University of Alberta

Normed Division Domains

by

Mark Goldenberg

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Normed Division Domains** submitted by **Mark Goldenberg** in partial fulfillment of the requirements for the degree of **Master of Science** in Mathematics.

Dr. J. E. (Ted) Lewis (Chair)

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Date: June 27, 2001

Abstract

Two examples of Normed Division Domains are examined.

The first example leads to graceful graphs and, in particular, graceful trees. A survey of results concerning the Graceful Tree Conjecture is given. Four approaches are presented:

(i) Construction of large graceful trees using smaller ones;

(ii) Assigning labels to particular vertices;

(iii) Establishing well-characterized classes of graceful trees;

(iv) Obtaining graceful trees by moving pairs of leaves of a given graceful tree.

A Computer-generated example, suggests that further progress may be made in the first approach. I developed some software that led me to believe that the last approach is very powerful.

The second example concerns the problem of tiling rectangular boards with polyominoes. Complete solution is given for polyominoes with up to 5 squares.

Acknowledgement

This thesis would not have made it to this point without help and support of many people.

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Chapter 0

Introduction

0.1 Normed Division Domains.

In his paper (see [17]) of 1978, Solomon W. Golomb introduced the notion of a normed division domain. It is a structure that is just enough to define divisibility, primes, greatest common divisor, least common multiple and some other notions of elementary number theory in a sensible way.

Definition 1.1. A triple $(S, <|, N)$ is called a *Normed Division Domain* (*NDD*) if:

(i) $(S, <|)$ is a partially ordered set;

(ii) N is a mapping from S to the set of non-negative integers;

(iii) if $a, b \in S$ and $a<|b$, then $N(a)|N(b)$.

(iv) if $e \in S$ and $N(e) = 1$, then $e<|a$ holds for all $a \in S$.

The function N is called a "norm" for S. N is not necessarily a norm in the full sense. We do not require addition or multiplication by scalars to be defined in S.

If $a, b \in S$ and $a<|b$, then we say that a *divides* b. If $e \in S$ and $N(e) = 1$, then e is called a unit in S.[1]

The following definitions are analogous to those in elementary number theory.

[1]We do not require uniqueness of the unit element.

Definition 1.2. Let $D = (S, <|, N)$ be an NDD. An element $p \in S$ is said to be *prime* in D if $N(p) > 1$ and there do not exist non-unit elements $a \in S$ with $a<|p$ and $N(a) \neq N(p)$.

Definition 1.3. Let $D = (S, <|, N)$ be an NDD and a, b be elements of S. An element d of S is said to be a *greatest common divisor* of a and b if $d<|a, b$ and $N(c) \leq N(d)$ for any $c \in S$ such that $c<|a, b$.

Definition 1.4. Let $D = (S, <|, N)$ be an NDD and a, b be elements of S. An element m of S is said to be a *least common multiple* of a and b if $a, b<|m$ and $N(c) \geq N(m)$ for any $c \in S$ such that $a, b<|c$.

In this thesis, we consider two examples of NDD's, both arising from graph theory.

0.2 Graphs with the E-norm.

We use standard terminology in graph theory (see, for example, [34]). Let S be the set of all finite, simple, connected, undirected graphs. For $G \in S$, we define the norm $N(G)$ as the number of edges in G. This is called the *E-norm* for the set S.

To form an NDD, we define a partial order on S that imitates divisibility. Let G, H be graphs in S. We say that H divides G and write $H<|G$ if G can be decomposed into edge-disjoint isomorphic copies of H, that is, if G can be covered completely by copies of H such that every edge of G is covered by an edge of exactly one copy of H. A vertex, on the other hand, can appear in several copies of H. Below is a simple example:

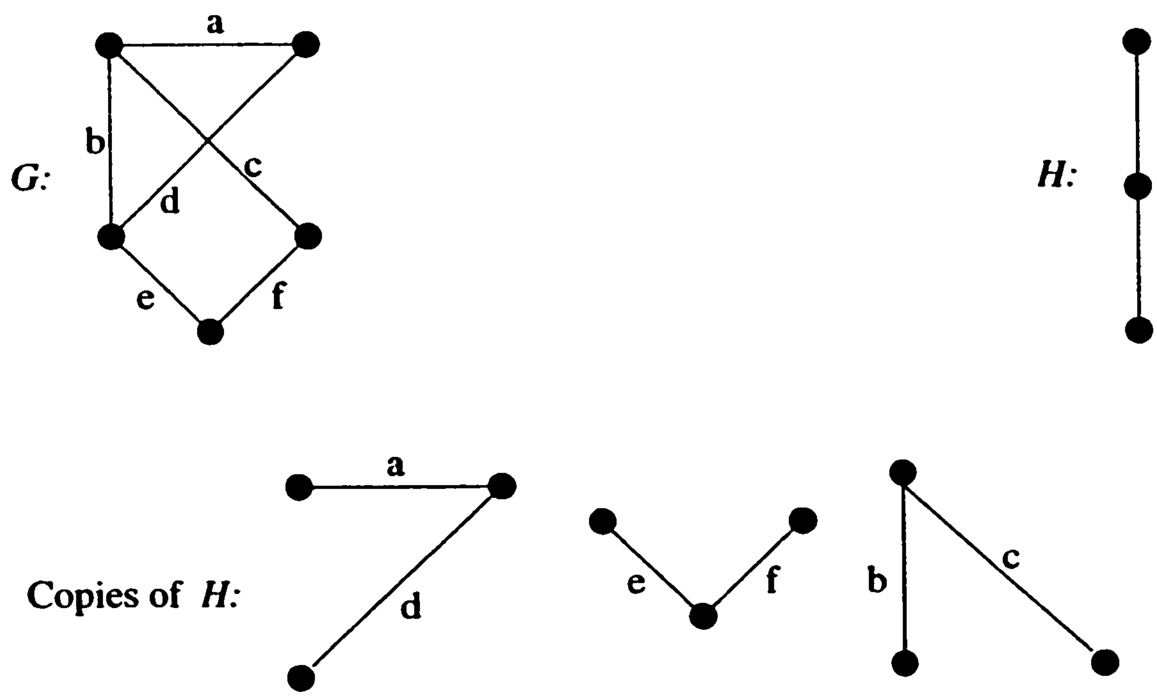


Figure 0.1.1: Illustration of divisibility for the E-norm.

