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Decision procedures for propositional fuzzy logic

by

Christopher Lepock ©

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Name of Author: Christopher Lepock

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10814 – 134 St.
Edmonton, Alberta
T5M 1H8

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Decision procedures for propositional fuzzy logic" submitted by Christopher Lepock in partial fulfillment of the requirements for the degree of Master of Arts.

Dr. F. Jeffry Pelletier

Dr. Bernard Linsky

Dr. W. David Sharp

Dr. Piotr Rudnicki

Date: 9/9/02

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Abstract

A major response to vagueness and sorites paradoxes is fuzzy logic, or logic with an infinite number of truth values. This thesis presents a procedure for constructing algebraic tableaux for RPL△, which is Łukasiewicz's $Ł_{\aleph}$ extended with constant truth functions and a determinacy operator. It appears that no tableau procedure has yet been developed for this particular system. The procedure is then extended to cover the finitely many-valued members of the Łukasiewicz family, product logic, and Zadeh's hedges. The soundness of the procedure for each of these systems is proven.

I

Fuzzy reasoning

In this thesis, I will present a tableau procedure for RPL△, a system of infinite-valued propositional logic, and prove its soundness. These will be called *algebraic* tableaux, in order to highlight certain differences between these tableaux and standard semantic tableaux. RPL△ is an extension of Łukasiewicz's $Ł_{\aleph}$, the most prominent system of infinite-valued logic; RPL△ adds to $Ł_{\aleph}$ the additional apparatus necessary to talk about the truth values of formulas within the system itself. It is thus a very powerful and useful system of fuzzy logic.

This chapter gives a brief introduction to propositional fuzzy logic and its applications to vagueness and control systems. Chapter II gives a detailed analysis of RPL△, the rationale for its truth functions, some of its interesting metalogical properties, and discusses some of the tautologies and valid inferences of the system. Chapter III discusses the extant decision procedures for propositional fuzzy logics. Chapter IV presents the tableau procedure and proves its soundness. Chapter V extends algebraic tableaux to cover several other prominent many-valued logics, and discusses some areas for future research. By and large, chapters I and III are informal and in English, the rest of the paper formal and more thick with symbolism. The reader who wishes to understand the general conclusions without slogging through too much formal language can read chapters I and III and sections V.5 and V.6. The reader interested only in the main results presented here, and already familiar with fuzzy logic, can read chapters III through V.

I will have to begin this chapter with a disclaimer. I will try to give an account of the motivations behind adopting an infinite-valued logic, in order to give the reader a better sense of what the logic is for. I will not, however, make a serious attempt to argue that we should adopt an infinite-valued logic, nor will I give a serious discussion of its merits and demerits. Such a project would be far beyond the scope of this work, and will have to be left for another occasion.

Finally, a brief note on terminology is in order. The term "fuzzy logic" is used to refer to many very different logical systems. Besides the systems of interest here, "fuzzy logic" includes systems in which the truth values of formulas are subintervals of [0,1], systems in which the truth values are only partially ordered, systems in which the truth values are themselves fuzzy sets, etc. Furthermore, commercial products using "fuzzy logic" generally use large finitely many-valued systems, not infinite-valued logics at all. The term's ambiguity seems to have led to confusion on at least one occasion.[1] Here, I will use "fuzzy logic" with what seems to be its standard denotation, to refer to any logical system with a whole lot of truth values. Systems in which the truth values are the real numbers in [0,1] will be referred to as infinite-valued logics; many-valued logics are systems where the truth values are at least three in number.

1. Worries about bivalence

One of the fundamental principles of classical logic is the law of bivalence—every proposition is either true or false. Similar laws are *tertium non*

[1] Susan Haack attacks Zadeh's FL (in which the truth values themselves are fuzzy sets), and does not challenge infinite-valued systems like Ł$_{\aleph}$ and RPL$\triangle$ (1996: 232-3). (Moreover, her arguments do not apply in any serious way to the latter systems.) Strangely, she calls FL "fuzzy logic", and refers to the latter systems as the "base logics". At least one response to her paper (Entemann 2002) seems to have interpreted her as criticizing infinite-valued logics in general (and having interpreted her arguments that way, refutes them without difficulty).

datur, or "there is no third [truth value]", and the law of excluded middle, the tautology $p \vee \neg p$. These are not always equivalent—for instance, on a supervaluationist account not every proposition is true or false, but $p \vee \neg p$ is a tautology. The differences are, however, not relevant to the present work, and so for current purposes the various laws may be run together. Although bivalence has generally been regarded as a fundamental law of logic, it has some bizarre consequences, which have worried logicians throughout history.

The first logician to question bivalence may very well have been Aristotle. In *De interpretatione* 9, he might have proposed that propositions about contingent future events are neither true nor false before those events occur, in response to an argument indicating that determinism followed from bivalence and disquotationality. (See Haack 1996: 73-90 for a discussion.) Philosophers in the Middle Ages extensively discussed this argument and its implications for the law of excluded middle.

Modern work has focused on problems caused by the vague terms that make up most of natural language. Vagueness is itself at best vaguely defined, and likely includes a number of different linguistic (and possibly epistemic and ontological) phenomena. The serious threat to bivalence comes from the fact that vague predicates have borderline cases, in which the predicate seems neither really to apply nor not apply. Suppose Phil, who is 5'10" tall, is standing in the doorway wearing a shirt of a shade midway between red and orange. Neither "tall" nor "not tall", nor either "in the room" or "not in the room" really apply to Phil, and neither "red" nor "not red" really apply to his shirt. The law of excluded middle, $\phi \vee \neg\phi$, seems to fail for such borderline cases. One is inclined to say that Phil is neither really tall nor not tall, and neither really in the room nor not in the room; but since $\neg p \wedge \neg\neg p$ is equivalent to $\neg(p \vee \neg p)$, this move seems to directly contradict bivalence.

Furthermore, there are a number of paradoxes associated with vagueness. These are not immediately counterexamples to bivalence, but arise from attempts to apply classical logic to vague predicates. The ancients called the most important of these *sorites* or *falakros* paradoxes—the former from *soros*, meaning "heap", the latter meaning "bald man". Obviously,

[S0] A man with 100 000 hairs on his head is not bald.

Just as obviously,

[S1] If a man with n hairs on his head is not bald, a man with n - 1 hairs is not bald.

One hair either way makes no difference. [S0] and [S1] imply

[S2] A man with 99 999 hairs is not bald,

which is perfectly reasonable; [S1] and [S2] imply

[S3] A man with 99 998 hairs is not bald,

which is, again, quite right. But if we continue reasoning in this manner, after 100 000 applications of modus ponens we get

[S10^5] A man with 1 hair on his head is not bald.

This conclusion is obviously false, although it might be comforting to some. We can additionally run the sorites in the other direction and reach the conclusion that regardless of how many hairs they have, all men are bald.

Sorites paradoxes come up (far too often) in argumentation in the form of slippery slope fallacies, and a fundamental part of successful informal reasoning is the ability to avoid being misled by such arguments. Hence, the sorites is not just a clever puzzle that can be readily ignored; its seeming classical validity shows that classical logic is not an adequate representation of cogent reasoning.

2. Major responses to vagueness

There has been a number of responses to the problem of vagueness and sorites paradoxes. I will (much too briefly) summarize the most important of these, to give the reader a better sense of what problems fuzzy logic was developed to avoid.

The epistemic view. The gist of this response is, interestingly, to deny that predicates actually have borderline cases, and likewise to deny that [S1] is true at all. For any predicate, there is an exact boundary between the objects of which it is true and the objects of which its negation is true. But no one can ever know just where this boundary is. (See Williamson 1992 and 1994 for a discussion and defense.) This response has the advantage of preserving the law of excluded middle—every man is either bald or not, although in some cases we cannot know the answer. Postulating linguistic facts no speaker of the language could possibly know, and which have no causal link to actual or possible speech behaviour, is accompanied by a litany of disadvantages, as the reader can imagine.

Supervaluations. A number of thinkers have proposed that vague expressions lack truth value—by failing to be precise, an expression fails to be true or false. The most popular version of this position is supervaluationism. The basic idea is that vague expressions may be made precise ("sharpened") in any number of ways. A *supervaluation* is an evaluation of every admissible sharpening of the vague elements of a proposition. A vague proposition is true if every admissible sharpening is true, false if every admissible sharpening is false, and neither if some sharpenings are true and some false. (See Kamp 1975, or Williamson 1994: 142-64.) Supervaluationism denies bivalence, since some propositions are neither true nor false, but $\phi \vee \neg\phi$ is a theorem, since on every sharpening, either ϕ or $\neg\phi$ is true. Supervaluations handle the sorites paradox by maintaining that on every sharpening, there is an exact, one-hair boundary between being bald and not being bald. Hence,

$$\exists n(\text{a man with n hairs is not bald} \wedge \text{a man with n} - 1 \text{ hairs is bald})$$

is true. However, since different sharpenings mark this line in different places, there is no number *k* such that instantiating *n* by *k* is true.

The supervaluationist approach has the serious defect of ignoring the phenomenon of higher-order vagueness. There is no boundary between borderline cases and non-borderline cases, or between borderline-borderline cases and borderline cases, etc. That is, any sharp boundary between the true, false, and neither cases of a predicate will be just as arbitrary as a sharp line between the extension of the predicate and the extension of its negation. A further problem is that there is variation within borderline cases. A proposition true on every sharpening except one is certainly in a better position than a proposition false on every sharpening except one, yet on a supervaluationist approach both are categorized the same way, as neither true nor false.

Three-valued logic. Instead of saying that vague expressions are neither true nor false, some authors have proposed assigning them a third value between true and false. The result is a logic with three truth values. There are a number of logical systems that might be used for this purpose, the details of which I will omit—Łukasiewicz's $Ł_3$, Kleene's K_3, and Bochvar's B_3 are the most likely candidates. (See Rescher 1969: 22-36 on these systems.) This approach has the same limitations as supervaluations—it does not account for higher-order vagueness or variation between borderline cases. The three-valued response to sorites paradoxes would be to draw two discrete lines, one between bald and borderline bald, and another between borderline bald and not bald. This is not much of an improvement over the two-valued response.

One might try to remedy these problems by moving to, say, a five-valued logic with the values "true", "borderline true/borderline", "borderline", "borderline false/borderline", "false". But this would not eliminate the problem; the lines between the five cases would have to be arbitrarily drawn. There is no principled way to divide vague assertions up into discrete packages, however many packages we try to use.

Precisification. A fourth approach is to throw up one's hands in disgust at the unclarity of natural language and "precisify" vague terms, replacing them with precisely defined counterparts. Frege and Carnap favoured this project, and Russell memorably wrote

> All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence...logic takes us nearer to heaven than other studies (1923: 65).

So rather than worry whether Phil is tall or his shirt red, we might define a new predicate TALL, such that x is TALL iff x is more than 5'11" tall, or specify a new predicate RED that holds of objects that reflect light in a precisely specified part of the spectrum under precisely specified standard conditions. This approach does not deny that [S1] is obviously true of the natural language predicate "bald". The apparatus of formal logic, however, can only be applied to precise terms with definite boundaries; if BALD is a precisification of "bald", the analogue to [S1]

If a man with n hairs is not BALD, a man with n - 1 hairs is not BALD, is actually false.

In some cases, instead of precisifying predicates, we can abandon them altogether in favour of descriptions of the states of affairs the property to which the vague term refers supervenes on. In this example, we would forego using tall and red and instead talk directly about Phil's height and the wavelength of the light reflected from his shirt under standard conditions. This is not, however, generally feasible, as vague terms frequently refer to properties that supervene on states of affairs so complex or disparate that referring to them with a special term is unavoidable. (Most important biological concepts, for instance, cannot be reduced to purely physical language.)

Precisifying vague terms preserves bivalence, which some might see as an advantage. There are two serious disadvantages, however. First, precisifications of complex terms are frequently extremely complex, since each borderline case must be accounted for without losing sight of the core meaning of the term. Before developing fuzzy set theory, Lotfi Zadeh worked on successively refining the meaning of terms like "state" and "adaptive" in accordance with this response to vagueness (see Gaines 1976: 624). These experiences led to his later claim that "complexity and precision bear an inverse relation to one another in the sense that, as the complexity of a problem increases, the possibility of analyzing it in precise terms diminishes." (Zadeh 1972: 150)

A second disadvantage is that precisified predicates are less informative than their vague counterparts, *because* they have no borderline cases. Suppose we say that an object is RED iff under standard conditions, it reflects light of a wavelength between 630 and 670 nm. Suppose A reflects light at 670 nm, B at 630 nm, and C at 629 nm. A and B are RED, but C is not RED. Nonetheless, the colours of B and C are indistinguishable without complex equipment, while A and B have very different shades. We express this sort of information in natural language by hedging claims about the colour of objects; we might say that A is "definitely red" but B and C are only "more-or-less red". But it is no more possible to be "more-or-less RED" than to be more or less an even integer, and so this option is not available.[2] Thus knowing whether a precisified predicate applies to either of two objects in many cases does not allow us to infer anything at all about the relations between those objects. Precisifications attempt to legislate away variation in borderline cases, but can only achieve this at the cost of reducing the expressive power of the language.

3. The motivation for fuzziness

Here, I will give what seem to me to be the central considerations motivating a move to an infinite-valued logic. The reader should note, as already indicated above, that these should not be taken as arguments *per se* for adopting fuzzy logic as a response to vagueness. What it means for a proposition to be "true to degree n" is obvious for some and inscrutable for others; my aim is simply to try to explain what we are trying to represent with such locutions, in the hope of giving the reader a better intuitive understanding of the basic concepts behind the logic.

The central intuition is that borderline cases just are borderline cases. A reddish-orange patch is a shade roughly halfway between red and orange on the spectrum of visible light; a man who is 5'11" is a man who is slightly above the average height. Whether we categorize the patch as "red" or "not red", the man as "tall" or "not tall", we lose the information that the patch is just as close on the spectrum to red as to orange, and that the man is not as tall as someone who is clearly tall. But any finite characterization of a property like redness or tallness will ignore the similarity between cases on the boundary between categories, and

[2] On the basis of our background knowledge about the objects to which the predicate applies, *we* can say that 630 nm is a borderline case of RED; but the predicate itself says nothing of the sort. Similarly, we know that Des Moines is "more American" than El Paso, but US law applies no less to the latter than to the former.

the differences between cases at opposite extremes of the same category. Hence, we are driven to adopt a continuum of degrees of applicability of a predicate.

More directly, we can note that the manifestations of many properties can be linearly ordered. Colour patches, obviously, can be ordered by their similarity to pure red, and men can be ranked in order of their baldness or height. Other, more interesting, examples are readily found. Compounds can be ordered in terms of toxicity from those that are harmless, through those that cause illnesses of progressively increasing severity, to those that cause immediate death. If we take democracy to be popular control of legislative structures, then we can order political systems in terms of the amount of control the populace has over the government. Drawing lines across such orderings is usually incredibly complicated, and arbitrarily divides borderline cases. A truly rigourous description of the phenomenon will have to talk directly about such orderings, and this is what infinite-valued logic is for.

Although fuzzy logic is *formally* a generalization of finitely many-valued approaches to vagueness, it is best understood as an extension of the program of precisifying vague terms (cf. Gaines 1976: 627-31); it is primarily an attempt to adapt vague predicates for use in science and engineering. In this sense, it is a continuation of Russell and Frege's project of cleansing the language of vagueness, ambiguity, and imprecision, in order to meet the needs of rigourous discourse and precise representation. The attempt to model how vague expressions are actually used in language is only a secondary goal. In particular, infinite-valued logic does not attempt to model the imprecision in the use of, and indeterminacy of extension of, vague predicates in natural language.[3] So right away we should note that we are not discussing a "logic of vagueness", if we

[3] There are many systems of fuzzy logic that do attempt to model other aspects of vagueness. For instance, Nguyen et al. (1996) model our inability to determine what the precise truth value of a proposition should be by taking the truth values of propositions to be subintervals of [0,1] rather than points in [0,1]. Zadeh's (1975) FL goes one step further and models the truth values themselves as fuzzy sets on [0,1]. Such systems are best seen as extensions on the basic infinite-valued logic with which we are concerned here.

understand this as meaning an exhaustive formal characterization of the behaviour of vague expressions. Rather, infinite-valued logic is an attempt to allow precise, rigourous language while avoiding the defects described above in the precisificationist approach—it is a logical framework for rigourous communication using, rather than avoiding, predicates with borderline cases.

4. The basics of infinite-valued logic

The foundation of all systems of fuzzy logic is that instead of just the traditional two truth values, there are infinitely many degrees between absolute true and absolute false. I hope the reader will allow me to avoid discussing the question of what exactly degrees of truth are, and what relation they bear to traditional (particularly, metaphysical) conceptions of truth. This question is far too complicated to address here; for now, let us set the metaphysics aside and take fuzzy truth as a purely formal structure to increase the expressive power of our logic. In this paper, I will use $/\phi/$ to represent the truth value of ϕ. The truth values will be taken to be all the points in the interval [0,1]. A formula is a tautology iff it always takes the value 1. Similarly, a set of premises Σ semantically implies a conclusion ϕ, written as always $\Sigma \vDash \phi$, iff on every valuation on which $\forall\psi \in \Sigma$, $/\psi/ = 1$, we have $/\phi/ = 1$.

We want, of course, to be able to talk about approximately true formulas and approximate inferences. This is commonly done by defining a subinterval of [0,1] as the *designated* ("truth-like", assertible, belief-worthy, or what have you) values. We then take a tautology to be a formula that always takes designated values, and a valid argument to be one where the conclusion takes a designated value whenever all the premises do. (See e.g. Priest 2001: 216-7.) The logic developed here allows us to state the truth values of formulas within the system, which in turn allows for both a very general, context-sensitive characterization of approximate validity, and a full syntactic treatment of the logic. Throughout, I will use the symbol ð for the greatest lower bound of the designated values.

As should be obvious, any denumerable set of connectives for a continuum-valued logic is functionally incomplete. Our choice of connectives thus determines what logical system we end up with, and is thereby very important. There is a very wide variety of different connectives used by different people working in the field, and thus a wide variety of infinite-valued logics. The versions of the classical connectives discussed in this section, and on which this thesis focuses, are the most widely used overall,[4] and also retain the greatest number of classical validities. I will leave off discussing the arguments for using these particular connectives until II.3-4, since the more general reasoning with which we are concerned in this section does not really depend on the specific choice of connectives. The system I will discuss will be called RPL△. It is Łukasiewicz's infinite-valued system $Ł_{\aleph}$ extended with constants and a determinacy operator, and was first proposed by C. G. Morgan and F. J. Pelletier (1977).

We define conjunction, disjunction, and negation as

$$/(\phi \vee \psi)/ = \max (/\phi/, /\psi/),$$
$$/(\phi \wedge \psi)/ = \min (/\phi/, /\psi/),$$
$$/\neg\phi/ = 1 - /\phi/.$$

That is, a conjunction has the same value as the least true of its conjuncts, and a disjunction takes the value of the most true of its conjuncts. Negation "flips" the truth value of a proposition; the distance of $\neg\phi$ from 0 is the same as the distance of ϕ from 1.

[4] For instance, Zadeh (1975) uses these connectives as the basis for his FL; the Kenevan Truth Interval Logic and Nguyen et al.'s (1996) interval-valued systems use these definitions of conjunction, disjunction, and negation; Machina (1976) uses these connectives in his logic of vague terms; etc.

