MINT 709 Capstone Project Report

AUC based Spectrum Sensing for Cognitive LTE Networks in Random Field of Interferers

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Abstract

Long-term evolution (LTE) networks are expanding worldwide due to their capability to provide high data rates to fulfill the growing data needs of mobile subscribers. RF spectrum is limited but its underutilization means that more subscribers can be accommodated in the LTE network. The technology cognitive radio (CR) can increase the utilization of RF spectrum making more spectrum available to LTE networks. Utilization of RF spectrum fluctuates depending on time and location, meaning, specific spectrum is idle in some places and during some time intervals. This underutilized spectrum, called spectrum holes, is dynamically sensed by CR and used for communication. A spectrum sensing technique used in LTE should be simple to implement and highly reliable for accurately sensing spectrum holes. Energy detection (ED) is one of simplest and popular technologies used for spectrum sensing. However, energy detector performance degrades when interfering signals from random number of users are present. In contrast to the traditional detection performance evaluation using receiver operating characteristic (ROC) curves; the area under ROC curves (AUC) is a single figure of merit for ED performance evaluation. Thus, AUC based performance evaluation of ED is used in this report to quantify ED performance in the presence of fading and interfering signals from spectrum users. An increase in the number of antennas (antenna diversity) capturing the signal was used to take advantage of multipath signal propagation and was found to improve ED performance significantly.

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List of Acronyms

3GPP 3rd Generation Partnership Project
CR Cognitive Radio
PU Primary user
ED Energy Detector
SNR Signal-to-noise ratio
DTV Digital Television
UTMS Universal Mobile Telecommunication System
OFDMA Orthogonal Frequency Division Multiple Access
ISI Inter-symbol interference
OFDM Orthogonal Frequency Division Multiplexing
SC-FDMA Single Carrier-Frequency Division Multiple Access
MIMO Multiple-input and Multiple-output
UE User equipment
HeNB Home evolved Node B
AWGN additive white Gaussian noise
AWGN additive white Gaussian noise
MIMO Multiple-input Multiple-output
ROC Receiver Operating Characteristic Curve
AUC Area Under ROC Curve

List of Symbols

Listed below are the n	isted below are the notations used throughout the report.	
Ω	Average fading power	
H_0	Hypothesis 0	
H_1	Hypothesis 1	
K	Rice factor	
λ	Detection threshold	
L	Total number of receiver antennas	
N	Total number of signal samples	
n(t)	Additive white Gaussian noise	
P_d	Probability of detection	
P_f	Probability of false alarm	
P_e	Probability of error in decision making	
P_{md}	Probability of missed detection	
p(r)	Probability Density Function (PDF)	
p	<i>p</i> -norm detection parameter	
h	Channel fading amplitude	
x(t)	Signal transmitted by PU	
T	Detector decision variable	
y(t)	Signal observed by CR	
Y_k	Energy from kth square-law device	

Chapter 1

Introduction

The importance of wireless communication is increasing globally throughout the world due to advances in cellular and broadband technologies. Every mobile subscriber wants to be connected all the time to the outside world through the best network. Smartphone capabilities and combination of networks with high spectral efficiency results in better user experience and thus increased mobile subscribers on the network [1]. There will be over 9.2 billion devices connected wirelessly by 2019, including computer to computer mobile connections [2]. Mobile broadband is also growing fast; by 2016 there are expected to be close to five billion mobile broadband subscriptions worldwide [1]. [1] and [2] show that the mobile connected devices will be more than the world population by end of 2014 and the need for high speed data connections will increase. Table 1.1 gives a forecast of the expected increase in the number of mobile devices connected worldwide by 2019. There is a 55% compound annual growth rate in long term evolution (LTE) uasge. Thus, this also indicates that the majority of these data hungry devices will most likely be served by LTE networks. The compound annual growth rate for LTE networks is high as LTE delivers 300 Mbps in downlink and 75 Mbps in uplink, assuming a bandwidth of 20 MHz [3]. To accommodate more mobile subscribers in the network, more RF spectrum is needed which will lead to further scarcity of limited radio spectrum which has already been allocated to specific services by government agencies around the world.

However, to accommodate the increasing number of users in a network and fulfill their increasing data demands, available RF spectrum should be used efficiently. Studies done by the FCC's Spectrum Policy task force show that spectrum usage varies significantly depending on time of day, frequency, geographical location and thus the RF spectrum is currently being underutilized [4]. For example, spectrum measurements taken in New York City show a maximum of 13.1% occupancy in the 30 MHz to 3 GHz band, which means that 86.9% of the spectrum is underutilized in this radio spectrum band [5].