The following question is a natural one to ask: "Given graphs G, H in S, can we tell whether H divides G or not?" This question seems difficult in general. The following special case has not been resolved:

Conjecture 2.1. (Ringel, 1963) The complete graph on $2n + 1$ vertices is divisible by every tree on $n + 1$ vertices.

We now introduce an apparently unrelated notion:

Definition 2.2. A tree on n vertices is said to be *graceful* if its vertices can be assigned numbers from 0 to $n-1$ (in a one-to-one manner) such that if we label the edges with the absolute differences of the numbers assigned to the end-vertices of the edges, then all the edges will have different numbers from 1 to $n - 1$.

The following is called the *Graceful Tree Conjecture* (GTC).

Conjecture 2.3. (see [20]) All trees are graceful.

It is remarkable how easily and elegantly Ringel's conjecture follows from the GTC. Suppose that all trees are graceful. Let n be a positive integer and a let T_{n+1} be a tree on $n+1$ vertices together with the numbering of its vertices that shows that this tree is graceful. We show how to decompose the clique K_{2n+1} into $2n+1$ copies of T_{n+1}. Choose $n+1$ consecutive vertices of the clique and number them from 0 to n. Mark the edge between two vertices u and v bold if and only if the vertices with numbers assigned to u and v are adjacent in T_{n+1}. The picture below shows this for n=4 and a gracefully labeled tree T_5 on the left.

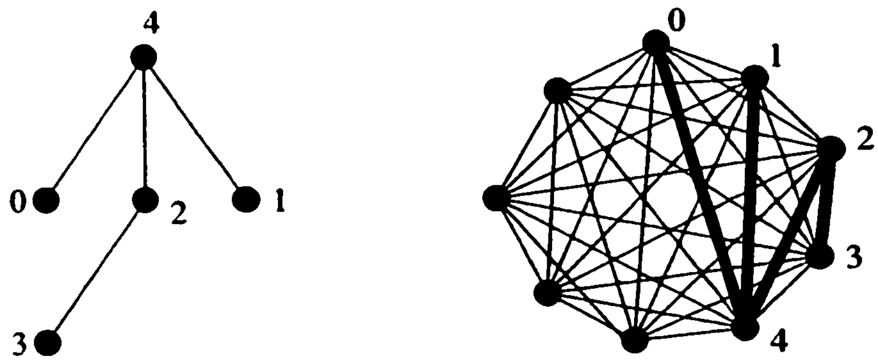


Figure 0.1.2: Ringel's Conjecture follows from the GTC.

By our construction, we mark bold exactly one edge of each of the lengths 1,2, ... ,n (the length of an edge is the distance between vertices that this edge connects in the chordless cycle formed by the vertices of K_{2n+1} and edges connecting consecutive vertices). Now we shift our $n+1$ chosen vertices by one position clockwise and perform the same procedure. Evidently, after doing this $2n+1$ times, we will have all edges of K_{2n+1} marked bold, which completes the proof.

In the first chapter of this thesis, we give the rigorous definition of a graceful graph together with some general results about them and survey the main approaches to the GTC.

0.3 Graphs with the V-norm

Again, let S be the set of all finite, simple, connected, undirected graphs. For $G \in S$, we now define the norm $N(G)$ as the number of vertices in G. This is called the *V-norm* for the set S.

To form an NDD, we define a partial order on S that imitates divisibility. Let G, H be graphs in S. We say that H divides G and write $H <| G$ if G can be decomposed into vertex-disjoint isomorphic copies of H, that is, if G can be covered by copies of H such that every vertex of G is covered by a vertex of exactly one copy of H. Note that not every edge of G has to be an edge of any copy of H. Below is a simple example:

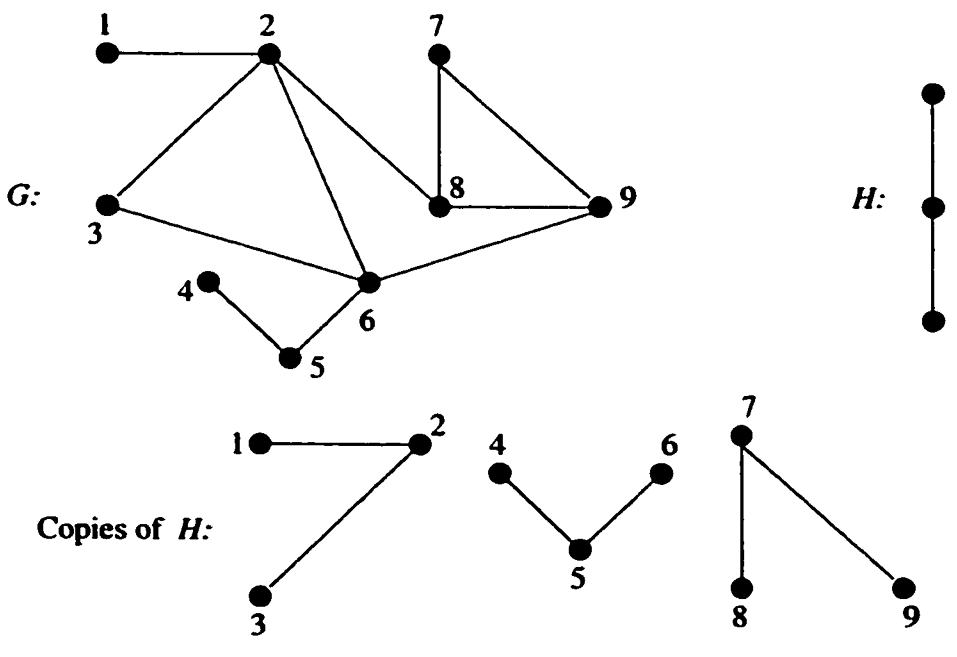


Figure 0.2.1: Illustration of divisibility for the V-norm.

