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DICHOTOMIES WITHOUT BOUNDED ANGULAR SEPARATION

BY

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Abstract

Dichotomies for systems of ordinary differential equations may be viewed as bounds on a Green function of a particular form. This view is used to provide an introduction to dichotomies.

The advantage of this treatment is the natural inclusion of angular separation conditions. These are related to bounds on part of the Green function which can be ignored for time-invariant systems and other systems with bounded angular separation of supplementary subspaces of solutions. By not ignoring it, angular separation enters naturally. Systems can be treated without requiring bounded angular separation.

The first three chapters describe the system to which future arguments will apply in some detail. In addition to familiarizing the reader with the notation used here, they are intended to help the beginner. After that, a brief survey is given of the current role of dichotomies in stability theory of non-linear and linear systems.

Next, the relationship between dichotomies and the asymptotic behaviour of solutions of linear homogeneous systems is established in sufficient generality to accommodate unbounded angular separation. As part of this program, angular separation is related to the norm of the difference between supplementary projections. This result is proved for any Banach space, without reference to ordinary differential equations. For general Banach spaces, "best possible" bounds are provided. These are compared to the usual bounds which relate angular separation

to the norm of each projection. For Hilbert spaces an exact formula is provided.

Some of the current kinds of dichotomies are then described. The relationship between dichotomies and the solutions of linear inhomogeneous systems is treated briefly. Finally, a guide to the search for new kinds of dichotomies is given. The advantage of comparing Green functions of two different systems is outlined.

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1. DESCRIPTION OF THE SYSTEM

A linear system of ordinary differential equations has the form

$$\mathbf{x}' - \mathbf{A}(t)\mathbf{x} = \mathbf{f}(t), \quad t \in J \tag{1}$$

where $J = [a, b]$ is a real interval which may be bounded or unbounded.

There is a homogeneous system corresponding to (1).

$$\mathbf{x}' - \mathbf{A}(t)\mathbf{x} = 0, \quad t \in J \tag{2}$$

When $\mathbf{A}$ is constant, the system is called time invariant (stationary, autonomous). When $\mathbf{A}$ varies with t, the system is called time varying (non-stationary).

The vectors $\mathbf{x}$, $\mathbf{f}$ and $\mathbf{A}\mathbf{x}$ may vary with t. At each t, they take their values in some Banach space $\mathbf{B}$.

In particular, $\mathbf{B}$ could be $\mathbf{R}^n$ or $\mathbf{C}^n$. For those spaces, an n-dimensional basis may be fixed. The linear operator $\mathbf{A}$ then becomes an $n \times n$ matrix which varies with t. *This matrix may become unbounded as $t \to a$ or $t \to b$.*

Only operators $\mathbf{A}$ which are linear are considered. That is, $\mathbf{A}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) = a_1\mathbf{A}\mathbf{x}_1 + a_2\mathbf{A}\mathbf{x}_2$ for all scalar a_1 and a_2. This has two immediate consequences.

Superposition Principle 1.1. *If $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of the homogeneous system (2), so is $a_1\mathbf{x}_1 + a_2\mathbf{x}_2$. Thus, the solutions of (2) form a linear subset.*

Superposition Principle 1.2. $\mathbf{x}_1$ and $\mathbf{x}_2$ *both satisfy the inhomogeneous system (1) if and only if one of them does and their difference solves (2). Thus adding* *each solution of (2) in turn to any one particular solution of (1) will produce all solutions of (1).*

Further restrictions on **f** and **A** are discussed in the next section.

2. THE SYSTEM IS DEFINED BY CONTINUOUS FUNCTIONS

Given any $(t_0, \mathbf{c}_0) \in J \times \mathbf{B}$, it is usually possible to prove there is a unique solution $\mathbf{x}(t)$ satisfying (1) the graph of which passes through $(t_0, \mathbf{c}_0)$, that is, with $\mathbf{x}(t_0) = \mathbf{c}_0$. For any system of ordinary differential equations, whether linear like (1) or non-linear, such existence and uniqueness of solutions follow from the Banach fixed point theorem once it is first shown that successive approximations converge. This first step is where linearity can help. Roughly speaking, assumptions are made to ensure J can be covered by finite intervals, chosen so that for each interval K in the cover, $\mathbf{f}|_K$ is continuous and $\int_K \|\mathbf{A}\|$ is bounded. Under these conditions, successive approximations converge.

One distinctive feature of the results discussed later is that they apply even when $\mathbf{A}$ is not bounded as $t \to a$ or $t \to b$, the endpoints of J. At points where $\mathbf{A}$ is not bounded, the method of proof outlined above sometimes still works, since it depends on $\int \|\mathbf{A}\|$ and not on $\|\mathbf{A}\|$ itself. In general though, solutions may fail to be unique or even exist at finite endpoints where $\mathbf{A}$ is not bounded, so these are removed from consideration in the following theorem.

Definition. *J' is the interval obtained from J by omitting any endpoints where $\mathbf{A}$ is not bounded.*

Theorem 2.1. *Assume $\mathbf{f}(t)$ is continuous for $t \in J'$ and $\mathbf{A}(t)\mathbf{c}$ is continuous for $(t, \mathbf{c}) \in J' \times \mathbf{B}$. Then given $(t_0, \mathbf{c}_0) \in J' \times \mathbf{B}$, system (1) has a unique solution*

$\mathbf{x}(t)$ *defined for all* $t \in J'$, *with* $\mathbf{x}(t_0) = \mathbf{c}_0$.

Corollary 2.2. *Assume* $\mathbf{A}(t)\mathbf{c}$ *is continuous for* $(t, \mathbf{c}) \in J' \times \mathbf{B}$, *as above. Then given* $(t_0, \mathbf{c}_0) \in J' \times \mathbf{B}$, *system (2) also has a unique solution* $\mathbf{x}(t)$ *defined for all* $t \in J'$, *with* $\mathbf{x}(t_0) = \mathbf{c}_0$.

Sketch of proof.

When **A** and **f** are continuous like this, the above conditions for covergence of successive approximations are satisfied by *any* cover of J by finite intervals.

The corollary follows since (2) is a special case of (1) with $\mathbf{f}(t) = 0$ for all $t \in J$. ∎

A byproduct of applying the Banach fixed point theorem is the knowledge that $\mathbf{x}(t)$ is continuous. From (1), it may be inferred that each solution $\mathbf{x}(t)$ will be one notch more differentiable than **A** and **f** are. For instance, when **A** and **f** are continuous, **x** will be continuously differentiable. In that case, (1) is satisfied at every $t \in J'$.

Thanks to these results, the assumption that **A** and **f** be continuous keeps the discussion simple, but is it necessary? The main line of argument still works for any linear system for which successive approximations converge on finite intervals in (a, b), giving solutions which extend to the whole of (a, b). It is known that not all linear systems have these characteristics. Precisely which systems have them is still an open question. However, systems which satisfy "Carathéodory conditions" are known to have them. These conditions allow not only all systems which are defined by continuous functions as described above, but also systems of practical interest which are defined by some kinds of functions which are not continuous, such as step functions. Dichotomies have been discussed under Carathéodory conditions. Technical digressions then obscure the main line of argument, which is the same as in the continuous case. This introduction will therefore treat only the continuous case.

3. FUNDAMENTAL SOLUTION

It is necessary to list the results of this section to have them in the explicit form needed for reference by later sections. The form of the results is the same whether general Banach spaces are considered or whether attention is restricted to the case when B is $\mathbf{R}^n$ or $\mathbf{C}^n$. Since no extra work is entailed, the proofs have been given for general Banach spaces.

Theorem 3.1. *The set* S *of all solutions of a linear system (1) is in one-one correspondence with* B.

Proof.

The idea of the proof is to select a t_0, then label each solution $\mathbf{x} \in \mathbf{S}$ by its value $\mathbf{x}(t_0) \in \mathbf{B}$.

Pick a $t_0 \in J'$, and define Φ.

$$\Phi: \mathbf{S} \to \mathbf{B}$$

$$\Phi: \mathbf{x} \mapsto \mathbf{x}(t_0) \tag{4}$$

Φ is called an initial value map. It will now be shown that Φ is a bijection, as a consequence of Theorem 2.1.

A solution $\mathbf{x}(t)$ of (1) is defined for some $t \in J'$. According to Theorem 2.1, $\mathbf{x}(t)$ is then a function that exists at all $t \in J'$, hence at t_0. That is, every solution extends to t_0.

(i) Since *every solution is a function which extends to* t_0, Φ is a well-defined function.

(ii) Since *solutions are unique when* $\mathbf{x}(t_0)$ *is given*, Φ is one-one.

(iii) Since *solutions exist with* $\mathbf{x}(t_0) = \mathbf{c}$ *for every* $\mathbf{c} \in \mathbf{B}$, Φ is onto. ▮

This correspondence applies to any system of ordinary differential equations, whether linear or not, for which the properties emphasized in (i), (ii), and (iii) in the proof can be established.

Although the set S of all solutions and the space B are in one-one correspondence, they may be very different in their structure. An important distinction is that B is a linear space, closed under vector addition and scalar multiplication, while S usually is not, as the following theorem shows.

Theorem 3.2. S *is a linear space* $\iff$ *the linear system is homogeneous.*

Proof.

$\Rightarrow$: Assume S is a linear space. It therefore includes the zero function, $\mathbf{x}(t) = 0$ for all $t \in J$. Since this must satisfy (1), $\mathbf{f}(t) = 0$ for all $t \in J$, and the system is homogeneous (2).

$\Leftarrow$: The converse is just the superposition principle 1.1. ▮

The significance of Theorem 3.2 is that it is possible for the set S to have the same structure as the space B only when the system is homogeneous. For the homogeneous system, S and B turn out to be isomorphic as vector spaces.

Theorem 3.3. *The set S of all solutions of a homogeneous linear system (2) is isomorphic as a vector space to the space* B.

Proof.

Finding one isomorphism is enough to prove this. Pick a $t_0 \in J'$. Construct Φ as in the proof of Theorem 3.1. There it was shown to be a bijection. Now we want to show that when the system is homogeneous, Φ is also linear.

$$a_1\Phi(\mathrm{x}) + a_2\Phi(\mathrm{y}) = \Phi(a_1\mathrm{x} + a_2\mathrm{y}) \qquad \text{for all } \mathrm{x}, \mathrm{y} \in \mathrm{S} \text{ and all scalar } a_1 \text{ and } a_2$$

$$a_1\mathrm{x}(t_0) + a_2\mathrm{y}(t_0) = \big(a_1\mathrm{x} + a_2\mathrm{y}\big)(t_0) \tag{5}$$

The left hand side of (5) is precisely how the right hand side is defined, so Φ is linear. Remember it was necesssary to restrict the domain of Φ to S for it to be a bijection. Therefore it is important to check that $a_1\mathrm{x} + a_2\mathrm{x}$ is in the domain of Φ in this equation. It is, as we know from the superposition principle 1.1. (This is pointed out since this is where the proof can fail for non-homogeneous or non-linear systems, even if Φ is bijective.) Since Φ is a linear bijection between linear spaces, it is an isomorphism, and proof of Theorem 3.3 is complete. ∎

This theorem is not enough to show that S and B are isomorphic as Banach spaces under Φ. To do that solutions must be considered "close" in some norm if and only if their initial values are close.

This coincides with the ordinary meaning of the word "close" when finite t intervals are considered. That result is called local continuity of solutions with respect to initial conditions and may be proved even for more general systems than ours by estimating the solutions using Gronwall's inequality. (In our case, it is also a consequence of Banach's theorem).

It can differ from the ordinary meaning of the word "close" when infinite t intervals are considered. The simplest illustration is this scalar system.

$$x' - x = 0, \qquad t \in [0, \infty], \quad x(t) \in \mathbf{R}$$

The difference between any two distinct solutions becomes unbounded as $t \to \infty$, no matter how small the difference in their initial values. Said another way, in this example Φ^{-1} is not continuous, and $\mathbf{S}$ and $\mathbf{B}$ are not isomorphic as Banach spaces, unless an unusual notion of closeness is defined for the solutions. Stability theory is the art of describing what can happen to solutions with close initial values.

The reader has been warned not to assume automatically that Φ^{-1} is continuous. This warning applies also to all fundamental solutions, which will now be introduced, when they are considered as maps from $\mathbf{B}$ to $\mathbf{S}$.

Definition. *A fundamental solution for the linear homogeneous system (2) is a (vector space) isomorphism* $\mathbf{X}$ *from the space* $\mathbf{B}$ *to the set* $\mathbf{S}$ *of solutions.*

The two principle results of this section follow immediately.

Theorem 3.4. Existence of a fundamental solution. *When* $\mathbf{A}(t)$ *is continuous, as described in Corollary 2.2, a fundamental solution for the homogeneous system (2) exists for* $t \in J'$.

Proof.

Pick a $t_0 \in J'$, and consider the map Φ^{-1}, the inverse of the map defined in Theorem 3.1. Theorem 3.3 proved $\Phi : \mathbf{S} \to \mathbf{B}$ is a vector space isomorphism. Therefore $\Phi^{-1} : \mathbf{B} \to \mathbf{S}$ is an isomorphism, and Φ^{-1} is one example of a fundamental solution for the homogeneous system (2). ∎

Theorem 3.5. The use of a fundamental solution to solve the homogeneous system (2). *Solutions* $\mathbf{x}$ *of the homogeneous system (2) are always of the form* $\mathbf{x} = \mathbf{X}\mathbf{c}$ *where* $\mathbf{X}$ *is a fundamental solution and* $\mathbf{c} \in \mathbf{B}$ *is a constant vector.*

Proof.

It is only necessary to explain the notation to see this is no more than a rephrasing of the definition of a fundamental solution.

For any linear map $\mathbf{X}$ from $\mathbf{B}$ to $\mathbf{S}$, the notation

$$\mathbf{x} = \mathbf{X}\mathbf{c} \tag{6}$$

shows $\mathbf{X}$ taking a constant $\mathbf{c} \in \mathbf{B}$ as input, and giving a function $\mathbf{x}$, satisfying (2), as output. No parentheses are put around $\mathbf{c}$ because $\mathbf{X}$ is linear. In the special case where $\mathbf{X}$ is a fundamental solution, each solution $\mathbf{x} \in \mathbf{S}$ is matched to exactly one $\mathbf{c} \in \mathbf{B}$ because $\mathbf{X}$ is a bijection. ∎

It is possible to characterize Φ^{-1} and the other fundamental solutions in terms of a differential equation they must satisfy. The following theorem is that characterization, which is shown to be an equivalent definition for a fundamental solution.

