University of Alberta

Techniques for Proving Nontileability of Planar Regions

by

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Abstract

Tiling problems have been a mainstay in recreational mathematics and they often lead to deep results. It can be shown that the question of whether an area can be tiled by a given set of tiles is part of the computational complexity class of problems known as NP complete.

At times it is possible to answer tiling problems in the negative using numbering arguments. This is precisely the case when not even a signed tiling of the area exists. As a result, some of the most difficult non-existence tiling problems are those in which it can be shown that a signed tiling does exist. More advanced methods involving combinatorial group theory have been developed to solve a handful of these types of problems.

This paper is a survey of the existing literature. One of the highlights of this thesis is an approach by John Conway and Jeff Lagarias. The material is not organized chronologically, but in such a way that makes it easier for the reader to comprehend some of the main results of interest through carefully chosen examples.

Acknowledgements

I wish to foremost thank my supervisor and mentor, Dr. Andy Liu, for introducing me to the study of Combinatorics and the topic of region tilings on the plane. Dr. Liu's engaging character, constructive guidance and tremendous generosity is deeply appreciated. I am also grateful for the support of Dr. Ted Lewis in the early stages of my Master of Science career. Dr. Lewis's belief in me and his constant words of support unquestionably assisted me in attaining my goals. Further, I am especially grateful to my family, especially my wife, Dawn, and my parents, for their constant support and encouragement to follow my heart. I sincerely appreciate all that they do for me.

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List of Symbols (in order of appearance)

Σ - Set of tiles

R - Region on the plane to be tiled

G - Group

a, b, g, h, x, y - Arbitrary group elements

$\circ$ - Group operation

e – Group identity element

z – Arbitrary complex number

C_n - The cyclic group of order n

F_n - The free group on n elements

$\Im$ - The set of integers

Q - The set of rational numbers

$\Re$ - The set of real numbers

C – The set of complex numbers

H – Subgroup of G

Hg – The right coset of H generated by g

G/S - Quotient group of G by the normal subgroup S

$G = \langle \cdots | \cdots \rangle$ - A presentation for G

$[x, y]$ - The commutator of x and y

G' - The commutator subgroup of G

S_n - The symmetric group of order n

A_n - The alternating group of order n

G^{ab} - The abelianization of G

$\Gamma(G)$ - Graph (Cayley Diagram) of G

$[\partial R]$ - Combinatorial boundary of region R

∂R - Topological boundary of region R

$\partial R(e)$ - The boundary word of the region R with leading edge e

W – Word in the free group

$W(P)$ - Word in the free group corresponding to the path P

$P(W)$ - Path corresponding to the word W

τ - A specific tiling of a region R

π - A polygon on the lattice

$I(\pi)$ - Image of $W(\pi)$

T_N - Hexagonal triangle (staircase region) of order N

$\phi(V)$ - Hexagonal signed area of the closed path V

A – Free abelian group of all cells in the lattice

$B(\Sigma)$ - Subgroup of A generated by all elements corresponding to tile placements

$H(\Sigma)$ - The Tile Homology Group of Σ

$a_{i,j}$ - The cell in the square lattice with lower left corner at lattice point (i, j)

$\bar{a}_{i,j}$ - The image of the cell $a_{i,j}$ in $H(\Sigma)$

C – The subgroup of F_n consisting of closed words

$\pi(\Sigma)$ - The Tile Homotopy Group of Σ

CHAPTER 1

TILING PROBLEMS

§1.1 PLANE AND REGION TILINGS

In mathematics, the concept of tiling is very similar to the tiling of a section of floor or tiling a mosaic. The difference, however, is that often when you are tiling a floor or mosaic you will cut a tile to make it fit the region you are tiling. Mathematical *tiles* are geometric figures so that identical copies of them can be used to cover a specified region without gaps or overlaps, and without cutting any of the tiles. Tiling problems can be categorized into two types: plane tiling problems and region tiling problems.

A *plane tiling* is a placement of tiles which covers the entire two-dimensional plane without gaps or overlaps. A *region tiling* is such a placement which covers a specific region of the plane without gaps, overlaps, or extra area outside of the region left over from any tile.

Problems concerning the tiling of regions of the plane are some of the most popular in combinatorics and recreational mathematics. Tilings used in art have been documented for centuries and there are still many unsolved tiling problems that mathematicians and computer scientists are working on today. Two tiling problems that have been of particular interest over the last 50 years are as follows:

(1) Given a finite set Σ of tiles, does Σ tile the entire plane?
(2) Given a finite set Σ of tiles and a region R, does Σ tile R?

The development of the interest in these problems is primarily credited to Hao Wang. Moreover, Wang along with Kahr, Buchi, and Moore contributed vastly to the solution of the decidability of problem (1) above, until it was proved in 1966 to be undecidable by R. Berger. This means that there is no algorithm that can be used to determine whether a tiling of the plane exists given a set of tiles. To prove this, Berger found a set of tiles which covers the plane aperiodically, but not periodically. Berger's set consisted of over twenty-thousand tiles. In 1969 R.M. Robinson found such a set consisting of only six tiles, and in 1974, Roger Penrose discovered a remarkable pair of tiles that cover the plane aperiodically, but not periodically. These are now known as Penrose kites and darts and Penrose constructed these tiles with the help of the golden ratio, Phi (see Example 1.3 below).

It is clear that problem (2) above is decidable through exhausting all possibilities. However, this problem is in the computational complexity class NP-complete which means that for many problems of this type, it is not feasible, even for a computer, to exhaust all possible tile patterns. The remainder of this paper will focus on problems of type (2) with certain restrictions on the types of tiles in Σ as well as the regions R to be tiled.

Examples:

1.1 The only regular polygons that can tile the plane are the equilateral triangle, square, and regular hexagon. These tilings are referred to as the triangle, square, and hexagon lattices. A tile in one of these lattices is referred to as a *cell.*

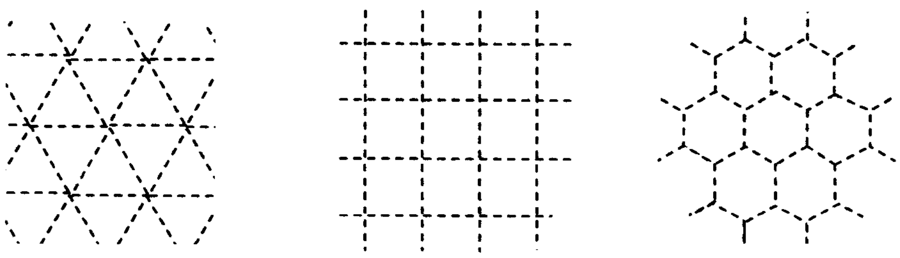


FIG 1.1 – The three regular polygon tilings

1.2 In the mid-1900s, M.C. Escher used different tiling patterns in several of his works of art. Note that the entire plane can be tiled by the fish shaped tiles used below.

FIG 1.2 – Fish – M.C. Escher

1.3 In 1974, Roger Penrose discovered a remarkable pair of tiles that can tile the plane aperiodically, but not periodically; these are known as Penrose kites and darts. The tiles can be prevented from tiling periodically by putting notches and tabs on the edges of the tiles, but a more aesthetic approach is to color the tiles as shown and require the edges to match.

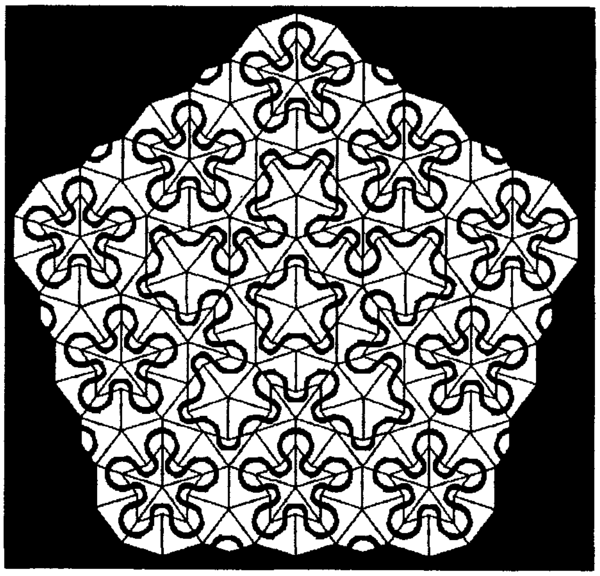
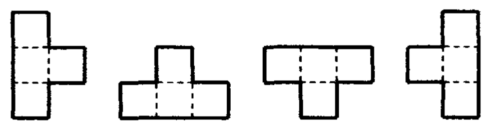


FIG 1.3 – Penrose Kites and Darts

1.4 An example of a region tiling problem is as follows:

Set of tiles: $\Sigma = \{$  $\}$

Region to be tiled: $R =$ rectangle of dimensions $m \times n$.

This problem was solved by D.W. Walkup [13]. In a paper published in 1964, Walkup proved that in order for an $m \times n$ rectangle to be tiled using only tiles in Σ, both m and n must be divisible by four.

§1.2 POLYOMINOES, POLYIAMONDS, POLYHEXES

From this point on, we restrict both the tiles in the set Σ as well as the regions R to be tiled to what are commonly known as polyominoes, polyiamonds and polyhexes. A polyomino is an extension of the domino. It is a connected region on the square lattice where any unit square in the polyomino must have at least one edge in common with another unit square in the polyomino. Further, the polyominoes that we will consider will not have any holes (an experienced reader will recognize this as precisely the definition of a simply connected region on the square lattice). Polyiamonds are extensions of the diamond; they are the analogue of polyominoes on the triangle lattice. Polyhexes are the analogue on the hexagon lattice.

We will consider a polyomino that can be obtained by another by means of reflection and/or rotation as equivalent to the first. See Figure 1.4 for the complete list of inequivalent monominoes, dominoes, trominoes and tetrominoes. An exercise for the reader is to construct the complete set of twelve inequivalent pentominoes. A further exercise is to construct the complete sets of polyiamonds and polyhexes comprised of five or less triangles and hexagons respectively. Some incredible results have been discovered with the tiling of polyominoes, polyiamonds and polyhexes and I highly recommend [6] from the list of references for the interested reader.

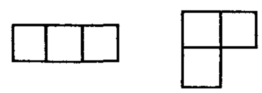
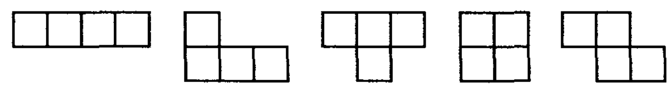
Monominoes	
Dominoes	
Trominoes	
Tetrominoes	

FIG 1.4 – *Table of inequivalent polyominoes using four or less squares*

In the next section we will also need the notion of a *signed tiling*. Here, the individual tiles are given weights of either +1 or -1. When determining the existence of a signed tiling, the entire lattice is considered; a lattice region R is said to have a signed tiling with tiles from Σ if there is a placement of weighted tiles (with overlapping allowed) such that each cell in R has a total weighted sum of +1 and each cell not in R has a total weighted sum of zero. In order to distinguish a signed tiling from a tiling, we will often refer to a signed tiling of R that does not use any negatively weighted tiles as a *perfect tiling*.

Examples:

1.5 A 3×3 square cannot be tiled by dominoes. This is clear since the area of a domino does not divide the area of a 3×3 square. In fact, an $m \times n$ rectangle is tileable by dominoes precisely when either m or n is divisible by 2.

1.6 The following is an example of a tiling of a hexagonal triangle of side length 9, using only the triangle 3-hex tile. An interesting exercise is to see which hexagonal triangles with side length 12 or less can be tiled by the triangle 3-hex tile.

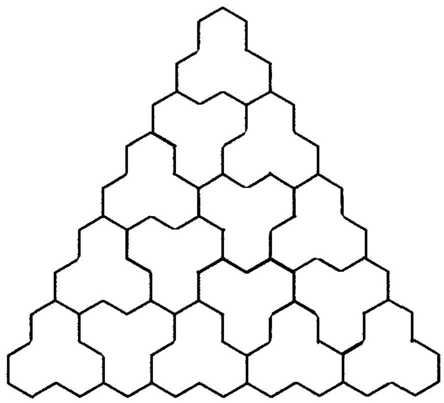
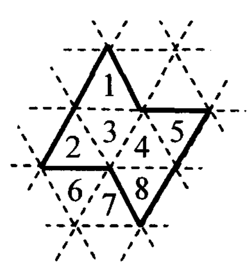


FIG 1.5 – Tiling of the hexagonal triangle with side length 9 with copies of the triangle 3-hex tile

1.7 The following region on the triangle lattice has a signed tiling by diamonds, but clearly doesn't have a perfect tiling.



Positive weighted tiles: [1,3], [4,5], [2,6], [7,8]

Negative weighted tiles: [6,7]

FIG 1.6 – Example of a region with a signed tiling but no perfect tiling

CHAPTER 2
METHODS FOR PROVING NONTILEABILITY OF REGIONS

§2.1 COLORING AND NUMBERING ARGUMENTS

It is clear how to show that a region can be tiled with a given set Σ of tiles: by producing a valid tiling. However it is often very difficult to show that a region cannot be tiled. One method is to exhaust all possible patterns, however as the regions to be tiled become larger and more complicated, this method quickly becomes impractical even for the fastest computers. Several more practical methods have been developed for showing the nonexistence of tilings of regions on regular lattices given a particular Σ.

The first of these types of arguments is the *coloring argument*. Say for example that we are to tile an 8×8 square region with horizontal and vertical domino tiles. This is not a difficult problem at all and one of the many possible solutions is shown in Figure 2.1.

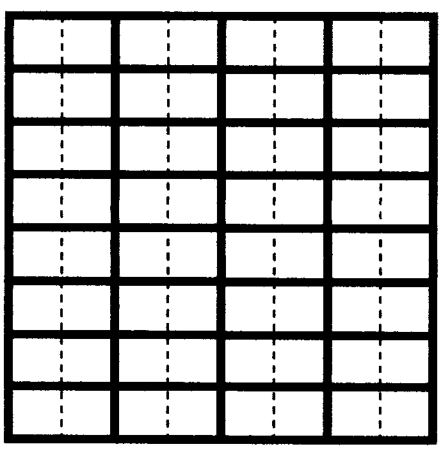


FIG 2.1 - Domino tiling of an 8×8 *square*

However, now if we chop off the top-left and bottom-right corners of the region, as in Figure 2.2(a), the problem becomes more difficult. It would take a very long time to try every possible pattern in order to determine if a solution exists. Yet, if we color alternating squares red and white as in Figure 2.2(b), and notice that each domino tile must cover exactly one red square and one white square, then if a tiling does exist, there must be the same amount of red squares as white squares. However, counting the squares resolves that there are 32 red squares but only 30 white. We can thus conclude that a tiling of the region with dominoes does not exist.