This divisibility problem, like the one in the preceding subsection, seems difficult in general.

We consider the special class of graphs, where vertices are lattice points and the edges have unit length. We ask whether a rectangular graph, such as G below, is divisible by an arbitrary graph in this class.

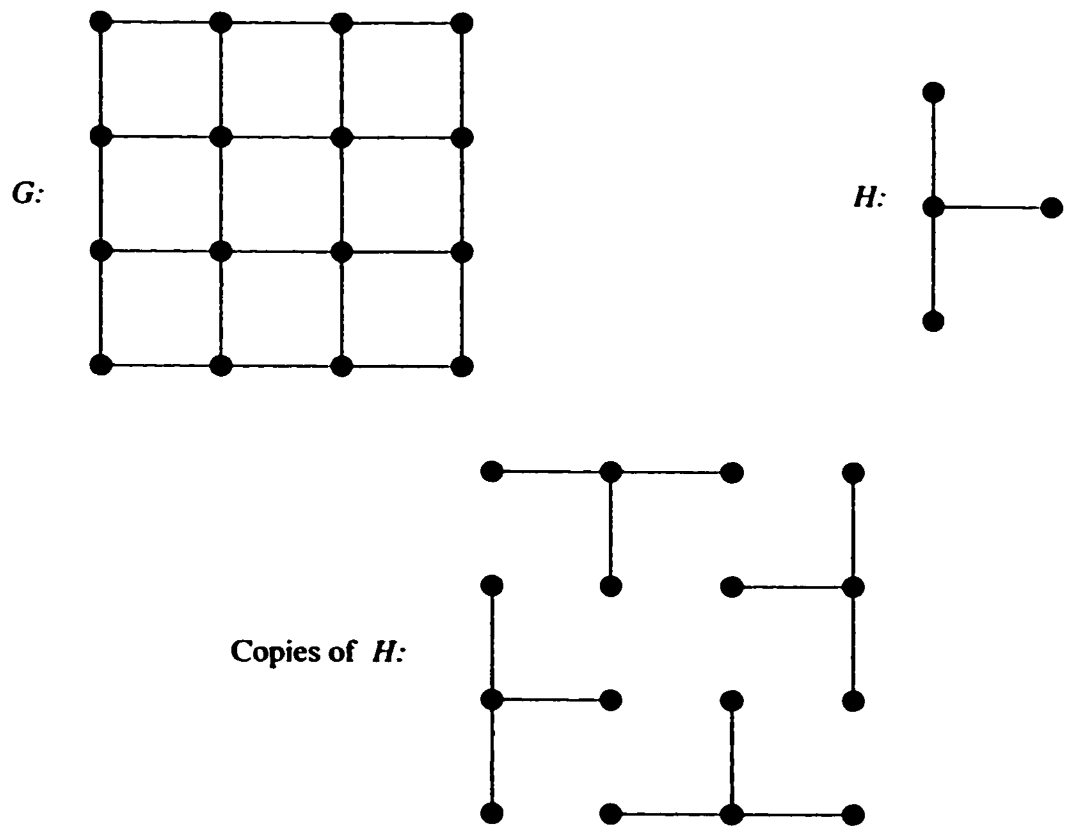


Figure 0.2.2: The kinds of dividends and divisors that we consider.

It is more convenient to represent such graphs as figures formed of unit squares connected "edge to edge." Such a figure is called a polyomino. The example in Figure 0.2.2 is then represented as follows:

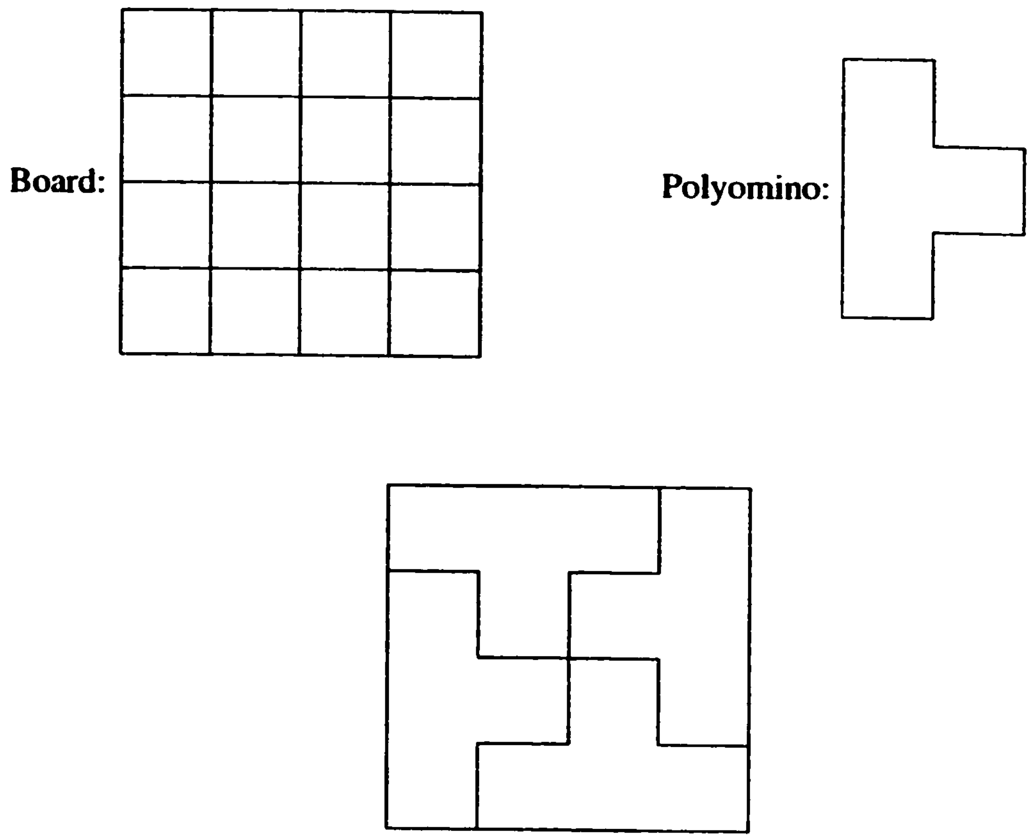


Figure 0.2.3: The introductory example of tiling with polyominoes.

The problem that we will be concerned with in Chapter 2 is: "Given a rectangular board, is it possible to tile it with a particular polyomino consisting of up to 5 squares?"

Chapter 1

Graceful Trees

1.1 Introduction

In this chapter, we study the class of *graceful trees*. In the introductory chapter, we showed that Ringel's conjecture follows from the *Graceful Tree Conjecture*. This chapter mainly deals with results that may lead us closer to the ultimate resolution of this famous conjecture.

In the rest of Section 1, we give the definition of a more general class called *graceful graphs*, and some of the general results about them.

In Section 2, we examine a number of ways of building large graceful trees from smaller ones.

In the beginning of Section 3, our concern shifts from the trees themselves to trying to find out how a graceful labeling of a tree should (or may) look like. We also look at some well-characterized classes of trees that have been shown to be graceful. The proof of gracefulness for some of those classes uses the labeling technique in the beginning of the section.

In Section 4, we study a very powerful method for obtaining graceful trees together with graceful labelings. As an illustration of this method, we describe the proof of gracefulness of trees of diameter 4 and derive a proof of gracefulness of a family of trees of diameter 5.

In Section 5, we make some concluding remarks.

We denote a finite simple undirected connected graph by $G = (V, E)$, where V is the set of vertices of G and E is the set of edges. Also, denote $|V|$ by n and denote $|E|$ by m. That is, n will always stand for the number of vertices and m will stand for the number of edges in the graph.

Definition 1.1. A mapping $\theta : V \to \mathbb{Z}^+ \cup \{0\}$ is said to be a *labeling* if it is one-to-one. Such a mapping induces another mapping which is defined on the set of edges E as follows: for an edge $uv \in E$, let $\theta(uv) = |\theta(u) - \theta(v)|$.

Remark. It will always be clear from the context which θ is meant, so no confusion is possible.

Definition 1.2. A labeling θ is said to be *graceful* if $\theta(V) \subset \{0, 1, \ldots, m\}$ and $\theta(E) = \{1, 2, \ldots, m\}$. A graph with such a labeling is said to be *graceful.*

There is also a stronger version of gracefulness.

Definition 1.3. A labeling θ is said to be *strongly graceful* and have *strength* k if it is graceful and there is an integer k such that for every edge $uv \in E$ either $\theta(u) \leq k < \theta(v)$ or $\theta(v) \leq k < \theta(u)$ holds. A graph G with such a labeling is said to be *strongly graceful.* Whenever we refer to a labeling with a strength, it is understood that the labeling is strongly graceful.

Remark. It is clear from the latter definition that all strongly graceful graphs are graceful.

Results about gracefulness are useful in advanced areas of modern high technology (see [2] for an example).

Historically, both classes were introduced by Aleksander Rosa in 1966 (see [29]) and became a popular topic of research after the paper by Solomon Golomb (see [16]) in 1972.[1]

[1]The terminology that is used in this thesis is different from what Rosa and Golomb use in the referenced papers. Instead, the terminology of more recent papers is adopted.

Here are some almost trivial observations.

Lemma 1.4. If θ is a graceful labeling of G, then there are vertices $u, v \in V(G)$ with $\theta(u) = 0$ and $\theta(v) = m$. Moreover, $uv \in E(G)$.

Proof. The lemma follows from the fact that, by the definition of a graceful labeling, there has to be an edge in G labeled m.

■

Lemma 1.5. Not all graceful graphs are strongly graceful.

Proof. A smallest counterexample is the cycle K_3 (triangle). Two labels are forced by the previous lemma. Any of the two assignments for the third vertex give us a graceful labeling which is not strongly graceful.

■

Here is an important necessary condition for a graph to be strongly graceful. We show later that this condition is not sufficient.

Lemma 1.6. (see [29]) If G is strongly graceful, then it is bipartite.

Proof. Let θ be a graceful labeling of G with strength k. We merely divide the vertex set V of G into two sets, call them X and Y. A vertex $v \in V$ belongs to X if $\theta(V) \leq k$; otherwise, we have $v \in Y$. Thus $X \cap Y = \emptyset$ and $X \cup Y = V$. It follows from the definition of a strongly graceful labeling that X is an independent set, and so is Y.

■

Quoting an unpublished result of Erdős, Gallian (see [14]) states that although most graphs are not graceful, most graphs that have some sort of regularity of structure are graceful. In fact, negative results (those proving that some classes of graphs are not graceful) are quite rare.

We now look at some general results about graceful graphs.

The next two theorems appeared in the first paper on the subject by Rosa. They are still the most general proved statements in the area. The first of these theorems also reappeared in the work by Golomb and we give his proof here.