Theorem 3.6. $\mathbf{X}$ *is a fundamental solution for the homogeneous system (2)* $\Longleftrightarrow$

X *is a solution of*

$$\mathbf{X}' - \mathbf{A}(t)\mathbf{X} = 0, \qquad t \in J' \tag{7}$$

and $\mathbf{X}(t_0)$ *is a bijection for some (equivalently, all)* $t_0 \in J'$.

Proof.

From the definition, a fundamental solution is a linear map from **B** to **S** which is a bijection. Lemma 3.7 will show that saying **X** is a linear map from **B** to **S** is equivalent to saying **X** solves (7). Lemma 3.8 will complete the proof by showing that **X** is a bijection if and only if $\mathbf{X}(t_0)$ is a bijection.

Lemma 3.7. **X** *is a linear map from* **B** *to* **S** $\iff$ **X** *is a solution of (7)*

Proof of Lemma.

X is linear map from **B** to **S** means that for any $\mathbf{c} \in \mathbf{B}$, $\mathbf{Xc} \in \mathbf{S}$. That is, **Xc** is a solution of the homogeneous system (2). Since $\mathbf{c} \in \mathbf{B}$ does not depend on t, formally we have the following.

$$(\mathbf{Xc})' - \mathbf{A}(t)(\mathbf{Xc}) = 0, \qquad t \in J', \text{ for all } \mathbf{c} \in \mathbf{B} \tag{8}$$

$$\iff \qquad \mathbf{X}'\mathbf{c} - \mathbf{A}(t)\mathbf{Xc} = 0, \qquad t \in J', \text{ for all } \mathbf{c} \in \mathbf{B} \tag{9}$$

$$\iff \qquad (\mathbf{X}' - \mathbf{A}(t)\mathbf{X})\mathbf{c} = 0, \qquad t \in J', \text{ for all } \mathbf{c} \in \mathbf{B} \tag{10}$$

A linear **X** satisfying (10) is what is meant by a solution to (7), so it only remains to justify this formal procedure. To do that, **X**′ must be suitably defined. When **B** is finite dimensional, **X** is a matrix of functions, and **X**′ is obtained by differentiating each entry. Now we will justify this, and at the same time see how **X**′ can be defined when **B** is not finite dimensional.

The time dependence in (6) may be shown explicitly. In (6), $\mathbf{x}$ is a linear function of $\mathbf{c}$. This means that at each fixed $t_0 \in J'$, $\mathbf{x}(t_0)$ is a linear function of $\mathbf{c}$. This function is defined to be $\mathbf{X}(t_0)$.

$$\mathbf{x}(t_0) = \mathbf{X}(t_0)\mathbf{c}, \quad \text{for fixed } t_0 \in J' \tag{11}$$

Here $\mathbf{X}(t_0)$ is a linear map from $\mathbf{B}$ to $\mathbf{B}$.

It is possible to define $\mathbf{X}(t)$ for each $t \in J'$, since t_0 was arbitrary. Thus (11) may be rewritten as:

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}, \quad \text{for all } t \in J' \tag{12}$$

From Corollary 2.2, $\mathbf{x}'(t)$ exists everywhere on J'. Using (12) in the usual definition of $\mathbf{x}'(t)$ as a limit shows $\mathbf{x}'(t)$ is a linear function of $\mathbf{c}$ at each fixed t. This function is defined to be $\mathbf{X}'(t)$.

$$\begin{aligned} \mathbf{x}'(t) &= (\mathbf{X}(t)\mathbf{c})' \\ &= \mathbf{X}'(t)\mathbf{c}, \quad \text{for all } t \in J' \end{aligned}$$

That is, $(\mathbf{X}\mathbf{c})' = \mathbf{X}'\mathbf{c}$, showing (8) and (9) are equivalent. The linearity of $\mathbf{X}'$ shows (9) and (10) are equivalent. ▮

Lemma 3.8. *Suppose* $\mathbf{X}$ *is linear and for all* $\mathbf{c} \in \mathbf{B}$, $\mathbf{X}\mathbf{c}$ *is a solution of the homogeneous system (2). Then the following are equivalent.*

(i) $\mathbf{X}$ is a bijection.

(ii) $\mathbf{X}(t_0)$ is a bijection for some $t_0 \in J'$.

(iii) $\mathbf{X}(t)$ is a bijection for all $t \in J'$.

Proof of lemma.

First we will show (i) $\iff$ (ii).

Using the same t_0 as in (ii), Φ may be constructed following the procedure in Theorem 3.1. Then $\Phi : \mathbf{x} \mapsto \mathbf{x}(t_0)$, for any solution $\mathbf{x}$ of (2). Pick any $\mathbf{c} \in \mathbf{B}$. $(\Phi \circ \mathbf{X})\mathbf{c} = \Phi(\mathbf{X}\mathbf{c}) = (\mathbf{X}\mathbf{c})(t_0) = \mathbf{X}(t_0)\mathbf{c}$. The choice of $\mathbf{c}$ was arbitrary, so this shows $\Phi \circ \mathbf{X} = \mathbf{X}(t_0)$. Because Φ is a bijection, this shows $\mathbf{X}$ is a bijection if and only if $\mathbf{X}(t_0)$ is a bijection. This completes the proof that (i) $\iff$ (ii).

The choice of t_0 in the above argument was arbitrary, so (i) $\Rightarrow$ (iii). Finally, (iii) $\Rightarrow$ (ii) since (ii) is a particular instance of (iii). This completes the proof of Lemma 3.8. ▮

To complete the proof of Theorem 3.6, it must be checked that the supposition that starts Lemma 3.8 is satisfied. When proving Theorem 3.6 in the forward direction, this is a direct consequence of the definition of the fundamental solution. When proving the converse, this follows from the fact that $\mathbf{X}$ solves (7), as was seen in Lemma 3.7. ▮

For clarification, some further results are now mentioned. These are not needed in the sequel, so we do not dwell on them.

By definition, the fundamental solutions which are of the form $\mathbf{X} = \Phi^{-1}$ for some t_0 have $\mathbf{X}(t_0) = \mathbf{I}$. Therefore they are the subset of the solutions of (7) with this initial value. When the unique existence of solutions of (7) for any initial value is established, we can show that any fundamental solution $\mathbf{Y}$ of (2) is of the form $\mathbf{Y} = \mathbf{XC}$ where $\mathbf{X}$ is any solution of (7) for which $\mathbf{X}(t_0)$ is an isomorphism for some (equivalently, all) $t_0 \in J'$, and $\mathbf{C}$ is a constant isomorphism. We can think of $\mathbf{C}$ as a constant linear change of variables on the domain of the fundamental solution. For finite dimensional spaces, this means any two fundamental solutions are the same, apart from a constant change of basis for their domains, and are equivalent for many purposes.

To establish the unique existence of solutions of (7), we may regard it as a system of the form (2), where $\mathbf{X}'$ and $\mathbf{A}(t)\mathbf{X}$ take their values in $[\mathbf{B}]$, the Banach space of bounded linear maps from $\mathbf{B}$ to $\mathbf{B}$. (For example, when $\mathbf{B}$ is $\mathbf{R}^n$ or $\mathbf{C}^n$, $[\mathbf{B}]$ is the set of all $n \times n$ constant matrices, with two matrices being close when corresponding entries are close). The left multiplication by $\mathbf{A}(t)$ indicated in (7) is linear in $\mathbf{X}$. It was to make this analogy between (2) and (7) clear that the letter $\mathbf{X}$ was used for the fundamental solutions. The unique existence of solutions is then given by Corollary 2.2, reading $\mathbf{X}$ in place of $\mathbf{x}$. This is a consequence of the fact that $\mathbf{A}(t)\mathbf{c}$ is continuous for all $(t, \mathbf{c}) \in J' \times \mathbf{B}$ implies $\mathbf{A}(t)\mathbf{C}$ is continuous for all $(t, \mathbf{C}) \in J' \times [\mathbf{B}]$. The smoothness of the solutions $\mathbf{X}(t)$ as a function of t also follows.

For finite dimensional $\mathbf{B}$, all fundamental solutions take their values in $[\mathbf{B}]$ and nothing is lost by considering (7) to take place in $[\mathbf{B}]$. For infinite dimensional systems, not all fundamental solutions as previously defined take their values in

[B], so considering (7) to take place in [B] only describes some of the fundamental solutions. Either a more restricted definition of a fundamental solution must be used, or we must be content with results which directly pertain only to some fundamental solutions and indirect inferences about the others. Alternatively, we can use a different method to establish the unique existence of solutions of (7).

In the case where $\mathbf{A}(t)$ is bounded for each t, which we are considering in this paper, choosing only $\mathbf{X}$ for which $\mathbf{X}(t_0)$ is bounded for some (equivalently, all) $t_0 \in J'$ is natural and quite satisfactory. That is, we may consider $\mathbf{X}(t) \in [\mathbf{B}]$ for each $t \in J'$. The real issue is whether this convenient assumption should be made for infinite dimensional systems when $\mathbf{A}(t)$ is unbounded at fixed t [14]. Such systems of ordinary differential equations crop up in the study of partial differential equations.

Summary.

This section established two principal results.

I Existence of fundamental solutions. *Under the hypotheses of continuity detailed in Corollary 2.2, fundamental solutions for the homogeneous system (2) exist for $t \in J'$.*

II The use of fundamental solutions to solve the homogeneous system (2). *Solutions $\mathbf{x}$ of the homogeneous system (2) are always of the form*

$$\mathbf{x} = \mathbf{X}\mathbf{c}$$

where $\mathbf{X}$ is a fundamental solution and $\mathbf{c} \in \mathbf{B}$ is a constant vector.

4. REASON FOR INTEREST IN DICHOTOMIES

The purpose of this chapter is to outline only those parts of the stability theory of ordinary differential equations which have been related to dichotomies. Introductions to the subject which do not suffer this distortion are Coppel [5] and Daleckiĭ and Kreĭn [7]. In particular, we want to focus attention on the role played by systems of the form (1).

It is not usually possible to find explicit solutions of the general ordinary differential equation.

$$\mathbf{y}' = \mathbf{g}(\mathbf{y}, t) \tag{13}$$

Instead, a qualitative theory has developed: What kind of solutions exist? How do they behave?

Linear systems are used to investigate how solutions of (13) are affected by smooth variations in some parameter, such as initial conditions. One reason such investigation is important is this. When a dynamical system is engineered for the real world, to operate reliably it must be stable with respect to initial conditions and disturbances.

The pioneering work, based on the variational equation for (13), was done by Poincaré and Liapunov (Liapunov's first method). For the work of Poincaré, see Hartman [12] and the introduction at the beginning of Abraham and Marsden

[1]. The bibliography in [1] lists the relevant references. Liapunov's work has been reprinted [15]. Coppel [5, p. 53] describes part of this work and gives references to later simplified treatments.

Systems of the form (13) are classified as follows. Each class is a subset of the one that follows.

(i) *linear systems with bounded constant* **A**

(ii) *linear systems with time-varying* $\mathbf{A}(t)$

(iii) *weakly non-linear systems (linear systems with small non-linear perturbations)*

(iv) *quasi-linear systems*

(v) *general non-linear systems (13)*

Brief remarks are next made about each of these in turn, for later discussion about dichotomies. As we proceed down this list, less and less is currently known about the behaviour of the solutions of the systems.

(i) *linear systems with bounded constant* **A**

(Note in finite dimensions, constant **A** are always bounded, so this stipulation only restricts infinite dimensional **A**). The theory of linear systems with constant bounded **A** is well advanced. This includes spectral theory and the usual Laplace transform methods.

(ii) *linear systems with time-varying* $\mathbf{A}(t)$

These are systems of the form (1) which are the focus of attention in this paper. The restrictions we have made on the space **B** (it must be a Banach space) and the coefficients $\mathbf{A}(t)$ and $\mathbf{f}(t)$ (they must be continuous) mean that we are not treating the problem in full generality.

(iii) *weakly non-linear systems (linear systems with small non-linear perturbations)*

The study of non-linear systems (13) which deviate from linear systems in a small way was instigated by Liapunov. Now this is the best developed branch of the theory of non-linear systems. It includes the case where $\mathbf{g}(\mathbf{x}, t)$ is analytic. Study of these systems is especially useful since many systems are designed with the ideal (i) in mind, and small non-linearities result from the imperfection of the implementation.

(iv) *quasi-linear systems*

These are systems similar in form to linear systems (1), except that the coefficients $\mathbf{A}$ and $\mathbf{f}$ are allowed to depend on the solutions $\mathbf{x}$ and not only on t.

$$\mathbf{x}' = \mathbf{A}(\mathbf{x}, t)\mathbf{x} + \mathbf{f}(\mathbf{x}, t) \tag{14}$$

Many systems (13) can be put in this form, with well-behaved $\mathbf{A}$ and $\mathbf{f}$. For example, we can take a first order Taylor expansion (with remainder) of $\mathbf{g}$ in $\mathbf{x}$ alone, or in $\mathbf{x}$ and t. This can be done, for instance, when $\mathbf{g}$ is continuously differentiable in the variable(s) of expansion. It leads to systems of the form (iii). Here is a specific example of this kind of expansion. When $\mathbf{g}$ is continuously differentiable in $\mathbf{x}$, we may write $\mathbf{g}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t)\mathbf{x} + \mathbf{g}(0, t)$. Let us briefly show this. Consider $\mathbf{h}(s) = \mathbf{g}(s\mathbf{x}, t)$ so that $\mathbf{h}'(s) = \mathbf{J}(s\mathbf{x}, t)\mathbf{x}$ where $\mathbf{J} = \dfrac{\partial \mathbf{g}}{\partial \mathbf{x}}$. Since

$\mathbf{h}(1) - \mathbf{h}(0) = \int_0^1 \mathbf{h}'(s)\,ds$

$$\mathbf{g}(\mathbf{x},t) - \mathbf{g}(\mathbf{0},t) = \mathbf{A}(\mathbf{x},t)\mathbf{x}$$

where $\mathbf{A}(\mathbf{x},t) = \int_0^1 \mathbf{J}(s\mathbf{x},t)\,ds$.

