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Invariant subspaces of certain classes of operators

by

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To my family

Abstract

The first part of the thesis studies invariant subspaces of strictly singular operators. By a celebrated result of Aronszajn and Smith, every compact operator has an invariant subspace. There are two classes of operators which are close to compact operators: strictly singular and finitely strictly singular operators. Pełczyński asked whether every strictly singular operator has an invariant subspace. This question was answered by Read in the negative. We answer the same question for finitely strictly singular operators, also in the negative. We also study Schreier singular operators. We show that this subclass of strictly singular operators is closed under multiplication by bounded operators. In addition, we find some sufficient conditions for a product of Schreier singular operators to be compact.

The second part studies almost invariant subspaces. A subspace Y of a Banach space is almost invariant under an operator T if $TY \subseteq Y + F$ for some finite-dimensional subspace F ("error"). Almost invariant subspaces of weighted shift operators are investigated. We also study almost invariant subspaces of algebras of operators. We establish that if an algebra is norm closed then the dimensions of "errors" for the operators in the algebra are uniformly bounded. We obtain that under certain conditions, if an algebra of operators has an almost invariant subspace then it also has an invariant subspace. Also, we study the question of whether an algebra and its closure have the same almost invariant subspaces.

The last two parts study collections of positive operators (including positive matrices) and their invariant subspaces. A version of Lomonosov theorem about dual algebras is obtained for collections of positive operators. Properties of indecomposable (i.e., having no common invariant order ideals) semigroups of nonnegative matrices are studied. It is shown that the "smallness" (in various senses) of some entries of matrices in an indecomposable semigroup of positive matrices implies the "smallness" of the entire semigroup.

Many of the results presented in this thesis were obtained jointly with other people. The thesis is based on several papers published by the author of this thesis and his co-authors. Among those papers are two single-author papers and five joint papers.

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Contents

Chapter 1

Introduction

Many of results in this thesis were obtained jointly with other people. I give appropriate credits in every section. The thesis is based on two papers of mine [97] and [98] and five joint papers with my collaborators [13, 38, 53, 99, 100].

1.1 Invariant Subspace Problem

This thesis is concerned with the study of linear bounded operators and collections of such operators on various Banach spaces. A *Banach space* is a complete normed space. Banach spaces were introduced by Banach in 1920-s; they form one of the central objects of study in Functional analysis. A *linear bounded operator* between Banach spaces X and Y is a linear continuous map from X to Y. In this thesis, we will simply write *operator* for a linear bounded operator between Banach spaces. An *operator on a Banach space* X is an operator from X to X. The set of all operators from X to Y is denoted by $L(X,Y)$. We write $L(X)$ for $L(X,X)$.

A fundamental question in the study of operators on Banach spaces is understanding the structure of their invariant subspaces. A subspace Y (in this thesis, a *subspace* of a Banach space always means a closed vector subspace; subspaces which are not necessarily closed will be referred to as *linear subspaces*) of a Banach space X is called *invariant* under a bounded linear operator $T \in L(X)$ if $Ty \in Y$ for every $y \in Y$. Invariant subspace is a natural replacement for a fundamental concept of analysis of matrices – eigenvector.

The set of all subspaces which are invariant for a given operator $T \in L(X)$ is denoted by *Lat* T. It is easy to see that *Lat* T forms a lattice under the operations $Y \wedge Z = Y \cap Z$ and $Y \vee Z = \overline{\text{span}}\,(Y \cup Z)$.

It is clear that every operator on a Banach space X has at least two invariant subspaces: X and $\{0\}$. These two subspaces are called *trivial*. An invariant subspace is called *non-trivial* if it is different from $\{0\}$ and X. A fundamental question for the study of operators is the *Invariant Subspace Problem* which was originally stated as the follows: *Does every bounded operator on a Banach space have a closed non-trivial invariant subspace?*

Throughout this thesis, whenever we write *invariant subspace*, we always mean (unless otherwise is stated explicitly) a non-trivial invariant subspace.

One can also ask about the existence of invariant subspaces for collections of operators. Namely, given a collection $\mathcal{C}$ of operators acting on a Banach space X, one might be interested if there exists a closed non-trivial subspace of X which is invariant under *every* operator in $\mathcal{C}$. This question is of particular interest when the collection $\mathcal{C}$ has certain properties, for example, forms an algebra of operators. In particular, one could consider the algebra of all operators which commute with a given operator T. This algebra is called the *commutant* for T and is denoted by $\{T\}'$. Any invariant subspace for this algebra is called a *hyperinvariant* subspace for T.

It is easy to show that if the underlying space X is not separable then the problem of existence of invariant subspaces for any operator on X has an immediate affirmative answer. Indeed, if $T : X \to X$ is any operator and $x \in X$ is an arbitrary non-zero vector then the space $Y = \overline{\text{span}}\{T^n x \colon n \geq 0\}$ is a non-trivial closed invariant subspace for T. Another useful simple fact is the following proposition (for proof, see, e.g., [1, Theorem 10.8]).

Proposition 1.1.1. *If an operator T on a Banach space or its adjoint T^* has an eigenvalue then T has an invariant subspace. If T is non-scalar (i.e., T is an operator not of form αI) then this subspace is even hyperinvariant.*

In particular, this shows that the question about invariant subspaces has an immediate affirmative answer for any operator on a finite-dimensional complex space of dimension greater than one. It can also be shown that any operator on a real finite-dimensional space of dimension greater than two has an invariant subspace, too.

Many results in the invariant subspace research are related in one way or another to compact operators. This direction was started by von Neumann who proved in the early thirties that every compact operator acting on a Hilbert space has a closed non-trivial invariant subspace (his work, however, was never published). Aronszajn and Smith [17] showed in 1954 that the same applies to arbitrary Banach spaces. The technique used in these results is an approximation method which allows to construct an invariant subspace for a compact operator T from a sequence of invariant subspaces for finite-dimensional operators closely related to T.

The approximation method of von Neumann–Aronszajn–Smith has been used and refined by many mathematicians. In 1966, Bernstein and Robinson proved that every polynomially compact operator has invariant subspaces:

Theorem 1.1.2. [26] *Let T be an operator on a Banach space. If there is a non-zero polynomial p such that $p(T)$ is compact then T has an invariant subspace.*

Bernstein and Robinson used Nonstandard Analysis in their work. Halmos in [63] reproved this fact without using Nonstandard Analysis. A further generalization of this result, using the same set of ideas, was given by Arveson and Feldman [18] who proved that if for an operator T on the Hilbert space $\lim \|T^n x\|^{1/n} = 0$ for some $x \neq 0$ and the norm closed algebra generated by T contains a nonzero compact operator then T has an invariant subspace. This result was extended to Banach spaces by Apostol [14] and Gillespie [54].

A very remarkable result which includes all of the aforementioned results and is considered a breakthrough in the Invariant Subspace Problem appeared in 1973. It is due to Lomonosov.

Theorem 1.1.3. [75] *If a non-scalar operator T on a complex Banach space commutes with a non-zero compact operator, then T has a non-trivial closed hyperinvariant subspace.*

Originally, it was not clear whether this theorem solves the Invariant Subspace Problem in affirmative. However, Hadwin, Nordgren, Radjavi and Rosenthal [61] found an example of an operator that fails the assumptions of Lomonosov's theorem. Their operator, however, had plenty of invariant subspaces.

Lomonosov used a technique of fixed points of continuous functions on compact sets. This technique proved to be very useful and has subsequently been employed by other authors. Hilden [84] proved a weaker form of Lomonosov's theorem using quite different "ping-pong" technique. He used the fact that, when dealing with the Invariant Subspace Problem, one may assume that the compact operator is quasinilpotent.

An interesting generalization of Lomonosov's theorem was obtained by Daughtry in [39]. He proved that if the commutator of operator T with a compact operator has rank one then T has an invariant subspace.

It should be noted that in 2009, Argyros and Haydon [16] constructed a separable Banach space X such that every operator on X can be written in the form $\lambda I + K$ where I is the identity operator and K is a compact operator. By the result of Aronszajn and Smith, every operator on this space has an invariant subspace. This is the first known example of an infinite dimensional separable Banach space with this property.

Examples of operators without closed non-trivial invariant subspaces (these operators are called *transitive operators*) were constructed by Enflo [48, 49] and Read [106]. Another example was published by Beauzamy [21] who simplified Enflo's example. Also, some simplifications of Read's construction have been published by Davie [22] and recently by Sirotkin [121, 122]. Atzmon [19] showed that there is an operator without invariant subspaces on a nuclear Fréchet space (see, e.g., [52] for more information about nuclear spaces).

After constructing his first counterexample, Read produced different versions of his operator which satisfy some additional properties. In [107] he constructed a transitive operator on ℓ_1. It was hoped that Hilden's technique mentioned above could be modifyed to apply to arbitrary quasinilpotent operators. Read showed in [109] this to be false by constructing an example of a transitive quasinilpotent operator. Another remarkable result of Read was a construction [108] of an operator which is *hypercyclic* at every non-zero vector, i.e., an operator without even invariant closed non-trivial *subsets*.

It was conjectured by Pełczyński that every strictly singular operator (see Chapter 2 for the definition) has an invariant subspace. The motivation for this conjecture is the fact that strictly singular operators and compact operators share a lot of important properties. For example, both these classes of operators form closed operator ideals; operators from both classes have identical spectral theory. Despite all these similarities, Read [110] constructed an example of a strictly singular operator without invariant subspaces.

The operators for which all the invariant subspaces are known are very few. Important examples are the right shift operator on ℓ_2, Donoghue operators and the Volterra operator.

Definition 1.1.4. An operator $S : \ell_2 \to \ell_2$, $T : e_i \mapsto e_{i+1}$, is called the ***right shift operator***. A weighted left shift operator $D : \ell_2 \to \ell_2$, $De_0 = 0, \quad De_i = w_i e_{i-1}, \quad i \in \mathbb{N}$, is called a ***Donoghue operator*** if the weights w_i are non-zero and satisfy the condition that the sequence $(|w_i|)_{i=1}^{\infty}$ is monotone and is in ℓ_2. An operator $V : L_2[0,1] \to L_2[0,1]$ is called the ***Volterra operator*** if $(Vf)(x) = \int_0^x f(y)dy$ for all $f \in L_2[0,1]$.

The right shift operator on ℓ_2 is one of the simplest operators; however the structure of its invariant subspaces is not obvious at all. The characterization of the invariant subspaces of the right shift operator was obtained by Beurling [28] in 1949. In his result, Beurling used a particular representation of the right shift operator which we will describe now. We use exposition from [102].

Consider the space $L_2(C,\mu)$ where C is the unit circle and μ is the normalized Lebesgue measure on C. For each $n \in \mathbb{Z}$, denote $e_n(z) = z^n$, $e_n \in L_2(C,\mu)$. Then $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L_2(C,\mu)$. Denote $H_2 = \overline{\text{span}}\{e_n : n \geqslant 0\} \subseteq L_2(C,\mu)$. The particular representation of the right shift operator used by Beurling is the operator of multiplication by the independent variable in H_2. That is, $(Sf)(z) = f(z) \cdot z$ for all $f \in H_2$.

Theorem 1.1.5. [28] *A non-zero subspace M of H_2 is invariant under the right shift operator S on $L_2(C,\mu)$ if and only if there is a function $\phi \in H_2$ such that $|\phi(z)| = 1$ a.e. on C and*

$$M = \phi H_2 := \{\phi f : f \in H_2\}.$$

Moreover, $\phi_1 H_2 = \phi_2 H_2$ with $|\phi_1(z)| = |\phi_2(z)| = 1$ a.e. on C if and only if ϕ_1/ϕ_2 is equal a.e. to a constant function.

It is a surprising corollary that the right shift operator has invariant subspaces which are infinite dimensional and infinite codimensional simultaneously.

Theorem 1.1.6. (see, e.g. [102, Corollary 3.15]) *If $\lambda \in \mathbb{C}$ is such that $|\lambda| = 1$ and $\phi(z) = \exp\big((\lambda + z)/(\lambda - z)\big)$ then the space $M = \phi H_2$ is an invariant subspace for the right shift operator having both infinite dimension and infinite codimension.*

Donoghue operators and the Volterra operator are examples of so-called unicellular operators. An operator T on a Banach space X is called *unicellular* if the lattice of its invariant subspaces $LatA$ is totally ordered (that is, if $Y, Z \in Lat\ A$ then either $Y \subseteq Z$ or $Z \subseteq Y$).

The following theorem is due to Donoghue [43] in the case the weights of a Donoghue operator from Definition 1.1.4 are $w_i = 2^{-i}$, and in full generality to Nikol'skiĭ [89], Parrot (unpublished), and Shields (unpublished).

Theorem 1.1.7. *A subspace M of ℓ_2 is invariant under a Donoghue operator if and only if M is of form* $\operatorname{span}\{e_k : k = 0, \dots, n\}$ *for some $n \in \mathbb{N}$.*

The following theorem was developed by Dixmier [42], Donoghue [43], and Brodskiĭ [29].

Theorem 1.1.8. *If V is the Volterra operator, then*

$$Lat\ V = \{M_\alpha : \alpha \in [0,1]\},$$

where $M_\alpha = \{f \in L_2[0,1] : f = 0 \text{ a.e. on } [0,\alpha]\}$.

Despite all the negative results, the Invariant Subspace Problem is far from being closed. For Hilbert spaces, the problem is open. Even in the Banach space setting, numerous questions remain unanswered. For example, it is not known whether there are transitive operators on reflexive spaces. With regard to this question, we would like to mention a conjecture proposed by Lomonosov which asked whether every adjoint operator has an invariant subspace. Notably, Schlumprecht and Troitsky showed in [118] that the Read's example in ℓ_1 is not adjoint.

In the rest of this introductory subsection, we will list some of the known results about invariant subspaces which we will not use directly in the thesis. This list is very incomplete and reflects the interests of the author of this thesis.

Theorem of Lomonosov shows in particular that if an operator T commutes with a non-scalar operator S which in turn commutes with a non-zero compact operator K then T has an invariant subspace. A natural question to ask in this regard is whether it is possible to make the chain of operators $T \leftrightarrow S \leftrightarrow K$ longer. This question was answered in the negative by Troitsky in [123].

Halmos introduced in [64] the notion of a quasitriangular operator. An operator T on Hilbert space is called *quasitriangular* if there exists a sequence (P_n) of finite-dimensional projections increasing to the identity operator such that $P_nTP_n - TP_n \to 0$ in norm. It is a surprising result of Apostol, Foiaş, and Voiculescu [15] that the adjoints of non-quasitriangular operators have eigenvalues, hence the operators themselves have hyperinvariant subspaces (see Proposition 1.1.1).

An operator $T \in L(X)$ is called *doubly power bounded* if $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. It was proved by Lorch [78] that every non-scalar doubly power bounded operator on a complex Banach space has a hyperinvariant subspace. This result, in particular, implies that every linear isometry on a Banach space has invariant subspaces.

The result of Lorch was generalized by Wermer to a class of non-quasianalytic operators. An invertible operator $T \in L(X)$ is called *quasianalytic* if $\sum_{n=-\infty}^{\infty} \frac{|\log\|T^n\||}{1+n^2} = \infty$. It is clear that every doubly power bounded operator is not quasianalytic. Wermer [127] showed that every non-quasianalytic operator whose spectrum $\sigma(T)$ is not a singleton has a hyperinvariant subspace. In particular, it follows from this theorem that if $T \in L(X)$ is such that $\sigma(T)$ is not a singleton and there exists a constant M such that $\|T^n\| \leqslant M|n|^k$ for $n = \pm 1, \pm 2, \dots$, then T has a hyperinvariant subspace. The result of Wermer was subsequently generalized and improved by many authors in various directions.

An operator $T : L_p(C,\mu) \to L_p(C,\mu)$, where C is the unit circle, μ the normalized Lebesgue measure on C, and $p \in [1,\infty]$, is called a *weighted composition operator* if $(Tf)(x) = w(x)f(\alpha x)$ where $\alpha \in C$ and w is a weight function. A weighted composition operator is called a *Bishop operator* if $w(x) = x$ for all x. Davie [40] showed that a Bishop operator has an invariant subspace provided $\frac{\alpha}{\pi}$ is not a Liouville number (recall that a number a is called a *Liouville number* if there exists a sequence $(\frac{m_n}{k_n})$ of pairwise distinct rational numbers such that $(m_n, k_n) = 1$ and $|a - \frac{m_n}{k_n}| < \frac{1}{n(k_n)^n}$). Generalizations of Davie's result were obtained by MacDonald [79, 80].

In Hilbert spaces, the Spectral Theorem for normal operators provides a large stock of invariant subspaces for such operators. In fact, Fuglede [51] showed that if T is a normal operator and E is its spectral measure then the range of the projection $E(S)$ is T-hyperinvariant for every Borel subset S of $\mathbb{C}$

(see also [101]). Dunford [47] generalized this theorem to Banach spaces.

It has been an open question for a long time whether subnormal operators (introduced by Halmos in [62]) on Hilbert spaces have invariant subspaces. An operator T on the Hilbert space H is called *subnormal* If it is the restriction of a normal operator to an invariant subspace. Brown [30] proved that every subnormal operator has an invariant subspace. Ideas developed by Brown were used in many other results.

A classical source about the invariant subspace problem is the book [102] by Radjavi and Rosenthal (which is primarily focused on Hilbert spaces). General treatments of this problem can be found in the book [22] by Beauzamy and in the book [1] by Abramovich and Aliprantis. There is also a big survey [90] of methods in the invariant subspace research with an extensive bibliography of 1253 items by Nikol'skiĭ.

1.2 Transitive algebras

Recall that an operator has a hyperinvariant subspace if its commutant has an invariant subspace. Observe that the commutant of an operator is an algebra of operators. Similarly to the commutants, one can ask the question of existence of a common invariant subspace for an arbitrary algebra of operators. Analogously to the case of single operators, if $\mathfrak{A}$ is an algebra of operators, we write $Lat\ \mathfrak{A}$ for the lattice of its invariant subspaces (including the trivial ones).

In what follows, we will use several standard topologies in the space of operators on a Banach space.

Definition 1.2.1. If X is a Banach space, $T \in L(X)$, and T_α is a net in $L(X)$, then we say that T_α converges to T in the ***strong operator topology*** and write $T_\alpha \xrightarrow{SOT} T$ if $T_\alpha x \xrightarrow{\|\cdot\|} x$ for all $x \in X$. We say that T_α converges to T in the ***weak operator topology*** and write $T_\alpha \xrightarrow{WOT} T$ if $T_\alpha x \xrightarrow{w} x$ for all $x \in X$. If X is a dual Banach space (i.e., $X = Z^*$ for some Banach space Z)

then we say that T_α converges to T in the ***weak*** * ***operator topology*** and write $T_\alpha \xrightarrow{W^*OT} T$ if $T_\alpha x \xrightarrow{w^*} x$ for all $x \in X$.

It is clear that WOT is weaker than SOT, and SOT is weaker than the norm topology in $L(X)$. It is a very useful fact that the closures of a convex subset of $L(X)$ in SOT and in WOT are the same. In particular, an algebra of operators is closed in SOT if and only if it is closed in WOT.

When working with invariant subspaces of algebras of operators, one can always assume that the given algebra is closed in WOT. The following proposition is well-known.

Proposition 1.2.2. *The invariant subspaces of an algebra* $\mathfrak{A}$ *and its closure* $\overline{\mathfrak{A}}^{WOT}$ *are the same.*

Definition 1.2.3. An algebra of operators acting on a Banach space X is called ***transitive*** if $Lat\ \mathfrak{A} = \big\{\{0\}, X\big\}$.

An example of a transitive algebra is the algebra of all operators on a Banach space X. The *Transitive Algebra Problem* is the analogue to the Invariant Subspace Problem for the algebras of operators. It was stated originally as: if $\mathfrak{A}$ is a WOT-closed transitive algebra of operators acting on a Banach space, must $\mathfrak{A}$ be equal to the algebra of all operators on this space?

If the underlying space is finite dimensional then the transitive algebra problem has an affirmative answer. This is the classical Burnside's theorem.

Theorem 1.2.4. (see, e.g., [103, Theorem 1.2.2]) *Every proper subalgebra of* $M_n(\mathbb{C})$ *is not transitive.*

For infinite dimensional spaces, the situation is more complicated. In the case of algebras of operators acting on general Banach spaces, the examples of Enflo and Read provide negative solution to the transitive algebra problem (this follows from the observation that the WOT-closed algebra generated by a single operator must be commutative). Regarding the algebras of operators acting on Hilbert spaces, the question is open. There are many partial affirmative results, both in the Hilbert and Banach spaces setting. We refer the reader to [102, Chapter 8] for a detailed introduction to the topic. We would like to mention a few results more closely related to our exposition.

The following result can be viewed as a generalization of Burnside's theorem. It is due to Lomonosov.

Theorem 1.2.5. [73] *Let X be a complex Banach space and let $\mathfrak{A}$ be a WOT-closed subalgebra of $L(X)$. If $\mathfrak{A}$ is transitive and contains a non-zero compact operator, then $\mathfrak{A} = L(X)$.*

Now we would like to present a "quantized" version of the preceding result. It is also due to Lomonosov and applies to algebras consisting of adjoint operators. First, we mention a simple criterion of transitivity of an algebra of operators which is a folklore in the theory of transitive algebras.

Proposition 1.2.6. *For an algebra $\mathcal{A}$ of operators on a Banach space X, the following statements are equivalent.*

(i) The algebra $\mathcal{A}$ is non-transitive.

(ii) There exists a non-zero vector $x \in X$ and a non-zero linear functional $f \in X^*$ such that for each $T \in \mathcal{A}$ we have $\langle f, Tx \rangle = 0$.

Now suppose that X is a dual space; that is, $X = Y^*$ for some Banach space Y. If $T \in L(X)$ is a bounded adjoint operator on X then there is a unique operator $S \in L(Y)$ such that $S^* = T$. We will write $S = T_*$; there is no ambiguity in this notation as T_* is taken with respect to Y. We will write $\|T\|_e$ for the essential norm of T, i.e., the distance from T to the space of compact operators. Note that in general, for an adjoint operator T, one has $\|T\|_e \leqslant \|T_*\|_e$. See [20] for an example of T such that $\|T\|_e < \|T_*\|_e$. The next theorem is due to Lomonosov.

Theorem 1.2.7. [76] *Let X be a dual Banach space and $\mathcal{A}$ a proper W^*OT-closed subalgebra of $L(X)$ consisting of adjoint operators. Then there exist non-zero $x \in X$ and $f \in X^*$ such that $\big|\langle f, Tx\rangle\big| \leqslant \|T_*\|_e$ for all $T \in \mathcal{A}$.*

A collection of operators on a Banach space is said to be a *Volterra collection* if each operator in the collection is compact and quasinilpotent.

The following theorem about Volterra algebras is due to Shulman.

Theorem 1.2.8. [119] *Each non-zero Volterra algebra has a non-trivial closed hyperinvariant subspace.*

It was a long-standing question whether the same is true for Volterra semigroups of operators. It was solved by Turovskii in affirmative in 1999 by proving that the algebra generated by a multiplicative Volterra semigroup is a Volterra algebra.

Theorem 1.2.9. [125] *Each non-zero multiplicative Volterra semigroup has a non-trivial closed hyperinvariant subspace.*

1.3 Overview of the results

The general thread of this thesis is the study of invariant subspaces of operators and collections of operators. We investigate the question of existence of invariant subspaces of various operators. Also, we consider variations of the notion of invariant subspace. First variation is algebraic (we call it almost invariant subspace), the other is topological and originates from the theory of transitive algebras. Finally, we study collections of operators (in particular, semigroups of such operators) having no invariant subspaces of special kind, called closed invariant ideals.

Chapter 2 of the thesis is concerned with the study of invariant subspaces of variations of strictly singular operators. It was mentioned in Section 1.1 that the class of strictly singular operators is similar to the class of compact operators in many respects. However, even though compact operators behave very well with respect to the invariant subspace problem, there exist strictly singular operators without invariant subspaces.

The class of finitely strictly singular operators (see Chapter 2 for the definition) is another class of operators close to the class of compact operators. In fact, finitely strictly singular operators "sit" between the compact and the strictly singular operators. They were introduced in 1970 by Milman and have been studied by many authors since then. It is a natural question whether every finitely strictly singular operator has an invariant subspace. In this thesis (Section 2.2), we answer this question in the negative by showing that the strictly singular operator without invariant subspaces constructed by Read in [110] is, in fact, finitely strictly singular. As an intermediate result, we prove that the formal inclusion operator from J_p to J_q with $1 \leqslant p < q < \infty$ is finitely strictly singular. This result, in turn, follows from the following lemma: every k-dimensional subspace in $\mathbb{R}^n$ contains a vector x whose coordinates "oscillate a lot": $|x_k| \leqslant 1$ for all k, and there are at least k coordinates whose value are the alternating 1 and -1, i.e., there is a sequence of coordinates k_i such that $x_{k_i} = (-1)^i$. We develop an original approach which uses combinatorial properties of polytopes in n-dimensional spaces to prove this lemma.

Another subclass of finitely strictly singular operators which we study in this thesis is the class of $\mathcal{S}_\xi$-singular operators. It was introduced in [11] in order to generalize the result of Milman [86] asserting that a product of two strictly singular operators on $L_p[0,1]$ or $C[0,1]$ is compact. All $\mathcal{S}_\xi$-singular operators are strictly singular, and each strictly singular operator is $\mathcal{S}_\xi$-singular for some ξ; also each finitely strictly singular operator is $\mathcal{S}_\xi$-singular for *all* ξ (see [11] for these results). It is a natural question whether $\mathcal{S}_\xi$-singular operators form an operator ideal. In this thesis, we show that the class of $\mathcal{S}_\xi$-singular operators is closed under the left and right multiplication by bounded operators. The main theorem of [11] asserts that under certain conditions on the underlying Banach space X, a product of finitely many $\mathcal{S}_\xi$-singular operators is compact. A consequence of this theorem is the fact that a power of any such operator is compact, and, hence, the operator has invariant subspaces by Theorem 1.1.2. We show that the conditions in this theorem can be slightly relaxed. Also, we exhibit a simple example of a finitely strictly singular operator (hence, an $\mathcal{S}_\xi$-singular operator for any ξ) which is not polynomially compact.

In Chapter 3, we study the notion of an almost invariant subspace for an operator. This notion was introduced in [13] by the author of this thesis and his co-authors. A subspace Y of a Banach space is almost invariant under an operator T if $TY \subseteq Y + F$ for some finite-dimensional subspace F ("error"). We develop an original technique of constructing almost invariant subspaces. In particular, this technique works with a large class of weighted shift operators. A significant part of Chapter 3 is concerned with the study of almost invariant subspaces of algebras of operators. We establish that if an algebra of operators is norm closed then the dimensions of "errors" corresponding to the given subspace Y for the operators in the algebra are uniformly bounded. We obtain that if a norm closed singly generated algebra has a non-trivial almost invariant subspace then it has an invariant subspace. This is achieved by developing an original geometric method of constructing an invariant subspace from an almost invariant subspace. This method is also generalized to norm closed algebras generated by finitely many commuting operators. Finally, we study whether an algebra and its closure have the same almost invariant subspaces. It turns out that the situation here is dramatically different from the case of invariant subspaces.

In Chapter 4 we obtain a version of Theorem 1.2.7 of Lomonosov [76] about algebras of adjoint operators for collections of positive operators on Banach lattices. We find conditions on a collection $\mathcal{C}$ of positive adjoint operators which guarantee the same conclusion as that of Theorem 1.2.7: there exist non-zero $x \in X$ and $f \in X^*$ such that $|\langle f, Tx\rangle| \leqslant \|T_*\|_e$ for all $T \in \mathcal{A}$, where T_* is the predual for T. We show that in the case of collections of positive operators, both x and f can be chosen positive. Our work has been inspired by the paper [45] by Drnovšek. In our argument, we adapt techniques from [72] and [84] to the Banach lattice setting. We also show that our result is in a sense sharp: there exists a collection of positive operators satisfying the conditions of our theorem which nevertheless fails to have an invariant subspace.

Chapter 5 is concerned with the study of properties of multiplicative semigroups of indecomposable semigroups of nonnegative matrices (see Chapter 5 for the definition). The central question for this chapter is: *if a certain property holds for a semigroup locally, then does it hold globally?* In Section 5.2 we show that if a positive functional is bounded on an indecomposable semigroup of nonnegative matrices then, after a similarity via a positive diagonal matrix, all entries of all matrices are in $[0, 1]$. We also obtain the following topological version of this result. It is easy to see that the requirement that a positive functional is small on an indecomposable semigroup of nonnegative matrices is not enough for saying that the semigroup is topologically small itself. However, we show that if we insist that all the diagonal entries of the matrices in the semigroup are between 0 and ε then, after a simultaneous diagonal similarity, all entries of the matrices in the semigroup are between 0 and $\sqrt[n]{\varepsilon}$ where n is the size of the matrices. We exhibit examples showing that our results are sharp.

In Section 5.3 we study indecomposable semigroups of nonnegative matrices with diagonal elements coming from a fixed finite set. We call such semigroups *semigroups with finite diagonals*. We are concerned with the question: when such semigroups are finite? In general, there exist infinite indecomposable semigroups with finite diagonals. The main results assert that if a semigroup with finite diagonals is self-adjoint or is constant-rank (see Section 5.3 for the definitions) then it is finite. The first result does not even require indecomposability. We also study the possible values that can appear on the diagonal positions of semigroups with finite diagonals.

Chapter 2

Strictly singular operators

In this chapter we show that finitely strictly singular operators need not have invariant subspaces. Also, we study properties of $\mathcal{S}_\xi$-singular operators. The results in this chapter have been published in paper [97] by the author of this thesis and in the joint paper [38] by the author of this thesis and his co-authors.

2.1 Overview

Strictly singular operators were introduced by Kato in [68] while he was studying certain questions in Perturbation Theory. Later, these operators and variations of them have been investigated by Pełczyński [94], Milman [85, 86], and others. Strictly singular operators turned out to be an interesting and important class of linear operators.

Definition 2.1.1. [68] Let X and Y be Banach spaces. An operator $T \in L(X, Y)$ is called ***strictly singular*** if for any infinite-dimensional closed subspace Z of X, the restriction of T to Z is not an isomorphism.

In this chapter, we will always assume that the Banach space X is infinite-dimensional whenever we talk about strictly singular operators.

It is easily seen that every compact operator on X is strictly singular. We will see that the converse is not true; the relationship between compact and strictly singular operators is more intricate.

Theorem 2.1.2. [68] *A bounded linear operator $T : X \to Y$ between Banach spaces with X infinite-dimensional is strictly singular if and only if for each infinite-dimensional closed subspace X_1 of X there exists an infinite-dimensional closed subspace X_2 of X_1 such that $T|_{X_2}$ is compact.*

It is a classical result that the perturbation of a Fredholm operators by a compact operator preserves the index of the operator. The importance of strictly singular operators stems in particular from the fact that they enjoy the same property (see, for example, [1, Theorem 4.63]):

Theorem 2.1.3. *Let $S, T : X \to Y$ be two bounded linear operators between Banach spaces with X infinite-dimensional. If T is Fredholm and S is strictly singular then $T + S$ is again a Fredholm operator and $i(T + S) = i(T)$.*

Another nice property of strictly singular operators is that they form a closed algebraic ideal. Precisely (see, e.g., [1, Corollary 4.62]),

Theorem 2.1.4. *The collection of strictly singular operators between Banach spaces X and Y is a closed subspace of $L(X, Y)$. Moreover, if in a scheme of bounded operators $X \xrightarrow{S} Y \xrightarrow{T} Z$ between Banach spaces either S or T is strictly singular, then TS is strictly singular.*

Let us recall another classical result. Calkin [34] showed that the only proper non-trivial closed ideal of $L(\ell_2)$ is the ideal of compact operators $\mathcal{K}(\ell_2)$. Gohberg, Markus and Feldman [55] showed that the same is true for ℓ_p with $1 \leq p < \infty$ and c_0. Combined with the theorem above, this showes that in ℓ_p, $1 \leq p < \infty$, and c_0 every strictly singular operator is compact.

A simple example of a non-compact strictly singular operator can be found in sequence spaces. Namely, if $1 \leq p_1 < p_2 < \infty$ then every operator $T : \ell_{p_1} \to \ell_{p_2}$ is strictly singular; however, the natural embedding (also referred to as the formal identity) $i : \ell_{p_1} \to \ell_{p_2}$ is clearly not compact.

The definition of strictly singular operators above can be reformulated in the following form: $T : X \to Y$ is strictly singular if and only if for every infinite-dimensional closed subspace Z of X and for every $\varepsilon > 0$ there is $x \in Z$ such that $\|Tx\| < \varepsilon\|x\|$. This can be considered as a motivation for the following definition.

Definition 2.1.5. [86] We say that an operator $T : X \to Y$ between Banach spaces is ***finitely strictly singular*** (term *superstrictly singular* is also used in the literature) if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace W of X with $\dim W \geq n$ there exists $z \in W$ such that $\|Tz\| < \varepsilon\|z\|$.

Finitely strictly singular operators first appeared implicitly in [88], then explicitly in [86]. Clearly, the class of finitely strictly singular operators is a subclass of strictly singular operators. It is also not very difficult to show that every compact operator is finitely strictly singular. The following fact (due to Mascioni [81]) is important, though simple, as it allows to transfer certain properties of strictly singular operators to finitely strictly singular operators.

Theorem 2.1.6. [81] *An operator $T : X \to Y$ between Banach spaces is finitely strictly singular if and only if for any free ultrafilter $\mathcal{U}$, the operator $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is strictly singular*[1].

In particular, similarly to strictly singular operators, finitely strictly singular operators from X to Y form a closed subspace of $L(X, Y)$, and the class of finitely strictly singular operators is stable under multiplication by bounded operators.

Milman showed in [86] that all the classes of operators mentioned above (compact, finitely strictly singular, and strictly singular operators) are in general different from each other by providing simple examples in spaces of sequences. More examples of such operators can be found in the work of Plichko [96].

To finish the section, we would like to mention some negative results. It is well-known that the dual of any compact operator is again compact (see [117]).

[1] Here $X_{\mathcal{U}}$ and $Y_{\mathcal{U}}$ are ultraproducts of X and Y, respectively, and $T_{\mathcal{U}}$ is defined by $T_{\mathcal{U}}([x_i]_{\mathcal{U}}) = [T(x_i)]_{\mathcal{U}}$; see [65] for more information about ultraproducts of Banach spaces.

Unfortunately, this nice property fails for both strictly singular and finitely strictly singular operators. A simple example of a strictly singular operator with non-strictly singular dual was found by Goldberg and Thorp [56]. It can actually be shown that there are finitely strictly singular operators whose adjoints are not even strictly singular (see [96] for details).

2.2 Invariant subspaces of finitely strictly singular operators

The results of this section have been published in [38]. They were obtained simultaneously and independently by two research groups. One group included Vladimir Troitsky and the author of this thesis; the other group consisted of Isabelle Chalendar, Emmanuel Fricain, and Dan Timotin. The proofs obtained by these two groups are very different; we will present the proofs invented by the first group in this thesis.

2.2.1 Read's strictly singular operator

Throughout this subsection, T will stand for the Read's strictly singular operator [110]. We start by describing the construction of T in [110]. In order to outline the construction, we will need James' p-spaces. James' p-space is a generalization of the classical James' example of a non-reflexive space isomorphic to its second dual [66]. The James' p-space J_p is a sequence space consisting of all sequences $x = (x_n)$ in c_0 satisfying $\|x\|_{J_p} < \infty$ where

$$\|x\|_{J_p} = \sup\Big\{\Big(\sum_{i=1}^{n-1} |x_{k_{i+1}} - x_{k_i}|^p\Big)^{\frac{1}{p}} \;:\; 1 \leqslant k_1 < \cdots < k_n,\ n \in \mathbb{N}\Big\}$$

is the norm in J_p. Under the norm $\|\cdot\|_{J_p}$, the space J_p is a Banach space. For more information about these and similar spaces, we refer the reader to [35], [71], [82], [120].

Definition 2.2.1. Let $1 \leqslant p < q < \infty$. The mapping $i_{p,q} : J_p \to J_q$ is called the ***formal identity mapping*** and is defined by

$$i_{p,q}\Big((x_i)_{i=1}^{\infty}\Big) = (x_i)_{i=1}^{\infty} \in J_q$$

for each $(x_i)_{i=1}^{\infty} \in J_p$.