To address the underutilization, technology called Cognitive Radio (CR) has emerged to utilize the available spectrum efficiently and deal with spectrum scarcity generated by inefficient spectrum use. CR is a radio device that can sense the spectral environment in its vicinity over a large frequency range and use this knowledge to opportunistically access these bands to meet user communications requirements. CR allows licensed radio frequency (RF) to be shared with unlicensed users in an opportunistic manner; that is, an unlicensed user can use the licensed frequency band while the licensed Primary user (PU) is not using it. Use by the secondary user when the frequency band is idle does not effect the PUs performance and access to the band, thus, the spectrum is used efficiently [6]. The main functions of CR are sensing spectrum holes, accessing idle channels, sharing the channel, and vacating the channel when the PU becomes active. Among these functions, spectrum sensing is the first and most important step toward using an underutilized spectrum. Techniques used to sense spectrum holes include cyclostationary detection, matched filter detection, local oscillator detection, and Energy Detector (ED) [7]. Among these methods ED is used the most because of its ease of implementation and fast sensing decision requiring no in-depth information about the signal to be detected [8]. Although ED is easy to implement, like other detection methods it has some demerits: ED is effected by receiver noise, multipath fading (due to scattering, diffraction, reflection of the signal resulting in multiple copies of same signal being received), and ED cannot distinguish between these different type of signals. A CR deployed in LTE may be effected by interference from other eNBs, or a stronger signal from macro- or microcells can be interfering with a weaker signal in femtocells and picocells [7]. Therefore, performance of ED deployed in LTE in the presence of random number of interfering nodes needs to be analysed. To evaluate energy detection performance in the presence of randomely interfering nodes, the following objectives will be pursued in this research.

1.1 Problems

Problem1. ED is the most popular and widely used dynamic spectrum access technique. But its performance degrades due to multipath fading and interference from other nodes whose both position from sensing CR and number are random. The performance of ED is traditionally

Mobile subscriptions	2013 (millions)	2019 (millions)
Worldwide mobile	6700	9200
Smartphones	1900	5600
Mobile broadband	2200	7600
WCDMA/HSPA	1600	4500
LTE	200	2600
Mobile PC, tablet and mobile router	300	700

Table 1.1: Expected increase in worldwide mobile subscriptions by 2019 [2].

characterized using the Receiver Operating Characteristic Curve (ROC) curves. Although the ROC curves fully characterize the performance of an ED, it requires careful selection of the detection threshold, which is an overhead to the sensing task. Instead, it is desirable to have a single figure of merit which avoid such overhead. Such a measure is the Area Under ROC Curve (AUC). The AUC is used to measure the detection capability of ED [9]. In first half of this report, ED performance will be quantified in an LTE network in the presence of interference from interfering nodes (which can be signal from a different eNB or a different cell) using AUC based analysis.

Problem2. Multipath fading is known to have detrimental impact wireless signal reception. To mitigate this problem, antenna diversity can be used. Antenna diversity enhances the performance of a communication system, as multiple copies of same signal are received at the receiver and these copies are combined to get the transmitted signal. To study the possible benefits of antenna diversity on AUC performance, we extend the problem 1 to a scenario where ED is equipped with multiple antennas.

1.2 Objectives

The major objectives of this project include:

- 1. Analysis of AUC based spectrum sensing in a random field of interferers.
- 2. An AUC based analysis of the performance of ED equipped with multiple antennas.

1.3 Potential impact and significance of the research

The increasing number of mobile subscribers and their appetite for data puts a burden on existing networks in terms of needed frequency spectrum. As the available spectrum is limited, cognitive radios with dynamic spectrum sensing are important in mitigating the effect of increasing

subscribers on networks. CRs sense and use the underutilized frequency bands. ED is the most utilized spectrum sensing technique as it has low complexity and ease of implementation. However, ED performance degrades with multipath fading scenarios. ED performance needs to be improved without unduly increasing its complexity. This research aims to contribute to better ED performance in high multipath fading scenarios by analysing ED performance using AUC, where ED is equipped with multiple antennas to possibly mitigate the effects of multipath fading. Better ED performance will improve user experience on LTE mobile networks in terms of high download, upload speeds available, and connectivity in rural and highly dense areas like city centres.

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Chapter 2

Literature Review

2.1 Need for LTE and CR

In early times mobile networks were used primarily to serve voice traffic generated by voice calls. In the past decade voice calls monopolized the traffic in mobile networks. The use of mobile data was initially low, but in the years leading up to 2010 mobile data use started to increase dramatically. Measurements of the total traffic being transfered by mobile networks around the world show data usage is in petabytes (million gigabytes) per month [10]. Reports show that the volume of data traffic increased by over 500 times between 2007 and 2013 and this trend is going to continue [1], [2]. This growth in data demand has been increased by availability of 3G communication technologies. The introduction of smartphones was a big factor in growing the mobile subscriber base. Smartphones are more useful and attractive than last generation phones which only supported calling. Smartphones support various applications created by developers causing an exponential increase in the number of applications being used by mobile subscribers. Thus, the majority of traffic in mobile networks is generated by smart devices running a variety of applications and by videos being watched on mobile devices. As more smart devices, applications, and videos evolve, data traffic on the network will increase. This increased traffic began to result in congestion of 2G and 3G networks around 2010 [11]. The need to serve the increasing demand for data has led mobile network operators to invest in network modernization by adopting new technologies such as high speed packet access (HSPA) and LTE for better performance [2]. Recent FCC studies show that the radio spectrum available is not being used efficiently [4]. For example, bandwidth allocated to a Digital Television (DTV) subscriber is not always used which makes the bandwidth allocation schemes inefficient. To take advantage of network underutilization, a technique called cognitive radio (CR) has come into existence. CRs are capable of identifying spectrum holes (idle spectrum) and utilizing these spectrum holes by not effecting the PUs performance. Thus CRs are meeting the growth in demand for radio spectrum [12].