The conditional will be defined as

$$/(\phi \supset \psi)/ = \min(1, 1 - /\phi/ + /\psi/),$$

or equivalently,

$$/(\phi \supset \psi)/ = \begin{cases} 1 \text{ if } /\phi/ \leq /\psi/, \\ 1 - (/\phi/ - /\psi/) \text{ otherwise.} \end{cases}$$

The truth value of a conditional is one minus the amount of truth lost when proceeding from the antecedent to the consequent. If $/\phi/ \leq /\psi/$, no truth is lost when going from ϕ to ψ, and so the value of the conditional is 1. The primary inference rule for the conditional is *fuzzified modus ponens*; given $/\phi/$ and $/\phi \supset \psi/$, we can determine a lower bound for $/\psi/$, since

$$/\psi/ \geq /\phi/ + /\phi \supset \psi/ - 1.$$

Fuzzified modus ponens will be discussed further in II.4.

Equivalence is defined as

$$/\phi \equiv \psi/ = 1 - |/\phi/ - /\psi/|.$$

The truth value of the biconditional is one minus the difference between the truth values of the two formulas. As usual, $\phi \equiv \psi$ is equivalent to $(\phi \supset \psi) \wedge (\psi \supset \phi)$.

We can define two other useful connectives in terms of the conditional,

$$(\phi \veebar \psi) =_{df} (\neg\phi \supset \psi),$$
$$(\phi \barwedge \psi) =_{df} \neg(\phi \supset \neg\psi).$$

These are called *bounded sum* and *bounded difference*, respectively, and are the Łukasiewicz t-norm and t-conorm (discussed in II.4). They take the values

$$/(\phi \veebar \psi)/ = \min(1, /\phi/ + /\psi/),$$
$$/(\phi \barwedge \psi)/ = \max(0, /\phi/ + /\psi/ - 1).$$

Additionally, we will want to be able to talk about the truth values themselves in the system. To do this, we introduce a denumerable infinity of constants:

$$/C_i/ = i, \text{ for each rational number } i \in [0,1].$$

We do not introduce constants for all the reals in [0,1] simply to avoid having an indenumerable set of primitive truth functions. The constants will be essential in characterizing approximate validity and approximate truth, as we will see in II.5.

Finally, we introduce a determinacy operator J_1, which we use to indicate whether or not a formula is absolutely true:

$$/J_1\phi/ = 1 \text{ if } /\phi/ = 1,$$
$$0 \text{ otherwise.}$$

The symbol '$\triangle$' is also widely used for this operator. We can define several other useful connectives in terms of J_1:

$$(\phi \rightarrow \psi) =_{df} J_1(\phi \supset \psi),$$
$$(\phi \leftrightarrow \psi) =_{df} J_1(\phi \equiv \psi),$$
$$J_i\phi =_{df} \phi \leftrightarrow C_i, \text{ for each } C_i,$$

which take the values

$/(\phi \rightarrow \psi)/ =$ 1 if $/\phi/ \leq /\psi/$,
0 otherwise;

$/(\phi \leftrightarrow \psi)/ =$ 1 if $/\phi/ = /\psi/$,
0 otherwise;

$/J_i\phi/ =$ 1 if $/\phi/ = i$,
0 otherwise.

The truth functions of RPL△ are also given at the very end of this thesis, for easy reference while reading.

5. The fuzzy approach to vagueness

Having set out the basics of the logic, we can now see how it can be applied to various sorts of problems. Borderline cases of predicates are assigned truth values between absolute true and absolute false that correspond to the relations they bear to other cases of the predicate. For instance, a patch of colour equidistant on the spectrum between red and orange would be considered red to degree 0.5; a patch that is red with just a slight tinge of orange might be red to degree 0.9; a patch of orange with a tinge of red might be red to degree 0.1, and so on.

Note that assignments of truth values to vague expressions must be made relative to particular contexts. The truth value of "Phil is tall", where Phil is 5'11", depends on the context in question. If we are using the locution in the context of tall men, "Phil is tall" might be true to degree 0.6. If the context includes both men and women, "Phil is tall" would have a somewhat higher truth value. If the context is professional basketball players, "Phil is tall" would take a value very close to 0.

The fuzzy analysis of sorites paradoxes is particularly elegant. The central idea is that the conditional

If a man with n hairs is not bald, a man with n – 1 hairs is not bald,

is not perfectly true. It is so close to being true that the difference between it and a perfectly true conditional is scarcely appreciable; the difference only manifests itself in long chains of conditionals. To see this, let

[S0] /¬(A man with 100 000 hairs is bald)/ = 1,

[S1] /¬(A man with n hairs is bald) ⊃ ¬(A man with n – 1 hairs is bald)/ = 0.99999.

(Since the system we are using does not have quantifiers,[5] we can take [S1] to be a schema of premises, one for each n up to 100 000, and all with the same truth value.) [S0] and [S1] entail

[S2] /¬(A man with 99 999 hairs is bald)/ = 0.99999,

That [S2] is not perfectly true is scarcely noticeable, and likewise for

[S3] /¬(A man with 99 998 hairs is bald)/ = 0.99998.

But if we iterate the argument 99 999 times, we get

[5] /∀xϕx/ is normally defined as the greatest lower bound of /ϕx/ for all values of x. So the quantified version of [S1],

∀n[¬(A man with n hairs is bald) ⊃ ¬(A man with n – 1 hairs is bald)],

takes the greatest lower bound of all the results of substituting an appropriate integer for n. On this solution to the paradox, every such formula has a truth value of (say) 0.99999; thus, the quantified version of the formula also takes the value 0.99999.

[$S10^5$] /¬(A man with 1 hair is bald)/ = 0.00001.

So we can see that fuzzy logic provides a powerful response to the sorites paradox; it avoids the false conclusion while at the same time explaining why we consider [S1] to be obviously true.

This response depends neither on the particular characterization of the conditional we have used nor the exact truth values assigned to sentences. To see this, let us look at a resolution of the paradox involving a different fuzzy conditional, the *Goguen implication* ⋑ (discussed further in II.4), defined as:

$$/\phi \Supset \psi/ = 1 \text{ if } /\phi/ \leq /\psi/,$$
$$/\psi/ \div /\phi/ \text{ otherwise.}$$

Let

[S1′] /¬(A man with n hairs is bald) ⋑ ¬(A man with n – 1 hairs is bald)/ = 0.99995.

If we run through the paradox using [S1′] instead of [S1], we get

[S2′] /¬(A man with 99 999 hairs is bald)/ = 0.99995,

[S3′] /¬(A man with 99 998 hairs is bald)/ ≈ 0.99990,

until we eventually reach

[$S10^5$′] /¬(A man with 1 hair is bald)/ ≈ 0.00674.

The response to the sorites using the Goguen conditional differs only marginally from that using the Łukasiewicz conditional. Note that if we were to set $[S1']$ = 0.99999, we would have $/[S10^{5}{'}]/ \approx 0.368$, which is surely wrong; it should be at most imperceptibly true that a man with only one hair is not bald. Fuzzy theorists do not mean to suggest, however, that sentences *actually* have precise truth values in [0,1], so that we might be *wrong* in assigning $[S1']$ a value of 0.99995 instead of 0.99999. Rather, the claim is just that [S1] is almost completely true. There are infinitely many value assignments that satisfy this requirement, and we choose which particular value to assign to [S1] primarily by the standards of convenience.[6]

6. Fuzzy control systems

The primary success of fuzzy logic has come not from its approach to vagueness, but rather from control systems based on it. The central principle between fuzzy control systems is easy to see, and does not require too many details about the exact workings of the system. (See Kosko 171ff. for a more thorough discussion.)

The strength of a system's response to an environmental factor should depend on how strong the environmental factor is. If the car is slightly too cold, one turns up the heat slightly; if it is extremely cold in the car, one puts the heat on higher; and so forth. When filling a glass of water, one pours the water quickly at first and more slowly as the glass becomes full, to prevent spilling. Translating this sort of reactivity to circumstances into rules for a control system

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[6] A similar view has been proposed for the truth functions themselves. As we have seen for the sorites paradox, different characterizations of the fuzzy connectives frequently lead to almost identical results in actual applications. This has led Gaines to propose that there may not be uniquely correct characterizations of the fuzzy connectives at all, that "the operations may be fuzzy as well as the data." (1976: 636)

is very complicated, however. If one were designing a thermostat, one would need a whole host of rules along the lines of

> if the temperature is 2° below target, motor speed should be at 100 rpm;
> if the temperature is 4° below target, motor speed should be at 150 rpm;

and so forth.

The basic idea of a fuzzy control system is that the same rule can be applied to different degrees depending on the strength of the relevant environmental factor. A rule for a thermostat might be of the form

> if the temperature is LOW, motor speed should be FAST.

The system would regard different temperatures as LOW to different degrees, and similarly for FAST. Hence, the rule will be invoked to different degrees depending on how high the temperature is. For instance, suppose 15° is LOW to degree 0.4. Then if the temperature is 15°, the rule is invoked to degree 0.4, and the output will be a motor speed that is FAST to degree 0.4. A single rule applied to different degrees can take the place of a large number of rules each of which applies fully or not at all.

The result is that fuzzy control systems allow for very efficient computation. Most of its applications have been in commercial products—fuzzy controllers are used in washing machines, cameras, air conditioners, stock traders, etc. (See Kosko 1993: 184-90 for a seemingly exhaustive list.) This track record tends to underwhelm the detractors of fuzzy logic, since the tasks these fuzzy systems perform can also be done by systems based on classical logic or Bayesian probability theory. By increasing the efficiency of computation, however, it becomes possible to perform tasks for which the computational resources were not previously available; hence, we can expect that there will be tasks that fuzzy control systems can manage that cannot be done by any other sort of system. This prediction has already been realized in at least one case; Michio Sugeno has developed a control system that can stabilize a helicopter that loses a rotor blade in flight, something no human or previous mathematical model has been able to do (Kosko 1993: 170). We can thus only expect fuzzy systems to be applied to more and more problems in the future.

7. Decision procedures

A decision procedure for a system of logic is an algorithm by which one can determine whether or not an arbitrary formula is a tautology, and whether a certain set of premises entails a certain conclusion. A third aim is to be able to determine if two given formulas have the same truth value on every valuation.[7] In RPL$\triangle$, this third aim is satisfied by any procedure that satisfies the first, since $/\phi \equiv \psi/ = 1$ on every valuation iff ϕ and ψ always take the same truth value. A decision procedure must, of course, terminate after a finite number of steps.

For any finite-valued truth-functional system, truth tables are an effective decision procedure—writing out the full truth table for any formula will indicate whether or not the formula is tautologous, and since formulas can only be finitely long, the truth table will always be finite. This procedure is not exactly practicable for an infinite-valued system.

There are a number of reasons why having a decision procedure is important for any logical system, and particularly for infinite-valued logic. The main interest will be, of course, for people applying the system in question to actual problems, and particularly those designing systems based on infinite-valued logic. A procedure to determine what formulas are tautologies or equivalent on all valuations, and what entailments are valid, is of paramount value in developing control systems, computer programs, and the like, based on the logic in question.

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[7] This is what Gehrke et al. (1999) require a decision procedure to accomplish.

Beyond this, however, a decision procedure for infinite-valued systems is of specifically philosophical interest. Gehrke et al. (1999) note the importance of a decision procedure for determining whether a system is acceptable; with a decision procedure for the logic, we can determine what formulas are tautologous or not and thus what inferences we sanction or abandon if we adopt the logic. A decision procedure for RPL△ will lead to a better understanding of how the logic models vagueness and approximately valid inferences. In particular, a number of theorists have proposed that fuzzy logic can be used to model much of approximate and inductive reasoning (see e.g. Gaines 1976: 649-56). A fuller understanding of how arguments work in RPL△ will help us to determine just what role infinite-valued logic should play in the formal study of these subjects.

Finally, there are particular advantages that come from having a tableau procedure for a logical system, which is the sort of procedure I will present in chapter IV. Knowing the definitions of the connectives of a formal system is not enough to gain a real understanding of the system; one must in addition be able to work in the system, to understand how the connectives relate to each other and the role they play in deduction. Reviewing a list of tautologies and nontautologies of the system is not sufficient for really understanding the system either; one needs to prove tautologies and give counterexamples oneself in order to get a sense of why certain formulas are valid in the system and not others. The extant axiomatizations of infinite-valued logic are not particularly perspicuous, and it is difficult to get a feel for the system by working with them. By facilitating deductions in RPL△, a tableau procedure should make it easier to understand the system, and might lead to a wider understanding of infinite-valued propositional logic.

To be of any use at all, a decision procedure must be *sound*; that is, whatever the decision procedure indicates as valid must actually be valid in the semantics. Using, of course, $\Sigma \vdash \phi$ to indicate that ϕ is entailed by Σ according to the decision procedure, the procedure is sound iff $\Sigma \vdash \phi$ implies $\Sigma \vDash \phi$. A true decision procedure should furthermore be *complete*, which is the case if every semantic validity can be shown to be such with the procedure; i.e., the procedure is complete iff $\Sigma \models \phi$ implies $\Sigma \vdash \phi$.

I will prove (in IV.7) that algebraic tableaux are sound for RPL$\triangle$. I will not, however, prove that the procedure is complete. Furthermore, I will not give an algorithm for constructing algebraic tableaux. Instead, I will give a slightly more informal procedure for constructing tableaux that can be readily used by anyone familiar with basic algebra. Since the procedure I will present in chapters IV and V is not a sound and complete algorithm for finding tautologies in infinite-valued logic, it is not strictly speaking a decision procedure for these logics. It should not be difficult to develop a true decision procedure based on algebraic tableaux, but this final step will not be taken in this thesis (as will be discussed further in V.5). However, so far as I am aware, no tableau procedures for RPL$\triangle$ have been previously developed (although procedures for $Ł_{\aleph}$ do exist), and a procedure that has not been shown to be complete is still better than no procedure at all.

II

An overview of infinite-valued logic

The system with which we are concerned is best understood as being made up of four subsystems: VSS, $Ł_{\aleph}$, RPL, and RPL$\triangle$. Each of the last three is an extension on its predecessor in the series. In this chapter, I will look at each system in turn, giving a general overview of the system, its valid formulas and inferences, and some interesting properties. I will state theorems without proving them, but cite the locations of the proofs for the interested (or suspicious) reader.

There are two reasons for examining each fragment of the main system separately. First, examining each system independently makes it easier to see the motivations for moving to the more powerful system. Second, each system is of interest in its own right and has been independently studied; thus, it is worthwhile to give a brief overview of each system.

As indicated in the last chapter, the truth values are all the real numbers in the interval [0,1]. The notions of formula, valuation, etc., are defined as usual, brackets are dropped according to the usual conventions, and so forth.

1. VSS

VSS (for Variant Standard System) is the most basic infinite-valued system. The only connectives are conjunction, disjunction, and negation, defined as:

$/\neg\phi/ = 1 - /\phi/,$

$/(\phi \vee \psi)/ = \max(/\phi/, /\psi/),$

$/(\phi \wedge \psi)/ = \min(/\phi/, /\psi/).$

These truth functions are propositional analogues of the basic operations of fuzzy set theory, in which objects belong to sets to varying degrees; that is, for each set S and object α in the domain, $\alpha \in S$ is assigned a truth value in [0,1]. Union, intersection, and complement are defined in this fuzzy set theory as:

$/\alpha \in A \cup B/ = \max(/\alpha \in A/, /\alpha \in B/),$

$/\alpha \in A \cap B/ = \min(/\alpha \in A/, /\alpha \in B/),$

$/\alpha \in A'/ = 1 - /\alpha \in A/.$

VSS is a very limited language. The most serious problem is that every formula takes the value 0.5 whenever all its propositional variables take 0.5. Hence, there is no reasonable notion of validity according to which VSS has any logical truths. Entemann (2000) takes a "fuzzy tautology" to be a formula that uniformly takes a value greater than 0.5, but then has to restrict valuations so that no propositional variable takes the value 0.5. As a result, a stronger system is greatly desirable.

2. $Ł_{\aleph}$

$Ł_{\aleph}$ is the result of adding a particular sort of conditional to VSS. (I will discuss infinite-valued conditionals in more detail in II.4, after describing $Ł_{\aleph}$.) $Ł_{\aleph}$ is a member of Łukasiewicz's family of many-valued logics. These logics all have the same primitive functions:

$/(\phi \supset \psi)/ = \min(1, 1 - /\phi/ + /\psi/),$

$/\neg\phi/ = 1 - /\phi/,$

and various derived truth functions, the most important of which are:

$\phi \vee \psi =_{df} (\phi \supset \psi) \supset \psi,$

$\phi \wedge \psi =_{df} \neg(\neg\phi \vee \neg\psi),$

$\phi \equiv \psi =_{df} (\phi \supset \psi) \wedge (\psi \supset \phi),$

$(\phi \veebar \psi) =_{df} (\neg\phi \supset \psi),$

$(\phi \barwedge \psi) =_{df} \neg(\phi \supset \neg\psi),$

the truth values of which are given at the end of this work. 1 is the only designated value. Each logic in the family then has a different number of truth values. The values of any system $Ł_n$ can be represented as the set of fractions $\{^0/_{n-1}, ^1/_{n-1}, \ldots, ^{n-1}/_{n-1}\}$. $Ł_2$, with the two truth values 0 and 1, is equivalent to classical logic. $Ł_3$ is the most extensively studied finitely many-valued member of the family. (See Rescher 1969: 22-8.)

At the other end of the scale are $Ł_{\aleph 0}$ and $Ł_{\aleph 1}$, in which the truth values are the rationals and reals, respectively, in the unit interval. $Ł_{\aleph 0}$ and $Ł_{\aleph 1}$ have the same set of tautologies (Malinowski 1993: 37), and so are considered variants of a single system $Ł_{\aleph}$.[8]

The Łukasiewicz family of systems has the interesting property that if n is an integer multiple of m, (i.e. there is an integer k such that km = n), then all the valid implications of the system $Ł_{n+1}$ are valid implications of $Ł_{m+1}$. Additionally, if n > m, there are valid implications of $Ł_{m+1}$ that are not valid in $Ł_{n+1}$ (Ackerman 1967: 60-3).

[8] I do not know of any proof, however, that all extensions on $Ł_{\aleph 0}$ and $Ł_{\aleph 1}$—for instance, RPL as opposed to a variant of RPL where the truth values are just the rationals in [0,1]—have the same tautologies. I will follow what seems to be standard convention, however, and speak of RPL as an extension of $Ł_{\aleph}$ instead of $Ł_{\aleph 1}$.

Since every integer is divisible by 1, we can see that the tautologies of any $Ł_n$ are proper subsets of the tautologies of $Ł_2$ (the classical tautologies). The tautologies of $Ł_{\aleph}$ are the intersection of the tautologies of every finite-valued member of the Łukasiewicz family (Malinowski 1993: 37-8). Because of this, any decision procedure that is sound for $Ł_{\aleph}$ is also sound for any other member of the Łukasiewicz family (including $Ł_2$). I will make use of this fact to provide tableau procedures for the members of the Łukasiewicz family in V.2.

Since 1 is the only designated value in $Ł_{\aleph}$, modus ponens is valid; i.e.,

$$\{\phi, \phi \supset \psi\} \vDash \psi.$$

Having only one designated value has the disadvantage, however, that we cannot represent approximately valid arguments in $Ł_{\aleph}$. For this purpose, we will have to move to a stronger system, as will be discussed further below in II.5.