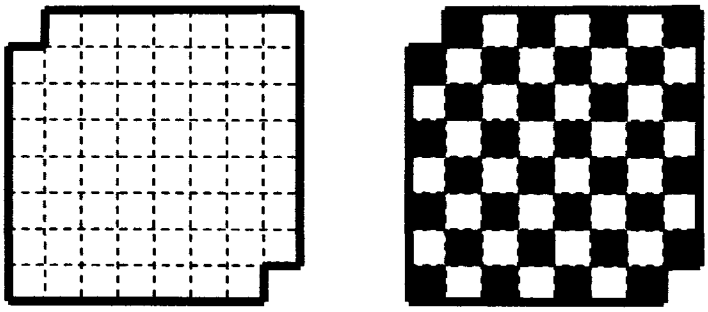


FIG 2.2(a),(b) – Coloring of an 8×8 mutilated square

A second method that is often used to prove nonexistence of tilings is called the *numbering argument.* Assume that we are to tile our 8×8 square region with copies of the L-tetromino tile; we are allowed to rotate and reflect the tiles as we wish. Anyone who has had some experience with the popular game Tetris will not have trouble solving this problem (see Figure 2.3). However the problem becomes much more difficult when we enlarge our region to a 10×10 square.

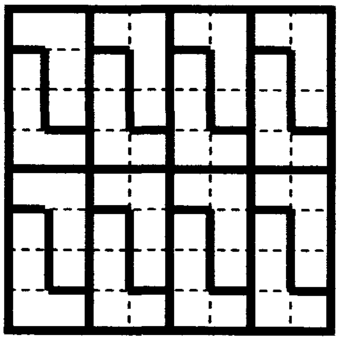


FIG 2.3 – Tiling of an 8×8 square with L-tetrominoes

Again it would be very impractical and time consuming to try every possible pattern. However, if we number the cells in alternating rows with ones and fives as in Figure 2.4, and notice that each tile will cover either three ones and a five (sum of eight), or three fives and a one (sum of 16), we can see that each tile will cover a total sum of a multiple of eight. This means that if a tiling were to exist, the total sum of all cells in the 10×10 square must also be a multiple of eight. Since the total sum is 300, which is not a multiple of eight, we can conclude there does not exist a perfect tiling of the 10×10 square region by L-tetrominoes.

1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5

FIG 2.4- Numbering of a 10×10 *square*

Numbering arguments are often useful in proving the nonexistence of tilings of a region by a set of tiles. However, there are region tiling problems for which perfect tilings do not exist, although it cannot be verified by any numbering argument. This is shown in the following theorem.

Theorem 2.1:

1) Any numbering argument that proves nonexistence of a tiling of a region also proves nonexistence of any signed tiling of that region.
2) If a simply connected region has no signed tiling by simply connected tiles, then there is a numbering argument proving that no signed tiling exists.

Proof:

1) This can be done by extending the numbering pattern over the entire lattice.
2) This will be done in the extension after Chapter 5

■

Important Consequence:

Theorem 2.1 shows that numbering arguments cannot prove non-existence of perfect tilings in regions that do have signed tilings. Therefore, these are often the most difficult tiling problems to solve.

§2.2 OTHER METHODS

Another method that can be used to determine tileability of regions given a certain set of tiles is called *integer programming*. In this method, each possible tile placement is given a variable x_i, which represents how many times that tile placement occurs in the tiling. For example, if we want to tile the region in Figure 2.5 on the square lattice with dominoes, then the ten possible tile placements are given the variables $x_1, x_2, \cdots, x_{10}$.

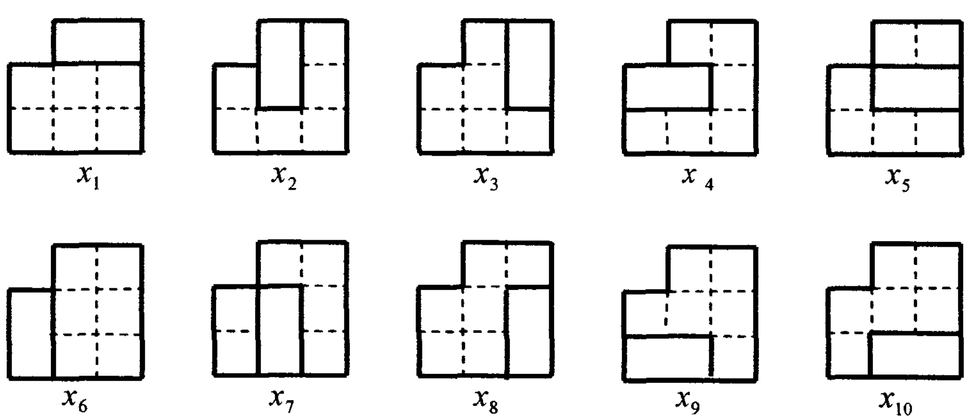


FIG 2.5 – Possible tile placements

Now, for each cell in the region, there is an equation which indicates that the cell is covered exactly once. Thus for this example, we get the following system of equations.

$$
\begin{array}{cccccccccccccccccccccc}
x_1 & + & x_2 & & & & & & & & & & & & & & & & & & = & 1 \\
x_1 & & & + & x_3 & & & & & & & & & & & & & & & & = & 1 \\
& & & & & & x_4 & & & + & x_6 & & & & & & & & & & = & 1 \\
& & x_2 & & & + & x_4 & + & x_5 & & & + & x_7 & & & & & & & & = & 1 \\
& & & & x_3 & & & + & x_5 & & & & & + & x_8 & & & & & & = & 1 \\
& & & & & & & & & & x_6 & & & & & + & x_9 & & & & = & 1 \\
& & & & & & & & & & & & x_7 & & & + & x_9 & + & x_{10} & & = & 1 \\
& & & & & & & & & & & & & & x_8 & & & + & x_{10} & & = & 1
\end{array}
$$

A tiling then corresponds to a solution to the above system. Further, a solution to the system where each x_i is either 0 or 1 corresponds to a tiling. A solution to the system where some of the x_i's are not 0 or 1 corresponds to a signed tiling. Thus the region tiling problem is equivalent to the zero-one integer programming problem and hence falls into the computational complexity class NP-complete.

Several other methods have been used to show the nontileability of regions with a given set of tiles. Some of them include exhausting all possible patterns and using subtle geometric arguments like the method that Walkup used in his 1964 paper [13].

In a paper published in 1988, John Conway and Jeff Lagarias describe an especially creative method of determining when a region does not have a perfect tiling. Their method proves to be stronger than any numbering argument: at times their method can prove nontilablity in regions that possess signed tilings, but not perfect tilings. Conway and Lagarias' method lies in the mathematical territory of combinatorial group theory, and thus included in this paper is the following section on the theory of groups. We will return to Conway and Lagarias' ingenious method in Chapter 4.

CHAPTER 3

GROUP THEORY

§3.1 ELEMENTS OF GROUP THEORY

The Theory of Groups is a very captivating and diverse subject in mathematics. It is with group theoretical methods that Abel proved that a general formula to solve for the roots of a quintic equation does not exist, and it is with the theory of groups that it was proved that it is not possible to trisect an arbitrary angle using only a compass and straightedge. One of the greatest mathematical feats of the 20^{th} century was the classification of all finite simple groups.

Only the basics of the theory of groups that is required for this paper is included here. The reader who is familiar with group theory may skip over this section and at times, if needed, refer to certain concepts. The interested reader is strongly encouraged to study this topic further; a good textbook for the beginner is [1] in the list of references.

Definition:

A *group* is an ordered pair $(G,\circ)$ where G is a set of elements and $\circ$ is a binary operation on G satisfying the following laws:

(i) For all $a,b \in G$, $a \circ b$ is still in G and is called the *product* of a and b.

(ii) For all $a,b,c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$

(iii) There exists an element $e \in G$ such that $e \circ g = g \circ e = g$ for all $g \in G$. e is called the *identity element* of the group.

(iv) For each $a \in G$ there is a unique element $a^{-1} \in G$, called the *inverse* of a, such that $a \circ a^{-1} = a^{-1} \circ a = e$

Further, the group $(G,\circ)$ is called *abelian* if $a \circ b = b \circ a$ for all $a,b \in G$.

When the operation $\circ$ is clear, we use the notation G instead of writing the ordered pair $(G,\circ)$. Further, we regularly use the notation gh for $g \circ h$.

In each of the examples below, the reader is encouraged to verify that the group described does, in fact, satisfy the above four laws in the definition of a group.

Examples:

3.1 The sets of integers $\Im$, rational numbers Q, real numbers $\Re$, and complex numbers C are all groups under the operation addition: zero is the identity element, and for any element x, the inverse of x is $-x$. Further, these groups are all abelian. The set of irrational numbers is not a group under addition since there is no identity element.

3.2 The set of integers with the operation multiplication is not a group since not every integer has a multiplicative inverse within the set of integers. However, the sets of non-zero rational numbers, real numbers and complex numbers are all groups under multiplication. The identity element is 1; a non-zero complex number $z = a + bi$ has multiplicative inverse $z^{-1} = \frac{a}{a^2+b^2} - \left(\frac{b}{a^2+b^2}\right)i$. Note that zero cannot be included in these groups since there is no z such that $0 \circ z = z \circ 0 = 1$.

3.3 The preceding examples are all examples of infinite groups. That is, there are an infinite number of elements in the group. Consider the set $C_n = \{a^0, a^1, a^2, \cdots a^{n-1}\}$ with the operation $\circ$, defined so that $a^k \circ a^m = a^{k+m}$, and the relation $a^n = a^0$. Then C_n is a group with n elements. The identity element in this group is a^0, and the inverse of the element a^k is a^{n-k}. This group is abelian since $a^{m+k} = a^{k+m}$. C_n is called the cyclic group of order n.

For small groups, it is often useful to construct what is called the *group table*. The group table illustrates what the product of any two elements in the group is. The group table for C_4 is shown below.

$\circ$	a^0	a^1	a^2	a^3
a^0	a^0	a^1	a^2	a^3
a^1	a^1	a^2	a^3	a^0
a^2	a^2	a^3	a^0	a^1
a^3	a^3	a^0	a^1	a^2

3.4 Take the set of integers and separate them into classes defined by their remainder when they are divided by five. For example 3, 8, -2, -7 would all belong to the same class while -2,-1,0,1,2 would all belong to different classes. Now label the five distinct classes with $\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}$ depending what the remainder is when an integer from that class is divided by five. Let $G=(\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\},\circ)$ where $\circ$ is defined so that $\bar{x}\circ\bar{y}=\overline{x+y}$ (for example, $\bar{3}\circ\bar{4}=\bar{7}=\bar{2}$). It follows that G is a group with identity element $\bar{0}$, and the inverse of $\bar{x}$ is $\overline{5-x}$. As an exercise, construct the group table for G. Notice from the group table that G is actually the same as C_5 from Example 3.3. Any of the C_n can be constructed in a similar way.

3.5 Another example of a finite group is the set of symmetries of a square (see Appendix B). A square can be rotated $0°,90°,180°,270°$ or it can be reflected on its vertical axis, horizontal axis, or either of its diagonals. Thus this group has a total of eight elements. The identity element is the rotation of $0°$. It is left as an exercise for the reader to compute the inverses for each of the eight symmetries. Is this group abelian?

3.6 The set of all even integers is a group under addition. Further, the set of all integers divisible by an integer n is a group under addition.

3.7 The *group* F_2 is the set of all possible *words* created with two elements and their inverses. For example, if the generators are g and h, then some elements of F_2

would be gh^2gh^{-4} and $hg^{12}h^{-5}ghg$. The operation of the group is concatenation. That is, the product of the two previous elements is $gh^2gh^{-3}g^{12}h^{-5}ghg$. (What is the inverse of this element?) This group is not abelian. F_2 is called the free group on two generators; the free group on *n* generators can be defined similarly.

3.8 The set of $n \times n$ matrices with entries from $\Im$ and non-zero determinant forms a group under matrix multiplication. This group is often called the *general linear group of degree n.* It is often given the notation $GL_n(\Im)$.

Remark: One useful property of groups is that for every $g,h \in G$, $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$

Often a group *G* will have "smaller" groups embedded inside of it that use the same operation as *G*. This is the notion of a *subgroup.*

Definition:

Let $(G,\circ)$ be a group. The subset *H* of *G* is a *subgroup* of *G* if *H* is nonempty and *H* is itself a group under $\circ$.

Examples:

3.9 The subsets $\{e\}$ and G are always subgroups of a group *G*.

3.10 The set of integers is a subgroup of the rational numbers under addition.

3.11 The set of non-zero integers is not a subgroup of the non-zero rational numbers under multiplication.

3.12 Consider the group $C_6 = \{a^0, a^1, \cdots, a^5\}$. Then the subset $\{a^0, a^2, a^4\}$ is a subgroup of C_6. Further, if we relabel a^0, a^2, a^4 as b^0, b^1, b^2 respectively, then we see that this subgroup is actually C_3. In fact C_m is a subgroup of C_n as long as *n* is divisible by *m*.

3.13 The rotations of $0°, 90°, 180°, 270°$ is a subgroup of the group of symmetries of a square. In fact, this subgroup is C_4.

3.14 The set of all integers divisible by 6 is a subgroup of the group of even integers under addition.

Once we have a subgroup of a group *G*, we can take an arbitrary element from *G* and multiply each element in the subgroup by that element. What results is another subset of the group that is not necessarily a group itself, but it has other interesting properties.

Definition:

Let *G* be a group, *H* be a subgroup of *G*, and *g* be an arbitrary element of *G*. The *right coset of H generated by g* is the subset of *G* defined by $Hg = \{h \circ g \mid h \in H\}$. Similarly, the *left coset* of *H* generated by *g* is the subset of *G* defined by $gH = \{g \circ h \mid h \in H\}$.

Examples:

3.15 Let *G* be the additive group of integers, let *H* be the subgroup of integer multiples of 3 and let $g_1 = 1$. Then Hg_1 = the subset of all integers that are one more than a multiple of 3. Note that if $g_2 = 2$, then $G = H \cup Hg_1 \cup Hg_2$.

3.16 Let G be the group of rational numbers *Q* under the operation addition; let *H* be the group of the set of integers $\Im$, and let $g_1 = 3, g_2 = \frac{1}{2}$. Then $Hg_1 = H$ and $g_2H = \left\{\frac{a}{2} \mid a \text{ is odd}\right\}$ the latter is not a group since it doesn't contain an identity element.

3.17 Consider the group of symmetries of the square in Appendix B. The set $H = \{0°$ rotation, reflection on the vertical line of symmetry$\}$ is a subgroup of this group. If we let *r* be the counterclockwise rotation of $90°$, then $rH = \{r,$ first diagonal reflection$\}$ while $Hr = \{r,$ second diagonal reflection$\}$.

We should take note of a few properties of right cosets. Firstly, $a \in Ha$ since *e*, the identity element of *G*, is in *H*. Secondly, if *H* contains *m* elements, then so does *Ha*, indeed *Ha* contains at most *m* elements and if $h_1 a = h_2 a$ for any $h_1, h_2 \in H$ then $h_1 = h_2$. It should also be clear that if $a \in H$ then $Ha = H$. Consider now two right cosets *Ha* and *Hb*, $a \neq b$, of *H* in *G*. Suppose that *c* is a common element of these two cosets so that for some $h_1, h_2 \in H$ we have $c = h_1 a = h_2 b$. Then $a = h_1^{-1} \circ (h_2 \circ b) = (h_1^{-1} \circ h_2) \circ b$ and since $h_1^{-1} \circ h_2 \in H$, it follows that $a \in Hb$ and $Ha = Hb$. Thus the set of right cosets of *H* in *G* partition *G* into distinct sets of equal size. These properties are also true for left cosets.