Let $1 \leqslant p < q < \infty$. Observe that if $x \in \ell_p$ then $x \in \ell_q$ and $\|x\|_{\ell_q} \leqslant \|x\|_{\ell_p}$. Indeed, this statement is trivial if $x = 0$. If $x \neq 0$, then define $y = \frac{x}{\|x\|_{\ell_p}}$. In particular, y satisfies $|y_i| \leqslant 1$ for all i, so that $|y_i|^q \leqslant |y_i|^p$. Then $\|y\|_{\ell_q}^q = \sum_{i=1}^{\infty}|y_i|^q \leqslant \sum_{i=1}^{\infty}|y_i|^p = 1$. Hence $\|y\|_{\ell_q} \leqslant 1 = \|y\|_{\ell_p}$, and therefore $\|x\|_{\ell_q} \leqslant \|x\|_{\ell_q}$.

Since $\|x\|_{J_p}$ is defined as the supremum of ℓ_p-norms of certain sequences, we can conclude that $J_p \subseteq J_q$ whenever $1 \leqslant p < q < \infty$, and the formal inclusion operator $i_{p,q} : J_p \to J_q$ is well-defined and has norm 1.

The underlying space for the Read's strictly singular operator T is defined as the ℓ_2-direct sum[1] of ℓ_2 and Y, $X = (\ell_2 \oplus Y)_{\ell_2}$, where Y itself is the ℓ_2-direct sum of an infinite sequence of J_p-spaces $Y = \big(\bigoplus_{i=1}^{\infty} J_{p_i}\big)_{\ell_2}$, with (p_i) a certain strictly increasing sequence in $(2, +\infty)$. The operator T is a compact perturbation of $0 \oplus W_1$, where $W_1 \colon Y \to Y$ acts as a weighted right shift, that is,

$$W_1(x_1, x_2, x_3, \dots) = (0, w_1x_1, w_2x_2, w_3x_3, \dots), \quad x_i \in J_{p_i}$$

with the weights $w_i \to 0$. Note that one should rather write $w_i i_{p_i,p_{i+1}} x_i$ instead of $w_i x_i$.

The main result of the section is

Theorem 2.2.2. *Read's strictly singular operator without invariant subspaces is finitely strictly singular.*

The main step in proving Theorem 2.2.2 is the following theorem.

Theorem 2.2.3. *If $1 \leqslant p < q < \infty$ then the formal identity operator $i_{p,q}$: $J_p \to J_q$ is finitely strictly singular.*

[1]Recall that ℓ_2-direct sum of Banach spaces X and Y is the space $X \times Y$ endowed with the norm $\|(x, y)\| = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$.

This theorem will be proved in the next subsection. We will now show how Theorem 2.2.2 follows from Theorem 2.2.3.

Proof of Theorem 2.2.2. Since finitely strictly singular operators form a closed algebraic ideal in $L(X)$, to prove that the Read's operator is finitely strictly singular, it is enough to prove that the operator W_1 is finitely strictly singular.

For $n \in \mathbb{N}$, define $V_n \colon Y \to Y$ via

$$V_n(x_1, x_2, x_3, \dots) = (0, w_1 x_1, \dots, w_n x_n, 0, 0 \dots), \quad x_i \in J_{p_i}.$$

It is clear that $\|V_n - W_1\| \leqslant \sup\{|w_i| \,:\, i \geqslant n+1\}$. Since $w_i \to 0$, we obtain $\|V_n - W_1\| \to 0$. Therefore it is enough to show that V_n is finitely strictly singular for every n. Given $n \in \mathbb{N}$, one can write

$$V_n = \sum_{i=1}^{n} w_i j_{i+1} i_{p_i, p_{i+1}} P_i,$$

where $P_i \colon Y \to J_{p_i}$ is the canonical projection, $P(x_1, x_2, \dots) = x_i$, and $j_i \colon J_{p_i} \to Y$ is the canonical inclusion, $j_i(x) = (\underbrace{0, \dots, 0}_{i-1}, x, 0, \dots)$. Since finitely strictly singular operators are closed under multiplication by bounded operators, V_n is a sum of finitely strictly singular operators by Theorem 2.2.3, hence is finitely strictly singular itself. □

2.2.2 Zigzag vectors

In this subsection we will prove Theorem 2.2.3. We use ideas of Milman [86] which he developed to prove the following fact.

Theorem 2.2.4. [86, 112] *If $1 \leqslant p < q < \infty$, then the formal identity operator $i : \ell_p \to \ell_q$, $i\big((x_i)\big) = (x_i)$, is finitely strictly singular.*

Milman's proof is based on the fact that every k-dimensional subspace E of $\mathbb{R}^n$ contains a vector x with sup-norm one having (at least) k coordinates equal in modulus to 1. For such a vector, one has $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$. The proof of our result is based on the following refinement of this observation. We will show that x can be chosen so that these k coordinates have alternating signs. For this "highly oscillating" vector x one has $\|x\|_{J_q} \ll \|x\|_{J_p}$.

Definition 2.2.5. A finite or infinite sequence of real numbers in $[-1, 1]$ is called a ***zigzag*** of order k if it has a subsequence of the form $(-1, 1, -1, 1, \dots)$ of length k.

Our proof of Theorem 2.2.3 will be based on the following lemma.

Lemma 2.2.6. *For every $k \leqslant n$, every k-dimensional subspace of $\mathbb{R}^n$ contains a zigzag of order k.*

This lemma can be found in the paper [126] of Voigt. Neither of the groups working on the project [38] (mentioned in the beginning of this section) was aware of the result of Voigt at the time; each of the groups developed their own original proof of this fact. In what follows, we will prove Voigt's lemma using a technique involving combinatorial properties of n-dimensional polytops. However, before presenting the general proof, we would like to show an elementary proof of the partial case $k = n-1$ which only uses linear algebra (this proof has not been published before).

Proof of Lemma 2.2.6 for $k = n - 1$. Let $E \subseteq \mathbb{R}^n$ be a subspace of dimension $n - 1$. Pick a functional $f = (f_i)_{i=1}^n$ such that $\ker f = E$. For each $1 \leqslant m \leqslant n$ such that $f_m \neq 0$, define a projection $P_m : \mathbb{R}^n \to E$ by

$$P_m(x) = x - \frac{1}{f_m} f(x) e_m,$$

where (e_i) is the standard basis for $\mathbb{R}^n$. Let $x^{(m)}$ stands for the vector defind by

$$x^{(m)} = (-1, 1, \dots, (-1)^{m-1}, 0, (-1)^m, \dots, (-1)^{n-1}).$$

Then

$$P_m(x^{(m)}) = \Big(-1, 1, \dots, (-1)^{m-1}, -\frac{1}{f_m} f(x^{(m)}), (-1)^m, \dots, (-1)^{n-1} \Big).$$

The m-th coordinate of this vector satisfies

$$\Big|\frac{1}{f_m}f(x^{(m)})\Big| = \Big|\frac{1}{f_m}\Big(\sum_{i=1}^{m-1}(-1)^i f_i - \sum_{i=m+1}^{n}(-1)^i f_i\Big)\Big|.$$

Define $g_i = (-1)^i f_i$, then the absolute value of the m-th coordinate is simply $\Big|\frac{1}{g_m}\Big(\sum_{i=1}^{m-1} g_i - \sum_{i=m+1}^{n} g_i\Big)\Big|$.

We will show by induction on n that there is m for which this expression is less than 1. Then for this m the vector $P_m(x^{(m)})$ is a k-zigzag in E.

If $n = 2$ then the statement is obvious, because one of the inequalities $\big|\frac{g_1}{g_2}\big| \leqslant 1$, $\big|\frac{g_2}{g_1}\big| \leqslant 1$ always holds.

Suppose the statement is true for $n \geqslant 2$. Let's establish it for $n+1$. We can assume that neither of g_i-s is zero as otherwise we can just drop the zero elements and get a smaller set of numbers. Define $h_1, \ldots, h_n \in \mathbb{R}$ by

$$h_i = \begin{cases} g_i, & \text{if } i \leqslant n-1, \\ g_n + g_{n+1}, & \text{if } i = n. \end{cases}$$

By the induction assumption, there is j such that $\Big|\frac{1}{h_j}\Big(\sum_{i=1}^{j-1} h_i - \sum_{i=j+1}^{n} h_i\Big)\Big| \leqslant 1$. If $i < n$ we are done. If $i = n$ then we get $\Big|\frac{1}{g_n+g_{n+1}}\sum_{j=1}^{n-1} g_j\Big| \leqslant 1$ (and, in particular, $g_n + g_{n+1} \neq 0$).

Claim. If b, c, and $b+c$ are non-zero real numbers, and $\big|\frac{a}{b+c}\big| \leqslant 1$ then either $\big|\frac{a-c}{b}\big| \leqslant 1$ or $\big|\frac{a+b}{c}\big| \leqslant 1$.

The proof of this *Claim* is a tedious check of several cases, each of which is an elementary inequality, so we omit it. The lemma follows from the *Claim* by letting $a = \sum_{j=1}^{n-1} g_j$, $b = g_n$, $c = g_{n+1}$. □

We will now introduce some objects needed for the general proof of Lemma 2.2.6.

Definition 2.2.7. A ***polytope*** in $\mathbb{R}^k$ is the convex hull of a finite set.

A set is a polytope if and only if it is bounded and can be constructed as the intersection of finitely many closed half-spaces (see [59] or [130] for more details about properties of polytopes).

Definition 2.2.8. Let P be a polytope. A ***supporting hyperplane*** for P is a hyperplane H such that P is entirely contained in one of the two closed half-spaces determined by the H and $P \cap H \neq \varnothing$. The intersection of a supporting hyperplane with the polytope P is called a ***face*** of the P. A ***facet*** of P is a face of (affine) dimension $k-1$.

A polytope P is centrally symmetric (i.e., $x \in P \iff -x \in P$) if and only if it can be represented as the absolutely convex hull of its vertices. That is, $P = \operatorname{conv}\{\pm\bar{u}_1, \ldots, \pm\bar{u}_n\}$ where $\pm\bar{u}_1, \ldots, \pm\bar{u}_n$ are the vertices of P. Clearly, if P is centrally symmetric and $P = \cap_{i=1}^n H_i$ where each H_i is a half-space, then $P = \cap_{i=1}^n H_i \bigcap \cap_{i=1}^n(-H_i)$. So, for a centrally cymmetric polytope P, there are vectors $\bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^k$ such that $\bar{u} \in P \iff -1 \leqslant \langle \bar{u}, \bar{a}_i \rangle \leqslant 1$ for all $i = 1, \ldots, m$, and the facets of P are described by $\big\{u \in P \,:\, \langle \bar{u}, \bar{a}_i \rangle = 1\big\}$ or $\big\{u \in P \,:\, \langle \bar{u}, -\bar{a}_i \rangle = 1\big\}$ as $i = 1, \ldots, m$.

Definition 2.2.9. A ***simplex*** in $\mathbb{R}^k$ is the convex hull of $k+1$ points. A polytope P in $\mathbb{R}^k$ is ***simplicial*** if all its faces are simplices (equivalently, if all the facets of P are simplices).

Remark 2.2.10. Every polytope can be perturbed into a simplicial polytope by an iterated "pulling" procedure, see e.g., [59, Section 5.2] for details. We will outline a slight modification of this procedure that also preserves the property of a polytope of being centrally symmetric. Suppose that P is a centrally symmetric polytope with vertices, say $\pm\bar{u}_1, \ldots, \pm\bar{u}_n$. Pull $\bar{u}_1$ "away from" the origin, but not too far, so that it does not reach any affine hyperplane spanned by the facets of P not containing $\bar{u}_1$; denote the resulting point $\bar{u}_1'$. Let $Q = \operatorname{conv}\{\bar{u}_1', -\bar{u}_1, \pm\bar{u}_2, \ldots, \pm\bar{u}_n\}$. By [59, 5.2.2, 5.2.3] this procedure does not affect the facets of P not containing $\bar{u}_1$, while all the facets of Q containing $\bar{u}_1'$ become pyramids having apex at $\bar{u}_1'$. Note that no facet of P contains both $\bar{u}_1$ and $-\bar{u}_1$. Hence, if we put $R = \operatorname{conv}\{\pm\bar{u}_1', \pm\bar{u}_2, \ldots, \pm\bar{u}_n\}$, then, by symmetry, all the facets of R containing $-\bar{u}_1'$ become pyramids with apex at $-\bar{u}_1'$, while the rest of the facets (in particular, the facets containing

Figure 2.1: Pulling out the first pair of vertices.

$\bar{u}'_1$) are not affected.

Now iterate this procedure with every other pair of opposite vertices. Let P' be the resulting polytope, $P' = \text{conv}\{\pm\bar{u}'_1, \ldots, \pm\bar{u}'_n\}$. Clearly, P' is centrally symmetric and simplicial as in [59, 5.2.4]. It also follows from the construction that if F is a facet of P' then all the vertices of P corresponding to the vertices of F belong to the same facet of P.

Definition 2.2.11. A polytope P is called ***marked*** if the following conditions are satisfied:

(i) P is simplicial, centrally symmetric, and has a non-empty interior.

(ii) Every vertex is assigned a natural number, called its ***index***, such that two vertices have the same index if and only if they are opposite to each other.

(iii) All the vertices of P are painted in two colors, say, black and white, so that opposite vertices have opposite colors.

Definition 2.2.12. A face of a marked polytope is said to be ***happy*** if, when one lists its vertices in the order of increasing indices, the colors of the vertices alternate.

For example, the front top facet of the marked polytope in the right hand side of Figure 2.2 is happy. See Figure 2.3 for more examples of happy faces.

Figure 2.2: Examples of marked polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Figure 2.3: Examples of happy simplices in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 2.2.13. The ***color code*** of a face F of a marked polytope P is the list of the colors of its vertices in the order of increasing indices.

For example, the color codes of the simplices in Figure 2.3 are (wbw) and $(bwbw)$ (where b and w correspond to "black" and "white", respectively).

Definition 2.2.14. A face in P is said to be a b***-face*** if its color code starts with b and a w***-face*** otherwise.

Definition 2.2.15. Suppose that R is a set of facets of a k-dimensional polytope P. The ***face boundary*** of R is the set of all $(k-2)$-dimensional faces E of P satisfying $E = F \cap G$ for some facets F and G such that $F \in R$ and $G \notin R$. We will denote the face boundary of R by $\tilde{\partial} R$. If F is a single facet, we put $\tilde{\partial} F = \tilde{\partial}\{F\}$.

Lemma 2.2.16. *Suppose that F is a facet of a marked polytope P. The following are equivalent:*

(i) F is happy;

(ii) $\tilde{\partial} F$ *contains exactly one happy b-face;*

(iii) $\tilde{\partial} F$ *has an odd number of happy b-faces;*

Proof. Observe that since F is a simplex, every face of F can be obtained by dropping one vertex of F and taking the convex hull of the remaining vertices. Hence, the color code of the face is obtained by dropping one symbol from the color code of F.

(i)⇒(ii) Suppose that F is happy, then its color code is either $(bwbw\dots)$ or $(wbwb\dots)$. In the former case, the only happy b-face of F is obtained by dropping the last vertex, while in the latter case the only happy b-face of F is obtained by dropping the first vertex.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Suppose that $\tilde{\partial} F$ has an odd number of happy b-faces. Let E be a happy b-face in $\tilde{\partial} F$. Then the color code of E is the sequence $(bwbw\dots)$ of length $k-1$. The color code of F is obtained by inserting one extra symbol into this sequence. Note that inserting the extra symbol should not result in two consecutive b's or w's, as in this case F would have exactly two happy b-faces (corresponding to removing each of the two consecutive symbols), which would contradict the assumption. Hence, the color code of F should be an alternating sequence, so that F is happy. □

Lemma 2.2.17. *If R is a set of facets of a marked polytope P, then the number of happy facets in R and the number of happy b-faces in $\tilde{\partial} R$ have the same parity.*

Proof. For a set Q of facets of P, denote the parity of the number of happy b-faces in $\tilde{\partial} Q$ by $p(Q)$. It is easy to see that if Q and S are two disjoints sets of facets of P, then $p(Q \cup S) = \big(p(Q) + p(S)\big) \pmod 2$. Hence $p(R) = \sum_{F \in R} p(\{F\}) \pmod 2$. By Lemma 2.2.16, this is equal to the parity of the number of happy facets in R. □

If F is a face of P, then we write $-F$ for the opposite face. If R is a set of facets, we write $-R = \{-F \,:\, F \in R\}$.

Lemma 2.2.18. *Every marked polytope has a happy facet.*

Proof. We will prove a stronger statement: *every marked polytope in $\mathbb{R}^k$ has an odd number of happy b-facets.* The proof is by induction on k. For $k = 1$, the statement is obvious. Let $k > 1$ and let P be a marked polytope in $\mathbb{R}^k$. Denote by $\mathcal{F}$ the set of all facets of P.

For every facet $F \in \mathcal{F}$, let $\bar{n}_F$ be the normal vector of F, directed outwards of P. Fix a vector $\bar{v}$ of length one such that $\bar{v}$ is not parallel to any of the facets of P (equivalently, not orthogonal to $\bar{n}_F$ for any facet F); it is easy to see that such a vector exists. By rotating P we may assume without loss of generality that $\bar{v} = (0, \ldots, 0, 1)$. Let T be the projection from $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$ defined by $T(x_1, \ldots, x_{k-1}, x_k) = (x_1, \ldots, x_{k-1})$. Denote $Q = T(P)$. Since T is linear and surjective, Q is a centrally symmetric convex polytope in $\mathbb{R}^{k-1}$ with a non-empty interior.

Figure 2.4: The images $T(P)$ of the polytopes in Figure 2.2.

It follows from our choice of $\bar{v}$ that the k-th coordinate of $\bar{n}_F$ is non-zero for every facet F. Define a set of facets R by

$$R = \left\{ F \in \mathcal{F} \,:\, \text{the } k\text{-th coordinate of } \bar{n}_F \text{ is positive} \right\}.$$

Clearly, a facet F is in $-R$ if and only if the k-th coordinate of $\bar{n}_F$ is negative. Hence, $-R \cap R = \varnothing$ and $-R \cup R = \mathcal{F}$. Observe that $\tilde{\partial} R = \tilde{\partial}(-R)$; hence $\tilde{\partial} R$ is centrally symmetric. Clearly, every vertical line (i.e., a line parallel to $\bar{v}$) that intersects the interior of P meets the boundary of P at exactly two points and meets the interior of Q at exactly one point. It follows that the restriction of T to $\bigcup R$ is a bijection between $\bigcup R$ and Q. The same is also true for $-R$. Therefore, the restriction of T to $\bigcup \tilde{\partial} R$ is a face-preserving bijection between $\bigcup \tilde{\partial} R$ and the boundary of Q. Under this bijection, the faces in $\tilde{\partial} R$ correspond to the facets of Q. Hence, this bijection induces a structure of a marked polytope on the boundary of Q, making Q into a marked polytope. It follows, by the induction hypothesis, that the boundary of Q has an odd number of happy b-facets. Hence, $\tilde{\partial} R$ has an odd number of happy b-faces. It follows from Lemma 2.2.17 that R has an odd number of happy facets.

Denote the number of all happy b-facets in R by m, and the number of all w-facets in R by ℓ. Then $m + \ell$ is odd. It is clear that that F is a happy b-facet if and only if $-F$ is a happy w-facet. Therefore $-R$ contains ℓ happy b-facets and m happy w-facets. Hence, the total number of happy b-facets of P is $m + \ell$, which we proved to be odd. $\square$

We are now ready to prove the main lemma.

Proof of Lemma 2.2.6. Suppose that $k \leqslant n$ and E is a subspace of $\mathbb{R}^n$ with $\dim E = k$. Let $\{\bar{b}_1, \dots, \bar{b}_k\}$ be a basis of E. We need to find a linear combination of these vectors $\bar{x} := a_1\bar{b}_1 + \cdots + a_k\bar{b}_k$ such that $\bar{x}$ is a zigzag. Let B be the $n \times k$ matrix with columns $\bar{b}_1, \dots, \bar{b}_k$, and let $\bar{u}_1, \dots, \bar{u}_n$ be the rows of B. If $\bar{a} = (a_1, \dots, a_k)$, then $x_i = \langle \bar{u}_i, \bar{a} \rangle$ as $i = 1, \dots, n$. Thus, it suffices to find $\bar{a} \in \mathbb{R}^k$ such that the vector $\big(\langle \bar{u}_i, \bar{a} \rangle\big)_{i=1}^n$ is a zigzag of order k.

Consider the centrally symmetric convex polytope P spanned by $\bar{u}_1, \dots, \bar{u}_n$, i.e., $P = \operatorname{conv}\{\pm\bar{u}_1, \dots, \pm\bar{u}_n\}$. Let $\pm\bar{u}_{m_1}, \dots, \pm\bar{u}_{m_r}$ be the smallest sequence of vectors such that $P = \operatorname{conv}\{\pm\bar{u}_{m_1}, \dots, \pm\bar{u}_{m_r}\}$. Then, in particular, each u_{m_i} (and $-u_{m_i}$) is a vertex of P. Using the "pulling" procedure (see Remark 2.2.10), construct a simplicial centrally symmetric polytope $P' = \operatorname{conv}\{\pm\bar{u}'_{m_1}, \dots, \pm\bar{u}'_{m_r}\}$. Every vertex of P' is either $\bar{u}'_{m_i}$ or $-\bar{u}'_{m_i}$ for some i. Paint the vertex white in the former case and black in the latter case (this defines the color uniquely by the minimality of the chosen sequence of vertices); assign index i to this vertex. This way we make P' into a marked polytope.

By Lemma 2.2.18, P' has a happy facet. This facet (or the facet opposite to it) is spanned by some $-\bar{u}'_{m_{i_1}}$, $\bar{u}'_{m_{i_2}}$, $-\bar{u}'_{m_{i_3}}$, $\bar{u}'_{m_{i_4}}$, etc, for some $1 \leqslant i_1 <$

$\cdots < i_k \leqslant r$ (recall that any facet is a $(k-1)$-simplex, so that there are exactly k indices in this "chain"). It follows that $-\bar{u}_{m_{i_1}}$, $\bar{u}_{m_{i_2}}$, $-\bar{u}_{m_{i_3}}$, $\bar{u}_{m_{i_4}}$, etc, are all contained in the same facet of P. Hence, they are contained in an affine hyperplane, say L, such that P "sits" between L and $-L$. Let $\bar{a}$ be the vector defining L, that is, $L = \big\{\bar{u} \,:\, \langle \bar{u}, \bar{a}\rangle = 1\big\}$. Since P is between L and $-L$, we have $-1 \leqslant \langle \bar{u}, \bar{a}\rangle \leqslant 1$ for every $\bar{u}$ in P. In particular, $-1 \leqslant x_i = \langle \bar{u}_i, \bar{a}\rangle \leqslant 1$ for $i = 1, \dots, n$. On the other hand, it follows from $-\bar{u}_{m_{i_1}}, \bar{u}_{m_{i_2}}, -\bar{u}_{m_{i_3}}, \bar{u}_{m_{i_4}}, \dots \in L$ that $x_{m_{i_1}} = -1$, $x_{m_{i_2}} = 1$, $x_{m_{i_3}} = -1$, $x_{m_{i_4}} = 1$, etc. Hence, $\bar{x}$ is a zigzag of order k. □

Corollary 2.2.19. *Let $k \in \mathbb{N}$, then every k-dimensional subspace of c_0 contains a zigzag of order k.*

Proof. Let F be a subspace of c_0 with $\dim F = k$. For every $n \in \mathbb{N}$, define $P_n \colon c_0 \to \mathbb{R}^n$ via $P_n \colon (x_i)_{i=1}^{\infty} \mapsto (x_i)_{i=1}^{n}$. We claim that there exists m such that every vector in F attains its norm on the first m coordinates. Indeed, define $g \colon F \setminus \{0\} \to \mathbb{N}$ via $g(x) = \max\big\{i \,:\, |x_i| = \|x\|_\infty\big\}$. Then g is upper semi-continuous, hence bounded on the unit sphere of F, so that we put $m = \max\big\{g(x) \,:\, x \in F, \|x\| = 1\big\}$.

If $x \in F$ is a nonzero vector then $\|x\|$ is attained on some x_i with $1 \leqslant i \leqslant m$, so that $x_i \neq 0$. In particular, P_m is one-to-one on F. Therefore $P_m(F)$ is a k-dimensional subspace of $\mathbb{R}^m$. Hence, by Theorem 2.2.6, there exists $x \in F$ such that $P_m x$ is a zigzag of order k. It follows that x is a zigzag of order k in F. □

Proof of Theorem 2.2.3. If $x \in J_p$, then $|x_i - x_j|^q \leqslant \big(2\|x\|_\infty\big)^{q-p}|x_i - x_j|^p$ for every $i, j \in \mathbb{N}$. Therefore $\|x\|_{J_q} \leqslant \big(2\|x\|_\infty\big)^{1-\frac{p}{q}}\|x\|_{J_p}^{\frac{p}{q}}$. Fix an arbitrary $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $(k-1)^{\frac{1}{p}-\frac{1}{q}} > \frac{1}{\varepsilon}$. Suppose that E is a subspace of J_p with $\dim E = k$. By Corollary 2.2.19, there is a zigzag $z \in E$ of order k. By the definition of norm in J_p, we have $\|z\|_{J_p} \geqslant 2(k-1)^{\frac{1}{p}}$.

Put $y = \frac{z}{\|z\|_{J_p}}$. Then $y \in E$ with $\|y\|_{J_p} = 1$. Obviously, $\|y\|_\infty \leqslant \frac{1}{2}(k-1)^{-\frac{1}{p}}$, so that

$$\|i_{p,q}(y)\|_{J_q} = \|y\|_{J_q} \leqslant (k-1)^{\frac{1}{q}-\frac{1}{p}} \|y\|_{J_p}^{\frac{p}{q}} < \varepsilon.$$

Hence, $i_{p,q}$ is finitely strictly singular. □

2.3 $\mathcal{S}_\xi$-singular operators

In this section, we investigate the class of $\mathcal{S}_\xi$-singular operators. The results of this section were published in [97].

2.3.1 Definition and basic properties

Milman proved in [86] that a product of two strictly singular operators on $L_p[0,1]$ ($1 \leqslant p < \infty$) or on $C[0,1]$ is compact. The importance of this result follows from the fact that compact operators are well-understood and have many nice properties. In particular, this result and Theorem 1.1.2 imply that any strictly singular operator on these spaces has an invariant subspace.

The $\mathcal{S}_\xi$-singular operators are defined in terms of ***Schreier families*** $\mathcal{S}_\xi$ which were originally introduced in [9]. These families are defined inductively for every ordinal $\xi < \omega_1$ in the following way. Set

$$\mathcal{S}_0 = \big\{\{n\} \,:\, n \in \mathbb{N}\big\} \cup \{\varnothing\}.$$

After defining $\mathcal{S}_\xi$ for some $\xi < \omega_1$, set

$$\mathcal{S}_{\xi+1} = \big\{\cup_{i=1}^n F_i \,:\, n \in \mathbb{N}, n \leqslant F_1 < \dots < F_n, F_i \in S_\xi\big\}.$$

Here by $A < B$ where A and B are two finite subsets of $\mathbb{N}$ we mean $\max A < \min B$, and by $n \leqslant A$ we mean $n \leqslant \min A$. It is a general convention that $\varnothing < F$ and $F < \varnothing$ for any non-empty finite set $F \subseteq N$. If $\xi < \omega_1$ is a limit ordinal and $\mathcal{S}_\alpha$ has been defined for all $\alpha < \xi$ then fix a sequence $\xi_n \nearrow \xi$ and define

$$\mathcal{S}_\xi = \big\{F \,:\, n \leqslant F \text{ and } F \in S_{\xi_n} \text{ for some } n \in \mathbb{N}\big\}.$$

Remark 2.3.1. In general, different choices of sequences $\xi_n \nearrow \xi$ in the last part of the definition of the Schreier families produce different families. Whereas this is not important for many constructions in the literature using Schreier families, we will assume in our exposition that particular sequences ξ_n have been fixed.

Definition 2.3.2. A family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called ***spreading*** if whenever $\{n_1, \ldots, n_k\} \in \mathcal{S}_\xi$ with $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_k$ satisfies $n_i \leqslant m_i$ for $i = 1, \ldots, k$, then $\{m_1, \ldots, m_k\} \in \mathcal{F}$.

It can be shown that each class $\mathcal{S}_\xi$ is spreading. Also, it is obvious that $\mathcal{S}_\xi \subseteq \mathcal{S}_{\xi+1}$. However, $\xi < \zeta$ doesn't generally imply $\mathcal{S}_\xi \subseteq \mathcal{S}_\zeta$.

For a sequence (x_n) in a Banach space and $A \subseteq \mathbb{N}$ we will write $[x_i]_{i \in A}$ for the closed linear span of $\{x_i\}_{i \in A}$.

Definition 2.3.3. [11] Let X and Y be two Banach spaces and $\xi < \omega_1$. We say that an operator $T \in L(X, Y)$ is $\mathcal{S}_\xi$***-singular*** and write $T \in \mathcal{SS}_\xi(X, Y)$ if for every $\varepsilon > 0$ and every basic sequence (x_n) in X there exist a set $F \in \mathcal{S}_\xi$ and a vector $z \in [x_i]_{i \in F}$ such that $\|Tz\| < \varepsilon \|z\|$. If $X = Y$ then we write $T \in \mathcal{SS}_\xi(X)$.

Remark 2.3.4. It was pointed out in [11] that T is $\mathcal{S}_\xi$-singular if and only if for every normalized basic sequence (x_n) and $\varepsilon > 0$ there exist a subsequence (x_{n_k}), $F \in \mathcal{S}_\xi$ and $w \in [x_{n_k}]_{k \in F}$ such that $\|Tw\| < \varepsilon \|w\|$.

It is easy to see that if X and Y are Banach spaces then for every $1 \leqslant \xi < \omega_1$ we have the following chain of inclusions:

$$\mathcal{K}(X, Y) \subseteq \mathcal{FSS}(X, Y) \subseteq \mathcal{SS}_\xi(X, Y) \subseteq \mathcal{SS}(X, Y).$$

It has already been mentioned in Section 2.1 that compact, strictly singular, and finitely strictly singular operators form closed operator ideals. It is a natural question whether $\mathcal{S}_\xi$-singular operators have the same property. It was shown in [11] that if the underlying Banach space X is reflexive then the class $\mathcal{SS}_\xi(X)$ is closed under the right and left multiplication by bounded operators. We will extend this result to non-reflexive spaces. We will use the following deep result of Rosenthal [111].

Theorem 2.3.5. [111] *Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space X. Then $(x_n)_{n=1}^\infty$ has a subsequence $(x_{n_k})_{k=1}^\infty$ satisfying one of the two mutually exclusive alternatives:*

(i) $(x_{n_k})_{k=1}^\infty$ is equivalent to the unit vector basis of ℓ_1;

(ii) $(x_{n_k})_{k=1}^\infty$ is a weak Cauchy sequence.

The following corollary of Theorem 2.3.5 is a standard fact. We present its proof for the convenience of the reader.

Lemma 2.3.6. *Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space X. Then there is a subsequence $(x_{n_k})_{k=1}^\infty$ such that one of the following conditions hold.*

(i) $(x_{n_k})_{k=1}^\infty$ converges;

(ii) $(x_{n_k})_{k=1}^\infty$ is equivalent to the unit vector basis of ℓ_1;

(iii) The difference sequence $(d_k)_{k=1}^\infty$ defined by $d_k = x_{n_{k+1}} - x_{n_k}$ has a seminormalized weakly null basic subsequence. Moreover, if X has a basis then this subsequence can be chosen to be equivalent to a block sequence of the basis.

Proof. If $(x_n)_{n=1}^\infty$ does not satisfy condition (ii) then $(x_n)_{n=1}^\infty$ has a weakly Cauchy subsequence $(x_{n_k})_{k=1}^\infty$ by Theorem 2.3.5. Then the sequence of differences $(x_{n_{k+1}} - x_{n_k})_{k=1}^\infty$ is weakly null. If $(x_n)_{n=1}^\infty$ does not contain a subsequence satisfying condition (i) then $(x_{n_{k+1}} - x_{n_k})_{k=1}^\infty$ is seminormalized. The condition (iii) now follows from the Bessaga-Pełczyński selection principle [27]. □

If F is a finite set, we will use the symbol $\#F$ for the number of elements in F.

We will use the following two technical lemmas.

Lemma 2.3.7. *Suppose that* $1 \leqslant \xi < \omega_1$*,* $A \in \mathcal{S}_\xi$*, and put* $A^{\times 2} = \{2i, 2i+2 : i \in A\}$*. Then* $A^{\times 2}$ *is also in* $\mathcal{S}_\xi$*.*

Proof. The proof is by induction on ξ. For $\xi = 1$, if $A \in \mathcal{S}_1$ then $\#A \leqslant \min A$. But then $\#(A^{\times 2}) \leqslant 2(\#A) \leqslant 2 \min A = \min A^{\times 2}$, so that $A^{\times 2} \in \mathcal{S}_\xi$. Suppose that we have already proved the statement for ξ, and let $A \in \mathcal{S}_{\xi+1}$. Then

$$A = \bigcup_{i=1}^{n} F_i \text{ where } n \in \mathbb{N},\ n \leqslant F_1 < \cdots < F_n, \text{ and } F_i \in \mathcal{S}_\xi \text{ for each } i.$$

It follows that

$$A^{\times 2} = \bigcup_{i=1}^{n} F_i^{\times 2}, \text{ and } F_i^{\times 2} \in \mathcal{S}_\xi \text{ for each } i \text{ by the induction hypothesis.}$$

Let $G_1 = F_1^{\times 2}$, $G_i = F_i^{\times 2} \backslash F_{i-1}^{\times 2}$ for $2 \leqslant i \leqslant n$, then

$$A^{\times 2} = \bigcup_{i=1}^{n} G_i, \quad G_i \in \mathcal{S}_\xi \text{ for each } i, \text{ and } n < 2n \leqslant G_1 < \cdots < G_n,$$

so that $A^{\times 2} \in \mathcal{S}_{\xi+1}$. Finally, suppose that ξ is a limit ordinal and $A \in \mathcal{S}_\xi$. Then $A \in \mathcal{S}_{\xi_n}$ and $n \leqslant A$ for some $n \in \mathbb{N}$. It follows from the induction hypothesis that $n < 2n \leqslant A^{\times 2} \in \mathcal{S}_{\xi_n}$, so that $A^{\times 2} \in \mathcal{S}_\xi$. □

Lemma 2.3.8. *Let* (n_i) *be a strictly increasing sequence,* $1 \leqslant \xi \leqslant \omega_1$*, and* $A \in \mathcal{S}_\xi$*. The the set* $A^{\times 2}_{(n_i)} = \{2n_i, 2n_{i+1} : i \in A\}$ *belongs to* $\mathcal{S}_\xi$*.*

Proof. The lemma follows from Lemma 2.3.7 and the fact that $\mathcal{S}_\xi$ is spreading.

□

Now we are ready to prove that the class of $\mathcal{S}_\xi$-singular operators is stable under left and right multiplications.

Theorem 2.3.9. *Suppose that* X *and* Y *are two Banach spaces and* $1 \leqslant \xi < \omega_1$*. If* $T \in \mathcal{SS}_\xi(X,Y)$*,* $A \in L(Y,V)$*, and* $B \in L(U,X)$ *for some Banach spaces* U *and* V*, then* $ATB \in \mathcal{SS}_\xi(U,V)$*.*

Proof. Assume that A and B are non-zero, as otherwise the statement is trivial. Let (x_n) be a basic sequence in X, and $\varepsilon > 0$. Since $A \neq 0$, there exists $F \in \mathcal{S}_\xi$ and $z \in [x_n]_{n\in F}$ such that $\|Tz\| < \frac{\varepsilon}{\|A\|}\|z\|$. Thus $\|ATz\| \leqslant \|A\|\|Tz\| < \|A\|\frac{\varepsilon}{\|A\|}\|z\| = \varepsilon\|z\|$, so that $AT \in \mathcal{SS}_\xi(X,V)$, hence $\mathcal{SS}_\xi$ is stable under the left multiplication.

