Logarithmic Hopf Link Invariants for the Unrolled Restricted Quantum Group of sl(2)

by

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Abstract

Little is known about Vertex Operator Algebras (VOAs) which are neither semi-simple nor rational, and most of the work on such VOAs has been focused around specific examples such as the Singlet VOA $\mathcal{W}(2, 2r-1)$. In this thesis, the relationship between subcategories of the module categories of the Singlet VOA and the unrolled restricted quantum group associated to $\mathfrak{sl}(2)$, $\overline{U}_q^H(\mathfrak{sl}(2))$ at $2r$-th root of unity $q = e^{\pi i/r}$ with $r \geq 2$ is studied. A family of deformable modules X_ϵ is used to efficiently compute open Hopf Links for $\overline{U}_q^H(\mathfrak{sl}(2))$ and particular $(1,1)$-tangle invariants colored with projective modules of $\overline{U}_q^H(\mathfrak{sl}_2)$. These tangle invariants are extensions of the Alexander invariants defined by Murakami. It is also shown that normalized modified traces of open Hopf links for $\overline{U}_q^H(\mathfrak{sl}(2))$-modules correspond exactly with the asymptotic quantum dimensions for certain $\mathcal{W}(2, 2r-1)$-modules.

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1 Introduction

1.1 Motivation

A conformal field theory (CFT) is a quantum field theory invariant under conformal transformations, essentially, a quantum field theory with added symmetry. This added symmetry makes many problems in CFT exactly solvable in two dimensions unlike most quantum field theories. Conformal field theory plays a significant role in string theory as well as statistical mechanics in the study of higher order phase transitions. Vertex operator algebras (VOAs) were first introduced by Richard Borcherds [34] in an attempt to formalize CFT. They provide a mathematical formulation of the symmetry algebras for conformal field theory, called chiral algebras, and provide a rigorous mathematical approach to string theory and two-dimensional conformal field theory. VOAs have found applications to many other areas of mathematics such as Lie theory, algebraic geometry, topology, and modular forms, and have played an important role in connecting seemingly distant areas of mathematics. One such example is given by monstrous moonshine, a term used by John Conway and Simon Norton [35] in 1979 to describe the unexpected connections between modular forms and the monster group. The conjectures of Conway and Norton were proven by Richard Borcherds [40] in 1992 using the monster VOA constructed by Igor Frenkel, James Lepowsky, and Arne Meurman [36].

Much of the study of vertex operator algebras has been focused on VOAs which are C_2-cofinite or rational. It is known that the representation category of a rational and C_2-cofinite (regular) VOA is semi-simple, and also modular and ribbon given some weak assumptions [11,12]. In this case, the VOA also carries three coinciding actions of the modular group: one on the linear span of torus one point functions, a categorical action given by twists and Hopf links, and an action diagonalizing fusion rules (this is only a S-matrix). C_2-cofinite VOAs have representation categories with only finitely many isomorphism classes of simple objects. Clearly, it is desirable to extend our understanding to those VOAs which have infinitely many simple objects. However, once we drop the C_2-cofiniteness assumption, our VOAs become much more difficult to understand because their representation categories are so large. One approach to this problem has been to restrict ourselves to a smaller, more manageable subcategory. It was shown in [16] by Huang, Lepowsky, and Zhang that if all objects in an abelian subcategory $\mathcal{C}$ of generalized V-modules for a VOA V are C_1-cofinite with some additional requirements, then $\mathcal{C}$ is braided and monoidal. One of the fundamental results for modularity of rational VOAs was given by Zhu in [23], where it was shown that characters are closed under modular transformations. This property and the action of the modular group on the space of one-point functions on the torus can both be preserved for irrational C_2-cofinite VOAs given some additional assumptions [24]. Furthermore, if the category of modules for a C_2-cofinite VOA is ribbon, then there is also an action of the modular group in the category [15,25].

VOAs which are rational and C_2-cofinite (regular) are fairly well understood through modularity of characters and the automorphicity of chiral blocks, as well as the categorical interpretation of VOA modules. These concepts are connected through the Verlinde formula which relates fusion product coefficients in the category of modules to the dimensions of chiral blocks. We noted above that all modules of a regular VOA are completely reducible and that there are only finitely many inequivalent simple modules. Let $M_0 = V, M_1, ..., M_n$ denote the inequivalent simple modules for a regular VOA V. Then, there exists a tensor product satisfying

$$M_i \otimes M_j \simeq \bigoplus_{k=0}^{n} \mathcal{N}_{ij}^k M_k.$$

For any $\tau \in \mathbb{H}$ (the upper half of the complex plane) and $v \in V$, the 1-point function F_{M_i} is defined as

$$F_{M_i}(\tau, v) := \mathrm{tr}_{M_i}(o(v) q^{L_0 - \frac{c}{24}}),$$

where $o(v)$ is the zero mode of the field associated to v. When $v = 1$, F_{M_i} is called the character of M_i and is denoted $\mathrm{ch}[M_i](\tau)$. The modular S-transformation $\tau \mapsto -1/\tau$ defines a matrix S^χ by

$$F_{M_i}(-1/\tau, v) = \tau^{wt(v)} \sum_{j=0}^{n} S_{ij}^\chi F_{M_j}(\tau, v).$$

This matrix and the tensor coefficients are related through the Verlinde formula [11,41,42]:

$$\mathcal{N}_{ij}^k = \sum_{\ell=0}^{n} \frac{S_{i\ell}^\chi S_{j\ell}^\chi (S^{\chi^{-1}})_{\ell k}}{S_{0\ell}^\chi}.$$

The Verlinde formula is one of the deepest results for regular VOAs and has been studied by many authors. It is natural to wonder if one could prove results for irrational non C_2-cofinite VOAs analogous to existing results for rational and C_2-cofinite VOAs, and in particular, if there is a generalization of the Verlinde formula. For C_2-cofinite VOAs some progress has been made by T. Creutzig and T. Gannon on the case of the triplet VOA in [39]. Logarithmic Hopf link invariants of $\overline{U}_q(\mathfrak{sl}(2))$ were shown to give a Verlinde formula for certain structure coefficients for the tensor ring, which together with some additional information completely determines the tensor ring. The modular S-matrix of the triplet vertex algebra is compared to the logarithmic Hopf link invariants and are found to be in agreement, and a comparison of Jordan blocks with open Hopf link operators of the triplet VOA is drawn. We would like to perform a similar analysis for VOAs which are not C_2-cofinite. Little is known about the general theory of irrational non C_2-cofinite VOAs, and most of the work on such VOAs has been on developing and understanding specific examples, with a particularly nice choice of example being the singlet VOA $\mathcal{W}(2, 2r-1)$. The singlet is a desirable choice for the study of irrational non C_2-cofinite VOAs as all of its irreducible modules are C_1-cofinite [38], and has been studied by many authors [2,4,5,31,32,33].

The characters of irrational non C_2-cofinite VOAs are not closed under the modular S-transformation, so we cannot construct the matrix S^χ as we did for regular VOAs. Instead, we will use quantities called regularized quantum dimensions. The quantum dimension of a regular VOA is defined to be

$$\mathrm{qdim}[X] := \lim_{\tau \to 0^+} \frac{\mathrm{ch}[X](\tau)}{\mathrm{ch}[M_0](\tau)}.$$

Applying this to $X = M_i$, we see that

$$\begin{aligned}
\mathrm{qdim}[M_i] &= \lim_{\tau \to 0^+} \frac{\mathrm{ch}[M_i](\tau)}{\mathrm{ch}[M_0](\tau)} \\
&= \lim_{\tau \to i\infty} \frac{\mathrm{ch}[M_i](-1/\tau)}{\mathrm{ch}[M_0](-1/\tau)} \\
&= \lim_{\tau \to i\infty} \frac{\sum_{j=0}^{n} S_{ij}^\chi \mathrm{ch}[M_j](\tau)}{\sum_{j=0}^{n} S_{0j}^\chi \mathrm{ch}[M_j](\tau)} \\
&= \lim_{\tau \to i\infty} \frac{\sum_{j=0}^{n} S_{ij}^\chi \mathrm{tr}_{M_j}(q^{L_0 - \frac{c}{24}})}{\sum_{j=0}^{n} S_{0j}^\chi \mathrm{tr}_{M_j}(q^{L_0 - \frac{c}{24}})}
\end{aligned}$$

where $q = e^{2\pi i \tau}$ tends to zero as tau tends to $i\infty$. It is conjectured that there exists an M_r with dominating conformal dimension among the M_i, which gives

$$\mathrm{qdim}[M_i] = S_{ir}^\chi / S_{0r}^\chi.$$

So the quantum dimensions are closely related to the matrix S^χ for regular VOAs, and although we cannot construct S^χ for irrational non C_2-cofinite VOAs, we can construct "regularized" quantum dimensions, which we expect to play the role of S^χ in the irrational non C_2-cofinite setting.

This thesis studies primarily the category $\mathcal{C}$ of finite dimensional highest weight modules for the unrolled restricted quantum group of $\mathfrak{sl}(2)$, $\overline{U}_q^H(\mathfrak{sl}(2))$, and its connections to the singlet VOA. The quantum group $\overline{U}_q^H(\mathfrak{sl}(2))$ is the associative algebra over $\mathbb{C}$ with five generators $\{E, F, K, K^{-1}, H\}$ and relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$HK^{\pm 1} = K^{\pm 1} H, \qquad [H, E] = 2E, \qquad [H, F] = -2F, E^r = F^r = 0.$$

with additional coalgebra structure. It is conjectured that some subcategory of the representation category of the singlet is (at least) monoidally equivalent to $\mathcal{C}$, so we hope to use $\overline{U}_q^H(\mathfrak{sl}(2))$ as a

toy model to better understand the singlet. In this thesis, we motivate this conjecture and provide a new approach to obtaining the tensor structure of $\mathcal{C}$ which will be more useful when trying to understand the singlet (and perhaps more sophisticated quantum groups). Constantino, Geer, and Patureau-Mirand determined the tensor ring structure of $\mathcal{C}$ in [1] through the use of characters and other more direct means. They then used this to determine the open Hopf links. For regular VOAs, the Verlinde formula follows as a direct consequence of open Hopf links being representations of the tensor ring of modules. However, the tensor structure of the category of modules for the singlet is difficult to determine. We hope to use $\overline{U}_q^H(\mathfrak{sl}(2))$ as a toy model for the singlet, so it is natural to wonder if we could determine the open Hopf links without prior knowledge of tensor product decompositions for modules, and we will show that this is indeed possible.

It has been shown previously in examples that modular-like properties of characters [2,27-30] and their quantum dimensions [2,31] relate to the fusion ring of the singlet and other vertex (super) algebras. In particular, the asymptotic quantum dimensions of characters relate to representations of the tensor ring of modules for the singlet, and it was conjectured that these quantum dimensions should have a categorical interpretation. We provide this interpretation via Hopf link invariants of $\overline{U}_q^H(\mathfrak{sl}(2))$. This also gives a new interpretation from the perspective of quantum topology of the Jacobi variable introduced in [2] as a regularization parameter for the classical false theta functions.

1.2 Results

The proof of the Verlinde formula for regular VOAs from Hopf links requires the identity

$$S_{ij}^{\chi}/S_{0,j}^{\chi} = S_{ij}^{\infty} \tag{1}$$

where S_{ij}^{∞} is the closed Hopf link colored with the simple modules M_i, M_j for a regular VOA. The primary results of this paper are to motivate the conjecture that some subcategory of the category of modules of $\mathcal{W}(2, 2r-1)$ is monoidally equivalent and to the category $\mathcal{C}$ of finite dimensional weight modules of $\overline{U}_q^H(\mathfrak{sl}(2))$ and to generalize equation (1), to improve upon existing techniques for computing Hopf link invariants for $\mathcal{C}$, and to extend Murakami's definition of Alexander Invariant given in [8]. The first is done by constructing a map between simple modules of $\overline{U}_q^H(\mathfrak{sl}(2))$ and $\mathcal{W}(2, 2r-1)$, and giving a comparison of normalized open Hopf link invariants for $\mathcal{C}$ and the regularized quantum dimensions for modules of $\mathcal{W}(2, 2r-1)$. We show that the ring generated by normalized Hopf link invariants is identical to the ring generated by the regularized quantum dimensions and hence is isomorphic to the Verlinde algebra of characters for $\mathcal{W}(2, 2r-1)$ as follows:

The most important classes of modules in $\mathcal{C}$ are:

- n+1 dimensional simple modules S_n with $n \in \{0, .., r-1\}$.
- r-dimensional modules V_α with $\alpha \in \mathbb{C}$.

- 2r-dimensional projective indecomposable modules P_i with $i \in \{0, .., r-1\}$.
- 1-dimensional modules $\mathbb{C}^H_{\ell r}$ with $\ell \in \mathbb{Z}$.

These modules are important because they appear in the classification of finite dimensional weight modules as shown by Constantino, Geer, and Patureau-Mirand [1]:

- Every simple module in $\mathcal{C}$ is isomorphic to $S_n \otimes \mathbb{C}^H_{\ell r}$ for some $n \in \{0, ..., r-1\}$ and $\ell \in \mathbb{Z}$ or V_α for some $\alpha \in \mathbb{C} - r\mathbb{Z}$.
- The module P_i is projective and indecomposable, and any projective indecomposable weight module with integer highest weight $(\ell+1)r - i - 2$ is isomorphic to $P_i \otimes \mathbb{C}^H_{\ell r}$.

A general classification for irreducible modules of the singlet VOA is not currently known, but the $\mathbb{Z}_{\geq 0}$-graded irreducible modules fall into one of two families:

- An uncountable family of typical modules F_λ, $\lambda \in \mathbb{C}$ with $\lambda \neq -\frac{t-1}{2}\sqrt{2r} + \frac{s-1}{\sqrt{2r}}$ for any $t \in \mathbb{Z}$ and $1 \leq s \leq r$.
- A countable family of atypical modules $M_{t,s}$ with $t \in \mathbb{Z}$ and $1 \leq s \leq r-1$.

We first note that the endomorphism ring of $P_i \otimes \mathbb{C}^H_{\ell r}$ is two dimensional and spanned by $\{id, x_{i,\ell}\}$, and that $\Phi_{Z, P_i \otimes \mathbb{C}^H_{\ell r}} \in \mathrm{End}(P_i \otimes \mathbb{C}^H_{\ell r})$.

Theorem 1.

1. *Let $\alpha \in (\mathbb{C} - \mathbb{Z}) \cup r\mathbb{Z}$. Then the map $\varphi : V_\alpha \mapsto F_{\frac{\alpha+r-1}{\sqrt{2r}}}, S_i \otimes \mathbb{C}^H_{kr} \mapsto M_{1-k,i+1}$ between simple modules in $\overline{U}^H_q(\mathfrak{sl}(2))$ and $\mathcal{W}(2, 2r-1)$, respectively, is a bijection and a morphism of rings up to equality of characters.*
2. *Let $\epsilon \in \mathbb{S}(k, j+1+p(k+1))$, then*

$$qdim[\varphi(X)^\epsilon] = \frac{t_{P_j \otimes \mathbb{C}^H_{\ell r}}(\Phi_{X, P_j \otimes \mathbb{C}^H_{\ell r}} \circ x_{j,\ell})}{t_{P_j \otimes \mathbb{C}^H_{\ell r}}(\Phi_{S_0, P_j \otimes \mathbb{C}^H_{\ell r}} \circ x_{j,\ell})}$$

and if $Re(\epsilon) > 0$ then

$$qdim[\varphi(X)^{\frac{-i\alpha}{\sqrt{2r}}}] = \frac{t_{V_\alpha}(\Phi_{X,V_\alpha})}{t_{V_\alpha}(\Phi_{S_0,V_\alpha})}.$$

Here, t_X is a modified trace on the ideal of projective modules (see sections 2.1, 4.3). The Verlinde algebra of characters for $\mathcal{W}(2, 2r-1)$ and the normalized Hopf links for $\overline{U}^H_q(\mathfrak{sl}(2))$ capture a great deal of information about the representation categories of $\mathcal{W}(2, 2r-1)$ and $\overline{U}^H_q(\mathfrak{sl}(2))$ respectively, so their equivalence tells us that these categories should be the same or very similar. The equality

of regularized quantum dimensions and normalized traces of Hopf links is the irreducible non C_2-cofinite analogue of equation (1).

The second result is shown by deriving an isomorphism which allows us to determine the action of Hopf links on projective indecomposable modules from their action on simple modules. We associate to $\overline{U}_q^H(\mathfrak{sl}(2))$ a positive integer parameter r. For each fixed choice of this parameter, the ideal of projective modules in $\mathcal{C}$ is generated by r-dimensional simple modules V_α with $\alpha \in (\mathbb{C} \cup r\mathbb{Z}) - \mathbb{Z}$ and $2r$-dimensional projective indecomposable modules $P_i \otimes \mathbb{C}_{\ell r}^H$ with $i \in \{0, ..., r-1\}$ and $\ell \in \mathbb{Z}$. In a manner similar to the ideas of Murakami and Nagatomo [19,20], one can define a family of modules X_ϵ with $\epsilon \in (-\frac{1}{2}, \frac{1}{2})$ such that

$$X_\epsilon \cong \begin{cases} V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon} & \text{if } \epsilon \neq 0 \\ P_i \otimes \mathbb{C}_{\ell r}^H & \text{if } \epsilon = 0 \end{cases}$$

So the projective indecomposable modules can be thought of as the limit of a direct sum of simple modules. From this, and other considerations, we find our main result:

Proposition 1. *Let Z be a weight module in $\mathcal{C}$. Then, the Hopf link $\Phi_{Z,P_i\otimes\mathbb{C}_{\ell r}^H} = a_Z Id_{P_i\otimes\mathbb{C}_{\ell r}^H} + b_Z x_{i,\ell}$ is given by:*

$$a_Z = \lim_{\epsilon\to 0} \left(2\lambda_{Z,V_{1+i-r+\ell r+\epsilon}} - \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}\right)$$
$$b_Z = \lim_{\epsilon\to 0} \frac{-1}{[1+i][\epsilon]} (\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}} - \lambda_{Z,V_{1+i-r+\ell r+\epsilon}})$$

where $\lambda_{Z,1+i-r+\ell r+\epsilon}$ and $\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}$ are the constants by which $\Phi_{Z,V_{1+i-r+\ell r+\epsilon}}$ and $\Phi_{Z,V_{-1-i+r+\ell r+\epsilon}}$ act respectively.

This proposition allows us to determine the action of the Hopf link $\Phi_{Z,-}$ for any weight module $Z \in \mathcal{C}$ on the projective indecomposable modules from its action on the simple V_α. The action of Hopf links on simple modules is easy to find as they act as scalars on them. Hence, we do not need to know the tensor structure of $\mathcal{C}$ at all, a classification of its modules is sufficient.

The proof of the above proposition can be adapted to extend the definition of Murakami's Alexander invariant T_λ in [8]. This invariant is a framed version of the invariant defined in [7], and is constructed by assigning the braiding, twist, and duality morphisms to ribbon graphs in the usual way, and coloring the graph with the typical $\overline{U}_q^H(\mathfrak{sl}(2))$-modules V_α. We show that this definition can be extended to allow the ribbon graphs to be colored with the projective indecomposable modules $P_i \otimes \mathbb{C}_{\ell r}^H$ as follows:

Theorem 2. *The colored* $(1,1)$*-ribbon graph* $T_{P_i\otimes\mathbb{C}^H_{\ell r}}$ *satisfies*

$$t_{P_i\otimes\mathbb{C}^H_{\ell r}}\left(T_{P_i\otimes\mathbb{C}^H_{\ell r}}\right)=\lim_{\epsilon\to 0}\left(t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)+t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)\right)$$

and

$$T_{P_i\otimes\mathbb{C}^H_{\ell r}}=aId_{P_i\otimes\mathbb{C}^H_{\ell r}}+bx_{i,\ell r}$$

with coefficients

$$a=\lim_{\epsilon\to 0}\frac{t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)}{\boldsymbol{d}\left(V_{-1-i+r+\ell r+\epsilon}\right)}=\lim_{\epsilon\to 0}\frac{t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)}{\boldsymbol{d}\left(V_{1+i-r+\ell r+\epsilon}\right)}$$

and

$$\begin{aligned}b&=\lim_{\epsilon\to 0}\frac{-1}{[1+i][\epsilon]}\left(\frac{t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)}{\boldsymbol{d}\left(V_{-1-i+r+\ell r+\epsilon}\right)}-\frac{t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)}{\boldsymbol{d}\left(V_{1+i-r+\ell r+\epsilon}\right)}\right)\\&=\frac{r}{2\pi i\{1+i\}}\left(\frac{d}{d\lambda}\frac{t_{V_\lambda}\left(T_\lambda\right)}{\boldsymbol{d}\left(V_\lambda\right)}\Big|_{\lambda=\ell r+r-i-1}-\frac{d}{d\lambda}\frac{t_{V_\lambda}\left(T_\lambda\right)}{\boldsymbol{d}\left(V_\lambda\right)}\Big|_{\lambda=i+i-r+\ell r}\right).\end{aligned}$$

2 Braided Tensor Categories

We will first recall the definition of a braided tensor category. Our main references for this section are [9] and [10]. A category $\mathcal{C}$ is a collection of objects $\mathrm{Ob}(\mathcal{C})$, and of morphisms $\mathrm{Hom}(\mathcal{C})$ such that for any morphism $f\in\mathrm{Hom}(\mathcal{C})$, there are associated objects $s(f)$ and $t(f)$ called the source and target of f respectively, and we use the notation $f:s(f)\to t(f)$. A category is equipped with a composition operation $f\circ g$ which is defined whenever $s(f)=t(g)$. We require that the operation be associative and that for any $V\in\mathrm{Ob}(\mathcal{C})$, there is a morphism $Id_V:V\to V$ such that $f\circ Id_V=f$ and $Id_V\circ g=g$ whenever $f\in\mathrm{Hom}(V,W)$, $g\in\mathrm{Hom}(U,V)$ for any $U,W\in\mathrm{Ob}(\mathcal{C})$. An isomorphism in a category is, as usual, a morphism $f:U\to V$ such that there exists a morphism $g:V\to U$ satisfying $f\circ g=Id_V$ and $g\circ f=Id_U$.

A tensor product on a category $\mathcal{C}$ is an operation $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ such that

- For every pair of objects $U,V\in\mathrm{Ob}(\mathcal{C})$, there is an associated object $U\otimes V\in\mathrm{Ob}(\mathcal{C})$.
- For every pair of morphisms $f:U_1\to V_1, g:U_2\to V_2$, there is an associated morphism $f\otimes g:U_1\otimes V_1\to U_2\otimes V_2$.
- For every additional pair of morphisms $f':V_1\to W_1, g':V_2\to W_2$, we have

$$(f'\otimes g')\circ(f\otimes g)=(f'\circ f)\otimes(g'\circ g).$$

- For every pair of objects $U,V\in\mathrm{Ob}(\mathcal{C})$, the identity morphism satisfies $Id_{U\otimes V}=Id_U\otimes Id_V$.

