UNIVERSITY OF ALBERTA

OPTIMAL RELIABILITY DESIGN OF SERIES CONSECUTIVE-k-OUT-OF-n:G SYSTEMS

By

JIANKAI SHEN

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE.

IN

ENGINEERING MANAGEMENT

DEPARTMENT OF MECHANICAL ENGINEERING

EDMONTON, ALBERTA

SPRING, 1993

ISBN 0-315-82190-6

UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: JIANKAI SHEN

TITLE OF THESIS: **OPTIMAL RELIABILITY DESIGN OF SERIES CONSECUTIVE-k-OUT-OF-n:G SYSTEMS**

DEGREE: MASTER OF SCIENCE IN ENGINEERING MANAGEMENT

YEAR THIS DEGREE GRANTED: Spring, 1993

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ABSTRACT

Consecutive-k-out-of-n systems have been intensively studied in the last ten years. A complete list of the invariant optimal designs for linear and circular consecutive-k-out-of-n systems is available in Zuo and Kuo [16]. It is a natural extension that consecutive-k-out-of-n systems may be subsystems of a complex system. Du and Hwang [2] first studied the optimal design of series linear and circular consecutive-2-out-of-n:F systems. They reported that there are invariant optimal designs for series linear consecutive-2-out-of-n:F systems which consist of n_1 consecutive-2-out-of-3:F subsystems, n_2 consecutive-2-out-of-$2m$:F subsystems and n_3 consecutive-2-out-of-$2(m-1)$:F subsystems. They also found that there are invariant optimal designs for series circular consecutive-2-out-of-n:F systems consisting of subsystems which are all circular consecutive-2-out-of-$2m$:F subsystems.

In this study the invariant optimal designs for series linear consecutive-k-out-of-n:G systems have been investigated. It is proved that there is invariant optimal design for a series linear consecutive-k-out-of-n:G system which consists of l consecutive-k-out-of-$2k$:G subsystems and m consecutive-$(k-1)$-out-of-$2(k-1)$:G subsystems. It is also noticed that the optimal partition of components for a series linear consecutive-k-out-of-n:G system with l consecutive-k-out-$2k$:G subsystems and m consecutive-$(k-1)$-of-$2(k-1)$:G subsystems is the same as the best partition of components for a series linear consecutive-2-out-of-n:F with l consecutive-2-out-$2k$ subsystems and m consecutive-2-out-of-$2(k-1)$:F subsystems.

ACKNOWLEDGMENTS

First of all I wish to express my sincerest gratitude to my supervisor, Dr. Mingjian Zuo, for his patient guidance, encouragement, and most of all his friendship as a person of my generation. I am also indebted to Dr. Laurel Travis and Dr. Don O. Koval for their significant substantive comments on the draft of the thesis. Thanks also go to other professors and technicians in the Department of Mechanical Engineering, who have created a very friendly and pleasant environment for study.

At last I would never forget to say thanks to my grandmother and my parents who always pray for me when I am away from home.

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ASSUMPTIONS AND NOTATION

ASSUMPTIONS

1. All components are either working or failed.

2. All components are statistically independent.

3. Any system or subsystem is either working or failed.

4. All the components in the system are interchangeable.

5. All the components in the system are distinct, i.e., no two components have the same reliability.

6. The reliability of any component in the system is not 0 or 1 if it is not stated.

NOTATION

$\lfloor x \rfloor$ - integer part of x

x_i - component reliability

p_i - component reliability

q_i - component unreliability, $p_i + q_i = 1$

P - a perfect component, $p(P) = 1, q(P) = 0$

D - a dummy component, $p(D) = 0, q(D) = 1$

X	-	an ordered set or arrangement of n components $x_1, x_2, \ldots, x_n$, $X = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$
$\overline{X}$	-	reversed set of X
$R(X)$		system reliability of arrangement X
$Q(X)$	-	system unreliability of arrangement X
W, V	-	unordered sets
A	-	an event
$\overline{A}$	-	the complementary event of A
$Pr(A)$	-	probability of event A
ϕ	-	impossible event, $Pr(\phi) = 0$
$A \cap B$	-	intersection of events A and B
$A \cup B$	-	union of events A and B
$X \cap Y$ or $X \bullet Y$	-	X and Y are connected in series
$X \cup Y$	-	X and Y are connected in parallel
(X, P, Y) or $(X, 1, Y)$	-	an ordered set or arrangement of components obtained by joining X and Y together through a perfect component
(X, D, Y) or $(X, 0, Y)$	-	an ordered set or arrangement of components obtained by joining X and Y together through a dummy component
$X_F(k, n)$	-	a consecutive-k-out-of-n:F system with arrangement X
$X_G(k, n)$	-	a consecutive-k-out-of-n:G system with arrangement X

$X_F(k_1, n_1) \cap Y_F(k_2, n_2)$	-	a system with $X_F(k_1, n_1)$ and $Y_F(k_2, n_2)$ connected in series
$X_G(k_1, n_1) \cap Y_G(k_2, n_2)$	-	a system with $X_G(k_1, n_1)$ and $Y_G(k_2, n_2)$ connected in series

CHAPTER 1. INTRODUCTION

The consecutive-k-out-of-n system was first studied by Kontoleon in 1980 [9]. A linear consecutive-k-out-of-n:F(G) system is a system with n components arranged on a line and the system fails (works) if and only if at least k consecutive components fail (work). A circular consecutive-k-out-of-n:F(G) system is a system with n components arranged on a circle and the system fails (works) if and only if at least k consecutive components fail (work). A consecutive-k-out-of-n:F system becomes a series system when $k = 1$ and a parallel system when $k = n$. A consecutive-k-out-of-n:G system becomes a series system when $k = n$ and a parallel system when $k = 1$.

In the last decade there have been extensive studies on the consecutive-k-out-of-n systems. The reason for this is the fact that consecutive-k-out-of-n systems are widely applicable as well as of great mathematical interest. For example, the spokes of a bicycle wheel constitute a circular consecutive-k-out-of-n:F system. A bicycle can work properly even there are some spokes broken as long as there are no k consecutive spokes broken. A long distance oil pipeline can be treated as a consecutive-k-out-of-n:F system if each pumping station has the power to pump oil directly to the next k stations. The pipeline in Figure 1.1 is a consecutive-2-out-of-7:F system. A microwave communications system is a consecutive-k-out-of-n:F system [1]. As shown in Figure 1.2, there are eight relay stations. Each station can transmit signals to the next two stations. Stations 2, 3, ..., 8 constitute a consecutive-2-out-of-7:F system which is connected with station 1 in series. Figure 1.3 shows a car parking lot. If here comes a bus which needs three consecutive lots to park, then the parking lots can be treated as a consecutive-3-out-of-n:G system ($n = 10$ in the figure). It was reported that the vacuum system of the accelerator at Brookhaven National Laboratory is a consecutive-k-out-of-n:F system [8]. The core of the accelerator consists of a great number of identical components. The operation of the accelerator will not be interrupted until there are a specific number of consecutive components fail.

There are many other applications of the consecutive-k-out-of-n systems. Computer network systems [6], integrated circuits [7], and statistical sampling are a few more examples.

Series and parallel consecutive-k-out-of-n systems are natural extensions of consecutive-k-out-of-n systems. For example, in the communications system of Figure 1.4, each relay station can transmit signals directly to the next two stations except that station 6 can only pass signals to station 7 as there is a big obstacle (a lake, for example) between stations 6 and 7. In the figure, Stations 2, 3, 4 and 5 constitute a consecutive-2-out-of-4:F subsystem, stations 8, 9, 10 and 11 constitute another consecutive-2-out-of-4:F subsystem. Station 1, station 6, station 7 and the two consecutive-2-out-of-4:F subsystems are connected in series.

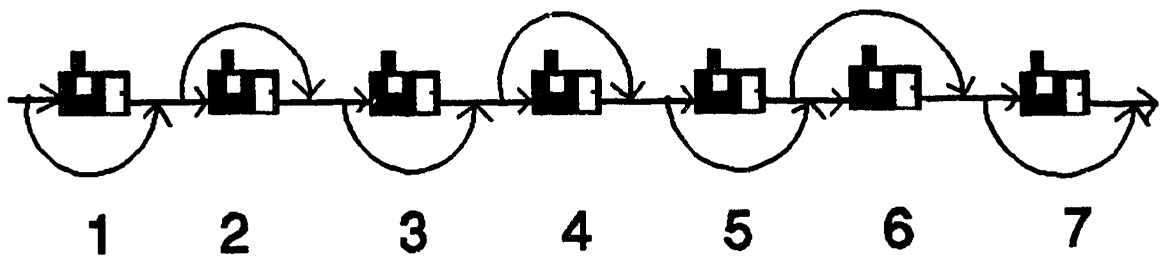


Figure 1.1: An Oil Pipeline

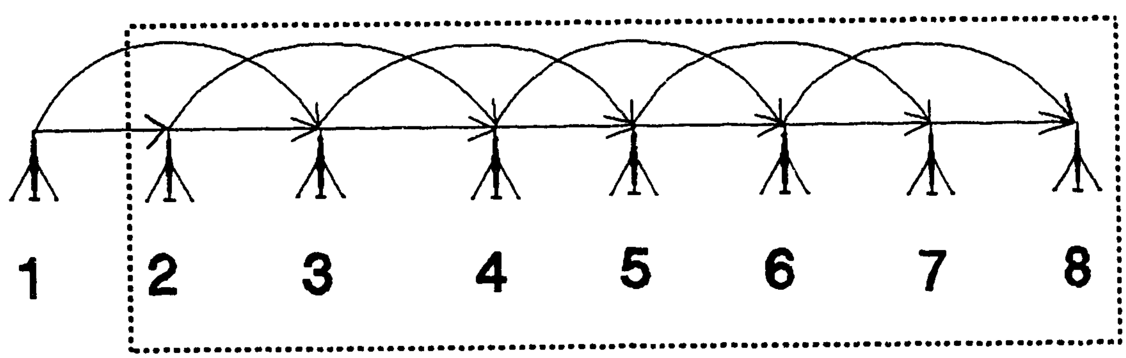
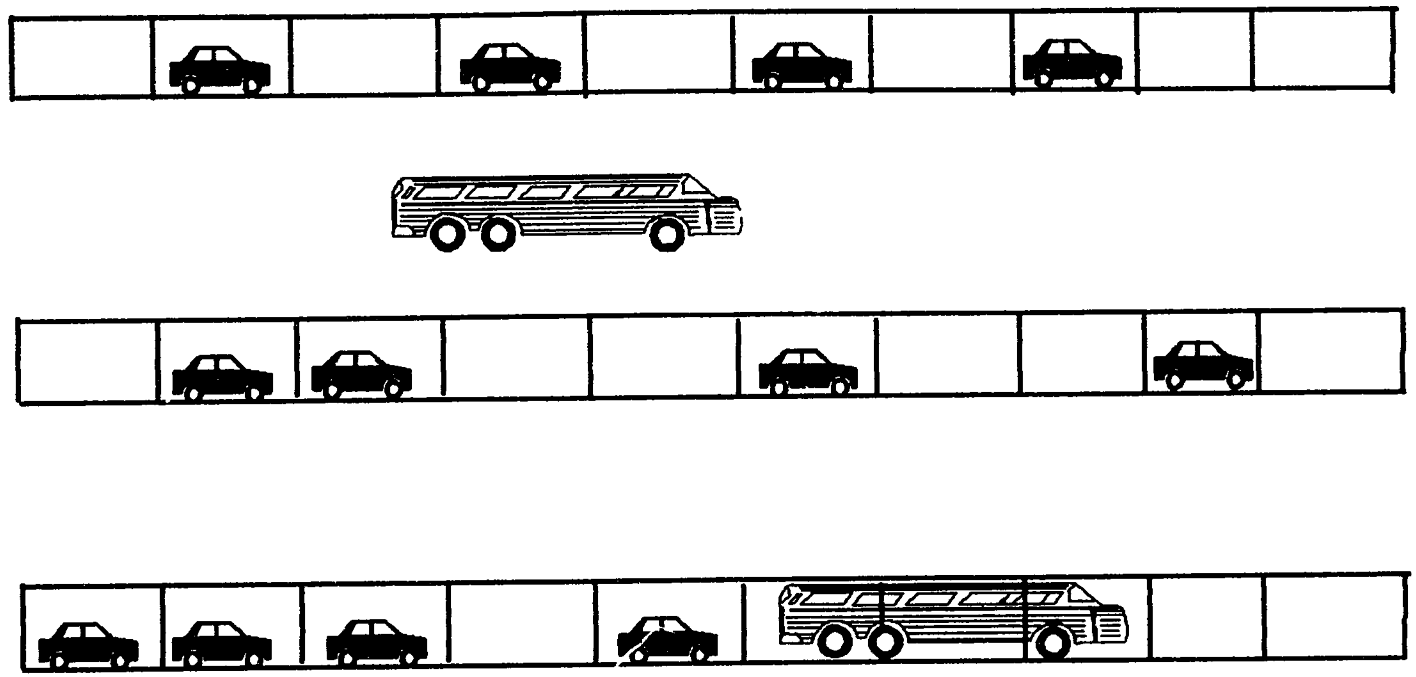