There is a standard trick for studying quasi-linear systems using linear systems. First note that for any given function $\mathbf{y} : J \to \mathbf{B}$

$$\mathbf{x}' = \mathbf{A}(\mathbf{y}(t),t)\mathbf{x} + \mathbf{f}(\mathbf{y}(t),t) \tag{15}$$

is a linear system of the form (1). (In this paper we consider only continuous $\mathbf{A}$ and $\mathbf{f}$. To make use of this system, we would first have to show $\mathbf{A}$ and $\mathbf{f}$ are continuous, or use results more general than those presented here). If we choose a solution of the system (14) to be $\mathbf{y}$ in (15), it is seen to also be a solution of (15).

Suppose we know properties common to the solutions of the different systems (15) formed by substituting different $\mathbf{y}$ in a family Y which includes all (or a subset of) solutions of (14). Then all (or the subset of) solutions of (14) have the common property.

A better approach is available when (15) has a unique solution in Y for each $\mathbf{y} \in Y$. Then we can think of the map $\mathbf{y} \mapsto \mathbf{x}$ which takes any function $\mathbf{y}$ which we put into (15) to the unique solution $\mathbf{x}$ in Y which results from (15). The fixed points of this map are the solutions of (14) in Y [2] [13].

(v) *general non-linear systems (13)*

There is a method of exploring the global behaviour of solutions which applies to all systems (13) and not only those in (i)–(iv) above. This is Liapunov's second or discussed there have since been surpassed). A more general introduction is given by Hartman [12] and Kartsatos [13, pp. 79–92]. The method is developed for systems satisfying Carathéodory conditions in Massera and Schäffer [18, pp. 311–320]. The basic reference is Yoshizawa [23].

Now the brief description of the classes of ordinary differential equations is finished. Much work has been done in relating these classes, so that knowledge of the better understood systems near the top of the list can be applied to systems at the bottom of the list.

In this scheme, the systems of the form (1) in (ii) are an interesting middle ground. The discussion of quasi-linear systems in (iv) provided a sketch of one way to make the transition from (v) to (ii). Let us now see how (i) and (ii) are related. At the same time, we will introduce dichotomies and present the scope of the present paper. Then we will return to indicate how dichotomies have been related to the other ideas outlined above, as well as a few others which have not been mentioned yet.

Because the spectral theory is so successful for time-invariant systems (i), attempts have been made to modify this idea to work for the time-varying systems (ii). When $A(t)$ is not constant, it is possible to consider its spectrum (set of eigenvalues) at each t. For different t this set can shuffle around instead of being static as it is when **A** is constant. This idea does not work out as hoped.

Example. [6, p. 3]

When **A** is constant and all eigenvalues are -1, spectral theory tells us that the system is uniformly asymptotically stable. In particular, all solutions tend to zero for increasing t. To see how badly eigenvalues fail as a diagnostic tool when $\mathbf{A}(t)$ is not constant, here is an example where all eigenvalues are -1 at every t, and yet the system has solutions that are unbounded!

Take $\mathbf{A}(t) = \mathbf{R}^{-1}(t)\mathbf{A}_0\mathbf{R}(t)$, where

$$\mathbf{A}_0 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{R}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

$\mathbf{A}_0$ is triangular so the eigenvalues appear on the diagonal. These eigenvalues are not affected by a rotation **R**.

Here is a fundamental matrix for the corresponding homogenous system (2).

$$\mathbf{X}(t) = \begin{pmatrix} e^{t}(\cos t + \frac{1}{2}\sin t) & e^{-3t}(\cos t - \frac{1}{2}\sin t) \\ e^{t}(\sin t - \frac{1}{2}\cos t) & e^{-3t}(\sin t + \frac{1}{2}\cos t) \end{pmatrix}$$

Every solution **Xc** where **c** has non-zero first component, becomes unbounded as $t \to \infty$. Thus understanding the behaviour of solutions completely in terms of the spectrum of $\mathbf{A}(t)$ when $\mathbf{A}(t)$ is not constant is doomed to failure. ∎

Nonetheless, in special circumstances, part of the spectral theory for static systems (i) can be applied to time-varying systems (ii). Floquet theory for periodic $\mathbf{A}(t)$ is a major advance. Coppel [5] summarizes what can be salvaged. A more fruitful approach than this is described next.

In retrospect, we may view developments the following way. For motivation, we look at the spectral theory for constant A. What kind of bounds on the solutions or the Green function arise from the spectrum and are used in the theory? To develop the analogue for time-varying $A(t)$, we consider bounds of the same form. These can be established without any reference to the moving spectrum of $A(t)$, although it is interesting to relate the bounds to the moving spectrum when this can be done. Finally, we forget about the spectrum altogether, and try to generalize the forms of the bounds so they can be applied to a broader class of systems.

Two developments of this kind are matrix measures and dichotomies. Matrix measures are used to estimate the growth rates of the norms of solutions. Introductions to matrix measures (logarithmic 'norms' of Lozinskiĭ) are Coppel [5, pp. 41,58] and Kartsatos [13, pp. 65–70].

As early as the work of Perron [21] the germ of the idea of a dichotomy can be seen. However, the word dichotomy was apparently first used in this connection by Massera and Schäffer, who give a thorough treatment of ordinary and exponential dichotomies in [18]. The introduction and chapter end notes in [18] describe the historical evolution of the idea, and give references to earlier work. A more accessible introduction is Coppel [6].

In the present paper, a dichotomy is considered to be a kind of bound on a particular form of the Green function of the system (1). Exponential and ordinary dichotomies came first. Bounds of these forms arise from consideration of the spectrum for constant bounded $\mathbf{A}$, and can be used to find properties of the solutions of (1) in that case. Without reference to the spectrum, criteria are found which will ensure bounds of the same form when $\mathbf{A}(t)$ is not constant.

For this, theorems 6.2 and 6.4 are developed. They relate the bounds on the Green function to the behaviour of the solutions of the homogeneous system (2). (Maĭzel' [16] first noticed this kind of connection in a simple case). In this way dichotomies are related to different asymptotic behaviour of two subspaces of solutions of (2). Dichotomies may therefore be viewed as a kind of conditional stability.

The program is carried out in sufficient generality to accommodate a broader class of bounds than ordinary and exponential dichotomies. Some of these, namely (μ_1, μ_2)-dichotomies [17] [20], are introduced. Other generalizations exist. A brief review is included in Elaydi and Hajek [10]. The present paper ends with further suggestions.

The relation between dichotomies and the asymptotic behaviour of solutions of (2) can be used to determine conditions for the existence of a dichotomy directly from $\mathbf{A}(t)$. Coppel [6] provides an introduction to this. For ordinary and exponential dichotomies in finite dimensional Banach spaces (with real scalars) Massera and Schäffer [18] introduce auxiliary functions akin to Liapunov functions suited to this purpose. Muldowney [20], who uses auxiliary functions of a similar kind and matrix measures, relates properties of $\mathbf{A}(t)$ to (μ_1, μ_2)-dichotomy conditions for $\mathbf{B} = \mathbf{R}^n$ and $\mathbf{B} = \mathbf{C}^n$. Further results on Liapunov functions and dichotomies appear in Coppel [4].

Dichotomies can be used to establish the uniqueness of solutions of the kind needed in (15) to establish the map $y \mapsto x$ described there. Much valuable information about this is presented in Massera and Schäffer [18], but they intentionally avoid discussing non-linear systems, and do not relate their work to this application. In this way, dichotomies are related to the study of quasi-linear systems. The relationship between non-linear systems and dichotomies is also being pursued in terms of the variational equation [10]. (A related formulation is touched on below).

Dichotomies describe the asymptotic behaviour of two subspaces of solutions. There are generalizations that describe three (trichotomies) or more subspaces of solutions. It would be interesting to phrase these as bounds on a Green function of a form different than that considered here.

The work of Perron has also grown in a different setting, manifolds with a Riemannian metric instead of Banach spaces [9]. Exponential dichotomies and exponential trichotomies arise there. They are not conditions on subspaces of solutions like ours.

In stable manifold theory, they assume in advance that the solutions of interest always have their values restricted to some compact subset. Then they try to investigate more closely what can happen. Our questions are different. In fact, we would be quite happy if solutions starting in a compact subset left the subset at some future time, provided we could describe how this happens. Despite this difference in setting and purpose, it is useful to compare this work with ours. For example, "dichotomies in variation" may be better formulated in terms of the derived flow.

Exponential trichotomy conditions ("hyperbolicity requirements") were used by Smale [22] in proving the existence of centre-stable manifolds. For an outline of these results see Abraham and Marsden [1 , pp. 528–531]. No doubt more general dichotomies will be used in that setting.

5. VARIATION OF CONSTANTS FORMULA AND GREEN FUNCTIONS

We have seen in (15) that it may be useful to consider systems of the form (1) for which $\mathbf{A}(t)$ or $\mathbf{f}(t)$ or both may not be explicitly determined in advance, although we may know some restrictions on them. What restrictions are necessary and sufficient to determine the behaviour or unique existence of solutions? Only part of the problem is considered. In the remainder of this paper, we find conditions on the solutions of the homogeneous system (2) which, in the presence of mild restrictions on $\mathbf{f}$, allow us to describe the behaviour of solutions of (1).

The problem of solving the inhomogeneous system (1) is now considered.

$$\mathbf{x}' - \mathbf{A}(t)\mathbf{x} = \mathbf{f}(t), \qquad t \in J \tag{16}$$

A change of variables simplifies this equation. Call the new variable $\mathbf{y}$.

$$\mathbf{x} = \mathbf{X}\mathbf{y} \tag{17}$$

Here $\mathbf{X}$ is a fundamental solution for the corresponding homogeneous system (2). Note that $\mathbf{X}$ always exists, but is not usually known explicitly, so the subsequent process is valuable for finding properties of the solutions of (1), but not usually the solutions explicitly.

This substitution (17) uses the variable $\mathbf{y}$ to replace the constant $\mathbf{c}$ in (6), and is therefore known as "variation of constants" or "variation of parameters". This last comes from the finite dimensional case where the components of $\mathbf{c}$ may be thought of as parameters used to determine which solution fits given initial conditions. (For any one solution of (2) though, the components of c are constant.)

Substitute (17) in (16) and solve for $\mathbf{y}'$.

$$\mathbf{y}' = \mathbf{X}^{-1}\mathbf{f} \tag{18}$$

Here, $(\mathbf{X}' - \mathbf{A}\mathbf{X})\,\mathbf{y} = \mathbf{0}$ from equation (10) has been used. The benefit of the change of variable is that this can be directly integrated.

$$\mathbf{y} = \int \mathbf{X}^{-1}\mathbf{f} \tag{19}$$

The idea of a dichotomy arises because it is advantageous to separate this integral into two pieces. Let us continue without doing so, to see what the problem is. This means we will eventually write (19) as this definite integral

$$\mathbf{y} = \mathbf{c} + \int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds \qquad \text{for any } t_0 \text{ in } J$$

where $\mathbf{c}$ is an arbitrary constant in $\mathbf{B}$. Before using this, let us continue from (19) for one more step.

The plan is to apply $\mathbf{X}$ to both sides to transform back to the original variable, then make the time dependence explicit again.

$$\mathbf{x} = \mathbf{X}\int \mathbf{X}^{-1}\mathbf{f}$$

$$\mathbf{x}(t) = \mathbf{X}(t)\left(\mathbf{c} + \int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds\right)$$

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds \tag{20}$$

This solution consists of two terms. Using different constants c from B, the left term produces each solution of the homogeneous system (2). The formula is therefore of the form expected from the superposition principle 1.2.

This result is known as the variation of constants formula, and may be verified by differentiation provided the integral exists.

The existence or boundedness of the solution (20) can be ensured for a variety of **f** by requiring the system to meet certain stability requirements.

Example.

Here is a sketch, without proof, of how these ideas develop when there is a bound of this form.

$$\left\|\mathbf{X}(t)\mathbf{X}^{-1}(s)\right\| \leq L\exp\left[m(t-s)\right] \qquad t \geq s \tag{21}$$

Here L is a positive real constant and m is a strictly negative real constant.

Systems meeting this condition are said to be uniformly asymptotically stable for increasing t, and the reader may already be aware of sufficient conditions on $\mathbf{A}(t)$ which guarantee this bound.

This gives a bound for the integral in (20).

$$\left\|\mathbf{X}(t)\int_{t_0}^{t}\mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds\right\| \leq \int_{t_0}^{t}\left\|\mathbf{X}(t)\mathbf{X}^{-1}(s)\right\|\,\left\|\mathbf{f}(s)\right\|\,ds$$

$$\leq \int_{t_0}^{t} L\exp\left[m(t-s)\right]\,\left\|\mathbf{f}(s)\right\|\,ds \qquad t \geq t_0 \tag{22}$$

Even when b, the right endpoint of J, is infinite, (22) will remain bounded as $t \to b$, provided some minor restriction is placed on $\mathbf{f}$. For example, this remains bounded when $\mathbf{f}$ is a locally integrable function for which $\int_t^{t+1} \|\mathbf{f}(\tau)\| \, d\tau < K$, where K is the same constant for any choice of t. Provided $\mathbf{f}$ meets this requirement, the estimate (21) thus ensures the right term in (20) is bounded.

The estimate (21) also ensures the left term in (20), which is a solution of the homogeneous system, will be bounded as $t \to b$. Thus if the system is uniformly asymptotically stable for increasing t, all solutions of the inhomogeneous system will be bounded as $t \to b$ for any $\mathbf{f}$ which meets the above requirements. No detailed knowledge of $\mathbf{f}$ is needed. ∎

This example is of limited practical use. The problem is that very few systems ever meet condition (21) because it requires all solutions to be well-behaved as $t \to b$. The stroke of genius is to harness separately the converging power of the solutions which are well-behaved as $t \to b$ and that of solutions which are well-behaved as $t \to a$. Let us return to (19) and separate the integral into two pieces.

Consider $\mathbf{B}$, the domain of the fundamental solution $\mathbf{X}$. (If $\mathbf{X}(t_0) = \mathbf{I}$, this domain may be considered to be the different possible initial values solutions may have at t_0). Let us assume there are two supplementary projections, $\mathbf{P}_1$ and $\mathbf{P}_2$, on $\mathbf{B}$ such that solutions corresponding to points in the subspace $\mathbf{P}_1\mathbf{B}$ and those corresponding to points in the subspace $\mathbf{P}_2\mathbf{B}$ have different behaviour. We use these projections to separate the integral in (19).

$$\mathbf{y} = \mathbf{b} + \int_a^t \mathbf{P}_1 \mathbf{X}^{-1}\mathbf{f} + \int_b^t \mathbf{P}_2 \mathbf{X}^{-1}\mathbf{f}$$

Here $\mathbf{b}$ is a constant in $\mathbf{B}$. It is not in general the same as $\mathbf{c}$ in (20), hence the introduction of the new symbol. This generalization includes (20) as a special case, since we may take $a = t_0$, $\mathbf{P}_1 = \mathbf{I}$, and $\mathbf{P}_2 = \mathbf{0}$. The lower limits of each integral are called a and b because henceforth they will be thought of as the endpoints of J, the interval of interest.