Let us show that that $TB \in \mathcal{SS}_\xi(U,Y)$, so that $\mathcal{SS}_\xi(X,Y)$ is stable under the right multiplication. Let (x_n) be a basic sequence in U and $\varepsilon > 0$. We can assume by Remark 2.3.4 that (x_n) is normalized. First, suppose that (Bx_n) has a basic subsequence, say, (Bx_{n_k}). Then there exists $G \in \mathcal{S}_\xi$ and $y = \sum_{k\in G}\alpha_k Bx_{n_k}$ such that $\|Ty\| \leqslant \frac{\varepsilon}{\|B\|}\|y\|$. Put $x = \sum_{k\in G}\alpha_k x_{n_k}$, then $Bx = y$ and $x \in [x_i]_{i\in F}$ where $F = \{n_k \,:\, k \in G\}$. We obtain

$$\|TBx\| = \|Ty\| \leqslant \frac{\varepsilon}{\|B\|}\|y\| \leqslant \varepsilon\|x\|.$$

Finally, observe that $F \in \mathcal{S}_\xi$ since $\mathcal{S}_\xi$ is spreading.

Now suppose that (Bx_n) has no basic subsequences. Then (Bx_{2n}) has no basic subsequences. By Lemma 2.3.6, (x_{2n}) has a subsequence (x_{2n_k}) such that either (Bx_{2n_k}) converges in norm or $(Bx_{2n_{k+1}} - Bx_{2n_k})$ has a basic subsequence. Denote $y_k = Bx_{2n_{k+1}} - Bx_{2n_k}$. Then either $y_k \to 0$ in norm or (y_k) has a basic subsequence.

Suppose that (y_{k_i}) is basic. Then there exists $G \in \mathcal{S}_\xi$ and $z = \sum_{i\in G}\alpha_i y_{k_i}$ such that $\|Tz\| \leqslant \frac{\varepsilon}{\|B\|}\|z\|$. Put $x = \sum_{i\in G}\alpha_i(x_{2n_{k_i+1}} - x_{2n_{k_i}})$. Observe that the set $A = \{k_i \colon i \in G\}$ belongs to $\mathcal{S}_\xi$ since $\mathcal{S}_\xi$ is spreading. Hence $x \in [x_i]_{i\in F}$ where $F = A^{\times 2}_{(n_k)} \in \mathcal{S}_\xi$ by Lemma 2.3.8. Also, $z = Bx$, so that

$$\|TBx\| = \|Tz\| \leqslant \frac{\varepsilon}{\|B\|}\|z\| \leqslant \varepsilon\|x\|.$$

Finally, suppose that $y_k \to 0$. Then we can find $m \in \mathbb{N}$ such that $\{m\} \in \mathcal{S}_\xi$ and $\|y_m\| < \frac{\varepsilon}{C\|T\|}$, where C is the basis constant of (x_n). Put $x = x_{2n_{m+1}} - x_{2n_m}$. Note that $\{2n_m, 2n_{m+1}\} = \{m\}^{\times 2}_{(n_k)} \in \mathcal{S}_\xi$ and $1 = \|x_{2n_m}\| \leqslant C\|x_{2n_{m+1}} - x_{2n_m}\|$, so that

$$\|TBx\| = \|Ty_m\| \leqslant \frac{\varepsilon}{C} \leqslant \varepsilon\|x_{2n_{m+1}} - x_{2n_m}\| = \varepsilon\|x\|.$$

□

Remark 2.3.10. It is known (see, for instance, the remark before Lemma 5 in [96]) that there exists a finitely strictly singular operator whose adjoint is not strictly singular. This already shows that the class of $\mathcal{S}_\xi$-singular operators is not stable under taking adjoints.

2.3.2 Products of $\mathcal{S}_\xi$-singular operators

In this subsection, we will slightly improve Theorem 4.1 of [11] by relaxing the assumption in that theorem. We will use some concepts of the asymptotic theory of Banach spaces.

Definition 2.3.11. Given two sequences $(x_n)_{n=1}^\infty$ and $(\tilde{x}_n)_{n=1}^\infty$ in Banach spaces X and Y, respectively, we say that $(\tilde{x}_n)_{n=1}^\infty$ is a ***spreading model*** for (x_n) if there exists a sequence of reals (ε_n) such that $\varepsilon_n \searrow 0$ and

$$\Big| \|\sum_{i=1}^n a_i x_{k_i}\| - \|\sum_{i=1}^n a_i \tilde{x}_i\| \Big| < \varepsilon_n$$

whenever $n \leqslant k_1 < \cdots < k_n$ and $a_1, \ldots, a_n \in [-1, 1]$.

Definition 2.3.12. A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called ***spreading*** if $\| \sum_{n=1}^m a_n x_n\| = \| \sum_{n=1}^m a_n x_{n_k}\|$ for all scalars a_n and all $n_1 < n_2 < \cdots < n_m$.

It follows from Definition 2.3.11 that spreading model for any sequence is necessarily spreading. For more information about spreading models, we refer the reader to [23], [31], [32], [91], [12], and [44].

Definition 2.3.13. A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called ***suppression 1-unconditional*** if, whenever $A \subseteq B$ are two finite sets of natural numbers, $\| \sum_A a_n x_n\| \leqslant \| \sum_B a_n x_n\|$ for all scalars a_n.

Theorem 2.3.14. [23, 31, 32] *If X is a Banach space then every bounded sequence $(x_n)_{n=1}^\infty$ in X with no convergent subsequences has a subsequence with a spreading model. Moreover, the spreading model is suppression 1-unconditional if $(x_n)_{n=1}^\infty$ is weakly null.*

In particular, Theorem 2.3.14 applies to seminormalized basic sequences. Following [12], we will denote the set of spreading models of all seminormalized weakly null basic sequences of a Banach space X by $\mathrm{SP}_w(X)$.

Definition 2.3.15. [11] We say that a seminormalized basic sequence $(x_n)_{n=1}^\infty$ in a Banach space X is ***Schreier spreading***, if there exists $1 \leqslant K < \infty$ such that for every $F \in \mathcal{S}_1$, increasing sequence $n_1 < n_2 < \cdots$ of positive integers, and scalars $(a_i)_{i\in F}$ we have

$$\frac{1}{K}\|\sum_{i\in F} a_i x_{n_i}\| \leqslant \|\sum_{i\in F} a_i x_i\| \leqslant K\|\sum_{i\in F} a_i x_{n_i}\|.$$

The following lemma follows from Theorem 2.3.14.

Lemma 2.3.16. [11] *Every bounded sequence without convergent subsequences has a Schreier spreading subsequence.*

Definition 2.3.17. We call two basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in Banach spaces X and Y, respectively, ***equivalent*** and write $(x_n)_{n=1}^\infty \approx (y_n)_{n=1}^\infty$ if $\sum_{n=1}^\infty a_n x_n$ converges if and only if $\sum_{n=1}^\infty a_n y_n$ converges. We will write $(x_n)_{n=1}^\infty \preceq (y_n)_{n=1}^\infty$ if the convergence of $\sum_{n=1}^\infty a_n y_n$ implies the convergence of $\sum_{n=1}^\infty a_n x_n$. If $(x_n)_{n=1}^\infty \preceq (y_n)_{n=1}^\infty$ and $(x_n)_{n=1}^\infty$ and $(x_n)_{n=1}^\infty \not\approx (y_n)_{n=1}^\infty$ then we will write $(x_n)_{n=1}^\infty \prec (y_n)_{n=1}^\infty$.

Following [11], we will denote the set of seminormalized weakly null Schreier spreading basic sequences of a Banach space X by $\mathrm{SP}_{1,w}(X)$. The set of equivalence classes of elements of $\mathrm{SP}_{1,w}(X)$ with respect to the equivalence relation in Definition 2.3.17 will be denoted by $\mathrm{SP}_{1,w,\approx}(X)$. Clearly, the relation $\preceq$ from Definition 2.3.17 defines a partial order relation on the set $\mathrm{SP}_{1,w,\approx}(X)$.

Definition 2.3.18. Let X be a Banach space and $1 \leqslant \xi < \omega_1$. Supppose that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are two Schreier spreading sequences in X. For $K \geqslant 1$ we write $(x_n)_{n=1}^\infty \overset{K}{\approx}_\xi (y_n)_{n=1}^\infty$ if for every $F \in \mathcal{S}_\xi$ and scalars $(a_i)_{i\in F}$ we have

$$\frac{1}{K}\Big\|\sum_{i\in F} a_i y_i\Big\| \leqslant \Big\|\sum_{i\in F} a_i x_i\Big\| \leqslant K\Big\|\sum_{i\in F} a_i y_i\Big\|.$$

We write $(x_n)_{n=1}^\infty \approx_\xi (y_n)_{n=1}^\infty$ if $(x_n)_{n=1}^\infty \overset{K}{\approx}_\xi (y_n)_{n=1}^\infty$ for some $K \geqslant 1$. Similarly, we write $(x_n)_{n=1}^\infty \preceq_\xi (y_n)_{n=1}^\infty$ if there is a constant $K > 0$ such that

$$\Big\|\sum_{i\in F} a_i x_i\Big\| \leqslant K\Big\|\sum_{i\in F} a_i y_i\Big\|$$

for every $F \in \mathcal{S}_\xi$ and every sequence $(a_n)_{n=1}^\infty$ of reals. Also, we write $(x_n)_{n=1}^\infty \prec_\xi (y_n)_{n=1}^\infty$ if $(x_n)_{n=1}^\infty \preceq_\xi (y_n)_{n=1}^\infty$ and $(x_n)_{n=1}^\infty \not\approx_\xi (y_n)_{n=1}^\infty$

We will denote the set of equivalence classes of elements of $\mathrm{SP}_{1,w}(X)$ with respect to the relation $\approx_\xi$ by $\mathrm{SP}_{1,w,\xi}(X)$. It is easy to see that "$\preceq_\xi$" induces a partial order relation on $\mathrm{SP}_{1,w,\xi}(X)$, which we will still denote "$\preceq_\xi$".

Remark 2.3.19. Suppose that $T \in L(X)$ and $(x_n)_{n=1}^\infty$ is in $\mathrm{SP}_{1,w}(X)$ such that $(Tx_n)_{n=1}^\infty$ is also in $\mathrm{SP}_{1,w}(X)$. Then $(Tx_n)_{n=1}^\infty \preceq_\xi (x_n)_{n=1}^\infty$ as $\big\|\sum_{i\in F} a_i T x_i\big\| \leqslant \|T\|\big\|\sum_{i\in F} a_i x_i\big\|$ for every $F \in \mathcal{S}_\xi$. Furthermore, if $T \in \mathcal{SS}_\xi(X)$ then $(Tx_n)_{n=1}^\infty \prec_\xi (x_n)_{n=1}^\infty$ as otherwise there would exist $C > 0$ such that $\big\|\sum_{i\in F} a_i T x_i\big\| \geqslant C\big\|\sum_{i\in F} a_i x_i\big\|$ for every $F \in \mathcal{S}_\xi$ and real $(a_n)_{n=1}^\infty$, which would contradict T being $\mathcal{S}_\xi$-singular.

Our next goal will be to present an extension of the following result (Theorem 4.1 from [11]).

Theorem 2.3.20. *Let X be a Banach space, and $N \in \mathbb{N} \cup \{0\}$ be such that the set $\mathrm{SP}_{1,w,\approx}(X)$ contains exactly N elements. Then the product of any $(N+1)$ strictly singular operators on X is compact. Moreover, if ℓ_1 does not isomorphically embed in X then the product of any N strictly singular operators on X is compact.*

If $1 \leqslant \xi < \omega$ and the set $\mathrm{SP}_{1,w,\xi}(X)$ contains exactly N elements, then for any $T_1, \ldots, T_N \in \mathcal{SS}_\xi(X)$ and every strictly singular operator S the composition $T_N T_{N-1} \ldots T_1 S$ is compact. Moreover, if ℓ_1 does not isomorphically embed in X then $T_N T_{N-1} \ldots T_1$ is compact.

The following lemma is a part of the proof of Theorem 4.1 in [11].

Lemma 2.3.21. *Let X be a Banach space and $N \geqslant 0$. Let $T_1, \ldots, T_N \in L(X)$ and $S \in \mathcal{SS}(X)$. If $T_N T_{N-1} \ldots T_1 S$ is not compact or $\ell_1 \not\hookrightarrow X$ and $T_N T_{N-1} \ldots T_1$ is not compact then there are $N+1$ sequences $(x_n^{(1)})_{n=1}^{\infty}, \ldots, (x_n^{(N+1)})_{n=1}^{\infty} \in \mathrm{SP}_{1,w}(X)$ such that $x_n^{(k)} = T_{k-1} \ldots T_1 x_n^{(1)}$, for each $k \leqslant N+1$, $n \in \mathbb{N}$.*

Now we will present the extension of Theorem 4.1 of [11]. We show that the conclusion of this theorem stays valid when the set of equivalence classes of spreading models of Schreier spreading sequences is infinite but contains no infinite chains. This approach was motivated by studies of the order structure of the set of spreading models of a Banach space in [12] and [44]. We would like to thank Professor Troitsky for sharing his ideas about this approach.

Theorem 2.3.22. *Let X be a Banach space and $N \geqslant 0$ be such that the partially ordered set $\big(\mathrm{SP}_{1,w,\approx}(X), \preceq\ \big)$ contains no chains of length greater than N. Then the product of any $(N+1)$ strictly singular operators on X is compact. Moreover, if ℓ_1 does not isomorphically embed in X then the product of any N strictly singular operators on X is compact.*

If $\big(\mathrm{SP}_{1,w,\xi}, \preceq_\xi\big)$ contains no chains of length greater than N, then for any $T_1, \ldots, T_N \in \mathcal{SS}_\xi(X)$ and every strictly singular operator S the composition $T_N T_{N-1} \ldots T_1 S$ is compact. Moreover, if ℓ_1 does not isomorphically embed in X then $T_N T_{N-1} \ldots T_1$ is compact.

Proof. Suppose that the conclusion of the theorem is not true. When $N = 0$, we just simply have that $\mathrm{SP}_{1,w} = \varnothing$, and this case was considered in [11].

Assume that $N \neq 0$. By Lemma 2.3.21, there are $N+1$ sequences $(x_n^{(1)}), \ldots, (x_n^{(N+1)}) \in \mathrm{SP}_{1,w}(X)$ such that $x_n^{(k)} = T_{k-1} \ldots T_1 x_n^{(1)}$, for each $k \leqslant N+1$, $n \in \mathbb{N}$. It is easy to see that $(x_n^{(N+1)}) \prec (x_n^{(N)}) \prec \cdots \prec (x_n^{(1)})$ in $\mathrm{SP}_{1,w,\approx}(X)$ because $T_1, \ldots, T_N$ are strictly singular. This gives a chain of length $N+1$ in $\mathrm{SP}_{1,w,\approx}(X)$, a contradiction.

The proof of the second part of the statement is analogous. □

To present another version of the last theorem, we will need two lemmas. The first lemma is a standard fact, see e.g., [11, Lemma 3.1].

Lemma 2.3.23. *Suppose that* $(x_n)_{n=1}^\infty$ *is a seminormalized basic sequence with a spreading model* $(\tilde{x}_n)_{n=1}^\infty$. *Then for every* $K > 1$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\frac{1}{K}\Big\|\sum_{i=1}^n a_i x_{k_i}\Big\| \leqslant \Big\|\sum_{i=1}^n a_i \tilde{x}_i\Big\| \leqslant K\Big\|\sum_{i=1}^n a_i x_{k_i}\Big\|$$

whenever $n_0 \leqslant n \leqslant k_1 < \cdots < k_n$ *and* $a_1, \ldots, a_n \in \mathbb{R}$.

Lemma 2.3.24. *Suppose that* $(x_n)_{n=1}^\infty$ *and* $(y_n)_{n=1}^\infty$ *are two Schreier spreading sequences in* $\mathrm{SP}_{1,w}(X)$ *with spreading models* $(\tilde{x}_n)_{n=1}^\infty$ *and* $(\tilde{y}_n)_{n=1}^\infty$, *respectively. Then* $(\tilde{x}_n)_{n=1}^\infty \preceq (\tilde{y}_n)_{n=1}^\infty$ *if and only if* $(x_n)_{n=1}^\infty \preceq_1 (y_n)_{n=1}^\infty$.

Proof. Let $(\tilde{x}_n)_{n=1}^\infty \preceq (\tilde{y}_n)_{n=1}^\infty$. Then there is $0 < C < \infty$ such that

$$\Big\|\sum_{i=1}^n a_i \tilde{x}_i\Big\| \leqslant C\Big\|\sum_{i=1}^n a_i \tilde{y}_i\Big\|$$

for every $a_i \in \mathbb{R}$. By Lemma 2.3.23, there is $n_0 \in \mathbb{N}$ such that

$$\Big\|\sum_{i=1}^n a_i \tilde{x}_i\Big\| \overset{2}{\approx} \Big\|\sum_{i=1}^n a_i x_{k_i}\Big\| \quad \text{and} \quad \Big\|\sum_{i=1}^n a_i \tilde{y}_i\Big\| \overset{2}{\approx} \Big\|\sum_{i=1}^n a_i y_{k_i}\Big\|$$

whenever $n_0 \leqslant n \leqslant k_1 < \cdots < k_n$ and $a_1, \ldots, a_n \in \mathbb{R}$.

Since $(x_i)_{n=1}^\infty$ and $(y_i)_{n=1}^\infty$ are Schreier spreading, we have

$$(x_i)_{n=1}^\infty \overset{K}{\approx}_1 (x_{n_0+i})_{n=1}^\infty \quad \text{and} \quad (y_i)_{n=1}^\infty \overset{K}{\approx}_1 (y_{n_0+i})_{n=1}^\infty$$

for some K.

Let $F = \{n_1, \ldots, n_m\} \in \mathcal{S}_1$, and $a_1, \ldots, a_m \in \mathbb{R}$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^m a_i x_{n_i}\Big\| \leqslant K\Big\|\sum_{i=1}^m a_i x_{n_0+n_i}\Big\| \leqslant 2K\Big\|\sum_{i=1}^m a_i \tilde{x}_i\Big\| \leqslant 2CK\Big\|\sum_{i=1}^m a_i \tilde{y}_i\Big\| \\ \leqslant 4CK\Big\|\sum_{i=1}^m a_i y_{n_0+n_i}\Big\| \leqslant 4CK^2\Big\|\sum_{i=1}^m a_i y_{n_i}\Big\|.\end{aligned}$$

For the converse, let now $(x_n)_{n=1}^\infty \preceq_1 (y_n)_{n=1}^\infty$. Again, using Lemma 2.3.23, we get: there is $n_0 \in \mathbb{N}$ such that

$$\Big\|\sum_{i=1}^m a_i \tilde{x}_i\Big\| \leqslant 2\Big\|\sum_{i=1}^m a_i x_{n_i}\Big\| \leqslant 2C\Big\|\sum_{i=1}^m a_i y_{n_i}\Big\| \leqslant 4C\Big\|\sum_{i=1}^m a_i \tilde{y}_i\Big\|$$

for every $m \in \mathbb{N}$, every $a_1, \ldots, a_m \in \mathbb{R}$, and $n_0 \leqslant m \leqslant n_1 < \cdots < n_m$. $\square$

Corollary 2.3.25. *Let X be a Banach space, and $n \geqslant 0$ be such that $(\mathrm{SP}_w(X)/\approx, \preceq)$ contains no chains of length greater than n. Then for any $T_1, \dots, T_n \in \mathcal{SS}_1(X)$ and every strictly singular operator S the composition $T_n T_{n-1} \dots T_1 S$ is compact. Moreover, if ℓ_1 does not isomorphically embed in X then $T_n T_{n-1} \dots T_1$ is compact.*

To finish the section, we would like to exhibit an example showing that in general there exist $\mathcal{S}_\xi$-operators on Banach spaces whose powers are never compact. The results of Section 2.2.6 imply that the Read's strictly singular operator without invariant subspaces would be such an example. Indeed, by Theorem 2.2.2, Read's operator is finitely strictly singular and, hence, $\mathcal{S}_\xi$-singular for any ξ. By Theorem 1.1.2, it even cannot be polynomially compact. We would like, however, to present a much simpler example.

Example 2.3.26. There is a separable Banach space X and a finitely strictly singular operator $T\colon X \to X$ such that no non-zero polynomial of T is compact.

Proof. Fix a strictly increasing sequence or reals $1 \leqslant p_1 < p_2 < \dots$, and put $X = \left(\bigoplus_{n=1}^\infty \ell_{p_n}\right)_{c_0}$. It can be easily verified that X is separable. Throughout the proof, whenever we consider $x \in X$, we assume that $x = (x_1, x_2, \dots)$ where $x_k \in \ell_{p_k}$ as $k \in \mathbb{N}$.

Let $T\colon X \to X$ be defined via the following formula

$$T\colon (x_1, x_2, \dots) \mapsto (0, x_1, \tfrac{x_2}{2}, \tfrac{x_3}{3}, \dots).$$

Recall that the formal identity $i : \ell_p \to \ell_q$ is finitely strictly singular if $1 \leqslant p < q < \infty$ (see [86]). Thus, by a reasoning analogous to the proof of Theorem 2.2.3, T is finitely strictly singular.

Let $Q(t) = \sum_{k=0}^n a_k t^k$, where $a_n \neq 0$, be a non-zero polynomial. Suppose for the sake of contradiction that $Q(T)$ is compact. Consider bounded operators $A\colon \ell_{p_1} \to X$ and $B\colon X \to \ell_{p_{n+1}}$ given by $A\colon h \mapsto (h, 0, 0, \dots)$ and $B\colon x \mapsto x_{n+1}$. For every $h \in \ell_{p_1}$ we have

$$Q(T)\colon (h, 0, 0, \dots) \mapsto (a_0 h, \tfrac{a_1}{1!} h, \tfrac{a_2}{2!} h, \dots, \tfrac{a_n}{n!} h, 0, 0, \dots),$$

so that $BQ(T)A(h) = \frac{a_n}{n!}h$. It follows that the compact operator $\frac{n!}{a_n}BQ(T)A$ equals the formal identity operator from ℓ_{p_1} to $\ell_{p_{n+1}}$, which is not compact since $p_1 < p_{n+1}$, contradiction. □

Chapter 3

Almost invariant subspaces

In this chapter we introduce and study the notion of an almost invariant subspace for an operator or a collection of operators. The results of this chapter have been published in the joint paper of the author of this thesis and Androulakis, Tcaciuc, and Troitsky [13] and in the single-author paper [98]. Sections 3.1 and 3.2 contain results from [13] whereas section 3.3 contains results from [98].

3.1 Definitions and simple properties

The following definition due to Tcaciuc introduces a generalization of the notion of an invariant subspace. We will study this generalization throughout the chapter.

Definition 3.1.1. If $T \in \mathcal{L}(X)$ and Y is a subspace of X, then Y is called ***almost invariant*** under T, or T***-almost invariant***, if there exists a finite dimensional subspace F of X such that $T(Y) \subseteq Y + F$.

One can ask the following natural version of the Invariant Subspace Problem: does every operator have an almost invariant subspace? This question, however, has a very simple answer. Indeed, it is easy to see that if the subspace Y of a Banach space X is of finite dimension or of finite codimension then Y is almost invariant under every operator on X. In order to avoid this triviality, we introduce a special type of a subspace.

Definition 3.1.2. A subspace Y of a Banach space X is called a ***half-space*** if it is both of infinite dimension and of infinite codimension in X.

Then we can ask the following question which we will refer as the ***almost invariant half-space problem***: *Does every operator on an infinite dimensional Banach space have an almost invariant half-space?* We will obtain partial solutions to this problem in the current section.

The natural question whether the usual right shift operator acting on a Hilbert space has almost invariant half-spaces has an affirmative answer. In fact, this operator has even invariant half-spaces (see Theorem 1.1.6).

It is natural to consider Donoghue operators as candidates for counterexamples to the almost invariant half-space problem, as their invariant subspaces are all finite-dimensional (see Theorem 1.1.7). Recall (cf. Definition 1.1.4) that a Donoghue operator $D \in \mathcal{L}(\ell_2)$ is an operator defined by

$$De_0 = 0, \quad De_i = w_i e_{i-1}, \quad i \in \mathbb{N},$$

where $(w_i)_{i=1}^{\infty}$ is a sequence of non-zero complex numbers such that $\big(|w_i|\big)_{i=1}^{\infty}$ is monotone decreasing and in ℓ_2. We will show in Section 3.2 that Donoghue operators have almost invariant half-spaces.

The following result is a characterization of almost invariant half-spaces of an operator in terms of invariant subspaces of its finite-rank perturbations.

Proposition 3.1.3. *Let $T \in \mathcal{L}(X)$ and $H \subseteq X$ be a half-space. Then H is almost invariant under T if and only if H is invariant under $T + K$ for some finite-rank operator K.*

Proof. Suppose that T has an almost invariant half-space H. Let F be a subspace of the smallest dimension satisfying the condition in Definition 3.1.1. Then $H \cap F = \{0\}$. Define $P\colon H + F \to F$ by $P(h + f) = f$. Since P is a finite-rank operator, we can extend it to a finite-rank operator on X using Hahn-Banach theorem. That is, there exists $\widetilde{P}\colon X \to F$ such that $\widetilde{P}|_{H+F} = P$. Define $K\colon X \to X$ by $K := -i\widetilde{P}T$ where $i : F \to X$ is the embedding. Clearly

K has finite rank. Also, for any $h \in H$ the image $Th \in H + F$, so we can write $Th = h' + f$ for some $h' \in H$ and $f \in F$. Thus

$$(T + K)(h) = Th - i\widetilde{P}Th = h' + f - i\widetilde{P}(h' + f) = h' + f - f = h'.$$

Hence $(T + K)H \subseteq H$.

Conversely, if H is invariant under $(T + K)(H) \subseteq H$ for some finite-rank operator K, then $T(H) \subseteq H + K(H)$. Since $\dim\bigl(K(H)\bigr) < \infty$, H is an almost invariant half space for T. □

Now we will study almost invariant half-spaces of adjoints of operators. We will need two simple lemmas. Both of them are standard; we include proofs for the convenience of the reader.

Lemma 3.1.4. *Let X be a Banach space and Y be a subspace of X. Then Y is infinite codimensional if and only if $Y^\perp$ is of infinite dimension. Thus Y is a half-space if and only if both Y and $Y^\perp$ are of infinite dimension.*

Proof. Suppose Y is of infinite codimension in X, that is, $\dim(X/Y) = \infty$. If $n \in \mathbb{N}$ then there exists a sequence $(x_k)_{k=1}^n \subseteq X$ such that $\{[x_k] : k = 1, \dots, n\}$ is linearly independent in X/Y, where $[x]$ denotes the equivalence class in X/Y containing x. Denote $F = [x_k]_{k=1}^n$. For each $k \in \{1, \dots, n\}$, define $f_k : Y + F \to \mathbb{F}$ by $f_k(x_i) = \delta_{ik}$ and $f_k(y) = 0$ for all $y \in Y$. Since $\dim(F) < \infty$, each f_k is bounded. Extend f_k from $Y + F$ to X by the Hahn-Banach theorem. It is easy to see, that each $f_k \in Y^\perp$ and the set $\{f_k : k = 1, \dots, n\}$ is linearly independent.

On the other hand, suppose that $\dim(X/Y) < \infty$, say, $\dim(X/Y) = m$. Fix a basis $\{[x_i] : i = 1, \dots, m\}$ for X/Y. Construct linear functionals $f_k \in Y^\perp$ $(k = 1, \dots, m)$ as before. If $f \in Y^\perp$ is arbitrary, define $c_k = f(x_k)$. It is easy to see that $f = c_1 f_1 + \dots + c_m f_m$, so that $\{f_k : k = 1, \dots, m\}$ forms a basis for $Y^\perp$. □

Lemma 3.1.5. *A subspace Y of X is a half-space if and only if $Y^\perp$ is a half-space in X^*.*

Proof. Suppose Y is a half-space. By Lemma 3.1.4, $Y^\perp$ must be infinite-dimensional. Also $(Y^\perp)^\perp \supseteq j(Y)$ where $j\colon X \to X^{**}$ denotes the natural embedding. Thus $(Y^\perp)^\perp$ is infinite dimensional. Now Lemma 3.1.4 yields that $Y^\perp$ is a half-space.

Let's assume that $Y^\perp$ is a half-space. Since $Y^\perp$ is infinite codimensional, Y must be infinite-dimensional (see, e.g. [8, Theorem 5.110]). On the other hand, since $Y^\perp$ is infinite dimensional, by Lemma 3.1.4 we obtain that Y is of infinite codimension, thus a half-space. □

Remark 3.1.6. The statement dual to that of Lemma 3.1.5 is not true in general. That is, if Z is a half-space in X^* then $Z_\perp$ need not be a half-space. For example, c_0 is a half-space in ℓ_∞ while $(c_0)_\perp = \{0\} \subseteq \ell_1$ is not.

Proposition 3.1.7. *Let T be an operator on a Banach space X. If T has an almost invariant half-space then so does its adjoint T^*.*

Proof. Let Y be a half-space in X such that Y is almost invariant under T, and F be a finite-dimensional subspace of X of smallest dimension such that $TY \subseteq Y + F$. Then $Y \cap F = \{0\}$. Denote $Z = (Y+F)^\perp$. By Lemma 3.1.5 Z is a half-space in X^*. For every $z \in Z$ and $y \in Y$ we have

$$\langle y, T^*z\rangle = \langle Ty, z\rangle = 0$$

since $Ty \in Y + F$. Therefore $T^*Z \subseteq Y^\perp$.

Let's prove that there exists a finite-dimensional subspace K of X^* such that $Y^\perp \subseteq Z + K$. This will show that Z is T^*-almost invariant: $T^*Z \subseteq Y^\perp \subseteq Z + K$. Let $f_1, \dots, f_n$ be the vectors of a basis of F^*. For each k, extend f_k to $Y \oplus F$ by putting $f_k|_Y = 0$, then extend the obtained functional to all of X by the Hahn-Banach theorem. Denote the latter extension by $\widetilde{f_k}$.

Put $K = \operatorname{span}\{\widetilde{f_1}, \dots, \widetilde{f_n}\}$. Let $f \in Y^\perp$ be arbitrary. Denote $g = f|_F$. Write $g = \sum_{i=1}^n a_i f_i$, where a_i are some numbers. Now put $\widetilde{g} = \sum_{i=1}^n a_i \widetilde{f_i}$. Clearly $\widetilde{g} \in K$. Observe that $f - \widetilde{g} \in Z$. Indeed, if $u + v \in Y \oplus F$ with $u \in Y$, $v \in F$ then $(f - \widetilde{g})(u+v) = (f-\widetilde{g})(u) + (f-\widetilde{g})(v) = f(u) + \big(f(v) - f(v)\big) =$

$f(u) = 0$ since $f \in Y^\perp$. Thus, for every $f \in Y^\perp$, there exist $f_1 \in Z$ and $f_2 \in K$ such that $f = f_1 + f_2$. □

3.2 Almost invariant subspaces of a single operator

All Banach spaces in this section are assumed to be complex. For a subset A of $\mathbb{C}$, we will write $A^{-1} = \left\{\frac{1}{\lambda} \colon \lambda \in A, \lambda \neq 0\right\}$. For a Banach space X and $T \in \mathcal{L}(X)$, we will use symbols $\sigma(T)$ for the spectrum of T, $r(T)$ for the spectral radius of T, and $\rho(T)$ for the resolvent set of T.

3.2.1 Weighted shift operators

Here we will exhibit a technique of constructing almost invariant half-spaces for operators developed in [13]. This technique is used to show that Donoghue operators have almost invariant subspaces.

Definition 3.2.1. For a nonzero vector $e \in X$ and $\lambda \in \rho(T)^{-1}$, define a vector $h(\lambda, e)$ in X by

$$h(\lambda, e) := \big(\lambda^{-1} I - T\big)^{-1}(e).$$

Note that if $|\lambda| < \frac{1}{r(T)}$ then[1] Neumann's formula (see, e.g., [1, Theorem 6.12]) yields

$$h(\lambda, e) = \lambda \sum_{n=0}^{\infty} \lambda^n T^n e. \tag{3.1}$$

Also, observe that $\big(\lambda^{-1} I - T\big) h(\lambda, e) = \big(\lambda^{-1} I - T\big)\big(\lambda^{-1} I - T\big)^{-1}(e) = e$ for every $\lambda \in \rho(T)^{-1}$, so that

$$T h(\lambda, e) = \lambda^{-1} h(\lambda, e) - e. \tag{3.2}$$

The last identity immediately yields the following result.

[1] In case $r(T) = 0$ we take $\frac{1}{r(T)} = +\infty$.

Lemma 3.2.2. *Let X be a Banach space, $T \in \mathcal{L}(X)$, $0 \neq e \in X$, and $A \subseteq \rho(T)^{-1}$. Put*

$$Y = \overline{\operatorname{span}}\big\{h(\lambda, e)\colon \lambda \in A\big\}.$$

Then Y is a T-almost invariant subspace (which is not necessarily a half-space), with $TY \subseteq Y + \operatorname{span}\{e\}$.

Lemma 3.2.3. *For any nonzero vector e in a Banach space X, we have*

$$h(\lambda, e) - h(\mu, e) = (\mu^{-1} - \lambda^{-1})h\big(\lambda, h(\mu, e)\big)$$

whenever $\lambda, \mu \in \rho(T)^{-1}$.

Proof. Indeed,

$$\begin{aligned} h(\lambda, e) - h(\mu, e) &= \big[(\lambda^{-1}I - T)^{-1} - (\mu^{-1}I - T)^{-1}\big](e) \\ &= (\mu^{-1} - \lambda^{-1})(\lambda^{-1}I - T)^{-1}(\mu^{-1}I - T)^{-1}(e) \\ &= (\mu^{-1} - \lambda^{-1})(\lambda^{-1}I - T)^{-1}h(\mu, e) \\ &= (\mu^{-1} - \lambda^{-1})h\big(\lambda, h(\mu, e)\big) \end{aligned}$$

□

Lemma 3.2.4. *Suppose that $T \in \mathcal{L}(X)$ has no eigenvectors. Then, for any nonzero vector $e \in X$ the set $\big\{h(\lambda, e) \,:\, \lambda \in \rho(T)^{-1}\big\}$ is linearly independent.*

Proof. We will use induction on n to show that for any nonzero vector $e \in X$ and any distinct $\lambda_1, \lambda_2, \ldots, \lambda_n \in \rho(T)^{-1}$ the set

$$\big\{h(\lambda_1, e), h(\lambda_2, e), \ldots, h(\lambda_n, e)\big\}$$

is linearly independent. The statement is obvious if $n = 1$; we assume it is true for $n - 1$ and will prove it for n.

Fix $e \in X$ and distinct $\lambda_1, \lambda_2, \ldots, \lambda_n \in \rho(T)^{-1}$. Let $a_1, a_2, \ldots, a_n$ be scalars such that $\sum_{k=1}^{n} a_k h(\lambda_k, e) = 0$. It follows from formula (3.2) that

$$0 = T\Bigg(\sum_{k=1}^{n} a_k h(\lambda_k, e)\Bigg) = \sum_{k=1}^{n} a_k \lambda_k^{-1} h(\lambda_k, e) - \sum_{k=1}^{n} a_k e.$$

If $\sum_{k=1}^n a_k \neq 0$ then $e \in \operatorname{span}\big\{h(\lambda_k, e)\big\}_{k=1}^n$, so that $\operatorname{span}\big\{h(\lambda_k, e)\big\}_{k=1}^n$ is T-invariant by (3.2). Since this subspace is finite-dimensional, it follows that T has an eigenvalue, which is a contradiction. Therefore $\sum_{k=1}^n a_k = 0$, so that $a_1 = -\sum_{k=2}^n a_k$.