An associativity constraint a for a tensor product is a family of isomorphisms assigning an isomorphism $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ to each triple of objects $U, V, W \in \mathrm{Ob}(\mathcal{C})$ such that the diagram

$$\begin{array}{ccc} (U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) \\ \downarrow{\scriptstyle (f \otimes g) \otimes h} & & \downarrow{\scriptstyle f \otimes (g \otimes h)} \\ (U' \otimes V') \otimes W' & \xrightarrow[a_{U',V',W'}]{} & U' \otimes (V' \otimes W') \end{array}$$

commutes for any morphisms f, g, h. The associativity constraint is said to satisfy the pentagon axiom if for every collection of objects $U, V, W, X \in \mathrm{Ob}(\mathcal{C})$, the diagram

$$\begin{array}{ccccc} ((U \otimes V) \otimes W) \otimes X & \xrightarrow{a_{U,V,W} \otimes Id_X} & ((U \otimes V) \otimes W) \otimes X & & \\ & & & \searrow^{a_{U \otimes V, W, X}} & \\ \downarrow{\scriptstyle a_{U, V \otimes W, X}} & & & & (U \otimes V) \otimes (W \otimes X) \\ & & & \swarrow_{a_{U,V,W \otimes X}} & \\ U \otimes ((V \otimes W) \otimes X) & \xrightarrow[Id_U \otimes a_{V,W,X}]{} & U \otimes (V \otimes (W \otimes X)) & & \end{array}$$

commutes. Let $I \in \mathrm{Ob}(\mathcal{C})$ be the unit object in $\mathcal{C}$. Then, a left unit constraint l in $\mathcal{C}$ is a natural family of isomorphisms assigning to each object U an isomorphism $l_U : I \otimes U \to U$ such that for any morphisms $f : U \to V$, the diagram

$$\begin{array}{ccc} I \otimes U & \xrightarrow{l_U} & U \\ \downarrow{\scriptstyle Id_I \otimes f} & & \downarrow{\scriptstyle f} \\ I \otimes V & \xrightarrow{l_V} & V \end{array}$$

commutes. Right unit constraints are defined similarly. The associativity, left, and right constraints satisfy the triangle axiom if the diagram

$$\begin{array}{ccc} (U \otimes I) \otimes V & \xrightarrow{a_{U,I,V}} & U \otimes (I \otimes V) \\ & {\scriptstyle r_U \otimes Id_V} \searrow \quad \swarrow {\scriptstyle Id_U \otimes l_V} & \\ & U \otimes V & \end{array}$$

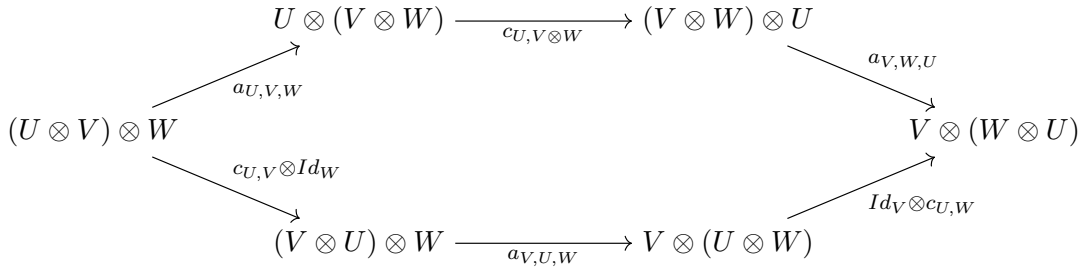
commutes for every pair of object $U, V \in \mathrm{Ob}(\mathcal{C})$. Then, a tensor (or monoidal) category is a category $\mathcal{C}$ equipped with a tensor product, associativity constraint satisfying the pentagon axiom, and left and right unit constraints satisfying the triangle axiom. The category is said to be strict if the associativity and unit constraints are the identity map. Strictness may sound like a very strong condition for a monoidal category, but it is not. Mac Lane's coherence theorem (see [9] for proof) states that for every monoidal category, there exists an equivalent strict monoidal category. It is important to note that this is not true in general for categories with additional structure. The

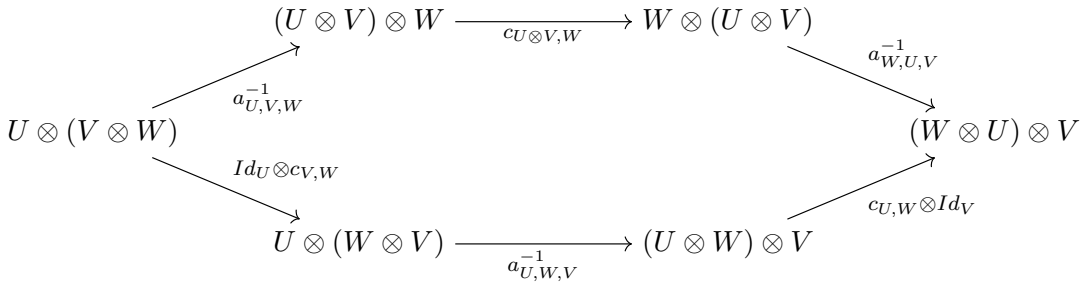
definition of a monoidal category can be extended to that of a braided tensor category by requiring that $\mathcal{C}$ also carries a braiding which is defined as follows:

A commutativity constraint is a family of isomorphisms $c_{U,V} : U \otimes V \to V \otimes U$ for each pair of objects $U, V \in \mathrm{Ob}(\mathcal{C})$ such that

$$\begin{array}{ccc} U_1 \otimes V_1 & \xrightarrow{c_{U_1,V_1}} & V_1 \otimes U_1 \\ \downarrow{\scriptstyle f\otimes g} & & \downarrow{\scriptstyle g\otimes f} \\ U_2 \otimes V_2 & \xrightarrow{c_{U_2,V_2}} & V_2 \otimes U_2 \end{array}$$

commutes for any morphisms $f : U_1 \to U_2, g : V_1 \to V_2$. A braiding is a commutativity constraint which also satisfies the Hexagon Axiom, which is the commutativity of the following diagrams:





A twist in a tensor category is a family of morphisms $\theta = \{\theta_V : V \to V\}$ such that

$$\theta_{U\otimes V} = c_{V,U} \circ c_{U,V} \circ (\theta_U \otimes \theta_V).$$

and for any morphism $f : U \to V$, we have $\theta_V \circ f = f \circ \theta_U$. If for a given $V \in \mathcal{C}$ there is an associated object V^* and "duality morphisms"

$$\overrightarrow{\mathrm{coev}}_V : \mathbb{1} \to V \otimes V^*, \qquad \overrightarrow{\mathrm{ev}}_V : V^* \otimes V \to \mathbb{1}.$$

satisfying the commutative diagrams:

$$\begin{array}{ccccc} V & \xrightarrow{v\mapsto 1\otimes v} & \mathbb{1}\otimes V & \xrightarrow{\overrightarrow{\mathrm{coev}}_V\otimes id_V} & (V\otimes V^*)\otimes V \\ \downarrow{\scriptstyle id_V} & & & & \downarrow{\scriptstyle a_{V,V^*,V}} \\ V & \xleftarrow[v\otimes 1\mapsto v]{} & V\otimes\mathbb{1} & \xleftarrow[id_V\otimes\overrightarrow{\mathrm{ev}}_V]{} & V\otimes(V^*\otimes V) \end{array}$$

$$\begin{array}{ccccc} V^* & \xrightarrow{f\mapsto f\otimes 1} & V^*\otimes\mathbb{1} & \xrightarrow{id_{V^*}\otimes\overrightarrow{\mathrm{coev}}_V} & V^*\otimes(V\otimes V^*) \\ \downarrow{\scriptstyle id_{V^*}} & & & & \downarrow{\scriptstyle a^{-1}_{V^*,V,V^*}} \\ V^* & \xleftarrow[1\otimes f\mapsto f]{} & \mathbb{1}\otimes V^* & \xleftarrow[\overrightarrow{\mathrm{ev}}_V\otimes id_{V^*}]{} & (V^*\otimes V)\otimes V^* \end{array}$$

which are equivalent to the relations

$$\begin{aligned} (id_V\otimes\overrightarrow{\mathrm{ev}}_V)\circ a_{V,V^*,V}\circ(\overrightarrow{\mathrm{coev}}_V\otimes id_V) &= id_V, \\ (\overrightarrow{\mathrm{ev}}_V\otimes id_{V^*})\circ a^{-1}_{V^*,V,V^*}\circ(id_{V^*}\otimes\overrightarrow{\mathrm{coev}}_V) &= id_{V^*}, \end{aligned}$$

then V^* is said to be left dual to V. If such a dual exists for every $V\in\mathcal{C}$ then $\mathcal{C}$ is called left rigid. The duality morphisms are said to be compatible with the braiding and twist if the relation

$$(\theta_V\otimes id_{V^*})\circ\overrightarrow{\mathrm{coev}}_V=(id_V\otimes\theta_{V^*})\circ\overrightarrow{\mathrm{coev}}_V.$$

holds for all objects V in $\mathcal{C}$. This compatibility ensures some nice properties for the category including a canonical isomorphism $(V^*)^*\simeq V$. Any category with this property is called pivotal. A category is called right rigid if it yields "right duality morphisms" given by

$$\begin{aligned} \overleftarrow{\mathrm{coev}}_V &:= (Id_{V^*}\otimes\theta_V)\circ c_{V,V^*}\circ\overrightarrow{\mathrm{coev}}_V\in\mathrm{Hom}(\mathbb{1},V^*\otimes V), \\ \overleftarrow{\mathrm{ev}}_V &:= \overrightarrow{\mathrm{ev}}_V\circ c_{V,V^*}\circ(\theta_V\otimes Id_{V^*})\in\mathrm{Hom}(V\otimes V^*,\mathbb{1}). \end{aligned}$$

which satisfy the diagrams

$$\begin{array}{ccccc} V & \xrightarrow{v\mapsto v\otimes 1} & V\otimes\mathbb{1} & \xrightarrow{id_V\otimes\overleftarrow{\mathrm{coev}}_V} & V\otimes(V^*\otimes V) \\ \downarrow{\scriptstyle id_V} & & & & \downarrow{\scriptstyle a^{-1}_{V,V^*,V}} \\ V & \xleftarrow[1\otimes v\mapsto v]{} & \mathbb{1}\otimes V & \xleftarrow[\overleftarrow{\mathrm{ev}}_V\otimes id_V]{} & (V\otimes V^*)\otimes V \end{array}$$

$$\begin{array}{ccccc} V^* & \xrightarrow{f\mapsto 1\otimes f} & \mathbb{1}\otimes V^* & \xrightarrow{\overleftarrow{\mathrm{coev}}_V\otimes id_{V^*}} & (V^*\otimes V)\otimes V^* \\ \downarrow{\scriptstyle id_{V^*}} & & & & \downarrow{\scriptstyle a_{V^*,V,V^*}} \\ V^* & \xleftarrow[f\otimes 1\mapsto f]{} & V^*\otimes\mathbb{1} & \xleftarrow[id_{V^*}\otimes\overleftarrow{\mathrm{ev}}_V]{} & V^*\otimes(V\otimes V^*) \end{array}$$

which are equivalent to the relations

$$(\overleftarrow{\mathrm{ev}}_V\otimes id_V)\circ a^{-1}_{V,V^*,V}\circ(id_V\otimes\overleftarrow{\mathrm{coev}}_V)=id_V,$$
$$(id_{V^*}\otimes\overleftarrow{\mathrm{ev}}_V)\circ a_{V^*,V,V^*}\circ(\overleftarrow{\mathrm{coev}}_V\otimes id_{V^*})=id_{V^*}.$$

A category is called rigid if it is both left and right rigid. A braided tensor category with twist and compatible duality is called ribbon.

2.1 Traces and Hopf Links

We will now construct the key objects of this thesis, the open Hopf links. The definitions in this subsection are taken from [1] and [37]. An ideal of $\mathcal{C}$ is a subcategory $\mathcal{I}$ which absorbs products and is closed under retracts. That is,

- If $U\in\mathcal{I}$ and $V\in\mathcal{C}$, then $U\otimes V\in\mathcal{I}$.
- If $U\in\mathcal{I}$ and $\alpha:V\to U$, $\beta:U\to V$ morphisms such that $\beta\circ\alpha=Id_V$, then $V\in\mathcal{I}$.

Definition 1. For any $U,V\in\mathcal{C}$ and any $f\in\mathrm{End}(U\otimes V)$, we define the left and right partial trace on $\mathrm{End}(V)$ and $\mathrm{End}(U)$ respectively as

$$\mathrm{ptr}_L(f)=(\overrightarrow{\mathrm{ev}}_U\otimes Id_V)\circ(Id_{U^*}\otimes f)\circ(\overleftarrow{\mathrm{coev}}_U\otimes Id_V)\in\mathrm{End}(V),$$
$$\mathrm{ptr}_R(f)=(Id_U\otimes\overleftarrow{\mathrm{ev}}_V)\circ(f\otimes Id_{V^*})\circ(Id_U\otimes\overrightarrow{\mathrm{coev}}_V)\in\mathrm{End}(U).$$

A modified trace on $\mathcal{I}$ is a family of linear functions indexed by the objects V in $\mathcal{I}$

$$\{t_V:\mathrm{End}(V)\to\mathbb{1}\}$$

such that

- If U $\in \mathcal{I}$ and $V \in \mathcal{C}$ then for any $f \in \text{End}(U \otimes V)$,
  $t_{U\otimes V}(f) = t_U(\text{ptr}_R(f))$
- If $U, V \in \mathcal{I}$ then for any morphisms $f : V \to U$ and $g : U \to V$ we have,
  $t_V(g \circ f) = t_U(f \circ g)$

The main focus of this paper will be on Hopf links:

Definition 2. If $U, V \in \mathcal{C}$, then the generalized Hopf link $\Phi_{U,V} \in \text{End}(V)$ is defined as

$$\Phi_{U,V} = \text{ptr}_R(c_{U,V} \circ c_{V,U}) = (Id_V \otimes \overleftarrow{\text{ev}}_U) \circ ((c_{U,V} \circ c_{V,U}) \otimes Id_{U^*}) \circ (Id_V \otimes \overrightarrow{\text{coev}}_U)$$

where $c_{U,V}$ is the braiding.

2.2 Ribbon Graphs and Graphical Calculus

In this section we review ribbon graphs and methods for constructing categorical morphisms from them. This is necessary for the construction of the Alexander invariant given by Murakami in [8]. Our primary references for this section are [9] and [10]. We begin by defining the fundamental constituents of a ribbon graph: bands, annuli, and coupons. A band is an embedding in $\mathbb{R}^3$ of the square $[0,1] \times [0,1]$, an annulus is an embedding of the cylinder $\mathcal{S}_1 \times [0,1]$, and a coupon is a band with a distinguished base. The image of the interval $\{\frac{1}{2}\} \times [0,1]$ in a band is called the core of the band and the image of $\mathcal{S}_1 \times \{\frac{1}{2}\}$ in an annulus is called the core of the annulus. Bands and annuli are called directed if their cores are oriented, and the orientation of the core is referred to as its direction.

Definition 3. Let m and n be nonnegative integers. Then a ribbon graph T of type (n, m) in $\mathbb{R}^3$ is an oriented surface Σ embedded in $\mathbb{R}^2 \times [0,1]$ which can be decomposed into a finite union of bands, coupons, and annuli such that:

- Σ meets the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ orthogonal along the line segments which are bases for certain bands in Σ:

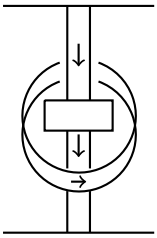
$$\{[i - (1/10), i + (1/10)] \times \{0\} \times \{0\} | i = 1, ..., n\},$$

$$[\{j - (1/10), j + (1/10)] \times \{0\} \times \{1\} | j = 1, ..., m\}.$$

- All other bases of bands which are not in the above line segments are along coupons and the bands, coupons, and annuli are otherwise disjoint.

- The bands and annuli of Σ are directed.

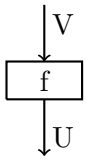
A ribbon graph is called homogeneous if in any neighbourhood of the above line segments the attached band is oriented upwards. A simple example of a (1,1)-ribbon graph containing bands, annuli, and a coupon is the graph



For the sake of simplicity, from this point on we neglect to draw the width of bands and the boundary. Some examples of important ribbon graphs are crossings

and curves

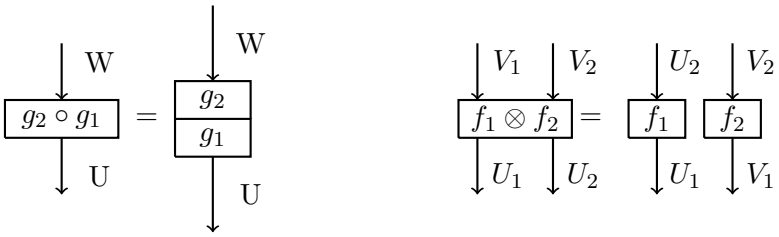
There is a way to represent morphisms in a category by "coloring" ribbon graphs with objects and morphisms. Let $\mathcal{C}$ be a strict tensor category. Then a morphism $f : U \to V$ is represented graphically as



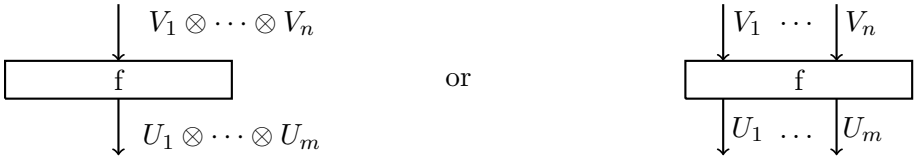
By labeling the arrows with U, V we are said to be coloring the bands with the objects $U, V \in \mathrm{Ob}(\mathcal{C})$, and the coupon with the morphism f. When the crossings and curves are colored with objects U, V, we denote them by $X^+_{U,V}, X^-_{U,V}, \cup_U, \overleftarrow{\cup}_U, \cap_U, \overleftarrow{\cap}_U$. These ribbon graphs are important because the category of homogeneous ribbon graphs is generated by these six graphs and some relations between them (see [21]). Here, a ribbon graph is called homogeneous if the intersection points of bands with the boundaries have the same orientation. For example, the band with one twist

is not homogeneous, but the band with two twists

is homogeneous. The composition of two morphisms $g_1 : U \to V$, $g_2 : V \to W$ and tensor product of morphisms $f_1 : U_1 \to V_1$, $f_2 : U_2 \to V_2$ are represented as



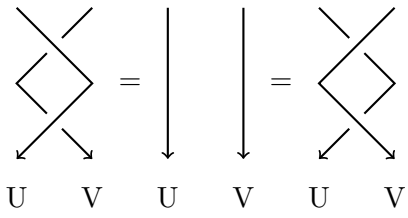
and a morphism on tensor products, $f : U_1 \otimes \cdots \otimes U_m \to V_1 \otimes \cdots \otimes V_n$ can be represented as



If $\mathcal{C}$ is a braided tensor category with braiding $c_{V,W}$, then we represent the braiding by the crossing graphs $X^+_{U,V}$ and $X^-_{U,V}$:

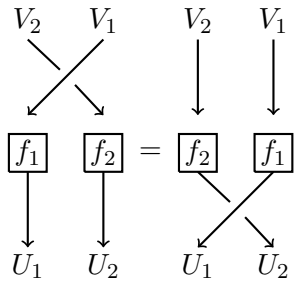
The invertibility of the braiding is then represented as

$$c^{-1}_{U,V} \circ c_{U,V} = Id_{U \otimes V} = c_{V,U} \circ c^{-1}_{V,U}$$

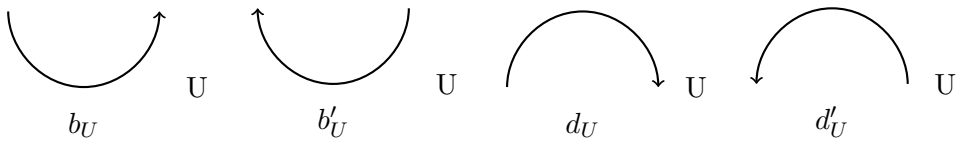


and if $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ are morphisms, then the naturality of the braiding is given by

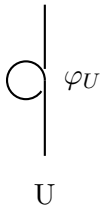
$$c_{V_1,V_2} \circ (f_1 \otimes f_2) = (f_2 \otimes f_1) \circ c_{U_1,U_2}$$



The graphs for the left and right duality morphisms b_U, d_U, b'_U, d'_U, are represented graphically by $\cup_U$, $\overleftarrow{\cup}_U, \cap_U$, and $\overleftarrow{\cap}_U$:

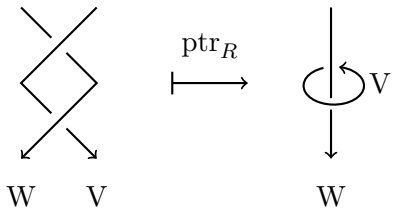


and the twist is represented as



An appropriately colored (n, l)-ribbon graph can be reduced to a $(n-1, l-1)$ ribbon graph through the action of the partial trace functions described in section 3.1. The right partial trace acts on a colored ribbon graph by joining the right-most ends, and the left partial trace acts by joining the left-most ends. For example, the Hopf link is defined in section 3.1 as $\Phi_{V,W} = \mathrm{ptr}_R(c_{V,W} \circ c_{W,V})$. Graphically, this is represented as

$c_{V,W} \circ c_{W,V}$ $\qquad$ $\Phi_{V,W} = \mathrm{ptr}_R(c_{V,W} \circ c_{W,V})$



Colored ribbon graphs will be used to construct Murakami's Alexander invariants defined in [8].

3 The Singlet VOA

In this section we will recall the definition and elementary facts about VOAs and the singlet as they are described in [17] and [2] respectively.

3.1 Vertex Operator Algebras

Given a ring R, we denote by $R[z]$, $R[[z]]$, and $R((z))$ the space of formal R-valued polynomials, Taylor series, and Laurent series respectively. That is,

$$R[z] = \left\{ \sum_{i=0}^{n} r_i z^i | r_i \in R, n \in \mathbb{Z}_+ \right\}$$
$$R[[z]] = \left\{ \sum_{i=0}^{\infty} r_i z^i | r_i \in R \right\}$$
$$R((z)) = \left\{ \sum_{i=-m}^{\infty} r_i z^i | r_i \in R, m \in \mathbb{Z}_+ \right\}$$

Let V be a complex vector space, and End(V) the collection of linear operators $f : V \to V$. The formal power series

$$A(z) = \sum_{i \in \mathbb{Z}} A_i z^{-i}$$

with coefficients $A_i \in \text{End}(V)$ is called a field if for all $v \in V$, $A(z)v$ is a Laurent series, so

$$A(z)v = \sum_{i \in \mathbb{Z}} A_i(v) z^{-i} \in V((z)).$$

This is equivalent to stating that for all $v \in V$, $A_n v = 0$ for sufficiently large $n(v)$. A $\mathbb{Z}_+$-graded vector space is a vector space V such that $V = \bigoplus_{i=0}^{\infty} V_i$ where each V_i is itself a vector space. A linear operator $f \in \text{End}(V)$ on a graded vector space V is said to be homogeneous of degree m if $f(V_m) \subset V_{n+m}$ for all $n \in \mathbb{Z}_+$, and we denote the degree by $\deg f$. A field $A(z)$ is then said to be homogeneous of conformal dimension m if each operator A_i is homogeneous of degree $m - i$.

Definition 4. Two fields $A(z)$ and $B(w)$ are said to be local iff there exists an $N \in \mathbb{Z}_+$ such that

$$(z-w)^N [A(z), B(w)] := (z-w)^N \left(A(z)B(w) - B(w)A(z) \right) = 0.$$

Equivalently, (see [17]) requiring that for every $v \in V$ and $\varphi \in V^*$, the Laurent series

$$\varphi(A(z)B(w)v) = \sum_{i \in \mathbb{Z}_+} \left(\sum_{j \in \mathbb{Z}_+} \varphi(A_i(B_j(v))) z^{-i} \right) w^{-j}$$
$$\varphi(B(w)A(z)v) = \sum_{j \in \mathbb{Z}_+} \left(\sum_{i \in \mathbb{Z}_+} \varphi(B_j(A_i(v))) w^{-j} \right) z^{-i}$$

are expansions in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ respectively of the same element

$$f_{v,\varphi} \in \mathbb{C}[[z,w]][z^{-1},w^{-1},(z-w)^{-1}].$$

Definition 5. A vertex operator algebra (VOA) is a $\mathbb{Z}_+$-graded vector space $V = \bigoplus_{i=0}^{\infty} V_i$ where $\dim V_i < \infty$ for each $i \in \mathbb{Z}_+$ equipped with the following objects:

- A distinguished vector $|0\rangle \in V_0$ called the vacuum vector.
- A linear operator $T : V \to V$ of degree 1 called the translation operator.
- A linear operator $Y(-,z) : V \to \mathrm{End}V[[z^{\pm 1}]]$ which sends each element $v \in V_m$ to a field of conformal dimension m: $Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$ with $\deg v_{(n)} = -n+m-1$.

These objects are subject to the following constraints:

- For any $v \in V$, we have $Y(v,z)|0\rangle \in V[[z]]$ where $Y(v,z)|0\rangle|_{z=0} = v$ and $Y(|0\rangle,z) = Id_V$.
- For any $v \in V$, $[T, Y(v,z)] = \partial_z Y(v,z)$ and $T|0\rangle = 0$.
- All fields $Y(v,z)$ are local with respect to each other.