Note that the limits have been chosen in each case so that the integral is taken over different intervals. As will be apparent shortly, this enables advantage to be taken of the two different behaviours. The derivative is still $\mathbf{P}_1 \mathbf{X}^{-1}\mathbf{f} + \mathbf{P}_2 \mathbf{X}^{-1}\mathbf{f} = (\mathbf{P}_1 + \mathbf{P}_2)\mathbf{X}^{-1}\mathbf{f} = \mathbf{X}^{-1}\mathbf{f}$ as (18) requires.

Applying $\mathbf{X}$ to transform back to the original variable now results in this form for (20).

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{X}(t)\mathbf{b} + \mathbf{X}(t)\int_a^t \mathbf{P}_1 \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds + \mathbf{X}(t)\int_b^t \mathbf{P}_2 \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds \\ &= \mathbf{X}(t)\mathbf{b} + \int_a^t \mathbf{X}(t)\mathbf{P}_1 \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds + \int_t^b -\mathbf{X}(t)\mathbf{P}_2 \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds \end{aligned} \tag{24}$$

Since t is a common limit of the integrals, they may be joined. This results in the following form.

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{b} + \int_a^b \mathbf{G}(t,s)\mathbf{f}(s)\,ds \tag{25}$$

$$\text{where}\quad \mathbf{G}(t,s) = \begin{cases} \mathbf{X}(t)\mathbf{P}_1\mathbf{X}^{-1}(s), & t > s \\ \mathbf{X}(t)\,(\mathbf{P}_1 - \mathbf{P}_2)\,\mathbf{X}^{-1}(s), & t = s \\ -\mathbf{X}(t)\mathbf{P}_2\mathbf{X}^{-1}(s), & t < s \end{cases} \tag{26}$$

A function which behaves as G does in (25) is called a Green function.

It has been shown that solutions of (1) are given by (25) when the integral exists. The general solution of the inhomogenous system (1) given by (25) is still of the form expected by the superposition principle 1.2. Using each b in $\overline{\mathbf{B}}$ in the term $\mathbf{X}(t)\mathbf{b}$ gives each solution of the homogeneous system (2), and the term $\int_a^b \mathbf{G}(t,s)\mathbf{f}(s)\,ds$ is a particular solution of the inhomogeneous system.

Any choice of supplementary pair of projections $\mathbf{P}_1$ and $\mathbf{P}_2$ in the definition of G (26) is acceptable, provided it permits proof that the integral in (25) exists. For any choice of projections, (26) shows G depends only on X, a fundamental solution of the homogeneous system (2), and not on f. The term $\mathbf{X}(t)\mathbf{b}$ in (25) also depends only on X and not on f. By careful choice of projections, it is thus possible to describe the solutions (25) of the inhomogeneous system from knowledge of the homogeneous system. The task is complicated by the fact that a fundamental solution X for (2) is seldom known explicitly, but conditions on $\mathbf{A}(t)$ can be formulated which ensure X and G have desired properties.

6. BOUNDS ON THE GREEN FUNCTION

Bounds on the Green function are equivalent to bounds on the solutions of the homogeneous system. To see this, we start by phrasing the definition of the Green function (26) in terms of the solutions of the homogeneous system (2).

In the previous chapter, **x** was used in the more general sense as a solution of the inhomogeneous system. When $\mathbf{f}(t)$ is set to 0, all results there reduce to the corresponding result for the homogeneous system. Now we are reverting to using **x** only in the more special sense as a solution of the homogeneous system.

Every solution **x** of the homogeneous system is given by $\mathbf{x} = \mathbf{X}\mathbf{c}$ for some **c** in **B**. (When $\mathbf{X}(t_0) = \mathbf{I}$ for some t_0 in J, **c** is the initial value of the solution at t_0.) The projections $\mathbf{P}_1$ and $\mathbf{P}_2$ split **c** into two components, namely $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$ where $\mathbf{c}_1 = \mathbf{P}_1\mathbf{c}$ and $\mathbf{c}_2 = \mathbf{P}_2\mathbf{c}$. These lie in two supplementary subspaces of **B**, namely $\mathbf{B}_1 = \mathbf{P}_1\mathbf{B}$ comprised of all possible $\mathbf{c}_1$ components, and $\mathbf{B}_2 = \mathbf{P}_2\mathbf{B}$ comprised of all possible $\mathbf{c}_2$ components.

By using the isomorphism **X**, $\mathbf{c}_1$ and $\mathbf{c}_2$ correspond to two component solutions, namely $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1 = \mathbf{X}\mathbf{c}_1$ and $\mathbf{x}_2 = \mathbf{X}\mathbf{c}_2$. These lie in two supplementary subspaces of the set of solutions **S**, namely $\mathbf{S}_1 = \mathbf{X}\mathbf{B}_1$ comprised of all possible $\mathbf{x}_1$ components, and $\mathbf{S}_2 = \mathbf{X}\mathbf{B}_2$ comprised of all possible $\mathbf{x}_2$ components. (When $\mathbf{X}(t_0) = \mathbf{I}$ for some t_0, $\mathbf{B}_1$ is a subspace of initial values, and $\mathbf{S}_1$ is the subspace of solutions that pass through them at t_0. These may stray far from

B_1 at other times. The corresponding remarks hold for B_2 and S_2.)

If we know $x(s)$ for some s, the component solution x_1 can be found this way.

$$x_1 = XP_1X^{-1}(s)x(s) \tag{27}$$

This is a result of the fact that $x(s) = X(s)c$ for all s, so the constant c can be determined by knowing the value $x(s)$ at any s, from $c = X^{-1}(s)x(s)$. Starting from $x(s)$, (27) represents the following operations.

$$x(s) \xrightarrow{X^{-1}(s)} c \xrightarrow{P_1} c_1 \xrightarrow{X} x_1$$

Note that for any one solution x, the same c and hence the same function x_1 results, independent of the s for which $x(s)$ is known.

From (27), for all t in J

$$x_1(t) = X(t)P_1X^{-1}(s)x(s)$$

From the same line of reasoning for the other component x_2, we get this result.

$$x_2(t) = X(t)P_2X^{-1}(s)x(s)$$

This allows us to rewrite (26) this way.

$$G(t,s)x(s) = \begin{cases} x_1(t) & t > s \\ x_1(t) - x_2(t) & t = s \\ -x_2(t) & t < s \end{cases} \tag{28}$$

When told what $\mathbf{x}(s)$ is at any present time s, the $\mathbf{x}_1$ component is known for future times, the $\mathbf{x}_2$ component is known for past times, and the difference between them is known at the present. Knowing these three for all solutions $\mathbf{x}$, that is, for all $\mathbf{x}(s) \in \mathbf{B}$, is equivalent to knowing the Green function. This much information is seldom available. However, bounds on the Green function will imply bounds on these, and bounds on these will imply bounds on the Green function.

The norm of the Green function may now be written in the style of (26) or in the style of (28).

$$\|\mathbf{G}(t,s)\| = \begin{cases} \|\mathbf{X}(t)\mathbf{P}_1\mathbf{X}^{-1}(s)\| & t > s \\ \|\mathbf{X}(t)\ (\mathbf{P}_1 - \mathbf{P}_2)\ \mathbf{X}^{-1}(s)\| & t = s \\ \|\mathbf{X}(t)\mathbf{P}_2\mathbf{X}^{-1}(s)\| & t < s \end{cases} \tag{29}$$

$$\|\mathbf{G}(t,s)\| = \sup \frac{\|\mathbf{G}(t,s)\mathbf{x}(s)\|}{\|\mathbf{x}(s)\|} = \begin{cases} \sup \dfrac{\|\mathbf{x}_1(t)\|}{\|\mathbf{x}(s)\|} & t > s \\ \sup \dfrac{\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|}{\|\mathbf{x}(s)\|} & t = s \\ \sup \dfrac{\|\mathbf{x}_2(t)\|}{\|\mathbf{x}(s)\|} & t < s \end{cases} \tag{30}$$

where the sup is taken over all $\mathbf{x}(s) \in \mathbf{B}$, $\mathbf{x}(s) \neq \mathbf{0}$ or, equivalently, over all $\mathbf{x} \in \mathbf{S}$, $\mathbf{x} \neq \mathbf{0}$.

For the case $t = s$, we consider $\|\mathbf{G}(t,t)\|$ at a fixed t: $\mathbf{G}(t,t) = \mathcal{P}_1 - \mathcal{P}_2$ where $\mathcal{P}_k = \mathbf{X}(t)\mathbf{P}_k\mathbf{X}^{-1}(t)$. Thus $\|\mathbf{G}(t,t)\| = \|\mathcal{P}_1 - \mathcal{P}_2\|$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are supplementary projections on $\mathbf{B}$. In general, $\|\mathcal{P}_1 - \mathcal{P}_2\|$, the norm of the difference between two supplementary projections on a Banach space **B**, is a measure of the separation of the two corresponding subspaces $P_1\mathbf{B}$ and $P_2\mathbf{B}$. Before proceeding with our particular case, let us look at the general result, which relates $\|P_1 - P_2\|$ to two other common measures of angular separation.

Since $1 = \|(P_1 - P_2)^2\| \leq \|P_1 - P_2\|^2$, we have $\|P_1 - P_2\| \geq 1$. Note that $\|P_1 - P_2\|$ is defined even when one of the projections P_k is 0 so the other is I (equivalently when one of the subspaces $P_k\mathbf{B}$ is $\{0\}$ so the other is **B**). In that case $\|P_1 - P_2\|$ has the value 1. (This value occurs in other situations, notably when the subspaces are non-zero but orthogonal, as we will see). In contrast, the other two measures of angular separation are not usually defined unless both projections are non-zero (equivalently, both subspaces are not $\{0\}$, so contain non-zero vectors and hence unit vectors). This shortcoming can be patched up by adopting conventions, but this is not useful. Here are the definitions when both subspaces are not $\{0\}$.

Definition. *This is angular distance* $\mathrm{Sn}(\mathcal{R}, \mathcal{S})$ *between two non-zero subspaces* $\mathcal{R}$ *and* $\mathcal{S}$ *of a Banach space.*

$$\mathrm{Sn}(\mathcal{R}, \mathcal{S}) = \inf_{\substack{\mathbf{r}\in\mathcal{R},\, \mathbf{r}\neq 0 \\ \mathbf{s}\in\mathcal{S},\, \mathbf{s}\neq 0}} \left\| \frac{\mathbf{r}}{\|\mathbf{r}\|} - \frac{\mathbf{s}}{\|\mathbf{s}\|} \right\|$$

It measures the least distance between any two unit vectors, one from $\mathcal{R}$ *and the other from* $\mathcal{S}$.

Definition. *The angle* θ *between two non-zero subspaces* $\mathcal{R}$ *and* $\mathcal{S}$ *of a Banach space can be defined only when the angle between vectors is defined.* **B** *could* *be $\mathbf{R}^n$ or $\mathbf{C}^n$ with the Euclidean norm, or any other Hilbert space. For non-zero vectors,*

$$\angle(\mathbf{r},\mathbf{s}) = \arccos \Re \left\langle \frac{\mathbf{r}}{\|\mathbf{r}\|}, \frac{\mathbf{s}}{\|\mathbf{s}\|} \right\rangle \qquad \angle(\mathbf{r},\mathbf{s}) \in [0,\pi]$$

where $\langle\ ,\ \rangle$ is the hermitean inner product associated with the space, and $\Re$ indicates the real part. (For $\mathbf{R}^n$, this is the the usual angle between vectors). The angle θ between the two non-zero subspaces $\mathcal{R}$ and $\mathcal{S}$ is defined this way.

$$\theta = \inf_{\substack{\mathbf{r}\in\mathcal{R},\, \mathbf{r}\neq 0 \\ \mathbf{s}\in\mathcal{S},\, \mathbf{s}\neq 0}} \angle(\mathbf{r},\mathbf{s})$$

It measures the least angle between any two non-zero vectors (equivalently, between any two unit vectors), one from $\mathcal{R}$ and the other from $\mathcal{S}$.

Lemma 6.1. Angular separation and the norm of the difference of supplementary projections. *Let $\mathbf{B}$ be any Banach space, and $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two supplementary projections on $\mathbf{B}$, so that $\mathcal{P}_1\mathbf{B}$ and $\mathcal{P}_2\mathbf{B}$ are the corresponding supplementary subspaces. The norm of the difference between the projections $\|\mathcal{P}_1 - \mathcal{P}_2\|$ is a measure of the separation of the subspaces $\mathcal{P}_1\mathbf{B}$ and $\mathcal{P}_2\mathbf{B}$. When both of the subspaces are not $\{0\}$, it is related to the other measures of angular separation as follows.*

(i) general Banach spaces

$$\frac{2}{1+\|\mathcal{P}_1-\mathcal{P}_2\|} \leq \mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) \leq \frac{4}{1+\|\mathcal{P}_1-\mathcal{P}_2\|}$$

(ii) Hilbert spaces only

$$\|P_1 - P_2\| = \cot\frac{\theta}{2}$$

$$\mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) = 2\sin\frac{\theta}{2} = \frac{2}{\sqrt{1 + \|P_1 - P_2\|^2}}$$

The bounds $\dfrac{2}{1 + \|P_1 - P_2\|}$ or $\dfrac{4}{1 + \|P_1 - P_2\|}$ are actually attained for some Banach spaces and projections. For specific Banach spaces they may be improved, as the result for Hilbert spaces shows.

Proof.

Before proving lemma 6.1 for general Banach spaces, two examples will be treated. The first example is a description without proof for $\mathbf{B} = \mathbf{R}^2$ with the Euclidean norm. For proof, the reader can look at the second example, which includes the first as a special case. The second example treats Hilbert spaces and provides an outlined proof. It gives the result in part (ii) of the lemma. Then the result for general Banach spaces is proved.

$$\|P_1 - P_2\| = \sup_{\mathbf{c}\neq\mathbf{0}} \frac{\|(P_1 - P_2)\,\mathbf{c}\|}{\|\mathbf{c}\|} = \sup_{\mathbf{c}\neq\mathbf{0}} \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|}$$

where $\mathbf{c}_k = P_k\mathbf{c}$.