By Lemma 3.2.3, we get

$$\begin{aligned} 0 = \sum_{k=1}^{n} a_k h(\lambda_k, e) &= \Big(-\sum_{k=2}^{n} a_k\Big) h(\lambda_1, e) + \sum_{k=2}^{n} a_k h(\lambda_k, e) \\ &= \sum_{k=2}^{n} a_k \big(h(\lambda_k, e) - h(\lambda_1, e)\big) \\ &= \sum_{k=2}^{n} a_k (\lambda_1^{-1} - \lambda_k^{-1}) h\big(\lambda_k, h(\lambda_1, e)\big). \end{aligned}$$

By the induction hypothesis, the set $\Big\{h\big(\lambda_k, h(\lambda_1, e)\big)\Big\}_{k=2}^n$ is linearly independent, hence $a_k(\lambda_1^{-1} - \lambda_k^{-1}) = 0$ for any $2 \leqslant k \leqslant n$. It follows immediately that $a_k = 0$ for any $1 \leqslant k \leqslant n$, and this concludes the proof. $\square$

The discussion above results in the following approach of constructing almost invariant subspaces for an operator. Suppose that T has no eigenvalues. Pick a non-zero $e \in X$ and a sequence of distinct numbers (λ_n) in $\rho(T)^{-1}$. Then the space $Y = \big[h(\lambda_n, e)\big]_{n=1}^\infty$ is almost invariant under T and is of infinite dimension. If one can show that e can be chosen so that Y is of infinite codimension then Y is a T-almost invariant half-space.

We will use the following numerical lemma.

Lemma 3.2.5. *Given a sequence (r_i) of positive reals, there exists a sequence (c_i) of positive reals such that the series $\sum_{i=0}^\infty c_i r_{i+k}$ converges for every k.*

Proof. For every i take $c_i = \frac{1}{2^i} \min\{\frac{1}{r_1}, \dots, \frac{1}{r_{2i}}\}$. For every $i \geqslant k$ we have $k + i \leqslant 2i$, so that $c_i r_{i+k} \leqslant \frac{1}{2^i}$. It follows that

$$\sum_{i=0}^{\infty} c_i r_{i+k} \leqslant \sum_{i=0}^{k-1} c_i r_{i+k} + \sum_{i=k}^{\infty} \tfrac{1}{2^i} < +\infty.$$

$\square$

Recall that a sequence (x_i) in a Banach space is called ***minimal*** if $x_k \notin [x_i]_{i\neq k}$ for every k (see, e.g., [71, section 1.f]). It is easy to see that this is equivalent to saying that for every k, the biorthogonal functional x_k^* defined on $\operatorname{span}\{x_i\}$ by $x_k^*\bigl(\sum_{i=0}^n \alpha_i x_i\bigr) = \alpha_k$ is bounded. Indeed, suppose that (x_k) is minimal. For $x = \sum_{i=0}^n \alpha_i x_i$, we have: $\|x\| \geqslant \alpha_k \operatorname{dist}(x_k, [x_i]_{i\neq k})$, so that $\|x_k^*\| \leqslant 1/\operatorname{dist}(x_k, [x_i]_{i\neq k})$. On the other hand, if $x_k \in [x_i]_{i\neq k}$ for some k then for each $\varepsilon > 0$ we can find $z \in \operatorname{span}\{x_i : i \neq k\}$ such that $\|x_k - z\| < \varepsilon$. But then $x_k^*(x_k - z) = x_k^*(x_k) = 1$, hence $\|x_k^*\| \geqslant 1/\varepsilon$.

Theorem 3.2.6. *Let X be a Banach space and $T \in \mathcal{L}(X)$ satisfying the following conditions:*

(i) T has no eigenvalues.

(ii) The unbounded component of $\rho(T)$ contains $\{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$.

(iii) There is a vector whose orbit is a minimal sequence.

Then T has an almost invariant half-space.

Proof. Let $e \in X$ be such that $(T^i e)_{i=0}^\infty$ is minimal. For each i put $x_i = T^i e$. Then for each k, the biorthogonal functional x_k^* defined on $\operatorname{span}\{x_i : i \in \mathbb{N}\}$ by $x_k^*\bigl(\sum_{i=0}^n \alpha_i x_i\bigr) = \alpha_k$ is bounded. Let $r_k = \|x_k^*\|$. Let (c_i) be a sequence of positive real numbers as in Lemma 3.2.5, so that $\beta_k := \sum_{i=0}^\infty c_i r_{i+k} < +\infty$ for every k. By making c_i's even smaller, if necessary, we may assume that $\sqrt[i]{c_i} \to 0$.

Consider a function $F : \mathbb{C} \to \mathbb{C}$ defined by $F(z) = \sum_{i=0}^\infty c_i z^i$. Then F is entire. We claim that without loss of generality, the set $\bigl\{z \in \mathbb{C} : F(z) = 0\bigr\}$ is infinite. Indeed, since $F(z)$ is entire and not constant, F has an essential singularity at infinity. Hence, by the Picard Theorem there exists a negative real number d such that the set $\bigl\{z \in \mathbb{C} : F(z) = d\bigr\}$ is infinite. Now replace c_0 with $c_0 - d$. This doesn't affect our other assumptions on the sequence (c_i).

Fix a sequence of distinct complex numbers (λ_n) such that $F(\lambda_n) = 0$ for every n. Since F is non-constant, the sequence (λ_n) has no accumulation points. Hence, $|\lambda_n| \to +\infty$.

Since the set $\{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$ belongs to $\rho(T)$, we may assume, by passing to a subsequence, if necessary, that each $\lambda_n \in \rho(T)^{-1}$. In particular, $h(\lambda_n, e)$ is defined for each n. Put $Y = [h(\lambda_n, e)]_{n=1}^{\infty}$. Then Y is almost invariant under T by Lemma 3.2.2 and $\dim Y = \infty$ by Lemma 3.2.4. We will prove that Y is actually a half-space by constructing a sequence of linearly independent functionals (f_n) such that every f_n annihilates Y.

For every $k = 0, 1, \dots,$ put $F_k(z) = z^k F(z)$. Let's write $F_k(z)$ in the form of Taylor series, $F_k(z) = \sum_{i=0}^{\infty} c_i^{(k)} z^i$. Then

$$c_i^{(k)} = \begin{cases} 0 & \text{if } i < k, \text{ and} \\ c_{i-k} & \text{if } i \geqslant k. \end{cases}$$

Define a functional f_k on $\operatorname{span}\{T^i e\}_{i=0}^{\infty}$ via $f_k(T^i e) = c_i^{(k)}$. Since T has no eigenvalues, the orbit of T is linearly independent thus f_k is well-defined. We will show now that f_k is bounded. Let $x \in \operatorname{span}\{T^i e\}_{i=0}^{\infty}$, then $x = \sum_{i=0}^{n} x_i^*(x) T^i e$ for some n, so that

$$\begin{aligned} |f_k(x)| = \Big| f_k\Big(\sum_{i=0}^{n} x_i^*(x) T^i e\Big)\Big| &\leqslant \Big(\sum_{i=0}^{n} \|x_i^*\| c_i^{(k)}\Big)\|x\| \\ &= \Big(\sum_{i=k}^{n} r_i c_{i-k}\Big)\|x\| \leqslant \Big(\sum_{i=k}^{\infty} r_i c_{i-k}\Big)\|x\| = \beta_k \|x\|, \end{aligned}$$

so that $\|f_k\| \leqslant \beta_k$. Hence, f_k can be extended by continuity to a bounded functional on $[T^i e]_{i=1}^{\infty}$, and then by the Hahn-Banach Theorem to a bounded functional on all of X.

Now we show that each f_k annihilates Y. Fix k. Denote by $\mathcal{C}$ the unbounded component of $\rho(T)^{-1}$. Then $\mathcal{C}$ contains a neighborhood of zero. Recall that for each $\lambda \in \rho(T)^{-1}$ such that $|\lambda| < \frac{1}{r(T)}$ we have $h(\lambda, e) = \lambda \sum\limits_{i=0}^{\infty} \lambda^i T^i e$. Therefore

$$f_k\big(h(\lambda, e)\big) = f_k\Big(\lambda \sum_{i=0}^{\infty} \lambda^i T^i e\Big) = \lambda \sum_{i=0}^{\infty} \lambda^i c_i^{(k)} = \lambda F_k(\lambda) = \lambda^{k+1} F(\lambda).$$

for every $\lambda \in \mathcal{C}$ such that $|\lambda| < \frac{1}{r(T)}$. The map $\lambda \mapsto h(\lambda, e)$ and, therefore, the map $\lambda \mapsto f_k\big(h(\lambda, e)\big)$, is analytic on the set $\rho(T)^{-1}$. Hence, by the principle of uniqueness of analytic function, the functions $f_k\big(h(\lambda, e)\big)$ and $\lambda^{k+1}F(\lambda)$ must agree on $\mathcal{C}$. Since $\lambda_n \in \mathcal{C}$ for all n, we have $f_k\big(h(\lambda_n, e)\big) = \lambda_n^{k+1}F(\lambda_n) = 0$ for all n. Thus, Y is annihilated by every f_k.

It is left to prove the linear independence of $\{f_k\}_{k=1}^{\infty}$. Observe that $f_k \neq 0$ for all k since $f_k(T^i e) \neq 0$ for $i \geqslant k$. Suppose that $f_N = \sum_{k=M}^{N-1} a_k f_k$ with $a_M \neq 0$. However $f_N(T^M e) = 0$ by definition of f_N while $\sum_{k=M}^{N-1} a_k f_k(T^M e) = a_M c_0 \neq 0$, contradiction. □

Remark 3.2.7. Note that condition (ii) of Theorem 3.2.6 is satisfied by many important classes of operators. For example, it is satisfied if $\sigma(T)$ is finite (in particular, if T is quasinilpotent) or if 0 belongs to the unbounded component of $\rho(T)$.

Corollary 3.2.8. *Suppose that $X = \ell_p$ $(1 \leqslant p < \infty)$ or c_0 and $T \in \mathcal{L}(X)$ is a weighted right shift operator with weights converging to zero but not equal to zero. Then both T and T^* have almost invariant half-spaces.*

Proof. It can be easily verified that T is quasinilpotent. Clearly, T has no eigenvalues, and the orbit of e_1 is evidently a minimal sequence. By Theorem 3.2.6 and Remark 3.2.7, T has almost invariant half-spaces. Finally, Proposition 3.1.7 yields almost invariant half-spaces for T^*. □

The following statement is a special case of Corollary 3.2.8.

Corollary 3.2.9. *If D is a Donoghue operator then both D and D^* have almost invariant half-spaces.*

3.2.2 Operators with many almost invariant subspaces

Let X be a Banach space and $T\colon X \to X$ be a bounded operator. It is easy to show that if every subspace of X is invariant under T then T must be a multiple of the identity. In fact, it is enough that every one-dimensional subspace is invariant under T. Indeed, if every one-dimensional subspace is T-invariant then for each $x \in X$ there is $\lambda_x \in \mathbb{F}$ such that $Tx = \lambda_x x$. If λ_x is not the same for all nonzero x, say, $\lambda_x \neq \lambda_y$ for some nonzero $x, y \in X$, then $T(x+y) = \lambda_{x+y}(x+y) = \lambda_x x + \lambda_y y$, so that $x = \alpha y$, which is a contradiction.

In this subsection we will obtain a result of the same spirit for almost invariant half-spaces. The ideas introduced in the proof will show to be useful in Section 3.3 where they are developed to a much greater depth.

Proposition 3.2.10. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that every half-space of X is almost invariant under T. Then T has a non-trivial invariant subspace of finite codimension. Iterating, one can get a chain of such subspaces.*

Proof. Let's assume that T has no non-trivial invariant subspaces of finite codimension. We will construct a half-space that is not almost invariant under T.

Let $f \in X^*$ be a nonzero functional. For each $n \in \mathbb{N}$, define $f_n = f \circ T^{n-1}$, and put $Y_n = \cap_{k=1}^n \ker f_k$. Denote also $Y_0 = X$. Then clearly $Y_{n+1} \subseteq Y_n$ for all n. Also, if $y \in Y_{n+1}$ then $f(T^k y) = f(T^{k-1}Ty) = 0$ for all $k = 1, 2, \dots, n$. Hence $f\big(T^k(Ty)\big) = 0$ for all $k = 0, 1, \dots, n-1$. Therefore $TY_{n+1} \subseteq Y_n$. In particular, since $\operatorname{codim} Y_n < \infty$ for all n, we get: $Y_{n+1} \neq Y_n$.

For each $n = 0, 1, \dots$, pick a vector $z_n \in Y_n \setminus Y_{n+1}$. Then, in particular, $Y_{n+1} \cup [z_n] = Y_n$, and $Y_{n+1} \cup [z_k]_{k=0}^n = X$. Also, it is easy to see that $Y_{n+1} \cap [z_k]_{k=0}^n = \{0\}$, so that $X = Y_{n+1} \oplus [z_k]_{k=0}^n$. Let P_n be the projection along $[z_k]_{k=0}^n$ onto Y_{n+1}. If $g_n = f_{n+1} \circ P_{n-1}$ then clearly $g_i(z_k) = \delta_{ki}$ for all k, i. Put $Z = [z_{2k}]_{k=1}^\infty$. Then obviously $\dim Z = \infty$ and $g_{2k-1}|_Z = 0$ for all $k \in \mathbb{N}$. Thus, Z is actually a half-space.

Suppose that there exists F with $\dim F < \infty$ such that $TZ \subseteq Z + F$. For each $k \in \mathbb{N}$, pick $u_k \in Z$ and $v_k \in F$ such that $Tz_{2k} = u_k + v_k$. We claim that if $k \in \mathbb{N}$, then $g_{2k-1}(v_k) \neq 0$ and for all $1 \leqslant i < k$, $g_{2i-1}(v_k) = 0$. This will lead to a contradiction because $\dim F < \infty$.

To prove the claim, observe that since $z_{2k} \in Y_{2k}$ and $Tz_{2k} \in Y_{2k-1}$, we obtain $0 \neq g_{2k}(z_{2k}) = f_{2k}(z_{2k}) = f_{2k-1}(Tz_{2k}) = g_{2k-1}(Tz_{2k})$. Since $g_{2k-1}|_Z = 0$, it follows that $g_{2k-1}(v_k) \neq 0$. By the same reason, for all $1 \leqslant i < k$, we get $0 = g_{2i}(z_{2k}) = f_{2i}(z_{2k}) = f_{2i-1}(Tz_{2k}) = g_{2i-1}(Tz_{2k})$, so that $g_{2i-1}(v_k) = 0$.

□

3.3 Almost invariant subspaces of algebras of operators

3.3.1 Introduction

This section is concerned with the study of almost invariant subspaces of collections of operators, most notably, algebras of operators. Just as the studies of transitive algebras generalize the Invariant Subspace Problem for a single operator, we introduce and study the notion of a subspace that is simultaneously *almost* invariant under every operator in a given algebra of operators. The results of this section were published by the author of this thesis in [98].

Definition 3.3.1. Let $\mathcal{C} \subseteq L(X)$ be an arbitrary collection of operators and $Y \subseteq X$ a subspace of X. We call Y ***almost invariant under*** $\mathcal{C}$, or $\mathcal{C}$***-almost invariant*** if Y is almost invariant under every operator in $\mathcal{C}$.

That is, a subspace is $\mathcal{C}$-almost invariant if for each $T \in \mathcal{C}$ there exists a finite-dimensional subspace F_T of X ("error") such that

$$TY \subseteq Y + F_T. \tag{3.3}$$

Like in the case of a single operator, every subspace that is not a half-space is automatically almost invariant under every collection $\mathcal{C}$ of operators on X.

In Subsection 3.3.2, we study the finite-dimensional "errors" F_T appearing in formula (3.3). We prove that if $\mathfrak{A}$ is an algebra without invariant half-spaces then for an $\mathfrak{A}$-almost invariant half-space Y these finite-dimensional subspaces cannot be the same (Proposition 3.3.4). On the other hand, we prove (Theorem 3.3.9) that if $\mathfrak{A}$ is norm closed then these finite-dimensional subspaces cannot be "too far apart"; the dimensions of these subspaces must be uniformly bounded.

In Subsection 3.3.3, the invariant subspaces of algebras having almost invariant half-spaces are investigated. It is proved that if $\mathfrak{A}$ is a norm closed algebra generated by a single operator then existence of an $\mathfrak{A}$-almost invariant half-space implies existence of an $\mathfrak{A}$-invariant half-space (Theorem 3.3.13). This theorem then is generalized to the case of a commutative algebra generated by a finite number of operators (Theorem 3.3.15).

Finally, the last subsection is concerned with the question of whether the almost invariant half-spaces of an algebra and its closure in various topologies are the same. It turns out that the situation here is dramatically different from the case of invariant subspaces.

3.3.2 Analysis of "errors"

Observe that the finite-dimensional subspace F_T appearing in the equation (3.3) is by no means unique. However we can consider the minimal dimension of a subspace satisfying this condition. We will find out that subspaces of minimal dimension satisfying (3.3) are of big importance. We collect some properties of such subspaces in the following lemma (see case (iii)). This lemma also contains some facts used in Proposition 3.3.4.

Lemma 3.3.2. *Let $Y \subseteq X$ be a subspace, $\mathcal{C}$ be a collection of bounded operators on X and $G \subseteq X$ be a finite-dimensional space of the smallest dimension such that $TY \subseteq Y + G$ for all $T \in \mathcal{C}$. Then*

(i) $Y + G = Y \oplus G$;

(ii) if $P : Y \oplus G \to G$ *is the projection along* Y *then*

$$\operatorname{span} \bigcup_{T\in\mathcal{C}} PT(Y) = G;$$

(iii) if $\mathcal{C}$ consists of a single operator, that is $\mathcal{C} = \{T\}$, and if $P : Y \oplus G \to G$ is the projection along Y then

$$PT(Y) = G.$$

Moreover, in this case G can be chosen so that $G \subseteq TY$.

Proof. (i) Suppose that there exists a non-zero $g \in G \cap Y$. Build $g_2, \dots g_n \in G$ such that $\{g, g_2, \dots, g_n\}$ is a basis of G. Denote $G_1 = \operatorname{span}\{g_2, \dots, g_n\}$. It is clear that $TY \subseteq Y + G_1$ for all $T \in \mathcal{C}$. However $\dim G_1 < \dim G$.

(ii) Define $F = \operatorname{span}\{g \in G \colon v + g \in TY \text{ for some } v \in Y, T \in \mathcal{C}\}$. Clearly $F = \operatorname{span} \bigcup_{T \in \mathcal{C}} PT(Y)$.

We claim that $TY \subseteq Y + F$ for all $T \in \mathcal{C}$. Indeed, if $y \in Y$ and $T \in \mathcal{C}$ then $Ty = v + g$ for some $v \in Y$ and $g \in G$. By definition of F we get: $g \in F$, hence $Ty \in Y + F$.

Since $F \subseteq G$ and G has the smallest dimension among the spaces with the property $TY \subseteq Y + G$ for all $T \in \mathcal{C}$, we get $G = F$.

(iii) The first part of this statement follows immediately from (ii). Let's prove the "moreover" part. Let $g_1, \dots, g_n$ be a basis of G. By (ii), there exist $u_1, \dots, u_n$ and $y_1, \dots, y_n$ in Y such that $Tu_i = y_i + g_i$ $(i = 1, \dots, n)$. Put $f_i = Tu_i$ and $F = [f_i]_{i=1}^n$. Then clearly $F \subseteq TY$. Also $Y + F = Y + G$, so that $TY \subseteq Y + F$. □

The following example shows that $\bigcup_{T \in \mathcal{C}} PT(Y)$ may not be a linear space even in the case when $\mathcal{C}$ is an algebra of operators, so that it is necessary to consider $\operatorname{span} \bigcup_{T \in \mathcal{C}} PT(Y)$ in the statement (ii) of Lemma 3.3.2.

Example 3.3.3. Let $X = \ell_2(\mathbb{Z})$. Define $T, S \in L(X)$ by

$$Te_0 = e_1, \quad Te_{-1} = e_2, \quad Te_i = 0 \text{ if } i \neq 0, -1,$$

and

$$Se_0 = e_3, \quad Se_i = 0 \text{ if } i \neq 0.$$

Since $T^2 = S^2 = TS = ST = 0$, the algebra $\mathfrak{A}$ generated by T and S consists exactly of the operators of form $aT + bS$ where a and b are arbitrary scalars.

Let $Y = [e_i]_{i\leqslant 0}$. Then clearly $\mathfrak{A}Y \subseteq Y + F$ where $F = \text{span}\{e_1, e_2, e_3\}$, and F is the space of the smallest dimension satisfying this condition. If $P : Y \oplus F \to F$ is the projection along Y then $\bigcup_{R\in\mathfrak{A}} PR(Y)$ is not a linear space. If it were, it would have been equal to F, by (ii) of Lemma 3.3.2. However the vector $e_2 + e_3$ is not in this union. Indeed, if $P(aT + bS)(\sum_{i=0}^{\infty} x_i e_{-i}) = (aT + bS)(x_0e_0 + x_1e_{-1}) = e_2 + e_3$ then $bx_0 = 1$ and $ax_1 = 1$. Hence $x_0 \neq 0$ and $a \neq 0$, so that $e_1^*[(aT + bS)(x_0e_0 + x_1e_{-1})] \neq 0$, which is impossible.

Suppose Y is a half-space that is almost invariant under a collection $\mathcal{C}$ of operators on X, that is, formula (3.3) holds for every operator T in $\mathcal{C}$ with some F_T. One may ask if it is possible that F_T does not depend on T. The following simple reasoning shows that in the case of algebras of operators, this can only happen if the algebra already has a common invariant half-space.

Proposition 3.3.4. *Let $Y \subseteq X$ be a half-space and $\mathfrak{A}$ an algebra of operators. Suppose that there exists a finite-dimensional space F such that for each $T \in \mathfrak{A}$ we have $TY \subseteq Y + F$. Then $\mathfrak{A}$ has a common invariant half-space.*

Moreover, if G is a space of the smallest dimension such that $TY \subseteq Y + G$ for all $T \in \mathfrak{A}$ then $Y + G$ is $\mathfrak{A}$-invariant.

Proof. Clearly, it is enough to prove the "moreover" part. Let G be a space of the smallest dimension such that $TY \subseteq Y + G$ for all $T \in \mathfrak{A}$.

Denote $\mathfrak{A}(Y) = \bigcup_{T\in\mathfrak{A}} TY$. Then $\mathfrak{A}(Y)$ is $\mathfrak{A}$-invariant. Hence, so is span $\mathfrak{A}(Y)$. Denote $Z = Y + \text{span}\mathfrak{A}(Y)$. Since $TY \subseteq \text{span}\mathfrak{A}(Y)$ for every $T \in \mathfrak{A}$, we obtain that Z is invariant under $\mathfrak{A}$.

Obviously, $Z \subseteq Y + G$. By Lemma 3.3.2, $Y + G = Y \oplus G$, and if $P : Y \oplus G \to G$ is a projection along Y then $P\big(\text{span}\mathfrak{A}(Y)\big) = G$. Hence $Y \oplus G = Y \oplus P\big(\text{span}\mathfrak{A}(Y)\big) = Y + \text{span}\mathfrak{A}(Y) = Z$, so that $Y \oplus G$ is invariant under $\mathfrak{A}$. $\square$

Definition 3.3.5. Let $T \in L(X)$ be an operator and $Y \subseteq X$ be a linear subspace. We will write $d_{Y,T}$ for the smallest n such that there exists F with

$TY \subseteq Y + F$ and $\dim F = n$.

The following observation is obvious.

Lemma 3.3.6. *Let $T \in L(X)$ be an operator and $Y \subseteq X$ be a subspace. Let $q : X \to X/Y$ be a quotient map. Then Y is T-almost invariant if and only if $(qT)|_Y$ is of finite rank. Moreover, $\dim(qT)(Y) = d_{Y,T}$.*

To proceed, we need the following two auxiliary lemmas.

Lemma 3.3.7. *Let $Y \subseteq X$ be a linear subspace and $\{u_i\}_{i=1}^N$ be a collection of linearly independent vectors in X such that $[u_i]_{i=1}^N \cap Y = \{0\}$. Let $\{v_i\}_{i=1}^N \subseteq X$ be arbitrary. Then for all but finitely many α the set $\{v_i + \alpha u_i\}_{i=1}^N$ is linearly independent, and $[v_i + \alpha u_i]_{i=1}^N \cap Y = \{0\}$.*

Proof. Let $F = \operatorname{span}\{u_i, v_i \colon i = 1, \dots, N\}$. Let $G = \big(Y + F\big)/Y$. Denote $x_i = u_i + Y \in G$, $z_i = v_i + Y \in G$. Then the set $\{x_i\}_{i=1}^N$ is linearly independent. Clearly, to establish the lemma it is enough to prove that the set $\{z_i + \alpha x_i\}_{i=1}^N$ is linearly independent for all but finitely many α.

Denote $M = \dim G$. Let $\{b_i\}_{i=1}^M$ be a basis of G such that $b_i = x_i$ for all $1 \leqslant i \leqslant N$. Denote the coordinates of vectors z_i in this basis by z_{ij}. Let A be the $M \times M$-matrix with the first N rows consisting of the coordinates of z_i $(i = 1, \dots, N)$, and the last $M - N$ rows being consisting of zeros:

$$A = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1,M-1} & z_{1M} \\ \vdots & & \cdots & & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{N,M-1} & z_{NM} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since the spectrum of A is finite, $\det(A + \alpha I) \neq 0$ for all but finitely many α. For these α, the rows of $A + \alpha I$ must be linearly independent. In particular, the first N rows are linearly independent. However, the first N rows of $A + \alpha I$ are exactly the representations of the vectors $z_i + \alpha x_i$ in the basis $\{b_i\}_{i=1}^M$. $\square$

Lemma 3.3.8. *Let $Y \subseteq X$ be a linear subspace and $T \in B(X)$. Suppose that $f_1, \dots, f_n \in TY$ are such that no non-trivial linear combination of $\{f_1, \dots, f_n\}$ belongs to Y. Then $n \leqslant d_{Y,T}$.*

Proof. Let $q : X \to X/Y$ be the quotient map. Then $qf_1, \dots, qf_n$ are linearly independent. Since $qf_1, \dots, qf_n \in (qT)(Y)$, we get $n \leqslant \dim(qT)(Y) = d_{Y,T}$ by Lemma 3.3.6. $\square$

The following theorem is the main statement in this section. Recall that, according to our convention, the term "subspace" stands for a norm closed subspace, while a "linear subspace" need not be closed.

Theorem 3.3.9. *Let $\mathfrak{S}$ be a subspace of $L(X)$. Suppose that Y is a linear subspace of X that is almost invariant under $\mathfrak{S}$. Then*

$$\sup_{S \in \mathfrak{S}} d_{Y,S} < \infty.$$

Proof. For every $S \in \mathfrak{S}$, fix a subspace $F_S \subseteq X$ such that $SY \subseteq Y + F_S$ and $\dim F_S = d_{Y,S}$. By Lemma 3.3.2, $Y + F_S$ is a direct sum. Fix $P_S : Y \oplus F_S \to F_S$ the projection along Y. Also fix a basis $(f_i^S)_{i=1}^{d_{Y,S}}$ of F_S and vectors $(g_i^S)_{i=1}^{d_{Y,S}}$ in Y such that $(P_S S)g_i^S = f_i^S$ (this can be done by Lemma 3.3.2(iii)).

Suppose that the statement of the theorem is not true. Then there exists a sequence of operators $(S_k) \subseteq \mathfrak{S}$ such that the sequence $(d_{Y,S_k})_{k=1}^{\infty}$ is strictly increasing. Without loss of generality, $\|S_k\| = 1$.

We will inductively construct a sequence (a_k) of scalars such that the following two conditions are satisfied for every m.

(i) If $T_m = \sum\limits_{k=1}^{m} a_k S_k$ then $N_m := d_{Y,T_m} \geqslant d_{Y,S_m}$.

(ii) Let

$$C_m = \sup_{b_1,\dots,b_{N_m} \in [-1,1]} \Big\| \sum_{i=1}^{N_m} b_i g_i^{T_m} \Big\| \cdot \max_{i=1,\dots,N_m} \big\|(f_i^{T_m})^*\big\|,$$

where $(f_i^{T_m})^*$ is the i-th biorthogonal functional for $(f_i^{T_m})_{i=1}^{N_m}$ in $F_{T_m}^*$, and

$$D_m = \min\Big\{1, \frac{1}{C_1 \cdot \|P_{T_1}\|}, \dots, \frac{1}{C_m \cdot \|P_{T_m}\|}\Big\}.$$

Then $0 < a_1 \leqslant \frac{1}{2}$ and $0 < a_{m+1} < \frac{1}{2^{m+1}} D_m \leqslant \frac{1}{2^{m+1}}$ for all $m \geqslant 1$.

Indeed, on the first step put $a_1 = \frac{1}{2}$. Suppose that $a_1, \dots, a_m$ have been constructed. Define D_m as in (ii). Denote for convenience $N = d_{Y,S_{m+1}}$. Let $u_i = f_i^{S_{m+1}}$ and $v_i = T_m g_i^{S_{m+1}}$, $i = 1, \dots, N$. By Lemma 3.3.7 we can find $0 < \alpha < \frac{1}{2^{m+1}} D_m$ such that no non-trivial linear combination of vectors from the set $\{v_i + \alpha u_i\}_{i=1}^N$ is contained in Y. Put $a_{m+1} = \alpha$. This makes both conditions (i) and (ii) satisfied for $m+1$. Indeed, condition (ii) is satisfied immediately. Let's check condition (i). For each $i = 1, \dots, N$, we have $T_{m+1} g_i^{S_{m+1}} = T_m g_i^{S_{m+1}} + \alpha S_{m+1} g_i^{S_{m+1}} = v_i + \alpha u_i + w_i$ where w_i is some vector in Y. Since no linear combination of $\{v_i + \alpha u_i\}_{i=1}^N$ is contained in Y, the same is true for $\{T_{m+1} g_i^{S_{m+1}}\}_{i=1}^N$. Condition (i) now follows from Lemma 3.3.8.

Denote $S = \sum\limits_{k=1}^{\infty} a_k S_k$. By condition (ii), $a_k \leqslant \frac{1}{2^k}$ for all $k \in \mathbb{N}$, so that S is well-defined. For every $m \in \mathbb{N}$, denote $R_m = \sum\limits_{k=m+1}^{\infty} a_k S_k$, so that $S = T_m + R_m$. By condition (ii), we get: $\|R_m\| < \frac{1}{C_m \cdot \|P_m\|}$ for all $m \in \mathbb{N}$.

Clearly, $S \in \mathfrak{S}$. By assumptions of the theorem, $SY = Y \oplus F_S$. Denote $n = \dim F_S < \infty$. Pick $m \in \mathbb{N}$ such that $N_m > n$ and put $z_i = S g_i^{T_m}$, $i = 1, \dots, N_m$. Since $N_m > n$, there exists a sequence $(b_i)_{i=1}^{N_m}$ of scalars such that $\max_i |b_i| = 1$ and

$$z := \sum_{i=1}^{N_m} b_i z_i \in Y.$$

Consider $y = \sum\limits_{i=1}^{N_m} b_i g_i^{T_m}$. We have

$$T_m y = Sy - R_m y = z - R_m y,$$

hence

$$(P_{T_m} T_m) y = -(P_{T_m} R_m) y.$$

Clearly, for each $i = 1, \dots, N_m$, we have $b_i = (f_i^{T_m})^* (P_{T_m} T_m y)$. Let k be such that $|b_k| = 1$. Then

$$1 = |b_k| = \left|(f_k^{T_m})^* (P_{T_m} T_m\, y)\right| \leqslant \left\|(f_k^{T_m})^*\right\| \cdot \|P_{T_m} T_m\, y\| =$$

$$= \left\|(f_k^{T_m})^*\right\| \cdot \|P_{T_m} R_m\, y\| \leqslant \left\|(f_k^{T_m})^*\right\| \cdot \|P_{T_m}\| \cdot \|R_m\| \cdot \left\|\sum_{i=1}^{N_m} b_i g_i^{T_m}\right\| \leqslant$$

$$\leqslant \|P_{T_m}\| \cdot \|R_m\| \cdot C_m < \|P_{T_m}\| \frac{1}{C_m \cdot \|P_{T_m}\|} C_m = 1$$

which is a contradiction. $\square$

3.3.3 When algebras with almost invariant subspaces have invariant subspaces

Now we will turn our attention to the study of invariant subspaces of algebras having almost invariant subspaces.

The following lemma is obvious.

Lemma 3.3.10. *Let X and Y be Banach spaces and $T \in L(X,Y)$ be of finite rank. Then* $\dim(\text{Range } T) = \operatorname{codim}(\ker T)$.

Let $T \in L(X)$ be an operator and $Y \subseteq X$ be a half-space. Consider two procedures of constructing new linear spaces:

$$\begin{array}{ll} D_T(Y) = \{y \in Y : Ty \in Y\} & \text{“going downwards”}, \\ U_T(Y) = Y + TY & \text{“going upwards”}. \end{array}$$

Clearly $D_T(Y) \subseteq Y \subseteq U_T(Y)$.

If Y and Z are two subspaces of X and $Y \subseteq Z$ then the symbol $\operatorname{codim}_Z Y$ will stand for the codimension of Y in Z.

Lemma 3.3.11. *Let $Y \subseteq X$ be a half-space and $T \in L(X)$. If Y is T-almost invariant then both $D_T(Y)$ and $U_T(Y)$ are half-spaces. Moreover,* $\operatorname{codim}_Y D_T(Y) = \operatorname{codim}_{U_T(Y)} Y = d_{Y,T}$.

Proof. The statement about $U_T(Y)$ follows immediately from the definition of an almost invariant subspace. Let's verify the statement about $D_T(Y)$. Obviously, we only need to verify the “moreover” part.

Let $TY \subseteq Y + F$ where F is such that $\dim F = d_{Y,T}$. By Lemma 3.3.2(i), we have $Y + F = Y \oplus F$. Let $P : Y \oplus F \to F$ be the projection onto F along Y. It is easy to see that $D_T(Y) = \ker(PT|_Y)$. By Lemma 3.3.10, we obtain $\operatorname{codim}_Y D_T(Y) = \dim(\text{Range } PT|_Y) = \dim F = d_{Y,T}$. $\square$

The following lemma is the key statement of this section. In this lemma, the symbols $D_T^k(Y)$ and $U_T^k(Y)$ are defined for all $k \geqslant 2$ by $D_T^k(Y) = D_T(D_T^{k-1}(Y))$

and $U_T^k(Y) = U_T(U_T^{k-1}(Y))$, and for $k = 1$ by $D_T^1(Y) = D_T(Y)$ and $U_T^1(Y) = U_T(Y)$.

Lemma 3.3.12. *Suppose that Y is a half-space in X that is almost invariant under an operator $T \in L(X)$. If $d_{Y,T} > 0$ and $d_{D_T^k(Y),T} \geqslant d_{Y,T}$ and $d_{U_T^k(Y),T} \geqslant d_{Y,T}$ for all $k \in \mathbb{N}$ then $d_{Y,T^m} \geqslant m$ for all $m \in \mathbb{N}$.*

Proof. Denote for convenience $N = d_{Y,T}$. Let $F \subseteq X$ be such that $TY \subseteq Y + F$ and $\dim F = N$. Fix a basis $f_1^0, \ldots, f_N^0 \in X$ of F.

Suppose that $d_{D_T^k(Y),T} \geqslant N$ and $d_{U_T^k(Y),T} \geqslant N$ for all $k \in \mathbb{N}$. Denote $Y_0 = Y$ and $Y_k = D_T^k(Y)$, $k \geqslant 1$. Since $Y_k = D_T(Y_{k-1})$ for all $k \geqslant 1$, it follows that $Y_k \subseteq Y_{k-1}$ and $TY_k \subseteq Y_{k-1}$ as $k \geqslant 1$.

We claim that for each $k \geqslant 1$ there exists an N-tuple $(f_1^k, \ldots, f_N^k)$ in Y_{k-1} such that

(i) $Y_k \oplus F_k = Y_{k-1}$ where $F_k = [f_i^k]_{i=1}^N$, and

(ii) if $P_k : Y_k \oplus F_k \to F_k$ is the projection onto F_k along Y_k (we also put $P_0 : Y \oplus F \to F$ the projection onto F along Y), then $(P_{k-1}T)f_i^k = f_i^{k-1}$ for all $i = 1, \ldots, N$.

Let $k = 1$. By Lemma 3.3.2(iii), for each $i = 1, \ldots, N$, we can find $f_i^1 \in Y$ such that $(P_0 T)f_i^1 = f_i^0$. Then (ii) is satisfied. Write $F_1 = [f_i^1]_{i=1}^N$. Since $Y_1 \cap F_1 = \{0\}$ by definition of Y_1 and $\dim F_1 = N = \operatorname{codim}_Y Y_1$ by Lemma 3.3.11, $Y_1 \oplus F_1 = Y$.

