nu_j \to \mu\nu$ then $\mu \in G$.

ii. If $\mu \in LUC(G)^*$ and $\mu\nu_j \to \mu\nu$ then $\mu \in M(G)$.

iii. If $\mu \in L^1(G)^{**}$ and $\mu\nu_j \to \mu\nu$ then $\mu \in L^1(G)$.

If we prefer to have the nets lying in G, then two nets are necessary.

Theorem 5.1.2. *Let G be a locally compact, non-compact group. Then there exist in G two nets (y_j) and (y'_j) with limits γ and γ' in G^{LUC}, respectively such that*

i. If $\mu \in G^{LUC}$ and both $\mu y_j \to \mu\gamma$ and $\mu y'_j \to \mu\gamma'$ then $\mu \in G$.

ii. If $\mu \in LUC(G)^*$ and both $\mu y_j \to \mu\gamma$ and $\mu y'_j \to \mu\gamma'$ then $\mu \in M(G)$.

iii. If $\mu \in L^1(G)^{**}$ and both $\mu y_j \to \mu\gamma$ and $\mu y'_j \to \mu\gamma'$ then $\mu \in L^1(G)$.

Note that this does not yield dtc sets for $L^1(G)$ since the nets are in the wrong space. However, they get around this using a Lemma that allows them to replace the net in G^{LUC} with one in $L^1(G)$ and obtain the following:

Theorem 5.1.3. *Let G be a locally compact, non-compact group. Then there exist two nets (h_j) and (h'_j) in $L^1(G)$ with limits H and H' in $L^1(G)^{**}$, respectively such that if $\mu \in L^1(G)^{**}$ and both $\mu h_j \to \mu H$ and $\mu h'_j \to \mu H'$ then $\mu \in L^1(G)$.*

Dales and Dedania tackled Beurling algebras on discrete groups in [9] and obtained the the following.

Theorem 5.1.4. *Let G be a countable discrete group, ω be a weight on G which is diagonally bounded by C on an infinite (and hence dispersed) subset S of G, and $n \in \mathbb{N}$ with $n > C$ (we could take $n = \lceil C \rceil$). Then there is a set $\mathfrak{D} \subset \beta G \setminus G \subset \ell^1(G,\omega)^{**}$ with $|\mathfrak{D}| = n$ such that if left multiplication by $m \in \ell^1(G,\omega)^{**}$ is continuous at the points in $\mathfrak{D}$ then $m \in \ell^1(G,\omega)$, that is, $\mathfrak{D}$ is an n-point set which determines the topological centre of $\ell^1(G,\omega)^{**}$.*

5.2 Determining sets for weighted objects

In this section we build towards nets which determine the topological centres of G^{LUC}_ω, $LUC(G,\omega^{-1})^*$, and $L^1(G,\omega)^{**}$, drawing influences from [9], [10], and [42] as well. We will spend most of time working in G and the unweighted world and only move into $\widetilde{G}$, G^{LUC}_ω, and the weighted side at the last minute.

Of course we begin with a technical lemma. The following result is exactly the same as [6, Proposition 4.2], except that we take our t_K in a dispersed subset of G. First first a compact symmetric neighbourhood V of $e \in G$ and then an open symmetric neighbourhood W of e with $W^4 \subseteq V$.

Lemma 5.2.1. *Let $\mathcal{K}$ be a family of compact subsets of the locally compact group G with nonempty interior, which is closed under finite unions, and with $|\mathcal{K}| = \kappa(G)$. Let S be a dispersed subset of G. Then there is a family $(t_K)_{K\in\mathcal{K}}$ of elements of S such that $t_K \notin K$ and whenever $H, K, H', K' \in \mathcal{K}$ with $H \neq K$, $H' \neq K'$, and $(H, K) \neq (H', K')$ we have*

$$VKt_Kt_H^{-1} \cap VK't_{K'}t_{H'}^{-1} = \varnothing.$$

In particular, all of the VKt_K are disjoint.