What we are trying to do is understand the geometric significance of the ratio.

$$K = \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \qquad \mathbf{c} \neq \mathbf{0} \tag{31}$$

The set of all possible K will be denoted as $\mathcal{K}$.

$$\mathcal{K} = \left\{ \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \;\middle|\; \mathbf{c} \in \mathbf{B}, \mathbf{c} \neq \mathbf{0} \right\}$$

$$\|\mathcal{P}_1 - \mathcal{P}_2\| = \sup \mathcal{K} \tag{32}$$

Since $\mathbf{c}_1 - \mathbf{c}_2$ is a non-zero vector in $\mathbf{B}$ whenever $\mathbf{c}_1 + \mathbf{c}_2$ is, changing the sign of $\mathbf{c}_2$ in (31) results in another element of $\mathcal{K}$. Thus $\mathcal{K}$ is a non-empty set of strictly positive real numbers that contains all its reciprocals. Any finite upper bound on this set is thus equivalent to a strictly positive lower bound. The two bounds are reciprocals. Furthermore, any upper bound must be at least one, and any lower bound must be at most one and imposes no restriction unless it is greater than zero.

Example. $\mathbf{B} = \mathbf{R}^2$ *with the Euclidean norm*

The components $\mathbf{c}_1$ and $\mathbf{c}_2$ belong to two supplementary subspaces of $\mathbf{B}$, namely $\mathcal{P}_1\mathbf{B}$ and $\mathcal{P}_2\mathbf{B}$. For $\mathbf{B} = \mathbf{R}^2$ with the Euclidean norm, the possible supplementary pairs of subspaces are of two kinds. First, one of the subspaces could be $\{\mathbf{0}\}$ and the other $\mathbf{R}^2$. This is equivalent to one of $\mathcal{P}_1$ or $\mathcal{P}_2$ being zero. Since one of the $\mathbf{c}_k$ in (31) is zero and the other is not, $K = 1$ always. $\|\mathcal{P}_1 - \mathcal{P}_2\| = \sup \mathcal{K} = 1$. This conforms to the discussion before the lemma was stated. Parts (i) and (ii) of the lemma do not apply to this case.

The second kind of supplementary pair of subspaces occurs when each subspace is a straight line. The lines intersect at one point, the origin. This is equivalent to both $\mathcal{P}_1$ and $\mathcal{P}_2$ being non-zero. For each non-zero $\mathbf{c} \in \mathbf{R}^2$, we can think of the parallelogram that shows $\mathbf{c}$ as the vector sum of the two components $\mathbf{c}_1$ and $\mathbf{c}_2$, each on a different one of these lines. Then K in (31) is the ratio of the diagonals. It is a natural measure of the angular separation between $\mathbf{c}_1$ and $\mathbf{c}_2$.

When the two lines are orthogonal, the parallelogram is a rectangle and the ratio $K = 1$ always. When the two lines are not orthogonal, for different $\mathbf{c}$ this ratio will vary, but it attains its extremes when the parallelogram is a rhombus. For these extreme cases, it is easy to see that the ratio is $\tan\frac{\phi}{2}$ where ϕ is the angle in the parallelogram at the origin. For a rhombus in a quadrant where this angle is acute, the ratio attains its minimum. For a rhombus in a quadrant where this angle is obtuse, the ratio attains its maximum. These two rhombuses are similar, but the roles of the diagonals are reversed. The corresponding ratios K are therefore reciprocals. For future reference, the different $\mathbf{c}$ values for each of these two rhombuses will be called A and B.

For every parallelogram, whether it is a rhombus or not, there is a similar parallelogram in an adjacent quadrant for which the roles of the diagonals are reversed. The corresponding ratios K are reciprocals. This illustrates the point that the set K contains all its reciprocals.

If θ is the acute angle between the two lines, for the rhombuses $\phi = \theta$ or $\phi = \pi - \theta$, so the extreme values of K are $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$. Thus from (32), $\|P_1 - P_2\| = \sup K = \cot\frac{\theta}{2}$.

There are only two possible angles, θ and $\pi - \theta$ between two unit vectors, one on each line. The least distance between any two such unit vectors is easily seen to be $\mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) = 2\sin\frac{\theta}{2}$.

This verifies the statement of part (ii) of the lemma for this case. However, it is interesting that each possible $K \in \left[\tan\frac{\theta}{2}, \cot\frac{\theta}{2}\right]$ is actually attained. This may be seen by considering a path from A to B which avoids the origin. Thus $\mathcal{K} = \left[\tan\frac{\theta}{2}, \cot\frac{\theta}{2}\right]$. ∎

Recall $P_1\mathbf{B}$ is the subspace of $\mathbf{B}$ which consists of all possible values of $\mathbf{c}_1$. The corresponding remark holds for $P_2\mathbf{B}$. In this example, there were only two possible angles ϕ between non-zero values of $\mathbf{c}_1$ and $\mathbf{c}_2$. The supplementary subspaces of $\mathbf{B}$, $P_1\mathbf{B}$ and $P_2\mathbf{B}$, were lines.

The situation may be different in other spaces. For example, take $\mathbf{B} = \mathbf{R}^3$ with the Euclidean norm. Assume one of the subspaces, say $P_1\mathbf{B}$, is a plane containing the origin while the other subspace is a line which intersects this plane only at the origin but is not perpendicular to it. The angle ϕ varies as different non-zero vectors, one on the plane the other on the line, are considered.

Apart from allowing for more possible values for ϕ, generalization to spaces where angles make sense (that is, when $\mathbf{B}$ is $\mathbf{R}^n$ or $\mathbf{C}^n$ with the Euclidean norm or any other Hilbert space) is straightforward.

Example. *Hilbert spaces. Proof of (ii) in lemma 6.1.*

For any vector $\mathbf{c} \in \mathbf{B}$, $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$ where $\mathbf{c}_k \in P_k\mathbf{B}$. The two supplementary subspaces $P_1\mathbf{B}$ and $P_2\mathbf{B}$ of $\mathbf{B}$ are comprised respectively of all possible values of $\mathbf{c}_1$ and $\mathbf{c}_2$.

We can write $\mathbf{c}_1 = a\mathbf{u}_1$ where $\mathbf{u}_1$ is a unit vector in $P_1\mathbf{B}$, and a is a real scalar. When $\mathbf{c}_1 \neq 0$, $\mathbf{u}_1 = \pm \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|}$ and $a = \pm \|\mathbf{c}_1\|$. When $\mathbf{c}_1 = 0$, we use the hypothesis that $P_1 \neq 0$ (that is, $P_1\mathbf{B} \neq \{0\}$) to show some unit vector $\mathbf{u}_1$ in $P_1\mathbf{B}$ exists, and write $\mathbf{c}_1 = a\mathbf{u}_1$ where $a = 0\ (= \pm \|\mathbf{c}_1\|)$. By repeating this construction we can write $\mathbf{c}_2 = b\mathbf{u}_2$ where $b = \pm \|\mathbf{c}_2\|$ and $\mathbf{u}_2$ is a unit vector in $P_2\mathbf{B}$. $\mathbf{c} = \mathbf{0}$ if and only if $a = b = 0$.

$$\mathbf{B} - \{0\} = \{a\mathbf{u}_1 + b\mathbf{u}_2 \mid \mathbf{u}_k \in P_k\mathbf{B},\ \|\mathbf{u}_k\| = 1,\ k = 1,2,\ a, b \in \mathbf{R} \text{ not both zero}\}$$

Here $\mathbf{u}_1$ and $\mathbf{u}_2$ are not fixed, but roam over all possible unit vectors in $P_1\mathbf{B}$ and $P_2\mathbf{B}$ respectively.

$$K = \left\{ \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \;\middle|\; \mathbf{c} \in \mathbf{B}, \mathbf{c} \neq \mathbf{0} \right\}$$

$$= \left\{ \frac{\|a\mathbf{u}_1 - b\mathbf{u}_2\|}{\|a\mathbf{u}_1 + b\mathbf{u}_2\|} \;\middle|\; \mathbf{u}_k \in P_k\mathbf{B},\ \|\mathbf{u}_k\| = 1,\ k = 1,2,\ a, b \in \mathbf{R} \text{ not both zero} \right\}$$

Since a hermitean inner product $\langle\ ,\ \rangle$ is available, the usual cosine rule for the sum of two vectors applies. For $\mathbf{c}_1 + \mathbf{c}_2 = a\mathbf{u}_1 + b\mathbf{u}_2$ we have

$$\begin{aligned}\|a\mathbf{u}_1 + b\mathbf{u}_2\|^2 &= \langle a\mathbf{u}_1 + b\mathbf{u}_2, a\mathbf{u}_1 + b\mathbf{u}_2\rangle \\ &= \langle a\mathbf{u}_1, a\mathbf{u}_1\rangle + \langle b\mathbf{u}_2, b\mathbf{u}_2\rangle + \langle a\mathbf{u}_1, b\mathbf{u}_2\rangle + \langle b\mathbf{u}_2, a\mathbf{u}_1\rangle \\ &= a^2 + b^2 + 2ab\cos\angle(\mathbf{u}_1, \mathbf{u}_2)\end{aligned}$$

where $\cos\angle(\mathbf{u}_1,\mathbf{u}_2) = \frac{1}{2}[\langle\mathbf{u}_1,\mathbf{u}_2\rangle + \langle\mathbf{u}_2,\mathbf{u}_1\rangle] = \Re\langle\mathbf{u}_1,\mathbf{u}_2\rangle$. $\Re$ indicates the real part. We may take $\angle(\mathbf{u}_1,\mathbf{u}_2) = \arccos\Re\langle\mathbf{u}_1,\mathbf{u}_2\rangle$ to lie in $(0,\pi)$.

This enables us to write

$$K^2 = \frac{a^2 + b^2 - 2ab\cos\phi}{a^2 + b^2 + 2ab\cos\phi}, \qquad a, b \text{ not both zero} \tag{33}$$

where a, b are any real numbers. The set of all possible angles ϕ which occur here is Φ.

$$\Phi = \{\angle(\mathbf{u}_1, \mathbf{u}_2) \mid \mathbf{u}_k \in P_k\mathbf{B}, \|\mathbf{u}_k\| = 1, k = 1,2\}$$

$$= \{\angle(\mathbf{c}_1, \mathbf{c}_2) \mid \mathbf{c}_k \in P_k\mathbf{B}, \mathbf{c}_k \neq 0, k = 1,2\}$$

For any fixed $\phi \in \Phi$, (33) has extreme values when $a = \pm b$. Since K is positive, the extreme values of K are $\tan\frac{\phi}{2}$ and $\cot\frac{\phi}{2}$.

As previously defined, the angle between the subspaces $P_1\mathbf{B}$ and $P_2\mathbf{B}$ is $\theta = \inf\Phi$. Note $0 \leq \theta \leq \frac{\pi}{2}$ because Φ is symmetric about $\frac{\pi}{2}$. To see this, pick any $\phi \in \Phi$. Then $\phi = \angle(\mathbf{u}_1, \mathbf{u}_2)$ for some $\mathbf{u}_1$ and $\mathbf{u}_2$. Since $\pi - \phi = \angle(\mathbf{u}_1, -\mathbf{u}_2)$, and $-\mathbf{u}_2$ is a unit vector in $P_2\mathbf{B}$ whenever $\mathbf{u}_2$ is, $\pi - \phi \in \Phi$.

For all $\phi \in \Phi$, $\max\{\tan\frac{\phi}{2}, \cot\frac{\phi}{2}\} = \cot\frac{\varphi}{2}$ where $\varphi = \min\{\phi, \pi - \phi\}$. The set of all possible φ is $\Phi' = \{\min\{\phi, \pi - \phi\} \mid \phi \in \Phi\} = \Phi \cap (0, \frac{\pi}{2}]$ where the symmetry of Φ has been used to show φ is in Φ when it is $\pi - \phi$.

$$\|P_1 - P_2\| = \sup K = \sup_{\varphi\in\Phi'} \cot\frac{\varphi}{2} = \cot \inf_{\varphi\in\Phi'} \frac{\varphi}{2} = \cot \inf_{\phi\in\Phi} \frac{\phi}{2} = \cot\frac{\theta}{2}$$

The square of the distance between two unit vectors, one in $P_1\mathbf{B}$, the other in $P_2\mathbf{B}$, is $2 - 2\cos\phi$ for $\phi \in \Phi$. Since $\inf_{\phi\in\Phi}(2 - 2\cos\phi) = 2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}$, we have $\mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) = 2\sin\frac{\theta}{2}$. Since $\|P_1 - P_2\| = \cot\frac{\theta}{2}$

$$\mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) = \frac{2}{\sqrt{1 + \|P_1 - P_2\|^2}}$$

This completes the proof of (ii) in lemma 6.1.

It is interesting that the set K can be found explicitly. This is not required for the proof of the lemma, so results are outlined.

$$\mathcal{K} = \begin{cases} \left[\tan\frac{\theta}{2}, \cot\frac{\theta}{2}\right], & \theta \in \Phi \\ \left(\tan\frac{\theta}{2}, \cot\frac{\theta}{2}\right), & \theta \notin \Phi \end{cases}$$

The fact that all intermediate values are attained results from considering paths from $a = +b$ to $a = -b$ which do not intersect $a = b = 0$, for a sequence of ϕ with limit θ. The parameters a and b were designed to permit this.

For finite dimensional Hilbert spaces, such as $\mathbf{R}^n$ or $\mathbf{C}^n$ with the Euclidean norm, $\mathcal{K}$ is always the closed interval. For infinite dimensional Hilbert spaces, cases where $\mathcal{K}$ is the open interval and cases where it is the closed interval both occur. ▮

Now we take up the proof of part (i) of lemma 6.1.

$$\frac{2}{1+\|\mathcal{P}_1+\mathcal{P}_2\|} = \frac{2}{1+\left[\sup\limits_{c\neq 0}\dfrac{\|\mathbf{c}_1-\mathbf{c}_2\|}{\|\mathbf{c}_1+\mathbf{c}_2\|}\right]}$$

$$= \inf_{c\neq 0} \frac{2\|\mathbf{c}_1+\mathbf{c}_2\|}{\|\mathbf{c}_1+\mathbf{c}_2\|+\|\mathbf{c}_1-\mathbf{c}_2\|}$$

$$\leq \inf_{\mathbf{u}_k\in U_k} \frac{2\|\mathbf{u}_1+\mathbf{u}_2\|}{\|\mathbf{u}_1+\mathbf{u}_2\|+\|\mathbf{u}_1-\mathbf{u}_2\|}$$

where $U_k = \{\mathbf{u}_k \in \mathcal{P}_k\mathbf{B} \mid \|\mathbf{u}_k\| = 1\}$, $k = 1,2$. In each case, U_k is the set of all unit vectors in the corresponding subspace. A subset of $\mathbf{B} - \{0\}$ is now indicated in the inf, so it cannot be smaller than before.