2.2 LTE

LTE is designed by a join effort of national and regional telecommunications standards bodies called 3rd Generation Partnership Project (3GPP) [1] and in full it is called 3GPP Long-Term Evolution. LTE has emerged from a prior 3GPP system called the Universal Mobile Telecommunication System (UTMS), which emerged from the Global System for Mobile Communications (GSM). LTE supports flexible carrier bandwidths, provides law latency, high data rates, and good spectral efficiency for fully Internet protocol (IP) based networks. LTE supports a fully packet switched IP network which is different from previous circuit switched networks and being a IP based network it has the ability to manage and provide high quality service. LTE provides seamless IP connection between user equipment (UE) and the LTE core network, called the packed data network (PDN), without any disruption to the user during mobility [13].

LTE offers better spectral efficiency than 2G and 3G technologies; that means more bits of information can be transferred per Hertz of spectrum per second. LTE offers 300 Mbps peak downlink and 75 Mbps peak uplink speeds in a 20 MHz channel. LTE supports variable carrier bandwidth deployments, ranging from 1.4 MHz up to 20 MHz; network operators can choose a frequency from this pool depending on the availability and the needs of the network. Equipment in LTE supports multiple-input multiple-output (MIMO) capacity to take advantage of multipath propagation which allows an eNB to transmit multiple data streams on the same channel concurrently. LTE supports handover and roaming services to existing 2G and 3G mobile networks [14].

2.2.1 OFDMA

LTE uses Orthogonal Frequency Division Multiple Access (OFDMA) as a mechanism for multiple media access only in downlink. This helps to achieve high transfer rates, and decrease of the influence of multipath. OFDMA is a multiplexing/access technique that multiplexes data flows from different users using different OFDM subchannels [15]. OFDM is used to overcome Inter-symbol interference (ISI). In Orthogonal Frequency Division Multiplexing (OFDM) a nonlinear frequency response channel is divided into multiple small flat fading channels, which helps OFDM to mitigate ISI in wireless communication channel data streams caused by a multipath or nonlinear response of the channel. The data to be transmitted is modulated and distributed between the subcarriers. A OFDM symbol is the collection of all subcarriers that can be transmitted in the available bandwidth [16].

For the LTE uplink Single Carrier-Frequency Division Multiple Access (SC-FDMA) is used. SC-FDMA is a technique with an architecture similar to that of OFDMA, and similar advantages against the fading due to multipath signal which is one of the most important features of OFDM. A waveform OFDMA employs hundreds of subcarriers that can be placed over the entire bandwidth, benefiting users through the frequency diversity. However, a SC-FDMA user can only get frequency diversity by jumping from symbol to symbol. OFDM is highly scalable. The distance between subcarriers is 15 kHz for LTE. LTE is flexible because it has a wide bandwidth (1.4, 3, 5, 10, 15, or 20 MHz) [17].

2.2.2 MIMO

As the data quotient is growing, it is necessary for mobile communication systems to develop new techniques to answer the ever increasing demand for spectrum. A so called multiantenna technology, Multiple-input Multiple-output (MIMO), can increase the amount of information transmitted. MIMO is currently used in LTE, WLAN and WiMax. Multiple antennas, both at the eNodeB and on the User equipment (UE), together with sophisticated signal processing improves the performance of the wireless link, even under the worst case scenarios without line of sight and with mobile users.

In a MIMO system there are two fundamental parameters to consider, diversity and degrees of freedom. Multiple antennas are used to increase the diversity or the degrees of freedom in a mobile communication system. The more independently faded channels there are, the higher the diversity of the system will be. That effect is used for transmitting the same information through independent channels and averaging the received signal. The probability of error decreases and the Signal-to-noise ratio (SNR) is improved with the use of MIMO [18]. The signals coming from different directions provide various degrees of freedom in communications. The same effect is received when the antennas are located nearby if there is dispersion. Essentially, if the channel response between pairs of transmit-receive antennas fades independently, it can be said that there are multiple independent spatial channels. Independent information is transmitted in each of these channels and it increases the transmission rate [16].

