University of Alberta

# Analysis and Design of Nonuniform Multirate Systems

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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**To my parents**

**Abstract**

In this thesis, we study periodic and multirate systems. In particular we are interested in rate changers, nonuniform filter banks, and nonuniform transmultiplexers.

We consider the general characteristics of multirate building blocks and the approximation of these systems by systems that have a different but compatible period. In the $\mathcal{H}_2$ model-matching case, we show how to obtain the optimal approximant. Then, we use these multirate building blocks in the design of nonuniform filter banks and transmultiplexers.

Contrary to uniform filter banks, for nonuniform filter banks, if we use LTI filters in the analysis and synthesis filter banks, it may not be possible to achieve perfect-reconstruction. But, by using general multirate building blocks, we can eliminate some of the design constraints.

We assume that the analysis filter bank consists of finite impulse response (FIR) filters. The initial FIR analysis filters are designed according to the characteristics of the input. By the design procedure, the FIR synthesis filters are found so that the norm of the error system is minimized over all synthesis filters that have a prespecified order. Then, the synthesis filters obtained in the previous step are fixed and the analysis filters are found similarly. By iteration, the norm of the error system decreases until it converges to its final value.

In the $\mathcal{H}_2$ design of nonuniform filter banks, the problem is formulated in the frequency domain. In this approach, the frequency selectivity of filters can also be considered by

adding penalty terms. At each iteration, the least squares solution of a system of linear equations yields the local optimal filters.

In the $\mathcal{H}_\infty$ design of nonuniform filter banks, the problem is formulated in the state-space framework. The problem is then posed as an optimization problem based on linear matrix inequalities, which can be solved effectively by interior point methods.

Nonuniform transmultiplexers can be used for the conversion of signals that have different sampling rates, between the time-division multiplexing (TDM) format and the frequency-division multiplexing (FDM) format. Here, we have posed the design of nonuniform transmultiplexers as an iterated semidefinite programming problem.

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# Symbols

| | |
|---|---|
| $\mathbf{R}$ | set of real numbers |
| $\mathcal{H}_2$ | Hardy space on the open unit disc in the complex plane |
| $\mathcal{H}_\infty$ | ditto |
| $\mathcal{F}$ | Fourier transform operator |
| $z^{-1}$ | unit-delay (operator) |
| $\delta$ | Dirac-delta function |
| $\underline{\mathrm{H}}$ | lifted transfer matrix of $H$ |
| $\tilde{H}$ | alias-component matrix of $H$ |
| $A^T$ | transpose of $A$ |
| $A^*$ | conjugate transpose of $A$ |
| $I_r$ | $r \times r$ identity matrix |

# Abbreviations

| | |
|---|---|
| AC | alias-component |
| FDM | frequency-division multiplexing |
| FIR | finite impulse response |
| IIR | infinite impulse response |
| lcm | least common multiple |
| LTI | linear time-invariant |
| LPTV | linear periodically time-varying |
| LSTV | linear switching time-varying |
| QMF | quadrature mirror filter bank |
| TDM | time-division multiplexing |

# Chapter 1

# Introduction

Multirate systems are systems that handle data at several sampling rates. Multirate systems typically consist of downsamplers that decrease the sampling rate, upsamplers which increase the sampling rate, and linear systems for filtering of signals.

One of the applications of multirate signal processing is in changing the sampling rates of signals. In digital audio industry different devices are operating at different sampling rates, i.e., the broadcast rate is 32 kHz, the sampling rate for CDs is 44.1 kHz and for studio work is 48 kHz. Therefore it is necessary to convert the signals from one sampling rate to another, so that the spectrums of the signals are similar in the frequency bands of interest [11, 24, 32, 31].

Another application of multirate signal processing is in the design and implementation of filters. In particular, in the design of finite impulse response (FIR) filters with sharp transition bands, instead of direct implementation, it is possible to use multirate structures, and perform some of the computations at lower sampling rates [34].

The other important issue in multirate signal processing is the design of multirate filter banks [34, 18, 25, 6, 17]. Multirate filter banks are mainly used for subband coding of signals. Statistically, although audio and visual signals have most of their energy concentrated in a particular frequency band, the suppression of other bands may result in a degradation of the quality. In subband coding, by using a multirate filter bank, the input signal is split into different subbands. In critically sampled case, the number of input samples per unit time is equal to the sum of the number of samples per unit time of the subbands. Subband signals are encoded according to the statistical or perceptual properties, and transmitted or stored according to the application. It is usually possible to represent the subband signals with fewer number of bits than the input signal. After transmission/retrieval, subband signals are processed, and the input signal is reconstructed. Therefore by using filter banks it is possible to represent signals more efficiently, and without any perceivable loss in quality.

A related topic is the design of transmultiplexers [33, 35, 19, 26, 1]. Transmultiplexers are used for the conversion of signals between the time-division multiplexing (TDM) format and the frequency-division multiplexing (FDM) format. Here, we are interested in the design of nonuniform transmultiplexers. These systems can be used in cases where the sampling rates of the input signals are not necessarily the same. As we will see most of the methods developed here for the filter banks can be modified for use in nonuniform transmultiplexers.

In the rest of this section, we will discuss rate changers, filter banks, and transmultiplexers in more detail, and will give an outline of the thesis.

## 1.1 Rate Changers

In digital signal processing, different devices might be operating at different sampling rates. It is often necessary to convert a signal with a given sampling rate to another signal that has a different sampling rate so that the frequency characteristics of the signal remains similar [11]. One way to do that is to convert the digital signal back to a continuous signal and resample the signal at the sampling rate of interest. This process is expensive, inherently introduces noise, and is usually not adopted.

A better approach is to convert the discrete time signal directly from one sampling rate to the other. This can be done by the system shown in Figure 1.1. This system consists of an upsampler by $p$, $\uparrow p$, a linear system $G$, and a downsampler by $q$, $\downarrow q$. The upsampler by $p$ inserts $p-1$ zeros between each two consecutive samples, and the downsampler by $q$ deletes $q-1$ samples between each consecutive $q$ samples at $kq$ and $(k+1)q$, for any integer $k$. For such a system, if we delay the input sequence by $q$ samples, the output will be delayed by $p$ samples. We call a system with this property a $(p,q)$-periodic system. In a real-time implementation, the time corresponding to $q$ samples in the input sequence of the system is equal to the time corresponding to $p$ samples of the output of the system. In other words the system is a sample rate changer, and the sampling rate of the output of the system $\omega$ is related to the sampling rate of the input $\omega_i$, by

$$\omega_i = \frac{q}{p}\omega.$$

As we argue later, by choosing the linear time-invariant (LTI) system $G$ to be a lowpass filter with the cutoff frequency $\omega_c = \min(\pi/p,\ \pi/q)$, it is possible to expand (if $p < q$) or contract (if $p > q$) the spectrum of the input over the frequency range of interest. Therefore the system retains the shape of the spectrum in the given frequency range, while it changes the sampling rate.

## 1.2 Filter Banks

By proper connection of rate changers that were described above, a multirate filter bank can be formed. A simple nontrivial case is the 2-channel quadrature mirror filter bank (QMF)

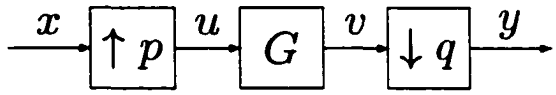


Figure 1.1: A sample-rate changer.

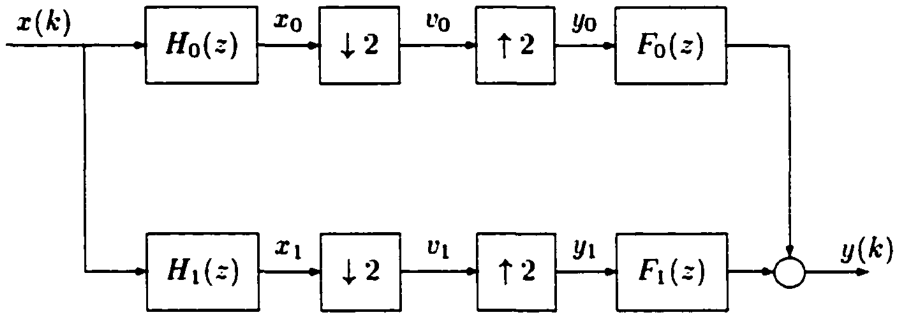


Figure 1.2: QMF filter bank.

shown in Figure 1.2. The filters $H_0$ and $H_1$ are typically lowpass and highpass respectively. Thus $x_0$ and $x_1$ are the low-frequency and high-frequency components of $x$. Since $x_0$ and $x_1$ are band-limited, if they are downsampled aliasing will be negligible.

Assume that the input to the filter bank $x(n)$, has most of its energy in the lower half of the spectrum, but the energy of the higher half of the spectrum is not negligible. Denote the subband signals that are obtained by filtering and downsampling, $v_0$ and $v_1$, respectively. As most of the energy of the signal is in $v_0$, we may allocate fewer bits to $v_1$ than to $v_0$. Thus the number of bits per second needed to transmit $v_0$ and $v_1$ collectively, is less than that needed for the transmission of $x(k)$, and it is usually more efficient to transmit or store the subband signals $v_0$ and $v_1$ instead of $x$. After transmitting/retrieving, the subband signals are upsampled by a factor of 2. Then the output of the upsamplers are processed by filters $F_0$ and $F_1$, and added together so that a replica of the original signal is reconstructed with a prespecified delay. Since the lower half of the spectrum is transmitted on the first channel and the higher half on the second channel, the filters $F_0$ and $F_1$ should be lowpass and high pass filters respectively.

If we delay the input to the filter bank by 2 samples, its output will be delayed by 2 samples. Thus the QMF is a periodically time-varying system with period 2. By blocking the inputs and outputs of this periodic system a 2 by 2 LTI system results [34]. As we will show later, the blocked transfer matrix of a filter bank can be obtained by the multiplication of the transfer matrix of the analysis filter bank and that of the synthesis filter bank. Because it is desirable to reconstruct the input of the multirate filter bank, the transfer matrix of the filter bank should be as close as possible to a blocking of a pure delay transfer function.

The delay depends on the order of the filters. The order of the filters in turn depends on the stop-band ripples and the transition bands of filters.

Assume that all analysis and synthesis filters are finite impulse response filters. The simultaneous design of analysis and synthesis filters in a multirate filter bank is a nonlinear optimization problem, but if either analysis or synthesis filters are known and the other set is to be designed, the problem can be solved efficiently: The initial FIR analysis filters may be designed according to the characteristics of the input. By the model-matching theory, the FIR synthesis filters are found so that the norm of the error system is minimized over all synthesis filters that have a prespecified order [7, 6, 27]. Then, the synthesis filters obtained in the previous step are fixed, and the analysis filters are found similarly. By iteration, the norm of the error system decreases until it converges to its final value.

Nonuniform filter banks are obtained by using fractional rate changers instead of integer rate changers as their building blocks. In a nonuniform filter bank, the sampling rates are not the same in all channels, and thus filters in different channels may have different bandwidths, which in turn may result in a higher efficiency in coding. The objective is to recover the input signal at the output of the filter bank, while the efficiency of the transmission is improved.

The design of nonuniform filter banks is the main focus of this thesis. It can be shown that the design of a nonuniform filter bank can be converted to the design of a uniform filter bank, subject to some structural constraints. The structural constraints cause some of the coefficients of the analysis and synthesis filters to appear in more than one entry of the transfer matrices of the analysis and synthesis filter banks, respectively. In other words, the design of nonuniform filter banks can be posed as a model-matching problem with a constraint on the elements of the transfer matrices. Most of the methods that were previously proposed for the design of nonuniform filter banks were applicable only when there were no constraints on the elements of the equivalent uniform filter bank. Here we pose the design of a nonuniform filter bank as a model-matching problem, and find the optimal FIR filter bank iteratively. Our approach is general and handles structural constraints. A related multirate system is a transmultiplexer, which we shall introduce next.

## 1.3 Transmultiplexers

Assume that we intend to transmit signals $x_0(n)$ and $x_1(n)$ through a channel. Take the transmultiplexer shown in Figure 1.3. Let $F_0 = H_1 = 1$, and $F_1 = H_0 = z^{-1}$. By this choice, we have $v(2n) = x_0(n)$, $v(2n+1) = x_1(n)$, $y_0(n) = x_0(n-1)$, and $y_1(n) = x_1(n-1)$. In other words, the transmultiplexer transmits the signal $v$ in the TDM format, and the

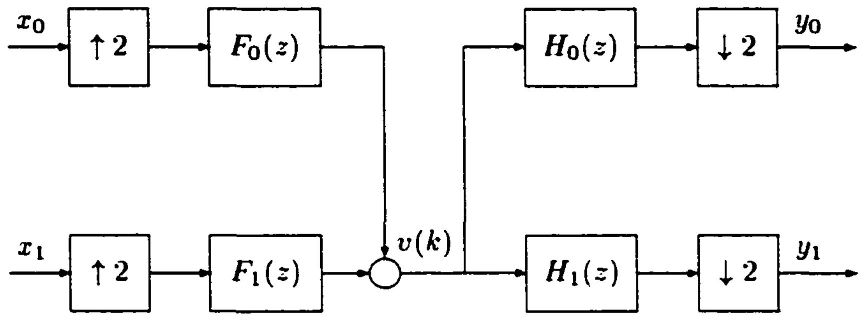


Figure 1.3: A 2-channel transmultiplexer.

outputs of the transmultiplexer are the delayed versions of the inputs. Note that here, the transmultiplexer is not doing any processing, and merely serves as a symbolic representation of the inputs in the TDM format.

Alteratively, we may transmit the signals in the FDM format, and recover the original signals at the outputs: The upsamplers by 2 compress the spectrum of the signals by a factor 2. For example, the spectrum of the $x_0$ in $[-\pi, \pi]$ is compressed on $[-\pi/2, \pi/2]$ and repeated outside this interval. Let $F_0$ and $F_1$ be ideal lowpass and highpass filters respectively. Then, the spectrum of the transmission signal $v$ is the compressed version of the spectrum of $x_0$ on $[-\pi/2, \pi/2]$, and that of $x_1$ on $[\pi/2, \pi]$. Therefore in this case, $v$ is the FDM format representation of the inputs of the transmultiplexer. If $H_0$ and $H_1$ are ideal lowpass and highpass filters, the upper channel and lower channel give the compressed versions of the spectrum of the input signals $x_0$ and $x_1$, respectively. As these are now band-limited, downsamplers by 2 expand the spectrums by 2, and yield the spectrum of original signals. Hence this transmultiplexer can be used for the conversion of the input signals between the TDM and the FDM formats.

By comparing transmultiplexers and filter banks, we see that they are the dual of each other, and that a transmultiplexer can be obtained by interchanging the analysis and synthesis parts of a filter bank. Note that the above mentioned transmultiplexer is an LTI system, because a unit shift in the input sequences results in a unit shift in the output sequences. Thus, we can define a transfer matrix for the transmultiplexer. The transfer matrix of a transmultiplexer can be obtained by post-multiplying the transfer matrix of the synthesis part by the transfer matrix of the analysis part. Therefore, the model-matching design of filter banks can be extended to transmultiplexers.

Nonuniform transmultiplexer were proposed recently [10, 20], and are obtained by interchanging the positions of the analysis and the synthesis parts of nonuniform filter banks. Contrary to a uniform transmultiplexer, a nonuniform transmultiplexer is not an LTI system, because the input of the different channels may have different sampling rates. A nonuniform transmultiplexer may be used for the conversion of input signals that have different sampling rates between the TDM and the FDM formats.

There are similarities between the design of nonuniform filter banks and the design of transmultiplexers. In the last part of the thesis, we obtain the alias-component (AC) matrix of nonuniform transmultiplexers. Using the AC matrix, we show that by minimizing the $\mathcal{H}_\infty$ norm of the error system, various distortion measures of a nonuniform transmultiplexer are minimized. Then, we pose the $\mathcal{H}_\infty$ design of a nonuniform transmultiplexer as an iterated semidefinite programming problem.

## 1.4 Scope of the Thesis

In Chapter 2, we discuss some issues related to the representation of linear periodically time-varying (LPTV) systems. These systems which we also refer to as periodic systems are relevant to our study because multirate filter banks are LPTV systems, furthermore it is sometimes straightforward to generalize the concepts from LPTV systems to multirate systems. We study two different representations of LPTV systems, namely, representations based on the alias-component matrices [30] and linear switching time-varying (LSTV) systems [28]. We show how to obtain these representations, and then we use the LSTV representation to address the issue of optimal approximation of an LPTV system by an LTI system or by an LPTV system with a period that is arbitrary but different from the original system. This is a generalization of the problem of approximating an LPTV system by an LTI system which was solved using the AC matrices in [8].

In Chapter 3, we study multirate building blocks such as upsamplers, downsamplers, and rate changers in general. We discuss the blocking of rate changers, which is used in the study of nonuniform filter banks and transmultiplexers in the following chapters. We also use the LSTV representation of rate changers to generalize the optimal approximation problem studied in Chapter 2 to cover rate changers. In other words, we study the optimal approximation of a multirate system by another multirate system that has a given but compatible period, i.e., has the same input and output sampling rates.

Chapters 4 and 5 are about the optimal design of nonuniform filter banks. In these chapters, we use the more general blocks in the synthesis filter bank and usual filters in the analysis filter bank. The general blocks eliminate some of the design constraints, and result in lower reconstruction errors.

In Chapter 5, we formulate the $\mathcal{H}_2$ model-matching problem in the frequency domain [27]. In this approach, the frequency selectivity of filters can also be considered by adding a penalty term. We follow an iterative solution, and at each iteration the results are obtained by finding the least squares solution of a system of linear equations.

In Chapter 6, we formulate the $\mathcal{H}_\infty$ model-matching problem in the state-space framework [29]. We propose an iterative solution similar to the one proposed in the $\mathcal{H}_2$ model-matching method. At each iteration, the problem is posed as an optimization based on linear matrix inequalities, and is solved by MATLAB LMI toolbox.

In Chapter 7, we extend the concept of alias-component matrices to nonuniform transmultiplexers. Then we pose the model-matching design of nonuniform transmultiplexers in the state-space domain. Based on the AC matrices, we show that the model-matching design yields transmultiplexers with low distortions. The problem is then solved by semidefinite programming.

The last chapter summerizes the work in the thesis, and outlines some possible future research directions.

# Chapter 2

# Representation of Linear Periodically Time-Varying Systems

A linear periodically time-varying discrete-time system with period $p$ is a system for which a shift in the input sequence by $p$ samples results in a shift of $p$ samples in the output sequence. We shall also refer to LPTV systems as periodic systems. Periodic systems are usually simpler to deal with than multirate systems, and some of the results for periodic systems can be directly generalized to general multirate systems.

LPTV systems appear frequently in control systems and signal processing: LPTV controllers are used to improve certain performance specifications of control systems [16]; and multirate filter banks, which are inherently LPTV, find application in digital communications [31, 34]. There are many ways to represent LPTV systems; common ones include state-space models with periodically time-varying coefficients, equivalent linear time-invariant models obtained by the blocking (lifting) technique [16], and alias-component matrices which are popular in signal processing [31, 34].

Here, we first study the alias-component matrices of LPTV systems and then examine two different structures for LPTV systems, each consisting of a switch, either at the input or the output, which periodically switches to several LTI blocks. These structures are called linear switching time-varying systems. Due to the simplicity of LSTV systems, they are very useful in studying a class of problems associated with LPTV systems, one of which is related to analyzing aliasing and interconnections of LPTV systems, namely, how to approximate a given LPTV system $G$ with period $p$ by an LPTV system $\tilde{G}$ with period $\tilde{p}$? The optimal choice, measured using the $\mathcal{H}_2$ norm of the error system $G - \tilde{G}$, turns out to be simple using the LSTV representations. We remark that the approximation problem we study in this chapter is quite general in that $p$ and $\tilde{p}$ are arbitrary integers; a special case when $\tilde{G}$ is LTI ($\tilde{p} = 1$) was studied in [8] using alias-component matrices; see also the work in [4] for some approximation results.

The chapter is organized as follows. In Section 2.1, we will study the frequency response and the alias-component matrix of an LPTV system. First, we use the Fourier series to find the steady-state response of discrete-time linear periodically time-varying systems to periodic inputs. Then, in order to obtain the response of LPTV systems to general inputs, we use the Fourier transform and establish a direct link between alias-component matrices and LPTV systems modeled by periodically time-varying difference equations; this is given in terms of the Fourier series of the parameters in the model. As we argue, the steady-state response of periodic systems can be calculated directly based on the alias-component matrices.

In Section 2.2, we study some basic properties for the two LSTV structures, including their time-domain and frequency-domain relationships and questions such as how to calculate system norms ($\ell_1$ and $\mathcal{H}_2$ norms), and how to characterize LSTV systems with period $p_1$ in the class of $p_1p_2$ periodic systems, where both $p_1$ and $p_2$ are positive integers. This will lead us to the solution of optimal approximation problem of an LPTV system with period $p$ by another LPTV system that has a given period $q$.

In Section 2.3, we present the optimal approximation of an LPTV system $G$ with period $p$ by an LPTV system $\tilde{G}$ with a pre-specified but arbitrary period $\tilde{p}$, so that the $\mathcal{H}_2$ norm of $G - \tilde{G}$ is minimized. Using the tools developed in Section 2.3, we show that when $\tilde{p}$ is not a divisor of $p$, the optimal approximant turns out to be LPTV with period $\bar{p}$, the greatest common divisor of $p$ and $\tilde{p}$. In other words, the class of LPTV systems with period $\tilde{p}$ will not yield any advantage over the class of LPTV systems with period $\bar{p}$ in achieving a lower approximation error.

Finally, in Section 2.4 we provide some concluding remarks.

## 2.1 Fourier Analysis of LPTV Systems

Complex exponential functions are the eigenfunctions of LTI systems. For an LPTV system with period $p$, the subspaces spanned by the functions $\{e^{j\omega n}, e^{j(\omega+\frac{1}{p})n)}, \cdots, e^{j(\omega+\frac{p-1}{p})n)}\}$ are invariant subspaces. In other words, the response of an LPTV system to inputs in the subspace will remain in it. By defining the relationship of the inputs and the outputs, a representation which is called the alias-component representation of the LPTV system is obtained.

As reported in [22], we may use the blocked model of an LPTV system, and obtain the alias-component matrix of the system. Here, we obtain an alternative, yet more direct formulation that relates the alias-component matrix to the Fourier series of the time-varying coefficients of the system. Then, we show that by using the alias-component matrix, the steady-state response to periodic inputs can be found directly.

In the next part, we use the Fourier series to compute the response of an LPTV system to periodic inputs. Then, by using the Fourier transforms, we find the response of an LPTV system to general inputs. This will lead to alias-component matrices of LPTV systems.