Figure 1.2: Relay Communications System I



Car Parking Lots

Figure 1.3 A Consecutive-3-out-of-10:G System

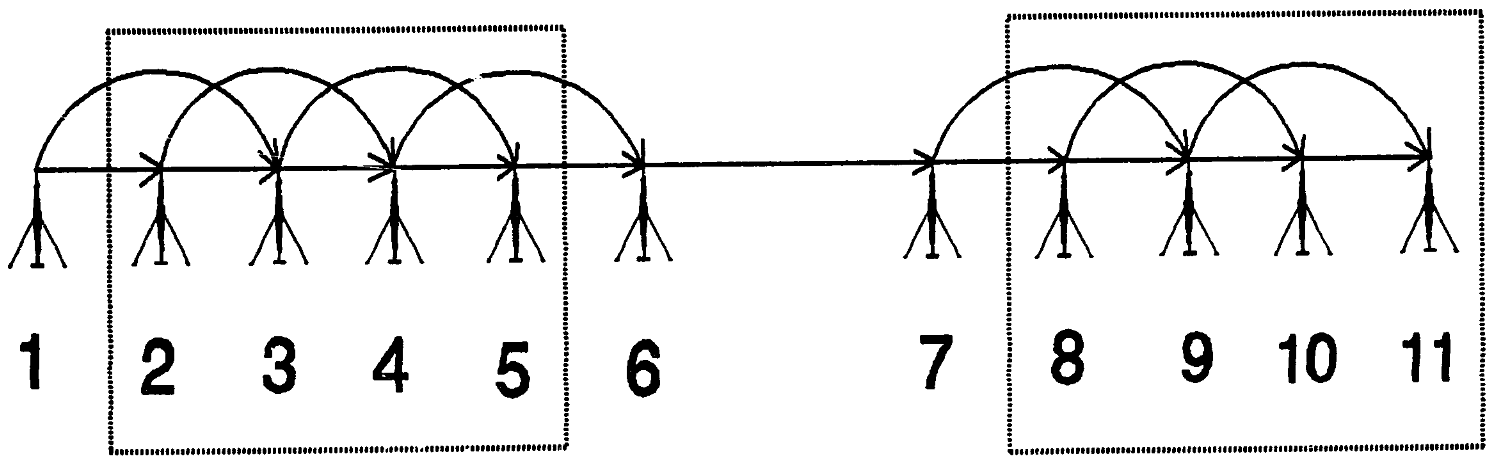


Figure 1.4: Relay Communications System II

CHAPTER 2. REVIEW OF CONSECUTIVE-k-OUT-OF-n SYSTEMS

DEFINITIONS

Greedy:

If X is an arrangement of the following n components:

$$p_1 < p_2 < \ldots < p_{n-1} < p_n,$$

then, arrangement X is called greedy if

$$X = (p_1, p_n, p_3, p_{n-2}, \ldots, p_{n-3}, p_4, p_{n-1}, p_2).$$

Regular:

Let

$$X = (x_1, x_2, \ldots, x_n).$$

X is called regular if

$$(x_i - x_{n-i+1})(x_{i+1} - x_{n-i}) < 0, \text{ for all } i,\ 1 \le i \le [\frac{n}{2}].$$

Singular:

Let X be an ordered set or arrangement of n components, where

$$X = (x_1, x_2, \ldots, x_n).$$

Then set or arrangement X is called singular if

$$(x_i - x_{n-i+1})(x_{i+1} - x_{n-i}) \; > \; 0, \text{ for all } i, \; 1 \leq i \leq [\frac{n}{2}].$$

If X is not singular, we can obtain a singular arrangement X' from X by the following operations:

1. Set $i = 1$.
2. If $(x_i - x_{n-i+1})(x_{i+1} - x_{n-i}) < 0$, interchange x_{i+1} and x_{n-i}.
3. If $i < [\frac{n}{2}]$, set $i = i + 1$ and go to step 2.
4. Stop.

In the following component x_i refers to the component with reliability value x_i.

2.1 System Reliability Evaluation

Series and parallel systems are the two most basic reliability models. For a series system consisting of n components with reliabilities $p_1, p_2, \ldots, p_n$, its system reliability R_S is:

$$R_S \; = \; p_1 p_2 \ldots p_n = \prod_{i=1}^{n} p_i. \tag{2.1}$$

For a parallel system consisting of n components with unreliabilities $q_1, q_2, \ldots, q_n$, its system unreliability Q_P is:

$$Q_P \; = \; q_1 q_2 \ldots q_n = \prod_{i=1}^{n} q_i. \tag{2.2}$$

In reality most systems are complex systems which are combinations of series, parallel, and even correlated components or subsystems. There are many methods and algorithms available to evaluate the system reliability of a complex system. The following reviews the concepts and methods that will be used in this study.

A path is a set of components such that the system works if all the components in the set work. A minimal path is a path which has no proper subset of components whose working will ensure the system's working. A minimal path is said to work if all its components work. A minimal path set is a set having all the minimal paths of the system.

A cut is a set of components such that the system fails if all the components in the set fail. A minimal cut is a cut which has no proper subset of components whose failure will cause system failure [15]. A minimal cut is said to fail if all its components fail. A minimal cut set is a set having all the minimal cuts of the system.

Two systems will have the same reliability if they are constructed from the same set of components and have exactly the same minimal path set or minimal cut set.

Denote R and Q the system reliability and unreliability, respectively. Assume we have a system which has n minimal paths $W_1, W_2, \ldots, W_n$ and m minimal cuts $V_1, V_2, \ldots, V_m$. The minimal path set W and the minimal cut set V of the system are

$$\begin{aligned} W &= (W_1, W_2, \ldots, W_n), \\ V &= (V_1, V_2, \ldots, V_m). \end{aligned}$$

Denote A_i the event that minimal path W_i works $(i = 1, 2, \ldots, n)$, and B_i the event that minimal cut V_i fails $(i = 1, 2, \ldots, m)$. Then,

$$R = Pr(A_1 \cup A_2 \cup \ldots \cup A_n), \tag{2.3}$$

$$Q = Pr(B_1 \cup B_2 \cup \ldots \cup B_m). \tag{2.4}$$

Now we introduce another method, Sum of Disjoint Products (SDP) method. The basic concept of SDP method is very simple. To find the probability of an event A, we could divide A into n subevents $C_1, C_2, \ldots, C_n$ such that

$$Pr(A) = Pr(C_1 \cup C_2 \cup \ldots \cup C_n), \tag{2.5}$$

$$C_i \cap C_j = \phi, \text{ for all } i,\ j,\ 1 \leq i,\ j \leq n \text{ and } i \neq j. \tag{2.6}$$

Then,

$$Pr(A) = Pr(C_1) + Pr(C_2) + \cdots + Pr(C_n).$$

The system reliability R in equation (2.3) can be evaluated by using SDP method. Let

$$\begin{aligned} C_1 &= A_1, \\ C_2 &= \overline{A}_1 \cap A_2, \\ C_3 &= \overline{A}_1 \cap \overline{A}_2 \cap A_3, \\ &\vdots \\ C_n &= \overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \ldots \cap \overline{A}_{n-1} \cap A_n. \end{aligned}$$

Then,

$$\begin{aligned} R &= Pr(A_1 \cup A_2 \ldots \cup A_n) \\ &= Pr(C_1 \cup C_2 \ldots \cup C_n) \\ &= Pr(C_1) + Pr(C_2) + \cdots + Pr(C_n). \end{aligned} \tag{2.7}$$

A detailed discussion on SDP method was conducted by Locks in [11].

2.2 Reliability Evaluation of Consecutive-k-out-of-n Systems

Let $X = (p_1, p_2, \ldots, p_n)$. Denote R_G the system reliability of consecutive-k-out-of-n:G system $X_G(k,n)$ with $n \leq 2k$, and Q_F the system unreliability of consecutive-k-out-of-n:F system $X_F(k,n)$ with $n \leq 2k$. The following formulas are available from Zuo and Kuo [16]:

$$R_G = \sum_{i=1}^{n-k+1} \left(q_{k+i} \prod_{j=i}^{k+i-1} p_j \right), \tag{2.8}$$

$$Q_F = \sum_{i=1}^{n-k+1} \left(p_{k+i} \prod_{j=i}^{k+i-1} q_j \right), \tag{2.9}$$

where $q_{n+1} \equiv 1$ in (2.8), $p_{n+1} \equiv 1$ in (2.9). From equations (2.8) and (2.9), R_G and Q_F can be reformulated as,

$$R_G = \sum_{i=1}^{l} \left(q_{k+i} \prod_{j=i}^{k+i-1} p_j \right) + R(p_{l+1}, p_{l+2}, \ldots, p_n), \tag{2.10}$$

$$Q_F = \sum_{i=1}^{l} \left(p_{k+i} \prod_{j=i}^{k+i-1} q_j \right) + Q(p_{l+1}, p_{l+2}, \ldots, p_n), \tag{2.11}$$

where $1 \leq l \leq n - k + 1$.

By decomposing on component p_i, R_G can be written as

$$\begin{aligned} R_G &= R(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n) \\ &= p_i R(p_1, p_2, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n) + q_i R(p_1, p_2, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_n). \end{aligned} \tag{2.12}$$

Equation (2.12) is the system reliability formula of $X_G(k,n)$ expanded on component p_i. It can be further expanded on component p_j. Let

$$\begin{aligned} R_1 &= R(p_1, p_2, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n), \\ R_2 &= R(p_1, p_2, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n), \\ R_3 &= R(p_1, p_2, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n), \\ R_4 &= R(p_1, p_2, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n). \end{aligned}$$

Then,

$$R_G = p_i p_j R_1 + p_i q_j R_2 + q_i p_j R_3 + q_i q_j R_4. \tag{2.13}$$

The formulas of R_G in (2.12) and (2.13) hold for any values of n and k, whereas (2.10) and (2.11) are based on the assumption that $n \leq 2k$.

Similarly we can get the unreliability formula Q_F expanded on p_i and p_j for consecutive-k-out-of-n:F system $X_F(k,n)$:

$$Q_F = p_i p_j Q_1 + p_i q_j Q_2 + q_i p_j Q_3 + q_i q_j Q_4, \tag{2.14}$$

where

$$\begin{aligned} Q_1 &= Q(p_1, p_2, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n), \\ Q_2 &= Q(p_1, p_2, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n), \\ Q_3 &= Q(p_1, p_2, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n), \\ Q_4 &= Q(p_1, p_2, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n). \end{aligned}$$

The formula of Q_F in (2.14) holds for any values of k and n.

2.3 Optimal Design of Consecutive-k-out-of-n Systems

A system is said to have invariant optimal design if its optimal design can be determined by knowing only the ranking of the reliabilities of its components. Such a system is called an invariant system, and its optimal design is called an invariant optimal design. A system is said to have variant optimal design if it does not have an invariant optimal design.

2.3.1 Optimal Design of Series-parallel and Parallel-series Systems

A simple series or parallel system has the same system reliability no matter how its components are arranged. If redundancy is applied to increase system reliability, different arrangements of components may have different system reliabilities.

A series-parallel system is a system with two or more subsystems (which will be called stations later) connected in series. Each of these stations is a parallel subsystem. The number of components in each station is called the redundancy of the station. Figure 2.1 shows a series-parallel system with S1, S2, S3 and S4 four stations. The four stations have redundancies of 2, 4, 1 and 3, respectively.

A parallel-series system is a system with two or more subsystems (which will be called lines later) connected in parallel. Each of these lines is a series subsystem. The number of components on each line is called the size of the line. The number of the lines is called the redundancy of the system. Figure 2.2 is a parallel-series system which has four lines or a redundancy of four. The four lines have sizes of 4, 3, 1 and 2, respectively.