2.2.3 Heterogeneous Network

Operators can meet the demands of the increasing subscriber base by either increasing capacity of existing networks by adding RF spectrum or by implementing more efficient modulation and coding schemes and deploying multiantenna techniques for better spectral efficiency. However, these measures alone are not sufficient in crowded regions like city centres or at cell boundaries which are prone to signal degradation due to a low SNR. Addition of small cells like femtocells or picocells in the existing macrocell and microcell networks to distribute traffic loads can help to solve these problems. Expanding the existing macronetwork, while preserving it as a homogeneous network, by adding more cells per eNB or deploying more macro eNBs will also increase network capacity. But adding more macrosites to a network is difficult and expensive, especially in populated and dense areas like city centres. An effective way to increase network capacity is to introduce small cells by adding low power eNBs to existing macro or micro-eNBs. Small cells like picocells and femtocells are added to increase the data rate of a network in extremely crowded areas with high user demand and these cells also fill regions that are not covered by the macronetwork either outdoors or indoors. Small cells improve spectral efficiency and expand indoor coverage within the network in a cost-effective way. Networks with macro, micro, pico, and femtocells provide increased bitrate per unit area [19], [20]. As the area covered by pico or femto cells is very small, the same frequency band can be reused several times in different small cells improving the spectral efficiency of the network. Even in LTE, a frequency reuse of 1 is used. Small cells covering areas between 40 to 75 meters are used to give better coverage in low SNR areas and to remove coverage holes in homogeneous systems. Femtocells which cover very small areas (10 to 29 meters) are used in remote places where no macrocells exist; for example, in remote countryside areas or in office premises. Femtocells are also called home base stations [21]. The Home evolved Node B (HeNB) was introduced in LTE Release 9. HeNB is a low power base station that provides coverage to indoor subscribers, like hotspots or coverage in underground stations [19]. A heterogeneous network increases the complications in networks, especially in LTE where a frequency reuse of 1 is used. Cognitive radio techniques can be helpful in deploying heterogeneous networks in LTE where small cell (femtocell, picocell) base stations can make use of spectrum sensing techniques to use the spectrum holes in the radio spectrum and thus serving subscribers efficiently with efficient RF spectrum utilization.



Figure 2.1: Hetrogeneous network [21].

2.3 Cognitive Radio

Cognitive radio (CR) has emanated as an auspicious technology to overcome the spectrum shortage issue in communication networks which has resulted due to poor spectrum utilization and allocation. FCC reports show that spectrum is underutilized in some geographical areas and during various times leading to an overall scarcity of radio spectrum [4]. Reports indicate that some areas of the RF spectrum are idle most of the time and whereas, some parts of RF spectrum are only used partially . For instance, the portion of licensed spectrum allocated to a digital television user is in use only when the user is watching TV, leaving spectrum holes during other times. A spectrum hole is a part of an RF spectrum assigned to a PU that is not in use by the licensed PU [22]. Cognitive radio, based on a technique called dynamic spectrum sensing, can be employed by unlicensed, or secondary, users (SU) to dynamically sense and locate unused RF channels and opportunistically communicate through the unused spectrum segments. The architecture of a CR transceiver is shown in Figure 2.2. Two important parts of a CR transceiver are the analog front-end that collects the radio signal and performs analog to digital conversion and the baseband processing unit. These components can be configured to adapt to different frequencies in the RF spectrum. The received signal at the CR is amplified and converted into digital form in the analog front end. In baseband processing block, the digital signal is modulated, encrypted or demodulated, decrypted at transmitter and receiver sides, respectively. CRs must have the capability of tuning to a large range of frequency spectrum, therefore, CR transceivers can sense a wide range of RF spectrum [23]. To understand how a CR works we need to understand the following terms.



Figure 2.2: CR transceiver block diagram [24].

2.3.1 **RF** spectrum

The radio frequency band of the electromagnetic spectrum extends from 3 kHz to about 300 GHz. Radio waves are used in communication systems. The International Telecommunication Union (ITU) distributes this RF spectrum to various agencies which then allocate particular slots of RF spectrum for purposes like radio, television broadcasts, mobile device communication, satellite communication, etc. Different frequencies have different properties and, accordingly, different uses. As low frequencies generally reach longer distances than high frequencies and can penetrate physical objects such as trees, buildings, and rain, low frequencies are used to broadcast AM radio. High frequencies have higher data-carrying capacity than low frequencies but are limited with respect to distance travelled and object penetration. Therefore, very high frequencies are generally used in line-of-sight communication. In mobile communication, frequencies between

400 MHz to 4 GHz are used because this range provides a trade-off between high data carrying capacity and long range transmission.

2.3.2 Spectrum holes

Spectrum holes represent potential opportunities for noninterfering spectrum use. Most bands are underused most of the time depending on spatial, geographical, and temporal conditions. A band of spectrum can be considered underutilized if it can accommodate transmissions without harming the operation of the primary user of the band. Figure 2.3 provides an overview of spectrum holes. Figure 2.3 shows that some of frequency bands in the RF spectrum are in use and some are unused, depending on frequency and time. Unused spectrum (spectrum holes) which is licensed can be used by SU's when the primary user is idle. SUs vacate the band when the PU becomes active without affecting its performance [23].



Figure 2.3: Spectrum holes in the RF spectrum [25].

2.4 Spectrum Sensing

Spectrum sensing is the first and fundamental part of a cognitive radio system. It is the task of gaining knowledge about RF spectrum usage and the presence of licensed users in the vicinity of the CR. To make optimum use of available spectrum, the CR is required to accurately sense when the licensed PU is not using the channel and to vacate the channel when the PU becomes active

again. Spectrum sensing techniques are mainly characterized into two classes: firstly primary transmitter detection which is performed based on the detection of a weak signal from a primary transmitter. And in primary receiver detection, the receivers local oscillator power is used to identify PUs that are active within the range of the CR as the local oscillator radiates some power back to the antenna. The most commonly used spectrum sensing techniques are described below.