### 2.1.1 Response to Periodic Inputs

Assume that $G$ is a periodic system with period $p$. As we discuss in the next section, if a periodic input with period $p$ is applied, the output at the steady-state will be periodic with the same period $p$. If we represent the input in terms of its Fourier series, as the system is periodic, the parameters of the system will also be periodic in time with the same period. Therefore if we use the Fourier representation, we can obtain the coefficients of the Fourier series representation of the output of the system. Assume that $G$ is described by the difference equation

$$a_0(n)y(n) + a_1(n)y(n-1) + \cdots + a_k(n)y(n-k) = b_0(n)u(n) + \cdots + b_l(n)u(n-l). \quad (2.1)$$

The condition that the system is periodic yields:

$$a_i(n) = a_i(n-p), \quad \text{for } i = 0, 1, \cdots, k$$

$$b_i(n) = b_i(n-p), \quad \text{for } i = 0, 1, \cdots, l.$$

Because $a_i$ and $b_i$ are periodic with period $p$, they can be expressed in terms of their Fourier series as

$$a_i(n) = \sum_{m=0}^{p-1} \bar{a}_{i,m} e^{2\pi jnm/p}, \quad \text{for } i = 0, 1, \cdots, k$$

and

$$b_i(n) = \sum_{m=0}^{p-1} \bar{b}_{i,m} e^{2\pi jnm/p}, \quad \text{for } i = 0, 1, \cdots, l.$$

Since the input and output are periodic with period $p$, we have:

$$u(n) = \sum_{m=0}^{p-1} \bar{u}_m e^{2\pi jnm/p},$$

and

$$y(n) = \sum_{m=0}^{p-1} \bar{y}_m e^{2\pi jnm/p}.$$

The terms in the difference equation are periodic with period $p$, and we have

$$b_i(n)u(n-i) = \sum_{r=0}^{p-1}\sum_{\alpha=0}^{p-1} \bar{b}_{i,r-\alpha}\bar{u}_\alpha e^{-ji2\pi\alpha/p} e^{jn2\pi r/p}.$$

Note that we have assumed that $\bar{b}_{i,m}$ have period $p$ in their second indices. Therefore equation (2.1) can be replaced by its Fourier representation:

$$\sum_{r=0}^{p-1}\left(\sum_{\alpha=0}^{p-1}\sum_{i=0}^{k} \bar{a}_{i,r-\alpha}\bar{y}_\alpha e^{-ji2\pi\alpha/p} - \sum_{\alpha=0}^{p-1}\sum_{i=0}^{l} \bar{b}_{i,r-\alpha}\bar{u}_\alpha e^{-ji2\pi\alpha/p}\right) e^{jn2\pi r/p} = 0. \tag{2.2}$$

By setting the coefficients of $e^{jn2\pi r/p}$ (for $r = 0$ to $p-1$) to zero in (2.2), we obtain $p$ equations. By solving these equations for $\bar{y}_0$ to $\bar{y}_{p-1}$, the output, $y(n)$, can be found. It should be clear that this method is applicable only when the system is stable.

If the input has a period $p_u$ that is different from the period of the system $p_s$, we can show that the output of the system will be periodic with a period $\tilde{p}$ that is given by the least common multiple of $p_u$ and $p_s$, i.e., $\tilde{p} = \mathrm{lcm}(p_u, p_s)$: As $a_i$, $b_i$ and $u$ are all periodic with period $\tilde{p}$, we can find the coefficients of their Fourier series. Then by solving (2.2), we get the coefficients of the Fourier series of the output. The following example shows the results for a case where $p_s$ and $p_u$ are different.

Take the difference equations with period $p_s = 2$:

$$y(n) + 0.5y(n-1) - 0.4y(n-2) = 0.2u(n) + 0.3u(n-1) \quad \text{for even } n,$$

$$y(n) + 0.6y(n-1) + 0.3y(n-2) = 0.1u(n) + 0.1u(n-1) \quad \text{for odd } n,$$

and assume that the input is periodic with period $p_u = 9$:

$$u(n) = 0.37 - 0.1\sin\left(\frac{2\pi n}{9}\right) - 0.38\cos\left(\frac{8\pi n}{9}\right).$$

The output can be obtained by expressing the input and the parameters of the system in terms of their Fourier series with period $\tilde{p} = 18$ (least common multiple of 2 and 9). The output of the system as found directly from the difference equation with the initial conditions, $y(0) = 0$, and $y(1) = 0$, is shown by the solid line in Figure 2.1. The dashed line in the Figure shows the periodic part of the output as obtained by solving (2.2). As shown here, the transients die out in less than 20 samples. In the next section, we study the response of LPTV systems to aperiodic inputs.

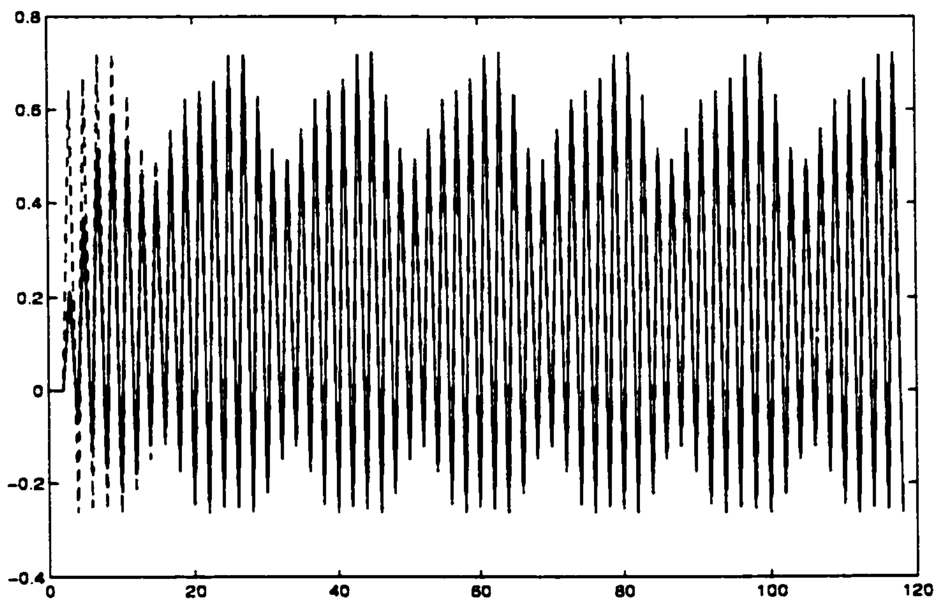


Figure 2.1: The steady-state (dashed) and actual (solid) outputs of the periodic system in the example, to the periodic input.

### 2.1.2 Alias-Component Matrices

In order to obtain the output to a general input, we use the Fourier transform. As in the previous section, we treat the parameters of the system as a periodic modulating signal.

For $\omega \in [0\ 2\pi]$, the Fourier transforms of $a_i$ are given by

$$A_i(\omega) = \mathcal{F}\{a_i(n)\} = 2\pi \sum_{m=0}^{p-1} \bar{a}_{i,m}\delta(\omega - \frac{2\pi m}{p}),$$

where $\delta$ is the Dirac-delta function. Similarly, we can express the Fourier transforms of $b_i(n)$ by

$$B_i(\omega) = 2\pi \sum_{m=0}^{p-1} \bar{b}_{i,m}\delta(\omega - \frac{2\pi m}{p}).$$

Applying the Fourier transform to the terms of the difference equation, we have

$$\mathcal{F}\{a_i(n)y(n-i)\} = \frac{1}{2\pi}\int_0^{2\pi} A_i(\Theta)Y(\omega - \Theta)e^{-(\omega-\Theta)ji}d\Theta,$$

where $Y$ is the Fourier transform of the output. Substituting $A_i$ by their Fourier series, we get

$$\mathcal{F}\{a_i(n)y(n-i)\} = \sum_{m=0}^{p-1} \bar{a}_{i,m}e^{-(\omega-\frac{2\pi m}{p})ji}Y(\omega - \frac{2\pi m}{p}). \tag{2.3}$$

Therefore the Fourier transform of (2.1) is

$$\sum_{m=0}^{p-1}\sum_{i=0}^{k} e^{-(\omega-\frac{2\pi m}{p})ji}\bar{a}_{i,m}Y(\omega - \frac{2\pi m}{p}) = \sum_{m=0}^{p-1}\sum_{i=0}^{l} e^{-(\omega-\frac{2\pi m}{p})ji}\bar{b}_{i,m}U(\omega - \frac{2\pi m}{p}),$$

where $U$ is the Fourier transform of the input of the system. Defining

$$\bar{U}(\omega) = \left[U(\omega)\ U(\omega - \frac{2\pi}{p})\ \cdots\ U(\omega - \frac{2\pi(p-1)}{p})\right]^T$$

and

$$\bar{Y}(\omega) = \left[Y(\omega)\ Y(\omega - \frac{2\pi}{p})\ \cdots\ Y(\omega - \frac{2\pi(p-1)}{p})\right]^T.$$

we have

$$\bar{A}(\omega)\bar{Y}(\omega) = \bar{B}(\omega)\bar{U}(\omega), \tag{2.4}$$

where the $(0, m)$ elements of $\bar{A}$ are given by

$$\bar{A}^{(0,m)}(\omega) = \sum_{i=0}^{k} e^{-(\omega-\frac{2\pi m}{p})ji}\bar{a}_{i,m},$$

and the rest of the elements are found using the pseudo-circulant property of $\bar{A}$. Note that the numbering starts from zero. Similarly the elements of the first row of the matrix $\bar{B}$ can be obtained as

$$\bar{B}^{(0,m)}(\omega) = \sum_{i=0}^{l} e^{-(\omega - \frac{2\pi m}{p})ji} \bar{b}_{i,m}.$$

From (2.4) we get

$$\bar{Y}(\omega) = \bar{A}^{-1}(\omega)\bar{B}(\omega)\bar{U}(\omega). \tag{2.5}$$

The matrix $\bar{A}^{-1}(\omega)\bar{B}(\omega)$ is called the alias-component matrix of the system $G$ and relates the input and the output of the system in the frequency domain.

Noting that the Fourier transform of a periodic input $u(n)$ is given by

$$U(\omega) = 2\pi \sum_{m=0}^{p-1} \bar{u}_m \delta(\omega - \frac{2\pi m}{p}),$$

and substituting in equation (2.4), we see that the output will be periodic, and its Fourier coefficients can be obtained by solving

$$\bar{A}(0)\bar{Y} = \bar{B}(0)\bar{U},$$

where $\bar{U} = [\bar{u}_0 \ \bar{u}_{p-1} \ \bar{u}_{p-2} \ \cdots \ \bar{u}_1]^T$, and $\bar{Y} = [\bar{y}_0 \ \bar{y}_{p-1} \ \bar{y}_{p-2} \ \cdots \ \bar{y}_1]^T$. This is equation (2.2) in the matrix form, as we expected. In other words, the steady-state output of the system can be calculated directly by using the alias-component matrix at $\omega = 0$. From here, we see that equation (2.2) will be singular, if and only if matrix $\bar{A}$ does not have full rank at $\omega = 0$.

In the next section, we shall consider the representation of LPTV system by LSTV systems.

## 2.2 LSTV Systems

An LPTV system with period $p$ can be modeled by a system consisting of $p$ LTI systems and a periodic switch at the output (Figure 2.2) or input (Figure 2.3). The LTI blocks

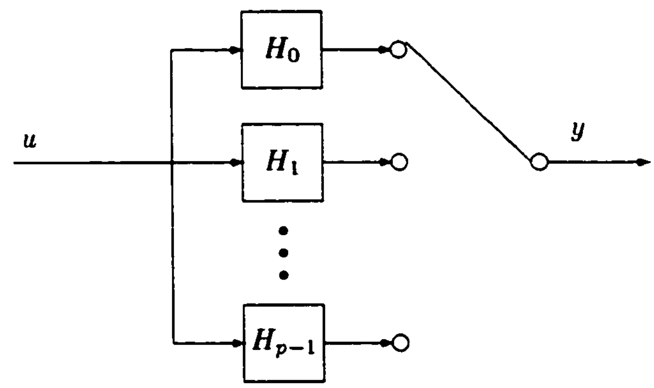


Figure 2.2: The structure of an LSTV system with a switch at the output.

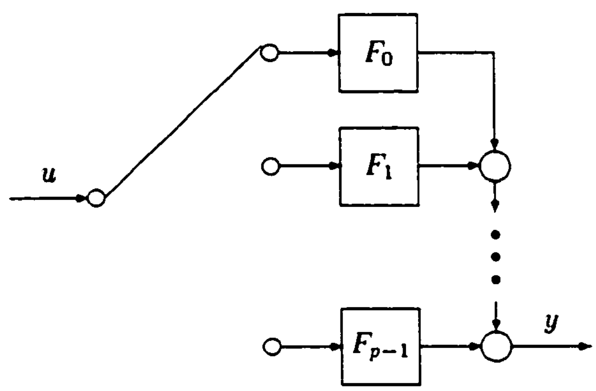


Figure 2.3: The structure of an LSTV system with a switch in the input.

in Figure 2.2 can be obtained as follows. The input-output relationship of a linear causal time-varying system is given by

$$y(k) = \sum_{l=-\infty}^{k} g(k,l)u(l), \tag{2.6}$$

where $y(k)$ is the output at time $k$, $u(l)$ the input at time $l$, and $g(k,l)$ is the output of the system at time $k$ when the input is a unit impulse at time $l$. For an LPTV system with period $p$, the shift invariance property results

$$g(k+p, l+p) = g(k, l).$$

By substituting

$$h(l, k) = g(k, k-l) \tag{2.7}$$

into (2.6), we have

$$y(k) = \sum_{l=-\infty}^{k} h(k-l, k)u(l).$$

If we assume that the LTI system $H_i$ has the impulse response $h(l, i)$, then $y(i)$ is the output of $H_i$ at time $i$ to the input sequence $u(l)$. The $p$-periodic condition on $g$ gives

$$h(l, i) = h(l, i+p).$$

Thus LTI blocks $H_i$ and $H_{i+p}$ are the same, and only $p$ LTI systems $H_0, H_1, ..., H_{p-1}$ are sufficient. Figure 2.2 shows the LSTV system that results. The switch connects the input to $H_0$ at time $k = 0$, then to $H_1$ at $k = 1$, and so on; at $k = p$, it connects back to $H_0$ and repeats. When the switch is not connected to a block, the input to the block is assumed to be zero. For ease of reference, we will represent the LSTV system in Figure 2.2 by $\{H_0, H_1, \cdots, H_{p-1}\}_o$, the subscript indicating the switch at the output.

From (2.7) it follows that there is a one-to-one correspondence between LPTV systems and LSTV systems. In other words there is no redundancy in the LSTV representation. The outputs of LTI blocks, $H_i$ are the same as the output of the periodic system $G$ at time $i + lk$, for an integer $l$, thus $G$ is stable if and only if $H_i$ are stable.

It can be shown similarly that an LPTV system with period $p$ can be represented by a system consisting of $p$ LTI systems and a periodic switch at the input, as shown in Figure 2.3. We will represent the system in Figure 2.3 by $\{F_0, F_1, \cdots, F_{p-1}\}_i$, the subscript indicating the switch at the input. Although the two LSTV systems are equivalent, for a specific application, one may be more useful than the other.

The frequency response of $G$ can be related to that of the LTI systems in the LSTV structure in Figure 2.3. Let $G = \{F_0, F_1, \cdots, F_{p-1}\}_i$, as shown in Figure 2.3. The input to $F_i$ block, $u_i(n)$ is equal to the input $u(n)$ at time $n = i + lp$, for any integer $l$, and zero otherwise. Thus we can write

$$u_i(n) = u(n)\left(\frac{1}{p}\sum_{k=0}^{p-1} W_p^{-(n-i)k}\right),$$

where $W_p = e^{-2\pi j/p}$. Taking the $z$-transform of both sides of this equation, we have

$$U_i(z) = \frac{1}{p}\sum_{k=0}^{p-1} U(zW_p^k)W_p^{ik}.$$

The output of $G$ is

$$\begin{aligned} Y(z) &= \sum_{i=0}^{p-1} U_i(z)F_i(z) \\ &= \frac{1}{p}\sum_{i=0}^{p-1}\sum_{k=0}^{p-1} F_i(z)W_p^{ik}U(zW_p^k) \\ &= \sum_{k=0}^{p-1}\left[\frac{1}{p}\sum_{i=0}^{p-1} F_i(z)W_p^{ik}\right]U(zW_p^k). \end{aligned}$$

The system $G$ is an LTI system, if $Y(z)$ depends only on $U(z)$. Thus $G$ is LTI, if and only if $\sum_{i=0}^{p-1} F_i(z)W_p^{ik}$ is equal to zero for $0 < k < p$. This is true if and only if all of $F_i$ are equal.

Similarly it can be shown that an LSTV system in Figure 2.2 becomes an LTI system if and only if all $H_i$ are equal.

Such a condition can be generalized to find a subclass of LPTV systems within a class of LPTV systems. Let $p_1$ and $p_2$ be positive integers. An LPTV system with period $p_1$, $G = \{\tilde{H}_0, \tilde{H}_1, \cdots, \tilde{H}_{p_1-1}\}_o$, is in the class of LPTV systems with period $p_1p_2$; and thus it can be represented by an LSTV structure $G = \{H_0, H_1, \cdots, H_{p_1p_2-1}\}_o$, consisting of $p_1p_2$ LTI blocks and a periodic switch. Since a shift of $p_1$ samples in the input sequence results in a shift of $p_1$ samples in the output, for $0 \leq i < p_1$ and $0 \leq j < p_2$, the systems $H_i$ and $H_{i+jp_1}$ are the same, and for $0 \leq i < p_1$, $H_i = \tilde{H}_i$. In other words, in the class of LPTV systems with period $p_1p_2$, the subclass of LPTV systems with period $p_1$ are those satisfying

$$H_{i+jp_1} = \tilde{H}_i, \quad j = 0, 1, \cdots, p_2 - 1; \quad i = 0, 1, \cdots, p_1 - 1.$$

Turning to norms for LPTV systems, either the structure in Figure 2.2 or the one in Figure 2.3 may be more useful. For example, in the calculation of the $\ell_1$ norm of an LPTV system, the structure in Figure 2.2 is simpler to use: The following lemma gives the $\ell_1$ norm of an LPTV system.

**Lemma 1** *The $\ell_1$ norm of an LPTV system $G = \{H_0, H_1, \cdots, H_{p-1}\}_o$ is given by*

$$\|G\|_1 = \max_{0 \leq i < p} \{\|H_i\|_1\}.$$

Let $\ell_\infty$ be the discrete-time signal space of bounded signals; the norm of $u$, denoted $\|u\|_\infty$, is defined as the least upper bound of the sequence $|u(n)|$. Recall that the $\ell_1$ norm of a linear system $G$ is $\ell_\infty$ induced, i.e., if $y = Gu$, then

$$\|G\|_1 = \sup_{u \neq 0} \frac{\|y\|_\infty}{\|u\|_\infty}.$$

If we represent $G$ by $\{H_0, H_1, \cdots, H_{p-1}\}_o$ as in Figure 2.2, then since the output of G is switched from the outputs of $H_i$, we have

$$\|G\|_1 \leq \max_{0 \leq i < p} \{\|H_i\|_1\}.$$

Now, let $u$ be a maximizing input for $H_i$, i.e., $\|u\| = 1$ and $\|H_i u\|_\infty = \|H_i\|_1$. (Here, we assume that the supremum is actually obtainable by a maximizing input; otherwise, we have to use a maximizing input sequence $\{u_k\}$ and the argument below is still valid.) Using this $u$ with possibly some delay as an input to $G$ we get that $\|Gu\|_\infty \geq \|H_i u\|_\infty = \|H_i\|_1$. Hence

$$\|G\|_1 \geq \max_{0 \leq i < p} \{\|H_i\|_1\}.$$

Therefore, the result in Lemma 1 follows.

Next, we turn to $\mathcal{H}_2$ norms of LPTV systems. To define the $\mathcal{H}_2$ norm of an LPTV system with period $p$, $p$ unit impulses at time 0 through time $p-1$ are applied to the input of the system. The $\mathcal{H}_2$ norm is the square root of the average of the squares of the 2-norms of the outputs to these inputs. The $\mathcal{H}_2$ norm can be related to the LTI blocks in Figure 2.3 as follows.

**Lemma 2** *The $\mathcal{H}_2$ norm of an LPTV system $G = \{F_0, F_1, \cdots, F_{p-1}\}_i$ is given by*

$$\|G\|_2 = \left(\frac{1}{p}\sum_{i=0}^{p-1} \|F_i\|_2^2\right)^{1/2}. \tag{2.8}$$

**Proof** By applying the impulse $\delta_{k-i}$ to the input of the LSTV system $G$, because of the switch, only the $i$-th block $F_i$ has an input not identically equal to zero. The output of the system to this input is

$$Y_i(z) = F_i(z)z^{-i},$$

and the 2-norm of this output is equal to $\|F_i\|_2$. By averaging the squares of the norms of the outputs of $G$ due to impulses at time 0 through $p-1$, the lemma follows. $\Box$

The $\mathcal{H}_2$ norm of an LPTV system can be also related to the LTI blocks in Figure 2.2.

**Lemma 3** *The $\mathcal{H}_2$ norm of an LPTV system $G = \{H_0, H_1, \cdots, H_{p-1}\}_o$ is given by*

$$\|G\|_2 = \left(\frac{1}{p}\sum_{j=0}^{p-1}\|H_j\|_2^2\right)^{1/2}. \tag{2.9}$$

**Proof** Since the switch is at the output, we need to use the polyphase decomposition [34] of the LTI blocks $H_j$:

$$H_j(z) = \sum_{l=0}^{p-1}\hat{H}_{jl}(z^p)z^{-l}.$$

The output of this block to the impulse at time $k = i$ is $\sum_{l=0}^{p-1}\hat{H}_{j,l}(z^p)z^{-(l+i)}$. The output of the system at sampling times $j + rp$ ($r$ is an integer) are taken from the $j$-th block; so if we define $\bar{Y}_j^i(z)$ as the contribution of the $j$-th block due to impulse input $\delta_{k-i}$, we have

$$Y_{ji}(z) = \begin{cases} \hat{H}_{j,p+j-i}(z^p)z^{-(j+p)}, & \text{for } j < i, \\ \hat{H}_{j,j-i}(z^p)z^{-j}, & \text{otherwise.} \end{cases}$$

Thus, the overall output due to input $\delta_{k-i}$ is

$$Y^i = \sum_{j=0}^{p-1}Y_{ji}(z)$$

and so

$$\|Y^i\|_2^2 = \sum_{j=0}^{p-1}\|\hat{H}_{j,(j-i)\ \mathrm{mod}\ p}\|_2^2,$$

where "$a$ mod $p$" gives the remainder of the division of $a$ by $p$. From the definition of the $\mathcal{H}_2$ norm it follows that

$$\|G\|_2^2 = \frac{1}{p}\sum_{i=0}^{p-1}\|Y^i\|^2 = \frac{1}{p}\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\|\hat{H}_{j,(j-i)\ \mathrm{mod}\ p}\|_2^2.$$

Since all of the indices of polyphase components of $H_j$ appear once in the above, we have

$$\|G\|_2^2 = \frac{1}{p}\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} \|\hat{H}_{j,i}\|_2^2.$$

The lemma follows immediately by noting

$$\|H_j\|_2^2 = \sum_{i=0}^{p-1} \|\hat{H}_{j,i}\|_2^2.$$

□

Based on (2.8) and (2.9), for an LSTV system, the minimization of the $\mathcal{H}_2$ norm of the system can be done in terms of the $\mathcal{H}_2$ norms of the LTI blocks. In the next section, these results are applied to find explicit solutions of some optimal approximation problem.

## 2.3 Optimal Approximation

In this section, we approximate an LPTV system $G$ with period $p$ by an LPTV system $\tilde{G}$ with period $\tilde{p}$ so that the impulse response of $\tilde{G}$ is as close as possible to that of $G$. The objective is to minimize the $\mathcal{H}_2$ norm of the error system, $G - \tilde{G}$, as shown in Figure 2.4.

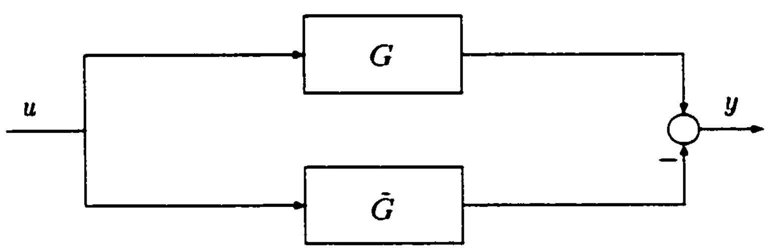


Figure 2.4: The approximation scheme.