Example 3.17 shows that given a subgroup *H*, the set of left cosets of *H* generated by an element are not always the same as the set of right cosets of *H* generated by the same element. It is actually very special when a subgroup has the property that its left coset is the same as its right coset regardless of what element is used to generate the cosets. A subgroup with this property is called *normal*.

Definition:

Let *H* be a subgroup of *G*. *H* is called *normal in G* provided that $gH = Hg$ for every $g \in G$. Another way to say this is that a subgroup *H* is normal in *G* provided, for every element $h \in H$ and every $g \in G$, we have $g^{-1}hg \in H$.

Examples:

3.18 The subgroups $\{e\}$ and *G* of a group *G* are always normal in *G*.

3.19 Any subgroup N of an abelian group G is normal since for all $g \in G$ and $n \in N$, $g^{-1}ng = g^{-1}gn = n \in N$.

3.20 The subgroup defined in Example 3.17 above is not normal, but the subgroup of rotations in the group of symmetries of a square is. In fact, any subgroup that contains half the elements of the entire group is normal.

Now that we have the concept of cosets, we can define operations between cosets. However when we define these operations, we have to be careful to ensure that what we are defining is actually *well-defined.*

Definition:

An operation $\circ$ is said to be *well-defined* if whenever $a = a'$ and $b = b'$, we have that $a \circ b = a' \circ b'$.

Examples:

3.21 Multiplication of rational numbers is well-defined. For example if $a = \frac{1}{2}$, $a' = \frac{2}{4}$, $b = \frac{2}{3}$, $b' = \frac{6}{9}$ we have $\frac{1}{2} \cdot \frac{2}{3} = \frac{2}{4} \cdot \frac{6}{9} = \frac{1}{3}$.

3.22 If we define the operation $\oplus$ on the rational numbers as $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$ then $\oplus$ is not well-defined. For example $\frac{1}{2} \oplus \frac{2}{3} = \frac{3}{5}$ and $\frac{2}{4} \oplus \frac{6}{9} = \frac{8}{13}$ but $\frac{3}{5} \neq \frac{8}{13}$.

3.23 Consider the group of the symmetries of the square in Appendix B and let r be the group element "$90°$ clockwise rotation" and let s be the element "reflection on the vertical line of symmetry". Let H be the subgroup defined in Example 3.17; then notice that $rH = sr^3H = \{r, sr^3\}$ and $r^3H = srH = \{r^3, sr\}$. If we define the operation $\circ$ on these cosets as $aH \circ bH = (a \circ b)H$, we have that

$rH \circ r^3H = (r \circ r^3)H = eH = H$, but $sr^3H \circ srH = (sr^3 \circ sr)H = r^2H = \{r^2, sr^2\}$.

So the operation $\circ$ is not well-defined with this set of cosets

Theorem 3.1:

The multiplication of cosets defined by $Ha \circ Hb = H(a \circ b)$ is well-defined if and only if *H* is a normal subgroup of *G*.

Proof:

$\Rightarrow$) We are given that $aH \circ bH = (a \circ b)H$ is well-defined. In particular, for any $h \in H$ we have that $eH = hH$, and consequently, for all $b \in G$ $hH \circ bH = eH \circ bH$. That is, $hbH = ebH = bH$. In other words, $b^{-1}hb \in H$ for all $b \in G$. So *H* is normal.

$\Leftarrow$) Now we are given that *H* is normal in *G*. Let $a'H = aH$. We must show that for any bH, we have that $a'H \circ bH = aH \circ bH$, that is, $a'bH = abH$. This is true provided that $(ab)^{-1}(a'b) \in H$. Further, $(ab)^{-1}(a'b) = b^{-1}a^{-1}a'b = b^{-1}(a^{-1}a')b$. But $a^{-1}a' \in H$ since $a'H = aH$. So $(ab)^{-1}(a'b) \in H$ since *H* is normal. We have shown that aH can be replaced with $a'H$ without affecting the value of any product. Similar calculations would show that bH can be replaced with any equal $b'H$. Hence, whenever $a'H = aH$ and $b'H = bH$ we have that $a'Hb'H = abH$ as desired.

Definition:

Let *G* be a group and *S* be a normal subgroup of G. The set of all distinct cosets of *S* in *G* is call the *quotient group of G by S* and is denoted by G/S. The group operation is defined as $Sg_1 \circ Sg_2 = S(g_1 \circ g_2)$. Note that *S* becomes the identity element in the quotient group G/S.

Examples:

3.24 With G and H as in Example 3.15 above, we have $G/H = \{H, H1, H2\}$. Note that we have the following group table.

$\circ$	***H***	***H1***	***H2***
H	*H*	*H1*	*H2*
H1	*H1*	*H2*	*H*
H2	*H2*	*H*	*H1*

Therefore in G/H, H is the identity element; $H^{-1} = H, H1^{-1} = H2, H2^{-1} = H1$; G/H is abelian. When dealing with a quotient group, it is common to use a single element from each coset to represent the coset. So $G/H = \{0,1,2\}$. Note that G/H is actually the same as C_3.

3.25 $Q/\mathfrak{Z} = \{q \in Q \mid q \in [0,1)\}$. This is an infinite abelian group.

Definition:

Let c and b be elements of a group G. Then c is called a *conjugate* of b provided that $c = a^{-1}ba$ for some $a \in G$. In this case, we also say that c and b are *conjugate.*

Examples:

3.26 In an abelian group, the only element conjugate to any element g is itself since if $c = a^{-1}ga$ then $c = ga^{-1}a = g$.

3.27 In the group of symmetries of the square, the counterclockwise rotation of 90° is conjugate to the counterclockwise rotation of 270° since if you reflect the square vertically, then rotate it 270° counterclockwise, then reflect vertically again, it is the same as rotating the square 90° counterclockwise.

3.28 Two elements in a free group that are circular permutations of each other are conjugate. For example, in F_3 the elements $x^3y^{-4}xzy^{-1}$, $y^{-4}xzy^{-1}x^3$, and $y^{-2}xzy^{-1}x^3y^{-2}$ are all conjugate.

Definition:

Let g be an element of a group G. The *conjugacy class* of g in G is $\{c \in G \mid c = a^{-1}ga \text{ for some } a \in G\}$. That is, the conjugacy class of g in G is the set of all elements in G that are conjugate to g.

Examples:

3.29 In an abelian group, each element is its own conjugacy class. This is because the only element conjugate to an element g in an abelian group is itself.

3.30 The identity element in a group is always its own conjugacy class.

3.31 The five conjugacy classes of the group of symmetries of the square are $\{$ 0° rotation $\}$, $\{$ 180° rotation $\}$, $\{$ 90° rotation , 270° rotation $\}$, $\{$ vertical reflection , horizontal reflection $\}$, $\{$ up-diagonal reflection , down-diagonal reflection $\}$. The conjugacy classes of a group always partition the group. That is, each element is in exactly one conjugacy class.

3.32 The conjugacy class of an element g in the free group F_n is $\{hgh^{-1} \mid h \in F_n\}$.

Definition:

A subset S of elements of a group G with the property that every element of G can be written as a (finite) product of elements of S and their inverses is called a *set of generators for G.* The notation used to say this is $G = \langle S \rangle$, and we say that *G is generated by S.* Any equation in the group that the generators satisfy is called a *relation in G.* If a group G is generated by a subset S and there is a collection of relations, say

$R_1, R_2, \cdots, R_n$ such that any relation among the elements of S can be deduced from these, we call these generators and relations a *presentation* of G and write $G = \langle S \mid R_1, R_2, \cdots, R_n \rangle$.

Examples:

3.33 The integer 1 is a generator for the additive group $\mathfrak{I}$ since every integer is a finite sum of +1's and -1's; so $\mathfrak{I} = \langle 1 \rangle$.

3.34 A presentation for the cyclic group $C_n = \langle a \mid a^n = 1 \rangle$.

3.35 Any of the square's eight symmetries can be obtained from the rotation of 90° and the vertical reflection elements. It is thus generated by these two elements; we will call them *r* and *s* respectively. Further, we have the relations $r^4 = 1$, $s^2 = 1$, $rs = sr^{-1}$. It can be shown that any other relation in this group can be derived from these three, and thus this group has the presentation $\langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle$.

Remark: A free group doesn't have any relations. This is why it is termed "free": it is "free" of relations.

§3.2 GRAPHS OF GROUPS (CAYLEY DIAGRAMS)

Suppose that you are setting up a soccer tournament with six teams, and you want each team to play three of the other teams. One way to describe this is with the diagram in Figure 3.1(a) where the six teams are represented by points and two teams are connected by a line if they are going to play each other in the tournament. Neither the placement of the six points, nor the lengths of the lines are important. We could just as well describe this situation with the diagram in Figure 3.1(b)

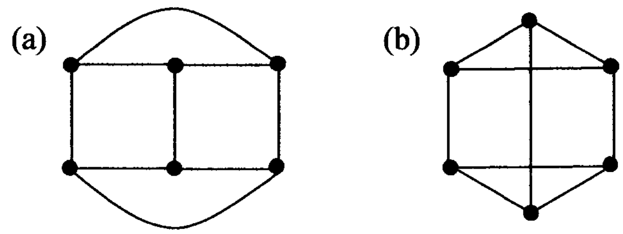


FIG 3.1 – Example of equivalent graphs

The diagrams in Figure 3.1 are examples of *graphs*. A graph is a collection of points (called *vertices*) and lines (called *edges*) connecting some of the points. Neither the locations of the vertices nor the lengths of the edges are important; what is important is which vertices are connected. The two graphs in the figure above are called *equivalent*. That is, we can label the vertices in each graph in such a way that two vertices that are connected in one graph are also connected in the second, and two vertices that aren't connected in the first are also not connected in the second (see Figure 3.2).

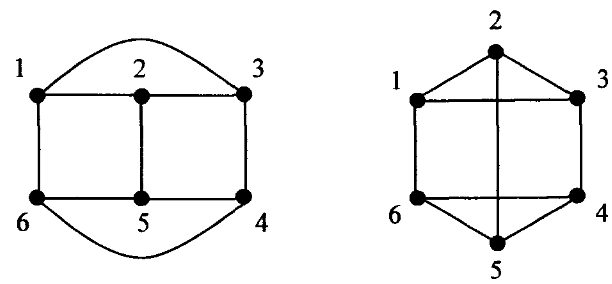


FIG 3.2 – Labeling of corresponding vertices

A *digraph* is a graph whose edges are given a direction as in the following diagram. Digraphs can be used to represent a situation like which airports are connected by one way flights. Further, a digraph with the property that any vertex can be reached from any other vertex by following the directed edges is said to be *strongly connected.*

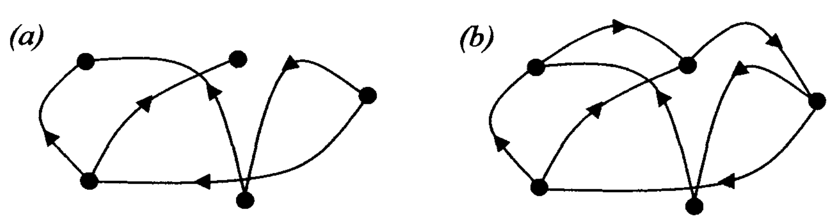


FIG 3.3 – (a) Digraph that is not strongly connected (b) Digraph that is strongly connected

Graphs and digraphs can be used to represent all sorts of situations: tournaments, flights connecting airports, electric networks, family trees, etc. Any type of situation where we are concerned about how a set of points is connected can be represented by a graph. Even a group can be represented by a graph.

The *graph of a group* is a digraph such that each vertex represents an element in the group and each edge represents the action of one of the group's generators on the vertex it's adjacent to. That is, if a is a generator of a group G and g is an arbitrary element of G, and $g \circ a = h \in G$, then there is a directed edge from the vertex representing g to the vertex representing h. Thus, any vertex v in the graph of a group with n generators will have $2n$ directed edges adjacent to it. n edges will be directed away from v (one for each generator) and n edges will be directed towards v (one for the inverse of each generator). In the graph of a group, the vertex which represents the identity element is arbitrary. That is, we can choose any vertex to be the identity element and we will still have the graph of the same group. Graphs of groups are also called *Cayley Diagrams*.

Example:

3.45 The graph (or Cayley diagram) of C_n is a polygon with n vertices. Each edge in the graph is directed with the same orientation (either clockwise or counterclockwise). Notice that each vertex has one edge directed away from it and one edge directed towards it.

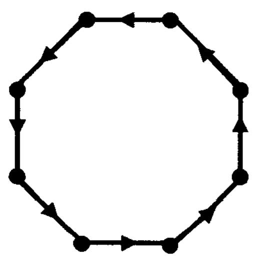


FIG 3.4 – The graph of the group C_8

3.46 The graph of an infinite group has an infinite number of vertices. For example the group of integers under addition has the following Cayley diagram. The group of integers is often thought of as the infinite cyclic group C_∞.

FIG 3.5 – The graph of $\Im$

3.47 The previous examples are graphs of groups with only one generator. The group of symmetries of a square has two generators. The following is its Cayley diagram. Here the red edges represent the action of the $90°$ rotation generator, and the blue edges represent the action of the reflection generator. When there are directed edges to and from a pair of vertices we usually represent this with a single non-directed edge. This means that the generator has the property that its square is the identity. Further, corners in a graph are interpreted as vertices. We can even think of the Cayley diagram for the group of symmetries of the square as the skeleton of a cube.

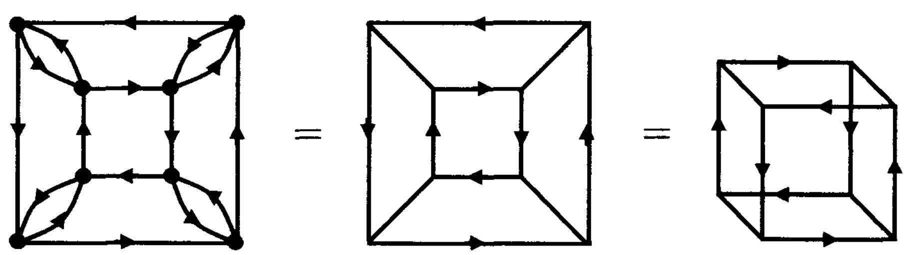


FIG 3.6 – The graph of the group of symmetries of the square

Definition:

Let Γ be the graph of a group *G*. Suppose any word in the generators of *G* that describes a closed path beginning and terminating at a vertex *v* of Γ also describes a closed path regardless of which vertex we choose to start our path from, then Γ is said to be *homogeneous.*

Notice that the three Cayley diagrams in the examples above are all homogeneous. In fact the graph of a group is always homogeneous. Further, any strongly connected homogeneous graph that can be "colored" with *n* colors in such a way that each vertex has exactly one incoming edge and one outgoing edge of each color is the graph of a group. This means that both of the graphs in Figure 3.7 are graphs of groups. Notice that the first is the graph of a group with 10 elements and two generators; this graph can be thought of as the skeleton of a pentagonal prism. The second is the graph of an infinite group with two generators.