Proof. The construction is a straightforward application of transfinite induction. Write $\mathcal{K} = \{K_\alpha : \alpha < \kappa(G)\}$ and order by inclusion. Let t_{K_0} be any element of S and then having chosen $t_\beta := t_{K_\beta}$ for $\beta < \alpha$ take

$$t_{K_\alpha} \in S \setminus \left[K_\alpha \cup \left(\bigcup_{\beta,\gamma,\delta<\alpha} K_\alpha^{-1}V^2K_\beta t_\beta t_\gamma^{-1}t_\delta\right) \cup \left(\bigcup_{\beta,\gamma,\delta<\alpha} t_\delta^{-1}K_\delta^{-1}V^2K_\beta t_\beta t_\gamma^{-1}\right)\right]$$

which is possible since the union is of strictly fewer than $\kappa(G)$ compact sets and so does not cover S since it is dispersed. □

We now employ a trick due to Neufang and called "Neufang's Family Breeding Technique" by [6], to produce several simultaneously disjoint families satisfying the hypotheses of Lemma 5.2.1. First fix a family of compact subsets of G with $|\mathcal{K}| = \kappa(G)$. Instead of applying the Lemma to $\mathcal{K}$, for every $H \in \mathcal{K}$, we can let $\mathcal{K}_H = \{HK : K \in \mathcal{K}\}$. Then the $\mathcal{K}_H$ are all families of compact subsets of G with $|\mathcal{K}_H| = \kappa(G)\kappa(G) = \kappa(G)$ so we may apply the Lemma to each $\mathcal{K}_H$, yielding nets $(t_{H,K})_{K\in\mathcal{K}}$ in S for each H satisfying that the $VHKt_{H,K}t_{H',K'}$ are distinct for distinct quadruplets (H, K, H', K').

Let $\mathcal{K}$ be a family of compact subsets of G as in the Lemma. Take $n > C$ and let $H_1, \ldots, H_n \in \mathcal{K}$ be distinct. For $1 \leq i \leq n$, construct the nets $(t_{H_i,K})_{K\in\mathcal{K}}$ using the trick after Lemma 5.2.1. Let $(y_{i,K}) = (t_{H_i,K})$ and now regard these nets in $\widetilde{G}$.

Since $G_\omega^{LUC} = \overline{\widetilde{G}}$ is compact, we may assume (by replacing them with convergent subnets) that the $(y_{i,K})$ converge to cluster points $\gamma_i \in G_\omega^{LUC}$. These nets should determine the topological centre.

Since ω is diagonally bounded on the $y_{i,K}$, which converge to γ_i

$$|\langle \mu\gamma_i, \omega\rangle| = \tilde{\Omega}(\mu, \gamma_i) = \|\mu\gamma_i\| \geq \frac{1}{C} > \frac{1}{n}$$

it remains to show the opposite inequality for at least one j in $1, \ldots, n$, which should follow from the carefully constructed disjointness property in Lemma 5.2.1.

Recently Filali and Salmi [20] have found sets which determine the topological centre of $LUC(G, \omega^{-1})$ with the $\lfloor C \rfloor + 1$ many points as we expect using their technique of slowly oscillating functions developed. They also construct determining sets for $L^1(G, \omega)^{**}$ when G is σ-compact and SIN (has small invariant neighbourhoods) with one more point as well as for some other weighted objects. A determination result using our techniques would be an improvement over their result for $LUC(G, \omega^{-1})^*$ since we only require continuity against a single net which is explicitly constructed.

Chapter 6

Isomorphisms of Beurling algebras

6.1 Equivalent of weights

Two weights on a locally group G are said to be *equivalent* if the corresponding Beurling algebras are isomorphic. Weights ω_1 and ω_2 are equivalent if $\exists c, C > 0$ such that $c\omega_1 \leq \omega_2 \leq C\omega_1$ pointwise, but this condition cannot be necessary since if ω is a weight on $\mathbb{R}$ then so is $x \mapsto e^x\omega(x)$. More generally, multiplying any weight by any homomorphism yields an equivalent weight. Unfortunately, there is no nice characterization of equivalent weights in terms of values of the weights or of the associated Ω functions.

Asking only that the Beurling algebras are isomorphic cannot determine all the important features, since for example the group algebras on any two countable groups are isomorphic. So we propose that we insist that the Beurling algebras are moreover *isometrically* isomorphic, which is the ultimate measure of identity for Banach algebras and offer the following definition.

Definition 6.1.1. *Let ω_1 and ω_2 be weights on a locally compact group G with corresponding functions $\Omega_1 := \Omega_{\omega_1}$ and $\Omega_2 := \Omega_{\omega_2}$. We call ω_1 and ω_2 strongly* *equivalent if $\Omega_1 = \Omega_2$. By cross multiplication, this is equivalent to the quotient $\frac{\omega_1}{\omega_2}$ being a homomorphism.*

It is clear that this notion of equivalence is in fact an equivalence relation.