Let

$$\phi^n =_{df} \phi \barwedge \ldots \barwedge \phi, \text{ with n instances of } \phi.$$

The Deduction Theorem holds for $Ł_{\aleph}$ in the following form (Hájek 1998: 43):

$$\Sigma \cup \{\phi\} \vDash \psi \text{ iff } \exists n \text{ such that } \Sigma \vDash \phi^n \supset \psi.$$

Since modus ponens is valid, of course, if $\Sigma \vDash \phi \supset \psi$, then $\Sigma \cup \{\phi\} \vDash \psi$.

An important property of classical logic that fails for $Ł_{\aleph}$ is compactness; if $\Sigma \vDash \phi$, there is not necessarily a finite $\Delta \subseteq \Sigma$ such that $\Delta \vDash \phi$. For instance, let

$$n\phi =_{df} \phi \veebar \ldots \veebar \phi, \text{ with n instances of } \phi.$$

Then consider this set of premises (from Hájek 1998: 75):

$\{np \supset q \mid n \text{ natural}\} \cup \{\neg p \supset q\}$.

/q/ = 1 on every model of this theory. If /p/ = 0, then /¬p/ = 1, so by the last premise /q/ = 1. If there is a k such that /p/ > $^1/_k$, then since for any given k,

$/k\phi/ = 1$ if $/\phi/ \geq {}^1/_k$,

/kp ⊃ q/ = 1 only if /q/ = 1. However, every finite subset of the premises has a model in which /q/ < 1. Compactness also fails for $Ł_{\aleph}$'s extensions RPL and RPL△.

Since compactness fails and a syntactic decision procedure can use only finitely many members of a set of premises, there are conclusions that are semantically implied by infinite sets of premises but not deducible by any syntactic method. As a result, there can be no syntactic decision procedure for $Ł_{\aleph}$ (Hájek 1998: 75) or RPL△ (Morgan & Pelletier 1977: 88-90) that is complete for infinite sets of premises.[9] Decision procedures for these systems can be complete, however, in the form

for finite Σ, $\Sigma \vdash \phi$ iff $\Sigma \vDash \phi$.

Since no stronger result is possible, I will henceforth use "complete" *simpliciter* to mean "complete for finite sets of premises". In this sense of "complete", there are complete axiomatizations of both $Ł_{\aleph}$ (Hájek 1998: 64, 70-5) and RPL△ (Hájek 1998: 57-61, 79-82).

[9] There is, however, an axiom set for RPL ($Ł_{\aleph}$ with constants but without the determinacy operator J_1) that is complete for infinite sets of premises. This is because of the way provability is defined in the syntax, a complex issue into which we need not enter. (See Hájek 1998: 80-2.) Similar results cannot be obtained for $Ł_{\aleph}$ or RPL△.

3. The rationale for the VSS connectives

We should pause momentarily in the exposition of the formal properties of the system to look at the motivations behind the definitions of the connectives. The first requirement for the definition of any many-valued connective is that it 'honour classicality' by yielding classical outputs for classical inputs; restricted to the truth values 0 and 1, the connectives should be equivalent to the classical connectives. The justification for this condition is just that if a many-valued connective C' does not agree with a classical connective C when restricted to the classical truth values, C' just isn't a generalization of C. Suppose, for instance, that we were to define

$$/\phi \sqcap \psi/ = (/\phi/ + /\psi/) \div 2.$$

Regardless of what interesting properties this operator has, since $/1 \sqcap 0/ = ½$ instead of 0, we cannot claim that this is a many-valued generalization of *conjunction*; we must view it as a new operator, one not analogous to any classical truth function.

It is more difficult to specify how to assign outputs for nonclassical inputs. Williamson proposes that any many-valued analogue of conjunction should satisfy these three principles (1994: 115-6):

$/\phi/ \leq /\phi \wedge \phi/$,

$/\phi \wedge \psi/ \leq /\phi/$ and $/\phi \wedge \psi/ \leq /\psi/$,

if $/\phi_1/ \leq /\phi_2/$ and $/\psi_1/ \leq /\psi_2/$, then $/\phi_1 \wedge \psi_1/ \leq /\phi_2 \wedge \psi_2/$.

He then proves that the truth function uniquely satisfying these principles is min $(/\phi/, /\psi/)$. Similarly, he argues that disjunction must satisfy

$$/\phi/ \geq /\phi \vee \phi/,$$

$$/\phi \vee \psi/ \geq /\phi/ \text{ and } /\phi \vee \psi/ \geq /\psi/,$$

$$\text{if } /\phi_1/ \leq /\phi_2/ \text{ and } /\psi_1/ \leq /\psi_2/, \text{ then } /\phi_1 \vee \psi_1/ \leq /\phi_2 \vee \psi_2/;$$

and shows that these conditions are uniquely satisfied by max $(/\phi/, /\psi/)$. (Gaines 1976: 631-2 presents a similar argument to the same conclusion.)

For negation, we can reason as follows. The extent to which ϕ is true should be the same as the extent to which $\neg\phi$ is false. The distance of $/\phi/$ from 0 should be the same as the distance of $/\neg\phi/$ from 1; i.e.,

$$|1 - /\phi/| = |0 - /\neg\phi/|,$$

which is equivalent to

$$/\neg\phi/ = 1 - /\phi/.$$

We can thus see that there are strong intuitive grounds for seeing the truth-functions of VSS as the intuitively strongest generalizations of classical conjunction, disjunction, and negation.

4. Conditionals in infinite-valued logic

The choice of the Łukasiewicz conditional is less obvious. There are more intuitively plausible ways of generalizing the classical conditional, and no truth function that satisfies all the intuitively plausible conditions we can come up with. The easiest way to see what constraints a suitable account of fuzzy implication should satisfy is to look at the development of generalized fuzzy logic—general conditions that are satisfied by a number of different specific systems (following Hájek 27-32).

Two of the basic principles underlying classical material implication are *ex falso quodlibet*, that a falsehood implies everything, and *verum ex quodlibet*, that everything implies a truth. The natural way to extend these principles to the infinite-valued case is to specify that a conditional is true whenever the antecedent is less true than the consequent—that is, if $/\phi/ \leq /\psi/$, $/\phi \supset \psi/ = 1$.

Another central requirement is that fuzzified modus ponens hold; given $/\phi/$ and $/\phi \supset \psi/$, we should be able to compute a lower bound of $/\psi/$. Furthermore, we will want this rule to be as strong as possible; from $/\phi/$ and $/\phi \supset \psi/$, we want to be able to compute a lower bound of $/\psi/$ as close as possible to the true value of $/\psi/$.

It is important to distinguish fuzzified modus ponens from ordinary modus ponens, which in fuzzy logic becomes the rule that if ϕ and $\phi \supset \psi$ are designated, ψ is designated. Fuzzified modus ponens is not concerned with preserving designation; given the values of the conditional and the antecedent, fuzzified modus ponens places restrictions on the value of the consequent regardless of what truth values are designated in a particular context. It thus separates the use of a conditional to make inferences about the truth value of its consequent from the use of a conditional to make inferences from designated premises to designated conclusions. As we will see in II.5, levels of designation are not fixed, but rather vary from context to context. Fuzzified modus ponens does not guarantee that ψ is designated if ϕ and $\phi \supset \psi$ are. But (except for the case where ϕ is not designated and $/\phi \supset \psi/ = 1$), fuzzified modus ponens guarantees that we can always determine whether ψ is designated for the particular level of designation used in that context.

As we will see below, ordinary modus ponens is not valid for designated values other than 1. This might seem odd. But when we remember that a sorites paradox is just a long chain of applications of modus ponens, we can see that an appropriate response to sorites paradoxes requires that ordinary modus ponens not always be valid.

The requirement that given $/\phi/$ and $/\phi \supset \psi/$, we should be able to compute a lower bound of $/\psi/$ can be formally expressed by saying that there should be some operation AND such that

$$/\phi \text{ AND } (\phi \supset \psi)/ \leq /\psi/.$$

We can state that we want this rule to be as strong as possible by saying that given this operation AND, we want

$$/\phi \supset \psi/ = \max(/\chi/ \mid /\phi \text{ AND } \chi/ \leq /\psi/).$$

(We assume that for any $/\phi/$, $/\psi/$, there is a $/\chi/$ that satisfies this condition.)

To make our account of the conditional as general as possible, we will only assume that AND is recognizable as a form of conjunction. A *continuous t-norm* is a binary operator $\star$ on [0,1] satisfying the following properties:

$$/\phi \star \psi/ = /\psi \star \phi/,$$
$$/\phi \star (\psi \star \chi)/ = /(\phi \star \psi) \star \chi/,$$
$$\text{if } /\phi_1/ \leq /\phi_2/, \text{ then } /\phi_1 \star \psi/ \leq /\phi_2 \star \psi/,$$
$$\text{if } /\psi_1/ \leq /\psi_2/, \text{ then } /\phi \star \psi_1/ \leq /\phi \star \psi_2/,$$
$$/1 \star \phi/ = /\phi/,$$
$$/0 \star \phi/ = 0.$$

A continuous t-norm $\star$ is a form of fuzzy conjunction. The dual of a t-norm is a *t-conorm*, +, and is a form of fuzzy disjunction.[10]

[10] A t-conorm must satisfy the first four conditions for a t-norm; the last two conditions are changed to $/1 + \phi/ = 1$ and $/0 + \phi/ = /\phi/$.

The *residuum* of a t-norm, $\Rightarrow$, is a form of fuzzy implication, defined (as the reader will have guessed) as

$$/\phi \Rightarrow \psi/ = \max\ (/\chi/ \mid /\phi \star \chi/ \leq /\psi/).$$

Given the restrictions on t-norms, the residuum will satisfy these intuitively plausible conditions for implication:

$$/\phi \Rightarrow \psi/ = 1 \text{ iff } /\phi/ \leq /\psi/,$$
$$/\phi \Rightarrow \phi/ = 1,$$
$$/1 \Rightarrow \phi/ = /\phi/.$$

(The second, of course, follows from the first.)

Finally, we introduce a constant function C_0, where $/C_0/ = 0$. The *precomplement* of a t-norm, –, is a form of fuzzy negation, defined as

$$-\phi =_{df} \phi \Rightarrow C_0.$$

These conditions are fairly general, and so there are a number of different functions that satisfy the requirements for a t-norm. We will look at the three of the most notable here, along with their t-conorms, residua, and precomplements:

Gödel t-norm: $/\phi \star_G \psi/ = \min\ (/\phi/, /\psi/)$

Gödel t-conorm: $/\phi +_G \psi/ = \max\ (/\phi/, /\psi/)$

Gödel residuum: $/\phi \Rightarrow_G \psi/ = $ 1 for $/\phi/ \leq /\psi/$,
$/\psi/$ otherwise

Gödel precomplement: $/-_G\phi/ = $ 1 for $/\phi/ = 0$,
0 otherwise

Goguen t-norm[11]: $/\phi \odot \psi/ = /\phi/ \times /\psi/$

Goguen t-conorm[12]: $/\phi \oplus \psi/ = /\phi/ + /\psi/ - (/\phi/ \times /\psi/)$

Goguen residuum: $/\phi \Supset \psi/ =$ 1 for $/\phi/ \leq /\psi/$,

$/\psi/ \div /\phi/$ otherwise

Goguen precomplement: $/-_G\phi/ =$ 1 for $/\phi/ = 0$,

0 otherwise

Łukasiewicz t-norm: $/(\phi \barwedge \psi)/ = \max(0, /\phi/ + /\psi/ - 1)$

Łukasiewicz t-conorm: $/(\phi \veebar \psi)/ = \max(1, /\phi/ + /\psi/)$

Łukasiewicz residuum: $/(\phi \supset \psi)/ = \min(1, 1 - /\phi/ + /\psi/)$

Łukasiewicz precomplement: $/\neg\phi/ = 1 - /\phi/$

I introduce different symbols for each set of truth functions for ease of later reference; the system defined by the Goguen connectives, *product logic*, will be discussed again in V.3.

Any system defined in this manner contains the conjunction and disjunction of VSS. For any continuous t-norm and its residuum, if we define

$$\phi \wedge \psi =_{df} \psi \star (\phi \Rightarrow \psi),$$

$$\phi \vee \psi =_{df} [(\phi \Rightarrow \psi) \Rightarrow \psi] \wedge [(\psi \Rightarrow \phi) \Rightarrow \phi],$$

we have

$$/(\phi \wedge \psi)/ = \min(/\phi/, /\psi/),$$

$$/(\phi \vee \psi)/ = \max(/\phi/, /\psi/).$$

[11] Generally called *algebraic product.*
[12] Generally called *algebraic sum.*

Hence, any system of generalized fuzzy logic contains those valuable connectives.

Each of these three conditionals can be seen as minimally acceptable accounts of fuzzy implication; they are not, however, all equal. An intuitively very plausible requirement for the conditional is that when $/\phi/ > /\psi/$, $/\phi \supset \psi/$ should be in some way proportionate to how different $/\phi/$ and $/\psi/$ are. The Gödel conditional $\Rightarrow_G$ does not satisfy this requirement. For instance, let $/\phi/ = 0.8$, $/\psi/ = 0.3$, and $/\chi/ = 0.2$. Then $/\phi \Rightarrow_G \chi/ = /\psi \Rightarrow_G \chi/ = 0.2$; but since $/\phi/$ is much greater than $/\chi/$, whereas $/\psi/$ is only slightly greater than $/\chi/$, we would expect that the truth value of "ϕ implies χ" would be much lower than that of "ψ implies χ". So $\Rightarrow_G$ is not really acceptable as a fuzzy conditional.[13]

The Goguen and Łukasiewicz conditionals both avoid this problem. Both sets of truth functions are used fairly widely. (See e.g., Gaines 1976 for discussion.) There are several considerations, however, militating towards viewing the Łukasiewicz conditional as the proper generalization of material implication. For the reasons given in II.3, the Łukasiewicz precomplement $\neg$ is a much stronger candidate for a generalization of classical negation than the Goguen precomplement $\neg_G$. Adopting the Łukasiewicz conditional along with its precomplement gives us

$$/\phi \supset \psi/ = /\neg\psi \supset \neg\phi/,$$
$$/\neg\phi/ = /\phi \supset C_0/.$$

For the Goguen conditional $\Supset$, the latter holds only for the Goguen precomplement $\neg_G$, and the former holds with neither negation.

The Goguen conditional is very similar to the rule governing conditional probability; because of this, it can be expected to be very useful in some

[13] The primary theoretical interest in Gödel's system comes from its relationship to intuitionistic logic. The logic produced by these truth functions is the intuitionistic logic IPC extended by the axiom $(\phi \supset \psi) \vee (\psi \supset \phi)$. (See Hájek 1998: 98-9.)

applications. The Łukasiewicz conditional is, however, a much more plausible generalization of the classical conditional. Similarly, fuzzy theorists also frequently use the Goguen t-norm $\odot$ and t-conorm $\oplus$. $\wedge$ and $\vee$ are, as noted in the last section, better generalizations of classical conjunction and disjunction. In particular, idempotency fails for both $\odot$ and $\oplus$; for $0 < /\phi/ < 1$,

$$/\phi \odot \phi/ < /\phi/,$$
$$/\phi \oplus \phi/ > /\phi/.$$

However, these two operators are widely seen as preferable to $\wedge$ and $\vee$ for some applications, precisely *because* idempotency fails (Gaines 1976: 635-6).[14] So the Goguen truth functions, although not suitable as a generalization of classical logic, are interesting in their own right.

Although I will focus on RPL$\triangle$, in order to make the decision procedure presented here as general as possible, it will be desirable for it to be applicable to all of these three systems. The Gödel truth functions are all definable in RPL$\triangle$:

$$\phi \star_G \psi =_{df} \phi \wedge \psi,$$
$$\phi +_G \psi =_{df} \phi \vee \psi,$$
$$\phi \Rightarrow_G \psi =_{df} (\phi \rightarrow \psi) \vee \psi,$$
$$-_G\phi =_{df} J_1\neg\phi.$$

[14] There are certainly linguistic phenomena that need to be formalized with a form of conjunction or disjunction that takes account of the truth values of both inputs, not just the lesser or greater, respectively—and idempotency fails for such connectives. Suppose there are two balls, A and B, such that

/A is red/ = 1,	/B is red/ = ½,
/A is small/ = ½,	/B is small/ = ½.

/A is red $\wedge$ A is small/ = /B is red $\wedge$ B is small/ = ½. But if one asks, "Bring me a ball that is red and small," A is a much better candidate than B (Edgington 1996: 304). Since /A is red $\odot$ A is small/ = ½ whereas /B is red $\odot$ B is small/ = ¼, using algebraic product instead of max yields the desired result.

There is thus no reason to introduce new tableau rules for the Gödel truth functions. The Goguen truth functions (except for the precomplement $\neg_G$) are not definable in RPL$\triangle$. Hence, tableau rules for these connectives will be presented, and proved to be sound, in V.3.

5. RPL

We do not want to restrict ourselves to a logic that admits of the possibility of infinitely many degrees of truth, but only allows us to *talk* about perfectly true or perfectly false formulas. Furthermore, we are interested in modeling *approximate* truth, and thus need a notion of what it is for a proposition to be "true enough" for the purposes at hand. The former problem is solved by augmenting the language so that we can talk about the truth values themselves in the system; the latter problem requires being able to identify more truth values as *designated*, or "truth-like". Both tasks can be accomplished by a single extension on $Ł_{\aleph}$.

The designated truth values of a many-valued system are, roughly, those that are sufficiently true to be asserted or believed. The designated values are all the reals in an interval [ð,1] or (ð,1]. Given a particular choice of designated values, we will be interested in formulas that uniformly take designated values and arguments where, on every valuation giving all the premises a designated value, the conclusion is also designated. Correspondingly, the *antidesignated* values are the "false-like", or approximately false values. In the systems with which we are concerned, we can say that ϕ is antidesignated iff $\neg\phi$ is designated.

The degree of truth necessary for a statement to be designated will vary, however, depending on the context. Suppose I report "I put 50 grams of powder in the water," when I actually mixed 55g of powder into the water. If I were making instant coffee, my statement would be true enough to be perfectly acceptable; if I were performing a chemical experiment, my statement would be dangerously inaccurate. Hence, we cannot specify a single level of designation that applies to all situations. To maximize the generality and context-independence of our logic, we want to be able to talk about what would be true given any level of designation. The only categorical limitation to put on the level of designation is to require that the value 0.5 not be designated; otherwise, both a formula and its negation could be designated.

We also want our logic to be able to represent approximately valid arguments. That is, we want to be able to distinguish between an argument that is not perfectly true, but which has a very high degree of validity, such as

A man with 100 000 hairs is not bald.
Therefore, a man with 99 999 hairs is not bald.

and an argument that has a very low degree of validity, like

A man with 100 000 hairs is not bald.
Therefore, a man with 9 hairs is not bald.

RPL, or Rational Pavelka Logic, provides us with the vocabulary to represent both varying levels of designated values and approximate validity. RPL introduces, for each rational $i \in [0,1]$, a constant truth function C_i, where

$$/C_i/ = i.$$

We do not introduce constants for all the reals to avoid having an indenumerable set of primitive truth functions. This extension allows us to make statements about the truth values of formulas with values other than 1 or 0, since

$$/C_i \supset \phi/ = 1 \text{ iff } /\phi/ \geq i,$$

$/\phi \supset C_i/ = 1$ iff $/\phi/ \leq i$.