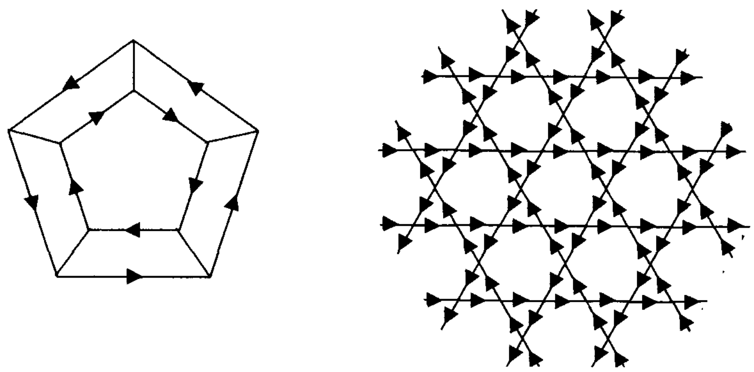


FIG 3.7 – Examples of homogeneous graphs

A section of the graph of the free group with two generators is shown in Figure 3.8. Recall that the free group does not have any relations, and thus there are no cycles in the graph of the free group. In this figure, the horizontal lines represent the action of the generator *a*, and the vertical lines represent the action of the generator *b*.

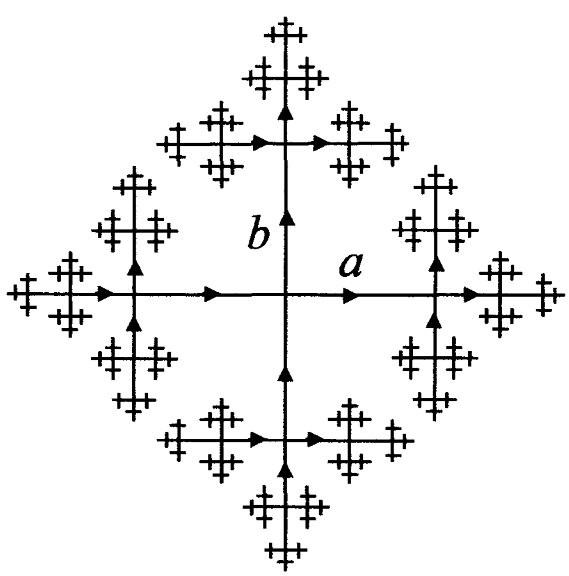


FIG 3.8 – Part of the graph of the free group F_2

CHAPTER 4

GROUP THEORETICAL APPROACH TO TILING PROBLEMS

§4.1 THE TILE PATH GROUP AND COMBINATORIAL BOUNDARY OF A REGION

We now return to solving tiling existence problems. In order to see how group theory can be used to demonstrate the nontileability of a lattice region R with tiles from a set Σ, we begin by defining the *combinatorial boundary* $[\partial R]$ of R, the *tile group* $T(\Sigma)$ of a set of tiles Σ, and the *tile path group* $P(\Sigma)$ of a set of tiles Σ. The criterion that Conway and Lagarias discovered is that in order for a region R to have a perfect tiling by tiles in Σ, the combinatorial boundary $[\partial R]$ of R must be equivalent to the identity element in the tile path group $P(\Sigma)$ of Σ. The combinatorial boundary, tile group and tile path group can be defined for any lattice region. However, for now we will restrict our definitions to the square lattice.

The square lattice consists of *lattice points*, *edges* and *cells*. Lattice points are the ordered pairs (x, y) with both x and y being integers; two lattice points are *neighbors* if they are distance one from each other. An edge is a line segment connecting two neighboring lattice points. A cell is the set of all points making up the interior and boundary of a square of area one having its four vertices at lattice points.

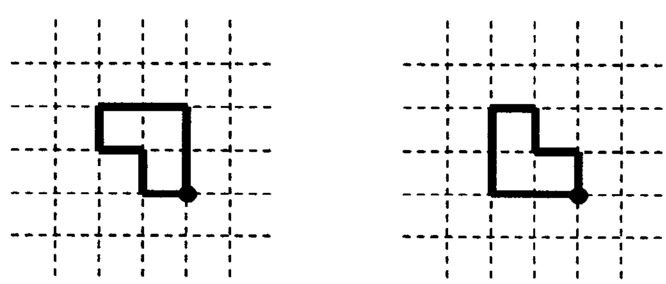
A *directed path* P in the square lattice is a sequence of directed edges specified by a sequence of lattice points $\{(x_i, y_i) \mid 0 \le i \le n\}$, where the i^{th} directed edge connects the lattice point (x_{i-1}, y_{i-1}) to the lattice point (x_i, y_i). P is said to be *closed* if

$(x_0, y_0) = (x_n, y_n)$. A directed path is called *simple* if no edge appears twice and if it does not cross itself.

The *topological boundary* ∂C of a cell consists of its four edges oriented counterclockwise. The topological boundary ∂R of a region *R* is a (unordered) set of directed edges found by taking the set of edges in ∂C for all *C* in *R,* and discarding any edges that occur twice with opposite orientations. A region *R* is *simply connected* if its complement is connected and if its boundary edges can be ordered to form a simple closed path.

A simple closed path bounding a simply connected region *R* can be uniquely identified by its *leading edge e*. We call this the *oriented boundary of R with leading edge e* and denote it by $\partial R(e)$. We define the first vertex in $\partial R(e)$ as the *base point.*

Directed paths in the square lattice can be described by words in the free group $F_2 = \langle a, u \rangle$ on two generators. Here, *a* symbolizes an across (to the right) oriented edge, and *u* symbolizes an upwards oriented edge. To the path $P = \{(x_i, y_i) \mid 0 \le i \le n\}$ we assign the word $W = W(P)$ given by $W = g_1 g_2 \cdots g_n$, where $g_i \in \{a, a^{-1}, u, u^{-1}\}$. Figure 4.1 gives the words associated to certain simply connected regions in the square lattice. The base points are symbolized by the highlighted vertices. Note that the paths around the regions are oriented counterclockwise.



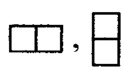
$R_1 = u^2 a^{-2} (u^{-1} a)^2$ $\qquad$ $R_2 = (ua^{-1})^2 u^{-2} a^2$

FIG 4.1 – Words in the free group $F_2 = \langle a, u \rangle$ *associated to simply connected regions* R_1, R_2

In the same light, given a word $W \in F_2$, we can assign the directed path $P = P(W)$, starting from the base point $(0,0)$ in the square lattice, obtained by reading W from left to right.

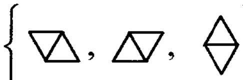
Note that all of the words in the set $\{\partial R(e) \mid e \text{ an edge of } \partial R\}$ are cyclic permutations of each other and hence are conjugate in F_2. The *combinatorial boundary* $[\partial R]$ of a simply connected region R is the conjugacy class in F_2 containing all of the oriented boundaries $\partial R(e)$ of R. That is, $[\partial R] = \{W \partial R(e) W^{-1} \mid W \in F_2\}$. The *tile group* $T(\Sigma)$ is defined as the smallest normal subgroup of F_2 containing the combinatorial boundaries $[\partial \Sigma_i]$ for each tile $\Sigma_i \in \Sigma$. That is, $T(\Sigma) = \langle W \partial \Sigma_i(e_i) W^{-1} \mid W \in F_2 \rangle$. The *tile path group* $P(\Sigma)$ is the quotient group ${F_2}/{T(\Sigma)}$. Recall that the group operation in F_2 (and resultantly in $T(\Sigma)$ and $P(\Sigma)$) is concatenation. The tile group and tile path group can be defined similarly for regions on the other lattices.

Examples:

4.1 Let $\Sigma = \{$  $\}$, then the tile group

$T(\Sigma) = \langle W a^2 u a^{-2} u^{-1} W^{-1}, V a u^2 a^{-1} u^{-2} V^{-1} \mid W, V \in F_2 \rangle$, and the tile path group

$P(\Sigma) = \langle a, u \mid a^2 u a^{-2} u^{-1} = a u^2 a^{-1} u^{-2} = 1 \rangle = \langle a, u \mid u a^2 = a^2 u, u^2 a = a u^2 \rangle$.

4.2 If we label the edges of the triangle lattice as a, b, c where a points at $0°$, b points at $120°$, and c points at $240°$, and if $\Sigma = \{$  $\}$, then

$P(\Sigma) = \langle a, b, c \mid c a c^{-1} a^{-1} = b c b^{-1} c^{-1} = a b a^{-1} b^{-1} = 1 \rangle = \langle a, b, c \mid ab = ba, ac = ca, cb = ca \rangle$

§4.2 CONWAY AND LAGARIAS' THEOREM

Now that we have defined the combinatorial boundary of a region and the tile group of a set of tiles, we can state our long awaited theorem.

Theorem 4.1: (Conway and Lagarias)
A necessary condition that a simply connected region R have a perfect tiling by tiles in a set Σ is that the combinatorial boundary $[\partial R]$ of R is in the tile group $T(\Sigma)$ of Σ.

Before proving this theorem, we will need the following lemma:

Lemma:
Given a tiling τ of a simply connected region R, there exists a decomposition $R = R^{*} \cup R^{**}$ such that R^{*}, R^{**} are both nonempty simply connected regions which can be tiled by Σ. Moreover, there are directed edges e_1 of ∂R^{*} and e_2 of ∂R^{**} so that $\partial R(e_1) = \partial R^{*}(e_1)\partial R^{**}(e_2)$.

Proof of Lemma:
Assume that τ is a tiling of *R* with $k \geq 2$ tiles. Choose any tile *S* in τ such that ∂R and ∂S have an edge *e* in common. Then one can partition ∂R into $2n$ parts where *n* is the number of times ∂R and ∂S have a set of consecutive edges in common. Refer to Figure 3.2 for an example of such a decomposition. Denote the parts of ∂R and ∂S as ∂R_i and ∂S_i respectively, starting with $\partial R_1 = \partial S_1$ so that ∂R_1 contains *e*. Continue labeling ∂R_{i+1} and ∂S_{i+1} in a counter-clockwise manner.

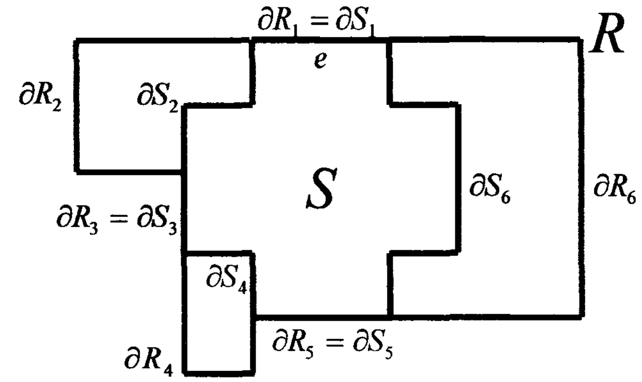


FIG 4.2 – Decomposition of ∂R into 2n parts where n is the number of times ∂R and ∂S have a set of consecutive edges in common

Now denote ∂R^* as $\partial R_2 \cup \partial S_2$ with the orientation of the edges in ∂S_2 reversed. Then ∂R^* is a simple closed path that encloses a nonempty simply connected region R^*. Let $R^{**} = R - R^*$, then R^{**} is also simply connected. Further, notice that S separates the tiles of τ in R^* from the tiles in $R^{**} - S$. So all the tiles of $\tau - \{S\}$ in $R - S$ lie in either R^* or R^{**}, hence R^* and R^{**} are both nonempty simply connected regions which can be tiled by Σ.

Further, if we let e_1 be the first edge in ∂R_2 and let e_2 be the first edge in ∂S_2, then we have that $\partial R(e_1) = \partial R^*(e_1)\partial R^{**}(e_2)$.

■

Example 4.3

Consider the following tiling of the 8×8 square region R by T-tetrominoes, and let the yellow tile be S:

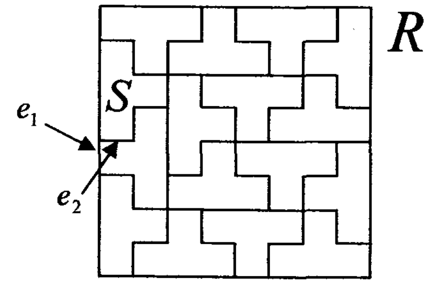


FIG 4.3 – Decomposition of R into R^* and R^{**}

Then $R^* = R \setminus S$ and $R^{**} = S$. Then we have that $R = R^* \cup R^{**}$ and both R^* and R^{**} can be tiled by T-tetrominoes. Further if we let e_1 and e_2 be the edges labeled in Figure 4.3, then we have that

$$\partial R(e_1) = u^{-4}a^8u^8a^{-8}u^{-4} = \partial R^*(e_1)\partial R^{**}(e_2) = \left(u^{-4}a^8u^8a^{-8}u^{-1}au^{-1}au^{-1}a^{-1}u^{-1}a^{-1}\right)\left(auaua^{-1}ua^{-1}u^{-3}\right)$$

Proof of Theorem 4.1:

The proof is by induction on the number of tiles in a tiling τ of R with tiles from Σ.

The result is clear if R can be tiled with a single tile. Indeed the combinatorial boundary of R is in the tile group since the tile group is defined as the smallest normal subgroup that contains the combinatorial boundaries of all tiles in Σ.

Assume that it is true for a tiling τ of R with $k \geq 2$ tiles. That is, that the combinatorial boundary of R is contained in the tile group of Σ.

We want to show now that for a tiling τ of R with $k+1$ tiles, $[\partial R]$ is contained in $T(\Sigma)$.

Assume that τ is a tiling of R with $k+1$ tiles. Then from the lemma, we can partition R into R^* and R^{**} where both R^* and R^{**} can be tiled with $m \leq k$ tiles, and hence both $\partial R^*(e_1)$ and $\partial R^{**}(e_2)$ are in $T(\Sigma)$ by the induction hypothesis. From the moreover part of the lemma and the group structure of $T(\Sigma)$ we have that $\partial R(e_1) \in T(\Sigma)$.

■

Theorem 4.2:

A necessary condition that a simply connected region R have a perfect tiling by tiles in a set Σ is that the combinatorial boundary $[\partial R]$ of R is trivial in the tile path group $P(\Sigma)$ of Σ.

Proof:

By the definition of $P(\Sigma)$ and Theorem 4.1.

Although it is easy to find a presentation for the tile path group of a set of tiles, it is often very difficult to work with a group given only a presentation of it. Hence, it is very difficult to decide whether or not the combinatorial boundary of a region is contained in the tile path group. However at times we can use the Cayley diagram of the tile path group to help show the nontileability of a region.

§4.3 THURSTON'S DIAMOND TILINGS

In a paper published in 1990 [12], William T. Thurston, from Princeton University, used Conway and Lagarias' tile group method to show when a region on the equilateral triangle lattice cannot be tiled by diamonds.

Thurston first labeled the lattice so that the set of edges parallel to the x-axis (at 0°) was labeled a; he labeled the set of edges pointing at 120° as b, and the set of edges pointing at 240° as c (see Figure 4.4). He noted that since this labeling is strongly connected and homogeneous, the labeled, directed lattice represents the graph of a group which he called A.