We use the LSTV structure in Figure 2.2 for $G$ and $\tilde{G}$:

$$G = \{H_0, H_1, \cdots, H_{p-1}\}_o, \quad (2.10)$$

$$\tilde{G} = \{\tilde{H}_0, \tilde{H}_1, \cdots, \tilde{H}_{\tilde{p}-1}\}_o. \quad (2.11)$$

The following theorem gives the $\mathcal{H}_2$-optimal approximation in the general case when $p$ and $\tilde{p}$ are two arbitrary positive integers.

**Theorem 1** *Let $\bar{p}$ be the greatest common divisor of $p$ and $\tilde{p}$. Write $p = q\bar{p}$. Then the optimal approximant is in fact LPTV with period $\bar{p}$: If we represent $\tilde{G}^{opt} = \{\tilde{H}_0^{opt}, \tilde{H}_1^{opt}, \cdots, \tilde{H}_{\bar{p}-1}^{opt}\}_o$, then*

$$\tilde{H}_r^{opt} = \frac{1}{q}\sum_{m=0}^{q-1} H_{r+m\bar{p}}, \quad r = 0, 1, \cdots, \bar{p}-1. \tag{2.12}$$

The proof of this theorem involves the following result from the number theory.

**Lemma 4** *(Chinese Remainder Theorem [3])*

*Let $q$ and $\tilde{q}$ be two coprime positive integers. For each pair of integers $m$ and $\tilde{m}$ satisfying $0 \leq m < q$ and $0 \leq \tilde{m} < \tilde{q}$, there exists one and only one integer $k$ with $0 \leq k < q\tilde{q}$ such that*

$$m = k \bmod q, \quad \tilde{m} = k \bmod \tilde{q}. \tag{2.13}$$

This lemma establishes a one-to-one correspondence between the integer set $\{k : 0 \leq k < q\tilde{q}\}$ and the set of integer pairs $\{(m, \tilde{m}) : 0 \leq m < q,\ 0 \leq \tilde{m} < \tilde{q}\}$ while the relations in (2.13) are satisfied.

**Proof of Theorem 1** Write $\tilde{p} = \tilde{q}\bar{p}$ and $G' = G - \tilde{G}$. It follows that $G'$ is LPTV with period $p' = q\tilde{q}\bar{p}$, the least common multiple of the periods for $G$ and $\tilde{G}$. Hence $G'$ has an LSTV structure $G' = \{H_0', H_1', \cdots, H_{p'-1}'\}_o$. To relate $H_i'$ to LTI systems in (2.10) and (2.11), we note that $G$ also has an LSTV structure with period $p'$, as was discussed in the preceding section:

$$G = \{H_0, H_1, \cdots, H_{p-1}, H_0, H_1, \cdots, H_{p-1}, \cdots, H_0, H_1, \cdots, H_{p-1}\}_o.$$

Similarly, $\tilde{G}$ can be represented by an LSTV structure with period $p'$:

$$\tilde{G} = \{\tilde{H}_0, \tilde{H}_1, \cdots, \tilde{H}_{\tilde{p}-1}, \tilde{H}_0, \tilde{H}_1, \cdots, \tilde{H}_{\tilde{p}-1}, \cdots, \tilde{H}_0, \tilde{H}_1, \cdots, \tilde{H}_{\tilde{p}-1}\}_o.$$

Thus the LTI blocks in $G'$ can be written:

$$H'_i = H_{i \bmod p} - \tilde{H}_{i \bmod \tilde{p}}, \quad i = 0, 1, \cdots, p' - 1.$$

By the definition of the $\mathcal{H}_2$ norm, our optimization problem is to find LTI systems $\tilde{H}_i$ to minimize

$$J = \frac{1}{p'} \sum_{i=0}^{p'-1} \|H'_i\|_2^2 = \frac{1}{p'} \sum_{i=0}^{p'-1} \|H_{i \bmod p} - \tilde{H}_{i \bmod \tilde{p}}\|_2^2. \tag{2.14}$$

In the above summation $i$ goes from 0 to $p' - 1$; this is the same as $r + k\tilde{p}$ where $r$ goes from 0 to $\tilde{p} - 1$ and $k$ from 0 to $q\tilde{q} - 1$, since $p' = q\tilde{q}\tilde{p}$. So the quantity $J$ in (2.14) can be written as

$$J = \frac{1}{p'} \sum_{k=0}^{q\tilde{q}-1} \sum_{r=0}^{\tilde{p}-1} \|H_{(r+k\tilde{p}) \bmod p} - \tilde{H}_{(r+k\tilde{p}) \bmod \tilde{p}}\|_2^2. \tag{2.15}$$

In order to simplify further, we invoke Lemma 4: For every $k$ with $0 \leq k \leq q\tilde{q} - 1$, there exist integers $m$ and $\tilde{m}$ satisfying

$$m = k \bmod q, \quad \tilde{m} = k \bmod \tilde{q},$$

or equivalently, for some integers $l$ and $\tilde{l}$,

$$k = m + lq = \tilde{m} + \tilde{l}\tilde{q}; \tag{2.16}$$

moreover, as $k$ goes from 0 to $q\tilde{q} - 1$, $m$ goes from 0 to $q - 1$ and $\tilde{m}$ from 0 to $\tilde{q} - 1$. Substituting (2.16) into (2.15) and noting

$$\begin{aligned} (r + k\tilde{p}) \bmod p &= [r + (m + lq)\tilde{p}] \bmod p = r + m\tilde{p}, \\ (r + k\tilde{p}) \bmod \tilde{p} &= [r + (\tilde{m} + \tilde{l}\tilde{q})\tilde{p}] \bmod \tilde{p} = r + \tilde{m}\tilde{p}, \end{aligned}$$

we get

$$\begin{aligned} J &= \frac{1}{p'} \sum_{\tilde{m}=0}^{\tilde{q}-1} \sum_{m=0}^{q-1} \sum_{r=0}^{\bar{p}-1} \|H_{r+m\bar{p}} - \tilde{H}_{r+\tilde{m}\bar{p}}\|_2^2 \\ &= \frac{1}{p'} \sum_{\tilde{m}=0}^{\tilde{q}-1} \sum_{r=0}^{\bar{p}-1} \left[ \sum_{m=0}^{q-1} \|H_{r+m\bar{p}} - \tilde{H}_{r+\tilde{m}\bar{p}}\|_2^2 \right] \end{aligned}$$

Note that each $\tilde{H}_{r+\tilde{m}\bar{p}}$ only appears $q$ times in $J$ in the inner summation; hence the optimization problem of minimizing $J$ over $\tilde{H}_{r+\tilde{m}\bar{p}}$ can be broken down into $\tilde{p}$ ($\tilde{p} = \tilde{q}\bar{p}$) sub-problems of minimizing

$$J_{r+\tilde{m}\bar{p}} = \sum_{m=0}^{q-1} \|H_{r+m\bar{p}} - \tilde{H}_{r+\tilde{m}\bar{p}}\|_2^2 \quad (0 \leq r < \bar{p}, \;\; 0 \leq \tilde{m} < \tilde{q})$$

over only a single LTI block. From here, it follows that the optimal $\tilde{H}_{r+\tilde{m}\bar{p}}$ is given by

$$\tilde{H}^{opt}_{r+\tilde{m}\bar{p}} = \frac{1}{q} \sum_{m=0}^{q-1} H_{r+m\bar{p}}, \quad r = 0, 1, \cdots, \bar{p}-1; \;\; \tilde{m} = 0, 1, \cdots, \tilde{q}-1. \tag{2.17}$$

It is evident that this optimal approximant is in fact LSTV with period $\bar{p}$ and hence it can be represented with only the first $\bar{p}$ LTI systems in (2.17); this proves (2.12). □

In Theorem 1, if $p$ and $\tilde{p}$ are coprime, or more specifically, if $\tilde{p} = 1$ (in this case we would like to approximate an LPTV system $G$ by an LTI one), then the optimal approximant is LTI and is given by the average of the LTI blocks of $G$. The optimal approximation of LPTV systems by LTI systems was reported in [8] in terms of aliasing component matrix representation of LPTV systems.

## 2.4 Summary

In this chapter, we studied the alias-component matrices and switching representation of LPTV systems. For an LPTV system the alias-component matrix yields the response to periodic and aperiodic inputs: For periodic inputs, the Fourier series is used, and the outputs are found by solving systems of linear equations. For aperiodic inputs, the Fourier transform is used and the alias-component matrix of the system is obtained. This results in expressing the alias-component matrix in terms of the Fourier series of the parameters of the system. As we showed, the steady-state response may be obtained using the alias-component matrix at $\omega = 0$ directly. Although the results are obtained for difference equations, they can be extended to state-space models readily.

In the second part of the chapter, we have discussed representations of LPTV systems using two linear switching time-varying structures: Basic properties have been derived and applied to optimal approximation problems involving LPTV systems. The approximation results are useful in analyzing aliasing effects in LPTV systems in a more refined manner: For example, for an LPTV system with period $p = p_1 p_2$, we can decompose the system into its LTI component, its component corresponding to an LPTV system with period $p_1$, and so on. The errors involved in representing the LPTV system by its various component systems can be easily analyzed based on the results of this chapter. In the next chapter, we introduce rate changers, and generalize the results in Section 2.3 to multirate systems.

# Chapter 3

# Rate changers

Rate changers are used as interface between systems that are operating at different sampling rates. Rate changers also serve as building blocks in more complex multirate systems such as nonuniform filter banks and transmultiplexers.

We will start this chapter by reviewing some of the fundamentals of upsamplers, downsamplers, and rate changers in Section 3.1. Then in Section 3.2, we shall discuss how to obtain the transfer matrix of a rate changer, which we need in the next chapters. In Section 3.3, we extend the results presented in Section 2.3 to a more general setup involving multirate periodic systems.

## 3.1 Fundamentals

A rate changer is shown in Figure 3.1. It consists of three main elements, an upsampler by $p$ which increases the input sampling rate by a factor of $p$, an LTI system that processes the data, and a downsampler by $q$ which decreases the sampling rate of the signal by $q$. If either $p$ or $q$ are equal to 1, a rate changer that changes the sampling rate by an integer factor is obtained.

The upsampler by $p$ inserts $p-1$ zeros between each two consecutive samples, thus its

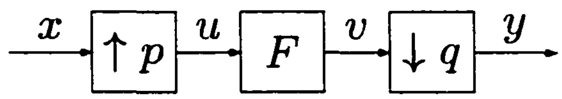


Figure 3.1: A sample-rate changer.

output $u$ is related to its input $x$ by:

$$u(k) = \begin{cases} x(k/p), & \text{if } p \text{ divides } k, \\ 0, & \text{otherwise.} \end{cases}$$

From this it follows that in the frequency domain we have

$$U(z) = X(z^p).$$

Therefore as shown in Figure 3.2, the upsampler contracts the input spectrum in $[-\pi, \pi]$ to $[-\frac{\pi}{p}, \frac{\pi}{p}]$, and the spectrum of the output becomes periodic with period $\frac{2\pi}{p}$. By filtering this output, we may eliminate the portion of spectrum outside $[-\frac{\pi}{p}, \frac{\pi}{p}]$.

The downsampler by $q$ deletes $q-1$ samples between each consecutive $q$ samples at $kq$ and $(k+1)q$ for any integer $k$. The output of the downsampler $y$ is related to its input $v$, by

$$y(k) = v(k\,q).$$

In the frequency domain, we have

$$Y(z^q) = \frac{1}{q}\sum_{i=0}^{q-1} V(zW_q^i),$$

where $W_q = e^{2\pi j/q}$. The terms $V(zW_q^i)$ are the shifted copies of $V(z)$ by the frequency shift $\frac{2\pi i}{q}$. If the signal is not band-limited to the region $|\omega| < \pi/q$, when these terms are added, they overlap in a region, and therefore aliasing will result. Because of the aliasing, it will not be possible to recover the original signal from the downsampled signal anymore. For

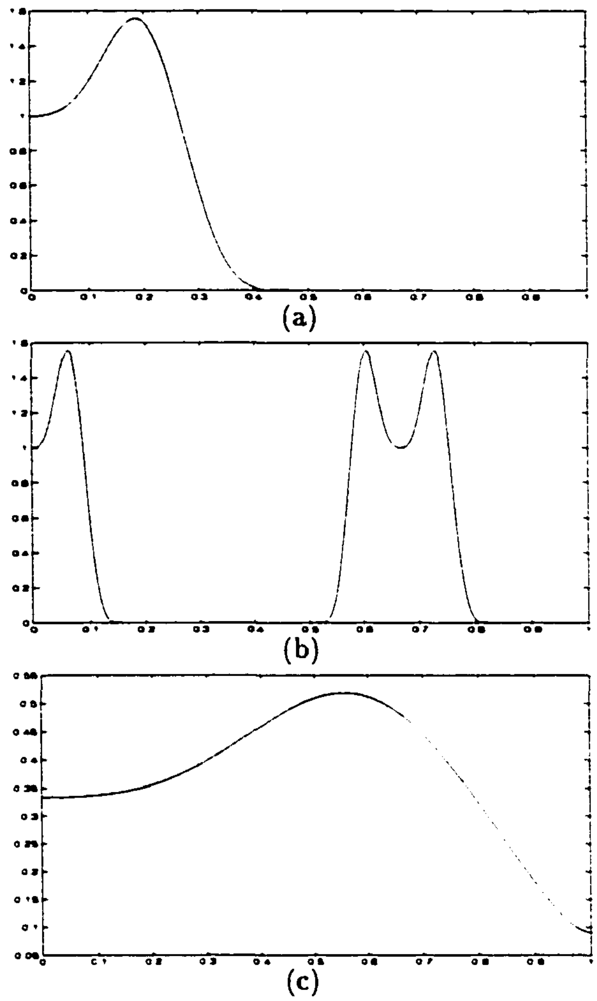


Figure 3.2: a) Input signal, b) upsampled signal by 3 ($p = 3$), c) downsampled signal by 3 ($q$=3), amplitude versus normalized frequency.

example if we downsample the signal in Figure 3.2a, as the bandwidth of the signal is more than $\pi/3$, the translated versions will overlap, and the spectrum of the downsampled signal will not be the compressed version of the original signal. The effect of aliasing is shown in Figure 3.2c. In order to eliminate the effect of aliasing, we may pre-filter the input to the downsampler by a lowpass filter with a cutoff frequency of $\omega_c = 2\pi/q$.

The need for pre-filtering in case of sampling rate reduction, and post-filtering in case of sampling rate increase may be explained by an assessment of the Nyquist rate of signals in a multirate system. As we mentioned before, by rate changing we expect to obtain a signal that has the same characteristics of the original analog signal. An upsampler merely increases the sampling rate by inserting zeros between each two consecutive samples, but by filtering this signal we obtain a close replica of the original analog signal. The signal before upsampling contains only the portion of the analog spectrum with frequencies less than its Nyquist rate. Therefore the rest of the spectrum of the analog signal may not be recovered from the input signal. This results in an output signal with a spectrum that is identically zero for the frequency range $|\omega| > 2\pi/p$. Similarly, when a signal is downsampled, its Nyquist rate decreases, and the portion of spectrum that is beyond the Nyquist rate of the downsampled signal may not be kept. Therefore, the signal should be pre-filtered to avoid aliasing.

Take the multirate system in Figure 3.1. For $p > q$, as the upsampling factor is higher than the downsampling factor, the rate changer increases the sampling rate of the input by the fractional ratio of $p/q$. The upsampler contracts the spectrum of the input in $[-\pi\ ,\ \pi]$ to cover the range $[-\frac{\pi}{p}\ ,\ \frac{\pi}{p}]$ of the spectrum of $u$, and the rest of the spectrum of $u$ is a shifted version of this portion. If we assume that the kernel system $F$ is a lowpass filter with the cutoff frequency $\omega_c = \pi/p$, then the repeated part of the spectrum of $u$ which appears outside $[-\frac{\pi}{p}\ ,\ \frac{\pi}{p}]$ is deleted by the filter. The downsampler contracts the input spectrum in $[-\pi\ ,\ \pi]$ to $[-\frac{q}{p}\pi\ ,\ \frac{q}{p}\pi]$, and the rest of the spectrum of $y$ is identically zero. For $p < q$, the rate changer decreases the sampling rate. Because in this case the rate changer expands a part of the input spectrum in the output, the rest of the input spectrum should be eliminated by the filter $F$ to avoid aliasing. If we take the filter $F$ to be a lowpass filter with the cutoff frequency $\omega_c = \pi/q$, then with an argument similar to the previous case, we can show that the rate-changer maps input spectrum in $[-\frac{p}{q}\pi\ ,\ \frac{p}{q}\pi]$ to cover the whole spectrum of the output $y$ in $[-\pi\ ,\ \pi]$.

In summary, if we assume that the kernel system $F$ is a lowpass filter with the cutoff frequency $\omega_c = \min(\pi/p, \pi/q)$, the rate changer will map the input spectrum in $[0\ ,\ \min(\pi,\ \frac{p}{q}\pi)]$ to cover the range $[0\ \ \min(\pi,\ \frac{q}{p}\pi)]$ of the spectrum of the output. Therefore, the filter $F$ serves to eliminate the unwanted effects due to aliasing or imaging, and the rate changer retains the shape of the spectrum in the frequency band of interest.

## 3.2 Transfer Matrices of the Blocked Systems

Consider the system shown in Figure 3.1, where $F$ is an LTI filter. For this system, a delay of $q$ samples in the input sequence results in a delay of $pq$ samples in the input to the filter; because the filter is time-invariant, its output is delayed by $pq$ samples, and consequently the output of the overall system is delayed by $p$ samples. Thus by defining the blocking of a signal $x$ as

$$\underline{x}(n) = [x(np)x(np+1)\cdots x(np+p-1)]^T,$$

we see that if we block the input sequence into blocks of length $q$ and the output sequence into blocks of length $p$, we will get a $p$ by $q$ LTI system.

The transfer matrix of the blocked system is obtained as follows [6]: A block of $q$ samples in the input corresponds to a block of $pq$ samples in the input of the filter. In the frequency domain, the relation between the input and output of the filter is given by

$$\underline{Y} = \hat{F}\underline{X},$$

where $\underline{X}$ is the $pq$ blocked input, $\underline{Y}$ is the $pq$ blocked output and $\hat{F}$ is the $pq$ by $pq$ blocked transfer matrix of the filter $F$. The matrix $\hat{F}$ is pseudo-circulant, and its components are related to the polyphase components $\hat{F}_i$ of $F$ in

$$F(z) = \sum_{i=0}^{pq-1} z^{-i} \hat{F}_i(z^{pq}),$$

as:

$$\hat{F} = \begin{bmatrix} \hat{F}_0 & z^{-1}\hat{F}_{pq-1} & \cdots & z^{-1}\hat{F}_1 \\ \hat{F}_1 & \hat{F}_0 & \cdots & z^{-1}\hat{F}_2 \\ \vdots & \vdots & & \vdots \\ \hat{F}_{pq-1} & \hat{F}_{pq-2} & \cdots & \hat{F}_0 \end{bmatrix}.$$

Because of the upsampler, all of the elements of $\underline{X}$ are zero except the 0, $p$, ..., $p(q-1)$ elements. Furthermore, because of the downsampler, only the elements with indices that are a factor of $q$ appear in the output, so if we take the $pq$ polyphase matrix of the filter, and set the inputs at sampling instances that are not a multiple of $p$ to zero, then the output of the filter at sampling times $0, q, ..., (p-1)q, \cdots$, appears in the output of the overall system at the sampling instances $0, 1, ..., p-1, \cdots$, respectively. Thus, the effect of the upsampler is equivalent to eliminating all of the columns of $\hat{F}$ matrix except the 0, $p$, ..., $p(q-1)$ columns (we assume that the numbering starts from 0), and the effect of the downsampler is equivalent to eliminating all of the rows except 0, $q$, ..., $p(q-1)$ rows. In other words, the $(k,l)$ element of the transfer matrix of the blocked system $\underline{F}$ is the $(kq, lp)$ element of the blocked transfer matrix of $\hat{F}$ or,

$$\underline{F}^{(k,l)} = \hat{F}^{(kq,lp)}, \qquad 0 \le k < p, \quad 0 \le l < q.$$

(The underline indicates a blocked transfer matrix.)

It is easy to show that each of the polyphase components of the filter $F$ appears only once in $\underline{F}$ if and only if $p$ and $q$ are coprime.

In the next chapter we use the results of this section in our analysis and design of nonuniform multirate filter banks. In the case the kernel system is not LTI, we get a system which is more general than the one we discussed in this section. We use these more general multirate systems in our design of nonuniform multirate systems. Because a system with an LPTV kernel has the same input and output sampling rates as an LTI system, one interesting problem would be to study the relationship of such systems with regard to a subclass of systems that have LTI kernels. We will study this issue further in the next section.

## 3.3 Optimal Approximation of General Multirate Systems

In this section, we extend the results that we obtained in Section 2.3 to multirate systems. Take the system shown in Figure 3.1. Let $F$ be an LTI filter. For this system the input $x$, and the output $y$ are related by

$$y(l) = \sum f(lq - kp)\, x(k).$$

From the above relationship we see that if the input sequence is delayed by $q$ samples, the output will be delayed by $p$ samples. Therefore, if we denote the system by $\mathcal{G}$ and the unit-delay operator by $z^{-1}$, we have

$$z^{-p}\mathcal{G} = \mathcal{G}z^{-q}. \tag{3.1}$$

In other words, a delay of $q$ samples in the input sequence results in a delay of $p$ samples in the output sequence. Thus, in a real-time implementation, the output sampling rate is $p/q$ times the input sampling rate. We will refer to this system as a $(p, q)$-periodic system.

We may compare two systems when they have the same input and output sampling rates, which means that they should have the same ratio $p/q$. When $p$ and $q$ are coprime a rate changer representation with an LTI kernel is the most general form. A $(p, q)$-periodic system is more interesting from an application viewpoint, if the integers $p$ and $q$ are not coprime [9]. In the general case, we may represent a $(p, q)$-periodic system similar to an LPTV system, i.e.,

$$y(l) = \sum g(l, k)x(k).$$

Because of the periodic condition on the system, the impulse response of the system has the property

$$g(k + p, l + q) = g(k, l), \quad \forall k, l.$$

Let $m$ be the greatest common divisor of $p$ and $q$, and write $p = \bar{p}m$ and $q = \bar{q}m$. We can represent a $(p, q)$-periodic system by the system shown in Figure 3.3, which involves the upsampler $\uparrow \bar{p}$, the downsampler $\downarrow \bar{q}$, $m$ LTI systems $F_0, F_1, \cdots, F_{m-1}$, and a periodic switch which connects each channel for $\bar{p}\bar{q}$ samples starting from time $k = 0$ and system $F_0$ [9].

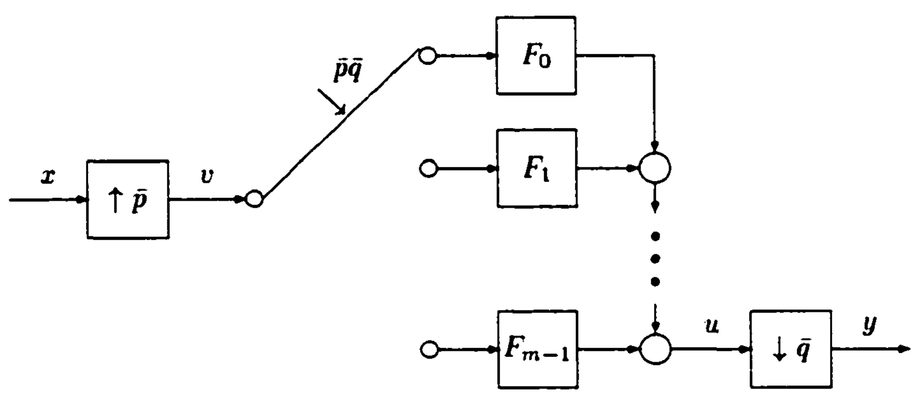


Figure 3.3: The structure of a $(p, q)$-periodic system.