Let $\mathfrak{h}$ denote the Heisenberg Lie algebra, so $\mathfrak{h}$ has vector space basis given by a central element $\mathbf{c}$ and b_n with $n \in \mathbb{Z}$. The commutation relations for $\mathfrak{h}$ are given by

$$[\mathbf{c}, b_n] = 0 \text{ for all } n \in \mathbb{Z} \text{ and } [b_n, b_m] = n\delta_{n+m,0}\mathbf{c}.$$

Let F_λ denote the usual Fock space of charge λ. As a vector space F_λ is the $\mathbb{C}$-span of polynomials in variables $\{b_{-1}, b_{-2}, ...\}$ and $\mathfrak{h}$ acts on F_λ by $b_0 = \lambda$, b_n for $n < 0$ acts by multiplication, $\mathbf{c}$ acts as the scalar λ, and b_n for $n > 0$ acts as $\lambda n \frac{\partial}{\partial b_{-n}}$.

Let $p \geq 2$ be a positive integer and let V_λ be the lattice vertex algebra (see [17]) associated to the lattice $L = \sqrt{2p}\mathbb{Z}$.

$$V_L = \bigoplus_{\lambda \in \sqrt{2p}\mathbb{Z}} F_\lambda.$$

Let e^γ denote the usual fields for lattice VOAs:

$$e^\gamma = S_\gamma z^{\gamma b_0} exp\left(-\gamma \sum_{n<0} \frac{b_n}{n} z^{-n}\right) exp\left(-\gamma \sum_{n>0} \frac{b_n}{n} z^{-n}\right),$$

where S_γ is the shift operator $F_\lambda \mapsto F_{\lambda+\gamma}$. If e^γ has Fourier expansion $e^\gamma(x) = \sum_{n\in\mathbb{Z}} e_n^\gamma x^{-n-1}$, define the "screening operator" $\tilde{Q} = e_0^{-\sqrt{\frac{2}{p}}}$. The singlet VOA is then defined to be $Ker_{F_0}\tilde{Q}$, which we will denote by $\mathcal{W}(2, 2p-1)$.

3.2 Representations of $\mathcal{W}(2, 2p-1)$

Let $\alpha_+ = \sqrt{2p}$, $\alpha_- = -\sqrt{2/p}$, $\alpha_0 = \alpha_+ + \alpha_-$, and $\alpha_{r,s} = -\frac{r-1}{2}\sqrt{2p} + \frac{s-1}{\sqrt{2p}}$. If we assume $r \in \mathbb{Z}$ and $1 \leq s \leq p$, then when $s = p$, the $\mathcal{W}(2, 2r-1)$-module $F_{\alpha_{r,s}}$ is irreducible as seen in [4]. For each $F_{\alpha_{r,s}}$ where $s \neq p$ we can associate an irreducible submodule $M_{r,s} := soc(F_{\alpha_{r,s}})$ which is defined to be the socle of $F_{\alpha_{r,s}}$.

A full classification of $\mathcal{W}(2, 2p-1)$-modules is not known, but a classification for irreducible $\mathbb{Z}_{\geq 0}$-graded modules was given in [5]. A $\mathbb{Z}_{\geq 0}$-graded $\mathcal{W}(2, 2r-1)$-module is called typical if it is also irreducible as a Virasoro module, and atypical if not. The irreducible F_λ are typical while the $M_{r,s}$ are atypical.

The regularized characters of these modules given by a parameter ϵ are defined as

$$\begin{aligned}
\mathrm{ch}[F^\epsilon_\lambda](\tau) &= e^{2\pi\epsilon(\lambda-\alpha_0/2)}\frac{z^{(\lambda-\alpha_0/2)^2/2}}{\eta(\tau)}, \\
\mathrm{ch}[M^\epsilon_{r,s}](\tau) &= \sum_{n=0}^{\infty} \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-1,p-s}}](\tau) - \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-2,s}}](\tau).
\end{aligned}$$

Given the terms

$$\begin{aligned}
S^\epsilon_{\lambda+\alpha_0/2,\mu+\alpha_0/2} &= e^{2\pi\epsilon(\lambda-\mu)}e^{-2\pi i\lambda\mu}, \\
S^\epsilon_{(r,s),\mu+\alpha_0/2} &= -e^{-2\pi\epsilon(\frac{1}{2}(r-1)\alpha_+ + \mu)}e^{\pi i(r-1)\alpha_+\mu}\frac{\sin(\pi s\alpha_-(\mu+i\epsilon))}{\sin(\pi\alpha_+(\mu+i\epsilon))},
\end{aligned}$$

the Verlinde algebra of characters $\mathcal{V}_{\mathrm{ch}}$ is then defined (in [2]) to be the algebra whose vector space structure is generated by the characters $\mathrm{ch}[V^\epsilon]$ with $V = F_\lambda, M_{r,s}$ and whose product is defined to be

$$\mathrm{ch}[V^\epsilon_a] \times \mathrm{ch}[V^\epsilon_b] := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{S^\epsilon_{a,\rho} S^\epsilon_{b,\rho} \overline{S^{-\epsilon}_{\rho,\mu}}}{S^\epsilon_{(1,1),\rho}} \mathrm{ch}[F^\epsilon_\mu] d\mu \right) d\rho.$$

It was shown in [2] that this integral is well defined for $V = F_\lambda, M_{r,s}$ and that the product for these modules is given by

$$\begin{aligned}
\mathrm{ch}[F^\epsilon_\lambda] \times \mathrm{ch}[F^\epsilon_\mu] &= \sum_{l=0}^{p-1} \mathrm{ch}[F_{\lambda+\mu+l\alpha_-}], \\
\mathrm{ch}[M^\epsilon_{r,s}] \times \mathrm{ch}[F^\epsilon_\mu] &= \sum_{\substack{l=-s+2 \\ l+s=0 \bmod 2}}^{s} \mathrm{ch}[F^\epsilon_{\mu+\alpha_{r,l}}], \\
\mathrm{ch}[M^\epsilon_{r,s}] \times \mathrm{ch}[M^\epsilon_{r',s'}] &= \sum_{\substack{l=|s-s'|+1 \\ l+s+s'=1 \bmod 2}}^{min\{s+s'-1,p\}} \mathrm{ch}[M^\epsilon_{r+r'-1,l}] \\
&\quad + \sum_{\substack{l=p+1 \\ l+s+s'=1 \bmod 2}}^{s+s'-1} \left(\mathrm{ch}[M^\epsilon_{r+r'-2,l-p}] + \mathrm{ch}[M^\epsilon_{r+r'-1,2p-l}] + \mathrm{ch}[M^\epsilon_{r+r',l-p}]\right).
\end{aligned}$$

The regularized asymptotic quantum dimensions are defined to be

$$\mathrm{qdim}[X^\epsilon] := \lim_{\tau \to 0^+} \frac{\mathrm{ch}[X^\epsilon](\tau)}{\mathrm{ch}[M^\epsilon_{1,1}](\tau)}.$$

As in [31], we introduce

$$B^r_\epsilon := -\min\left\{ \left| \frac{m}{\sqrt{2r}} - \mathrm{Im}(\epsilon) \right| \,\middle|\, m \in \mathbb{Z} - r\mathbb{Z} \right\}.$$

For $\mathrm{Re}(\epsilon) > B^r_\epsilon$, the regularized asymptotic dimensions are given by

$$\begin{aligned}
\mathrm{qdim}[F^\epsilon_\lambda] &= q_\epsilon^{2\lambda-\alpha_0} \frac{sin(-\pi\alpha_+\epsilon i)}{sin(\pi\alpha_-\epsilon i)} = q_\epsilon^{2\lambda-\alpha_0} \sum_{\substack{l=-p+1 \\ l+p=1 \bmod 2}}^{p-1} q_\epsilon^{\alpha_- l}, \\
\mathrm{qdim}[M^\epsilon_{r,s}] &= q_\epsilon^{-(r-1)\alpha_+} \frac{sin(\pi s\alpha_-\epsilon i)}{sin(\pi\alpha_-\epsilon i)} = q_\epsilon^{-(r-1)\alpha_+} \sum_{\substack{l=-s+1 \\ l+s=1 \bmod 2}}^{s-1} q_\epsilon^{\alpha_- l}.
\end{aligned}$$

For $\mathrm{Re}(\epsilon) < B^r_\epsilon$, we have $\mathrm{qdim}[F^\epsilon_\lambda] = 0$, and for $\epsilon \in \mathbb{S}(k,m)$, $k \in \mathbb{Z}$, $m = 0, ..., 2r-1$, we have

$$\mathrm{qdim}[M^\epsilon_{r,s}] = \begin{cases} (-1)^{m(t-1)} \dfrac{\sin(\pi ms/r)}{\sin(\pi m/r)} & \text{if } m \neq 0, r, \\ (-1)^{(m+1)(t-1)+\frac{m}{r}(s-1)} \dfrac{\sin(\pi s/r)}{\sin(\pi/r)} & \text{if } m = 0, r, \end{cases}$$

where

$$\mathbb{S}(k,m) = \left\{ \epsilon \in \mathbb{C} \,\middle|\, k + \frac{2m-1}{4r} < \frac{\mathrm{Im}(\epsilon)}{\sqrt{2r}} < k + \frac{2m+1}{4r} \right\}.$$

The algebra of regularized quantum dimensions $\mathcal{Q}$ is defined to be the algebra whose vector space structure is generated by the regularized quantum dimensions of typical and atypical modules with point-wise multiplication. It was shown in [2] that $\mathcal{V}_{\text{ch}}$ and $\mathcal{Q}$ are isomorphic. We will see that the normalized Hopf link invariants for $\overline{U}_q^H(\mathfrak{sl}(2))$ are exactly the regularized quantum dimensions for $\mathcal{W}(2, 2r-1)$. The normalized Hopf link invariants determine much of the structure of the representation category for $\overline{U}_q^H(\mathfrak{sl}(2))$ and it is conjectured ([2]) that $\mathcal{V}_{\text{ch}}$ also determines much of the monoidal structure of the representation category of $\mathcal{W}(2, 2r-1)$, strongly suggesting that $\overline{U}_q^H(\mathfrak{sl}(2))$ should make a good toy model for understanding representations of the singlet VOA.

4 The Unrolled Restricted Quantum Group of $\mathfrak{sl}(2)$ and its Representation Theory

We construct and review the unrolled restricted quantum group of $\mathfrak{sl}(2)$, $\overline{U}_q^H(\mathfrak{sl}(2))$, as in [1]. Let $r \geq 2$ be a positive integer and $q = e^{\frac{\pi i}{r}} \in \mathbb{C}$ a $2r^{th}$ root of unity. For any $x \in \mathbb{C}$, we fix the notation

$$\{x\} = q^x - q^{-x}, \quad [x] = \frac{\{x\}}{\{1\}}, \quad \text{and} \quad \{x\}! = \{x\}\{x-1\}...\{1\}.$$

4.1 Defining $\overline{U}_q^H(\mathfrak{sl}(2))$

Let $U_q(\mathfrak{sl}(2))$ be the associative algebra over $\mathbb{C}$ with generators K, K^{-1}, E, F, and relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

We can define a counit $\epsilon : U_q(\mathfrak{sl}(2)) \to \mathbb{C}$, coproduct $\Delta : U_q(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2)) \otimes U_q(\mathfrak{sl}(2))$, and antipode $S : U_q(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2))$ on $U_q(\mathfrak{sl}(2))$ by

$$\begin{aligned}
&\Delta(K) = K \otimes K, && \epsilon(K) = 1, && S(K) = K^{-1},\\
&\Delta(E) = 1 \otimes E + E \otimes K, && \epsilon(E) = 0, && S(E) = -EK^{-1},\\
&\Delta(F) = K^{-1} \otimes F + F \otimes 1, && \epsilon(F) = 0, && S(F) = -KF.
\end{aligned}$$

This gives $U_q(\mathfrak{sl}(2))$ the structure of a Hopf algebra. We define the unrolled quantum group of $\mathfrak{sl}(2)$, $U_q^H(\mathfrak{sl}(2))$, by extending $U_q(\mathfrak{sl}(2))$ through the addition of a fifth generator H with relations

$$HK^{\pm 1} = K^{\pm 1}H, \qquad [H, E] = 2E, \qquad [H, F] = -2F.$$

We extend the coproduct, counit, and antipode to this algebra by defining

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \epsilon(H) = 0, \qquad S(H) = -H.$$

The unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{sl}(2))$ is then obtained from $U_q^H(\mathfrak{sl}(2))$ by quotienting out the relations $E^r = F^r = 0$.

4.2 Representation theory of $\overline{U}_q^H(\mathfrak{sl}(2))$

We will now define the representation category of $\overline{U}_q^H(\mathfrak{sl}(2))$ which we are interested in. Our main reference for this section is [1].

For any finite dimensional $\overline{U}_q^H(\mathfrak{sl}(2))$-module V, an eigenvalue $\lambda \in \mathbb{C}$ of H is called a weight of V and its associated eigenspace is called a weight space. Any $v \in V$ in the eigenspace of λ ($Hv = \lambda v$) is called a weight vector of weight λ. V is called a weight module if it is a direct sum of its weight spaces (H acts semi-simply) and the element $K \in \overline{U}_q^H(\mathfrak{sl}(2))$ acts as q^H on V ($Kv = q^{Hv}$). Let $\mathcal{C}$ be the category of finite dimensional weight $\overline{U}_q^H(\mathfrak{sl}(2))$-modules.

$\mathcal{C}$ is a ribbon category: If V is an object in $\mathcal{C}$ with basis $\{v_1, ..., v_n\}$, then V has the natural dual vector space $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$ with basis $\{v_1^*, ..., v_n^*\}$, and the left duality morphisms are

$$\overrightarrow{\text{coev}}_V : \mathbb{C} \to V \otimes V^* \quad \text{and} \quad \overrightarrow{\text{ev}}_V : V^* \otimes V \to \mathbb{C}$$

where $\overrightarrow{\text{coev}}(1) = \sum_i^n v_i \otimes v_i^*$, and $\overrightarrow{\text{ev}}(f \otimes w) = f(w)$. Let $v_1, .., v_n$ denote a basis for a module V, and let $v = \sum_{j=1}^n \lambda_i v_i$ an element in V. Then,

$$\begin{aligned}(id_V \otimes \overrightarrow{\text{ev}}_V) \circ a_{V,V^*,V} \circ (\overrightarrow{\text{coev}}_V \otimes id_V)(1 \otimes v) &= (id_V \otimes \overrightarrow{\text{ev}}_V) \circ a_{V,V^*,V} \sum_i^n (v_i \otimes v_i^*) \otimes v \\ &= \sum_{i=1}^n \sum_{j=1}^n v_i \otimes \lambda_j v_i^*(v_j) \\ &= \sum_{i=1}^n \lambda_i v_i \otimes 1 = v \otimes 1.\end{aligned}$$

and if $f = \sum_{j=1}^{n} \lambda_j v_j^*$, then

$$(\overrightarrow{\text{ev}}_V \otimes id_{V^*}) \circ a^{-1}_{V^*,V,V^*} \circ (id_{V^*} \otimes \overrightarrow{\text{coev}}_V)(f \otimes 1) = (\overrightarrow{\text{ev}}_V \otimes id_{V^*}) \circ a^{-1}_{V^*,V,V^*} \sum_{i=1}^{n} f \otimes (v_i \otimes v_i^*)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_j v_j^*(v_i) \otimes v_i^*$$
$$= \sum_{j=1}^{n} \lambda_j v_j^* \otimes 1 = 1 \otimes f.$$

Which shows that $\mathcal{C}$ is left rigid. The braiding and twist were defined in [3] by Ohtsuki as follows: For each pair of objects V, W in $\mathcal{C}$ one can define an R-matrix operator on $V \otimes W$ as

$$R = q^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n \tag{2}$$

where $q^{H\otimes H/2}(v \otimes w) = q^{\lambda_v \lambda_w/2} v \otimes w$ for weight vectors v, w with weights λ_v and λ_w. The braiding on $\mathcal{C}$ is then given by a collection of maps $c_{V,W} : V \otimes W \to W \otimes V$ where $c_{V,W}(v \otimes w) = \tau(R(v \otimes w))$ where τ is the regular flip map $w \otimes v \mapsto v \otimes w$. Ohtsuki defined an operator on each $V \in \mathcal{C}$ as

$$\theta = K^{r-1} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} S(F^n) q^{-H^2/2} E^n. \tag{3}$$

The twist $\theta_V : V \to V$ was then defined as the operator $v \mapsto \theta^{-1} v$. $\mathcal{C}$ also admits compatible right duality morphisms

$$\overleftarrow{\text{ev}}_V : V \otimes V^* :\to \mathbb{C}, \qquad \overleftarrow{\text{ev}}_V(v \otimes f) = f(K^{1-r} v),$$
$$\overleftarrow{\text{coev}}_V : \mathbb{C} \to V^* \otimes V, \qquad \overleftarrow{\text{coev}}(1) = \sum_i K^{r-1} V_i \otimes v_i^*.$$

Showing $\mathcal{C}$ is right dual is identical to the proof for left duality.l The quantum dimension, $\text{qdim}(V)$ of an object $v \in \mathcal{C}$ is defined to be $\text{qdim}(V) = \sum_{i=1}^{n} v_i^*(K^{1-r} v_i)$. Notice that this is the constant by which $\overleftarrow{\text{ev}}_V \circ \overrightarrow{\text{coev}}_V$ acts on $\mathbb{C}$.

4.3 Classification of simple and projective $\overline{U}_q^H(\mathfrak{sl}(2))$-modules

A classification of simple and projective $\overline{U}_q^H(\mathfrak{sl}(2))$-modules was given in [1] as follows: for any $n \in \{0, ..., r-1\}$ let S_n be the simple highest weight $n+1$ dimensional module of weight n with basis $\{s_0, ..., s_n\}$ and action

$$Fs_i = s_{i+1}, \quad Es_i = [i][n+1-i]s_{i-1}, \quad Hs_i = (n-2i)s_i, \quad Es_0 = Fs_n = 0.$$

For any $\alpha \in \mathbb{C}$, let V_α be the r-dimensional highest weight $\overline{U}_q^H(\mathfrak{sl}(2))$-modules of highest weight $\alpha + r - 1$ whose action is defined on a basis $\{v_0, ..., v_{r-1}\}$ as

$$Ev_i = [i][i-\alpha]v_{i-1}, \quad Fv_i = v_{i+1}, \quad Hv_i = (\alpha + r - 1 - 2i)v_i, \quad Ev_0 = Fv_{r-1} = 0.$$

V_α is called typical if $\alpha \in \ddot{\mathbb{C}} := (\mathbb{C} - \mathbb{Z}) \cup r\mathbb{Z}$ and atypical otherwise. Notice that the typical V_α are simple since any element v_i in the basis generates the entire module through the action of E and F. If V_α is atypical, however, then we have $\alpha = rm + k$ for some $m \in \mathbb{Z}$ and $1 \leq k \leq r-1$. Then,

$$Ev_k = -[k][k - (rm + k)]v_{k-1} = [k][rm]v_{k-1} = 0.$$

Since $\{rm\} = q^{rm} - q^{-rm} = (-1)^m - (-1)^{-m} = 0$. So, when V_α is atypical, it contains a simple submodule generated by the basis elements $\{v_k, v_{k+1}, ..., v_{r-1}\}$.

For any $\ell \in \mathbb{Z}$, let $\mathbb{C}_{\ell r}$ be the one dimensional module on which E, F act as zero and H acts as ℓr. Then, we have the following proposition (proven in [1]):

Proposition 2. • *The typical V_α ($\alpha \in \ddot{\mathbb{C}}$) are projective in $\mathcal{C}$.*
• *Every simple module in $\mathcal{C}$ is isomorphic to $S_n \otimes \mathbb{C}_{\ell r}$ for some $n \in \{0, ..., r-1\}$ and $\ell \in \mathbb{Z}$ or V_α for some $\alpha \in \ddot{\mathbb{C}}$.*

Note that in [38], we consider an enlargement of the category $\mathcal{C}$ and the typical V_α are no longer projective in this larger category. For any $i \in \{0, ..., r-2\}$, let P_i be the 2r-dimensional vector space over $\mathbb{C}$ with basis
$\{\mathsf{w}_i^H, \mathsf{w}_{i-2}^H, ..., \mathsf{w}_{-i}^H, \mathsf{w}_{i+2}^R, ..., \mathsf{w}_{2r-2-i}^R, \mathsf{w}_{i+2-2r}^L, ..., \mathsf{w}_{-i-2}^L, \mathsf{w}_i^S, ..., w_{-i}^S\}$. Then P_i is a well defined $\overline{U}_q^H(\mathfrak{sl}(2))$-module ([1]) with action

$$\mathsf{w}_{i+2}^R = E\mathsf{w}_i^H, \qquad \mathsf{w}_i^S = F\mathsf{w}_{i+2}^R, \qquad \mathsf{w}_{-i-2}^L = F^{i+1}\mathsf{w}_i^H,$$

$$\mathsf{w}^H_{i-2k} = F^k\mathsf{w}^H_i \qquad \text{and} \qquad \mathsf{w}^S_{i-2k} = F^k\mathsf{w}^S_i \qquad \text{for } k \in \{0,...,i\},$$
$$\mathsf{w}^R_{i+2+2k} = E^k\mathsf{w}^R_{i+2} \qquad \text{and} \qquad \mathsf{w}^L_{i+2} = F^k\mathsf{w}^L_{j-r} \qquad \text{for } k \in \{0,...,j\},$$

$$H\mathsf{w}^X_k = k\mathsf{w}^X_k, \qquad K\mathsf{w}^X_k = q^k\mathsf{w}^X_k \text{ for } X \in \{L,R,H,S\},$$
$$E\mathsf{w}^R_k = \mathsf{w}^R_{k+2}, \qquad F\mathsf{w}^X_k = \mathsf{w}^X_{k-2} \text{ for } X \in \{H,S,L\},$$
$$F\mathsf{w}^H_{-i} = \mathsf{w}^L_{-i-2}, \qquad E\mathsf{w}^L_{-i-2} = \mathsf{w}^S_{-i}, \qquad E\mathsf{w}^R_{2r-2-i} = E\mathsf{w}^S_i = F\mathsf{w}^S_{-i} = F\mathsf{w}^L_{i+2-2r} = 0,$$

$$E\mathsf{w}^H_{i-2k} = -\gamma_{i,k}\mathsf{w}^H_{i-2(k-1)} + \mathsf{w}^S_{i-2(k-1)}, \qquad E\mathsf{w}^S_{i-2k} = \gamma i,k\mathsf{w}^S_{i-2(k-1)},$$
$$F\mathsf{w}^R_{i+2+2k} = -\gamma_{j,k}\mathsf{w}^R_{i+2+2(k-1)}, \qquad E\mathsf{w}^L_{-i-2-2k} = -\gamma_{j,k}\mathsf{w}^L_{-i-2-2(k-1)},$$

where $\gamma_{n,m} = \gamma_{n,n-m+1} = [m][n-m+1]$. The following proposition was proven in [1]:

Proposition 3. *The module P_i is projective and indecomposable, and any projective indecomposable weight module with integer highest weight $(\ell+1)r-i-2$ is isomorphic to $P_i \otimes \mathbb{C}^H_{\ell r}$. The endomorphism ring of $P_i \otimes \mathbb{C}^H_{\ell r}$ is two dimensional and we denote its basis $\{Id, x_{i,\ell}\}$ where the action of $x_{i,\ell}$ on $P_i \otimes \mathbb{C}^H_{\ell r}$ is determined by $x_{i,\ell}(\mathsf{w}^H_i \otimes v) = \mathsf{w}^S_i \otimes v$.*

It was shown in [1] that there exists a modified trace $\{t_V : \mathrm{End}(V) \to \mathbb{C}\}_{V \in Proj}$ on the ideal of projective modules, denoted *Proj*, in $\mathcal{C}$. This trace is unique up to scalar multiples. In particular, there is a trace which acts as $t_V(f) = (-1)^{r-1}\langle f \rangle$ on any element $f \in \mathrm{End}(V_0)$ where $\langle f \rangle$ is the scalar by which f acts on V_0. The modified quantum dimension is defined as

$$\mathbf{d} : \mathrm{Ob}(Proj) \to \mathbb{C} \qquad \mathbf{d}(V) = t_V(Id_V).$$