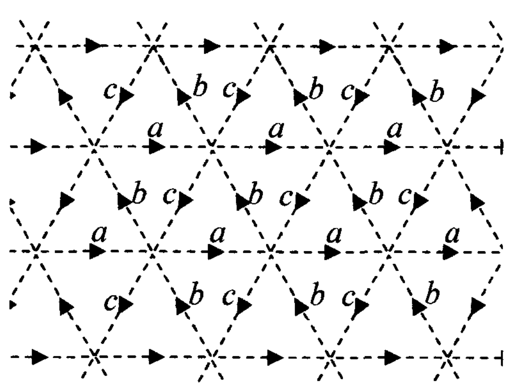


FIG 4.4 –Homogeneous labeling of the equilateral triangle lattice

Suppose we wanted to tile a region R on the triangle lattice bounded by a polygon π. π can be described by the sequence of edges that its boundary traces out; this is a word in the generators a,b,c of A. Moreover, we can think of it as an element $W(\pi)$ in the free group F_3 with generators a,b,c.

When a diamond is placed on the triangle lattice, its boundary traces out one of three paths: $D_1 = aba^{-1}b^{-1}$, $D_2 = bcb^{-1}c^{-1}$, $D_3 = cac^{-1}a^{-1}$. Depending on the leading edge of the diamonds' boundary, D_1, D_2, D_3 can be described as circular permutations as the previously stated D_i's. However, this will just give conjugate elements in F_3. The tile path group $P(\Sigma)$, which we will call D, is defined by these relations. That is, $D = \langle a,b,c \mid D_1 = D_2 = D_3 = 1 \rangle$. The reader may notice that the relations say that the three generators commute with one another, so D is actually $\Im^3$.

Observation:

By Theorem 4.2, if a region R bounded by a polygon π on the equilateral triangle lattice is to be tiled by diamonds, then the image $I(\pi)$ of $W(\pi)$ in D must be trivial.

Because of the convenience of D being isomorphic to $\Im^3$, there is a very nice geometric interpretation of this. Think of the Cayley diagram $\Gamma(D)$ of the group D as the skeleton of the canonical cubical tessellation of 3-space so the cubes are lined up face to face. Label the edges parallel to the x,y,z-axis with a,b,c respectively with positive orientation (see Figure 4.5). If the region R, on the triangle lattice, can be tiled with diamonds, then the word describing the boundary of R must correspond to a closed path in $\Gamma(D)$.

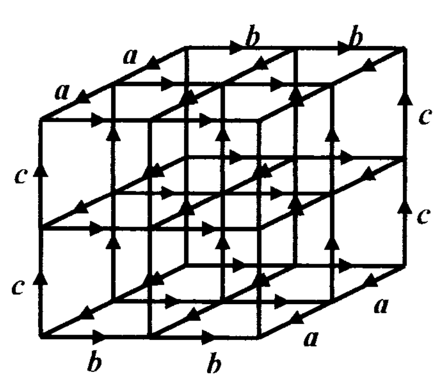


FIG 4.5 –Eight cells from $\Gamma(D)$

Examples:

4.4 The following region R does not have a tiling by diamonds since the word describing its boundary does not represent a closed path in $\Gamma(D)$.

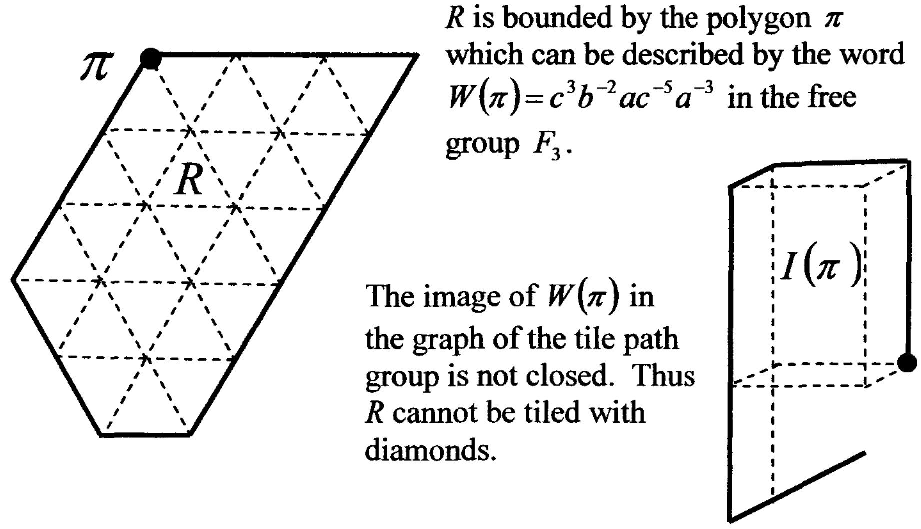


FIG 4.6 – Region on the triangle lattice untileable by diamonds

4.5 The following is a tiling of a region R by diamonds. Thus R is necessarily bounded by a polygon π which corresponds to a closed path in the graph of the tile path group.

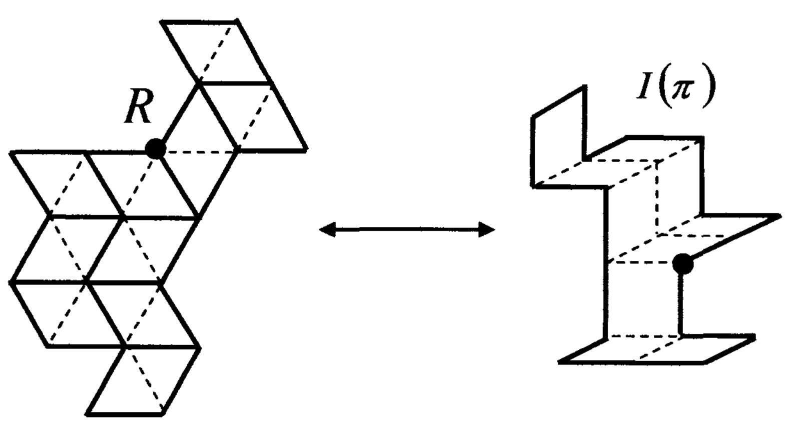
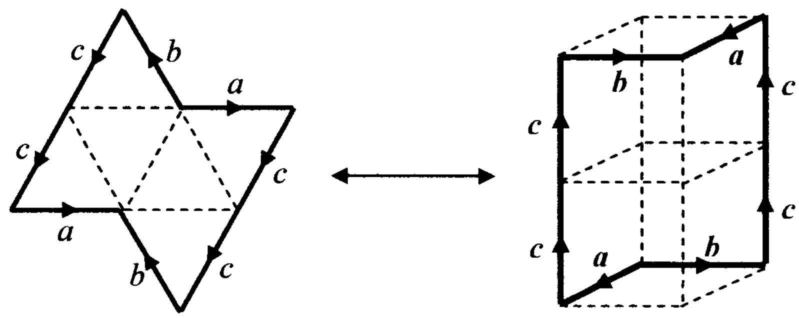


FIG 4.7 – Region on the triangle lattice tileable by diamonds

It is worth noting that the condition that $I(\pi)$ is trivial in D is not sufficient to guarantee a tiling by diamonds. There exist regions in the triangle lattice that satisfy Conway and Lagarias' condition, however no perfect diamond tiling exists (see Figure 4.8). Thus if the combinatorial boundary of R is trivial in the tile path group $P(\Sigma)$, we still cannot determine that a tiling of that region exists with tiles from Σ.



$$W(\pi) = c^2ab^{-1}c^{-2}a^{-1}b$$

FIG 4.8 –Region in the triangle lattice untileable by diamonds with boundary corresponding to the identity in the tile path group D

In the next chapter we will see some methods that at times prove useful in showing nontileability in situations when the boundary word method above seems insufficient.

§4.4 REID'S CONTRIBUTION

In a paper published in 2003 [10], Michael Reid gives the following extension of Conway's Theorem.

Theorem 4.3 (Reid):

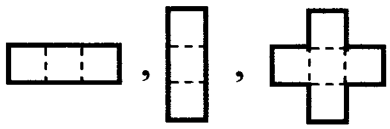
Suppose that the combinatorial boundary of each tile in Σ is trivial in a group G. If a simply connected region can be tiled by Σ, then its combinatorial boundary must also be trivial in G.

Proof:

This theorem can be proved by induction on the number of tiles in a tiling of R much the same way that Theorem 4.1 was proved.

What Reid's Theorem is saying is that it doesn't matter what group is used in Theorem 4.1. The tile path group works fine since all of the combinatorial boundaries of the tiles in Σ are trivial in $P(\Sigma)$. Theorem 4.3 is an extremely useful result: although it is quite easy to find a finite presentation for the tile path group, it is usually very difficult to work with the tile path group in this form. It may be easier to find a different group that the combinatorial boundaries of all tiles in Σ are trivial in.

Example 4.6:

Let $\Sigma = \Big\{$  $\Big\}$.

Consider the permutation group on five variables, S_5 (see appendix A). Let $a = (1,2,3) \in S_5$, and $u = (3,4,5) \in S_5$. Then, since $a^3ua^{-3}u^{-1} = au^3a^{-1}u^{-3} = auaua^{-1}ua^{-1}u^{-1}a^{-1}u^{-1}au^{-1} = (1) \in S_5$, the combinatorial boundaries of each of the tiles in Σ are equivalent to the identity element in S_5. Therefore, if a region is tileable by the tiles in Σ, its combinatorial boundary must also be equivalent to the identity in S_5. With this in mind, we can conclude that any $(3m+1)\times(3n+1)$ rectangle cannot be tiled by tiles from Σ. This can be seen since the boundary word of a $(3m+1)\times(3n+1)$ rectangle is:

$$a^{3n+1}u^{3m+1}a^{-(3n+1)}u^{-(3m+1)} = a^{3m}au^{3n}ua^{-3n}a^{-1}u^{-3m}u^{-1} = aua^{-1}u^{-1} = (123)(345)(132)(354) = (235) \neq 1$$

It can similarly be shown that the boundary words of $(3m+1)\times(3n+2)$ and $(3m+2)\times(3n+2)$ rectangles are also never trivial in S_5 and thus no rectangles of these dimensions can be tiled by tiles in Σ. It follows that if a rectangle is tileable by tiles in Σ, either m or n must be divisible by three.

It should be further noted that since a 1×1 square has a signed tiling with tiles from Σ, there is no numbering argument that could have proved this result.

CHAPTER 5

WINDING NUMBERS

§5.1 WINDING NUMBER OF A CLOSED PATH AND SIGNED AREA

We have seen how if the combinatorial boundary $[\partial R]$ of a region R does not correspond with the identity element in a group G that all of the boundaries of tiles in Σ are trivial in, then we can determine that a perfect tiling of R does not exist with tiles from Σ. However, if $[\partial R]$ does correspond to the identity element in G we cannot conclude that a perfect tiling exists. In this section we will introduce the concepts of winding numbers and signed area which at times can be used to determine that a region cannot be tiled, even though $[\partial R]$ is trivial in any such G we can find.

Definition:

Let p be a closed lattice path, and let P be a point on the plane that is not on p. If we draw a ray emanating from P, then the ***winding number of*** P is the number of times p crosses from one side of the ray to the other in the counterclockwise direction minus the number of times it crosses in the clockwise direction (this definition is consistent regardless of the direction of the ray). The ***winding number of a cell*** in the lattice can be defined as the winding number of an arbitrary point in the cell since the winding number of any point in a cell is the same as the winding number of any other point in the cell. Further, for any lattice and any closed path on the lattice, the winding numbers of the cells in the lattice are all zero except for a finite number of cells. Hence we can calculate the sum of the winding numbers of all the cells in the lattice. This is what we define as the *signed area of the path.*

Examples:

5.1 The closed path on the square lattice in Figure 5.1 can be described by the word $u^{-2}auaua^{-2}$ in F_2. Each of the cells labeled *A*, *B*, and *C* have winding number +1 while all other cells in the lattice have winding number zero. Thus the signed area of the path is +3. In fact, if the closed path is simple (no edge appears twice and the path does not cross itself), then the signed area is equal to plus or minus the number of cells enclosed by the path: positive if the path winds counterclockwise and negative if the path winds clockwise.

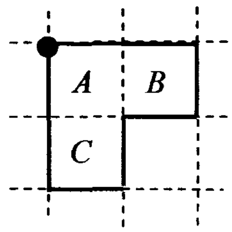


FIG 5.1 – closed path on the square lattice

5.2 The following closed path on the square lattice can be described by the word $a^2ua^{-1}u^{-1}a^{-2}uau^{-1}$ in F_2. The winding number of the cell labeled *A* is +1 and the winding number of the cell labeled *B* is -1. The winding numbers of all other cells are zero. Thus, the signed area of the path is zero.

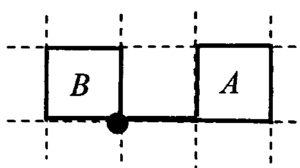


FIG 5.2 – closed path on the square lattice with signed area zero

Winding numbers and signed areas can be used to strengthen Theorem 4.3. At times, geometric subtleties arise from Cayley diagrams. Especially when the Cayley diagram is planar (it can be drawn on the plane without graph edges crossing).

Theorem 5.1

If the boundary word of each tile in Σ corresponds to a closed path enclosing a signed area of zero in a particular Cayley Diagram, then for a region R to be tileable by tiles in Σ, its boundary word must also correspond to a closed path enclosing a signed area of zero.

Proof:

The proof is done by induction on the number of tiles in a tiling of R much like the proof of Theorem 4.1

§5.2 TILING AZTEC DIAMONDS WITH SKEW TETROMINOES

Although at times, finding the tile path group of a set of tiles, and the Cayley diagram of that group can prove nontileablity of regions in the plane, generally it is very difficult to find a presentation for such a group that is easy to work with. Take for instance the tile group for the four orientations of the skew tetromino (see Figure 5.3). A presentation for this group is $\langle a,u \mid a^2uaua^{-2}u^{-1}a^{-1}u^{-1} = auau^2a^{-1}u^{-1}a^{-1}u^{-2} = au^{-1}au^2a^{-1}ua^{-1}u^{-2} = a^2ua^{-1}ua^{-2}u^{-1}au^{-1} = 1\rangle$; it is not at all easy to construct the Cayley diagram of this group given this presentation.

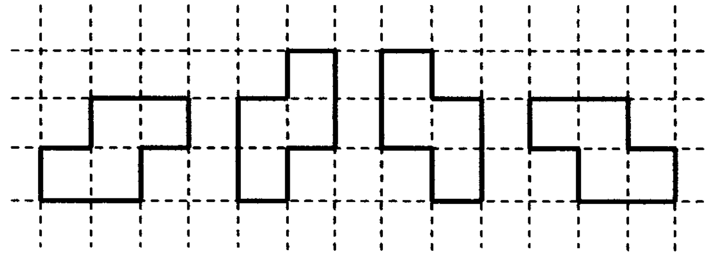


FIG 5.3 –The four orientations of the skew tetromino

However, thanks to Theorem 4.3, we don't actually need to find the tile path group itself. In a paper published in 1997 [9], James Propp uses a particular strongly connected homogeneous directed graph (thus a graph of a group) such that the boundary words of all orientations of the skew tetromino correspond to closed paths in order to prove that Aztec diamonds of any size cannot be perfectly tiled by skew tetrominoes (see Figure 5.4); we will call this graph Γ. Propp uses Γ to establish that if a region in the square lattice is tileable by skew tetrominoes, then the word that describes the boundary of that region must also correspond to a closed path in Γ. Further, Propp noticed that each of the orientations of the skew tetromino correspond to a path in Γ that encloses a signed area of zero. Thus, if a region in the square lattice is tileable by skew tetrominos, then the word describing the boundary of that region must also correspond to a closed path enclosing a signed area of zero in Γ.