A series-parallel system is called a complex series-parallel system when a component is replaced by a supercomponent or subsystem of more than one component. The system in Figure 2.3 is a complex series-parallel system in which component 2 is a consecutive-2-out-of-4:F subsystem. The four stations in the system have redundancies of 2, 4, 1 and 3, respectively. Station S1 in Figure 2.3 is called a complex station or a complex parallel subsystem. A complex parallel-series system can be defined similarly.

A series-parallel system has invariant optimal design only when each station has a constant redundancy of 2 [14]. The invariant optimal design is to:

1. *Assign the best component and the worst component to a station.*

2. *Assign the next best component and the next worst component to another station.*

3. *Continue the assignment until each station has been assigned two components.*

It is obvious that a complex series-parallel system has no invariant optimal design if there two or more stations which have a redundancy of greater than two.

All parallel-series systems have invariant optimal designs [4]. Assume a parallel-series system has n lines with size $m_1, m_2, \ldots, m_n$, respectively, where $m_1 \leq m_2 \ldots \leq m_n$. The invariant optimal design is to:

1. *Assign the best m_1 components to the line with size m_1.*

2. *Assign the next best m_2 components to the line with size m_2.*

3. *Continue the assignment until all lines have been assigned with the required number of components.*

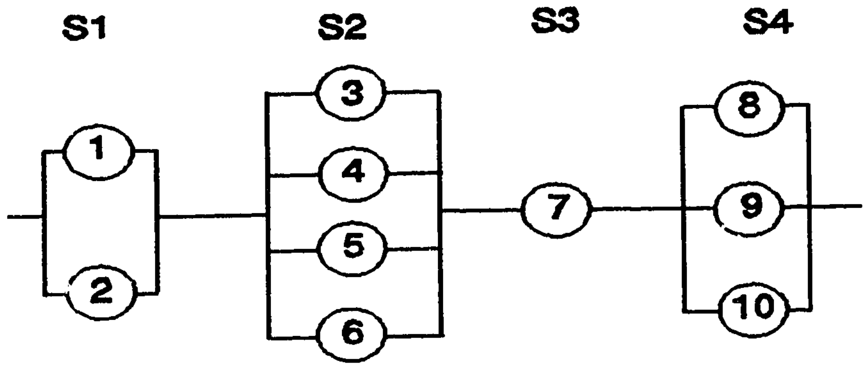


Figure 2.1: A Series-Parallel System

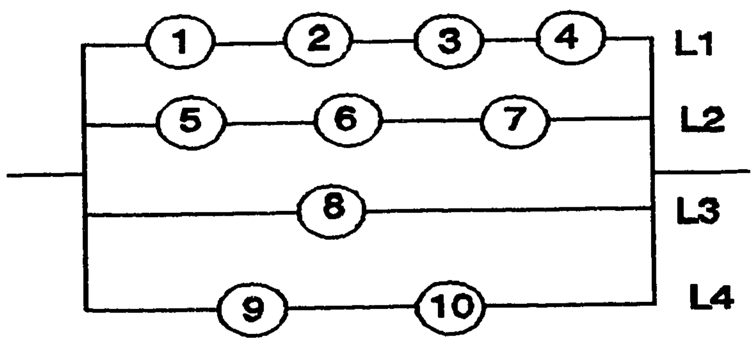


Figure 2.2: A Parallel-Series System

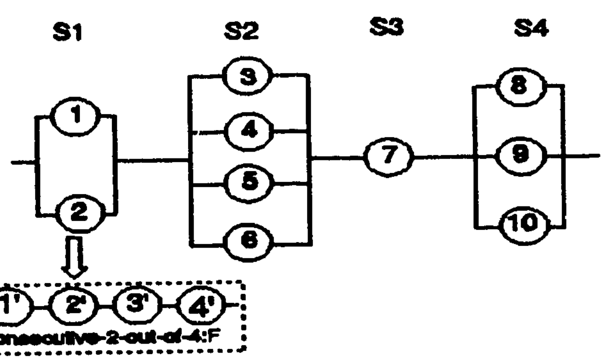


Figure 2.3: A Complex Series-Parallel System

2.3.2 Optimal Design of Consecutive-k-out-of-n Systems

The optimal design of a linear consecutive-2-out-of-n:F system was proved independently by Malon [12] and Du & Hwang [3]. The optimal arrangement of a linear consecutive-2-out-of-n:F system is greedy. Later Malon [13] investigated the optimal design of consecutive-k-out-of-n:F systems with $k > 2$. Du and Hwang also reported the optimal design of a circular consecutive-2-out-of-n:F system in [3]. The optimal design of linear and circular consecutive-k-out-of-n:F(G) systems with $k > 2$ was studied by Kuo et al [10] and Zuo & Kuo [16]. Table 2.1 is a list of the invariant optimal designs of linear consecutive-k-out-of-n systems. A complete list of the invariant optimal designs for both linear and circular consecutive-k-out-of-n systems can be found in Zuo and Kuo [16].

2.3.3 Optimal Design of Consecutive-k-out-of-n:G Systems

The optimal design of consecutive-k-out-of-$2k$:G systems was investigated by Kuo et al [10]. However, the proof of Theorem 2 in Kuo et al [10] is incorrect. A correct proof is provided below in three separate theorems.

Lemma 2.1: *Let*

$$X = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}),$$
$$F(X) = x_1 x_2 \ldots x_k + x_{k+1} x_{k+2} \ldots x_{2k}.$$

Then, X must be singular for $F(X)$ to be maximized, i.e.,

$$x_i < x_{2k-i+1}, \quad 1 \leq i \leq k. \tag{2.15}$$

Table 2.1: Optimal Design of Linear Consecutive-k-out-of-n Systems

k	consecutive-k-out-of-n:F system	consecutive-k-out-of-n:G system
1	(any arrangement)	(any arrangement)
2	(1, n, 3, n-2, ..., n-3, 4, n-1, 2)	no invariant design
$2<k<n/2$	no invariant design	no invariant design
$n/2 \leq k < n-2$	no invariant design	(1, 3, 5, ..., 2(n-k)-1, (any arrangement), 2(n-k), ..., 6, 4, 2)
$k=n-2$	(1, 4, (any arrangement), 3, 2)	(1, 3, (any arrangement), 4, 2)
$k=n-1$	(1, (any arrangement), 2)	(1, (any arrangement), 2)
$k=n$	(any arrangement)	(any arrangement)

The pair of components x_i and x_{2k-i+1} are called symmetric components, where $1 \leq i \leq k$.

Proof of Lemma 2.1:

Without loss of generality, we assume $x_1 < x_{2k}$. Assume further that X is not singular. Then there must be an integer $l\,(1 \leq l < 2k)$ such that

$$x_i < x_{2k-i+1}, \quad 1 \leq i \leq l, \tag{2.16}$$

$$x_{l+1} > x_{2k-l}. \tag{2.17}$$

We will show that $F(X)$ is not maximized. Let

$$\begin{aligned} X1 &= (x_1, \ldots, x_l, x_{2k-l}, x_{l+2}, \ldots, x_k, x_{k+1}, \ldots, x_{2k-l-1}, x_{l+1}, x_{2k-l+1}, \ldots, x_{2k}), \\ X2 &= (x_1, \ldots, x_l, x_{2k-l}, x_{2k-l-1}, x_{2k-l-2}, \ldots, x_{k+1}, x_k, \ldots, x_{l+1}, x_{2k-l+1}, \ldots, x_{2k}), \end{aligned}$$

where $X1$ is obtained from X by interchanging x_{l+1} and x_{2k-l}, and $X2$ is obtained from X by flipping the string of components $x_{l+1}, x_{l+2}, \ldots, x_{2k-l}$. Now we prove that either $X1$ or $X2$ improves X.

$$\begin{aligned} F(X1) &= x_1x_2\ldots x_lx_{2k-l}x_{l+2}x_{l+3}x_k \\ &\quad + x_{k+1}x_{k+2}\ldots x_{2k-l-2}x_{2k-l-1}x_{l+1}x_{2k-l+1}\ldots x_{2k}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} F(X2) &= x_1x_2\ldots x_lx_{2k-l}x_{2k-l-1}x_{2k-l-2}\ldots x_{k+1} \\ &\quad + x_k\ldots x_{l+1}x_{2k-l+1}x_{2k-l+2}\ldots x_{2k}. \end{aligned} \tag{2.19}$$

$$\begin{aligned} \Delta F1 &= F(X1) - F(X) \\ &= x_1x_2\ldots x_lx_{l+2}\ldots x_k(x_{2k-l} - x_{l+1}) \\ &\quad + x_{k+1}x_{k+2}\ldots x_{2k-l-1}x_{2k-l+1}\ldots x_{2k}(x_{l+1} - x_{2k-l}) \end{aligned}$$

$$= \left(x_{l+1} - x_{2k-l}\right)\left(x_{k+1}\dots x_{2k-l-1}x_{2k-l+1}\dots x_{2k} - x_1x_2\dots x_l x_{l+2}\dots x_k\right). \tag{2.20}$$

$$\begin{aligned}
\Delta F2 &= F(X2) - F(X) \\
&= x_{2k-l}x_{2k-l-1}\dots x_{k+1}(x_1x_2\dots x_l - x_{2k-l+1}x_{2k-l+2}\dots x_{2k}) \\
&\quad + x_{l+1}x_{l+2}\dots x_k(x_{2k-l+1}x_{2k-l+2}\dots x_{2k} - x_1x_2\dots x_l) \\
&= (x_{2k-l+1}x_{2k-l+2}\dots x_{2k} - x_1x_2\dots x_l)(x_{l+1}x_{l+2}\dots x_k - x_{k+1}x_{k+2}\dots x_{2k-l}).
\end{aligned} \tag{2.21}$$

Now we show either $\Delta F1 > 0$ or $\Delta F2 > 0$. From the assumption in (2.16) and (2.17) we have

$$x_{l+1} - x_{2k-l} > 0, \tag{2.22}$$

$$x_{2k-l+1}x_{2k-l+2}\dots x_{2k} - x_1x_2\dots x_l > 0 \tag{2.23}$$

If

$$x_{k+1}\dots x_{2k-l-1}x_{2k-l+1}\dots x_{2k} - x_1x_2\dots x_l x_{l+2}\dots x_k \;\geq\; 0,$$

then $\Delta F1 \geq 0$, i.e., $X1$ is at least as good as X.

If

$$\begin{aligned}
x_{k+1}\dots x_{2k-l-1}x_{2k-l+1}\dots x_{2k} - x_1x_2\dots x_l x_{l+2}\dots x_k &< 0, \quad \text{or} \\
(x_{k+1}\dots x_{2k-l-1})(x_{2k-l+1}\dots x_{2k}) - (x_1x_2\dots x_l)(x_{l+2}\dots x_k) &< 0,
\end{aligned}$$

then

$$x_{k+1}x_{k+2}\dots x_{2k-l-1} \;<\; x_{l+2}x_{l+3}\dots x_k \tag{2.24}$$

since

$$x_{2k-l+1}x_{2k-l+2}\dots x_{2k} \;>\; x_1x_2\dots x_l.$$

From (2.17), and (2.24) we have

$$x_{k+1}x_{k+2}\dots x_{2k-l-1}x_{2k-l} \;<\; x_{l+1}x_{l+2}\dots x_k,$$

or $\Delta F2 > 0$, i.e., $X2$ improves X. We conclude that either $X1$ or $X2$ improves X.

By a series operations of interchanging two symmetrical components or flipping the string of components ending with two symmetrical components in X, we can get a singular arrangement X' from X, such that

$$F(X') \;\geq\; F(X). \tag{2.25}$$

Note that $\overline{X'}$ is also singular, and $F(X') = F(\overline{X'})$.

Lemma 2.2: *Let*

$$\begin{aligned} X &= (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{2k}), \\ F(X) &= x_1x_2\dots x_k(1 - x_{k+1}) + (1 - x_k)x_{k+1}x_{k+2}\dots x_{2k}. \end{aligned}$$

Then, X must be singular for $F(X)$ to be maximized.

Proof of Lemma 2.2:

The proof of Lemma 2.2 is similar to the proof of Lemma 2.1.

Theorem 2.1: *Let*

$$X \;=\; (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{2k}),$$

then X must be singular for it to be an optimal design of system $X_G(k, 2k)$.

Proof of Theorem 2.1:

Theorem 2.1 indicates that the system reliability of $X_G(k,2k)$ can be improved if X is not singular. We prove the theorem by using induction on k in $X_G(k,2k)$.