It was shown in [1] that for any typical V_α, we have

$$\mathbf{d}(V_\alpha) = (-1)^{r-1}\prod_{j=1}^{r-1}\frac{\{j\}}{\{\alpha+r-j\}} = (-1)^{r-1}r\frac{\{\alpha\}}{\{r\alpha\}}.$$

4.4 Characters and Tensor Decompositions

We define the character of a weight module $V \in \mathcal{C}$ to be

$$\chi(V) = \sum_\alpha \dim(V(\lambda))z^\lambda$$

where $z \in \mathbb{C}$ and $V(\lambda)$ is the λ-eigenspace under the action of H. Characters record the eigenspace structure of a module. Since all of the modules in $\mathcal{C}$ are direct sums of their eigenspaces, the classification of projective modules above shows that projective modules are isomorphic if their

characters agree, which provides a nice method for determining tensor decompositions for projective modules. Let

$$[k]_z = z^{k-1} + z^{k-3} + ... + z^{-(k-1)} = \frac{z^k - z^{-k}}{z - z^{-1}}.$$

Then, if $\alpha \in \ddot{\mathbb{C}}$ and $i \in \{0, ..., r-1\}$ we have

$$\chi(V_\alpha) = z^\alpha [r]_z, \quad \chi(S_i) = [i+1]_z, \text{ and } \chi(P_i) = (z^{(r-1-i)} + z^{(1+i-r)})[r]_z.$$

Proposition 4. *The complete list of tensor decompositions of isomorphism classes is the following: For $i, j \in \{0, ..., r\}$, we have*

$$S_i \otimes S_j = \begin{cases} \bigoplus\limits_{\substack{l=|i-j| \\ by\ 2}}^{i+j} S_l & \text{if } i+j < r \\ \bigoplus\limits_{\substack{l=|i-j| \\ by\ 2}}^{2r-4-i-j} S_l \oplus \bigoplus\limits_{\substack{l=2r-2-i-j \\ by\ 2}}^{r-1} P_l & \text{if } i+j \geq r \end{cases}$$

where "by 2" here means we only count every second term in the sum. If $\alpha, \beta \in \ddot{\mathbb{C}}$ such that $\alpha + \beta \notin \mathbb{Z}$ then

$$V_\alpha \otimes V_\beta = \bigoplus_{\substack{l=1-r \\ by\ 2}}^{r-1} V_{\alpha+\beta+l}.$$

When $\alpha + \beta = n \in \mathbb{Z}$, set $n = i + kr$ with $i \in \{0, ..., r-1\}$ and $k \in \mathbb{Z}$. If $\alpha \in \ddot{\mathbb{C}}$ we have

$$V_\alpha \otimes V_{n-\alpha} = \left(\bigoplus_{\substack{l=r-1-i \\ by\ -2}}^{0} P_l \otimes \mathbb{C}^H_{kr} \right) \oplus \left(\bigoplus_{\substack{l=r-i \\ by\ 2}}^{r-1} P_l \otimes \mathbb{C}^H_{(k+1)r} \right)$$

and if $\alpha \in \mathbb{C} - \mathbb{Z}$, then we have

$$V_\alpha \otimes S_i = \bigoplus_{\substack{l=-i \\ by\ 2}}^{i} V_{\alpha+l},$$

$$V_{kr} \otimes S_i = V_0 \otimes S_i \otimes \mathbb{C}^H_{kr},$$

where $V_0 \otimes S_i = \bigoplus_{\substack{l=r-1-i \\ by\ 2}}^{r-1} P_l$. *The tensor decompositions for products of the projective* P_i *are*

$$P_i \otimes S_j = \left(\bigoplus_{\substack{l=|i-j| \\ by\ 2}}^{min(i+j,r-1)} P_l \right) \oplus \left(\bigoplus_{\substack{l=24-2-i-j \\ by\ 2}}^{r-1} P_l \right) \oplus \left(\bigoplus_{\substack{l=r+i-j \\ by\ 2}}^{r-1} \right),$$

$$P_i \otimes P_j = \left((\mathbb{C}_r^H \oplus \mathbb{C}_{-r}^H) \otimes (P_i \otimes S_{r-2-j}) \right) \oplus 2(P_i \otimes S_j),$$

$$V_0 \otimes P_j = \left(\bigoplus_{\substack{l=j+1 \\ by\ 2}}^{r-1} (\mathbb{C}_r^H \oplus \mathbb{C}_{-r}^H) \otimes P_l \right) \oplus \bigoplus_{\substack{l=r-1-j \\ by\ 2}}^{r-1} 2P_l,$$

where we assume the index on the P_i *to be in* $\{0, ..., r-2\}$ *while the index on the* S_j *are in* $\{0, ..., r-1\}$. *Finally, we have*

$$V_\alpha \otimes P_j = \bigoplus_{\substack{l=0 \\ by\ 2}}^{2(r-1)} V_{\alpha-j+l} \oplus \bigoplus_{\substack{l=0 \\ by\ 2}}^{2(r-1)} V_{\alpha+j-l}.$$

Proof. All of these identities except for $S_i \otimes S_j$ can be derived from their characters since every other product is projective. We only provide proofs for identities which are not given in [1].

$$\begin{aligned}
\chi(V_\alpha \otimes V_{n-\alpha}) &= z^{i+kr}[r]_z^2 = z^{kr}(z^{r-1+i} + z^{r-3+i} + ... + z^{1-r+i})[r]_z \\
&= z^{kr}\left((z^{1-r+i} + ... + z^{r-1-i}) + (z^{r+1-i} + ... + z^{r-1+i}) \right)[r]_z \\
&= z^{kr}\left(\sum_{\substack{l=r-1-i \\ \text{by -2}}}^{0} (z^l + z^{-l}) \right)[r]_z + z^{(k+1)r}[i]_z[r]_z \\
&= \left(\sum_{\substack{l=r-1-i \\ \text{by -2}}}^{0} \chi(P_l \otimes \mathbb{C}_{kr}^H) \right) + \chi(V_0 \otimes S_{i-1} \otimes \mathbb{C}_{(k+1)r}^H).
\end{aligned}$$

If $\alpha = kr$, we have $\chi(V_{kr} \otimes S_i) = z^{kr}[r]_z[i+1]_z = \chi(V_0 \otimes S_i \otimes \mathbb{C}_{kr}^H)$ and if $\beta \in \mathbb{C} - \mathbb{Z}$ then

$$\chi(V_\beta \otimes S_i) = z^\beta [i+1]_z [r]_z = \sum_{\substack{l=-i \\ \text{by } 2}}^{i} z^{\beta+l}[r]_z = \sum_{\substack{l=-i \\ \text{by } 2}}^{i} \chi(V_{\beta+l}).$$

Hence, for $\alpha \in \ddot{\mathbb{C}}$ and $n = i + kr$ with $i \in \{0, ..., r-1\}$ and $k \in \mathbb{Z}$ we have

$$V_\alpha \otimes V_{n-\alpha} = \left(\bigoplus_{\substack{l=r-1-i \\ \text{by -2}}}^{0} P_l \otimes \mathbb{C}^H_{kr} \right) \oplus \left(\bigoplus_{\substack{l=r-i \\ \text{by } 2}}^{r-1} P_l \otimes \mathbb{C}^H_{(k+1)r} \right)$$

and for $\alpha \in \mathbb{C} - \mathbb{Z}$, then we have

$$V_\alpha \otimes S_i = \bigoplus_{\substack{l=-i \\ \text{by } 2}}^{i} V_{\alpha+l},$$

$$V_{kr} \otimes S_i = V_0 \otimes S_i \otimes \mathbb{C}^H_{kr}.$$

We can determine the product $V_\alpha \otimes P_j$ for $\alpha \in \mathbb{C} - \mathbb{Z}$ from its character:

$$\begin{aligned}
\chi(V_\alpha \otimes P_j) &= z^\alpha [r]_z (z^{r-1-j} + z^{1+j-r})[r]_z \\
&= (z^{\alpha-j} + z^{\alpha-j+2} + ... + z^{\alpha-j+2(r-1)})[r]_z + (z^{\alpha+j} + z^{\alpha+j-2} + ... + z^{\alpha+j-2(r-1)})[r]_z \\
&= \sum_{\substack{l=0 \\ \text{by } 2}}^{2(r-1)} \chi(V_{\alpha-j+l}) + \sum_{\substack{l=0 \\ \text{by } 2}}^{2(r-1)} \chi(V_{\alpha+j-l}).
\end{aligned}$$

□

4.5 Hopf Link Computations

In this section we will construct the open Hopf links associated to $\mathcal{C}$ as was done in [1]. We will see that the method used to construct the open Hopf links on projective modules requires knowledge of the tensor ring structure. We will introduce later a new method for computing these Hopf links without using the tensor ring. Recall the Hopf links from definition 2:

$$\Phi_{U,V} = \mathrm{ptr}_R(c_{U,V} \circ c_{V,U}) = (Id_V \otimes \overleftarrow{\mathrm{ev}_U}) \circ ((c_{U,V} \circ c_{V,U}) \otimes Id_{U^*}) \circ (Id_V \otimes \overrightarrow{\mathrm{coev}}_U).$$

Proposition 5. *Let $i, j \in \{0, ..., r-2\}$ and $\alpha, \beta \in \ddot{\mathbb{C}} = (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$. Then, we have the following:*

$$\Phi_{V_\beta,V_\alpha}=\frac{(-1)^{r-1}r}{\boldsymbol{d}(V_\alpha)}q^{\alpha\beta}Id_{V_\alpha},\qquad \Phi_{P_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}=(-1)^{r-1}rq^{r\ell\alpha}\frac{q^{(r-1-i)\alpha}+q^{-(r-1-i)\alpha}}{\boldsymbol{d}(V_\alpha)}Id_{V_\alpha},$$
$$\Phi_{S_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}=q^{r\ell\alpha}\frac{\{(i+1)\alpha\}}{\{\alpha\}}Id_{V_\alpha},\quad \Phi_{V_\alpha,P_j\otimes\mathbb{C}^H_{\ell r}}=(-1)^{r(\ell+1)+\ell+j}q^{r\ell\alpha}r(q^{(r-j-1)\alpha}+q^{-(r-j-1)\alpha})x_{j,\ell},$$

$$\Phi_{P_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}=(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}2r(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)})x_{j,\ell},$$
$$\begin{aligned}\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}&=\frac{(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}}\{(i+1)(j+1)\}Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\\&\quad+\frac{(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}}(i\{(i+2)(j+1)\}-(i+2)\{i(j+1)\})x_{j,\ell}.\end{aligned}$$

Proof. This proposition is a modification of Lemma 6.7 in [1] and the proof here will follow theirs closely adjusting it only where necessary.

Let $V,W\in\mathcal{C}$ and $w\in W$ a highest weight vector of weight λ. Let $\{v_0,...,v_s\}$ be a basis for V and λ_k the weight of v_k. Then,

$$\begin{aligned}\Phi_{V,W}(w)&=(Id_W\otimes\overleftarrow{ev}_V)\circ(c_{V,W}\otimes Id_{V^*})\circ(c_{W,V}\otimes Id_{V^*})\circ(Id_W\otimes\overrightarrow{coev}_V)(w)\\&=\sum_{k=0}^{s}(Id_W\otimes\overleftarrow{ev}_V)\circ(c_{V,W}\otimes Id_{V^*})\circ(c_{W,V}\otimes Id_{V^*})(w\otimes v_k\otimes v_k^*)\\&=\sum_{k=0}^{s}(Id_W\otimes\overleftarrow{ev}_V)\circ(\tau(R(\tau(R(w\otimes v_k))))\otimes v_k^*)\\&=\sum_{k=0}^{s}(Id_W\otimes\overleftarrow{ev}_V)\circ(q^{H\otimes H}(w\otimes v_k)\otimes v_k^*)\\&=\sum_{k=0}^{s}(Id_W\otimes\overleftarrow{ev}_V)\circ(q^{\lambda\lambda_k}(w\otimes v_k)\otimes v_k^*)\\&=\sum_{k=0}^{s}q^{\lambda\lambda_k}w\otimes q^{(1-r)\lambda_k}\\&=\big(\sum_{k=0}^{s}q^{(\lambda+1-r)\lambda_k}\big)w.\end{aligned}$$

Define for any $\gamma\in\mathbb{C}$, $\Psi_\gamma:\mathbb{Z}[z]\to\mathbb{C}$ by $\Psi_\gamma(z^s)=q^{\gamma s}$. Then, one sees that $\Phi_{V,W}(w)=\Psi_{\lambda+1-r}(\chi(V))w$. When W is simple we then have

$$\Phi_{V,W}=\Psi_{\lambda+1-r}(\chi(V))Id_W \tag{4}$$

by Schur's Lemma. Since $\alpha \in \ddot{\mathbb{C}}$ V_α is typical, hence simple, we can compute the Hopf links on V_α directly.

$$\Phi_{V_\beta,V_\alpha} = \Psi_\alpha(\chi(V_\beta))Id_{V_\alpha} = \Psi_\alpha\left(z^\beta \frac{z^r - z^{-r}}{z - z^{-1}}\right)Id_{V_\alpha} = \frac{\{r\alpha\}}{\{\alpha\}}q^{\alpha\beta}Id_{V_\alpha} = \frac{(-1)^{r-1}r}{\mathbf{d}(V_\alpha)}q^{\alpha\beta}Id_{V_\alpha},$$
$$\Phi_{S_i,V_\alpha} = \Psi_\alpha(\chi(S_i))Id_{V_\alpha} = \Psi_\alpha\left(\frac{z^{i+1} - z^{-(i+1)}}{z - z^{-1}}\right)Id_{V_\alpha} = \frac{\{(i+1)\alpha\}}{\{\alpha\}}Id_{V_\alpha},$$

$$\begin{aligned}\Phi_{P_i,V_\alpha} &= \Psi_\alpha(\chi(P_i))Id_{V_\alpha} = \Psi_\alpha\left((z^{r-1-i} + z^{-r+1+i})\frac{z^r - z^{-r}}{z - z^{-1}}\right)Id_{V_\alpha} \\ &= (-1)^{r-1}r\frac{q^{(r-1-i)\alpha} + q^{-(r-1-i)\alpha}}{\mathbf{d}(V_\alpha)}Id_{V_\alpha},\end{aligned}$$

$$\Phi_{\mathbb{C}^H_{\ell r},V_\alpha} = \Psi_\alpha(\chi(z^{\ell r}))Id_{V_\alpha} = q^{r\ell\alpha}Id_{V_\alpha}.$$

For any $X, Y, Z \in \mathcal{C}$ we have $\Phi_{X\otimes Y,Z} = \Phi_{X,Z} \circ \Phi_{Y,Z}$ so this proves the proposition for Φ_{V_β,V_α}, $\Phi_{S_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}$, and $\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}$.

The endomorphism ring of $P_j \otimes \mathbb{C}^H_{\ell r}$ is two dimensional so any endomorphism of $P_j \otimes \mathbb{C}^H_{\ell r}$ has the form $aId_{P_j\otimes\mathbb{C}^H_{\ell r}} + bx_{j,\ell}$ for some $a, b \in \mathbb{C}$. So to determine $\Phi_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}$ it is enough to find a and b. $\mathsf{w}^S_j \otimes v \in P_j \otimes \mathbb{C}^H_{\ell r}$ is a highest weight vector, so

$$\begin{aligned}\Phi_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}(\mathsf{w}^S_j \otimes v) = \Psi_{j+\ell r+1-r}(\chi(S_1))\mathsf{w}^S_j \otimes v &= (q^{j+\ell r+1-r} + q^{-(j+\ell r+1-r)}\mathsf{w}^S_j \otimes v \\ &= (-1)^{\ell+1}(q^{j+1} + q^{-(j+1)})\mathsf{w}^S_j \otimes v.\end{aligned}$$

Hence, $a = (-1)^{\ell+1}(q^{j+1} + q^{-(j+1)})$. S_1 has a basis $\{s_0, s_1\}$ where $Es_0 = 0$, $Es_1 = s_0$, $Fs_0 = s_1$, $Fs_1 = 0$, and $Hs_i = (-1)^i s_i$. Recall that $c_{V,W} = \tau \circ R$ where τ is the flip map and $E^2 = F^2 = 0$ on S_1 so $R = q^{H\otimes H/2}(Id \otimes Id + (q - q^{-1})E \otimes F)$ here. For some $X, Y \in P_j \otimes \mathbb{C}^H_{\ell r}$, we have

$$\begin{aligned}c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}} \circ c_{P_j\otimes\mathbb{C}^H_{\ell r},S_1}(\mathsf{w}^H_j \otimes v \otimes s_0) &= c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}(q^{(j+\ell r)/2}s_0 \otimes \mathsf{w}^H_j \otimes v \\ &\quad + (q - q^{-1})q^{\ell r-(j+\ell r+2)/2}s_1 \otimes \mathsf{w}^R_{j+2} \otimes v) \\ &= (q^{(j+\ell r)}\mathsf{w}^H_j \otimes v \otimes s_0 + (q - q^{-1})^2 q^{\ell r-1}\mathsf{w}^S_j \otimes v \otimes s_0) + X \otimes s_1 \\ &= (-1)^\ell(q^j\mathsf{w}^H_j \otimes v \otimes s_0 + (q - q^{-1})^2 q^{-1}\mathsf{w}^S_j \otimes v \otimes s_0) + X \otimes s_1\end{aligned}$$

$$\begin{aligned} c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}\circ c_{P_j\otimes\mathbb{C}^H_{\ell r},S_1}(\mathsf{w}_j^H\otimes v\otimes s_1) &= c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}(q^{-(j+\ell r)/2}s_1\otimes\mathsf{w}_j^H\otimes v)\\ &= (q^{-(j+\ell r)}\mathsf{w}_j^H\otimes v\otimes s_1)+Y\otimes s_0\\ &= (-1)^\ell q^{-j}\mathsf{w}_j^H\otimes v\otimes s_1+Y\otimes s_0. \end{aligned}$$

Hence,

$$\begin{aligned} \Phi_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}(\mathsf{w}_j^H\otimes v) &= (Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overleftarrow{\mathrm{ev}}_{S_1})\circ((c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}\circ c_{P_j\otimes\mathbb{C}^H_{\ell r},S_1}\otimes Id_{S_1^*})\circ(Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overrightarrow{\mathrm{coev}}_{S_1})(\mathsf{w}_j^H\otimes v)\\ &= (Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overleftarrow{\mathrm{ev}}_{S_1})\circ((c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}\circ c_{P_j\otimes\mathbb{C}^H_{\ell r},S_1}(\mathsf{w}_j^H\otimes v\otimes s_0))\otimes s_0^*)\\ &+ (Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overleftarrow{\mathrm{ev}}_{S_1})\circ((c_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}\circ c_{P_j\otimes\mathbb{C}^H_{\ell r},S_1}(\mathsf{w}_j^H\otimes v\otimes s_1))\otimes s_1^*)\\ &= (-1)^\ell(q^j\mathsf{w}_j^H\otimes v+(q-q^{-1})^2q^{-1}\mathsf{w}_j^S\otimes v)\otimes s_0^*(K^{1-r}s_0)+X\otimes s_0^*(K^{1-r}s_1)\\ &+ (-1)^\ell q^{-j}\mathsf{w}_j^H\otimes v\otimes s_1^*(K^{1-r}s_1)+Y\otimes s_1^*(K^{1-r}s_0)\\ &= (-1)^{\ell+1}q(q^j\mathsf{w}_j^H\otimes v+(q-q^{-1})^2q^{-1}\mathsf{w}_j^S\otimes v)+(-1)^{\ell+1}q^{-j-1}\mathsf{w}_j^H\otimes v\\ &= (-1)^{\ell+1}(q^{j+1}+q^{-j-1})\mathsf{w}_j^H\otimes v+(-1)^{\ell+1}(q-q^{-1})^2\mathsf{w}_j^S\otimes v. \end{aligned}$$

For the sake of consistency with [1], we rescale $x_{j,\ell}$ by a constant $\frac{(q^{j+1}-q^{-(j+1)})^2}{(q-q^{-1})^2}$ so, we have $a=(-1)^{\ell+1}(q^{j+1}+q^{-j-1})$ and $b=(-1)^{\ell+1}(q^{j+1}-q^{-(j+1)})^2$. The S_i are classified up to isomorphism by their characters, so we obtain the relation $S_1\otimes S_i\cong S_{i+1}\oplus S_{i-1}$ from their characters. Therefore, we have

$$\Phi_{S_1,P_j\otimes\mathbb{C}^H_{\ell r}}\circ\Phi_{S_i,P_j\otimes\mathbb{C}^H_{\ell r}}=\Phi_{S_1\otimes S_i,P_j\otimes\mathbb{C}^H_{\ell r}}=\Phi_{S_{i+1},P_j\otimes\mathbb{C}^H_{\ell r}}+\Phi_{S_{i-1},P_j\otimes\mathbb{C}^H_{\ell r}}.$$

This gives the recurrence relation

$$\Phi_{S_{i+1},P_j\otimes\mathbb{C}^H_{\ell r}}=(a+bx_{j,\ell})\Phi_{S_i,P_j\otimes\mathbb{C}^H_{\ell r}}-\Phi_{S_{i-1},P_j\otimes\mathbb{C}^H_{\ell r}}$$

whose solution

$$\Phi_{S_i,P_j\otimes\mathbb{C}^H_{\ell r}}=\frac{(-1)^{i(\ell+1)}}{\{j+1\}}(\{(i+1)(j+1)\}+x_{j,\ell}(i\{(i+2)(j+1)\}-(i+2)\{i(j+1)\}))$$

can be found by induction. Since $S_{r-1}=V_0$, we have $\Phi_{V_0,P_j\otimes\mathbb{C}^H_{\ell r}}=(-1)^{r(\ell+1)+j+\ell}2rx_{j,\ell}$. As projective modules are classified up to isomorphism by their characters, we have the isomorphism of modules

$$V_0\otimes S_{r-i-1}=V_0\otimes S_{r-i-3}\oplus P_i.$$

Hence, we obtain the relation

$$\begin{aligned}\Phi_{P_i,P_j\otimes\mathbb{C}^H_{\ell r}} &= \Phi_{V_0,P_j\otimes\mathbb{C}^H_{\ell r}}\circ(\Phi_{S_{r-i-1},P_j\otimes\mathbb{C}^H_{\ell r}}-\Phi_{S_{r-i-3},P_j\otimes\mathbb{C}^H_{\ell r}})\\ &= \frac{(-1)^{i(\ell+1)+j+1}2r}{\{j+1\}}(\{(r-i)(j+1)\}-\{(r-i-2)(j+1)\})x_{j,\ell}\\ &= (-1)^{i(\ell+1)+j+1}2r(q^{(r-i-1)(j+1)}+q^{-(r-i-1)(j+1)})x_{j,\ell}\\ &= (-1)^{i(\ell+1)}2r(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)})x_{j,\ell}.\end{aligned}$$

To complete the proof for $\Phi_{P_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}$, we need only compute $\Phi_{\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}$ and take the appropriate compositions of Hopf links:

$$\begin{aligned}\Phi_{\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}(\mathsf{w}^H_j\otimes v) &= (Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overleftarrow{\mathrm{ev}}_{\mathbb{C}^H_{\ell' r}})\circ((c_{\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}c_{P_j\otimes\mathbb{C}^H_{\ell r},\mathbb{C}^H_{\ell' r}})\otimes Id_{\mathbb{C}^{H*}_{\ell' r}})\circ(\mathsf{w}^H_j\otimes v\otimes w\otimes w^*)\\ &= (Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overleftarrow{\mathrm{ev}}_{\mathbb{C}^H_{\ell' r}})\circ((c_{\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}c_{P_j\otimes\mathbb{C}^H_{\ell r},\mathbb{C}^H_{\ell' r}})(\mathsf{w}^H_j\otimes v\otimes w)\otimes w^*)\\ &= (Id_{P_j\otimes\mathbb{C}^H_{\ell r}}\otimes\overleftarrow{\mathrm{ev}}_{\mathbb{C}^H_{\ell' r}})\circ(q^{H\otimes H}(\mathsf{w}^H_j\otimes v\otimes w)\otimes w^*)\\ &= q^{(j+\ell r)\ell' r}(\mathsf{w}^H_j\otimes v)\otimes w^*(K^{1-r}w)\\ &= q^{(j+\ell r)\ell' r}(\mathsf{w}^H_j\otimes v)\otimes q^{\ell' r(1-r)}\\ &= (-1)^{(j+\ell r+1-r)\ell'}\mathsf{w}^H_j\otimes v.\end{aligned}$$

Notice that the weight of any vector $\mathsf{w}^X_i\otimes v\in P_j\otimes\mathbb{C}^H_{\ell r}$ differs from the weight of $\mathsf{w}^H_j\otimes v$ by a multiple of 2, so the exponent of -1 does not see this difference. Hence, we have

$$\Phi_{\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}} = (-1)^{(j+\ell r+1-r)\ell'}Id_{P_j\otimes\mathbb{C}^H_{\ell r}}.$$

We delay the proof of $\Phi_{V_\alpha,P_j\otimes\mathbb{C}^H_{\ell r}}$ until after the following corollary.