So for any set of designated values $[\eth,1]$, a formula ϕ is designated iff $/C_\eth \supset \phi/ = 1$, and a formula is a fuzzy tautology for that level of designation iff $\vDash C_\eth \supset \phi$. We can represent approximately valid arguments as well; the conditional [S1] from our solution of the sorites paradox in I.5 can be represented as

$\neg$(a man with n hairs is bald) $\supset [C_{0.99999} \supset \neg$(a man with n – 1 hairs is bald)].

In effect, once combined with constants, material implication can perform double duty as both a formal analogue of the conditional and a mechanism for stating the truth values of formulas.

RPL has the derived deduction rule (Hájek 1998: 80)

$\{C_i \supset \phi, C_j \supset (\phi \supset \psi)\} \vDash C_k \supset \psi$, where $k = i + j - 1$.

Thus, we can state fuzzified modus ponens in the system itself.

6. RPL△

The language of RPL is still somewhat limited. We cannot define bivalent truth functions in it—more generally, since all the operators are continuous, we cannot define any discontinuous truth functions. A further problem is that although we can state in RPL that $/\phi/ \leq i$, $/\phi/ \geq i$, or $/\phi/ = i$ (with $\phi \supset C_i$, $C_i \supset \phi$, and $\phi \equiv C_i$, respectively), we cannot state $/\phi/ < i$ or $/\phi/ > i$. For instance, we cannot state that $/\phi/ > 0.5$ with $\neg(\phi \supset C_{0.5})$. To make that assertion, we need a formula the value of which will be 1 iff $/\phi/ > 0.5$. But

$$/\neg(\phi \supset C_{0.5})/ = 1 - \min(1, 1 - /\phi/ + 0.5)$$
$$= 1 - \min(1, 1.5 - /\phi/)$$

Since $/\phi/ \leq 1$, $\min(1, 1.5 - /\phi/) \geq 0.5$; thus, $/\neg(\phi \supset C_i)/ \leq 0.5$. It is not hard to see that $/\neg[\phi \supset C_i]/ = 1$ only if $/\phi/ = 1$ and $i = 0$.

We can augment the language to eliminate this difficulty by adding the new operator

$$/J_1\phi/ = 1 \text{ if } /\phi/ = 1,$$
$$0 \text{ otherwise.}$$

The philosophical literature on vagueness generally uses the symbol 'Δ' for this operator (as does Hájek 1998), and thus the system may be more readily recognized if referred to as RPLΔ. I will refer to the basic operator as 'J_1', however, by analogy with the J-operators for other truth values, which can be defined using J_1:

$$J_i\phi =_{df} J_1(\phi \equiv C_i), \text{ for each } C_i.$$

We can also use J_1 to define the bivalent implication and equivalence mentioned above in I.4:[15]

$$(\phi \rightarrow \psi) =_{df} J_1(\phi \supset \psi),$$
$$(\phi \leftrightarrow \psi) =_{df} J_1(\phi \equiv \psi).$$

---

[15] Equivalently, one can take the bivalent implication $\rightarrow$ as primitive, and then define

$$J_i\phi =df (\phi \rightarrow C_i) \wedge (C_i \rightarrow \phi).$$

(See Morgan & Pelletier 1977: 86n.)

Since

$$/(\phi \rightarrow \psi)/ = 1 \text{ if } /\phi/ \leq /\psi/,$$
$$0 \text{ otherwise,}$$

we can readily see that $/\neg(C_i \rightarrow \phi)/ = 1$ iff $/\phi/ < i$ and $/\neg(\phi \rightarrow C_i)/ = 1$ iff $/\phi/ > i$. We can also use the bivalent conditional to give a syntactic account of open intervals of designated values; if the designated values are $(\eth, 1]$, then ϕ is designated iff $/\neg(\phi \rightarrow C_\eth)/ = 1$.

If we note that for any n, $J_1\phi \supset \phi^n$ (where 'ϕ^n' is, as above, $\phi \barwedge \ldots \barwedge \phi$, with n instances of ϕ), we can see that the Deduction Theorem holds for RPL$\triangle$ in the form

$$\Sigma \cup \{\phi\} \vDash \psi \text{ iff } \Sigma \vDash J_1\phi \supset \psi.$$

7. Tautologies and non-tautologies of RPL$\triangle$

I will close this chapter by discussing some tautologies of RPL$\triangle$ and some formulas which are tautologies of classical logic but not of RPL$\triangle$. This section will hopefully give the reader a better sense of what are and are not logical truths of the system with which we are concerned. As in the rest of this chapter, proofs will not be given here; proofs for most of the tautologies can be found in Hájek 1998, chapters 2-3.

Most of the standard properties of conjunction and disjunction hold, such as

$$\vDash p \wedge q \supset q \wedge p,$$
(commutativity)
$$\vDash p \vee q \supset q \vee p,$$

$$\models [p \wedge (q \wedge r)] \equiv [(p \wedge q) \wedge r],$$

(associativity)

$$\models [p \vee (q \vee r)] \equiv [(p \vee q) \vee r],$$

$$\models \neg(p \wedge q) \equiv \neg p \vee \neg q,$$

(DeMorgan's Laws)

$$\models \neg(p \vee q) \equiv \neg p \wedge \neg q,$$

$\models p \wedge q \supset p,$ (simplification)

$\models p \supset p \vee q,$ (addition)

$\models p \supset (q \supset p \wedge q),$ (adjunction)

$$\models p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r),$$

(distribution laws)

$$\models p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

As noted in II.1, the Łukasiewicz conditional is not definable in terms of $\wedge$, $\vee$, and $\neg$. We have

$$\models p \vee q \supset (\neg p \supset q),$$

$$\models \neg(p \wedge \neg q) \supset (p \supset q),$$

but the converses do not hold:

$$\not\models (p \supset q) \supset \neg p \vee q,$$

$$\not\models (p \supset q) \supset \neg(p \wedge \neg q).$$

Needless to say,

$$\not\models p \vee \neg p,$$

$$\not\models \neg(p \wedge \neg p).$$

We do have, however,

$$\models C_{0.5} \supset p \vee \neg p,$$

$$\models p \wedge \neg p \supset C_{0.5};$$

i.e., $/p \vee \neg p/ \geq 0.5$ and $/\neg(p \wedge \neg p)/ \leq 0.5$. Thus as long as the designated values are a (not necessarily proper) subinterval of (0.5, 1], $p \wedge \neg p$ will never be designated, and $p \vee \neg p$ will never be antidesignated.

Double negation holds, i.e.

$$\models p \equiv \neg\neg p.$$

Many properties of the classical conditional obtain for its fuzzy counterpart:

$\models (p \supset q) \equiv (\neg q \supset \neg p)$, (contraposition)

$\models [p \supset (q \supset r)] \equiv (p \wedge q \supset r)$, (exportation)

$\models [(p \supset r) \wedge (q \supset r)] \supset (p \vee q \supset r)$, (disjunctive syllogism)

$\models (p \supset r) \supset (p \wedge q \supset r)$, (strengthening antecedent)

$\models (p \supset q) \vee (q \supset p)$,

$\models q \supset (p \supset q)$, (paradoxes of material implication)

$\models p \supset (\neg p \supset q)$,

$\models C_0 \supset p$, (*ex falso quodlibet*)

$\models p \supset C_1$, (*verum ex quodlibet*)

$\models (C_1 \supset p) \equiv p$,

$$\models [p \supset (q \supset r)] \supset [q \supset (p \supset r)];$$

and similarly for the biconditional:

$$\models p \equiv p,$$
$$\models (p \equiv q) \supset (p \supset r) \equiv (q \supset r),$$
$$\models (p \equiv q) \supset (r \supset p) \equiv (r \supset q).$$

If the only designated value is 1, then the usual inference rules hold:

$\{p, p \supset q\} \models q$, (modus ponens)

$\{\neg q, p \supset q\} \models \neg p$, (modus tollens)

$\{p \supset q, q \supset r\} \models p \supset r$, (transitivity)

$\{p \vee q, \neg p\} \models q$, (modus tollendo ponens)

$\{p \wedge q\} \models p$,
(simplification)
$\{p \wedge q\} \models q$.

The first three, however, do not always hold if we designate more values than 1; i.e., if the designated values are [ð,1], we have

$$\{C_ð \supset p, C_ð \supset (p \supset q)\} \not\models C_ð \supset q,$$
$$\{C_ð \supset \neg q, C_ð \supset (p \supset q)\} \not\models C_ð \supset \neg p,$$
$$\{C_ð \supset (p \supset q), C_ð \supset (q \supset r)\} \not\models C_ð \supset (p \supset r).$$

For instance, let /p/ = /p ⊃ q/ = ð. Then /q/ = 2ð − 1 < ð. We have fuzzified versions of each of these inferences:

$$/q/ \geq /p/ + /p \supset q/ - 1,$$

$$/\neg p/ \geq /\neg q/ + /p \supset q/ -1,$$

$$/p \supset r/ \geq /p \supset q/ + /q \supset r/ - 1.$$

These allow us to compute a lower bound for the truth value of the consequent, although we cannot guarantee that the consequent will be designated whenever the premises are. The analogues for designated values [ð,1] of the latter two rules above are valid:

$$\{C_ð \supset p \vee q, \neg(C_ð \rightarrow p)\} \models C_ð \supset q,$$

$$\{C_ð \supset p \wedge q\} \models C_ð \supset p,$$

$$\{C_ð \supset p \wedge q\} \models C_ð \supset q.$$

A few important nontautologies of RPLΔ are

$\not\models p \wedge \neg p \supset q,$ (explosion)

$\not\models [(p \supset q) \supset p] \supset p,$ (Peirce's law)

$\not\models [p \supset (p \supset q)] \supset (p \supset q).$ (contraction)

Note, however, that as long as $ð > 0.5$,

$$\{C_ð \supset p \wedge \neg p\} \models q,$$

since on no valuation is the premise true.

Finally, there are a number of tautologies and inferences that depend on having a finite number of truth values that fail in RPLΔ. For instance, in any logic with a finite number n of truth values, there is a tautology of the form

$$(p_1 \equiv p_2) \vee \ldots \vee (p_1 \equiv p_{n+1}) \vee (p_2 \equiv p_{n+1}) \vee \ldots \vee (p_n \equiv p_{n+1})$$

with n + 1 propositional variables. In classical logic, for instance,

$$\vDash (p_1 \equiv p_2) \vee (p_1 \equiv p_3) \vee (p_2 \equiv p_3),$$

and in Ł$_3$,

$$\vDash (p_1 \equiv p_2) \vee (p_1 \equiv p_3) \vee (p_1 \equiv p_4) \vee (p_2 \equiv p_3) \vee (p_2 \equiv p_4) \vee (p_3 \equiv p_4).$$

In RPL△, however, for any finite n,

$$\nvDash (p_1 \equiv p_2) \vee \ldots \vee (p_1 \equiv p_n) \vee (p_2 \equiv p_n) \vee \ldots \vee (p_{n-1} \equiv p_n).$$

Similarly, in every finite-valued Ł$_n$,

$$\{p \veebar p^{n-1}\} \vDash p,$$

with p^k defined as in II.2. But in Ł$_{\aleph}$ (and RPL△), for any finite n,

$$\{p \veebar p^{n-1}\} \nvDash p.$$

If ð < 1, however, for any ð there is a finite k such that

$$\{C_ð \supset p \veebar p^k\} \vDash C_ð \supset p,$$

although for any k,

$$\{C_ð \supset p \veebar p^k\} \nvDash C_ð \supset J_1 p.$$

This should give the reader a sufficient understanding of RPL$\triangle$ to be able to see roughly the structure of the logic and the inferences it sanctions and does not sanction. We now turn to looking at decision procedures for this system.

III

Decision procedures for infinite-valued logic

Despite the extensive interest in it, fairly little work has been done on decision procedures for fuzzy logic. This chapter reviews what seems, to the best of my knowledge, to be all the decision procedures for infinite-valued logics that have been proposed. Two procedures have been proposed for VSS, and three for $Ł_{\aleph}$. A decision procedure covering most major propositional fuzzy logics has been proposed, and another procedure specifically for RPL△ has been proposed, but not formalized. All of these methods either do not cover all the truth functions of RPL△ or have significant limitations. As a result, there is a real need for a sound decision procedure for RPL△.

1. A general decision procedure for infinite-valued logics

Gehrke et al. (1999) have shown that algorithms exist for most forms of propositional fuzzy logic to determine if two formulas are equivalent on all valuations. They define an algebraic set as the union of finitely many sets $S_1, \ldots, S_n$, where each S_i consists of all tuples satisfying one or more conditions of the form

$P_k(x_1, \ldots, x_n) = Q_k(x_1, \ldots, x_n)$,

$P_k(x_1, \ldots, x_n) > Q_k(x_1, \ldots, x_n)$, or

$P_k(x_1, \ldots, x_n) \geq Q_k(x_1, \ldots, x_n)$, where P_k, Q_k are polynomials with rational coefficients. A function f is algebraic iff its graph, the set of tuples $\langle x_1, \ldots, x_n, f(x_1, \ldots, x_n)\rangle$, is algebraic; a propositional logic is algebraic iff its set of truth values is a algebraic set and all its logical operators are algebraic functions. They then show that the truth values of fuzzy propositional logic (including interval-valued logics[16]) are algebraic, and most major operators of fuzzy logic are algebraic, including those of RPL△.[17]

The question of whether two formulas F and G of a given propositional logic L are equivalent on all valuations can be stated as

$$\forall p_1 \ldots \forall p_n(/F/ = /G/),$$

where $p_1 \ldots p_n$ are all the propositional variables found in F or G. If L is algebraic, this formula will be a composition of algebraic functions. Tarski (1957) has given a decision procedure for all formulas of this form, and so this procedure will work for any algebraic propositional logic. Gehrke et al. (1999) go on to show that propositional logics the operators of which are nonalgebraic functions (e.g., by only being definable in terms of polynomials P_k, Q_k with irrational coefficients) are undecidable.

Where n is the number of propositional variables, Tarski's algorithm in some cases takes more than 2^{2^n} steps to run. Hence, despite the generality of Gehrke et al.'s (1999) procedure, it is too complex to be generally feasible as a decision procedure for RPL△. A less complex algorithm is greatly desirable.

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[16] In which the truth values of formulas are subintervals of [0,1], instead of points in [0,1]. Such logics try to model the indeterminacy of vague expressions by not specifying precise truth values for formulas (as discussed in I.3). Nguyen et al. (1996) have constructed such a system with the connectives $\wedge$, $\vee$, and $\neg$.

[17] Gehrke et al. (1999) show that bounded sum and difference, in terms of which the Łukasiewicz conditional may be defined, are algebraic. The graph of each C_i is $\{i\}$, and the graph of J_1 is

$$\{(x_1, x_2) \mid x_1 = 1 \wedge x_2 = 1\} \cup \{(x_1, x_2) \mid x_1 < 1 \wedge x_2 = 0\},$$

both of which are algebraic sets.

2. Kenevan truth interval logic

Gehrke et al. (1997) show that VSS is equivalent to a three-element algebra; since this algebra is finite, there is a finite algorithm that determines tautologousness in VSS.[18] A less erudite decision procedure for VSS is the Kenevan Truth Interval Logic (KTIL), a tableau procedure for VSS, which I will briefly discuss.

Kenevan et al. (1992) take the truth value of a formula to be a discrete point in [0,1], but do not assume that we know what the exact truth value is—the truth value of a formula may be represented as an interval [a,b] of [0,1]. If we know nothing about the truth value of a formula ϕ at all, we represent its truth value as being in the interval [0,1]. Each formula and the interval containing its truth value is represented in the tableau by

$$A: [a_0,a_1].$$

If we know /A/ = a, we represent this with the line

$$A: [a,a].$$

Open intervals can be represented in KTIL by assigning α to an interval $[a_0, a_1 - \varepsilon]$ or $[a_0 + \varepsilon, a_1]$, where we dictate that $a_1 - \varepsilon < a_1$, but for all values x found in the tableau such that $x \neq a_1-\varepsilon$ and $x < a_1$, $x < a_1 - \varepsilon$ (and similarly for $a_0 + \varepsilon$).

The decomposition rules for the connectives are:

[18] As well, Nguyen et al. (1996) show that the tautologies of their interval-valued system are a proper subset of the tautologies of VSS. Thus, the decision procedure for their system is a sound, though incomplete, decision procedure for VSS.

Disjunction:

$$
\begin{array}{c}
A: [a_0,a_1] \\
B: [b_0,b_1] \\
A \vee B: [p_0,p_1] \\
\swarrow \qquad \qquad \searrow
\end{array}
$$

$$
\begin{array}{ll}
A: [a_0, \min(a_1, p_1)] & A: [\max(a_0, p_0), \min(a_1, p_1)] \\
B: [\max(b_0, p_0), \min (b_1, p_1)] & B: [b_0, \min(b_1, p_1)] \\
A \vee B: [\max(a_0,b_0,p_0), \min(p_1, \max[a_1,b_1])] & A \vee B: [\max(a_0,b_0,p_0), \min(p_1, \max[a_1,b_1])]
\end{array}
$$

Conjunction:

$$
\begin{array}{c}
A: [a_0,a_1] \\
B: [b_0,b_1] \\
A \wedge B: [p_0,p_1] \\
\swarrow \qquad \qquad \searrow
\end{array}
$$

$$
\begin{array}{ll}
A: [\max(a_0, p_0), a_1] & A: [\max(a_0, p_0), \min(a_1, p_1)] \\
B: [\max(b_0, p_0), \min(b_1, p_1)] & B: [\max(b_0, p_0), b_1] \\
A \wedge B: [\max(p_0, \min[a_0, b_0]), \min(a_1,b_1,p_1)] & A \wedge B: [\max(p_0, \min[a_0,b_0]), \min(a_1,b_1,p_1)]
\end{array}
$$

Negation:

$$
\begin{array}{c}
A: [a_0,a_1] \\
\downarrow \\
\neg A: [1 - a_1, 1 - a_0]
\end{array}
$$

Intersection:

$$
\begin{array}{c}
A: [x_0,x_1] \\
A: [y_0,y_1] \\
\downarrow \\
A: [\max (x_0,y_0), \min (x_1,y_1)]
\end{array}
$$

If a branch of a tableau indicates that the truth value of a formula is in a null interval, then that branch closes (being impossible).

Proofs can be performed in KTIL by either of two methods. The first is "truth interval refinement" (Entemann 2000: 170), in which one enters as premises the smallest intervals in which the truth values of the formulas are known to be, and then applies the decomposition rules until the intervals either cease to contract or become null. Any open branches yield more accurate estimates of the truth values of the formulas; if all branches are closed, there are no valuations satisfying the conditions in the premises.

One can also perform "proofs by contradiction", modeled after the standard methods for tableaux, where one assumes that each of the premises (if any) is in the interval (0.5,1], and the conclusion is in [0,0.5). (Entemann takes the interval (0.5,1] to be the designated values and forbids propositional variables to take the value 0.5, a move I will discuss in a moment.) If all branches of the tableau close, then it is impossible for the premises to be designated and the conclusion undesignated. Any open branches give counterexamples to the inference.

Entemann has proven proofs by contradiction in KTIL to be sound and complete for all valuations of VSS on which no formula takes the value 0.5. Kenevan et al. (1992) have shown the "correctness" of the disjunction rule; i.e., that on any valuation on which the antecedent lines are true of the values of A, B, and $A \vee B$, the lines inferred will also be true for those formulas. I will prove the soundness of the truth interval refinement method in IV.6, by showing that the inference rules are valid in the tableau procedure given here, which is sound.