Suppose the claim is true for $k \geqslant 1$. Then $Y_k \oplus F_k = Y_{k-1}$. Since $TY_k \subseteq Y_{k-1}$ and $d_{Y_k,T} \geqslant N = \dim F_k$, we get $d_{Y_k,T} = N$. Then from Lemma 3.3.2(iii) for each $i = 1, \ldots, N$ there exists $f_i^{k+1} \in Y_k$ such that $(P_k T)f_i^{k+1} = f_i^k$, so that (ii) is satisfied for $k + 1$. To show (i), write $F_{k+1} = [f_i^{k+1}]_{i=1}^N$, and observe: $Y_{k+1} \cap F_{k+1} = \{0\}$ by definition of Y_{k+1} and $\dim F_{k+1} = N = \operatorname{codim}_{Y_k} Y_{k+1}$ by Lemma 3.3.11.

Observe that from condition (ii) of this claim we have: for each $k \geqslant 1$ there exists $y \in Y$ such that $T^k f_i^k = y + f_i^0$. That is, f_i^k is a k-th "preimage" of f_i^0. It follows that any $f \in F$ has a k-th "preimage" in Y_{k-1}.

Denote $Z_0 = Y$, $Z_k = U_T^k(Y)$, $k \geqslant 1$. That is, $Z_k = U_T(Z_{k-1})$. In particular, $TZ_{k-1} \subseteq Z_k$ for all $k \geqslant 1$. We claim that $Z_k = Y \oplus F \oplus TF \cdots \oplus T^{k-1}F$. Indeed, for $k = 0$ this is obvious. Suppose the claim is true for $k \geqslant 1$. Let's prove that $Z_{k+1} = Y \oplus F \oplus TF \cdots \oplus T^k F$. We have $Z_{k+1} = U_T(Z_k) = Z_k + TZ_k = (Y \oplus F \oplus TF \cdots \oplus T^{k-1}F) + (TY + TF + T^2F + \cdots + T^kF) = (Y \oplus F \oplus TF \cdots \oplus T^{k-1}F) + T^kF$ since $TY \subseteq Y \oplus F$. In particular, $Z_{k+1} = Z_k + T^kF$. We only have to prove that this sum is direct. We have $\dim T^kF \leqslant N$ since $\dim F = N$. On the other hand, $TZ_k \subseteq Z_{k+1} = Z_k + T^kF$. Since $d_{Z_k,T} \geqslant N$ for all $k \geqslant 0$, we get $\dim T^kF = N = d_{Z_k,T}$. By Lemma 3.3.2(i), the sum must be direct.

Observe that in particular, this means that if $f \in F$ is non-zero then $T^kf \in Z_{k+1} \setminus Z_k$ $(k \geqslant 0)$.

We will establish now that $d_{Y,T^m} \geqslant m$ by constructing a sequence $(w_k)_{k=1}^m$ in T^mY such that $w_k \in Z_{m-k+1} \setminus Z_{m-k}$. This, in particular, will mean that T^mY contains m vectors $\{w_k\}_{k=1}^m$ whose no non-zero linear combination belongs to Y. The statement of the lemma will then follow from Lemma 3.3.8.

Let $u \in F$ be a non-zero vector, $m \in \mathbb{N}$ be arbitrary, and $k \in \{1, \ldots, m\}$. Put u_k to be the k-th "preimage" of u, that is, such a vector in Y_{k-1} that $T^ku_k = v_k + u$ for some $v_k \in Y$. Then $T^mu_k = T^{m-k}T^ku_k = T^{m-k}v_k + T^{m-k}u$. Since $v_k \in Y$, it follows that $T^{m-k}v_k \in Z_{m-k}$. Also since $u \neq 0$, we get $T^{m-k}u \in Z_{m-k+1} \setminus Z_{m-k}$. It is left to put $w_k = T^m(u_k)$. □

As an immediate corollary we get the following theorem.

Theorem 3.3.13. *Let $T \in L(X)$ be an operator and $\mathfrak{A}$ the norm closed algebra generated by T. If $\mathfrak{A}$ has an almost invariant half-space then $\mathfrak{A}$ has an invariant half-space.*

Proof. If $d_{Y,T} = 0$ then there is nothing to prove. Let $d_{Y,T} > 0$. Since $\mathfrak{A}$ is norm closed, $\sup_{S\in\mathfrak{A}} d_{Y,S} < \infty$ by Theorem 3.3.9. In particular, $\sup_{m\in\mathbb{N}} d_{Y,T^m} < \infty$. By Lemma 3.3.12, it follows that either $d_{D_T^k(Y),T} < d_{Y,T}$ or $d_{U_T^k(Y),T} < d_{Y,T}$ for some $k \in \mathbb{N}$.

Applying this finitely many times we get a half-space Z such that $d_{Z,T} = 0$. Since Z is T-invariant, it is $\mathfrak{A}$-invariant. □

The method we developed can be used to generalize Theorem 3.3.13. We will need a small lemma.

Lemma 3.3.14. *Let $T \in L(X)$ and $Y \subseteq X$ is a T-almost invariant subspace. If $S \in L(X)$ is such that $TS = ST$ and Y is S-invariant then $D_T(Y)$ and $U_T(Y)$ are also S-invariant.*

Proof. If $y \in D_T(Y)$, then $T(Sy) = STy \in Y$ since $Ty \in Y$ and Y is S-invariant. Hence $Sy \in D_T(Y)$, so that $D_T(Y)$ is S-invariant.

Let $u + v \in U_T(Y) = Y + TY$, with $u \in Y$ and $v = Ty$ for some $y \in Y$. Then $S(u + v) = Su + STy = Su + TSy \in U_T(Y)$ since $Su, Sy \in Y$, so that $U_T(Y)$ is S-invariant. □

Theorem 3.3.15. *Let $\mathfrak{A}$ be the norm-closed algebra generated by a finite number of pairwise commuting operators. If $\mathfrak{A}$ has an almost invariant half-space then $\mathfrak{A}$ has an invariant half-space.*

Proof. Let $\mathfrak{A}$ be generated by pairwise commuting operators $T_1, \dots, T_n$ and let Y be an $\mathfrak{A}$-almost invariant half-space. We will prove that there exists a half-space which is invariant under each T_k $(k = 1, \dots, n)$.

Observe that if $T \in \mathfrak{A}$ then both $D_T(Y)$ and $U_T(Y)$ are $\mathfrak{A}$-almost invariant because $\operatorname{codim}_Y D_T(Y) < \infty$ and $\operatorname{codim}_{U_T(Y)} Y < \infty$ by Lemma 3.3.11. Clearly, Y is almost invariant under the norm closed algebra generated by T_1. Apply a finite sequence of procedures D_{T_1} and U_{T_1} to Y to obtain a T_1-invariant half-space Y_1, as in the proof of Theorem 3.3.13. Then, Y_1 is $\mathfrak{A}$-almost invariant. Apply a finite sequence of procedures D_{T_2} and U_{T_2} to Y_1 to obtain a T_2-invariant half-space. By Lemma 3.3.14, Y_2 is also T_1-invariant. Obviously, it is still $\mathfrak{A}$-almost invariant. Repeat this procedure $n - 2$ more times to get an $\mathfrak{A}$-invariant half-space. □

3.3.4 Almost invariant subspaces of closures of algebras

It is well-known that the invariant subspaces of an algebra and its WOT-closure are the same. We will now study this question for almost invariant subspaces.

First, we establish that if the algebra $\mathfrak{A}$ is norm closed then almost invariant subspaces of $\mathfrak{A}$ and $\overline{\mathfrak{A}}^{WOT}$ are the same. We need a lemma for this statement.

Lemma 3.3.16. *Let Y be a subspace of X and $\mathfrak{A}$ an algebra of operators acting on X. Let $N \in \mathbb{N}$ be such that $d_{Y,T} \leqslant N$ for all $T \in \mathfrak{A}$. Then $d_{Y,T} \leqslant N$ for all $T \in \overline{\mathfrak{A}}^{WOT}$.*

Proof. Suppose that the conclusion is not true. Let $T \in \overline{\mathfrak{A}}^{WOT}$ be an operator with $d_{Y,T} \geqslant N+1$. Let $F \subseteq X$ be such that $\dim F = d_{Y,T}$ and $TY \subseteq Y \oplus F$. Fix $N+1$ linearly independent vectors $(f_i)_{i=1}^{N+1}$ in F. By Lemma 3.3.2(iii), there exist $(u_i)_{i=1}^{N+1} \subseteq Y$ such that for each $i = 1, \ldots, N+1$ we have $Tu_i = y_i + f_i$ for some $y_i \in Y$. Since $Y \cap F = \{0\}$, $[Tu_i]_{i=1}^{N+1} \cap Y = \{0\}$ and $Tu_1, \ldots, Tu_{N+1}$ are linearly independent.

Since $\overline{\mathfrak{A}}^{WOT} = \overline{\mathfrak{A}}^{SOT}$, we can find a net $(T_\alpha) \subseteq \mathfrak{A}$ such that $T_\alpha \overset{SOT}{\to} T$. Let $q : X \to X/Y$ be the quotient map. Since $[Tu_i]_{i=1}^{N+1} \cap Y = \{0\}$ and $Tu_1, \ldots, Tu_{N+1}$ are linearly independent, the collection $\{(qT)u_i\}_{i=1}^{N+1}$ is linearly independent. Observe that if $\varepsilon > 0$ is sufficiently small then each collection $\{v_i\}_{i=1}^{N+1}$ satisfying $\|v_i - (qT)u_i\| < \varepsilon$ as $i = 1, \ldots, N+1$ is again linearly independent.

Fix α_0 such that for all $\alpha \geqslant \alpha_0$ we have $\|(T_\alpha - T)(u_i)\| < \varepsilon$ for all $i = 1, \ldots, N+1$. Then $\|(qT_\alpha - qT)(u_i)\| < \varepsilon$, so that the collection $\{(qT_\alpha)u_i\}_{i=1}^{N+1}$ is linearly independent for all $\alpha \geqslant \alpha_0$. By Lemma 3.3.6, this, however, implies that $d_{Y,T_\alpha} \geqslant N+1$ for all $\alpha \geqslant \alpha_0$ which contradicts the assumptions. $\square$

Corollary 3.3.17. *Let $\mathfrak{A}$ be a norm closed algebra of operators on X and Y be a half-space in X. Then Y is $\mathfrak{A}$-almost invariant if and only if Y is $\overline{\mathfrak{A}}^{WOT}$-almost invariant.*

Proof. If Y is $\mathfrak{A}$-almost invariant then by Theorem 3.3.9 there exists $N \in \mathbb{N}$ such that $d_{Y,S} < N$ for all $S \in \mathfrak{A}$. By Proposition 3.3.16 the same is true for all $S \in \overline{\mathfrak{A}}^{WOT}$. This implies that Y is $\overline{\mathfrak{A}}^{WOT}$-almost invariant. The converse statement is obvious. □

Remarkably, the condition that $\mathfrak{A}$ is norm closed is essential for the above result, as the next example shows. This example also exhibits that, unlike in the case of invariant subspaces, there exists an operator whose almost invariant half-spaces are different from those of the norm closed algebra generated by this operator.

Example 3.3.18. Let D be a Donoghue operator on ℓ_2. Put $\mathfrak{A} = \{p(D)\colon p$ is a polynomial such that $p(0) = 0\}$. By Corollary 3.2.9 D has an almost invariant half-space. Then $\mathfrak{A}$ has an almost invariant half-space. However, since all the invariant subspaces of D are finite dimensional and therefore $\mathfrak{A}$ has no invariant half-spaces, $\overline{\mathfrak{A}}^{\|\cdot\|}$ has no almost invariant half-spaces by Theorem 3.3.13.

Chapter 4

Transitive algebra techniques in Banach lattices

In this chapter, we use techniques from [72] and [84] to obtain a lattice version of Theorem 1.2.7 of Lomonosov which is valid for collections of positive operators. This is a joint work with Troitsky. The results from this chapter were inspired by the paper of Drnovšek [45] about invariant subspaces of collections of positive operators and were published in [100].

4.1 Invariant subspaces of positive operators

Throughout the chapter, X will stand for a Banach lattice. That is, X is a Banach space endowed with a lattice order which is compatible with the linear structure of X and satisfies the following two conditions:

(i) $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$;

(ii) $\big\||x|\big\| = \|x\|$ for all $x \in X$, where $|x| = x \vee (-x)$.

For an extensive treatment of Banach lattices, we refer the reader to the monographs [83], [114], and [1]. We will write $x > y$ if $x \geqslant y$ and $x \neq y$.

There are many indications that compatibility of an operator with the order structure can be useful in determining many properties of this operator. Perhaps the most well-known result in this regard is the Perron-Frobenius theorem [95, 50] (for a more accessible exposition, see, e.g., [1, Theorem 8.26]) which (in the simplest form) asserts that a square matrix with strictly positive entries has a unique largest real eigenvalue and the corresponding eigenvector is unique (up to scaling) and has strictly positive entries. Another classical result is the automatic continuity of positive linear mappings: every positive linear map between two Banach lattices is continuous.

A very satisfactory extension of the Perron-Frobenius theorem to positive compact operators on Banach lattices was obtained by Krein and Rutman [69].

Theorem 4.1.1. [69] *If T is a compact positive operator on a Banach lattice X with $r(T) > 0$ then the spectral radius $r(T)$ is an eigenvalue having a positive eigenvector $x > 0$.*

For non-compact positive operators on Banach lattices, the statement of Theorem 4.1.1 is not valid in general. For example, the right shift operator on ℓ_2 is a positive operator without eigenvalues. However, even for non-compact operators, positivity of the operator yields certain information about the spectrum.

Theorem 4.1.2. [69] *The spectral radius of a positive operator on a Banach lattice belongs to the spectrum of the operator.*

It has been conjectured (see, e.g., [7, Conjecture 1]) that every positive operator on a separable Banach lattice has an invariant subspace. This conjecture is still unsettled. None of the known examples of operators without invariant subspaces is positive. It should be noted that Sirotkin [122] was able to fine-tune the Read's operator acting on ℓ_1 from [107] in such a way that the resulting matrix has only one negative entry. Also, Troitsky showed in [124] that the modulus of Read's quasinilpotent operator [109] has an eigenvalue.

A classical result about invariant subspaces of operators acting on Banach lattices of continuous functions is due to Krein [69].

Theorem 4.1.3. [69] *If T is a positive operator on $C(K)$ where K is a compact Hausdorff space then the adjoint T^* of T has an eigenvalue.*

By Theorem 4.1.3, if T is a non-scalar operator on $C(K)$ then T has a hyperinvariant subspace (see Proposition 1.1.1). There are several proofs and modifications of Krein's theorem in the literature (see, e.g., [3, Theorems 6.3 and 7.1], [116, p.315], and [92]).

An important branch of the invariant subspace research for positive operators originates from the Ando-Krieger theorem. To proceed, we will need some terminology.

Definition 4.1.4. A subset A of a Banach lattice X is said to be ***solid*** if $|x| \leqslant |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of X is called an ***order ideal***.

Definition 4.1.5. A net x_α in a Banach lattice X converges in order to $x \in X$ (we write it as $x_\alpha \xrightarrow{o} x$) if there are two nets y_β and z_γ such that

(i) $y_\beta \uparrow x$ (i.e., y_β is increasing and $\sup y_\beta = x$) and $z_\gamma \downarrow x$;

(ii) for each β and γ there exists α_0 such that $y_\beta \leqslant x_\alpha \leqslant z_\gamma$ for all $\alpha \geqslant \alpha_0$.

Definition 4.1.6. A set A in a Banach lattice X is ***order closed*** if $x_\alpha \xrightarrow{o} x$ and $x_\alpha \in A$ for all α imply $x \in A$. An order closed ideal of X is called a ***band***.

According to this definition, we do not assume that an order ideal is a closed subspace in the given Banach lattice. However, it is a well-known fact that bands in Banach lattices are necessarily norm closed.

Definition 4.1.7. Let (Ω, Σ, μ) be a σ-finite measure space and $X = L_0(\Omega, \Sigma, \mu)$ the set of (equivalence classes of) all μ-measurable functions. Let $H \subseteq X$ be a Banach lattice with order and linear structure inherited from X. A positive operator $T : H \to H$ is called ***integral operator*** if there exists a $\mu \times \mu$-measurable function $k(\cdot, \cdot)$ such that

$$Tx(t) = \int k(s,t)x(s)d\mu(s)$$

for all $x \in H$ and for μ-almost all $t \in \Omega$.

The Ando-Krieger theorem is the following statement.

Theorem 4.1.8. [10, 70] (see also [1, Corollary 9.37]) *Each quasinilpotent positive integral operator has a non-trivial invariant band.*

Theorem 4.1.8 has been generalized by many mathematicians. See, for example, works of Caselles [36], Schaefer [113], and Zaanen [129, Section 136]. Grobler [57] found a simple proof of Theorem 4.1.8.

It was shown by Schaefer in [113] that the Ando-Krieger theorem is not valid for general positive operators on Banach lattices. He found an example of a quasinilpotent positive operator on $L_p(\mu)$ without non-trivial closed invariant order ideals. Nevertheless, it turned out that under additional conditions, versions of Theorem 4.1.8 hold for positive operators which are not integral. The following theorem is due to de Pagter [41].

Theorem 4.1.9. [41] *A positive compact quasinilpotent operator on a Banach lattice has a non-trivial closed invariant order ideal.*

The idea of de Pagter's proof is an adaptation of Hilden's technique [84] to the context of positive operators on Banach lattices. The paper by de Pagter caused a lot of subsequent research. Generalizations of Theorem 4.1.9 were obtained by Grobler [58], Schaefer [115], Caselles [37], Abramovich, Aliprantis, and Burkinshaw [4, 5], and others.

We would like to describe a generalization from [4] in more details. To proceed, we need some defintions.

Definition 4.1.10. An operator $T \in L(X)$ is ***quasinilpotent at a point*** $x \in X$ if $\lim\limits_{n\to\infty} \|T^n x\|^{1/n} = 0$.

Definition 4.1.11. Let X be a Banach lattice and $T \in L(X)$. We say that an operator $S \in L(X)$ is ***dominated*** by T (or that T ***dominates*** S) if $|Sx| \leqslant T(|x|)$ holds for all $x \in X$.

Observe that an operator T dominating any other operator is necessarily positive.

Definition 4.1.12. An operator $T : X \to Y$ between Banach lattices is called ***AM-compact*** if $T[a, b]$ is norm totally bounded for all $a, b \in X$.

It is clear that any compact operator is AM-compact. The converse is not true (see, e.g., [2, Problem 10.3.7]).

Theorem 4.1.13. [4] *Let X be a Banach lattice and $T \in L(X)$ be a positive operator. Suppose that there exists a positive operator $S \in L(X)$ such that:*

(i) $ST \leqslant TS$;

(ii) S is quasinilpotent at some $x_0 > 0$;

(iii) S dominates a non-zero AM-compact operator.

Then T has a non-trivial closed invariant ideal.

Compared to Theorem 4.1.9, this result, instead of imposing all conditions on one operator, shifts some of them to an operator the set $\{S\colon ST \leqslant TS\}$ (which is called super left commutant of T, see Definition 4.1.14). Therefore it can be considered as a lattice version of Lomonosov's theorem about compact operators (Theorem 1.1.3).

This theorem was generalized by Drnovšek to collections of positive operators in [45]. He used an alternative to the notion of quasinilpotence at a point which we will describe now.

Definition 4.1.14. Let $\mathcal{C}$ be a collection of positive operators on a Banach lattice X. The ***super right commutant*** $[\mathcal{C}\rangle$ is defined by

$$[\mathcal{C}\rangle = \{S \in L(X)\colon S \geqslant 0 \text{ and } ST - TS \geqslant 0 \text{ for all } T \in \mathcal{C}\}.$$

The ***super left commutant*** $\langle\mathcal{C}]$ is

$$\langle\mathcal{C}] = \{S \in L(X)\colon S \geqslant 0 \text{ and } ST - TS \leqslant 0 \text{ for all } T \in \mathcal{C}\}.$$

Definition 4.1.15. For two collections $\mathcal{C}$ and $\mathcal{D}$ of operators acting on the same space we write $\mathcal{C}\mathcal{D} = \{TS\colon T \in \mathcal{C}, S \in \mathcal{D}\}$. The symbol $\mathcal{C}^n$ is defined as the product of n copies of $\mathcal{C}$.

Definition 4.1.16. If U is a subset of X then we write $\|U\| = \sup\{\|x\| \colon x \in U\}$. We call a collection $\mathcal{C}$ of operators ***finitely quasinilpotent*** at a vector $x \in X$ if $\limsup_n \|\mathcal{F}^n x\|^{\frac{1}{n}} = 0$ for every finite subcollection $\mathcal{F}$ of $\mathcal{C}$.

Clearly, finite quasinilpotence at x implies local quasinilpotence at x of every operator in the collection.

Theorem 4.1.17. [45] *If $\mathcal{C}$ is a collection of positive operators on a Banach lattice X such that*

(i) $\mathcal{C}$ is finitely quasinilpotent at some positive non-zero vector, and

(ii) some operator in $\mathcal{C}$ dominates a non-zero AM-compact operator,

then $\mathcal{C}$ and $[\mathcal{C}\rangle$ have a common closed invariant order ideal.

Theorem 4.1.17 can be viewed to a certain extend as a lattice version of Theorem 1.2.5. Combining it with [125, Corollary 7], Drnovšek also obtained the following generalization of Theorem 4.1.9.

Theorem 4.1.18. *A multiplicative semigroup of quasinilpotent compact positive operators on a Banach lattice of dimension at least two has a nontrivial closed invariant ideal.*

We would like to finish this introductory section with the following result of Abramovich, Aliprantis, and Burkinshaw which applies, for example, to positive operators acting on ℓ_p or c_0.

Theorem 4.1.19. [6] *Let X be a Banach lattice whose order is defined by an unconditional basis, $T : X \to X$ be a positive operator. If T commutes with a non-zero positive operator that is quasinilpotent at some $x > 0$ then T has a non-trivial closed invariant ideal.*

4.2 A version of Lomonosov theorem for collections of positive operators

Observe that if a collection $\mathcal{C}$ of positive operators has a closed nontrivial invariant ideal then there exist non-zero positive $x \in X$ and $f \in X^*$ such that $\langle f, Tx\rangle = 0$ for all $T \in \mathcal{C}$. Indeed, let J be a closed nontrivial $\mathcal{C}$-invariant ideal. Pick a non-zero positive $x \in J$ and a non-zero $g \in X^*$ such that $g|_J = 0$. Put $f = |g|$. Then by the Riesz-Kantorovich formula (see, e.g., [1, p.16]) $\langle f, Tx\rangle = \sup\{g(Ty) : -x \leqslant y \leqslant x\} = \sup\{0 : -x \leqslant y \leqslant x\} = 0$ for all $T \in \mathcal{C}$.

If $\mathcal{C}$ is a semigroup then the converse is also true. Indeed, let $f \in X^*$ and $x \in X$ be positive, non-zero elements, such that $\langle f, Tx\rangle = 0$ for all $T \in \mathcal{C}$. Define J to be the ideal generated by the set $A = \{Tx : T \in \mathcal{C}\}$. It is easy to see that $J \subseteq \ker f$, hence $\overline{J}^{\|\cdot\|} \subseteq \ker f$, so that $\overline{J}^{\|\cdot\|}$ is non-trivial. Also, A is $\mathcal{C}$-invariant since $\mathcal{C}$ is a semigroup. The claim now follows from the following lemma.

Lemma 4.2.1. *Let $\mathcal{C}$ be a collection of positive operators. If a set A is $\mathcal{C}$-invariant then the norm closure of the ideal generated by A is also $\mathcal{C}$-invariant.*

Proof. Let J be the ideal generated by A. It is enough to prove that J is $\mathcal{C}$-invariant. Let $y \in J$. There exists $z \in A$ such that y is dominated by a scalar multiple of z. That is, $|y| \leqslant \alpha z$ for some $\alpha > 0$. If $T \in \mathcal{C}$ then $Tz \in A$. Hence $Ty \in J$, as $|Ty| \leqslant \alpha Tz$. So, J is T-invariant for all $T \in \mathcal{C}$. □

The goal of this section is to "quantize" Theorem 4.1.17 in the same manner that Theorem 1.2.5 was "quantized" into Theorem 1.2.7. Our proofs use ideas from [72] and [84].

In the rest of this section, X will be a real Banach lattice. We will also assume that X is a dual Banach space; that is, $X = Y^*$ for some (fixed) Banach space Y. We will start with a version of Theorem 1.2.7 for convex collections of positive operators. Recall that if $T \in L(X)$ is an adjoint operator then

$T_* \in L(Y)$ stands for a (unique) operator satisfying the condition $T = (T_*)^*$ (cf. page 11 of this thesis).

In the following two lemmas, we collect several standard facts that we will use later.

Lemma 4.2.2. *Let Z be a vector lattice, $x \in Z_+$. Then for each $y, z \in Z$ one has*

(i) $|x \wedge y - x \wedge z| \leqslant |y - z|$;

(ii) *if* $|y| \leqslant z$ *then* $|x - x \wedge z| \leqslant |x - x \wedge y|$;

(iii) $|x - x \wedge y| \leqslant |x - y|$.

Proof. It is easy to check that the formulas are valid in $\mathbb{R}$. The validity of the formulas in any Banach lattice follows from Yudin's theorem. □

Lemma 4.2.3. *If K is a compact operator on Y then K^* is w^*-$\|\cdot\|$ continuous on bounded sets.*

Proof. It is enough to show that if $x_\alpha \xrightarrow{w^*} 0$ and $\|x_\alpha\| \leqslant 1$ then $K^* x_\alpha \xrightarrow{\|\cdot\|} 0$. We have:

$$\|K^* x_\alpha\| = \sup_{y \in B_Y} |\langle K^* x_\alpha, y\rangle| = \sup_{y \in B_Y} |\langle x_\alpha, Ky\rangle|.$$

Fix $\varepsilon > 0$. Let $(y_i)_{i=1}^n \subseteq B_Y$ be such that $K(B_Y) \subseteq \cup_{i=1}^n B(Ky_i, \varepsilon)$. There exists $\overline{\alpha}$ such that $|\langle x_\alpha, Ky_i\rangle| < \varepsilon$ for all $\alpha \geqslant \overline{\alpha}$ and $i \in \{1, \ldots, n\}$.

For $y \in B_Y$, pick $i \in \{1, \ldots, n\}$ such that $\|Ky - Ky_i\| < \varepsilon$. Then

$$|\langle x_\alpha, Ky\rangle| = |\langle x_\alpha, Ky_i\rangle + \langle x_\alpha, Ky - Ky_i\rangle| \leqslant$$
$$\leqslant |\langle x_\alpha, Ky_i\rangle| + \|x_\alpha\| \cdot \|Ky - Ky_i\| < 2\varepsilon.$$

Hence $\|K^* x_\alpha\| < 2\varepsilon$ for all $\alpha \geqslant \overline{\alpha}$. □

Theorem 4.2.4. *Let $\mathcal{C}$ be a convex collection of positive adjoint operators on X. If there is $x_0 > 0$ such that every operator in $\mathcal{C}$ is locally quasinilpotent at x_0 then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \|T_*\|_e$ for all $T \in \mathcal{C}$.*

Proof of Theorem 4.2.4. Clearly, we may assume that $\|x_0\| = 1$. Also, without loss of generality, $\mathcal{C}$ is closed under taking positive multiples of its elements, otherwise we replace $\mathcal{C}$ with $\{\alpha T \colon T \in \mathcal{C}, 0 < \alpha \in \mathbb{R}\}$. Fix $0 < \varepsilon < \frac{1}{10}$. Define

$$\begin{aligned} \mathcal{C}_\varepsilon &= \big\{T \in \mathcal{C} \colon \|T_*\|_e < \varepsilon\big\}, \text{ and} \\ H_\varepsilon(x) &= \big\{z \in X \colon |z| \leqslant Tx \text{ for some } T \in \mathcal{C}_\varepsilon\big\}, \quad x \in X_+. \end{aligned}$$

Clearly, $H_\varepsilon(x)$ is convex and solid for all $x \in X_+$. Also, if $T \in \mathcal{C}_\varepsilon$, then $Tx \in H_\varepsilon(x)$. In what follows, we will consider two cases.

Case 1. Suppose that $\overline{H_\varepsilon(x)} \neq X$ for some nonzero $x \in X_+$. Since $H_\varepsilon(x)$ is convex, there is a nonzero $g \in X^*$ such that $g(y) \leqslant 1$ for all $y \in H_\varepsilon(x)$. Consider $h = |g| \in X^*$. Then for any $y \in H_\varepsilon(x)$ we have

$$h(y) \leqslant h\big(|y|\big) = \sup\big\{g(u) \colon -|y| \leqslant u \leqslant |y|\big\} \leqslant 1$$

since $H_\varepsilon(x)$ is solid. In particular, $\langle h, Tx\rangle \leqslant 1$ for all $T \in \mathcal{C}_\varepsilon$.

Put $f = \frac{\varepsilon}{2}h$. We claim that $\langle f, Tx\rangle \leqslant \|T_*\|_e$ for each $T \in \mathcal{C}$. Indeed, if T is compact, i.e., $\|T_*\|_e = 0$, then $\alpha T \in \mathcal{C}_\varepsilon$ for all $0 < \alpha \in \mathbb{R}$. Therefore $\langle h, \alpha Tx\rangle \leqslant 1$ for all $0 < \alpha \in \mathbb{R}$, so that $\langle f, Tx\rangle = \frac{\varepsilon}{2}\langle h, Tx\rangle = 0$. If T is not compact then $\frac{\varepsilon T}{2\|T_*\|_e} \in \mathcal{C}_\varepsilon$, whence

$$\langle f, Tx\rangle = \|T_*\|_e\Big\langle h, \frac{\varepsilon T}{2\|T_*\|_e}x\Big\rangle \leqslant \|T_*\|_e.$$

Case 2. Suppose that $\overline{H_\varepsilon(x)} = X$ for all nonzero $x \in X_+$. Then, in particular, for each $x \in X$ there is $y_x \in H_\varepsilon(x)$ such that $\|x_0 - y_x\| < \varepsilon$. Fix an operator $T_x \in \mathcal{C}_\varepsilon$ such that $|y_x| \leqslant T_x x$. Then (ii) and (iii) of Lemma 4.2.2 yield $\|x_0 - x_0 \wedge T_x x\| \leqslant \|x_0 - x_0 \wedge y_x\| \leqslant \|x_0 - y_x\| < \varepsilon$.

Let $U_0 = \big\{x \in X_+ \colon \|x - x_0\| \leqslant \frac{1}{2}\big\}$. Since $\big\|(T_x)_*\big\|_e < \varepsilon$, there is an adjoint compact operator $K_x \in \mathcal{K}(X)$ such that $\|K_x - T_x\| < \varepsilon$. By Lemma 4.2.3, each K_x is w^*-$\|\cdot\|$ continuous on norm bounded sets. It follows that there is a relative (to U_0) w^*- open neigborhood $W_x \subseteq U_0$ of x such that $\|K_x z - K_x x\| < \varepsilon$

whenever $z \in W_x$. Then using Lemma 4.2.2 (i), for every $z \in W_x$ we get:

$$\begin{aligned}\left\|x_0 - x_0 \wedge T_x z\right\| &\leqslant \left\|x_0 - x_0 \wedge T_x x\right\| + \left\|x_0 \wedge T_x x - x_0 \wedge K_x x\right\| \\ &\quad + \left\|x_0 \wedge K_x x - x_0 \wedge K_x z\right\| + \left\|x_0 \wedge K_x z - x_0 \wedge T_x z\right\| \\ &\leqslant \left\|x_0 - x_0 \wedge T_x x\right\| + \|T_x x - K_x x\| + \|K_x x - K_x z\| + \|K_x z - T_x z\| \\ &< \varepsilon + \varepsilon\|x\| + \varepsilon + \varepsilon\|z\| \leqslant 5\varepsilon < \tfrac{1}{2}.\end{aligned}$$

Together with $T_x \geqslant 0$ this yields $(x_0 \wedge T_x y) \in U_0$ for each $y \in W_x$.

Note that U_0 is w^*- compact since U_0 is the intersection of X_+ with a closed ball. Hence, we can find $x_1, \dots, x_n \in U_0$ such that $U_0 = \bigcup_{k=1}^{n} W_{x_k}$. Define $T = T_{x_1} + \cdots + T_{x_n} \in \mathcal{C}$. Then by Lemma 4.2.2(ii), we have $x_0 \wedge Tx \in U_0$ for every $x \in U_0$.

Define a sequence $(y_n) \subseteq U_0$ by $y_0 = x_0$ and $y_{n+1} = x_0 \wedge T y_n$. Clearly $0 \leqslant y_n$ for all n, and $y_n \leqslant T y_{n-1} \leqslant \ldots \leqslant T^n y_0 = T^n x_0$, so that $\|y_n\| \leqslant \|T^n x_0\|$. Thus $y_n \to 0$ as $n \to \infty$ by the local quasinilpotence at x_0. This is a contradiction by the definition of U_0. □

The next theorem shows that the conclusion of Theorem 4.2.4 is also true for some collections of operators which are not necessarily convex. We will, however, use a more restrictive quasinilpotence condition. We will need some additional definitions.

Let $\mathcal{C}$ be a collection of positive operators. Following [1, p. 412], define

$$\mathcal{D}_{\mathcal{C}} = \Big\{ D \in L(X)_+ \ : \ \exists T_1, \dots, T_k \in [\mathcal{C}\rangle \text{ and } S_1, \dots, S_k \in \bigcup_{n=1}^{\infty} \mathcal{C}^n \text{ such that } D \leqslant \sum_{i=1}^{k} T_i S_i \Big\}$$

(see Definition 4.1.14 for the definition of the super-right commutant $[\mathcal{C}\rangle$). In other words, $\mathcal{D}_{\mathcal{C}}$ is the smallest additive and multiplicative semigroup which contains the collection $[\mathcal{C}\rangle \cdot \mathcal{C}$ (see Definition 4.1.15 for the definition of a product of two sets) and such that $T \in \mathcal{D}_{\mathcal{C}}$ and $0 \leqslant S \leqslant T$ imply $S \in \mathcal{D}_{\mathcal{C}}$ (see [1, Lemma 10.41]).

Let $\mathcal{C}$ be a collection of positive adjoint operators on X. Define

$$\mathcal{E}_{\mathcal{C}} = \big\{T \in \mathcal{D}_{\mathcal{C}} \colon T = S^* \text{ for some } S \in L(Y)\big\}.$$

Since adjoint operators are stable under addition and multiplication, $\mathcal{E}_{\mathcal{C}}$ is an additive and multiplicative semigroup. It is also clear that $\mathcal{C} \subseteq \mathcal{E}_{\mathcal{C}}$.

Theorem 4.2.5. *Let $\mathcal{C}$ be a collection of positive adjoint operators on X. If $\mathcal{C}$ is finitely quasinilpotent at some $x_0 > 0$ then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \|T_*\|_e$ for all $T \in \mathcal{E}_{\mathcal{C}}$.*

Proof. Clearly $\mathcal{E}_{\mathcal{C}}$ is convex. Note that the finite quasinilpotence of $\mathcal{C}$ at x_0 implies the finite quasinilpotence of $\mathcal{D}_{\mathcal{C}}$ (and, therefore, of $\mathcal{E}_{\mathcal{C}}$) at x_0 (see, e.g., [1, Lemma 10.43]). Finally, apply Theorem 4.2.4 to $\mathcal{E}_{\mathcal{C}}$. □

Now suppose, in addition, that Y is itself a Banach lattice. Then we can improve the conclusion of Theorem 4.2.4.

Definition 4.2.6. An operator T on a Banach lattice Z is called ***AM-compact*** if $T([a,b])$ is compact for all $a \leqslant b$ in Z.

The following fact is definitely well-known. We include a proof of it for convenience of the reader.

Lemma 4.2.7. *The set of all AM-compact operators on a Banach lattice Z forms a closed subspace in $L(Z)$.*

Proof. It is clear that the set of AM-compact operators is a subspace in $L(Z)$. Suppose that (T_n) is a sequence of AM-compact operators such that $T_n \to T$ in norm. Let's prove that T is also AM-compact.