(i)

It is easy to prove that Theorem 2.1 holds when $k=2$. Let $X=(x_1,x_2,x_3,x_4)$. Assume $x_1 < x_4, x_2 > x_3$, i.e., X is not singular. Let $X'=(x_1,x_3,x_2,x_4)$, where X' is a singular arrangement obtained from X. We show that $R(X'_G(2,4)) > R(X_G(2,4))$.

$$R(X_G(2,4)) = x_1x_2 + x_2x_3 + x_3x_4 - x_1x_2x_3 - x_2x_3x_4, \tag{2.26}$$

$$R(X'_G(2,4)) = x_1x_3 + x_3x_2 + x_2x_4 - x_1x_3x_2 - x_3x_2x_4, \tag{2.27}$$

$$R(X'_G(2,4)) - R(X_G(2,4)) = (x_2 - x_3)(x_4 - x_1) > 0, \tag{2.28}$$

i.e., X' improves X.

(ii)

Assume Theorem 2.1 holds for $k < l$, we need to show it is also true when $k = l$. Let

$$X = (x_1, x_2, \ldots, x_l, x_{l+1}, x_{l+2}, \ldots, x_{2l}).$$

Assume X is not singular and X' is a singular arrangement obtained from X, where

$$X' = (x'_1, x'_2, \ldots, x'_l, x'_{l+1}, x'_{l+2}, \ldots, x'_{2l}).$$

Denote R_s, R'_s the system reliabilities of arrangements X and X', respectively. Then,

$$\begin{aligned} R_s = {} & x_l x_{l+1} R(x_1, x_2, \ldots, x_{l-1}, 1, 1, x_{l+2}, \ldots, x_{2l}) \\ & + x_l(1 - x_{l+1}) R(x_1, x_2, \ldots, x_{l-1}, 1, 0, x_{l+2}, \ldots, x_{2l}) \end{aligned}$$

$$+(1-x_l)x_{l+1}R(x_1,x_2,\ldots,x_{l-1},0,1,x_{l+2},\ldots,x_{2l})$$

$$+(1-x_l)(1-x_{l+1})R(x_1,x_2,\ldots,x_{l-1},0,0,x_{l+2},\ldots,x_{2l})$$

$$= x_lx_{l+1}R\overbrace{(x_2,\ldots,x_{l-1},x_{l+2},\ldots,x_{2l-1})}^{(l-2)-out-of-2(l-2):G\ system}$$

$$+x_1x_2\ldots x_{l-1}x_l(1-x_{l+1})+(1-x_l)x_{l+1}x_{l+2}\ldots x_{2l}+0, \quad (2.29)$$

$$R'_s = x'_lx'_{l+1}R\overbrace{(x'_2,\ldots,x'_{l-1},x'_{l+2},\ldots,x'_{2l-1})}^{(l-2)-out-of-2(l-2):G\ system}$$

$$+x'_1x'_2\ldots x'_{l-1}x'_l(1-x'_{l+1})+(1-x'_l)x'_{l+1}x'_{l+2}\ldots x'_{2l}+0. \quad (2.30)$$

Let

$$F(X) = x_1x_2\ldots x_{l-1}x_l(1-x_{l+1})+(1-x_l)x_{l+1}x_{l+2}\ldots x_{2l},$$

$$X1 = (x_2,\ldots,x_{l-1},x_{l+2},\ldots,x_{2l-1}),$$

$$F(X') = x'_1x'_2\ldots x'_{l-1}x'_l(1-x'_{l+1})+(1-x'_l)x'_{l+1}x'_{l+2}\ldots x'_{2l},$$

$$X'1 = (x'_2,\ldots,x'_{l-1},x'_{l+2},\ldots,x'_{2l-1}).$$

Then,

$$R_s = x_lx_{l+1}R(X1)+F(X), \quad (2.31)$$

$$R'_s = x'_lx'_{l+1}R(X'1)+F(X'). \quad (2.32)$$

From the induction assumption, Lemma 4.2 and the way X' was obtained from X we have

$$x'_lx'_{l+1} = x_lx_{l+1}, \quad (2.33)$$

$$R(X'1) > R(X1), \quad (2.34)$$

$$F(X') > F(X). \quad (2.35)$$

From (2.33), (2.34) and (2.35) we have

$$R_s' > R_s, \tag{2.36}$$

i.e., X' improves X.

Theorem 2.2: *Let*

$$X = (x_1, x_2, \ldots, x_k, p, x_{k+1}, x_{k+2}, \ldots, x_{2k}),$$

then X must be singular for it to be an optimal arrangement of system $X_G(k, 2k+1)$.

Proof of Theorem 2.2:

We will show that the system reliability of $X_G(k, 2k+1)$ can be improved if X is not singular. Denote X' a singular arrangement obtained from X, where

$$X' = (x_1', x_2', \ldots, x_k', p, x_{k+1}', \ldots, x_{2k}').$$

Let R_s, R_s' be the system reliabilities of $X_G(k, 2k+1)$ and $X_G'(k, 2k+1)$, respectively. Then

$$\begin{aligned} R_s &= pR(x_1, x_2, \ldots, x_k, 1, x_{k+1}, x_{k+2}, \ldots, x_{2k}) \\ &\quad +(1-p)R(x_1, x_2, \ldots, x_k, 0, x_{k+1}, \ldots, x_{2k}) \\ &= pR\overbrace{(x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k-1})}^{(k-1)-out-of-2(k-1):G\ system} \\ &\quad +(1-p)(x_1 x_2 \ldots x_k + x_{k+1} \ldots x_{2k} - x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots x_{2k}), \end{aligned} \tag{2.37}$$

$$\begin{aligned} R_s' &= pR(x_1', x_2', \ldots, x_k', 1, x_{k+1}', x_{k+2}', \ldots, x_{2k}') \\ &\quad +(1-p)R(x_1', x_2', \ldots, x_k', 0, x_{k+1}', \ldots, x_{2k}') \end{aligned}$$

$$= pR\overbrace{(x'_2, \ldots, x'_k, x'_{k+1}, x'_{k+2}, \ldots, x'_{2k-1})}^{(k-1)-out-of-2(k-1):G\ system}$$
$$+(1-p)(x'_1 x'_2 \ldots x'_k + x'_{k+1} \ldots x'_{2k} - x'_1 x'_2 \ldots x'_k x'_{k+1} x'_{k+2} \ldots x'_{2k}). \tag{2.38}$$

Let

$$\begin{aligned} X1 &= (x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k-1}), \\ F(X) &= x_1 x_2 \ldots x_k + x_{k+1} \ldots x_{2k} - x_1 x_2 \ldots x_k x_{k+1} \ldots x_{2k}, \\ X'1 &= (x'_2, \ldots, x'_k, x'_{k+1}, x'_{k+2}, \ldots, x'_{2k-1}), \\ F(X') &= x'_1 x'_2 \ldots x'_k + x'_{k+1} \ldots x'_{2k} - x'_1 x'_2 \ldots x'_k x_{k+1} \ldots x'_{2k}, \end{aligned}$$

then

$$R_s = pR(X1) + (1-p)F(X), \tag{2.39}$$

$$R'_s = pR(X'1) + (1-p)F(X'). \tag{2.40}$$

By the way X' was obtained from X we have

$$x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots x_{2k} = x'_1 x'_2 \ldots x'_k x'_{k+1} x'_{k+2} \ldots x'_{2k} = c,$$

where c is a constant with the component set given. By Lemma 2.1 and Theorem 2.1 we have

$$F(X') > F(X), \tag{2.41}$$

$$R(X'1) > R(X1). \tag{2.42}$$

Therefore

$$R'_s > R_s. \tag{2.43}$$

Theorem 2.3: *Let*

$$X = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}),$$

then, a necessary condition for X *to be an optimal arrangement of* $X_G(k, 2k)$ *is:*

$$x_{i+1} > x_{2k-i+1}, \quad 1 \leq i \leq k-1. \tag{2.44}$$

Proof of Theorem 2.3:

Denote R_s the system reliability of $X_G(k, 2k)$, then

$$\begin{aligned} R_s &= R\overbrace{(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k})}^{k-out-of-2k:G\ system} \\ &= R\overbrace{(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}, 0)}^{k-out-of-(2k+1):G\ system}. \end{aligned} \tag{2.45}$$

Let

$$X1 = (x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}, 0).$$

By Theorem 2.2 $X1$ must be singular for R_s to be maximized. Hence we have

$$\begin{aligned} x_1 &> 0, \\ x_{i+1} &> x_{2k-i+1}, \quad 1 \leq i \leq k-1. \end{aligned} \tag{2.46}$$

CHAPTER 3. OPTIMAL DESIGN OF SERIES CONSECUTIVE-k-OUT-OF-n:F SYSTEMS

A series consecutive-k-out-of-n:F system is a system with two or more consecutive-k-out-of-n:F subsystems connected in series. We first state a lemma.

Lemma 3.1: *Let*

$$\begin{aligned} X &= (p_1, p_2, \ldots, p_n), \\ X^1 &= (p_{n-k+1}, p_{n-k+2}, \ldots, p_k), \\ X^2 &= (p_1, p_2, \ldots, p_{n-k}, p_{k+1}, \ldots, p_n), \end{aligned}$$

where $n < 2k$. Denote X_P^1 the parallel system which consists of all the components in X^1. Then,

$$\begin{aligned} Q(X_F(k,n)) &= Q(X_P^1 \cup X_F^2(k_1, 2k_1)) \\ &= q_{n-k+1} q_{n-k+2} \cdots q_k Q(X_F^2(k_1, 2k_1)), \end{aligned} \tag{3.1}$$

where $k_1 = n - k$.

Proof of Lemma 3.1:

Lemma 3.1 is immediate since $X_F(k,n)$ and $X_P^1 \cup X_F^2(k_1, 2k_1)$ have the same minimal cut set.

Lemma 3.1 implies that a consecutive-k-out-of-n:F system with $n < 2k$ is equivalent to a complex parallel system in terms of system reliability evaluation. The equivalent complex parallel system has a redundancy of $2k - n + 1$: each of the $2k - n$ components in X^1 constitutes a line; and subsystem $X_F^2(k_1, 2k_1)$ is another line. For example, system $X_F(5,7)$, where $X = (p_1, p_2, p_3, p_4, p_5, p_6, p_7)$, is equivalent to a complex parallel system with a redundancy of four ($2 \times 5 - 7 + 1 = 4$). Each of components p_3, p_4 and p_5 contributes one to the redundancy; subsystem $X_F^2(2,4)$, where $X^2 = (p_1, p_2, p_6, p_7)$ adds another one to the redundancy.

For a series consecutive-k-out-of-n:F system to have invariant optimal design, each of its consecutive-k-out-of-n:F subsystems must have invariant optimal design. There is an invariant optimal design for consecutive-2-out-of-n:F, consecutive-$(n-2)$-out-of-n:F and consecutive-$(n-1)$-out-of-n:F systems (Table 2.1).

If a series consecutive-k-out-of-n:F system has invariant optimal design, the optimal design is readily known as long as we know the best partition of its components into each subsystem. In the following we will use optimal partition and optimal design interchangeably.

3.1 Series Consecutive-2-out-of-n:F System

Du and Hwang [2] developed the following necessary conditions for the optimal design of series consecutive-2-out-of-n:F system $X_F(2,l) \cap Y_F(2,m)$:

$$\text{Both } X_F(2,l) \text{ and } Y_F(2,m) \text{ must be of optimal arrangement,} \tag{3.2}$$

$$\text{All } (X,P,Y),\ (\overline{X},P,Y),\ (X,P,\overline{Y}) \text{ and } (\overline{X},P,\overline{Y}) \text{ must be regular.} \tag{3.3}$$

By using the above necessary conditions Du and Hwang found that there is invariant optimal design for systems:

(i) $X_F(2,3) \cap Y_F(2,n)$

(ii) $X_F(2,2k) \cap Y_F(2,2k)$

(iii) $X_F(2,2k) \cap Y_F(2,2(k-1))$

(iv) $\bigcap_{i=1}^{l} X_F^i(2,2k)$

(v) $\bigcap_{i=1}^{l} X_F^i(2,3) \bullet \bigcap_{i=1}^{m} Y_F^i(2,2k) \bullet \bigcap_{i=1}^{r} Z_F^i(2,2(k-1))$

(i) The optimal partition of components for system $X_F(2,3) \cap Y_F(2,n)$ is:

1. Assign the best one and the worst two components to X.

2. Assign the rest of the components to Y.

The optimal design holds for all $n > 1$. If $n = 1$, which is a trivial case, the worst component should be assigned to Y as $Y_F(2,1)$ is always a perfect subsystem with any assigned component.

