Decay of a Bound Muon to a Bound Electron

by

Anna Morozova

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Department of Physics
University of Alberta

Abstract

A bound muon in the presence of a nucleus can decay into an electron, which belongs to either continuous or discrete (bound) energy spectrum. The underlying physics of both cases differ a lot, and so does their importance. The Standard Model decay of a bound muon into an outgoing energetic electron provides a background in the experimental searches for the lepton-flavor-violating $\mu \to e$ conversions in the field of nucleus, whereas the decay into a bound electron for large value of Z has its analogy with the QCD due to the strong electromagnetic interaction. The present thesis focuses on the study of the latter case, i.e., the exclusive weak decay $(Z\mu) \to (Ze)\,\nu_\mu\bar{\nu}_e$. This decay proceeds through the muon decay $\mu \to e + \nu_\mu + \bar{\nu}_e$ in the presence of a spinless nucleus. We consider the setup where all the electrons were removed from the atom and there is only a muon in $1S$ state. The decay rates for $Z = 10$ and $Z = 80$ are calculated in two different approaches, namely, an Atomic Alchemy formalism developed by C. Greub et al., Phys. Rev. D52, 4028 (1995) and by modifying the one developed by A. Czarnecki et al., Phys. Rev. D84, 013006 (2011) for the decay of a bound muon into an outgoing energetic electron. We consider the interaction between electron and nucleus to be a Coulomb one and the spin of the nucleus is neglected. Point nucleus wave functions are used for numerical calculations of the decay rate and for the second formalism the case of a finite nucleus with the Fermi charge distribution is considered as well. It is found that the results for these approaches match for the small value of $Z\alpha$, however, they are different by 41 % in the large $Z\alpha$ limit. In order to see if the two approaches coincide in certain approximations, we have considered two limiting cases: the muon and electron masses being almost equal and the small $Z\alpha$ limit. Again, in these limiting cases a good agreement, both analytical and numerical, is found between the two formalisms.

Preface

Chapter 3 of this dissertation adapts the formalism from [1]. It contains corrections of errors made in the derivation of a bound muon to a bound electron decay rate. These mistakes were found and corrected by G. Zhang and me. The idea to compare these results via calculations of the same decay by modifying the formalism of ref. [2] belongs to A. Czarnecki. The program for these calculations was written by A. Czarnecki, X. Garcia Tormo and M. J. Aslam and was modified for our purposes by G. Zhang and M. J. Aslam.

The idea of studying the limiting cases presented in Chapter 4 belongs to A. Czarnecki. All the analytical and numerical calculations in this study were performed separately by G. Zhang, M. J. Aslam and me.

Appendix D contains a description for a program written by A. Volotka. My contribution consists in writing documentation for the program which can be used as a user guide for running and understanding its functionality. I also adjusted it for more precise calculations for the case of muonic wave functions. The program is available at ref. [3].

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List of Notations and Abbreviations

FF - form factors;

QED - Quantum Electrodynamics;

IR - Infrared Divergence;

UV - Ultraviolet Divergence;

$c = \hbar = 1$ - relativistic unit system;

$\alpha = e^2$ - fine-structure constant in relativistic units;

$\eta_{\mu\nu}$ - Minkowski metric;

p (roman style) - 4-vectors;

p (bold face) - 3-vectors;

$p = |\mathbf{p}|$ (italic style) - scalars;

$\not{p} = \mathrm{p}_\mu \gamma^\mu$ - operators contracted with gamma matrices;

$\gamma^\mu = (\gamma^0, \boldsymbol{\gamma}) = (\beta, \beta\boldsymbol{\alpha})$ - Dirac matrices;

$\bar{\psi} = \psi^\dagger \gamma^0$ - Dirac adjoint;

$\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$ - unit vector in the direction of **p**;

$|e\rangle$ - electron wave function;

$|\mu\rangle$ - muon wave function;

$|\delta\mu/e\rangle$ - perturbed wave function of muon/electron;

Indices:

Latin letters (i, j) run over $1, 2, 3$;

Greek letters (μ, ν) run over $0, 1, 2, 3$.

Chapter 1

Introduction

When bombarded by a high energy muon beam, the atom captures muons which, in turn, cascade rapidly to the $1S$ state, ejecting almost all the electrons in the atom [1]. The resulting exotic atom is called a muonic atom and is formed within 10^{-10} seconds after the penetration of the muonic beam inside the atom. Since muon and electron are both negatively charged leptons, their behavior is similar, apart from the following differences:

1. The mass of a muon is roughly 207 that of an electron, which results in the smaller Bohr radius (since it is inversely proportional to the mass $r \sim \frac{1}{m}$) and, therefore, the energy levels for a muon are more affected by the nucleus. This makes in turn the QED effects more substantial since the electromagnetic force becomes stronger as the distance between charged particles decreases;

2. Finite lifetime of a muon.

The lifetime of a free muon is about $2.2\,\mu s$, but it is significantly different from the lifetime of a bound one. In the latter case, the muon in its initial state has a different momentum distribution and less available energy, and the electron produced by the decay undergoes a strong Coulomb interaction with the nucleus. Also, for the large Z the finite size of the nucleus should be accounted for. All these factors contribute to the lifetime of a bound muon to differ from the free muon lifetime, which was first shown by Porter and Primakoff in ref. [4]. Later it was studied by Gilinski and Mathews in ref. [5] with the point nucleus approximation for the muon wave function and in [6] the electron energy spectrum is calculated accounting for the finite size of the heavy nucleus for the bound muon.

According to the Standard Model, a muon decays into an electron, muon neutrino and electron antineutrino:

$$\mu \to e + \nu_\mu + \bar{\nu}_e.$$

Nevertheless, the Standard Model is not complete so far. The experiments for finding the physics beyond it in muon decays are currently conducted at Fermilab [7] and COMET [8], both of which mainly focus on studying the Mu2e (muon-to-electron) conversion in the Coulomb field without the emission of neutrinos [2].

This dissertation focuses on studying the decay rate of a bound muon into a bound electron: $(Z\mu) \to (Ze)\,\nu_\mu\bar{\nu}_e$. Here, the parenthesis signifies a bound state (Z being the atomic number). This transition proceeds by a weak decay $B_1 \longrightarrow B_2 + \nu_\mu + \bar{\nu}_e$, where B_1

and B_2 are bound states that consist of $(Z\mu)$ and (Ze), respectively. We consider the setup when all the electrons were removed from the atom and there is only a muon in $1S$ state. The energy for a resulting bound electron is fixed since it is also in $1S$ state. If $1S$ state is not available for a bound electron produced in the decay it will occupy the L shell. The decay rate is then suppressed by a factor of $1/n^3$, where n is the principal quantum number. In comparison, in case of a decay into an outgoing electron, its energy spectrum is continuous and running from 0 to the muon mass.

In order to estimate the decay rate of a bound muon into a bound electron, we need to solve the Dirac equation in the presence of a central potential. This equation is solvable analytically only for the Coulomb potential. These exact solutions for a bound muon and electron are discussed in Chapter 2.

When calculating properties of weak decays of one electromagnetically bound state into another, it is important to take into account the relativistic corrections, which modify the decay rate of hydrogen-like systems. Also, the level shift scales like Z^4, which makes them significant for high-Z ions.

The calculations for the bound decay rate accounting for the relativistic corrections were performed in [1] both for point and finite nucleus wave functions in cases of $Z = 10$ and $Z = 80$ by accounting for the Coulomb interaction between the electron and the spinless nucleus. It is named "Atomic Alchemy" because in the process considered here the atomic species are changed from a muon to electron. In carrying out the calculations for $(Z\mu) \to (Ze)\,\nu_\mu\bar{\nu}_e$ transition, the first step is to solve the Dirac equation for the bound particle wave function $\Phi(\mathbf{k})$ with a point nucleus using the following expansion by frequencies

$$\Phi(\mathbf{k}) = \sum_r \left[A_r(\mathbf{k}) \frac{u_r(\mathbf{k})}{\sqrt{2\mathrm{k}^0}} + B_r^*(-\mathbf{k}) \frac{v_r(-\mathbf{k})}{\sqrt{2\mathrm{k}^0}} \right], \tag{1.0.1}$$

where r is the spin state. The terms proportional to B_r^* are neglected, since according to the normalization condition

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_r \left\{ |A_r(\mathbf{k})|^2 + |B_r(\mathbf{k})|^2 \right\} = 1, \tag{1.0.2}$$

the second term in Eq. (1.0.1) that corresponds to an antiparticle is estimated to contribute only 0.002 fraction of 1 and, hence, is ignored. Thus, the wave function in this approximation is defined as $\psi(\mathbf{k}) \equiv A_r(\mathbf{k})$ and the states are considered to be in $1S$.

Chapter 3 presents detailed calculations of $(Z\mu) \to (Ze)\,\nu_\mu\bar{\nu}_e$ in Atomic Alchemy's formalism [1]. After reviewing the derivation of the factorization formula, a detailed calculation of FF's is given. It is found that the sign of some terms of the FF's are not correct. The numerical values of decay branching ratios $\frac{\Gamma_{(Z\mu^-)\to(Ze^-)\nu_\mu\bar{\nu}_e}}{\Gamma_0}$ with the corrected FF's are then calculated for the cases of $Z = 10$ and $Z = 80$. Section 3.2 presents the calculations of the same decay rate by modifying the formalism developed for the bound muon decay into an outgoing electron in the presence of a nucleus [2]. In ref. [2] there is no such approximation as (1.0.1) in ref. [1], therefore, we find it illuminating to compare the numerical results given by these two approaches.

The formalism [2] treats both cases of the Coulomb interaction and a nucleus of finite size characterized by the Fermi distribution. The numerical values for decay ratios for $Z = 10$ and $Z = 80$ in formalisms of [1] and [2] show that the difference in values is as insignificant as few percents for small $Z\alpha$ and greater for larger values, where for $Z = 80$ the discrepancy is of about 41%.

In order to see if the two approaches are consistent we study two limiting cases in Chapter 4, namely, equal muon and electron masses ($m_\mu \approx m_e$) and small $Z\alpha$ with original masses retained. For the latter case it is convenient to redefine the decay ratios in terms of new FF's A_i. Also, both approaches [1, 2] are compared for the nearly equal mass limit in Table 4.7.1 which shows that they are in complete agreement. Finally, the dissertation is concluded in Chapter 5.

It is then followed by several Appendices, which discuss some properties of Dirac gamma matrices and the Dirac equation. Appendix D presents a detailed description for the software, which performs numerical calculations of self-energy shifts of a bound muon and bound electron. The inclusion of such corrections to the calculations of the bound muon rate is the next step in our research.

Chapter 2

External Field Dirac Equation

In order to solve the muon decay in the orbit one needs to know the relativistic wave functions of an initial state of the muon and final state of the electron in a relativistic theory. To obtain these wave functions it is needed to solve the Dirac equation in the central field. This equation can only be solved analytically for the case of the Coulomb potential. In the current Chapter we summarize the derivation of the Dirac wave functions for the Coulomb potential, the details of which can be found in [9-12].

2.1 Relativistic Electron in the Central Field

For an electron moving in a spherically symmetric field, the total angular momentum is given by

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{2.1.1}$$

where

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{2.1.2}$$

is the orbital momentum whose eigenfunctions are spherical harmonics $Y_l^m(\hat{\mathbf{r}})$:

$$\mathbf{L}^2 Y_l^m(\hat{\mathbf{r}}) = l(l+1) Y_l^m(\hat{\mathbf{r}}), \tag{2.1.3}$$

$$L_z Y_l^m(\hat{\mathbf{r}}) = m Y_l^m(\hat{\mathbf{r}}). \tag{2.1.4}$$

The operator

$$\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}, \tag{2.1.5}$$

is the spin momentum with the two-component spinors η_μ as its eigenfunctions

$$\mathbf{S}^2 \eta_\mu = \frac{1}{2}\left(\frac{1}{2}+1\right)\eta_\mu = \frac{3}{4}\eta_\mu, \tag{2.1.6}$$

$$S_z \eta_\mu = \mu \eta_\mu, \tag{2.1.7}$$

where $\mu = \pm\frac{1}{2}$.

In the presence of an electromagnetic field the stationary Dirac equation is given by

$$\left[\gamma^\mu \left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\Phi = 0. \tag{2.1.8}$$

From this equation the corresponding Dirac-Coulomb Hamiltonian which satisfies

$$\mathcal{H}_{DC}(\mathbf{r})\,\Phi(\mathbf{r}) = E\Phi(\mathbf{r})$$

is derived to be

$$\mathcal{H}_{DC}(\mathbf{r}) = \boldsymbol{\alpha}\cdot\mathbf{p} - \frac{e^2 Z}{|\mathbf{r}|} + m\beta, \tag{2.1.9}$$

(see Appendix B for a more detailed derivation and properties of this Hamiltonian). The wave function Φ in the component form is

$$\Phi(\mathbf{r}) = \begin{pmatrix} \Phi^u(\mathbf{r}) \\ \Phi^\iota(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} g_{Ejl}(r)\,\chi_{jlM}(\hat{\mathbf{r}}) \\ i f_{Ejl}(r)\,\chi_{j\bar{l}M}(\hat{\mathbf{r}}) \end{pmatrix}, \tag{2.1.10}$$

where the quantum number l defines the orbital angular momentum and $\bar{l}$ will be defined below. The functions $g_{Ejl}(r)$ and $f_{Ejl}(r)$ are the radial wave functions corresponding to upper and lower components, respectively, and the two-component functions $\chi_{jlM}(\hat{\mathbf{r}})$ have only the angular dependence. Since $\mathcal{H}_{DC}$ commutes with both operators $\mathbf{J}^2$ and J_z, the wave functions $\Phi(\mathbf{r})$, or more specifically their angular parts $\chi_{jlM}(\hat{\mathbf{r}})$, must be their eigenvectors as well:

$$\begin{cases} [\mathcal{H}_{DC}(\mathbf{r}), \mathbf{J}^2] = 0 \\ [\mathcal{H}_{DC}(\mathbf{r}), J_z] = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{J}^2\Phi(\mathbf{r}) = j(j+1)\,\Phi(\mathbf{r}) \\ J_z\Phi(\mathbf{r}) = M\Phi(\mathbf{r}) \end{cases}. \tag{2.1.11}$$

Knowing the eigenfunctions of orbital and spin momentum operators, the functions $\chi_{jlM}(\hat{\mathbf{r}})$ can be constructed explicitly out of them. $\chi_{jlM}(\hat{\mathbf{r}})$ satisfy the set of relations (2.1.11):

$$\begin{cases} \mathbf{J}^2\chi_{jlM}(\hat{\mathbf{r}}) = j(j+1)\,\chi_{jlM}(\hat{\mathbf{r}}) \\ J_z\chi_{jlM}(\hat{\mathbf{r}}) = M\chi_{jlM}(\hat{\mathbf{r}}) \end{cases}. \tag{2.1.12}$$

Therefore, the functions $\chi_{jlM}(\hat{\mathbf{r}})$ can be constructed as linear combinations of the spherical harmonics $Y_l^m(\hat{\mathbf{r}})$ and two-component spinors η_μ:

$$\chi_{jlM}(\hat{\mathbf{r}}) = \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m(\hat{\mathbf{r}})\,\eta_\mu, \tag{2.1.13}$$

where $C^{jm}_{j_1m_1j_2m_2}$ are the Clebsch-Gordan for which the following identities are satisfied:

$$|j_1 - j_2| \le j \le j_1 + j_2, \tag{2.1.14}$$

$$m = m_1 + m_2. \tag{2.1.15}$$

The spherical spinors form a complete set of orthonormalized functions

$$\int d\Omega \left(\chi_{jlM}\right)^{\dagger} \chi_{j'l'M'} = \delta_{jj'}\delta_{ll'}\delta_{MM'}, \tag{2.1.16}$$

and using Eq. (2.1.13) in Eqs. (2.1.3) and (2.1.6):

$$\begin{aligned}
\mathbf{L}^2\chi_{jlM}(\hat{\mathbf{r}}) &= \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu}\left[\mathbf{L}^2 Y_l^m(\hat{\mathbf{r}})\right]\eta_\mu \\
&= l(l+1)\sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m(\hat{\mathbf{r}})\,\eta_\mu = l(l+1)\,\chi_{jlM}(\hat{\mathbf{r}}), \qquad (2.1.17) \\
\mathbf{S}^2\chi_{jlM}(\hat{\mathbf{r}}) &= \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu}\left[\mathbf{S}^2 Y_l^m(\hat{\mathbf{r}})\right]\eta_\mu \\
&= \frac{3}{4}\sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m(\hat{\mathbf{r}})\,\eta_\mu = \frac{3}{4}\chi_{jlM}(\hat{\mathbf{r}}). \qquad (2.1.18)
\end{aligned}$$

As mentioned earlier, the quantum number l appearing in Eq. (2.1.10) defines the orbital momentum of the particle along with its parity. Consider the space inversion $\mathbf{P} : \mathbf{r} \to -\mathbf{r}$ in the Dirac equation (2.1.8). Such transformation will act on the position space on which the wave function (2.1.10) is defined in the following way

$$\Phi(t,\mathbf{r}) \to \Phi'(t,\mathbf{Pr}) = \mathcal{P}\Phi(t,\mathbf{r}), \tag{2.1.19}$$

where $\mathcal{P}$ is the linear operator which is to be determined and which should preserve the invariance of the Dirac equation:

$$\mathbf{P}\left[\gamma^\mu\left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\Phi'(t,\mathbf{Pr}) = 0 \tag{2.1.20}$$

Thus,

$$\begin{aligned}
&\mathbf{P}\left[\gamma^\mu\left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\mathcal{P}\Phi(t,\mathbf{r}) \\
&= \left[\mathbf{P}\left\{\gamma^0\left(\mathrm{p}_0 - eV\right)\right\} - \mathbf{P}\left\{\boldsymbol{\gamma}\cdot\left(\mathbf{p} - e\mathbf{A}\right) - m\right\}\right]\mathcal{P}\Phi(t,\mathbf{r}) \\
&= \left[\gamma^0\left(\mathrm{p}_0 - eV\right) + \boldsymbol{\gamma}\cdot\left(\mathbf{p} - e\mathbf{A}\right) - m\right]\mathcal{P}\Phi(t,\mathbf{r}) = 0. \qquad (2.1.21)
\end{aligned}$$

Since the last expression should reduce to $\left[\gamma^\mu\left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\Phi(t,\mathbf{r}) = 0$ it follows that

$$\gamma^0\mathcal{P} = \mathcal{P}\gamma^0,\ \boldsymbol{\gamma}\mathcal{P} = -\mathcal{P}\boldsymbol{\gamma}, \tag{2.1.22}$$

which can be satisfied by the choice of

$$\mathcal{P} = c_p \gamma^0, \tag{2.1.23}$$

where c_p is some c-number which depends on the particle's intrinsic parity. Now,

$$\mathcal{P}\Phi\left(t, \mathbf{Pr}\right) = c_p \gamma^0 \Phi\left(t, -\mathbf{r}\right) = c_p \begin{pmatrix} g_{Ejl}\left(r\right) \chi_{jlM}\left(-\hat{\mathbf{r}}\right) \\ -if_{Ejl}\left(r\right) \chi_{j\bar{l}M}\left(-\hat{\mathbf{r}}\right) \end{pmatrix}. \tag{2.1.24}$$

The space inversion in the spherical coordinates affects only the spherical harmonics $Y_l^m\left(\hat{\mathbf{r}}\right) = Y_l^m\left(\theta, \phi\right)$ in the following way

$$\mathbf{P}: \begin{cases} \theta \to \pi - \theta \\ \phi \to \pi + \phi \end{cases} \Rightarrow \mathbf{P} Y_l^m\left(\theta, \phi\right) = \left(-1\right)^l Y_l^m\left(\theta, \phi\right). \tag{2.1.25}$$

Therefore,

$$\chi_{jlM}\left(-\hat{\mathbf{r}}\right) = \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m\left(-\hat{\mathbf{r}}\right) \eta_\mu = \left(-1\right)^l \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu}\left(m\mu\right) Y_l^m\left(\hat{\mathbf{r}}\right) \eta_\mu = \left(-1\right)^l \chi_{jlM}\left(\hat{\mathbf{r}}\right). \tag{2.1.26}$$

Substituting this result in Eq. (2.1.24)

$$\mathcal{P}\Phi\left(t, \mathbf{Pr}\right) = c_p \begin{pmatrix} g_{Ejl}\left(r\right)\left(-1\right)^l \chi_{jlM}\left(\hat{\mathbf{r}}\right) \\ if_{Ejl}\left(r\right)\left(-1\right)^{\bar{l}+1} \chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right) \end{pmatrix}, \tag{2.1.27}$$

whose components should have the same parity just as they have in (2.1.10). Therefore, it follows that

$$l = \bar{l} + 1. \tag{2.1.28}$$

From the set of equations for the upper and lower components of the bispinor (for more details see Appendix B, Eq. (5.8.79)) it follows

$$\left(E + m\right) \Phi^\iota\left(\mathbf{p}\right) = \left(\boldsymbol{\sigma} \cdot \mathbf{p}\right) \Phi^u\left(\mathbf{p}\right), \tag{2.1.29}$$

$$\left(E - m\right) \Phi^u\left(\mathbf{p}\right) = \left(\boldsymbol{\sigma} \cdot \mathbf{p}\right) \Phi^\iota\left(\mathbf{p}\right). \tag{2.1.30}$$

Upon substitution of the explicit form of upper and lower components given in Eq. (2.1.10), the Eq. (2.1.29) can be rewritten as

$$\left(E + m\right) if_{Ejl}\left(r\right) \chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right) = p\left(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\right) g_{Ejl}\left(r\right) \chi_{jlM}\left(\hat{\mathbf{r}}\right). \tag{2.1.31}$$

Since under the spatial rotations the operator $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})$ acts in the same way as $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$, therefore,

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\chi_{jlM}(\hat{\mathbf{r}}) = c\chi_{j\bar{l}M}(\hat{\mathbf{r}})\,, \tag{2.1.32}$$

where c is some c-number. In order to find it let's multiply both sides of Eq. (2.1.32) by the Hermitian conjugate of $\chi_{j\bar{l}M}(\hat{\mathbf{r}})$ on the left and perform the angular integration. Using the orthonormality of the spherical spinors (2.1.16) it results in:

$$c = \int \chi^{\dagger}_{j\bar{l}M}(\hat{\mathbf{r}})\,(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\chi_{jlM}(\hat{\mathbf{r}})\,d\Omega. \tag{2.1.33}$$

To evaluate this integral, it is useful to express the unit vector components in terms of the spherical harmonics

$$\begin{aligned}\hat{r}_x =& \sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{1}\right),\\ \hat{r}_y =& i\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} + Y_1^{1}\right),\\ \hat{r}_z =& 2\sqrt{\frac{\pi}{3}}Y_1^{0},\end{aligned} \tag{2.1.34}$$

and then use the formula for integration of three spherical harmonics

$$\int d\Omega Y_{l_1}^{m_1*}Y_{l_2}^{m_2}Y_{l_3}^{m_3} = \sqrt{\frac{(2l_2+1)(2l_3+1)}{4\pi(2l_1+1)}}C^{l_1m_1}_{l_2m_2l_3m_3}C^{l_10}_{l_20l_30}. \tag{2.1.35}$$

Also, the Pauli matrices act on the two-component spinors in the following way

$$\eta^{\dagger}_{\mu_1}\sigma^{x}\eta_{\mu_2} = \delta_{\mu_1,-\mu_2}, \tag{2.1.36}$$
$$\eta^{\dagger}_{\mu_1}\sigma^{y}\eta_{\mu_2} = (-1)^{1-\mu_1}\,\delta_{\mu_1,-\mu_2}, \tag{2.1.37}$$
$$\eta^{\dagger}_{\mu_1}\sigma^{z}\eta_{\mu_2} = (-1)^{\frac{1}{2}-\mu_1}\,\delta_{\mu_1,\mu_2}. \tag{2.1.38}$$

Putting everything together in Eq. (2.1.33) and after some algebra the coefficient c is found to be -1. Next, multiplying Eq. (2.1.32) by $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ and using the fact that

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\chi_{jlM}(\hat{\mathbf{r}}) = -\chi_{j\bar{l}M}(\hat{\mathbf{r}})\,, \tag{2.1.39}$$
$$-\chi_{jlM}(\hat{\mathbf{r}}) = (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\chi_{j\bar{l}M}(\hat{\mathbf{r}})\,. \tag{2.1.40}$$

Substituting these results into the set of equations analogous to Eqs. (2.1.30) with the

Coulomb potential

$$
\begin{aligned}
(E - eV - m)\,\Phi^{u}(\mathbf{p}) - (\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{\iota}(\mathbf{p}) &= 0,\\
(E - eV + m)\,\Phi^{\iota}(\mathbf{p}) - (\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{u}(\mathbf{p}) &= 0.
\end{aligned}
\tag{2.1.41}
$$

we get the following equation for the lower component of the Dirac bispinor

$$
(\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{\iota}(\mathbf{p}) = i\,(\boldsymbol{\sigma}\cdot\mathbf{p})\,f_{Ejl}(r)\,\chi_{j\bar{l}M}(\hat{\mathbf{r}}) = -i\,(\boldsymbol{\sigma}\cdot\mathbf{p})\,(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,f_{Ejl}(r)\,\chi_{jlM}(\hat{\mathbf{r}})\,. \tag{2.1.42}
$$

Using $(\boldsymbol{\sigma}\cdot\mathbf{p})\,(\boldsymbol{\sigma}\cdot\mathbf{r}) = (\mathbf{p}\cdot\mathbf{r}) + i\boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{r}]$ in the Eq. (2.1.42) leads to

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{\iota}(\mathbf{p}) &= -\left\{i\,(\mathbf{p}\cdot\mathbf{r}) - \boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{r}]\right\}\frac{f_{Ejl}(r)}{r}\chi_{jlM}(\hat{\mathbf{r}})\\
&= -\left\{(\nabla\cdot\mathbf{r})\,\frac{f_{Ejl}(r)}{r} - \boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{r}]\,\frac{f_{Ejl}(r)}{r}\right\}\chi_{jlM}(\hat{\mathbf{r}})\\
&= -\left\{\mathbf{r}\nabla\left(\frac{f_{Ejl}(r)}{r}\right) + \frac{f_{Ejl}(r)}{r}\mathrm{div}\,(\mathbf{r}) + (\boldsymbol{\sigma}\cdot\mathbf{L})\,\frac{f_{Ejl}(r)}{r}\right\}\chi_{jlM}(\hat{\mathbf{r}})\,,
\end{aligned}
\tag{2.1.43}
$$

where we used Eq. (2.1.2). As $\mathrm{div}\,(\mathbf{r}) = 3$ and $(\mathbf{r}\nabla)\left(\frac{1}{r}\right) = -\frac{1}{r}$, Eq. (2.1.43) results in

$$
(\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{\iota}(\mathbf{p}) = -\left\{\frac{df_{Ejl}(r)}{dr} + \frac{2}{r}f_{Ejl}(r) + \frac{1}{r}\,(\boldsymbol{\sigma}\cdot\mathbf{L})\,f_{Ejl}(r)\right\}\chi_{jlM}(\hat{\mathbf{r}})\,. \tag{2.1.44}
$$

Next, consider the operator identity

$$
\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2 = \mathbf{L}^2 + 2\mathbf{S}\cdot\mathbf{L} + \mathbf{S}^2 \Rightarrow 2\mathbf{S}\cdot\mathbf{L} = \boldsymbol{\sigma}\cdot\mathbf{L} = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2, \tag{2.1.45}
$$

which upon acting on a spherical spinor $\chi_{jlM}(\hat{\mathbf{r}})$ gives

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\mathbf{L})\,\chi_{jlM}(\hat{\mathbf{r}}) &= \left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right)\chi_{jlM}(\hat{\mathbf{r}}) = \left[j\,(j+1) - l\,(l+1) - \frac{3}{4}\right]\chi_{jlM}(\hat{\mathbf{r}})\\
&\equiv -\,(1+\kappa_{jl})\,\chi_{jlM}(\hat{\mathbf{r}})\,,
\end{aligned}
\tag{2.1.46}
$$

where the quantum number κ_{jl} is defined as

$$
\kappa_{jl} = l\,(l+1) - j\,(j+1) - \frac{1}{4}. \tag{2.1.47}
$$

If $j = l - \frac{1}{2}$ then

$$
\kappa_{jl} = l\,(l+1) - \left(l - \frac{1}{2}\right)\left(l + \frac{1}{2}\right) - \frac{1}{4} = l^2 + l - l^2 + \frac{1}{4} - \frac{1}{4} = l, \tag{2.1.48}
$$

and if $j = l + \frac{1}{2}$

$$\kappa_{jl} = l\left(l+1\right) - \left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right) - \frac{1}{4} = l^2 + l - l^2 - 2l - \frac{3}{4} - \frac{1}{4} = -\left(l+1\right). \tag{2.1.49}$$

To sum up

$$\kappa_{jl} = \begin{cases} l, & \text{if } j = l - \frac{1}{2} \\ -\left(l+1\right), & \text{if } j = l + \frac{1}{2} \end{cases}, \text{ or } \kappa_{jl} = \begin{cases} j + \frac{1}{2}, & \text{if } j = l - \frac{1}{2} \\ -\left(j + \frac{1}{2}\right), & \text{if } j = l + \frac{1}{2} \end{cases}, \tag{2.1.50}$$

and

$$\kappa_{jl} = -\,\kappa_{j\bar{l}}, \tag{2.1.51}$$

$$\bar{l} = l - 1. \tag{2.1.52}$$

Now Eq. (2.1.44) can be written in terms of the newly defined quantum number κ_{jl}

$$\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^{\iota}\left(\mathbf{p}\right) = -\left\{\frac{df_{Ejl}\left(r\right)}{dr} + \frac{1-\kappa_{jl}}{r}\,f_{Ejl}\left(r\right)\right\}\chi_{jlM}\left(\hat{\mathbf{r}}\right). \tag{2.1.53}$$

Similarly, for the upper component of the Dirac bispinor

$$\begin{aligned} \left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^{u}\left(\mathbf{p}\right) &= \left(\boldsymbol{\sigma}\cdot\mathbf{p}\right) g_{Ejl}\left(r\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right) = -\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\left(\boldsymbol{\sigma}\cdot\mathbf{r}\right)\frac{g_{Ejl}\left(r\right)}{r}\chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right) \\ &= -\left\{-i\left(\nabla\cdot\mathbf{r}\right) + i\boldsymbol{\sigma}\cdot\left[\mathbf{p}\times\mathbf{r}\right]\right\}\frac{g_{Ejl}\left(r\right)}{r}\chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right) \\ &= i\left\{\frac{dg_{Ejl}\left(r\right)}{dr} + \frac{1+\kappa_{jl}}{r}g_{Ejl}\left(r\right)\right\}\chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right). \end{aligned} \tag{2.1.54}$$

After the substitution of the expressions (2.1.53) and (2.1.54) into (2.1.41) and canceling spherical spinors and a factor of i on both sides the set of equations for the radial wave functions is obtained to be

$$\begin{aligned} &\left(\tfrac{d}{dr} + \tfrac{1+\kappa_{jl}}{r}\right) g_{Ejl}\left(r\right) - \left(E - eV + m\right) f_{Ejl}\left(r\right) = 0, \\ &\left(\tfrac{d}{dr} + \tfrac{1-\kappa_{jl}}{r}\right) f_{Ejl}\left(r\right) + \left(E - eV - m\right) g_{Ejl}\left(r\right) = 0. \end{aligned} \tag{2.1.55}$$

2.2 Electron in Coulomb Field

We want to derive the wave functions in the point nucleus approximation, which is valid only for $Z\alpha \ll 1$. Consider the Coulomb potential $V\left(r\right) = -\frac{eZ}{r}$ for the set of the equations

(2.1.55). In the limit of $r \to 0$ these equations take the following form

$$\begin{array}{c} \left(\frac{d}{dr} + \frac{\kappa}{r}\right) G\left(r\right) - \frac{Z\alpha}{r} F\left(r\right) = 0 \\ \left(\frac{d}{dr} - \frac{\kappa}{r}\right) F\left(r\right) + \frac{Z\alpha}{r} G\left(r\right) = 0 \end{array}, \tag{2.2.1}$$

where the indices were dropped for brevity and the following change of variables was made

$$G\left(r\right) \equiv r g_{Ejl}\left(r\right), \; F\left(r\right) \equiv r f_{Ejl}\left(r\right). \tag{2.2.2}$$

In Eq. (2.2.1) the terms proportional to $E \pm m$ were neglected. Let's assume that the solutions of Eqs. (2.2.1) are of the form

$$G\left(r\right) = G_0 r^{\gamma}, \; F\left(r\right) = F_0 r^{\gamma}, \tag{2.2.3}$$

that upon substitution in Eq. (2.2.1) give

$$\begin{array}{l} G_0\left(\gamma + \kappa\right) - F_0 Z\alpha = 0, \\ G_0 Z\alpha + F_0\left(\gamma - \kappa\right) = 0. \end{array} \tag{2.2.4}$$

This system has non-trivial solutions only when

$$\begin{vmatrix} (\gamma + \kappa) & -Z\alpha \\ Z\alpha & (\gamma - \kappa) \end{vmatrix} = 0 \Rightarrow \gamma^2 = \kappa^2 - \left(Z\alpha\right)^2. \tag{2.2.5}$$

Let the solutions for the radial wave functions in Eq. (2.1.55) be of the form

$$g\left(x\right) = \sqrt{m + E} e^{-\frac{1}{2}x} x^{\gamma - 1}\left[W_1\left(x\right) + W_2\left(x\right)\right], \tag{2.2.6}$$

$$f\left(x\right) = -\sqrt{m - E} e^{-\frac{1}{2}x} x^{\gamma - 1}\left[W_1\left(x\right) - W_2\left(x\right)\right], \tag{2.2.7}$$

where the indices Ejl were dropped and the following change of variables was made

$$x = 2\lambda r, \; \lambda = \sqrt{m^2 - E^2}. \tag{2.2.8}$$

Upon substituting Eq. (2.2.6) into Eq. (2.1.55), the first equation becomes:

$$2\lambda\left(\frac{d}{dx} + \frac{1 + \kappa}{x}\right) g\left(x\right) - \left(E + m\right) f\left(x\right) - \frac{2Z\alpha\lambda}{x} f = 0. \tag{2.2.9}$$