Let $a \leqslant b \in Z$. Denote $M = \max\big\{\|a\|, \|b\|\big\}$. Without loss of generality, $M \neq 0$. Fix $\varepsilon > 0$. We will construct a finite ε-net in $T([a,b])$.

Let $m \in \mathbb{N}$ be such that $\|T_m - T\| < \frac{\varepsilon}{3M}$. Since T_m is AM-compact, there exists a finite set $\{x_1, \dots, x_n\} \subseteq [a,b]$ such that $\{T_m x_1, \dots, T_m x_n\}$ is an $\frac{\varepsilon}{3}$-net for $T_m([a,b])$. Let $x \in [a,b]$. Pick $i \in \{1, \dots, n\}$ such that $\|T_m x_i - T_m x\| < \frac{\varepsilon}{3}$.

Then $\|Tx_i - Tx\| \leqslant \|Tx_i - T_m x_i\| + \|T_m x_i - T_m x\| + \|T_m x - Tx\| \leqslant M\|T - T_m\| + \frac{\varepsilon}{3} + M\|T_m - T\| < \varepsilon$.

That is, $\{Tx_1, \dots, Tx_n\}$ is a finite ε-net in $T([a,b])$. $\square$

For an operator T acting on Y, define

$$\theta(T) = \inf\bigl\{\|T - K\| \colon K \text{ is AM-compact}\bigr\}.$$

Clearly, θ is a seminorm on $L(Y)$. Also, it follows from Lemma 4.2.7 that $\theta(T) = 0$ if and only if T is AM-compact.

For $\xi \in Y_+$, define a seminorm ρ_ξ on X via $\rho_\xi(x) = |x|(\xi)$.

Lemma 4.2.8. *If $\xi \in Y_+$ and $K \in L(Y)$ is AM-compact, then $K^* \colon (B_X, w^*) \to (X, \rho_\xi)$ is continuous.*

Proof. Let $x_\alpha \xrightarrow{w^*} x$, with x_α, $x \in B_X$. Write

$$\rho_\xi\bigl(K^*x_\alpha - K^*x\bigr) = \bigl|K^*x_\alpha - K^*x\bigr|(\xi) = \sup_{-\xi \leqslant \zeta \leqslant \xi} \langle x_\alpha - x, K\zeta\rangle = \sup_{\nu \in A} \langle x_\alpha - x, \nu\rangle,$$

where $A = K\bigl([-\xi, \xi]\bigr)$. By assumption, K is AM-compact, thus A is a $\|\cdot\|$-compact set.

For $\nu \in A$, fix α_ν such that $\bigl|\langle x_\alpha - x, \nu\rangle\bigr| < \frac{\varepsilon}{3}$ whenever $\alpha \geqslant \alpha_\nu$. If $\mu \in Y$ is such that $\|\mu - \nu\| < \frac{\varepsilon}{3}$ then for $\alpha \geqslant \alpha_\nu$ we have

$$\bigl|\langle x_\alpha - x, \mu\rangle\bigr| \leqslant \tfrac{\varepsilon}{3}\|x_\alpha - x\| + \bigl|\langle x_\alpha - x, \nu\rangle\bigr| < \tfrac{2\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

Pick $\nu_1, \dots, \nu_n \in A$ such that $A \subseteq \bigcup_{k=1}^{n} B(\nu_k, \frac{\varepsilon}{3})$. Then for every $\alpha \geqslant \max\{\alpha_{\nu_1}, \dots, \alpha_{\nu_n}\}$ we must have $\rho_\xi(K^*x_\alpha - K^*x) < \varepsilon$. $\square$

Definition 4.2.9. An operator $T \in L(X)$ will be said w^****-locally quasinilpotent*** at a pair (x_0, ξ_0), where $x_0 \in X$ and $\xi_0 \in Y$, if $\bigl|T^n x_0(\xi_0)\bigr|^{\frac{1}{n}} \to 0$.

Clearly, if T is locally quasinilpotent at x_0 then T is w^*-locally quasinilpotent at (x_0, ξ_0) for every $\xi_0 \in Y$.

Theorem 4.2.10. *Suppose that $X = Y^*$ for some Banach lattice Y, and $\mathcal{C}$ is a convex collection of positive adjoint operators on X. Suppose that there exists a pair $(x_0, \xi_0) \in X_+ \times Y_+$ such that $x_0(\xi_0) \neq 0$ and every operator from $\mathcal{C}$ is w^*-locally quasinilpotent at (x_0, ξ_0). Then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx \rangle \leqslant \theta(T_*)$ for all $T \in \mathcal{C}$.*

Proof. The proof of the theorem is similar to that of Theorem 4.2.4. We may assume that $\|x_0\| = 1$, $\|\xi_0\| = 1$, and $\mathcal{C}$ is closed under taking positive multiples. Put $\rho_{\xi_0}(x) = |x|(\xi_0)$. Evidently, $\rho_{\xi_0}(x) \leqslant \|x\|$ for all $x \in X$. It is also clear that $|x| \leqslant |y|$ implies $\rho_{\xi_0}(x) \leqslant \rho_{\xi_0}(y)$.

Fix $0 < \varepsilon < \frac{x_0(\xi_0)}{8}$. Define

$$\begin{aligned} \mathcal{C}_\varepsilon &= \{T \in \mathcal{C} \colon \theta(T_*) < \varepsilon\} \text{ and} \\ G_\varepsilon(x) &= \{z \in X \colon |z| \leqslant Tx \text{ for some } T \in \mathcal{C}_\varepsilon\}, \quad x \in X_+. \end{aligned}$$

Suppose that $G_\varepsilon(x)$ is not dense in X for some $x \in X_+$. Analogously to the proof of Theorem 4.2.4, we find a positive functional $h \in X_+^*$ such that $\langle h, Tx \rangle \leqslant 1$ for all $T \in \mathcal{C}_\varepsilon$. Considering separately the cases $\theta(T_*) = 0$ and $\theta(T_*) \neq 0$, we get the conclusion of the theorem.

Thus, we may assume that $\overline{G_\varepsilon(x)} = X$ for all $x > 0$. Define

$$U_0 = \{x \in X_+ \colon \|x\| \leqslant 1 \text{ and } \rho_{\xi_0}(x - x_0) \leqslant \frac{x_0(\xi_0)}{2}\}.$$

Clearly, U_0 is w^*-compact.

Let $x \in U_0$ be arbitrary. Since $\overline{G_\varepsilon(x)} = X$, we can find $T_x \in \mathcal{C}_\varepsilon$ such that $\rho_{\xi_0}(x_0 - x_0 \wedge T_x x) \leqslant \|x_0 - x_0 \wedge T_x x\| < \varepsilon$. Fix an operator K_x adjoint to an AM-compact operator such that $\|T_x - K_x\| < \varepsilon$. By Lemma 4.2.8, we can find a relative (to U_0) w^*-open neighborhood $V_x \subseteq U_0$ of x such that

$\rho_{\xi_0}(K_x x - K_x z) < \varepsilon$ for all $z \in V_x$. Then for an arbitrary $z \in V_x$, we have

$$\begin{aligned}\rho_{\xi_0}\big(x_0 - x_0 \wedge T_x z\big) &\leqslant \rho_{\xi_0}\big(x_0 - x_0 \wedge T_x x\big) + \rho_{\xi_0}\big(x_0 \wedge T_x x - x_0 \wedge K_x x\big)\\ &\quad + \rho_{\xi_0}\big(x_0 \wedge K_x x - x_0 \wedge K_x z\big) + \rho_{\xi_0}\big(x_0 \wedge K_x z - x_0 \wedge T_x z\big)\\ &< \varepsilon + \|T_x x - K_x x\| + \rho_{\xi_0}(K_x x - K_x z) + \|K_x z - T_x z\|\\ &< \varepsilon + \|T_x - K_x\| \cdot \|x\| + \varepsilon + \|T_x - K_x\| \cdot \|z\| < 4\varepsilon < \frac{x_0(\xi_0)}{2}.\end{aligned}$$

Take $x_1, \dots, x_m$ in U_0 such that $\bigcup\limits_{k=1}^{m} V_{x_k} = U_0$. Then $T = T_{x_1} + \dots + T_{x_k} \in \mathcal{C}$ satisfies $\rho_{\xi_0}\big(x_0 - x_0 \wedge Tz\big) \leqslant \frac{x_0(\xi_0)}{2}$ for all $z \in U_0$. Since $\|x_0 \wedge Tz\| \leqslant \|x_0\| = 1$, we have $x_0 \wedge Tz \in U_0$ for all $z \in U_0$.

Put $z_0 = x_0$ and $z_{n+1} = x_0 \wedge Tz_n$. By the w^*-local quasinilpotence of T at (x_0, ξ_0) we have $\rho_{\xi_0}(z_n) \leqslant \rho_{\xi_0}(T^n x_0) = \big|T^n x_0(\xi_0)\big| \to 0$ as $n \to \infty$ which is impossible by the definition of U_0. □

The following result is derived from Theorem 4.2.10 in the same way that Theorem 4.2.5 was deduced from Theorem 4.2.4.

Theorem 4.2.11. *Suppose that $X = Y^*$ for some Banach lattice Y, and $\mathcal{C}$ is a collection of positive adjoint operators on X. If $\mathcal{C}$ is finitely quasinilpotent at some $x_0 > 0$ then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \theta(T_*)$ for all $T \in \mathcal{E}_{\mathcal{C}}$.*

As every operator on ℓ_p ($1 \leqslant p < \infty$) is AM-compact, this theorem can be used as an alternative proof of the following (certainly known) result.

Corollary 4.2.12. *Every collection of positive operators on ℓ_p, $1 < p < \infty$, which is finitely quasinilpotent at a non-zero positive vector, has a non-trivial closed common invariant ideal.*

Of course, Corollary 4.2.12 follows easily from Theorem 4.1.17 when $1 \leqslant p < \infty$.

Corollary 4.2.13. *Every collection of positive adjoint operators on ℓ_∞ which is finitely quasinilpotent at a non-zero positive vector has a non-trivial closed common invariant ideal.*

The following example shows that the assumptions in Theorems 4.2.5 and 4.2.11 in general do not guarantee the existence of an invariant subspace.

Example 4.2.14. *There is a collection $\mathcal{C}$ of operators which satisfies all the conditions of Theorem 4.2.11 and has no common non-trivial invariant subspaces.* Namely, in [46], the authors constructed a multiplicative semigroup $\mathcal{S}_p$ of positive square-zero operators acting on $L_p[0,1]$, $1 \leqslant p < \infty$, having no common non-trivial invariant subspaces. The semigroup $\mathcal{S}_p$ is constructed to be the union of semigroups $\mathcal{S}_{p,n}$ $(n = 0, 1, \dots)$ satisfying the following two conditions: $\mathcal{S}_{p,n} \cdot \mathcal{S}_{p,n} = \{0\}$ for all n and $\mathcal{S}_{p,m} \cdot \mathcal{S}_{p,n} \subseteq \mathcal{S}_{p,n}$, $\mathcal{S}_{p,n} \cdot \mathcal{S}_{p,m} \subseteq \mathcal{S}_{p,n}$ for all $m < n$. We claim that $\mathcal{S}_p$ is in fact finitely quasinilpotent at every positive vector, so that for each $1 < p < \infty$, the collection of operators $\mathcal{C} = \mathcal{S}_p$ satisfies the conditions of Theorem 4.2.11.

Indeed, let us show that if $\mathcal{F} \subseteq \mathcal{S}_p$ has k elements, then $\mathcal{F}^{2^k} = \{0\}$. The proof is induction on k. If $k = 1$ then the conclusion is true since $\mathcal{S}_p$ consists of square-zero operators. Suppose the statement is true for $k-1$; let us prove it for k. Let $T_1 T_2 \dots T_{2^k}$ be a product of 2^k operators, each belonging to $\mathcal{F}$. Each T_i belongs to some $\mathcal{S}_{p,n_i}$. Pick an operator T_m from the set $\{T_1, \dots, T_{2^k}\}$ such that n_m is the biggest. Clearly, we can assume that the product $T_1 \dots T_{2^k}$ contains only one copy of T_m since otherwise this product belongs to $\mathcal{S}_{p,m} \cdot \mathcal{S}_{p,m} = \{0\}$. Define $\mathcal{F}_0 = \mathcal{F} \setminus \{T_m\}$. Then $\mathcal{F}_0$ contains $k-1$ elements. By the induction assumption, $\mathcal{F}_0^{2^{k-1}} = \{0\}$, so that product of more than 2^{k-1} operators from $\mathcal{F}_0$ is zero. It is easy to see that one of the products $T_1 \dots T_{m-1}$ or $T_{m+1} \dots T_{2^k}$ contains more than 2^{k-1} operators, and the operators in these two products all belong to $\mathcal{F}_0$.

Remark 4.2.15. Even though Theorem 1.2.5 is not a special case of Theorem 1.2.7, in the case of an algebra of adjoint operators the former can be easily deduced from the latter, see [76, Corollary 1]. Similarly, we will show that in case of adjoint operators, Theorem 4.1.17 can be deduced from Theorem 4.2.11. Indeed, suppose that $X = Y^*$ for some Banach lattice Y, and

$\mathcal{C}$ is a collection of positive adjoint operators which is finitely quasinilpotent at some $x_0 > 0$ and some operator in it dominates a non-zero AM-compact positive[1] adjoint operator K. We will show that there is a non-trivial closed ideal which is invariant under $\mathcal{C}$ and under all adjoint operators in $[\mathcal{C}\rangle$.

Clearly, $K \in \mathcal{E}_{\mathcal{C}}$. Let x and f be as in Theorem 4.2.11.

$$\begin{aligned} \mathcal{J}_1 &= \bigl\{z \in X \,:\, |z| \leqslant T_1 K T_2 x \text{ for some } T_1, T_2 \in \mathcal{E}_{\mathcal{C}}\bigr\}, \\ \mathcal{J}_2 &= \bigl\{z \in X \,:\, T|z| = 0 \text{ for all } T \in \mathcal{E}_{\mathcal{C}}\bigr\}, \text{ and} \\ \mathcal{J}_3 &= \bigl\{z \in X \,:\, |z| \leqslant Tx \text{ for some } T \in \mathcal{E}_{\mathcal{C}}\bigr\}. \end{aligned}$$

It is easy to see that $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ are ideals in X, invariant under $\mathcal{C}$ and under all adjoint operators in $[\mathcal{C}\rangle$. It is left to show that at least one of the three must be non-trivial. Clearly, $\mathcal{J}_2$ is closed and $\mathcal{J}_2 \neq X$. Suppose that $\mathcal{J}_2 = \{0\}$. In particular, $x \notin \mathcal{J}_2$. It follows that $\mathcal{J}_3 \neq \{0\}$. Suppose that $\mathcal{J}_3$ is dense in X. It follows from Theorem 4.2.11 that $\mathcal{J}_1 \subseteq \ker f$; hence $\overline{\mathcal{J}_1}$ is proper. Assume that $\mathcal{J}_1 = \{0\}$. Hence, $T_1 K T_2 x = 0$ for all $T_1, T_2 \in \mathcal{E}_{\mathcal{C}}$. Since $\mathcal{J}_2 = \{0\}$, it follows that K vanishes on $\mathcal{E}_{\mathcal{C}} x$ and, therefore, on $\mathcal{J}_3$. Since $\mathcal{J}_3$ is dense in X it follows that $K = 0$; a contradiction.

[1] Unlike in Theorem 4.1.17, we require that $K \geqslant 0$ here.

Chapter 5

Indecomposable semigroups of nonnegative matrices

In this chapter, we study properties of multiplicative semigroups of nonnegative matrices. Results from this chapter are joint results of the author of this thesis and his collaborators. They have been published in [53] and [99].

5.1 Introduction

The following general type of question has been of interest in various contexts, including linear representations of groups and semigroups: if a certain property about a group or a semigroup $\mathcal{S}$ holds locally, then does it hold globally? For example, it is well known that if $\mathcal{S}$ is an irreducible (multiplicative) group of matrices, and if the trace functional takes a finite number of values on $\mathcal{S}$, then $\mathcal{S}$ is itself finite (*irreducible* means no common invariant subspaces). Another example of this type is the original version of Burnside's theorem [33] (cf. Theorem 1.2.4):

Theorem 5.1.1. [33] *If $\mathcal{G}$ is a group of unitary matrices in $\mathbb{M}_n(\mathbb{C})$, and for each nonzero vector $v \in \mathbb{C}^n$, the set $\{Gv \colon G \in \mathcal{G}\}$ spans $\mathbb{C}^n$, then $\mathcal{G}$ spans $\mathbb{M}_n(\mathbb{C})$.*

There have been a number of more recent results of this nature. Okniński [93, Proposition 4.9] showed that if $\mathcal{S}$ is an irreducible semigroup of matrices and $\{\mathrm{tr} S \colon S \in \mathcal{S}\}$ is a finite set, then $\mathcal{S}$ is finite. Radjavi and Rosenthal [104] replaced the trace functional in this statement with an arbitrary non-zero functional. It is also shown in [104] that if $\mathcal{S}$ is an irreducible semigroup of matrices and for some non-zero functional f the set $\{f(S) \colon S \in \mathcal{S}\}$ is bounded then $\mathcal{S}$ is also bounded. An interesting result was obtained by Bernik, Mastnak, and Radjavi in [25]: every irreducible group of matrices with nonnegative diagonal entries is similar, via a positive diagonal matrix, to a group of matrices with nonnegative entries.

In this work, we will mostly be interested in semigroups of matrices with nonnegative entries.

Definition 5.1.2. A matrix $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ is called ***nonnegative*** if $a_{ij} \geqslant 0$ for all $1 \leqslant i, j \leqslant n$.

Definition 5.1.3. A collection $\mathcal{C}$ of positive operators is called ***indecomposable*** if $\mathcal{C}$ has no common closed non-trivial invariant order ideals (see Definition 4.1.4). An operator A is ***indecomposable*** if the set $\{A\}$ is indecomposable.

In $\mathbb{R}^n$, the order ideals are exactly subspaces spanned by a subset of the standard basis. So, a collection $\mathcal{C}$ of nonnegative matrices is indecomposable if any subspace spanned by a subset of the standard basis of $\mathbb{R}^n$ is not invariant under $\mathcal{C}$.

An example of a local-to-global result for semigroups of nonnegative matrices is the work of Livshits, MacDonald, and Radjavi [74] which shows that if the diagonal elements of all matrices in an indecomposable semigroup of nonnegative matrices come from the set $\{0, 1\}$, then the entire semigroup is similar, via a positive diagonal matrix, to a semigroup of matrices whose all entries come from the set $\{0, 1\}$; semigroups with this property are called *binary*. The authors also obtain generalizations to infinite dimensional lattices.

In this thesis we study structure of indecomposable semigroups of nonnegative matrices satisfying one of the following conditions:

- a non-zero positive functional is bounded on the semigroup;

- the diagonal entries of matrices in the semigroup are small (i.e., take values in $[0, \varepsilon]$ where $0 < \varepsilon \leqslant 1$);
- the diagonal entries of matrices in the semigroup come from a fixed finite set.

We will also obtain generalizations of our results to operators acting on infinite-dimensional Banach lattices.

In the rest of this section we will collect some relevant facts and definitions about the structure of indecomposable semigroups of nonnegative matrices. A good source about this topic is the book of Radjavi and Rosenthal [103]; many statements in this section come from this source.

The following proposition is rather standard. The directions "$(i) \longrightarrow (iii)$" and "$(ii) \longrightarrow (i)$" are obvious; for the direction "$(iii) \longrightarrow (ii)$", see, e.g., [103, Lemma 5.1.5].

Proposition 5.1.4. *Let $\mathcal{S}$ be a semigroup in $M_n^+(\mathbb{R})$. Then the following statements are equivalent.*

(i) $\mathcal{S}$ is indecomposable;

(ii) for every $i, j \leqslant n$ there exists $S \in \mathcal{S}$ with $(S)_{ij} > 0$;

(iii) no permutation of the basis reduces $\mathcal{S}$ to the block form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$.

The following fact which (in the case of matrices) can be found in many books about nonnegative matrices (see, e.g., [24, Theorem 2.2.7]) is useful in determining if a given matrix is indecomposable. It also applies to operators on ℓ_p or c_0 (see [105, Proposition 1.2]) which can be represented as infinite matrices. For a nonnegative matrix (t_{ij}), we say that there is an *arc* from i to j and write $i \to j$ if $t_{ij} \neq 0$. We say that there is a *path* from i to j if there is a sequence of arcs $i = k_0 \to k_1 \to \cdots \to k_n = j$.

Proposition 5.1.5. *Let $T = (t_{ij})$ be a matrix or a positive operator on ℓ_p $(1 \leqslant p < \infty)$ of c_0. Then T is indecomposable if and only if for every i, j, there is a finite path from i to j.*

The following theorem is a generalization of the well-known Perron-Frobenius theorem.

Theorem 5.1.6. [103, Corollary 5.2.13] *Let A be an indecomposable nonnegative matrix with $r(A) = 1$. Denote by r the minimal rank of nonzero members of the semigroup $\overline{\mathbb{R}^+\mathcal{S}}$ where $\mathcal{S}$ is the semigroup generated by A. Then the following holds:*

(i) There exists a nonnegative nonzero vector x, unique up to a scalar multiple, such that $Ax = x$;

(ii) the set $\{\lambda \in \sigma(A) \colon |\lambda| = 1\}$ consists precisely of all the r-th roots of unity; each member of this set is a simple eigenvalue (i.e., the corresponding eigenspace has dimension one);

(iii) $\sigma(A)$ is invariant under the rotation about the origin by the angle $2\pi/r$;

(iv) if $r > 1$ then there is a permutation matrix P such that $P^{-1}AP$ has the block form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & A_r \\ A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & A_{r-1} & 0 \end{bmatrix}$$

(with square diagonal blocks).

The following corollary will be used in Section 5.3. Though it is standard, we include the proof for convenience of the reader.

Corollary 5.1.7. *Let S be a matrix such that $r(S) = 1$. Then the modulus-one eigenvalues of S are roots of unity of degrees at most k where k is the number of such eigenvalues.*

Proof. If S is indecomposable then the statement follows immediately from Theorem 5.1.6. If S is not indecomposable then, after a permuation of the basic vectors, S can be represented in the upper block triangular form with indecomposable diagonal blocks (note that this procedure doesn't change the spectrum of a matrix). The result follows from the fact that the set of eigenvalues of the matrix $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is exactly the union of eigenvalues of matrices A and B which follows from considering the Jordan canonical forms of A and B. □

Definition 5.1.8. A nonnegative matrix is said to be ***row*** (***column***) ***stochastic*** if each of its rows (columns) sums to 1. A matrix is ***doubly stochastic*** if it is both row and column stochastic.

Definition 5.1.9. A collection of matrices $\mathcal{C}$ is ***block-monomial*** if each member has only one nonzero block in each block row and block column under a given block structure. We call $\mathcal{C}$ ***monomial*** if it is block-monomial relative to blocks of size 1×1.

Theorem 5.1.10. [103, Lemma 5.1.11] *If $\mathcal{G}$ is a group of invertible nonnegative matrices, then $\mathcal{G}$ is monomial. Furthermore, if $\mathcal{G}$ is bounded, then $\mathcal{G}$ is similar, via a positive diagonal matrix, to a group of permutation matrices.*

The next theorem is a very important structure result which will be used many times.

Theorem 5.1.11. [103, Lemma 5.1.9] *Let E be a nonnegative idempotent of rank r.*

(i) If E has no zero rows or columns then there exists a permutation matrix P such that $P^{-1}EP$ has the block-diagonal form

$$E_1 \oplus \cdots \oplus E_r,$$

where each E_i is an idempotent of rank one whose entries are all positive;

(ii) in general, there exists a permutation matrix P such that $P^{-1}EP$ has the block-triangular form

$$E = \begin{bmatrix} 0 & XF & XFY \\ 0 & F & FY \\ 0 & 0 & 0 \end{bmatrix},$$

with square diagonal blocks, where $F = F_1 \oplus \cdots \oplus F_r$ is an idempotent without zero rows or columns as in (i) and X and Y are two nonnegative matrices.

Definition 5.1.12. We call the (2,2) block, F, of the block representation of E from Theorem 5.1.11, the ***rigid part*** of E.

5.2 Bounded and topologically small semigroups

This section is concerned with the boundedness and topological smallness of semigroups of nonnegative matrices or positive operators on Banach lattices. All results from this section were published in [53].

5.2.1 Boundedness conditions

In this subsection we work with the question: if a non-zero positive functional is bounded on a semigroup, is the semigroup bounded? We start by collecting three simple statements about the diagonal similarities (Lemma 5.2.1, Lemma 5.2.4, and Lemma 5.2.6). All statements are standard and straightforward, so we do not include proofs.

Lemma 5.2.1. *Let $A, D \in M_n(\mathbb{R})$ such that D is diagonal and invertible, $A = (a_{ij})$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then the ij-th entry of $D^{-1}AD$ equals $\frac{d_j}{d_i} a_{ij}$. In particular, the diagonal entries of A and of $D^{-1}AD$ agree.*

Definition 5.2.2. The matrix whose ij-th entry is $\sup\{(S)_{ij} \;:\; S \in \mathcal{S}\}$ is denoted by $\sup \mathcal{S}$.

Remark 5.2.3. Let $\mathcal{S}$ be a semigroup in $M_n^+(\mathbb{R})$. Since $M_n(\mathbb{R})$ is finite-dimensional, the following are equivalent:

(i) $\mathcal{S}$ is norm bounded;

(ii) $\mathcal{S}$ is bounded entry-wise, i.e., $\sup\{(S)_{ij} \;:\; S \in \mathcal{S}\} < +\infty$ for every pair $i, j \leqslant n$;

(iii) $\mathcal{S}$ is order bounded, i.e., there exists $T \in M_n(\mathbb{R})$ such that $S \leqslant T$ for every $S \in \mathcal{S}$. In this case, we write $\mathcal{S} \leqslant T$.

It is clear that if (i)–(iii) are satisfied then $\sup \mathcal{S}$ is defined.

Lemma 5.2.4. *Let $\mathcal{S}$ be a bounded semigroup in $M_n^+(\mathbb{R})$ and D a diagonal matrix with positive diagonal entries. Then $D^{-1}\mathcal{S}D$ is again a bounded semigroup and* $\sup(D^{-1}\mathcal{S}D) = D^{-1}(\sup \mathcal{S})D$.

Definition 5.2.5. A matrix $T = (t_{ij})$ will be called ***compressed*** if $T \geqslant 0$ and $t_{ij}t_{jk} \leqslant t_{ik}$ for all i, j, and k.

Lemma 5.2.6. *(i) If $T = (t_{ij})$ is compressed then $t_{ii} \leqslant 1$ for all i.*

(ii) If $\mathcal{S}$ is a bounded semigroup in $M_n^+(\mathbb{R})$ then $T = \sup \mathcal{S}$ is compressed. In this case, $\mathcal{S}$ is indecomposable if and only if $t_{ij} > 0$ for all i and j.

(iii) Let T be a compressed matrix and D a diagonal matrix with positive diagonal entries. Then $D^{-1}TD$ is compressed.

Definition 5.2.7. Given $r > 0$, we write $M_n\big([0,r]\big)$ for the set of all $n \times n$ matrices with entries in $[0,r]$.

The next proposition shows that a compressed matrix can be made "smaller" by applying a suitable diagonal similarity.

Proposition 5.2.8. *Suppose that $r \geqslant 1$ and $T \in M_n\big([0,r]\big)$ is compressed. Then there exists $D = \operatorname{diag}(d_m)_{m=1}^n$ with $(d_m) \subset [\frac{1}{r}, r]$ such that $D^{-1}TD \in M_n([0,1])$.*

Proof. Let $T = (t_{ij})$. Since T is compressed, $t_{ii} \leqslant 1$ for all i. We will inductively construct $(d_m)_{m=1}^n$ in such a way that if for $m \leqslant n$ we put

$$D_m = \operatorname{diag}(d_1, \dots, d_m, 1, 1, \dots, 1),$$

then $D_m \in M_n([0,r])$, and the upper left $m \times m$-corner of D_m is in $M_m([0,1])$. Note that for every $m > 1$, $D_m^{-1}TD_m$ can be obtained from $D_{m-1}^{-1}TD_{m-1}$ by scaling the m-th column of the latter by d_m and the m-th row by $\frac{1}{d_m}$. It follows that if $i \neq m$ and $j \neq m$, then the ij-entries of $D_m^{-1}TD_m$ coincide with those of $D_k^{-1}TD_k$, where $k \geqslant m$. In particular, the upper left $m \times m$ corners are the same.

Put $d_1 = 1$. Suppose that $d_1, \dots, d_{m-1}$ have already been constructed (in the interval $[\frac{1}{r}, r]$) so that $U := D_{m-1}^{-1}TD_{m-1}$ is in $M_n([0,r])$ and its $(m-1) \times (m-1)$ upper left corner is in $M_{m-1}([0,1])$. Put $U = (u_{ij})$. Once we assign a value to d_m, we will write $V = D_m^{-1}TD_m$, $V = (v_{ij})$. Put

$$a = \max_{i=1,\dots,m-1} u_{im} \qquad \text{and} \qquad b = \max_{j=1,\dots,m-1} u_{mj}.$$

Suppose first that both a and b are less than or equal to 1. In this case, the $m \times m$ upper left corner of U is already in $M_m([0,1])$. Take $d_m = 1$; then $V = U$. Suppose now that $\max\{a,b\} > 1$.

Case 1: $a \geqslant b$. Then $1 < a \leqslant r$ and there exists $k < m$ such that $u_{km} = a$. In this case, we put $d_m = \frac{1}{a}$, then $\frac{1}{r} \leqslant d_m < 1$. Since the m-th column of V is obtained by dividing the m-th column of U by a (except v_{mm} which equals $t_{mm} \leqslant 1$), we have $v_{im} \leqslant 1$ as $i = 1, \dots, m$ and $v_{im} \leqslant u_{im} \leqslant r$ as $i > m$. Also, $v_{km} = 1$. Since V is compressed, for every $j \neq m$ we have

$$v_{mj} = v_{km}v_{mj} \leqslant v_{kj} = u_{kj},$$

because $k \neq m$. It follows that V is in $M_n([0,r])$ and its $m \times m$ upper left corner is in $M_m([0,1])$.

Case 2: $b > a$. In this case the statement is obtained by transposing U and applying Case 1. As a result, we choose the m-th entry of D_m^{-1} to be in $[\frac{1}{r}, 1]$. Hence, the m-th entry of D_m belongs to $[1, r]$. □

As a simple corollary, we get the following theorem about bounded semigroups of nonnegative matrices.

Theorem 5.2.9. *Let $r \geqslant 1$ and $\mathcal{S}$ be a semigroup in $M_n\big([0,r]\big)$. Then there exists $D = \operatorname{diag}(d_m)_{m=1}^n$ with $(d_m) \subset [\frac{1}{r}, r]$ such that $D^{-1}\mathcal{S}D \in M_n\big([0,1]\big)$.*

Proof. Let $T = \sup \mathcal{S}$. Then T is compressed. Let D be as in Proposition 5.2.8. Now Lemma 5.2.4 yields $D^{-1}\mathcal{S}D \leqslant D^{-1}TD$, and the result follows. □

The next statement shows that if a semigroup is indecomposable then its boundedness follows from boundedness of a positive functional on this semigroup.

Proposition 5.2.10. *Let $\mathcal{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$. Suppose that there exists a non-zero positive functional $\phi \in \big(M_n(\mathbb{R})\big)^*$ such that the set $\{\phi(S) : S \in \mathcal{S}\}$ is bounded. Then $\mathcal{S}$ is bounded.*

Proof. First, let us show that for some $k, l \in \{1, \dots, n\}$, the set $\{(S)_{kl} : S \in \mathcal{S}\}$ is bounded. Write

$$\phi(A) = \sum_{i,j=1}^{n} c_{ij} a_{ij}, \quad A = (a_{ij}),$$

where $c_{ij} \geqslant 0$. Since ϕ is non-zero, there exist k, l such that $c_{kl} \neq 0$. Since $\phi(A) \geqslant c_{kl} a_{kl}$ for every positive matrix $A = (a_{ij})$ and the set $\{\phi(S) : S \in \mathcal{S}\}$ is bounded, the set $\{(S)_{kl} : S \in \mathcal{S}\}$ is bounded, too.

Suppose that $\mathcal{S}$ is not bounded, that is, there exist two indices $i, j \leqslant n$ and a sequence (S_m) in $\mathcal{S}$ such that $(S_m)_{ij} \to \infty$ as $m \to \infty$. There are two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{S}$ such that $a_{ki} \neq 0$ and $b_{jl} \neq 0$. Then

$$a_{ki}(S_m)_{ij} b_{jl} \leqslant \big(AS_m B\big)_{kl} \leqslant \sup\{(S)_{kl} \colon S \in \mathcal{S}\} < \infty$$

holds for every $m \in \mathbb{N}$, which is impossible. □

Combining Theorem 5.2.9 with Proposition 5.2.10, we immediately get the following results which answer the question raised in the beginning of this section.

Corollary 5.2.11. *Let $\mathcal{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$ such that $\varphi(\mathcal{S})$ is bounded for some positive functional $\phi \in \big(M_n(\mathbb{R})\big)^*$. Then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}\mathcal{S}D \subseteq M_n\big([0,1]\big)$.*

The following corollary is a special case of Corollary 5.2.11

Corollary 5.2.12. *Let $\mathcal{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$ such that the set $\{(S)_{ij} \ : \ S \in \mathcal{S}\}$ is bounded for some pair (i,j). Then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}\mathcal{S}D \subseteq M_n([0,1])$.*

5.2.2 Topological smallness

We have proved that if $\mathcal{S}$ is bounded at a single entry then, after a positive diagonal similarity, all its entries are bounded by 1. Now, we will try to replace "bounded" with "small".

First, we show that the direct analogue of Corollary 5.2.11 is not valid in this setting.

Example 5.2.13. Let $\varepsilon > 0$. Generate a semigroup $\mathcal{S}$ by the following matrices:

$$A = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ \varepsilon & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $\mathcal{S}$ is indecomposable. Also, it can be easily checked that

$$B^2 = C^2 = AC = BA = DA = DB = CD = 0,$$

$AB \leqslant B$, $CA \leqslant C$, $BC \leqslant A$, $CB \leqslant D$, $BD = B$, $DC = C$, $A^2 \leqslant A$, and $D^2 = D$. Hence, $(S)_{11} \leqslant \varepsilon$, $(S)_{12} \leqslant \varepsilon$, and $(S)_{21} \leqslant \varepsilon$ for all $S \in \mathcal{S}$. Nevertheless, $(D)_{22} = 1$, and this cannot be made any smaller by applying a diagonal similarity since a diagonal similarity does not change diagonal entries of matrices.

The problem with Example 5.2.13 is that diagonal similarities do not change the diagonal entries. It turns out, however, that smallness of diagonal entries is all we need to ensure the semigroup is small itself.

Lemma 5.2.14. *Let $T = (t_{ij})$ be an $n \times n$ compressed matrix such that $t_{ij} > 0$ for all i and j. Let $\varepsilon > 0$. If $t_{ii} < \varepsilon$ for all i then there exists a nonnegative invertible diagonal matrix D such that $D^{-1}TD \in M_n([0, \sqrt[n]{\varepsilon}])$.*

Proof. If $\varepsilon = 1$, then the result follows immediately from Lemma 5.2.8. So we assume for the rest of the proof that $\varepsilon < 1$. Let

$$\delta = \inf_D \bigl\{\max_{i,j} (D^{-1}TD)_{ij}\bigr\}, \tag{5.1}$$

where the infimum is taken over all nonnegative invertible diagonal matrices D. Let $t_{\max} = \max_{i,j} t_{ij}$ and $t_{\min} = \min_{i,j} t_{ij}$. Note that $t_{\min} > 0$. Put

$$\mathcal{D} = \Bigl\{\operatorname{diag}(d_1, \dots, d_n) \ : \ 1 \leqslant d_i \leqslant \frac{t_{\max}}{t_{\min}} \text{ for all } i = 1, \dots, n\Bigr\}.$$

We claim that the infimum in (5.1) can be taken over all $D \in \mathcal{D}$. Indeed, let D be a diagonal matrix with positive diagonal entries. Note that scaling D by a positive scalar does not change $D^{-1}TD$. Therefore we may assume that $\min d_i = 1$. Let $i_0 \leqslant n$ be such that $d_{i_0} = 1$; put $V = D^{-1}TD$, $V = (v_{ij})$. If $d_j > \frac{t_{\max}}{t_{\min}}$ for some j then $v_{i_0 j} = t_{i_0 j}\frac{d_j}{d_{i_0}} > \frac{t_{i_0 j} t_{\max}}{t_{\min} d_{i_0}} = \frac{t_{i_0 j} t_{\max}}{t_{\min}} \geqslant t_{\max} = \max\limits_{i,j} (I^{-1}TI)_{ij}$. Since $I \in \mathcal{D}$, the claim follows.