- Energy detection: ED is a detection technique where decision depends on the PUs transmitted signal and no prior knowledge about PU signal is required. ED uses the power of the received signal to decide whether the PU is active or idle. ED suffers performance degradation in low SNR scenarios and under uncertain noise variance. Regardless of this, ED has a simple design and low complexity and thus is widely used for spectrum sensing [26].
- 2. Cyclostationary detection: As modulated signals have some periodicity, this technique uses the periodicity in the PUs transmitted signal to differentiate between the PUs signal and signals from SUs operating in same frequency region. This detection technique is more immune to noise variation than energy detection. However, cyclostationary detection is more complicated (than ED) to implement as it requires prior information, like the modulation scheme used in the PU signal [26].
- 3. Matched filter detection: If a CR already has all the information about a PUs signal such as, modulation type, pulse shaping, and packet format the mached filter technique is optimal. However, this technique requires the SU to have a different receiver antenna for each PU signal type which makes it impractical, specially when large chunks of RF spectrum accommodating different PU signal types need to be detected [26].

2.5 Hypothesis Testing

Depending on the idle state or busy state of a PU, in the presence of noise, signal detection at the secondary user can be modelled as a binary hypothesis problem given as; H_0 : primary user is idle and H_1 : primary user is operating.

When y(t) is the signal received at the cognitive radio, s(t) is the signal transmitted by the primary user, n(t) is the additive white Gaussian noise, and h is the complex channel gain of the

transmission channel between the cognitive radio and the primary user, y(t) can be written under two hypotheses: H_0 and H_1 [26],

$$y(t) = \begin{cases} n(t), & :H_0\\ hs(t) + n(t), & :H_1 \end{cases}$$
(2.1)

2.6 Energy Detector

ED is the simplest and most widely used spectrum sensing technique because of its ease of implementation. An energy detector calculates the energy associated with the received signal over a fixed time period and bandwidth. The measured value is compared with a properly selected threshold to conclude about the presence or absence of a primary signal. A block diagram of an energy detector is shown in Figure 2.4. The decision statistic for ED is given by,

$$Y = \frac{1}{N} \sum_{i=1}^{N} |y_i|^2 \stackrel{H_1}{\underset{H_0}{\geq}} \lambda,$$
(2.2)

where N is the samples received during the sampling time and y_i is the sampled input signal to the energy detector.

2.6.1 Performance analysis

To detect the PU signal, the signal received at the CR is first passed through a band pass filter then amplified to a desired level and compared with a predefined threshold. At the CR, the signal is observed in a fixed bandwidth W over a time window T. The time-bandwidth product is denoted by u = TW, which relates to the sample size as N = 2u [26]. The key parameters for detecting the PUs signal are P_d , P_f , P_e :

$$P_{d} = \mathbb{P}(Y > \lambda | H_{1})$$

$$P_{f} = \mathbb{P}(Y > \lambda | H_{0})$$

$$P_{e} = P_{f}.\mathbb{P}(H_{0}) + P_{m}.\mathbb{P}(H_{1})$$
(2.3)

A significant performance indicator for ED is the ROC. The ROC is obtained by plotting P_d against P_f [27].



Figure 2.4: Energy detector block diagram [28].

2.6.2 Area Under the ROC Curve

The AUC is a single figure of merit of an energy detector which fully characterizes its performance. The traditional detection performance evaluation method using ROC curves fully quantifies ED performance, but the AUC provides a single figure of merit for performance evaluation, avoiding the need to compute the threshold which is required in traditional ROC based evaluation. By definition the AUC is given by equation 2.4, AUC varies between 0.5 and 1. The larger the area under the ROC curve, the higher will be the capability of detection by the energy detector [29].

$$AUC = \int_0^1 P_d dP_f \tag{2.4}$$

2.7 Multipath Fading

In wireless communications, multipath is a propagation phenomenon which yields in radio signals reaching the receiver by two or more paths. In cellular networks multipath propagation may be due to refraction, reflection, diffraction or scattering from rain, trees, buildings, mountains or other objects that intrude between the signal from the eNB to user equipment (UE). A simple example of multipath propagation is shown in Figure 2.5. Thus, the waves arriving at the UE may be in phase with each other or may be out of phase. Signal received in phase added together to yield a powerful signal at the receiver. This is called constructive interference. RF signals which are received out of phase interfere with each other producing a weak or fading signal due to destructive interference [30]. The following models describe multipath fading in wireless propagation.

a. Rayleigh fading model: When signal fading is observed and no effective propagation is



Figure 2.5: Signal travelling through different paths to receiver [30].

present along the line of sight between the transmitting and receiving antennas, the fading is called Rayleigh fading [31]. Rayleigh fading model is used to analyze the cellular communication system in areas where the receiver is getting multiple copies of the same signal due to reflection, diffraction, or scattering from different objects; for example, in an urban area the signal might be reflected due to buildings in the signal path [30]. In the Rayleigh fading model the variation in magnitude of the signal, which has travelled through the communication channel, depends on the PDF of the channel. The PDF for the Rayleigh fading model is given by [30]

$$p(h) = \frac{2h}{\Omega} e^{\left(-h^2/\Omega\right)}, h > 0$$
(2.5)

where h denotes the channel fading amplitude and Ω denotes the average fading power.