Using the radial functions from Eq. (2.2.6) in the above equation results in

$$
\begin{aligned}
&e^{-\frac{1}{2}x}x^{\gamma-1} \\
&\quad\times\left\{\left(\frac{d}{dx}+\frac{1+\kappa}{x}\right)\sqrt{m+E}\left(W_1+W_2\right)+\frac{1}{2\lambda}\left[E+m+\frac{2Z\alpha\lambda}{x}\right]\sqrt{m-E}\left(W_1-W_2\right)\right\} \\
&\quad=\sqrt{m+E}\left\{-\frac{1}{2}e^{-\frac{1}{2}x}x^{\gamma-1}\left(W_1+W_2\right)+e^{-\frac{1}{2}x}\left(\gamma-1\right)x^{\gamma-2}\left(W_1+W_2\right)\right. \\
&\quad\left.+e^{-\frac{1}{2}x}x^{\gamma-1}\frac{d}{dx}\left(W_1+W_2\right)+\left(1+\kappa\right)e^{-\frac{1}{2}x}x^{\gamma-2}\left(W_1+W_2\right)\right\} \\
&\quad+\frac{1}{2\lambda}\left(E+m\right)\sqrt{m-E}e^{-\frac{1}{2}x}x^{\gamma-1}\left(W_1-W_2\right)+Z\alpha\sqrt{m-E}e^{-\frac{1}{2}x}x^{\gamma-2}\left(W_1-W_2\right)=0.
\end{aligned}
\tag{2.2.10}
$$

After canceling the extra powers of x and the exponent, the rearrangement gives

$$
\begin{aligned}
&-\frac{1}{2}x\left(W_1+W_2\right)+\left(\gamma-1\right)\left(W_1+W_2\right)+x\frac{d}{dx}\left(W_1+W_2\right)+\frac{x}{2}\left(W_1-W_2\right) \\
&+Z\alpha\sqrt{\frac{m-E}{m+E}}\left(W_1-W_2\right)+\left(1+\kappa\right)\left(W_1+W_2\right)=0.
\end{aligned}
\tag{2.2.11}
$$

This results in

$$
x\frac{d}{dx}\left(W_1+W_2\right)+\left(\gamma+\kappa\right)\left(W_1+W_2\right)-xW_2+Z\alpha\sqrt{\frac{m-E}{m+E}}\left(W_1-W_2\right)=0. \tag{2.2.12}
$$

Treating the second equation of Eq. (2.1.55) in the same way gives

$$
x\frac{d}{dx}\left(W_1-W_2\right)+\left(\gamma-\kappa\right)\left(W_1-W_2\right)+xW_2-Z\alpha\sqrt{\frac{m+E}{m-E}}\left(W_1+W_2\right)=0. \tag{2.2.13}
$$

Adding and subtracting Eqs. (2.2.12) and (2.2.13) leads to

$$
x\frac{dW_1}{dx}+\left(\gamma-\frac{Z\alpha E}{\lambda}\right)W_1+\left(\kappa-\frac{Z\alpha m}{\lambda}\right)W_2=0 \tag{2.2.14}
$$

$$
x\frac{dW_2}{dx}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)W_2+\left(\kappa+\frac{Z\alpha m}{\lambda}\right)W_1=0. \tag{2.2.15}
$$

From the first equation W_2 is given by

$$
W_2=\left[\left(\frac{Z\alpha E}{\lambda}-\gamma\right)W_1-x\frac{dW_1}{dx}\right]\left[\kappa-\frac{Z\alpha m}{\lambda}\right]^{-1}, \tag{2.2.16}
$$

and differentiating it with respect to x gives

$$\frac{dW_2}{dx} = \left[\left(\frac{Z\alpha E}{\lambda} - \gamma - 1 \right) \frac{dW_1}{dx} - x \frac{d^2 W_1}{dx^2} \right] \left[\kappa - \frac{Z\alpha m}{\lambda} \right]^{-1}. \tag{2.2.17}$$

Using these expressions in Eq. (2.2.19) gives

$$x \left[\left(\frac{Z\alpha E}{\lambda} - \gamma - 1 \right) \frac{dW_1}{dx} - x \frac{d^2 W_1}{dx^2} \right] + \left(\gamma + \frac{Z\alpha E}{\lambda} - \frac{Z\alpha E}{\lambda} \right) \left[\left(\frac{Z\alpha E}{\lambda} - \gamma \right) W_1 - x \frac{dW_1}{dx} \right] + \left[\kappa^2 - \left(\frac{Z\alpha m}{\lambda} \right)^2 \right] W_1 = 0, \tag{2.2.18}$$

or equivalently

$$x \frac{d^2 W_1}{dx^2} + (2\gamma + 1 - x) \frac{dW_1}{dx} - \left[\kappa^2 - \left(\frac{Z\alpha m}{\lambda} \right)^2 - \gamma^2 + \left(\frac{Z\alpha E}{\lambda} \right)^2 - x \left(\frac{Z\alpha E}{\lambda} - \gamma \right) \right] \frac{W_1}{x} = 0. \tag{2.2.19}$$

Noting that

$$\kappa^2 - \left(\frac{Z\alpha m}{\lambda} \right)^2 - \gamma^2 + \left(\frac{Z\alpha E}{\lambda} \right)^2 = \kappa^2 - \left(\frac{Z\alpha}{\lambda} \right)^2 (m^2 - E^2) - \gamma^2, \tag{2.2.20}$$

and using γ and λ from Eqs. (2.2.5) and (2.2.8), respectively, we can see that

$$\kappa^2 - \left(\frac{Z\alpha m}{\lambda} \right)^2 - \gamma^2 + \left(\frac{Z\alpha E}{\lambda} \right)^2 = 0. \tag{2.2.21}$$

Thus,

$$x \frac{d^2 W_1}{dx^2} + (2\gamma + 1 - x) \frac{dW_1}{dx} - \left(\gamma - \frac{Z\alpha E}{\lambda} \right) W_1 = 0. \tag{2.2.22}$$

From the second equation of the system (2.2.6), W_1 can be expressed as

$$W_1 = - \left[x \frac{dW_2}{dx} + \left(\gamma + \frac{Z\alpha E}{\lambda} - x \right) W_2 \right] \left[\kappa + \frac{Z\alpha m}{\lambda} \right]^{-1}. \tag{2.2.23}$$

and after differentiating it with respect to x gives

$$\frac{dW_1}{dx} = - \left[\frac{dW_2}{dx} + x \frac{d^2 W_2}{dx^2} + \left(\gamma + \frac{Z\alpha E}{\lambda} - x \right) \frac{dW_2}{dx} - W_2 \right] \left[\kappa + \frac{Z\alpha m}{\lambda} \right]^{-1}, \tag{2.2.24}$$

that upon substituting these values for W_1 and $\frac{dW_1}{dx}$ in Eq. (2.2.15) leads to

$$-x\left[\frac{dW_2}{dx}+x\frac{d^2W_2}{dx^2}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)\frac{dW_2}{dx}-W_2\right]$$
$$-\left(\gamma-\frac{Z\alpha E}{\lambda}\right)\left[x\frac{dW_2}{dx}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)W_2\right]+\left[\kappa^2-\left(\frac{Z\alpha m}{\lambda}\right)^2\right]W_2=0. \quad (2.2.25)$$

After rearranging the terms, we have

$$x\frac{d^2W_2}{dx^2}+(2\gamma+1-x)\frac{dW_2}{dx}-\left(1+\gamma-\frac{Z\alpha E}{\lambda}\right)W_2=0. \quad (2.2.26)$$

Each of the Eqs. (2.2.22) and (2.2.26) is of a form of the Kummer's equation

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0 \quad (2.2.27)$$

which has a confluent hypergeometric function

$$F(a,b;z)=1+\frac{a}{b}\frac{z}{1!}+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2!}+... \quad (2.2.28)$$

as its solution. Therefore, the functions W_1 and W_2 can be expressed in the form of confluent hypergeometric functions

$$W_1(x)=\alpha_0F\left(\gamma-\frac{Z\alpha E}{\lambda},2\gamma+1;x\right), \quad (2.2.29)$$
$$W_2(x)=\beta_0F\left(1+\gamma-\frac{Z\alpha E}{\lambda},2\gamma+1;x\right). \quad (2.2.30)$$

Using them in Eqs. (2.2.15) and (2.2.19) and setting $x=0$ gives the condition for coefficients α_0 and β_0:

$$\left(\kappa-\frac{Z\alpha m}{\lambda}\right)\beta_0=-\left(\gamma-\frac{Z\alpha E}{\lambda}\right)\alpha_0. \quad (2.2.31)$$

From the explicit form of the hypergeometric functions given in Eq. (2.2.28) it follows that in the limit $x\to\infty$ the functions W_1 and W_2 will also go to infinity. Therefore, the following condition for the series to terminate should be imposed:

$$\gamma-\frac{Z\alpha}{\lambda}=-n_r,\ n_r=\begin{cases}0,1,2,...\text{ if }\kappa<0\\1,2,3,...\text{ if }\kappa>0\end{cases}. \quad (2.2.32)$$

From the continuity equation (5.8.70) derived in Appendix B:

$$\frac{\partial \mathrm{j}_{\mu}(\mathrm{x})}{\partial \mathrm{x}_{\mu}} = 0, \tag{2.2.33}$$

the normalization for the stationary bound states follows as

$$\int \rho(\mathbf{r})\, d^3\mathbf{r} = \int \Phi^{\dagger}(\mathbf{r})\, \Phi(\mathbf{r})\, d^3\mathbf{r} = 1, \tag{2.2.34}$$

and the radial functions $g(r)$ and $f(r)$ are,therefore, normalized as follows

$$\int dr\, r^2 \left[g^2(r) + f^2(r)\right] = 1. \tag{2.2.35}$$

Using this condition together with (2.2.31) gives explicit expression for radial wave functions.

To sum up, the Dirac-Coulomb wave function is given by

$$\Phi(\mathbf{r}) = \begin{pmatrix} g_{nlj}(r)\chi_{ljM}(\hat{\mathbf{r}}) \\ i f_{nlj}(r)\chi_{\bar{l}jM}(\hat{\mathbf{r}}) \end{pmatrix}, \tag{2.2.36}$$

where $\bar{l} = 2j - l = l \pm 1$ and the radial wave functions

$$\begin{aligned} g_{nlj}(r) =& \frac{(2\lambda_n)^{\frac{3}{2}}}{\Gamma(2\gamma_n + 1)} \left[\frac{\left(1 + \frac{E_n}{m}\right)\Gamma(2\gamma_n + n_r + 1)}{4N_n(N_n - \kappa_n)\, n_r!}\right]^{\frac{1}{2}} (2\lambda r)^{\gamma_n - 1} e^{-\lambda_n r} \\ & \times \left\{(N_n - \kappa_n) F(-n_r, 2\gamma_n + 1; 2\lambda_n r) - n_r F(1 - n_r, 2\gamma_n + 1; 2\lambda_n r)\right\}, \end{aligned} \tag{2.2.37}$$

$$\begin{aligned} f_{nlj}(r) =& \frac{-(2\lambda_n)^{\frac{3}{2}}}{\Gamma(2\gamma_n + 1)} \left[\frac{\left(1 - \frac{E_n}{m}\right)\Gamma(2\gamma_n + n_r + 1)}{4N_n(N_n - \kappa_n)\, n_r!}\right]^{\frac{1}{2}} (2\lambda r)^{\gamma_n - 1} e^{-\lambda_n r} \\ & \times \left\{(N_n - \kappa_n) F(-n_r, 2\gamma_n + 1; 2\lambda_n r) + n_r F(1 - n_r, 2\gamma_n + 1; 2\lambda_n r)\right\}, \end{aligned} \tag{2.2.38}$$

where n is the principle quantum number and the energy levels are found from the Sommerfeld's formula

$$E_n = \sqrt{m^2 - \lambda_n^2}, \quad \lambda_n = \frac{1}{aN_n}, \quad a = \frac{1}{Z\alpha m}, \quad n_r = n - \kappa_n, \tag{2.2.39}$$

$$N_n = \sqrt{n^2 - 2n_r(|\kappa_n| - \gamma_n)}, \quad \gamma_n = \sqrt{\kappa_n^2 - (Z\alpha)^2}. \tag{2.2.40}$$

The quantum number κ_n is defined by

$$\kappa_n = \begin{cases} l, & j = l - \frac{1}{2} \\ -(l + 1), & j = l + \frac{1}{2} \end{cases}. \tag{2.2.41}$$

and the spherical spinors

$$\chi_{jlM}(\hat{\mathbf{r}}) = \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m(\hat{\mathbf{r}})\,\eta_\mu$$
$$= (-1)^{-l+\frac{1}{2}-M}\sqrt{2j+1}\sum_{m\mu}\begin{pmatrix} l & \frac{1}{2} & j \\ m & \mu & -M \end{pmatrix} Y_l^m(\hat{\mathbf{r}})\eta_\mu. \qquad (2.2.42)$$

2.3 $1S$ Wave Functions

In the forthcoming study only the wave functions in the state $1S$ are of interest to us. They can be obtained from Eq. (2.2.37) and Eq. (2.2.38) by setting the following values of the quantum numbers to $n = 1$, $l = 0$, $j = \frac{1}{2}$. Thus,

$$g_{1S_{\frac{1}{2}}}(r) \equiv g(r) = \left(\frac{2}{a}\right)^{\gamma+\frac{1}{2}}\sqrt{\frac{1+\gamma}{2\Gamma(2\gamma+1)}}\exp\left(-\frac{r}{a}\right)r^{\gamma-1}, \qquad (2.3.1)$$

$$f_{1S_{\frac{1}{2}}}(r) \equiv f(r) = -\sqrt{\frac{1-\gamma}{1+\gamma}}\,g_{1S_{\frac{1}{2}}}(r)\,. \qquad (2.3.2)$$

The spherical spinors with the spin-up $\left(M = \frac{1}{2}\right)$ are

$$\chi_{\frac{1}{2}0\frac{1}{2}}(\hat{\mathbf{r}}) = (-1)^{\frac{1}{2}-\frac{1}{2}}\sqrt{2}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} Y_0^0(\hat{\mathbf{r}})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{4\pi}}, \qquad (2.3.3)$$

$$\chi_{\frac{1}{2}1\frac{1}{2}}(\hat{\mathbf{r}}) = -\sqrt{2}\left\{\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} Y_1^0(\hat{\mathbf{r}})\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} Y_1^1(\hat{\mathbf{r}})\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$
$$= -\sqrt{2}\left\{\frac{1}{\sqrt{6}}\left[\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{3}}\left[\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$
$$= -\frac{1}{\sqrt{4\pi}}\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}. \qquad (2.3.4)$$

Therefore, the ground state wave function for the bound state in position space can be written as

$$\Phi(\mathbf{r}) = \begin{pmatrix} g_{1S_{\frac{1}{2}}}(r)\chi_{\frac{1}{2}0\frac{1}{2}}(\hat{\mathbf{r}}) \\ if_{1S_{\frac{1}{2}}}(r)\chi_{\frac{1}{2}1\frac{1}{2}}(\hat{\mathbf{r}}) \end{pmatrix}$$

$$= \frac{(2mZ\alpha)^{\gamma+\frac{1}{2}}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} r^{\gamma-1} \exp(-mZ\alpha r) \begin{pmatrix} 1 \\ 0 \\ \frac{i(1-\gamma)}{Z\alpha}\cos\theta \\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{i\phi} \end{pmatrix}. \tag{2.3.5}$$

The function in the momentum space is given by the Fourier transform:

$$\Phi(\mathbf{k}) = \int d^3\mathbf{r}\, \Phi(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{2.3.6}$$

It is useful to discuss the Fourier transformation of $\Phi(\mathbf{r})$ by considering one component at a time. For the zeroth component of the wave function one can write

$$\int d^3\mathbf{r} \exp(-i\mathbf{k}\cdot\mathbf{r})\, r^{\gamma-1} \exp(-mZ\alpha r)$$
$$= 2\pi \int_0^\infty dr\, r^2 r^{\gamma-1} \int_{-1}^{1} d(\cos\theta) \exp(-mZ\alpha r) \exp(-ikr\cos\theta)$$
$$= 2\pi \int_0^\infty dr\, r^2 r^{\gamma-1} \exp(-mZ\alpha r) \frac{2\sin(kr)}{kr}. \tag{2.3.7}$$

After changing the variables $r \to \frac{r}{mZ\alpha}$ and using the integral

$$\int_0^\infty ds\, \exp(-s)\, s^{a-1} \sin(ps) = \Gamma(a) \left(1+p^2\right)^{-a/2} \sin(a \arctan p), \tag{2.3.8}$$

Eq. (2.3.7) becomes

$$\frac{4\pi a^{\gamma+1}}{k} \int_0^\infty dr\, r^{(\gamma+1)-1} \exp(-r) \sin(akr)$$
$$= \frac{4\pi a^{\gamma+1}}{k} \Gamma(1+\gamma) \left(1+a^2k^2\right)^{-(1+\gamma)/2} \sin\left[(1+\gamma)\arctan(ak)\right]. \tag{2.3.9}$$

Making use of the following trigonometric identities

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}, \tag{2.3.10}$$

$$\arctan x = \arccos \frac{1}{\sqrt{1+x^2}}, \tag{2.3.11}$$

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a), \tag{2.3.12}$$

Eq. (2.3.9) becomes

$$\begin{aligned} &\frac{4\pi a^{\gamma+1}}{k}\Gamma(1+\gamma)\left(1+a^2k^2\right)^{-(1+\gamma)/2}\sin\left[(1+\gamma)\arctan(ak)\right] \\ &= \frac{4\pi a^{\gamma+1}}{k}\Gamma(1+\gamma)\left(1+a^2k^2\right)^{-\gamma/2-1}(\sin\rho + ak\cos\rho), \end{aligned} \tag{2.3.13}$$

where $\rho \equiv \gamma \arctan(ak)$ and $a = \frac{1}{mZ\alpha}$. Thus, the zeroth component becomes

$$\begin{aligned} g(k) &= \Gamma(1+\gamma)\frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha)^{\gamma-1}\frac{4\pi}{k}a^{\gamma+1}\frac{\sin\rho + ak\cos\rho}{(1+a^2k^2)^{1+\gamma/2}} \\ &= \frac{N}{k(1+a^2k^2)^{1+\gamma/2}}(\sin\rho + ak\cos\rho), \end{aligned} \tag{2.3.14}$$

where $N \equiv 2^{\gamma+1}\Gamma(1+\gamma)\sqrt{\frac{a\pi(1+\gamma)}{\Gamma(1+2\gamma)}}$ as defined in [1]. It is worth mentioning that the notations for the upper and lower radial components of radial functions defined here with the use of a different convention in comparison with [1].

The 2[nd] component is proportional to

$$\int d^3\mathbf{r}\exp(-ikr\cos\theta)\, r^{\gamma-1}\exp(-mZ\alpha r)\cos\theta. \tag{2.3.15}$$

The integration over the angle θ gives

$$I \equiv \int_{-1}^{1} d(\cos\theta)\exp(-ikr\cos\theta)\cos\theta = \frac{2\left[\sinh(ikr) - ikr\cosh(ikr)\right]}{(ikr)^2}, \tag{2.3.16}$$

and since $\sinh y = -i\sin(iy)$ and $\cosh y = \cos(iy)$ the term in the square parenthesis becomes

$$-\frac{4\pi i}{k^2}\int dr\, r^2\exp(-mZ\alpha r)\, r^{\gamma-1}\left(\frac{\sin(kr) - kr\cos(kr)}{r^2}\right). \tag{2.3.17}$$

Thus, the first part of this expression is

$$\begin{aligned}
&\int dr\, r^2 \exp\left(-mZ\alpha r\right)\sin\left(kr\right) r^{\gamma-3} = a^\gamma \int dr\, r^{\gamma-1}\exp\left(-r\right)\sin\left(kr\right) \\
&= a^\gamma \Gamma\left(\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2}\sin\left(\gamma \arctan\left(ak\right)\right),
\end{aligned} \tag{2.3.18}$$

and the second part

$$\int dr\, r^\gamma \exp\left(-mZ\alpha r\right)\cos\left(kr\right) \to a^{1+\gamma}\int dr\; \exp\left(-r\right)\cos\left(akr\right) r^\gamma. \tag{2.3.19}$$

Doing the the integration by parts

$$\begin{aligned}
&\int dr\; \exp\left(-r\right)\cos\left(akr\right) r^\gamma \\
&= -\frac{a^{1+\gamma}}{ak}\int dr\; \sin\left(akr\right) e^{-r}\left(\gamma r^{\gamma-1} - r^\gamma\right) \\
&= -\frac{a^\gamma}{k}\int dr\left[\gamma \sin\left(akr\right) e^{-r} r^{\gamma-1} - \sin\left(akr\right) e^{-r} r^{(1+\gamma)-1}\right] \\
&= -\frac{a^\gamma}{k}\left\{\gamma\Gamma\left(\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2}\sin\rho - \Gamma\left(1+\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2-1/2}\sin\left[\left(1+\gamma\right)\arctan\left(ak\right)\right]\right\} \\
&= -\frac{a^\gamma}{k}\left\{\gamma\Gamma\left(\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2}\sin\rho - \Gamma\left(1+\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2-1/2}\frac{\sin\rho + ak\cos\rho}{\sqrt{1+a^2k^2}}\right\}.
\end{aligned} \tag{2.3.20}$$

Putting Eqs. (2.3.18) and (2.3.20) together one gets

$$\begin{aligned}
I = &-\frac{4\pi i}{k^2}\left\{a^\gamma\Gamma\left(\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2}\sin\left(\rho\right) + a^\gamma\Gamma\left(1+\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2}\sin\left(\rho\right)\right. \\
&\left. - a^\gamma\Gamma\left(1+\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2-1}\left(\sin\rho + ak\cos\rho\right)\right\} \\
= &-\frac{4\pi i}{k^2}a^\gamma\Gamma\left(1+\gamma\right)\left(1+a^2k^2\right)^{-\gamma/2}\left\{\frac{\sin\rho}{\gamma} + \sin\rho - \frac{\sin\rho}{1+a^2k^2} - \frac{ak\cos\rho}{1+a^2k^2}\right\} \\
= &-\frac{4\pi i}{k^2}\frac{a^\gamma\Gamma\left(1+\gamma\right)}{\gamma\left(1+a^2k^2\right)^{1+\gamma/2}}\left\{\left[1+\left(1+\gamma\right)a^2k^2\right]\sin\rho - \gamma ak\cos\rho\right\}.
\end{aligned} \tag{2.3.21}$$

The 4th component in (2.3.6) is identically zero since

$$\int_0^{2\pi} d\phi \exp\left(i\phi\right) = 0. \tag{2.3.22}$$

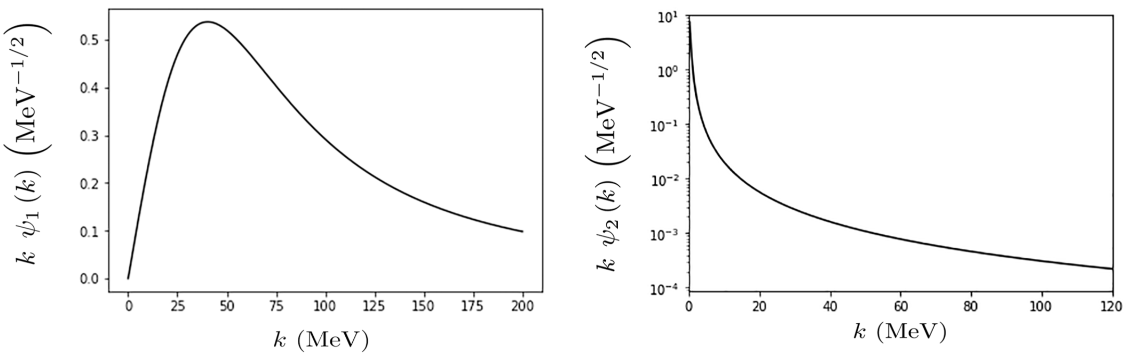

Figure 2.3.1: The Dirac wave functions for the point nucleus for muon (to the left) and electron (to the right). The electronic wave function has the same shape as the muonic one. But due to the proportionality of the larger spinor component to the Bohr radius ($a^{3/2}$) which is almost 200 times greater than that of a muon the electronic wave function has a sharper peak near the lower momentum region.

Thus, the lower component is proportional to

$$f(k) = \frac{Nm(1-\gamma)}{\gamma k^2 (1+a^2k^2)^{1+\gamma/2}} \left(\left[1+(1+\gamma)a^2k^2\right]\sin\rho - \gamma ak\cos\rho\right). \tag{2.3.23}$$

Next, the wave function $\Phi(\mathbf{k})$ can expanded in terms of free spinors $u_r(\mathbf{k})$ and $v_r(-\mathbf{k})$ by projecting it onto plane waves with positive and negative energies

$$\Phi(\mathbf{k}) = \sum_r \left[A_r(\mathbf{k})\frac{u_r(\mathbf{k})}{\sqrt{2\mathrm{k}^0}} + B_r^*(-\mathbf{k})\frac{v_r(-\mathbf{k})}{\sqrt{2\mathrm{k}^0}}\right]. \tag{2.3.24}$$

Here the integral $\int \frac{d^3\mathbf{k}}{(2\pi)^3}\sum_r |B_r(\mathbf{k})|^2$ corresponds to the probability of existing the state $(e^-e^-e^+)$ in the atom. And since for $Z = 80$

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\sum_r |A_r(\mathbf{k})|^2 = 0.998, \tag{2.3.25}$$

which almost equals to identity, the terms with the coefficient B_r^*can be neglected. Thus, the Dirac wave function $\psi(\mathbf{k})$ is defined as $\psi(\mathbf{k}) \equiv A_r(\mathbf{k})$ and the states are considered to be in $1S$ with the spin up: $r = +\frac{1}{2}$.

The final expression for the bound Dirac wave function for the spin up state $\psi(\mathbf{k}) = A_{\frac{1}{2}}(\mathbf{k})$ in terms of $g(k)$ and $f(k)$ becomes:

$$\psi(\mathbf{k}) = \sqrt{\frac{\mathrm{k}^0+m}{2\mathrm{k}^0}}\left(g(k) + \frac{k}{\mathrm{k}^0+m}f(k)\right). \tag{2.3.26}$$

	$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \left|\psi_1(\mathbf{k})\right|^2$	$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \left|\psi_2(\mathbf{k})\right|^2$
$Z = 10$	0.99999980	0.99999980
$Z = 80$	0.99840433	0.99840433

Table 2.3.1: The normalization for the muon and electron for $Z = 10$ and $Z = 80$.

The wave functions $k\psi_1(k)$ and $k\psi_2(k)$ for muon and electron, respectively, are plotted in Fig. (2.3.1). Numerical results for the normalization of the wave function (2.3.26) are presented in the Tab. 2.3.1 for the cases of $Z = 10$ and $Z = 80$

Chapter 3

Muon Decay

In this Chapter we consider the bound state transition $(Z\mu) \to (Ze)\,\nu_\mu\bar{\nu}_e$ (from here onward the parentheses for $(Z\mu)$ and (Ze) stand for the corresponding bound states) to proceed through the the weak decay $\mu \to e\nu_\mu\bar{\nu}_e$ in the presence of a nucleus charge Z, which is considered to be spinless. In two different formalisms developed in refs. [1] and [2]. The corresponding free muon decay is discussed in Appendix C.

3.4 Bound Muon Decay in the Atomic Alchemy Formalism

The formalism for the transition $B_1 \to B_2 + X$, where B_1 and B_2 are bound states, is developed in [1]. The Lagrangian describing the Fermi interaction for the standard muon to electron decay in the Fierz rearranged form is

$$\mathcal{L}_F = 2\sqrt{2}G_F\left(\bar{\Phi}_e\gamma^\rho L\Phi_\mu\right)\left(\bar{\Phi}_{\nu_\mu}\gamma_\rho L\Phi_{\bar{\nu}_e}\right) + \text{h.c.} \tag{3.4.1}$$

The wave function for the bound state B_1, which in our case is $(Z\mu)$ in its rest frame, is

$$|B_1, \mathbf{p}_{B_1} = \mathbf{0}\rangle = \sqrt{2m_{B_1}}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{\psi_1(\mathbf{k}_1)}{\sqrt{4\mathrm{k}_1^0\mathrm{k}_3^0}}b_\mu^\dagger(\mathbf{k}_1)\,a_Z^\dagger(\mathbf{k}_3)\,|0\rangle\,, \tag{3.4.2}$$

where $b_\mu^\dagger$ and $a_Z^\dagger$ are the creation operators for the muon and the nucleus, respectively, that act on the vacuum state $|0\rangle$, and

$$\mathrm{k}_1^0 = \sqrt{\mathbf{k}_1 + m_\mu^2},\ \mathrm{k}_3^0 = \sqrt{\mathbf{k}_3 + M^2},\ \mathbf{k}_3 = -\mathbf{k}_1 \tag{3.4.3}$$

are the corresponding energies and momenta of these particles. In this form the wave function is normalized in a covariant way

$$\langle B_1, \mathbf{0}\,|B_1, \mathbf{0}\rangle = 2m_{B_1}(2\pi)^3\,\delta^{(3)}(\mathbf{0})\,. \tag{3.4.4}$$

The probability amplitude $\psi_1(\mathbf{k}_1)$ to find a muon with momentum $\mathbf{k}_1$ in the atom is given by Eq. (2.3.26) and according to the approximation made in Eq. (2.3.25) it is taken to be normalized to unity:

$$\int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \left|\psi_1(\mathbf{k}_1)\right|^2 = 1. \tag{3.4.5}$$

The wave function of the final bound state in its center-of-mass frame is

$$|B_2, \mathbf{p}_{B_2} = \mathbf{0}\rangle = \sqrt{2m_{B_2}} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{\psi_2(\mathbf{k}_2)}{\sqrt{4\mathrm{k}_2^0\mathrm{k}_4^0}} b_e^\dagger(\mathbf{k}_2)\, a_Z^\dagger(\mathbf{k}_4)\, |0\rangle\,, \tag{3.4.6}$$

where $b_e^\dagger$ is the electron's creation operator, $\mathrm{k}_2^0 = \sqrt{\mathbf{k}_2 + m_e^2}$ is its energy and $\mathrm{k}_4^0 = \sqrt{\mathbf{k}_4 + M^2}$ is the energy of the nucleus. The masses used above are defined as

$$\begin{aligned} m_{B_1} &= M + m_1, \quad m_1 = m_\mu - E_{\text{bind},1}, \\ m_{B_2} &= M + m_2, \quad m_2 = m_e - E_{\text{bind},2}. \end{aligned} \tag{3.4.7}$$

After the decay of a muonic atom, in the center of mass frame of B_1 the particle B_2 is moving with a momentum $\mathbf{p}_{B_2} = -\mathbf{q}$. Thus, the boost $\Lambda(\mathbf{v})$ along the direction of $\mathbf{q}$ should be performed, and the corresponding boosted wave function for B_2 becomes

$$\begin{aligned} |B_2, \mathbf{p}_{B_2} = -\mathbf{q}\rangle &= \Lambda(\mathbf{v})\, |B_2, \mathbf{p}_{B_2} = -\mathbf{0}\rangle \\ &= \sqrt{2m_{B_2}} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{\psi_2(\mathbf{k}_2)}{\sqrt{4\mathrm{k}_2^0\mathrm{k}_4^0}} \left(\Lambda b_e^\dagger(\mathbf{k}_2)\,\Lambda^{-1}\right) \left(\Lambda a_Z^\dagger(\mathbf{k}_4)\,\Lambda^{-1}\right) |0\rangle\,, \end{aligned} \tag{3.4.8}$$

where $\mathbf{v} = \frac{\mathbf{q}}{m_{B_2}}$ stands for non-relativistic velocities. With the assumption that the axis of spin quantization is parallel to the boost, the transformation of fields is given by [13]:

$$\Lambda b_e^\dagger(\mathbf{k}_2)\,\Lambda^{-1} = \sqrt{\frac{\mathrm{k}_2^0}{\Lambda \mathrm{k}_2^0}}\, b_e^\dagger(\Lambda \mathbf{k}_2) \equiv \sqrt{\frac{\mathrm{k}_2^0}{\tilde{\mathrm{k}}_2^0}}\, b_e^\dagger\left(\tilde{\mathbf{k}}_2\right), \tag{3.4.9}$$

$$\Lambda a_Z^\dagger(\mathbf{k}_4)\,\Lambda^{-1} = \sqrt{\frac{\mathrm{k}_4^0}{\tilde{\mathrm{k}}_4^0}}\, a_Z^\dagger\left(\tilde{\mathbf{k}}_4\right), \tag{3.4.10}$$

where

$$\tilde{\mathbf{k}}_2 = \mathbf{k}_2 - \frac{m_e}{m_{B_2}}\mathbf{q},\ \tilde{\mathbf{k}}_4 = \mathbf{k}_4 - \frac{M}{m_{B_2}}\mathbf{q}. \tag{3.4.11}$$

Substituting expressions (3.4.9) and (3.4.10) in Eq. (3.4.8) the bound wave function is obtained to be

$$|B_2, \mathbf{p}_{B_2}\rangle = \sqrt{2m_{B_2}} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{\psi_2(\mathbf{k}_2)}{\sqrt{4\tilde{\mathrm{k}}_2^0\tilde{\mathrm{k}}_4^0}} b_e^\dagger\left(\tilde{\mathbf{k}}_2\right) a_Z^\dagger\left(\tilde{\mathbf{k}}_4\right) |0\rangle\,. \tag{3.4.12}$$