(ii) The invariant optimal partition of components for system $X_F(2,2k) \cap Y_F(2,2k)$ is:

1. Assign the best of the remaining components to X and the next best to Y.

2. Assign the best of the remaining components to Y and the next best to X.

3. Repeat steps 1 and 2 alternately until all components have been assigned.

(iii) The optimal partition of components for system $X_F(2,2k) \cap Y_F(2,2(k-1))$ is:

1. Assign both the best component and the worst component to X.

2. Assign the best of the remaining components to X and the next best to Y.

3. Assign the best of the remaining components to Y and the next best to X.

4. Repeat steps 2 and 3 alternately until all components have been assigned.

(iv) Assume we have $2km$ components with their reliabilities arranged in ascending order

$$p_1 < p_2 < \ldots < p_{2km-1} < p_{2km},$$

and a series consecutive-2-out-of-n:F system Z with m consecutive-2-out-of-$2k$:F subsystems, where

$$Z = \bigcap_{i=1}^{m} X_F^i(2,2k) = X_F^1(2,2k) \cap X_F^2(2,2k) \cap \ldots \cap X_F^m(2,2k).$$

Then the optimal partition of components for system Z is:

For $i = 1, 2, \ldots, k$

1. *Assign components $p_{2(i-1)m+1}, p_{2(i-1)m+2}, \ldots, p_{2(i-1)m+m}$ to $X^1, X^2, \ldots, X^m$, respectively.*

2. *Assign components $p_{(2i-1)m+1}, p_{(2i-1)m+2}, \ldots, p_{2im}$ to $X^m, X^{m-1}, \ldots, X^2, X^1$, respectively.*

(v) Assume a series consecutive-2-out-of-n system Z is

$$Z = \bigcap_{i=1}^{l} X_F^i(2,3) \bullet \bigcap_{i=1}^{m} Y_F^i(2,2k) \bullet \bigcap_{i=1}^{r} Z_F^i(2,2(k-1)).$$

From (i), (ii), (iii) and (iv) the optimal design for Z can be obtained as follows:

1. *Assign the best one and the worst two components to X^1, Assign the best one and the worst two of the remaining components to X^2. Continue this assignment until all the l $X_F^i(2,3)$ subsystems have been assigned with three components.*

2. Add r perfect components and r dummy components to the set of the $2k(m+r)-2r$ *remaining components in step 1. Optimally assign those* $2k(m+r)$ *components to the* $(m+r)$ *subsystems of system* $\bigcap_{i=1}^{m+r} Y_F^i(2,2k)$ *by using (iv).*

When there is no invariant optimal design, conditions (3.2) and (3.3) can be used to find out all the candidate optimal designs, one of which must be the optimal design. For example, the best arrangement for a series consecutive-2-out-of-n:F system $X_F(2,5) \cap Y_F(2,l)$ with $l > 1$ among three candidate optimal arrangements. Let

$$x_1 < x_2 < x_3 < x_4 < x_5,$$

$$y_1 < y_2 < \ldots < y_{l-1} < y_l.$$

By using conditions (3.2) and (3.3) we get the following three candidate optimal arrangements or ways to partition our $l+5$ components between X and Y.

$$x_1 < x_2 < x_3 < y_1 < y_2 < y_3 < \ldots < y_{l-1} < y_l < x_4 < x_5, \tag{3.4}$$

$$x_1 < x_2 < y_1 < x_3 < y_2 < y_3 < \ldots < y_{l-1} < y_l < x_4 < x_5, \tag{3.5}$$

$$x_1 < x_2 < y_1 < y_2 < x_3 < y_3 < \ldots < y_{l-1} < y_l < x_4 < x_5. \tag{3.6}$$

The optimal design of system $X_F(2,5) \cap Y_F(2,l)$ must be one of the three arrangements in (3.4), (3.5) and (3.6).

The following sections consider all other series consecutive-k-out-of-n:F systems which may have invariant optimal design. In the following we will assume all series consecutive-k-out-of-n:F systems have only two subsystems. The results can be easily extended to the systems with more than two subsystems. Let

$$X = (x_1, x_2, \ldots, x_n),$$

$$Y = (y_1, y_2, \ldots, y_n).$$

3.2 Series Consecutive-$(n-2)$-out-of-n:F Systems

When $n > 4$ there is no invariant optimal design for series consecutive-$(n-2)$-out-of-n:F systems. By Lemma 3.1 $X_F(n-2,n)$ and $Y_F(n-2,n)$ are equivalent to two complex parallel systems, each with a redundancy of $n-3$ $(= 2(n-2)-n+1)$. Thus, $X_F(n-2,n) \cap Y_F(n-2,n)$ is equivalent to a complex series-parallel system of two stations. A complex series-parallel system has no invariant optimal design when two or more of its stations has a redundancy of more than two. Therefore $X_F(n-2,n) \cap Y_F(n-2,n)$ cannot have invariant optimal design when $n-3 > 2$ or $n > 5$. It may have invariant optimal design only when $n-3 = 2$ or $n = 5$. But we found that $X_F(3,5) \cap Y_F(3,5)$ has no invariant optimal design (see a counter example in the Appendix).

3.3 Series Consecutive-$(n-1)$-out-of-n:F Systems

There is no invariant optimal design for series consecutive-$(n-1)$-out-of-n:F systems when $n > 3$. By Lemma 3.1 $X_F(n-1,n) \cap Y_F(n-1,n)$ can be treated as a complex series-parallel system in which each station has a redundancy of $n-1$ $(2(n-1)-n+1 = n-1)$. Therefore $X_F(n-1,n) \cap Y_F(n-1,n)$ cannot have invariant optimal design when $n-1 > 2$ or $n > 3$.

CHAPTER 4. OPTIMAL DESIGN OF SERIES CONSECUTIVE-k-OUT-OF-n:G SYSTEMS

DEFINITIONS

Co-singular:

Denote (X, Y) a set consisting of two ordered subsets X and Y, where

$$\begin{aligned} X &= (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}), \\ Y &= (y_1, y_2, \ldots, y_k, y_{k+1}, \ldots, y_{2k}). \end{aligned}$$

Then both (X, Y) and (Y, X) are called co-singular set or arrangement if

$$x_i < y_i, \quad 1 \le i \le k; \quad x_i > y_i, \quad k+1 \le i \le 2k.$$

If (X, Y) is not co-singular we can obtain a co-singular set (X', Y') from (X, Y) by the following algorithm:

1. Set $i = 1$.
2. If $x_i > y_i$, interchange x_i and y_i; and if $x_{k+i} < y_{k+i}$, interchange x_{k+i} and y_{k+i}.
3. If $i < k$, set $i = i + 1$ and go to step 2.
4. Stop.

A series consecutive-k-out-of-n:G system is a system with two or more consecutive-k-out-of-n:G subsystems connected in series. For a series consecutive-k-out-of-n:G system to have invariant optimal design, all its consecutive-k-out-of-n:G subsystems must have invariant optimal design. From Table 2.1 we know that a consecutive-k-out-of-n:G subsystem has invariant optimal design when $n \leq 2k$.

4.1 Optimal Design of Series Consecutive-k-out-of-n:G Systems

Similar to Lemma 3.1 in Chapter 3, we have Lemma 4.1.

Lemma 4.1: *Assume $n < 2k$ and let*

$$\begin{aligned} X &= (x_1, x_2, \ldots, x_n), \\ X^1 &= (x_{n-k+1}, x_{n-k+2}, \ldots, x_k), \\ X^2 &= (x_1, \ldots, x_{n-k}, x_{k+1}, \ldots, x_n). \end{aligned}$$

Denote X_S^1 a series system which is composed of all the components in X^1. Then

$$\begin{aligned} R(X_G(k,n)) &= R(X_S^1 \cap X_G^2(k_1, 2k_1)), \\ &= x_{n-k+1}x_{n-k+2}\ldots x_k R(X_G^2(k_1, 2k_1)), \end{aligned} \tag{4.1}$$

where $k_1 = n - k$.

Proof of Lemma 4.1:

$X_G(k,n)$ and $X_S^1 \cap X_G^2(k_1, 2k_1)$ have the same set of minimal paths. Lemma 4.1 follows immediately.

Lemma 4.1 means that a consecutive-k-out-of-n:G system with $n < 2k$ is equivalent to a series system which consists of $2k - n$ components and a consecutive-k_1-out-of-$2k_1$:G subsystem, where $k_1 = n - k$.

Lemma 4.2: *Assume we partition $2n$ components into two ordered sets X and Y, where*

$$\begin{aligned} X &= (x_1, x_2, \ldots, x_n), \\ Y &= (y_1, y_2, \ldots, y_n), \\ F(X,Y) &= \prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i. \end{aligned}$$

If $x_1 < y_1$, then a necessary condition for $F(X,Y)$ to be maximized is:

$$x_i < y_i, \quad 1 \leq i \leq n. \tag{4.2}$$

Proof of Lemma 4.2:

Lemma 4.2 can be proved similarly as Lemma 2.1 in Chapter 2.

Lemma 4.3: *Let Y be a system which has component x and a consecutive-k-out-of-$2k$:G subsystem connected in series. Then component x must be the most reliable component for Y to have maximum system reliability.*

Proof of Lemma 4.3:

Let $X = (x_1, x_2, \ldots, x_{2k})$ and assume $x_i > x$ for some $i, 1 \leq i \leq 2k$. We will show that the system reliability can be improved by interchanging component x and component x_i. Let R_s be the system reliability.

$$\begin{aligned} R_s &= xR(X_G(k, 2k)) \\ &= x[x_i R(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{2k}) \\ &\quad + (1 - x_i) R(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{2k})] \\ &= x x_i R(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{2k}) \\ &\quad + x(1 - x_i) R(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{2k}) \end{aligned} \tag{4.3}$$

Denote R'_s the system reliability after interchanging component x with component x_i.

$$\begin{aligned} R'_s &= x_i x R(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{2k}) \\ &\quad + x_i(1-x) R(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{2k}). \end{aligned} \tag{4.4}$$

Then,

$$R'_s - R_s = (x_i - x) R(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{2k}) > 0,$$

i.e., the system reliability can be improved by interchanging components x and x_i if there exists any x_i such that $x_i > x$. As a result, the optimal design of Y must have component x as the most reliable component.

Lemma 4.4: *Let*

$$\begin{aligned} X &= (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}), \\ Y &= (y_1, y_2, \ldots, y_k, y_{k+1}, \ldots, y_{2k}), \\ F(X,Y) &= a\left(\prod_{i=1}^{k} y_i\right) R(X) + b\left(\prod_{i=1}^{k} x_i\right) R(Y), \end{aligned}$$

where $a \geq b > 0$, $R(X)$ and $R(Y)$ are system reliabilities of $X_G(k, 2k)$ and $Y_G(k, 2k)$, respectively. Then (X, Y) must be co-singular for $F(X, Y)$ to be maximized, i.e.,

$$x_i < y_i,\ 1 \leq i \leq k; \quad x_i > y_i,\ k+1 \leq i \leq 2k. \tag{4.5}$$

Proof of Lemma 4.4:

We need to show that $F(X, Y)$ is not maximized if (X, Y) is not co-singular. We prove the lemma by using induction on k.