Since $\mathcal{D}$ is compact, it follows that the infimum in (5.1) is, actually, attained at some D. Let $D = \operatorname{diag}(d_1, \dots, d_n)$ and put $V = D^{-1}TD$, $V = (v_{ij})$. Then $\delta = \max_{i,j} v_{ij}$. Moreover, we may choose D so that the number of occurrences of δ in V is the smallest possible. Note that V is compressed by Lemma 5.2.6. It is left to show that $\delta \leqslant \sqrt[n]{\varepsilon}$. Suppose that, on the contrary, $\delta > \sqrt[n]{\varepsilon}$.

It follows that $\delta > \varepsilon$, so that δ never occurs on the diagonal of V. Hence, after a permutation of the basis, we may assume that $v_{12} = \delta$. We claim that $v_{2j} = \delta$ for some j. Indeed, otherwise, we could slightly decrease d_2 so that the non-diagonal entries in the second row of V increase but stay below δ, but then the non-diagonal entries in the second column of V would decrease, so that v_{12} would become less then δ; however, this would contradict our assumption that V has the smallest possible number of occurrences of δ. Since δ never occurs on the diagonal of V, we know that $j \neq 2$. Note also that $j \neq 1$ as, otherwise,

$$\delta^n \leqslant \delta^2 = v_{12}v_{21} \leqslant v_{11} = t_{11} \leqslant \varepsilon$$

would contradict our assumption that $\delta > \sqrt[n]{\varepsilon}$. Thus, $j > 2$. Again, by a permutation of the basis vectors $\vec{e}_3, \dots, \vec{e}_n$, we may assume that $j = 3$, so that $v_{23} = \delta$.

As in the preceding paragraph, we observe that $v_{3j} = \delta$ for some j. Again, we must have $j > 3$ because

$$\begin{array}{ll} \text{if } j = 1 & \text{then } \delta^n \leqslant \delta^3 = v_{12}v_{23}v_{31} \leqslant v_{11} = t_{11} \leqslant \varepsilon, \\ \text{if } j = 2 & \text{then } \delta^n \leqslant \delta^2 = v_{23}v_{32} \leqslant v_{22} = t_{22} \leqslant \varepsilon, \\ \text{if } j = 3 & \text{then } \delta^n \leqslant \delta = v_{33} = t_{33} \leqslant \varepsilon; \end{array}$$

each case contradicts $\delta > \sqrt[n]{\varepsilon}$. Again, by a permutation of the basis vectors $\vec{e}_4, \dots, \vec{e}_n$, we may assume that $j = 4$, so that $v_{34} = \delta$.

Proceeding inductively, we show that for each $m \leqslant n$ we have (after a permutation of the basis) $v_{12} = \cdots = v_{m-1,m} = \delta$, and that $v_{mj} = \delta$ for some j. Furthermore, $j > m$ as, otherwise, we would get $\delta^n \leqslant \delta^m \leqslant \varepsilon$. But this leads to a contradiction for $m = n$ as $j > n$ is impossible. □

Theorem 5.2.15. *Let $\mathcal{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$ and $\varepsilon > 0$. If all the diagonal entries in all the matrices in $\mathcal{S}$ are less than or equal to ε then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}\mathcal{S}D \subseteq M_n\big([0, \sqrt[n]{\varepsilon}]\big)$.*

Proof. By Proposition 5.2.10, $\mathcal{S}$ is bounded. Let $T = \sup \mathcal{S}$. Then T is positive and compressed by Lemma 5.2.6. By Lemma 5.2.14, there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}TD \in M_n\big([0, \sqrt[n]{\varepsilon}]\big)$. By Lemma 5.2.4, $D^{-1}\mathcal{S}D \leqslant D^{-1}TD$, so that $D^{-1}\mathcal{S}D \subseteq M_n\big([0, \sqrt[n]{\varepsilon}]\big)$. □

The following example shows that the estimate obtained in Theorem 5.2.15 is sharp.

Example 5.2.16. Take any $\varepsilon \in (0, 1]$ and put $\delta = \sqrt[n]{\varepsilon}$. Let

$$P = \begin{bmatrix} 0 & \delta & 0 & 0 & \dots & 0 \\ 0 & 0 & \delta & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \delta & 0 \\ 0 & 0 & \dots & \dots & 0 & \delta \\ \delta & 0 & \dots & \dots & 0 & 0 \end{bmatrix}.$$

Let $\mathcal{S} = \{P^k : k = 1, 2, \dots\}$. By Lemma 5.1.5, P is indecomposable, so that $\mathcal{S}$ is an indecomposable semigroup. The diagonal elements of P^k are all zeros for each $1 \leqslant k \leqslant n-1$, and $P^n = \delta^n I = \varepsilon I$. Also, $P^{k+n} = P^k P^n = \varepsilon P^k \leqslant P^k$. Thus, the maximal value for every diagonal element over all the matrices in $\mathcal{S}$ is ε. On the other hand, $(P)_{i,i+1} = (P)_{n,1} = \delta = \sqrt[n]{\varepsilon}$ for all $1 \leqslant i < n$. It is clear that this bound cannot be decreased by a positive diagonal similarity.

5.3 Semigroups with finite diagonals

In this section we will study semigroups of nonnegative matrices with the property that the diagonal entries of all matrices in the semigroup come from a fixed finite set. The results from this section have been published in the joint work of Radjavi, Williamson, and the author of this thesis [99].

5.3.1 Preliminary results

It has been shown recently in [74] that if all the diagonal entries of an indecomposable nonnegative semigroup consist of zeros and ones, then the semigroup is finite (and furthermore, all entries are in $\{0, 1\}$ after a suitable diagonal similarity). The indecomposability condition is clearly necessary for this result. For example, the semigroup of all upper-triangular nonnegative matrices whose diagonal elements are all equal to 1 is by no means finite.

We consider the following question: if all diagonal entries in an indecomposable nonnegative semigroup come from a fixed finite set, is the semigroup itself finite? The following example due to Williamson [128] shows that in general the answer is negative.

Example 5.3.1. Let

$$\mathcal{S} = \left\{ \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix}, \begin{bmatrix} 0 & T \\ E & 0 \end{bmatrix} \right\},$$

where $E = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and T runs over the set of all matrices of form $\begin{bmatrix} p & q \\ q & p \end{bmatrix}$ where $p, q \geqslant 0$, $p + q = 1$.

The semigroup in Example 5.3.1 is indecomposable and is not very far from having only zeros and ones on the diagonals: the set of all the diagonal entries of matrices in $\mathcal{S}$ is $\{0, 1/2\}$. However, $\mathcal{S}$ is not finite, and incidentally, consists of doubly stochastic matrices.

Although the answer in general is negative, we get affirmative results in two significant cases: that of a self-adjoint semigroup, and that of constant-rank semigroups.

Definition 5.3.2. A semigroup $\mathcal{S}$ of nonnegative matrices will be called a ***semigroup with finite diagonals*** if all the diagonal entries of all the matrices in $\mathcal{S}$ come from a finite set. We will call $\mathcal{S}$ a ***semigroup with finite trace set*** if the set $\{\operatorname{tr}(T) \colon T \in \mathcal{S}\}$ is finite.

The exposition about constant-rank semigroups is based in part on the work of Williamson [128]. Theorem 5.3.28 is stated there. However the proof of Theorem 5.3.28 in [128] contains a mistake; we present a different proof here.

The following lemma can be found in [128]; we chose to include a proof of it for the sake of completeness.

Lemma 5.3.3. *Let $\mathcal{S}$ be a semigroup of $n \times n$ matrices with finite trace set. If $S \in \mathcal{S}$ then all the nonzero eigenvalues of S are roots of unity of degrees at most n. In particular, $r(S) \leqslant 1$ for all $S \in \mathcal{S}$.*

Proof. By [77, Proposition 2.2], $r(S) \leqslant 1$. Let n be the size of S and $(\lambda_i)_{i=1}^n$ be the sequence of the eigenvalues of S (with multiplicities), ordered by

$$1 = |\lambda_1| = \cdots = |\lambda_k| > |\lambda_{k+1}| \geqslant \ldots \geqslant |\lambda_n| \geqslant 0,$$

where $0 \leqslant k \leqslant n$. By Corollary 5.1.7, the modulus-one eigenvalues of S are roots of unity of degree at most k. It is left to show that $\lambda_{k+1} = \cdots = \lambda_n = 0$.

Observe that the set $\{\Sigma_{i=1}^k \lambda_i^j \colon j \in \mathbb{N}\}$ is finite since sequences $(\lambda_i^j)_{j=1}^\infty$ are all periodic for each $i \in \{1, \ldots, k\}$. Also, since $|\lambda_i| < 1$ for all $i = k+1, \ldots, n$, for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $j \geqslant N$

$$\varepsilon > \Sigma_{i=k+1}^{n} |\lambda_i|^j \geqslant |\Sigma_{i=k+1}^{n} \lambda_i^j| \geqslant 0.$$

Therefore, the sequence $\left(|\Sigma_{i=k+1}^{n} \lambda_i^j|\right)_{j=1}^{\infty}$ either has a strictly decreasing subsequence or a constant zero tail. If the former were true, the set $\{\Sigma_{i=k+1}^{n} \lambda_i^j : j \in \mathbb{N}\}$ would be infinite. However, this set cannot be infinite because $\{\Sigma_{i=1}^{k} \lambda_i^j : j \in \mathbb{N}\}$ and $\{\Sigma_{i=1}^{n} \lambda_i^j : j \in \mathbb{N}\}$ are both finite. Thus, for some $r \in \mathbb{N}$ we have

$$\Sigma_{i=k+1}^{n} \lambda_i^{rj} = 0, \quad j \in \mathbb{N}.$$

By [67] (see also [103, Lemma 2.1.15(ii)]) this implies $\lambda_i^r = 0$, and hence $\lambda_i = 0$ for all $i = k+1, \ldots, n$. □

Lemma 5.3.4. *Let $\mathcal{S}$ be an indecomposable semigroup with finite trace set. Then each $S \in \mathcal{S}$ is similar to a matrix of the form*

$$\begin{bmatrix} U & 0 \\ 0 & N \end{bmatrix}$$

where U is a unitary diagonal matrix and N is a nilpotent matrix.

Proof. Let $S \in \mathcal{S}$ and let

$$\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}$$

be the Jordan form of S where J is an invertible matrix and N is a nilpotent matrix. Split J into the sum $D + M$ of a diagonal matrix D and an upper triangular matrix with zero diagonal M; note that $DM = MD$.

We claim that $M = 0$. Indeed, suppose $M \neq 0$. By Proposition 5.2.10, the semigroup $\mathcal{S}$ is bounded. Hence so is the set $\{J^m : m \in \mathbb{N}\}$. Let k be such that $M^k \neq 0$ and $M^{k+1} = 0$. Since $DM = MD$ we get

$$J^m = D^m + \binom{m}{1} D^{m-1} M + \cdots + \binom{m}{k} D^{m-k} M^k$$

for all $m \geqslant k$. By Lemma 5.3.3, all the diagonal entries of D are of absolute value 1, hence $\|D^m\| = 1$ for all m. This implies $\|J^m\| \to \infty$ as $m \to \infty$. Therefore $M = 0$.

This shows that S is similar to

$$\begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix}.$$

By Lemma 5.3.3, D is unitary. □

Corollary 5.3.5. *Let $\mathcal{S}$ be an indecomposable semigroup of $n \times n$ matrices with finite trace set. Then there exists $m \in \mathbb{N}$ such that S^m is an idempotent for each $S \in \mathcal{S}$.*

Proof. Put $m = n!$ and let $S \in \mathcal{S}$. By Lemma 5.3.4, S is similar to

$$\begin{bmatrix} U & 0 \\ 0 & N \end{bmatrix}$$

where U is a unitary diagonal matrix and N is a nilpotent matrix. By Lemma 5.3.3, every diagonal entry of U is a root of unity of degree at most n. Hence S^m is similar to

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is an identity matrix. Therefore S^m is an idempotent. □

In the rest of this subsection we record some simple partial results regarding the main problem. Theorems 5.3.6, 5.3.7, and 5.3.10 were obtained by Williamson in [128]. We present, however, different proofs of these statements.

Theorem 5.3.6. *Let $\mathcal{S}$ be a commutative finitely generated indecomposable semigroup with finite trace set. Then $\mathcal{S}$ is finite.*

Proof. Let $\{A_i\}_{i=1}^n$ be a set of generators of $\mathcal{S}$. By commutativity, each $S \in \mathcal{S}$ can be written as $S = \prod_{i=1}^n A_i^{k_i}$ for some $k_i \geqslant 0$. By Corollary 5.3.5, there is $m \in \mathbb{N}$ such that $E_i := A_i^m$ is an idempotent for each $i = 1, \ldots, n$. Thus k_j's can be chosen in $\{0, 1, \ldots, 2m-1\}$ and, therefore, $\mathcal{S}$ is finite. □

Theorem 5.3.7. *Let $\mathcal{S}$ be an indecomposable semigroup of invertible matrices with finite trace set. Then $\mathcal{S}$ is finite and after a diagonal similarity, is in fact a permutation group.*

Proof. First, we prove that $\mathcal{S}$ is actually a group of matrices. Indeed, clearly the only possible idempotent in $\mathcal{S}$ is the identity matrix. If $S \in \mathcal{S}$ then by Corollary 5.3.5 there is $m \in \mathbb{N}$ such that S^m is an idempotent, i.e., $S^m = I$. Then $S^{-1} = S^{m-1} \in \mathcal{S}$.

By Proposition 5.2.10, $\mathcal{S}$ is bounded. By Theorem 5.1.10, $\mathcal{S}$ is similar to a permutation group and thus is finite. □

It should be noted that if we replace the condition about the trace in the last theorem with the condition of finiteness of the diagonal entries of the members of $\mathcal{S}$ then Theorem 5.3.7 becomes a special case of Theorem 5.3.28 which will be proved in the next section. The following simple example shows that the finiteness of the trace does not in general imply the finiteness of all the diagonal entries.

Example 5.3.8.

$$\mathcal{S} = \left\{ \begin{bmatrix} p & q \\ p & q \end{bmatrix} : p + q = 1, p \geqslant 0, q \geqslant 0 \right\}.$$

Then $\{tr(S) \colon S \in \mathcal{S}\} = \{1\}$, but the diagonal entries of members of $\mathcal{S}$ take all values in $[0, 1]$.

Before introducing our general results, we consider the case of 2×2 and 3×3 matrices. We will need the following combinatorial lemma.

Lemma 5.3.9. *Let A be an infinite set and $n \in \mathbb{N}$. Suppose that the sets $B(i, j) \subseteq A$ (where $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n+1\}$) are such that for each j the following two conditions are satisfied:*

(i) $B(i, j) \cap B(k, j) = \emptyset$ for all $i \neq k$;

(ii) $\cup_{i=1}^{n} B(i, j) = A$.

Then there exist $j_1 \neq j_2$ in $\{1, \ldots, n+1\}$ and i in $\{1, \ldots, n\}$ such that $B(i, j_1) \cap B(i, j_2)$ is infinite.

Proof. The proof is by induction on n. If $n = 1$ then $B(1,1) = B(1,2) = A$ by the assumptions of the lemma, so that $B(1,1) \cap B(1,2)$ is infinite.

Suppose that the statement is true for $n-1$. Let's prove it for n. We have: $\cup_{i=1}^{n} B(i, n+1) = A$. Relabeling, if necessary, we can assume that $B(n, n+1)$ is infinite.

If the statement is not true then $B(n, n+1) \cap B(n, j)$ is finite for all $j \leqslant n$. Denote $A_1 = B(n, n+1) \setminus (\cup_{j=1}^{n} B(n, j))$. It is clear that A_1 is infinite. For each $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$, define $C(i, j) = A_1 \cap B(i, j)$. Then for each $j \in \{1, \ldots, n\}$, we have:

(i) if $i \neq k$ in $\{1, \ldots, n-1\}$ then $C(i, j) \cap C(k, j) \subseteq B(i, j) \cap B(k, j) = \emptyset$;

(ii) $\cup_{i=1}^{n-1} C(i, j) = A_1 \cap (\cup_{i=1}^{n-1} B(i, j)) = A_1 \cap (\cup_{i=1}^{n} B(i, j)) = A_1 \cap A = A_1$.

By the induction assumption, $C(i, j) \cap C(k, j)$ is infinite for some j and $i \neq k$. Hence $B(i, j) \cap B(k, j)$ is infinite, too. $\square$

Theorem 5.3.10. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals consisting of 2×2 or 3×3 matrices. Then $\mathcal{S}$ is finite.*

Proof. For two indices i and j and a subset X of $\mathcal{S}$, put $X_{ij} = \{S_{ij} \colon S \in X\}$ where S_{ij} stands for the (i, j)-th entry of S.

Assume $\mathcal{S}$ consists of 2×2 matrices. Suppose $\mathcal{S}$ is infinite. Then, without loss of generality, we can assume that the set $\mathcal{S}_{12}$ is infinite (since the diagonal entries are finite by the assumptions of the theorem). Fix $A \in \mathcal{S}$ such that $A_{21} \neq 0$. By the hypothesis, $(\mathcal{S}A)_{11} = \{S_{11}A_{11} + S_{12}A_{21} \colon S \in \mathcal{S}\}$ should be finite, which is impossible.

Now assume $\mathcal{S}$ consists of 3×3 matrices. Suppose $\mathcal{S}$ is infinite. Again, we can assume that $\mathcal{S}_{12}$ is infinite. Fix $A \in \mathcal{S}$ such that $A_{21} \neq 0$. Since $(\mathcal{S}A)_{11} = \{S_{11}A_{11} + S_{12}A_{21} + S_{13}A_{31} : S \in \mathcal{S}\}$ is finite, the set $\mathcal{S}_{13}$ is necessarily infinite. By considering $(B\mathcal{S})_{33}$ where $B_{31} \neq 0$, we see that $\mathcal{S}_{23}$ is infinite. Analogously, $\mathcal{S}_{21}$ is infinite.

Let $\mathcal{F} = \{S_{ii} \colon S \in \mathcal{S}, 1 \leqslant i \leqslant 3\}$ and $\mathcal{F}_1 = \{a - bc \colon a, b, c \in \mathcal{F}\}$. Since $(ST)_{11} \in \mathcal{F}$ and $(ST)_{11} - S_{11}T_{11} = S_{12}T_{21} + S_{13}T_{31}$ for all $T, S \in \mathcal{S}$, we have $S_{12}T_{21} + S_{13}T_{31} \in \mathcal{F}_1$ for all $T, S \in \mathcal{S}$.

Let n be the cardinality of $\mathcal{F}_1$. That is, $\mathcal{F}_1 = \{a_1, \ldots, a_n\}$. Define a set A to be such an infinite subset of $\mathcal{S}$ that satisfies the condition that $S_{12} \neq 0$ for all $S \in A$ and $S'_{12} \neq S''_{12}$ for all $S' \neq S''$ in A (such exists since $\mathcal{S}_{12}$ is infinite). Pick $T_1, \ldots T_{n+1} \in \mathcal{S}$ such that $(T_i)_{21} \neq 0$ for all i and $(T_i)_{21} \neq (T_j)_{21}$ for $i \neq j$ (this is possible to do since $\mathcal{S}_{21}$ is infinite). For every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n+1\}$, define

$$B(i, j) = \{S \in A \ : \ S_{12}(T_j)_{21} + S_{13}(T_j)_{31} = a_i\}.$$

It is clear that for all $j \in \{1, \ldots, n+1\}$ and all $i \neq k \in \{1, \ldots, n\}$, we have $B(i, j) \cap B(k, j) = \emptyset$. Also, $\cup_{i=1}^{n} B(i, j) = A$ for all j. By Lemma 5.3.9, it follows that for some $j_1 \neq j_2 \in \{1, \ldots, n+1\}$ and $i \in \{1, \ldots, n\}$, the set $B(i, j_1) \cap B(i, j_2)$ is infinite.

Denote $T' = T_{j_1}$, $T'' = T_{j_2}$, and $a = a_i$. It follows that the system of equations

$$\begin{aligned} S_{12}T'_{21} + S_{13}T'_{31} &= a, \\ S_{12}T''_{21} + S_{13}T''_{31} &= a \end{aligned} \tag{5.2}$$

is satisfied for infinitely many $S \in A$. Therefore the matrix

$$\begin{bmatrix} T'_{21} & T'_{31} \\ T''_{21} & T''_{31} \end{bmatrix}$$

is not invertible. This means that the second line of (5.2) is in fact a scalar multiple of the first line. However, since $S_{12} \neq 0$ for all $S \in A$ and $T'_{21} \neq 0$, $a \neq 0$. Therefore $T'_{21} = T''_{21}$, which is impossible. $\square$

5.3.2 Self-adjoint semigroups

In this subsection we show that if a semigroup with finite diagonals is self-adjoint then it is finite. Moreover, our argument reveals the structure of such semigroups. In contrast with most statements in the other sections, the semigroups in the present section are not assumed to be indecomposable.

Definition 5.3.11. A collection $\mathcal{C}$ of matrices is called ***self-adjoint*** if for each $S \in \mathcal{C}$ we have $S^* \in \mathcal{C}$. Note that for our purposes, S^* is just the transpose of S.

We start with two nice properties of self-adjoint semigroups with finite trace set.

Lemma 5.3.12. *Let $\mathcal{S}$ be a self-adjoint semigroup with finite trace set. Then for each $S \in \mathcal{S}$ the matrix SS^* is an idempotent.*

Proof. By Lemma 5.3.3, every eigenvalue of SS^* is either zero or a root of unity. Since SS^* is self-adjoint and positive definite, $\sigma(SS^*) \subseteq \{0, 1\}$. Since SS^* is also diagonalizable, the Lemma follows. $\square$

Lemma 5.3.13. *If $\mathcal{S}$ is a self-adjoint semigroup with finite trace set then each idempotent in $\mathcal{S}$ is self-adjoint.*

Proof. Let $E = E^2 \in \mathcal{S}$. Then E is unitarily similar to the matrix in the block form $\begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ where I is an identity matrix. With the same similarity, E^* is similar to $\begin{bmatrix} I & 0 \\ X^* & 0 \end{bmatrix}$. Then EE^* is similar to $\begin{bmatrix} I + XX^* & 0 \\ 0 & 0 \end{bmatrix}$. By Lemma 5.3.12, EE^* is an idempotent, hence $(I + XX^*)^2 = (I + XX^*)$. This, however, can only happen when $X = 0$. Indeed, $(I + XX^*)^2 = (I + XX^*)$ implies $XX^* = -(XX^*)^2$. Both XX^* and $(XX^*)^2$ are positive elements in the C*-algebra of all $n \times n$-matrices over $\mathbb{C}$. Since the intersection of the positive cone in a C*-algebra with its negation is zero, $XX^* = 0$. Thus $\|X\|^2 = \|XX^*\| = 0$, so that $X = 0$. $\square$

The next theorem is one of our main results in this section.

Theorem 5.3.14. *Let $\mathcal{S}$ be a (not necessarily indecomposable) semigroup with finite trace set. If $\mathcal{S}$ is self-adjoint then all the entries of all matrices in $\mathcal{S}$ are of the form $\sqrt{\xi\eta}$ where ξ and η are either diagonal values of some matrices in $\mathcal{S}$ or zero.*

Proof of Theorem 5.3.14. Let $\mathcal{F} = \{S_{ii} : S \in \mathcal{S}, i = 1, \ldots, N\}$. We will prove that every $S \in \mathcal{S}$ can be written in the block form

$$S = \Delta_1 \begin{bmatrix} u_1 v_1^* & 0 & \ldots & 0 \\ 0 & u_2 v_2^* & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & u_k v_k^* \end{bmatrix} \Delta_2^* \tag{5.3}$$

where Δ_1 and Δ_2 are each permutations and u_i, v_i are vectors whose entries are either of the form $\sqrt{\xi}$, with $\xi \in \mathcal{F}$ or are all zero (with no restrictions on the size of u_i and v_i; that is, the blocks $u_i v_i^*$ are in general rectangular).

Fix $S \in \mathcal{S}$. Set $P = SS^*$ and $Q = S^*S$. By Lemma 5.3.12, both P and Q are self-adjoint idempotents. Choose two permutations Γ_1 and Γ_2 such that the matrices $P_1 = \Gamma_1 P \Gamma_1^*$ and $Q_1 = \Gamma_2 Q \Gamma_2^*$ are block-diagonal with self-adjoint blocks of rank one or zero. Since $\operatorname{rank}(P) = \operatorname{rank}(Q) = \operatorname{rank}(S)$, we deduce that P_1 and Q_1 have the same number of nonzero blocks. Denote this number by r. That is, $P_1 = (P_1)_1 \oplus \cdots \oplus (P_1)_r \oplus 0$ and $Q_1 = (Q_1)_1 \oplus \cdots \oplus (Q_1)_r \oplus 0$, where either of the last zero entries could be absent.

Put $T = \Gamma_1 S \Gamma_2^*$. Then clearly $TT^* = P_1$ and $T^*T = Q_1$. Write T in the rectangular block form

$$T = \begin{bmatrix} T_{11} & \ldots & T_{1r} & T_{1r+1} \\ \vdots & & \vdots\vdots & \\ T_{r1} & \ldots & T_{rr} & T_{rr+1} \\ T_{r+11} & \ldots & T_{r+1r} & T_{r+1r+1} \end{bmatrix}$$

where the vertical sizes of blocks are those of the blocks of P_1 and the horizontal sizes are those of the blocks of Q_1, and the $(r+1)$-th row or $(r+1)$-th column, or both could be void.

Since $P_1 = TT^*$ has the same range as T, we get $P_1 T = T$. Analogously, $TQ_1 = T$. Therefore $P_1 T Q_1 = T$. Observe that in fact T is a partial isometry with corresponding projections P_1 and Q_1.

We claim that each block row and each block column of T has at most one nonzero block. Indeed, since TT^* is block-diagonal, we get $\sum_{k=1}^{r+1} T_{ik} T_{jk}^* = 0$ for all $i \neq j$. Hence for each k and $i \neq j$ we have $T_{ik} T_{jk}^* = 0$. This implies that if for some n and m the (n, m) entry of T_{ik} is not zero then the m-th column of each T_{jk} is zero for all $j \neq i$. Since $P_1 T Q_1 = T$ and the diagonal entries of P_1 and Q_1 are strictly positive or zero, the entries of all T_{ij} are either all zero or are all nonzero simultaneously. It follows that each block column of T can contain at most one nonzero block. Considering T^*T, we get the same conclusion about the block rows.

Changing the order of blocks in Q_1 (by changing Γ_2), if necessary, we can assume that T is block-diagonal with rectangular diagonal blocks:

$$T = \begin{bmatrix} T_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & T_r & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

where $T_i = (P_1)_i T_i (Q_1)_i$ for all $i = 1, \dots, r$. Also, $T_i T_i^* = (P_1)_i$ and $T_i^* T_i = (Q_1)_i$.

Recalling that every $(P_1)_i$ and $(Q_1)_i$ is a rank-one projection, write $(P_1)_i = x_i x_i^*$ and $(Q_1)_i = y_i y_i^*$ for some vectors x_i and y_i satisfying $\|x_i\| = \|y_i\| = x_i^* x_i = y_i^* y_i = 1$ $(i = 1, \dots, r)$. Clearly $\operatorname{rank}(T_i) = 1$ for all $i = 1, \dots, k$. Hence for each i there exist vectors u_i and v_i such that $T_i = u_i v_i^*$.

Fix i and denote for simplicity of notation $x = x_i$, $y = y_i$, $u = u_i$, and $v = v_i$. Since $P_1 T = T$ and $TQ_1 = T$, we get $xx^*uv^* = uv^*$ and $uv^*yy^* = uv^*$. Let $\alpha = x^*u$ and $\beta = v^*y$. Then $uv^* = \alpha x v^* = \beta u y^*$. This is only possible when $u = \alpha x$ and $v = \beta y$.

This shows that there is a scalar γ such that $uv^* = \gamma xy^*$. We claim that $\gamma = 1$. Indeed, from the the equality $TT^* = P_1$, we obtain $\gamma^2 (xy^*)(xy^*)^* = \gamma^2 xy^*yx^* = \gamma^2 xx^*$ is equal to xx^*. Since $\gamma \geqslant 0$, we get $\gamma = 1$.

We have shown that $T_i = x_i y_i^*$ for each $i = 1, \dots, r$. To establish formula (5.3), it is left to note that since for all i and j the numbers $(x_i)_j^2$ and $(y_i)_j^2$ are some diagonal entries of P_1 and Q_1, respectively, the entries of x_i and y_i are all of the form $\sqrt{\xi}$ with $\xi \in \mathcal{F}$. □

Corollary 5.3.15. *Every self-adjoint semigroup with finite diagonals is finite.*

5.3.3 Constant-rank semigroups

In this subsection we will prove that if all nonzero matrices in an indecomposable semigroup with finite diagonals have the same rank, then the semigroup must be finite. The key step in obtaining this result is proving that the idempotent matrices in such a semigroup form a finite set (Theorem 5.3.23). We will need a series of lemmas to prove this.

Recall (see Theorem 5.1.11) that if E is a nonnegative idempotent matrix with no zero rows or columns then there is a permutation of the basis which makes E block-diagonal with each diagonal block being a square rank-one idempotent matrix without zero entries.

Lemma 5.3.16. *Let $\mathcal{S}$ be an indecomposable semigroup with finite trace set and E a block-diagonal idempotent in $\mathcal{S}$ whose diagonal blocks are square rank-one idempotents without zero entries. Then the set $\mathcal{S}_E = \{A \in \mathcal{S} \colon \operatorname{rank}(A) = \operatorname{rank}(E)$ and $EAE = A\}$ is a finite group with identity E. This group is block-monomial relative to the block structure inherited from E.*

Proof. First, we will show that $\mathcal{S}_E$ is a group with identity E. Indeed, let $A, B \in \mathcal{S}_E$. Then clearly $EABE = AB$. Also, in some basis, E can be represented as $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Since $EAE = A$ and $EBE = B$, and $\operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(E)$, in this basis A and B will be represented as $\begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix}$, respectively, where A_0 and B_0 are invertible matrices. Then the representation of AB is $\begin{bmatrix} A_0B_0 & 0 \\ 0 & 0 \end{bmatrix}$, and A_0B_0 is again invertible, so that $\operatorname{rank}(AB) = \operatorname{rank}(E)$. This shows that $\mathcal{S}_E$ is a semigroup.

Let us show that $\mathcal{S}_E$ is closed under inverses. Let $A \in \mathcal{S}_E$ be arbitrary. By Corollary 5.3.5, there is $m \in \mathbb{N}$ such that A^m is an idempotent which we will denote by F. Since $\mathcal{S}_E$ is a semigroup, $F \in \mathcal{S}_E$. In particular, $\operatorname{rank}(F) = \operatorname{rank}(E)$ and $EFE = F$. Therefore $E = F$. Thus the matrix A^{m-1} is the inverse of A in $\mathcal{S}_E$, and hence $\mathcal{S}_E$ is a group.

Let $r = \operatorname{rank}(E)$. That is, $E = E_1 \oplus \cdots \oplus E_r$ where each E_i is a rank-one idempotent without zero entries. Applying a suitable diagonal similarity to $\mathcal{S}$, we can assume without loss of generality that E is row stochastic. In particular, since the blocks of E have rank one, each block of E is a strictly positive matrix having all rows equal to each other.

Let $K(r,s)$ stand for the $r\times s$ matrix having the value $1/\sqrt{rs}$ at each entry. A straightforward calculation shows that $K(r,s)K(s,t) = K(r,t)$ for all r, s, and $t \in \mathbb{N}$.

For each $i = 1,\dots,r$, denote the size of E_i by r_i. Since each E_i is row stochastic, we have $E_iK(r_i,r_j) = K(r_i,r_j)$ for all $i,j \in \{1,\dots,r\}$. Let $L_{ij} = E_iK(r_i,r_j)E_j{=}K(r_i,r_j)E_j$. Then

$$L_{ij}L_{jk} = E_iK(r_i,r_j)E_jE_jK(r_j,r_k)E_k = \\ = E_iK(r_i,r_j)K(r_j,r_k)E_k = E_iK(r_i,r_k)E_k = L_{ik}. \quad (5.4)$$

Let $A \in \mathcal{S}_E$ be arbitrary. Write A in the block form inherited from E:

$$A = \begin{bmatrix} A_{11} & \dots & A_{1r} \\ \vdots & & \vdots \\ A_{r1} & \dots & A_{rr} \end{bmatrix}.$$

Since $EAE = A$, we get $A_{ij} = E_iA_{ij}E_j$ for all $i,j \in \{1,\dots,r\}$. The ranks of E_i and E_j are equal to 1. Hence, L_{ij} and A_{ij} are rank-one matrices which correspond to operators having the same range and null space. Thus for each $i,j \in \{1,\dots,r\}$ there exists a nonnegative λ_{ij} such that $A_{ij} = \lambda_{ij}L_{ij}$.

This shows that every matrix $A \in \mathcal{S}_E$ can be represented as a numerical matrix $\widetilde{A} = (\lambda_{ij})_{i,j=1}^r$. By formula (5.4) we also conclude that $\widetilde{AB} = \widetilde{A}\widetilde{B}$. Observe also that $\widetilde{E}$ is the $r\times r$ identity matrix. Therefore the set $\mathcal{G} = \{\widetilde{A}\colon A \in \mathcal{S}_E\}$ is a group of nonnegative invertible matrices.

Since $\mathcal{S}$ is an indecomposable semigroup with bounded trace, by Proposition 5.2.10, $\mathcal{S}$ itself is bounded. In particular, $\mathcal{S}_E$ is bounded, and hence $\mathcal{G}$ is bounded. Therefore, by [103, Lemma 5.1.11], $\mathcal{G}$ is a finite monomial group. Hence $\mathcal{S}_E$ is finite and block-monomial. □

The next lemma is a technical statement that allows us to work with the upper-left corners of matrices in a semigroup.

Lemma 5.3.17. *Let $\mathcal{S}$ be an indecomposable semigroup of $n \times n$ matrices. Let $k \in \{1, \ldots, n\}$ and*

$$J_k = \{S \in \mathcal{S}\colon \text{ rows } k+1 \text{ through } n \text{ of } S \text{ are zero}\}.$$

Put $\mathcal{S}_k = \{A\colon A$ is the upper-left $k \times k$ corner of some $S \in J_k\}$. If $\mathcal{S}_k$ has no permanent zero rows, that is, if for each $i \in \{1, \ldots, k\}$ there is a matrix $A \in \mathcal{S}_k$ such that the i-th row of A is not zero, then $\mathcal{S}_k$ is an indecomposable semigroup.

Proof. A straightforward calculation shows that $\mathcal{S}_k$ is a semigroup for each k. We now establish the indecomposability statement.

We need to show that for each $i, j \in \{1, \ldots, k\}$ there is a matrix $A \in \mathcal{S}_k$ such that the (i, j) entry of A is different from zero. Pick a matrix $U \in \mathcal{S}_k$ whose i-th row is not zero, say $(U)_{im} \neq 0$ for some $m \in \{1, \ldots, k\}$. There is a matrix V such that $T := \begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix} \in J_k$. Since $\mathcal{S}$ is indecomposable, there is $S \in \mathcal{S}$ such that $(S)_{mj} \neq 0$. Then $(TS)_{ij} \neq 0$. Also, $TS \in J_k$. Clearly the upper-left $k \times k$ corner of TS has a nonzero (i, j) entry. □

The next lemma is the same statement about the lower-right corners of matrices in a semigroup. Its proof is analogous to that of Lemma 5.3.17, so we omit it.