3.4.1 Factorization Formula

The S-matrix for the decay can be constructed in the following way:

$$S = i(2\pi)^4 \delta^{(4)}(\mathrm{p}_1 - \mathrm{p}_2 - \mathrm{q}) \left\langle B_2, \mathbf{p}_{B_2}; \nu_\mu\left(\mathrm{p}_{\nu_\mu}\right) \bar{\nu}_e\left(\mathrm{p}_{\bar{\nu}_e}\right) \left|\mathcal{L}_\omega(0)\right| B_1, \mathbf{0} \right\rangle, \tag{3.4.13}$$

where $\mathcal{L}_\omega$ is the four-fermion interaction which obeys translational invariance:

$$\mathcal{L}_\omega(\mathrm{x}) = e^{i\mathrm{P}\cdot\mathrm{x}} \mathcal{L}_\omega(0) e^{-i\mathrm{P}\cdot\mathrm{x}}. \tag{3.4.14}$$

Substituting Eqs. (3.4.2) and (3.4.12) into the S-matrix, we get

$$S = i(2\pi)^4 \delta^{(4)}(\mathrm{p}_1 - \mathrm{p}_2 - \mathrm{q}) \sqrt{4m_{B_1} m_{B_2}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{\psi_1(\mathbf{k}_1)}{\sqrt{4\mathrm{k}_1^0 \mathrm{k}_3^0}} \int \frac{d^3 k_2}{(2\pi)^3} \frac{\psi_2^*(\mathbf{k}_2)}{\sqrt{4\tilde{\mathrm{k}}_2^0 \tilde{\mathrm{k}}_4^0}}$$
$$\times \left\langle \nu_\mu\left(\mathrm{p}_{\nu_\mu}\right) \bar{\nu}_e\left(\mathrm{p}_{\bar{\nu}_e}\right) \middle| b_e\left(\tilde{\mathbf{k}}_2\right) a_Z\left(\tilde{\mathbf{k}}_4\right) \mathcal{L}_\omega(0)\, b_\mu^\dagger(\mathbf{k}_1)\, a_Z^\dagger(\mathbf{k}_3) \middle| 0 \right\rangle. \tag{3.4.15}$$

Applying the commutation relation $\left[a_Z\left(\tilde{\mathbf{k}}_4\right), a_Z^\dagger(\mathbf{k}_3)\right] = (2\pi)^3\, 2\mathrm{k}_3^0 \delta^{(3)}\left(\mathbf{k}_3 - \tilde{\mathbf{k}}_4\right)$ and the fact that $a_Z|0\rangle = 0$ to the second line in (3.4.15), it becomes

$$(2\pi)^3 \sqrt{4\mathrm{k}_3^0 \tilde{\mathrm{k}}_4^0}\, \delta^{(3)}\left(\mathbf{k}_3 - \tilde{\mathbf{k}}_4\right) \left\langle \nu_\mu\left(\mathrm{p}_{\nu_\mu}\right) \bar{\nu}_e\left(\mathrm{p}_{\bar{\nu}_e}\right) \middle| b_e\left(\tilde{\mathbf{k}}_2\right) \mathcal{L}_\omega(0)\, b_\mu^\dagger(\mathbf{k}_1) \middle| 0 \right\rangle, \tag{3.4.16}$$

where the delta-function $\delta^{(3)}\left(\mathbf{k}_3 - \tilde{\mathbf{k}}_4\right)$ is given by

$$\delta^{(3)}\left(\mathbf{k}_3 - \tilde{\mathbf{k}}_4\right) = \delta^{(3)}\left(\mathbf{k}_2 - m_{\mathrm{red},2} \mathbf{v}_{\mathrm{rel},2}\right), \tag{3.4.17}$$

and $m_{\mathrm{red},2} \equiv \frac{M m_e}{M + m_e}$ is the reduced mass of B_2, $\mathbf{v}_{\mathrm{rel},2} \equiv \frac{\mathbf{k}_1}{m_{\mathrm{red},2}} - \frac{\mathbf{q}}{m_e}$ is the relative velocity of the final state particles. With the use of Eq. (3.4.17) the integration over $\mathbf{k}_2$ can be carried out in Eq. (3.4.15) which gives

$$S = i(2\pi)^4 \delta^{(4)}(\mathrm{p}_1 - \mathrm{p}_2 - \mathrm{q}) \sqrt{4m_{B_1} m_{B_2}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{\psi_1(\mathbf{k}_1)}{\sqrt{4\mathrm{k}_1^0 \tilde{\mathrm{k}}_2^0}} \psi_2^*\left(\mathbf{k}_1 - \frac{m_{\mathrm{red},2}}{m_e}\mathbf{q}\right)$$
$$\times \left\langle \nu_\mu\left(\mathrm{p}_{\nu_\mu}\right) \bar{\nu}_e\left(\mathrm{p}_{\bar{\nu}_e}\right) e\left(\tilde{\mathbf{k}}_2\right) \left|\mathcal{L}_\omega(0)\right| \mu(\mathbf{k}_1) \right\rangle. \tag{3.4.18}$$

Thus, the invariant amplitude can be written in terms of the amplitude for the free muon decay invariant amplitude $\mathcal{M}_{\mu\to e\nu_\mu\bar{\nu}_e}$ as

$$\mathcal{M}_{B_1\to B_2} = \sqrt{4m_{B_1}m_{B_2}}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{\psi_1(\mathbf{k}_1)}{\sqrt{4\mathrm{k}_1^0\tilde{\mathrm{k}}_2^0}}\psi_2^*(\mathbf{k}_1-\mathbf{q})\,\mathcal{M}_{\mu\to e\nu_\mu\bar{\nu}_e}, \tag{3.4.19}$$

$$\mathcal{M}_{\mu\to e\nu_\mu\bar{\nu}_e} = \left\langle\nu_\mu\left(\mathrm{p}_{\nu_\mu}\right)\bar{\nu}_e\left(\mathrm{p}_{\bar{\nu}_e}\right)e\left(\mathbf{k}_1-\mathbf{q}\right)\left|\mathcal{L}_\omega(0)\right|\mu\left(\mathbf{k}_1\right)\right\rangle, \tag{3.4.20}$$

where the approximation $\mathbf{k}_1-\frac{m_{\mathrm{red},2}}{m_e}\mathbf{q}=\mathbf{k}_1-\mathbf{q}+O(1/M)$ is used.

The formulas (3.4.19) and (3.4.20) can be rewritten in the notation of [1]

$$\mathcal{M}_{B_1\to B_2}\equiv\mathcal{M}_{sr}=\frac{4G_F}{\sqrt{2}}\sqrt{4m_{B_1}m_{B_2}}N_\rho S^\rho_{sr}, \tag{3.4.21}$$

where the neutrino current N_ρ is

$$N_\rho=\bar{u}\left(\mathrm{p}_{\nu_\mu}\right)\gamma_\rho L\upsilon\left(\mathrm{p}_{\bar{\nu}_e}\right), \tag{3.4.22}$$

and the matrix elements S^ρ_{sr} that correspond to to $(\mu Z)\to(eZ)$ transition is

$$S^\rho_{sr}=\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{\psi_1(\mathbf{k}_1)}{\sqrt{2\mathrm{k}_1^0}}\frac{\psi_2^*(\mathbf{k}_1-\mathbf{q})}{\sqrt{2\mathrm{k}_2^0}}\bar{u}_s(e;\mathbf{k}_1-\mathbf{q})\,\gamma^\rho Lu_r(\mu;\mathbf{k}_1)\,. \tag{3.4.23}$$

Here the indices s and r stand for spins of electron and muon, respectively, and $L=\frac{1-\gamma_5}{2}$ is the left projection operator.

3.4.2 Form Factors

3.4.2.1 Normalization for Bound Spinors

In order to find the normalized form of the bound spinors, it is useful to express them in terms of the Kronecker delta functions. This can be achieved in the following way

$$\begin{aligned}&\bar{u}_{r'}(\mu;\mathbf{k}_1)\,u_r(B_1;\mathbf{0})=\\&=\sqrt{\mathrm{k}_1^0+m_\mu}\sqrt{2m_{B_1}}\begin{pmatrix}\left(\chi^{r'}\right)^\dagger & \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_1)}{\mathrm{k}_1^0+m_\mu}\left(\chi^{r'}\right)^\dagger\end{pmatrix}\begin{pmatrix}\chi^r\\0\end{pmatrix}\\&=\sqrt{2m_{B_1}\left(\mathrm{k}_1^0+m_\mu\right)}\delta_{rr'}.\end{aligned} \tag{3.4.24}$$

This gives

$$\delta_{rr'}=\frac{\bar{u}_{r'}(\mu;\mathbf{k}_1)\,u_r(B_1;\mathbf{0})}{\sqrt{2m_{B_1}\left(\mathrm{k}_1^0+m_\mu\right)}}. \tag{3.4.25}$$

Similarly, for states with spins s and s' we find

$$
\begin{aligned}
&\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)u_{s'}\left(e;\mathbf{k}_1-\mathbf{q}\right)= \\
&=\sqrt{\mathrm{k}_2^0+m_e}\begin{pmatrix}\left(\chi^s\right)^\dagger & \frac{\left(\boldsymbol{\sigma}\cdot\mathbf{p}_{B_2}\right)}{E_{B_2}+m_{B_2}}\left(\chi^s\right)^\dagger\end{pmatrix}\sqrt{E_{B_2}+m_{B_2}}\begin{pmatrix}\chi^{s'} \\ \frac{-\boldsymbol{\sigma}\cdot\left(\mathbf{k}_1-\mathbf{q}\right)}{\mathrm{k}_2^0+m_e}\chi^{s'}\end{pmatrix} \\
&\approx\sqrt{2m_{B_2}\left(\mathrm{k}_2^0+m_e\right)}\left(1-\frac{\left(\boldsymbol{\sigma}\cdot\mathbf{p}_{B_2}\right)\left(\boldsymbol{\sigma}\cdot\left(\mathbf{k}_1-\mathbf{q}\right)\right)}{\left(E_{B_2}+m_{B_2}\right)\left(\mathrm{k}_2^0+m_e\right)}\right)\delta_{ss'} \\
&=\sqrt{2m_{B_2}\left(\mathrm{k}_2^0+m_e\right)}\delta_{ss'}+O\left(1/M\right),
\end{aligned}
\tag{3.4.26}
$$

and the relations for the Kronecker $\delta_{ss'}$ is extracted to be

$$
\delta_{ss'}=\frac{\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)u_{s'}\left(e;\mathbf{k}_1-\mathbf{q}\right)}{\sqrt{2m_{B_2}\left(\mathrm{k}_2^0+m_e\right)}},
\tag{3.4.27}
$$

where an approximation $E_{B_2}\approx m_{B_2}$ was made for non-relativistic velocities after the decay.

3.4.2.2 The Matrix Element

With the use of the normalization conditions (3.4.25) and (3.4.27), the Eq.(3.4.23) can be rewritten into a convenient form

$$
S_{sr}^{\rho}=\delta_{ss'}S_{s'r'}^{\rho}\delta_{r'r},
\tag{3.4.28}
$$

where the summation over s' and r' is implicit. Writing

$$
\bar{u}_s\left(e;\mathbf{k}_1-\mathbf{q}\right)=\sum_{s'}\delta_{ss'}\bar{u}_{s'}\left(e;\mathbf{k}_1-\mathbf{q}\right)==\frac{\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\left(\not{k}_1-\not{q}+m_e\right)}{\sqrt{2m_{B_2}\left(\mathrm{k}_2^0+m_e\right)}},
\tag{3.4.29}
$$

will lead to

$$
\begin{aligned}
S_{sr}^{\rho}=\int\frac{d^3k_1}{\left(2\pi\right)^3}\frac{\psi_1\left(\mathbf{k}_1\right)}{\sqrt{2\mathrm{k}_1^0}}\frac{\psi_2^*\left(\mathbf{k}_1-\mathbf{q}\right)}{\sqrt{2\mathrm{k}_2^0}}&\frac{\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)u_{s'}\left(e;\mathbf{k}_1-\mathbf{q}\right)}{\sqrt{2m_{B_2}\left(\mathrm{k}_2^0+m_e\right)}} \\
\times\bar{u}_{s'}\left(e;\mathbf{k}_1-\mathbf{q}\right)\gamma^\rho Lu_{r'}\left(\mu;\mathbf{k}_1\right)&\frac{\bar{u}_{r'}\left(\mu;\mathbf{k}_1\right)u_r\left(B_1;\mathbf{0}\right)}{\sqrt{2m_{B_1}\left(\mathrm{k}_1^0+m_\mu\right)}}.
\end{aligned}
\tag{3.4.30}
$$

Thus, the matrix element (3.4.23) becomes:

$$S^{\rho}_{sr}=\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{\psi_1(\mathbf{k}_1)}{\sqrt{2\mathrm{k}_1^0}}\frac{\psi_2^*(\mathbf{k}_1-\mathbf{q})}{\sqrt{2\mathrm{k}_2^0}}$$
$$\times\frac{\bar{u}_s(B_2;\mathbf{p}_{B_2})}{\sqrt{2m_{B_2}(\mathrm{k}_2^0+m_e)}}(\not{k}_1-\not{q}+m_e)\gamma^\rho L(\not{k}_1+m_\mu)\frac{u_r(B_1;\mathbf{0})}{\sqrt{2m_{B_1}(\mathrm{k}_1^0+m_\mu)}}. \qquad (3.4.31)$$

The expression (3.4.31) can be further simplified with the repeated usage of the Dirac equations

$$\gamma^0 u_r(B_1;\mathbf{0})=u_r(B_1;\mathbf{0}), \qquad (3.4.32)$$
$$\bar{u}_s(B_2;\mathbf{p}_{B_2})\gamma^0=\bar{u}_s(B_2;\mathbf{p}_{B_2})+O\left(\frac{1}{M}\right). \qquad (3.4.33)$$

3.4.2.3 Derivation of Form Factors F_i

For brevity the expression (3.4.31) will be written in the following form

$$S^{\rho}_{sr}\sim\int d^3\mathbf{k}_1\bar{u}_s(B_2;\mathbf{p}_{B_2})(\not{k}_1-\not{q}+m_e)\gamma^\rho L(\not{k}_1+m_\mu)u_r(B_1;\mathbf{0}), \qquad (3.4.34)$$

where the functions ψ and the factors in the denominator dropped, which will be restored at the end of the simplification of the expression for S^{ρ}_{sr}

The term proportional to $(\not{k}_1-\not{q})$ can be rewritten as $\gamma^0(\mathrm{k}_1-\mathrm{q})^0-\boldsymbol{\gamma}\cdot(\mathbf{k}_1-\mathbf{q})$. Also, using

$$\mathrm{k}_2^0=\mathrm{k}_1^0-\mathrm{q}^0, \qquad (3.4.35)$$

the expression (3.4.34) for S^{ρ}_{sr} becomes

$$S^{\rho}_{sr}\sim\int d^3\mathbf{k}_1\bar{u}_s(B_2;\mathbf{p}_{B_2})\left[\mathrm{k}_2-\boldsymbol{\gamma}\cdot(\mathbf{k}_1-\mathbf{q})+m_e\right]\gamma^\rho L\left[\mathrm{k}_1^0-\boldsymbol{\gamma}\cdot\mathbf{k}_1+m_\mu\right]u_r(B_1;\mathbf{0}). \qquad (3.4.36)$$

Let's simplify this expression term by term. First, consider the term proportional to $\boldsymbol{\gamma}\cdot\mathbf{q}=\gamma^0\mathrm{q}^0-\not{q}$, i.e.,

$$
\begin{aligned}
&\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)(\boldsymbol{\gamma}\cdot\mathbf{q})\,\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right)\\
&=\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\left(\gamma^0\mathrm{q}^0-\not{q}\right)\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right)\\
&=\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\left(\mathrm{q}^0-\not{q}\right)\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right)\\
&=\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\mathrm{q}^0\gamma^\rho L\left(\mathrm{k}_1+m_\mu\right)u_r\left(B_1;\mathbf{0}\right)-\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\not{q}\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right) \quad (3.4.37)\\
&=\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\left\{\mathrm{q}^0\gamma^\rho L\left(\mathrm{k}_1+m_\mu\right)-\left(2\mathrm{q}^\rho-\gamma^\rho\not{q}\right)L\left(\mathrm{k}_1^0+m_\mu\right)\right\}u_r\left(B_1;\mathbf{0}\right). \quad (3.4.38)
\end{aligned}
$$

where in the second line the identity (3.4.32) is used and the anticommutation relation for gamma matrices $\{\gamma^\rho,\gamma^\sigma\}=2\eta^{\rho\sigma}$ is applied to the last term in (3.4.37). The term proportional to q^ρ gives zero after being contracted with the neutrino tensor:

$$
\mathrm{q}^\rho\bar{\nu}_e\gamma_\rho L\nu_\mu=\bar{\nu}_e\not{q}L\nu_\mu=\bar{\nu}_e\left(\not{p}_1+\not{p}_2\right)L\nu_\mu=0. \quad (3.4.39)
$$

Therefore, we always drop the term proportional to q^ρ from now on. With this simplification Eq. (3.4.38) becomes

$$
\begin{aligned}
&\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)(\boldsymbol{\gamma}\cdot\mathbf{q})\,\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right)=\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\mathrm{q}^0\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right)\\
&+\bar{u}_s\left(B_2;\mathbf{p}_{B_2}\right)\left(\gamma^\rho\not{q}\right)L\left(\mathrm{k}_1^0+m_\mu\right)u_r\left(B_1;\mathbf{0}\right). \quad (3.4.40)
\end{aligned}
$$

Substituting Eq. (3.4.40) in Eq. (3.4.36) and dropping u_r and $\bar{u}_s$ further for brevity, we can write

$$
\begin{aligned}
S_{sr}^\rho\sim&\left[\mathrm{k}_2^0-\boldsymbol{\gamma}\cdot\left(\mathbf{k}_1-\mathbf{q}\right)+m_e\right]\gamma^\rho L\left[\mathrm{k}_1^0-\boldsymbol{\gamma}\cdot\mathbf{k}_1+m_\mu\right]\\
=&\left(\mathrm{k}_2^0+m_e\right)\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)-\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)+\left(\boldsymbol{\gamma}\cdot\mathbf{q}\right)\gamma^\rho L\left(\mathrm{k}_1^0+m_\mu\right)\\
&-\left(\mathrm{k}_2^0+m_e\right)\gamma^\rho L\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)+\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)\gamma^\rho L\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)-\left(\boldsymbol{\gamma}\cdot\mathbf{q}\right)\gamma^\rho L\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)\\
=&\left\{\left(\mathrm{k}_1^0+m_\mu\right)\left(\mathrm{k}_2^0+m_e\right)+\mathrm{q}^0\left(\mathrm{k}_1^0+m_\mu\right)\right\}\gamma^\rho L+\left(\mathrm{k}_1^0+m_\mu\right)\gamma^\rho\not{q}L-\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)\gamma^\rho L\left[\mathrm{k}_1^0+m_\mu\right]\\
&-\left(k_2^0+m_e\right)\gamma^\rho L\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)+\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)\gamma^\rho L\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right)-\left(\boldsymbol{\gamma}\cdot\mathbf{q}\right)\gamma^\rho L\left(\boldsymbol{\gamma}\cdot\mathbf{k}_1\right). \quad (3.4.41)
\end{aligned}
$$

Now consider the terms proportional to $\mathbf{k}_1$. Since both functions ψ_1 and ψ_2^* are scalar functions, from the rotational invariance it follows

$$
\int d^3\mathbf{k}_1\psi_1\left(\mathbf{k}_1\right)\psi_2^*\left(\mathbf{k}_1-\mathbf{q}\right)k_1^i=A_1\left(\mathbf{q}\right)q^i. \quad (3.4.42)
$$

Multiplying both parts of (3.4.42) by q_i, the coefficient A_1 is obtained to be

$$A_1(\mathbf{q}) = \frac{1}{\mathbf{q}^2}\int d^3\mathbf{k}_1 \psi_1(\mathbf{k}_1)\,\psi_2^*(\mathbf{k}_1-\mathbf{q})\,(\mathbf{k}_1\cdot\mathbf{q})$$
$$\equiv \int d^3\mathbf{k}_1 \psi_1(\mathbf{k}_1)\,\psi_2^*(\mathbf{k}_1-\mathbf{q})\,C, \tag{3.4.43}$$

where $C = \frac{(\mathbf{k}_1\cdot\mathbf{q})}{\mathbf{q}^2}$. Thus,

$$(\boldsymbol{\gamma}\cdot\mathbf{k}_1)\,\gamma^\rho L = C\,(\boldsymbol{\gamma}\cdot\mathbf{q})\,\gamma^\rho L = C\left(\mathrm{q}^0 - \not{q}\right)\gamma^\rho L = C\mathrm{q}^0\gamma^\rho L + C\gamma^\rho\not{q}L, \tag{3.4.44}$$

$$\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1) = C\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{q}) = C\gamma^\rho L\left(\mathrm{q}^0 - \not{q}\right) = C\mathrm{q}^0\gamma^\rho L - C\gamma^\rho\not{q}R. \tag{3.4.45}$$

Here $R = \frac{1+\gamma_5}{2}$ is the right projection operator which obeys $R\gamma^\rho = \gamma^\rho L$. After substituting Eqs. (3.4.44) and (3.4.45) in Eq. (3.4.41), it gives

$$\begin{aligned}
&S^\rho_{sr} \sim \\
&\left\{\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right) + \mathrm{q}^0(1-C)\left(\mathrm{k}_1^0 + m_\mu\right) - C\mathrm{q}^0\left(\mathrm{k}_2^0 + m_e\right)\right\}\gamma^\rho L \\
&+ C\left(\mathrm{k}_2^0 + m_e\right)\gamma^\rho\not{q}R + (1-C)\left(\mathrm{k}_1^0 + m_\mu\right)\gamma^\rho\not{q}L + (\boldsymbol{\gamma}\cdot\mathbf{k}_1)\,\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1) - (\boldsymbol{\gamma}\cdot\mathbf{q})\,\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1).
\end{aligned} \tag{3.4.46}$$

It is useful to simplify the last term of Eq. (3.4.46) as

$$\begin{aligned}
(\boldsymbol{\gamma}\cdot\mathbf{q})\,\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1) &= C\left(\mathrm{q}^0 - \not{q}\right)\gamma^\rho L\left(\mathrm{q}^0 - \not{q}\right) \\
&= C\left\{\left(\mathrm{q}^0\right)^2\gamma^\rho L - \mathrm{q}^0\gamma^\rho L\not{q} - \mathrm{q}^0\not{q}\gamma^\rho L + \not{q}\gamma^\rho\not{q}R\right\} \\
&= C\left\{\left(\mathrm{q}^0\right)^2\gamma^\rho L - \mathrm{q}^0\gamma^\rho\not{q}R + \mathrm{q}^0\gamma^\rho\not{q}L - \mathrm{q}^2\gamma^\rho R\right\},
\end{aligned} \tag{3.4.47}$$

which leads to

$$\begin{aligned}
S^\rho_{sr} \sim &\left\{\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right) + \mathrm{q}^0(1-C)\left(\mathrm{k}_1^0 + m_\mu\right) - C\mathrm{q}^0\left(\mathrm{k}_2^0 + m_e\right) - C\left(\mathrm{q}^0\right)^2\right\}\gamma^\rho L \\
&+ \left\{C\left(\mathrm{k}_2^0 + m_e\right) + C\mathrm{q}^0\right\}\gamma^\rho\not{q}R + \left\{(1-C)\left(\mathrm{k}_1^0 + m_\mu\right) - C\mathrm{q}^0\right\}\gamma^\rho\not{q}L \\
&+ C\mathrm{q}^2\gamma^\rho R + (\boldsymbol{\gamma}\cdot\mathbf{k}_1)\,\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1).
\end{aligned} \tag{3.4.48}$$

Again, for the term proportional to $\mathbf{k}_1^2$ the rotational invariance gives:

$$\int d^3\mathbf{k}_1\psi_1(\mathbf{k}_1)\,\psi_2^*(\mathbf{k}_1-\mathbf{q})\,(k_1)_i\,(k_1)_j = A_2'(\mathbf{q})\,\delta_{ij} + A_3'(\mathbf{q})\,q_i q_j$$
$$= A_2(\mathbf{q})\left(q_i q_j - \frac{1}{3}\delta_{ij}\mathbf{q}^2\right) + A_3(\mathbf{q})\,\delta_{ij}\mathbf{q}^2, \qquad (3.4.49)$$

where both terms in the last line are orthogonal and therefore the corresponding coefficients A_2 and A_3 can be obtained by consecutive contraction of the Eq. (3.4.49) with $\left(q_i q_j - \frac{1}{3}\delta_{ij}\mathbf{q}^2\right)$ and $\delta^{ij}\mathbf{q}^2$, respectively. This gives

$$A_2 = \frac{3}{2\left(\mathbf{q}^2\right)^2}\int d^3\mathbf{k}_1\psi_1(\mathbf{k}_1)\,\psi_2^*(\mathbf{k}_1-\mathbf{q})\left[(\mathbf{k}_1\cdot\mathbf{q})^2 - \frac{1}{3}\mathbf{k}_1^2\mathbf{q}^2\right]$$
$$\equiv \int d^3\mathbf{k}_1\psi_1(\mathbf{k}_1)\,\psi_2^*(\mathbf{k}_1-\mathbf{q})\,B, \qquad (3.4.50)$$

and

$$A_3 = \int d^3\mathbf{k}_1\psi_1(\mathbf{k}_1)\,\psi_2^*(\mathbf{k}_1-\mathbf{q})\,\frac{\mathbf{k}_1^2}{3\mathbf{q}^2}, \qquad (3.4.51)$$

where the coefficient A_2 was written in terms of $B = \frac{3(\mathbf{k}_1\cdot\mathbf{q})^2-\mathbf{k}_1^2\mathbf{q}^2}{2(\mathbf{q}^2)^2}$ defined in [1]. Then, the last term in Eq. (3.4.48) can be expressed as

$$(\boldsymbol{\gamma}\cdot\mathbf{k}_1)\,\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1) = \gamma^i\gamma^\rho L\gamma^j\left\{B\left(q_i q_j - \frac{1}{3}\delta_{ij}\mathbf{q}^2\right) + A_3\delta_{ij}\mathbf{q}^2\right\}$$
$$= B\boldsymbol{\gamma}\cdot\mathbf{q}\,(\gamma^\rho L)\,\boldsymbol{\gamma}\cdot\mathbf{q} - \frac{B}{3}\gamma^i\gamma^\rho L\gamma^i\mathbf{q}^2 + A_3\mathbf{q}^2\gamma^i\gamma^\rho L\gamma^i$$
$$= B\left(\mathrm{q}^0 - \not{q}\right)(\gamma^\rho L)\left(\mathrm{q}^0 - \not{q}\right) + \gamma^i\gamma^\rho\gamma^i R\left\{A_3\mathbf{q}^2 - \frac{B}{3}\mathbf{q}^2\right\}, \qquad (3.4.52)$$

where

$$\gamma^i\gamma^\rho L\gamma^i = -\gamma^0\gamma^\rho L\gamma^0 + \gamma^i\gamma^\rho L\gamma^i + \gamma^0\gamma^\rho L\gamma^0$$
$$= -\left(\gamma^0\gamma^\rho L\gamma_0 - \gamma^i\gamma^\rho L\gamma_i\right) + \gamma^0\gamma^\rho L\gamma^0$$
$$\to 2\gamma^\rho R + \gamma^\rho L. \qquad (3.4.53)$$

Here we have dropped the γ^0's due to the Dirac equations (3.4.32) and (3.4.33). Thus, expression (3.4.52) becomes

$$(\boldsymbol{\gamma}\cdot\mathbf{k}_1)\,\gamma^\rho L\,(\boldsymbol{\gamma}\cdot\mathbf{k}_1) =$$
$$(2\gamma^\rho R + \gamma^\rho L)\left(A_3\mathbf{q}^2 - \frac{B}{3}\mathbf{q}^2\right) + B\left\{\left(\mathrm{q}^0\right)^2\gamma^\rho L - \mathrm{q}^0\gamma^\rho\not{q}R + \mathrm{q}^0\gamma^\rho\not{q}L - \mathrm{q}^2\gamma^\rho R\right\} \qquad (3.4.54)$$

Also,

$$A_3\mathbf{q}^2 - \frac{B}{3}\mathbf{q}^2 = \frac{\mathbf{k}_1^2\mathbf{q}^2}{3\mathbf{q}^2} - \frac{\mathbf{q}^2}{3}\frac{3\left(\mathbf{k}_1\cdot\mathbf{q}\right)^2 - \mathbf{k}_1^2\mathbf{q}^2}{2\left(\mathbf{q}^2\right)^2} = \frac{\mathbf{k}_1^2\mathbf{q}^2 - \left(\mathbf{k}_1\cdot\mathbf{q}\right)^2}{2\mathbf{q}^2} \equiv A, \tag{3.4.55}$$

where A is a quantity defined in [1]. Putting everything together, the final expression for S_{sr}^{ρ} is

$$\begin{aligned} &S_{sr}^{\rho} \sim \\ &\left\{\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right) + \mathrm{q}^0\left(1-C\right)\left(\mathrm{k}_1^0 + m_\mu\right) - C\mathrm{q}^0\left(\mathrm{k}_2^0 + m_e\right) + \left(B-C\right)\left(\mathrm{q}^0\right)^2 + A\right\}\gamma^\rho L \\ &+ \left\{\left(C-B\right)\mathrm{q}^2 + 2A\right\}\gamma^\mu R + \left\{\left(1-C\right)\left(\mathrm{k}_1^0 + m_\mu\right) + \left(B-C\right)\mathrm{q}^0\right\}\gamma^\rho \not{q} L \\ &+ \left\{C\left(\mathrm{k}_2^0 + m_e\right) - \left(B-C\right)\mathrm{q}^0\right\}\gamma^\rho \not{q} R. \end{aligned} \tag{3.4.56}$$

This expression of the S-matrix can be written in a more convenient form as follows [1]:

$$S_{sr}^{\rho} = \frac{\bar{u}_s\left(B_2; \mathbf{p}_{B_2}\right) T^\rho u_r\left(B_1; \mathbf{0}\right)}{\sqrt{4m_{B_1}m_{B_2}}}. \tag{3.4.57}$$

where

$$T^\rho = F_1\left(\mathrm{q}^2\right)\gamma^\rho L + F_2\left(\mathrm{q}^2\right)\gamma^\rho R + F_3\left(\mathrm{q}^2\right)\gamma^\rho\frac{\not{q}}{m_\mu}L + F_4\left(\mathrm{q}^2\right)\gamma^\rho\frac{\not{q}}{m_\mu}R. \tag{3.4.58}$$

In the expression (3.4.58) we have put back everything that was dropped in writing Eq. (3.4.34). The FF's are given by

$$F_i\left(\mathrm{q}^2\right) = \int \frac{d^3\mathbf{k}_1}{\left(2\pi\right)^3}\psi_1\left(\mathbf{k}_1\right)\psi_2^*\left(\mathbf{k}_1 - \mathbf{q}\right)\frac{h_i}{\sqrt{4\mathrm{k}_1^0\mathrm{k}_2^0\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right)}}, \tag{3.4.59}$$

with

$$\begin{aligned} h_1 &= \left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right) + \mathrm{q}^0\left[\left(1-C\right)\left(\mathrm{k}_1^0 + m_\mu\right) - C\left(\mathrm{k}_2^0 + m_e\right)\right] \\ &\quad + \left(B-C\right)\left(\mathrm{q}^0\right)^2 + A, &(3.4.60) \\ h_2 &= \left(C-B\right)\mathrm{q}^2 + 2A, &(3.4.61) \\ h_3 &= \left[\left(1-C\right)\left(\mathrm{k}_1^0 + m_\mu\right) + \left(B-C\right)\mathrm{q}^0\right]m_\mu, &(3.4.62) \\ h_4 &= \left[C\left(\mathrm{k}_2^0 + m_e\right) - \left(B-C\right)\mathrm{q}^0\right]m_\mu. &(3.4.63) \end{aligned}$$

It is worth noting that in ref. [1] the expressions for h_1 and h_2 include the terms of A with the opposite sign. The Tables 3.4.1 and 3.4.2 present numerical comparison between F_i

$\lvert\mathbf{q}\rvert$ (MeV)	F_1	F'_1	F_2	F'_2
10	0.00479743	0.00502886	0.00018832	0.00065119
20	0.00430349	0.00451858	0.00013482	0.00056500
30	0.00365043	0.00384268	0.00007005	0.00045455
40	0.00298111	0.00314818	0.00001213	0.00034628
50	0.00238271	0.00252534	-0.00003053	0.00025473
60	0.00188767	0.00200827	-0.00005735	0.00018374
70	0.00149515	0.00159676	-0.00007188	0.00013136
80	0.00119033	0.00127599	-0.00007760	0.00009373

Table 3.4.1: The FF F_i are calculated according to [1]. F'_i represent the corresponding corrected FF with the correct sign in front of the coefficient A. The values of the parameters used for numerical calculations are $Z = 80$, $m_\mu = 105.658375 \pm 0.000002\,\text{MeV}$, $m_e = 0.510998946 \pm 0.000000003\,\text{MeV}$ and $\alpha = 1/137 \approx 0.00729735257$.