(i)

We first show that the lemma holds when $k = 1$. Let

$$\begin{aligned} X &= (x_1, x_2), \\ Y &= (y_1, y_2), \end{aligned}$$

then

$$\begin{aligned} F(X,Y) &= ay_1 R(x_1, x_2) + bx_1 R(y_1, y_2) \\ &= ay_1(x_1 + x_2 - x_1x_2) + bx_1(y_1 + y_2 - y_1y_2). \end{aligned} \tag{4.6}$$

Without loss of generality, assume $x_1 < y_1, x_2 < y_2$, i.e., (X,Y) is not co-singular. Let

$$\begin{aligned} X' &= (x_1, y_2), \\ Y' &= (y_1, x_2), \end{aligned}$$

where (X', Y') is a co-singular set obtained from (X, Y). Then

$$F(X',Y') = ay_1(x_1 + y_2 - x_1y_2) + bx_1(y_1 + x_2 - y_1x_2). \tag{4.7}$$

Now we prove that $F(X',Y') > F(X,Y)$.

$$\begin{aligned} \Delta F &= F(X',Y') - F(X,Y) \\ &= (y_2 - x_2)(ay_1 - bx_1 - ay_1x_1 + bx_1y_1) \\ &= (y_2 - x_2)[ay_1(1 - x_1) - bx_1(1 - y_1)] \; > \; 0. \end{aligned} \tag{4.8}$$

(ii)

Assume Lemma 4.4 holds for $k < l$, we prove that it is also true for $k = l$. Let

$$\begin{aligned} X &= (x_1, x_2, \ldots, x_l, x_{l+1}, \ldots, x_{2l}), \\ Y &= (y_1, y_2, \ldots, y_l, y_{l+1}, \ldots, y_{2l}), \end{aligned}$$

then

$$\begin{aligned}
F(X,Y) &= a\left(\prod_{i=1}^{l} y_i\right) R(X) + b\left(\prod_{i=1}^{l} x_i\right) R(Y) \\
&= a\left(\prod_{i=1}^{l} y_i\right) R(x_1, x_2, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_{2l}) \\
&\quad + b\left(\prod_{i=1}^{l} x_i\right) R(y_1, y_2, \ldots, y_{l-1}, y_l, y_{l+1}, \ldots, y_{2l}) \\
&= a\left(\prod_{i=1}^{l} y_i\right) x_l R(x_1, x_2, \ldots, x_{l-1}, 1, x_{l+1}, \ldots, x_{2l}) \\
&\quad + a\left(\prod_{i=1}^{l} y_i\right) (1 - x_l) \left(\prod_{i=1}^{l} x_{l+i}\right) \\
&\quad + b\left(\prod_{i=1}^{l} x_i\right) y_l R(y_1, y_2, \ldots, y_{l-1}, 1, y_{l+1}, \ldots, y_{2l}) \\
&\quad + b\left(\prod_{i=1}^{l} x_i\right) (1 - y_l) \left(\prod_{i=1}^{l} y_{l+i}\right) \\
&= x_l y_l \left[a\left(\prod_{i=1}^{l-1} y_i\right) R\overbrace{(x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system} \right. \\
&\quad \left. + b\left(\prod_{i=1}^{l-1} x_i\right) R\overbrace{(y_1, y_2, \ldots, y_{l-1}, y_{l+1}, \ldots, y_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system} \right] \\
&\quad + a\left(\prod_{i=1}^{l} y_i\right) (1 - x_l) \left(\prod_{i=1}^{l} x_{l+i}\right) \\
&\quad + b\left(\prod_{i=1}^{l} x_i\right) (1 - y_l) \left(\prod_{i=1}^{l} y_{l+i}\right) \qquad (4.9)
\end{aligned}$$

$$= x_l y_l c + d, \tag{4.10}$$

where

$$\begin{aligned}
c &= a\left(\prod_{i=1}^{l-1} y_i\right) R\overbrace{(x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system} \\
&\quad + b\left(\prod_{i=1}^{l-1} x_i\right) R\overbrace{(y_1, y_2, \ldots, y_{l-1}, y_{l+1}, \ldots, y_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system}, \\
d &= a\left(\prod_{i=1}^{l} y_i\right)(1 - x_l)\left(\prod_{i=1}^{l} x_{l+i}\right) \\
&\quad + b\left(\prod_{i=1}^{l} x_i\right)(1 - y_l)\left(\prod_{i=1}^{l} y_{l+i}\right).
\end{aligned}$$

Denote (X', Y') a co-singular set obtained from (X, Y), where

$$\begin{aligned}
X' &= (x'_1, x'_2, \ldots, x'_l, x'_{l+1}, \ldots, x'_{2l}), \\
Y' &= (y'_1, y'_2, \ldots, y'_l, y'_{l+1}, \ldots, y'_{2l}),
\end{aligned}$$

and

$$x'_i < y'_i, \ 1 \le i \le l; \quad x'_i > y'_i, \ l + 1 \le i \le 2l. \tag{4.11}$$

Then

$$\begin{aligned}
F(X', Y') &= x'_l y'_l \Bigg[a\left(\prod_{i=1}^{l-1} y'_i\right) R\overbrace{(x'_1, x'_2, \ldots, x'_{l-1}, x'_{l+1}, \ldots, x'_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system} \\
&\qquad + b\left(\prod_{i=1}^{l-1} x'_i\right) R\overbrace{(y'_1, y'_2, \ldots, y'_{l-1}, y'_{l+1}, \ldots, y'_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system} \Bigg] \\
&\quad + a\left(\prod_{i=1}^{l} y'_i\right)(1 - x'_l)\left(\prod_{i=1}^{l} x'_{l+i}\right)
\end{aligned}$$

$$+b\left(\prod_{i=1}^{l} x'_i\right)(1-y'_l)\left(\prod_{i=1}^{l} y'_{l+i}\right) \tag{4.12}$$

$$= x'_l y'_l c' + d', \tag{4.13}$$

where

$$c' = a\left(\prod_{i=1}^{l-1} y'_i\right) R\overbrace{(x'_1, x'_2, \ldots, x'_{l-1}, x'_{l+1}, \ldots, x'_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system}$$

$$+b\left(\prod_{i=1}^{l-1} x'_i\right) R\overbrace{(y'_1, y'_2, \ldots, y'_{l-1}, y'_{l+1}, \ldots, y'_{2l-1})}^{(l-1)-out-of-2(l-1):G\ system},$$

$$d' = +a\left(\prod_{i=1}^{l} y'_i\right)(1-x'_l)\left(\prod_{i=1}^{l} x'_{l+i}\right)$$

$$+b\left(\prod_{i=1}^{l} x'_i\right)(1-y'_l)\left(\prod_{i=1}^{l} y'_{l+i}\right).$$

By induction assumption, Lemma 4.2 and the way (X', Y') was obtained from (X, Y) we have

$$x'_l y'_l = x_l y_l, \tag{4.14}$$

$$c' > c, \tag{4.15}$$

$$d' > d. \tag{4.16}$$

Therefore

$$F(X', Y') > F(X, Y), \tag{4.17}$$

or Lemma 4.4 holds for $k = l$.

Lemma 4.5: *Let*

$$X = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}),$$

$$
\begin{aligned}
Y &= (y_1, y_2, \dots, y_k, y_{k+1}, \dots, y_{2k}), \\
X1 &= (x_2, x_3, \dots, x_k, x_{k+2}, \dots, x_{2k}), \\
Y1 &= (y_2, y_3, \dots, y_k, y_{k+2}, \dots, y_{2k}), \\
F(X,Y) &= x_{k+1}(1-y_{k+1})\left(\prod_{i=1}^{k} y_i\right) R(X1) + (1-x_{k+1})y_{k+1}\left(\prod_{i=1}^{k} x_i\right) R(Y1),
\end{aligned}
$$

where $R(X1)$ and $R(Y1)$ are the system reliabilities of $X1_G((k-1), 2(k-2))$ and $Y1_G((k-1), 2(k-1))$, respectively. Then, (X, Y) must be co-singular for $F(X,Y)$ to be maximized.

Proof of Lemma 4.5:

First of all, it is easy to see that $F(X,Y) = F(Y,X)$. Assume (X,Y) is not co-singular and let

$$
\begin{aligned}
X' &= (x'_1, x'_2, \dots, x'_k, x'_{k+1}, \dots, x'_{2k}), \\
Y' &= (y'_1, y'_2, \dots, y'_k, y'_{k+1}, \dots, y'_{2k}), \\
X'' &= (x''_1, x''_2, \dots, x''_k, x''_{k+1}, \dots, x''_{2k}), \\
Y'' &= (y''_1, y''_2, \dots, y''_k, y''_{k+1}, \dots, y''_{2k}).
\end{aligned}
$$

Both (X', Y') and $X'', Y'')$ are co-singular arrangement obtained from (X, Y) such that

$$x'_i < y'_i, \ \ 1 \le i \le k; \ \ x'_i > y'_i, \ \ k+1 \le i \le 2k, \tag{4.18}$$

$$x''_i > y''_i, \ \ 1 \le i \le k; \ \ x''_i < y''_i, \ \ k+1 \le i \le 2k, \tag{4.19}$$

By the way (X', Y') and (X'', Y'') were obtained from (X, Y) we have

$$X' = Y'', \quad Y' = X'', \quad \text{and}$$

$$F(X', Y') = F(Y'', X'') = F(X'', Y'').$$

Let

$$\begin{aligned} a &= x_{k+1}(1-y_{k+1})y_1, \\ b &= (1-x_{k+1})y_{k+1}x_1. \end{aligned}$$

We prove the lemma separately for two cases: $a \geq b$ and $a < b$.

(i) $x_{k+1}(1-y_{k+1})y_1 = a \geq b = (1-x_{k+1})y_{k+1}x_1$.

By Lemma 4.4 we have

$$\begin{aligned} &\overbrace{x_{k+1}(1-y_{k+1})y_1}^{a}\left(\prod_{i=2}^{k} y_i'\right) R(x_2', x_3', \ldots, x_k', x_{k+2}', \ldots, x_{2k}') \\ &+\overbrace{(1-x_{k+1})y_{k+1}x_1}^{b}\left(\prod_{i=2}^{k} x_i'\right) R(y_2', y_3', \ldots, y_k', y_{k+2}', \ldots, y_{2k}') \\ > &\overbrace{x_{k+1}(1-y_{k+1})y_1}^{a}\left(\prod_{i=2}^{k} y_i\right) R(x_2, x_3, \ldots, x_k, x_{k+2}, \ldots, x_{2k}) \\ &+\overbrace{(1-x_{k+1})y_{k+1}x_1}^{b}\left(\prod_{i=2}^{k} x_i\right) R(y_2, y_3, \ldots, y_k, y_{k+2}, \ldots, y_{2k}) = F(X,Y). \end{aligned} \tag{4.20}$$

By Lemma 4.3 and (4.18) we have

$$\begin{aligned} &\left(\prod_{i=2}^{k} y_i'\right) R(x_2', x_3', \ldots, x_k', x_{k+2}', \ldots, x_{2k}') \\ > &\left(\prod_{i=2}^{k} x_i'\right) R(y_2', y_3', \ldots, y_k', x_{k+2}', \ldots, x_{2k}') \\ > &\left(\prod_{i=2}^{k} x_i'\right) R(y_2', y_3', \ldots, y_k', y_{k+2}', \ldots, y_{2k}'). \end{aligned} \tag{4.21}$$

By Lemma 4.2, (4.18) and (4.21) we have

$$F(X',Y') = x_{k+1}'(1-y_{k+1}')y_1'\left(\prod_{i=2}^{k} y_i'\right) R(x_2', x_3', \ldots, x_k', x_{k+2}', \ldots, x_{2k}')$$

$$+(1-x'_{k+1})y'_{k+1}x'_1\left(\prod_{i=2}^{k}x'_i\right)R(y'_2,y'_3,\ldots,y'_k,y'_{k+2},\ldots,y'_{2k})$$

$$> \overbrace{x_{k+1}(1-y_{k+1})y_1}^{a}\left(\prod_{i=2}^{k}y'_i\right)R(x'_2,x'_3,\ldots,x'_k,x'_{k+2},\ldots,x'_{2k})$$

$$+\overbrace{(1-x_{k+1})y_{k+1}x_1}^{b}\left(\prod_{i=2}^{k}x'_i\right)R(y'_2,y'_3,\ldots,y'_k,y'_{k+2},\ldots,y'_{2k}). \quad (4.22)$$

From (4.20) and (4.22) we have

$$F(X',Y') \;>\; F(X,Y). \quad (4.23)$$

(ii) $x_{k+1}(1-y_{k+1})y_1 = a < b = (1-x_{k+1})y_{k+1}x_1$.

Similarly as in case (i) it can be proved that

$$F(X'',Y'') \;>\; F(X,Y). \quad (4.24)$$

Lemma 4.1 states that a general consecutive-k-out-of-n:G system with $n < 2k$ can be transformed into an equivalent system with a series subsystem and a consecutive-k_1-out-of-$2k_1$:G subsystem connected in series, where $k_1 = n-k$. Lemma 4.3 implies that the optimal design of a consecutive-k-out-of-n system with $n < 2k$ will have the most reliable components assigned to the series subsystem. By Lemma 4.1 and Lemma 4.3 the optimal design of system $X_G(k_1,l)\cap Y_G(k_2,m)$ with $l \le 2k_1, m \le 2k_2$ is obtained if the optimal design of system $X'_G((l-k_1),2(l-k_1))\cap Y'_G((m-k_2),2(m-k_2))$ is available. Thus we first study the optimal design of system $X_G(k,2k)\cap Y_G(k,2k)$.