6.2 Isometric isomorphisms of Beurling algebras

We intend the following Theorem to justify our preceding definition.

Theorem 6.2.1. *Let G and H be locally compact groups with weights ω_1 and ω_2 respectively.*

i. If $\varphi : G \cong H$ is an isomorphism of the groups G and H and $\omega_G \circ \varphi$ and ω_H are strongly equivalent, then the Beurling algebras $L^1(G, \omega_G)$ and $L^1(H, \omega_H)$ are isometrically isomorphic.

ii. If ω_1 and ω_2 are two weights on a locally compact group G, then the Beurling algebras $L^1(G, \omega_1)$ and $L^1(G, \omega_2)$ are isometrically isomorphic if and only if ω_1 and ω_2 are strongly equivalent.

Proof. i. If $G \cong H$ we might as well assume that $G = H$ so that ω_1 and ω_2 are strongly equivalent weights on G. Then $\frac{\omega_2}{\omega_1}$ is a homomorphism so the map

$$L^1(G, \omega_1) \to L^1(G, \omega_2), \quad f \mapsto f\frac{\omega_1}{\omega_2}$$

is an isometric isomorphism.

ii. Suppose that $L^1(G, \omega_1)$ and $L^1(G, \omega_2)$ are isometrically isomorphic (as algebras). Then so are $L^\infty(G, \omega_1^{-1}) = L^1(G, \omega_1)^*$ and $L^\infty(G, \omega_2^{-1}) = L^1(G, \omega_2)^*$, $LUC(G, \omega_1^{-1}) = L^\infty(G, \omega_1^{-1}) \cdot L^1(G, \omega_1)$ and $LUC(G, \omega^{-1}) = L^\infty(G, \omega_2^{-1}) \cdot L^1(G, \omega_2)$, and $LUC(G, \omega_1^{-1})^*$ and $LUC(G, \omega_2^{-1})^*$. Thus the semigroups $\overline{\langle G_{\omega_1}^{LUC} \rangle}$ and $\overline{\langle G_{\omega_2}^{LUC} \rangle}$ generated by the spectra $G_{\omega_1}^{LUC} = \Delta(LUC(G, \omega_1^{-1})$ and $G_{\omega_2}^{LUC} =$

$\Delta(LUC(G,\omega_2^{-1}))$ respectively are norm-preservingly isomorphic (as semigroups). Then $\Omega_1 = \Omega_2$ by Theorem 4.2.4 so w_1 and ω_2 are strongly equivalent. □

An examination of the proof of ii yields that if we can conclude that two groups are isomorphic if the semigroups generated by their weighted compactifications are, then we do not need to assume that the groups are the same. By Theorem 4.2.4, this is the case if the weighted compactifications are strongly Arens irregular. So we have proved the following.

Theorem 6.2.2. *If ω_G and ω_H are weights on locally compact groups G and H respectively such that the $G_{w_G}^{LUC}$ and $H_{w_H}^{LUC}$ are strongly Arens irregular (which is the case e.g. the weights are diagonally bounded on dispersed sets), then the Beurling algebras $L^1(G,\omega_G)$ and $L^1(H,\omega_H)$ are isometrically isomorphic if and only if G and H are isomorphic and ω_G and ω_H are strongly equivalent, up to composing one with the group isomorphism.*

Note the similarity between the relationship between the previous two results and Theorems 4.2.4 and 4.2.5.

The following conjecture is sufficient to remove the assumption that we can extract the groups from the weighted compactification.

Conjecture 6.2.3. *Let G be a locally compact group with weight ω. Then the topological centre of G_ω^{LUC} is*

$$\mathcal{Z}_t(G_\omega^{LUC}) = \tilde{G} \cup \{m \in G_\omega^{LUC} \setminus \tilde{G} : m(G_\omega^{LUC} \setminus \tilde{G}) = \{0\}\}$$

We say that a function $f : X \times Y \to \mathbb{C}$ defined on topological spaces X and Y 0-*clusters* if

$$\lim_\alpha \lim_\beta f(x_\alpha, y_\beta) = \lim_\beta \lim_\alpha f(x_\alpha, y_\beta) = 0$$

whenever (x_α) and (y_β) are nets of distinct elements in X and Y respectively such that both repeated limits exist.