$\lvert\mathbf{q}\rvert$(MeV)	F_1	F'_1	F_2	F'_2
10	0.0002029	0.00020454	$3.8015062{\cdot}10^{-5}$	$3.8866117{\cdot}10^{-5}$
20	$3.0509518{\cdot}10^{-5}$	$3.0537957{\cdot}10^{-5}$	$3.1366506{\cdot}10^{-6}$	$3.3410198{\cdot}10^{-6}$
30	$5.4156085{\cdot}10^{-6}$	$3.5115656{\cdot}10^{-6}$	$5.4680428{\cdot}10^{-7}$	$6.0649955{\cdot}10^{-7}$
40	$1.3494489{\cdot}10^{-6}$	$1.2862501{\cdot}10^{-6}$	$1.4164301{\cdot}10^{-7}$	$1.7638670{\cdot}10^{-7}$
50	$5.9038607{\cdot}10^{-7}$	$5.9913763{\cdot}10^{-7}$	$1.1037297{\cdot}10^{-8}$	$2.6913015{\cdot}10^{-8}$
60	$3.0022005{\cdot}10^{-7}$	$3.0539554{\cdot}10^{-7}$	$5.4042601{\cdot}10^{-11}$	$8.6634225{\cdot}10^{-9}$
70	$1.7108758{\cdot}10^{-7}$	$1.7428102{\cdot}10^{-7}$	$6.1553051{\cdot}10^{-9}$	$6.0009907{\cdot}10^{-10}$
80	$1.0984531{\cdot}10^{-7}$	$1.1206792{\cdot}10^{-7}$	$6.1962072{\cdot}10^{-9}$	$4.2160059{\cdot}10^{-9}$

Table 3.4.2: The FF F_i are calculated according to [1]. F'_i represent the corresponding corrected FF with the correct sign in front of the coefficient A. The values of the parameters used for numerical calculations are $Z = 10$, $m_\mu = 105.658375 \pm 0.000002\,\text{MeV}$, $m_e = 0.510998946 \pm 0.000000003\,\text{MeV}$ and $\alpha = 1/137 \approx 0.00729735257$.

and F'_i for cases of $Z = 80$ and $Z = 10$, respectively. Here: $F_1 \sim h_1$, with $-A$ and $F_2 \sim h_2$, with $-2A$ in their respective last terms for the form factors derived in [1], and F'_1 and F'_2 are with corrected sign of A terms in the FF's. The values for F_i and F'_i are of the same order for $Z = 10$, but for $Z = 80$ the difference is up to an order of magnitude.

3.4.3 The Decay Rate into the Bound Electron

The differential decay rate formula is given by [13]:

$$d\Gamma = \frac{1}{2m_{B_1}} d\Phi \left|\mathcal{M}_{sr}\right|^2, \quad d\Phi = \prod_f \frac{d^3\mathbf{p}_f}{2E_f\,(2\pi)^3}\,(2\pi)^4\,\delta^{(4)}\left(\mathrm{p}_i - \mathrm{p}_f\right), \tag{3.4.64}$$

where $d\Phi$ is the phase space and the invariant amplitude is given by Eq. (3.4.21). From this expression the decay rate of $(Z\mu) \to (Ze)$ is

$$\Gamma = \frac{1}{2m_{B_1}} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}} \frac{d^3\mathbf{p}_{\bar{\nu}_e}}{(2\pi)^3 2E_{\bar{\nu}_e}} (2\pi)^4 \delta^{(4)} \left(\mathrm{p}_1 - \mathrm{p}_2 - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) |\mathcal{M}_{sr}|^2 . \tag{3.4.65}$$

The last expression can be rewritten as the integral over 4-momentum q with the use of the extra delta function as

$$\begin{aligned} \Gamma = & \frac{1}{2m_{B_1}} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}} \frac{d^3\mathbf{p}_{\bar{\nu}_e}}{(2\pi)^3 2E_{\bar{\nu}_e}} \\ & \times \int d^4\mathrm{q}\, (2\pi)^4 \delta^{(4)} \left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) \delta^{(4)} \left(\mathrm{p}_1 - \mathrm{p}_2 - \mathrm{q}\right) |\mathcal{M}_{sr}|^2 . \end{aligned} \tag{3.4.66}$$

Substituting the expression for $\mathcal{M}_{sr}$ from Eq. (3.4.21) it becomes

$$\begin{aligned} \Gamma = & \frac{1}{2m_{B_1}} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}} \frac{d^3\mathbf{p}_{\bar{\nu}_e}}{(2\pi)^3 2E_{\bar{\nu}_e}} \int d^4\mathrm{q}\, (2\pi)^4 \delta^{(4)} \left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) \\ & \times \delta^{(4)} \left(\mathrm{p}_1 - \mathrm{p}_2 - \mathrm{q}\right) \left(\frac{4G_F}{\sqrt{2}} \sqrt{4m_{B_1} m_{B_2}}\right)^2 \sum_{rs} \left(N_\rho S^\rho_{sr}\right)^\dagger \left(N_\sigma S^\sigma_{sr}\right) . \end{aligned} \tag{3.4.67}$$

3.4.3.1 Integration over Neutrino Phase Space

The decay rate (3.4.67) should be first integrated over neutrino phase space. The integral is given by

$$N_{\rho\sigma} \equiv \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}} (2\pi)^4 \delta^{(4)} \left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) \left(\frac{4G_F}{\sqrt{2}}\right)^2 \sum_{\text{spins}} N^\dagger_\rho N_\sigma , \tag{3.4.68}$$

where $N_\rho = \bar{u}\left(\mathrm{p}_{\nu_\mu}\right) \gamma_\rho L v \left(\mathrm{p}_{\bar{\nu}_e}\right)$. The summation over the spins gives

$$\begin{aligned} \sum_{\text{spins}} N^\dagger_\rho N_\sigma &= \sum_{\text{spins}} \left[\bar{u}\left(\mathrm{p}_{\nu_\mu}\right) \gamma_\rho L v \left(\mathrm{p}_{\bar{\nu}_e}\right)\right]^\dagger \bar{u}\left(\mathrm{p}_{\nu_\mu}\right) \gamma_\sigma L v \left(\mathrm{p}_{\bar{\nu}_e}\right) \\ &= \sum_{\text{spins}} \bar{u}\left(\mathrm{p}_{\bar{\nu}_e}\right) \gamma_\rho L v \left(\mathrm{p}_{\nu_\mu}\right) \bar{u}\left(\mathrm{p}_{\nu_\mu}\right) \gamma_\sigma L v \left(\mathrm{p}_{\bar{\nu}_e}\right) \\ &= \mathrm{Tr}\left[\not{p}_{\bar{\nu}_e} \gamma_\rho L \not{p}_{\nu_\mu} \gamma_\sigma L\right] = \frac{1}{2} \mathrm{Tr}\left[\not{p}_{\bar{\nu}_e} \gamma_\rho \not{p}_{\nu_\mu} \gamma_\sigma\right] + 2i\epsilon_{\rho\sigma\alpha\beta} \mathrm{p}^\alpha_{\bar{\nu}_e} \mathrm{p}^\beta_{\nu_\mu} . \end{aligned} \tag{3.4.69}$$

The term containing the antisymmetric tensor $\epsilon_{\rho\sigma\alpha\beta}$ vanishes after the integration over the phase space and Eq. (3.4.69) becomes

$$\sum_{\text{spins}} N_\rho^\dagger N_\sigma = 2\left[(\mathrm{p}_{\bar{\nu}_e})_\rho \left(\mathrm{p}_{\nu_\mu}\right)_\sigma + (\mathrm{p}_{\bar{\nu}_e})_\sigma \left(\mathrm{p}_{\nu_\mu}\right)_\rho - \left(\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{p}_{\nu_\mu}\right)\eta_{\rho\sigma}\right]. \tag{3.4.70}$$

Next, the following integral should be evaluated

$$\begin{aligned} I_{\rho\sigma} &= \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q}-\mathrm{p}_{\nu_\mu}-\mathrm{p}_{\bar{\nu}_e}\right)(\mathrm{p}_{\bar{\nu}_e})_\rho \left(\mathrm{p}_{\nu_\mu}\right)_\sigma \\ &= D_1\left(\mathrm{q}^2\eta_{\rho\sigma}+2\mathrm{q}_\rho \mathrm{q}_\sigma\right) + D_2\left(\mathrm{q}^2\eta_{\rho\sigma}-2\mathrm{q}_\rho \mathrm{q}_\sigma\right), \end{aligned} \tag{3.4.71}$$

where the expected result has been written in the most general form as the sum of two orthogonal terms and it is understood that D_1 and D_2 are both functions of q^2. To find the coefficient D_2 contract both parts of (3.4.71) with $(\mathrm{q}^2\eta^{\rho\sigma} - 2\mathrm{q}^\rho \mathrm{q}^\sigma)$:

$$\begin{aligned} &4\mathrm{q}^4 D_2 = \\ &\int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q}-\mathrm{p}_{\nu_\mu}-\mathrm{p}_{\bar{\nu}_e}\right)\left\{\mathrm{q}^2\left(\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{p}_{\nu_\mu}\right) - 2\left(\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{q}\right)\left(\mathrm{p}_{\nu_\mu}\cdot \mathrm{q}\right)\right\} \\ &= 0, \end{aligned} \tag{3.4.72}$$

since in the massless neutrino limit

$$\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{q} = \mathrm{p}_{\bar{\nu}_e}\cdot\left(\mathrm{p}_{\bar{\nu}_e}+\mathrm{p}_{\nu_\mu}\right) = \mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{p}_{\nu_\mu}, \tag{3.4.73}$$

$$\mathrm{p}_{\nu_\mu}\cdot \mathrm{q} = \mathrm{p}_{\bar{\nu}_\mathrm{e}}\cdot \mathrm{p}_{\nu_\mu}, \tag{3.4.74}$$

$$\mathrm{q}^2 = \left(\mathrm{p}_{\bar{\nu}_e}+\mathrm{p}_{\nu_\mu}\right)^2 = 2\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{p}_{\nu_\mu}. \tag{3.4.75}$$

Contracting (3.4.71) with $(\mathrm{q}^2\eta^{\rho\sigma} + 2\mathrm{q}^\rho \mathrm{q}^\sigma)$:

$$\begin{aligned} &12\mathrm{q}^4 D_1 \\ &= \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q}-\mathrm{p}_{\nu_\mu}-\mathrm{p}_{\bar{\nu}_e}\right)\left\{\mathrm{q}^2\left(\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{p}_{\nu_\mu}\right) + 2\left(\mathrm{p}_{\bar{\nu}_e}\cdot \mathrm{q}\right)\left(\mathrm{p}_{\nu_\mu}\cdot \mathrm{q}\right)\right\} \\ &= \mathrm{q}^4 \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q}-\mathrm{p}_{\nu_\mu}-\mathrm{p}_{\bar{\nu}_e}\right). \end{aligned} \tag{3.4.76}$$

The phase space integration in the last line is given by

$$\int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(3)}\left(\mathbf{q} - \mathbf{p}_{\nu_\mu} - \mathbf{p}_{\bar{\nu}_e}\right) \delta\left(\mathrm{q}^0 - E_{\nu_\mu} - E_{\bar{\nu}_e}\right)$$
$$= 4\pi \int \frac{d\omega\, \omega^2}{(2\pi)^2\, 2\omega} \frac{1}{2\omega} \delta\left(\mathrm{q}^0 - 2\omega\right) = \int \frac{d\omega}{2\,(2\pi)^2} \delta\left(\omega - \frac{\mathrm{q}^0}{2}\right) = \frac{1}{8\pi}. \qquad (3.4.77)$$

Where in the last line the two neutrinos are considered in their center of mass frame, and thus, $\mathbf{p}_{\nu_\mu} = -\mathbf{p}_{\bar{\nu}_e}$ and their energy is $\omega = E_{\nu_\mu} = E_{\bar{\nu}_e}$. Therefore, the coefficient $D_1 = \frac{1}{12}\frac{1}{8\pi}$ and the integral $I_{\rho\sigma}$ is

$$I_{\rho\sigma} = \frac{1}{12}\frac{1}{8\pi}\left(\mathrm{q}^2\eta_{\rho\sigma} + 2\mathrm{q}_\rho \mathrm{q}_\sigma\right). \qquad (3.4.78)$$

Also, taking into account that

$$\int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) \left(\mathrm{p}_{\bar{\nu}_e}\right)_\rho \left(\mathrm{p}_{\nu_\mu}\right)_\sigma$$
$$= \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) \left(\mathrm{p}_{\bar{\nu}_e}\right)_\sigma \left(\mathrm{p}_{\nu_\mu}\right)_\rho. \qquad (3.4.79)$$

Finally, the integral over the neutrino phase space is

$$N_{\rho\sigma} \equiv 16G_F^2 \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3\, 2E_{\bar{\nu}_e}} \frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3\, 2E_{\nu_\mu}} (2\pi)^4\, \delta^{(4)}\left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right) \left[2\left(\mathrm{p}_{\bar{\nu}_e}\right)_\rho \left(\mathrm{p}_{\nu_\mu}\right)_\sigma - \frac{\mathrm{q}^2}{2}\eta_{\rho\sigma}\right]$$
$$= 16G_F^2 \frac{1}{8\pi}\left[\frac{2}{12}\left(\mathrm{q}^2\eta_{\rho\sigma} + 2\mathrm{q}_\rho\mathrm{q}_\sigma\right) - \frac{\mathrm{q}^2}{2}\eta_{\rho\sigma}\right] = \frac{2G_F^2}{3\pi}\left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right). \qquad (3.4.80)$$

3.4.3.2 Derivation of Decay Rate into Bound Electron

Substituting the result for the integral over neutrino phase space

$$N_{\rho\sigma} = \frac{2G_F^2}{3\pi}\left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right) \qquad (3.4.81)$$

in Eq. (3.4.67) and performing the integration over the 3-momentum $\mathbf{p}_2$ and q^0 gives

$$\Gamma = \frac{1}{2m_{B_1}} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3\, 2E_2} \int d^3\mathbf{q} \int d\mathrm{q}^0 \delta\left(m_{B_1} - m_{B_2} - \mathrm{q}^0\right)$$
$$\times\, \delta^{(3)}\left(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{q}\right)\left(4m_{B_1}m_{B_2}\right) \sum_{rs} \left(S^\rho_{sr}\right)^\dagger \left(S^\sigma_{sr}\right) N_{\rho\sigma}.$$
$$= \frac{1}{2m_{B_1}}\frac{1}{2m_{B_2}} \int \frac{d^3\mathbf{q}}{(2\pi)^3}\left(4m_{B_1}m_{B_2}\right) \sum_{rs} \left(S^\rho_{sr}\right)^\dagger \left(S^\sigma_{sr}\right) N_{\rho\sigma} = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \sum_{rs} \left(S^\rho_{sr}\right)^\dagger \left(S^\sigma_{sr}\right) N_{\rho\sigma}, \qquad (3.4.82)$$

where $E_2 = m_{B_2}$ was used and it is understood that $m_{B_2} = m_{B_1} - \mathrm{q}^0$. The decay rate formula, after averaging over the spin, becomes

$$\Gamma = \frac{1}{2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \sum_{rs} (S^\rho_{sr})^\dagger (S^\sigma_{sr}) N_{\rho\sigma}, \tag{3.4.83}$$

where the matrix element S^ρ_{sr} is defined in terms of T^ρ given by Eq. (3.4.58). Therefore,

$$\sum_{rs} (S^\rho_{sr})^\dagger (S^\sigma_{sr}) = \frac{1}{4m_{B_1}m_{B_2}} \sum_{rs} (\bar{u}_s(\mathrm{p}_2) T^\rho u_r(\mathrm{p}_1)) (\bar{u}_s(\mathrm{p}_2) T^\sigma u_r(\mathrm{p}_1))^\dagger, \tag{3.4.84}$$

Thus, differential decay rate is given by

$$\frac{d\Gamma}{d|\mathbf{q}|} = \frac{|\mathbf{q}|^2 G_F^2}{24\pi^3} \frac{1}{m_{B_1}m_{B_2}} \sum_{rs} (\bar{u}_s(\mathrm{p}_2) T^\rho u_r(\mathrm{p}_1)) (\bar{u}_s(\mathrm{p}_2) T^\sigma u_r(\mathrm{p}_1))^\dagger \left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2 \eta_{\rho\sigma}\right). \tag{3.4.85}$$

Using the expressions for T^ρ and T^σ from (3.4.58) in Eq. (3.4.85), we get the terms corresponding to FF's $F_i(\mathrm{q}^2)$. They can be disscussed one by one.

First consider the term proportional to $F_1^2(\mathrm{q}^2)$:

$$\mathrm{Term\,I} = \left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2 \eta_{\rho\sigma}\right) \sum_{rs} \bar{u}_s(\mathrm{p}_2) (\gamma^\rho L) u_r(\mathrm{p}_1) \bar{u}_r(\mathrm{p}_1) (\gamma^\sigma L) u_s(\mathrm{p}_2). \tag{3.4.86}$$

After summations over rs in (3.4.86) we get the trace which is not zero only for the even number of gamma matrices

$$\begin{aligned}
&\mathrm{Term\,I} = \\
&= \left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2 \eta_{\rho\sigma}\right) \mathrm{Tr}\left[(\not{p}_2 + m_{B_2}) \gamma^\rho L (\not{p}_1 + m_{B_1}) \gamma^\sigma L\right] \\
&= \frac{1}{4} \left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2 \eta_{\rho\sigma}\right) \\
&\times \left\{ \mathrm{Tr}\left[\not{p}_2 \gamma^\rho \left(1-\gamma^5\right) \not{p}_1 \gamma^\sigma \left(1-\gamma^5\right)\right] + m_{B_1} m_{B_2} \mathrm{Tr}\left[\gamma^\rho \left(1-\gamma^5\right) \gamma^\sigma \left(1-\gamma^5\right)\right] \right\}.
\end{aligned} \tag{3.4.87}$$

The second trace identity is

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^\rho \left(1-\gamma^5\right) \not{p}_1 \gamma^\sigma \left(1-\gamma^5\right)\right] = \mathrm{Tr}\left[\gamma^\rho \gamma^\sigma\right] + \mathrm{Tr}\left[\gamma^\rho \gamma^5 \gamma^\sigma \gamma^5\right] \\
&= \mathrm{Tr}\left[\gamma^\rho \gamma^\sigma\right] - \mathrm{Tr}\left[\gamma^\rho \gamma^5 \gamma^5 \gamma^\sigma\right] = \mathrm{Tr}\left[\gamma^\rho \gamma^\sigma\right] - \mathrm{Tr}\left[\gamma^\rho \gamma^\sigma\right] = 0,
\end{aligned} \tag{3.4.88}$$

where the facts that γ^5 anticommutes with all γ^ρ and $(\gamma^5)^2 = I$ were used. And the first trace identity is

$$\mathrm{Tr}\left[\not{p}_2\gamma^\rho\left(1-\gamma^5\right)\not{p}_1\gamma^\sigma\left(1-\gamma^5\right)\right] = 2\mathrm{Tr}\left[\not{p}_2\gamma^\rho\not{p}_1\gamma^\sigma\right] + 8i\varepsilon^{\rho\sigma\nu\beta}\mathrm{p}_{1\alpha}\mathrm{p}_{2\beta}. \tag{3.4.89}$$

The term containing $\varepsilon^{\rho\sigma\nu\beta}$ symbol disappears after contracting with the neutrino tensor $N_{\rho\sigma}$. Thus, the only term that is left

$$\begin{aligned}&\mathrm{Term\,I} = \\ &= \frac{1}{2}\left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[\not{p}_2\gamma^\rho\not{p}_1\gamma^\sigma\right] = \frac{1}{2}\left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)4\left[\mathrm{p}_1^\rho\mathrm{p}_2^\sigma + \mathrm{p}_2^\rho\mathrm{p}_1^\sigma - (\mathrm{p}_1\cdot\mathrm{p}_2)\,\eta^{\rho\sigma}\right] \\ &= 2\left\{2(\mathrm{q}\cdot\mathrm{p}_1)\left(\mathrm{q}\cdot\mathrm{p}_2\right) + \mathrm{q}^2\left(\mathrm{p}_1\cdot\mathrm{p}_2\right)\right\}.\end{aligned} \tag{3.4.90}$$

Since $\mathrm{p}_1 = (m_{B_1}, 0)$, $\mathrm{p}_2 = (m_{B_2}, -\mathbf{q})$, therefore,

$$\mathrm{Term\,I} = 2\left\{2\left(\mathrm{q}^0 m_{B_1}\right)\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right) + \mathrm{q}^2\left(m_{B_1}m_{B_2}\right)\right\}. \tag{3.4.91}$$

The term proportional to F_2^2 gives the same expression, since $\mathrm{Tr}\left[\not{p}_2\gamma^\rho R\not{p}_1\gamma^\sigma R\right] = \mathrm{Tr}\left[\not{p}_2\gamma^\rho L\not{p}_1\gamma^\sigma L\right]$.

The term proportional to $\left(\frac{F_3}{m_\mu}\right)^2$ is

$$\begin{aligned}\mathrm{Term\,II} &= \left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)\left(\gamma^\rho\not{q}L\right)u_r(\mathrm{p}_1)\left[\bar{u}_s(\mathrm{p}_2)\left(\gamma^\sigma\not{q}L\right)u_r(\mathrm{p}_1)\right]^\dagger \\ &= \left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)\left(\gamma^\rho\not{q}L\right)u_r(\mathrm{p}_1)\,\bar{u}_r(\mathrm{p}_1)\left(R\not{q}\gamma^\sigma\right)u_s(\mathrm{p}_2).\end{aligned} \tag{3.4.92}$$

The summation in (3.4.92) gives

$$\begin{aligned}\mathrm{Tr}\left[(\not{p}_2 + m_{B_2})\gamma^\rho\not{q}L\left(\not{p}_1 + m_{B_1}\right)R\not{q}\gamma^\sigma\right] &= \mathrm{Tr}\left[\not{p}_2\gamma^\rho\not{q}\not{p}_1 L\not{q}\gamma^\sigma\right] + m_{B_1}m_{B_2}\mathrm{Tr}\left[\gamma^\rho\not{q}LR\not{q}\gamma^\sigma\right] \\ &= \mathrm{Tr}\left[\not{p}_2\gamma^\rho\not{q}\not{p}_1 L\not{q}\gamma^\sigma\right],\end{aligned} \tag{3.4.93}$$

since $LR = 0$, $L^2 = L$ and $L\gamma^\rho = \gamma^\rho R$, $R\gamma^\rho = \gamma^\rho L$. The first trace in Eq. (3.4.93) gives

$$\mathrm{Tr}\left[\not{p}_2\gamma^\rho\not{q}\not{p}_1 L\not{q}\gamma^\sigma\right] = \mathrm{p}_{1\beta}\mathrm{p}_{2\lambda}\mathrm{q}_\kappa\mathrm{q}_\alpha\mathrm{Tr}\left[\gamma^\lambda\gamma^\rho\gamma^\alpha\gamma^\beta\left(\frac{1-\gamma^5}{2}\right)\gamma^\sigma\gamma^\kappa\right]. \tag{3.4.94}$$

The trace which includes γ^5 matrix is given by

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right] &= \eta^{\rho\alpha}\mathrm{Tr}\left[\gamma^\beta\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right] - \eta^{\rho\beta}\mathrm{Tr}\left[\gamma^\alpha\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right] \\ &+ \eta^{\alpha\beta}\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right] + i\epsilon^{\rho\alpha\beta\tau}\mathrm{Tr}\left[\gamma_\tau\gamma^5\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right],\end{aligned} \tag{3.4.95}$$

where was used the following identity from [14]

$$\gamma^\rho\gamma^\alpha\gamma^\beta = \eta^{\rho\alpha}\gamma^\beta - \eta^{\rho\beta}\gamma^\alpha + \eta^{\alpha\beta}\gamma^\rho + i\epsilon^{\rho\alpha\beta\tau}\gamma_\tau\gamma^5. \tag{3.4.96}$$

Applying to each term in (3.4.95) the identities

$$\mathrm{Tr}\left[\gamma^\beta\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right] = -4i\epsilon^{\lambda\kappa\sigma\beta}, \tag{3.4.97}$$

and

$$\mathrm{Tr}\left[\gamma_\tau\gamma^\sigma\gamma^\kappa\gamma^\lambda\right] = 4\left(\delta^\sigma_\tau\eta^{\kappa\lambda} - \delta^\kappa_\tau\eta^{\sigma\lambda} + \delta^\lambda_\tau\eta^{\sigma\kappa}\right), \tag{3.4.98}$$

to the last term in Eq. (3.4.95)

$$\begin{aligned}&\mathrm{Tr}\left[\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\kappa\gamma^\lambda\gamma^5\right] = \\ &= -4i\left\{\eta^{\rho\alpha}\epsilon^{\beta\sigma\kappa\lambda} - \eta^{\rho\beta}\epsilon^{\alpha\sigma\kappa\lambda} + \eta^{\alpha\beta}\epsilon^{\rho\sigma\kappa\lambda} + \eta^{\kappa\lambda}\epsilon^{\rho\alpha\beta\sigma} - \eta^{\sigma\lambda}\epsilon^{\rho\alpha\beta\kappa} + \eta^{\sigma\kappa}\epsilon^{\rho\alpha\beta\lambda}\right\},\end{aligned} \tag{3.4.99}$$

which completely disappears after contracting with the neutrino tensor. What left is then

$$\begin{aligned}\mathrm{Term\,II} &= \frac{1}{2}\left(\mathrm{q}_\rho\mathrm{q}_\kappa - \mathrm{q}^2\eta_{\rho\kappa}\right)\mathrm{p}_{1\beta}\mathrm{p}_{2\lambda}\mathrm{q}_\sigma\mathrm{q}_\alpha\mathrm{Tr}\left[\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\kappa\gamma^\lambda\right] \\ &= \frac{1}{2}\left(\mathrm{q}_\rho\mathrm{q}_\kappa - \mathrm{q}^2\eta_{\rho\kappa}\right)\mathrm{p}_{1\beta}\mathrm{p}_{2\lambda}\mathrm{q}_\sigma\mathrm{q}_\alpha\left\{\eta^{\rho\alpha}\mathrm{Tr}\left[\gamma^\beta\gamma^\sigma\gamma^\kappa\gamma^\lambda\right] - \eta^{\rho\beta}\mathrm{Tr}\left[\gamma^\alpha\gamma^\sigma\gamma^\kappa\gamma^\lambda\right]\right. \\ &\left.+\eta^{\rho\sigma}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\kappa\gamma^\lambda\right] - \eta^{\rho\kappa}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\lambda\right] + \eta^{\rho\lambda}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\kappa\right]\right\}.\end{aligned} \tag{3.4.100}$$

Applying to each term of the last expression the trace identity

$$\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta\right] = 4\left(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\sigma\beta} + \eta^{\rho\beta}\eta^{\sigma\alpha}\right) \tag{3.4.101}$$

and after some algebra, the overall coefficient for $\left(\frac{F_3}{m_\mu}\right)^2$ is given by

$$\mathrm{Term\,II} = 2\mathrm{q}^2\left\{4\mathrm{q}^0 m_{B_1}\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right) - \mathrm{q}^2 m_{B_1} m_{B_2}\right\}. \tag{3.4.102}$$

The term proportional to $\left(\frac{F_4}{m_\mu}\right)^2$ is given by $\mathrm{Tr}\left[(\not p_2 + m_2)\gamma^\rho \not q R(\not p_1 + m_1) L \not q \gamma^\sigma\right]$ and the same coefficient is obtained as in Eq. (3.4.102)

Now evaluate the cross terms: first, the one proportional to F_1F_2

$$\begin{aligned}\text{Term III} &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)(\gamma^\rho L)\,u_r(\mathrm{p}_1)\,\bar{u}_r(\mathrm{p}_1)(\gamma^\sigma R)\,u_s(\mathrm{p}_2)\\ &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[(\not{p}_2 + m_{B_2})\gamma^\rho L(\not{p}_1 + m_{B_1})\gamma^\sigma R\right]\\ &= \left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right) m_{B_1} m_{B_2}\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\left(1+\gamma^5\right)\right] = -12\mathrm{q}^2 m_{B_1} m_{B_2},\end{aligned} \tag{3.4.103}$$

where in the last line we used

$$\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\left(1+\gamma^5\right)\right] = \mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\right] - \mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\gamma^5\right] = 4\eta^{\rho\sigma}. \tag{3.4.104}$$

The coefficient in Eq. (3.4.103) will be doubled since the term proportional to F_2F_1 gives the same contribution.

Next, consider the term proportional to $\frac{F_3F_4}{m_\mu^2}$

$$\begin{aligned}\text{Term IV} &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)(\gamma^\rho \not{q} L)\,u_r(\mathrm{p}_1)\left[\bar{u}_s(\mathrm{p}_2)(\gamma^\sigma \not{q} R)\,u_r(\mathrm{p}_1)\right]^\dagger\\ &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[(\not{p}_2 + m_{B_2})\gamma^\rho \not{q} L(\not{p}_1 + m_{B_1}) L\not{q}\gamma^\sigma\right]\\ &= -12\mathrm{q}^4 m_{B_1} m_{B_2}.\end{aligned} \tag{3.4.105}$$

The term proportional to $\frac{F_1F_3}{m_\mu}$:

$$\begin{aligned}\text{Term V} =&2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)(\gamma^\rho L)\,u_r(\mathrm{p}_1)\left[\bar{u}_s(\mathrm{p}_2)(\gamma^\sigma \not{q} L)\,u_r(\mathrm{p}_1)\right]^\dagger\\ =&2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)(\gamma^\rho L)\,u_r(\mathrm{p}_1)\,\bar{u}_r(\mathrm{p}_1)(\gamma^\sigma \not{q} L)\,u_r(\mathrm{p}_2)\\ =&2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[(\not{p}_2 + m_{B_2})\gamma^\rho L(\not{p}_1 + m_{B_1}) L\gamma^\sigma \not{q}\right]\end{aligned}$$

$$\begin{aligned}\text{Term V} &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)(\gamma^\rho L)\,u_r(\mathrm{p}_1)\left[\bar{u}_s(\mathrm{p}_2)(\gamma^\sigma \not{q} L)\,u_r(\mathrm{p}_1)\right]^\dagger\\ &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\sum_{rs}\bar{u}_s(\mathrm{p}_2)(\gamma^\rho L)\,u_r(\mathrm{p}_1)\,\bar{u}_r(\mathrm{p}_1)(\gamma^\sigma \not{q} L)\,u_r(\mathrm{p}_2)\\ &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[(\not{p}_2 + m_{B_2})\gamma^\rho L(\not{p}_1 + m_{B_1}) L\gamma^\sigma \not{q}\right]\\ &= 2\left(\mathrm{q}_\rho \mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right) m_{B_1}\mathrm{Tr}\left[\not{p}_2\gamma^\rho L\gamma^\sigma \not{q}\right]\\ &= -12 m_{B_1}\mathrm{q}^2\left(\mathrm{q}\cdot \mathrm{p}_2\right) = -12 m_{B_1}\mathrm{q}^2\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right).\end{aligned} \tag{3.4.106}$$

For the term proportional to $\frac{F_2 F_4}{m_\mu}$ the trace is given by

$$\begin{aligned} \mathrm{Tr}\left[\left(\not{p}_2 + m_{B_2}\right)\gamma^\rho R\left(\not{p}_1 + m_{B_1}\right) R\gamma^\sigma \not{q}\right] &= \mathrm{q}_\alpha \mathrm{p}_{2\beta}\mathrm{Tr}\left[\gamma^\beta\gamma^\rho\gamma^\sigma\gamma^\alpha\right] \\ &= -12 m_{B_1}\mathrm{q}^2\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right). \end{aligned} \tag{3.4.107}$$

The term proportional to $\frac{F_1 F_4}{m_\mu}$ is

$$\begin{aligned} \text{Term VI} &= 2\left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[\left(\not{p}_2 + m_{B_2}\right)\gamma^\rho L\left(\not{p}_1 + m_{B_1}\right) R\gamma^\sigma \not{q}\right] \\ &= m_{B_2}\left(\mathrm{q}_\rho\mathrm{q}_\sigma - \mathrm{q}^2\eta_{\rho\sigma}\right)\mathrm{Tr}\left[\gamma^\rho\left(1-\gamma^5\right)\not{p}_1\gamma^\sigma\not{q}\right] \\ &= 12 m_{B_2}\mathrm{q}^2\left(\mathrm{q}\cdot\mathrm{p}_1\right), \end{aligned} \tag{3.4.108}$$

And the one proportional to $\frac{F_2 F_3}{m_\mu}$ has the same coefficient as in Eq. (3.4.108) by the same argument as in (3.4.106).