The serious limitation of KTIL is that it is only valid for the meager language of VSS. Entemann has to restrict his completeness theorem with the assumption that no formula takes the value 0.5 because without this assumption, VSS has no logical truths—every truth function yields 0.5 if all the inputs are 0.5. We now look at proof procedures for more powerful logics that yield logical truths for inputs with values anywhere in [0,1].

3. Constraint tableaux

So far as I am aware, only two decision procedures for practical use have been proposed specifically for $Ł_{\aleph}$,[19] by Hähnle (1994)[20] and Beavers (1993). Both procedures work by converting the problem of determining theoremhood in $Ł_{\aleph}$ into a problem in linear programming; there are a number of automatic procedures for solving the latter set of problems. Because the principles underlying the two systems are so similar, I will only look in detail at Hähnle's constraint tableaux.

Constraint tableaux are a variant on signed tableaux for finite-valued logics, which consist of formulas prefixed with signs indicating their truth values. (See D'Agostino et al. 1999: 538-48.) In constraint tableaux, rather than representing individual truth values, the signs place constraints on the truth values of their formulas. These constraints can include variables; for instance,

$$\mathbf{< 1}\,\phi$$

indicates that $/\phi/ < 1$, while

$$\leq \mathbf{i}\ \phi$$

indicates that $/\phi/ \leq i$.

The decomposition rules of constraint tableaux result in new constraints applied to the formulas of which the original was composed, and inequations

---

[19] Mundici (1987) also gives a procedure for determining if a formula of $Ł_{\aleph}$ is satisfiable (takes a value greater than 0 on some valuation). This procedure can be used to determine if a formula is a theorem of $Ł_{\aleph}$ by determining if its negation is satisfiable. Mundici's method, however, is only given in order to prove that satisfiability in $Ł_{\aleph}$ is in NP, and does not appear to be feasible for practical use.

[20] Constraint tableaux were first presented in Hähnle (1993); Hähnle (1994) provides a more detailed discussion of constraint tableaux specifically for $Ł_{\aleph}$.

representing the relations between the variables used in the signs. There are no branching rules; instead, Hähnle uses binary variables (variables that can only be 0 or 1), which represent the same information as new branches of the tableau would. For instance, the decomposition rules for $\supset$ are:

$\leq \mathbf{i}\ \phi \supset \psi$		$\geq \mathbf{i}\ \phi \supset \psi$	
$\downarrow$		$\downarrow$	
$\geq \mathbf{i_1}\phi$	$y \leq i, i_1 \leq 1 - y$	$\leq \mathbf{i_1}\phi$	
$\leq \mathbf{i_2}\psi$	$1 - i_1 + i_2 = i$	$\geq \mathbf{i_2}\psi$	$1 - i_1 + i_2 = i$

In the left-hand rule, y is a binary variable. The information contained in the signs can also be entered into the series of inequations; for instance, from

$$\leq \mathbf{i_2}\psi$$

we infer $/\psi/ \leq i_2$.

Applying the decomposition rules for constraint tableaux is simple, particularly since there are no branching rules. The difficult part is determining whether a branch is open or closed. To do this, Hähnle treats the series of inequations associated with a tableau as a problem in bounded mixed integer programming (bMIP). Roughly, a bMIP problem is an attempt to find the least set of variables that satisfies a set of inequations. If there is no set of variables that satisfies the inequations associated with a constraint tableau, the branch closes.

To prove that a formula ϕ is a theorem of $Ł_{\aleph}$, it is sufficient to assume

$$\leq \mathbf{i}\ \phi$$

decompose ϕ fully, and apply bMIP to the resulting inequations. ϕ is a theorem iff the least i that satisfies every inequation is 1.

Hähnle has proven constraint tableaux sound and complete for $Ł_{\aleph}$, and it seems that it would not be difficult to extend this result to RPL. However, as Hähnle (1994) notes, constraint tableaux cannot be extended to systems that contain discontinuous or non-linear truth functions, since these functions cannot be represented in bMIP. Hence, his system cannot be extended to cover RPL△ or product logic.

Constraint tableaux appear to have been designed specifically for implementation in automated theorem provers. Since the mathematical knowledge required to solve bMIP problems is not that widely found,[21] it seems that constraint tableaux are not feasible as a tableau system for general use. (*Mutatis mutandis*, this problem also afflicts the procedure presented in Beavers [1993]). Hence, there is still a need for a decision procedure that covers infinite-valued logics more powerful than $Ł_{\aleph}$, and that can be readily used without much mathematical expertise.

4. The method of hypotheses

Morgan and Pelletier (1977: 92-5) have sketched a procedure for finding tautologies of RPL△, the outline of which they attribute to David Lewis. Their "method of hypotheses" is based on the idea that since every formula is only finitely long, one need only check a finite number of points and intervals in [0,1] in order to determine if the formula is tautologous. Which points and intervals the formula needs to be evaluated on depend on the relations between the propositional variables in the formula. The method thus attempts to determine what points and intervals are relevant to the tautologousness of a formula, and then checks the truth value of the formula on all the possible assignments of truth values to relevant truth intervals.

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[21] E.g., among philosophers.

The method of hypotheses has not been formalized, and so I can only give a rough sketch of what a decision procedure based on it would look like. Let $<\phi>$ be the set of all subformulas of ϕ. (Since ϕ is finite, $<\phi>$ is finite.) The procedure then consists of the following steps:

(1) We first (tentatively) identify reference points and reference intervals. 0 and 1 are reference points, and if ϕ contains any constants C_i or J-operators J_i, i is a reference point. A reference point, or the interval (j,k) between any two reference points j and k, is a reference interval. If we are interested in an interval $[\delta,1]$ or $(\delta,1]$ of designated values, then instead of having 0 and 1 as reference points, we add the reference intervals $[0,1-\delta]$ and $[\delta,1]$ or $[0,1-\delta)$ and $(\delta,1]$ respectively.

(2) A hypothesis consists of an assignment of each member of $<\phi>$ to a reference interval. We construct all the hypotheses; i.e., all combinations of assignments of members of $<\phi>$ to reference intervals. This is done by first constructing all possible assignments of propositional variables to reference intervals; then, for each assignment of propositional variables, we add every possible assignment of members of $<\phi>$ with one connective; and so on through subformulas of ϕ of increasing complexity until we have constructed all possible hypotheses. The number of hypotheses may be, to say the least, large, but since $<\phi>$ is finite and the number of reference intervals is finite, there can only be finitely many hypotheses.

(3) For each formula α with one connective in each hypothesis, we determine whether it is possible for α to be in the reference interval to which it is assigned by that hypothesis, given the reference intervals the propositional variables in α occupy. We discard impossible hypotheses. For each hypothesis that remains once this is complete, we determine whether it is possible for each formula with

two connectives to be in the reference interval assigned to it given the reference intervals of its components, and so on until we have covered all members of $\langle \phi \rangle$.

The conditions under which a hypothesis is possible are not ideally clear, because the method has not been formalized. It seems that a hypothesis is possible iff every valuation on which the values of each member of $\langle \phi \rangle$ are within the intervals specified by the hypothesis is consistent. For instance, if p is assigned to (0.25,0.5), q is assigned to 0, and $p \vee q$ is assigned to 1, then the hypothesis is impossible, because given the values assigned to p and q, $/p \vee q/ < 0.5$.

The hypothesis may not contain sufficient information to determine whether it is possible. For instance, if p is assigned to (0.25,0.5) and $\neg p$ is assigned to (0.5,1), the hypothesis would be possible were $\neg p \in (0.5,0.75)$, but impossible were $\neg p \in [0.75,1)$. In such cases, we need to add new reference intervals that reflect all the information we need. In this example, we add the reference point 0.75, getting three new reference intervals—(0.5,0.75), 0.75, and (0.75,1). If we have divided a former reference interval (a,b) into new reference intervals (a,c), c, and (c,b), then for each hypothesis not already found to be impossible that assigns α, a member of $\langle \phi \rangle$, to (a,b), we construct three subhypotheses, assigning α to (a,c), c, and (c,b).

Successively refining reference intervals will not always be sufficient for determining if formulas are tautologies of RPLΔ. Consider a hypothesis assigning each of ϕ, ψ, and $\phi \supset \psi$ to (0,1). We do not know if this hypothesis is possible without more information about the truth values of ϕ and ψ; if $/\phi/ > /\psi/$, then $/\phi \supset \psi/ \in (0,1)$, but if $/\phi/ \leq /\psi/$, then $/\phi \supset \psi/ = 1$. Introducing a new reference point c that divides (0,1) into three reference intervals (0,c), c, and (c,1) does not help the problem. Knowing that $/\phi/$ and $/\psi/$ are in an interval (i,j) does not allow us to determine $/\phi \supset \psi/$; we need to know in addition whether $/\phi/ \leq /\psi/$ or $/\phi/ > /\psi/$. Rather, we would need to construct subhypotheses giving information about the relative values of formulas. In the case just described, for instance, we might add three subhypotheses, on which $/\phi/ < /\psi/$, $/\phi/ > /\psi/$, and $/\phi/ = /\psi/$.[22]

(4) If ϕ is assigned the reference interval 1 by every possible hypothesis, ϕ is a tautology. If we are working with designated values other than 1, then if every possible hypothesis assigns ϕ to $[\eth,1]$ (or $(\eth,1]$, as the case may be), ϕ is a fuzzy tautology.

The method of hypotheses has the disadvantage of having extremely high computational complexity. If m is the number of subformulas of ϕ and n is the number of reference intervals, the number of possible hypotheses is n^m. So for instance, let ϕ be

$$[C_{½} \supset (p \supset q)] \vee [C_{½} \supset (\neg p \supset q)].$$

ϕ has eight subformulas, and we need at least five reference intervals (and possibly more, if the hypotheses generated with those five intervals contain insufficient information). There are 5^8 possible assignments of subformulas of ϕ to reference intervals, and so there are 5^8 = 390 625 hypotheses that need to be checked. Hence, a decision procedure with lower complexity would be very advantageous.

5. Informal reasoning in RPL△

It would seem that the most fruitful method for finding a decision procedure for infinite-valued logic would be to attempt to formalize how people

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[22] Morgan and Pelletier (1977) does not mention forming hypotheses with this sort of information. Pelletier (personal communication) has indicated, however, that this was what he and Morgan had intended.

actually reason when doing informal proofs. There are several different ways of doing this (see Gaines 1976: 654 or Rescher 1969: 39 for examples), but given the strength of *reductio ad absurdum* proofs, we should pay particular attention to formalizing the informal rules people use when performing *reductiones* in infinite-valued logic. An example should show how such reasoning can be performed.

I will prove that $p \supset (q \supset p)$ is a tautology of RPL$\triangle$. To begin with, we can assume

[1] $/p \supset (q \supset p)/ < 1$;

if this assumption leads to a contradiction, we know the formula is a tautology. Looking at the definition of $/p \supset q/$, we can see that $/p \supset q/ < 1$ only if $/p/ > /q/$. So we can infer

[2] $/p/ > /q \supset p/$.

This is equivalent to

[3] $/q \supset p/ < /p/$.

Since $/p/ \leq 1$, this line tells us that

[4] $/q \supset p/ < 1$.

We know that whenever $/p \supset q/ < 1$,

[5] $/p \supset q/ = 1 - /p/ + /q/$;

thus, we can substitute the right-hand side of [5] into [3], which gives us

[6] $1 - /q/ + /p/ < /p/$.

Dropping /p/ from both sides and rearranging the terms, we see that this is equivalent to

[7] $/q/ > 1$.

But this is impossible; we know that the truth value of a formula can never exceed 1. Hence, $/p \supset (q \supset p)/ < 1$ is impossible, and $p \supset (q \supset p)$ is a tautology. The tableau system presented in the next chapter attempts to formalize this sort of reasoning about infinite-valued logic.

IV

Algebraic tableaux for RPL△

In this chapter, I will present algebraic tableaux for RPL△. The tableau procedure works by giving rules by which to resolve the assumptions into inequalities involving only numbers and the truth values of propositional variables, which are then solved using standard algebra. (Hence it is best to call them "algebraic", rather than "semantic", tableaux.) Instead of giving an algorithm that generates algebraic tableaux, I will present a more informal procedure that may be easily used by anyone familiar with basic algebra. Finding an algorithm for constructing algebraic tableaux is a project for future work.

The system is fairly complicated, due to the number of connectives in the system and the large number of cases that need to be accounted for. Even so, in order to avoid redundancies, I have avoided introducing separate rules for inferences that can be made using a combination of several different rules. As a result, the rules may seem counterintuitive or difficult to work with when considered independently. After presenting the basic system, I will explain informally how the rules can be combined in order to make the inferences one needs in the tableaux. I will also present a set of derived rules that simplify making tableaux, and show that they are merely shortcuts that add nothing to the basic rules of the system. In a similar vein, I will show that the decomposition rules of the Kenevan Truth Interval Logic are reducible to the rules of my procedure; since the latter is sound (as will be proved at the end of this chapter), the former is also sound. Before showing the validity of the derived rules and the soundness of KTIL, I will give a few examples of the tableau procedure in action, so that the reader can see how it is used. Finally, I will prove the soundness of the procedure for RPLΔ.

1. Tableau rules

We begin with a conclusion A, which we wish to prove, and a possibly empty set of premises Σ. The procedure has three steps.

Step 1. If 1 is the only designated value, the first line of the tableau should be

$$/A/ < 1.$$

Next, for any formula $\phi \in \Sigma$, a line of the form

$$/\phi/ = 1$$

should be entered in the tableau.

One may want to prove restrictions on the truth values of one's conclusion, or put restrictions on the truth values of the premises, other than that they are absolutely true. For instance, one may want to prove that the conclusion is designated, or assume that the premises are designated. More generally, one may want to show that whenever the truth value of the premises exceeds some n, the conclusion does also; this shows that whenever n is designated, the argument holds. There are two ways to do such proofs in the tableaux. The more intuitive way is to enter such restrictions directly into the tableau as inequalities. So for instance, if one wants to prove that $/A/ \geq n$, the first line of the tableau should be

$$/A/ < n.$$

(I.e., if one wants to prove that A is designated, the first line would be $/A/ < ð$; if one wanted to prove, say, that $/A/ \geq ½$, the first line would be $/A/ < ½$; etc.) To say that a premise B is designated, one would enter the line

$$/B/ \geq ð,$$

(or $/B/ > ð$, depending on whether the designated values form an open or closed interval).

The other way to represent restrictions on the truth values of premises is to state them directly in RPL△. Using this method, the two cases in the last paragraph would be represented as

$$/C_n \supset A/ < 1,$$

and

$$/C_ð \supset B/ = 1.$$

For every constant C_i found in any formula of the premises or conclusion, enter a line of the form

$$/C_i/ = i.$$

For future reference, this rule will be called "C".

For every formula or subformula ϕ found in the premises or conclusion that is negated (prefixed with $\neg$), enter a line of the form

$$/\neg\phi/ = 1 - /\phi/.$$

This rule will be called "N".

For instance, suppose we want to show that $/p \supset \neg q/ \geq 0.5$, and we are assuming $/p \supset r/ = 1$ and $/r \supset (q \supset C_{0.5})/ \geq 0.75$. Then the first few lines of the tableau should look like this:

$$
\begin{aligned}
&/p \supset \neg q/ < 0.5 \\
&/p \supset r/ = 1 \\
&/r \supset (q \supset C_{0.5})/ \geq 0.75 \\
&/\neg q/ = 1 - /q/ \\
&/C_{0.5}/ = 0.5
\end{aligned}
$$

If we want to state these inequalities in the language of RPL△, the first few lines should be

$$
\begin{aligned}
&/C_{0.5} \supset (p \supset \neg q)/ < 1 \\
&/p \supset r/ = 1 \\
&/C_{0.75} \supset [r \supset (q \supset C_{0.5})]/ = 1 \\
&/\neg q/ = 1 - /q/ \\
&/C_{0.5}/ = 0.5 \\
&/C_{0.75}/ = 0.75
\end{aligned}
$$

Step 2. The following rules may be applied at any point in the derivation, and come in three categories.

(a) Decomposition rules. These can be applied at any time to any formulas of the forms specified. Names are given above the rules for future reference.

∨GE

$/\phi \vee \psi/ \geq \chi$

↙ ↘

$/\phi/ \geq \chi$ $\qquad$ $/\psi/ \geq \chi$

∨LE

$/\phi \vee \psi/ \leq \chi$

↓

$/\phi/ \leq \chi$

$/\psi/ \leq \chi$

$\vee$SG

$/\phi \vee \psi/ > \chi$

$\swarrow \quad \searrow$

$/\phi/ > \chi \qquad /\psi/ > \chi$

$\vee$SL

$/\phi \vee \psi/ < \chi$

$\downarrow$

$/\phi/ < \chi$

$/\psi/ < \chi$

$\wedge$GE

$/\phi \wedge \psi/ \geq \chi$

$\downarrow$

$/\phi/ \geq \chi$

$/\psi/ \geq \chi$

$\wedge$LE

$/\phi \wedge \psi/ \leq \chi$

$\swarrow \quad \searrow$

$/\phi/ \leq \chi \qquad /\psi/ \leq \chi$

$\wedge$SG

$/\phi \wedge \psi/ > \chi$

$\downarrow$

$/\phi/ > \chi$

$/\psi/ > \chi$

$\wedge$SL

$/\phi \wedge \psi/ < \chi$

$\swarrow \quad \searrow$

$/\phi/ < \chi \qquad /\psi/ < \chi$

$\supset$GE

$/\phi \supset \psi/ \geq \chi$

$\downarrow$

$/\phi/ \leq /\psi/ - \chi + 1$

$\supset$LE

$/\phi \supset \psi/ \leq \chi$

$\swarrow \quad \searrow$

$/\phi/ \geq /\psi/ - \chi + 1 \qquad /\phi/ \leq /\psi/$

$\qquad\qquad\qquad\qquad\qquad \chi \geq 1$

$\supset$SG

$/\phi \supset \psi/ > \chi$

$\downarrow$

$/\phi/ < /\psi/ - \chi + 1$

$\supset$SL

$/\phi \supset \psi/ < \chi$

$\swarrow \quad \searrow$

$/\phi/ > /\psi/ - \chi + 1 \qquad /\phi/ \leq /\psi/$

$\qquad\qquad\qquad\qquad\qquad \chi > 1$

$\equiv$GE

$/\phi \equiv \psi/ \geq \chi$

$\downarrow$

$/\phi/ \leq /\psi/ - \chi + 1$

$/\psi/ \leq /\phi/ - \chi + 1$

$\equiv$LE

$/\phi \equiv \psi/ \leq \chi$

$\swarrow \quad \searrow$

$/\phi/ \geq /\psi/ - \chi + 1 \qquad /\psi/ \geq /\phi/ - \chi + 1$

$\equiv$SG

$$/\phi \equiv \psi/ > \chi$$

$$\downarrow$$

$$/\phi/ < /\psi/ - \chi + 1$$
$$/\psi/ < /\phi/ - \chi + 1$$

$\equiv$SL

$$/\phi \equiv \psi/ < \chi$$

$$\swarrow \qquad \searrow$$

$$/\phi/ > /\psi/ - \chi + 1 \qquad /\psi/ > /\phi/ - \chi + 1$$

$\veebar$GE

$$/\phi \veebar \psi/ \geq \chi$$

$$\downarrow$$

$$/\phi/ + /\psi/ \geq \chi$$

$\veebar$LE

$$/\phi \veebar \psi/ \leq \chi$$

$$\swarrow \qquad \searrow$$

$$/\phi/ + /\psi/ \leq \chi \qquad \begin{array}{l} /\phi/ + /\psi/ \geq 1 \\ \chi \geq 1 \end{array}$$