Theorem 1.7. (see [29, 16]) If G is an Eulerian graph with m edges and m is congruent to either 1 or 2 modulo 4, then it is impossible to label G gracefully.

Proof. Assume to the contrary that a graceful labeling θ of such an Eulerian graph exists. Let $(v_1, v_2, \ldots, v_{m+1} = v_0)$ be a trail in G such that every edge of G is listed among $v_i v_{i+1}$ exactly once, where i ranges over $1, 2, \ldots, m$. We claim that the sum of labels of edges of this trail is even. In fact, we have

$$\begin{aligned}\sum_{i=1}^{m} |\theta(v_i) - \theta(v_{i+1})| &\equiv \sum_{i=1}^{m} (\theta(v_i) - \theta(v_{i+1})) \\ &\equiv \theta(v_1) - \theta(v_{m+1}) \\ &\equiv 0 (\text{mod } 2)\end{aligned}$$

Thus the sum of all labels of the edges of G is even.

Now we will obtain a contradiction by proving that the same sum must be odd. Since θ is graceful, we have:

$$\sum_{i=1}^{m} |\theta(v_i) - \theta(v_{i+1})| = 1 + 2 + \ldots + m = \frac{m(m+1)}{2},$$

which is obviously odd if $m = 4k + 1$ or $m = 4k + 2$ for some non-negative integer k.

This contradiction completes the proof.

■

Right after the above theorem Rosa presents another result with a proof, which, with minor alterations, became a template for many proofs in the area of graceful labelings. We would like to show such a proof with all the details.

Theorem 1.8.(see [29]) A chordless cycle C_n is strongly graceful if and only if $n \equiv 0 \pmod 4$. If $n \equiv 3 \pmod 4$, then C_n can be labeled gracefully.

Proof. While the necessity part of the statement follows from Theorem 1.7 and Lemma 1.6, the sufficiency part is shown by giving explicit labelings.

Denote the vertices of C_n by $v_1, v_2, \ldots, v_n$.

Suppose that $n \equiv 0 \pmod 4$. The example below shows how we could label our graph for $n = 8$.

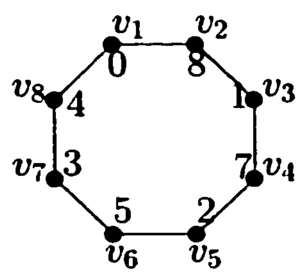


Figure 1.1.1: Labeling C_n strongly gracefully for $n \equiv 0 \pmod 4$.

In general, we define

$$\theta(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ n - \frac{i}{2} + 1 & \text{if } i \text{ is even and } i \leq \frac{n}{2}, \\ n - \frac{i}{2} & \text{if } i \text{ is even and } i > \frac{n}{2}. \end{cases}$$

It is evident that, for any $1 \leq i \leq n$, we have $0 \leq \theta(v_i) \leq n$. Let $k = \frac{n}{2} - 1$. Note that $\frac{i-1}{2}$ ranges over

$$0, 1, \ldots, k$$

for odd i. Also, $n - \frac{i}{2}$ ranges over

$$k+1, k+2, \dots, \frac{3n}{4} - 1$$

for even $i > \frac{n}{2}$, while $n - \frac{i}{2} + 1$ ranges over

$$\frac{3n}{4} + 1, \frac{3n}{4} + 2, \dots, n$$

for even $i \leq \frac{n}{2}$. Hence θ is a labeling of C_n with the desired range.

Now, for even $i > \frac{n}{2}$, except for $\theta(v_n v_1) = (n - \frac{n}{2}) - 0 = \frac{n}{2}$, we have

$$\theta(v_i v_{i-1}) = (n - \frac{i}{2}) - (\frac{i-2}{2}) = n - i + 1$$

and

$$\theta(v_i v_{i+1}) = (n - \frac{i}{2}) - (\frac{i}{2}) = n - i.$$

Thus $\theta(v_i v_{i+1})$ and $\theta(v_i v_{i-1})$ range over the edge labels $1, 2, \dots, \frac{n}{2}$.

For even $i \leq \frac{n}{2}$ we have

$$\theta(v_i v_{i-1}) = (n - \frac{i}{2} + 1) - (\frac{i-2}{2}) = n - i + 2$$

and

$$\theta(v_i v_{i+1}) = (n - \frac{i}{2} + 1) - (\frac{i}{2}) = n - i + 1.$$

Thus $\theta(v_i v_{i+1})$ and $\theta(v_i v_{i-1})$ range over the edge labels $\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n$.

Thus θ is graceful. Also, θ has strength k; this is easy to see.

Thus the proof for the case when $n \equiv 0 \pmod 4$ is complete.

Now, suppose that $n \equiv 3 \pmod 4$. The example below shows how we could label our graph for $n = 7$.

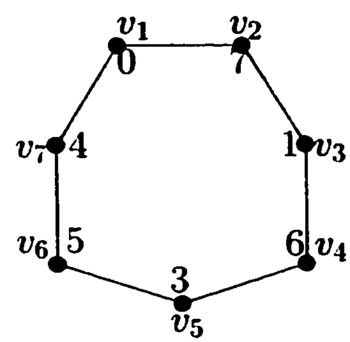


Figure 1.1.2: Labeling C_n gracefully for $n \equiv 3 (\text{mod } 4)$.

We define

$$\theta(v_i) = \begin{cases} n + 1 - \frac{i}{2} & \text{if } i \text{ is even,} \\ \frac{i-1}{2} & \text{if } i \text{ is odd and } i \leq \frac{n-1}{2}, \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } i > \frac{n-1}{2}. \end{cases}$$

The proof that θ is graceful is similar to the proof in the previous case.

■

One can see that the gracefulness of a graph G does not imply the same property about G's induced subgraphs. As a counterexample, let G be a chordless cycle C_5. By Theorem 1.7 we know that G is not graceful. However, G is an induced subgraph of the graph shown below together with its graceful labeling.