Collecting all the terms gives the differential decay rate

$$\begin{aligned} \frac{d\Gamma}{d\left|\mathbf{q}\right|} = \frac{\left|\mathbf{q}\right|^2}{24\pi^3}&\frac{G_F^2}{m_{B_1} m_{B_2}} \\ &\times\left\{2\left[2\left(\mathrm{q}^0 m_{B_1}\right)\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right) + \mathrm{q}^2\left(m_{B_1} m_{B_2}\right)\right]\left(F_1^2 + F_2^2\right)\right. \\ &+ \frac{2\mathrm{q}^2}{m_\mu^2}\left[4\mathrm{q}^0 m_{B_1}\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right) - \mathrm{q}^2 m_{B_1} m_{B_2}\right]\left(F_3^2 + F_4^2\right) \\ &- 12\mathrm{q}^2 m_{B_1} m_{B_2} F_1 F_2 - 12\mathrm{q}^4 m_{B_1} m_{B_2}\frac{F_3 F_4}{m_\mu^2} \\ &\left. - \frac{12 m_{B_1}\mathrm{q}^2}{m_\mu}\left(\mathrm{q}^0 m_{B_2} + \mathbf{q}^2\right)\left(F_1 F_3 + F_2 F_4\right) + \frac{12 m_{B_1}\mathrm{q}^2}{m_\mu}\left(F_1 F_4 + F_2 F_3\right)\right\}. \end{aligned} \tag{3.4.109}$$

Dropping in all the terms of the order $(1/M)$ and substituting $\mathrm{q}^0 = (m_1 - m_2)$ in Eq. (3.4.109), the differential rate becomes

$$\frac{d\Gamma}{d\left|\mathbf{q}\right|} = \frac{\left|\mathbf{q}\right|^2 G_F^2}{12\pi^3} K\left(\left|\mathbf{q}\right|\right) + \mathcal{O}\left(1/M\right), \tag{3.4.110}$$

where

$$\begin{aligned} K\left(\left|\mathbf{q}\right|\right) = &\left[\mathrm{q}^2 + 2\left(m_1 - m_2\right)^2\right]\left(F_1^2 + F_2^2\right) + \frac{\mathrm{q}^2}{m_\mu^2}\left[4\left(m_1 - m_2\right)^2 - \mathrm{q}^2\right]\left(F_3^2 + F_4^2\right) \\ &- 6\mathrm{q}^2\left[F_1 F_2 + \frac{\mathrm{q}^2}{m_\mu^2} F_3 F_4 + \frac{m_1 - m_2}{m_\mu}\left(F_1 - F_2\right)\left(F_3 - F_4\right)\right]. \end{aligned} \tag{3.4.111}$$

3.4.4 Decay Rate into the Free Electron

For the decay rate of a bound muon to a free electron: $(Z\mu) \to Ze\nu_\mu\bar{\nu}_e$, the S-matrix element in the rest frame of B_1 for such a process is

$$\begin{aligned} S &= \langle Z, e, \nu_\mu, \bar{\nu}_e | \, i \int d^4\mathrm{x} \mathcal{L}\left(\mathrm{x}\right) | B_1, \mathbf{p}_1 = 0\rangle \\ &= i \left(2\pi\right)^4 \delta^{(4)} \left(\mathrm{p}_1 - \mathrm{p}_\mathrm{Z} - \mathrm{p}_\mathrm{e} - \mathrm{q}\right) \langle Z, e, \nu_\mu, \bar{\nu}_e | \, \mathcal{L}\left(0\right) | B_1, \mathbf{p}_1 = 0\rangle \,, \end{aligned} \tag{3.4.112}$$

where p_Z is the final momentum of the nucleus and $\mathrm{q} = \mathrm{p}_{\nu_\mu} + \mathrm{p}_{\bar{\nu}_\mathrm{e}}$. The invariant amplitude of the decay is then

$$\mathcal{M} = \sqrt{2m_{B_1}} \int \frac{d^3\mathbf{k}_1}{\sqrt{2\mathrm{k}_1^0}} \psi_1\left(\mathbf{k}_1\right) \sqrt{2k_3^0} \delta^{(3)}\left(\mathbf{p}_Z - \mathbf{k}_3\right) \langle e, \nu_\mu, \bar{\nu}_e \left| \mathcal{L}\left(0\right) \right| \mu\rangle \,, \tag{3.4.113}$$

and the total decay rate is therefore given by

$$\begin{aligned} \Gamma_{(Z\mu)\to Ze\nu_\mu\bar{\nu}_e} &= \int \frac{d^3\mathbf{k}_1}{\left(2\pi\right)^3} \left|\psi_1\left(\mathbf{k}_1\right)\right|^2 \frac{2m_\mu}{2\mathrm{k}_1^0} \\ \times &\left[\frac{1}{2m_\mu} \int d\tilde{\Phi} \left(2\pi\right)^4 \delta^{(4)} \left(\mathrm{p}_1 - \mathrm{p}_\mathrm{Z} - \mathrm{p}_\mathrm{e} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_\mathrm{e}}\right) \left|\langle e, \nu_\mu, \bar{\nu}_e \left| \mathcal{L}\left(0\right) \right| \mu\rangle\right|^2\right], \end{aligned} \tag{3.4.114}$$

where $\mathrm{p}_\mu = \mathrm{p}_1 - \mathrm{p}_\mathrm{Z} = \mathrm{p}_1 - \mathrm{k}_3$, and the phase integral after the integration over p_Z is

$$d\tilde{\Phi} = \frac{d^3\mathbf{p}_e}{\left(2\pi\right)^3 2\mathrm{p}_e^0} \frac{d^3\mathbf{p}_{\nu_\mu}}{\left(2\pi\right)^3 2\mathrm{p}_{\nu_\mu}^0} \frac{d^3\mathbf{p}_{\bar{\nu}_e}}{\left(2\pi\right)^3 2\mathrm{p}_{\bar{\nu}_e}^0}. \tag{3.4.115}$$

In the formula (3.4.114) the momentum of the muon is $\mathbf{k}_1$, so the following approximation is made $|\mu\left(\mathbf{k}_1\right)\rangle \approx |\mu\left(\mathbf{p}_\mu\right)\rangle$ for the expression in the square parenthesis to be equal to the total decay rate of the muon. Also, for the bound muon

$$\mathrm{p}_\mu^2 = \left(\gamma m_\mu\right)^2 - \mathbf{k}_1^2 + \mathcal{O}\left(1/M\right) \neq m_\mu^2, \tag{3.4.116}$$

where $\gamma m_\mu \equiv m_\mu - E_{\mathrm{bind},1}$ and

$$\gamma = \sqrt{1 - \left(Z\alpha\right)^2}. \tag{3.4.117}$$

Therefore, this approximation makes the matrix element off-shell, but retains the conservation of momentum. In the rest frame of B_1: $\mathrm{p}_\mu^0 = \gamma m_\mu$ and $\mathbf{p}_\mu = \mathbf{k}_1$. Thus, the expression in the

	$\langle L^{-1}\rangle$
$Z = 10$	0.99735612
$Z = 80$	0.84619822

Table 3.4.3: The mean inverse Lorentz factor for the muon and electron for $Z = 10$ and $Z = 80$.

square parenthesis in (3.4.114) becomes

$$\frac{1}{2m_\mu}\int d\tilde{\Phi}\,(2\pi)^4\,\delta^{(4)}\left(\mathrm{p}_\mu - \mathrm{p}_\mathrm{e} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_\mathrm{e}}\right)\left|\langle e,\nu_\mu,\bar{\nu}_e\left|\mathcal{L}(0)\right|\mu(\mathbf{p}_\mu)\rangle\right|^2 \tag{3.4.118}$$

$$= \frac{1}{2m_\mu}\int d\tilde{\Phi}\,(2\pi)^4\,\delta\left(\gamma m_\mu - \mathrm{p}^0_\mathrm{e} - \mathrm{p}^0_{\nu_\mu} - \mathrm{p}^0_{\bar{\nu}_\mathrm{e}}\right)\delta^{(3)}\left(\mathbf{p}_e + \mathbf{p}_{\nu_\mu} + \mathbf{p}_{\bar{\nu}_e}\right) \tag{3.4.119}$$

$$\times\left|\langle e,\nu_\mu,\bar{\nu}_e\left|\mathcal{L}(0)\right|\mu(\mathbf{p}_\mu)\rangle\right|^2, \tag{3.4.120}$$

where in Eq. (3.4.118) the muon momentum is taken $\mathrm{p}^2_\mu = (\gamma m_\mu)^2$ and $\mathbf{k}^2_1$ is neglected. The Eq. (3.4.120) is the decay rate for the free muon with the effective mass γm_μ and can be related to the real muon decay by a scale transformation $\mathrm{p} \to \gamma\mathrm{p}$ for momenta of the electron and neutrinos. Performing this scale transformation for the parts of this integral gives

$$d\tilde{\Phi} \to \gamma^6 d\tilde{\Phi}, \tag{3.4.121}$$

$$\delta\left(\gamma m_\mu - \mathrm{p}^0_\mathrm{e} - \mathrm{p}^0_{\nu_\mu} - \mathrm{p}^0_{\bar{\nu}_\mathrm{e}}\right)\delta^{(3)}\left(\mathbf{p}_e + \mathbf{p}_{\nu_\mu} + \mathbf{p}_{\bar{\nu}_e}\right) \to \gamma^{-4}\delta\left(m_\mu - \mathrm{p}^0_\mathrm{e} - \mathrm{p}^0_{\nu_\mu} - \mathrm{p}^0_{\bar{\nu}_\mathrm{e}}\right)\delta^{(3)}\left(\mathbf{p}_e + \mathbf{p}_{\nu_\mu} + \mathbf{p}_{\bar{\nu}_e}\right). \tag{3.4.122}$$

Making use of Eqs. (3.4.120) - (3.4.122) gives

$$\Gamma_{(Z\mu)\to Ze\nu_\mu\bar{\nu}_e} \equiv \Gamma_0\gamma^2\left\langle L^{-1}\right\rangle,\ \gamma = \sqrt{1-(Z\alpha)^2}, \tag{3.4.123}$$

where

$$\Gamma_0 = \frac{1}{2m_\mu}\int d\tilde{\Phi}\delta\left(m_\mu - \mathrm{p}^0_\mathrm{e} - \mathrm{p}^0_{\nu_\mu} - \mathrm{p}^0_{\bar{\nu}_\mathrm{e}}\right)\delta^{(3)}\left(\mathbf{p}_e + \mathbf{p}_{\nu_\mu} + \mathbf{p}_{\bar{\nu}_e}\right)\left|\langle e,\nu_\mu,\bar{\nu}_e\left|\mathcal{L}(0)\right|\mu(\mathbf{p}_\mu)\rangle\right|^2 \tag{3.4.124}$$

is the free muon decay rate and

$$\left\langle L^{-1}\right\rangle = \int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\left|\psi_1(\mathbf{k}_1)\right|^2\frac{2m_\mu}{2\mathrm{k}^0_1}, \tag{3.4.125}$$

is the mean inverse Lorentz factor. Numerical results for the mean inverse Lorentz factor

(3.4.125) are presented in the Table 3.4.3 for the cases of $Z = 10$ and $Z = 80$.

It is worth pointing out that when $Z\alpha \to 0$, the muon wave function is spread over all of space uniformly. Then the probability of it to decay within the screened Coulomb field of the nucleus is zero. Thus, in this limit, we expect the decay rate $\Gamma \to \Gamma_0$. But for larger value of Z such approximation is no longer valid.

3.5 Bound Muon Decay Rate to an Outgoing Energetic Electron: Another Formalism

For the case when the decay of a bound muon results into a highly energetic outgoing electron a formalism was developed in ref. [2]. This approach can be modified to calculate the decay rate of a bound muon into a bound electron. But before suggesting this possible modification, a short summary of the approach developed [2] is presented below.

The Lagrangian describing the Fermi interaction for the muon decay is given by (3.4.1). The neutrino part of the phase space $W_{\rho\sigma}$ is given by the expression similar to (3.4.80) with some differences in constant coefficients:

$$W_{\rho\sigma} = -\frac{\pi}{3\,(2\pi)^3}\left(\mathrm{q}^2\eta_{\rho\sigma} - \mathrm{q}_\rho \mathrm{q}_\sigma\right), \tag{3.5.1}$$

and the bound muon decay rate is then

$$\Gamma_{(Z\mu)\to Ze\nu_\mu\bar{\nu}_e} = \frac{2G_F^2}{(2\pi)^6}\sum_{e\,\mathrm{spin}}\int d\mathrm{q}^2\frac{d^3\mathbf{q}}{E_q}\frac{d^3\mathbf{p}_e}{E_e^2}\,(2\pi)\,\delta\left(E_\mu - E_e - \mathrm{q}^0\right)J^\rho J^{\sigma\dagger}W_{\rho\sigma}, \tag{3.5.2}$$

$$J^\rho \equiv \left[\int d^3\mathbf{r}e^{-i\mathbf{q}\cdot\mathbf{r}}\bar{\Phi}_e\gamma^\rho L\Phi_\mu\right], \tag{3.5.3}$$

where the average over the muon spin is incorporated in the definition Φ_μ, which is the solution of the Dirac equation for the muon and Φ_e for the electron. The 4-momentum transferred to the neutrinos is $\mathrm{q} = (\mathrm{q}^0, \mathbf{q})$, $\mathbf{p}_e$ is the electron 3-momentum, E_μ and E_e being the muon and electron energies, respectively. The integration over q^2 and the angular parts of the currents J^ρ gives

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_e} = \sum_{K\kappa} \frac{4}{\pi m_\mu^5} (2j_\kappa + 1) \int_0^{E_\mu - E_e} d|\mathbf{q}|\ |\mathbf{q}|^2$$
$$\times \left\{ \left[(E_\mu - E_e)^2 - |\mathbf{q}|^2 \right] \left[\frac{|S_{K\kappa}^0|^2}{K(K+1)} + \frac{|S_{K\kappa}^{-1}|^2}{K(2K+1)} + \frac{|S_{K\kappa}^{+1}|^2}{(K+1)(2K+1)} \right] \right.$$
$$\left. + \left[(E_\mu - E_e)|\mathbf{q}| \right] 2\mathrm{Im} \left[\frac{S_{K\kappa}\left(S_{K\kappa}^{-1} + S_{K\kappa}^{+1}\right)^*}{2K+1} \right] + |\mathbf{q}|^2 \left[\frac{|S_{K\kappa}^{-1} + S_{K\kappa}^{+1}|^2}{(2K+1)^2} + |S_{K\kappa}|^2 \right] \right\}, \tag{3.5.4}$$

where the amplitudes are

$$S_{K\kappa}^0 = \begin{cases} -i(\kappa - 1)\langle j_K(|\mathbf{q}|r)(f_\kappa G + g_\kappa F)\rangle \\ (\kappa + 1)\langle j_K(|\mathbf{q}|r)(g_\kappa G + f_\kappa F)\rangle \end{cases}, \tag{3.5.5}$$

$$S_{K\kappa}^{-1} = \begin{cases} \langle j_{K-1}(|\mathbf{q}|r)\left[(\kappa - K - 1) g_\kappa G - (\kappa + K - 1) f_\kappa F\right]\rangle \\ -i\langle j_{K-1}(|\mathbf{q}|r)\left[(\kappa + K + 1) f_\kappa G + (\kappa - K + 1) g_\kappa F\right]\rangle \end{cases}, \tag{3.5.6}$$

$$S_{K\kappa}^{+1} = \begin{cases} \langle j_{K+1}(|\mathbf{q}|r)\left[(\kappa + K) g_\kappa G + (K - \kappa + 2) f_\kappa F\right]\rangle \\ -i\langle j_{K+1}(|\mathbf{q}|r)\left[(\kappa - K) f_\kappa G + (\kappa + K + 2) g_\kappa F\right]\rangle \end{cases}, \tag{3.5.7}$$

$$S_{K\kappa} = \begin{cases} i\langle j_K(|\mathbf{q}|r)(f_\kappa G - g_\kappa F)\rangle \\ \langle j_K(|\mathbf{q}|r)(g_\kappa G + f_\kappa F)\rangle \end{cases}, \tag{3.5.8}$$

and j_K stands for the spherical Bessel function of order K; g_κ, f_κ and G, F are the upper and lower components of the radial part of the solution of the Dirac equation in the Coulomb potential for the bound electron and muon, respectively. They are defined by (2.3.6) and (2.3.7). For the amplitudes S the following notation is used

$$\langle a\rangle \equiv \int_0^\infty a r^2 dr, \tag{3.5.9}$$

and the first (second) row in each entry corresponds to odd (even) values of $l_\kappa + K$, where $l_\kappa = j_\kappa + \frac{1}{2}\frac{\kappa}{|\kappa|}$. These quantum numbers are defined in [2]. In this case $j_\kappa = |\kappa| - \frac{1}{2}$ and for the given value of K, $\kappa \neq 0$ can only take values $\pm K$ and $\pm(K+1)$. Now K can have all the values from 0 to ∞, but K can not be zero in $S_{K\kappa}^{0,-1}$.

Note that in the decay rate Eq. (3.5.4) the outgoing electron can have energy from 0 to m_μ and is normalized in its final state as

$$\langle E, j, j_z | E', j', j_z'\rangle = \frac{1}{\rho(E)}\delta(E - E')\delta_{jj'}\delta_{j_z j_z'} = 2\pi\delta(E - E')\delta_{jj'}\delta_{j_z j_z'}, \tag{3.5.10}$$

$\frac{\Gamma_{(Z\mu^-)\to(Ze^-)\nu_\mu\bar{\nu}_e}}{\Gamma_0}$	$Z=10$	$Z=80$
Alchemy's Result [1]	$2.46\cdot 10^{-9}$	$2.01\cdot 10^{-5}$
Alchemy's Result [1]: Corrected	$1.42\cdot 10^{-8}$	$3.10\cdot 10^{-5}$
Coulomb [2]	$1.32\cdot 10^{-8}$	$1.83\cdot 10^{-5}$
Fermi [2]	$1.31\cdot 10^{-8}$	$2.63\cdot 10^{-5}$
Nuclear parameters for Fermi potential	$r_0=2.78$, $a=0.55$	$r_0=6.38$, $a=0.54$

Table 3.6.1: Numerical values for $\frac{\Gamma_{(Z\mu^-)\to(Ze^-)\nu_\mu\bar{\nu}_e}}{\Gamma_0}$ for $Z=10$ and $Z=80$ using the formalism of [1] and [2]. The first row presents results given in [1]. The last row gives the parameters used to perform calculations in the Fermi potential presented in the fourth row.

since the density of the states is $\rho(E)=\frac{1}{2\pi}$. If we want to modify (3.5.4) for the case when an electron electron stays bound, we should take into account that it can be only in the ground state (with the discrete energy γm_e). Therefore,

$$dE\rho(E)\to 1. \tag{3.5.11}$$

Also the values of the quantum numbers for the ground state of muon and electron in Eqs. (3.5.5)-(3.5.8) are taken to be $K=0$, 1 and $\kappa=-1$.

3.6 Numerical Results

It is convenient to present the numerical results for the decay rates as ratios $\frac{\Gamma_{(Z\mu)\to(Ze)\nu_\mu\bar{\nu}_e}}{\Gamma_0}$, where $\Gamma_{(Z\mu)\to(Ze)\nu_\mu\bar{\nu}_e}$ is the decay rate of a bound muon to a bound electron and Γ_0 is the free muon decay rate, i. e.

$$\Gamma_0=\frac{G_F^2 m_\mu^5}{192\pi^3}. \tag{3.6.1}$$

The calculations in the formalism [2] are performed by considering two kind of potentials: Coulomb and Fermi, where for the latter case the charge density is defined as

$$\rho(r)=\frac{\rho_0}{1+\exp\left(\frac{r-r_0}{a_0}\right)},\ \text{with} \int d^3\mathbf{r}\,\rho(\mathbf{r})=Ze. \tag{3.6.2}$$

It is worth emphasizing that in the case of the Fermi charge distribution, the Dirac equation for the muon and electron wave functions has to be solved numerically. To do this part, we used the mechanism developed in [2]. Now we can compare the numerical values presented in [1] with the values we get after correcting the signs in the expressions for the form factors h_1 and h_2 given in Eqs. (3.4.60) and (3.4.61) with the ones obtained from the modified formalism of [2]. The results are given in the Table. 3.6.1 we can conclude that changing FF as $F_i \to F_i'$ affects the decay rates only for as large Z as 80. Also, the differences in the numbers for these two approaches [1, 2] are as small as few percents for small $Z\alpha$ and are more significant for larger values of $Z\alpha$: for $Z = 80$ the discrepancy is of about 41% if the numbers are compared for the Coulomb potential.

Chapter 4

Some Limiting Cases

In order to see if the results in two approaches are consistent with each other it is useful to discuss the following limiting cases:

1. Nearly equal muon and electron masses;
2. Small $Z\alpha$ limit.

4.7 Nearly Equal Masses

4.7.1 Atomic Alchemy's Formalism [1]

Consider the case where the masses of muon and electron are almost equal, i.e.,

$$m_\mu - m_e = \epsilon m_\mu, \tag{4.7.1}$$

where ϵ is a small parameter. Under the limit $\epsilon \to 0$, we have for the decay rate of a bound muon into the bound electron to be

$$\begin{aligned} \Gamma &= \int_0^{m_1-m_2} d\left|\mathbf{q}\right| \frac{G_F^2\left|\mathbf{q}\right|}{12\pi^3} K\left(\left|\mathbf{q}\right|\right) \\ &= \gamma^5\epsilon^5 \frac{G_F^2 m_\mu^5}{15\pi^3}\left[\left(F_1\left(0\right) - F_2\left(0\right)\right)^2 + F_1\left(0\right)F_2\left(0\right)\right]. \end{aligned} \tag{4.7.2}$$

Since under such a limit the electron mass is almost that of the muon, its momentum is very small and it can not leave the atom. The momentum transferred to the neutrinos is $\mathbf{q} \to 0$, therefore, the FF's F_1 and F_2 in Eq. (4.7.2) are evaluated at $|\mathbf{q}| = 0$:

$$F_1\left(0\right) = \int \frac{d^3\mathbf{k}_1}{\left(2\pi\right)^3}\psi_1\left(\mathbf{k}_1\right)\psi_2^*\left(\mathbf{k}_1\right)\frac{2\mathrm{k}_1^0 + m_\mu}{3\mathrm{k}_1^0} = \frac{2}{3} + \frac{1}{3}\left\langle L^{-1}\right\rangle, \tag{4.7.3}$$

$$F_2\left(0\right) = \int \frac{d^3\mathbf{k}_1}{\left(2\pi\right)^3}\psi_1\left(\mathbf{k}_1\right)\psi_2^*\left(\mathbf{k}_1\right)\frac{\mathrm{k}_1^0 - m_\mu}{3\mathrm{k}_1^0} = \frac{1}{3} - \frac{1}{3}\left\langle L^{-1}\right\rangle, \tag{4.7.4}$$

where $\mathrm{k}_1^0 = \sqrt{\mathbf{k}_1^2 + m_\mu^2}$. Thus, the decay rate

$$\Gamma = \frac{64}{5}\epsilon^5\gamma^5\Gamma_0\left[\frac{1 + \left\langle L^{-1}\right\rangle + \left\langle L^{-1}\right\rangle^2}{3}\right]. \tag{4.7.5}$$

The numerical results for $\frac{\Gamma_{(Z\mu)\to(Ze)\nu_\mu\bar{\nu}_e}}{\Gamma_0}$ in this limiting case are presented in the Table 4.7.1.

Thus, as the mass of the electron approaches that of the muon the electron does not have enough energy to escape the atom, and the total decay rate of the the bound muon tends to the decay rate into the bound electron: $\Gamma_{\text{total}} \to \Gamma_{(Z\mu)\to(Ze)}$. Once $\epsilon = 0$ and $m_e = m_\mu$ the phase space for the neutrinos reduces to zero and, therefore, no decay can occur at all.

4.7.2 Another Approach

The decay rate for the bound final state of an electron according to (3.4.83) is

$$\Gamma = \int \frac{d^3\mathbf{q}}{(2\pi)^3} |\mathcal{M}|^2 , \tag{4.7.6}$$

where in the invariant amplitude

$$|\mathcal{M}|^2 = \frac{1}{2} \sum_{rs} J^{\alpha\beta} N_{\alpha\beta}, \tag{4.7.7}$$

$N_{\alpha\beta}$ is the integral over neutrino phase space and the tensor $J^{\alpha\beta}$ is created from particle currents in the following way

$$J^{\alpha\beta} = J^\alpha \left(J^\beta\right)^\dagger , \; J^\alpha \equiv \int d^3\mathbf{r} \bar{\Phi}_e(\mathbf{r}) \gamma^\alpha L \Phi_\mu(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}}. \tag{4.7.8}$$

In the case of nearly equal masses the neutrino momentum $\mathbf{q} \approx 0$, therefore, the exponent in Eq. (4.7.8) is $e^{-i\mathbf{q}\cdot\mathbf{r}} \approx 1$ and the current becomes

$$J^\alpha \equiv \int d^3\mathbf{r} \bar{\Phi}_e(\mathbf{r}) \gamma^\alpha L \Phi_\mu(\mathbf{r}) , \tag{4.7.9}$$

The ground state wave function for the bound state in the position space $\Phi(\mathbf{r})$ with the spin up is given in Eq. (2.3.5) and can be written in an alternative form as

$$\Phi_\uparrow(\mathbf{r}) \equiv \mathcal{G}(r) u_+(\hat{\mathbf{r}}) , \tag{4.7.10}$$

where

$$\mathcal{G}(r) \equiv \frac{(2mZ\alpha)^{\gamma+\frac{1}{2}}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} r^{\gamma-1} \exp(-mZ\alpha r) , \tag{4.7.11}$$

and the bispinor part is

$$u_+(\hat{\mathbf{r}}) \equiv \begin{pmatrix} 1 \\ 0 \\ \frac{i(1-\gamma)}{Z\alpha}\cos\theta \\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{i\phi} \end{pmatrix}. \tag{4.7.12}$$

Since in the spherical coordinates

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}, \tag{4.7.13}$$

the spinor u_+ can be written in terms of this scalar product of the Pauli matrices, unit vector and the eigenvectors of σ_z as follows

$$u_+(\hat{\mathbf{r}}) = \begin{pmatrix} \phi_+ \\ \frac{i(1-\gamma)}{Z\alpha}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\phi_+ \end{pmatrix},\ \phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.7.14}$$

Or, in terms of the Dirac matrices

$$\begin{aligned} u_+(\hat{\mathbf{r}}) &= \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} + \frac{i(1-\gamma)}{Z\alpha}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\begin{pmatrix} 0 \\ \phi_+ \end{pmatrix} = \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} - \frac{i(1-\gamma)}{Z\alpha}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} - \frac{i(1-\gamma)}{Z\alpha}(\boldsymbol{\gamma}\cdot\hat{\mathbf{r}})\begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} \equiv \varrho_\mu\gamma^\mu\begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} = \not{\varrho}\begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}, \end{aligned} \tag{4.7.15}$$

where the operator ϱ^μ is defined as

$$\varrho^\mu = \varrho^\mu(\hat{\mathbf{r}}) = \left(1, \frac{i(1-\gamma)}{Z\alpha}\hat{\mathbf{r}}\right). \tag{4.7.16}$$

The hermitian conjugate of u_+ is then

$$u_+^\dagger(\hat{\mathbf{r}}') = \left[\not{\varrho}'\begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}\right]^\dagger = \begin{pmatrix} \phi_+^\dagger & 0 \end{pmatrix}\not{\varrho}'^\dagger, \tag{4.7.17}$$

and the Dirac adjoint

$$\bar{u}_+(\hat{\mathbf{r}}') = \begin{pmatrix} \phi_+^\dagger & 0 \end{pmatrix}\varrho_\mu'^*\gamma^{\mu\dagger}\gamma^0 = \begin{pmatrix} \phi_+^\dagger & 0 \end{pmatrix}\gamma^0\varrho_\mu'^*\gamma^\mu = \begin{pmatrix} \phi_+^\dagger & 0 \end{pmatrix}\not{\varrho}'^*, \tag{4.7.18}$$

where

$$(\varrho'^\mu)^* = (\varrho^\mu)^*(\hat{\mathbf{r}}') = \left(1, -\frac{i(1-\gamma)}{Z\alpha}\hat{\mathbf{r}}'\right). \tag{4.7.19}$$

This gives the expression

$$u_+(\hat{\mathbf{r}})\,\bar{u}_+(\hat{\mathbf{r}}') = \not{\phi}\begin{pmatrix}\phi_+\\0\end{pmatrix}\begin{pmatrix}\phi_+^\dagger & 0\end{pmatrix}(\not{\phi}')^* = \not{\phi}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\not{\phi}'^*. \tag{4.7.20}$$

For the wave function with the spin down we get in the similar way

$$\Phi_\downarrow(\mathbf{r}) \equiv \mathcal{G}(r)\,u_-(\hat{\mathbf{r}}),\; u_-(\hat{\mathbf{r}}) = \begin{pmatrix}\phi_-\\ \frac{i(1-\gamma)}{Z\alpha}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\phi_-\end{pmatrix},\; \phi_- = \begin{pmatrix}0\\1\end{pmatrix}, \tag{4.7.21}$$

which is of the same form as for u_+. Thus, following the same steps we get

$$u_-(\hat{\mathbf{r}})\,\bar{u}_-(\hat{\mathbf{r}}') = \not{\phi}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\not{\phi}'^*, \tag{4.7.22}$$

and the summation over spins yields

$$\sum_{\text{spins}} u\bar{u} = u_+\bar{u}_+ + u_-\bar{u}_- = \not{\phi}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\not{\phi}'^* = \not{\phi}\left(\frac{I+\gamma^0}{2}\right)\not{\phi}'^*. \tag{4.7.23}$$

The current is then

$$J^\alpha \equiv \int d^3\mathbf{r}\,\mathcal{G}_e(r)\,\mathcal{G}_\mu(r)\,\bar{u}_e(\hat{\mathbf{r}})\,\gamma^\alpha L u_\mu(\hat{\mathbf{r}}). \tag{4.7.24}$$

Multiplying it with $\left(J^\beta\right)^\dagger$ and keeping the radial part separately gives

$$\begin{aligned} J^\alpha\left(J^\beta\right)^\dagger &\sim \frac{1}{2}\sum_{\text{spins}} \bar{u}_e(\hat{\mathbf{r}})\,\gamma^\alpha L u_\mu(\hat{\mathbf{r}})\,\bar{u}_\mu(\hat{\mathbf{r}}')\,\gamma^\sigma L u_e(\hat{\mathbf{r}}') \\ &= \frac{1}{2}\mathrm{Tr}\left[u_e(\hat{\mathbf{r}}')\,\bar{u}_e(\hat{\mathbf{r}})\,\gamma^\alpha L u_\mu(\hat{\mathbf{r}})\,\bar{u}_\mu(\hat{\mathbf{r}}')\,\gamma^\sigma L\right] \\ &= 4G_F^2\mathrm{Tr}\left[\not{\phi}\left(\frac{I+\gamma^0}{2}\right)\not{\phi}'^*\gamma^\alpha L\not{\phi}'\left(\frac{I+\gamma^0}{2}\right)\not{\phi}^*\gamma^\beta L\right]. \end{aligned} \tag{4.7.25}$$

This trace can be split into four terms each of which can be evaluated separately:

$$\text{Tr}\left[\not{\varepsilon}\gamma^0\not{\varepsilon}'^*\gamma^\alpha L\not{\varepsilon}'\not{\varepsilon}^*\gamma^\beta L\right] = 0, \tag{4.7.26}$$

$$\text{Tr}\left[\not{\varepsilon}\not{\varepsilon}'^*\gamma^\alpha L\not{\varepsilon}'\gamma^0\not{\varepsilon}^*\gamma^\beta L\right] = 0, \tag{4.7.27}$$

since traces with odd number of gamma matrices equal to zero. Also,

$$\text{Tr}\left[\not{\varepsilon}\not{\varepsilon}'^*\gamma^\alpha L\not{\varepsilon}'\not{\varepsilon}^*\gamma^\beta L\right] = \text{Tr}\left[\not{\varepsilon}\not{\varepsilon}'^*\gamma^\alpha\not{\varepsilon}'\not{\varepsilon}^*\gamma^\beta RL\right] = 0, \tag{4.7.28}$$

where the property $RL = 0$ was used. Thus, the only non-zero trace is

$$\text{Tr}\left[\not{\varepsilon}\gamma^0\not{\varepsilon}'^*\gamma^\alpha L\not{\varepsilon}'\gamma^0\not{\varepsilon}^*\gamma^\beta L\right] = \text{Tr}\left[\not{\varepsilon}\gamma^0\not{\varepsilon}'^*\gamma^\alpha\not{\varepsilon}'\gamma^0\not{\varepsilon}^*\gamma^\beta L\right], \tag{4.7.29}$$

and is evaluated with the help of Mathematica.