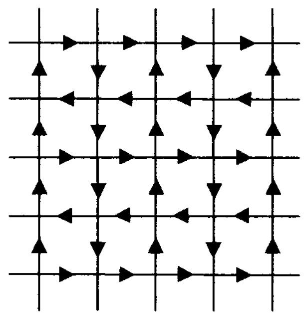


FIG 5.4 – Propp's Cayley diagram Γ for an infinite group with two generators – the horizontal and vertical directed edges represent the action of the generators a and u respectively. Note that the direction that a and u represent alternate each row and column respectively.

Definition:

An *Aztec diamond of order n* is a region in the square lattice consisting of $2n(n+1)$ unit squares arranged in centered rows of lengths $2,4,6,\cdots,2n-2,2n,2n,2n-2,\cdots,6,4,2$.

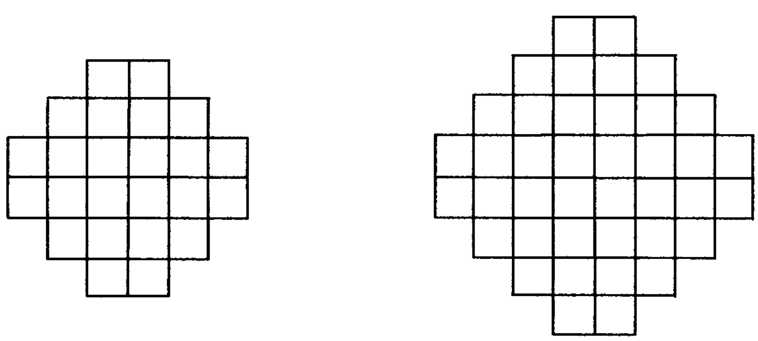


FIG 5.5 – Aztec diamonds of orders 3 and 4

Result:

An Aztec diamond of any order $n > 0$ cannot be tiled using the four orientations of the skew tetromino.

Proof: The combinatorial boundary of an Aztec diamond of order *n* is $\left\lfloor (ua)^n (au^{-1})^n (u^{-1}a^{-1})^n (a^{-1}u)^n \right\rfloor$. Although this corresponds to a closed path in the Cayley diagram in Figure 5.4, it corresponds to a closed path with a signed area of $2n$ or $-2n$ if *n* is even (depending on which vertex you use as your base point in the Cayley diagram), and to a closed path with signed area $2(n+1)$ or $-2(n+1)$ if *n* is odd.

Thus, by Theorem 5.1, since the signed area enclosed by the closed path corresponding to the boundary of the Aztec diamond is not zero for any value of $n > 0$, we have the desired result.

■

It is also worth noting that an Aztec diamond of order *n* has a signed tiling by skew tetrominoes whenever $n \equiv 0,3 (\mathrm{mod}\, 4)$, and thus (by the consequence of Theorem 2.1) there is no numbering argument that proves an Aztec diamond of this size cannot be tiled by skew tetrominoes. It is in this sense that we can say the group theoretical approach to tiling problems is stronger than the numbering argument approach.

Another way to prove Theorem 5.1 (courtesy of Michael Reid) using only information from Chapter 4 is to consider the group of 3×3 matrices with entries from $\Im$ and non-zero determinants under matrix multiplication; let $a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $u = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Then one can easily verify that each of the boundary words of the four orientations of the skew tetromino are trivial in this group, but the boundary word for the Aztec diamond of order *n* is not trivial for any value of $n > 0$.

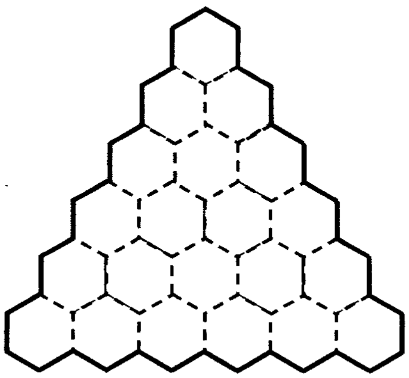
In the next section, we explore a problem where a solution using only information from Chapter 4 has not yet been discovered; it is conceivable that a solution using only methods from Chapter 4 does not exist.

§5.3 HEXAGONAL TRIANGLE TILING PROBLEM

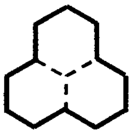
In this section, we study another example where Conway and Lagarias' method can be used to solve problems that coloring or numbering arguments cannot. We examine another problem of which a signed tiling exists but a perfect tiling does not.

The problem, in its original form, was stated on the hexagon lattice. We will state the problem in its original form and then show how it is equivalent to a similar problem on the square lattice.

For this problem, we introduce the notation T_N with N being a non-negative integer. T_N denotes the triangular region with N hexagons along each side. See Figure 5.6.



T_6



T_2

FIG 5.6 – Examples of T_N for N=6 and N=2

Problem 5.1: (hexagon lattice)

For which values of N can T_N be tiled with copies of T_2 if we are allowed to rotate tiles as we wish?

Figure 5.7 shows a perfect tiling of T_9 with copies of T_2.

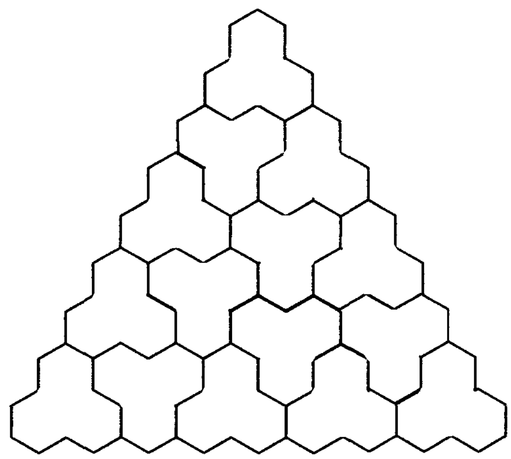
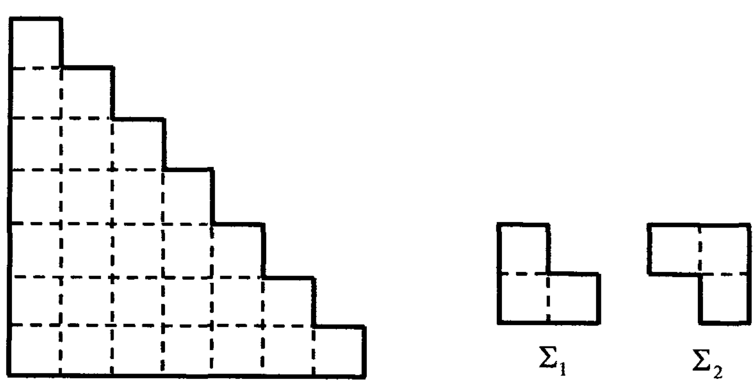


FIG 5.7 – Tiling of T_9 with copies of T_2

Problem 5.1 can be converted into the following problem on the square lattice. Firstly, T_N now denotes the staircase region with N steps as in Figure 5.8(a).

Problem 5.2: (square lattice)

For which values of N can the staircase region T_N be tiled with copies of T_2 if the only orientations of T_2 allowed, are the ones displayed in Figure 5.8(b).



(a) (b)

FIG 5.8 – (a) T_N on the square lattice for N=7 (b) Allowable orientations of T_2

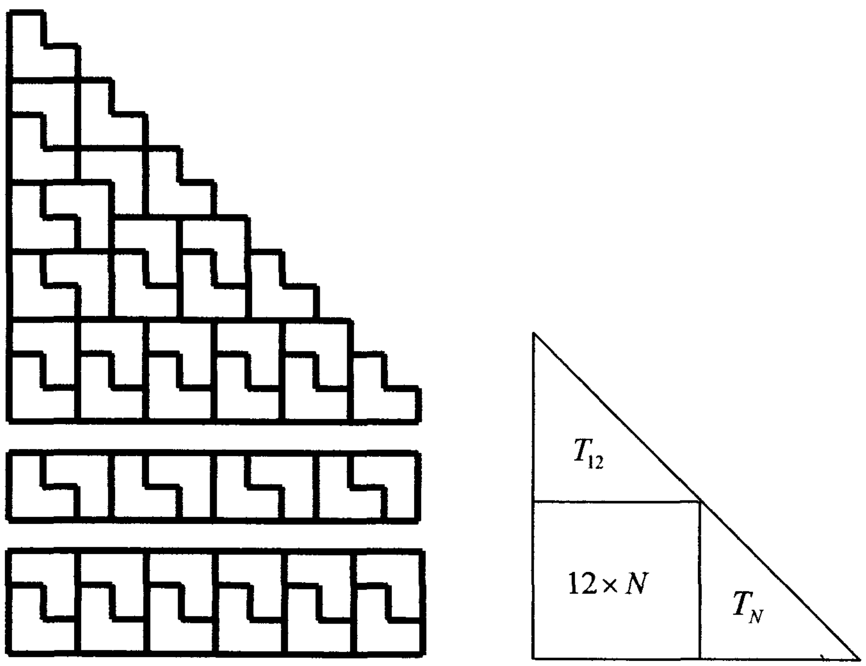
It is useful, at this point, to take note of a few details.

Note #1:

Notice that in order for either a signed tiling or a perfect tiling to exist, clearly T_N must have an area of a multiple of three. Since T_N has an area of $A = \frac{N(N+1)}{2}$, it is an easy exercise to show that A is a multiple of three precisely when $N \equiv 0,2 \pmod{3}$.

Note #2:

Notice that the staircase region T_{12} as well as the 12×2 and 12×3 rectangles can be tiled as in Figure 5.9(a). This shows that if T_N has a perfect tiling, then so does T_{N+12}, see Figure 5.9(b).



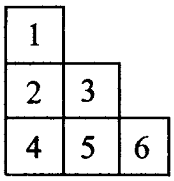
(a) *Perfect tilings of* T_{12} *and* 12×2 *&* 12×3 *rectangles*

(b) *Method of tiling* T_{N+12} *from a given tiling of* T_N

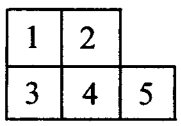
FIG 5.9

Note #3:

Notice that the both the staircase region T_3 and the 3×1 rectangle have signed tilings as in Figure 5.10(a). This shows that if T_N has a signed tiling, then so does T_{N+3}, see Figure 5.10(b).

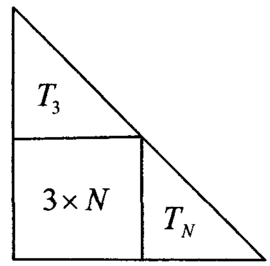


Positive tiles: [1,2,3],[2,4,5],[3,5,6]
Negative tiles: [2,3,5]



Positive tiles: [1,3,4],[2,4,5]
Negative tiles: [1,2,4]

(a) *Signed tilings of T_3 and 3×1 rectangle*



(b) *Method of constructing a signed tiling for T_{N+3} from a given signed tiling of T_N*

FIG 5.10

Note #4:

The staircase regions T_2, T_9, T_{11}, T_{12} have perfect tilings (refer to Figure 5.11 for such tilings). The staircase regions T_2, T_3 have signed tilings (it is obvious for T_2 and it was shown for T_3 in Figure 5.10(a)).

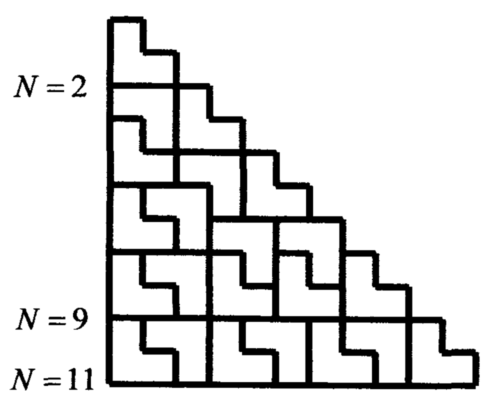


FIG 5.11 - Perfect tilings of T_2, T_9, T_{11}. The tiling for T_{12} was shown in Fig 5.9(a)

In summary, we have shown that the staircase region T_N can be perfectly tiled if $N = 0,2,9,11 \pmod{12}$, and that it cannot be tiled if $N = 1,4,7,10 \pmod{12}$. Therefore the only values of N that are still left to be determined are $N = 3,5,6,8 \pmod{12}$. Moreover, since T_N has a signed tiling when $N = 3,5,6,8 \pmod{12}$, Theorem 2.1 tells us that no numbering argument can be used to solve this problem for these cases.

In order to determine whether or not the staircase region T_N has a perfect tiling with the tiles shown in Figure 5.8 for $N = 3,5,6,8 \pmod{12}$, we define a special subgroup H of F_2 by the property that the group $G = F_2/H$ has the Cayley diagram shown in Figure 5.12. If $F_2 = \langle a,u \rangle$ then the edges corresponding to the generators a and u are red and blue respectively. We know that this is the graph of a group since it is strongly connected and homogeneous.

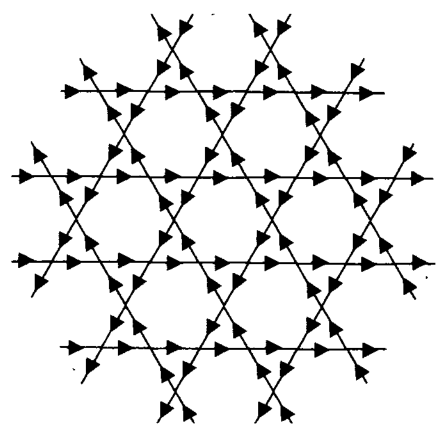


FIG 5.12 – Cayley diagram $\Gamma(G)$ the red and blue edges represent the action of the generators a and u respectively

The relevance of H is due to the following claim:

Claim:

The combinatorial boundary $[\partial T_N]$ for $N = 0,2 \pmod 3$, as well as $[\partial \Sigma_1]$ and $[\partial \Sigma_2]$ are contained in H.

Proof:

It is enough to show that individual representatives of $[\partial\Sigma_1]$, $[\partial\Sigma_2]$ and $[\partial T_N]$ are contained in H (when $N \equiv 0,2(\text{mod}\,3)$). Notice first that $\partial\Sigma_1 = u^{-2}a^2(ua^{-1})^2$ and $\partial\Sigma_2 = (u^{-1}a)^2 u^2 a^{-2}$; it is easy to check that both of these correspond to closed paths in $\Gamma(G)$ and thus are contained in H. Further, $\partial T_N = u^{-N}a^N(ua^{-1})^N$, and it is easy to check that a^3, u^{-3} and $(ua^{-1})^3$ all correspond to closed paths in $\Gamma(G)$; this means that if ∂T_N is in H, then so is ∂T_{N+3} since $u^{-(N+3)}a^{N+3}(ua^{-1})^{N+3} = u^{-N}u^{-3}a^N a^3 (ua^{-1})^N (ua^{-1})^3$. Also note that ∂T_2 and ∂T_3 correspond to closed paths in G and thus are in H which completes the proof.