From the triangle inequality, $\|\mathbf{u}_1+\mathbf{u}_2\| + \|\mathbf{u}_1-\mathbf{u}_2\| \geq \|2\mathbf{u}_1\| = 2$.

$$\inf_{\mathbf{u}_k \in U_k} \frac{2\|\mathbf{u}_1 + \mathbf{u}_2\|}{\|\mathbf{u}_1 + \mathbf{u}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\|} \leq \inf_{\mathbf{u}_k \in U_k} \|\mathbf{u}_1 + \mathbf{u}_2\|$$

Finally we have the required result.

$$\frac{2}{1 + \|\mathcal{P}_1 + \mathcal{P}_2\|} \leq \inf_{\mathbf{u}_k \in U_k} \|\mathbf{u}_1 + \mathbf{u}_2\| = \mathrm{Sn}(\mathcal{P}_1\mathbf{B}, \mathcal{P}_2\mathbf{B})$$

Next we find the upper bound on $\mathrm{Sn}(\mathcal{P}_1\mathbf{B}, \mathcal{P}_2\mathbf{B})$.

$$\inf_{\mathbf{u}_k \in U_k} \|\mathbf{u}_1 - \mathbf{u}_2\| \leq \frac{4}{1 + \|\mathcal{P}_1 - \mathcal{P}_2\|}$$

Let $D = \|\mathbf{c}_1 + \mathbf{c}_2\| + \|\mathbf{c}_1 - \mathbf{c}_2\|$.

$$\left\| \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} + \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} \right\| \leq \frac{1}{D} \left\| 2(\mathbf{c}_1 + \mathbf{c}_2) + \frac{D - 2\|\mathbf{c}_1\|}{\|\mathbf{c}_1\|}\mathbf{c}_1 + \frac{D - 2\|\mathbf{c}_2\|}{\|\mathbf{c}_2\|}\mathbf{c}_2 \right\|$$

$$\leq \frac{1}{D}\Big(2\|\mathbf{c}_1 + \mathbf{c}_2\| + |\, D - 2\|\mathbf{c}_1\| \,| + |\, D - 2\|\mathbf{c}_2\| \,| \Big)$$

$$= \frac{1}{D}\Big(4\|\mathbf{c}_1 + \mathbf{c}_2\| + 2\|\mathbf{c}_1 - \mathbf{c}_2\| - 2\left[\|\mathbf{c}_1\| + \|\mathbf{c}_2\|\right] \Big)$$

$$\leq \frac{4\|\mathbf{c}_1 + \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\| + \|\mathbf{c}_1 - \mathbf{c}_2\|}$$

Here $\mathbf{c}_k = \mathcal{P}_k\mathbf{c}$ and $\mathbf{c}_k \neq \mathbf{0}$. Note $D \geq 2\|\mathbf{c}_k\|$ by the triangle inequality, so the absolute value of a non-negative value is considered in each case.

$$\inf_{\mathbf{c}_k \neq \mathbf{0}} \left\| \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} + \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} \right\| \leq \inf_{\mathbf{c}_k \neq \mathbf{0}} \frac{4\|\mathbf{c}_1 + \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\| + \|\mathbf{c}_1 - \mathbf{c}_2\|} = \frac{4}{1 + \sup\limits_{\mathbf{c}_k \neq \mathbf{0}} \dfrac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|}}$$

Since $\{\|\mathbf{u}_1 - \mathbf{u}_2\| \mid \mathbf{u}_k \in U_k\} = \left\{ \left\| \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} + \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} \right\| \;\middle|\; \mathbf{c}_k \neq \mathbf{0} \right\}$ the left term is $\mathrm{Sn}(\mathcal{P}_1, \mathcal{P}_2)$ as required. Now we want to relate the right term to $\|\mathcal{P}_1 - \mathcal{P}_2\|$.

$$\sup \left\{ \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \;\middle|\; \mathbf{c}_k \neq \mathbf{0} \right\} = \sup \left\{ \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \;\middle|\; \mathbf{c} \neq \mathbf{0} \right\} = \|\mathcal{P}_1 - \mathcal{P}_2\|$$

To see this, first note the set $S = \left\{ \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \;\middle|\; \mathbf{c}_k \neq \mathbf{0} \right\}$ is not empty since the subspaces $P_k\mathbf{B}$ contain non-zero elements by hypothesis. Pick any $a \in S$. Then a and $\frac{1}{a}$ are both in S. Therefore S contains an element not less than 1. The set S is a subset of the other set $S' = \left\{ \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{\|\mathbf{c}_1 + \mathbf{c}_2\|} \;\middle|\; \mathbf{c} \neq \mathbf{0} \right\}$. The only difference between S and S' is contributions from elements for which precisely one of the $\mathbf{c}_k$ is zero. These contribute the value 1. Since this is equalled or exceeded by a value in S, the two sets have the same supremum.

This completes the proof of the bound.

$$\mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) \leq \frac{4}{1 + \|P_1 - P_2\|}$$

The following bounds have been obtained.

$$\frac{2}{1 + \|P_1 - P_2\|} \leq \mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) \leq \frac{4}{1 + \|P_1 - P_2\|}$$

Can we hope to do better? The following examples show that these bounds are actually attained for some Banach spaces and projections, so for arbitrary Banach spaces the constants which appear here are the best possible. (This does not preclude better bounds of a different form). For particular spaces we can do better, as shown by the previous results for Hilbert spaces.

Example. $\mathrm{Sn}(P_1\mathbf{B}, P_2\mathbf{B}) = \frac{4}{1 + \|P_1 - P_2\|}$

We consider $\mathbf{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$, and take P_k to be the canonical projections, so that $P_1(x_1, x_2) = (x_1, 0)$ and $P_2(x_1, x_2) = (0, x_2)$. Then $\|P_1 - P_2\| = \sup_{|x_1|+|x_2|=1} \|(x_1, 0) - (0, x_2)\| = 1$. $U_1 = \{(1,0), (-1,0)\}$ and $U_2 =$

$\{(0,1),(0,-1)\}$, so $\mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) = \inf_{\mathbf{u}_k \in U_k} \|\mathbf{u}_1 - \mathbf{u}_2\| = \min\{\|(\pm1,\pm1)\|\}$, where the choices of sign are independent. Thus $\mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) = 2$ and

$$\mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) = 2 = \frac{4}{2} = \frac{4}{1+\|\mathcal{P}_1-\mathcal{P}_2\|}$$

Example. $\mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) = \dfrac{2}{1+\|\mathcal{P}_1-\mathcal{P}_2\|}$

We keep the same space R^2 and same projections $\mathcal{P}_k$ as in the previous example, but change the norm to $\|(x_1,x_2)\| = \max\{|x_1|,|x_2|\}$. The U_k are the same as before. $\|\mathcal{P}_1-\mathcal{P}_2\| = \sup_{\max\{|x_1|,|x_2|\}=1} \|(x_1,0)-(0,x_2)\| = 1$, but now $\mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) = \min\{\|(\pm1,\pm1)\|\} = 1$. Thus

$$\mathrm{Sn}(\mathcal{P}_1\mathbf{B},\mathcal{P}_2\mathbf{B}) = 1 = \frac{2}{2} = \frac{2}{1+\|\mathcal{P}_1-\mathcal{P}_2\|}$$

This completes the proof of lemma 6.1. ▮

Let us now apply this general lemma to the particular case where $\|\mathcal{P}_1-\mathcal{P}_2\|$ is the norm of the Green function $\|\mathbf{G}(t,t)\|$ for fixed t in (29) or (30). If $\mathbf{x}(t)$ is the value at t of some solution $\mathbf{x}$ that has non-zero components $\mathbf{x}_1 \in \mathbf{S}_1$ and $\mathbf{x}_2 \in \mathbf{S}_2$, it is seen that this norm is a measure of the least possible angular separation between $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, since $\mathcal{P}_1\mathbf{x}(t) = \mathbf{x}_1(t)$ and $\mathcal{P}_2\mathbf{x}(t) = \mathbf{x}_2(t)$. (If one of the $\mathbf{S}_k$ is $\{0\}$, this norm is one, but angular separation is not defined). Equivalently, this norm measures the least angular separation between the subspaces $\mathcal{P}_1\mathbf{B} = \mathbf{X}(t)\mathbf{P}_1\mathbf{B}$ and $\mathcal{P}_2\mathbf{B} = \mathbf{X}(t)\mathbf{P}_2\mathbf{B}$, comprised of all possible values at t taken by solutions in $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. Armed with this interpretation, we can now proceed to the main results of this chapter.

Theorem 6.2. *For a Green function of the form (26), suppose* $\|\mathbf{G}(t,s)\| \leq g(t,s)$ *where* $g(s,t)$ *is a (non-negative) real valued function. The behaviour of solutions of the homogeneous system is then restricted as follows.*

$$g(t,s) \text{ for } t > s \Longrightarrow \begin{cases} \mathbf{P}_1 = 0 & \text{no restraint on any solution} \\ \mathbf{P}_1 \neq 0 & \text{these restraints on the solutions } \mathbf{x}_1 \in \mathbf{S}_1 \\ & \|\mathbf{x}_1(T)\| \leq g(T,t_0)\|\mathbf{x}_1(t_0)\| \quad \text{for } T > t_0 \\ & \|\mathbf{x}_1(\tau)\| \geq \dfrac{1}{g(t_0,\tau)}\|\mathbf{x}_1(t_0)\| \quad \text{for } \tau < t_0 \end{cases}$$

$$g(t,s) \text{ for } t = s \Longrightarrow \begin{cases} \text{this restraint on the solutions } \mathbf{x} \in \mathbf{S},\ \mathbf{x} \neq \mathbf{0} \\ \dfrac{1}{g(t,t)} \leq \dfrac{\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|}{\|\mathbf{x}_1(t) + \mathbf{x}_2(t)\|} \leq g(t,t) \quad \text{for all } t \\ \text{That is,} \\ \qquad \mathrm{Sn}(t) \geq \dfrac{2}{1 + g(t,t)} \\ \text{where } \mathrm{Sn}(t) = \inf\limits_{\mathbf{x}_k \neq 0} \left\| \dfrac{\mathbf{x}_1(t)}{\|\mathbf{x}_1(t)\|} - \dfrac{\mathbf{x}_2(t)}{\|\mathbf{x}_2(t)\|} \right\| \\ \text{measures the least angular separation between} \\ \text{the values } \mathbf{x}_1(t) \text{ and } \mathbf{x}_2(t) \text{ for non-zero } \mathbf{x}_k \in \mathbf{S}_k. \\ \text{For Hilbert spaces,} \\ \qquad \angle\,(\mathbf{x}_1(t), \mathbf{x}_2(t)) \geq 2\mathrm{arccot}\, g(t,t) \end{cases} \tag{34}$$

$$g(t,s) \text{ for } t < s \Longrightarrow \begin{cases} \mathbf{P}_2 = 0 & \text{no restraint on any solution} \\ \mathbf{P}_2 \neq 0 & \text{these restraints on the solutions } \mathbf{x}_2 \in \mathbf{S}_2 \\ & \|\mathbf{x}_2(T)\| \geq \dfrac{1}{g(t_0,T)}\|\mathbf{x}_2(t_0)\| \quad \text{for } T > t_0 \\ & \|\mathbf{x}_2(\tau)\| \leq g(\tau,t_0)\|\mathbf{x}_2(t_0)\| \quad \text{for } \tau < t_0 \end{cases}$$

Proof.

Here is how these have been obtained from (30). Consider the first case, when $t > s$. It will be shown that the top form in (30) implies the two inequalities in the top case in (34). It is seen that when $\mathbf{P}_1 = 0$, or equivalently $\mathbf{x}_1 = 0$ for all solutions $\mathbf{x} \in \mathbf{S}$, this says $0 \leq g(t,s)$ and imposes no restraint. Otherwise $\mathbf{S}_1$ contains non-zero solutions $\mathbf{x}_1$, and writing the top form in (30) for them yields this.

$$\|\mathbf{x}_1(t)\| \leq g(t,s)\|\mathbf{x}_1(s)\| \qquad t > s \tag{35}$$

Thus $g(t,s)$ is strictly positive. This allows us to take its reciprocal later. Note (35) also holds for the zero solution in $\mathbf{S}_1$.

Letting the roles of t and s in (35) be taken by T and t_0 respectively immediately gives the first inequality in the top case of (34). Letting the roles of t and s in (35) be taken by t_0 and r respectively, gives the second inequality in the top case of (34).

The case $t > s$ is now finished. Similar reasoning applies to the case $t < s$, so only the case $t = s$ needs further discussion. Note that $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2$ is a (non-zero) solution in $\mathbf{S}$ if and only if $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ is a (non-zero) solution. Therefore $\dfrac{\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|}{\|\mathbf{x}_1(t) + \mathbf{x}_2(t)\|} \leq g(t,t)$ remains true when the sign of $\mathbf{x}_2$ is reversed. Since the left side (hence the right side) of this inequality is strictly positive, this is equivalent to $\dfrac{1}{g(t,t)} \leq \dfrac{\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|}{\|\mathbf{x}_1(t) + \mathbf{x}_2(t)\|}$, and we see $g(t,t) \geq 1$.

The remaining inequalities follow from lemma 6.1 and the discussion following it. Take $\mathcal{P}_k = \mathbf{X}(t)\mathbf{P}_k\mathbf{X}^{-1}(t)$. Then $\mathbf{G}(t,t) = \mathcal{P}_1 - \mathcal{P}_2$ and $\mathrm{Sn}(t) = \mathrm{Sn}(\mathcal{P}_1\mathbf{B}, \mathcal{P}_2\mathbf{B})$. Note that when one of the $\mathbf{P}_k$ is zero, these remaining inequalities say nothing, since it is not possible to find non-zero $\mathbf{x}_k \in \mathbf{S}_k$. The ratio $\dfrac{\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|}{\|\mathbf{x}_1(t) + \mathbf{x}_2(t)\|} = 1$ in this case. ∎

By considering the conditions in this theorem as $\tau \to a$ or $T \to b$ where $J = [a, b]$ is the interval of definition of the system, it is seen that the asymptotic behaviour of solutions in each subspace is described. This is determined by the asymptotic form of $g(t,s)$.