The structure in Figure 3.3 is written

$$\mathcal{G} = \{F_0, F_1, \cdots, F_{m-1}\}^{\bar{p}, \bar{q}}. \tag{3.2}$$

For a linear $(p,q)$-periodic system, the output of the system is known for any input, if the response of the system is known to unit impulses at time 0 through $q-1$. In order to define the $\mathcal{H}_2$ norm of linear $(p,q)$-periodic systems, we apply $q$ unit impulses at time 0 through $q-1$. The $\mathcal{H}_2$ norm is the square root of the average of the squares of the 2-norms of the outputs to these inputs. With an argument similar to that in Lemma 2 and Lemma 3, the $\mathcal{H}_2$ norm of $\mathcal{G}$ can be related to the LTI blocks in its representation in Figure 4.5:

$$\|\mathcal{G}\|_2^2 = \frac{1}{q}\sum_{i=0}^{m-1}\|F_i\|_2^2.$$

When $p$ and $q$ are coprime ($m=1$), the representation has a simple form; otherwise for an integer $1 \le \tilde{m} < m$, we can pose the following approximation problem: Given a $(p,q)$-periodic system as in (3.2), find its $\mathcal{H}_2$-optimal approximant $\tilde{\mathcal{G}}$ which is $(\tilde{m}\bar{p},\tilde{m}\bar{q})$-periodic. Note that such $\tilde{\mathcal{G}}$ leaves the sampling conversion ratio unchanged.

The more interesting case is when $\tilde{m}$ is a divisor of $m$; in this case we write $m = n\tilde{m}$; such a system $\tilde{\mathcal{G}}$ can be represented by

$$\tilde{\mathcal{G}} = \{\tilde{F}_0, \tilde{F}_1, \cdots, \tilde{F}_{\tilde{m}-1}\}^{\tilde{p},\tilde{q}}.$$

It is clear that this system is also $(p,q)$-periodic and hence can be also represented by

$$\tilde{\mathcal{G}} = \{\tilde{F}_0, \tilde{F}_1, \cdots, \tilde{F}_{\tilde{m}-1}, \tilde{F}_0, \tilde{F}_1, \cdots, \tilde{F}_{\tilde{m}-1}, \cdots, \tilde{F}_0, \tilde{F}_1, \cdots, \tilde{F}_{\tilde{m}-1}\}^{\bar{p},\bar{q}},$$

where the total number of the LTI blocks is $m$. Now the error system $\mathcal{G}' = \mathcal{G} - \tilde{\mathcal{G}}$ is $(p,q)$-periodic with $\mathcal{G}' = \{F_0', F_1', \cdots, F_{m-1}'\}^{\bar{p},\bar{q}}$; the LTI blocks are given by

$$F_i' = F_i - \tilde{F}_{i \bmod \tilde{m}}, \quad i = 0, 1, \cdots, m-1.$$

So the optimal approximation problem is to select LTI systems $\tilde{F}_0', \tilde{F}_1', \cdots, \tilde{F}_{\tilde{m}-1}'$ to minimize

$$\|\mathcal{G}'\|_2^2 \;=\; \frac{1}{q}\sum_{i=0}^{p-1}\|F_i - \tilde{F}_{i \bmod \tilde{m}}\|_2^2$$

$$= \frac{1}{q}\sum_{r=0}^{\tilde{m}-1}\sum_{l=0}^{n-1}\|F_{r+l\tilde{m}} - \tilde{F}_r\|_2^2.$$

Thus it follows that the optimal approximant $\tilde{G}$ has the following LTI blocks

$$\tilde{F}_r^{opt} = \frac{1}{n}\sum_{l=0}^{n-1} F_{r+l\tilde{p}}, \quad r = 0, 1, \cdots, \tilde{m} - 1.$$

## 3.4 Summary

In this section we studied some issues related to rate changers. We reviewed the response of rate changers in the frequency domain, and discussed their behavior when the kernel filter in a lowpass filter. Then we presented the transfer matrix of a blocked rate changer. At the end of the chapter, we studied general periodic systems and the approximation of them by simpler multirate systems. In the next chapters we will use rate changers in the design of nonuniform filter banks and transmultiplexers.

# Chapter 4

# $\mathcal{H}_2$ Optimal Design of Nonuniform Filter Banks

In the previous chapters, we studied some issues related to periodic systems, rate changers, and general periodic systems. In this chapter, we will use some of the materials presented in the previous chapters to design nonuniform filter banks using the model-matching method. By some examples, we show that good filter banks can be designed for cases where even alias cancellation is not possible.

This chapter is organized as follows. In section 4.1, we will introduce the nonuniform filter banks. Sections 4.2 and 4.3 describe the solution method. In Section 4.2, the transfer matrix of the blocked system is discussed and in Section 4.3, the design problem is transformed to the least squares solution of a system of linear equations. Section 4.4 provides some design examples. Section 4.5 concludes the chapter.

## 4.1 Introduction

A nonuniform multirate filter bank is shown in Figure 4.1. If the ensemble average of the energy of the signal varies greatly in different frequency bands, a high coding gain can be achieved. In an $m$-channel maximally decimated uniform filter bank all channels have the same downsampling factor $q_i = m$ and upsampling factor $p_i = 1$. The design techniques for these filter banks are well developed, and it is generally possible to find filters so that the exact replica of the input is reconstructed in the output [34]. In a maximally decimated uniform filter bank, because of the Nyquist sampling theorem, the bandwidth of the filters should be equal to $\frac{\pi}{m}$. In a nonuniform filter bank, the sampling rates are not the same in all channels, and thus filters in different channels may have different bandwidths.

The nonuniform filter bank splits the signal into frequency bands:

$$\begin{aligned} &\left[0, \tfrac{p_0}{q_0}\pi\right], \\ &\left[\textstyle\sum_{k=0}^{i-1} \tfrac{p_k}{q_k}\pi \,,\, \sum_{k=0}^{i} \tfrac{p_k}{q_k}\pi\right] \quad i = 1, \cdots, m-1. \end{aligned} \tag{4.1}$$

The higher bound for the last channel should be equal to $\pi$, i.e.,

$$\sum_{k=0}^{m-1} \frac{p_k}{q_k} = 1.$$

For a fixed number of channels, the extra degree of freedom results in a situation where

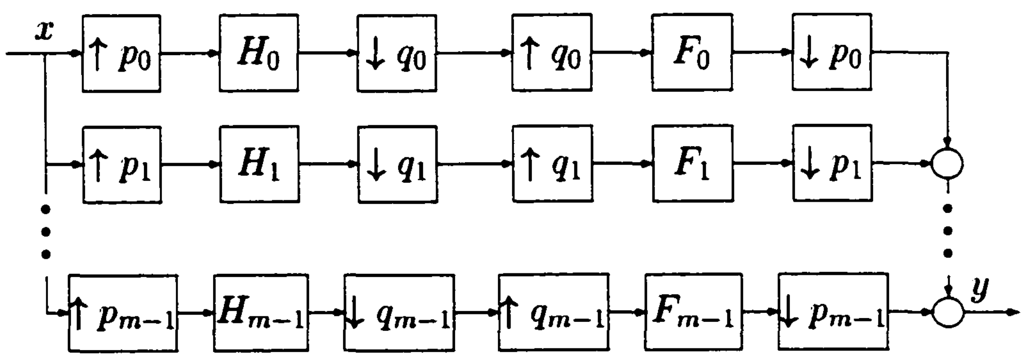


Figure 4.1: A general nonuniform filter bank.

we can select the bandwidth of filters so that the variance of the energy of the signals in different channels is further increased; this in turn increases the coding gain. In some cases, the problem of the design of a nonuniform filter bank may either be converted to the problem of the design of a uniform filter bank [18] or solved by a method developed for uniform filter banks [6, 9], but generally structural constraints may arise [9, 12] which make the design problem different from that of a uniform filter bank [7, 13, 14, 34].

Figure 4.2 shows a 3-channel nonuniform maximally decimated filter bank. As mentioned in [12], this filter bank can be converted to a uniform 6-channel filter bank as shown in Figure 4.3, where the first two and the last three analysis and synthesis filters are related to each other. Therefore, the design problem of this nonuniform filter bank is reduced to that of a uniform filter bank, where some filters are related to each other.

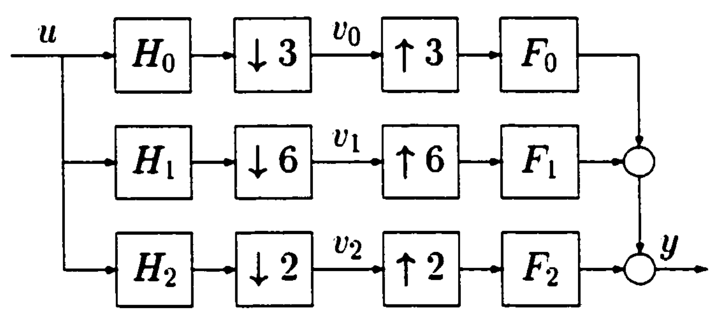


Figure 4.2: A 3-channel nonuniform filter bank.

By substituting filters $F_0$ and $H_0$ by 2-periodic systems and filters $F_2$ and $H_2$ by 3-periodic systems, we are able to eliminate the interdependence between the filters in Figure 4.3. For example, consider the first channel of the nonuniform filter bank shown in Figure 4.2 and assume that $H_0$ is an LTI block, but $F_0$ is a 2-periodic block. The combination of downsampler and upsampler by a factor of 3 can be represented by the subsystem shown in Figure 4.4, and the 2-periodic subsystem $F_0$ can be represented by a subsystem consisting of a switch and two LTI blocks, as shown in Figure 4.5. The switch is periodic and connects to $\Phi_{00}$ initially, to $\Phi_{01}$ at the next sampling time, back to $\Phi_{00}$ and repeats.

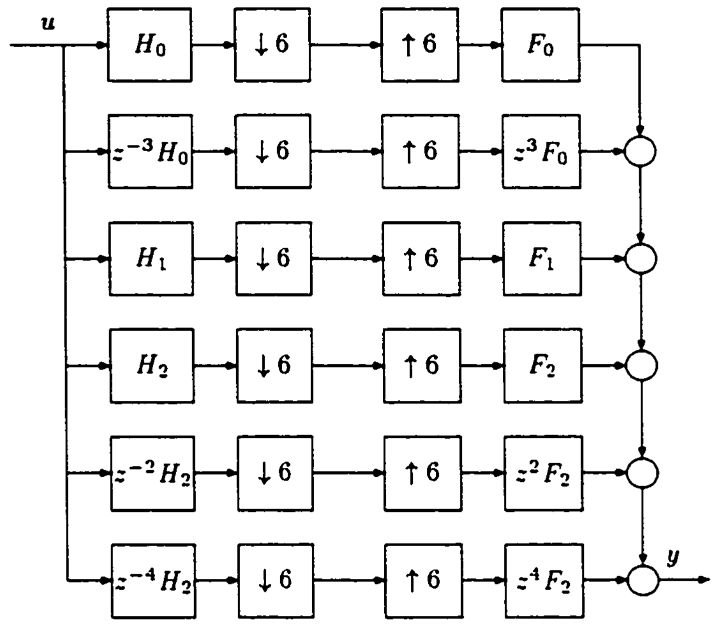


Figure 4.3: The equivalent uniform filter bank.

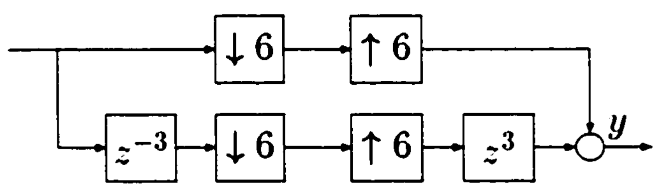


Figure 4.4: An equivalent representation of a downsampler followed by an upsampler by a factor of 3.

The output is formed by adding the outputs of $\Phi_{00}$ and $\Phi_{01}$. Now, if we cascade these two equivalent subsystems, because of the presence of the periodic switch, the output of the first channel in the equivalent representation of Figure 4.4 does not affect the second block $\Phi_{01}$ and the output of the second channel does not affect the first block $\Phi_{00}$. Therefore, we can eliminate the summation and the switch. By cascading $H_0$ and the input block of Figure 4.4, we obtain a representation similar to the first two channels in Figure 4.3, but the $F_0$ block is substituted by $\Phi_{00}$ and the $z^3F_0$ block is substituted by $z^3\Phi_{01}$. Since $\Phi_{00}$

and $\Phi_{01}$ are not interrelated, the interdependence between the synthesis filters in the first two channels of the equivalent representation is eliminated.

Similarly, by substituting $H_0$ by a 2-periodic system, we can show that the interdependence between the analysis filters can be eliminated. However, in this case we should use another equivalent representation of 2-periodic systems which consists of two LTI blocks with common inputs and a periodic switch in the output as discussed in Section 2.2. The constraints for other channels can be eliminated likewise.

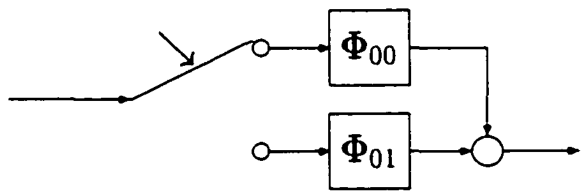


Figure 4.5: The equivalent representation of a 2-periodic system.

In order to deal with the constraints in a general setting, we will formulate the design problem as an $\mathcal{H}_2$ optimization problem: We attempt to minimize the $\mathcal{H}_2$-norm of the error system $T$ formed by subtracting the output of a pure delay transfer function from the output of the filter bank. This results in a filter bank that has an impulse response close to the desired impulse response, i.e., a pure delay. This is a nonlinear optimization problem. When either analysis or synthesis filters are fixed, the solution can be obtained by the least squares solution of a linear system of equations. In order to design the filter bank, we use an iterative solution: The initial analysis filters are designed according to the frequency characteristics of the input. At each iteration, the analysis filters are first fixed and the synthesis filters are found so that the $\mathcal{H}_2$-norm of the error system is minimized over all synthesis filters of a prespecified order. Then, similarly, the synthesis filters obtained in the previous step are fixed and the analysis filters are found so that the $\mathcal{H}_2$-norm of the error system is minimized over all analysis filters of a prespecified order. By repeating the above procedure the $\mathcal{H}_2$-norm decreases until it reaches its final value. If the frequency characteristics of the filters are not satisfactory, instead of the $\mathcal{H}_2$-norm of the error system, we may consider an objective function that takes into account the stopband attenuation of the filters and the $\mathcal{H}_2$-norm of the error system.

In the rest of the chapter, we use the results of Section 3.2, and find the the transfer matrix of a blocked nonuniform filter bank. Then, we will formulate the model-matching design of filter banks, and give some design examples.

## 4.2 Transfer Matrices of Blocked Multirate Filter Banks

The analysis and synthesis parts of a nonuniform filter bank usually consist of the subsystem shown in Figure 4.6, where $G$ is an LTI filter. It is possible to use more general blocks as discussed in Section 3.3, and [24, 9, 15], but when $p$ and $q$ are coprime this building block is quite general [6], and in this section we will restrict our study to filter banks using this basic building block.

As discussed in Section 3.2, if we block the input sequence into blocks of length $q$ and the output sequence into blocks of length $p$, we will get a $p$ by $q$ LTI system. We denote the $p$ by $q$ transfer matrix of this blocked system $\underline{G}$. It is easy to show that each of the

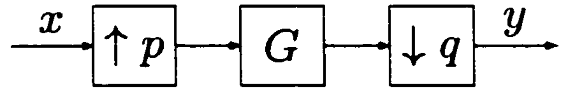


Figure 4.6: A sample-rate changer.

polyphase components of the filter $G$ appears only once in $\underline{G}$ if and only if $p$ and $q$ are coprime.

As is shown in Figure 4.1, the $i$-th channel of a nonuniform filter bank consists of a $(p_i, q_i)$ periodic subsystem cascaded with a $(q_i, p_i)$ periodic subsystem. A delay of $q_i$ samples in the input sequence results in a delay of $p_i$ samples in the output of the first subsystem, which in turn results in a delay of $q_i$ samples in the output of the channel. Therefore, the $i$-th channel is periodic with period $q_i$, and by postmultiplying the transfer matrix of the analysis part $\underline{H_i}$ by the transfer matrix of the synthesis subsystem $\underline{F_i}$, we can obtain the transfer matrix for the $q_i$ blocked representation of the cascaded subsystems. This matrix is $q_i$ by $q_i$. The output of the filter bank is obtained by adding the outputs of the channels. The filter bank is periodic with period $q = \text{lcm}(q_0, q_1, ..., q_{m-1})$ (lcm is the least common multiple). Therefore, if we block $q$ inputs and outputs of the filter bank, an LTI system will result, and the transfer matrix for the blocked system can be obtained using $q$ blocks in the input instead of $q_i$ blocks in the input of each channel. This is equivalent to eliminating all of the columns and rows that are not a multiple of $p_i$ and $q_i$, respectively, of the $qp_i$-fold polyphase matrix of analysis filters $H_i$. The polyphase representation of $F_i$ may be obtained similarly. By postmultiplying $\underline{H_i}$ by $\underline{F_i}$ the $q$ blocked transfer matrix of the channels is obtained. Because the outputs of all channels are added, the blocked transfer matrix of the filter bank $\underline{T}$ can be found by adding the transfer matrix of the channels, i.e.,

$$\underline{T} = \sum_{i=0}^{m-1} \underline{F}_i \underline{H}_i = \underline{F}\,\underline{H}, \tag{4.2}$$

where

$$\underline{H} = \left[ \underline{H}_0^* \;\; \underline{H}_1^* \;\; \cdots \;\; \underline{H}_{m-1}^* \right]^*,$$

$$\underline{F} = \begin{bmatrix} \underline{F}_0 & \underline{F}_1 & \cdots & \underline{F}_{m-1} \end{bmatrix}.$$

Here superscript $*$ represents the conjugate transpose. Depending on the upsampling and downsampling factors in the filter bank, the elements of the matrices $\underline{F}$ and $\underline{H}$ may or may not be interdependent. When all of the $q_i$ are not equal, some of the polyphase components appear at two different entries of $\underline{H}$ and $\underline{F}$, which means that some of the elements of these matrices are interdependent. The design method should be capable of handling these structural dependencies.

## 4.3 $\mathcal{H}_2$ Optimization

The use of $\mathcal{H}_2$ optimization in the time domain was discussed in [25]. In this section, we discuss the optimization in the frequency domain. As was mentioned before, we take all of the filters to be FIR. We will first show that when the analysis filters are fixed, the synthesis filters that minimize the $\mathcal{H}_2$-norm of the error system are found by the least squares solution of a linear system of equations. Then we will point out the dual problem of finding the analysis filters when the synthesis filters are fixed. In the last part, we will use the first two cases and discuss an iterative procedure for designing filter banks.

In the $\mathcal{H}_2$ optimization, the coefficients are found so that the impulse response of the filter bank is close to the delayed version of the input sequence, i.e., we would like to have a filter bank that is close to the LTI system:

$$T_d(z) = z^{-\bar{k}}.$$

Because the filter bank is a periodically time-varying system with period $q$, the impulse responses of the filter bank for impulses at time 0 through time $q-1$ are compared with the respective desired outputs. Thus, the optimal filter bank minimizes

$$J_e = \left[\sum_{l=0}^{q-1} ||y_l - T_d u_l||_2^2\right]^{1/2} = \left[\sum_{l=0}^{q-1} ||y_l - z^{-(\bar{k}+l)}||_2^2\right]^{1/2},$$

where $u_l(z) = z^{-l}$ and $y_l$ are the inputs and the outputs, respectively. $J_e$ is equal to the $\mathcal{H}_2$ norm of the error system $\underline{R} = \underline{F}\,\underline{H} - \underline{T_d}$, where, as discussed in the previous section, $\underline{F}$ is the blocked transfer matrix of the synthesis filter bank, $\underline{H}$ is the blocked transfer matrix of the analysis filter bank, and $\underline{T_d}$ is obtained by blocking $T_d(z) = z^{-\bar{k}}$: Writing $\bar{k} = \bar{l}q + r$ with $0 \leq r < q$, we have

$$\underline{T_d}(z) = z^{-\bar{l}} \begin{bmatrix} 0 & z^{-1} I_r \\ I_{q-r} & 0 \end{bmatrix},$$

where $I_r$ is the $r \times r$ identity matrix.

In the frequency domain the $\mathcal{H}_2$-norm of the error system is given by:

$$||\underline{R}||_2 = \left\{\frac{1}{2\pi}\int_0^{2\pi} \text{trace}\left[\underline{R}(e^{j\omega})^* \underline{R}(e^{j\omega})\right] d\omega\right\}^{1/2},$$

where the trace of a square matrix is the sum of the diagonal elements. By expanding the trace, we can relate the $\mathcal{H}_2$-norm of the error system to the $\mathcal{H}_2$-norm of the elements of its blocked transfer matrix $\underline{R}$ as:

$$J_e = ||\underline{R}||_2 = \left\{\sum_{k=0}^{q-1}\sum_{l=0}^{q-1} ||\underline{R}^{(k,l)}||_2^2\right\}^{1/2}. \tag{4.3}$$

We assume that the filter bank is FIR, thus the elements of $\underline{R}$ are also FIR and can be expressed as

$$\underline{R}^{(k,l)} = \sum_{n=0}^{n_{k,l}} R_{k,l,n} z^{-n},$$

where $n_{k,l}$ is the highest degree of the polynomial of the $(k,l)$ component of $\underline{R}$. The $\mathcal{H}_2$-norm of the elements of $\underline{R}$ is related to the coefficients of the polynomial as:

$$||\underline{R}^{(k,l)}||_2^2 = \sum_{n=0}^{n_{k,l}} R_{k,l,n}^2.$$

Substituting in (4.3), we have

$$J_e = ||R||_2 = \left\{ \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} \sum_{n=0}^{n_{k,l}} R_{k,l,n}^2 \right\}^{1/2} . \tag{4.4}$$

As can be seen in (4.4), $R_{k,l,n}$ is the coefficient of $z^{-n}$ in the $(k,l)$ element of the blocked transfer matrix of the error system $\underline{R}$. These coefficients are the coefficients of polynomials which result by deducting the $(k,l)$ element of $\underline{T_d}$ from the multiplication of the $k$-th row of $\underline{F}$ by the $l$-th column of $\underline{H}$. Since the coefficients of the analysis filters are multiplied by the coefficients of the synthesis filters in (4.4), minimization of this expression is a nonlinear optimization problem. In order to solve this problem, we use an iterative approach. We start with a set of frequency-selective analysis filters $H_i^{(0)}$ having prespecified orders $\bar{n}_i$, and find the $\mathcal{H}_2$ optimal synthesis filters $F_i^{(1)}$ over all FIR filters $F_i$ having orders $n_i$. Since the coefficients of the analysis filter bank are fixed, the expression in (4.4) becomes a quadratic expression and can be found as the least squares solution of a set of linear equations that yield $R_{k,l,n}$ for $0 \leq k < q$, $0 \leq l < q$, $0 \leq n < n_{k,l}$.

Assume that the filters $F_i$ have the following representation

$$F_i(e^{-j\omega}) = \sum_{l=0}^{n_i} f_{il} e^{-j\omega l}.$$

By defining a vector whose elements consist of the coefficients of the filter $F_i$ as,

$$\phi_i = [f_{i0} \; f_{i1} \; \cdots \; f_{in_i}]^*,$$

and defining $\phi$ as the vector of coefficients of the synthesis filter bank

$$\phi = [\phi_0^* \; \phi_1^* \; \cdots \; \phi_{m-1}^*]^*,$$

we have

$$J_e^2 = (A\phi - C)^*(A\phi - C).$$

The matrix $A$ and vector $C$ depend on the delay of the filter bank and the coefficients of the analysis filter bank. Thus the least squares solution of the systems of equations

$$A\phi = C$$

will result in the coefficients of the synthesis filters that minimize the $\mathcal{H}_2$-norm of the error system for the given set of the analysis filters. Since the coefficients of different synthesis filters do not appear in the same equation, the system of linear equations can be decomposed into $m$ sets of linear equations:

$$A_i\phi_i = C_i \quad 0 \leq i < m$$

where $A_i$ and $C_i$ are submatrices of $A$ and $C$, respectively. Thus the objective function reduces to

$$J_e^2 = \sum_{i=0}^{m-1} (A_i\phi_i - C_i)^*(A_i\phi_i - C_i).$$

Note that the structural dependencies can be handled without any difficulty.