■

Finally we consider winding numbers to solve problem 5.2 for $N = 3,5,6,8(\text{mod}\,12)$.

First we define the *hexagonal signed area of a closed path V in* $\Gamma(G)$, and denote it by $\phi(V)$. Let $V \in H$ be the word that describes a closed path in the Cayley diagram $\Gamma(G)$ above. Then $\phi(V)$ is the sum of the winding numbers of all the hexagons in $\Gamma(G)$.

We know that $\partial T_N \in H$ (for $N = 3,5,6,8(\text{mod}\,12)$), and since $\partial T_n = u^{-N}a^N(ua^{-1})^N$, we have that $\phi(\partial T_N) = \left\lfloor \frac{N+1}{3} \right\rfloor$ in those cases. Further, it is easy to calculate $\phi(\partial\Sigma_1) = 1$ and $\phi(\partial\Sigma_2) = -1$, and by the homogeneousity of $\Gamma(G)$ we have that $\phi([\partial\Sigma_1]) = 1$ and $\phi([\partial\Sigma_2]) = -1$. Suppose τ is a tiling of T_N by tiles in Σ, then we have that $\phi(\partial T_N)$ will be an odd number if there is an odd number of tiles in τ, and an even number if there is an even number of tiles. In other words, if m is the number of tiles in τ, then $\left\lfloor \frac{N+1}{3} \right\rfloor \equiv m(\text{mod}\,2)$. Next, notice that since each tile has an area of three, we know that the area of T_N is an odd number if there is an odd number of tiles in τ and an even number if there is an even number of tiles. In other words, if *m* is the number of tiles in τ, then $\frac{N(N+1)}{2} \equiv m(\mathrm{mod}\,2)$.

Therefore we have that in a tiling τ of T_N, $\left\lfloor \frac{N+1}{3} \right\rfloor \equiv \frac{N(N+1)}{2}(\mathrm{mod}\,2)$. Note that both sides of this congruence are periodic modulo 12, and it is easily checked that this congruence does not hold for $N = 3,5,6,8(\mathrm{mod}\,12)$, proving that $\partial T_N \notin T(\Sigma)$ in these cases, and hence...

Result:

The triangle region T_N can be tiled perfectly by copies of T_2 if and only if $N = 0,2,9,11(\mathrm{mod}\,12)$.

The preceding argument is how Conway and Lagarias proved this result.

It should also be noted that a more elementary (although perhaps more tedious) solution to this problem was given by Donald C. West in 1991 and can currently be viewed on Donald West's homepage [14].

§5.4 CUMULATIVE SUMMARY

Once a set of tiles Σ has been given, a region *R* can fall into one of three categories. It can (i) have a perfect tiling (ii) have no signed tiling (and hence no perfect tiling) or (iii) have a signed tiling but no perfect tiling. If the region falls into category (i) we can prove that it has a tiling by producing a valid tiling for it. If the region falls into category (ii) then there exists a numbering argument that proves a signed tiling doesn't exist. If the region falls into category (iii) then there is no numbering argument that can be used to show this, so these are often the difficult tiling problems to solve.

At times, Reid's Theorem (Theorem 4.3) can be used to solve some of the problems in which the region falls into category (iii). If we can find a group such that the combinatorial boundaries of all the tiles in Σ are trivial in, but the combinatorial boundary of R is not, then we can conclude that R cannot be tiled. This is easier said than done, but in fact we don't actually need to find a specific group. If we can find a homogeneous directed graph Γ with the property that all the boundary words of the tiles in Σ correspond to closed paths in Γ, but the boundary word of R does not correspond to a closed path, then we can conclude that R cannot be tiled. We can further use the concept of winding numbers and signed area of a closed path to strengthen our arguments.

We can always find a finite presentation for the tile path group $P(\Sigma)$. This group, by definition, has the property that all the combinatorial boundaries of tiles in Σ are trivial in it. However, it is usually very difficult to work with a group given only a finite presentation. In the next chapter we introduce a few other groups that can help to determine when a region does not have a perfect tiling.

EXTENSION

TILE HOMOLOGY AND HOMOTOPY GROUPS

In this extension we enter further into the realm of Group Theory and define the tile homology and tile homotopy groups of a set of tiles. Some of the concepts here are from the more advanced theory of abstract algebra and algebraic topology, and it is assumed that the reader has some experience in the area. The group theory included in Chapter 3 does not cover what is done in this extension. Nevertheless, we encourage the less experienced reader to continue reading to gain an idea of how the theory of groups can assist in solving tiling problems. We also recommend Michael Reid's paper [10] for someone who is interested in this area. Reid's work in this area is most impressive.

Although the concepts in this chapter can be applied to any of the regular lattices, we will be restricting ourselves to the square lattice.

§E.1 TILE HOMOLOGY GROUP

Let A be the free abelian group of all the cells in the lattice. That is, every element of A has an integer-entried coordinate corresponding to each cell in the lattice; the group operation is coordinatewise addition. The generators of A are the elements with a 1 in exactly one coordinate and zeros elsewhere. Each generator corresponds to a cell on the lattice. To a placement of a tile in Σ, we associate the element of A which is 1 in those coordinates whose cell is covered by the tile placement, and is 0 in all other coordinates. In a similar way, to a region, we can associate an element of A.

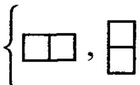
Let $B(\Sigma) \subseteq A$ be the subgroup of A generated by all elements corresponding to tile placements of tiles in Σ. The *tile homology group of* Σ is the quotient group $H(\Sigma) = A/B(\Sigma)$. A region R then has a signed tiling by Σ precisely when its image is the identity element in $H(\Sigma)$. Further, R has a perfect tiling if and only if its corresponding element in A is the sum of elements in A corresponding solely to positive tile placements.

Let's go back to our discussion on quotient groups to get a better understanding of the structure of $H(\Sigma)$. Consider a region R that does not possess a signed tiling by tiles from Σ (assuming one exists). This region corresponds to an element in A that is not in $B(\Sigma)$. Using this element, a separate coset in A is formed when we multiply (in this case sum coordinatewise) this element with the elements of $B(\Sigma)$. We can choose an element to represent this coset which has only 0's and 1's in its coordinates such that any other element in this coset with only 0's and 1's has no less 1's than this element. In other words, we choose an element that has the "best" possible signed tiling to represent this coset (there will be infinitely many). This means that any element in this coset can be altered by adding and subtracting elements representing tile placements until it is this representing element.

Finally recall that the elements of $H(\Sigma)$ are precisely the distinct cosets of $B(\Sigma)$ in A. It is in this sense that we say $H(\Sigma)$ measures the obstruction to having a signed tiling by tiles is Σ. $H(\Sigma)$, in a way, counts the number of cells that a region R will have left over when the best possible signed tiling of R has been constructed. This is why $H(\Sigma)$ has so cleverly been named the ***tile homology group***: it can be used to measure the number of holes left in an attempted signed tiling of a region R.

Generally, the tile homology group is defined by infinitely many generators and infinitely many relations. However, in some cases, the tile homology group of a given set of tiles Σ can be shown to be finitely generated. For now, we will restrict ourselves to the square lattice and introduce some notation. Following the ideal of Michael Reid, the cell with lower left corner at the point (i, j) will be called the (i, j) cell, and the generator element of A corresponding to this cell will be called $a_{i,j}$. The image of $a_{i,j}$ in $H(\Sigma)$ will be denoted by $\bar{a}_{i,j}$.

Examples:

E.1 Let $\Sigma = \{$  $\}$, then the tile homology group $H(\Sigma)$ is defined by

Generators: all $\bar{a}_{i,j}$ with $i, j \in \Im$

Relations:

$$\bar{a}_{i,j} + \bar{a}_{i+1,j} = 0$$

$$\bar{a}_{i,j} + \bar{a}_{i,j+1} = 0$$

Note that we have the following two tile placements exhibited in Figure E.1 which yield the relations $\bar{a}_{i,j} - \bar{a}_{i+1,j-1} = 0$ and $\bar{a}_{i,j} - \bar{a}_{i+1,j+1} = 0$ respectively, and thus $\bar{a}_{i,j} = \bar{a}_{i+1,j-1}$ and $\bar{a}_{i,j} = \bar{a}_{i+1,j+1}$.

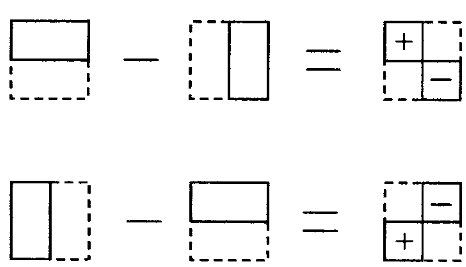
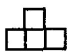


FIG E.1 – Tile placements corresponding to translations of cells by one diagonal unit

These relations show that all of $H(\Sigma)$ is actually generated by the two elements $\bar{a}_{0,0}$ and $\bar{a}_{0,1}$, and the original relations collapse into the one relation $\bar{a}_{0,0} + \bar{a}_{0,1} = 0$. So $\bar{a}_{0,1}$ is the inverse of $\bar{a}_{0,0}$. It is not too difficult to see that $H(\Sigma)$ is isomorphic to $\Im$. A precise isomorphism can be defined by $[R] \mapsto w - r$ where w and r are the number of white and red squares in R when the lattice has been given the checkerboard coloring as in Section 2.1. It is left as an exercise to show this mapping is indeed an isomorphism.

E.2 Let $\Sigma = \{$  $\}$ with all four orientations allowed.

First, note that we have the following tile placements which yield the relation that $\overline{a}_{i,j} = \overline{a}_{i+1,j+1}$. This means that any cell in the lattice can be translated diagonally; it is not hard to see that this is true for any diagonal direction. Therefore $H(\Sigma)$ is generated by the two elements $\overline{a}_{0,0}$ and $\overline{a}_{0,1}$

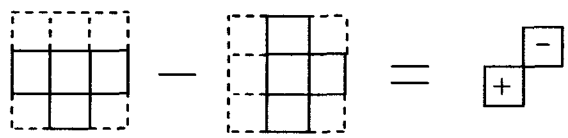


FIG E.2 – Translation of a cell one diagonal unit

Secondly, note that we have the following tile placement which yields the relation that $\overline{a}_{0,1} = (\overline{a}_{0,0})^5$. This is due to the fact that all five "holes" in the tiling are bishopwise related to each other, and the extra cell outside of the 4×4 rectangle is not. Recall that the 4×4 rectangle is tileable with T-tetrominoes; we thus have that $(\overline{a}_{0,0})^5 - (\overline{a}_{0,1}) = 0$. Therefore, the tile homology group is generated by the single element $\overline{a}_{0,0}$.

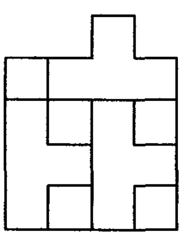


FIG E.3 – Tile placement showing $(\overline{a}_{0,0})^5 = (\overline{a}_{0,1})$

Thirdly, note that the following tile placement yields the relation $(\overline{a}_{0,0})^8 = 0$ since each of the eight "holes" in this tiling are bishopwise related; we know that a 4×8 rectangle can be tiled with T-tetrominoes.

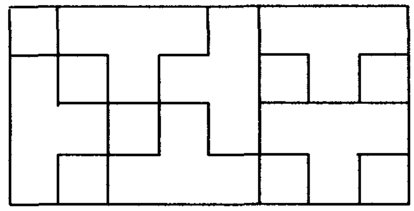


FIG E.4 – Tile placement showing $\left(\bar{a}_{0,0}\right)^{8} = 0$

Finally, $\bar{a}_{0,0}$ doesn't have order 4 since $\bar{a}_{0,0} \neq \left(\bar{a}_{0,0}\right)^{5} = \bar{a}_{0,1}$.

So the tile homology group is generated by a single element that has order eight. We can conclude that the tile homology group for the T-tetromino is C_8.

We can use this information to prove the following result…

Result:

An $m \times n$ rectangle has a signed tiling with $\boldsymbol{h}$ holes provided that property $\boldsymbol{P}$ is true in the following table.

h	***P***
0	$mn \equiv 0 \pmod 8$
1	$mn \equiv 1 \pmod 4$
2	$mn \equiv 2 \pmod 4$
3	$mn \equiv 3 \pmod 4$
4	$mn \equiv 4 \pmod 8$

We can use this to conclude, for example, that a 10×10 square does not have a perfect tiling with T-tetrominoes.

We are now in a position to prove Theorem 2.1(2). That is, if a region does not have a signed tiling by tiles in Σ, then there is a numbering argument showing this.

Proof of Theorem 2.1(2):

Suppose R is a region that does not have a signed tiling. *Let* $r \in H(\Sigma)$ be the image of the region R in the tile homology group. Let $\langle r \rangle \subseteq H(\Sigma)$ be the cyclic subgroup generated by r. Note that there is a homomorphism $\phi : \langle r \rangle \to Q/\Im$ with $\phi(r) \neq 0$. For example, if r has infinite order, then ϕ may be defined by $\phi(r) = \frac{1}{2} \bmod \Im$, while if r has finite order $n > 1$, then we may take $\phi(r) = \frac{1}{n} \bmod \Im$.

$Q/\Im$ is a divisible abelian group (injective $\Im$-module), and the sequence $0 \to \langle r \rangle \xrightarrow{i} H(\Sigma)$ is exact, so the homomorphism ϕ extends to a homomorphism $H(\Sigma) \to Q/\Im$, which is defined on all of $H(\Sigma)$; we will call this homomorphism Φ. Further, since A is a free abelian group (projective $\Im$-module), and since the sequence $Q \xrightarrow{\pi} Q/\Im \to 0$ is exact, the composite map $A \xrightarrow{\pi} A/B(\Sigma) = H(\Sigma) \xrightarrow{\Phi} Q/\Im$ lifts to a homomorphism $\Psi : A \to Q$ such that the following square commutes (where the

$$
\begin{tikzcd}
A \arrow[r, "\Psi"] \arrow[d] & Q \arrow[d] \\
A/B(\Sigma) = H(\Sigma) \arrow[r, "\Phi"] & Q/\Im
\end{tikzcd}
$$

vertical arrows are the natural projections). Then Ψ defines a numbering of the cells with rational numbers (we send each basis element of A to an element in Q). Moreover, $B(\Sigma)$ is the kernel of $A \to Q/\Im$, which means that every tile placement covers an integral total. Also, R covers a total that is not an integer because $\Phi(r) \neq 0$.

■

The tile homology group is useful for detecting whether a region has a signed tiling, or how many "holes" are left after the region has been given its best possible signed tiling. However, it is not very usefully in detecting when a group with a signed tiling does not have a perfect tiling. The next group we introduce, the tile homotopy group, can at times prove useful in this task.

§E.2 TILE HOMOTOPY GROUP

Another group closely related to the tile homology group is the ***tile homotopy group,*** defined as follows. Let F_2 be the free group with elements being words describing paths through the square lattice. Recall that for a set of tiles Σ, if we let $T(\Sigma)$ be the smallest normal subgroup of F_2 containing all possible boundary words of tiles in Σ, then the *tile path group of* Σ, denoted $P(\Sigma)$ is the quotient group $F_2/T(\Sigma)$. If a region R has a tiling by Σ, then its boundary word is trivial in $P(\Sigma)$.