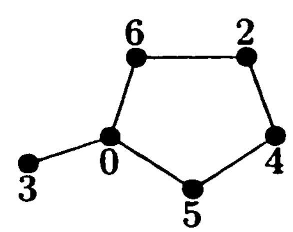


Figure 1.1.3: A subgraph of a graceful graph need not be graceful.

Once we have gained some vision of the problem of gracefulness for graphs in general, let us go to the main subject of this chapter of the thesis.

For trees, the definition of gracefulness can be stated more concisely as follows. Let T be a tree on n vertices. A labeling θ of the vertices of T is graceful if the $n-1$ edges of T are labeled $1, 2, \ldots, n-1$ under θ.

Probably everyone (who ever gets to knowing the notion of graceful graphs) knows the construction of the proof of the following theorem, whose format has definitely influenced all the subsequent terminology of the subject.

Definition 1.9. A tree T is called a *caterpillar* if removal of all its leaves leaves a path. This path is called the *backbone* of the caterpillar.

Theorem 1.10.(see [29]) All caterpillars are strongly graceful.

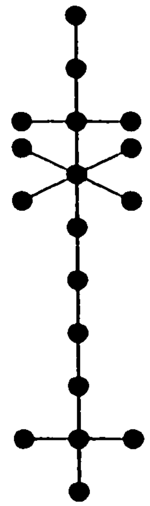
Proof. We draw vertices of the two independent sets of the given caterpillar on the left and the right side as follows. Start by drawing the backbone (the sides alternate starting from the left). After that, for each vertex of the backbone, we draw its leaf-neighbors on the opposite side so that edges do not cross.

Label vertices consecutively – start with 0 at the top vertex on the left side, go down increasing the vertex labels each time by 1. Then proceed to the right side going from the bottom vertex up.

It follows that the edges get consecutive labels starting from 1 at the bottom edge.

■

Example: Here is a demonstration of the above theorem.



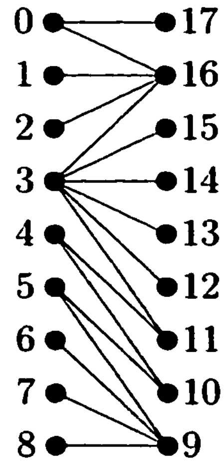


Figure 1.1.4: Graceful labeling for caterpillars.

The left part shows a tree on 18 vertices, while the right part shows its graceful labeling with strength $k = 8$.

It is logical to give the negative counterpart of Theorem 1.10 here. In Section 4, we will prove that all trees of diameter 4 are graceful. From Theorem 1.10, we know that caterpillars of diameter 4 are strongly graceful. Our next theorem verifies that there are no other strongly graceful trees of diameter 4. This result is very important and will be used in the sections to follow.

We prove a useful lemma first.

Lemma 1.11. (see [20]) Let T be a caterpillar of diameter 4 and let θ be its strongly graceful labeling. Suppose that x is the center vertex[2] of T. Then $\theta(x) \neq 0$.

Proof. If T is a path on five vertices, then the only graceful labeling (up to symmetry) with the center vertex labeled 0 is shown below. It is easy to see that this labeling is not strongly graceful.

[2] Note that a tree of an even diameter d has a unique center vertex, i.e. a unique vertex whose distance to any other given vertex of the tree is at most $\frac{d}{2}$.

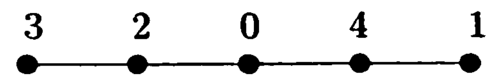


Figure 1.1.5: The only graceful labeling of P_5 with the center vertex labeled 0.

Now, assume that T is not a path. Then we can represent it as follows.

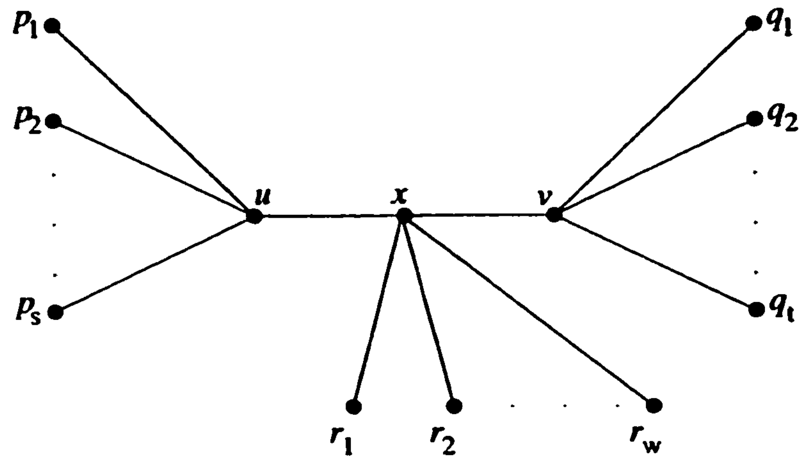


Figure 1.1.6: Canonical representation of non-path caterpillars of diameter 4.

Assume to the contrary that $\theta(x) = 0$. Let k be the strength of θ. Since θ is strongly graceful, $\theta(u)$, $\theta(v)$ and all of $\theta(r_i)$ are greater than k. It then follows that all of $\theta(p_i)$ and $\theta(q_i)$ are not greater than k. Therefore, the only edges with labels greater than k are the ones incident on x.

Suppose that $\theta(q_1) = 1$. Then we must have $\theta(v) = k + 1$. Let j be the smallest positive integer such that $\theta(q_i) \neq j$ for any i. Then all the labels

$$(k+1)-(j-1), (k+1)-(j-2), \ldots, (k+1)-1,$$

that is, all the edge labels from $k - j + 2$ to k, appear among the labels of the edges vq_i. Then all the edge labels of up_i must be less than $k - j + 2$. On the other hand, we have $\theta(p_l) = j$ for some l. Hence $\theta(p_l u) = \theta(u) - j < k - j + 2$, which implies $\theta(u) < k + 2$, which is impossible, since we know that $\theta(u) > k$ and $\theta(v) = k + 1$. This contradiction completes the proof.