It appears that in passing from the conditions on the left side of (34) to the conclusions on the right, some information has been lost. With the exception of the middle "angular separation" condition ($t = s$), we have retained information only about solutions which are purely in $\mathbf{S}_1$ or purely in $\mathbf{S}_2$. Via (30), the condition $\|\mathbf{G}(t,s)\| \le g(t,s)$ also restricts solutions which are a mixture of the two kinds. When one of the subspaces $\mathbf{S}_k$ is $\{0\}$, there is no problem. Otherwise, if we start from the right side of (34) it is not possible in general to to reverse the implications completely. However, it turns out that enough information about the way solutions mix is contained in the angular separation condition to allow us to recover from the right side of (34) somewhat weaker bounds on the norm of the Green function. For this we need the following tool.

Lemma 6.3. Angular separation and norms of supplementary projections. *Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are supplementary projections on a Banach space* $\mathbf{B}$. *The following estimate is valid when $\mathcal{P}_k \neq \mathbf{0}$.*

$$\frac{1}{\|\mathcal{P}_k\|} \leq \mathrm{Sn}(\mathcal{P}_1\mathbf{B}, \mathcal{P}_2\mathbf{B}) \leq \frac{2}{\|\mathcal{P}_k\|}$$

Proof.

This is lemma 1.1 on page 156 in Daleckiĭ and Kreĭn [7]. ∎

Note that $1 + \|\mathcal{P}_1 - \mathcal{P}_2\| = \|\mathcal{P}_1 + \mathcal{P}_2\| + \|\mathcal{P}_1 - \mathcal{P}_2\| \geq 2\|\mathcal{P}_k\|$ so (when $\mathcal{P}_k \neq 0$), the order of the bounds in lemmas 6.1 and 6.3 is this.

$$\frac{2}{1 + \|\mathcal{P}_1 - \mathcal{P}_2\|} \leq \frac{1}{\|\mathcal{P}_k\|} \leq \mathrm{Sn}(\mathcal{P}_1\mathbf{B}, \mathcal{P}_2\mathbf{B}) \leq \frac{4}{1 + \|\mathcal{P}_1 - \mathcal{P}_2\|} \leq \frac{2}{\|\mathcal{P}_k\|}$$

Theorem 6.4. *(Partial converse of theorem 6.2.) Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be two supplementary subspaces of solutions of a homogeneous system,* $\mathbf{S}_1 \neq \{0\}$ *and* $\mathbf{S}_2 \neq \{0\}$*, satisfying these conditions for some (non-negative) real-valued functions* $k(t,s)$ *and* $\alpha(t)$.

$$\|\mathbf{x}_1(t)\| \leq k(t,s)\|\mathbf{x}_1(s)\| \quad t > s,\ \mathbf{x}_1 \in \mathbf{S}_1$$

$$\mathrm{Sn}(t) \geq \alpha(t) > 0$$

$$\|\mathbf{x}_2(t)\| \leq k(t,s)\|\mathbf{x}_2(s)\| \quad s > t,\ \mathbf{x}_2 \in \mathbf{S}_2$$

where for each fixed t, $\mathrm{Sn}(t) = \inf_{\mathbf{x}_k \neq 0} \left\| \frac{\mathbf{x}_1(t)}{\|\mathbf{x}_1(t)\|} - \frac{\mathbf{x}_2(t)}{\|\mathbf{x}_2(t)\|} \right\|$ *measures the least angular distance between the values* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *for non-zero* $\mathbf{x}_k \in \mathbf{S}_k$*. Then the* *corresponding Green function of the form (26) satisfies* $\|\mathbf{G}(t,s)\| \le g(t,s)$ *where*

$$g(t,s) = \begin{cases} \dfrac{2k(t,s)}{\alpha(s)} & t > s \\ \dfrac{4}{\alpha(t)} - 1 & t = s \\ \dfrac{2k(t,s)}{\alpha(s)} & t < s \end{cases} \tag{36}$$

Here we have written the cases for $t > s$ and $t < s$ separately, since often in applications the form of $k(t,s)$ is different in these two cases. The conditions on the solutions here are like those on the right side of (34). They have been abbreviated by the omission of the reciprocal relations, which are equivalent.

Bounds on angular separation are often derived using differential inequalities. Traditionally Sn has been used even when angles are defined. Its derivatives are easier. For this reason, the angular separation condition has been phrased in terms of Sn. Note this condition could also be written in the style $\dfrac{\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|}{\|\mathbf{x}_1(t) + \mathbf{x}_2(t)\|} \le k(t,t)$, immediately giving a bound on $\|\mathbf{G}(t,t)\|$ and indirectly providing a factor in place of $\alpha(s)$ in the other terms.

Proof.

First take $\mathcal{P}_k = \mathbf{X}(s)\mathbf{P}_k\mathbf{X}^{-1}(s)$,

so that $\mathbf{x}_k(s) = \mathcal{P}_k\mathbf{x}(s)$ and $\mathrm{Sn}(s) = \mathrm{Sn}(\mathcal{P}_1\mathbf{B}, \mathcal{P}_2\mathbf{B})$.

$\|\mathbf{x}_1(s)\| = \|\mathcal{P}_1\mathbf{x}(s)\| \le \|\mathcal{P}_1\|\,\|\mathbf{x}(s)\|$. Since $\mathbf{S}_1 \ne \{0\}$, it follows that $\mathbf{P}_1 \ne 0$ and $\mathcal{P}_1 \ne 0$. Thus $\|\mathbf{x}(s)\| \ge \dfrac{1}{\|\mathcal{P}_1\|}\|\mathbf{x}_1(s)\|$. When $\mathbf{x}_1 \ne 0$

$$\frac{\|\mathbf{x}_1(t)\|}{\|\mathbf{x}(s)\|} \le \frac{2\|\mathbf{x}_1(t)\|}{\dfrac{2}{\|\mathcal{P}_1\|}\|\mathbf{x}_1(s)\|} \le \frac{2\|\mathbf{x}_1(t)\|}{\mathrm{Sn}(s)\|\mathbf{x}_1(s)\|} \le \frac{2k(t,s)}{\alpha(s)} \qquad t > s$$

Here lemma 6.3. has been applied.

For $t > s$, (30) shows $\|G(t,s)\| = \sup_{x \neq 0} \frac{\|x_1(t)\|}{\|x(s)\|}$. Since the ratio is 0 whenever $x_1(t) = 0$, the inequality

$$\frac{\|x_1(t)\|}{\|x(s)\|} \leq \frac{2k(t,s)}{\alpha(s)} \qquad t > s$$

is always satisfied when $x \neq 0$, and

$$\|G(t,s)\| = \sup_{x \neq 0} \frac{\|x_1(t)\|}{\|x(s)\|} \leq \frac{2k(t,s)}{\alpha(s)} \qquad t > s$$

as required.

The case $t < s$ is proved by replacing subscript 1 by 2 throughout. The case $t = s$ follows from lemma 6.1. Take $\mathcal{P}_k = X(t)P_kX^{-1}(t)$. Then $G(t,t) = \mathcal{P}_1 - \mathcal{P}_2$ and $\mathrm{Sn}(t) = \mathrm{Sn}(\mathcal{P}_1B, \mathcal{P}_2B)$. ∎

To pin down these ideas, let us apply them to some simple cases.

Example. *Ordinary dichotomy.* $\|G(t,s)\| \leq L$

Here we are considering $g(t,s) = L$, a real constant. Since we know $\|G(t,t)\| \geq 1$, this is only possible for $L \geq 1$.

Case 1. $P_1 = 0, P_2 = I$

When $P_1 = 0$, the subspace $S_1 = \{0\}$. The angular distance between the values at t of two solutions, one in S_1 and the other in S_2, is not defined. The conditions in the last case in (34) show that for the subspace $S_2 = S$,

$$\|x_2(T)\| \geq \frac{1}{L}\|x_2(t_0)\| \qquad \text{for } T > t_0 \tag{37}$$

so all non-zero solutions are bounded away from zero for increasing time, and

$$\|\mathbf{x}_2(\tau)\| \leq L\|\mathbf{x}_2(t_0)\| \qquad \text{for } \tau < t_0 \tag{38}$$

so all non-zero solutions are bounded for decreasing time.

Conversely, if all non-zero solutions are bounded away from zero for increasing time and bounded for decreasing time, in a uniform way so that (37), or equivalently (38), holds with the same L for all solutions, then the Green function of the form (26) with $\mathbf{P}_1 = \mathbf{0}$ is bounded.

Case 2. $\mathbf{P}_1 = \mathbf{I}, \mathbf{P}_2 = \mathbf{0}$

When $\mathbf{P}_2 = \mathbf{0}$, the analysis is similar to case 1, but the behaviour is complementary in the sense that the roles of increasing and decreasing times are reversed.

Case 3. $\mathbf{P}_1 \neq \mathbf{0}, \mathbf{P}_2 \neq \mathbf{0}$

When both $\mathbf{P}_1 \neq \mathbf{0}$ and $\mathbf{P}_2 \neq \mathbf{0}$, the angular distance between the values at t of two non-zero solutions, one in $\mathbf{S}_1$ and the other in $\mathbf{S}_2$, is defined. At each fixed t, $\mathrm{Sn}(t)$ measures the least such distance. From the middle case in (34), $\mathrm{Sn}(t) \geq \dfrac{2}{1+L}$. Not only must the non-zero solutions in $\mathbf{S}_1$ always be at strictly positive angular distance from any non-zero solutions in $\mathbf{S}_2$, but also this distance is bounded away from zero both for increasing and decreasing t. That is, it cannot approach zero even for extreme times. This last is a consequence of the fact that $L = g(t,t)$ is constant. (It will also hold for other kinds of $g(t,t)$ which are uniformly bounded for all t. It may or may not hold for $g(t,t)$ which are unbounded for extreme times, since this does not determine whether $\|\mathbf{G}(t,t)\|$ remains bounded or not).

The analysis of the behaviour of solutions in S_1 repeats that in case 1 and case 2 above. This leads to the following summary of results (34) of thereom 6.2 for the case $P_1 \neq 0, P_2 \neq 0$.

Ordinary Dichotomy conditions. *There are supplementary subspaces of solutions S_1 and S_2 with these properties.*

(i) *All solutions in S_1 are bounded for increasing t and all non-zero solutions in S_1 are bounded away from zero for decreasing t. More precisely, the norm of non-zero solutions in S_1 cannot experience gain more than a factor of L for any interval of increasing t no matter how long. (Equivalently, the norm of non-zero solutions in S_1 cannot diminish by more than a factor of $\frac{1}{L}$ for any interval of decreasing t no matter how long). Here $L \geq 1$ does not depend on which solutions are considered.*

(ii) *All solutions in S_2 are bounded for decreasing t and all non-zero solutions in S_2 are bounded away from zero for increasing t. More precisely, the norm of non-zero solutions in S_2 cannot diminish by more than a factor of $\frac{1}{L}$ for any interval of increasing t no matter how long. (Equivalently, the norm of non-zero solutions in S_2 cannot experience gain more than a factor of L for any interval of decreasing t no matter how long). Here $L \geq 1$ does not depend on which solutions are considered.*

(iii) *At each fixed t, the non-zero solutions in $\mathbf{S}_1$ are always at strictly positive angular distance from all non-zero solutions in $\mathbf{S}_2$, and even in the limit as $t \to a$ or $t \to b$ this angular distance is bounded away from zero. Here $J = [a, b]$ is the interval of definition of the system.*

Note the behaviours in (i) and (ii) are complementary, in the sense that the roles of increasing and decreasing t are reversed.

The word dichotomy means a cutting in two. Here the space of solutions S is cut into two parts $\mathbf{S}_1$ and $\mathbf{S}_2$ with complementary behaviour. This kind of dichotomy is called ordinary to distinguish it from other kinds which will soon be introduced.

So far we have seen that systems with a Green function of the form (26) which is bounded by a constant satisfy these ordinary dichotomy conditions. This was a consequence of theorem 6.2. These conditions were specifically stated for the case 3 where $\mathbf{P}_1 \neq 0$ and $\mathbf{P}_2 \neq 0$. However, rereading them for case 1 and case 2 shows that the statement is still true for them. In case 1, $\mathbf{S}_1 = \{0\}$. The non-zero solutions in $\mathbf{S}_1$ talked about in (i) and (iii) do not exist. These conditions say nothing and are harmlessly true. In case 2, $\mathbf{S}_2 = \{0\}$. The non-zero solutions in $\mathbf{S}_2$ talked about in (ii) and (iii) do not exist. These conditions say nothing and are harmlessly true.

Now we want to see that when any homogeneous system satisfies these ordinary dichotomy conditions, there is a Green function (for the corresponding inhomogeneous system) of the form (26) which is bounded by a constant. This has already been seen for case 1 and case 2. It [illegible]s to show this for case 3 by using theorem 6.4.

Suppose there are two supplementary subspaces of solutions $\mathbf{S}_1$ and $\mathbf{S}_2$. Let us use (i) in the ordinary dichotomy conditions to formulate the required behaviour of solutions in $\mathbf{S}_1$. The solutions in $\mathbf{S}_1$ do not experience gain more than a factor of K_1 for any interval of increasing t no matter how long. (Equivalently, the norm of non-zero solutions in $\mathbf{S}_1$ does not diminish by more than a factor of $\frac{1}{K_1}$ for any interval of decreasing t no matter how long). Here $K_1 \geq 1$ does not depend on which solutions are considered. Thus we have

$$\|\mathbf{x}_1(t)\| \leq K_1 \|\mathbf{x}_1(s)\| \qquad t > s, \mathbf{x}_1 \in \mathbf{S}_1 \tag{39}$$

Next let us use (ii) in the ordinary dichotomy conditions to formulate the required behaviour of solutions in $\mathbf{S}_2$. The norm of non-zero solutions in $\mathbf{S}_2$ does not diminish by more than a factor of $\frac{1}{K_2}$ for any interval of increasing t no matter how long. (Equivalently, the norm of non-zero solutions in $\mathbf{S}_2$ does not experience gain more than a factor of K_2 for any interval of decreasing t no matter how long). Here $K_2 \geq 1$ does not depend on which solutions are considered.

$$\|\mathbf{x}_2(t)\| \leq K_2 \|\mathbf{x}_2(s)\| \qquad t < s, \mathbf{x}_2 \in \mathbf{S}_2 \tag{40}$$

Note that different constants K_1 and K_2 have been used for each subspace of solutions. Both (39) and (40) remain true when K_1 and K_2 are replaced by larger constants. Therefore, if these are true for some K_1 and K_2, replacing both K_1 and K_2 by $K = \max\{K_1, K_2\}$ or any larger constant will bring them strictly into accord with the ordinary dichotomy conditions previously stated. However, improved bounds on the Green function will result if we allow K_1 and K_2 to be individually as small as possible. This is not helpful in the present search for a single constant bound on the Green function. We will effectively consider their maximum anyway. However, with an eye to other applications for which improved bounds might be helpful, we defer this as long as possible.