$\veebar$SG

$$/\phi \veebar \psi/ > \chi$$

$$\downarrow$$

$$/\phi/ + /\psi/ > \chi$$

$\veebar$SL

$$/\phi \veebar \psi/ < \chi$$

$$\swarrow \qquad \searrow$$

$$/\phi/ + /\psi/ < \chi \qquad \begin{array}{l} /\phi/ + /\psi/ \geq 1 \\ \chi > 1 \end{array}$$

$\barwedge$GE

$$/\phi \barwedge \psi/ \geq \chi$$

$$\swarrow \qquad \searrow$$

$$/\phi/ + /\psi/ \geq \chi + 1 \qquad \begin{array}{l} /\phi/ + /\psi/ \leq 1 \\ \chi \leq 0 \end{array}$$

$\barwedge$LE

$$/\phi \barwedge \psi/ \leq \chi$$

$$\downarrow$$

$$/\phi/ + /\psi/ \leq \chi + 1$$

$\barwedge$SG

$$/\phi \barwedge \psi/ > \chi$$

$$\swarrow \qquad \searrow$$

$$/\phi/ + /\psi/ > \chi + 1 \qquad \begin{array}{l} /\phi/ + /\psi/ \leq 1 \\ \chi < 0 \end{array}$$

$\barwedge$SL

$$/\phi \barwedge \psi/ < \chi$$

$$\downarrow$$

$$/\phi/ + /\psi/ < \chi + 1$$

$\rightarrow$SG

$$/\phi \rightarrow \psi/ > 0$$

$$\downarrow$$

$$/\phi/ \leq /\psi/$$

$\rightarrow$SL

$$/\phi \rightarrow \psi/ < 1$$

$$\downarrow$$

$$/\phi/ > /\psi/$$

↔SG

$/\phi \leftrightarrow \psi/ > 0$

↓

$/\phi/ = /\psi/$

↔SL

$/\phi \leftrightarrow \psi/ < 1$

↙ ↘

$/\phi/ > /\psi/$ $\qquad$ $/\phi/ < /\psi/$

JSG

$/J_i\phi/ > 0$

↓

$/\phi/ = i$

JSL

$/J_i\phi/ < 1$

↙ ↘

$/\phi/ > i$ $\qquad$ $/\phi/ < i$

Note that 'χ' can be a number, the truth value of a formula, some combination of these, or so forth. For instance, the following is a proper use of the rule ⊃LE:

$$/p \supset q/ \leq /p \rightarrow q/$$

↙ ↘

$/p/ \geq /q/ - /p \rightarrow q/ + 1$ $\qquad$ $/p/ \leq /p \rightarrow q/$, $/p \rightarrow q/ \geq 1$

(b) Inequality rule: Any inequality that is, in elementary algebra, equivalent to or deducible from those entered in the derivation, may be entered as any line in the tableau. E.g.,

1.	$/p/ > /q/ - /p \vee q/ + 1$	
2.	$/p \vee q/ = 1$	
3.	$/p \vee q/ > /q/ - /p/ + 1$	(from [1])
4.	$/q/ - /p/ + 1 < 1$	(from [2] and [3])
5.	$/q/ < /p/$	(from [5])

We need add only one rule to match reasoning using inequalities to the tableau format:

$$/\phi/ \neq \chi$$

↙ ↘

$/\phi/ > \chi$ $\qquad$ $/\phi/ < \chi$

Both sorts of inequality rules will be referred to as "A" (for "algebra").

(c) Boundary rules. These rules allow one to make use of the fact that $0 \leq /\phi/ \leq 1$. For any formula ϕ of RPL△, a line of the form

$$/\phi/ \leq 1$$

or

$$/\phi/ \geq 0$$

may be entered at any point in the tableau. This rule will be called "B".

Step 3. A branch of the tableau terminates when it contains a line or lines asserting an inequation that is not possible on any valuation. To be more specific, a branch terminates when it contains either (a) a line or lines indicating a mathematically impossible inequation (e.g. $/p/ < /p/$, $1 > 1$), (b) a line asserting that the truth value of a formula of RPL△ is greater than 1 or less than 0, or (c) lines asserting that the value of some formula occupies both of two disjoint intervals (e.g., $/p/ > /q/$, $/p/ \leq /q/$). (Strictly speaking, (b) is reducible to (c) by rule B, and (c) is reducible to (a) by rule A, but both are worth mentioning separately.) The derivation is complete when every branch has terminated; then, the argument has been shown to be valid, the formula a theorem, or more generally, the restrictions on /A/ have been shown to hold.

In ordinary tableaux, a branch of a derivation also terminates when no more rules can be applied to it. I have not, however, given an algorithm for determining when rule A (any inequality following from previous lines by elementary algebra is admissible) no longer applies. Until such an algorithm is added to the procedure (an important area for future research on the system), a branch can only be said to terminate when it asserts something that cannot possibly be true of the truth values of the formulas referred to on that branch. (This will be discussed further in V.5.)

To make writing the tableaux less tedious, we will adopt the convention that one may drop the slashes around formulae. As nothing in the tableaux hinges on distinguishing formulae from their truth values, this convention will not lead to any confusion. The tableaux given from here on will follow this convention, lest I wear out my '/' key.

2. Using the tableau rules

Here, I will give some indications of how the different rules can be combined when doing tableaux. The reader will certainly have noticed that there are no decomposition rules for negation. This is because the rules N and A can be combined to mimic decomposition rules, and a single rule governing negation is easier to remember than four. For instance, from $\neg p \geq ð$ we reason:

$\neg p \geq ð$	
$\neg p = 1 - p$	[N]
$1 - p \geq ð$	[A]
$p \leq 1 - ð$	[A]

There are also no rules governing equalities in the basic system. Equalities can be treated as follows:

$\phi = \chi$	
$\phi \leq \chi$	[A]
$\phi \geq \chi$	[A]

and then the various GE and LE rules can be applied. This is really too tedious to be practical; hence, derived rules governing equalities will be given below and shown to be abbreviations of uses of the A, GE, and LE rules in the manner just shown. The rules governing equalities are not presented as basic rules in order to make proving the soundness of the procedure simpler.

The rules for $\rightarrow$, $\leftrightarrow$, and the J-operators are all based on the same basic principle. If, for instance, $/\phi \rightarrow \psi/ < 1$, then since $\rightarrow$ is a bivalent connective, $/\phi \rightarrow \psi/ = 0$, and thus we can infer that $/\phi/ > /\psi/$. The other rules are justified similarly. Hence, GE and LE rules are not of much use for these connectives; given, for instance, that $/\phi \rightarrow \psi/ \geq \chi$, we cannot infer anything about $/\phi \rightarrow \psi/$ unless we additionally know that $\chi > 0$. In lieu of GE and LE rules, we can use reasoning along these lines:

$$\begin{array}{ll} J_1 p \leq q & \\ q < 1 & \\ J_1 p < 1 & \text{[A]} \end{array}$$

Finally, rule B allows one to enter into the tableaux the information that the truth values of formulas are bounded. We can then apply rule A to make use of this information; e.g.,

$$\begin{array}{ll} p > q & \\ q \geq 0 & \text{[B]} \\ p > 0 & \text{[A]} \end{array}$$

The remainder of the tableau rules should be (relatively speaking) self-explanatory.

Because of the large number of tableau rules, it is easy to get caught up in complicated tangents that turn out not to lead to closed branches. To avoid this, it is best to apply just the decomposition rules, using rule A only to simplify expressions, until all complex formulas have been decomposed into propositional variables. Then one applies the boundary rules and more complex algebra to close any remaining branches. One can, of course, frequently find shorter tableaux by applying rules A and B before all complex formulas have been decomposed, but as a general rule it is best not to do so unless it is obvious that this will lead directly to the closure of a branch.

3. Derived rules

These decomposition rules can be proven from the rules above, which will be done for some of them in IV.5. They are thus not strictly necessary for the system, but provide useful shortcuts when making tableaux. Tableaux given as examples will use these rules, and will refer to them by DR plus the number of the rule.

1.

$$\begin{array}{ccc} & \phi \vee \psi = \chi & \\ \swarrow & & \searrow \\ \phi = \chi & & \psi = \chi \\ \psi \leq \chi & & \phi \leq \chi \end{array}$$

2.

$$\begin{array}{ccc} & \phi \vee \psi = 1 & \\ \swarrow & & \searrow \\ \phi = 1 & & \psi = 1 \end{array}$$

3.

$$\begin{array}{ccc} & \phi \wedge \psi = \chi & \\ \swarrow & & \searrow \\ \phi = \chi & & \psi = \chi \\ \psi \geq \chi & & \phi \geq \chi \end{array}$$

4.

$$\begin{array}{c} \phi \wedge \psi = 1 \\ \downarrow \\ \phi = 1 \\ \psi = 1 \end{array}$$

5.

$$\begin{array}{ccc} & \phi \supset \psi = \chi & \\ \swarrow & & \searrow \\ \phi = \psi - \chi + 1 & & \phi \leq \psi \\ & & \chi = 1 \end{array}$$

6.

$$\begin{array}{c} \phi \supset \psi = 1 \\ \downarrow \\ \phi \leq \psi \end{array}$$

7.

$$\begin{array}{ccc} & \phi \equiv \psi = \chi & \\ \swarrow & & \searrow \\ \phi = \psi - \chi + 1 & & \psi = \phi - \chi + 1 \end{array}$$

8.

$$\begin{array}{c} \phi \equiv \psi = 1 \\ \downarrow \\ \phi = \psi \end{array}$$

9.

$$\phi \veebar \psi = \chi$$
$$\swarrow \qquad \searrow$$
$$\phi + \psi = \chi \qquad \begin{array}{l} \phi + \psi \geq 1 \\ \chi = 1 \end{array}$$

10.

$$\phi \veebar \psi = 1$$
$$\downarrow$$
$$\phi + \psi \geq 1$$

11.

$$\phi \barwedge \psi = \chi$$
$$\swarrow \qquad \searrow$$
$$\phi + \psi = \chi + 1 \qquad \begin{array}{l} \phi + \psi \leq 1 \\ \chi = 0 \end{array}$$

12.

$$\phi \barwedge \psi = 1$$
$$\downarrow$$
$$\begin{array}{l} \phi = 1 \\ \psi = 1 \end{array}$$

13.

$$\phi \supset \psi < 1$$
$$\downarrow$$
$$\phi > \psi$$

14.

$$\phi \veebar \psi < 1$$
$$\downarrow$$
$$\phi + \psi < 1$$

15.

$$C_i \supset \phi = 1$$
$$\downarrow$$
$$\phi \geq i$$

16.

$$C_i \supset \phi < 1$$
$$\downarrow$$
$$\phi < i$$

4. Examples

I will now give a few examples of the tableaux in action. In the tableaux given in the rest of this chapter, I will adopt the practice of repeating lines of the tableaux, so that new lines will always follow from applications of the rules to the lines immediately preceding them. I will also introduce lines in accordance with the N and C rules immediately before their first use, rather than at the beginning of the tableau, as is strictly correct. This should make the tableaux easier for the reader to follow and avoid the need for extensive commentary.

Proof that /p ∨ ¬p/ ≥ ½:

$p \vee \neg p < \frac{1}{2}$	[hyp.]
$p < \frac{1}{2}$	[∨SL]
$\neg p < \frac{1}{2}$	[∨SL]
$\neg p = 1 - p$	[N]
$1 - p < \frac{1}{2}$	[A]
$p > \frac{1}{2}$	[A]
$p < \frac{1}{2}$	[repetition]
×	

Proof that (¬q ⊃ ¬p) ⊃ (p ⊃ q) is a tautology:

$(\neg q \supset \neg p) \supset (p \supset q) < 1$	[hyp.]
$\neg q \supset \neg p > p \supset q$	[DR13]
$\neg q < \neg p - (p \supset q) + 1$	[⊃SG]
$\neg p = 1 - p$	[N]
$\neg q < 2 - p - (p \supset q)$	[A]
$\neg q = 1 - q$	[N]
$1 - q < 2 - p - (p \supset q)$	[A]
$p \supset q < q - p + 1$	[A]

↙ ↘

[⊃SL]	$p > q - (q - p + 1) + 1$
[A]	$p > p$
	×

$p \leq q$	[⊃SL]
$q - p + 1 > 1$	[⊃SL]
$\neg q \supset \neg p > p \supset q$	[repetition]
$p \supset q < 1$	[B]
$p > q$	[DR13]
$p \leq q$	[repetition]
×	

5. The validity of the derived rules

I will now show the validity of five of the derived rules presented in IV.3, by showing that a use of any of them can be replaced by a sequence using only the basic rules of the system. Proofs for the other derived rules are sufficiently similar to the ones given that the reader should be able to easily satisfy herself that those rules are also valid.

(DR1)

$\phi \vee \psi = \chi$	[prem.]
$\phi \vee \psi \geq \chi$	[A]
$\phi \vee \psi \leq \chi$	[A]
$\phi \leq \chi$	[$\vee$LE]
$\psi \leq \chi$	[$\vee$LE]
$\phi \vee \psi \geq \chi$	[repetition]

↙ ↘

[$\vee$GE]	$\phi \geq \chi$	$\psi \geq \chi$	[$\vee$GE]
[repetition]	$\phi \leq \chi$	$\psi \leq \chi$	[repetition]
[A]	$\phi = \chi$	$\psi = \chi$	[A]
[repetition]	$\psi \leq \chi$	$\phi \leq \chi$	[repetition]

(DR5)

$\phi \supset \psi = \chi$	[prem.]
$\phi \supset \psi \leq \chi$	[A]
$\phi \supset \psi \geq \chi$	[A]
$\phi \leq \psi - \chi + 1$	[$\supset$GE]
$\phi \supset \psi \leq \chi$	[repetition]

↙ ↘

[$\supset$LE]	$\phi \geq \psi - \chi + 1$	$\phi \leq \psi$	[$\supset$LE]
[repetition]	$\phi \leq \psi - \chi + 1$	$\chi = 1$	[$\supset$LE]
[A]	$\phi = \psi - \chi + 1$		

(DR6)

$\phi \supset \psi = 1$	[prem.]
$\phi \supset \psi \geq 1$	[A]
$\phi \leq \psi + 1 - 1$	[$\supset$GE]
$\phi \leq \psi$	[A]

(DR16)

$C_i \supset \phi < 1$	[prem.]

↙ ↘

[$\supset$SL]	$C_i > \phi - 1 + 1$	$C_i \leq \phi$	[$\supset$SL]
[A]	$C_i > \phi$	$1 > 1$	[$\supset$SL]
[C]	$C_i = i$	$\times$	
[A]	$\phi < i$		

(DR9)

$$
\begin{array}{ll}
\phi \veebar \psi = \chi & \text{[prem.]} \\
\phi \veebar \psi \leq \chi & \text{[A]} \\
\phi \veebar \psi \geq \chi & \text{[A]} \\
\phi + \psi \geq \chi & [\veebar\text{GE}] \\
\phi \veebar \psi \leq \chi & \text{[repetition]}
\end{array}
$$

$$
\swarrow \qquad \searrow
$$

$$
\begin{array}{llll}
[\veebar\text{LE}] & \phi + \psi \leq \chi & \phi + \psi \geq 1 & [\veebar\text{LE}] \\
\text{[repetition]} & \phi + \psi \geq \chi & \chi = 1 & [\veebar\text{LE}] \\
\text{[A]} & \phi + \psi = \chi & &
\end{array}
$$

6. Proof of soundness of KTIL

As mentioned in III.2, Entemann (2000) proved that proof by contradiction in KTIL was sound for VSS. He did not prove the soundness of the truth interval refinement method, in which one narrows down the possible truth values of a set of formulas instead of trying to prove them inconsistent. Kenevan et al. (1992) proved that the rules were "correct", but did not give a full proof of the soundness of the system. Since, as will be proved in IV.7, RPL△-tableaux are sound for RPL△, and VSS is a fragment of RPL△, showing the validity of KTIL in RPL△ demonstrates its soundness for VSS. I will show that the KTIL disjunction rule is equivalent to certain uses of the rules of RPL△-tableaux; similar tableau sequences indicate that the same holds for the conjunction rule. The soundness of the KTIL rules for intersection and negation is obvious.

The disjunction decomposition rule for KTIL is:

$$
\begin{array}{c}
A: [a_0,a_1] \\
B: [b_0,b_1] \\
A \vee B: [p_0,p_1] \\
\swarrow \qquad \searrow
\end{array}
$$

$$
\begin{array}{ll}
A: [a_0, \min(a_1, p_1)] & A: [\max(a_0, p_0), \min(a_1, p_1)] \\
B: [\max(b_0, p_0), \min(b_1, p_1)] & B: [b_0, \min(b_1, p_1)] \\
A \vee B: [\max(a_0,b_0,p_0), \min(p_1, \max[a_1,b_1])] & A \vee B: [\max(a_0,b_0,p_0), \min(p_1, \max[a_1,b_1])]
\end{array}
$$

The validity of the refinements of the intervals of A and B in the disjunction rule is shown by:

$$
\begin{array}{llll}
 & A \vee B \geq p_0 & & \text{[prem.]} \\
 & A \vee B \leq p_1 & & \text{[prem.]} \\
 & A \leq p_1 & & [\vee\text{LE}] \\
 & B \leq p_1 & & [\vee\text{LE}] \\
 & \swarrow & \searrow & \\
[\vee\text{GE}] & A \geq p_0 & B \geq p_0 & [\vee\text{GE}]
\end{array}
$$

If we additionally know that /A/ $\leq a_1$ and /B/ $\leq b_1$, then since /A/ $\leq p_1$ and /B/ $\leq p_1$ we know that /A/ $\leq \min(a_1, p_1)$ and /B/ $\leq \min(b_1, p_1)$; similarly, we know that either /A/ $\geq \max(a_0, p_0)$ or /B/ $\geq \max(b_0, p_0)$.

The next tableau shows that if /A/ $\geq a_0$, /A $\vee$ B/ $\geq a_0$:

$$
\begin{array}{ll}
A \geq a_0 & \text{[prem.]} \\
A \vee B < a_0 & \text{[hyp.]} \\
A < a_0 & [\vee\text{SL}] \\
\times &
\end{array}
$$

The proof that if /B/ $\geq b_0$, /A $\vee$ B/ $\geq b_0$ is the same, *mutatis mutandis*. As before, if we know /A $\vee$ B/ $\geq a_0$, /A $\vee$ B/ $\geq b_0$, and /A $\vee$ B/ $\geq p_0$, we know /A $\vee$ B/ $\geq \max(a_0, b_0, p_0)$.

Let $\max(a_1, b_1) = a_1$:

$$
\begin{array}{llll}
 & A \leq a_1 & & \text{[prem.]} \\
 & B \leq b_1 & & \text{[prem.]} \\
 & a_1 \geq b_1 & & \text{[prem.]} \\
 & A \vee B > a_1 & & \text{[hyp.]} \\
 & \swarrow & \searrow & \\
[\vee\text{SG}] & A > a_1 & B > a_1 & [\vee\text{SG}] \\
 & \times & B > b_1 & \text{[A]} \\
 & & \times &
\end{array}
$$

The tableau is the same, *mutatis mutandis*, for max $(a_1,b_1) = b_1$. Hence, the truth interval refinement method of proof in KTIL is sound if RPL△-tableaux are, which is about to be proven.