b. Rician fading model: The Rician fading model is used to describe a scenario in which one of the paths in a multipath propagation is a line of sight. This kind of fading is present in mobile satellite communications. A Rician Fading model is described by two key parameters K and Ω , where K defines ratio of power in a direct path to that of scattered paths,

called the Rice factor, and Ω denotes the average wave envelope power [30]. The PDF of the Rician fading model is given by [30]

$$f(h) = \frac{2(k+1)h}{\Omega} e^{(-K - \frac{(K+1)h^2}{\Omega})} I_0\left(2\sqrt{\frac{K(K+1)}{\Omega}}h\right)$$
(2.6)

c. Nakagami-m fading model: The Nakagami distribution characterizes brisk fading in high frequency channels. This model describes land mobile and indoor mobile multipath propagation [30] best. The Nakagami-m fading gives a probability of gamma distribution [32]. It has two parameters: a shape parameter m and controlling spread, Ω.

$$f(h;m,\omega) = \frac{2m^m}{\Gamma(m)\Omega^m} h^{2m-1} e^{\left(\frac{-m}{\Omega}h^2\right)}$$
(2.7)

The Nakagami fading model is used to fit empirical data, and it gives a closer match to some experimental data than either Rayleigh or Ricean fading models [30].

2.8 Antenna Diversity

In modern mobile communication systems, the basic common requirements are a high data transmission rate, low bit error probability, and high spectral efficiency. Increasing capacity and reducing multipath interference is an ongoing task in mobile communication systems. Multipath fading in wireless propagation is a major problem that can be overcome using antenna diversity techniques [33]. The use of antenna diversity increases the channel capacity, taking advantage of multipath propagation by using multiple antennas. This adds extra complexity and extra equipment cost in the network but additional RF spectrum is not needed [33]. Antenna diversity may be attained by using multiple transmitter and/ or receiver antennas. The receiver receives multiple copies of the same signal which has travelled different paths with different degrees of fading. The receiver can either choose the strongest signal or combine the signals received.

Among many antenna diversity combining techniques the following are well known schemes [33].

1. Selective combining (SC): Selective combining is very simple to implement. Here signals received in a signal burst at the various antennas are compared, the signal that has the highest SNR is selected, and all other received signals are discarded. Then the selected

antenna is used for rest of the burst duration. This approach is only useful if the channel does not change significantly during the signal burst [33].

- 2. Maximal ratio combining (MRC): In this method, signals from all of the antennas are amplified according to their channel gains and then added. This technique gives the highest achievable SNR at the receiver at different times [33].
- 3. Equal gain combining (EGC): This is a simple form of the maximal ratio combining technique. Here all the branch weights are set to unity and the signals from each branch are co-phased and combined at the receiver [33].
- 4. Square law combining (SLC): In this model, signals received at different antennas are passed through square law devices. In square law devices signal squaring and integration operations are performed and the output of these devices is added to get the decision statistics. Finally, the decision statistics are used to decide the absence or presence of a PU signal [33].

Among all these methods the maximal ratio combining method is optimal but has increased complexity in implementation. Moreover, MRC requires channel rate information at the receiver side. However, this is in contrast to the non-coherent operation of ED, as this does not need such information. And the equipment is simple SLC which gives it an edge to be used in mobile communication as its components are easy to implement inside the UE [30].

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Chapter 3

AUC based analysis of Energy Detector in presence of interference with antenna diversity

3.1 Introduction

In this chapter, AUC based analysis is used to study the performance of ED in the presence of multipath fading, noise, and interference. Here, we are analyzing the AUC for ED with different values for interferer density, samples collected, SINR, path-loss exponent and interfering signal power. In our model both the, number of interferes and their locations are random. To explore performance improvements, the scenario is extended to a multiple antenna situation to examine the effect of antenna diversity. Fading is characterized by Rayleigh fading. The performance is evaluated via numerical results obtained using Monte-Carlo simulations.

Sections 3.2 and 3.3 describe the system model and simulation model, respectively. The results are subjected to AUC analysis for which the ED performance is discussed in section 3.4. In section 3.5 using antenna diversity results are analyzed and the ED performance is discussed.

3.2 System model

In our system model we consider a wireless network with a PU, a sensing CR node and random number of interfering nodes, spread within the vicinity of the CR. The PU is the licensed user of the frequency band of interest, so the PU has priority to use the channel and the interfering users, which could be other CR nodes, eNBs, HeNBs, WiFis or WiMax nodes are operating such that their location and numbers are unknown.

As in Figure 3.1 the CR of interest, CR_0 , is at the centre of a circle of radius R. The interfering nodes in the circle are trying to sense and access the same frequency band. Here the CR network is noncooperating, that is, the CR units do not know the location, number, and active or idle

status of other cognitive radios in their vicinity. These nodes interfere with the ability of the CR_0 node to sense the channel. Figure 3.1 shows k other interfering nodes (I_1, I_2, I_k) present in our disk of radius R. The PU transmitter is at a fixed distance from CR_0 . These nodes are located randomly in the disk and it is assumed that the kth interfering node is located at distance r_k . Here we consider y(t) is the received signal at the CR_0 node, $i_k(t)$ is the interference signal from the kth node. Thus, the signal received can be given as follows [27] :



Figure 3.1: CR network with multiple interferers [8].