The $\mathcal{H}_2$-norm of the error system is related to the aliasing distortion and magnitude distortion of the filter bank. Defining the alias distortion $AD$, magnitude distortion $MD$, and phase distortion $PD$, as:

$$\begin{aligned} AD^2 &= \frac{1}{2\pi}\sum_{l=1}^{q}\int_0^{2\pi} |S_l(e^{j\omega})|^2 d\omega \\ MD^2 &= \frac{1}{2\pi}\int_0^{2\pi} (|S_0(e^{j\omega})| - 1)^2 d\omega \\ PD^2 &= \max_{\omega} |\angle S_0(j\omega) + \bar{k}\omega|, \end{aligned} \tag{4.5}$$

where for $l = 1, 2, \cdots, q-1$, $S_l$ are the alias components and $S_0$ is the linear component of the filter bank [34, 8], we can show that:

$$MD^2 + AD^2 < J_e^2/q.$$

Therefore, these distortions are bounded from the above by a factor $1/\sqrt{q}$ of the $\mathcal{H}_2$-norm of the error system.

The frequency selectivity of filters $F_0$ through $F_{m-1}$ can be taken into account by adding penalty terms for the stopband ripples of the filters to the objective function. Assume that the filters $F_i$ have the passband $[\omega_{2i}, \omega_{2i+1}]$. We define the following objective function

$$J^2 = J_e^2 + J_\phi^2,$$

where

$$J_\phi^2 = \sum_{i=0}^{m-1} \alpha_i [\int_0^{\omega_{2i}} F_i^*(e^{j\omega}) F_i(e^{j\omega}) d\omega + \int_{\omega_{2i+1}}^{\pi} F_i^*(e^{j\omega}) F_i(e^{j\omega}) d\omega]. \quad (4.6)$$

In a maximally decimated filter bank, if we minimize the norm of the error system subject to the constraint on the stop-bands, the minimization of the norm will cause the passbands to be almost flat in there passbands. Thus, we only need to consider a penalty term for the stopband attenuation of the filters.

By taking

$$\Omega = [1 \ \ e^{-j\omega} \ \cdots \ e^{-n_i j\omega}]^*,$$

we have,

$$F_i(e^{j\omega}) = \Omega^* \phi_i.$$

The penalty function can be simplified as follows:

$$J_\phi^2 = \sum_{i=0}^{m-1} \alpha_i \phi_i^* [\int_0^{\omega_{2i}} \Omega\Omega^* d\omega + \int_{\omega_{2i+1}}^{\pi} \Omega\Omega^* d\omega] \phi_i.$$

The term in the brackets can be substituted by a real positive definite matrix $\Gamma_i$ with the $(k, l)$ element given by:

$$\Gamma_i(k, l) = \begin{cases} \frac{1}{(k-l)}[sin(\omega_{2i}(k-l)) - sin(\omega_{2i+1}(k-l))], & \text{if } l \neq k \\ \pi + \omega_{2i} - \omega_{2i+1}. & \text{otherwise} \end{cases}$$

Thus, the objective function reduces to:

$$J^2 = \sum_{i=0}^{m-1} [(A_i\phi_i - C_i)^*(A_i\phi_i - C_i) + \alpha_i\phi_i^*\Gamma_i\phi_i].$$

To get the coefficients of the optimal filters, we set the derivative of $J^2$ with respect to the vectors $\phi_i$ equal to zero:

$$\frac{\partial J^2}{\partial \phi_i} = 2[(A_i^*A_i + \alpha_i\Gamma_i)\phi_i - A_i^*C_i] = 0.$$

Thus, if $A_i^*A_i + \alpha_i\Gamma_i$ is invertible, we have

$$\phi_i^{opt} = (A_i^*A_i + \alpha_i\Gamma_i)^{-1}A_i^*C_i.$$

If the synthesis filters that minimize the $\mathcal{H}_2$-norm of the error system do not result in satisfactory filters, we may proceed by finding the set of analysis filters that obtain the minimum objective function for the synthesis filters that were found in the previous step. The steps are similar and thus are omitted. By repeating this procedure, we may be able to design acceptable nonuniform filter banks. The algorithm for the iterative design of a nonuniform filter bank is summarized as follows (the number in parentheses in superscript refers to the iteration):

1. Use a conventional method like the Remez exchange algorithm to design the initial analysis filters $H_0^{(0)}$ through $H_{m-1}^{(0)}$ with desired order and frequency characteristics.

2. Find the synthesis filters $F_0^{(n)}$ through $F_{m-1}^{(n)}$ so that the nonuniform filter bank that results by cascading the synthesis filter bank to analysis filters is optimal in the $\mathcal{H}_2$ sense.

3. Find the analysis filters $H_0^{(n)}$ through $H_{m-1}^{(n)}$ so that the filter bank consisting of these filters and the synthesis filters $F_0^{(n)}$ through $F_{m-1}^{(n)}$, obtained in the previous step is optimal in the $\mathcal{H}_2$ sense.

4. Evaluate the objective function $J^{(n)}$. Terminate if $|J^{(n-1)} - J^{(n)}| < \epsilon$ for a prespecified positive value of $\epsilon$; otherwise increment $n$ by 1, and go to Step 2.

The algorithm minimizes an objective function defined by:

$$\bar{J}^2 = J_e^2(\phi, \psi) + J_\phi^2 + J_\psi^2,$$

where $\psi$ is a vector containing the coefficients of the analysis filters $H_i$, $J_\psi$ takes into account the stopband attenuation of analysis filters, and are defined similarly to $\phi$, $J_\phi$ in (4.3) and (4.6), respectively. In the algorithm, Step 2 minimizes $\bar{J}$ when $\psi$ is fixed, and Step 3 minimizes $\bar{J}$ when $\phi$ is fixed. In either case the minimum is found by solving a set of (over determined) linear equations. This optimization problem can also be solved by gradient-based optimization procedures. The gradient methods may or may not yield the global optimum. In every step of the method described here, the minimum has a unique analytic solution, making this method more suitable than gradient-based methods. Because of this, the method converges very fast during the initial iterations; in some cases the initial results might even be satisfactory, without any need for iteration. By this algorithm $\bar{J}$ decreases until it reaches the local minimum. The method may not converge to the global minimum, and it is necessary to start from suitable initial filters; otherwise, even when the $\mathcal{H}_2$-norm of the system is low, the sensitivity of the final filters to the noise might be too high. The initial filters may be obtained by considering the nonuniform perfect-reconstruction filter bank with ideal filters. This point is discussed further in the examples given in the next section.

## 4.4 Examples

A filter bank that divides the input signal into three consecutive bands as shown in Figure 4.2 is considered. It was shown in [12] that this filter bank can not cancel aliasing. As was discussed in [18], in some cases, if we use the same ideal filters for analysis and synthesis, we are not able to obtain a perfect-reconstruction filter bank. For example, if we assume that $H_0$, $H_1$ and $H_2$ have the passbands $[\frac{\pi}{2}\ ,\ \frac{5\pi}{6}]$, $[\frac{5\pi}{6}\ ,\ \pi]$ and $[0\ ,\ \frac{\pi}{2}]$ respectively, then the signals in the output of the analysis filter bank are aliased in such a way that it is not possible to recover them by ideal filters. In those cases, it may be possible to use the method presented here and obtain a reasonably low norm for the error system, but the frequency responses of the synthesis filters will not resemble those of ideal filters, and some of the filters may have high sensitivity to noise.

Now, assume that $H_0$, $H_1$, and $H_2$ split the input signal into the following bands: $[0\ ,\ \frac{\pi}{3}]$, $[\frac{\pi}{3}\ ,\ \frac{\pi}{2}]$, and $[\frac{\pi}{2}\ ,\ \pi]$. Following the procedure mentioned in the previous sections, the transfer matrix of the blocked system is $\underline{T} = \underline{F}\ \underline{H}$ with

$$\underline{H} = \begin{bmatrix} \hat{H}_{00} & z^{-1}\hat{H}_{05} & z^{-1}\hat{H}_{04} & z^{-1}\hat{H}_{03} & z^{-1}\hat{H}_{02} & z^{-1}\hat{H}_{01} \\ \hat{H}_{03} & \hat{H}_{02} & \hat{H}_{01} & \hat{H}_{00} & z^{-1}\hat{H}_{05} & z^{-1}\hat{H}_{04} \\ \hat{H}_{10} & z^{-1}\hat{H}_{15} & z^{-1}\hat{H}_{14} & z^{-1}\hat{H}_{13} & z^{-1}\hat{H}_{12} & z^{-1}\hat{H}_{11} \\ \hat{H}_{20} & z^{-1}\hat{H}_{25} & z^{-1}\hat{H}_{24} & z^{-1}\hat{H}_{23} & z^{-1}\hat{H}_{22} & z^{-1}\hat{H}_{21} \\ \hat{H}_{22} & \hat{H}_{21} & \hat{H}_{20} & z^{-1}\hat{H}_{25} & z^{-1}\hat{H}_{24} & z^{-1}\hat{H}_{23} \\ \hat{H}_{24} & \hat{H}_{23} & \hat{H}_{22} & \hat{H}_{21} & \hat{H}_{20} & z^{-1}\hat{H}_{25} \end{bmatrix},$$

$$\underline{F} = \begin{bmatrix} \hat{F}_{00} & z^{-1}\hat{F}_{03} & \hat{F}_{10} & \hat{F}_{20} & z^{-1}\hat{F}_{24} & z^{-1}\hat{F}_{22} \\ \hat{F}_{01} & z^{-1}\hat{F}_{04} & \hat{F}_{11} & \hat{F}_{21} & z^{-1}\hat{F}_{25} & z^{-1}\hat{F}_{23} \\ \hat{F}_{02} & z^{-1}\hat{F}_{05} & \hat{F}_{12} & \hat{F}_{22} & \hat{F}_{20} & z^{-1}\hat{F}_{24} \\ \hat{F}_{03} & \hat{F}_{00} & \hat{F}_{13} & \hat{F}_{23} & \hat{F}_{21} & z^{-1}\hat{F}_{25} \\ \hat{F}_{04} & \hat{F}_{01} & \hat{F}_{14} & \hat{F}_{24} & \hat{F}_{22} & \hat{F}_{20} \\ \hat{F}_{05} & \hat{F}_{02} & \hat{F}_{15} & \hat{F}_{25} & \hat{F}_{23} & \hat{F}_{21} \end{bmatrix},$$

where $\hat{H}_{ij}$ and $\hat{F}_{ij}$ are the $j$-th components in the 6-fold polyphase decompositions of $H_i$

and $F_i$, respectively. Note that the elements of $\underline{F}$ are interrelated and thus the optimization technique that was discussed in [6] through [9] for the minimization of the $\mathcal{H}_\infty$-norm of the error system is not applicable. The initial analysis filters are designed using the MATLAB function *fir1*. The length of the filters $H_0$, $H_1$ and $H_2$ are 15, 17, and 15, respectively. These analysis filters and the synthesis filters of length 30 are to be designed so that the filter bank approximates a delay of 19 samples. We take $\alpha_i = 0$, thus we do not consider any penalty term for the frequency characteristics of the filters. If we do not iterate and simply obtain the synthesis filters that result in the minimum norm of the error system with the initial analysis filters, the minimum attainable $\mathcal{H}_2$-norm of the error system will be 0.0842. By iterating, the $\mathcal{H}_2$-norm decreases to 0.035 after 200 iterations. Thus, the average of the norm of the error of the impulse response of the filter bank is $0.035/\sqrt{6} = 0.0143$. The $\mathcal{H}_2$-norm of the error system versus the iterations is shown in Figure 4.7.

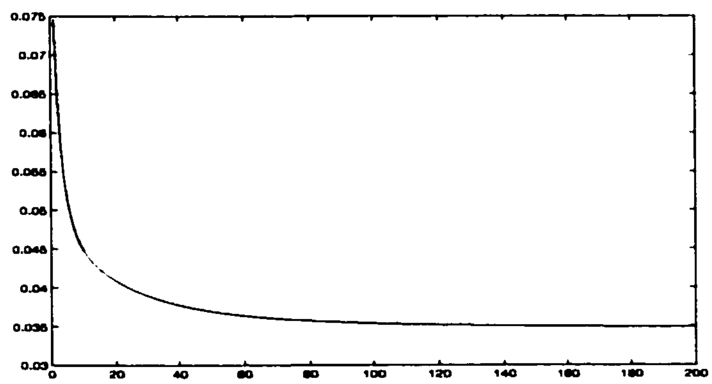


Figure 4.7: The $\mathcal{H}_2$-norm of the error system versus the number of iterations for a 3-channel filter bank with LTI blocks.

The final analysis filters are shown in Figure 4.8. The final synthesis filters are shown in Figure 4.9. It is interesting to note that, although no penalty for the frequency selectivity of filters is imposed, the final filters are frequency selective.

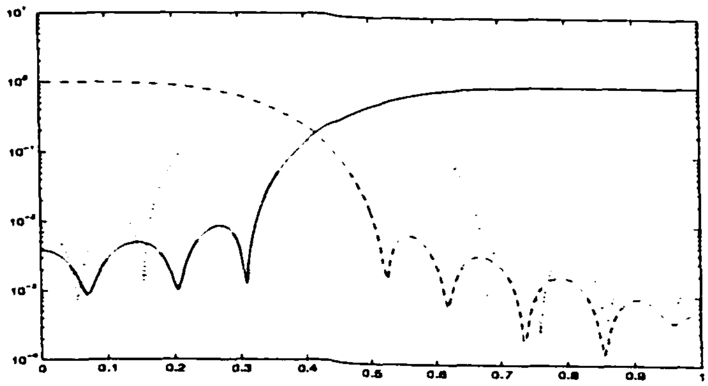

Figure 4.8: Analysis filters $H_0$ (dash), $H_1$ (dot), and $H_2$ (solid): amplitude versus normalized frequency.

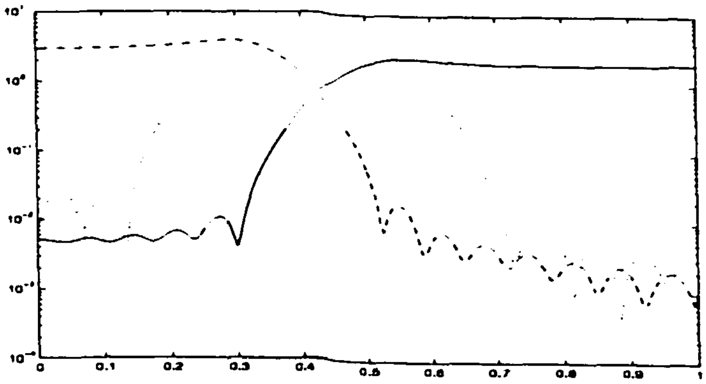

Figure 4.9: Synthesis filters $F_0$ (dash), $F_1$ (dot), $F_2$ (solid): amplitude versus normalized frequency.

A random input signal with a uniform distribution in $[-1, 1]$ was applied to the input of this filter bank. The error, which is the difference between the output signal and the delayed version of the input signal, is shown in Figure 4.10. By using synthesis filters that have higher orders and increasing the delay of the filter bank accordingly, it is possible to decrease the $\mathcal{H}_2$-norm of the error system.

If we allow the synthesis $F_0$ and $F_2$ to be 2-periodic and 3-periodic, respectively, while keeping analysis filters $H_0$ through $H_2$ as LTI blocks, for systems with the same order as in the previous part, we are able to achieve an $\mathcal{H}_2$-norm of 0.0013 for the error system,

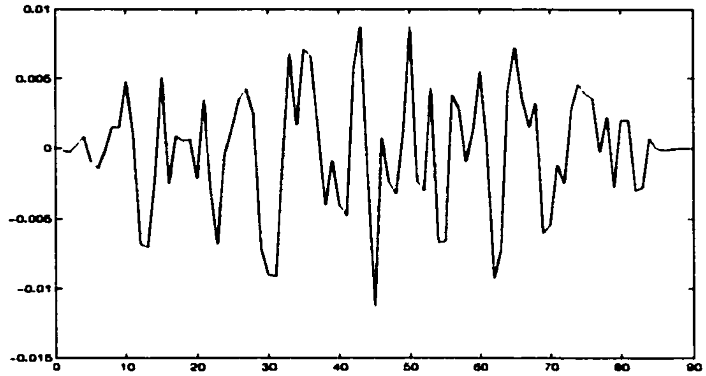


Figure 4.10: The error signal (versus time) of the filter bank with LTI blocks for a random input with a uniform distribution in $[-1, 1]$.

which is a significant improvement. The final analysis filters are shown in Figure 4.11. It is interesting to note that the analysis filters are frequency selective as before. The coefficients of the initial and final analysis filters are given in Table 4.1. There is not a significant change in the coefficients of the filters after iterating, which means that the method converges to the local minimum. Therefore, although the subband signals are similar to those in the previous part, the error in the output is significantly smaller.

The frequency characteristics of the LTI blocks in the switched representation of synthesis blocks, as per Figure 4.5, appear in Figures 4.12- 4.13. These filters have frequency characteristics similar to those of filters $F_0$ through $F_2$ in the filter bank with the LTI synthesis filters. The total of floating point operation count is $2.853 \times 10^9$. Thus, the floating point count per iteration is $14.2 \times 10^6$.

Similarly, a random input signal with a uniform distribution in $[-1, 1]$ was applied to the input of this filter bank. The error of the output of the filter bank is shown in Figure 4.14. Various distortions of the filter banks in the above examples are calculated as per (4.5). The results are summarized in Table 4.2.

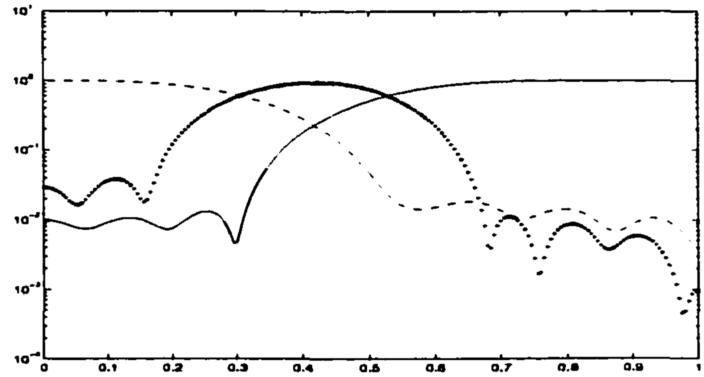

Figure 4.11: Analysis filters, for the filter bank with general synthesis blocks $H_0$ (dash), $H_1$ (dot), and $H_2$ (solid): amplitude versus normalized frequency.

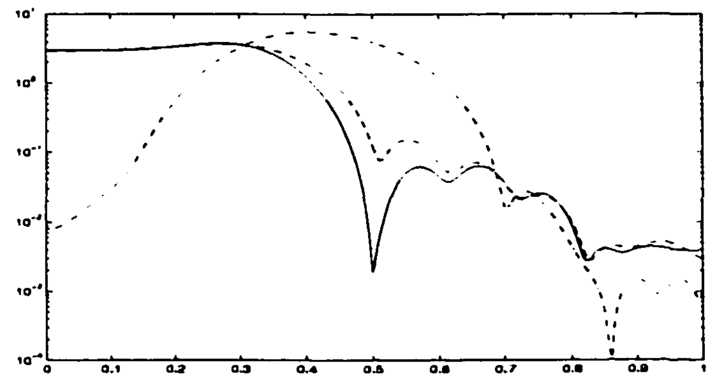

Figure 4.12: Synthesis filters for the filter bank with the general blocks, $\phi_{00}$ (solid), $\phi_{01}$ (dash), $F_1$ ('-.'): amplitude versus normalized frequency.

## 4.5 Summary

In this chapter, we have proposed the $\mathcal{H}_2$ optimal design of nonuniform filter banks. Our method handles the structural dependency in the design procedure and yields a solution that has an impulse response close to a pure delay. At each iteration, the least squares solution of a system of linear equations is found, allowing the final analysis and synthesis filters to be obtained easily. In the next chapter, we shall study the model-matching design of nonuniform filter banks by using the $\mathcal{H}_\infty$ norm of systems.

| $n$ | $h_{0n}^{(0)}$ | $h_{1n}^{(0)}$ | $h_{2n}^{(0)}$ |
|---|---|---|---|
| 0 | 0.0031 | -0.0043 | 0.0037 |
| 1 | 0.0000 | -0.0151 | 0.0000 |
| 2 | -0.0139 | 0.0000 | -0.0162 |
| 3 | -0.0300 | 0.0671 | 0.0000 |
| 4 | 0.0000 | 0.0578 | 0.0684 |
| 5 | 0.1133 | -0.1179 | 0.0000 |
| 6 | 0.2617 | -0.1851 | -0.3049 |
| 7 | 0.3315 | 0.0639 | 0.5019 |
| 8 | 0.2617 | 0.2587 | -0.3049 |
| 9 | 0.1133 | 0.0639 | 0.0000 |
| 10 | 0.0000 | -0.1851 | 0.0684 |
| 11 | -0.0300 | -0.1179 | 0.0000 |
| 12 | -0.0139 | 0.0578 | -0.0162 |
| 13 | 0.0000 | 0.0671 | 0.0000 |
| 14 | 0.0031 | 0.0000 | 0.0037 |
| 15 | | -0.0151 | |
| 16 | | -0.0043 | |

(a)

| $n$ | $h_{0n}^{(200)}$ | $h_{1n}^{(200)}$ | $h_{2n}^{(200)}$ |
|---|---|---|---|
| 0 | 0.0054 | 0.0060 | -0.0047 |
| 1 | 0.0012 | -0.0070 | -0.0075 |
| 2 | -0.0152 | -0.0057 | -0.0119 |
| 3 | -0.0312 | 0.0623 | 0.0032 |
| 4 | 0.0004 | 0.0613 | 0.0668 |
| 5 | 0.1141 | -0.1155 | -0.0011 |
| 6 | 0.2620 | -0.1871 | -0.3046 |
| 7 | 0.3315 | 0.0622 | 0.5020 |
| 8 | 0.2608 | 0.2594 | -0.3052 |
| 9 | 0.1105 | 0.0662 | -0.0001 |
| 10 | -0.0020 | -0.1838 | 0.0689 |
| 11 | -0.0270 | -0.1203 | 0.0009 |
| 12 | -0.0009 | 0.0493 | -0.0158 |
| 13 | 0.0088 | 0.0614 | -0.0019 |
| 14 | -0.0069 | 0.0041 | 0.0010 |
| 15 | | 0.0051 | |
| 16 | | 0.0119 | |

(b)

Table 4.1: Coefficients of (a) Initial analysis filters, (b) final analysis filters

| | Synthesis filters | Periodic synthesis blocks |
|---|---|---|
| $\mathcal{H}_2$-norm | 0.035 | 0.0013 |
| $AD$ | 0.0132 | 0.00046 |
| $MD$ | 0.0042 | 0.00018 |
| $PD$ (deg.) | 0.4033 | 0.0285 |

Table 4.2: The $\mathcal{H}_2$-norm and various distortions of filter banks in examples

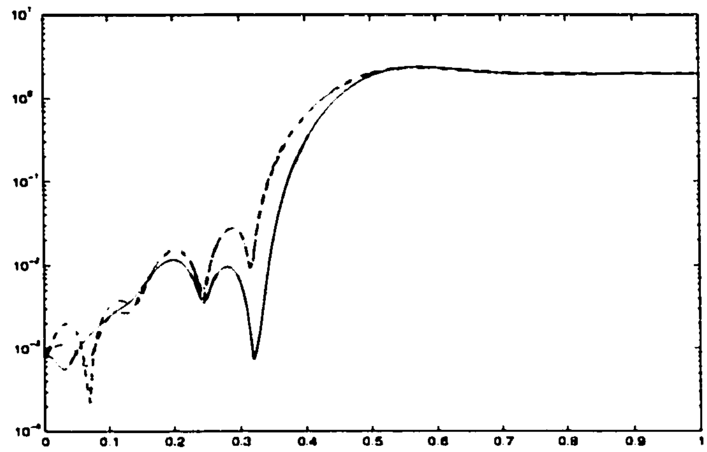


Figure 4.13: Synthesis filters for the filter bank with the general blocks, $\phi_{20}$ (solid), $\phi_{21}$ (dash), $\phi_{22}$ ('-.'): amplitude versus normalized frequency.