■

Theorem 1.12. (see [20]) No non-caterpillar tree T of diameter 4 has a strongly graceful labeling.

Proof. First we agree on the following convenient way of representing a given tree T of diameter 4. We align the set of vertices of T in 4 columns. The second column consists of the center vertex only, denoted a_0. In the first column are the leaves of T that are adjacent to a_0. In the third column are the non-leaf neighbors of a_0. Denote them by $a_1, a_2, \ldots, a_p$, such that vertices with even degrees (i.e. with odd numbers of leaf-neighbors) have labels with lower indices than vertices with odd degrees. In the fourth column, we have all the leaf-neighbors of vertices from the third column. We assign double indices to these vertices. Thus any tree of degree 4 is drawn as on the picture below.

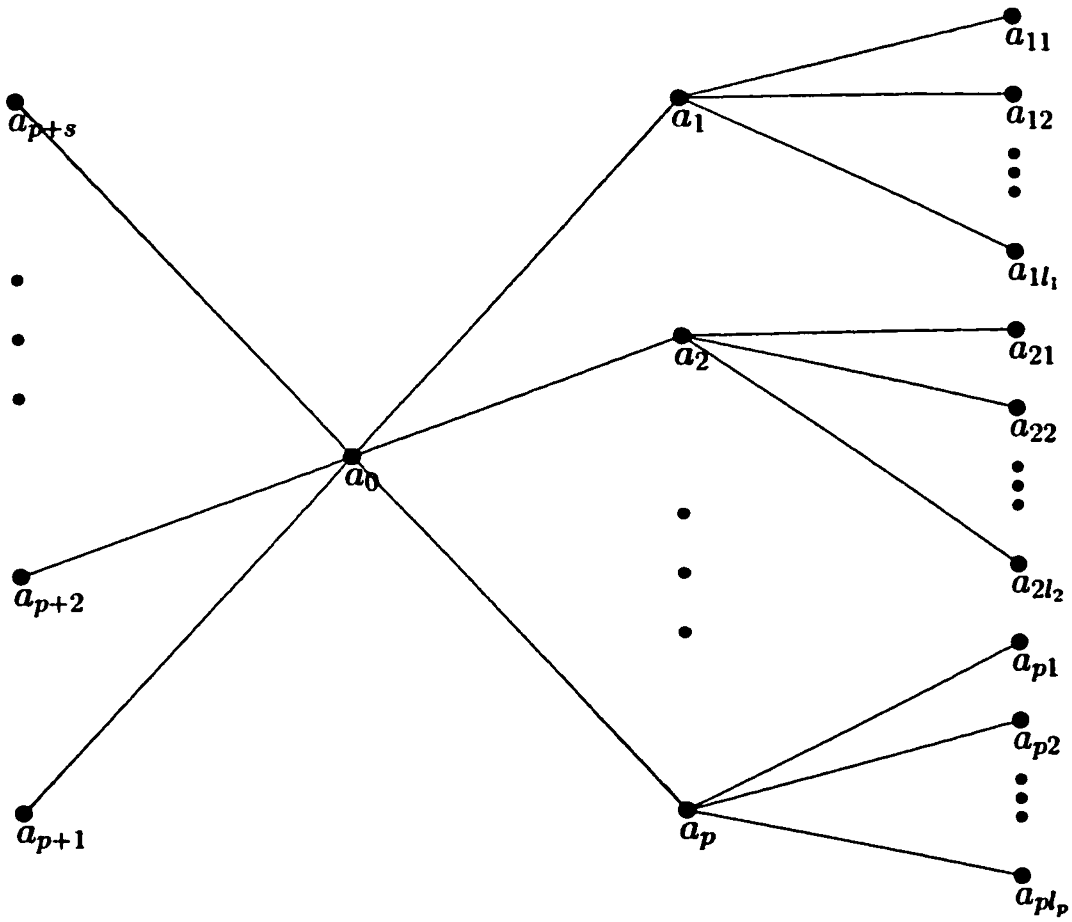


Figure 1.1.7: Canonical representation of trees of diameter 4.

Assume to the contrary that T is the smallest (in the number of edges) non-caterpillar tree of diameter 4 that admits a strongly graceful labeling. Let θ be a strongly graceful labeling of T with strength k. Let T' be the subtree of T that we get by removing all the leaves of T together with the edges incident upon these leaves.

We consider the edge h of T with $\theta(h) = m$. There are two cases.

Case 1. Assume that $h \in E(T')$. So, suppose that $h = a_0a_1$.

We can assume that $\theta(a_0) = n - 1$ and $\theta(a_1) = 0$[3]. Then all the labels from 0 to k appear among $\theta(a_i)$ $(i \geq 1)$. Therefore, a_0a_i get all the edge labels from $m - k$ to m.

Suppose by symmetry that $\theta(a_{21}) = n - 2$. We must have $\theta(a_2) = k$. Let j be the largest positive integer such that $\theta(a_{2\alpha}) \neq j$ for any $1 \leq \alpha \leq l_2$. Assume then, without loss of generality, that $\theta(a_{31}) = j$. We know that $\theta(a_3) \leq k - 1$. Therefore, $\theta(a_3a_{31}) = \theta(a_{31}) - \theta(a_3) \geq j - (k-1) = j - k + 1$, which is a contradiction, since a vertex with the label $j + 1$ appears among the neighbors of a_2.