Lemma 5.3.18. *Let $\mathcal{S}$ be an indecomposable semigroup of $n \times n$ matrices. Let $k \in \{1, \ldots, n\}$ and*

$$J'_k = \{S \in \mathcal{S}\colon \text{ columns } 1 \text{ through } k \text{ of } S \text{ are zero}\}.$$

Put $\mathcal{S}'_k = \{A\colon A$ is the lower-right $(n-k) \times (n-k)$ corner of some $S \in J'_k\}$. If $\mathcal{S}'_k$ has no permanent zero columns then $\mathcal{S}'_k$ is an indecomposable semigroup.

The following two lemmas provide additional information about $\mathcal{S}_k$ and $\mathcal{S}'_k$ in constant-rank semigroups.

Lemma 5.3.19. *Let $\mathcal{S}$ be an indecomposable semigroup of $n \times n$ matrices whose every non-zero member has rank r. Let $E = \begin{bmatrix} F & FX \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$ be an idempotent such that the matrix F does not contain zero rows, and k be the size of F. Then every non-zero member of $\mathcal{S}_k$ (as in Lemma 5.3.17) has rank r.*

Proof. First, observe that if $A \in \mathcal{S}_k$ is such that $FAF \neq 0$ then A has rank r. Indeed, pick a nonnegative matrix B such that $T := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ belongs to $\mathcal{S}$. Then $ETE = \begin{bmatrix} FAF & FAFX \\ 0 & 0 \end{bmatrix}$ is not zero, hence $\operatorname{rank}(ETE) = r$. However, this means that $\operatorname{rank}(FAF) = r$. Therefore $r = \operatorname{rank}(FAF) \leqslant \operatorname{rank}(A) \leqslant \operatorname{rank}(T) = r$, so that $\operatorname{rank}(A) = r$.

We claim that if $A \in \mathcal{S}_k$ is arbitrary then there are two matrices $C_1, C_2 \in \mathcal{S}_k$ such that $FC_1AC_2F \neq 0$. Indeed, by Lemma 5.3.17, $\mathcal{S}_k$ is indecomposable. Apply a permutation to $\mathcal{S}_k$ which makes F into a matrix of block form $\begin{bmatrix} 0 & ZG \\ 0 & G \end{bmatrix}$ where G is a block-diagonal matrix with diagonal blocks having no zero entries (see Theorem 5.1.11). If $D \in \mathcal{S}_k$ is represented as $\begin{bmatrix} K & L \\ M & N \end{bmatrix}$ then FDF is represented as $\begin{bmatrix} 0 & ZGMZG + ZGNG \\ 0 & GMGZ + GNG \end{bmatrix}$. Since the diagonal blocks of G do not have zero entries, the product FDF is only zero when $N = 0$. It follows from indecomposability of $\mathcal{S}_k$ that if $A \in \mathcal{S}_k$ is not zero then there exist $C_1, C_2 \in \mathcal{S}_k$ such that the $(2,2)$-block of C_1AC_2 is not zero.

This implies that for each non-zero $A \in \mathcal{S}_k$, there are $C_1, C_2 \in \mathcal{S}_k$ such that $\operatorname{rank}(C_1AC_2) = r$. Therefore $\operatorname{rank}(A) \geqslant r$. However the rank of any nonzero matrix in $\mathcal{S}$ is r, hence $\operatorname{rank}(A) \leqslant r$. □

The proof of the next lemma is analogous to that of Lemma 5.3.19, so we omit it.

Lemma 5.3.20. *Let $\mathcal{S}$ be an indecomposable semigroup of $n \times n$ matrices whose every non-zero member has rank r. Let $E = \begin{bmatrix} 0 & ZF \\ 0 & F \end{bmatrix} \in \mathcal{S}$ be an idempotent such that the matrix F does not contain zero columns, and $n - k$ be the size of F. Then every non-zero member of $\mathcal{S}'_k$ (as in Lemma 5.3.18) has rank r.*

Recall (see Theorem 5.1.11) that if E is a nonnegative idempotent matrix then, after a permutation, E can be written as

$$E = \begin{bmatrix} 0 & XF & XFY \\ 0 & F & FY \\ 0 & 0 & 0 \end{bmatrix} \tag{5.5}$$

where F is a nonnegative idempotent without zero rows or zero columns and X, Y are two nonnegative matrices. The block F is the rigid part of E (see Definition 5.1.12).

The following lemma is the first step in establishing the finiteness of the set of idempotents of an indecomposable constant-rank semigroup with finite diagonals. Note that this lemma requires neither indecomposability nor constancy of rank. In the proof of the lemma, we use ideas from [128].

Lemma 5.3.21. *Let $\mathcal{S}$ be a semigroup with finite diagonals. Then the set*

$$\Big\{F\colon F \text{ is the rigid part of some } E = E^2 \in \mathcal{S}\Big\}$$

is finite.

Proof. Let N be the size of matrices in $\mathcal{S}$. Fix three numbers $m, n, k \geqslant 0$ such that $m + n + k = N$. We will prove that the set

$$\mathcal{F} = \Big\{F\colon F \text{ is the rigid part of some } E = E^2 \in \mathcal{S} \\ \text{whose diagonal blocks are of size } m,\ n, \text{ and } k, \text{ respectively}\Big\}$$

is finite. For each $F \in \mathcal{F}$ there exists a permutation matrix P such that $P^{-1}FP = E_1 \oplus \cdots \oplus E_r$, where each E_i is an idempotent of rank one whose entries are all positive. There are only finitely many choices for the permutation P, the number of blocks, r, and the sizes of each block in this representation. Therefore it suffices to show that, after a fixed permutation P, there are only finitely many members in $\mathcal{F}$ having the same sequence of block sizes.

Let $F', F'' \in \mathcal{F}$ and a permutation P be such that $P^{-1}F'P = E_1' \oplus \cdots \oplus E_r'$, $P^{-1}F''P = E_1'' \oplus \cdots \oplus E_r''$ and the sizes of E_i' and E_i'' are the same for all

$i = 1, \dots, r$. Fix $i \in \{1, \dots r\}$. We will prove that if the sequences of the diagonal entries of E_i' and E_i'' are the same (that is, if $(E_i')_{jj} = (E_i'')_{jj}$ for all j) then $E_i' = E_i''$. Since there are only finitely many choices for such diagonal sequences, the conclusion will follow.

Relabel for convenience $E_i' = Q$, $E_i'' = R$. If Q and R have size 1, we are done. Hence we can assume that the size is at least 2. Write $Q = (q_{ij})$, $R = (r_{ij})$. Since Q and R are both positive rank-one matrices with equal diagonals, there is a positive diagonal matrix D such that $R = DQD^{-1}$ (for example, the matrix $D = \text{diag}(\frac{r_{1j}}{q_{1j}})$ will do the job). Also, since Q and R are both strictly positive, RQ is again of rank one. Thus, $\sigma(RQ) = \{tr(RQ), 0\}$. Let $D = \text{diag}(d_j)$. Then

$$tr(RQ) - 1 = tr(DQD^{-1}Q) - tr(Q^2) = \sum_{i,j} d_i d_j^{-1} q_{ij} q_{ji} - \sum_{i,j} q_{ij} q_{ji} =$$
$$= \sum_{i,j} (d_i d_j^{-1} - 1) q_{ij} q_{ji} = \sum_{i<j} (d_i d_j^{-1} + d_j d_i^{-1} - 2) q_{ij} q_{ji}.$$

We will be done if we prove that D is a multiple of the identity. Assume otherwise. Fix $i < j$ such that $d_i \neq d_j$. Observe that for $a > 0$ we have $a + a^{-1} \geqslant 2$ and the equality holds if and only if $a = 1$. Hence using $a = d_i d_j^{-1}$, we get $(d_i d_j^{-1} + d_j d_i^{-1} - 2) q_{ij} q_{ji} > 0$, by strict positivity of elements of Q. Thus $tr(RQ) > 1$, and therefore the spectral radius of RQ, $r(RQ) > 1$, so that $r(F''F') > 1$. This is impossible by Lemma 5.3.3. □

In the following lemma we establish finiteness of the set of idempotents of a special kind in constant-rank semigroups with finite diagonals.

Lemma 5.3.22. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals such that all nonzero members of $\mathcal{S}$ have the same rank and*

$$\mathcal{E} = \left\{ E = E^2 \in \mathcal{S} \colon E = \begin{bmatrix} F & X \\ 0 & 0 \end{bmatrix} \text{ for some block } X \right\}$$

where F is a fixed idempotent matrix without zero rows and columns. Then $\mathcal{E}$ is finite.

Proof. Denote by r the rank of all nonzero members of $\mathcal{S}$. Applying a suitable permutation to $\mathcal{S}$ we can assume that F is of the form $F = F_1 \oplus \cdots \oplus F_r$ where each F_i is an idempotent of rank 1 whose entries are all positive. Furthermore, applying a diagonal similarity, we can assume that F is row stochastic.

Let k be the size of F. Define J_k and $\mathcal{S}_k$ as in Lemma 5.3.17. Since F does not have zero rows, $\mathcal{S}_k$ is indecomposable. Also, by Lemma 5.3.19, all non-zero members of $\mathcal{S}_k$ have rank r. Clearly, F is a nonzero idempotent in $\mathcal{S}_k$. Define

$$\mathcal{S}_0 = F\mathcal{S}_k F.$$

By Lemma 5.3.16 we deduce that $\mathcal{S}_0$ is a finite group that is block-monomial relative to the block structure inherited from F.

Consider the set

$$\mathcal{X} = \left\{ X \colon X = FX \text{ and } \begin{bmatrix} F & X \\ 0 & 0 \end{bmatrix} \in \mathcal{S} \right\}.$$

To prove the lemma, we need to show that $\mathcal{X}$ is finite. Write every $X \in \mathcal{X}$ in a block form compatible with the block form of F:

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_r \end{bmatrix} = \begin{bmatrix} F_1 X_1 \\ \vdots \\ F_r X_r \end{bmatrix}.$$

To prove the lemma, we need to establish that every entry of X can only take finitely many values. We will prove this for the $(1,1)$-entry. For other entries, the argument is analogous.

Denote $\mathcal{X}_1 = \{X \in \mathcal{X} \colon X_{11} \neq 0\}$. To prove the Lemma, it is enough to show that $\{X_{11} \colon X \in \mathcal{X}_1\}$ is finite. Since $\mathcal{S}$ is indecomposable, there exists a matrix $S = \begin{bmatrix} H & K \\ R & Q \end{bmatrix}$ in $\mathcal{S}$ such that the $(1,1)$ entry of R, R_{11}, is not zero. For each $X \in \mathcal{X}_1$ the upper-left block of the product

$$\begin{bmatrix} F & X \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} H & K \\ R & Q \end{bmatrix} \cdot \begin{bmatrix} F & X \\ 0 & 0 \end{bmatrix}$$

belongs to $\mathcal{S}_0$ and is equal to $F(H + XR)F$. Since $\mathcal{S}_0$ is block monomial with respect to the block structure of F, so is the set $\{F(H + XR)F \colon X \in$

$\mathcal{X}_1\}$. However, H is fixed and all matrices in this expression are nonnegative. Therefore the set $\mathcal{Y}_1 = \{FXRF \colon X \in \mathcal{X}_1\}$ is finite and has the property that every row of blocks in each matrix in $\mathcal{Y}_1$ has at most one nonzero block.

Write RF in a block form, conforming to the block columns of F:

$$RF = \begin{bmatrix} L_1 & \dots & L_r \end{bmatrix}.$$

For each $X \in \mathcal{X}_1$ we have

$$FXRF = \begin{bmatrix} X_1L_1 & \dots & X_1L_r \\ \vdots & & \vdots \\ X_rL_1 & \dots & X_rL_r \end{bmatrix}.$$

Since $X_{11} \neq 0$ for all $X \in \mathcal{X}_1$, the block $X_1L_1 \neq 0$. Therefore $X_1L_i = 0$ for all $i \in \{2, \dots, r\}$. Again, by $X_{11} \neq 0$ this implies that the first row of each L_i is equal to zero ($i = 2, \dots, r$). Observe that, since the blocks of F are row stochastic and have rank one, all rows of each F_i and, hence, of each $X_i = F_iX_i$ are the same ($i = 1, \dots, r$). In particular, the first entry in every row of $X_1 = F_1X_1$ is equal to X_{11}. Thus the leading entry of $RX = L_1X_1 + \dots + L_rX_r$ is equal to $s \cdot X_{11}$ where s is the sum of elements from the first row of L_1. Observe that $s \neq 0$ by the choice of R. Also, RX is the lower-right block of the product

$$\begin{bmatrix} H & K \\ R & Q \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F & X \\ 0 & 0 \end{bmatrix}$$

which belongs to $\mathcal{S}$. Therefore there are only a finite number of values for $s \cdot X_{11}$. Since s is independent of the choice of X, the set $\{X_{11} \colon X \in \mathcal{X}_1\}$ is finite which completes the proof. □

Theorem 5.3.23. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals. If all nonzero elements of $\mathcal{S}$ have the same rank r then the set of idempotents in $\mathcal{S}$ is finite.*

Proof. Each idempotent in $\mathcal{S}$ is in the form of (5.5) after a suitable permutation. Since the number of possible permutations is finite, it is enough to prove that for each permutation P, the indecomposable semigroup $P^{-1}\mathcal{S}P$ contains finitely many idempotents in the form of (5.5).

Relabeling, if necessary, we can assume that the permutation P has already been applied to $\mathcal{S}$. For a fixed nonnegative idempotent F without zero rows or zero columns, define

$$\mathcal{E}_F = \{E = E^2 \in \mathcal{S}\colon \text{ the } (2,2) \text{ block of } E \text{ in the block form of (5.5) is } F\},$$

$$\mathcal{X}_F = \{XF\colon XF \text{ is the } (1,2) \text{ block in the form of (5.5) for some } E \in \mathcal{E}_F\},$$

$$\mathcal{Y}_F = \{FY\colon FY \text{ is the } (2,3) \text{ block in the form of (5.5) for some } E \in \mathcal{E}_F\}.$$

Fix the $(2,2)$-block F. By Lemma 5.3.21, it suffices to show that $\mathcal{X}_F$ and $\mathcal{Y}_F$ are finite. Let us prove first that $\mathcal{Y}_F$ is finite. We assume that $\mathcal{Y}_F$ is not empty, as otherwise there is nothing to prove.

Denote by k the number of rows in the $(1,1)$ block of the representation (5.5). Define J'_k and $\mathcal{S}'_k$ as in Lemma 5.3.18. We consider two cases.

Case 1. There exists $FY \in \mathcal{Y}_F$ which has no zero columns. Then there is an idempotent in $\mathcal{S}'_k$ which has no zero columns. Then by Lemma 5.3.18 and Lemma 5.3.20, the semigroup $\mathcal{S}'_k$ is a constant rank indecomposable semigroup. By Lemma 5.3.22, the idempotents in $\mathcal{S}'_k$ form a finite set. Hence $\mathcal{Y}_F$ is finite.

Case 2. All FY in $\mathcal{Y}_F$ have zero columns. Let n be the number of columns in each FY. For each $C \subseteq \{1, \ldots, n\}$, define $\mathcal{Y}_{F,C} = \{FY \in \mathcal{Y}_F \ :$ exactly columns numbered by members of C are zero $\}$. We will prove that each $\mathcal{Y}_{F,C}$ is finite.

There is a permutation Q such that if $FY \in \mathcal{Y}_{F,C}$ then $Q^{-1}\begin{bmatrix} F & FY \\ 0 & 0 \end{bmatrix}Q$ is of form $\begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix}$ where E_1 is of the form $\begin{bmatrix} F & FY_1 \\ 0 & 0 \end{bmatrix}$ and Y_1 has no zero columns. Now the finiteness of $\mathcal{Y}_{F,C}$ follows from the argument in *Case 1* applied to the semigroup $Q^{-1}\mathcal{S}Q$.

The finiteness of each $\mathcal{X}_F$ is established by applying an analogous argument to $\mathcal{S}^*$. □

The following example shows that the condition on the rank is important in Theorem 5.3.23.

Example 5.3.24. An indecomposable semigroup with finite diagonals having infinitely many idempotents.

$$\mathcal{S} = \left\{ \begin{bmatrix} I & S \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix}, \begin{bmatrix} E & E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ E & E \end{bmatrix} \right\}$$

where $E = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and S runs over all matrices of form $\begin{bmatrix} p & q \\ q & p \end{bmatrix}$, with $p + q = 1$, $p, q \geqslant 0$.

Even though the next lemma is standard, we decided to include a proof of it.

Lemma 5.3.25. *Let N be a nonnegative $n \times n$ matrix such that $N^2 = 0$. Then there exists a permutation of the basis vectors such that N can be written as $N = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ (with square diagonal blocks). Moreover, if N is nonzero then A can be chosen to contain no zero columns or (alternatively) no zero rows.*

Proof. Let $\mathcal{F} = \{i \colon Ne_i = 0\}$ where (e_i) is the standard unit vector basis. We will first show that $\mathcal{F}$ cannot be empty. Suppose otherwise. Then $Ne_1 = (a_1 e_1 + \cdots + a_n e_n)$ for some nonnegative a_i, where at least one, say a_k, is positive. Then, by the nonnegativity of N and since $N a_k e_k \neq 0$, $\|N^2 e_1\| \geq \|N(a_k e_k)\| > 0$, which is a contradiction. Therefore, applying a suitable permutation, we can assume that $\mathcal{F} = \{1, \ldots, k\}$ for some k. Since $N^2 = 0$, for each $i \in \{k+1, \ldots, n\}$ we have $Ne_i = \sum_{j \in \mathcal{F}} a_{ij} e_j$ for some nonnegative a_{ij}. This shows that N can be represented in the desired form with A having no zero columns (provided $N \neq 0$). If A has zero rows then, applying a permutation and partitioning the first diagonal block into two diagonal subblocks, we obtain a new A with no zero rows (but some zero columns). $\square$

Lemma 5.3.26. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals. If all nonzero members of $\mathcal{S}$ have the same rank, then the set $\{N \in \mathcal{S} \colon N \text{ is nilpotent}\}$ is finite.*

Proof. Denote by r the rank of the nonzero elements in $\mathcal{S}$. The proof is by induction on the size n of matrices in $\mathcal{S}$. If $n = 1$ then there are no nonzero nilpotent matrices in $\mathcal{S}$. Let $n > 1$.

Clearly, since the rank of all nonzero elements of $\mathcal{S}$ is the same, if $N \in \mathcal{S}$ is nilpotent then $N^2 = 0$. By Lemma 5.3.25, after a permutation of the basis, we can write $N = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ for some nonnegative matrix A without zero rows. Since the number of possible permutations is finite, it is enough, as in Theorem 5.3.23, to show that $\mathcal{S}$ contains only finitely many nilpotent matrices in this block form.

Define

$$\mathcal{N}_k = \left\{ \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \in \mathcal{S} \colon A \text{ has } k \text{ nonzero rows and no zero rows} \right\}.$$

(Note that we have to allow A to have zero columns in the definition above, because the diagonal blocks have to be square.) For a matrix $N \in \mathcal{N}_k$, we will denote by a_N the leading entry, A_{11}, of the block A. As in the proof of Theorem 5.3.23, it is enough to show that the set $\{a_N \neq 0 \colon N \in \mathcal{N}_k\}$ is finite.

Pick any matrix $M = \begin{bmatrix} H & L \\ J & K \end{bmatrix} \in \mathcal{S}$ such that the leading entry of J is different from zero. If $a_N \neq 0$ then NM is not nilpotent, and hence a power of NM is a nonzero idempotent by Corollary 5.3.5. Denote this idempotent by E_N. Since N and E_N have the same range, $E_N N = N$. In particular, the zero rows of E_N and N are the same. Hence in the block form inherited from N we get $E_N = \begin{bmatrix} Q & Z \\ 0 & 0 \end{bmatrix}$. Clearly $Q = Q^2$ and $Z = QZ$, so that Q has no zero rows.

Case 1. Suppose that E_N and N have common zero columns. After a suitable permutation the matrices E_N and N can be written in the block form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & XF & XFY \\ 0 & 0 & F & FY \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively, where F has no zero columns and the fourth block column in each of the two matrices has no common zero columns. Since $E_N N = N$, we get $B = XFC$ and $C = FC$. In particular FY and C have no common zero columns. Let j be the number of zero columns in the first two block columns. Define $\mathcal{S}'_j$ as in Lemma 5.3.18. Then $\mathcal{S}'_j$ is an indecomposable semigroup. We will show now that the rank of nonzero elements in $\mathcal{S}'_j$ is equal to r.

Let $\tilde{F} = \begin{bmatrix} F & FY \\ 0 & 0 \end{bmatrix}$, $\tilde{X} = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$, and $\tilde{C} = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$, then $E_N = \begin{bmatrix} 0 & \tilde{X}\tilde{F} \\ 0 & \tilde{F} \end{bmatrix}$ and $N = \begin{bmatrix} 0 & \tilde{X}\tilde{C} \\ 0 & \tilde{C} \end{bmatrix}$. Let $V \in \mathcal{S}'_j$ be nonzero. Then there exists $T = \begin{bmatrix} 0 & U \\ 0 & V \end{bmatrix} \in \mathcal{S}$. Consider the products $E_N T = \begin{bmatrix} 0 & \tilde{X}\tilde{F}V \\ 0 & \tilde{F}V \end{bmatrix}$ and $NT = \begin{bmatrix} 0 & \tilde{X}\tilde{C}V \\ 0 & \tilde{C}V \end{bmatrix}$. Since $V \neq 0$ and the matrices $\tilde{F}$ and $\tilde{C}$ have no common zero columns, one of the matrices $E_N T$ or NT is different from zero and hence has rank r. It is left to note that $\operatorname{rank}(E_N T) = \operatorname{rank}(\tilde{F}V)$, $\operatorname{rank}(NT) = \operatorname{rank}(\tilde{C}V)$, and $r = \operatorname{rank}(T) \geqslant \operatorname{rank}(V) \geqslant \operatorname{rank}(\tilde{F}V) \vee \operatorname{rank}(\tilde{C}V) = \operatorname{rank}(E_N T) \vee \operatorname{rank}(NT) = r$.

So, the semigroup $\mathcal{S}'_j$ is an indecomposable semigroup with finite diagonals whose nonzero elements have constant rank. Also, the size of matrices in $\mathcal{S}'_j$ is smaller than n. Thus, by the induction hypothesis, there are finitely many nilpotent matrices in $\mathcal{S}'_j$. Therefore the matrix $\tilde{C}$ comes from a finite set. By Theorem 5.3.23, there are finitely many idempotents in $\mathcal{S}$, hence the matrix $\tilde{X}$ also comes from a finite set. Hence so does the matrix N.

Case 2. Suppose E_N and N have no common zero columns. Then in particular Q is an idempotent without zero rows and zero columns.

Write $Q = Q_1 \oplus \cdots \oplus Q_r$ where each Q_i is a rank-one idempotent without zero entries. In this block structure, write

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix} = \begin{bmatrix} Q_1 A_1 \\ \vdots \\ Q_r A_r \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} J_1 & \dots & J_r \end{bmatrix}.$$

Applying a suitable diagonal similarity (note that these diagonal similarities come from a finite set since they depend on E_N only, and the set of idempotents in $\mathcal{S}$ is finite by Theorem 5.3.23), we can assume that Q is row stochastic. Then the rows of A_1 are all the same. Write $NM = \begin{bmatrix} AJ & AK \\ 0 & 0 \end{bmatrix}$. Clearly

$Q(AJ) = (AJ)Q$ and, since $E_N N = N$, $Q(AJ) = AJ$. The size of Q is $n-k$. Let $\mathcal{S}_{n-k}$ be as in Lemma 5.3.17. Then $\mathcal{S}_{n-k}$ is indecomposable. Therefore the matrix AJ is block monomial by Lemma 5.3.16.

We have

$$AJ = \begin{bmatrix} A_1 J_1 & \dots & A_1 J_r \\ \vdots & & \vdots \\ A_r J_1 & \dots & A_r J_r \end{bmatrix}.$$

The leading block of AJ is different from zero. Hence $A_1 J_i = 0$ for all $i \in \{2, \dots, r\}$. The leading entry of A_1 is nonzero. Hence the first row of each J_i $\big(i \in \{2, \dots, r\}\big)$ is zero. Denote the sum of elements in the first row of J_1 by s. By analyzing the product of $\begin{bmatrix} H & L \\ J & K \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$, we get: the value sa_N is on the diagonal of this product, and hence can only take finitely many values. Since s is independent of N and is different from zero, this shows that a_N can only take finitely many values, too. □

Lemma 5.3.27. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals such that all nonzero members of $\mathcal{S}$ have the same rank. Let $E \in \mathcal{S}$ be a nonzero idempotent. Then the set $\mathcal{S}_E = \{S \in \mathcal{S} \colon ESE = S\}$ is a finite group with unit E.*

Proof. By Lemma 5.3.3, $r(T) = 1$ for all $T \in \mathcal{S}_E$. So, the statement follows from [103, 5.2.2(iv)]. The condition in [103, 5.2.2(iv)] that $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ is not essential since it is only used to establish that $\mathcal{S}_E$ is bounded (which follows from Proposition 5.2.10) and that for each $S \in \mathcal{S}_E$ a sequence of powers of S converges to an idempotent in $\mathcal{S}_E$ (which follows from Lemma 5.3.4). □

Theorem 5.3.28. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals. If all nonzero members of $\mathcal{S}$ have the same rank, then $\mathcal{S}$ is finite.*

Proof. Let $\mathcal{E}$ be the set of all nonzero idempotents in $\mathcal{S}$. For each $E \in \mathcal{E}$, denote $\mathcal{S}_E = \{S \in \mathcal{S} \colon ESE = S\}$. By Lemma 5.3.27, $\mathcal{S}_E$ is a finite group with unit E. We claim that each non-nilpotent member of $\mathcal{S}$ belongs to $\bigcup_{E \in \mathcal{E}} \mathcal{S}_E$. Indeed, by Lemma 5.3.4, each $S \in \mathcal{S}$ is represented in some basis as $\begin{bmatrix} U & 0 \\ 0 & N \end{bmatrix}$

where U is a unitary diagonal matrix and N is a nilpotent matrix. If S is not nilpotent then $N = 0$ because the rank of all nonzero elements of $\mathcal{S}$ is the same. Therefore a power S^m of any non-nilpotent $S \in \mathcal{S}$ is a nonzero idempotent E. In this basis, E is represented as $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, and hence $ESE = S$.

Since the set $\mathcal{E}$ is finite by Theorem 5.3.23, this shows that the set of non-nilpotent matrices in $\mathcal{S}$ is finite. The finiteness of nilpotent elements in $\mathcal{S}$ is shown in Lemma 5.3.26. □

The natural (in view of Theorem 5.3.7) question whether the finiteness of diagonal entries in the statement of Theorem 5.3.28 can be replaced with finiteness of the trace has a negative answer, as Example 5.3.8 shows. In fact, the semigroup in that example consists of idempotents only, so that the corresponding question asked about Theorem 5.3.23 would already have a negative answer.

5.3.4 Admissible diagonal values

In this subsection we analyze what values there could be on the diagonal positions of a semigroup with finite diagonals.

Theorem 5.3.29. *Let $\mathcal{S}$ be an indecomposable semigroup with finite diagonals. Then for each $S \in \mathcal{S}$ the sequence (S_{ii}) can be partitioned into subsequences each of which either adds up to 1 or consists of zeros.*

Proof. Let $S \in \mathcal{S}$ be fixed. By Lemma 5.3.3, the possible eigenvalues of S are roots of unity and zero. After a permutation, S can be decomposed into a block triangular form whose diagonal blocks are indecomposable matrices $S_1, \dots, S_k$. Pick any $i \in \{1, \dots, k\}$ and denote for convenience $T = S_i$. It is enough to prove that the statement of Theorem is valid for T.

Since T is indecomposable, T is not nilpotent. Let $r \geqslant 1$ be the number of nonzero eigenvalues (counting multiplicities) of T. Then $r = \operatorname{rank}(T)$. By Corollary 5.3.5, the minimal rank of nonzero matrices in the norm closed semigroup generated by T is r. Hence by the Theorem 5.1.6 (iv), after a permutation, T can be written in the block form

$$T = \begin{bmatrix} 0 & \dots & 0 & T_r \\ T_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & T_{r-1} & 0 \end{bmatrix}.$$

If $r > 1$ then all the diagonal elements are zero, since permutations only change the order of diagonal elements. If $r = 1$ then zero has multiplicity $n-1$ (where n is the size of T). Since $1 \in \sigma(T)$, we get $\operatorname{tr}(T) = 1$, hence the sum of the diagonal elements of T is 1. $\square$

Definition 5.3.30. A finite set $\mathcal{F} \subseteq \mathbb{R}_+$ is called ***admissible*** if $\mathcal{F}$ can be written as a (not necessarily disjoint) union $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ where each $\mathcal{F}_k = \{x_1, \dots, x_{i_k}\}$ satisfies the condition that

$$\sum_{j=1}^{i_k} m_j x_j = 1$$

for some $m_j \in \mathbb{N}$ $(j = 1, \dots, i_k)$.

Example 5.3.31. The set $\{\frac{1}{5}, \frac{1}{3}, \frac{2}{9}, \frac{2}{3}\}$ is admissible since $5 \cdot \frac{1}{5} = 1$, $\frac{1}{3} + 3 \cdot \frac{2}{9} = 1$, and $\frac{2}{3} + \frac{1}{3} = 1$. The sets $\{0\}$ and $\{\frac{3}{7}, \frac{2}{5}\}$ are not admissible.

The following lemma is obvious.

Lemma 5.3.32. *A finite union of admissible sets is admissible.*

Theorem 5.3.33. *Let $\mathcal{F} \subseteq \mathbb{R}$ be such that $0 \in \mathcal{F}$. Then $\mathcal{F}$ is admissible if and only if there exists an indecomposable semigroup $\mathcal{S}$ with finite diagonals such that the set of diagonal values of all the matrices in $\mathcal{S}$ is equal to $\mathcal{F}$.*

Proof. If $\mathcal{S}$ is an indecomposable semigroup with finite diagonals and $S \in \mathcal{S}$, then the set $\mathcal{F}_S$ of all the diagonal entries of S is admissible by Theorem 5.3.29. Since $\mathcal{S}$ is a semigroup with finite diagonals, there are only finitely many choices for the set $\mathcal{F}_S$. Therefore $\mathcal{F} = \cup_{S \in \mathcal{S}} \mathcal{F}_S$ is admissible by Lemma 5.3.32.

Let $\mathcal{F}$ be admissible. Write $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ as in the definition of an admissible set. We will show that there exists a semigroup $\mathcal{S}$ as in the statement of the theorem.

For each $k \in \{1,\ldots,n\}$, write $\mathcal{F}_k = \{x_1^{(k)},\ldots,x_{i_k}^{(k)}\}$ and fix $m_1^{(k)},\ldots,m_{i_k}^{(k)}$ such that $\sum_{j=1}^{i_k} m_j^{(k)} x_j^{(k)} = 1$. Put $N_k = \sum_{j=1}^{i_k} m_j^{(k)}$ and define a vector $y^{(k)} = (y_i^{(k)})_{i=1}^{N_k} \in \mathbb{R}^{N_k}$ by

$$y^{(k)} = \Big(\underbrace{x_1^{(k)},\ldots,x_1^{(k)}}_{m_1^{(k)}},\underbrace{x_2^{(k)},\ldots,x_2^{(k)}}_{m_2^{(k)}},\ldots,\underbrace{x_n^{(k)},\ldots,x_n^{(k)}}_{m_n^{(k)}}\Big).$$

Clearly, $\sum_{i=1}^{N_k} y_i^{(k)} = 1$. For each $i,j \in \{1,\ldots,n\}$, define a rank-one $N_j \times N_i$ matrix

$$T_{ij} = \begin{bmatrix} y_1^{(i)} & \ldots & y_{N_i}^{(i)} \\ \vdots & & \vdots \\ y_1^{(i)} & \ldots & y_{N_i}^{(i)} \end{bmatrix}.$$

Since each T_{ij} is row stochastic, a routine check shows that for all $i,j,k \in \{1,\ldots,n\}$ we have $T_{ij}T_{jk} = T_{ik}$.

Now let E_{ij} be the block matrix with n vertical and n horizontal blocks such that the (k,l) block of E_{ij} is equal to the $N_k \times N_l$ zero matrix if $k \neq i$ or $l \neq j$ and is equal to T_{ij} if $k = i$ and $l = j$. Define

$$\mathcal{S} = \{E_{ij} \colon 1 \leqslant i,j \leqslant n\} \cup \{\mathbf{0}\}.$$

Then clearly $\mathcal{S}$ is an indecomposable semigroup whose set of diagonal elements is $\mathcal{F}$. □

The last statement to be proved in this section is the assertion that if an admissible set $\mathcal{F} \subseteq \mathbb{R}_+$ does not contain zero, then there may not be an indecomposable semigroup of matrices whose diagonal entries form a set which is exactly $\mathcal{F}$. It will need an auxiliary lemma which may be of some independent interest.

Lemma 5.3.34. *Let $\mathcal{S}$ be a semigroup with finite diagonals such that no member of $\mathcal{S}$ has zero on the diagonal. If the minimal rank $m_{\mathcal{S}}$ of nonzero elements in $\mathcal{S}$ is not one, then $\mathcal{S}$ is decomposable.*

Proof. Suppose $\mathcal{S}$ is indecomposable and $m_{\mathcal{S}} \geqslant 2$. Fix a minimal idempotent $E \in \mathcal{S}$. Since E has no zeros on the diagonal, $E = E_1 \oplus \cdots \oplus E_{m_{\mathcal{S}}}$ where each E_i is a strictly positive idempotent.

Let $S \in \mathcal{S}$ be an arbitrary matrix. By Corollary 5.3.5, there is $m \in \mathbb{N}$ such that $(ESE)^m$ is an idempotent which we will denote by F. Clearly $EF = FE = F$. Since the diagonal values of matrices in $\mathcal{S}$ do not admit zeros, $E = F$ by minimality of E.

We claim that up to a permutation similarity, S is block-diagonal relative to the block-structure inherited from E. Indeed, suppose that S is not block-diagonal, say, $(1, 2)$-block of S is not zero. Write $(ESE)^m = ES[E(ESE)^{m-1}]$. Since members of $\mathcal{S}$ can only have non-zero elements on the diagonal, there exist $\lambda > 0$ and $\mu > 0$ such that $E \geqslant \lambda I$ and $E(ESE)^{m-1} > \mu I$. Then $E \geqslant \lambda\mu S$. That is, E has a non-zero $(1, 2)$-block, which is impossible. □

Proposition 5.3.35. *If $\mathcal{F} = \{\frac{1}{2}, \frac{1}{3}\}$ then there is no indecomposable semigroup, $\mathcal{S}$, such that the set of diagonal entries of matrices in $\mathcal{S}$ is equal to $\mathcal{F}$.*

Proof. Suppose such a semigroup, $\mathcal{S}$, exists. By Lemma 5.3.34, $\mathcal{S}$ contains an idempotent E of rank one. Since E cannot have zeros on the diagonal, E must be strictly positive. Since also $\mathrm{tr}(E) = 1$, there are, up to a diagonal similarity, only two choices for E:

$$\text{either} \quad E = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \quad \text{or} \quad E = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3. \end{bmatrix}$$

That is, $\mathcal{S}$ consists of either 2×2 matrices or 3×3 matrices. We will consider these two cases separately.

Assume the size of matrices in $\mathcal{S}$ is 2. Let A be a matrix having $1/3$ on the diagonal. That is, up to a permutation, $A = \begin{bmatrix} 1/3 & a \\ b & c \end{bmatrix}$ for some a, b, and c. By Lemma 5.3.3, the eigenvalues of A are either zero or roots of unity of degree at most 2. Also, $\mathrm{tr}(A) \geqslant 0$. Therefore the only possible values for $\mathrm{tr}(A)$ are 0, 1, and 2. In either case, c cannot belong to $\mathcal{F}$.

Now let the size of matrices in $\mathcal{S}$ be 3. Again, fix a matrix A with $1/2$ on the diagonal. Denote the two other diagonal entries of A by a and b. Observe that in this case, the only possible values for $\mathrm{tr}(A)$ are 0, 1, 2 and 3, none of which can be achieved by choosing a and b in $\mathcal{F}$. □

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