□

Corollary 1. *The modified trace of the Hopf links are given by:*

$$t_{V_\alpha}(\Phi_{V_\beta,V_\alpha}) = (-1)^{r-1}rq^{\alpha\beta},\qquad t_{V_\alpha}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}) = (-1)^{r-1}q^{r\ell\alpha}\frac{r\{(i+1)\alpha\}}{\{r\alpha\}},$$

$$t_{P_i\otimes\mathbb{C}^H_{\ell r}}(\Phi_{V_\alpha,P_i\otimes\mathbb{C}^H_{\ell r}}) = t_{V_\alpha}(\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}) = (-1)^{r-1}q^{r\ell\alpha}r(q^{(r-1-i)\alpha}+q^{-(r-1-i)\alpha}),$$

$$t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{P_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}) = (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}2r(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)}),$$
$$t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}) = (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}(i+1)(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)}),$$
$$t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}\circ x_{j,\ell}) = (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}\frac{\{(i+1)(j+1)\}}{\{j+1\}}.$$

Proof. The Hopf links on V_α are easy to compute:

$$t_{V_\alpha}(\Phi_{V_\beta,V_\alpha}) = \frac{(-1)^{r-1}r}{\mathbf{d}(V_\alpha)}q^{\alpha\beta}t_{V_\alpha}(Id_{V_\alpha}) = (-1)^{r-1}rq^{\alpha\beta},$$
$$t_{V_\alpha}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}) = q^{r\ell\alpha}\frac{\{(i+1)\alpha\}}{\{\alpha\}}t_{V_\alpha}(Id_{V_\alpha}) = (-1)^{r-1}q^{r\ell\alpha}\frac{r\{(i+1)\alpha\}}{\{r\alpha\}},$$
$$t_{V_\alpha}(\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}) = \frac{(-1)^{r-1}q^{r\ell\alpha}r}{\mathbf{d}(V_\alpha)}(q^{(r-1-i)\alpha}+q^{-(r-1-i)\alpha})\mathbf{d}(V_\alpha).$$

To compute the trace of the Hopf links on $P_j\otimes\mathbb{C}^H_{\ell r}$, we need to understand how the modified trace acts on the identity and $x_{j,\ell}$. Using the decomposition derived for $V_0\otimes S_i$ in Proposition 4, one easily obtains the following isomorphism:

$$V_0\otimes S_{r-j-1}\otimes\mathbb{C}^H_{\ell r}\simeq V_0\otimes S_{r-j-3}\otimes\mathbb{C}^H_{\ell r}\oplus P_j\otimes\mathbb{C}^H_{\ell r}.$$

Using the properties of the modified trace given in definition 1, we see that the modified trace of the identities of these modules yields the relation

$$\mathbf{d}(P_j\otimes\mathbb{C}^H_{\ell r}) = \mathbf{d}(V_0)(\mathrm{qdim}(S_{r-j-1}\otimes\mathbb{C}^H_{\ell r})-\mathrm{qdim}(S_{r-j-3}\otimes\mathbb{C}^H_{\ell r}))$$

where

$$\mathrm{qdim}(S_n\otimes\mathbb{C}^H_{\ell r}) = \sum_{i=0}^{n}(s_i\otimes v)^*(K^{1-r}s_i\otimes v) = \sum_{i=0}^{n}q^{(n-2i+\ell r)(1-r)}$$
$$= q^{\ell r(1-r)}\frac{\{n(1-r)+1\}}{\{1\}} = (-1)^{\ell(1-r)+n}\frac{\{n+1\}}{\{1\}}.$$

Hence,

$$\begin{aligned}\mathbf{d}(P_j\otimes\mathbb{C}^H_{\ell r}) &= \mathbf{d}(V_0)(\mathrm{qdim}(S_{r-j-1}\otimes\mathbb{C}^H_{\ell r})-\mathrm{qdim}(S_{r-j-3}\otimes\mathbb{C}^H_{\ell r}))\\ &= \frac{(-1)^{r-1}}{\{1\}}\big((-1)^{\ell(1-r)+r-j-1}\{r-j\}-(-1)^{\ell(1-r)+r-j-3}\{r-j-2\}\big)\\ &= (-1)^{\ell(1-r)+j}((q^{r-j-1}+q^{r-j-3}+\ldots+q^{-(r-j-1)})-(q^{r-j-3}+\ldots+q^{-(r-j-3)}))\\ &= (-1)^{\ell(1-r)+j+1}(q^{j+1}+q^{-(j+1)}).\end{aligned}$$

Now, $t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{V_0,P_j\otimes\mathbb{C}^H_{\ell r}}) = t_{V_0}(\Phi_{P_j\otimes\mathbb{C}^H_{\ell r},V_0})$ gives the relation

$$2r(-1)^{r(\ell+1)+\ell+j}t_{P_j\otimes\mathbb{C}^H_{\ell r}}(x_{j,\ell}) = 2r(-1)^{r-1}.$$

Hence, $t_{P_j\otimes\mathbb{C}^H_{\ell r}}(x_{j,\ell}) = (-1)^{\ell(r+1)+j+1}$. We can now compute the Hopf links on the $P_j\otimes\mathbb{C}^H_{\ell r}$ modules using Proposition 5:

$$\begin{aligned}t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{P_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}) &= (-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}2r(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)})(-1)^{\ell(r+1)+j+1}\\ &= (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}2r(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)}),\end{aligned}$$

$$\begin{aligned}t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}) &= \frac{(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}}\{(i+1)(j+1)\}(-1)^{\ell(1-r)+j+1}(q^{j+1}+q^{-(j+1)})\\ &\quad+\frac{(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}}(i\{(i+2)(j+1)\}-(i+2)\{i(j+1)\})(-1)^{\ell(r+1)+j+1})\end{aligned}$$

$$\begin{aligned}&= \frac{(-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}}{\{j+1\}}\Big(\{(i+1)(j+1)\}(q^{j+1}+q^{-(j+1)})+(i\{(i+2)(j+1)\}-(i+2)\{i(j+1)\})\Big)\\ &= (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}(i+1)(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)}),\end{aligned}$$

where the last equality is found by expanding the brackets and simplifying. The last trace is computed using the fact that $x^2_{j,\ell}=0$.

$$\begin{aligned}t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}\circ x_{j,\ell}) &= (-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}\frac{\{(i+1)(j+1)\}}{\{j+1\}}t_{P_j\otimes\mathbb{C}^H_{\ell r}}(x_{j,\ell}))\\ &= (-1)^{\ell(i+r+1)+(j+\ell r+1-r)\ell'+i+j+1}\frac{\{(i+1)(j+1)\}}{\{j+1\}}.\end{aligned}$$

□

To find $\Phi_{V_\alpha,P_j\otimes\mathbb{C}^H_{\ell r}} = aId_{P_j\otimes\mathbb{C}^H_{\ell r}} + bx_{j,\ell}$ we first compute the action of this Hopf link on the highest weight vector $\mathsf{w}^S_j \otimes v \in P_j \otimes \mathbb{C}^H_{\ell r}$ of weight $j + \ell r$:

$$\Phi_{V_\alpha,P_j\otimes\mathbb{C}^H_{\ell r}}(\mathsf{w}^S_j \otimes v) = \Psi_{j+\ell r+1-r}(\chi(V_\alpha))\mathsf{w}^S_j \otimes v = q^{(j+\ell r+1-r)\alpha}\frac{\{r(j+\ell r+1-r)\}}{\{j+\ell r+1-r\}} = 0.$$

So, we must have $a = 0$. We can now solve for b using the symmetry property of the modified trace and the value of $t_{V_\alpha}(\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},V_\alpha})$ computed above.

$$(-1)^{r-1}q^{r\ell\alpha}r(q^{(r-j-1)\alpha} + q^{-(r-j-1)\alpha}) = t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{V_\alpha,P_j\otimes\mathbb{C}^H_{\ell r}}) = t_{P_j\otimes\mathbb{C}^H_{\ell r}}(bx_{j,\ell}) = b(-1)^{\ell(r+1)+j+1}$$

Hence, $\Phi_{V_\alpha,P_j\otimes\mathbb{C}^H_{\ell r}} = (-1)^{r(\ell+1)+\ell+j}q^{r\ell\alpha}r(q^{(r-j-1)\alpha} + q^{-(r-j-1)\alpha})x_{j,\ell}$.

5 Results

Proposition 6. *We have for any $\alpha \in \{1, ..., r-1\}$ the short exact sequences of modules*

$$0 \to V_{r-1-i+\ell r} \to P_i \otimes \mathbb{C}^H_{\ell r} \to V_{1+i-r+\ell r} \to 0$$
$$0 \to S_{r-1-\alpha} \otimes \mathbb{C}^H_{\ell r} \to V_{\alpha+\ell r} \to S_{\alpha-1} \otimes \mathbb{C}^H_{(\ell+1)r} \to 0$$

and corresponding identities of Hopf links for the projective modules $X = V_\alpha, P_j \otimes \mathbb{C}^H_{\ell r}$:

$$\Phi_{V_{\alpha+\ell r},X} = \Phi_{S_{r-1-\alpha}\otimes\mathbb{C}^H_{\ell r},X} + \Phi_{S_{\alpha-1}\otimes\mathbb{C}^H_{(\ell+1)r},X},$$
$$\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},X} = \Phi_{V_{r-1-i+\ell r},X} + \Phi_{V_{1+i-r+\ell r},X}.$$

Proof. $V_{\alpha+\ell r}$ is atypical so it contains a highest weight submodule generated by standard basis vector v_α with basis $\{v_\alpha, ..., v_{r-1}\}$. This submodule has dimension $r - \alpha$ and highest weight $r - 1 - \alpha + \ell r$ so it isomorphic to $S_{r-1-\alpha} \otimes \mathbb{C}^H_{\ell r}$. The quotient module $V_{\alpha+\ell r} / < v_\alpha >$ has dimension α and highest weight $\alpha - 1 + (\ell + 1)r$ so it is isomorphic to $S_{\alpha-1} \otimes \mathbb{C}^H_{(\ell+1)r}$. We therefore have the short exact sequence

$$0 \to S_{r-1-\alpha} \otimes \mathbb{C}^H_{\ell r} \to V_{\alpha+\ell r} \to S_{\alpha-1} \otimes \mathbb{C}^H_{(\ell+1)r} \to 0$$

The element $\mathsf{w}^S_i \otimes v \in P_i \otimes \mathbb{C}^H_{\ell r}$ generates a highest weight submodule of dimension $i+1$ and highest weight $i+\ell r$ hence is isomorphic to the submodule $< v_{r-1-i} >$ of $V_{r-1-i+\ell r}$ as both of these modules are isomorphic to $S_i \otimes \mathbb{C}^H_{\ell r}$. So there is an isomorphism $\widetilde{\phi} :< v_{r-1-i} > \to < \mathsf{w}^S_i \otimes v >$. We can ex-

tend this to an injective homomorphism of $V_{r-1-i+\ell r}$ by defining a map $\phi : V_{r-1-i+\ell r} \to P_i \otimes \mathbb{C}^H_{\ell r}$ by

$$\phi(v_n) = \begin{cases} \mathsf{w}^R_{j+r} \otimes v & : n = 0 \\ (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} \mathsf{w}^R_{j+r-2n} \otimes v & : 0 < n < r-1-i \\ \widetilde{\phi}(v_n) & : n \geq r-1-i \end{cases}$$

and extending linearly over sums, where we have set $j = r-2-i$ for simplicity. This map is clearly bijective and linear so we need only show it is a module morphism for $n < r-1-i$. When $n < r-1-i$ we have

$$\phi(Hv_n) = (2r-2-i+\ell r-2n)\phi(v_n) = (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} H(\mathsf{w}^R_{j+r-2n} \otimes v) = H\phi(v_n),$$

$$\phi(Fv_n) = \phi(v_{n+1}) = (-1)^{n+1} \prod_{s=0}^{n} \gamma_{j,j-s}(\mathsf{w}^R_{j+r-2(n+1)} \otimes v) = (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} F(\mathsf{w}^R_{j+r-2n} \otimes v) = F\phi(v_n).$$

To show that ϕ commutes with the action of E we first note the identity

$$[n][n-r+1+i-\ell r] = (-1)^{\ell+1}[r-n-1-i][n] = (-1)^{\ell+1}[j-(n-1)][n] = (-1)^{\ell+1}\gamma_{j,j-(n-1)}.$$

Hence,

$$\begin{aligned} \phi(Ev_n) = [n][n-r+1+i-\ell r]\phi(v_{n-1}) &= (-1)^n \prod_{s=0}^{n-2} \gamma_{j,j-s}(-1)^{\ell} \gamma_{j,j-(n-1)}(\mathsf{w}^R_{j+r-2(n-1)} \otimes v) \\ &= (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} E(\mathsf{w}^R_{j+r-2n} \otimes v) = E\phi(v_n). \end{aligned}$$

This shows that $V_{r-1-i+kr}$ can be embedded in $P_i \otimes \mathbb{C}^H_{\ell r}$ by ϕ. By a similar argument, one can see that $P_i \otimes \mathbb{C}^H_{\ell r}/\phi(V_{r-1-i+\ell r}) \backsimeq V_{1+i-r+\ell r}$ which proves the second short exact sequence. The Hopf link identities are proven by direct computation using Proposition 5:

$$\begin{aligned} \Phi_{V_{r-1-i+\ell r},V_\alpha} + \Phi_{V_{1+i-r+\ell r},V_\alpha} - \Phi_{P_i \otimes \mathbb{C}^H_{\ell r},V_\alpha} &= \frac{(-1)^{r-1} r}{\mathbf{d}(V_\alpha)} q^{\ell r \alpha}(q^{(r-1-i)\alpha} + q^{(1+i-r)\alpha}) \\ &\quad - \frac{(-1)^{r-1} r}{\mathbf{d}(V_\alpha)} q^{\ell r \alpha}(q^{(r-1-i)\alpha} + q^{-(r-1-i)\alpha}) \\ &= 0, \end{aligned}$$

$$\begin{aligned}\Phi_{S_i\otimes\mathbb{C}^H_{\ell r},V_\alpha}+\Phi_{S_{r-i-2}\otimes\mathbb{C}^H_{(\ell+1)r},V_\alpha}-\Phi_{V_{r-1-i+\ell r},V_\alpha}&=\frac{q^{r\ell\alpha}}{\{\alpha\}}\Big(\{(i+1)\alpha\}+q^{r\alpha}\{(r-i-1)\alpha\}-q^{(r-1-i)\alpha}\{r\alpha\}\Big)\\&=\frac{q^{r\ell\alpha}}{\{\alpha\}}\Big(\{(i+1)\alpha\}-\{(i+1)\alpha\}-q^{(r-1-i)\alpha}(q^{r\alpha}-q^{-r\alpha})\Big)\\&=\frac{q^{r\ell\alpha}}{\{\alpha\}}\Big(q^{(r-1-i)\alpha}(q^{r\alpha}-q^{-r\alpha})\Big)\\&=\frac{q^{r\ell\alpha}}{\{\alpha\}}\Big(q^{-(i+1)\alpha}-q^{-(i+1)\alpha}\Big)=0.\end{aligned}$$

By expanding $\Phi_{V_{r-1-i+\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}+\Phi_{V_{1+i-r+\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}$, we obtain

$$\begin{aligned}&(-1)^{r(\ell+1)+k+j}rq^{r\ell(r-1-i+\ell' r)}(q^{(r+j+1)(r-1-i+\ell' r)}+q^{-(r+j+1)(r-1-i+\ell' r)})x_{j,\ell}\\&+(-1)^{r(\ell+1)+\ell+j}rq^{r\ell(1+i-r+\ell' r)}(q^{(r+j+1)(1+i-r+\ell' r)}+q^{-(r+j+1)(1+i-r+\ell' r)})x_{j,\ell}\\&=(-1)^{\ell+j+\ell(1+i+\ell' r)+1+i+\ell' r}r(q^{(j+1)(r-1-i+\ell' r)}+q^{-(j+1)(r-1-i+\ell' r)})x_{j,\ell}\\&+(-1)^{\ell+j+\ell(1+i+\ell' r)+1+i+\ell' r}r(q^{(j+1)(1+i-r+\ell' r)}+q^{-(j+1)(1+i-r+\ell' r)})x_{j,\ell}\\&=(-1)^{\ell+j+\ell(1+i+\ell' r)+1+i+\ell' r+(j+1)(\ell'+1)}r(q^{(j+1)(1+i)}+q^{-(j+1)(1+i)})x_{j,\ell}\\&+(-1)^{\ell+j+\ell(1+i+\ell' r)+1+i+\ell' r+(j+1)(\ell'+1)}r(q^{(j+1)(1+i)}+q^{-(j+1)(1+i)})x_{j,\ell}\\&=(-1)^{i(\ell+1)+\ell'(r+1+j+\ell r)}2r(q^{(j+1)(1+i)}+q^{-(j+1)(1+i)})x_{j,\ell}\\&=\Phi_{P_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}.\end{aligned}$$

Recall that the Hopf links on $P_j\otimes\mathbb{C}^H_{\ell r}$ have the form $aId_{P_j\otimes\mathbb{C}^H_{\ell r}}+bx_{j,\ell}$. The coefficient of the identity of $\Phi_{V_{\alpha+\ell' r,P_j\otimes\mathbb{C}^H_{\ell r}}}$ is zero and the coefficient of the identity for $\Phi_{S_{r-1-\alpha}\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}+\Phi_{S_{\alpha-1}\otimes\mathbb{C}^H_{(\ell'+1)r},P_j\otimes\mathbb{C}^H_{\ell r}}$ is

$$\begin{aligned}&\frac{(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}}\{(r-\alpha)(j+1)\}+\frac{(-1)^{(\alpha-1)(\ell+1)+(j+\ell r+1-r)(\ell'+1)}}{\{j+1\}}\{\alpha(j+1)\}\\&=-\frac{(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1-r)\ell'+j+1}}{\{j+1\}}\{\alpha(j+1)\}-\frac{(-1)^{(-\alpha-1)(\ell+1)+(j+\ell r+1-r)(\ell'+1)}}{\{j+1\}}\{\alpha(j+1)\}\\&=\frac{(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1-r)\ell'+j+1}}{\{j+1\}}\big(\{\alpha(j+1)\}-\{\alpha(j+1)\}\big)=0.\end{aligned}$$

(Note that α is an integer in this computation)

So, $\Phi_{S_{r-1-\alpha}\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}+\Phi_{S_{\alpha-1}\otimes\mathbb{C}^H_{(\ell'+1)r}}-\Phi_{V_{\alpha+\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}=bx_{j,\ell}$ and we also have the identity:

$$
\begin{aligned}
&\frac{(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1+r)\ell'}}{\{j+1\}}\big((r-1-\alpha)\{(r+1-\alpha)(j+1)\}-(r+1-\alpha)\{(r-1-\alpha)(j+1)\}\\
&+\frac{(-1)^{(\alpha-1)(\ell+1)+(j+\ell r+1+r)(\ell'+1)}}{\{j+1\}}(\alpha-1)\{(\alpha+1)(j+1)\}-(\alpha+1)\{(\alpha-1)(j+1)\}\big)\\
&=\frac{(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1+r)\ell'+j+1}}{\{j+1\}}\big((r-1-\alpha)\{(1-\alpha)(j+1)\}-(r+1-\alpha)\{(-1-\alpha)(j+1)\}\\
&+\frac{(-1)^{(r-\alpha-1)(\ell+1)+(j+\ell r+1+r)\ell'+j+1}}{\{j+1\}}(\alpha-1)\{(\alpha+1)(j+1)\}-(\alpha+1)\{(\alpha-1)(j+1)\}\big)\\
&=\frac{(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1+r)\ell'+j+1}}{\{j+1\}}r\left(\{(\alpha+1)(j+1)\}-\{(\alpha-1)(j+1)\}\right)\\
&=(-1)^{(r-1-\alpha)(\ell+1)+(j+\ell r+1+r)\ell'+j+1}r\left(q^{\alpha(j+1)}+q^{-\alpha(j+1)}\right).
\end{aligned}
$$

The left hand side of this equation is the constant coefficient for $x_{j,\ell}$ in $\Phi_{S_{r-1-\alpha}\otimes\mathbb{C}^H_{\ell r},P_j\otimes\mathbb{C}^H_{\ell r}}+\Phi_{S_{\alpha-1}\otimes\mathbb{C}^H_{(\ell'+1)r}}$ and the right hand side is the constant coefficient for $x_{j,\ell}$ in $\Phi_{V_{\alpha+\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}$.
This shows that $b=0$ and completes the proof.

□

The equations

$$
\begin{aligned}
0\to V_{r-1-i+\ell r}\to P_i\otimes\mathbb{C}^H_{\ell r}\to V_{1+i-r+\ell r}\to 0\\
0\to S_{r-1-\alpha}\otimes\mathbb{C}^H_{\ell r}\to V_{\alpha+\ell r}\to S_{\alpha-1}\otimes\mathbb{C}^H_{(\ell+1)r}\to 0
\end{aligned}
$$

describe the structure $P_i\otimes\mathbb{C}^H_{\ell r}$ and $V_{\alpha+\ell r}$ respectively in terms of simpler modules. It is easy to see from the definition that $P_i\otimes\mathbb{C}^H_{\ell r}$ has Loewy diagram as in Figure 1:

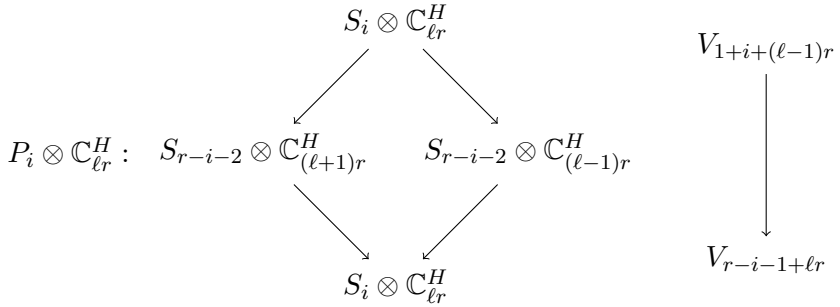


Figure 1: Loewy diagram of $P_i\otimes\mathbb{C}^H_{\ell r}$

Proposition 7. *The tensor ring of projective weight modules is determined up to isomorphism by*

the modified traces of Hopf links and the isomorphism is given by $\phi : V \mapsto \frac{t_{V_\alpha}(\Phi_{V,V_\alpha})}{\mathbf{d}(V_\alpha)}$.