Contracting the neutrino tensor

$$N_{\alpha\beta} = \frac{2G_F^2}{3\pi}\left(\mathrm{q}_\alpha\mathrm{q}_\beta - \mathrm{q}^2\eta_{\alpha\beta}\right) \tag{4.7.30}$$

with the angular part of $J^{\alpha\beta}$ leads to

$$\begin{aligned} J^{\alpha\beta}N_{\alpha\beta} \sim \frac{G_F^2}{48\pi}\Bigg\{(\mathrm{q}^0)^2\left[\frac{8}{3}\left(\frac{1-\gamma}{Z\alpha}\right)^4 - \frac{16}{3}\left(\frac{1-\gamma}{Z\alpha}\right)^2 + 64\left(\frac{1-\gamma}{Z\alpha}\right)^2\hat{r}_i\hat{r}'_i + 24\right] \\ +\mathbf{q}^2\left[-\frac{8}{9}\left(\frac{1-\gamma}{Z\alpha}\right)^4 + 16\left(\frac{1-\gamma}{Z\alpha}\right)^2 - 64\left(\frac{1-\gamma}{Z\alpha}\right)^2\hat{r}_i\hat{r}'_i - 8\right]\Bigg\}, \end{aligned} \tag{4.7.31}$$

where the following property was used

$$\hat{r}'_i\hat{r}'_j = \hat{r}_i\hat{r}_j = \frac{\delta_{ij}}{3}. \tag{4.7.32}$$

Using the expression for $Z\alpha$ and the fact that after the angular integration over $d\Omega$ and $d\Omega'$ the terms proportional to $\hat{r}_i\hat{r}'_i$ disappear. Finally, we have

$$\begin{aligned} J^{\alpha\beta}N_{\alpha\beta} \sim \frac{G_F^2}{6\pi}\Bigg\{(\mathrm{q}^0)^2\left[\frac{1}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 - \frac{2}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 + 3\right] \\ +\mathbf{q}^2\left[-\frac{1}{9}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 + 2\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 - 1\right]\Bigg\}. \end{aligned} \tag{4.7.33}$$

Now the invariant amplitude becomes

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{2}\sum_{rs} J^{\alpha\beta}N_{\alpha\beta}, \\
&= \frac{G_F^2}{6\pi}\left\{\left(\mathrm{q}^0\right)^2\left[\frac{1}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 - \frac{2}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 + 3\right]\right. \\
&\left. +\mathbf{q}^2\left[-\frac{1}{9}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 + 2\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 - 1\right]\right\}\left[\int d^3\mathbf{r}\,\mathcal{G}_e\left(r\right)\mathcal{G}_\mu\left(r\right)\right]^2.
\end{aligned} \tag{4.7.34}
$$

Finally, the decay rate becomes

$$
\begin{aligned}
\Gamma &= \int d\left|\mathbf{q}\right|\mathbf{q}^2\frac{|\mathcal{M}|^2}{2\pi^2} \\
&= \frac{G_F^2\mathbf{q}^2}{12\pi^3}\left\{\left(\mathrm{q}^0\right)^2\left[\frac{1}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 - \frac{2}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 + 3\right]\right. \\
&\left. +\mathbf{q}^2\left[-\frac{1}{9}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 + 2\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 - 1\right]\right\}\left[\int d^3\mathbf{r}\,\mathcal{G}_e\left(r\right)\mathcal{G}_\mu\left(r\right)\right]^2.
\end{aligned} \tag{4.7.35}
$$

Which after the integration over the 3-momentum $|\mathbf{q}|$, takes the form

$$
\begin{aligned}
\Gamma &= \frac{G_F^2}{12\pi^3}\left\{\frac{\mathbf{q}^3\left(\mathrm{q}^0\right)^2}{3}\left[\frac{1}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 - \frac{2}{3}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 + 3\right]\right. \\
&= \left. +\frac{\mathbf{q}^5}{5}\left[-\frac{1}{9}\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 + 2\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2 - 1\right]\right\}\left[\int d^3\mathbf{r}\,\mathcal{G}_e\left(r\right)\mathcal{G}_\mu\left(r\right)\right]^2.
\end{aligned} \tag{4.7.36}
$$

In the nearly equal masses limit the components of the neutrino 4-momentum are $\mathrm{q}^0 = |\mathbf{q}| = \epsilon\gamma m_\mu$, therefore,

$$
\begin{aligned}
\Gamma &= \frac{G_F^2}{135\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^4 + 2\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)^2\right]\left[\int d^3\mathbf{r}\,\mathcal{G}_e\left(r\right)\mathcal{G}_\mu\left(r\right)\right]^2 \\
&= \frac{G_F^2}{135\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\frac{2\gamma^2+4\gamma+3}{\left(1+\gamma\right)^2}\right]\left[\int d^3\mathbf{r}\,\mathcal{G}_e\left(r\right)\mathcal{G}_\mu\left(r\right)\right]^2.
\end{aligned} \tag{4.7.37}
$$

Next, evaluating integral over the position space gives

$$\int d^3\mathbf{r}\,\mathcal{G}_e(r)\,\mathcal{G}_\mu(r) = \frac{(2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma+\frac{1}{2}}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int d^3\mathbf{r}\,r^{2\gamma-2}\exp\left[-\left(m_e+m_\mu\right)Z\alpha r\right]$$
$$= (2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma+\frac{1}{2}}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int dr\,r^{2\gamma}\exp\left[-\left(m_e+m_\mu\right)Z\alpha r\right]$$
$$= (2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma+\frac{1}{2}}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\frac{\Gamma(1+2\gamma)}{\left[\left(m_e+m_\mu\right)Z\alpha\right]^{2\gamma+1}}$$
$$= \left(\frac{2}{m_e+m_\mu}\right)^{2\gamma+1}(m_e m_\mu)^{\gamma+\frac{1}{2}}\left(\frac{1+\gamma}{2}\right). \tag{4.7.38}$$

Upon substituting into the last expression the electron mass in the limit $m_e = (1-\epsilon)\,m_\mu$

$$\int d^3\mathbf{r}\,\mathcal{G}_e(r)\,\mathcal{G}_\mu(r) = \left(\frac{1+\gamma}{2}\right)\left[\frac{(1-\epsilon)^{\gamma+\frac{1}{2}}}{(1-\epsilon/2)^{2\gamma+1}}\right] \approx \left(\frac{1+\gamma}{2}\right). \tag{4.7.39}$$

If we do not drop the terms $\mathcal{O}(\epsilon^2)$, the expression for the Eq. (4.7.39) would still remains the same. Consider the normalization condition for the radial wave functions Eqs. (2.3.1) and (2.3.2)

$$\int dr\,r^2\left[f^2(r)+g^2(r)\right] = 1, \tag{4.7.40}$$

where

$$f(r) = -\sqrt{\frac{1-\gamma}{1+\gamma}}g(r), \tag{4.7.41}$$

therefore,

$$\left(1+\frac{1-\gamma}{1+\gamma}\right)\int dr\,r^2 g^2(r) = 1 \Rightarrow \int dr\,r^2 g^2(r) = \left(\frac{1+\gamma}{2}\right). \tag{4.7.42}$$

Note that the function $\mathcal{G}(r)$ is defined as

$$\mathcal{G}_\mu(r) \equiv \frac{g(r)}{\sqrt{4\pi}}. \tag{4.7.43}$$

Substituting Eq. (4.7.39) into the decay rate formula and dividing over Γ_0 gives the ratio

$$\frac{\Gamma}{\Gamma_0} = \frac{16\epsilon^5\gamma^5}{45}\left[2\gamma^2+4\gamma+3\right]. \tag{4.7.44}$$

The numerical values found from this formula are compared to the calculations performed using the Atomic Alchemy's formalism [1] in the same limit and those are presented in the Table 4.7.1. We can consider that two approaches are fully consistent in this nearly equal masses limit.

Z	Alchemy [1]	Eq. (4.7.5)	Eq. 4.7.44
10	$1.25 \cdot 10^{-9}$	$1.26 \cdot 10^{-9}$	$1.26 \cdot 10^{-9}$
80	$3.85 \cdot 10^{-10}$	$3.83 \cdot 10^{-10}$	$3.80 \cdot 10^{-10}$

Table 4.7.1: Numerical values for $\frac{\Gamma_{(Z\mu)\to(Ze)\nu_\mu\bar{\nu}_e}}{\Gamma_0}$ in cases of $Z = 10$ and $Z = 80$ for the parameter $\epsilon = 0.01$. The second column correspond to the calculations in formalisms of [1] and the third and fourth correspond to calculations according to formulas (4.7.5) and (4.7.44), derived for this formalism.

4.8 Small $Z\alpha$ Limit

The purpose of this section is to calculate the decay rate for small $Z\alpha$ limit in approaches [1, 2] without considering equal muon and electron masses.

The matrix element (3.4.23) can be rewritten if we define the wave function in the momentum space in the following way

$$\Phi_r(\mathbf{k}) \equiv G(k) \begin{pmatrix} \chi_r \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{k})}{\mathrm{k}^0+m}\chi_r \end{pmatrix}, \tag{4.8.1}$$

where

$$G(k) \equiv \frac{\mathrm{k}^0+m}{2\mathrm{k}^0}\left(g(k) + \frac{k}{\mathrm{k}^0+m}f(k)\right). \tag{4.8.2}$$

Then

$$S^\rho_{sr} = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\bar{\Phi}_s(e;\mathbf{k}_1-\mathbf{q})\,\gamma^\rho L\Phi_r(\mu;\mathbf{k}_1). \tag{4.8.3}$$

The sudden approximation works well for small values of $Z\alpha$ since the muon is weakly bound in this case. To study for which magnitudes of $\mathbf{q}$ the matrix element $S^\rho_{sr}(\mathbf{q})$ gives the biggest contribution it can be first rewritten as the Fourier Transform of (4.8.3):

$$S^\rho_{sr}(\mathbf{q}) = \int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1-\mathbf{k}_2)}\bar{\Phi}_s(e;\mathbf{k}_2)\,\gamma^\rho L\Phi_r(\mu;\mathbf{k}_1). \tag{4.8.4}$$

Taking the limit of small $Z\alpha$ sets $\gamma \approx 1$ which makes the spin dependent part of the Dirac wave function negligible and Eq. (4.8.2) then becomes

$$G(k) \approx \frac{\mathrm{k}^0+m}{2\mathrm{k}^0}g(k), \tag{4.8.5}$$

and

$$\Phi_r(\mathbf{k}) \approx \frac{\mathrm{k}^0 + m}{2\mathrm{k}^0} g(k) \begin{pmatrix} \chi_r \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{k})}{\mathrm{k}^0+m}\chi_r \end{pmatrix}. \tag{4.8.6}$$

The corresponding decay rate is given by

$$\Gamma = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{2} \sum_{sr} (S^\rho_{sr})^\dagger S^\sigma_{sr} N_{\rho\sigma}. \tag{4.8.7}$$

After contracting the matrix elements S^ρ_{sr} with $N_{\rho\sigma}$ it becomes

$$\Gamma = \frac{G_F^2}{3\pi} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \times \sum_{sr} \left\{ \left(\mathrm{q}^0\right)^2 \bar{S}^\dagger_{sr} \cdot \bar{S}_{sr} + |\mathbf{q}|^2 \left[\left|S^0_{sr}\right|^2 - \bar{S}^\dagger_{sr} \cdot \bar{S}_{sr} + \left|\hat{\mathbf{q}} \cdot \bar{S}_{sr}\right|^2 \right] - 2\mathrm{q}^0 |\mathbf{q}| \,\mathrm{Re}\left(\left(S^0_{sr}\right)^\dagger \hat{\mathbf{q}} \cdot \bar{S}_{sr} \right) \right\}. \tag{4.8.8}$$

Each of the terms in the last expression can be written in terms of form factors A_i, that result from the rotational invariance of the following integrals

$$\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1-\mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \equiv A_0\left(\mathbf{q}^2\right), \tag{4.8.9}$$

$$\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1-\mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \frac{k_1^i}{\mathrm{k}_1^0 + m_\mu} \equiv A_1\left(\mathbf{q}^2\right) \hat{q}^i, \tag{4.8.10}$$

$$\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1-\mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \frac{k_2^i}{\mathrm{k}_2^0 + m_e} \equiv A_2\left(\mathbf{q}^2\right) \hat{q}^i, \tag{4.8.11}$$

$$\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1-\mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \frac{k_1^i}{\mathrm{k}_1^0 + m_\mu} \frac{k_2^j}{\mathrm{k}_2^0 + m_e} \equiv \delta^{ij} A_3\left(\mathbf{q}^2\right) + \left(\hat{q}^i \hat{q}^j - \frac{1}{3}\delta^{ij} \right) A_4\left(\mathbf{q}^2\right). \tag{4.8.12}$$

Thus, the explicit form of A_i is

$$A_i\left(\mathbf{q}^2\right) \equiv \int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1)\, a_i(\mathbf{k}_1, \mathbf{k}_2), \tag{4.8.13}$$

where

$$a_0 = 1,\ a_1 = \frac{(\hat{\mathbf{q}} \cdot \mathbf{k}_1)}{\mathrm{k}_1^0 + m_\mu},\ a_2 = \frac{(\hat{\mathbf{q}} \cdot \mathbf{k}_2)}{\mathrm{k}_2^0 + m_e},$$

$$a_3 = \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)}{3\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right)},\ a_4 = \frac{3}{2}\frac{(\hat{\mathbf{q}} \cdot \mathbf{k}_1)(\hat{\mathbf{q}} \cdot \mathbf{k}_2) - \frac{1}{3}(\mathbf{k}_1 \cdot \mathbf{k}_2)}{\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right)}. \tag{4.8.14}$$

Now, consider the zeroth component of S_{sr}^{μ}

$$\begin{aligned}
S_{sr}^{0}(\mathbf{q}) &= \int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1 - \mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \\
&\times \begin{pmatrix} \chi_s \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_2)}{\mathrm{k}_2^0+m}\chi_s \end{pmatrix}^{\dagger} \gamma^0\gamma^0 L \begin{pmatrix} \chi_r \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_1)}{\mathrm{k}_1^0+m}\chi_r \end{pmatrix} \\
&= \frac{1}{2}\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1 - \mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \\
&\times \chi_s^{\dagger}\left\{1 - \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_1)}{\mathrm{k}_1^0 + m_\mu} - \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_2)}{\mathrm{k}_2^0 + m_e} + \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_1)(\boldsymbol{\sigma}\cdot\mathbf{k}_2)}{\left(\mathrm{k}_1^0 + m_\mu\right)\left(\mathrm{k}_2^0 + m_e\right)}\right\}\chi_r.
\end{aligned} \tag{4.8.15}$$

The last line can be rewritten using the following property of the Pauli Matrices $\boldsymbol{\sigma}$

$$(\boldsymbol{\sigma}\cdot\mathbf{k}) = (\hat{\mathbf{q}}\cdot\mathbf{k})(\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}), \tag{4.8.16}$$

which gives

$$S_{sr}^{0}(\mathbf{q}) = \frac{1}{2}\left\{(A_0 + 3A_3)\chi_s^{\dagger}\chi_r - (A_1 + A_2)\chi_s^{\dagger}(\hat{\mathbf{q}}\cdot\boldsymbol{\sigma})\chi_r\right\}, \tag{4.8.17}$$

where A_i can be obtained from Eq. (4.8.13). The 3-vector component of S_{sr}^{μ}:

$$\begin{aligned}
S_{sr}^{k}(\mathbf{q}) &= \frac{1}{2}\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1 - \mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \\
&\times \begin{pmatrix} \chi_s \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_2)}{\mathrm{k}_2^0+m}\chi_s \end{pmatrix}^{\dagger} \gamma^0\gamma^k L \begin{pmatrix} \chi_r \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_1)}{\mathrm{k}_1^0+m}\chi_r \end{pmatrix},
\end{aligned}$$

which after the multiplication of the bispinors becomes

$$\begin{aligned}
S_{sr}^{k}(\mathbf{q}) &= \frac{1}{2}\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{r}(\mathbf{k}_1 - \mathbf{k}_2)} G(e;\mathbf{k}_2)\, G(\mu;\mathbf{k}_1) \\
&\times \chi_s^{\dagger}\left\{-\sigma^k + \sigma^k\frac{(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})(\mathbf{k}_1\cdot\hat{\mathbf{q}})}{\mathrm{k}_1^0 + m_\mu} + \frac{(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})(\mathbf{k}_2\cdot\hat{\mathbf{q}})}{\mathrm{k}_2^0 + m_e}\sigma^k - \frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_2)\,\sigma^k\,(\boldsymbol{\sigma}\cdot\mathbf{k}_1)}{\left(\mathrm{k}_2^0 + m_e\right)\left(\mathrm{k}_1^0 + m_\mu\right)}\right\}\chi_r \\
&= \frac{1}{2}\chi_s^{\dagger}\left\{\left(-A_0 + A_3 - \frac{1}{3}A_4\right)\sigma^k + A_1\sigma^k(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\right. \\
&\left. + A_2(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\sigma^k - (\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\sigma^k(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})A_4\right\}\chi_r.
\end{aligned} \tag{4.8.18}$$

Now all the amplitudes in Eq. (4.8.8) can be evaluated one by one. Consider

$$\begin{aligned}
\sum_{sr}\left|S^0_{sr}(\mathbf{q})\right|^2 &= \sum_{sr}\left(S^0_{sr}(\mathbf{q})\right)^\dagger S^0_{sr}(\mathbf{q}) \\
&= \frac{1}{4}\sum_{sr}\left\{(A_0+3A_3)^*\chi_r^\dagger\chi_s-(A_1+A_2)^*\chi_r^\dagger(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\chi_s\right\} \\
&\times\left\{(A_0+3A_3)\chi_s^\dagger\chi_r-(A_1+A_2)\chi_s^\dagger(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\chi_r\right\} \\
&= \frac{1}{4}\left\{2\left|A_0+3A_3\right|^2+2\left|A_1+A_2\right|^2\right\},
\end{aligned} \tag{4.8.19}$$

where in the second line we used $\sum_r \chi_r\chi_r^\dagger = I$ and

$$\begin{aligned}
&\sum_s \chi_s^\dagger(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\chi_s \\
&= \sum_s \chi_s^\dagger\begin{pmatrix}\hat{q}^3 & \hat{q}^1-i\hat{q}^2\\ \hat{q}^1+i\hat{q}^2 & -\hat{q}^3\end{pmatrix}\chi_s \\
&= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\hat{q}^3 & \hat{q}^1-i\hat{q}^2\\ \hat{q}^1+i\hat{q}^2 & -\hat{q}^3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\hat{q}^3 & \hat{q}^1-i\hat{q}^2\\ \hat{q}^1+i\hat{q}^2 & -\hat{q}^3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \\
&= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\hat{q}^3\\ \hat{q}^1+i\hat{q}^2\end{pmatrix}+\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\hat{q}^1-i\hat{q}^2\\ -\hat{q}^3\end{pmatrix}=\hat{q}^3+\left(-\hat{q}^3\right)=0.
\end{aligned} \tag{4.8.20}$$

Likewise,

$$\begin{aligned}
&\sum_{sr}\mathbf{S}^\dagger_{sr}\cdot\mathbf{S}_{sr}=\sum_{sr}\left(S^k_{sr}\right)^\dagger\cdot S^k_{sr} \\
&= \frac{1}{4}\sum_r\chi_r^\dagger\left\{\left(-A_0+A_3-\frac{1}{3}A_4\right)^*\sigma^k+A_1^*(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\sigma^k+A_2^*\sigma^k(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})-(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\sigma^k(\bar{\sigma}\cdot\hat{q})A_4^*\right\} \\
&\times\left\{\left(-A_0+A_3-\frac{1}{3}A_4\right)\sigma^k+A_1\sigma^k(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})+A_2(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\sigma^k-(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\sigma^k(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})A_4\right\}\chi_r. \\
&= \frac{1}{4}\left\{6\left[\left|A_0-A_3+\frac{1}{3}A_4\right|^2+|A_1|+|A_2|^2+|A_4|^2\right]-4\mathrm{Re}\left[\left(A_0-A_3+\frac{1}{3}A_4\right)^*A_4+A_1^*A_2\right]\right\}.
\end{aligned} \tag{4.8.21}$$

The cross terms are calculated in the following way

$$\hat{q}^k\cdot S^k_{sr}=\frac{1}{2}\chi_s^\dagger\left\{\left(-A_0+A_3-\frac{4}{3}A_4\right)(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})+A_1+A_2\right\}\chi_r, \tag{4.8.22}$$

after squaring and summing over s, r

$$\sum_{sr} |\hat{\mathbf{q}} \cdot \mathbf{S}_{sr}|^2 = \frac{1}{4} \left\{ 2 \left| A_0 - A_3 + \frac{4}{3} A_4 \right|^2 + 2 \left| A_1 + A_2 \right|^2 \right\}, \tag{4.8.23}$$

and the last amplitude in Eq. (4.8.8) gives

$$\begin{aligned} &\sum_{sr} \mathrm{Re}\left[\hat{\mathbf{q}} \cdot \mathbf{S}_{sr}^{\dagger} S_{sr}^0\right] = \sum_{sr} \mathrm{Re}\left[\hat{q}^k \cdot \left(S_{sr}^k\right)^{\dagger} S_{sr}^0\right] \\ &= \frac{1}{4} \sum_r \chi_r^{\dagger} \left\{ \left(-A_0 + A_3 - \frac{4}{3} A_4\right)^* (\boldsymbol{\sigma} \cdot \hat{\mathbf{q}}) + A_1^* + A_2^* \right\} \left\{ (A_0 + 3A_3) - (A_1 + A_2)(\boldsymbol{\sigma} \cdot \hat{\mathbf{q}}) \right\} \chi_r \\ &= \frac{1}{4} \left\{ 2\mathrm{Re}\,(A_1 + A_2)^* \left(2A_0 + 2A_3 + \frac{4}{3} A_4\right) \right\}. \end{aligned} \tag{4.8.24}$$

To sum up, the obtained decay rate is

$$\begin{aligned} \Gamma = \frac{G_F^2}{3\pi} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \sum_{sr} \Big\{ &\left(\mathrm{q}^0\right)^2 \mathbf{S}_{sr}^{\dagger} \cdot \mathbf{S}_{sr} + |\mathbf{q}|^2 \left[\left|S_{sr}^0\right|^2 - \mathbf{S}_{sr}^{\dagger} \cdot \mathbf{S}_{sr} + |\hat{\mathbf{q}} \cdot \mathbf{S}_{sr}|^2 \right] \\ &-2\mathrm{q}^0 |\mathbf{q}| \,\mathrm{Re}\left(\left(S_{sr}^0\right)^{\dagger} \hat{\mathbf{q}} \cdot \mathbf{S}_{sr} \right) \Big\}, \end{aligned} \tag{4.8.25}$$

where the amplitudes are

$$\begin{aligned} \sum_{sr} \mathbf{S}_{sr}^{\dagger} \cdot \mathbf{S}_{sr} &= \frac{1}{4} \Big\{ 6 \left[\left| A_0 - A_3 + \frac{1}{3} A_4 \right|^2 + |A_1| + |A_2|^2 + |A_4|^2 \right], \\ &\quad -4\mathrm{Re}\left[\left(A_0 - A_3 + \frac{1}{3} A_4 \right)^* A_4 + A_1^* A_2 \right] \Big\}, \end{aligned} \tag{4.8.26}$$

$$\sum_{sr} |\hat{\mathbf{q}} \cdot \mathbf{S}_{sr}|^2 = \frac{1}{4} \left\{ 2 \left| A_0 - A_3 + \frac{4}{3} A_4 \right|^2 + 2 \left| A_1 + A_2 \right|^2 \right\}, \tag{4.8.27}$$

$$\sum_{sr} \mathrm{Re}\left[\hat{\mathbf{q}} \cdot \mathbf{S}_{sr}^{\dagger} S_{sr}^0\right] = \frac{1}{4} \left\{ 2\mathrm{Re}\,(A_1 + A_2)^* \left(2A_0 + 2A_3 + \frac{4}{3} A_4\right) \right\}, \tag{4.8.28}$$

$$\sum_{sr} \left|S_{sr}^0\right|^2 = \frac{1}{4} \left\{ 2 \left| A_0 + 3A_3 \right|^2 + 2 \left| A_1 + A_2 \right|^2 \right\}. \tag{4.8.29}$$

To evaluate which values of q give the most significant contributions to amplitudes we evaluate the matrix element S_{sr}^0. For [2] its form is the following

$$S_{sr}^0 (\mathbf{q}) |_{full} = \frac{1}{2} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} f_e (\mathbf{k}_2) f_\mu (\mathbf{k}_1) \chi_s^{\dagger} \chi_r, \tag{4.8.30}$$

and the corresponding expression for ref. [1] is

$$S^0_{sr}(\mathbf{q}) = \frac{1}{2}\left\{(A_0 + 3A_3)\,\chi^\dagger_s\chi_r - (A_1 + A_2)\,\chi^\dagger_s(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\,\chi_r\right\}. \tag{4.8.31}$$

In order to compare these two expressions numerically set $s = r = \frac{1}{2}$ and $\mathbf{q} = q\hat{\mathbf{z}}$:

$$S^0_{\frac{1}{2}\frac{1}{2}}(\mathbf{q})\,|_{full} = \frac{1}{2}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}g_e(k_2)\,g_\mu(k_1), \tag{4.8.32}$$

$$S^0_{\frac{1}{2}\frac{1}{2}}(\mathbf{q}) = \frac{1}{2}\{(A_0 + 3A_3) - (A_1 + A_2)\}. \tag{4.8.33}$$

The numerical values for $Z = 1$ are presented in the Table 4.8.1. It can be seen that the main contribution comes from small values of q and also that the results match each other very well in that region. Thus, the form factors $A_i = 0$, for $i = 1, ..., 4$ since their integrants are proportional to $\mathbf{k}_1$ or $\mathbf{k}_2$. The decay rate then becomes

$$\frac{\Gamma}{\Gamma_0} = \frac{16}{m^5_\mu}\int_0^{\mathrm{q}^0} d\,|\mathbf{q}|\;|\mathbf{q}|^2\,|A_0|^2\left[3\left(\mathrm{q}^0\right)^2 - |\mathbf{q}|^2\right],\; Z\alpha \to 0. \tag{4.8.34}$$

The FF A_0 in Eq. (4.8.34) can be transformed in the following way

$$\begin{aligned}
A_0 &= \int d^3\mathbf{r}\,e^{-i\mathbf{q}\cdot\mathbf{r}}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}e^{i\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k}_2)}G(e;\mathbf{k}_2)\,G(\mu;\mathbf{k}_1)\\
&\simeq \int d^3\mathbf{r}\,e^{-i\mathbf{q}\cdot\mathbf{r}}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}e^{i\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k}_2)}\frac{\mathrm{k}^0_1 + m_\mu}{2\mathrm{k}^0_1}g(\mu;\mathbf{k}_1)\,\frac{\mathrm{k}^0_2 + m_e}{2\mathrm{k}^0_2}g(e;\mathbf{k}_2)\\
&\simeq \int d^3\mathbf{r}\,e^{-i\mathbf{q}\cdot\mathbf{r}}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}e^{i\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k}_2)}g(\mu;\mathbf{k}_1)\,g(e;\mathbf{k}_2)\\
&= \int_0^\infty dr\,r^2 j_0(qr)\,g(\mu;\mathbf{k}_1)\,g(e;\mathbf{k}_2),
\end{aligned} \tag{4.8.35}$$

where in the last line we used the plane wave expansion of the exponent $e^{-i\mathbf{q}\cdot\mathbf{r}}$. This analytical result can be compared to [2] by setting the smaller components of wave function to zero:

$$\frac{\Gamma}{\Gamma_0} = \frac{16}{m^5_\mu}\int_0^{q^0} d\,|\mathbf{q}|\;|\mathbf{q}|^2\,\langle j_0 g_e g_\mu\rangle^2\left[3\left(\mathrm{q}^0\right)^2 - |\mathbf{q}|^2\right],\; Z\alpha \to 0, \tag{4.8.36}$$

$\lvert\mathbf{q}\rvert$ (MeV)	$S^0_{\frac{1}{2}\frac{1}{2}}(\mathbf{q})$	$S^0_{\frac{1}{2}\frac{1}{2}}(\mathbf{q})\rvert_{full}$
0.0	$1.33 \cdot 10^{-3}$	$1.33 \cdot 10^{-3}$
0.5	$7.16 \cdot 10^{-7}$	$7.17 \cdot 10^{-7}$
1.0	$4.62 \cdot 10^{-8}$	$4.64 \cdot 10^{-8}$
5.0	$7.48 \cdot 10^{-11}$	$7.51 \cdot 10^{-11}$
100	$4.03 \cdot 10^{-16}$	$4.70 \cdot 10^{-16}$

Table 4.8.1: Numerical values for $S^0_{\frac{1}{2}\frac{1}{2}}(\mathbf{q})$ for $Z = 1$. The second column presents results calculated according to formalism in [1], the third - calculations according to [2].

$$\langle j_0 g_e g_\mu \rangle = \int_0^\infty dr\, r^2 j_0(qr)\, g_e(r)\, g_\mu(r)\,. \tag{4.8.37}$$

Thus, under the small $Z\alpha$ limit, the approaches considered in [1] and [2] give the same result for the decay rate.

Chapter 5

Conclusion

We have calculated the decay rate of a bound muon to a bound electron using Dirac wave functions for different values of Z in two different formalisms developed in [1] and [2]. For the latter we performed calculations both for the Coulomb potential and Fermi distribution.

While reproducing the calculation presented in Atomic Alchemy [1], we got different signs of the A term's in the expressions for h_1 and h_2 given by (3.4.60) and (3.4.61), respectively. After making the corresponding corrections and modifying the approach developed for the case of bound muon to an outgoing electron in presence of a nucleus [2], we compared the numerical results of the ratio of bound to free muon decay rate in both approaches. The two limiting cases of nearly equal masses and small $Z\alpha$ are considered as well. We conclude that for small $Z\alpha$ the differences in the values of branching ratio for both approaches are insignificant. However, the values start to differ considerably with increasing values of $Z\alpha$ and we find that for $Z = 80$ the difference is about 41%. As the formalism developed in [1] is missing the part of wave function that corresponds to possibility of creation of particle-antiparticle pair, we consider the approach developed in [2] to be more suitable for all values of $Z\alpha$.

These calculations are going to serve as a base to add the radiative corrections, which are to be performed according to the formalism developed in ref. [15]. To evaluate them one can use a code in Fortran whose documentation is provided in Appendix D. A detailed description of numerical calculations of various Feynman diagrams required to incorporate these radiative corrections is also given in the same Appendix.

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Appendix A: Pauli and Dirac Matrices

From the commutation and anticommutation relations for the Pauli matrices,

$$[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l,\ \{\sigma_j, \sigma_k\} = 2\delta_{jk}, \tag{5.8.38}$$

it follows

$$\sigma_j\sigma_k = \delta_{jk} + i\varepsilon_{jkl}\sigma_l. \tag{5.8.39}$$

Multiplying Eq. (5.8.39) by a_k and a_j in turn,

$$\begin{cases} \sigma_j\left(\sigma_k a_k\right) = a_j + i\varepsilon_{jkl}a_k\sigma_l \\ \left(\sigma_j a_j\right)\sigma_k = a_k + i\varepsilon_{jkl}a_j\sigma_l = a_k + i\varepsilon_{klj}\sigma_l a_j \end{cases}, \tag{5.8.40}$$

or in the vector form

$$\begin{cases} \boldsymbol{\sigma}\left(\boldsymbol{\sigma}\cdot\mathbf{a}\right) = \mathbf{a} + i\left[\mathbf{a}\times\boldsymbol{\sigma}\right] \\ \left(\boldsymbol{\sigma}\cdot\mathbf{a}\right)\boldsymbol{\sigma} = \mathbf{a} + i\left[\boldsymbol{\sigma}\times\mathbf{a}\right] \end{cases}. \tag{5.8.41}$$

Thus, from Eq. (5.8.41) it follows that

$$\begin{aligned} \left(\boldsymbol{\sigma}\cdot\mathbf{a}\right)\left(\boldsymbol{\sigma}\cdot\mathbf{b}\right) &= \left(\mathbf{a}\cdot\mathbf{b}\right) + i\left[\boldsymbol{\sigma}\times\mathbf{a}\right]\cdot\mathbf{b} \\ &= \left(\mathbf{a}\cdot\mathbf{b}\right) + i\boldsymbol{\sigma}\cdot\left[\mathbf{a}\times\mathbf{b}\right]. \end{aligned} \tag{5.8.42}$$

Properties of Dirac Gamma Matrices in D-dimensional Space

Here are some properties of γ-matrices in D-dimensional space, where $D = 4 - 2\epsilon$, $\epsilon \to 0$. The commutation algebra is $\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu = 2\eta^{\mu\sigma}$.

$$\gamma_\mu\gamma^\mu = 4 - 2\epsilon = 2\left(2-\epsilon\right), \tag{5.8.43}$$

$$\gamma_\mu\gamma^\sigma\gamma^\mu = \gamma_\mu(2\eta^{\mu\sigma} - \gamma^\mu\gamma^\sigma) = 2(\epsilon-1)\gamma^\sigma, \tag{5.8.44}$$

$$\gamma_\mu\gamma^\lambda\gamma^\sigma\gamma^\mu = \gamma_\mu\gamma^\lambda\left(2\eta^{\mu\sigma} - \gamma^\mu\gamma^\sigma\right) = 2\gamma^\sigma\gamma^\lambda + 2\left(2-\epsilon\right)\gamma^\lambda\gamma^\sigma = 4\eta^{\lambda\sigma} - 2\epsilon\gamma^\lambda\gamma^\sigma \tag{5.8.45}$$

$$\gamma_\mu\gamma^\lambda\gamma^\sigma\gamma^\beta\gamma^\mu = 2\gamma^\beta\gamma^\lambda\gamma^\sigma - \left(4\eta^{\lambda\sigma} - 2\epsilon\gamma^\lambda\gamma^\sigma\right)\gamma^\beta = -2\gamma^\beta\gamma^\sigma\gamma^\lambda + 2\epsilon\gamma^\lambda\gamma^\sigma\gamma^\beta. \tag{5.8.46}$$

Different Representations of Gamma Matrices

Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{5.8.47}$$

$$\beta = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \alpha^k = \gamma^0\gamma^k = \begin{pmatrix} -\sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{5.8.48}$$

A transformation from the Weyl representation to the Dirac representation is a unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -I & I \\ I & I \end{pmatrix}, \tag{5.8.49}$$

which gives:

$$\Phi_D = \begin{pmatrix} f \\ g \end{pmatrix} = U\psi_W = U \begin{pmatrix} \Phi_L \\ \Phi_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_R - \Phi_L \\ \Phi_R + \Phi_L \end{pmatrix}, \tag{5.8.50}$$

where the index D stands for the Dirac representation and W for the Weyl.

The transformation of γ matrices:

$$\gamma_D = U\gamma_W U^{-1} \tag{5.8.51}$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \gamma^k = \gamma^0\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{5.8.52}$$

The Dirac α and β matrices:

$$\beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}. \tag{5.8.53}$$

The chiral projections are defined as:

$$L = \frac{1}{2}\left(1 - \gamma^5\right) = \frac{1}{2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{5.8.54}$$

Appendix B: Dirac Equation for Particle in Electromagnetic Field

The following Appendix summarizes some properties of the free Dirac equation. A useful reference for this part is [9].