If this conjecture is true, then we have the following.

Corollary 6.2.4. *G_ω^{LUC} is Arens regular iff Ω 0-clusters on $G \times G$.*

Which would generalize [10, Theorem 8.11], and then we also have the the following, as desired.

Conjecture 6.2.5. *The Beurling algebras $L^1(G, \omega_1)$ and $L^1(H, \omega_2)$ are isometrically isomorphic (as algebras) if and only if there is a topological isomorphism (of groups) $\varphi : G \cong H$ and ω_1 and $\omega_2 \circ \varphi$ are strongly equivalent.*

Chapter 7

The Fourier algebra of the Fell group

7.1 The Fourier algebra

In 1964, Eymard [15] introduced the Fourier algebra $A(G)$ of a locally compact group G. First we need the following definitions.

Definition 7.1.1. *Let G be a locally compact group.*

i. *$C^*(G)$, the* group C*-algebra *of G, is the completion of $L^1(G)$ with respect to the norm*

$$\|f\|_c := \sup_{\pi} \|\pi(f)\|$$

where the supremum is taken over all non-degenerate $*$-representations π of $L^1(G)$ on a Hilbert space.

ii. *$C^*_\rho(G)$, the* reduced group C*-algebra *of G, is the closure of $\rho(L^1(G))$ in $B(L^2(G))$ where ρ is the extension of the left regular representation of G on $L^2(G)$, that is, for $f \in L^1(G)$ and $h \in L^2(G)$,*

$$\rho(f)h = f * h.$$

iii. *A continuous complex-valued function f on G is called* positive definite *if for any $x_1, \dots, x_n \in G$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$*

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} f(x_i^{-1} x_j) \geq 0$$

The collection of positive definite functions on G is denoted $P(G)$.

iv. *$B(G)$, the* Fourier-Stieltjes algebra *of G, is the linear span of $P(G)$ with the norm induced by the duality $B(G) = C^*(G)^*$.*

v. *$B_\rho(G)$, the* reduced Fourier-Stieltjes algebra *of G, is the weak*-closure of the collection of functions in $B(G)$ having compact support. It is a closed ideal in $B(G)$ and as the notation suggests, we have $B_\rho(G) = C^*_\rho(G)^*$.*

vi. *$VN(G)$, the* group von Neumann algebra *of G, is the WOT-closed subalgebra of $B(L^2(G))$ generated by $\rho(G)$, or equivalently, the bicommutant $\rho(G)''$ in $B(L^2(G))$.*

We now define the Fourier algebra.

Definition 7.1.2. *Let G be a locally compact group. The* Fourier algebra *of G can defined by any of the following equivalent conditions:*

i. *$A(G) := \{h * \tilde{k} : h, k \in L^2(G)\}$ where $\tilde{k}(x) := \overline{k(x^{-1})}$.*

ii. *$A(G)$ is the space of coefficients of the left regular representation of G, that is*

$$A(G) = \{\text{functions } G \ni g \mapsto \langle \rho(g)\xi, \eta \rangle : \xi, \eta \in L^2(G)\}$$

iii. *$A(G)$ is the closed ideal in $B(G)$ generated by the functions in $B(G)$ with compact support.*

$A(G)$ can be identified with the predual of the von Neumann algebra $VN(G)$, from which it inherits a norm, but for any $u \in A(G)$ we can find $\xi, \eta \in L^2(G)$ such that $u(g) = \langle \rho(g)\xi, \eta \rangle$ and $\|u\| = \|\xi\|_2 \|\eta\|_2$. With this norm, $A(G)$ is a commutative Banach algebra with the pointwise product.

If G is abelian, then $A(G) \cong L_1(\hat{G})$ via the Fourier transform, where $\hat{G}$ is the dual group of G.

We will see that understanding of a related algebra will be essential towards the study of the centre of $A(G)^{**}$.