Proof. ϕ is bijective and clearly preserves the additive structure as the trace is linear and $\Phi_{V\oplus W,V_\alpha} = \Phi_{V,V_\alpha} + \Phi_{W,V_\alpha}$. So, ϕ is an isomorphism if we have $\phi(V \otimes W) = \phi(V)\phi(W)$ for any weight modules V, W. This is equivalent to

$$t_{V_\alpha}(\Phi_{V\otimes W,V_\alpha}) = \frac{t_{V_\alpha}(\Phi_{V,V_\alpha})t_{V_\alpha}(\Phi_{W,V_\alpha})}{\mathbf{d}(V_\alpha)}.$$

V_α is simple so the Hopf links act as scalars and there exist $\lambda_V, \lambda_W \in \mathbb{C}$ such that $\Phi_{V,V_\alpha} = \lambda_V Id_{V_\alpha}$ and $\Phi_{W,V_\alpha} = \lambda_W Id_{V_\alpha}$. Hence,

$$\begin{aligned}
t_{V_\alpha}(\Phi_{V\otimes W,V_\alpha}) = t_{V_\alpha}(\Phi_{V,V_\alpha} \circ \Phi_{W,V_\alpha}) &= t_{V_\alpha}(\lambda_V\lambda_W Id_{V_\alpha}) \\
&= \lambda_V\lambda_W t_{V_\alpha}(Id_{V_\alpha}) \\
&= \lambda_V\lambda_W \frac{t_{V_\alpha}(Id_{V_\alpha})^2}{t_{V_\alpha}(Id_{V_\alpha})} \\
&= \frac{t_{V_\alpha}(\lambda_V Id_{V_\alpha})t_{V_\alpha}(\lambda_W Id_{V_\alpha})}{\mathbf{d}(V_\alpha)} \\
&= \frac{t_{V_\alpha}(\Phi_{V,V_\alpha})t_{V_\alpha}(\Phi_{W,V_\alpha})}{\mathbf{d}(V_\alpha)}
\end{aligned}$$

which completes the proof. □

5.1 Comparison

Proposition 8. *Let* $\alpha \in \ddot{\mathbb{C}}$. *Then the map* $\varphi : V_\alpha \mapsto F_{\frac{\alpha+r-1}{\sqrt{2r}}}$, $S_i \otimes \mathbb{C}^H_{\ell r} \mapsto M_{1-\ell,i+1}$ *extended linearly over direct sums for* $\alpha \in \ddot{\mathbb{C}}$ *is a morphism up to equality of characters.*

Proof. We can compute the relations using the tensor decompositions computed in Proposition 4:

$$\begin{aligned}
\mathrm{ch}[\varphi(V_\alpha \otimes V_\beta)] = \sum_{\substack{k=1-r \\ k+r=1 \bmod 2}}^{r-1} \mathrm{ch}[\varphi(V_{\alpha+\beta+k})] &= \sum_{\substack{k=1-r \\ k+r=1 \bmod 2}}^{r-1} \mathrm{ch}\left[F_{\frac{\alpha+\beta+k+r-1}{\sqrt{2r}}}\right] \\
&= \sum_{\substack{k=1-r \\ k+r=1 \bmod 2}}^{r-1} \mathrm{ch}\left[F_{\frac{\alpha+r-1}{\sqrt{2r}}+\frac{\beta+r-1}{\sqrt{2r}}+\frac{k+1-r}{\sqrt{2r}}}\right] \\
&= \sum_{l=0}^{r-1} \mathrm{ch}\left[F_{\frac{\alpha+r-1}{\sqrt{2r}}+\frac{\beta+r-1}{\sqrt{2r}}-\frac{2l}{\sqrt{2r}}}\right] \\
&= \mathrm{ch}\left[F_{\frac{\alpha+r-1}{\sqrt{2r}}}\right] \times \mathrm{ch}\left[F_{\frac{\beta+r-1}{\sqrt{2r}}}\right] \\
&= \mathrm{ch}[\varphi(V_\alpha)] \times \mathrm{ch}[\varphi(V_\beta)],
\end{aligned}$$

$$\begin{aligned}
\mathrm{ch}[\varphi((S_i \otimes \mathbb{C}^H_{\ell r}) \otimes V_\alpha)] &= \sum_{\substack{l=-i \\ l+i=0 \bmod 2}}^{i} \mathrm{ch}[\varphi(V_{\alpha+\ell r+l})] \\
&= \sum_{\substack{l=-i \\ l+i=0 \bmod 2}}^{i} \mathrm{ch}\left[F_{\frac{\alpha+\ell r+l+r-1}{\sqrt{2r}}}\right] \\
&= \sum_{\substack{l=-(i+1)+2 \\ l+i=0 \bmod 2}}^{i+1} \mathrm{ch}\left[F_{\frac{\alpha+r-1}{\sqrt{2r}}+\frac{\ell\sqrt{r}}{\sqrt{2}}+\frac{l-1}{\sqrt{2r}}}\right] \\
&= \mathrm{ch}[M_{1-\ell,i+1}] \times \mathrm{ch}\left[F_{\frac{\alpha+r-1}{\sqrt{2r}}}\right] \\
&= \mathrm{ch}[\varphi(S_i \otimes \mathbb{C}^H_{\ell r})] \times \mathrm{ch}[\varphi(V_\alpha)].
\end{aligned}$$

The Hopf link identities $\Phi_{V_{\alpha+\ell r},X} = \Phi_{S_{r-1-\alpha}\otimes\mathbb{C}^H_{\ell r},X} + \Phi_{S_{\alpha-1}\otimes\mathbb{C}^H_{(\ell+1)r},X}$ and $\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},X} = \Phi_{V_{r-1-i+\ell r},X} + \Phi_{V_{1+i-r+\ell r},X}$ motivate the definition

$$\begin{aligned}
\varphi(P_i \otimes \mathbb{C}^H_{\ell r}) &= \varphi(V_{r-1-i+\ell r}) \oplus \varphi(V_{1+i-r+\ell r}) \\
&= \varphi(S_i \otimes \mathbb{C}^H_{\ell r}) \oplus \varphi(S_{r-i-2} \otimes \mathbb{C}^H_{(\ell+1)r}) \oplus \varphi(S_{r-2-i} \otimes \mathbb{C}^H_{(\ell-1)r}) \oplus \varphi(S^H_i \otimes \mathbb{C}^H_{\ell r}) \\
&= M_{1-\ell,i+1} \oplus M_{\ell,r-i-1} \oplus M_{2-\ell,r-i-1} \oplus M_{1-\ell,i+1}.
\end{aligned}$$

It is easy to see using Proposition 4, the fact that S_{r-1} is projective, and projective modules being classified by characters that

$$S_1 \otimes S_j = \begin{cases} S_1 & if\ j = 0 \\ S_{j+1} \oplus S_{j-1} & if\ 1 \le j \le r-2 \\ P_{r-2} & if\ j = r-1 \end{cases}$$

If j=0, then $\mathrm{ch}[\phi(S_1 \otimes S_0)] = \mathrm{ch}[M_{1,2}] = \mathrm{ch}[M_{1,2}] \times \mathrm{ch}[M_{1,1}] = \mathrm{ch}[\varphi(S_1)] \times \mathrm{ch}[\varphi(S_0)]$. If $1 \le j \le r-2$, then we have

$$\begin{aligned}
\mathrm{ch}[\varphi(S_1 \otimes \mathbb{C}^H_{\ell r} \otimes S_j \otimes \mathbb{C}^H_{\ell' r})] &= \mathrm{ch}[\varphi(S_{j-1} \otimes \mathbb{C}^H_{(\ell+\ell')r}) \oplus \varphi(S_{j+1} \otimes \mathbb{C}^H_{(\ell+\ell')r})] \\
&= \mathrm{ch}[M_{1-(\ell+\ell'),j}] + \mathrm{ch}[M_{1-(\ell+\ell'),j+2}] \\
&= \mathrm{ch}[M_{1-\ell,1}] \times \mathrm{ch}[M_{1-\ell',j+1}] \\
&= \mathrm{ch}[\varphi(S_1 \otimes \mathbb{C}^H_{\ell r})] \times \mathrm{ch}[\varphi(S_j \otimes \mathbb{C}^H_{\ell' r})].
\end{aligned}$$

The more general case $\mathrm{ch}[\varphi(S_i \otimes \mathbb{C}^H_{\ell r} \otimes S_j \otimes \mathbb{C}^H_{\ell' r})] = \mathrm{ch}[\varphi(S_i \otimes \mathbb{C}^H_{\ell r})] \times \mathrm{ch}[\varphi(S_j \otimes \mathbb{C}^H_{\ell' r})]$ then follows inductively using the equation $S_1 \otimes S_i \otimes S_j = (S_{i+1} \otimes S_j) \oplus (S_{i-1} \otimes S_j)$.

□

Proposition 9. *The quantum dimensions of the typical and atypical modules for the singlet vertex algebra are in agreement with the modified traces of their corresponding $\overline{U}^H_q(\mathfrak{sl}_2)$ modules in the following sense:*

$$\mathrm{qdim}[F^{\frac{-i\alpha}{\sqrt{2r}}}_{\frac{\beta+r-1}{\sqrt{2r}}}] = \frac{t_{V_\alpha}(\Phi_{V_\beta,V_\alpha})}{t_{V_\alpha}(\Phi_{S_0,V_\alpha})} \qquad qdim[M^{\frac{-i\alpha}{\sqrt{2r}}}_{1-\ell,j+1}] = \frac{t_{V_\alpha}(\Phi_{S_j\otimes\mathbb{C}^H_{\ell r},V_\alpha})}{t_{V_\alpha}(\Phi_{S_0\otimes V_\alpha})}$$
$$\mathrm{qdim}[M^{\epsilon}_{1-\ell',i+1}] = \frac{t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}\circ x_{j,\ell})}{t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_0,P_j\otimes\mathbb{C}^H_{\ell r}}\circ x_{j,\ell})} \text{ for } Re(\epsilon) < 0)$$

Proof. The proof follows directly by computation:

$$\begin{aligned}
\mathrm{qdim}[F^{\frac{-i\alpha}{\sqrt{2r}}}_{\frac{\beta+r-1}{\sqrt{2r}}}] &= e^{\pi\frac{-i\alpha}{\sqrt{2r}}(2\frac{\beta+r-1}{\sqrt{2r}}-\alpha_0)}\frac{(e^{-\pi\sqrt{2r}\frac{-i\alpha}{\sqrt{2r}}i^2} - e^{\pi\sqrt{2r}\frac{-i\alpha}{\sqrt{2r}}i^2})}{(e^{-\pi\frac{\sqrt{2}}{\sqrt{r}}\frac{-i\alpha}{\sqrt{2r}}i^2} - e^{\pi\frac{\sqrt{2}}{\sqrt{r}}\frac{-i\alpha}{\sqrt{2r}}i^2})} \\
&= e^{\frac{-\pi i\alpha\beta}{r}}\frac{(e^{\frac{-\pi i\alpha r}{r}} - e^{\frac{\pi i\alpha r}{r}})}{(e^{\frac{-\pi i\alpha}{r}} - e^{\frac{\pi i\alpha}{r}})} \\
&= q^{\alpha\beta}\frac{\{r\alpha\}}{\{\alpha\}} \\
&= \frac{t_{V_\alpha}(\Phi_{V_\beta,V_\alpha})}{t_{V_\alpha}(\Phi_{S_0,V_\alpha})},
\end{aligned}$$

$$\begin{aligned}
\mathrm{qdim}[M^{\frac{-i\alpha}{\sqrt{2r}}}_{1-\ell,j+1}] &= e^{\pi\ell\sqrt{2r}\frac{-i\alpha}{\sqrt{2r}}}\frac{(e^{-\pi(j+1)\frac{\sqrt{2}}{\sqrt{r}}\frac{-i\alpha}{\sqrt{2r}}i^2} - e^{\pi(j+1)\frac{\sqrt{2}}{\sqrt{r}}\frac{-i\alpha}{\sqrt{2r}}i^2})}{(e^{-\pi\frac{\sqrt{2}}{\sqrt{r}}\frac{-i\alpha}{\sqrt{2r}}i^2} - e^{\pi\frac{\sqrt{2}}{\sqrt{r}}\frac{-i\alpha}{\sqrt{2r}}i^2})} \\
&= e^{-\pi\ell i\alpha}\frac{(e^{\frac{-\pi(j+1)i\alpha}{r}} - e^{\frac{\pi(j+1)i\alpha}{r}})}{(e^{\frac{-\pi i\alpha}{r}} - e^{\frac{\pi i\alpha}{r}})} \\
&= q^{r\ell\alpha}\frac{\{(j+1)\alpha\}}{\{\alpha\}} \\
&= \frac{t_{V_\alpha}(\Phi_{S_j\otimes\mathbb{C}^H_{\ell r},V_\alpha})}{t_{V_\alpha}(\Phi_{S_0\otimes V_\alpha})}.
\end{aligned}$$

For $\mathrm{Re}(\epsilon) < 0$,

$$\begin{aligned}\frac{t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_i\otimes\mathbb{C}^H_{\ell' r},P_j\otimes\mathbb{C}^H_{\ell r}}\circ x_{j,\ell})}{t_{P_j\otimes\mathbb{C}^H_{\ell r}}(\Phi_{S_0,P_j\otimes\mathbb{C}^H_{\ell r}}\circ x_{j,\ell})} &= (-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}\frac{\{(i+1)(j+1)\}}{\{j+1\}}\\ &= (-1)^{(j+\ell r+1-r)\ell'}\frac{\{(i+1)(j+1+r(\ell+1))\}}{\{j+1+r(\ell+1)\}}\\ &= \mathrm{qdim}[M^{\epsilon}_{1-\ell',i+1}]\end{aligned}$$

with $\epsilon \in \mathbb{S}(\ell, j+1+r(\ell+1))$. □

5.2 Hopf Links

Recall that the Hopf links on the simple V_α are given by

$$\Phi_{V_\beta,V_\alpha} = \frac{(-1)^{r-1}r}{\mathbf{d}(V_\alpha)}q^{\alpha\beta}Id_{V_\alpha} \qquad \Phi_{S_i\otimes\mathbb{C}^H_{\ell r},V_\alpha} = q^{r\ell\alpha}\frac{\{(i+1)\alpha\}}{\{\alpha\}}Id_{V_\alpha}$$

$$\Phi_{P_i\otimes\mathbb{C}^H_{\ell r},V_\alpha} = (-1)^{r-1}rq^{r\ell\alpha}\frac{q^{(r-1-i)\alpha}+q^{-(r-1-i)\alpha}}{\mathbf{d}(V_\alpha)}Id_{V_\alpha}$$

Theorem 3. *Let $\epsilon \in (-\frac{1}{2},\frac{1}{2})$, $i \in \{0,...,r-2\}$ and let $\ell \in \mathbb{Z}$. Denote by X_ϵ the module with vector space basis $\{w^L_{i+2-2r}, w^L_{i+4-2r}, ..., w^L_{-i-2}, w^H_{-i}, ..., w^H_i, w^S_{-i}, ..., w^S_i, w^R_{i+2}, ..., w^R_{2r-2-i}\}$ and action given by*

$$w^R_{i+2} = (-1)^\ell E w^H_i, \qquad w^L_{-i-2} = F w^H_{-i}, \qquad F w^R_{i+2} = w^S_i + [1+i][\epsilon]w^H_i,$$

$$\begin{aligned} w^H_{i-2k} &= F^k w^H_i \quad \text{and} \quad w^S_{i-2k} = F^k w^S_i && \text{for } k \in \{0,...,i\},\\ w^L_{-i-2-2k} &= F^k w^L_{-i-2} \quad \text{and} \quad w^R_{i+2+2k} = (-1)^{k\ell}E^k w^R_{i+2} && \text{for } k \in \{0,...,r-2-i\},\end{aligned}$$

$$\begin{aligned} Hw^X_k &= (k+\ell r+\epsilon)w^X_k, \quad Kw^X_k = (-1)^\ell q^{k+\epsilon}w^X_k, && \text{for } X \in \{L,H,S,R\},\\ Ew^R_k &= (-1)^\ell w^R_{k+2}, \quad Fw^X_k = w^X_{k-2}, && \text{for } X \in \{L,S,H\},\\ Fw^S_{-i} &= [1+i][\epsilon]w^L_{-i-2}, \quad Ew^S_i = 2(-1)^{\ell+1}[1+i][\epsilon]w^R_{i+2}, && Ew^R_{2r-2-i} = Fw^L_{i+2-2r} = 0,\end{aligned}$$

$$Ew^L_{-i-2} = 2(-1)^\ell[i+1][\epsilon]w^H_{-i} + (-1)^\ell w^S_{-i}, \quad Ew^L_{-i-2-2k} = (-1)^{\ell+1}[1+i+k][k-\epsilon]w^L_{-i-2+2(k-1)},$$
$$Fw^R_{i+2+2k} = -[1+i+k][k+\epsilon]w^R_{i+2+2(k-1)},$$

$$Ew^H_{i-2k} = (2[1+i-k+\epsilon][k] - [1+i-k][k-\epsilon])(-1)^\ell w^H_{i-2(k-1)} + (-1)^\ell w^S_{i-2(k-1)},$$
$$Ew^S_{i-2k} = (2[1+i-k][k-\epsilon] - [1+i-k+\epsilon][k])(-1)^\ell w^S_{i-2(k-1)} + 2(-1)^{\ell+1}[1+i]^2[\epsilon]^2 w^H_{i-2(k-1)}.$$

$$\textit{Then, } X_\epsilon = \begin{cases} V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon} & \text{if } \epsilon \neq 0 \\ P_i \otimes \mathbb{C}^H_{\ell r} & \text{if } \epsilon = 0 \end{cases}$$

Proof. Let $\epsilon \in (-\frac{1}{2}, \frac{1}{2}) - \{0\}$. Let $\{x_o, ..., x_{r-1}\}$ denote the standard basis for $V_{1+i-r+\ell r+\epsilon}$ and $\{y_0, ..., y_{r-1}\}$ the standard basis for $V_{-1-i+r+\ell r+\epsilon}$. Define a new basis $\{\mathsf{w}^L_{i+2-2r}, \mathsf{w}^L_{i+4-2r}, ..., \mathsf{w}^L_{-i-2}, \mathsf{w}^H_{-i}, ..., \mathsf{w}^H_i, \mathsf{w}^S_{-i}, ..., \mathsf{w}^S_i, \mathsf{w}^R_{i+2}, ..., \mathsf{w}^R_{2r-2-i}\}$ for $V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon}$ by

$$\mathsf{w}^H_{i-2k} = 2x_k - \frac{1}{2[1+i][\epsilon]} y_{r-1-i+k}, \qquad \mathsf{w}^S_{i-2k} = -2[1+i][\epsilon]x_k + y_{r-1-i+k},$$
$$\mathsf{w}^L_{-i-2-2k} = 2x_{1+i+k}, \qquad \mathsf{w}^R_{i+2+2k} = \frac{-1}{2[1+i][\epsilon]} \left(\prod_{s=0}^{k} [1+i+s][-s-\epsilon] \right) y_{r-2-i-k}.$$

We will first prove the statement for $\epsilon \neq 0$. Recall that the action on the standard basis element $v_k \in V_\alpha$ is given by

$$Hv_k = (\alpha + r - 1 - 2k)v_k, \qquad Ev_k = [k][k-\alpha]v_{k-1}, \qquad Fv_k = v_{k+1}.$$

By direct computation, we obtain the following:

$$\begin{aligned} E\mathsf{w}^H_i &= 2Ex_0 - \frac{1}{2[1+i][\epsilon]} Ey_{r-1-i} = -\frac{[r-1-i][-\ell r-\epsilon]}{2[1+i][\epsilon]} y_{r-2-i} \\ &= \frac{1}{2}(-1)^\ell y_{r-2-i} = (-1)^\ell \mathsf{w}^R_{i+2}, \end{aligned}$$

$$F\mathsf{w}^H_{-i} = 2Fx_i - \frac{1}{2[1+i][\epsilon]} Fy_{r-1} = 2x_{i+1} = \mathsf{w}^L_{-i-2},$$

$$\mathsf{w}^S_i + [1+i][\epsilon]\mathsf{w}^H_i = -2[1+i][\epsilon]x_0 + y_{r-1-i} + 2[1+i][\epsilon]x_0 - \frac{1}{2}y_{r-1-i} = \frac{1}{2}y_{r-1-i} = \frac{1}{2}Fy_{r-2-1} = F\mathsf{w}^R_{i+2}.$$

Hence, we have shown $\mathsf{w}^R_{i+2} = (-1)^\ell E\mathsf{w}^H_i$, $\mathsf{w}^L_{-i-2} = F\mathsf{w}^H_{-i}$, and $F\mathsf{w}^R_{i+2} = \mathsf{w}^S_i + [1+i][\epsilon]\mathsf{w}^H_i$. It is

easily seen that $F\mathsf{w}_k^X = \mathsf{w}_{k-2}^X$ for all $X \in \{L, S, H\}$ and that

$$\begin{aligned}
E\mathsf{w}_{i+2+2k}^R &= \frac{-1}{2[1+i][\epsilon]} \left(\prod_{s=0}^{k} [1+i+s][-s-\epsilon] \right) Ey_{r-2-i-k} \\
&= \frac{-[r-2-i-k][-1-k-\ell r-\epsilon]}{2[1+i][\epsilon]} \left(\prod_{s=0}^{k} [1+i+s][-s-\epsilon] \right) y_{r-2-i-(k+1)} \\
&= \frac{-(-1)^\ell}{2[1+i][\epsilon]} \left(\prod_{s=0}^{k+1} [1+i+s][-s-\epsilon] \right) y_{r-2-i-(k+1)} = (-1)^\ell \mathsf{w}_{i+2+2(k+1)}^R
\end{aligned}$$

so $E\mathsf{w}_k^R = (-1)^\ell \mathsf{w}_{k+2}^R$, which gives $\mathsf{w}_{i+2+2k}^R = (-1)^{k\ell} E^k \mathsf{w}_{i+2}^R$, $\mathsf{w}_{i-2k}^H = F^k \mathsf{w}_i^H$, $\mathsf{w}_{i-2k}^S = F^k \mathsf{w}_i^S$, and $\mathsf{w}_{-i-2-2k}^L = F^k \mathsf{w}_{-i-2}^L$.