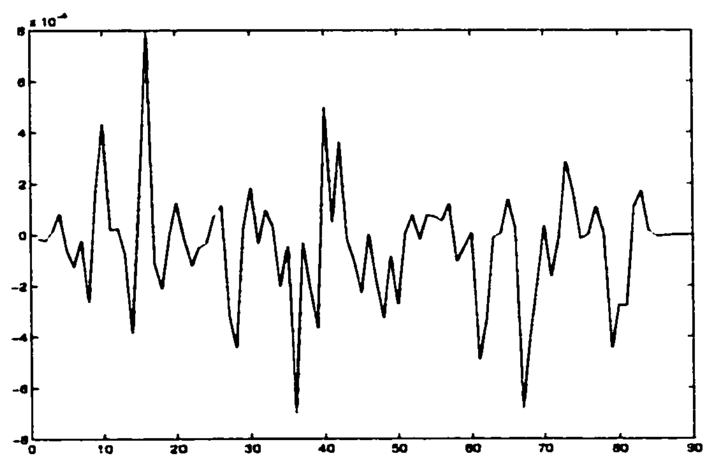


Figure 4.14: The error signal (versus time) of the filter bank with general blocks for a random input with a uniform distribution in $[-1, 1]$.

# Chapter 5

# $\mathcal{H}_\infty$ Design of Nonuniform Filter Banks

In the previous chapter we formulated the $\mathcal{H}_2$ model-matching design of nonuniform filter banks in the frequency domain. In this chapter, we give another formulation. This formulation attempts to minimize the $\mathcal{H}_\infty$ model-matching problem in the time domain.

## 5.1 Introduction

As we mentioned in the previous chapter, it is usually possible to relate a nonuniform filter bank to a uniform filter bank with possibly interrelated filters. Thus, the design of a nonuniform filter bank can be converted to the design of a uniform filter bank subject to some structural constraints. The design process should be capable of handling these constraints. For example, methods presented in [6, 18] are suitable for the cases where no structural constraints are present. In [7], it was shown that filter banks may be designed by model-matching, i.e., by minimizing the $\mathcal{H}_\infty$ norm of an error system, formed by subtracting the output of a pure delay transfer function from the output of the filter bank. By such a design method the analysis filters are designed in advance and IIR synthesis filters are found so that the $\mathcal{H}_\infty$ norm of the error system is minimized, then the IIR filters are approximated by FIR filters. The filters are found by solving two Riccati equations. Because of the approximation, the final filters are suboptimal. Moreover, this method can not accommodate the structural constraints.

In this chapter, we follow an iterative approach. At each iteration, we use semidefinite programming and obtain the FIR synthesis filters for a given set of FIR analysis filters or vise versa. The problem is a convex optimization problem, and since no approximation is involved, at each iteration the solution is optimal, i.e., the FIR synthesis (analysis) filters are optimal for the given analysis (synthesis) filters. Here, we consider the $\mathcal{H}_\infty$ norm as the optimality criterion. Thus the designed filter bank is closest to the desired ideal system in the worst case scenario. As we will see, the constraints will not pose any difficulty in the design process.

The semidefinite programming (SDP) problem is the optimization problem of a linear function subject to the constraint that a matrix be positive definite. In other words, the following problem is a semidefinite programming problem:

$$\begin{array}{ll} \text{minimize} & c^T\alpha, \\ \text{subject to} & G(\alpha) > 0, \end{array}$$

where

$$G(\alpha) = G_0 + \sum_{i=1}^{m} \alpha_i G_i,$$

and the matrices $G_0, \cdots, G_m \in \mathbf{R}^{n\times n}$ are symmetric. Here, for real symmetric matrices $A$ and $B$, $A > B$ whenever $A - B$ is positive definite. The inequality $G(\alpha) > 0$ is called a linear matrix inequality (LMI). The SDP problems are convex optimization problems and can be solved using interior point methods. Thus SDP problems are polynomial time solvable if an *a priori* bound on their solution is known [2, 5].

The chapter is organized as follows: In Section 5.2, we discuss the model-matching formulation for filter banks. In Section 5.3, this problem is then converted to an SDP problem. In Section 5.4, we give an example for the design of a three channel nonuniform filter bank. The example involves periodic blocks in the synthesis filter bank and frequency selective filters as the analysis filters. Finally, Section 5.5 provides some concluding remarks.

## 5.2 Formulation

A nonuniform filter bank as shown in Figure 5.1 is considered. In this section, we will discuss how a model-matching problem for the design of multirate filter banks can be obtained.

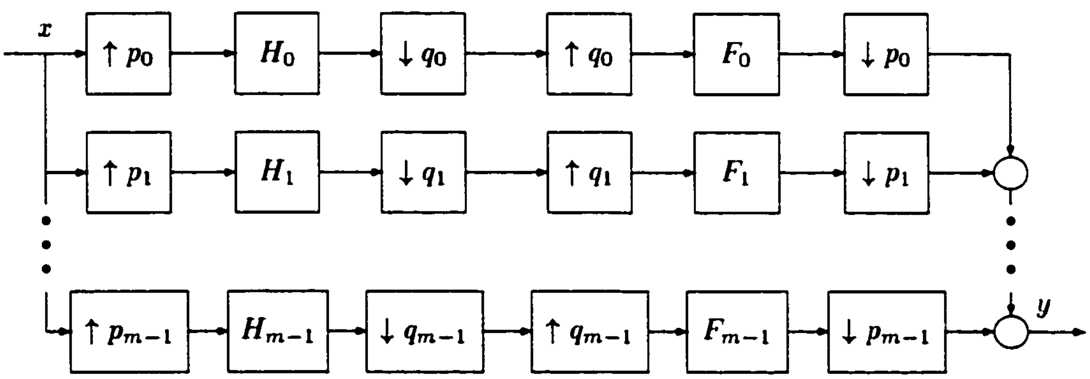


Figure 5.1: A general nonuniform filter bank.

As we discussed in the previous chapter, this nonuniform filter bank is a periodic system with period $q = \text{lcm}(q_0, q_1, \cdots, q_{m-1})$, where $q_i$ are the downsampling factors. By blocking the input and output signals, a multi-input multi-output $q$ by $q$ LTI system results. The transfer matrix of the blocked filter bank $\underline{T}(z)$ is the tandem connection of the transfer matrix of the blocked analysis filter bank $\underline{H}(z)$ and that of synthesis filter bank, $\underline{F}(z)$, i.e.,

$$\underline{T}(z) = \underline{F}(z)\underline{H}(z).$$

We are interested in a system which has a transfer function close to a pure delay. Since blocking is norm-preserving, we can compare the blocked transfer function of a filter bank with that of a pure delay.

The $q$ blocked transfer matrix, $\underline{T_d}(z)$, of the pure delay by $\bar{k} = \bar{l}q + r$ with $0 \leq r < q$, is:

$$\underline{T_d}(z) = z^{-l} \begin{bmatrix} 0 & z^{-1} I_r \\ I_{q-r} & 0 \end{bmatrix}.$$

Assume $\underline{F}$, $\underline{H}$ and $\underline{T_d}$ have the state-space realizations:

$$\underline{F} = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right],$$

$$\underline{H} = \left[\begin{array}{c|c} A_h & B_h \\ \hline C_h & D_h \end{array}\right],$$

$$\underline{T_d} = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right].$$

We use the controllable canonical form and thus the coefficient of filters appear in the matrices $C$ and $D$ above and the structures of the matrices $A$ and $B$ in the realizations of transfer matrices are similar. It is straightforward to show, $\underline{T}(z) = \underline{F}(z)\,\underline{H}(z)$ has the following realization:

$$\underline{T}(z) = \left[\begin{array}{cc|c} A_h & 0 & B_h \\ B_f C_h & A_f & B_f D_h \\ \hline D_f C_h & C_f & D_f D_h \end{array}\right].$$

Since we assume that the filter bank is FIR, all of the $A$ matrices in the realization of $\underline{F}$, $\underline{H}$, $\underline{T_d}$, and $\underline{T}(z)$ have all of their eigenvalues equal to zero.

The $\mathcal{H}_\infty$ norm of the error system is by definition the $\ell_2$ induced norm of $\underline{R} = \underline{F}\,\underline{H} - \underline{T_d}$. The realization of $\underline{R}$ can be obtained with little effort. In order to reduce the realization, note that the (1,2) block of the realization $A$ matrix of $T$ is 0. When the order of $\underline{T_d}$ is less than or equal to that of $\underline{H}$, we have

$$\underline{R}(z) = \left[\begin{array}{cc|c} A_h & 0 & B_h \\ B_f C_h & A_f & B_f D_h \\ \hline D_f C_h - [C_d \ 0] & C_f & D_f D_h - D_d \end{array}\right].$$

A similar relationship can be written for the case that the order of $\underline{T_d}$ is higher than that of $\underline{H}$, but such a realization may not be minimal (this will not pose any problems).

The coefficients of the analysis filters appear in $C_h$ and $D_h$ and those of synthesis filters appear in $C_f$ and $D_f$. In uniform filter banks there are no structural constraints on the elements of $C_f$ and $D_f$, but in nonuniform filter banks some of the coefficients of filters may appear in more than one entry of these matrices. Thus the model-matching problem should be solved subject to these structural constraints.

When the delay and the analysis filters are given and the synthesis filters that minimize the $\mathcal{H}_\infty$ norm of the error system are to be found, $\underline{T_d}(z)$ and $\underline{H}(z)$ are known and the coefficients of the synthesis filters appear only in $\underline{F}(z)$. Therefore, $C_f$ and $D_f$ are variable and the rest of the matrices are fixed. Note that in this case, $A$ and $B$ matrices of the realization of $\underline{R}$ are fixed and only $C$ and $D$ matrices are variable. Furthermore the coefficients of the synthesis filters appear directly in the above matrices. For this case, the design of a multirate filter bank can be cast as the following optimization problem:

> Find matrices $C_f$ and $D_f$ (or equivalently the coefficients of filters $F_0$ through $F_{m-1}$) so as to minimize $\| \underline{R}(z) \|_\infty$.

Since $C_f$ and $D_f$ appear directly in $\underline{R}$, as discussed in the next section, using the bounded real lemma, this problem can be converted into semidefinite programming. Once the problem is converted, it can be solved by interior point methods.

The dual problem of finding the FIR analysis filters that result in the minimum $\mathcal{H}_\infty$ norm of the error system can be solved by minimizing the norm of the transpose of the error system $\underline{R}^T(z) = \underline{H}^T(z)\,\underline{F}^T(z) - \underline{T_d}^T(z)$. This can be done since the $\mathcal{H}_\infty$ norm of $\underline{R}^T(z)$ is equal to that of $\underline{R}$ and the variable coefficients appear in $C$ and $D$ matrices of the realization of $\underline{H}^T$.

Using the above mentioned cases, we are able to design a filter bank iteratively. The algorithm for the iterative design of a nonuniform filter bank is summarized as follows (the number in the parenthesis in superscript refers to the iteration):

1. Use a conventional method like Remez exchange algorithm to design the initial analysis filters $H_0^{(0)}$ through $H_{m-1}^{(0)}$ with desired order and frequency characteristics.

2. Find the synthesis filters $F_0^{(n)}$ through $F_{m-1}^{(n)}$ so that the nonuniform filter bank that results by cascading the synthesis filter bank to analysis filters $H_0^{(n-1)}$ through $H_{m-1}^{(n-1)}$ is optimal in the $\mathcal{H}_\infty$ sense.

3. Find the analysis filters $H_0^{(n)}$ through $H_{m-1}^{(n)}$ so that the filter bank consisting of these filters and the synthesis filters $F_0^{(n)}$ through $F_{m-1}^{(n)}$, obtained in the previous step, is optimal in the $\mathcal{H}_\infty$ sense.

4. Evaluate the objective function $||\underline{R}^{(n)}||_\infty$. Terminate if $||\underline{R}^{(n-1)}||_\infty - ||\underline{R}^{(n)}||_\infty < \epsilon$ for a prespecified positive value of $\epsilon$; otherwise increment $n$ by 1, and go to step 2.

Note that the order of the filters are fixed throughout the design procedure.

## 5.3 Conversion to LMI

In order to perform the minimization, we apply the bounded real lemma [5]. A discrete time system

$$\underline{R}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{5.1}$$

has an $\ell_2$ induced norm less than or equal to $\gamma$ if and only if there exists a positive definite symmetric matrix $P$ so that the following LMI is satisfied:

$$\begin{bmatrix} A^TPA - P + C^TC & C^TD + A^TPB \\ D^TC + B^TPA & D^TD - \gamma^2 I + B^TPB \end{bmatrix} \leq 0. \tag{5.2}$$

Since the products $C^TC, D^TD, C^TD$ and $D^TC$ appear in the above relationship, it is not a standard LMI. However, using Schur complement, it can be transformed to a standard form: Since

$$\begin{bmatrix} A^TPA - P + C^TC & C^TD + A^TPB \\ D^TC + B^TPA & D^TD - \gamma^2 I + B^TPB \end{bmatrix} = \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB - \gamma^2 I \end{bmatrix} - \begin{bmatrix} C^T \\ D^T \end{bmatrix} (-I^{-1}) \begin{bmatrix} C & D \end{bmatrix},$$

using Schur complements [2], (5.2) is satisfied if and only if

$$\begin{bmatrix} A^TPA - P & A^TPB & C^T \\ B^TPA & B^TPB - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix} \leq 0.$$

We are interested in finding FIR filters $F_0$ through $F_{m-1}$ so that the $\ell_2$ induced norm of $\underline{R}$ is minimized. This is equivalent to the following optimization problem: Assuming a state-space model for $\underline{R}(z)$ as given in (5.1)

$$\begin{aligned} &\text{minimize} \quad \gamma \\ &\text{subject to} \quad \begin{bmatrix} A^TPA - P & A^TPB & C^T \\ B^TPA & B^TPB - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix} \leq 0, \\ &\qquad\qquad\quad P > 0. \end{aligned}$$

In steps 2 and 3 of the algorithm, the coefficients of the filters that are to be designed appear directly in the $C$ and $D$ matrices. Thus, by finding the solution of the above problem subject to some constraints on the elements of $C$ and $D$, the coefficients of the filters can be obtained. This is a standard LMI problem and can be solved readily.

## 5.4 Example

A nonuniform filter bank that divides the input signal into three consecutive bands as shown in Figure 5.1 is considered, with $p_0 = p_1 = p_2 = 1$ and $q_0 = 3$, $q_1 = 6$ and $q_2 = 2$. It was shown in [12] that in this case, alias cancellation is not achievable if non-ideal filters are used.

Assume that $H_0$, $H_1$ and $H_2$ split the input signal into the following bands $[0\ ,\ \frac{\pi}{3}]$, $[\frac{\pi}{3}\ ,\ \frac{\pi}{2}]$ and $[\frac{\pi}{2}\ ,\ \pi]$. The initial analysis filters are designed using the MATLAB function *fir1*. The lengths of the filters $H_0$, $H_1$ and $H_2$ are 15, 17 and 15, respectively. The synthesis filters of length 24 and analysis blocks of lengths given above are to be designed so that the filter bank approximates a delay of 18 samples.

We allow the synthesis $F_0$ and $F_2$ to be 2-periodic and 3-periodic, respectively, while keeping synthesis filter $F_1$ and the analysis filters $H_0$ through $H_2$ as LTI blocks (filters). By this choice, in step 3 of the algorithm, the coefficients of the the matrices $C_h$ and $D_h$ are interrelated, and in step 2, the coefficients of synthesis filters are independent [6].

The $\mathcal{H}_\infty$ norm of the error system reduces to 0.0383 after 5 iterations. The final analysis filters are shown in Figure 5.2. The frequency characteristics of the LTI blocks in the switched representation of synthesis blocks, as was shown in Figure 4.5, appear in Figures 5.3- 5.4. Th filters in the LSTV representation of the synthesis filters have parameters

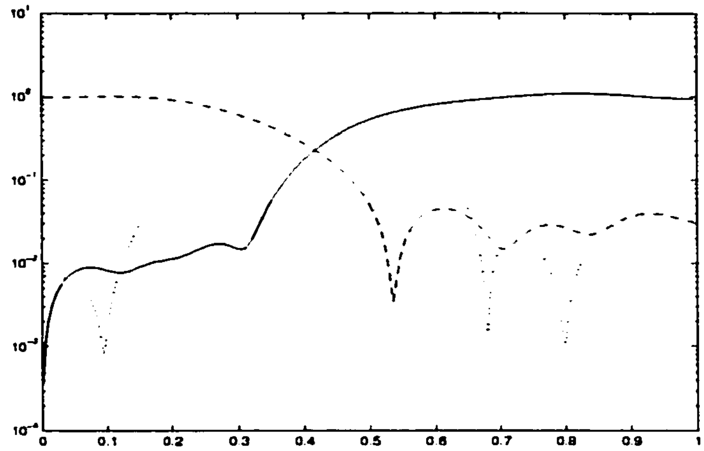


Figure 5.2: Analysis filters $H_0$ (dash), $H_1$ (dot), and $H_2$ (solid): amplitude versus normalized frequency.

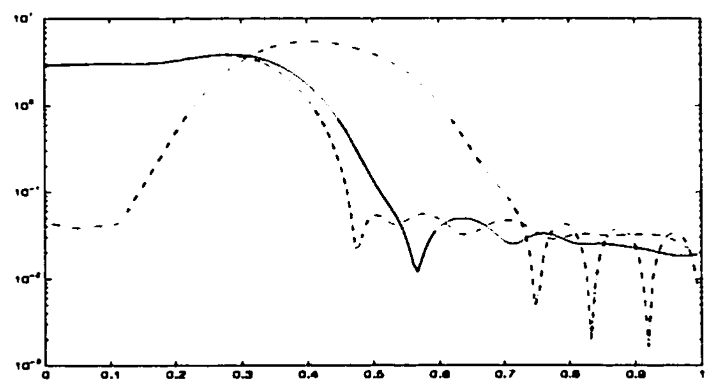

Figure 5.3: Synthesis filters for the filter bank with the general blocks, $\phi_{00}$ (solid), $\phi_{01}$ (dash), $F_1$ ('-.'): amplitude versus normalized frequency.

that are close to each other, which makes their frequency characteristics similar.

A random input signal with a uniform distribution in $[-1, 1]$ is applied to the input of this filter bank. The error which is the difference between the output signal and the delayed version of the input signal is shown in Figure 5.5. This shows that the results are satisfactory.

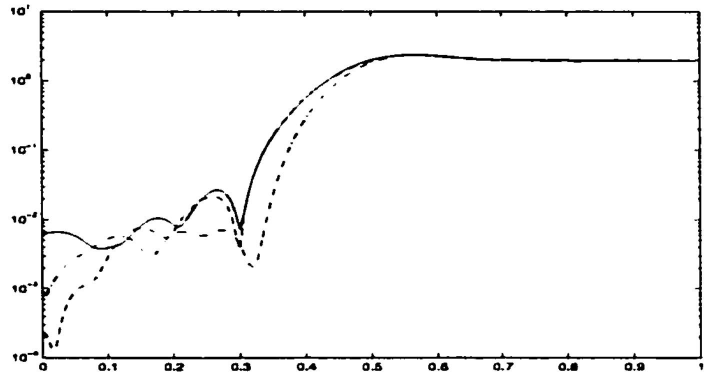

Figure 5.4: Synthesis filters for the filter bank with the general blocks, $\phi_{20}$ (solid), $\phi_{21}$ (dash), $\phi_{22}$ ('-.'): amplitude versus normalized frequency.

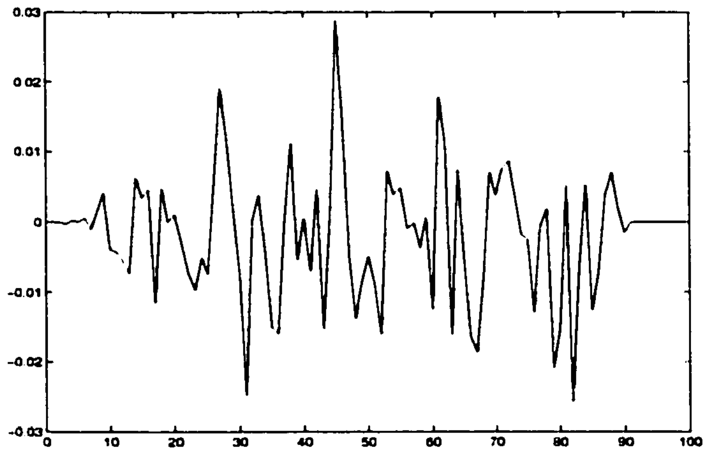

Figure 5.5: The error signal of the filter bank for a random input with a uniform distribution in $[-1, 1]$.

## 5.5 Concluding Remarks

By the design procedure given here, the $\mathcal{H}_\infty$-norm of the error system decreases until it reaches some local minimum. The design procedure does not guarantee convergence to the global minimum. But if we start from suitable analysis filters, after a couple of iterations the filter coefficients will converge. Furthermore, as no constraints on frequency selectivity of filters are imposed, after some iterations, the frequency characteristics of the filters may become undesirable. In this case, it might be desirable to impose constraints on the stopband of the filters. This can be done by adding some LMIs as discussed in [21]. The $\mathcal{H}_\infty$ model-matching formulation presented in this chapter is computationally more intensive than the $\mathcal{H}_2$ model-matching problem given in the previous chapter. Furthermore, the results seem to give filters that are slightly less frequency selective that those obtained by the $\mathcal{H}_2$ model-matching design. In the next chapter, we study a nonuniform transmultiplexer that is obtained by interchanging the positions of the analysis and the synthesis parts of a nonuniform filter bank.

# Chapter 6

# Nonuniform Transmultiplexers: Analysis and Design

In the previous chapters, we studied the design of nonuniform filter banks. In this chapter we discuss a multirate system which is the dual of a nonuniform filter bank, i.e, a nonuniform transmultiplexer.

## 6.1 Introduction

A general nonuniform $m$ channel transmultiplexer is shown in Figure 6.1. It has two parts, the synthesis part and the analysis part. The synthesis part transforms signals that have different sampling rates from the time-division multiplexing format to the frequency-division multiplexing format. The FDM signal is then transmitted. At the receiving end, the analysis part of the transmultiplexer reconstructs the original signals.