$A(G)$ acts on $VN(G)$ via for $u, v \in A(G)$ and $T \in VN(G)$,

$$\langle u \cdot T, v \rangle := \langle T, uv \rangle.$$

For $T \in VN(G)$, let

$$\operatorname{supp} T := \{g \in G : u \cdot T = 0 \ \Rightarrow \ u(g) = 0 \quad \forall u \in A(G)\}$$

and let

$$UCB(\hat{G}) = \overline{\{T \in VN(G) : \operatorname{supp} T \text{ is compact}\}}.$$

$UCB(\hat{G})$ is a C*-subalgebra of $VN(G)$ equal to the closure of $A(G) \cdot VN(G)$. $UCB(\hat{G})$ is furthermore an involutive subalgebra of $VN(G)$ i.e. for $n \in UCB(\hat{G})^*$ and $T \in UCB(\hat{G})$, the element $n \cdot T \in VN(G)$ defined by for $u \in A(G)$

$$\langle n \cdot T, u \rangle := \langle n, u \cdot T \rangle$$

lies in $UCB(\hat{G})$. Then $UCB(\hat{G})^*$ is itself a Banach algebra under the product

$$\langle m \cdot n, T \rangle := \langle m, n \cdot T \rangle$$

where $m, n \in UCB(\hat{G})^*$ and $T \in UCB(\hat{G})$, which will appear often in the study of the centre of $A(G)^{**}$.

There is an extensive theory for the Fourier algebra which we do not have space to include here. The best place to start is still [15], see e.g. [12] for additional results.

7.2 The topological centre of the Fourier algebra

Recall that "AR" stands for "Arens regular" and "SAI" stands for "strongly Arens irregular".

Unlike group algebras, the topological centre of the Fourier algebra is not well understood. Of course, if G is abelian then $A(G) \cong L_1(\hat{G})$ so $A(G)$ is SAI, but there are only partial results in the nonabelian case.

Theorem 7.2.1 (6.5 in [31])**.** *If G is discrete and amenable then $A(G)$ is SAI.*

Theorem 7.2.2 (3.2 in [21])**.** *If G is a locally compact group such that $A(G)$ is AR, then G is discrete.*

Combining these results we obtain an earlier result [34, Proposition 5.3].

Corollary 7.2.3. *If G is an amenable locally compact group, then $A(G)$ is AR iff G is finite.*

It is tempting to conjecture that $A(G)$ is SAI if G is amenable. However, Losert has shown that the Fourier algebra of the compact group $SU(3)$ is not SAI. Little else is known about the compact case. Surprisingly, Losert has also shown that if G is a locally compact group containing a free subgroup, then G is not SAI [35], but even if such G is discrete, it can still not be AR by [21, Corollary 3.8].

Beyond this, it is necessary to consider $UCB(\hat{G})^*$. We define the *topological centre of* $UCB(\hat{G})^*$ to be

$$\mathfrak{Z}_t(UCB(\hat{G})^*) := \{m \in UCB(\hat{G})^* \mid \textit{the map } UCB(\hat{G})^* \ni n \mapsto m{\cdot}n \textit{ is } w^*\textit{-}w^*\textit{-continuous}\}.$$

$B_\rho(G)$ embeds into $UCB(\hat{G})^*$ (see [31, §4]) and $B_\rho(G) \subseteq \mathfrak{Z}_t(UCB(\hat{G})^*)$ [31, Proposition 4.5]. It is often easier to deal with $UCB(\hat{G})^*$ than $A(G)^{**}$, and knowledge of the centre of $UCB(\hat{G})^*$ can be translated to $A(G)^{**}$. For example, we have the following.

Theorem 7.2.4 (6.4 in [31])**.** *Let G be an amenable locally compact group. If we have $\mathfrak{Z}_t(UCB(\hat{G})^*) = B_\rho(G)$, then $\mathcal{Z}_t(A(G)) = A(G)$ (i.e. $A(G)$ is SAI).*

Then we have the following two results from Hu [25].

Theorem 7.2.5. *Let G be a metrizable locally compact group such that $\overline{[G,G]}$ is not open in G. Then*

i. $\mathfrak{Z}_t(UCB(\hat{G})^*) = B_\rho(G)$

ii. $\mathcal{Z}_t(A(G)^{**}) = A(G)$ *if G is amenable.*

Theorem 7.2.6. *Let $G = G_0 \times \prod_{i=1}^{\infty} G_i$, where each G_i, $i \geq 0$ is a metrizable locally compact group and G_i is compact and nontrivial for $i \geq 1$. Then*

i. $\mathfrak{Z}_t(UCB(\hat{G})^*) = B_\rho(G)$

ii. $\mathcal{Z}_t(A(G)) = A(G)$ *if G_0 is amenable.*

These results extend results of Lau and Losert in [31] and [32], where they assume the group to be second countable rather than metrizable.