$$y(t) = \begin{cases} n(t) + \sum_{k=1}^{K} g_k i_k(t), & :H_0\\ hx(t) + n(t) + \sum_{k=1}^{K} g_k i_k(t), & :H_1 \end{cases}$$
(3.1)

Where x(t) is the signal form PU when it is in active state, n(t) is the additive white Gaussian noise (AWGN) observed at CR node, $i_k(t)$ is the interfering signal from kth CR which is accessing or trying to access same channel. H_0 and H_1 represent the absence and presence of PU user signal respectively. To conclude about absence or presence of PUs signal, the signal y(t) is sampled and fed to ED. If N is the number of samples collected then decision variable is given by equation (2.2) [27]

3.3 Description of simulation model

The numerical results in this report are obtained using a semi-Monte Carlo simulation, an iterative process that relies on repeated random sampling of the unknown probabilistic entity to obtain numerical results using a wide range of computational algorithms [34].

To acquire simulations, iterations are repeated large number of times to average over the random variables. As interference is conditional for variables K and $r(r_1, r_2, ..., r_k)$, we need the distribution of $r(r_1, r_2, ..., r_k)$. For this purpose, we assume that $r(r_1, r_2, ..., r_k)$ is uniformly distributed within the circle of radius R with area πR^2 as [8]

$$f_{r_k}(x) = \begin{cases} \frac{2x}{R^2} & 0 < r_k < D\\ 0 & otherwise \end{cases}$$
(3.2)

The performance of ED is evaluated using AUC based analysis, for which the detection variables under H_1 and H_0 denoted by Y_{H_1} and Y_{H_0} respectively, are obtained as

$$AUC = \mathbb{P}\left\{Y_{H_1} \geqslant Y_{H_0}\right\} \tag{3.3}$$

3.3.1 Antenna Diversity combining scheme

As already discussed in section 2.8, antenna diversity may be used in a CR network for performance improvement in the network. In the system model, here we assume there are L antennas at the detector. The SLC mechanism is used to combine the signal received at different antennas and to make the appropriate decision about the active or idle state of a PU. Figure 3.2 shows the SLC combining technique, where the signal received at each antenna is passed through a square law device, each signal is scaled equally, and the output of these devices is added to get the decision statistics to decide about the absence or presence of a PU signal. If N is the number of samples, L is the number of receiver antennas, and $y_{i,l}$ is the signal from each square law device, then the overall decision variable is given by

$$Y_{SLC} = \frac{1}{NL} \sum_{i=1}^{N} \sum_{i=1}^{L} |y_{i,l}|^2, \qquad (3.4)$$

AUC is obtained as

$$AUC_{SLC} = \mathbb{P}\left\{Y_{SLC,H_1} \ge Y_{SLC,H_0}\right\}$$
(3.5)

where Y_{SLC,H_1} is Y_{SLC} conditioned on H_1 and Y_{SLC,H_0} is Y_{SLC} conditioned on H_0 .



Figure 3.2: Antenna Diversity.

3.4 Results for AUC based analysis of ED in the presence of interference

In this section, the ED performance is analyzed in presence of interference using AUC. The ROC is a plot of P_d versus P_f . The larger the AUC, the higher will be the capability of energy detection. To obtain graphical results for CR performance, several parameters that affect CR performance are varied, as described in sections 3.4.1 to 3.5.3.

3.4.1 Effect of varying N on AUC

Figure 3.3 is a plot of AUC versus SINR for different values of samples collected at the CR. It shows that the performance of ED improves as the number of samples collected at CR are increased. When the number of samples are increased from 10 to 20 at SINR = -15 dB AUC increases from 0.607 to 0.6511, which is a 7.27% increase in AUC. Similarly there is a 33% increase in AUC when number of samples collected are increased from 10 to 100. Therefore,

the simulation results show that there is an improvement in ED performance if more samples are collected.



Figure 3.3: AUC vs SINR for varying $N, R = 100, \beta = 0.0001, P_i = 5 \text{ dB}, \alpha = 2$.

3.4.2 Effect of β on AUC

Figure 3.4 is a plot of AUC versus SINR for different interferer densities in the CR network. Interferer density β represents the concentration of other interfering nodes active in the network. The simulation results indicate that as interferer density β is increased the AUC decreases significantly. A smaller AUC means that the probability of a false alarm will be high. Figure 3.4 shows that when β is increased from 0.001 to 0.01 at SINR value of -5 dB there is a 15% decrease in the AUC and this trend is increased to 22.43% for a change in β from 0.01 to 0.1. Instantly, there is a 36.58% decrease in the AUC with a change in beta from 0.0001 to 0.1. So we can conclude that the lower the interferer (SU) density, the better will be the CR performance

3.4.3 Effect of α on AUC

Figure 3.5 is a plot of AUC versus SINR for various values of path loss exponent α . Indoor, outdoor, urban, suburban values for the path loss exponent are considered 1.7, 2, 3, 4 respectively. In urban areas the propagated signal has obstacles in its path that indoor signals do not, so the value of α is higher for the urban path than for the indoor path. Figure 3.5 shows that when α is



Figure 3.4: AUC vs SINR for varying β with R = 50, N = 4, $P_i = 5 \text{ dB}$, $\alpha = 2$.

changed from 1.7 for the indoor path to a value of 4 for the outdoor path at an SINR of -6 dB the AUC value changes from 0.7219 to 0.7474, a 3.42% increase in CR performance. Therefore, a change in α has very little effect on the CR performance.