Next, let us use (iii) in the ordinary dichotomy conditions to formulate the required behaviour of the angular distance between the non-zero solutions in $\mathbf{S}_1$ and all non-zero solutions in $\mathbf{S}_2$. This is bounded away from zero.

$$\left\| \frac{\mathbf{x}_1(t)}{\|\mathbf{x}_1(t)\|} - \frac{\mathbf{x}_2(t)}{\|\mathbf{x}_2(t)\|} \right\| \geq \beta > 0 \qquad \text{for all } \mathbf{x}_k \in \mathbf{S}_k,\, \mathbf{x}_k \neq \mathbf{0}$$

That is,

$$\inf_{\mathbf{x}_k \neq \mathbf{0}} \left\| \frac{\mathbf{x}_1(t)}{\|\mathbf{x}_1(t)\|} - \frac{\mathbf{x}_2(t)}{\|\mathbf{x}_2(t)\|} \right\| = \mathrm{Sn}(t) \geq \beta > 0$$

(This is equivalent to $\inf_{t \in J'} \mathrm{Sn}(t) > 0$ and the condition is sometimes stated that way).

Application of theorem 6.4. is now straightforward. Take $k(t,s) = K_1$ for $t > s$, $k(t,s) = K_2$ for $t < s$, and $\alpha(t) = \beta$. Then (36) shows $\|\mathbf{G}(t,s)\| \leq g(t,s)$ where

$$g(t,s) = \begin{cases} \dfrac{2K_1}{\beta} & t > s \\ \dfrac{4}{\beta} - 1 & t = s \\ \dfrac{2K_2}{\beta} & t < s \end{cases}$$

That is, $\|\mathrm{G}(t,s)\| \leq L$ where $L = \max\left\{\frac{2K_1}{\beta}, \frac{4}{\beta} - 1, \frac{2K_2}{\beta}\right\}$ is a constant. This concludes the proof that when any homogeneous system satisfies the ordinary dichotomy conditions, there is a Green function of the form (26) which is bounded by a constant.

In summary, the solutions of a homogeneous system satisfy the ordinary dichotomy conditions if and only if there is a Green function of the form (26) the norm of which is bounded by a constant. This finishes the example. ∎

Example. (μ_1, μ_2)-*dichotomy.* $$\|\mathrm{G}(t,s)\| \leq \begin{cases} L\exp\left(\int_s^t \mu_1\right) & t > s \\ L & t = s \\ L\exp\left(\int_s^t \mu_2\right) & t < s \end{cases}$$

Here μ_1 and μ_2 are integrable functions of t, usually assumed continuous. L is a real constant, and from the case $t = s$ we know only $L \geq 1$ are possible. We can write the condition in this compressed form.

$$\|\mathrm{G}(t,s)\| \leq g(t,s) = L\exp\left(\int_s^t \mu_k\right), \quad \text{if } (-1)^k(s-t) \geq 0, k = 1,2 \tag{41}$$

Elsewhere, these conditions are written in the form

$$\|\mathbf{X}(t)\mathbf{P}_k\mathbf{X}^{-1}(s)\| \leq L_k \exp\left(\int_s^t \mu_k\right), \quad \text{if } (-1)^k(s-t) \geq 0, k = 1,2 \tag{42}$$

where L_1 and L_2 are positive constants.

The two forms (41) and (42) are not the same. However bounds of the form (41) imply bounds of the form (42), and bounds of the form (42) imply bounds of the form (41). Let us prove this.

First suppose bounds of the form (41) are given. Let $\mathbf{X}(t)\mathbf{P}_k\mathbf{X}^{-1}(t) = \mathcal{P}_k$. From the triangle inequality

$$\|\mathcal{P}_k\| \le \frac{\|\mathcal{P}_1 + \mathcal{P}_2\| + \|\mathcal{P}_1 - \mathcal{P}_2\|}{2} = \frac{1 + \|\mathcal{P}_1 - \mathcal{P}_2\|}{2} \le \frac{1+L}{2} \le L$$

where we have used the fact that $\mathbf{G}(t,t) = \mathcal{P}_1 - \mathcal{P}_2$ to show $\|\mathcal{P}_1 - \mathcal{P}_2\| \le L$ and to show only $L \ge 1$ are possible, so $\frac{1+L}{2} \le L$. Therefore, (42) holds with $L_1 = L$ and $L_2 = L$.

Second suppose bounds of the form (42) are given. $\|\mathcal{P}_1 - \mathcal{P}_2\| \le \|\mathcal{P}_1\| + \|\mathcal{P}_2\|$, so (41) holds with $L = L_1 + L_2$. Note that a constant upper bound for $\|\mathbf{G}(t,t)\|$ is thus inherent in (42). When both $\mathbf{P}_k$ are not zero, this is equivalent to a positive lower bound on the angular distance between non-zero solutions, one in each supplementary subspace of solutions of the homogeneous system.

Therefore we may assume the conditions are in the form (41), or more explicitly, in the form given in the example heading. Using theorem 6.2, these translate into (μ_1, μ_2)-dichotomy conditions on the solutions of the homogeneous system analagous to the ordinary dichotomy conditions (i), (ii) and (iii), developed in the previous example. Using $g(t,s) = L\exp\left(\int_s^t \mu_1\right)$ in the top case of (34), gives the analogue of (i). Using $g(t,s) = L\exp\left(\int_s^t \mu_2\right)$ in the bottom case of (34), gives the analogue of (ii). Considering the limits of these analogues of (i) and (ii) as

$t \to a$ or $t \to b$ reveals the asymptotic behaviour of solutions in S_1 and S_2. The condition (iii) remains unchanged since $g(t,t) = L$ is a constant.

If the solutions of a homogeneous system satisfy these (μ_1, μ_2)-dichotomy conditions, bounds of the form (41) can be recovered, either directly when one of the $\mathbf{P}_k = 0$, or by using theorem 6.4.

Thus the bounds on the Green function (41) determine and are determined by the (μ_1, μ_2)-dichotomy conditions on the behaviour of the solutions of the homogeneous system.

Here is a catalogue of some (μ_1, μ_2)-dichotomies that have been studied extensively. For them, the functions $\mu_k(t) = m_k$ where m_k is constant, k=1,2. Therefore (41) may be specialized.

$$\|\mathbf{G}(t,s)\| \le L \exp\left[m_k(t-s)\right], \qquad \text{if } (-1)^k(s-t) \ge 0, k = 1,2 \tag{43}$$

Definition. *If condition (43) is satisfied with:*

(i) $m_1 = m_2 = 0$, *the system has an ordinary dichotomy.*

(ii) $m_1 < 0 < m_2$, *the system has an exponential dichotomy.*

(iii) $m_1 < m_2$, *the system has an exponential splitting.*

Ordinary dichotomies have already been met in the preceding example. Now they are viewed as a special kind of (μ_1, μ_2)-dichotomy. The three ideas (i), (ii) and (iii) are related as follows, though we do not prove this here.

Let O be the set of all systems that have an ordinary dichotomy, E be the set of all systems that have an exponential dichotomy, and S be the set of all systems with an exponential splitting. O and S may be pictured as two overlapping circles in a Venn diagram, with E in their intersection. The sets $O - S$ and $S - O$ are not empty, and E is a proper subset of $O \cap S$.

Let $M_{\mathbf{B}}$ denote the set of all systems defined on a Banach space **B** or a Banach space isomorphic to **B** that have a (μ_1, μ_2)-dichotomy (for any continuous μ_1 and μ_2). Similarly, we define $O_{\mathbf{B}}$, $E_{\mathbf{B}}$ and $S_{\mathbf{B}}$, which are all subsets of $M_{\mathbf{B}}$. Suppose we start with a system in one of these four classes, and perform a change of variables. Will the new system be in the same class or in a different class?

This depends on the type of change of variables allowed. If smooth linear changes of variable are allowed, *any* system of the form (1) can be transformed into any other system of the form (1) (in the same space **B** or one isomorphic to it). However, if the change of variables and its inverse are in addition required to be bounded as functions of t, in which case the transform is known as a kinematic similarity, this is not possible. The sets $M_{\mathbf{B}}$, $O_{\mathbf{B}}$, $E_{\mathbf{B}}$ and $S_{\mathbf{B}}$ are in fact each stable under kinematic similarities [7, p. 166].

In some circumstances, when $\mathbf{A}(t)$ in the definition of a system is perturbed, the new system will have an exponential dichotomy if the old one did. For this reason, the property of having an exponential dichotomy is said to be rough or robust [4, p. 62] [6, p. 34] [7] [18].

Finally, we can return to (22). This can now be recognized as a system with an exponential dichotomy, with $\mathbf{P}_1 = \mathbf{I}$, $\mathbf{P}_2 = \mathbf{0}$, and $m_1 = m$. The argument in that example can be carried out for the form (25) of the solution instead of the form (21). In the next chapter we will try to elucidate the nature of such arguments. This may be viewed as an imitation of the argument for (22) and similar examples in a more general context which includes them as special cases.

This concludes the example of (μ_1, μ_2)-dichotomies. ∎

We have not made full use of theorems 6.2 and theorems 6.4. To discuss these examples time-varying bounds on $\|G(t,t)\|$ were not needed, since all systems mentioned in them satisfy the angular distance property, as detailed in (iii) of the ordinary dichotomy conditions. The use of time-varying bounds on $\|G(t,t)\|$ only becomes apparent in the next chapter.

Now we return to the study of the solutions of the inhomogeneous system in the form (26).

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{b} + \int_a^b \mathbf{G}(t,s)\mathbf{f}(s)\,ds \tag{44}$$

Here the symbol $\mathbf{x}$ represents a solution to the inhomogeneous system. This differs from its use in chapter 6 to represent solutions of the homogeneous system, but conforms to the earlier use in chapter 5.

Theorem 6.2 in the previous chapter showed bounds on the Green function determine some behaviour of the solutions of the homogeneous system. This is the term $\mathbf{X}(t)\mathbf{b}$ in (44). Let us see how conditions on $\mathbf{G}$ are useful for determining properties of the right term in (44). Here is a simple example.

Example. *Suppose the system (2) with $J = [0, \infty]$ has an exponential dichotomy. Then for any bounded continuous* $\mathbf{f}$, $\int_0^\infty \mathbf{G}(t,s)\mathbf{f}(s)\,ds$ *in (44) is bounded.*

$$\int_0^\infty \mathbf{G}(t,s)\mathbf{f}(s)\,ds = \int_0^t \mathbf{X}(t)\mathbf{P}_1\mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds - \int_t^\infty \mathbf{X}(t)\mathbf{P}_2\mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds$$

We can show the integrals exist by showing $\int_0^t \|\mathbf{X}(t)\mathbf{P}_1\mathbf{X}^{-1}(s)\|\,\|\mathbf{f}(s)\|\,ds$ and $\int_t^\infty \|\mathbf{X}(t)\mathbf{P}_2\mathbf{X}^{-1}(s)\|\,\|\mathbf{f}(s)\|\,ds$ are bounded. This can be shown using the estimates provided by the exponential dichotomy condition (namely (43) with $m_1 <$

$0 < m_2$). These also show $\left\| \int_0^\infty G(t,s)f(s)\,ds \right\| \le L\left(\frac{1}{|m_1|} + \frac{1}{m_2}\right) \sup_{t \ge 0} \|f(t)\|$. Thus $\int_0^\infty G(t,s)f(s)\,ds$ in (44) is bounded.

This result has an interesting converse. If for all bounded continuous **f**, the system (1) with $J = [0, \infty]$ has a bounded solution then the system has an exponential dichotomy. To prove this, some of the possible **f** are used as test probes. ∎

There are many similar results relating dichotomies, the nature of the solutions and the nature of the different possible **f**. For exponential and ordinary dichotomies, they are described in detail in Massera and Schäffer [18].

Here is a guide to which bounds on $\|G(t,s)\|$ may be useful.

1. *The bounds must be established from* $A(t)$. *This practical problem imposes limitations on which bounds can be used.*

The Green function is usually not explicitly known. Bounds must be inferred from $A(t)$ in the homogeneous system (2). In the case of (μ_1, μ_2)-dichotomies two general methods have been developed [20] for finite dimensional **B**. First, the bounds on $\|G(t,s)\|$ can be related to $A(t)$ using matrix measures. Second, bounds on $\|G(t,s)\|$ can be related to $A(t)$ using pairs of functions akin to Liaponov functions. Using pairs of functions allows us to accomodate the different behaviour of solutions in each subspace. This is in contrast to usual use of Liapunov functions (Liapunov's second method) in which all solutions are shown to have common asymptotic nature. References to other work are included in the previous discussion of this in chapter 4.

2. *The bounds should help describe the asymptotic behaviour of solutions of the homogeneous system.*

The limiting form of the bounds $g(t,s)$ as $t \to a$ or $t \to b$ is involved here.

3. *The bounds should help estimate the integral in (44), to prove existence and find the behaviour of solutions of the inhomogeneous system.*

Note that in the preceding example the exponential dichotomy bounds are themselves the norm of a Green function. (Any (μ_1, μ_2)-dichotomy bounds can be thought of this way, but the idea is only helpful when properties of the underlying system are known). This suggests we should use Green functions (kernels) of other systems for comparison. Properties of the other system can then help show properties of the integral in (44). The other system does not need to be of the same dimension as the original system.

Systems which do not satisfy the angular separation critereon cannot have (μ_1, μ_2)-dichotomies. On the contrary, for them $\|G(t,t)\|$ must become unbounded for extreme times. By looking at the kernels of simple systems (for example, two dimensional systems) we may find advantageous forms for comparison of these systems, in contrast to using $L = L(t)$ in (41) chosen some other way. The full generality of theorems 6.2 and 6.4 will be exploited.

Finally, by investigating the kernels of other systems, the appropriate form of the functions akin to Liapunov functions may be found, which may then be tried on the original system. This overcomes the "loss of direction" inherent in direct comparison of the norms of the kernels of the two systems.

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