7.3 The Fourier algebra of the Fell group

Let G be a group with a subgroup $H \leq G$ and a normal subgroup $N \trianglelefteq G$. Given a homomorphism $\varphi : H \to \mathrm{Aut}(N)$, the group of automorphisms of N, for $h \in H$ write $\varphi(h) = \varphi_h \in \mathrm{Aut}(N)$. Then $N \times H$ is a group under the operation

$$(n_1, h_1) * (n_2, h_2) = (n_1\varphi_{h_1}(n_2), h_1 h_2)$$

called the semidirect product of N and H and denoted $N \rtimes H$.

Now let p be a prime number and N denote the field of p-adic numbers. Also let K be the set of p-adic numbers of valuation $p^0 = 1$. Then K is a compact group under multiplication which acts on N by multiplication.

The p-adic numbers are a different (from $\mathbb{R}$) completion of $\mathbb{Q}$ (in fact, the only other one) with respect to a metric defined via prime factorizations.

Although it is very interesting, we do not develop their theory here, since we will not need to get our hands dirty with them anyway. A gentle reference is [23].

Definition 7.3.1 (Baggett [4])**.** *The* Fell group *is the semidirect product* $N \rtimes K$*.*

The Fell group is totally disconnected (i.e. the only connected components are singletons), amenable, and unimodular [3]. Furthermore, the dual object $\hat{G}$ of G (the set of equivalence classes of irreducible unitary representations of G), which fails to a be a group since G is nonabelian) is countable [4, Theorem 4.5]. Mauceri computed the dual object.

Lemma 7.3.2 (Mauceri [37])**.** *Let G be the Fell group. Then $\hat{G} = \hat{G}_1 \cup \hat{G}_2$ where $\hat{G}_1 = \{\pi_j : j \in \mathbb{Z}\}$ where the π_j represent G on $L_2(K)$ via*

$$\pi_j(m, l)f(k) = \exp(2\pi i p^j km)f(kl)$$

and $\hat{G}_2 = \{\pi_\theta : \theta \in \hat{K}\}$ where the π_θ are the characters of K,

$$\pi_\theta(m, l) = \theta(l).$$

So the characters of G are the $\{\pi_\theta : \theta \in \hat{K}\}$.

We may now describe the topological centre of $A(G)$.

Theorem 7.3.3. *The Fourier algebra of the Fell group is strongly Arens irregular.*

Proof. $G/N \cong K$ which is abelian, so $[G, G] \subseteq N$ and hence $\overline{[G, G]} \subseteq N$. Since $\overline{[G, G]}$ coincides with the intersection of the kernels of the characters of G, which do not depend on N, $N \subseteq \overline{[G,G]}$. Hence $\overline{[G,G]} = N$ is not open in G. Then since G is metrizable and amenable, $A(G)$ is strongly Arens irregular by a Theorem 7.2.5. $\square$

This is the first non-discrete non-abelian totally disconnected example.

Chapter 8

Remarks and further problems

We present here some open problems concerning the theory we have developed here.

1. **Minimality of dtc sets.** Are the dtc sets obtained in Chapter 5 minimal? While it is expected that the size of dtc sets for weighted objects should depend on the diagonal bound on the weight, we have no results along these lines. There are essentially no minimality results for dtc sets in general, except that dtc sets for commutative Banach algebras must have at least two elements (there are one-point dtc sets for non-commutative algebras). If the weighted is not diagonally bounded, can we still find (possibly infinite) dtc sets?

2. **Topological centres of weighted objects.** What exactly are the topological centres of G_ω^{LUC}, $L^1(G,\omega)^1(G,\omega)$, $LUC(G,\omega^{-1})^*$, etc. when the weight is not diagonally bounded (on a dispersed subset of the group)?

5. **Determining sets in the weighted case.** We must complete the extension of Budak, Işık, and Pym's results to the weighted case, begun in section 5.2.

4. Can we prove the conjecture in section 6.2?

5. **Isomorphisms of the weighted compactification.** What can be said about homomorphisms of the semigroups $\langle G_\omega^{LUC} \rangle$? Can they be classified via transformations of the group and relations on the weights (or at least the Ω)? It is not even clear in what category the semigroup generated by these semigroups should be

considered. They inherit some linear and norm structure from $LUC(G, \omega^{-1})^*$, but it is difficult to refer to these structures when considering the semigroups abstractly.

6. See the ends of [9] and [10] for many more problems, most of which are still open.

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