3.4.4 Effect of α on AUC for a fixed SINR

Figure 3.6 is a plot of AUC versus path loss exponent α . As α is changed from 2 to 6 there is an increase in the AUC which shows that as α increases the CR performance improves. There is noticeable increase in the AUC as the path loss exponent α increases from 2 to 3.5; on further increases in the path loss exponent the AUC value remains almost constant. Therefore, we can conclude that a CRs performance is better in urban and suburban areas where the path loss exponent is high.

3.5 Performance of Energy Detector using Antenna Diversity

3.5.1 Effect of Interfering signal power P_i on ED performance

Figure 3.7 depicts the effect of interfering signals from secondary users on the performance of a CR. Figure 3.7 is a plot of AUC versus P_s which is the PUs signal power for different values of interfering signal power (P_i). As P_i increases the AUC decreases. As P_i changes from 30 dB to



Figure 3.5: AUC vs SINR for varying α with R = 50, $P_i = 5 \text{ dB}$, $\beta = 0.001$.



Figure 3.6: AUC vs α with R = 150, $P_i = 5 \text{ dB}$, $\beta = 0.001$, SINR = -15 dB.

20 dB, the AUC increases from 0.5676 to 0.7697, a 35.6% change. A further decrease in P_i from 30 dB to 10 dB corresponds to a 57% increase in the AUC. Because a higher area under the ROC

curve predicts a better CR performance, we can conclude that a lower value of P_i will improve a CRs performance.



Figure 3.7: AUC vs. P_s dB for varying P_i dB with N = 25, L = 5, R = 150, $\beta = 0.0001$.

3.5.2 Effect of *L* on ED performance

Figure 3.8 show the effect of using multiple antennas on CRs performance, this is a plot between AUC and P_s with different number of antennas used at CR. Here P_s is the PU signal power. We can observe from the plot that as we increase L from 2 to 4 there is 10% increase in AUC and increase in L from 4 to 8 boosts AUC by 8.63% when signal power is -10 dB. Therefore, there is 20.44% escalation in AUC with change in antennas from 2 to 8. Thus it can be concluded that CR performance improves significantly using antenna diversity and if more antennas are used performance is better.

3.5.3 Effect of β over AUC for varying L

Figure 3.9 shows the influence of β over AUC versus L. From plot we can see that as L increases AUC increases and with different values of interferer density AUC changes. Here, we can see that at L = 4 when β decreases form 0.1 to 0.0001 AUC increases by 8.5% and for L = 8with same change in β the increase in AUC is 10.22%. So it can be concluded that as interferer density decreases AUC increase and AUC has higher value for higher value of L. Therefore, it



Figure 3.8: AUC vs. $P_s \,\mathrm{dB}$ for L with $R = 150, \beta = 0.001, P_i = 5 \,\mathrm{dB}.$



Figure 3.9: AUC vs. L for varying β with R = 150, $P_i = 5 \text{ dB}$.

is desirable to have higher value of receiver antennas L and lower value of interferer density β for better CR performance in a network with multiple interferers operating, trying to sense and

access same frequency band.



Figure 3.10: CAUC vs. P_s dB for varying β with R = 150, N = 25, $P_i = 5$ dB.

3.5.4 Effect of β over CAUC for varying P_s

The Figure 3.10 shows the behaviour of plot between CAUC and PsdB with variation in β . Lower value of CAUC means better CR performance, here we can notice that as the power of signal transmitted by PU increases CAUC decreases which implies that CRs capability to sense the channel correctly is increasing. When β is increased CAUC is increasing, as we can see in plot when β changes from 0.0001 to 0.01 CAUC increases by 64.45% from 0.1823 to 0.2998 at P_s -8dB. Form our numerical analysis we can conclude that higher PU signal power P_s and lower interferer debsity β is beneficial for better performance of CR.

3.6 Conclusion

From our results we conclude that ED performance is better if more samples are collected and ED performance degrades as interfering node density increases and as interfering signal power increases. Further analysis of ED equipped with multiple antennas showed that ED performance improves significantly as more antennas are deployed.

Chapter 4

Conclusion

This project presents an analysis of ED in addressing spectrum sensing problems in CR. However, as opposed to the traditional P_d and P_f based ROC analysis, our analysis is based on the AUC, thus the results are based on AUC analysis because AUC provides a single figure of merit for ED performance. The increase in mobile subscribers and their data demands on mobile networks results in a need for more RF spectrum. LTE networks provide high data rates, high spectral efficiency, and low latency attributes that help LTE meet the growing data demands. To further improve the spectral efficiency of LTE, spectrum holes can be exploited using cognitive radio techniques. ED is the mostly used spectrum sensing technique because of its low complexity and ease of installation. However, ED performance degrades when the SNR is low, and when interfering nodes are present in the vicinity of a sensing node. ED performance is also degraded when multipath signal propagation is present. To overcome these drawbacks and improve energy detection, multiple antennas that exploit multipath propagation are considered to improve ED performance.

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