It turns out that we don't need to work with a group as "big" as the tile path group, we can consider a "smaller" group. Let $C \subseteq F_2$ be the subgroup of closed words. That is words in F_2 that correspond to closed paths in the square lattice. The ***tile homotopy group of*** Σ is the quotient group $\pi(\Sigma) = C/T(\Sigma)$. In fact, the tile homology group is the abelianization of the tile homotopy group; Michael Reid gives a proof of this in his paper [10].

Note that all of the boundary words of the tiles in a given set Σ are trivial in $\pi(\Sigma)$. So we can use $\pi(\Sigma)$ with Theorem 4.3. Further $\pi(\Sigma)$ (if accessible) is "smaller" than $P(\Sigma)$ and hence can show nontileability in more regions than $P(\Sigma)$.

With the help of the computer algebra system GAP and the following two theorems, Reid uses tile homotopy methods to prove that certain sets of tiles cannot tile various rectangles in the square lattice although these rectangles possess signed tilings. The results are very impressive considering how difficult these finitely presented groups can be to work with.

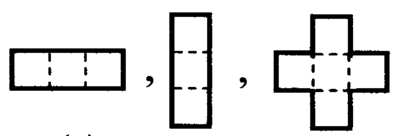
Theorem E.3:

If G is a finitely presented group, and $H \subseteq G$ is a subgroup of finite index in G, then H is also finitely presented. Further, a presentation of H can be computed explicitly.

Theorem E.4:

Suppose that a^m and u^n are central in the tile path group $P(\Sigma)$ for some $m, n > 0$. Then $\pi(\Sigma)$ has index mn inside $P(\Sigma)/\langle a^m, u^n \rangle$.

Theorems E.3 and E.4 are useful tools in determining the structure of $\pi(\Sigma)$.

Example: Let $\Sigma = \{$  $\}$.

A finite presentation for $P(\Sigma)$ is

$$\langle a, u \mid a^3ua^{-3}u^{-1} = au^3a^{-1}u^{-3} = auaua^{-1}ua^{-1}u^{-1}a^{-1}u^{-1}au^{-1} = 1 \rangle$$

The first two tiles in Σ show us that a^3 and u^3 are central in $P(\Sigma)$; a finite presentation for $P(\Sigma)/\langle a^m, u^n \rangle$ is $\langle a, u \mid a^3 = u^3 = auaua^{-1}ua^{-1}u^{-1}a^{-1}u^{-1}au^{-1} = 1 \rangle$.

Theorem E.4 says that in this example, $\pi(\Sigma)$ is a subgroup of $P(\Sigma)/\langle a^m, u^n \rangle$ of index $mn = 9$. Moreover, Theorem E.3 ensures that we can explicitly compute a finite presentation for $\pi(\Sigma)$. With the assistance of the computer algebra system GAP, Michael Reid found that $\left| P(\Sigma) / \langle a^m, a^n \rangle \right| = 1080$, hence $|\pi(\Sigma)| = \frac{1080}{9} = 120$, and its structure can completely be determined.

§E.3 CLOSING REMARK

Although Conway's group theoretical method proves to be stronger than coloring or numbering arguments, this method has only been applied in a handful of cases. In the appendix of Michael Reid's paper [10], he gives several examples where the tile homotopy group method, along with Theorems E.3 and E.4, has proved useful. It is an incredibly impressive list of examples. However, as previously stated, the region tiling problem on the square lattice falls into the computational complexity class NP-complete; this means that we still have a terribly long way to go before we can solve all region tiling problems.

APPENDIX A

PERMUTATIONS

The Set S_n

Let $S = \{1,2,3,\cdots,n\}$ and denote S_n as the set of $n!$ permutations of these n symbols. For example, if $S = \{1,2,3\}$ we have that $S_3 = \{123,132,213,231,312,321\}$. Let $i_1, i_2, i_3, \cdots, i_n$ be some arrangement of the elements of S. We can use the two-line notation to for this permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}$. This is read, "1 is replaced with i_1, 2 is replaced with i_2, 3 is replaced with i_3, $\cdots$, n is replaced with i_n". For example the arrangement 2,1,3,5,4 in S_5 can be noted $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$.

The Group $(S_n, \circ)$

Let $\alpha, \beta \in S_n$, by the product $\alpha \circ \beta$ we mean α and β are to be performed in that order. Now a permutation of a permutation of the elements of S is another permutation of these elements and thus $\alpha \circ \beta \in S_n$. For example in S_5, if $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$ and

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$ then $\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix} \in S_5$.

Note that the permutation $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$ can also be written as

$\beta = \begin{pmatrix} 2 & 1 & 3 & 5 & 4 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$. That is, the order of the columns is not important. With this in mind, if we let $\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{pmatrix}$, and

$\gamma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ k_1 & k_2 & k_3 & \cdots & k_n \end{pmatrix}$.

Then $(\alpha \circ \beta) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix} \circ \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_n \\ \beta(i_1) & \beta(i_2) & \beta(i_3) & \cdots & \beta(i_n) \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \beta(i_1) & \beta(i_2) & \beta(i_3) & \cdots & \beta(i_n) \end{pmatrix}$ and

$$(\alpha \circ \beta) \circ \gamma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \beta(i_1) & \beta(i_2) & \beta(i_3) & \cdots & \beta(i_n) \end{pmatrix} \circ \begin{pmatrix} \beta(i_1) & \beta(i_2) & \beta(i_3) & \cdots & \beta(i_n) \\ \gamma(\beta(i_1)) & \gamma(\beta(i_2)) & \gamma(\beta(i_3)) & \cdots & \gamma(\beta(i_n)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \gamma(\beta(i_1)) & \gamma(\beta(i_2)) & \gamma(\beta(i_3)) & \cdots & \gamma(\beta(i_n)) \end{pmatrix}$$

Similarly,

$$\alpha \circ (\beta \circ \gamma) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix} \circ \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_n \\ \gamma(\beta(i_1)) & \gamma(\beta(i_2)) & \gamma(\beta(i_3)) & \cdots & \gamma(\beta(i_n)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \gamma(\beta(i_1)) & \gamma(\beta(i_2)) & \gamma(\beta(i_3)) & \cdots & \gamma(\beta(i_n)) \end{pmatrix}.$$

And so $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$; we say that $\circ$ is *associative.*

The permutation $e = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$ is the identity permutation since clearly for any permutation α, we have that $\alpha \circ e = e \circ \alpha = \alpha$.

Finally, interchanging the two rows of a permutation α gives us the inverse permutation α^{-1} since $\alpha\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix} \circ \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} = e$

and $\alpha^{-1}\alpha = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} = e$.

Therefore the set S_n satisfies all of the axioms for a group under the operation $\circ$. Note that if $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, then $\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ so $\alpha\beta \neq \beta\alpha$. Therefore in general, S_n is a non-abelian group.

Cycle Notation

The permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$ can be written as (13254) where the cycle (13254) is to be read, "1 is replaced by 3, 3 is replaced by 2, 2 by 5, 5 by 4, and 4 by 1". The permutation $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{pmatrix}$ can be written as (354). The symbols 1 and 2 are missing from the cycle, we are to interpret this as 1 and 2 are unchanged by this permutation. The permutation $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$ can be written $(25)\circ(34)$. The interpretation is clear, "2 and 5 are interchanged, 3 and 4 are interchanged, and 1 is left unchanged". Note that these cycles (25) and (34) are disjoint. That is that they do not have any symbols in common. In cycle notation, each element should be written either as a single cycle or as a product of two or more disjoint cycles. In this notation, the identity element e will be denoted by (1).

Transpositions

A cycle which involves the interchange of only two of the n symbols of $S = \{1,2,3,\cdots,n\}$ is called a *transposition*. Any permutation can be expressed as a product of transpositions. For example, the permutation (12345) of S_5 can be written $(12)\circ(13)\circ(14)\circ(15)$ and the permutation $(12)\circ(345)$ can be written as $(12)\circ(34)\circ(35)$. A permutation is called *even* if it can be expressed as a product of an even number of transpositions, and it is called *odd* if it can be expressed as a product of an odd number of transpositions.

Theorem A.1: *Of the* $n!$ *permutations on n symbols, half are even and half are odd.*

Proof: Denote the even permutations by $p_1, p_2, p_3, \cdots p_u$ and the odd permutations by $q_1, q_2, q_3, \cdots, q_v$. Let *t* be any transposition. Now $t \circ p_1, t \circ p_2, t \circ p_3, \cdots, t \circ p_u$ are odd permutations on *n* symbols and are unique since $p_1, p_2, p_3, \cdots p_u$ are unique; thus $u \le v$. Similarly, $t \circ q_1, t \circ q_2, t \circ q_3, \cdots, t \circ q_v$ are distinct and even; thus $v \le u$. Hence $u = v = \frac{n!}{2}$.

■

The Subgroup $(A_n, \circ)$

The subset of all even permutations of S_n is in fact a subgroup of S_n. We call this subgroup the *alternating group of order n,* denoted A_n. Indeed the product of two even permutations is an even permutation, the inverse of an even permutation is an even permutation, and the identity permutation is an even permutation.

From Theorem A.1 we know that exactly half of the elements of S_n are in A_n. This means that there are only two cosets of A_n in S_n; these cosets, whether left or right, are the same: they are A_n and the complement of A_n. And thus, A_n is a normal subgroup of S_n.

APPENDIX B

DIHEDRAL GROUPS

Motions of Symmetry

A *motion of symmetry* of a figure is a reflection or rotation so that the figure perfectly lands on top of itself. For example if we rotate a square by $90°$ it will land on top of itself; this is one of the eight different symmetries of the square. Similarly, if we reflect an equilateral triangle about the line of symmetry that intersects a vertex *v* and the midpoint of the side opposite *v* then it will land on itself. The equilateral triangle has a total of six symmetries.

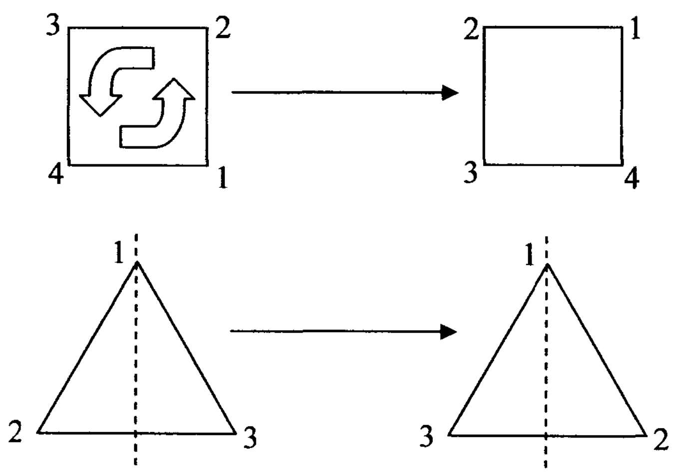


FIG B.1 – motions of symmetry

For any regular polygon with *n* sides, $n \geq 3$, the set of motions of symmetry of the figure; forms a group; we call this group the ***dihedral group of order n*** denoted D_n. If we label the vertices of the polygon $1,2,3,\cdots,n$ then an element of D_n will be a permutation of the n symbols. For example, if we label the vertices of the square $1,2,3,4$ as in Figure B.2, there are eight different symmetries that arise. These symmetries are shown in the table below the figure.

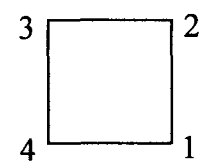


FIG B.2 – Labeling of the corners of the square

Motion of symmetry	**Corresponding permutation**
$0°$ counterclockwise rotation	(1)
$90°$ counterclockwise rotation	(1234)
$180°$ counterclockwise rotation	$(13)(24)$
$270°$ counterclockwise rotation	(1432)
Reflection about the vertical axis	$(14)(23)$
Reflection about the horizontal axis	$(12)(34)$
Reflection about the main diagonal	(13)
Reflection about the other diagonal	(24)

The Group D_n

Let a and b be elements of D_n, we define $a \circ b$ to simply mean we perform the motion a first and b second. For example in D_4, if a is the $90°$ counterclockwise rotation and b is the reflection about the vertical axis, then we have the following progression shown in Figure B.3.

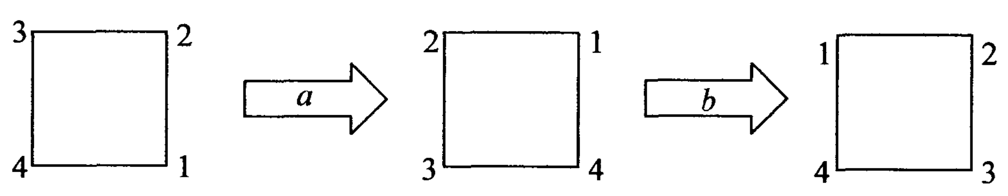


FIG B.3 – Multiplication of motions in D_4

Notice that rotating counterclockwise by $90°$ and then reflecting about the vertical axis is the same as reflecting about the diagonal that slopes up to the right. In fact any motion of symmetry followed by a second motion of symmetry is itself a motion of symmetry. Therefore the set D_n is closed under the operation $\circ$.

Further, since motions of symmetry of a polygon with n sides can be seen as permutations on n symbols, it follows that the operation $\circ$ is associative (see appendix A). That is, if a, b, and c are elements of D_n we have that $(a \circ b) \circ c = a \circ (b \circ c)$.

The inverse of any reflection is repeating the same reflection, and the inverse of a rotation of $x°$ counterclockwise is a rotation of $360 - x°$ counterclockwise. Finally, the rotation of $0°$ is the identity element. Therefore all of the group axioms are satisfied and indeed D_n is a group under $\circ$.

In fact since $\circ$ in D_n is actually the same as multiplication of permutations in S_n, we have that D_n is a subgroup of S_n.

Presentation of D_n

Any motion of symmetry of a regular n-gon can be expressed as a finite product of the motions of symmetry "rotation of $(360/n)°$ counterclockwise" and "reflection about the vertical axis". So we say that the group D_n is *generated* by these two elements. For example in D_4, if we let r be the rotation of $90°$ counterclockwise and s be the reflection about the vertical axis, then we have the following table (the rotation of $0°$ is notated "1").

Motion of Symmetry	**In terms of *r* and *s***
$0°$ counterclockwise rotation	1
$90°$ counterclockwise rotation	r
$180°$ counterclockwise rotation	r^2
$270°$ counterclockwise rotation	r^3
Reflection about the vertical axis	s
Reflection about the horizontal axis	sr^2
Reflection about the main diagonal	sr^3
Reflection about the other diagonal	sr

The $2n$ elements of D_n can be found similarly. Let r be the rotation of $(360/n)^{\circ}$ counterclockwise and let s be the reflection about the vertical axis. Then the elements of D_n are $\{1, r, r^2, r^3, \cdots, r^{n-1}, s, sr, sr^2, sr^3, \cdots, sr^{n-1}\}$. Since performing s twice results in the identity element as does performing r n times, we have that $s^2 = r^n = 1$. Further, we have the very important relation that $rsrs = 1$ which gives us that $rs = sr^{-1}$. All other relations in this group can be expressed as a product of these three relations, therefore we have the presentation $D_n = \langle r, s \mid s^2 = r^n = 1, rs = sr^{-1} \rangle$.

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