In the design of nonuniform transmultiplexers, it is desired that the outputs of channels be delayed versions of the inputs of the channels, i.e.,

$$y_i(n) = x_i(n - d_i) \quad \text{for} \ \ i = 0, ..., m-1,$$

where $d_i$ is the delay of the $i$-th channel. In an ideal maximally decimated transmultiplexer,

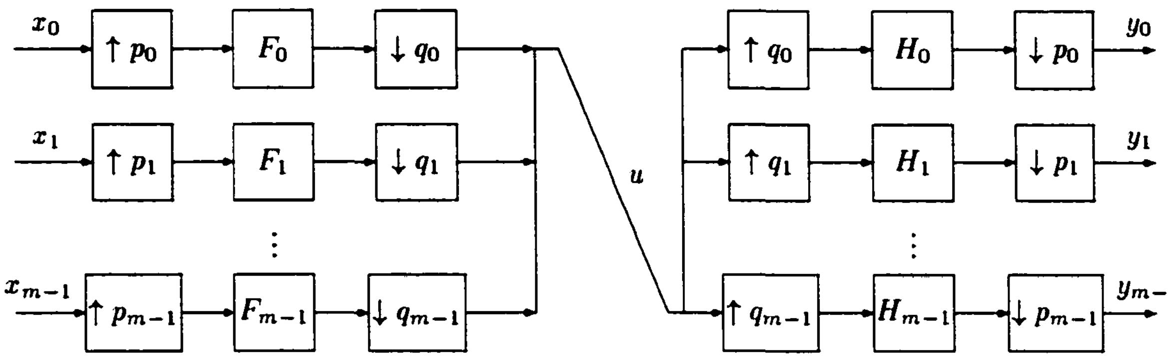


Figure 6.1: A general nonuniform transmultiplexer.

the spectrum of the input signal $x_i$ appears on the frequency bands

$$\begin{aligned} &\left[0, \tfrac{q_0}{p_0}\pi\right], && \text{for } i = 0, \\ &\left[\textstyle\sum_{k=0}^{i-1} \tfrac{q_k}{p_k}\pi \,,\, \sum_{k=0}^{i} \tfrac{q_k}{p_k}\pi\right] && \text{for } i = 1, \cdots, m-1, \end{aligned} \tag{6.1}$$

of the transmitted signal $u$. The higher bound for the component due to the last channel should be equal to $\pi$, i.e.,

$$\sum_{k=0}^{m-1} \frac{q_k}{p_k} = 1.$$

As we mentioned in previous chapters, it may not be possible to allocate frequency bands arbitrarily. Rather the $i$-th frequency band has to be located at $[\frac{s_i}{p_i}, \frac{s_i+1}{p_i}]$ for some $s_i \in \{0, \cdots, p_i\}$ [18].

In a nonuniform transmultiplexer, the synthesis part of the transmultiplexer consists of rate-changer subsystems described by the upsampler by $p_i$, the linear filter $H_i$ with an impulse response $h_i(n)$, and the downsampler by $q_i$. The $i$-th subsystem of the synthesis part maps the spectrum of the inputs of channels to the frequency bands given by (6.1). Because all outputs of these subsystems are added to obtain the resulting FDM signal $u$, the sampling rate of the outputs of all these subsystems are the same. Thus, the input sampling rates $\omega_i$, and the output sampling rate $\omega$ of all synthesis channels are related by

$$\omega_i = \frac{q_i}{p_i}\omega.$$

At the other end, the analysis part of the transmultiplexer receives the signal $u$, and reconstructs the inputs to the transmultiplexer. In summary, a nonuniform transmultiplexer may be used for the conversion of signals that have different sampling rates, between the TDM and the FDM formats.

As we will discuss later, a nonuniform transmultiplexer can be converted to a uniform transmultiplexer. Similar to nonuniform filter banks [12], some of the filters in the equivalent representation of a nonuniform transmultiplexer may be interrelated. Thus, it can be shown that in some cases, it may not even be possible to eliminate cross talk distortions in nonuniform transmultiplexers. The interdependency between filters in the equivalent representation translates into interdependency between elements of the transfer matrices of the synthesis or analysis parts. If we use linear periodically time-varying systems with appropriate periods instead of linear time-invariant filters, we obtain an equivalent uniform transmultiplexer with no interdependence between filters [27, 9].

The use of LPTV systems may not be acceptable for the filters in the synthesis part, because by using LPTV systems aliasing will be present in the channels, and it will be hard to ensure that the output of the synthesis part are in the FDM format. In this case, a transmultiplexer with LTI filters in the synthesis part, and LPTV subsystems in the analysis part eliminates the design constraints on the analysis part in the equivalent representation, and provides the TDM to FDM conversion.

The theory and design of transmultiplexers have been studied extensively, see [33, 35, 19, 26, 34, 1], and the references therein. Nonuniform transmultiplexers have been proposed recently [10, 20]. Once the design of nonuniform transmultiplexers is converted to the design of the uniform transmultiplexers, if there are no constraints on filters, a method developed for the design of uniform transmultiplexers may be used. One of the methods of designing multirate systems is the model-matching method. Similar to nonuniform filter banks [12, 18, 25], in nonuniform transmultiplexers, it may not be possible to achieve perfect-reconstruction. The model-matching method results in a transmultiplexer that behaves almost like a perfect-reconstruction transmultiplexer. In [20], the blocking method was used, and the $\mathcal{H}_2$ model-matching which minimizes the norm of the impulse response of an error system, was proposed for the design of transmultiplexers. The $\mathcal{H}_2$ model-matching problem results in the solution of a linear system of equations. Although the design is simple in the $\mathcal{H}_2$ model-matching, for the worst case scenario the errors may be too high.

The model-matching method was used in [6, 7, 14, 9] for the design of filter banks for the worst case scenario by solving two Riccati equations. The use of two Riccati equations cannot deal with the structural constraints that may appear in the design of nonuniform transmultiplexers. But, by the semidefinite programming approach [5, 29], we can design nonuniform transmultiplexers in the most general case.

In this chapter, we study the alias-component matrices for non-uniform transmultiplexers, and use semidefinite programming for the design of transmultiplexers in the worst case scenario. As we show, the distortions in transmultiplexers can be related to the $\mathcal{H}_\infty$ norm of some error systems. Therefore, a design based on the minimization of the $\mathcal{H}_\infty$ norm of error systems results in transmultiplexers with low distortions.

The chapter is organized as follows: In Section 6.2, it is first shown that a nonuniform transmultiplexer can be transformed into a uniform transmultiplexer, and then the transfer matrix of nonuniform transmultiplexer is obtained. In Section 6.3, the alias-component matrices of transmultiplexers are studied. In Section 6.4, the model-matching design of transmultiplexers is disussed. Some design examples are given in Section 6.5. Section 6.6 concludes the chapter.

## 6.2 Transfer Matrices of Nonuniform Transmultiplexers

Take $p = \text{lcm}(p_0, p_1, \cdots, p_{m-1})$, and set $n_i = p/p_i$. A delay in the input sequence by $n_i q_i$ samples in the $i$-th input of nonuniform transmultiplexer results in a delay of $p$ samples in $u$. Such a delay in $u$, in turn results in a delay of $n_i q_i$ samples in the output sequence of the $i$-th channel. Therefore the transmultiplexer is a general linear periodically time-varying system for which a delay of $n_i q_i$ in the input sequence of channel $i$, results in a delay of the same duration in the output to the channel. Because the inputs to the transmultiplexer should be reconstructed at the outputs of the channels, the channel response should be close in some sense to a pure delay. Therefore, although the nonuniform transmultiplexer is $n_i q_i$-periodic from the $i$-th input to the $i$-th output, it should behave similar to an LTI system with a pure delay transfer function. Furthermore there should be no cross talk between channels.

As we show in the following, a nonuniform transmultiplexer can be converted to a uniform transmultiplexer with upsampling factor $p = \text{lcm}(p_0, p_1, \cdots, p_{m-1})$. Take $n_i = p/p_i$, and for the simplicity of discussion, for the rest of the chapter, we assume that $q_i = 1$. In order to obtain the equivalent representation, we place the $n_i$-channel perfect-reconstruction filter bank in Figure 6.2 in front of the upsampler by $p_i$. Moving the upsampler by $p_i$, and $F_i$ inside the delay ladder and noting that the cascade connection of $\uparrow p_i$ and $\uparrow n_i$ is $\uparrow p$, we

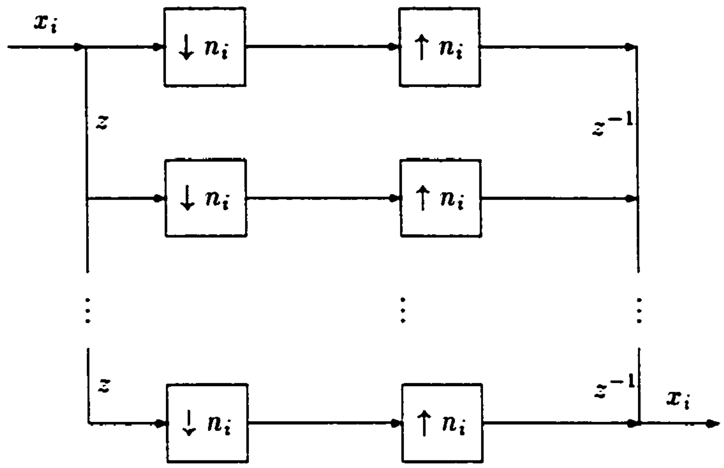


Figure 6.2: An $n_i$-channel maximally decimated perfect-reconstruction filter bank.

see that the $i$-th channel of the transmultiplexer is replaced by the left side of Figure 6.3. Similarly, by cascading the output of the $i$-th channel of the analysis part to the input of the perfect-reconstruction filter bank shown in Figure 6.2, we obtain the right side of Figure 6.3.

In Figure 6.3, the inputs to the upsamplers by $p$ are the $n_i$-polyphase representation of $x_i$. Similarly the outputs of the downsamplers by $p$ yield the $n_i$ polyphase decomposition of the output $y_i$. Therefore the part between upsampler by $p$ and downsampler by $p$ results in an equivalent $p$ channel uniform transmultiplexer, and the inputs and outputs of this equivalent transmultiplexer are the polyphase components of the inputs and outputs of the nonuniform transmultiplexer.

The $n_i$ filters in the equivalent analysis part and synthesis part of Figure 6.3 are related to the analysis filter $H_i$ and synthesis filter $F_i$, respectively. Because of this interdependence between filters, it can be shown that in some cases of nonuniform transmultiplexers, similar to nonuniform filter banks [12], it may not even be possible to eliminate cross talk distortions.

If we allow $F_i$ and $H_i$ to be LPTV systems with period $p$ instead of LTI filters, the interdependence between filters in the synthesis and analysis parts of the equivalent representation of the $i$-th channel will be eliminated. Take the periodic system with the representation shown in Figure 6.4, where $\tilde{F}_{ik}$ are LTI systems, and the switch connects to $\tilde{F}_{i0}$, $\tilde{F}_{i1}$, ..., $\tilde{F}_{i,n_i-1}$ at sampling times 0, $p_i$, ... $p_i(n_i-1)$, and repeats. For the $k$-th subchannel of the equivalent representation of the $i$-th synthesis subsystem of Figure 6.3, the input to $F_i$ is obtained by first upsampling a polyphase component of the input signal, and then delaying it by $kp_i$ samples. As the switch is periodic with period $p = p_i n_i$ and is connected to $\tilde{F}_{i0}$ at sampling time equal to zero, for this subchannel only $\tilde{F}_{ik}$ will have an input not identically equal to zero, and for the $k$-th subchannel, we can replace the block $z^{-kp_i}F_i$ by $z^{-p_i}\tilde{F}_{ik}$. Thus, the use of this switching system can eliminate the interdependency between filters in the synthesis part.

Similarly, let $H_i$ be an LPTV system with period $p$ shown in Figure 6.5, where $\tilde{H}_{ik}$ are LTI systems, and the switch connects to $\tilde{H}_{i0}$, $\tilde{H}_{i1}$, ..., $\tilde{H}_{i,n_i-1}$ at sampling times 0, $p_i$, ..., $p_i(n_i-1)$, and repeats. We can show that $z^{kp_i}H_i$ in Figure 6.3 will be replaced by $z^{kp_i}\tilde{H}_{ik}$ in the equivalent representation of the $i$-th channel. This will result in the elimination of interdependency between the filters in the equivalent representation of the $i$-th analysis subsystem.

Therefore, by using proper LPTV systems with period $p$, the design of a nonuniform transmultiplexer converts into the design of a uniform transmultiplexer. In particular, if $\tilde{F}_{i,k}$ and $\tilde{H}_{i,k}$ are frequency selective, and the equivalent transmultiplexer achieves perfect-reconstruction, the nonuniform transmultiplexer will also be perfect-reconstruction, and the spectrum of the $i$-th signal will be mapped into the frequency band given by (6.1). However, the spectrum of the output of synthesis part will not simply be the concatenation of the spectrum of the inputs of the nonuniform transmultiplexer as is the case for a uniform transmultiplexer. Rather, the spectrum of polyphase components of the input signal will be mapped into different subbands within the frequency band of the $i$-th channel in (6.1). If we want to have the TDM to FDM conversion, we should use LTI synthesis filters $F_i$. In that case, it may not even be possible to eliminate the cross talk between channels. In order to solve the problem in the case where $H_i$ and/or $F_i$ are LTI, we first discuss the transfer matrix of a transmultiplexer, and then formulate the model-matching problem in the following.

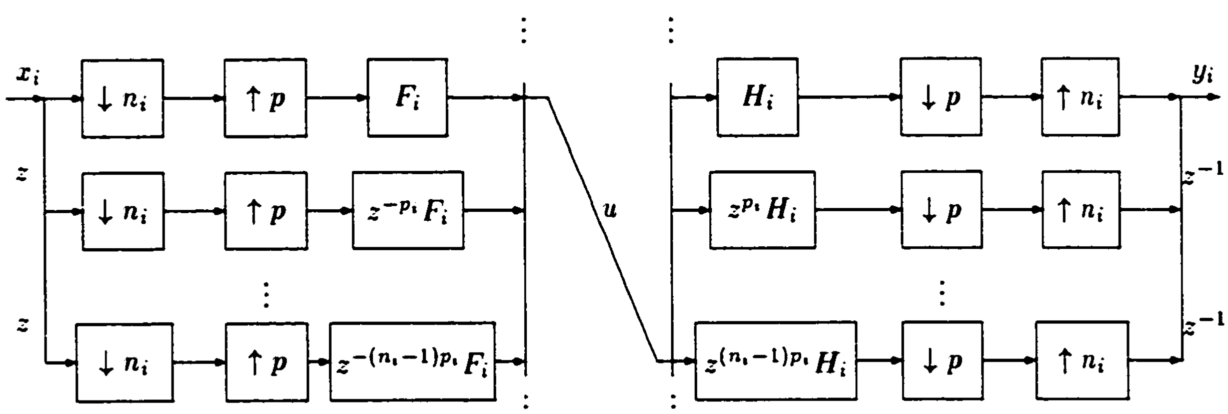


Figure 6.3: The equivalent representation of the general nonuniform transmultiplexer. The representation of the $i$-th channel is shown.

If we block the input and output signals, a multi-input multi-output $p$ by $p$ LTI system results. By blocking this equivalent representation, we obtain the input-output relationship of the system: Assume that $F_i$ and $H_i$ are LTI. Take the polyphase components of the

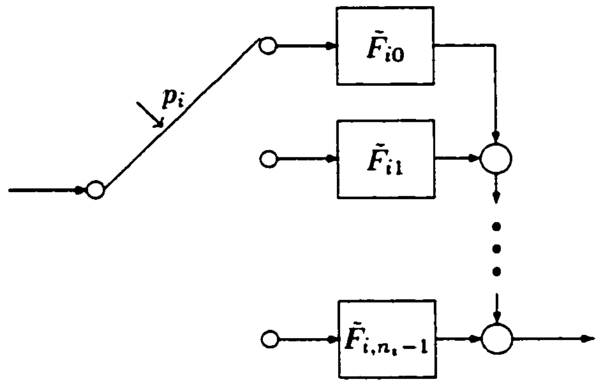


Figure 6.4: The representation of $F_i$.

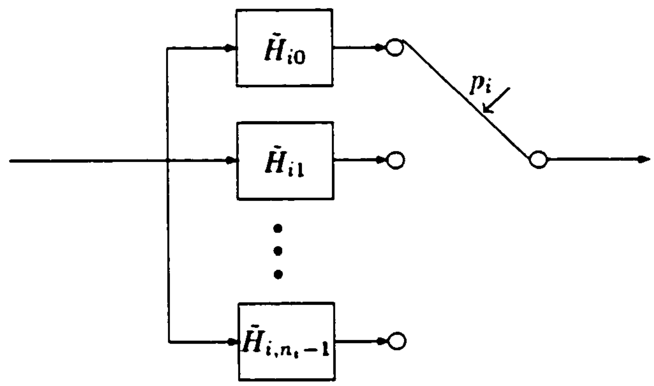


Figure 6.5: The representation of $H_i$.

channel inputs $x_i$, and channel outputs $y_i$:

$$X_i(z) = \sum_{k=0}^{n_i-1} \hat{X}_{ik}(z^{n_i})z^{-k}, \tag{6.2}$$

$$Y_i(z) = \sum_{k=0}^{n_i-1} \hat{Y}_{ik}(z^{n_i})z^{-k}. \tag{6.3}$$

Defining the $p$-polyphase versions

$$\hat{X}_i(z) = [\hat{X}_{i0}(z) \;\; \hat{X}_{i1}(z) \;\cdots\; \hat{X}_{i,(n_i-1)}(z)]^T,$$

and

$$\hat{Y}_i(z) = [\hat{Y}_{i0}(z) \;\; \hat{Y}_{i1}(z) \;\cdots\; \hat{Y}_{i,(n_i-1)}(z)]^T,$$

we see that the equivalent uniform transmultiplexer has inputs and outputs given by $\hat{X}$ and $\hat{Y}$, where

$$\hat{X} = [\hat{X}_0^T \; \hat{X}_1^T \; \cdots \; \hat{X}_{m-1}^T]^T,$$

and

$$\hat{Y} = [\hat{Y}_0^T \; \hat{Y}_1^T \; \cdots \; \hat{Y}_{m-1}^T]^T.$$

The synthesis filters in the equivalent uniform transmultiplexer are

$$G_{k_0} = z^{-k_0 p_0} F_0, \quad \text{for } 0 \leq k_0 < n_0,$$

and

$$G_k = z^{-k_i p_i} F_i, \quad k = k_i + \sum_{l=0}^{i-1} n_l, \quad \text{for } 0 \leq k_i < n_i, \text{ and } \quad 0 < i < m.$$

Similarly, the analysis filters in the equivalent uniform transmultiplexer are

$$E_{k_0} = z^{k_0 p_0} H_0, \quad \text{for } 0 \leq k_0 < n_0,$$

and

$$E_k = z^{k_i p_i} H_i, \quad k = k_i + \sum_{l=0}^{i-1} n_l, \quad \text{for } 0 \leq k_i < n_i, \quad \text{and} \quad 0 < i < m.$$

The transfer matrix of the nonuniform transmultiplexer can be found in terms of the polyphase components of the equivalent filters $E_i$ and $G_i$ as follows: Let

$$G_i(z) = \sum_{k=0}^{p-1} \hat{G}_{ik}(z^p) z^{-k}.$$

A delay of $n_i$ samples in the input of $i$-th channel of the nonuniform transmultiplexer for $i = 0, \cdots, m-1$, corresponds to a unit delay of the inputs to the equivalent uniform transmultiplexer, and results in a delay of $p$ samples at the output of the synthesis part. Thus by decomposing the output of the synthesis part $u$, in terms of its $p$-polyphase components as

$$U(z) = \sum_{k=0}^{p-1} \hat{U}_k(z^p) z^{-k},$$

and defining the blocked version of $u$

$$\hat{U}(z) = [\hat{U}_0(z) \;\; \hat{U}_1(z) \;\; \cdots \;\; \hat{U}_{(p-1)}(z)]^T,$$

we see that the polyphase components of $u$ and $x$ are related by

$$\hat{U}(z) = \underline{F}(z)\hat{X}(z), \tag{6.4}$$

where elements of $\underline{F}$ are given by

$$\underline{F}^{(i,l)} = \hat{G}_{l,i} \quad \text{for} \quad 0 \leq i < p, \;\; 0 \leq l < p.$$

(Note that we start the numbering from 0, and the superscript shows the index of elements of matrices.) Therefore, all of the polyphase components of filters $F_i$ appear on every row equal or higher than $\sum_{l=0}^{i-1} n_l$ and lower than $\sum_{l=0}^{i} n_l$. In other words, there is a structural constraint on the elements of the rows of these matrices, as they are not totally free to change, and are interrelated.

Similarly, the outputs of the transmultiplexer can be found in terms of the polyphase elements of the synthesis filters $\hat{E}_{i,k}$

$$E_i(z) = \sum_{k=0}^{p-1} \hat{E}_{ik}(z^p) z^{-k},$$

as

$$\hat{Y}(z) = \underline{H}(z)\hat{U}(z), \tag{6.5}$$

where the elements of $p$ by $p$ matrix $\underline{H}$ are given by

$$\underline{H}^{(i,0)} = \hat{E}_{i0}, \qquad \text{for } 0 \leq i < p,$$

$$\underline{H}^{(i,l)} = z^{-1}\hat{E}_{i,p-l} \quad \text{for } 0 \leq i < p, \quad 0 < l < p.$$

Again we see that the elements of some of the columns of $\underline{H}$ are interdependent. Substituting (6.4) in (6.5), we have

$$\hat{Y}(z) = \underline{H}(z)\underline{F}(z)\hat{X}(z). \tag{6.6}$$

This relationship gives the polyphase components of the output of the nonuniform transmultiplexer in terms of the polyphase components of the input of the transmultiplexer. Based on the presentation, we see that unless all of the upsampling factors $p_i$ are equal, the filters in the equivalent representation will be interrelated, which results in interdependence between the elements of some of the rows of $\underline{F}$ or some of the columns of $\underline{H}$. If $\underline{H}(z)\underline{F}(z)$ is a proper blocking of pure time-delay transfer functions from inputs to outputs of each channel, the transmultiplexer will be perfect-reconstruction. Because of the structural constraints, in general it may not be possible to have a perfect-reconstruction nonuniform transmultiplexer.