The Dirac Equation for a free electron (in natural units $\hbar = c = 1$)

$$i\frac{\partial}{\partial t}\Phi\left(t,\mathbf{r}\right) = \mathcal{H}\left(\mathbf{r}\right)\Phi\left(t,\mathbf{r}\right), \tag{5.8.55}$$

$$\mathcal{H}_0\left(\mathbf{r}\right) = \left(\boldsymbol{\alpha}\cdot\mathbf{p}\right) + \beta m, \tag{5.8.56}$$

where $\boldsymbol{\alpha}$ and β are 4×4 matrices in the Dirac-Pauli representation

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5.8.57}$$

with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ being the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.8.58}$$

and I is a 2×2 identity matrix. α and β satisfy the following properties

$$\alpha_i^2 = \beta^2 = \mathbb{I},\ \{\alpha_i, \alpha_j\} = 2\delta_{ij}\mathbb{I},\ \{\alpha_i, \beta\} = 0, \tag{5.8.59}$$

$$\alpha_i^\dagger = \alpha_i,\ \beta^\dagger = \beta, \tag{5.8.60}$$

where $\mathbb{I}$ is the 4×4 identity matrix and δ_{ij} is the Kroneker delta symbol. In terms of the conventional gamma matrices

$$\gamma^\mu = \left(\gamma^0, \boldsymbol{\gamma}\right) \equiv \left(\beta, \beta\boldsymbol{\alpha}\right),\ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}. \tag{5.8.61}$$

In Eq. (5.8.56) $\mathbf{p} = -i\nabla$ is the momentum operator, which is the spatial part of the 4-gradient $\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$

$$\mathrm{p}^\mu \equiv i\partial^\mu = \left(p^0, \mathbf{p}\right) = \left(i\frac{\partial}{\partial t}, -i\nabla\right), \tag{5.8.62}$$

and $\Phi(t,\mathbf{r})$ is a 4-component bispinor

$$\Phi(t,\mathbf{r}) = \begin{pmatrix} \Phi^{u}(t,\mathbf{r}) \\ \Phi^{\iota}(t,\mathbf{r}) \end{pmatrix}, \tag{5.8.63}$$

where $\Phi^{u}(t,\mathbf{r})$ and $\Phi^{\iota}(t,\mathbf{r})$ are the upper and lower components, respectively. Multiplying both parts of the Dirac equation (5.8.55) by β

$$i\beta\frac{\partial}{\partial t}\Phi(t,\mathbf{r}) = \beta\left[(\boldsymbol{\alpha}\cdot\mathbf{p}) + \beta m\right]\Phi(t,\mathbf{r}) = \left[(\boldsymbol{\gamma}\cdot\mathbf{p}) + m\right]\Phi(t,\mathbf{r}), \tag{5.8.64}$$

$$\Rightarrow \left[i\beta\frac{\partial}{\partial t} - (\boldsymbol{\gamma}\cdot\mathbf{p}) - m\right]\Phi(t,\mathbf{r}) = \left[i\gamma^{0}\frac{\partial}{\partial t} + i(\boldsymbol{\gamma}\cdot\nabla) - m\right]\Phi(t,\mathbf{r}) = 0,$$

$$\left[i\partial\!\!\!/ - m\right]\Phi(t,\mathbf{r}) = 0, \tag{5.8.65}$$

where $\partial\!\!\!/ \equiv \gamma^{\mu}\partial_{\mu}$ and Eq. (5.8.65) is the Dirac equation for a free particle. Its conjugate gives

$$\bar{\Phi}(t,\mathbf{r})\left[i\gamma^{\mu}\partial_{\mu} + m\right] = 0, \tag{5.8.66}$$

where the Dirac conjugate $\bar{\Phi}(t,\mathbf{r})$ is defined as

$$\bar{\Phi}(t,\mathbf{r}) = \Phi^{\dagger}(t,\mathbf{r})\gamma^{0}, \tag{5.8.67}$$

$$\Phi^{\dagger}(t,\mathbf{r}) = \begin{pmatrix} \Phi^{u\dagger}(t,\mathbf{r}) & \Phi^{\iota\dagger}(t,\mathbf{r}) \end{pmatrix}. \tag{5.8.68}$$

After multiplying Eq. (5.8.65) by $\bar{\Phi}$ on the right and Eq. (5.8.66) by Φ on the left and then adding them together, we have

$$\bar{\Phi}\left[i\partial\!\!\!/ - m\right]\Phi + \bar{\Phi}\left[i\partial\!\!\!/ + m\right]\Phi = 0, \tag{5.8.69}$$

or

$$\bar{\Phi}\gamma^{\mu}\left(p\!\!\!/_{\mu}\Phi\right) + \left(p\!\!\!/_{\mu}\bar{\Phi}\right)\gamma^{\mu}\Phi = 0. \tag{5.8.70}$$

Defining the probability current density as

$$\mathrm{j}_{\mu}(t,\mathbf{r}) \equiv i\bar{\Phi}(t,\mathbf{r})\gamma^{\mu}\Phi(t,\mathbf{r}), \tag{5.8.71}$$

Eq. (5.8.70) can be rewritten as

$$\frac{\partial \mathrm{j}_{\mu}(\mathrm{x})}{\partial \mathrm{x}_{\mu}} = 0, \tag{5.8.72}$$

that is just the continuity equation. The zeroth component of Eq. (5.8.71) can be used to define the probability density for $\rho(r)$, which is the charge density divided by the charge of the field, and it can be written as

$$-i\mathrm{j}_0(t,\mathbf{r}) = \rho(t,\mathbf{r}) = \Phi^\dagger(t,\mathbf{r})\,\Phi(t,\mathbf{r})\,. \tag{5.8.73}$$

The Dirac equation in the momentum representation

$$i\frac{\partial}{\partial t}\Phi(t,\mathbf{p}) = \mathcal{H}_0(\mathbf{p})\,\Phi(t,\mathbf{p})\,, \tag{5.8.74}$$

where $\mathcal{H}_0(\mathbf{p})$ is defined in Eq. (5.8.56). Substituting expressions (5.8.57) and (5.8.63) for $\boldsymbol{\alpha}$, β and Φ, respectively, in Eq. (5.8.74) leads to

$$i\frac{\partial}{\partial t}\begin{pmatrix}\Phi^u(t,\mathbf{p})\\ \Phi^\iota(t,\mathbf{p})\end{pmatrix} = \begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\mathbf{p}\begin{pmatrix}\Phi^u(t,\mathbf{p})\\ \Phi^\iota(t,\mathbf{p})\end{pmatrix} + \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}\Phi^u(t,\mathbf{p})\\ \Phi^\iota(t,\mathbf{p})\end{pmatrix}m, \tag{5.8.75}$$

which splits into a system of two equations

$$i\frac{\partial\Phi^u(t,\mathbf{p})}{\partial t} = (\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^\iota(t,\mathbf{p}) + m\Phi^u(t,\mathbf{p})\,, \tag{5.8.76}$$

$$i\frac{\partial\Phi^\iota(t,\mathbf{p})}{\partial t} = (\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^u(t,\mathbf{p}) - m\Phi^\iota(t,\mathbf{p})\,. \tag{5.8.77}$$

When the states are stationary $\Phi_s(t,\mathbf{p}) = \Phi_s(\mathbf{p})\exp(-iE_s t)$ each of them can be labeled with the index s and the Dirac equation is then

$$\mathcal{H}_0(\mathbf{p})\,\Phi_s(\mathbf{p}) = E_s\Phi_s(\mathbf{p})\,. \tag{5.8.78}$$

Equations (5.8.76) and (5.8.77) for such cases become

$$\begin{aligned}(E-m)\,\Phi^u(\mathbf{p}) - (\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^\iota(\mathbf{p}) = 0,\\ (E+m)\,\Phi^\iota(\mathbf{p}) - (\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^u(\mathbf{p}) = 0,\end{aligned} \tag{5.8.79}$$

where the indices s, which however are understood, were dropped for brevity. In order to have non-zero solutions for (5.8.79), the following condition has to be satisfied

$$\begin{vmatrix}(E-m) & -(\boldsymbol{\sigma}\cdot\mathbf{p})\\ -(\boldsymbol{\sigma}\cdot\mathbf{p}) & (E+m)\end{vmatrix} = 0 \Rightarrow E^2 - m^2 - (\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p}) = 0. \tag{5.8.80}$$

The last term in Eq. (5.8.80) is $(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p}) = \mathbf{p}^2$ according to the Pauli matrices' property

(5.8.42):

$$(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b}) + i\boldsymbol{\sigma}[\mathbf{a}\times\mathbf{b}], \tag{5.8.81}$$

and the whole expression for the energy is

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}. \tag{5.8.82}$$

Thus, the Dirac equation (5.8.55) has both positive and negative solutions.

The Dirac equation for an electron in electromagnetic field can be obtained from Eq. (5.8.65) by the following substitution for the 4-gradient

$$\partial^\mu \to \mathrm{D}^\mu = \partial^\mu + ie\mathrm{A}^\mu, \tag{5.8.83}$$

or by

$$\mathrm{p}^\mu \to \mathrm{p}^\mu - e\mathrm{A}^\mu, \tag{5.8.84}$$

in the momentum representation, where A_μ is a 4-vector potential

$$\mathrm{A}^\mu = (V, \mathbf{A}), \tag{5.8.85}$$

with V being the electric potential and $\mathbf{A}$ being the magnetic potential. Equation (5.8.65) then becomes

$$[\gamma^\mu(\mathrm{p}_\mu - e\mathrm{A}_\mu) - m]\,\Phi = 0. \tag{5.8.86}$$

Multiplying the last expression by β and remembering that $\gamma^\mu\beta = (I, \boldsymbol{\alpha})$, we get

$$\left\{(I,\boldsymbol{\alpha})\begin{pmatrix} i\frac{\partial}{\partial t} \\ -\mathbf{p}\end{pmatrix} - e(I,\boldsymbol{\alpha})\begin{pmatrix} V \\ -\mathbf{A}\end{pmatrix} - m\beta\right\}\Phi = 0, \tag{5.8.87}$$

which can be rewritten in terms of the Hamiltonian for the electromagnetic field $H_{EM}(\mathbf{r})$

$$i\frac{\partial\Phi}{\partial t} = \mathcal{H}_{EM}(\mathbf{r})\,\Phi, \tag{5.8.88}$$

$$\mathcal{H}_{EM}(\mathbf{r}) = \boldsymbol{\alpha}\cdot(\mathbf{p} - e\mathbf{A}) + eV + m\beta. \tag{5.8.89}$$

The Coulomb field at the point located by $|\mathbf{r}|$ away from a nucleus with a charge Z is given by

$$\mathbf{A} = 0, \quad V(r) = -\frac{eZ}{|\mathbf{r}|}, \tag{5.8.90}$$

and the Dirac-Coulomb Hamiltonian is therefore

$$\mathcal{H}_{DC}\left(\mathbf{r}\right) = \boldsymbol{\alpha}\cdot\mathbf{p} - \frac{e^2 Z}{|\mathbf{r}|} + m\beta. \tag{5.8.91}$$

The solutions for the stationary equation (5.8.78) with $\mathcal{H}_{DC}\left(\mathbf{r}\right)$ are given by continuous spectrum of energy $E \in (-\infty, m] \cup [m, +\infty)$ for free electron states. The solutions corresponding to a bound electron (finite motion) are given by the discrete spectrum in the range $E \in (-m, m)$.

Appendix C: Free Muon Decay

The invariant amplitude for the Feynman diagram (5.8.1) is given according to Feynman rules from [13] (Section 4.7) by

$$\mathcal{M} = L^{\rho} L^{\sigma} D_{\rho\sigma} \left(\mathrm{p}_1 - \mathrm{p}_{\nu_\mu}\right), \tag{5.8.92}$$

where L^{ρ} is the vertex between the incoming muon and outgoing muon neutrino:

$$L^{\rho} = \bar{u}\left(\mu, \mathrm{k}_1\right) \left[-i\frac{g}{\sqrt{2}}\gamma^{\rho}\frac{1-\gamma^5}{2}\right] u\left(\mathrm{p}_{\nu_\mu}\right), \tag{5.8.93}$$

and L^{σ} is the vertex between the outgoing electron and electron antineutrino

$$L^{\sigma} = \bar{u}\left(e, \mathrm{k}_2\right) \left[-i\frac{g}{\sqrt{2}}\gamma^{\sigma}\frac{1-\gamma^5}{2}\right] v\left(\mathrm{p}_{\bar{\nu}_e}\right),$$

The propagator of the mediating W^- boson is

$$D_{\rho\sigma}\left(\mathrm{k}\right) = -i\frac{\eta_{\rho\sigma} - \mathrm{k}_\rho \mathrm{k}_\sigma}{\mathrm{k}^2 - M_W^2} \approx \frac{i\eta_{\rho\sigma}}{M_W^2}, \tag{5.8.94}$$

where was used an approximation which follows from the fact that the boson's mass $M_W \simeq$ 80.4 GeV is much greater than that of a muon $m_\mu \simeq 0.106$ GeV. Substitution of the corresponding expressions into the invariant amplitude gives

$$\mathcal{M} = \frac{g^2\eta_{\rho\sigma}}{8M_W^2}\bar{u}\left(\mu, \mathrm{k}_1\right)\gamma^{\rho}\left(1-\gamma^5\right) u\left(\mathrm{p}_{\nu_\mu}\right)\bar{u}\left(e, \mathrm{k}_2\right)\gamma^{\sigma}\left(1-\gamma^5\right) v\left(\mathrm{p}_{\bar{\nu}_e}\right). \tag{5.8.95}$$

Also, the Fermi coupling constant G_F can be used

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{5.8.96}$$

to bring the amplitude to the form

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\bar{u}\left(\mu, \mathrm{k}_1\right)\gamma^{\rho}\left(1-\gamma^5\right) u\left(\mathrm{p}_{\nu_\mu}\right)\bar{u}\left(e, \mathrm{k}_2\right)\gamma_{\rho}\left(1-\gamma^5\right) v\left(\mathrm{p}_{\bar{\nu}_e}\right), \tag{5.8.97}$$

where the gamma matrices were contracted with the metric $\eta_{\rho\sigma}$. The squared amplitude is

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^{\dagger}, \tag{5.8.98}$$

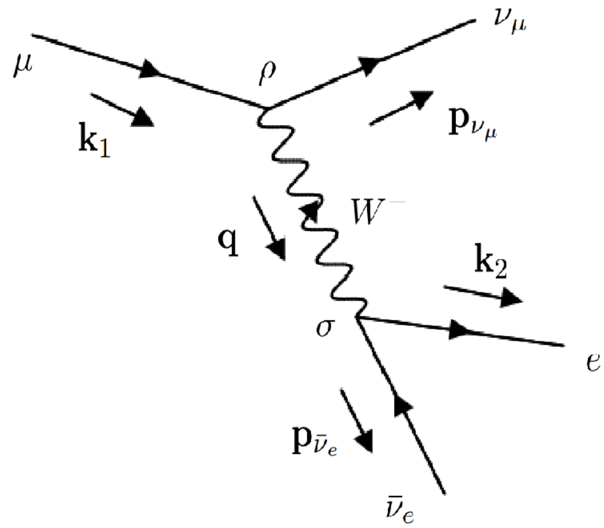

Figure 5.8.1: Feynman diagram for muon decay in the Standard Model.

where the hermitian conjugate is derived to be

$$\begin{aligned}\mathcal{M}^\dagger &= \left[\frac{G_F}{\sqrt{2}}\bar{u}\left(\mu,\mathrm{k}_1\right)\gamma^\rho\left(1-\gamma^5\right)u\left(\mathrm{p}_{\nu_\mu}\right)\bar{u}\left(e,\mathrm{k}_2\right)\gamma_\rho\left(1-\gamma^5\right)v\left(\mathrm{p}_{\bar{\nu}_e}\right)\right]^\dagger \\ &= \frac{G_F}{\sqrt{2}}v^\dagger\left(\mathrm{p}_{\bar{\nu}_e}\right)\left(1-\left(\gamma^5\right)^\dagger\right)\gamma_\rho^\dagger\gamma^0u^\dagger\left(e,\mathrm{k}_2\right)u^\dagger\left(\mathrm{p}_{\nu_\mu}\right)\left(1-\left(\gamma^5\right)^\dagger\right)\left(\gamma^\rho\right)^\dagger\gamma^0u\left(\mu,\mathrm{k}_1\right). \quad (5.8.99)\end{aligned}$$

Using the following properties of gamma matrices

$$\left(\gamma^5\right)^\dagger=\gamma^5,\ \left\{\gamma^5,\gamma^0\right\}=\gamma^5\gamma^0+\gamma^0\gamma^5=0,\ \left(\gamma^\lambda\right)^\dagger\gamma^0=\gamma^0\gamma^\lambda, \quad (5.8.100)$$

brings the invariant amplitude to the form

$$\mathcal{M}^\dagger=\frac{G_F}{\sqrt{2}}\bar{v}\left(\mathrm{p}_{\bar{\nu}_e}\right)\left(1+\gamma^5\right)\gamma_\rho u\left(e,\mathrm{k}_2\right)\bar{u}\left(\mathrm{p}_{\nu_\mu}\right)\left(1+\gamma^5\right)\gamma^\rho u\left(\mu,\mathrm{k}_1\right). \quad (5.8.101)$$

Next, the squared amplitude should be averaged over the initial states and summed over the final ones:

$$\begin{aligned}\overline{|\mathcal{M}|^2} = \frac{G_F^2}{4}\sum_{\text{spins}}&\bar{v}\left(\mathrm{p}_{\bar{\nu}_e}\right)\left(1+\gamma^5\right)\gamma_\rho u\left(e,\mathrm{k}_2\right)\bar{u}\left(\mathrm{p}_{\nu_\mu}\right)\left(1+\gamma^5\right)\gamma^\rho u\left(\mu,\mathrm{k}_1\right) \\ &\times\bar{u}\left(\mu,\mathrm{k}_1\right)\gamma^\sigma\left(1-\gamma^5\right)u\left(\mathrm{p}_{\nu_\mu}\right)\bar{u}\left(e,\mathrm{k}_2\right)\gamma_\sigma\left(1-\gamma^5\right)v\left(\mathrm{p}_{\bar{\nu}_e}\right). \quad (5.8.102)\end{aligned}$$

The inner part of the last expression can be considered separately:

$$
\begin{aligned}
&\sum_{\text{spins}} \bar{u}\left(\mathrm{p}_{\nu_\mu}\right)\left(1+\gamma^5\right)\gamma^\rho u\left(\mu,\mathrm{k}_1\right)\bar{u}\left(\mu,\mathrm{k}_1\right)\gamma^\sigma\left(1-\gamma^5\right)u\left(\mathrm{p}_{\nu_\mu}\right)\\
&=\sum_{\text{spins}} u\left(\mathrm{p}_{\nu_\mu}\right)\bar{u}\left(\mathrm{p}_{\nu_\mu}\right)\left(1+\gamma^5\right)\gamma^\rho u\left(\mu,\mathrm{k}_1\right)\bar{u}\left(\mu,\mathrm{k}_1\right)\gamma^\sigma\left(1-\gamma^5\right)\\
&=\mathrm{Tr}\left[\not{p}_{\nu_\mu}\left(1+\gamma^5\right)\gamma^\rho\left(\not{k}_1+m_\mu\right)\gamma^\sigma\left(1-\gamma^5\right)\right], \qquad (5.8.103)
\end{aligned}
$$

where was used the completeness relation for Dirac spinors

$$\sum_{\text{spins}} u\left(\mathrm{p}\right)\bar{u}\left(\mathrm{p}\right)=\not{p}+m. \qquad (5.8.104)$$

Since the trace is just a number it can be taken out from the middle of the expression (5.8.102):

$$
\begin{aligned}
\overline{|\mathcal{M}|^2}=&\frac{G_F^2}{4}\mathrm{Tr}\left[\not{p}_{\nu_\mu}\left(1+\gamma^5\right)\gamma^\rho\left(\not{k}_1+m_\mu\right)\gamma^\sigma\left(1-\gamma^5\right)\right]\\
&\times\sum_{\text{spins}}\bar{v}\left(\mathrm{p}_{\bar{\nu}_e}\right)\left(1+\gamma^5\right)\gamma_\rho u\left(e,\mathrm{k}_2\right)\bar{u}\left(e,\mathrm{k}_2\right)\gamma_\sigma\left(1-\gamma^5\right)v\left(\mathrm{p}_{\bar{\nu}_e}\right), \qquad (5.8.105)
\end{aligned}
$$

and the same method can be applied to the sum that is left

$$
\begin{aligned}
\overline{|\mathcal{M}|^2}=&\frac{G_F^2}{4}\mathrm{Tr}\left[\not{p}_{\nu_\mu}\left(1+\gamma^5\right)\gamma^\rho\left(\not{k}_1+m_\mu\right)\gamma^\sigma\left(1-\gamma^5\right)\right]\\
&\times\mathrm{Tr}\left[\not{p}_{\bar{\nu}_e}\left(1+\gamma^5\right)\gamma_\rho\left(\not{k}_2+m_e\right)\gamma_\sigma\left(1-\gamma^5\right)\right]. \qquad (5.8.106)
\end{aligned}
$$

The terms in the traces proportional to the masses disappear since they contain the odd number of gamma matrices. Thus, the last expression simplifies to

$$
\begin{aligned}
\overline{|\mathcal{M}|^2}=&\frac{G_F^2}{4}\mathrm{Tr}\left[\not{p}_{\nu_\mu}\gamma^\rho\left(1-\gamma^5\right)\not{k}_1\gamma^\sigma\left(1-\gamma^5\right)\right]\\
&\times\mathrm{Tr}\left[\not{p}_{\bar{\nu}_e}\gamma_\rho\left(1-\gamma^5\right)\not{k}_2\gamma_\sigma\left(1-\gamma^5\right)\right]. \qquad (5.8.107)
\end{aligned}
$$

where was used the fact that $\left(1+\gamma^5\right)\gamma^\lambda=\gamma^\lambda\left(1-\gamma^5\right)$. The rules from [19] p. 263 for evaluation of such traces give

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{G_F^2}{4}\left[2\mathrm{Tr}\left(\gamma^\rho \not{k}_1 \gamma^\sigma \not{p}_{\nu_\mu}\right) + 8i\varepsilon^{\rho\alpha\sigma\beta}\left(\mathrm{p}_{\nu_\mu}\right)_\alpha (\mathrm{k}_1)_\beta\right] \\ &\times \left[2\mathrm{Tr}\left(\gamma_\rho \not{k}_2 \gamma_\sigma \not{p}_{\bar{\nu}_e}\right) + 8i\varepsilon_{\rho\gamma\sigma\delta}\mathrm{k}_2^\gamma \mathrm{p}_{\bar{\nu}_e}^\delta\right] \\ &= 16G_F^2\left[\mathrm{k}_1^\rho \mathrm{p}_{\nu_\mu}^\sigma + \mathrm{k}_1^\sigma \mathrm{p}_{\nu_\mu}^\rho - \left(\mathrm{k}_1 \cdot \mathrm{p}_{\nu_\mu}\right)\eta^{\rho\sigma} + i\varepsilon^{\rho\alpha\sigma\beta}\left(\mathrm{p}_{\nu_\mu}\right)_\alpha (\mathrm{k}_1)_\beta\right] \\ &\times \left[(\mathrm{k}_2)_\rho \left(\mathrm{p}_{\bar{\nu}_e}\right)_\sigma + (\mathrm{k}_2)_\sigma \left(\mathrm{p}_{\bar{\nu}_e}\right)_\rho - \left(\mathrm{k}_2 \cdot \mathrm{p}_{\nu_\mu}\right)\eta_{\rho\sigma} + i\varepsilon_{\rho\gamma\sigma\delta}\mathrm{k}_2^\gamma \mathrm{p}_{\bar{\nu}_e}^\delta\right]. \qquad (5.8.108)\end{aligned}$$

Both expressions in the square brackets in two last lines contain three terms of symmetric tensors and the forth antisymmetric term. Thus, noting that the product of an antisymmetric and symmetric tensor yields zero, the invariant amplitude becomes

$$\begin{aligned}\overline{|\mathcal{M}|^2} = 16G_F^2 &\left\{2\left[(\mathrm{k}_1 \cdot \mathrm{k}_2)\left(\mathrm{p}_{\nu_\mu} \cdot \mathrm{p}_{\bar{\nu}_e}\right) + \left(\mathrm{k}_1 \cdot \mathrm{p}_{\bar{\nu}_e}\right)\left(\mathrm{k}_2 \cdot \mathrm{p}_{\nu_\mu}\right)\right]\right. \\ &\left. -\varepsilon^{\rho\sigma\alpha\beta}\varepsilon_{\rho\sigma\gamma\delta}\left(\mathrm{p}_{\nu_\mu}\right)_\alpha (\mathrm{k}_1)_\beta \mathrm{k}_2^\gamma \mathrm{p}_{\bar{\nu}_e}^\delta\right\}. \qquad (5.8.109)\end{aligned}$$

Applying the following identity

$$\varepsilon^{\rho\sigma\alpha\beta}\varepsilon_{\rho\sigma\gamma\delta} = -2\delta_\gamma^\alpha \delta_\delta^\beta + 2\delta_\delta^\alpha \delta_\gamma^\beta, \qquad (5.8.110)$$

and putting together all the terms finally gives

$$\overline{|\mathcal{M}|^2} = 64G_F^2\left(\mathrm{k}_1 \cdot \mathrm{p}_{\bar{\nu}_e}\right)\left(\mathrm{k}_2 \cdot \mathrm{p}_{\nu_\mu}\right). \qquad (5.8.111)$$

The general formula for the differential decay rate is given in [13] p. 107 in Eq. (4.83). According to it the differential rate for the free muon decay is

$$d\Gamma = \frac{1}{2m_\mu}\overline{|\mathcal{M}|^2}d\Phi, \qquad (5.8.112)$$

where $d\Phi$ is the phase space of the outgoing particles

$$d\Phi = (2\pi)^4\delta^{(4)}\left(\mathrm{k}_1 - \mathrm{k}_2 - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right)\frac{d^3\mathbf{k}_2}{(2\pi)^3 2E_e}\frac{d^3\mathbf{p}_{\bar{\nu}_e}}{(2\pi)^3 2E_{\bar{\nu}_e}}\frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}. \qquad (5.8.113)$$

Next, the integration over the neutrino momenta is easily performed

$$\begin{aligned}I^{\alpha\beta} &= \int \frac{d^3\mathbf{p}_{\nu_e}}{(2\pi)^3 2E_{\bar{\nu}_e}}\frac{d^3\mathbf{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}(2\pi)^4\delta^{(4)}\left(\mathrm{q} - \mathrm{p}_{\nu_\mu} - \mathrm{p}_{\bar{\nu}_e}\right)\mathrm{p}_{\bar{\nu}_e}^\alpha \mathrm{p}_{\nu_\mu}^\beta \\ &= \frac{1}{12}\frac{1}{8\pi}\left(\mathrm{q}^2\eta^{\alpha\beta} + 2\mathrm{q}^\alpha \mathrm{q}^\beta\right), \qquad (5.8.114)\end{aligned}$$

since it is of the same form as in Eq. (3.4.71).

Since it it is assumed that the electron is massless $|\mathbf{k}_2| = E_e$ and then the differential decay rate is

$$d\Gamma = \frac{2G_F^2}{3m_\mu\,(2\pi)^3}\left[(\mathrm{k}_1\cdot\mathrm{k}_2)\,\mathrm{q}^2 + 2(\mathrm{k}_1\cdot\mathrm{q})(\mathrm{k}_2\cdot\mathrm{q})\right]E_e dE_e. \tag{5.8.115}$$

In the rest frame of the muon $\mathrm{k}_1 = (m_\mu, \mathbf{0})$, the kinematic relations are

$$(\mathrm{k}_1\cdot\mathrm{k}_2)\,\mathrm{q}^2 = m_\mu E_e\left(m_\mu^2 - 2m_\mu E_e\right), \tag{5.8.116}$$

$$(\mathrm{k}_1\cdot\mathrm{q})(\mathrm{k}_2\cdot\mathrm{q}) = \left(\mathrm{k}_1^2 - \mathrm{k}_1\cdot\mathrm{k}_2\right)\left(\mathrm{k}_1\cdot\mathrm{k}_2 - \mathrm{k}_2^2\right) = \left(m_\mu^2 - m_\mu E_e\right)m_\mu E_e, \tag{5.8.117}$$

$$d\Gamma = \frac{G_F^2 m_\mu}{12\pi^3}\left[3m_\mu - 4E_e\right]E_e^2 dE_e. \tag{5.8.118}$$

Next, consider the energy of the muon neutrino

$$E_{\nu_\mu} = \left|\mathbf{p}_{\nu_\mu}\right| = \left|\mathbf{p}_{\bar{\nu}_e} + \mathbf{k}_2\right| = \sqrt{E_{\bar{\nu}_e}^2 + E_e^2 + 2E_{\bar{\nu}_e}E_e\cos\theta}. \tag{5.8.119}$$

It follows, that depending on values of the angle θ the muon neutrino's energy is in the following range

$$|E_{\bar{\nu}_e} - E_e| \le E_{\nu_\mu} \le E_{\bar{\nu}_e} + E_e, \tag{5.8.120}$$

or, after applying the conservation of energy

$$|E_{\bar{\nu}_e} - E_e| \le m_\mu - E_e - E_{\bar{\nu}_e} \le E_{\bar{\nu}_e} + E_e \Rightarrow E_e \le \frac{m_\mu}{2}. \tag{5.8.121}$$

The last expression gives the limits for integration over E_e in Eq. (5.8.118). After performing the integration, the free muon decay rate is obtained:

$$\Gamma_0 = \frac{g^2 m_\mu^5}{6144\pi^3 M_W^4} = \frac{G_F^2 m_\mu^5}{192\pi^3}. \tag{5.8.122}$$

Appendix D: Documentation of Fortran Code for Bound Muon Decay

When measuring properties of atoms, it is important to take into account effects of quantum electrodynamics (QED). Of those effects the most important one is the radiative self-energy. For hydrogen-like systems, the level shift scales like Z^4 which makes it significant for high-Z ions. For $\alpha Z \ll 1$ this effect has been usually calculated as a series expansion of αZ. But for $Z > 10$ this method is no longer valid since the terms of higher order are not sufficiently small any more. Hence, αZ should not be treated as a perturbative expansion parameter any more.

Basic Formalism

The following section briefly summarizes results presented in ref. [15]. The energy shift of a particle in a bound state a to the first order in α is:

$$\triangle E_a = 2i\alpha \int\limits_{-\infty}^{\infty} d\omega \int d\mathbf{r}' d\mathbf{r} \Phi_a^\dagger(\mathbf{r}')\alpha^\rho G(E-\omega;\mathbf{r}',\mathbf{r}) D_{\rho\sigma}(\omega,\mathbf{r}'-\mathbf{r})\alpha^\sigma \Phi_a(\mathbf{r})$$
$$-\delta m \int d\mathbf{r} \bar{\Phi}_a(\mathbf{r})\Phi_a(\mathbf{r}), \qquad (5.8.123)$$

where $\Phi_a(\mathbf{r})$ are the bound state solutions of the Dirac equation for the Dirac-Coulomb Hamiltonian (5.8.91) given by Eq. (2.1.10) and δm is a mass counterterm. The Dirac-Coulomb Green's function $G(\omega;\mathbf{r}',\mathbf{r}) = \frac{1}{\omega - \mathcal{H}_{DC}}$ can be expanded as series in the free Green's function $G^0(\omega;\mathbf{r}'-\mathbf{r}) = \frac{1}{\omega-\mathcal{H}}$ [1] using the identity first introduced in [17].

$$\frac{1}{\omega-\mathcal{H}_{DC}} = \frac{1}{\omega-\mathcal{H}_0} + \frac{1}{\omega-\mathcal{H}_0}V(\mathbf{r})\frac{1}{\omega-\mathcal{H}_0} + \frac{1}{\omega-\mathcal{H}_0}V(\mathbf{r})\frac{1}{\omega-\mathcal{H}}V(\mathbf{r})\frac{1}{\omega-\mathcal{H}_0}. \qquad (5.8.124)$$

Here $\mathcal{H}_0$ is the free Dirac Hamiltonian defined in (5.8.56). Inserting expansion (5.8.124) into (5.8.123) we get the three more terms which we label to be zero-potential, one-potential, and many-potential terms respectively:

$$\triangle E_a^{(1)} = \triangle E_{\text{zero}} + \triangle E_{\text{one}} + \triangle E_{\text{many}} - \delta m \int d\mathbf{r}\bar{\Phi}_a(\mathbf{r})\Phi_a(\mathbf{r}). \qquad (5.8.125)$$

[1] Since it does not contain the interaction with the nucleus it has the translation symmetry.

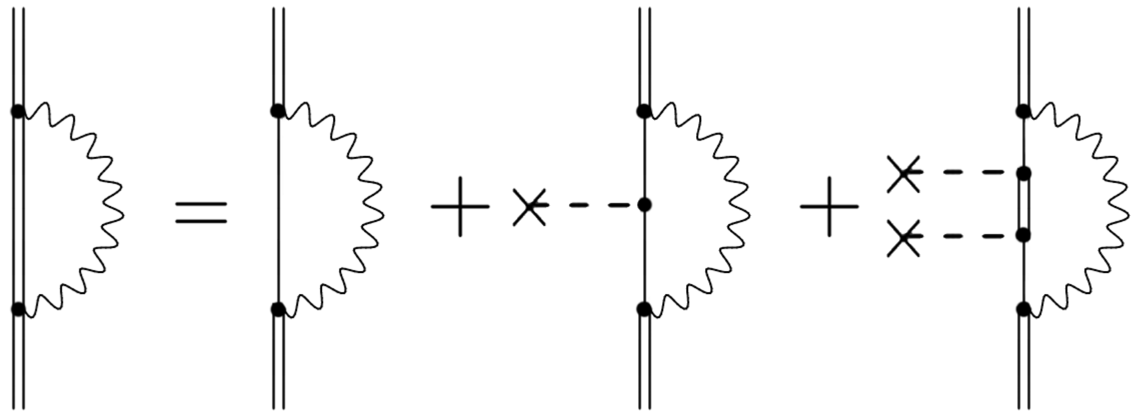

Figure 5.8.2: The self-energy shift expanded as the sum of zero-, one- and many potential terms, respectively. The double line corresponds to the bound propagator and dotted ones to interactions with the nucleous.

Figure (5.8.2) depicted the Feynman diagrams corresponding to this expansion. The expression for the zero-potential term follows immediately from (5.8.123) by substituting the free Green's function in place of a bound one.

$$\triangle E_{\text{zero}} = 2i\alpha \int\limits_{-\infty}^{\infty} d\omega \int d\mathbf{r}' d\mathbf{r} \Phi_a^\dagger(\mathbf{r}')\alpha^\rho G^0(E-\omega; \mathbf{r}'-\mathbf{r})\alpha^\sigma D_{\rho\sigma}(\omega; \mathbf{r}'-\mathbf{r})\Phi_a(\mathbf{r}). \qquad (5.8.126)$$

The one-potential term corresponds to one interaction with the nucleus, so the expression for it in terms of the free Green's functions is

$$\triangle E_{\text{one}} = 2i\alpha \int\limits_{-\infty}^{\infty} d\omega \int d\mathbf{r}'' d\mathbf{r}' d\mathbf{r} \Phi_a^\dagger(\mathbf{r}')\,\alpha^\rho G^0(E_a-\omega, \mathbf{r}'-\mathbf{r}'')$$
$$\times V(|\mathbf{r}''|)\, G^0(E_a-\omega, \mathbf{r}''-\mathbf{r})\alpha^\sigma D_{\rho\sigma}(\omega; \mathbf{r}'-\mathbf{r})\,\Phi_a(\mathbf{r})\,. \qquad (5.8.127)$$

It is convenient to convert the zero-potential and one-potential terms into the momentum space and the energy shift (5.8.123) becomes

$$\triangle E_a^{(1)} = \int \frac{d\mathbf{p}}{(2\pi)^3} \bar{\Phi}_a(\mathbf{p}) \left(\Sigma^0(\mathrm{p}) - \delta m\right) \Phi_a(\mathbf{p})$$
$$+ \int \frac{d\mathbf{p}'}{(2\pi)^3} \frac{d\mathbf{p}}{(2\pi)^3} \bar{\Phi}_a(\mathbf{p}') \Gamma^0(\mathrm{p}',\mathrm{p}) V(|\mathbf{p}-\mathbf{p}'|) \Phi_a(\mathbf{p}) + \triangle E_{\mathrm{many}}, \tag{5.8.128}$$

where

$$\Sigma^{(0)}(\mathrm{p}) = -4\pi i\alpha \int \frac{d^4\mathrm{k}}{(2\pi)^4} \frac{1}{\mathrm{k}^2} \gamma_\sigma \frac{\not{p} - \not{k} + m}{(\mathrm{p}-\mathrm{k})^2 - m^2} \gamma^\sigma \tag{5.8.129}$$

is the self-energy correction, and

$$\Gamma^\mu(\mathrm{p}',\mathrm{p}) = -4\pi i\alpha \int \frac{d^4\mathrm{k}}{(2\pi)^4} \frac{1}{\mathrm{k}^2} \gamma_\sigma \frac{\not{p}' - \not{k} + m}{(\mathrm{p}'-\mathrm{k})^2 - m^2} \gamma^\mu \frac{\not{p} - \not{k} + m}{(\mathrm{p}-\mathrm{k})^2 - m^2} \gamma_\sigma \tag{5.8.130}$$

is the vertex correction,

$$\triangle E_{\mathrm{many}} = 2i\alpha \int d\omega \int d\mathbf{r}' d\mathbf{r} \Phi_a^\dagger(\mathbf{r}') \alpha^\rho G^{2+}(E-\omega; \mathbf{r}'-\mathbf{r}) \alpha^\sigma D_{\rho\sigma}(\omega; \mathbf{r}'-\mathbf{r}) \Phi_a(\mathbf{r}). \tag{5.8.131}$$

Both zero-potential and one-potential term should be dimensionaly regularized. These energy shifts are solved by using the program, which functioning is described in the following section.