Theorem 4.1: *Let*

$$X = (x_1,x_2,\ldots,x_k,x_{k+1},\ldots,x_{2k}),$$
$$Y = (y_1,y_2,\ldots,y_k,y_{k+1},\ldots,y_{2k}).$$

Then (X,Y) *must be co-singular for it to be an optimal arrangement of system* $X_G(k,2k)\cap Y_G(k,2k)$, *i.e.,*

$$x_i < y_i,\ \ 1\le i\le k;\ \ \ x_i > y_i,\ \ k+1\le i\le 2k. \tag{4.25}$$

Proof of Theorem 4.1:

We need to show that the system reliability of $X_G(k,2k)\cap Y_G(k,2k)$ is not maximized if (X,Y) is not co-singular or does not satisfy (4.25). We prove the theorem by using induction on k. When $k=1$ $X_G(1,2)\cap Y_G(1,2)$ is a series-parallel system with each station having a redundancy of 2. As stated in Chapter 2, the optimal design of $X_G(1,2)\cap Y_G(1,2)$ has the best and the worst components in X, and the rest two components in Y. Thus the theorem holds for $k=1$. Assume Theorem 4.1 holds for $k<l$. We will demonstrate that the theorem is also true when $k=l$. Let

$$\begin{aligned} X &= (x_1,x_2,\ldots,x_l,x_{l+1},\ldots,x_{2l}),\\ Y &= (y_1,y_2,\ldots,y_l,y_{l+1},\ldots,y_{2l}). \end{aligned}$$

Assume (X,Y) is not co-singular and (X',Y') is obtained from (X,Y) and is co-singular, where

$$\begin{aligned} X' &= (x'_1,x'_2,\ldots,x'_l,x'_{l+1},\ldots,x'_{2l}),\\ Y' &= (y'_1,y'_2,\ldots,y'_l,y'_{l+1},\ldots,y'_{2l}). \end{aligned}$$

Denote R_s the system reliability of $X_G(l,2l)\cap Y_G(l,2l)$, R'_s the system reliability of $X'_G(l,2l)\cap Y'_G(l,2l)$. We show that $R'_s > R_s$.

$$R_s = \left[x_{l+1}R(x_1,x_2,\ldots,x_l,1,x_{l+2},\ldots,x_{2l}) + \left(\prod_{i=1}^{l} x_i\right)(1-x_{l+1})\right]$$

$$
\begin{aligned}
&\times \left[y_{l+1} R(y_1, y_2, \ldots, y_l, 1, y_{l+2}, \ldots, y_{2l}) + \left(\prod_{i=1}^{l} y_i \right) (1 - y_{l+1}) \right] \\
=\ & x_{l+1} y_{l+1} R \overbrace{(x_2, x_3, \ldots, x_l, x_{l+2}, \ldots, x_{2l})}^{(l-1)-out-of-2(l-1):G\ system} R \overbrace{(y_2, y_3, \ldots, y_l, y_{l+2}, \ldots, y_{2l})}^{(l-1)-out-of-2(l-1):G\ system} \\
& + x_{l+1}(1 - y_{l+1}) \left(\prod_{i=1}^{l} y_i \right) R \overbrace{(x_2, x_3, \ldots, x_l, x_{l+2}, \ldots, x_{2l})}^{(l-1)-out-of-2(l-1):G\ system} \\
& + (1 - x_{l+1}) y_{l+1} \left(\prod_{i=1}^{l} x_i \right) R \overbrace{(y_2, y_3, \ldots, y_l, y_{l+2}, \ldots, y_{2l})}^{(l-1)-out-of-2(l-1):G\ system} \\
& + (1 - x_{l+1})(1 - y_{l+1}) \left(\prod_{i=1}^{l} x_i \right) \left(\prod_{i=1}^{l} y_i \right) \qquad (4.26) \\
=\ & x_{l+1} y_{l+1} c + d + e, \qquad (4.27)
\end{aligned}
$$

where

$$
\begin{aligned}
c &= R \overbrace{(x_2, x_3, \ldots, x_l, x_{l+2}, \ldots, x_{2l})}^{(l-1)-out-of-2(l-1):G\ system} R \overbrace{(y_2, y_3, \ldots, y_l, y_{l+2}, \ldots, y_{2l})}^{(l-1)-out-of-2(l-1):G\ system}, \\
d &= x_{l+1}(1 - y_{l+1}) \left(\prod_{i=1}^{l} y_i \right) R \overbrace{(x_2, x_3, \ldots, x_l, x_{l+2}, \ldots, x_{2l})}^{(l-1)-out-of-2(l-1):G\ system} \\
&\quad + (1 - x_{l+1}) y_{l+1} \left(\prod_{i=1}^{l} x_i \right) R \overbrace{(y_2, y_3, \ldots, y_l, y_{l+2}, \ldots, y_{2l})}^{(l-1)-out-of-2(l-1):G\ system}, \\
e &= (1 - x_{l+1})(1 - y_{l+1}) \left(\prod_{i=1}^{l} x_i \right) \left(\prod_{i=1}^{l} y_i \right).
\end{aligned}
$$

$$
R'_s = x'_{l+1} y'_{l+1} c' + d' + e', \qquad (4.28)
$$

where

$$
\begin{aligned}
c' &= R \overbrace{(x'_2, x'_3, \ldots, x'_l, x'_{l+2}, \ldots, x'_{2l})}^{(l-1)-out-of-2(l-1):G\ system} R \overbrace{(y'_2, y'_3, \ldots, y'_l, y'_{l+2}, \ldots, y'_{2l})}^{(l-1)-out-of-2(l-1):G\ system}, \\
d' &= x'_{l+1}(1 - y'_{l+1}) \left(\prod_{i=1}^{l} y'_i \right) R \overbrace{(x'_2, x'_3, \ldots, x'_l, x'_{l+2}, \ldots, x'_{2l})}^{(l-1)-out-of-2(l-1):G\ system}
\end{aligned}
$$

$$+(1-x'_{l+1})y'_{l+1}\left(\prod_{i=1}^{l}x'_i\right)R\overbrace{(y'_2,y'_3,\ldots,y'_l,y'_{l+2},\ldots,y'_{2l})}^{(l-1)-out-of-2(l-1):G\ system},$$

$$e' = (1-x'_{l+1})(1-y'_{l+1})\left(\prod_{i=1}^{l}x'_i\right)\left(\prod_{i=1}^{l}y'_i\right).$$

By induction assumption, the way (X', Y') is obtained from (X, Y) and Lemma 4.5 we have

$$x'_{l+1}y'_{l+1} = x_{l+1}y_{l+1}, \tag{4.29}$$

$$c' > c, \tag{4.30}$$

$$d' > d, \tag{4.31}$$

$$e' = e. \tag{4.32}$$

From (4.29), (4.30), (4.31) and (4.32) we have proved

$$R'_s > R_s. \tag{4.33}$$

Theorem 4.2: *Let*

$$X = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k}),$$

$$Y = (y_1, y_2, \ldots, y_k, y_{k+1}, \ldots, y_{2k}).$$

Then (4.34) and (4.35) below are necessary and sufficient conditions for the optimal design of system $X_G(k, 2k) \cap Y_G(k, 2k)$.

$$\textit{Both } X \textit{ and } Y \textit{ are optimal,} \tag{4.34}$$

$$(X, Y) \textit{ is co-singular.} \tag{4.35}$$

Proof of Theorem 4.2:

It is obvious that (X, Y) is not optimal if any of X and Y is not optimal, i.e., (4.34) is a necessary condition for the optimal design of $X_G(k,2k) \cap Y_G(k,2k)$. In Theorem 4.1 we have already proved that (4.35) is a necessary condition for the optimal design of $X_G(k,2k) \cap Y_G(k,2k)$. We need only to show that (4.34) and (4.35) are also sufficient conditions for the optimal design of system $X_G(k,2k) \cap Y_G(k,2k)$. We can show that there is only one configuration which satisfies both (4.34) and (4.35). In other words (4.34) and (4.35) can uniquely determine the optimal arrangement of components for system $X_G(k,2k) \cap Y_G(k,2k)$. Assume we have $4k$ components with their reliabilities arranged in ascending order:

$$p_1 < p_2 < \ldots < p_{4k-1} < p_{4k}$$

Let

$$x_1 < x_2 < \ldots < x_{2k},$$

$$y_1 < y_2 < \ldots < y_{2k}.$$

By (4.34), both X and Y must be optimal, we have,

$$X = (x_1, x_3, \ldots, x_{2k-1}, x_{2k}, x_{2(k-1)}, \ldots, x_2), \quad (4.36)$$

$$Y = (y_1, y_3, \ldots, y_{2k-1}, y_{2k}, y_{2(k-1)}, \ldots, y_2). \quad (4.37)$$

Without loss of generality, we assume that $x_1 < y_1$. By (4.35) we have

$$x_1 < y_1, \;\; x_3 < y_3, \;\; \ldots, \;\; x_{2k-3} < y_{2k-3}, \;\; x_{2k-1} < y_{2k-1},$$

$$x_2 > y_2, \;\; x_4 > y_4, \;\; \ldots, \;\; x_{2k-2} > y_{2k-2}, \;\; x_{2k} > y_{2k}.$$

From the inequalities above we get

$$x_1 < y_1 < y_2 < x_2 < x_3 < y_3 < \ldots < y_{2k-3} < y_{2k-2} < x_{2k-2} < x_{2k-1} < y_{2k-1} < y_{2k} < x_{2k}.$$

With this string of inequalities we obtain the best design:

$$x_1 = p_1, x_2 = p_4, x_3 = p_5, x_5 = p_8, \ldots, x_{2k-1} = p_{4k-3}, x_{2k} = p_{4k},$$

$$y_1 = p_2, y_2 = p_3, y_3 = p_6, y_5 = p_7, \ldots, y_{2k-1} = p_{4k-2}, y_{2k-1} = p_{4k-1}.$$

Therefore, the invariant optimal partition of components for system $X_G(k, 2k) \cap Y_G(k, 2k)$ is to:

1. *Assign the best of the remaining components to X and the next best to Y.*
2. *Assign the best of the remaining components to Y and the next best to X.*
3. *Repeat steps 1 and 2 alternately until all components have been assigned.*

Theorem 4.3: *The optimal partition of components for system $X_G(k, 2k) \cap Y_G((k-1), 2(k-1))$ is to:*

1. *Assign the best component and the worst component to X.*
2. *Assign the best of the remaining components to X and the next best to Y.*
3. *Assign the best of the remaining components to Y and the next best to X.*
4. *Repeat steps 2 and 3 until all components have been assigned.*

Proof of Theorem 4.3

Theorem 4.3 is a special case of Theorem 4.2. The result in Theorem 4.2 holds for any values of p_i $(1 \leq i \leq 4k)$ such that $0 \leq p_i \leq 1$. Let $p_1 = 0$ and $p_{4k} = 1$.

The optimal design obtained by Theorem 4.2 becomes the optimal design of system $X_G(k,2k) \cap Y_G((k-1),2(k-1))$ as both the dummy and the perfect components must be assigned in the same subsystem, which is actually a consecutive-$(k-1)$-out-of-$2(k-1)$:G subsystem under optimal arrangement.