7. Proof of soundness of RPL△-tableaux

Some definitions will be necessary to prove the soundness of the procedure. A valuation v of propositional variables of RPL△ is *faithful* to a branch b of a derivation iff any assertion about the truth values of a formula made by any line of b states a truth about the value v assigns to that formula. Equivalently, v is faithful to b iff every assertion on b about the truth values of formulas is consistent with the values assigned to propositional variables by v.

Let $\Sigma \vdash A$ indicate that there is a tableau showing that $/A/ = 1$ can be deduced from the assumption that for every $\phi \in \Sigma$, $/\phi/ = 1$.

Soundness lemma. Suppose a valuation v is faithful to a branch b of a derivation. If any tableau rule is applied to any line or lines of b, v will be faithful to at least one of the branches generated.

Unfortunately, the lemma must be proven separately for each rule.

A. Since this rule allows only mathematically valid inferences, any result of applying it will be entailed by or equivalent to previous lines in b, and thus v will be faithful to the new lines in the branch as well.

Recall that A also includes the rule:

$$/\phi/ \neq \chi$$

$$\swarrow \qquad \searrow$$

$$/\phi/ > \chi \qquad\qquad /\phi/ < \chi$$

Suppose $/\phi/ \neq \chi$; then either $/\phi/ > \chi$ or $/\phi/ < \chi$, and so v will be faithful to one of the new branches formed by applying this part of the rule.

B. Since any valuation can only assign formulas truth values in [0,1], where ϕ is a formula of RPL$\triangle$, a line of the form $/\phi/ \leq 1$ or $/\phi/ \geq 0$ states a truth about $/\phi/$ on every valuation.

C. Follows immediately from the definition of $/C_i/$.

N. Follows immediately from the definition of $/\neg/$.

$\vee$SG. If $/\phi \vee \psi/ > \chi$, then max $(/\phi/, /\psi/) > \chi$. If $/\phi/ \geq /\psi/$, then $/\phi/ > \chi$; if $/\psi/ \geq \chi$, then $/\psi/ > \chi$.

$\vee$GE. Analogous to the proof for $\vee$SG; just replace '>' with '≥'.

$\vee$SL. If $/\phi \vee \psi/ < \chi$, then max $(/\phi/, /\psi/) < \chi$. But if the larger of $/\phi/$ and $/\psi/$ is less than χ, $/\phi/ < \chi$ and $/\psi/ < \chi$. (The same, *mutatis mutandis*, for $\vee$LE.)

$\wedge$SG. If $/\phi \wedge \psi/ > \chi$, then min $(/\phi/, /\psi/) > \chi$. So both $/\phi/ > \chi$ and $/\psi/ > \chi$. (The same, *mutatis mutandis*, for $\wedge$GE.)

$\wedge$SL. If $/\phi \wedge \psi/ < \chi$, then min $(/\phi/, /\psi/) < \chi$. Suppose $/\phi/ \leq /\psi/$. Then min $(/\phi/, /\psi/) = /\phi/$, so $/\phi/ < \chi$. Similarly for $/\phi/ \geq /\psi/$. (The same, *mutatis mutandis*, for $\wedge$LE.)

⊃SG. Suppose $/\phi/ \leq /\psi/$. $\chi < 1$, since otherwise $/\phi \supset \psi/ > 1$, which is impossible. So $1 - \chi > 0$; $/\phi/ < /\psi/ - \chi + 1$ immediately follows. Recall that implication can be defined as

$$/\phi \supset \psi/ = 1 \text{ if } /\phi/ < /\psi/,$$
$$1 - /\phi/ + /\psi/ \text{ otherwise.}$$

So suppose $/\phi/ > /\psi/$; then $/\phi \supset \psi/ = 1 - /\phi/ + /\psi/$. So $1 - /\phi/ + /\psi/ > \chi$, which is equivalent to $/\phi/ < /\psi/ - \chi + 1$.

⊃GE. If $/\phi/ \leq /\psi/$, then if $\chi \leq 1$, $1 - \chi \geq 0$, so $/\phi/ \leq /\psi/ - \chi + 1$. If $\chi > 1$, then $/\phi \supset \psi/ > 1$, which is impossible. If $/\phi/ > /\psi/$, then $/\phi \supset \psi/ = 1 - /\phi/ + /\psi/$ and $1 - /\phi/ + /\psi/ \geq \chi$, which is equivalent to $/\phi/ \leq /\psi/ - \chi + 1$.

⊃SL. Suppose $/\phi/ > /\psi/$; then $/\phi \supset \psi/ = 1 - /\phi/ + /\psi/$. So $1 - /\phi/ + /\psi/ < \chi$, which is equivalent to $/\phi/ > /\psi/ - /\rho/ + 1$. Suppose $/\phi/ \leq /\psi/$; then by the above definition $/\phi \supset \psi/ = 1$, so $\chi > 1$.

⊃LE. Suppose $/\phi/ \leq /\psi/$. Then by the above definition $/\phi \supset \psi/ = 1$, so $\chi \geq 1$. Suppose $/\phi/ > /\psi/$. Then $/\phi \supset \psi/ = 1 - /\phi/ + /\psi/$ and $1 - /\phi/ + /\psi/ \leq \chi$, so $/\phi/ \geq /\psi/ - \chi + 1$.

≡SG. If $/\phi \equiv \psi/ > \chi$, $1 - |/\phi/ - /\psi/| > \chi$, and $|/\phi/ - /\psi/| < 1 - \chi$. Suppose $/\phi/ > /\psi/$. Then $/\phi/ - /\psi/ < 1 - \chi$, which is equivalent to $/\phi/ < /\psi/ - \chi + 1$. $\chi < 1$, or else $/\phi \equiv \psi/ > 1$, so $1 - \chi > 0$. Thus, since $/\psi/ \leq /\phi/$, $/\psi/ < /\phi/ - \chi + 1$. Similarly for $/\phi/ < /\psi/$. Suppose $/\phi/ = /\psi/$. Since $1 - \chi > 0$, both $/\phi/ < /\psi/ - \chi + 1$ and $/\psi/ < /\phi/ - \chi + 1$ immediately follow.

$\equiv$GE. If $/\phi \equiv \psi/ \geq \chi$, as above, $|/\phi/ - /\psi/| \leq 1 - \chi$. Suppose $/\phi/ \geq /\psi/$. Then $/\phi/ - /\psi/ \leq 1 - \chi$, which is equivalent to $/\phi/ \leq /\psi/ - \chi + 1$. Since $/\psi/ \leq /\phi/$ and $\chi \leq 1$ (or else $/\phi \equiv \psi/ > 1$), $/\psi/ \leq /\phi/ - \chi + 1$. Analogous for $/\phi/ \leq /\psi/$.

$\equiv$SL. If $/\phi \equiv \psi/ < \chi$, $|/\phi/ - /\psi/| > 1 - \chi$. Suppose $/\phi/ \geq /\psi/$. Then $/\phi/ - /\psi/ > 1 - \chi$, which is equivalent to $/\phi/ > /\psi/ - \chi + 1$. Analogous for $/\phi/ \leq /\psi/$.

$\equiv$LE. If $/\phi \equiv \psi/ \leq \chi$, $|/\phi/ - /\psi/| \geq 1 - \chi$. Suppose $/\phi/ \geq /\psi/$. Then $/\phi/ - /\psi/ \geq 1 - \chi$, which is equivalent to $/\phi/ \geq /\psi/ - \chi + 1$. Analogous for $/\phi/ \leq /\psi/$.

$\underline{\vee}$GE. If $/\phi \underline{\vee} \psi/ \geq \chi$, $\min(1, /\phi/ + /\psi/) \geq \chi$. If $/\phi/ + /\psi/ > 1$, since $\chi \leq 1$ (or else $/\phi \underline{\vee} \psi/ > 1$), $/\phi/ + /\psi/ > \chi$, so $/\phi/ + /\psi/ \geq \chi$. If $/\phi/ + /\psi/ \leq 1$, $\min(1, /\phi/ + /\psi/) = /\phi/ + /\psi/$, so $/\phi/ + /\psi/ \geq \chi$. (The same, *mutatis mutandis*, for $\underline{\vee}$SG.)

$\underline{\vee}$LE. Suppose $/\phi/ + /\psi/ \geq 1$. Then $\min(1, /\phi/ + /\psi/) = 1$, so $\chi \geq 1$. Suppose $/\phi/ + /\psi/ < 1$. Then $\min(1, /\phi/ + /\psi/) = /\phi/ + /\psi/$, so $/\phi/ + /\psi/ \leq \chi$.

$\underline{\vee}$SL. If $/\phi/ + /\psi/ < 1$, then $\min(1, /\phi/ + /\psi/) = /\phi/ + /\psi/$, and $/\phi/ + /\psi/ < \chi$. If $/\phi/ + /\psi/ \geq 1$, then $\min(1, /\phi/ + /\psi/) = 1$, so $\chi > 1$.

$\overline{\wedge}$GE. If $/\phi \overline{\wedge} \psi/ \geq \chi$, $\max(0, /\phi/ + /\psi/ - 1) \geq \chi$. Suppose $/\phi/ + /\psi/ \leq 1$. Then $\max(0, /\phi/ + /\psi/ - 1) = 0$, so $\chi \leq 0$. Suppose $/\phi/ + /\psi/ > 1$. Then $\max(0, /\phi/ + /\psi/ - 1) = /\phi/ + /\psi/ - 1$. Therefore, $/\phi/ + /\psi/ - 1 \geq \chi$, and $/\phi/ + /\psi/ \geq \chi + 1$.

$\overline{\wedge}$SG. If $/\phi \overline{\wedge} \psi/ > \chi$, $\max(0, /\phi/ + /\psi/ - 1) > \chi$. Suppose $/\phi/ + /\psi/ > 1$. Then $\max(0, /\phi/ + /\psi/ - 1) = /\phi/ + /\psi/ - 1$, $/\phi/ + /\psi/ - 1 > \chi$, and $/\phi/ + /\psi/ > \chi + 1$. Suppose $/\phi/ + /\psi/ \leq 1$. Then $\max(0, /\phi/ + /\psi/ - 1) = 0$, so $\chi < 0$.

$\barwedge$LE. If $/\phi \barwedge \psi/ \leq \chi$, max $(0, /\phi/ + /\psi/ - 1) \leq \chi$. Suppose $/\phi/ + /\psi/ \leq 1$. $\chi \geq 0$, or else $/\phi \barwedge \psi/ < 0$, which is impossible; so $/\phi/ + /\psi/ \leq \chi + 1$. Suppose $/\phi/ + /\psi/ > 1$. Then max $(0, /\phi/ + /\psi/ - 1) = /\phi/ + /\psi/ - 1$, so $/\phi/ + /\psi/ - 1 \leq \chi$ and $/\phi/ + /\psi/ \leq \chi + 1$. (The same, *mutatis mutandis*, for $\barwedge$SL.)

$\rightarrow$SG. If $/\phi \rightarrow \psi/$ is strictly greater than 0, since '$\rightarrow$' is a bivalent connective, $/\phi \rightarrow \psi/ = 1$. So $/\phi/ \leq /\psi/$.

$\rightarrow$SL. If $/\phi \rightarrow \psi/$ is strictly less than 1, since '$\rightarrow$' is a bivalent connective, $/\phi \rightarrow \psi/ = 0$. So $/\phi/ > /\psi/$.

JSG. Since the J-operators are bivalent, if $/J_i\phi/$ is strictly greater than 0, $/J_i\phi/ = 1$. So $/\phi/ = i$.

JSL. Since the J-operators are bivalent, if $/J_i\phi/$ is strictly less than 1, $/J_i\phi/ = 0$. So $/\phi/ \neq i$; i.e. $/\phi/ < i$ or $/\phi/ > i$.

$\leftrightarrow$SG. Since '$\leftrightarrow$' is a bivalent connective, if $/\phi \leftrightarrow \psi/$ is strictly greater than 0, $/\phi \leftrightarrow \psi/ = 1$. So $/\phi/ = /\psi/$.

$\leftrightarrow$SL. Since '$\leftrightarrow$' is a bivalent connective, if $/\phi \leftrightarrow \psi/$ is strictly less than 1, $/\phi \leftrightarrow \psi/ = 0$. So $/\phi/ \neq /\psi/$; i.e., $/\phi/ > /\psi/$ or $/\phi/ < /\psi/$.

Special soundness theorem.

If $\Sigma \vdash A$, then $\Sigma \vDash A$.

Proof: We prove the contrapositive. Suppose $\Sigma \nvDash A$. Then there is a valuation v such that for all $\phi \in \Sigma$, $/\phi/ = 1$, and $/A/ < 1$. Consider a tableau in which for each

$\phi \in \Sigma$, '$\phi = 1$' has been entered as a line, '$A < 1$' has been entered as a line, and every branch of which has terminated. v is faithful to the initial segment of this derivation. By repeated applications of the Soundness Lemma, v must be faithful to every initial section of some branch b generated from this initial list by repeated applications of the tableau rules. So v is faithful to the entirety of branch b. If b were closed, it would have to contain some impossible inequality. But this cannot be the case, since v is faithful to b, and an impossible inequality cannot state a truth about the values of formulas on v. So there cannot be any such tableau, and $\Sigma \nvDash A$.

Let us introduce a further notational convention in order to prove the soundness of the tableaux when we assume restrictions on the truth values of the premises other than that they are equal to 1, or prove facts about the truth value of the conclusion other than it equals 1. Let $\phi_1, \phi_2, \ldots, \phi_n, \psi$ be formulas and $I_1, I_2, \ldots, I_n$, J be subintervals, or unions of disjoint subintervals, of [0,1]; then

$$/\phi_1/ \in I_1, /\phi_2/ \in I_2, \ldots, /\phi_n/ \in I_n \vDash /\psi/ \in J$$

means that for every valuation on which $/\phi_1/ \in I_1, /\phi_2/ \in I_2, \ldots, /\phi_n/ \in I_n$, $/\psi/ \in J$. Similarly, let

$$/\phi_1/ \in I_1, /\phi_2/ \in I_2, \ldots, /\phi_n/ \in I_n \vdash /\psi/ \in J$$

mean that there is a tableau showing that $/\psi/ \in J$ from the assumptions that $/\phi_1/ \in I_1, /\phi_2/ \in I_2, \ldots, /\phi_n/ \in I_n$. (We cannot, of course, directly state lines like $/\phi/ \in [a,b]$ in the tableaux. This should be interpreted as shorthand for the pair of lines indicating $/\phi/ \geq a$, $/\phi/ \leq b$. Where the lower bound of the interval in which $/\phi/$ is placed is 0 or the upper bound 1, we may not have entered this information explicitly in the tableau, but by rule B, we may do so at any point.) For both forms of notation, if no restrictions on the values of any propositional variables are assumed, we state nothing on the left side of the turnstile.

A few examples should make the idea clearer. As noted in II.8, $/p \vee \neg p/ \geq 0.5$. Using this notation, we state this as

$$\vDash /p \vee \neg p/ \in [0.5,1].$$

Similarly, suppose we have a tableau showing that if $\neg p$ is designated, we can deduce that $p \supset q$ is designated; we state this as

$$/\neg p/ \in [\eth,1] \vdash /p \supset q/ \in [\eth,1].$$

Now we can establish the more general result.

General soundness theorem.

If $/\phi_1/ \in I_1, \ldots, /\phi_n/ \in I_n \vdash /A/ \in J$, then $/\phi_1/ \in I_1, \ldots, /\phi_n/ \in I_n \vDash /A/ \in J$.

Proof: We prove the contrapositive. Suppose $/\phi_1/ \in I_1, \ldots, /\phi_n/ \in I_n \nvDash /A/ \in J$. Then there is a valuation v such that $/\phi_1/ \in I_1, \ldots, /\phi_n/ \in I_n$, and $/A/ \notin J$. Consider a tableau such that for each ϕ_k, the assumption that $/\phi_k/ \in I_k$ has been entered as one or more lines, a line has been entered indicating that $/A/ \notin J$, and every branch of which has terminated. v is faithful to the initial segment of this derivation. By repeated applications of the Soundness Lemma, v must be faithful to every initial section of some branch b generated from this initial list by repeated applications of the tableau rules. So v is faithful to the entirety of branch b. If b were closed, it would have to contain some impossible inequality. But this cannot be the case, since v is faithful to b, and an impossible inequality cannot state a truth about the values of formulas on v. So there cannot be any such tableau, and $/\phi_1/ \in I_1, \ldots, /\phi_n/ \in I_n \nvdash /A/ \in J$.

V

Extensions on algebraic tableaux

The value of the tableau system presented in the last chapter can be easily seen—it provides a sound procedure for finding validities and tautologies of RPL△ that can be used without much difficulty or mathematical knowledge. This chapter looks at a few remaining areas of interest regarding the procedure. I will first point out that since one can refer to the truth values of formulas within the tableaux, one does not strictly need the ability to refer to truth values in the formal language itself. Hence, instead of working in RPL△, one can use the more intuitive, simpler language of $Ł_{\aleph}$. Then, I will present three extensions to the tableau rules. One set of rules augments the procedure to cover the other systems in the Łukasiewicz family (including $Ł_2$, classical propositional logic). The second covers another important infinite-valued system, product logic (discussed in II.4), and the third covers Zadeh's accounts of linguistic hedges. Finally, I will discuss some problems for future research, the most important of which are automating the tableau procedure and proving its completeness.

1. $Ł_{\aleph}$-tableaux

As I have previously noted, not much work has been done on decision procedures for propositional fuzzy logic. The only completely formalized procedure covering RPL△, discussed in III.1, is too complicated for use in practice. Several reasonably simple procedures have been developed for finding theorems of $Ł_{\aleph}$. But as we saw in II.5, $Ł_{\aleph}$ is too weak to model vagueness and approximate reasoning. If our language cannot talk about intermediate truth values, "the apparent many-valuedness is only illusory, since we cannot *say* anything in a many-valued way" (Morgan & Pelletier 1997: 86, emphasis in original).

The lack of a suitable decision procedure for RPL$\triangle$ is, therefore, a significant absence, and the tableau system I present here should represent a significant contribution to the study, application, and dissemination of fuzzy logic. The procedure is certainly much more complex than most tableau systems, although more intuitive than the constraint tableaux discussed in III.3. This is because of the complexity of the logic itself—despite appearances, the tableaux greatly simplify working with infinite-valued logic. I will briefly discuss one aspect of this simplification.

In this section and V.2, we will consider just the rules of the tableau procedure for the language of $Ł_{\aleph}$; that is, we ignore the rules for constants, J-operators, $\rightarrow$, and $\leftrightarrow$ and consider only A, B, N, and the rules for $\vee$, $\wedge$, $\supset$, and $\equiv$. Algebraic tableaux using only these rules will be called $Ł_{\aleph}$-tableaux. As we saw in chapter II, $Ł_{\aleph}$ is the basic logic in which we are interested; we added constants and the determinacy operator only in order to allow us to refer to the truth values in the language itself. RPL$\triangle$ is, however, less than perfectly perspicuous; it is fairly counterintuitive that $\phi \supset \psi$ means "ϕ implies ψ", but $C_{\delta} \supset \psi$ means "ψ is designated".

A more intuitively appealing way of talking about the truth values of formulas is to do so in $Ł_{\aleph}$-tableaux, rather than in the formal language itself. Consider, for instance, this commonsense argument that cannot be represented in classical logic:

Anyone who can do logic is interesting.
Wilfred is kind of boring.

Therefore, Wilfred isn't very good at logic.