General Structure of the Program

This program is developed by A. Volotka. The purpose of this section is to explain its main functional. The general structure of the program is shown in the block-diagram (5.8.3) and the different parts of the program are explained below.

Gaussian Quadrature and B-Splines

The directory **bsplines** contains files in which solutions of the Dirac equation in a cavity of radius **xcav** are approximated with piecewise polynomials. For that purpose file **de_boor.f** contains a subroutine **BSPLV** which generates values of the B-splines $B_{i,k}(x)$ of order k at x for the knot sequence $\{t_i\}$, $i = 1, 2, \ldots$.

$$B_{i,1}(x) = \begin{cases} 1, \text{ if } t_i \le x < t_{i+1} \\ 0, \quad \text{otherwise} \end{cases}, \tag{5.8.132}$$

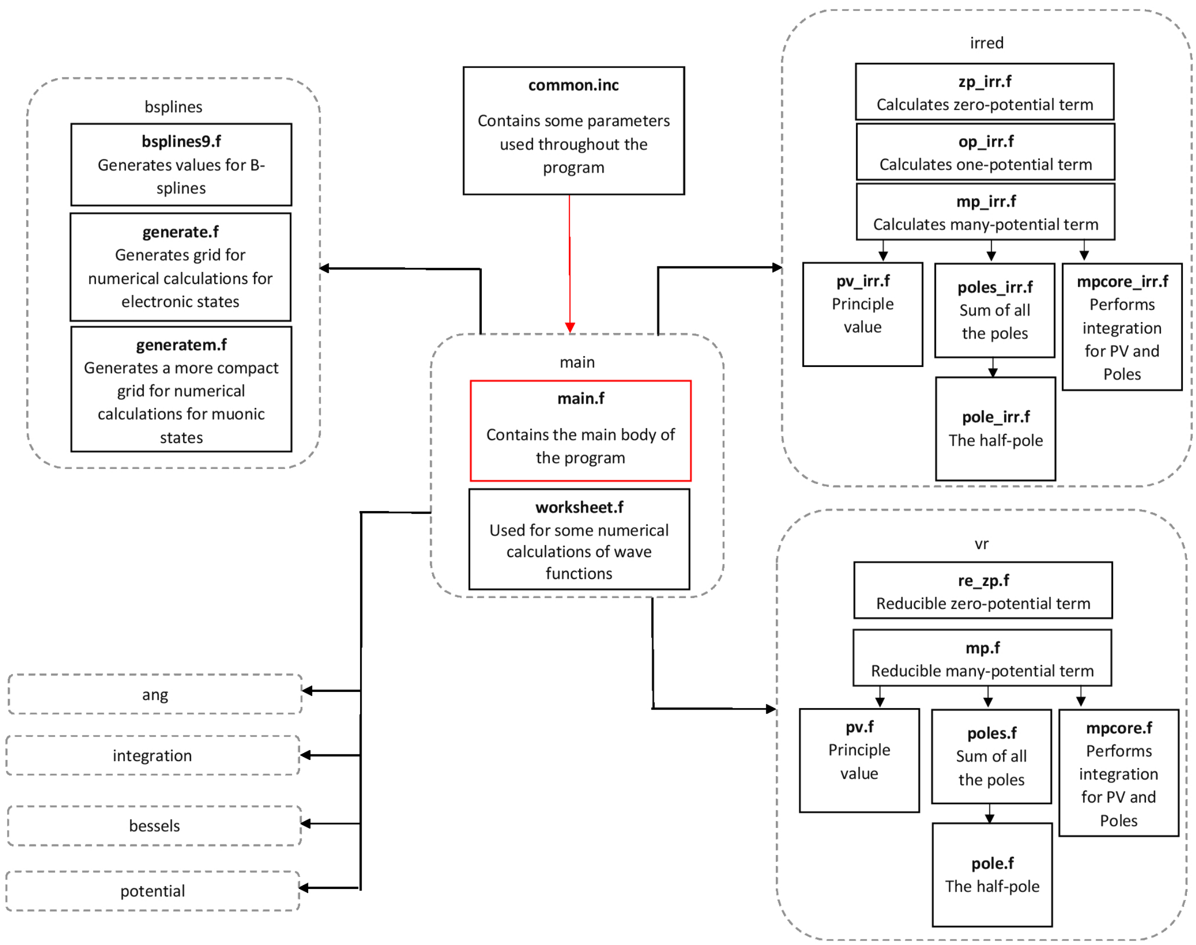

Figure 5.8.3: Directories and files contained in them.

$$B_{i,k}(x) = \frac{x - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(x), \tag{5.8.133}$$

where $k = 2, 3...$.

The file **bsplines9.f** stores grids for integration for given parameter **na** which is the number of internal knots of the grid. The files **generate.f** and **generatem.f** both generate the necessary basis sets with the difference that the first one does it for electronic states and the second one - for muonic. As a result, whenever the integration is done with the help of the Gaussian Quadrature two choices of grid are possible: **grid** - a more sparse one or **gridm** - a denser one, but defined over a smaller region. The latter grid is needed to be introduced due to the following fact: the wave function of a muon is localized over smaller position space than that of an electron, therefore, the grid fit for electronic wave function does not have enough knots to make integration as precise. The relative difference in the intervals between the consecutive knot of these two grids can be described be the relation: $\mathbf{grid}(i) - \mathbf{grid}(i-1) = \mathbf{amass\_a}[\mathbf{gridm}(i) - \mathbf{gridm}(i-1)]$, where **amass_a** is the muon mass.

The Gaussian Quadrature rules which are used by the splines later for integration:

$$\int_0^\infty f(r)\,dr \to \int_0^{r_{\max}} f(r)\,dr \approx \sum_{ii=0}^{n_a} \int_{r_{ii}}^{r_{ii+1}} f(r)\,dr, \tag{5.8.134}$$

then in the last integral the following substitution is made $r = r_{ii} + (r_{ii+1} - r_{ii})\,x$:

$$\sum_{ii=0}^{n_a} \int_{r_{ii}}^{r_{ii+1}} f(r)\,dr \to \sum_{ii=0}^{n_a} \int_0^1 f(x)\,dx \approx \sum_{ii=0}^{n_a} \sum_{jj} \omega_{jj} f(x_{jj}). \tag{5.8.135}$$

Here ω_{jj} are the weights which are stored in the program in $\mathbf{cc(jj, iNstor/iNstme)}$, the values of x_{jj} are stored in $\mathbf{xx(jj, iNstor/iNstme)}$ and $\mathbf{jj = 1, Nstor(iNstme)}$. Both $\mathbf{cc(jj, iNstor/iNstme)}$ and $\mathbf{xx(jj, iNstor/iNstme)}$ are generated by the subroutine **D01BAZ** contained in the file with the same name in the directory **intagration**. And values for both **iNstor** and **iNstme** are stored in the **Block Data** at the end of the file **main.f** in the directory **main**. Variables r_{ii} and r_{ii+1} are given in the program by $\mathbf{r0 = grid(ii)}$ and $\mathbf{r1 = grid(ii+1)}$, where **ii** runs from **0** to **na**. The parameter $\mathbf{xcav = 1/(amas\_a/b)}$ defines the extent of the cavity over which the grid for the numerical integration is stretched. It is also defined in **main.f** and **amas_a/b** correspond to the muon/electron mass, respectively.

The file **generate.f** contains the following useful functions:

1. **getgf**(gg, ff, ix, kax, i_ab, ib, r) gives $g(r)$ and $f(r)$ as the output **gg** and **ff**, re-

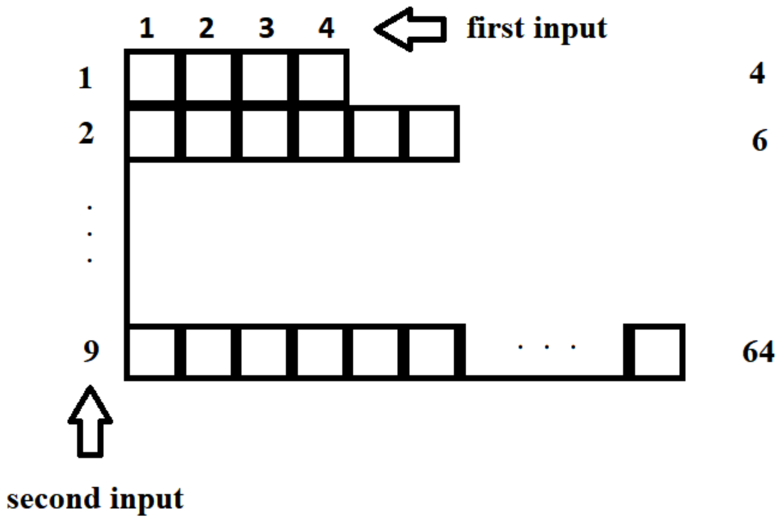

Figure 5.8.4: The structure of the stored data for the Gaussian Quadrature integration.

spectively. The input parameters: **r** is a variable defining position, **kax** is κ_n given in (2.1.47), **ix** is the number in the spline basis **nn_a/b** which is given by **nr2nn**,

2.
$$\text{i\_ab} = \begin{cases} 0, & \text{for muon} \\ 1, & \text{for electron} \end{cases},$$

3.
$$\text{ib} = \begin{cases} 0, & \text{free state} \\ 1, & \text{bound state} \\ 2, & \text{perturbed state} \end{cases}.$$

4. **getenn**(ix, kax, i_ab,ib). Gives the energy of the state corresponding to the parameter **ix**, which is the principle quantum number n.

5. **getern**(nr, kax, i_ab). Gives the energy of the state corresponding to the parameter n_r, which is the radial quantum number.

The file **generatem.f** contains the functions **getgfm**, **getennm**, **getternm** which do the same thing as the functions listed above only on the muonic grid **gridm**.

The file **bsplines9.f** contains a function **nr2nn**(nr, kax) which generates principal quantum numbers from the radial quantum number n_r and the number κ_n.

Calculation of Self-Energy Corrections

The self-energy correction amplitude $\mathcal{A}$ can be written as the sum of the the zeroth-order

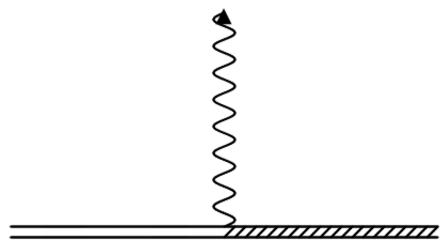

Figure 5.8.5: The zeroth order correction amplitude $\mathcal{A}^{(0)}$.

and first-order correction amplitudes:

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)}, \tag{5.8.136}$$

where $\mathcal{A}^{(1)} = \mathcal{A}^{(1,\text{irred})} + \mathcal{A}^{(1,\text{red})} + \mathcal{A}^{(1,\text{vertex})}$ is the sum of irreducible, reducible and vertex contributions. The expression (5.8.136) can be rewritten:

$$\mathcal{A} = \mathcal{A}^{(0)} \left(1 + \frac{\alpha}{\pi}\delta^{(1)}\right), \tag{5.8.137}$$

where $\delta^{(1)} = \frac{\mathcal{A}^{(1)}}{\mathcal{A}^{(0)}}\frac{\pi}{\alpha}$. The program calculates numerical values for these δ's in the modules **irred** and **vr**.

The Zeroth Order Correction

The amplitude for the bound muon decay (5.8.5) into the bound electron to the zeroth order:

$$\mathcal{A}^{(0)} = \langle e \left\| \delta V_l \right\| \mu \rangle = (-1)^l \int dr\, j_l\left(\omega r\right) \left[G_e G_\mu G_l\left(\kappa_e, \kappa_\mu\right) + F_e F_\mu G_l\left(-\kappa_e, -\kappa_\mu\right)\right], \tag{5.8.138}$$

where l is the orbital quantum number, j_l is the Spherical Bessel Function, $\omega = E_\mu - E_e$ is the difference between the muon's and electron's energies, respectively, δV_l is the interaction with the nucleus

$$\delta V_l = j_l\left(\omega r\right) \sqrt{\frac{4\pi}{2l+1}} Y_l^0\left(\hat{\mathbf{r}}\right), \tag{5.8.139}$$

where $\langle e \left\| \delta V_l \right\| \mu \rangle$ is a reduced matrix element, which doesn't depend on the projections of angular quantum numbers and follows from the Wigner-Eckart theorem:

$$\langle e \left\| \delta V_l \right\| \mu \rangle = \left[(-1)^{j_e - m_e} \begin{pmatrix} j_e & l & j_\mu \\ -m_e & m & m_\mu \end{pmatrix}\right]^{-1} \langle e \left| \delta V_l \right| \mu \rangle, \tag{5.8.140}$$

Figure 5.8.6: Irreducible correction amplitudes. The double line corresponds to the bound state, the dashed part of the diagram is the muonic state.

where functions $G_{\mu/e}$ and $F_{\mu/e}$ are defined in Eq. (2.2.2), and the function $G_l\left(\kappa_1,\kappa_2\right)$ is

$$G_l\left(\kappa_1,\kappa_2\right) = (-1)^{j_2+\frac{1}{2}}\sqrt{\left(2j_1+1\right)\left(2j_2+1\right)\left(2l_1+1\right)\left(2l_2+1\right)} \times \begin{pmatrix} l_1 & l & l_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} j_1 & l & j_2 \\ l_2 & \frac{1}{2} & l_1 \end{Bmatrix}. \qquad (5.8.141)$$

This quantity is calculated in the **main**.**f** in the module Matrix element: <b | | j_l(wr) sqrt(4\pi/(2l+1)) Y_l | | a>. Later this value is assigned to the variable **pvalue_red** for calculations of $\delta^{(1)}$.

Irreducible Parts

$$\mathcal{A}^{(1,\text{irred})} = \left\{\sum_{n\neq n_e}\frac{\left\langle e\left|\Sigma_e\left(E_e\right)\right|n\right\rangle\left\langle n\left|\delta V_l\right|\mu\right\rangle}{E_e-E_n} + \sum_{n\neq n_\mu}\frac{\left\langle e\left|\delta V_l\right|n\right\rangle\left\langle n\left|\Sigma_\mu\left(E_\mu\right)\right|\mu\right\rangle}{E_e-E_n}\right\} \times \left[(-1)^{j_e-\frac{1}{2}}\begin{pmatrix} j_e & l & j_\mu \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}\right]^{-1} \equiv \left\langle e\left\|\Sigma_e\left(\varepsilon_e\right)\right\|\delta\mu\right\rangle + \left\langle \delta e\left\|\Sigma_\mu\left(\varepsilon_\mu\right)\right\|\mu\right\rangle, \qquad (5.8.142)$$

where

$$\Sigma_{e/\mu}\left(E_{e/\mu}\right) = 2\alpha i\int d\omega\gamma_\mu G\left(E_{e/\mu}-\omega\right)\gamma^\mu D^{\mu\nu}\left(\omega\right) \qquad (5.8.143)$$

The expression (5.8.142) gives the irreducible contributions of the sum of the diagrams (5.8.6) (or so-called wave function corrections). In the program they are calculated and located in the directory **irred** and start in the file **main**.**f** in the modules called < b | SIGMA | da > and < db | SIGMA | a >, respectively. To both terms of Eq. (5.8.142) the potential expansion as in (5.8.2) is applied and the calculations of the corresponding zero- and one-potential parts are done in the files **zp_irr**.**f** and **op_irr**.**f** with the only difference in the starting parameters.

Zero-Potential Term

File **zp_irr**.**f** in the directory **irred**.

For < b | SIGMA | da > the amplitude $\mathcal{A}^{(1,\mathrm{irred},1)}$ is given by function **dE_zp_irr** in the program and has the following form

$$\mathcal{A}^{(1,\mathrm{irred},1)} = \frac{\alpha}{4\pi}\int\limits_0^\infty p^2 dp \times \left\{ a(\rho_e)\left(\widetilde{g}_e\widetilde{g}_{\delta\mu} - \widetilde{f}_e\widetilde{f}_{\delta\mu}\right) + b(\rho_e)E_e\left(\widetilde{g}_e\widetilde{g}_{\delta\mu} + \widetilde{f}_e\widetilde{f}_{\delta\mu}\right) + p\left(\widetilde{g}_e\widetilde{f}_{\delta\mu} - \widetilde{f}_e\widetilde{g}_{\delta\mu}\right)\right\}, \tag{5.8.144}$$

where the integral is given by the function **zp_func_irr** (**p**), and

$$a(\rho) = 2m\left(1 + \frac{2\rho}{1-\rho}\ln\rho\right), \tag{5.8.145}$$

$$b\left(\rho\right) = -\frac{2-\rho}{1-\rho}\left(1 + \frac{\rho}{1-\rho}\ln\rho\right), \tag{5.8.146}$$

where $\rho = 1 - \frac{\mathrm{p}^2}{m^2}$.

For the part < db | SIGMA | a >

$$\mathcal{A}^{(1,\mathrm{irred},2)} = \frac{\alpha}{4\pi}\int\limits_0^\infty p^2 dp \times \left\{ a(\rho_\mu)\left(\widetilde{g}_{\delta e}\widetilde{g}_{\mu} - \widetilde{f}_{\delta e}\widetilde{f}_{\mu}\right) + b(\rho_\mu)E_\mu\left(\widetilde{g}_{\delta e}\widetilde{g}_{\mu} + \widetilde{f}_{\delta e}\widetilde{f}_{\mu}\right) + p\left(\widetilde{g}_{\delta e}\widetilde{f}_{\mu} - \widetilde{f}_{\delta e}\widetilde{g}_{\mu}\right)\right\}. \tag{5.8.147}$$

The integration from 0 to ∞ in the momentum space is calculated with the employment of the following substitution of variables and splitting of the interval by $p_0 = \alpha Z m_{a/b}$:

$$\begin{aligned}\int\limits_0^\infty dp\, f\left(p\right) &= \left(\int\limits_0^{p_0}\left[\begin{array}{c} p \to p_0 t^2 \\ dp \to 2p_0 t dt\end{array}\right] + \int\limits_{p_0}^\infty\left[\begin{array}{c} p = \frac{p_0}{t} \\ dp = -\frac{p_0 dt}{t^2}\end{array}\right]\right) dp\, f\left(p\right) \\ &= \left(\int\limits_0^1 2p_0 t + \int\limits_0^1 \frac{p_0}{t^2}\right) dt\, f\left(t\right),\end{aligned} \tag{5.8.148}$$

where f stands for some function.

The radial functions g and f are converted to the momentum space with the use of the

Spherical Bessel Transformation (SBT):

$$\tilde{g}(p) = 4\pi \int_0^\infty dr r^2 g(r) j_l(pr), \tag{5.8.149}$$

$$\tilde{f}(p) = 4\pi \int_0^\infty dr r^2 f(r) j_{\bar{l}}(pr). \tag{5.8.150}$$

In the program this is done with the use of the function **ftr_wf**(p,l,key,$f(r)$) contained in the file **ftr**.**f** in the directory **integration**. It performs SBT $j_l(x)$ of order l for the function $f(x)$ into the momentum space p:

$$\int_0^\infty f(x) j_l(x) dx = \int_0^\infty f(x) [A(x)\sin(\omega x) + B(x)\cos(\omega x)] dx,$$

$$\text{key} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ g & f & g_\delta & f_\delta \end{pmatrix}.$$

The coefficients $A(x)$ and $B(x)$ are defined in the program as **sph_jl_sf** and **sph_jl_cf**, respectively, and contained in the file **sph_j**.**f** in the directory **bessels**. And the function **anf_wf_a**/**b**(r) gives $\frac{G/F}{r} j_l(pr) r^2 \sqrt{\frac{2}{\pi}} p$. They both use the subroutine **d01anf** (f, a, b, ω, KEY, EPSABS, EPSREL, RESULT, ABSERR, 1 W, LW, IW, LIW, IFAIL) which is located in the directory **integration** and it calculates an approximation to the sine or the cosine transform of a function f over the interval $[a, b]$:

KEY=1:

$$\text{RESULT} = \int_a^b f(x)\cos(\omega x) dx,$$

KEY=2:

$$\text{RESULT} = \int_a^b f(x)\sin(\omega x) dx.$$

One-Potential Term

File **op_irr**.**f** in the directory **irred**.

For $< \mathrm{b} \mid \mathrm{SIGMA} \mid \mathrm{da} >$

$$\mathcal{A}^{(1,\mathrm{irred},3)} = \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^3} \bar{\Phi}_e(\mathbf{p}_1)\,\Gamma^0_R(\mathrm{p}_1,\mathrm{p}_2) V(|\mathbf{p}_1 - \mathbf{p}_2|)\,\Phi_{\delta\mu}(\mathbf{p}_2)$$
$$= \frac{\alpha^2}{2(2\pi)^3} \int_0^\infty \int_0^\infty dp_1 dp_2 \int_{-1}^{1} d\xi\, \frac{p_1^2 p_2^2}{q^2} \left\{ \mathcal{F}_1(p_1,p_2,\xi)\, P_l(\xi) + \mathcal{F}_2(p_1,p_2,\xi)\, P_{\bar{l}}(\xi) \right\}, \tag{5.8.151}$$

and for the part $< \mathrm{db} \mid \mathrm{SIGMA} \mid \mathrm{a} >$

$$\mathcal{A}^{(1,\mathrm{irred},4)} = \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \bar{\Phi}_{\delta e}(\mathbf{p}_1)\,\Gamma^0(\mathrm{p}_1,\mathrm{p}_2) V(|\mathbf{p}_1 - \mathbf{p}_2|)\,\Phi_\mu(\mathbf{p}_2), \tag{5.8.152}$$

which are given by

$$\text{dE\_op\_irr} = \frac{\alpha^2}{2(2\pi)^3} \text{op\_int\_irr}\,().$$

The program gives results in the form

$$\delta^{(1,1)} = \frac{\mathcal{A}^{(1,\mathrm{irred})}}{\mathcal{A}^{(0)}} \frac{\pi}{\alpha} = \frac{\text{dE\_op\_irr}}{\text{pvalue\_red}} \times \text{pi} \times \text{acl}$$

To perform the double integration in the momentum space the following change of variables is made:

$$\left\{ \begin{array}{l} p_1 \equiv \frac{1}{\sqrt{2}}(p+r) \\ p_2 \equiv \frac{1}{\sqrt{2}}(p-r) \end{array} \right. \Rightarrow \frac{\partial(p_1,p_2)}{\partial(p,r)} = 1 \Rightarrow \int_0^\infty \int_0^\infty dp_1 dp_2 = \int_0^\infty dp \int_{-p}^{p} dr,$$

and

$$\int_{-p}^{p} f(r)\, dr = \int_0^p [f(-r) + f(r)]\, dr.$$

For the integration in the momentum space p the substitution of variables is done according to (5.8.148):

$$\text{op\_int\_irr}\,() = \left(\int_0^{p_0} + \int_{p_0}^{\infty} \right) dp\, \text{op\_p\_irr}\,(),$$

where $p_0 = 2\alpha Z m_{a/b}$ and

$$\text{op\_int\_irr}\,() = \int_0^p [\text{op\_pr\_irr}\,(r) + \text{op\_pr\_irr}\,(-r)]\, dr,$$

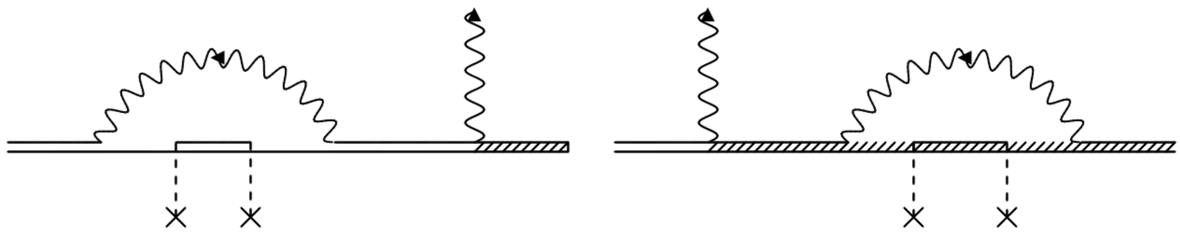

Figure 5.8.7: Irreducible correction amplitudes for many potential terms.

where the substitution $r = pt^2$, $dr = 2ptdt$ is done. Now

$$\text{op\_pr\_irr}\,() = \text{op\_pp\_irr}\,() \times p_1 \times p_2$$

$$\text{op\_pp\_irr}\,() = \int_{-1}^{1} \text{op\_ppz\_irr}\,()\, dz,$$

where $\mathrm{z} = 1 - 2t^2$ is the cosine of the angle ξ in (5.8.151), and

$$\text{op\_ppz\_irr}\,() = \left(\text{F\_1} \times \text{P\_l} + \text{F\_2} \times \text{P\_}\bar{\text{l}}\right) \times \text{V\_p1\_p2}\,()$$

with P_l being the Legendre polynomials defined up to $l = 2$, and V_p1_p2 () is the potential contained in the file **potential.f**.

Many-Potential Term

Files **mp_irr.f**, **mpcore_irr.f**, **pv_irr.f**, **poles_irr.f**, **pole_irr.f**, in the directory **irred**.

For < b | SIGMA | da >

$$\mathcal{A}^{(1,\text{irred},\text{mp},1)} = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_{\alpha_e,\beta_e,i_e} \frac{\langle e\beta_e \left| I(\omega) \right| \alpha_e \delta\mu \rangle \langle \alpha_e \left| V \right| i_e \rangle \langle i_e \left| V \right| \beta_e \rangle}{(E_e - \omega - E_{\beta_e})(E_e - \omega - E_{\alpha_e})(E_e - \omega - E_{i_e})} \tag{5.8.153}$$

$$\mathcal{A}^{(1,\text{irred},\text{mp},2)} = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_{\alpha_\mu,\beta_\mu,i_\mu} \frac{\langle \delta e\beta_\mu \left| I(\omega) \right| \alpha_\mu \mu \rangle \langle \alpha_\mu \left| V \right| i_\mu \rangle \langle i_\mu \left| V \right| \beta_\mu \rangle}{\left(E_\mu - \omega - E_{\beta_\mu}\right)\left(E_\mu - \omega - E_{\alpha_\mu}\right)\left(E_\mu - \omega - E_{i_\mu}\right)}, \tag{5.8.154}$$

where $I(\omega) = e^2 \alpha^\rho \alpha^\sigma D_{\rho\sigma}(\omega) = \alpha\,(1 - \boldsymbol{\alpha} \cdot \boldsymbol{\alpha})\, \frac{\exp\left(i\sqrt{\omega^2 + i\delta}\right) r_{12}}{r_{12}}$.

After the Wick's rotation each expression can be split into the sum of three terms

$$\mathcal{A}^{(1,\text{irred,mp})} = \mathcal{A}^{(1,\text{irred,PV})} + \frac{1}{2}\mathcal{A}^{(1,\text{irred,pole})} + \sum_{\substack{i_{e/\mu} \\ E_i < E_{e/\mu}}} \mathcal{A}^{(1,\text{irred,poles})}, \tag{5.8.155}$$

which are the principle value, half-pole on the the imaginary axis and sum of the remaining poles.

Important Notes

In the file **main**.**f** after the 424th line, the calculation of the wave functions perturbed by the decay begins. In this case, the perturbed muon wave function (da) corresponds to the wave function of the virtual electron with angular quantum numbers j_b, l_b, and conversely, the perturbed electron wave function (db) corresponds to the virtual wave function of the muon. The spline decomposition coefficients are stored in the array **gfcoef**(-,-,-,nn_b,ka_b,1,2) for da (430-433 lines) and in **gfcoef**(-,-,-,nn_a,ka_a,0,2) for db (443-446 lines). These wave functions are calculated in the momentum space by the function ftr_wf which takes (p, l_a/2, 2, anf_wf_a) as its arguments for the state db. This can be seen if refer to the file **ftr**.**f** where in this case the function **getgf** (gga, ffa, nn_a, ka_a, 0, 2, r) is called, for which the parameters 0 and 2 are the parameters of the function db.

Make File

Here are listed some important inputs and outputs of the program.

1. Directory **main**:
   (a) File **main**.**f**:
      i. subroutine **inp_data** () calls for initial values of some parameters;
      ii. subroutine **rad_init** () calls for generation of the grid and potential values for corresponding grid knots;
      iii. subroutine **ext_init** () checks normalization of wave functions in the coordinate and momentum spaces, calculates the zeroth order of amplitude (5.8.138). It calculates and stores values for perturbed wave functions |da> and <db|;
      iv. Calculates the irreducible parts for < db | SIGMA | a > and < b | SIGMA | da > by calling subroutines **ZP_IRR** (), **OP_IRR** () and **MP_IRR** ();
      v. **Block data** stores vales for **Nstor**, **iNst**, **iNstm** and **iNstv**.

(b) File **worksheet.f**:

i. subroutine **worksheet_1** () calculates energies and wave functions of **a** and **b** states;

ii. subroutine **worksheet_2** () calculates matrix elements $\sqrt{\langle a\,|r^2|\,a\rangle}$, $\langle a\,|[\mathbf{r}\times\bar{\alpha}]_z|\,a\rangle$ and $\langle a\,|G\,(\varepsilon,\mathbf{r_1},\mathbf{r_2})|\,a\rangle$;

iii. subroutine **worksheet_3** () calculates the zeroth order of amplitude (5.8.138), calculates and stores values for perturbed wave functions |da> and <db|.

2. Directory **bsplines**:

(a) File **bsplines9.f**:

i. subroutine **bs_grid** () generates a grid inside the nucleous, exponential and non-exponential grid;

ii. function **nr2nn**(nr, kax) which generates principal quantum numbers from the radial quantum number nr $\equiv n_r$ and the number kax $\equiv \kappa_n$.

(b) File **generate.f**:

i. **getgf**(gg, ff, ix, kax, i_ab, ib, r) gives $g(r)$ and $f(r)$ as the output **gg** and **ff**, respectively. The input parameters: **r** is a variable defining position, **kax** is κ_n given in (2.1.47), **ix** is the number in the spline basis **nn_a/b** which is given by **nr2nn**,

$$\text{i\_ab} = \begin{cases} 0, & \text{for muon} \\ 1, & \text{for electron} \end{cases},$$

$$\text{ib} = \begin{cases} 0, & \text{free state} \\ 1, & \text{bound state} \\ 2, & \text{perturbed state} \end{cases};$$

ii. **getenn**(ix, kax, i_ab,ib). Gives the energy of the state corresponding to the parameter **ix**, which is the principle quantum number n;

iii. **getern**(nr, kax, i_ab). Gives the energy of the state corresponding to the parameter n_r, which is the radial quantum number.

(c) File **generatem.f** (do the same calculations as **generate.f** only for muonic grid):

i. **getgfm**(gg, ff, ix, kax, i_ab, ib, r);

ii. **getennm**(ix, kax, i_ab,ib);

iii. **geternm**(nr, kax, i_ab).

(d) File **de_boor.f** generates values of the B-splines.

3. Directory **irred**:

   (a) File **zp_irred.f** contains subroutine **ZP_IRR**() which calculates self-enegy corrections (5.8.129) for < db | SIGMA | a > and < b | SIGMA | da >;

   (b) File **op_irred.f** contains subroutine **OP_IRR**() which calculates vertex corrections (5.8.130) for < db | SIGMA | a > and < b | SIGMA | da >;

   (c) File **mp_irred.f** contains subroutine **MP_IRR**() which calculates corrections for many-potential interactions (5.8.131) for < db | SIGMA | a > and < b | SIGMA | da >;

   (d) File **pv_irred.f** contains subroutine **PV_IRR**() which calculates the principle value of the integral (5.8.131) and is called in the file **mp_irred.f**;

   (e) File **poles_irred.f** contains subroutine **POLES_IRR**() which calculates the integral (5.8.131) for poles not lying on the imaginary axis after the Wick's rotation and is called in the file **mp_irred.f**;

   (f) File **pole_irred.f** contains subroutine **POLE_IRR**() which calculates the integral (5.8.131) for a half-pole lying on the imaginary axis after the Wick's rotation and is called in the file **mp_irred.f**.

4. Directory **integration**:

   (a) File **ftr.f** calculates Fourier Transforms into momentum space;

   (b) File **d01anf.f** contains subroutine **d01anf** (f, a, b, ω, KEY, EPSABS, EPSREL, RESULT, ABSERR, 1 W, LW, IW, LIW, IFAIL) which calculates an approximation to the sine or the cosine transform of a function.

5. Directory **bessels**:

   (a) File **sph_j.f** contains coefficient for the Spherical Bessel Transformation.