H acts on a standard basis vector $v_k \in V_\alpha$ by $Hv_k = (\alpha + r - 1 - 2k)v_k$, so we have

$$\begin{aligned}
Hx_k &= (1+i-r+\ell r+\epsilon+r-1-2k)x_k = (i-2k+\ell r+\epsilon)x_k, \\
Hy_{r-1-i+k} &= (-1-i+r+\ell r+\epsilon+r-1-2(r-1-i+k))y_{r-1-i+k} = (i-2k+\ell r+\epsilon)y_{r-1-i+k}, \\
Hx_{1+i+k} &= (1+i-r+\ell r+\epsilon+r-1-2(1+i+k))x_{1+i+k} = (-i-2-2k+\ell r+\epsilon)x_{1+i+k}, \\
Hy_{r-1-i-(k+1)} &= (-1-i+r+\ell r+\epsilon+r-1-2(r-1-i-(k+1))y_{r-1-i-(k+1)}, \\
&= (i+2+2k+\ell r+\epsilon)y_{r-1-i-(k+1)}.
\end{aligned}$$

From this, it immediately follows that

$$\begin{aligned}
H\mathsf{w}_{i-2k}^H &= (i-2k+\ell r+\epsilon)\mathsf{w}_{i-2k}^H, \\
H\mathsf{w}_{i-2k}^S &= (i-2k+\ell r+\epsilon)\mathsf{w}_{i-2k}^S, \\
H\mathsf{w}_{-i-2-2k}^L &= (-i-2-2k+\ell r+\epsilon)\mathsf{w}_{-i-2-2k}^L, \\
H\mathsf{w}_{i+2+2k}^R &= (i+2+2k+\ell r+\epsilon)\mathsf{w}_{i+2+2k}^R,
\end{aligned}$$

and K acts as q^H, so we have shown that $H\mathsf{w}_k^X = (k+\ell r+\epsilon)\mathsf{w}_k^X$ and $K\mathsf{w}_k^X = q^{k+\ell r+\epsilon}\mathsf{w}_k^X = (-1)^\ell q^{k+\epsilon}\mathsf{w}_k^X$ for all $X \in \{L, H, S, R\}$. It is easy to see that $E\mathsf{w}_{2r-2-i}^R = F\mathsf{w}_{i+2-2r}^L = 0$ as $Fx_{r-1} = Ey_0 = 0$. We also have

$$F\mathsf{w}^S_{-i} = -2[1+i][\epsilon]Fx_i + Fy_{r-1} = -2[1+i][\epsilon]x_{i+1} = -[1+i][\epsilon]\mathsf{w}^L_{-i-2},$$
$$E\mathsf{w}^S_{i} = -2[1+i][\epsilon]Ex_0 + Ey_{r-1-i} = [1+i][-\ell r-\epsilon]y_{r-2-i} = 2(-1)^{\ell+1}[1+i][\epsilon]\mathsf{w}^R_{i+2},$$

$$E\mathsf{w}^L_{-i-2-2k} = 2Ex_{1+i+k} = 2[1+i+k][r+k-\ell r-\epsilon]x_{i+k} = (-1)^{\ell+1}[1+i+k][k-\epsilon]\mathsf{w}^L_{-i-2-2(k-1)},$$

$$\begin{aligned} F\mathsf{w}^R_{i+2+2k} &= \frac{-1}{2[1+i][\epsilon]}\left(\prod_{s=0}^{k}[1+i+s][-s-\epsilon]\right)y_{r-1-i-k} \\ &= \frac{-[1+i+k][-k-\epsilon]}{2[1+i][\epsilon]}\left(\prod_{s=0}^{k-1}[1+i+s][-s-\epsilon]\right)y_{r-2-i-(k-1)} \\ &= -[1+i+k][k+\epsilon]\mathsf{w}^R_{i+2+2(k-1)}, \end{aligned}$$

$$\begin{aligned} (-1)^\ell\left(\mathsf{w}^S_{-i} + 2[1+i][\epsilon]\mathsf{w}^H_{-i}\right) &= (-1)^\ell\left(-2[1+i][\epsilon]x_i + y_{r-1} + 4[1+i][\epsilon]x_i - y_{r-1}\right) \\ &= 2(-1)^\ell[1+i][\epsilon]x_i = 2[1+i][\ell r+\epsilon]x_i = E\mathsf{w}^L_{-i-2}. \end{aligned}$$

From the definition of w^H_{i-2k} and w^S_{i-2k} it is easy to show that

$$\begin{aligned} x_k &= \mathsf{w}^H_{i-2k} + \frac{1}{2[1+i][\epsilon]}\mathsf{w}^S_{i-2k}, \\ y_{r-1-i+k} &= 2\mathsf{w}^S_{i-2k} + 2[1+i][\epsilon]\mathsf{w}^H_{i-2k}. \end{aligned}$$

From this, we find

$$\begin{aligned} E\mathsf{w}^H_{i-2k} &= 2Ex_k - \frac{1}{2[1+i][\epsilon]}Ey_{r-1-i+k} = 2[k][1+i-k+\ell r+\epsilon]x_{k-1} - \frac{[1+i-k][k-\ell r-\epsilon]}{2[1+i][\epsilon]}y_{r-2-i+k} \\ &= 2[k][1+i-k+\epsilon](-1)^\ell\left(\mathsf{w}^H_{i-2(k-1)} + \frac{1}{2[1+i][\epsilon]}\mathsf{w}^S_{i-2(k-1)}\right) \\ &- \frac{[1+i-k][k-\epsilon]}{2[1+i][\epsilon]}(-1)^\ell\left(2\mathsf{w}^S_{i-2(k-1)} + 2[1+i][\epsilon]\mathsf{w}^H_{i-2(k-1)}\right) \\ &= \left(2[k][1+i-k+\epsilon] - [1+i-k][k-\epsilon]\right)(-1)^\ell\mathsf{w}^H_{i-2(k-1)} \\ &+ \left(\frac{[k][1+i-k+\epsilon] - [1+i-k][k-\epsilon]}{[1+i][\epsilon]}\right)(-1)^\ell\mathsf{w}^S_{i-2(k-1)}, \end{aligned}$$

$$
\begin{aligned}
E\mathsf{w}^S_{i-2k} &= -2[1+i][\epsilon]Ex_k + Ey_{r-1-i+k} \\
&= -2[1+i][\epsilon][k][1+i-k+\ell r+\epsilon]x_{k-1} + [1+i-k][k-\ell r-\epsilon]y_{r-2-i+k} \\
&= -2[1+i][\epsilon][k][1+i-k+\epsilon](-1)^\ell \left(\mathsf{w}^H_{i-2(k-1)} + \frac{1}{2[1+i][\epsilon]}\mathsf{w}^S_{i-2(k-1)}\right) \\
&+ [1+i-k][k-\epsilon](-1)^\ell \left(2\mathsf{w}^S_{i-2(k-1)} + 2[1+i][\epsilon]\mathsf{w}^H_{i-2(k-1)}\right) \\
&= \left(2[1+i-k][k-\epsilon] - [1+i-k+\epsilon][k]\right)(-1)^\ell \mathsf{w}^S_{i-2(k-1)} \\
&+ 2[1+i][\epsilon]\left([1+i-k][k-\epsilon] - [1+i-k+\epsilon][k]\right)(-1)^\ell \mathsf{w}^H_{i-2(k-1)}.
\end{aligned}
$$

However, by expanding the brackets we get the identity

$$[k][1+i-k+\epsilon] - [1+i-k][k-\epsilon] = [1+i][\epsilon].$$

Hence, the above equations give

$$
\begin{aligned}
\mathsf{w}^H_{i-2k} &= \left(2[k][1+i-k+\epsilon] - [1+i-k][k-\epsilon]\right)(-1)^\ell \mathsf{w}^H_{i-2(k-1)} + (-1)^\ell \mathsf{w}^S_{i-2(k-1)}, \\
\mathsf{w}^S_{i-2k} &= \left(2[1+i-k][k-\epsilon] - [1+i-k+\epsilon][k]\right)(-1)^\ell \mathsf{w}^S_{i-2(k-1)} + 2(-1)^{\ell+1}[1+i]^2[\epsilon]^2 \mathsf{w}^H_{i-2(k-1)},
\end{aligned}
$$

as desired. This proves that $X_\epsilon = V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon}$ when $\epsilon \neq 0$. As $\epsilon \to 0$, it is easy to see that the action on X_0 is exactly the action on $P_i \otimes \mathbb{C}^H_{\ell r}$ (see [1]) by identifying $\mathsf{w}^X_k \in X_0$ with $\mathsf{w}^X_k \otimes v \in P_i \otimes \mathbb{C}^H_{kr}$. Hence, we have shown

$$
X_\epsilon = \begin{cases} V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon} & \text{if } \epsilon \neq 0 \\ P_i \otimes \mathbb{C}^H_{\ell r} & \text{if } \epsilon = 0 \end{cases}
$$

□

Corollary 2. *The action of the twist on $P_j \otimes \mathbb{C}^H_{\ell r}$ is given by $aId_{P_j \otimes \mathbb{C}^H_{\ell r}} + bx_{j,\ell}$ where*

$$
\begin{aligned}
a &= q^{-(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))} \\
b &= -\frac{\{1\}^2}{\{1+j\}}(r-1-j)q^{-(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}
\end{aligned}
$$

Proof. The twist acts as the inverse of

$$\theta = K^{r-1}\sum_{n=0}^{r-1}\frac{\{1\}^{2n}}{\{n\}!}q^{n(n-1)/2}S(F^n)q^{-H^2/2}E^n$$

which acts as $K^{r-1}q^{-H^2/2}$ on highest weight vectors. Endomorphisms act on simple modules by scalars, so it is enough to determine the scalar by which θ acts on the highest weight vector in a simple module. So, θ acts on $V_{1+j-r+\ell r+\epsilon}$ and $V_{-1-j+r+\ell r+\epsilon}$ as $q^{(r-1)(j+\ell r+\epsilon)}q^{-(j+\ell r+\epsilon)^2/2}$ and $q^{(r-1)(2r-2-j+\ell r+\epsilon)}q^{-(2r-2-j+\ell r+\epsilon)^2/2}$ respectively. These values agree in the limit $\epsilon \to 0$ since

$$\begin{aligned}
q^{(r-1)(2r-2-j+\ell r)}q^{-(2r-2-j+\ell r)^2/2} &= q^{(2r-2-j+\ell r)(r-1-\frac{1}{2}(2r-2-j+\ell r))} = q^{-\frac{1}{2}(2r-2-j+\ell r)(-j+\ell r)} \\
&= q^{-\frac{1}{2}((2r-2)(-j+\ell r)+(-j+\ell r)^2)} = q^{(r-1)(j-\ell r)}q^{-(-j+\ell r)^2/2} \\
&= (-1)^{j-\ell r}q^{-j+\ell r}q^{-(\frac{1}{2}j^2-j\ell r+\frac{1}{2}k^2r^2)} = (-1)^{j+\ell r}q^{-j-\ell r}q^{-(\frac{1}{2}j^2+j\ell r+\frac{1}{2}\ell^2r^2)} \\
&= q^{(r-1)(j+\ell r)}q^{-(j+\ell r)^2/2}.
\end{aligned}$$

Note that

$$\begin{aligned}
q^{(r-1)(j+\ell r+\epsilon)}q^{-(j+\ell r+\epsilon)^2/2} &= q^{(j+\ell r+\epsilon)(r-1-\frac{1}{2}(j+\ell r+\epsilon)}, \\
q^{(r-1)(2r-2-j+\ell r+\epsilon)}q^{-(2r-2-j+\ell r+\epsilon)^2/2} &= q^{(j-\ell r-\epsilon)(r-1-(j-\ell r-\epsilon)/2)}.
\end{aligned}$$

So, if we denote the action of θ on $P_j \otimes \mathbb{C}^H_{\ell r}$ by $a'Id_{P_j\otimes\mathbb{C}^H_{\ell r}} + b'x_{j,\ell}$ then we have

$$\begin{aligned}
a' &= \lim_{\epsilon\to 0}\left(2q^{(j+\ell r+\epsilon)(r-1-(j+\ell r+\epsilon)/2)} - q^{(j-\ell r-\epsilon)(r-1-(j-\ell r-\epsilon)/2)}\right) \\
&= \left(2q^{(j+\ell r)(r-1-(j+\ell r)/2)} - q^{(j+\ell r)(r-1-(j+\ell r)/2)}\right) = q^{(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}
\end{aligned}$$

and

$$b' = \lim_{\epsilon\to 0}\frac{-1}{[1+j][\epsilon]}(q^{(j-\ell r-\epsilon)(r-1-(j-\ell r-\epsilon)/2)} - q^{(j+\ell r+\epsilon)(r-1-(j+\ell r+\epsilon)/2)})$$

We can compute b' using L'Hopitals rule.
The derivative of the numerator is

$$\begin{aligned}
&\ln q\left((r-1-(j-\ell r-\epsilon)/2) - \frac{1}{2}(j-\ell r-\epsilon)\right)q^{(j-\ell r-\epsilon))(r-1-(j-\ell r-\epsilon)/2)} \\
&+\ln q\left(r-1-(j+\ell r+\epsilon)/2) - \frac{1}{2}(j+\ell r+\epsilon)\right)q^{(j+\ell r+\epsilon)(r-1-(j+\ell r+\epsilon)/2)}
\end{aligned}$$

which gives

$$2\ln q(r-1-j)q^{(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}$$

when evaluated at zero. The derivative of the denominator evaluated at zero is $\frac{2[1+j]}{\{1\}}\ln q$, so $b' = \frac{\{1\}^2}{\{1+j\}}(r-1-j)q^{(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}$.

If we denote the twist as $aId_{P_j\otimes\mathbb{C}^H_{\ell r}} + bx_{j,\ell}$ then since it is the inverse of θ, we have

$$Id_{P_j\otimes\mathbb{C}^H_{\ell r}} = (aId_{P_j\otimes\mathbb{C}^H_{\ell r}} + bx_{j,\ell}) \circ (a'Id_{P_j\otimes\mathbb{C}^H_{\ell r}} + b'x_{j,\ell}) = aa'Id_{P_j\otimes\mathbb{C}^H_{\ell r}} + (ab' + a'b)x_{j,\ell}.$$

Hence, $aa' = 1$ and $ab' + a'b = 0$. So, $a = a'^{-1} = q^{-(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}$ and

$$b = -a'^{-1}ab' = -a^2b' = -\frac{\{1\}^2}{\{1+j\}}(r-1-j)q^{-(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}.$$

□

Lemma 1. $\lim_{\epsilon\to a} AX_\epsilon = A\lim_{\epsilon\to a} X_\epsilon \quad \forall A \in \overline{U}_q^H(\mathfrak{sl}(2))$

Proof. The lemma holds for E, F, H, and K by construction of X_ϵ and hence holds for all polynomials in E, F, H, and K. □

Corollary 3. *The partial trace commutes with limits over functions in End(X_ϵ).*

Proof. The partial trace acts on elements of $\mathrm{End}(X_\epsilon)$ as a polynomial in E, F, H, and K with complex coefficients hence commutes with limits by the Lemma. □

Proposition 10. $\lim_{\epsilon\to 0} \boldsymbol{d}(X_\epsilon) = \boldsymbol{d}(X_0)$

Proof. Set $\lambda = 1 + i - r$, then we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\mathbf{d}(X_\epsilon) = \lim_{\epsilon\to 0}\mathbf{d}(V_{\lambda+\ell r+\epsilon}\oplus V_{-\lambda+\ell r+\epsilon}) = \lim_{\epsilon\to 0}(\mathbf{d}(V_{\lambda+\ell r+\epsilon}) + \mathbf{d}(V_{-\lambda+\ell r+\epsilon}))\\
&= (-1)^{r-1}r\lim_{\epsilon\to 0}\left(\frac{\{\lambda+\ell r+\epsilon\}}{\{r(\lambda+\ell r+\epsilon)\}} + \frac{\{-\lambda+\ell r+\epsilon\}}{\{r(-\lambda+\ell r+\epsilon)\}}\right)\\
&= (-1)^{\ell(r-1)+r-1}r\lim_{\epsilon\to 0}\left(\frac{(q^{\lambda+\epsilon}-q^{-(\lambda+\epsilon)})(q^{r(-\lambda+\epsilon)}-q^{-r(-\lambda+\epsilon)}) + (q^{-\lambda+\epsilon}-q^{-(-\lambda+\epsilon)})(q^{r(\lambda+\epsilon)}-q^{-r(\lambda+\epsilon)}}{(q^{r(\lambda+\epsilon)}-q^{-r(\lambda+\epsilon)})(q^{r(-\lambda+\epsilon)}-q^{-r(-\lambda+\epsilon)})}\right).
\end{aligned}$$

The derivative of the numerator of the fraction evaluated at zero is given by

$$\begin{aligned}
&\ln q\left((q^{\lambda}+q^{-\lambda})(q^{-r\lambda}-q^{r\lambda})+r(q^{\lambda}-q^{-\lambda})(q^{-r\lambda}+q^{r\lambda})\right)\\
&+\ln q\left((q^{-\lambda}+q^{\lambda})(q^{r\lambda}-q^{-r\lambda})+r(q^{-\lambda}-q^{\lambda})(q^{r\lambda}+q^{-r\lambda})\right)\\
&=\ln q\left(2(q^{\lambda}+q^{-\lambda})(q^{r\lambda}-q^{-r\lambda})+2r(q^{r\lambda}+q^{-r\lambda})(q^{\lambda}-q^{\lambda}+q^{-\lambda}-q^{-\lambda})\right)\\
&=2\ln q(q^{\lambda}+q^{-\lambda})(q^{r\lambda}-q^{-r\lambda})=0.
\end{aligned}$$

Where we have used the fact that $q^r = q^{-r}$. The derivative of the denominator evaluated at zero is given by

$$\ln q\left(r(q^{r\lambda}+q^{-r\lambda})(q^{-r\lambda}-q^{r\lambda})+r(q^{r\lambda}-q^{-r\lambda})(q^{-r\lambda}+q^{r\lambda})\right)=2r\ln q(q^{r\lambda}+q^{-r\lambda})(q^{r\lambda}-q^{-r\lambda})=0.$$

So, we apply L'Hopitals a second time and after simplifying we obtain a numerator of $8r(-1)^{1+i-r}(q^{1+i-r}+q^{-(1+i-r)})$, and a denominator of $8r^2$. Hence,

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathbf{d}(X_{\epsilon})=(-1)^{\ell(r-1)+r-1}r\left(\frac{8r(-1)^{1+i-r}(q^{1+i-r}+q^{-(1+i-r)})}{8r^2}\right)&=(-1)^{\ell(r-1)+i+1}(q^{i+1}+q^{-i-1})\\
&=\mathbf{d}(P_i).
\end{aligned}$$

□

Proposition 11. *Let Z be a weight module in $\mathcal{C}$. Then, the Hopf link $\Phi_{Z,P_i\otimes\mathbb{C}^H_{\ell r}} = a_Z Id_{P_i\otimes\mathbb{C}^H_{\ell r}} + b_Z x_{i,\ell}$ is given by:*

$$\begin{aligned}
a_Z&=\lim_{\epsilon\to 0}\left(2\lambda_{Z,V_{1+i-r+\ell r+\epsilon}}-\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}\right)\\
b_Z&=\lim_{\epsilon\to 0}\frac{-1}{[1+i][\epsilon]}(\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}-\lambda_{Z,V_{1+i-r+\ell r+\epsilon}})
\end{aligned}$$

where $\lambda_{Z,1+i-r+\ell r+\epsilon}$ and $\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}$ are the constants by which $\Phi_{Z,V_{1+i-r+\ell r+\epsilon}}$ and $\Phi_{Z,V_{-1-i+r+\ell r+\epsilon}}$ act respectively. That is,

$$\begin{aligned}
\lambda_{Z,V_{1+i-r+\ell r+\epsilon}}&=\Psi_{1+i-r+\ell r+\epsilon}(\chi(Z))\\
\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}&=\Psi_{-1-i+r+\ell r+\epsilon}(\chi(Z))
\end{aligned}$$

Proof. Denote by $\lambda_{Z,\gamma}$ the constant by which Φ_{Z,V_γ} acts, and let $a_\epsilon = 2, b_\epsilon = \frac{-1}{2[1+i][\epsilon]}, c = b^{-1}, d = 1$ as in the proposition. By construction, $\lim_{\epsilon\to 0} c_{Z,X_\epsilon\otimes\mathbb{C}^H_{\ell r}} = c_{Z,P_i\otimes\mathbb{C}^H_{\ell r}}$ and $\lim_{\epsilon\to 0} c_{X_{\epsilon\otimes\mathbb{C}^H_{\ell r}},Z} = c_{P_i\otimes\mathbb{C}^H_{\ell r},Z}$ since the action on X_ϵ converges to the action on P_i in the limit $\epsilon\to 0$. Let $\{x_o,...,x_{r-1}\}$ denote the standard basis for $V_{1+i-r+\epsilon}$ and $\{y_0,...,y_{r-1-i}\}$ the standard basis for $V_{-1-i+r+\epsilon}$ and let $\mathsf{w}_i^H = a_\epsilon x_0 + b_\epsilon y_{r-1-i}$ and $\mathsf{w}_i^S = c_\epsilon x_0 + d_\epsilon y_{r-1-i}$ as in the proposition above. Then w_i^H generates

X_ϵ and if v is the basis element for $\mathbb{C}^H_{\ell r}$, then $\mathsf{w}_i^H \otimes v = a_\epsilon x_0 \otimes v + b_\epsilon y_{r-1-i} \otimes v$ generates $X_\epsilon \otimes \mathbb{C}^H_{\ell r}$. Hence, by Corollary

$$\begin{aligned}\lim_{\epsilon\to 0} \Phi_{Z,X_\epsilon}(\mathsf{w}_i^H \otimes v) &= \lim_{\epsilon\to 0} \mathrm{ptr}_R(c_{Z,X_\epsilon\otimes\mathbb{C}^H_{\ell r}} \circ c_{X_\epsilon\otimes\mathbb{C}^H_{\ell r},Z})(\mathsf{w}_i^H \otimes v)\\ &= \mathrm{ptr}_R(c_{Z,P_i\otimes\mathbb{C}^H_{\ell r}} \circ c_{P_i\otimes\mathbb{C}^H_{\ell r},Z})(\mathsf{w}_i^H \otimes v) = \Phi_{Z,P_i}(\mathsf{w}_i^H \otimes v).\end{aligned}$$

Notice that $x_0 = d_\epsilon \mathsf{w}_i^H - b_\epsilon \mathsf{w}_i^S$ and $y_{r-1-i} = a_\epsilon \mathsf{w}_i^S - c_\epsilon \mathsf{w}_i^H$. We can now compute the coefficients a, b of $\Phi_{Z,P_i} = a_Z Id_{P_i} + b_Z x_{i,0}$:

$$\begin{aligned}\Phi_{Z,P_i}(\mathsf{w}_i^H \otimes v) &= \lim_{\epsilon\to 0} \Phi_{Z,X_\epsilon\otimes\mathbb{C}^H_{\ell r}}(\mathsf{w}_i^H \otimes v) = \lim_{\epsilon\to 0} \Phi_{Z,X_\epsilon\otimes\mathbb{C}^H_{\ell r}}(a_\epsilon x_0 \otimes v + b_\epsilon y_{r-1-i} \otimes v)\\ &= \lim_{\epsilon\to 0} \left(a_\epsilon \lambda_{Z,V_{1+i-r+\ell r+\epsilon}} x_0 \otimes v + b_\epsilon \lambda_{Z,V_{-1-i+r+kr+\epsilon}} y_{r-1-i} \otimes v\right)\\ &= \lim_{\epsilon\to 0} \left(a_\epsilon \lambda_{Z,V_{1+i-r+\ell r+\epsilon}} (d_\epsilon \mathsf{w}_i^H - b_\epsilon \mathsf{w}_i^S) \otimes v + b_\epsilon \lambda_{Z,V_{-1-i+r+\epsilon}} (a_\epsilon \mathsf{w}_i^S - c_\epsilon \mathsf{w}_i^H) \otimes v\right)\\ &= \lim_{\epsilon\to 0} (a_\epsilon d_\epsilon \lambda_{Z,V_{1+i-r+\ell r+\epsilon}} - b_\epsilon c_\epsilon \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}) \mathsf{w}_i^H \otimes v\\ &\quad + (a_\epsilon b_\epsilon \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}} - a_\epsilon b_\epsilon \lambda_{Z,V_{1+i-r+\ell r+\epsilon}}) \mathsf{w}_i^S \otimes v\\ &= \lim_{\epsilon\to 0} (2\lambda_{Z,V_{1+i-r+\ell r+\epsilon}} - \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}) \mathsf{w}_i^H \otimes v\\ &- \frac{1}{[1+i][\epsilon]} (\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}} - \lambda_{Z,V_{1+i-r+\ell r+\epsilon}}) \mathsf{w}_i^S \otimes v.\end{aligned}$$

$V_{1+i-r+\ell r+\epsilon}$ and $V_{-1-i+r+\ell r+\epsilon}$ are simple, so

$$\begin{aligned}\lambda_{Z,V_{1+i-r+\ell r+\epsilon}} &= \Psi_{1+i-r+\ell r+\epsilon}(\chi(Z)),\\ \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}} &= \Psi_{-1-i+r+\ell r+\epsilon}(\chi(Z)).\end{aligned}$$

□

Corollary 4. *The Hopf links $\Phi_{Z,P_j\otimes\mathbb{C}^H_{\ell r}}$ with $Z \in S_i, V_\alpha, P_i$ are given by $\Phi_{Z,P_j\otimes\mathbb{C}^H_{\ell r}} = a_Z Id_{P_j\otimes\mathbb{C}^H_{\ell r}} + b_Z x_{j,\ell}$ where*

$$a_{P_i} = a_{V_\alpha} = 0, \quad a_{S_i\otimes\mathbb{C}^H_{\ell' r}} = (-1)^{i(\ell+1)+\ell'(1+j-r+\ell r)} \frac{\{(i+1)(j+1)\}}{\{j+1\}},$$

and

$$b_{S_i\otimes\mathbb{C}^H_{\ell' r}} = (-1)^{i(\ell+1)+\ell'(1+j-r+\ell r)}\frac{i\{(i+2)(j+1)\}-(i+2)\{i(j+1)\}}{[j+1]^2\{j+1\}},$$
$$b_{V_\alpha} = \frac{(-1)^{r(\ell+1)+\ell+j}}{[j+1]^2} r q^{r\ell\alpha}(q^{(r-1-j)\alpha}+q^{-(r-1-j)\alpha}),$$
$$b_{P_i\otimes\mathbb{C}^H_{\ell' r}} = \frac{(-1)^{i(\ell+1)+\ell'(1+j-r+\ell r)}}{[j+1]^2} 2r(q^{(i+1)(j+1)}+q^{-(i+1)(j+1)}).$$