(Such arguments are of significant interest for fuzzy theorists; see e.g. Gaines 1976.) One way to formalize this argument in RPL$\triangle$[23] is as follows (assuming we have some principled way of assigning these particular truth values):

P1. $C_{0.9} \supset$ (Wilfred can do logic $\supset$ Wilfred is interesting)

P2. $C_{0.5} \supset \neg$(Wilfred is interesting)

C. Wilfred can do logic $\supset C_{0.6}$

These lines assert that

P1. /Wilfred can do logic $\supset$ Wilfred is interesting/ ≥ 0.9

P2. /$\neg$(Wilfred is interesting)/ ≥ 0.5

C. /Wilfred can do logic/ ≤ 0.6

But we can enter *these* assertions directly into $Ł_{\aleph}$-tableaux; we do not need to translate them into the language of RPL$\triangle$. That we can do so is what the general soundness theorem of IV.7 proves. That theorem states that however we place restrictions on the truth values of the premises, any restrictions on the truth value of the conclusion that we deduce in a tableau will obtain, for any valuation on which the truth values of the premises are in the intervals we specified. In this example, we can make a (very simple) tableau to show that on any valuation where $/p \supset q/ \geq 0.9$ and $/\neg q/ \geq 0.5$, $/p/ \leq 0.6$. The tableaux are sound whether we state such restrictions on the truth values of the premises in RPL$\triangle$ or directly as lines in a tableau. Most people will probably find working with $Ł_{\aleph}$-tableaux

[23] Instead of taking "kind of" and "not very" as placing lower and upper bounds on the truth values of propositions, we could also formalize these locutions as hedges, a method arguably more faithful to the original language. Hedges will be briefly discussed in V.4.

easier than working with RPL△; hence, the tableau procedure should facilitate working with the approximately valid arguments we have heretofore needed RPL△ to formalize.

2. Tableaux for the Łukasiewicz family

As noted in II.2, the tautologies of $Ł_{\aleph}$ are the intersection of the tautologies of all the $Ł_n$ in the Łukasiewicz family. Hence, the tableau rules given in the last chapter are sound for all members of the Łukasiewicz family (including $Ł_2$, which is classical propositional logic).

We can get a stronger tableau procedure for members of the Łukasiewicz family by adding two new rules to reflect the fact that any $Ł_n$ has only finitely many truth values. Where *n* is the number of truth values in the logic and *k* is an integer less than *n*,

$$/\phi/ < {}^{k}/_{n-1} \quad\quad\quad\quad /\phi/ > {}^{k}/_{n-1}$$
$$\downarrow \quad\quad\quad\quad\quad\quad \downarrow$$
$$/\phi/ \leq {}^{k-1}/_{n-1} \quad\quad\quad\quad /\phi/ \geq {}^{k+1}/_{n-1}$$

In $Ł_2$, these rules allow the inferences

$$p < 1 \quad\quad\quad\quad p > 0$$
$$\downarrow \quad\quad\quad\quad \downarrow$$
$$p \leq 0 \quad\quad\quad\quad p \geq 1$$

In $Ł_3$, we have four inferences:

$$p < 1 \quad\quad p < ½ \quad\quad p > 0 \quad\quad p > ½$$
$$\downarrow \quad\quad \downarrow \quad\quad \downarrow \quad\quad \downarrow$$
$$p \leq ½ \quad\quad p \leq 0 \quad\quad p \geq ½ \quad\quad p \geq 1$$

One can, of course, easily show that we can infer $p = 0$ and $p = 1$ instead of $p \leq 0$ and $p \geq 1$. Inferring the former equalities rather than the latter inequalities would be a suitable derived rule for use in practice.

The soundness of these rules should be easy to see, and does not need a formal proof. Since any valuation v of $Ł_n$ can only assign a formula a value divisible by n - 1, if v is faithful to a line of the form $/\phi/ < {}^{k}/_{n-1}$, v must assign ϕ some value divisible by n - 1 and less than ${}^{k}/_{n-1}$, the greatest of which is ${}^{k-1}/_{n-1}$. The reasoning is similar for the second new rule. Having established that a valuation faithful to the first line is also faithful to the line that the rule allows us to infer, the remainder of the soundness proof proceeds exactly as before.

3. Tableaux for product logic

As discussed in II.4 above, another very prominent infinite-valued system is product logic. This system replaces $\barwedge$, $\veebar$, and $\supset$ with $\odot$, $\oplus$, and $\Supset$ respectively, with the latter truth functions defined as:

$$/\phi \odot \psi/ = /\phi/ \times /\psi/,$$

$$/\phi \oplus \psi/ = /\phi/ + /\psi/ - /\phi/ \times /\psi/,$$

$$/\phi \Supset \psi/ = 1 \text{ for } /\phi/ \leq /\psi/, \quad /\psi/ \div /\phi/ \text{ otherwise.}$$

(The Goguen precomplement $-_G\phi$, equivalent to $J_1\neg\phi$ in RPL$\triangle$, is sometimes used for negation, but the Łukasiewicz negation $\neg$ seems to be more commonly used.[24]) A number of theorists consider these versions of the connectives to be

[24] E.g., in Gaines (1976), Goguen (1969), and Gehrke et al. (1999).

superior to those of Ł$_{\aleph}$ for some purposes (see e.g. Gaines 1976: 635-6, Goguen 1969: 346-7). Although the Łukasiewicz connectives should be regarded as the proper generalizations of the classical connectives (because, as noted in II.4, idempotency fails for $\odot$ and $\oplus$, and contraposition for $\Supset$), the Goguen connectives are still of substantial importance. So, I will give a set of additional tableau rules to cover these alternative connectives.

For any formula or part consisting of two subformulas ϕ, ψ conjoined by $\odot$, a line of the form

$$/\phi \odot \psi/ = /\phi/ \times /\psi/$$

should be entered into the tableau. This rule will be called GC.

For any formula or part consisting of two subformulas ϕ, ψ disjoined by $\oplus$, a line of the form

$$/\phi \oplus \psi/ = /\phi/ + /\psi/ - /\phi/ \times /\psi/$$

should be entered into the tableau. This rule will be called GD.

Decomposition rules for $\Supset$ are:

$\Supset$GE

$/\phi \Supset \psi/ \geq \chi$

↓

$/\psi/ \geq /\phi/ \times \chi$

$\Supset$LE

$/\phi \Supset \psi/ \leq \chi$

↙ ↘

$/\psi/ \leq /\phi/ \times \chi$	$/\phi/ \leq /\psi/$ $\chi \geq 1$

$\Supset$SG

$/\phi \Supset \psi/ > \chi$

↓

$/\psi/ > /\phi/ \times \chi$

$\Supset$SL

$/\phi \Supset \psi/ < \chi$

↙ ↘

$/\psi/ < /\phi/ \times \chi$	$/\phi/ \leq /\psi/$ $\chi > 1$

The proof of the soundness of these rules for product logic is exactly similar to the soundness proof of IV.7; all that needs to be added are the requisite clauses of the Soundness Lemma that pertain to the new rules:

GC. Follows immediately from the definition of $/\odot/$.

GD. Follows immediately from the definition of $/\oplus/$.

$\Supset$GE. Suppose $/\phi \Supset \psi/ = 1$. Then $/\phi/ \leq /\psi/$. Since $/\phi \Supset \psi/ \geq \chi$, $\chi \leq 1$, so $/\phi/ \times \chi \leq /\psi/$; i.e., $/\psi/ \geq /\phi/ \times \chi$. Suppose $/\phi \Supset \psi/ < 1$. Then $/\phi \Supset \psi/ = /\psi/ \div /\phi/$, so $/\psi/ \div /\phi/ \geq \chi$ and $/\psi/ \geq /\phi/ \times \chi$. (The same, *mutatis mutandis*, for $\Supset$SG.)

$\Supset$LE. Suppose $/\phi \Supset \psi/ = 1$. Then $/\phi/ \leq /\psi/$ and since $/\phi \Supset \psi/ \leq \chi$, $\chi \geq 1$. Suppose $/\phi \Supset \psi/ < 1$. Then $/\phi \Supset \psi/ = /\psi/ \div /\phi/$, so $/\psi/ \div /\phi/ \leq \chi$ and $/\psi/ \leq /\phi/ \times \chi$. (The same, *mutatis mutandis*, for $\Supset$SL.)

4. Hedges

A final important group of truth functions in fuzzy logic are *hedges*. These are locutions like "very", "slightly", "not too much", and so forth, which we use to qualify the truth values of our sentences. For instance, in ordinary speech we might distinguish between the height of A, who is 6'6", and of B, who is 5'11", by saying that A is *very* tall, whereas B is *sort of* tall. Fuzzy theorists interpret the former locution as indicating that the truth value of "A is tall" is very close to 1; the latter locution is interpreted as indicating that B is close to being a borderline case of tallness.

The hedges *concentration* and *dilation*, representing "very" and "slightly", respectively, are generally taken to be basic; other natural language hedges can be defined in terms of these. (See Gaines 1976: 650-3.) There is a way to represent these hedges in $Ł_{\aleph}$; we can define "very ϕ" as $\phi \barwedge \phi$, and "slightly ϕ" as $\phi \veebar \phi$, since these take the values

$/\phi \barwedge \phi/ = 2 \times /\phi/ - 1$ for $/\phi/ \geq 0.5$,
0 otherwise;

$/\phi \veebar \phi/ = 1$ for $/\phi/ \geq 0.5$,
$2 \times /\phi/$ otherwise.

These definitions may not be entirely satisfactory. A plausible constraint on an account of concentration is that if $/\phi/ > /\psi/$, /very ϕ/ > /very ψ/. However, let $/\phi/ = 0.4$ and $/\psi/ = 0.1$; then $/\phi \barwedge \phi/ = /\psi \barwedge \psi/ = 0$.

Potentially more satisfactory hedges can be obtained in product logic by defining "very ϕ" as $\phi \odot \phi$ and "slightly ϕ" as $\phi \oplus \phi$, since

$/\phi \odot \phi/ = /\phi/^2$,
$/\phi \oplus \phi/ = 2 \times /\phi/ - /\phi/^2$.

The most commonly used hedges, however, seem to be those proposed by Zadeh (see Gaines 1976: 643, 650-2). Using $\gamma\phi$ for "very ϕ" and $\delta\phi$ for "slightly ϕ", we have

$/\gamma\phi/ = /\phi/^2$,
$/\delta\phi/ = /\phi/^{0.5}$.

It is very simple to augment algebraic tableaux to cover Zadeh's hedges. For any formula or subformula ϕ in the scope of an operator γ, we enter into the tableau a line of the form

$$/\gamma\phi/ = /\phi/^2.$$

Likewise, for any formula or subformula ϕ in the scope of an operator δ, we enter into the tableau a line of the form

$$/\delta\phi/ = /\phi/^{0.5}.$$

It is also easy to see that these rules are sound for Zadeh's hedges; the necessary soundness lemmas proceed immediately from the definitions of the connectives. The tableau procedure is thus now fully developed, as it covers all the most important operators used in propositional fuzzy logic.

5. Problems for future research

I have presented algebraic tableaux as an informal system. A number of different rules will typically apply to any given line of a proof, and the user may choose to apply whichever of them seems most likely to lead to the desired conclusion. This way of presenting the tableaux makes them easier for people to use, but has the disadvantage of not providing an algorithm for determining when an open branch of a tableau is finished. There is thus no algorithm as of yet for determining if an open branch of a tableau furnishes a counterexample to the inference in question. Branches that *seem* open do provide hypotheses for counterexamples, the accuracy of which may be easily checked. So far, in work with the tableaux, branches to which no more rules seem to apply have always furnished counterexamples. Inductive evidence is not, of course, a proof; but it does suggest that a proof is possible.

In addition, it seems that the completeness of the system (for finite sets of premises, as noted in II.2) cannot be proven without an algorithm for determining when no more rules can be fruitfully applied to open branches. Thus, providing an algorithm for constructing algebraic tableaux will provide a true decision procedure for infinite-valued logic. There should be no difficulty in finding such an algorithm for generating algebraic tableaux for RPL△ and the Łukasiewicz family, since the fragment of arithmetic with just addition and subtraction is known to be decidable. It is not clear, however, if it is possible to provide an algorithm for constructing tableaux for the Goguen connectives and Zadeh's hedges, since not all arithmetical statements involving multiplication and division are decidable. Automating the tableaux—giving an algorithm for making tableaux—and proving the completeness of the procedure (or an augmented version, if the current procedure is not complete) will be the main aims of future work on algebraic tableaux.

A further question is the computational complexity of the procedure. This issue has not yet been studied, and sets a research agenda for those fuzzy logicians interested in the theory of their subject.

6. Conclusion

The main thrust of this work has been to present a tableau procedure for RPL△, an extension on $Ł_{\aleph}$, the most commonly used infinite-valued logic. I have then extended the system to cover the finite-valued members of the Łukasiewicz family, product logic, and Zadeh's hedges. The tableau procedure is sound for all these systems, but has not been proven complete. In addition, in order to increase the reader's familiarity with infinite-valued logic in general and RPL△ in particular, I have given an overview of the use of fuzzy logic in vagueness, sorites paradoxes, and control systems, an account of the central principles motivating fuzzy logic, and a summary of the most important technical details of the systems in question. Thus, the reader will hopefully be able to see what infinite-valued logic has to offer in areas that classical logic addresses only poorly (if at all), and what algebraic tableaux have to offer for the study and application of fuzzy logic.

Works cited

Ackerman, Robert (1967). *Introduction to many-valued logics.* London: Routledge.

Aristotle (1997). *De interpretatione.* J. L. Ackrill, trans. Excerpted in Richard N. Bosley and Martin Tweedale, eds. *Basic issues in medieval philosophy.* Peterborough, Ontario: Broadview.

Beavers, Gordon (1993). "Automated theorem proving for Łukasiewicz logics." *Studia Logica* 52: 183-96.

D'Agostino, Marcello, Dov M. Gabbay, Reiner Hähnle, and Joachim Posegga, eds. (1999) *Handbook of tableau methods*. Dordrecht: Kluwer.

Entemann, Carl W. (2000) "A fuzzy logic with interval truth values". *Fuzzy sets and systems* 113: 161-83.

———— (2002). "Fuzzy logic: Misconceptions and clarifications". *Artificial Intelligence Review* 17: 65-84.

Fine, Kit (1996). "Vagueness, truth and logic". In Rosanna Keefe and Peter Smith, eds. *Vagueness: A reader.* Cambridge, Mass.: MIT, 119-50.

Gaines, B. R. (1976) "Foundations of fuzzy reasoning". *International Journal of Man-Machine Studies* 8, 623-68.

Gehrke, Mai, Carol Walker, and Elbert Walker (1997). "A mathematical setting for fuzzy logics". *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 5:3, 223-38.

Gehrke, Mai, Vladik Kreinovich, and Bernadette Bouchon-Meunier (1999). "Propositional fuzzy logics: decidable for some (algebraic) operators; undecidable for more complicated ones". *International Journal of Intelligent Systems* 14:9 (Sept. 1999), 935-47.

Goguen, J. A. (1969) "The logic of inexact concepts". *Synthese* 19: 325-73.

Haack, Susan (1996). *Deviant logic, fuzzy logic: Beyond the formalism.* Chicago: University of Chicago.

Hähnle, Reiner (1993). *Automated deduction in multiple-valued logics.* Oxford: Oxford.

———— (1994). "Many-valued logic and mixed integer programming". *Annals of Mathematics and Artificial Intelligence* 12: 231-63.

Hájek, Petr (1998). *Metamathematics of fuzzy logic.* Dordrecht: Kluwer.

Kamp, J. A. W. (1975) "Two theories about adjectives". In E. L. Keenan, ed. *Formal semantics of natural language.* Cambridge: Cambridge, 123-55.

Kenevan, James R. and Richard E. Neapolitan (1992). "A model theoretic approach to propositional fuzzy logic using Beth Tableaux". In Lotfi A. Zadeh and Janusz Kacprzyk, eds. *Fuzzy logic for the management of uncertainty.* New York: Wiley, 141-57.

Kosko, Bart (1993). *Fuzzy thinking: The new science of fuzzy logic.* New York: Hyperion.

Machina, Kenton F. (1976) "Truth, belief and vagueness". *Journal of Philosophical Logic* 5: 47-78.

Malinowski, Grzegorz (1993). *Many-valued logics.* Oxford: Oxford.

Morgan, C. G., and F. J. Pelletier (1977). "Some notes concerning fuzzy logics". *Linguistics and Philosophy* 1: 79-97.

Mundici, Daniele (1987). "Satisfiability in many-valued sentential logic is NP-complete." *Theoretical Computer Science* 52: 145-53.

Nguyen, Hung T., Olga M. Kosheleva, and Vladik Kreinovich (1996). "Is the success of fuzzy logic really paradoxical? Toward the actual logic behind expert systems". *International Journal of Intelligent Systems* 11: 295-326.

Priest, Graham (2001). *An introduction to non-classical logic.* Cambridge, UK: Cambridge.

Rescher, Nicholas (1969). *Many-valued logic.* New York: McGraw-Hill.

Russell, Bertrand (1923). "Vagueness". *Australasian Journal of Philosophy and Psychology* 1: 84-92. Page references are to the reprint in Rosanna Keefe and Peter Smith, eds. (1996) *Vagueness: A reader.* Cambridge, Mass.: MIT, 61-8.

Tarski, Alfred (1957). *A decision method for elementary algebra and geometry.* Santa Monica, CA: Rand.

Williamson, Timothy (1992). "Vagueness and ignorance". *Proceedings of the Aristotelian Society, Supp. Vol.* 66: 145-62.

——— (1994). *Vagueness*. London: Routledge.

Zadeh, Lotfi A. (1972) "Fuzzy languages and their relation to human and machine intelligence". *Proceedings of the International Conference on Man and Computer, Bordeaux, France* 130-65. Page references are to the reprint in George J. Klir and Bo Yuan, eds. (1996) *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A. Zadeh.* Singapore: World Scientific, 148-79.

——— (1975). "Fuzzy logic and approximate reasoning". *Synthese* 30: 407-28.

Appendix: major truth functions used in the text

VSS:

$/\neg\phi/ = 1 - /\phi/$

$/(\phi \vee \psi)/ = \max(/\phi/, /\psi/)$

$/(\phi \wedge \psi)/ = \min(/\phi/, /\psi/)$

$Ł_{\aleph}$:

$/(\phi \supset \psi)/ = \min(1, 1 - /\phi/ + /\psi/)$

$/(\phi \veebar \psi)/ = \min(1, /\phi/ + /\psi/)$

$/(\phi \barwedge \psi)/ = \max(0, /\phi/ + /\psi/ - 1)$

$/(\phi \equiv \psi)/ = 1 - |/\phi/ - /\psi/|$

RPL:

$/C_i/ = i$, for each rational $i \in [0,1]$

RPL△:

$/J_i\phi/ = $ 1 if $/\phi/ = i$, for each rational $i \in [0,1]$
0 otherwise

$/(\phi \rightarrow \psi)/ = $ 1 if $/\phi/ \leq /\psi/$
0 otherwise

$/(\phi \leftrightarrow \psi)/ = $ 1 if $/\phi/ = /\psi/$
0 otherwise

Product logic:

$/\phi \odot \psi/ = /\phi/ \times /\psi/$

$/\phi \oplus \psi/ = /\phi/ + /\psi/ - /\phi/ \times /\psi/$

$/\phi \Supset \psi/ = $ 1 if $/\phi/ \leq /\psi/$
$/\psi/ \div /\phi/$ otherwise