## 6.3 Alias-Component Matrices

Consider the $m$-channel nonuniform transmultiplexer in Figure 6.1. As we discussed in Section 6.2, we can transform a nonuniform transmultiplexer to a uniform transmultiplexer subject to some possible constraints on the filters. In order to be consistent with the results of the previous section, we assume that $q_i = 1$. The general case can be obtain by slight modifications, i.e., by replacing $n_i$ by $n_i q_i$ in the following. The inputs and outputs of this system are the $n_i$ polyphase components of the inputs and outputs of the nonuniform transmultiplexer. Thus, if we block the input and output signals, a multi-input multi-output $p$ by $p$ LTI system results. By blocking this equivalent representation, we obtain the input-output relationship of the system: Defining $\Gamma_i(e^{j\omega}) = \text{diag}(1, e^{-j\omega}, \cdots e^{-j\omega(n_i-1)})$, and a 1 by $n_i$ vector of ones as $1_{n_i} = [1 \;\; 1 \;\; \cdots \;\; 1]$, equation (6.2) can be written in terms of $n_i$ blocked version of input $X_i$ given by

$$X_i(e^{j\omega}) = 1_{n_i}\Gamma_{n_i}(e^{j\omega})\hat{X}_i(e^{jn_i\omega}).$$

For $k = 0, \cdots, n_i - 1$, we have

$$X_i(e^{j(\omega+2\pi\frac{k}{n_i})}) = \Lambda_{n_i}^{(k)}\Gamma_{n_i}(e^{j\omega})\hat{X}_i(e^{jn_i\omega}),$$

where $\Lambda_{n_i}^k$ is the $k$-th row of the $n_i$ by $n_i$ DFT matrix $\Lambda_{n_i}$ with the $(l,k)$ element given as $e^{-j2\pi(l+k)/n_i}$. By defining

$$\bar{X}_i(e^{j\omega}) = [X_i(e^{j\omega}) \; X_i(e^{j(\omega+2\pi\frac{1}{n_i})}) \; \cdots \; X_i(e^{j(\omega+2\pi\frac{n_i-1}{n_i})})]^T,$$

and

$$\bar{X}(e^{j\omega}) = [\bar{X}_0^T(e^{j\omega}) \; \bar{X}_1^T(e^{j\omega}) \; \cdots \; \bar{X}_{m-1}^T(e^{j\omega})]^T,$$

we see that the blocked version of input $X_i$, can be calculated in terms of the input as

$$\bar{X}_i(e^{j\omega}) = \Lambda_{n_i}\Gamma_{n_i}(e^{j\omega})\hat{X}_i(e^{jn_i\omega}).$$

Defining

$$\Lambda = \text{diag}\left\{\Lambda_0, \Lambda_1, \cdots, \Lambda_{n_{m-1}}\right\},$$

and

$$\Gamma(e^{j\omega}) = \text{diag}\left\{\Gamma_0(e^{j\frac{\omega}{n_0}}), \Gamma_1(e^{j\frac{\omega}{n_1}}), \cdots, \Gamma_{m-1}(e^{j\frac{\omega}{n_{m-1}}})\right\},$$

we have

$$\bar{X}(e^{j\omega}) = \Lambda\Gamma(e^{j\omega})\hat{X}(e^{j\omega}). \tag{6.7}$$

Similarly, let

$$\bar{Y}_i(e^{j\omega}) = [Y_i(e^{j\omega})\ Y_i(e^{j(\omega+2\pi\frac{1}{n_i})})\ \cdots\ Y_i(e^{j(\omega+2\pi\frac{n_i-1}{n_i})})]^T$$

and

$$\bar{Y}(e^{j\omega}) = [\bar{Y}_0^T(e^{j\omega})\ \bar{Y}_1^T(e^{j\omega})\ \cdots\ \bar{Y}_{m-1}^T(e^{j\omega})]^T.$$

for the output, we can write

$$\bar{Y}_i(e^{j\omega}) = \Lambda_{n_i}\Gamma_{n_i}(e^{j\omega})\hat{Y}_i(e^{jn_i\omega}).$$

and we have

$$\bar{Y}(e^{j\omega}) = \Lambda\Gamma(e^{j\omega})\hat{Y}(e^{j\omega}). \tag{6.8}$$

Substituting (6.7) and (6.8) in (6.6), we obtain the alias-component representation of the transmultiplexer as

$$\bar{Y} = \Lambda\Gamma\underline{H}\,\underline{F}\Gamma^{-1}\Lambda^{-1}\bar{X},$$

where the dependence of the matrices on the frequency is suppressed. The matrix

$$\bar{T} = \Lambda\Gamma\underline{H}\,\underline{F}\Gamma^{-1}\Lambda^{-1}$$

is the alias-component matrix of the nonuniform transmultiplexer.

In a transmultiplexer, cross-talk between channels is not desirable. Furthermore, the output of each channel should be close to a delayed version of the input to that channel. Therefore, if we take

$$\bar{T}_{di} = \text{diag}\left\{e^{j\omega d_i},\ e^{j(\omega+\frac{2\pi}{n_i})d_i},\ \cdots,\ e^{j(\omega+\frac{2\pi(n_i-1)}{n_i})d_i}\right\},$$

and

$$\bar{T}_d = \text{diag}\left\{\bar{T}_{d0},\ \bar{T}_{d1},\ \cdots,\ \bar{T}_{d,m-1}\right\},$$

then $\bar{T}$ should approximate $\bar{T}_d$.

The alias-component matrix is $p$ by $p$. If we partition the matrix according to the dimensions of the inputs and outputs in the equivalent representation, we obtain the following representation

$$\bar{T} = \begin{bmatrix} \bar{T}_{00} & \bar{T}_{01} & \cdots & \bar{T}_{0,m-1} \\ \bar{T}_{10} & \bar{T}_{11} & \cdots & \bar{T}_{1,m-1} \\ \vdots & \vdots & & \vdots \\ \bar{T}_{m-1,0} & \bar{T}_{m-1,1} & \cdots & \bar{T}_{m-1,m-1} \end{bmatrix}.$$

The $n_i$ by $n_l$ submatrices $T_{il}$ gives the relationship of the output of the $i$-th channel to the input of the $l$-th channel. When $i$ and $l$ are different, this represents the leakage or cross talk between channels. We may define, the cross talk distortion of the $i$-th channel $CD_i$, as the contribution of the inputs of other channels to the $i$-th output. In terms of the elements of the transfer matrix, we can write

$$CD_i^2 = \max_{\omega} \sum_{l=1,l\neq i}^{m-1} \sum_{k=0}^{n_l-1} |\bar{T}_{il}^{(0,k)}(e^{j\omega})|^2,$$

where the indices in the parenthesis represent the elements of the matrices. We can show that:

$$CD_i^2 = \sup \frac{||y_i||_2^2}{\sum_{k=1,k\neq i}^{n} ||x_k||_2^2}, \quad \text{subject to} \quad x_i = 0.$$

where the 2-norm of the signal is given by the square root of its energy.

The $\bar{T}_{ii}$ submatrix is $n_i$ by $n_i$. Its $(0,0)$ element, $\bar{T}_{ii}^{(0,0)}$ represent the transfer function between the input of the $i$-th channel and its output. If the system is LTI from the $i$-th input to the $i$-th output, the matrix $\bar{T}_{ii}$ will be diagonal. In the general case $q_i \neq 1$, the off diagonal elements of $\bar{T}_{ii}$ may not be equal to zero, because even if $F_i$ and $H_i$ are LTI, the nonuniform transmultiplexer is $q_i$ periodic from its $i$-th input to its $i$-th output. Thus an aliasing distortion may be present. We can define the aliasing distortion, $AD$, as

$$AD_i^2 = \max_{\omega} \sum_{k=1}^{n_i-1} |\bar{T}_{ii}^{(0,k)}(e^{j\omega})|^2.$$

Deviation of the diagonal elements of $\bar{T}_{ii}$ from those of the matrices $T_{di}$ represent the amplitude and the phase distortion of the $i$-th channel. The magnitude distortion $MD$ can define as

$$MD_i^2 = ||\bar{T}_{ii}^{(0,0)} - e^{j\omega d_i}||_{\infty}^2.$$

The $\mathcal{H}_\infty$ norm of a row submatrix of any matrix, is less than the $\mathcal{H}_\infty$ norm of the matrix. Taking the $l$-th row of the alias-component matrix of the error system $T - T_d$ yields

$$||\bar{T}^{(l,0)} - \bar{T}_d^{(l,0)} \quad \bar{T}^{(l,1)} - \bar{T}_d^{(l,1)} \quad \cdots \quad \bar{T}^{(l,p-1)} - \bar{T}_d^{(l,p-1)}||_\infty \leq ||\bar{T} - \bar{T}_d||_\infty.$$

Writing this in terms of the partition of the alias-component matrices gives

$$|\bar{T}_{ii}^{(0,0)}(e^{j\omega}) - \bar{T}_{di}^{(0,0)}(e^{j\omega})|^2 + \sum_{k=1}^{n_i-1} |\bar{T}_{ii}^{(0,k)}(e^{j\omega})|^2 + \sum_{l=0,l\neq i}^{m-1} \sum_{k=0}^{n_l-1} |\bar{T}_{il}^{(0,k)}(e^{j\omega})|^2 \leq ||\bar{T} - \bar{T}_d||_\infty^2, \quad \forall\omega.$$

Maximizing over $\omega$ yields

$$\max(MD_i, AD_i, CD_i) < ||\bar{T} - \bar{T}_d||_\infty.$$

Therefore, the distortions are bounded from above by the infinity norm of the error system. In other words, by minimizing the norm of the error system, a transmultiplexer results that has distortion measures less than the infinity norm of the error system. In the next section, we study a method for the design of transmultiplexers based on the minimization of the $\mathcal{H}_\infty$ norm of the error system.

## 6.4 Model-Matching Formulation

The $\mathcal{H}_\infty$ model-matching design of nonuniform transmultiplexers is similar to that of the nonuniform filter banks discussed in the previous chapter. In this section, we will only discuss the parts of the model-matching design that are specific to nonuniform transmultiplexers, and refer the reader to the previous chapter for the rest of the details.

As we mentioned in the previous section, the $p$ by $p$ transfer matrix of the blocked nonuniform transmultiplexer, $\underline{T}(z)$, is given by postmultiplying the transfer matrix of the blocked representation of the synthesis part $\underline{F}(z)$, by that of the analysis part, $\underline{H}(z)$, i.e.,

$$\underline{T}(z) = \underline{H}(z)\underline{F}(z).$$

The desired transmultiplexer is a system which has a transfer function close to a pure delay from the input to the output of each channel, and zero leakage between different channels. Since blocking is norm-preserving, we may compare the blocked transfer matrix of the transmultiplexer, with a transfer matrix $\underline{T_d}(z)$, which corresponds to a proper blocking of a zero transfer function from the inputs to the outputs of different channels, and a pure delay $d_i$, from the input to the output of channels $i = 0$ to $m-1$. If the desired delay for the $i$-th channel is $d_i$ samples, the $n_i$ blocked transfer matrix of the channel, $\underline{T_{di}}(z)$, can be obtained by writing $d_i = \bar{l}_i n_i + r_i$, with $0 \leq r_i < n_i$, as

$$\underline{T_{di}}(z) = z^{-\bar{l}_i} \begin{bmatrix} 0 & z^{-1} I_{r_i} \\ I_{n_i - r_i} & 0 \end{bmatrix}.$$

The $p$ blocked transfer matrix of the nonuniform transmultiplexer is

$$\underline{T_d}(z) = \mathrm{diag}(\underline{T_{d0}}, \underline{T_{d1}}, \cdots, \underline{T_{d,m-1}}).$$

Assuming state-space for transfer matrices $\underline{H}$, $\underline{F}$, and $\underline{T_d}$, we can formulate the model-matching design of transmultiplexers as discussed in Section 5.2 and Section 5.3. We will discuss this procedure further in the next example.

## 6.5 Design Example

Take the 3-channel nonuniform transmultiplexer with upsampling factors $p_0 = 3$, $p_1 = 6$, $p_2 = 2$, and down sampling factors $q_0 = q_1 = q_2 = 1$. For this transmultiplexer, we can show that, it is not possible to cancel cross-talk distortions, thus we proceed with the method discussed in previous sections to minimize the error measure from the inputs of the transmultiplexer to its outputs.

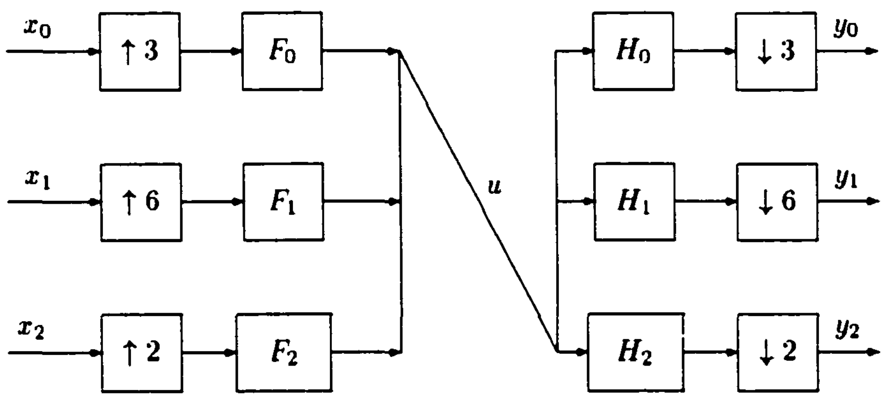


Figure 6.6: A three-channel nonuniform transmultiplexer.

Let the sampling rate of the transmitted signal be $\omega$, the transmultiplexer takes signals with sampling rates $\omega/3$, $\omega/6$ and $\omega/2$ as the inputs to the channels, upsamples the signals and filters the signals so that the output of the channels of synthesis filter bank, occupy the frequency bands : $[0 , \frac{\pi}{3}]$, $[\frac{\pi}{3} , \frac{\pi}{2}]$ and $[\frac{\pi}{2} , \pi]$ of the transmitted signal, respectively. At the receiving end, if we use filters with the same frequency characteristics as the synthesis filters and downsample the signals, we may obtain signals with a spectrum close to the original signals. But here, instead of LTI filters in the analysis part, we use periodic blocks with period 2 and 3, for the first and last channel, and an LTI filter for the second channel. By this choice, the structural constraints in the design of analysis bank are eliminated, and thus we can achieve a transmultiplexer that has a low error.

We expect the transmultiplexer to have a delay of 6, 3, and 9 samples from the inputs of the first, second and third channels to the outputs of these channels, respectively. In order to design the transmultiplexer, we chose FIR equiripple filters with orders 15, 15, 17 as the initial synthesis filters. We design periodic analysis blocks with the above mentioned periods and with order 24, and LTI synthesis filters with the same order as the initial filters iteratively, and according to the method discussed in the previous sections. The final synthesis filters are shown in Figure 6.7. The analysis subsystem for each channel consists of LTI systems with a switch at the output as per Figure 6.5. The LTI blocks for the switching realizations of the LPTV systems are shown in Figures 6.8 to 6.10. As can be seen, the LTI blocks in the switching representation are all frequency selective, and have similar frequency characteristics. The coefficients of the filters in the representation of each block are close to each other. Therefore the aliasing caused by the blocks are relatively small, and the LPTV blocks $H_0$ and $H_1$ are close to LTI. The $\mathcal{H}_\infty$ norm of the error system is 0.017.

Figure 6.11, shows the error of the first channel of the nonuniform transmultiplexer, when random inputs with uniform distribution in [-1, 1] is applied to the inputs of the transmultiplexer. The error in the output of other channels are comparable in magnitude.

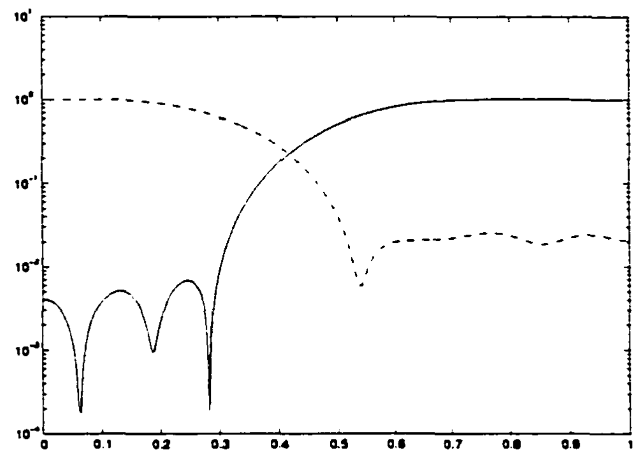

Figure 6.7: Synthesis filters $F_0$ (dash) , $F_1$ (dot), $F_2$ (solid): amplitude versus normalized frequency

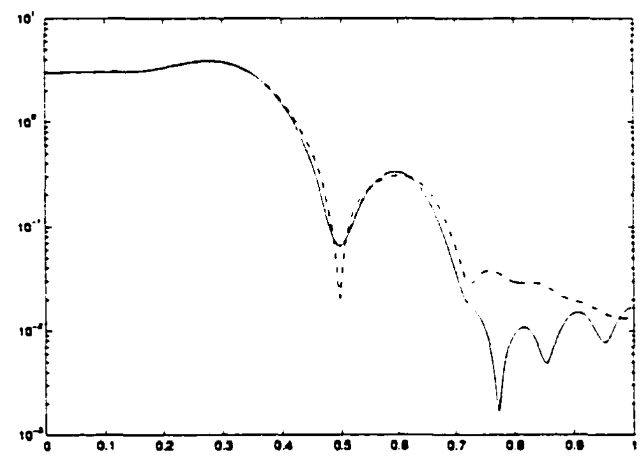

Figure 6.8: Analysis filters $\tilde{H}_{0,0}$ (solid), $\tilde{H}_{0,1}$ (dash): amplitude versus normalized frequency

## 6.6 Concluding Remarks

We have studied nonuniform transmultiplexers, and presented a design based on the minimization of the $\mathcal{H}_\infty$ norm of the error systems. Here, the presentation was based on the assumption that downsamplers are not present, i.e., $q_i = 1$ for $i = 0, 1, \cdots, m-1$. For the generalization to the case where for some or all of channels $q_i \neq 1$, we should consider $n_i q_i$-polyphase decomposition of inputs $x_i$ and outputs $y_i$. For this case, we can show that

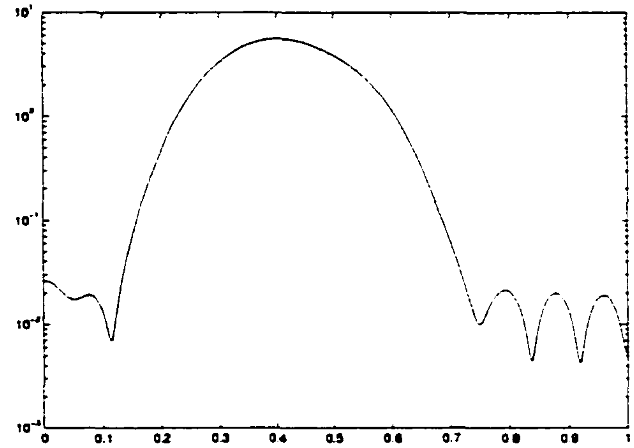

Figure 6.9: Analysis filters $\tilde{H}_{1,0}$: amplitude versus normalized frequency

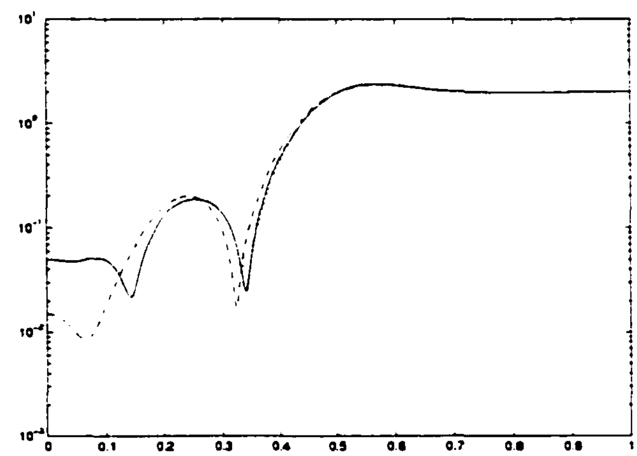

Figure 6.10: Analysis filters $\tilde{H}_{2,0}$ (solid), $\tilde{H}_{2,1}$ (dash), $\tilde{H}_{2,2}$ (dot): amplitude versus normalized frequency

the delayed versions of $q_i$-polyphase components of $F_i$ and $H_i$ will appear as the filters in the equivalent $p$-band uniform transmultiplexer. The structural constraints will still be present, and the rest of the work will be fairly similar.

The case where $F_i$ and $H_i$ are LPTV rather than LTI was also discussed. As pointed out, this case will result in equivalent uniform transmultiplexers with no interdependency between the analysis and synthesis filters of the equivalent uniform transmultiplexer. In the last section, we furnished an $\mathcal{H}_\infty$ design example with LTI synthesis filters $F_i$, and LPTV

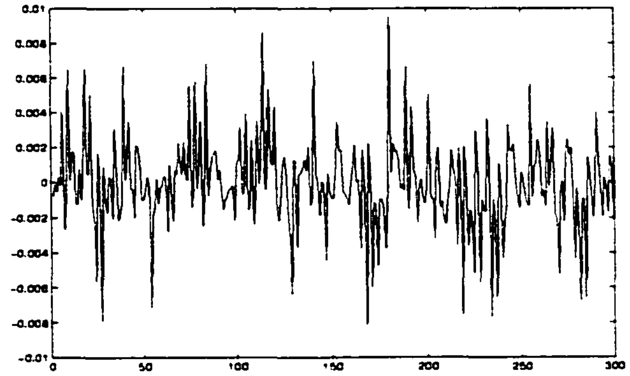


Figure 6.11: Error in the response of the first channel of the transmultiplexer systems $H_i$.

# Chapter 7

# Conclusion and Future Extensions

This chapter summarizes the results reported in this thesis, and proposes some possible future research directions.

## 7.1 Summary

In this thesis, we have studied periodic and multirate systems. We discussed the response of of a periodic system to periodic and aperiodic inputs, and showed that by using the Fourier series, we can find the response of a periodic system to periodic inputs. For general inputs, we used the Fourier transform, and obtained the alias-component matrix of the system. As we argued, by using the alias-component matrix at $\Omega = 0$, we can find the response of the system to an input with the same period as that of the system. We discussed the LSTV representations of periodic systems. Using these representations, we solved the problem of approximating an LPTV system by another LPTV system that has a given but different period so that the $\mathcal{H}_2$ norm of the error system is minimized. The LSTV representations were then used for extending these results to multirate systems.

The rest of the thesis deals with the design of nonuniform filter banks and transmultiplexers. The design of FIR nonuniform filter banks and transmultiplexers was posed as a model-matching problem, and an iterative method was proposed. For example in the design of a nonuniform filter bank, the initial analysis filters are designed according to the characteristics of the input. By the design procedure, the FIR synthesis filters are found so that the norm of the error system is minimized over all synthesis filters that have a prespecified order. Then, the synthesis filters obtained in the previous step are fixed and the analysis filters are found similarly. By iteration, the norm of the error system decreases until it converges to its final value. This algorithm only finds the local optimal solutions of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the error systems. Based on the results it can be argued that the $\mathcal{H}_2$ norm is computationally less intensive, and yields filters that have better frequency characteristics.

In order to reduce the error in these systems, we used LPTV filters. Based on the results, we see that in the LSTV representation of any of these LPTV systems, the blocks have frequency characteristics that are similar to each other, and since their coefficients are also close to each other, these LPTV systems behave close to LTI systems.

In the next section, we give some possible future research directions.

## 7.2 Future Extensions

In this thesis, we have presented a general design procedure based on model-matching for nonuniform filter banks and transmultiplexers. One of the crucial issues is the selection of the initial analysis or synthesis filters in a nonuniform filter bank or transmultiplexer, respectively. If the initial filters are not selected properly, the final results will not be satisfactory. Furthermore, we have not directly attempted to take bit allocation into consideration. Instead, this was done indirectly by the choice of the initial filters.

A direct improvement in the method would be to optimize the coding gain for a filter bank. For a nonuniform filter bank, we may also generalize the optimization criteria to take into account the statistical characteristics of the signals. This has been done in [17] for perfect-reconstruction filter banks. But, in general, for a given set of decimation factors, as we discussed before, it may not be possible to have perfect-reconstruction filter banks. Thus, the design of nonuniform filter banks should include the effects of the coding gain and the norm of the error system.

In the methods presented here, the final designs are only locally optimal, as at each iteration, the method presented here finds the best set of analysis filters for the given synthesis filters or vice versa. Finding the global solution of the problem should be the next step, but without taking into account the frequency characteristics of the filters, the global optimal solution might not even be acceptable. Therefore, a plausible future direction would be to effectively include the desired frequency characteristics of the filters into the design procedure, and attempt to find the global optimal solution to the nonlinear optimization problem that results.

Another interesting extension to the present work is to study general multirate system. Consider the nonuniform transmultiplexer shown in Figure 6.1. As we discussed in the thesis, the transmultiplexer can be considered as a multi-input multi-output general multirate system, where a delay of $d_i$, in the input sequence $x_i$, results in a delay of $d_i$ in the output sequence $y_i$, respectively. A more general system is an $m$ input $n$ output system where a delay of $q_0, q_1, \ldots, q_{m-1}$ in the input sequence results in a delay of $\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_{n-1}$ in the output sequence. For the single-input single-output case, realization of general multirate blocks were studied in [8]. By blocking these systems, LTI multi-input multi-output systems are obtained. We can also define an aliasing-component matrices for these systems. Therefore, an interesting topic for the future work is the realization and structural characteristics of these general multi-input multi-output multirate systems along with some implementable examples for possible new applications.

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