Proof. For $Z\in\{S_i,P_i,V_\alpha\}$, we have

$$\lambda_{\mathbb{C}^H_{\ell' r},V_{1+j-r+\ell r+\epsilon}} = \Psi_{1+j-r+\ell r+\epsilon}(\chi(\mathbb{C}^H_{\ell' r})) = (-1)^{\ell'(1+j-r+\ell r+\epsilon)},$$
$$\lambda_{\mathbb{C}^H_{\ell' r},V_{-1-j+r+\ell r+\epsilon}} = \Psi_{-1-j+r+\ell r+\epsilon}(\chi(\mathbb{C}^H_{\ell' r})) = (-1)^{\ell'(-1-j+r+\ell r+\epsilon)},$$

$$\begin{aligned}\lambda_{S_i,V_{1+j-r+\ell r+\epsilon}} &= \Psi_{1+j-r+\ell r+\epsilon}(\chi(S_i)) = \frac{\{(i+1)(1+j-r+\ell r+\epsilon)\}}{\{1+j-r+\ell r+\epsilon\}}\\ &= (-1)^{i(\ell+1)}\frac{\{(i+1)(1+j+\epsilon)\}}{\{1+j+\epsilon\}},\end{aligned}$$

$$\begin{aligned}\lambda_{S_i,V_{-1-i+r+\ell r+\epsilon}} &= \Psi_{-1-j+r+\ell r+\epsilon}(\chi(S_i)) = \frac{\{(i+1)(-1-j+r+\ell r+\epsilon)\}}{\{-1-j+r+\ell r+\epsilon\}}\\ &= (-1)^{i(\ell+1)}\frac{\{(i+1)(-1-j+\epsilon)\}}{\{-1-j+\epsilon\}},\end{aligned}$$

$$\lambda_{V_\alpha,V_{1+j-r+\ell r+\epsilon}} = \Psi_{1+j-r+\ell r+\epsilon}(\chi(V_\alpha)) = q^{(1+j-r+\ell r+\epsilon)\alpha}\frac{\{r(1+j-r+\ell r+\epsilon)\}}{\{1+j+r+\ell r+\epsilon\}},$$
$$\lambda_{V_\alpha,V_{-1-i+r+\ell r+\epsilon}} = \Psi_{-1-j+r+\ell r+\epsilon}(\chi(V_\alpha)) = q^{(-1-j+r+\ell r+\epsilon)\alpha}\frac{\{r(-1-j+r+\ell r+\epsilon)\}}{\{-1-j+r+\ell r+\epsilon\}},$$

$$\begin{aligned}\lambda_{P_i,V_{1+j-r+\ell r+\epsilon}} &= \Psi_{1+j-r+\ell r+\epsilon}(\chi(P_i))\\ &= (q^{(r-i-1)(1+j-r+\ell r+\epsilon)}+q^{(1+i-r)(1+j-r+\ell r+\epsilon)})\frac{\{r(1+j-r+\ell r+\epsilon)\}}{\{1+j-r+\ell r+\epsilon\}},\end{aligned}$$

$$\begin{aligned}\lambda_{P_i,V_{-1-j+r+\ell r+\epsilon}} &= \Psi_{-1-j+r+\ell r+\epsilon}(\chi(P_i))\\ &= (q^{(r-i-1)(-1-j+r+\ell r+\epsilon)}+q^{(1+i-r)(-1-j+r+\ell r+\epsilon)})\frac{\{r(-1-j+r+\ell r+\epsilon)\}}{\{-1-j+r+\ell r+\epsilon\}}.\end{aligned}$$

Clearly, $a_{V_\alpha} = a_{P_i} = 0$ since $\frac{\{r(1+j-r+\ell r)\}}{\{1+j-r+\ell r\}} = \frac{\{r(-1-j+r+\ell r)\}}{\{-1-j+r+\ell r\}} = 0$, and

$$a_{S_i} = \lim_{\epsilon\to 0}\left(2\lambda_{S_i,V_{1+i-r+\ell r+\epsilon}} - \lambda_{S_i,V_{-1-i+r+\ell r+\epsilon}}\right) = (-1)^{i(\ell+1)}\frac{\{(i+1)(j+1)\}}{\{j+1\}}.$$

By L'Hopitals rule, we can compute the other terms:

$$\begin{aligned} b_{S_i} &= \lim_{\epsilon\to 0}\frac{(-1)^{i(\ell+1)+1}}{[j+1][\epsilon]}\left(\frac{\{(i+1)(-1-j+\epsilon)\}}{\{-1-j+\epsilon\}} - \frac{\{(i+1)(1+j+\epsilon)\}}{\{1+j+\epsilon\}}\right) \\ &= \lim_{\epsilon\to 0}\frac{(-1)^{i(\ell+1)+1}}{[j+1][\epsilon]\{-1-j+\epsilon\}\{1+j+\epsilon\}}\Big(\{(i+1)(-1-j+\epsilon)\}\{1+j+\epsilon\} \\ &\qquad - \{(i+1)(1+j+\epsilon)\}\{-1-j+\epsilon\}\Big). \end{aligned}$$

The term in brackets can be expanded as

$$\begin{aligned} &(q^{-(i+1)(1+j-\epsilon)} - q^{(i+1)(1+j-\epsilon)})(q^{1+j+\epsilon} - q^{-1-j-\epsilon}) - (q^{(1+i)(1+j+\epsilon)} - q^{-(1+i)(1+j+\epsilon)})(q^{-1-j+\epsilon} - q^{1+j-\epsilon}) \\ &= q^{-i(j+1)}q^{(i+2)\epsilon} - q^{(i+2)(j+1)}q^{-i\epsilon} - q^{-(i+2)(j+1)}q^{i\epsilon} + q^{i(j+1)}q^{-(i+2)\epsilon} \\ &- q^{i(j+1)}q^{(i+2)\epsilon} + q^{-(i+2)(j+1)}q^{-i\epsilon} + q^{(i+2)(j+1)}q^{i\epsilon} - q^{-i(j+1)}q^{-(i+2)\epsilon} \end{aligned}$$

whose derivative evaluated at $\epsilon = 0$ is $2\ln q(i\{(i+2)(j+1)\} - (i+2)\{i(j+1)\})$

$$\begin{aligned} [\epsilon]\{-1-j+\epsilon\}\{1+j+\epsilon\} &= \frac{1}{\{1\}}(q^\epsilon - q^{-\epsilon})(q^{2\epsilon} - q^{2(j+1)} - q^{-2(j+1)} + q^{-2\epsilon}) \\ &= \frac{1}{\{1\}}(q^{3\epsilon} - q^{2(j+1)}q^\epsilon - q^{-2(j+1)}q^\epsilon + q^{-\epsilon} - q^\epsilon + q^{2(j+1)}q^{-\epsilon} + q^{-2(j+1)}q^{-\epsilon} - q^{-3\epsilon}) \end{aligned}$$

whose derivative evaluated at $\epsilon = 0$ is

$$\frac{1}{\{1\}}(4 - 2q^{2(+1)} - 2^{-2(j+1)}) = -2\ln q\frac{\{j+1\}^2}{\{1\}}.$$

So,

$$b_{S_i} = (-1)^{i(\ell+1)}\frac{i\{(i+2)(j+1)\} - (i+2)\{i(j+1)\}}{[j+1]^2\{j+1\}}$$

Similarly, we find:

$$b_{V_\alpha} = \frac{(-1)^{r(\ell+1)+\ell+j}}{[j+1]^2} r q^{rk\alpha}(q^{(r-1-j)\alpha} + q^{-(r-1-j)\alpha}),$$
$$b_{P_i} = \frac{(-1)^{i(\ell+1)}}{[j+1]^2} 2r(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)}).$$

It is easy to show that $a_{\mathbb{C}^H_{\ell' r}} = (-1)^{\ell'(1+j-r+\ell r)} Id_{P_j \otimes \mathbb{C}^H_{\ell r}}$ and $b_{\mathbb{C}^H_{\ell' r}} = 0$, so the proposition follows from $\Phi_{X\otimes Y, P_j \otimes \mathbb{C}^H_{\ell r}} = \Phi_{X, P_j \otimes \mathbb{C}^H_{\ell r}} \circ \Phi_{Y, P_j \otimes \mathbb{C}^H_{\ell r}}$.

□

5.3 Alexander Invariants

Let L be a be a (n,n)-ribbon graph colored by modules in $\mathcal{C}$ such that the bands at the $i-th$ intersections are colored by the same module, and at least one of the colors is colored by a simple V_λ. Let T_λ be the $(1,1)$-ribbon graph obtained by closing the other ends through the action of the left and right partial trace. Then the re-normalized Reshetikhin-Turaev link invariant (see [6]) is

$$F'(L) := t_{V_\lambda}(T_\lambda).$$

Note that we are identifying T_λ with the corresponding morphism obtained by identifying tangles with combinations of the braiding, duality morphisms, and twist as described in section 3.2. These invariants were shown to coincide with Murakami's Alexander invariants described in [8]. We now show that these results can be extended to any projective module, not just the simple V_λ. Let L be constructed as above but with at least one of the colors being $P_i \otimes \mathbb{C}^H_{\ell r}$, and let $T_{P_i \otimes \mathbb{C}^H_{\ell r}}$ be the colored $(1,1)$-ribbon graph obtained by closing the other ends, and let T_λ be the ribbon graph obtained by re-coloring the open $P_i \otimes \mathbb{C}^H_{\ell r}$-colored band with V_λ. Then,

Theorem 4. *The colored $(1,1)$-ribbon graph $T_{P_i \otimes \mathbb{C}^H_{\ell r}}$ satisfies*

$$t_{P_i \otimes \mathbb{C}^H_{\ell r}}\left(T_{P_i \otimes \mathbb{C}^H_{\ell r}}\right) = \lim_{\epsilon \to 0}\left(t_{V_{1+i-r+\ell r+\epsilon}}(T_{1+i-r+\ell r+\epsilon}) + t_{V_{-1-i-r+\ell r+\epsilon}}(T_{-1-i-r+\ell r+\epsilon})\right)$$

and

$$T_{P_i \otimes \mathbb{C}^H_{\ell r}} = a Id_{P_i \otimes \mathbb{C}^H_{\ell r}} + b x_{i,\ell r}$$

with coefficients

$$a = \lim_{\epsilon \to 0} \frac{t_{V_{-1-i-r+\ell r+\epsilon}}(T_{-1-i-r+\ell r+\epsilon})}{\boldsymbol{d}(V_{-1-i+r+\ell r+\epsilon})} = \lim_{\epsilon \to 0} \frac{t_{V_{1+i-r+\ell r+\epsilon}}(T_{1+i-r+\ell r+\epsilon})}{\boldsymbol{d}(V_{1+i-r+\ell r+\epsilon})}$$

and

$$b = \lim_{\epsilon\to 0} \frac{-1}{[1+i][\epsilon]} \left(\frac{t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)}{\boldsymbol{d}\left(V_{-1-i+r+\ell r+\epsilon}\right)} - \frac{t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)}{\boldsymbol{d}\left(V_{1+i-r+\ell r+\epsilon}\right)} \right)$$
$$= \frac{r}{2\pi i\{1+i\}} \left(\frac{d}{d\lambda} \frac{t_{V_\lambda}\left(T_\lambda\right)}{\boldsymbol{d}\left(V_\lambda\right)}\Big|_{\lambda=\ell r+r-i-1} - \frac{d}{d\lambda} \frac{t_{V_\lambda}\left(T_\lambda\right)}{\boldsymbol{d}\left(V_\lambda\right)}\Big|_{\lambda=i+i-r+\ell r} \right).$$

Before proving this theorem, we remark that this result nicely relates to the work of Murakami and Nagatomo on logarithmic link invariants for different quantum groups [8],[19],[20]. We also note that

$$T_\lambda = \frac{t_{V_\lambda}\left(T_\lambda\right)}{\mathbf{d}\left(V_\lambda\right)} Id_{V_\lambda}.$$

Proof. For the first statement, we use the identity $\lim_{\epsilon\to 0} V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon} = P_i \otimes \mathbb{C}^H_{\ell r}$ which gives

$$t_{P_i\otimes\mathbb{C}^H_{\ell r}}\left(T_{P_i\otimes\mathbb{C}^H_{\ell r}}\right) = \lim_{\epsilon\to 0} T_{V_{1+i-r+\ell r+\epsilon}\oplus V_{-1-i+r+\ell r+\epsilon}}\left(T_{V_{1+i-r+\ell r+\epsilon}\oplus V_{-1-i+r+\ell r+\epsilon}}\right)$$

where we pulled the limit out of the function using the fact that limits commute with the partial trace and the action of $\overline{U}_q^H(\mathfrak{sl}(2))$. The relation follows since coloring with the direct sum of two objects $X \oplus Y$ amounts to computing the sum of the individually colored components

$$t_{X\oplus Y}\left(T_{X\oplus Y}\right) = t_X\left(T_X\right) + t_Y\left(T_Y\right).$$

For the second statement, since w_i^H generates $P_i \otimes \mathbb{C}^H_{\ell r}$, it is enough to find the action of $T_{P_i\otimes\mathbb{C}^H_{\ell r}}$ on w_i^H. For this we compute the action of $T_{V_{1+i-r+\ell r+\epsilon}\oplus V_{-1-i+r+\ell r+\epsilon}}$ on w_i^H and then take the limit ϵ to zero. Recall that we have $\mathsf{w}_i^H = a_\epsilon x_0 + b_\epsilon y_{r-1-i}$ and $\mathsf{w}_i^S = c_\epsilon x_0 + d_\epsilon y_{r-1-i}$ where x_0, y_{r-1-i}, $a_\epsilon = 2$, $b_\epsilon = -\frac{1}{2[1+i][\epsilon]}$, $c_\epsilon = -2[1+i][\epsilon]$, and $d_\epsilon = 1$ are as in the construction of X_ϵ. Notice that $x_0 = d_\epsilon \mathsf{w}_i^H - b_\epsilon \mathsf{w}_i^S$ and $y_{r-1-i} = a_\epsilon \mathsf{w}_i^S - c_\epsilon \mathsf{w}_i^H$. We can now compute the action of $T_{P_i\otimes\mathbb{C}^H_{\ell r}}$ on w_i^H:

$$T_{P_i\otimes\mathbb{C}^H_{\ell r}}(\mathsf{w}_i^H) = T_{P_i\otimes\mathbb{C}^H_{\ell r}}\left(\lim_{\epsilon\to 0} a_\epsilon x_0 + b_\epsilon y_{r-1-i}\right) = \lim_{\epsilon\to 0}\left(a_\epsilon T_{1+i-r+\ell r+\epsilon}(x_0) + b_\epsilon T_{-1-i+r+\ell r+\epsilon}(y_{r-1-i})\right)$$
$$= \lim_{\epsilon\to 0}\left(a_\epsilon \frac{t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)}{\mathbf{d}\left(V_{1+i-r+\ell r+\epsilon}\right)}(d_\epsilon \mathsf{w}_i^H - b_\epsilon \mathsf{w}_i^S) + \right.$$
$$\left. b_\epsilon \frac{t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)}{\mathbf{d}\left(V_{-1-i+r+\ell r+\epsilon}\right)}(a_\epsilon \mathsf{w}_i^S - c_\epsilon \mathsf{w}_i^H) \right).$$

It follows that

$$a = \lim_{\epsilon\to 0}\left(a_\epsilon d_\epsilon \frac{t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)}{\mathbf{d}\left(V_{1+i-r+\ell r+\epsilon}\right)} - b_\epsilon c_\epsilon \frac{t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)}{\mathbf{d}\left(V_{-1-i+r+\ell r+\epsilon}\right)} \right)$$
$$= \lim_{\epsilon\to 0}\left(2\frac{t_{V_{1+i-r+\ell r+\epsilon}}\left(T_{1+i-r+\ell r+\epsilon}\right)}{\mathbf{d}\left(V_{1+i-r+\ell r+\epsilon}\right)} - \frac{t_{V_{-1-i-r+\ell r+\epsilon}}\left(T_{-1-i-r+\ell r+\epsilon}\right)}{\mathbf{d}\left(V_{-1-i+r+\ell r+\epsilon}\right)} \right)$$

and

$$b = -\lim_{\epsilon\to 0}\left(a_\epsilon b_\epsilon \frac{t_{V_{1+i-r+\ell r+\epsilon}}(T_{1+i-r+\ell r+\epsilon})}{\mathbf{d}(V_{1+i-r+\ell r+\epsilon})} - a_\epsilon b_\epsilon \frac{t_{V_{-1-i-r+\ell r+\epsilon}}(T_{-1-i-r+\ell r+\epsilon})}{\mathbf{d}(V_{-1-i+r+\ell r+\epsilon})}\right)$$
$$= \lim_{\epsilon\to 0}\frac{1}{[1+i][\epsilon]}\left(\frac{t_{V_{1+i-r+\ell r+\epsilon}}(T_{1+i-r+\ell r+\epsilon})}{\mathbf{d}(V_{1+i-r+\ell r+\epsilon})} - \frac{t_{V_{-1-i-r+\ell r+\epsilon}}(T_{-1-i-r+\ell r+\epsilon})}{\mathbf{d}(V_{-1-i+r+\ell r+\epsilon})}\right).$$

Evaluating the limits (using L'Hôpital's rule for b) give the result. □

6 Future Work

There is a family of non C_2-cofinite W-algebras denoted $W^0(Q)_r$ where $r \geq 2$ and Q is the root lattice of a simply-laced simple Lie algebra $\mathfrak{g}$. This family is a generalization of the singlet VOA $\mathcal{M}(r)$. The representation categories for these W-algebras are expected to have connections to representation categories of certain unrolled quantum groups $\overline{U}_q^{\mathfrak{h}}(\mathfrak{g})$ at a 2r-th root of unity. These quantum groups have tensor structure much more complicated than that of $\overline{U}_q^H(\mathfrak{sl}(2))$ so direct methods for computing the tensor ring are difficult to manage. The Hopf link point of view provides a different approach to understanding these representation categories. In particular, this point of view may be of value to the study of $\overline{U}_q^{\mathfrak{h}}(\mathfrak{sl}(3))$ where the tensor ring, and even module classifications are unknown.

7 References

[1] F. Constantino, N. Geer, B. Patureau-Mirand - *Some Remarks on the Unrolled Quantum Group of* $\mathfrak{sl}(2)$, J. Pure Appl. Algebra 219 (2015), no. 8, 3238–3262.

[2] T. Creutzig, A. Milas - *False Theta Functions and the Verlinde formula*, Advances in Mathematics, 262 (2014), 520-545.

[3] T. Ohtsuki - *Quantum invariants. A study of knots, 3-manifolds and their sets*, Series on Knots and Everything, 29. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[4] D. Adamović, A. Milas - *Logarithmic intertwining operators and* $\mathcal{W}(2, 2r-1)$*-algebras*, J. Math. Phys. 48 (2007) 073503.

[5] D. Adamović - *Classification of irreducible modules of certain subalgebras of free boson vertex algebra*, J. Algebra 270 (2003) 115-132.

[6] N. Geer, B. Patureau-Mirand, V. Turaev - *Modified Quantum Dimensions and Re-Normalized Link Invariants*, Compositio Mathematica, 145 (2009), no. 1, 196-212.

[7] Y. Akutsu, T. Deguchi, and T. Ohtsuki - *Invariants of colored links*, J. Knot Theory Ramifications I (1992), no.2, 161-184.

[8] J. Murakami - *Colored Alexander Invariants and Cone-Manifolds*, Osaka J. Math. 45 no.2 (2008), 541-564.

[9] C. Kassel - *Quantum Groups*, Springer-Verlag, 1995.

[10] Vladimir Turaev - *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18.

[11] Y.Z. Huang - *Vertex operator algebras and the Verlinde conjecture*, Commun. Contemp. Math. 10 (2008) 103-154.

[12] Y.Z Huang - *Rigidity and modularity of vertex tensor categories*, Commun. Contemp. Math. 10 (2008) 871-911.

[13] M. Miyamoto -C_1*-cofiniteness and Fusion Products of Vertex Operator Algebras*, In: Huang, Y.-Z. (ed.) Conformal field theories and tensor categories held in 2011. Springer Proceedings in Mathematics & Statistics. arXiv:1305.3008 (2013).

[14] V. Lyubashenko -*Modular transformations for tensor categories*, J. Pure Appl. Algebra 98 (1995), no. 3, 279-327.

[15] V. Lyubashenko -*Modular properties of ribbon abelian categories*, Symposia Gaussiana, Proc. of the 2nd Gauss Symposium, Munich, 1993, Conf. A (Berlin, New York), Walter de Ruyter, 1995, pp. 529-579 arXiv:hep-th/9405168.

[16] Y.Z. Huang, J. Lepowsky, L. Zhang - *Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I-VIII*, arXiv:1012.4193, 1012.4196, 1012.4197, 1012.4198, 1012.4199, 1012.4202, 1110.1929, 1110.1931.

[17] E. Frenkel, D. Ben-Zvi - *Vertex Algebras and Algebraic Curves*, American Mathematical Society, 2001.

[18] T. Gannon - *Moonshine Beyond the Monster*, Cambridge University Press, 2006.

[19] J. Murakami - *From colored Jones invariants to logarithmic invariants.* arXiv:1406.1287.

[20] J. Murakami and K. Nagatomo - *Logarithmic knot invariants arising from restricted quantum groups.* Internat. J. Math. 19 (2008), no. 10, 1203–1213.

[21] N. Yu Reshetikhin and V.G. Turaev - *Ribbon Graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990), no. 1, 1–26.

[22] P. Etingof, S. Gelaki, D.i Nikshych, V. Ostrik - *Tensor Categories*, American Mathematical Society, Mathematical surveys and monographs ; volume 205.

[23] Y. Zhu - *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. 9 (1996) 237-302.

[24] M. Miyamoto - *Modular invariance of vertex operator algebra satisfying C_2-cofiniteness*, Duke Math. J. 122 (2004) 51-91.

[25] V. Lyubashenko - *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun.Math. Phys. 172 (1995) 467-516.

[26] T. Creutzig and A. Milas - *Higher rank partial and false theta functions and representation theory*, arXiv:1607.08563

[27] C. Alfes and T. Creutzig - *The mock modular data of a family of super algebras.*, Proc. Amer.

Math. Soc. 142 (2014), no. 7, 2265-2280.

[28] T. Creutzig and D. Ridout - *Relating the archetypes of logarithmic conformal field theory*, Nuclear Phys. B 872 (2013), no. 3, 348-391.

[29] T. Creutzig and D. Ridout - *Modular data and Verlinde formulae for fractional level WZW models I*, Nuclear Phys. B 865 (2012), no. 1, 83-114.

[30] T. Creutzig and D. Ridout - *Modular data and Verlinde formulae for fractional level WZW models II*, Nuclear Phys. B 875 (2013), no. 2, 423-458.

[31] T. Creutzig, A. Milas, and S. Wood, - *On regularized quantum dimensions of the singlet vertex operator algebra and false theta functions*, to appear in IMRN, arXiv:1411.3282

[32] D. Adamović and A. Milas - *On the triplet vertex algebra $\mathcal{W}(p)$*, Adv. Math. 217 (2008), 2664-2699.

[33] D. Adamović and A. Milas - *Some applications and constructions of intertwining operators in LCFT*, submitted.

[34] R. E. Borcherds - *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA 83 (1986), 3068-3071.

[35] J. Conway and S. Norton - *Monstrous moonshine*, Bull. London Math. Soc. 11 (1979), no. 3, 308-339.

[36] Igor Frenkel, James Lepowsky, and Arne Meurman - *Vertex operator algebras and the Monster*, Pure and Applied Mathematics 134, Boston, (1988) MA: Academic Press, pp. liv+508 pp, ISBN 978-0-12-267065-7

[37] N. Geer, J. Kujawa, B. Patureau-Mirand - *Generalized trace and modified dimension functions on ribbon categories.* Selecta Math. (N.S.) 17 (2011), no. 2, 453-504

[38] T. Creutzig, A. Milas, and M. Rupert - *Logarithmic link invariants of $\overline{U}_q^H(\mathfrak{sl}(2))$ and asymptotic dimensions of singlet vertex algebras*, arXiv:1605.05634

[39] T. Creutzig, T. Gannon - *Logarithmic conformal field theory, log-modular tensor categories and modular forms* arXiv:1605.04630

[40] R. Borcherds - *Monstrous moonshine and monstrous Lie superalgebras* Inventiones Mathematicae 1992 109:405. doi:10.1007/BF01232032.

[41] E. Verlinde - *Fusion rules and modular transformations in two-dimensional conformal field theory* Nucl. Phys. B 300 (1988), 360-376.

[42] G. Moore, N. Seiberg - *Classical and quantum conformal field theory* Comm. Math. Phys. Volume 123, Number 2 (1989), 177-254.