Theorem 4.4: *Assume we have a series consecutive-k-out-of-n:G system X_M which consists of m consecutive-k-out-2k:G subsystems:*

$$X_M = X_G^1(k,2k) \cap X_G^2(k,2k) \cap \cdots \cap X_G^m(k,2k).$$

If we have 2mk components with their reliabilities arranged in ascending order:

$$p_1 < p_2 < \ldots < p_{2mk-1} < p_{2mk},$$

then the optimal partition of components for X_M can be obtained by the following algorithm:

1. *Set $i = 1$.*
2. *Assign $p_{2(i-1)m+1}, p_{2(i-1)m+2}, \cdots, p_{2(i-1)m+m}$ to $X^1, X^2, \ldots, X^m$, respectively.*
3. *Assign $p_{(2i-1)m+1}, p_{(2i-1)m+2}, \cdots, p_{2im}$ to $X^m, X^{m-1}, \ldots, X^1$, respectively.*
4. *If $i < k$, set $i = i+1$ and go to step 2; stop when $i = k$.*

Proof of Theorem 4.4

Let

$$x_1^1 < x_2^1 < \ldots < x_{2k}^1,$$
$$x_1^2 < x_2^2 < \ldots < x_{2k}^2,$$
$$\vdots$$
$$x_1^m < x_2^m < \ldots < x_{2k}^m,$$

First of all every subsystem X^i $(1 \le i \le m)$ must be optimal, or

$$X^1 = (x_1^1, x_3^1, \ldots, x_{2k-1}^1, x_{2k}^1, \ldots, x_4^1, x_2^1), \tag{4.38}$$

$$X^2 = (x_1^2, x_3^2, \ldots, x_{2k-1}^2, x_{2k}^2, \ldots, x_4^2, x_2^2), \tag{4.39}$$

$$\vdots \tag{4.40}$$

$$X^m = (x_1^m, x_3^m, \ldots, x_{2k-1}^m, x_{2k}^m, \ldots, x_4^m, x_2^m). \tag{4.41}$$

Without loss of generality we assume

$$x_1^1 < x_1^2 < \ldots < x_1^m.$$

If X_M is optimal all pairs of subsystems $X^i \cap X^{i+1}$ for $i = 1, 2, \ldots, m-1$, and $X^m \cap X^1$ must be optimal. By (4.35) we have

$$x_1^i < x_2^i < \ldots < x_m^i, \quad for\ i = 1, 3, \ldots, 2k-1, \tag{4.42}$$

$$x_1^i > x_2^i > \ldots > x_m^i, \quad for\ i = 2, 4, \ldots, 2k, \tag{4.43}$$

Theorem 4.4 follows immediately from (4.42) and (4.43).

Theorem 4.5: *Assume there is a series consecutive-k-out-n:G system with l consecutive-k-out-of-2k:G subsystems and m consecutive-(k−1)-out-of-2(k−1):G subsystems. The optimal partition of components for the system can be obtained as follows:*

1. *Add m perfect components and m dummy components to the set of components available.*

2. *Do optimal design as if for a series consecutive-k-out-of-n:G system with (l+m) consecutive-k-out-of-2k:G subsystems.*

Proof of Theorem 4.5:

Theorem 4.5 is a special case of Theorem 4.4 as Theorem 4.3 is a special case of Theorem 4.2.

The optimal design obtained by Theorem 4.5 has $(l+m)$ consecutive-k-out-of-$2k$:G subsystems, of which m subsystems have both a perfect and a dummy components. Each of these m subsystems is actually a consecutive-$(k-1)$-out-of-$2(k-1)$:G subsystem under optimal design. Therefore, the optimal design obtained by Theorem 4.5 is the optimal design for a series consecutive-k-out-of-n:G system with l consecutive-k-out-of-$2k$:G subsystems and m consecutive-$(k-1)$-out-of-$2(k-1)$:G subsystems.

4.2 Relationship Between Consecutive-k-out-of-$2k$:G System and Consecutive-2-out-of-$2k$:F System

From Section 3.1 and Section 4.3 we found that given a set of components the optimal partition of these components for a series consecutive-k-out-of-n:G system $X_G(k,2k) \cap Y_G(k,2k)$ is the same as that of a series consecutive-2-out-of-n:F system $X_F(2,2k) \cap Y_F(2,2k)$. We take examples to demonstrate this relationship.

First consider the optimal design of series consecutive-2-out-of-4:G system $X_G(2,4) \cap Y_G(2,4)$, which is the simplest case. Assume we have eight components with reliabilities:

$$p_1 < p_2 < \ldots < p_8,$$

and

$$X = (x_1, x_2, x_3, x_4),$$

$$Y = (y_1, y_2, y_3, y_4).$$

The set of minimal cuts of system $X_G(2,4)$ is:

$$(1)\ x_1, x_3 \quad (2)\ x_2, x_3 \quad (3)\ x_2, x_4.$$

Now we look at system $X'_F(2,4)$, where $X' = (x_1, x_3, x_2, x_4)$. Its minimal cut set is:

$$(1)\ x_1, x_3 \quad (2)\ x_2, x_3 \quad (3)\ x_2, x_4.$$

$X_G(2,4)$ and $X'_F(2,4)$ have exactly the same set of minimal cuts. The two systems are therefore equivalent in terms of system reliability evaluation although they are different kinds of systems. Similarly, system $Y_G(2,4)$ is equivalent to system $Y'_F(2,4)$ where $Y' = (y_1, y_3, y_2, y_4)$. Thus, system $X_G(2,4) \cap Y_G(2,4)$ is equivalent to system $X'_F(2,4) \cap Y'_G(2,4)$. In Chapter 3 we have already known that $X'_F(2,4) \cap Y'_F(2,4)$ has invariant optimal design,

$$X' = (p_1, p_8, p_5, p_4), \tag{4.44}$$

$$Y' = (p_2, p_7, p_6, p_3). \tag{4.45}$$

From (4.44) and (4.45) we directly obtain the optimal design for system $X_G(2,4) \cap Y_G(2,4)$,

$$X = (p_1, p_5, p_8, p_4), \tag{4.46}$$

$$Y = (p_2, p_6, p_7, p_3). \tag{4.47}$$

When $k > 2$ the relationship between $X_G(k,2k)$ and $X'_F(2,2k)$ is not as simple as that between $X_G(2,4)$ and $X'_F(2,4)$. Consider a consecutive-4-out-of-8:G system

$X_G(4,8)$. Let

$$\begin{aligned} X &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8), \\ X^1 &= (x_1, x_5, x_2, x_6, x_3, x_7, x_4, x_8), \\ X^2 &= (x_3, x_5, x_4, x_6). \end{aligned}$$

The minimal cut set of $X_G(4,8)$ is:

(x_1,x_5), (x_2,x_5), (x_2,x_6), (x_3,x_5), (x_3,x_6), (x_3,x_7), (x_4,x_5), (x_4,x_6), (x_4,x_7), (x_4,x_8).

The minimal cut set of $X_F^1(2,8)$ is:

$$(x_1,x_5),\ (x_5,x_2),\ (x_2,x_6),\ (x_6,x_3),\ (x_3,x_7),\ (x_7,x_4),\ (x_4,x_8).$$

The minimal cut set of $X_F^2(2,4)$ is:

$$(x_3,x_5),\ (x_5,x_4),\ (x_4,x_6).$$

It's obvious that the minimal cut set of $X_G(4,8)$ is the union of the minimal cut sets of $X_F^1(2,8)$ and $X_F^2(2,4)$. Denote A the event that system $X_G(4,8)$ works, A_1 the event that $X_F^1(2,8)$ works and A_2 the event that $X_F^2(2,4)$ works. Then

$$R(X_G(4,8)) = Pr(A) = Pr(A_1 \cap A_2) = R(X_F^1(2,8) \cap X_F^2(2,4)). \quad (4.48)$$

$X_G(4,8)$ can be converted to an equivalent system $X_F^1(2,8) \cap X_F^2(2,4)$, where X^1 is a different arrangement of the components in X and can be determined by X; $X_F^2(2,4)$ is a correlated subsystem of $X_F^1(2,8)$ and X^2 can be determined by X^1 or X. If system $X_G(4,8)$ is of optimal design both $X_F^1(2,8)$ and $X_F^2(2,4)$ are of optimal design. This relationship can be stated conversely. Given an arrangement of system $X_F^1(2,8)$, we can find a correlated subsystem $X_F^2(2,4)$ such that system $X_F^1(2,8) \cap X_F^2(2,4)$ is equivalent to system $X_G(4,8)$. Optimal arrangement of components in system $X_F^1(2,8)$ will result in the optimal arrangement of components in subsystem $X_F^2(2,4)$ and therefore the optimal design of system $X_F^1(2,8) \cap X_F^2(2,4)$ or system $X_G(4,8)$. There is the same relationship between $Y_G(4,8)$, $Y_F^1(2,8)$ and $Y_F^2(2,4)$, where

$$\begin{aligned} Y &= (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8), \\ Y^1 &= (y_1, y_5, y_2, y_6, y_3, y_7, y_4, y_8), \\ Y^2 &= (y_3, y_5, y_4, y_6). \end{aligned}$$

Denote B the event that system $Y_G(4,8)$ works, B_1 the event that $Y_F^1(2,8)$, B_2 the event that $Y_F^2(2,4)$ works. Then

$$R(Y_G(4,8)) = Pr(B) = Pr(B_1 \cap B_2) = R(Y_F^1(2,8) \cap Y_F^2(2,4)). \tag{4.49}$$

Now we observe system $X_G(4,8) \cap Y_G(4,8)$.

$$\begin{aligned} R(X_G(4,8) \cap Y_G(4,8)) &= Pr(A \cap B) \\ &= Pr(A_1 \cap A_2 \cap B_1 \cap B_2) \\ &= Pr((A_1 \cap B_1) \cap (A_2 \cap B_2)). \end{aligned} \tag{4.50}$$

In (4.50), $Pr(A_1 \cap B_1)$ is the reliability of series consecutive-2-out-of-8:F system $X_F^1(2,8) \cap Y_F^1(2,8)$, and $Pr(A_2 \cap B_2)$ is the reliability of system $X_F^2(2,4) \cap Y_F^2(2,4)$. The arrangement of components in system $X_G(4,8) \cap Y_G(4,8)$ determines the arrangements of components in system $X_F^1(2,8) \cap Y_F^1(2,8)$ and system $X_F^2(2,4) \cap Y_F^2(2,4)$.

Both systems $X_F^1(2,8) \cap Y_F^1(2,8)$ and $X_F^2(2,4) \cap Y_F^2(2,4)$ are of optimal design if system $X_G(4,8) \cap Y_G(4,8)$ is optimally designed. Conversely, given a design of system $X_F^1(2,8) \cap Y_F^1(2,8)$, a correlated subsystem $X_F^2(2,4) \cap Y_F^2(2,4)$ can be determined such that system $X_F^1(2,8) \cap Y_F^1(2,8) \cap X_F^2(2,4) \cap Y_F^2(2,4)$ is equivalent to a series consecutive-4-out-of-8:G system $X_G(4,8) \cap Y_G(4,8)$. The optimal design of system $X_F^1(2,8) \cap Y_F^1(2,8)$ will result in the optimal design of subsystem $X_F^2(2,4) \cap Y_F^2(2,4)$, and therefore the optimal design of system $X_F^1(2,8) \cap Y_F^1(2,8) \cap X_F^2(2,4) \cap Y_F^2(2,4)$ or system $X_G(4,8) \cap Y_G(4,8)$.

As shown early, the proof of the optimal design of system $X_F(2,2k) \cap Y_F(2,2k)$ is totally different from the way of finding the optimal design of system $X_G(k,2k) \cap Y_G(k,2k)$ although the two systems have exactly the same best partition of components. The optimal design of system $X_G(k,2k) \cap Y_G(k,2k)$ with $k > 2$ may be derived directly from the optimal design of system $X_F(2,2k) \cap Y_F(2,2k)$, or vice versa, as in the case where $k = 2$. We need to gain deeper understanding of the relationship between systems $X_G(k,2k)$ and $X_F(2,2k)$ before we could reach any conclusion.

CONCLUSION

We studied optimal design of systems with linear consecutive-k-out-of-n:G subsystems connected in series, and identified and proved the invariant optimal design of series consecutive-k-out-of-n:G systems with $n \leq 2k$. A system is called a series circular consecutive-k-out-of-n:G system if it has circular consecutive-k-out-of-n:G subsystems connected in series. The optimal designs of series circular consecutive-k-out-of-n:G systems are yet to be investigated.

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APPENDIX A.

A Counter Example

There is no invariant optimal design for system $X_F(3,5) \cap Y_F(3,5)$. Assume we have ten components with reliabilities

$$p_1 < p_2 < \ldots < p_{10}.$$

When

$$p_1 = 0.042, p_2 = 0.214, p_3 = 0.330, p_4 = 0.573, p_5 = 0.596,$$

$$p_6 = 0.649, p_7 = 0.653, p_8 = 0.661, p_9 = 0.747, p_{10} = 0.861,$$

the optimal design for $X_F(3,5) \cap Y_F(3,5)$ is:

$$X = (p_1, p_6, p_{10}, p_5, p_2),$$

$$Y = (p_3, p_8, p_9, p_7, p_4).$$

While when

$$p_1 = 0.219, p_2 = 0.312, p_3 = 0.734, p_4 = 0.744, p_5 = 0.746,$$

$$p_6 = 0.802, p_7 = 0.878, p_8 = 0.934, p_9 = 0.947, p_{10} = 0.958,$$

the optimal design for $X_F(3,5) \cap Y_F(3,5)$ is:

$$X = (p_3, p_5, p_{10}, p_7, p_2),$$

$$Y = (p_1, p_8, p_9, p_6, p_4).$$