Case 2. We now assume that $h \in E(T)\backslash E(T')$. Suppose that $h = yz$, where y is a leaf. Assume that $\theta(y) = m$ and $\theta(z) = 0$[4]. Consider the tree T'' with $V(T'') = V(T)\backslash\{y\}$ and $E(T'') = E(T)\backslash\{yz\}$. Then the labeling θ'' defined on $V(T'')$ by $\theta''(v) = \theta(v)$ is obviously a strongly graceful labeling of T'' with the same strength k as the strength of θ. By the minimality of T, T'' must be a caterpillar; also, $z = a_i$ for some $i \leq p$. We can get a strongly graceful labeling of T'' where z is labeled $n - 2$[5]. But then we can obtain still a new tree T''' similarly as we obtained T'' from T, which leads us to a strongly graceful labeling of that tree with a_0 labeled 0, which contradicts Lemma 1.9.

Thus the proof is complete.

■

1.2 Some constructions.

All the papers surveyed in this section are characterized by building a larger graceful tree (or a family of graceful trees) using other graceful (and/or strongly graceful) trees.

---

[3]We are running a little bit ahead here. That this assumption does not lead to loss of generality will be proved in the next Section, see Lemma 2.1.

[4]See the preceding footnote.

[5]See the preceding footnote.

We proceed in chronological order. Some of the results described below are proper generalizations of the constructions described earlier in the section, by which their authors were inspired.

We preface the main discussion of this section by several useful lemmas which we will also use in other sections.

Lemma 2.1. (see [30]) Let T be a graceful tree on n vertices and let θ be a graceful labeling of T. Then the labeling $\overline{\theta}$ defined by $\overline{\theta}(v) = n - 1 - \theta(v)$ for every $v \in V(T)$ is also graceful. If θ has strength k, then $\overline{\theta}$ has the same strength.

Proof. It is easy to see that $\overline{\theta}(V(T)) = \theta(V(T))$ and that for all edges uv of T we have $\overline{\theta}(uv) = \theta(uv)$.

■

Lemma 2.2. (see [30]) Let T be a strongly graceful tree and let θ be a labeling for T with strength k. Then the labeling $\hat{\theta}$ defined by

$$\hat{\theta}(v) = \begin{cases} k - \theta(v) & \text{if } \theta(v) \leq k, \\ n - (\theta(v) - k) & \text{if } \theta(v) > k \end{cases}$$

for every $v \in V(T)$ also has strength k.

Proof. It is easy to see that $\hat{\theta}(V(T)) = \theta(V(T))$ and that for all edges uv of T we have $\hat{\theta}(uv) = n - \theta(uv)$.

■

Remark: The labeling $\overline{\theta}$ is called the *complementary* labeling for θ. The labeling $\hat{\theta}$ is called the *reverse* labeling for θ.[6] In the

[6]The notation introduced in the two lemmas is kept throughout this part of the thesis; so, $\overline{\theta}$ and $\hat{\theta}$ will stand for the complementary labeling and the reverse labeling for θ respectively.

proof of Lemma 2.2, we define $\hat{\theta}$ by taking the "complementary" labeling in each of the independent sets of our tree.

The following observation is due to D.Sheppard, in 1976.

Lemma 2.3. (see [32]) Let T be a strongly graceful tree. If θ is a labeling for T with strength k and $uv \in E(T)$ is the edge with $\theta(uv) = 1$, then $k = min\{\theta(u), \theta(v)\}$.

1.2.1 The "Canonical Amalgamation" construction.

One of the first constructions was due to C. Huang, A. Kotzig and A. Rosa.

Theorem 2.4. (see [20]) Let T_1 and T_2 be graceful trees and let θ_1 and θ_2 be their respective graceful labelings. Let $v_1^* \in V(T_1)$ and $v_2^* \in V(T_2)$ be such that $\theta_1(v_1^*) = \theta_2(v_2^*) = 0$. If θ_2 has strength k, then the tree T obtained from T_1 and T_2 by identifying v_1^* and v_2^* is graceful. If, in addition, θ_1 has strength k', then θ has strength $k' + k$.

Proof. We define a labeling θ on $V(T)$ as follows. For a vertex $v \in V(T)$, let θ be determined by

$$\theta(v) = \begin{cases} \theta_1(v) + k & \text{if } v \in V(T_1), \\ \hat{\theta}_2(v) & \text{if } v \in V(T_2) \text{ and } \theta_2(v) \leq k, \\ \hat{\theta}_2(v) + |V(T_1)| - 1 & \text{if } v \in V(T_2) \text{ and } \theta_2(v) > k. \end{cases}$$

First we show that $\theta(v_1^*) = \theta(v_2^*)$. We have:

$$\theta(v_1^*) = \theta_1(v_1^*) + k = 0 + k = k$$

and

$$\theta(v_2^*) = \hat{\theta}_2(v_2^*) = k - \theta_2(v_2^*) = k - 0 = k.$$

Also, θ is trivially a labeling for T and edges of T_1 keep their original labels while labels of T_2 grow by $|V(T_1)|-1$, which means that edges of T get all the labels from 1 to $|V(T_1)| + |V(T_2)| - 1$.

Suppose now that θ_1 has strength k'. Note that the set of edge labels of T_1 is not changed by our construction. Hence the edge labeled 1 stays in T_1. By Lemma 2.3 and since all the vertex labels of T_1 are raised by k, the proof is complete.

■

The construction in Theorem 2.4 is called the *Canonical Amalgamation* construction.[7]

Let us give another very useful construction which is very similar to the one just described and has the same preconditions.

Theorem 2.5.[8] Let T_1 and T_2 be graceful trees and let θ_1 and θ_2 be their respective graceful labelings. Let $v_1^* \in V(T_1)$ and $v_2^* \in V(T_2)$ be such that $\theta_1(v_1^*) = \theta_2(v_2^*) = 0$. If θ_2 has strength k, then the tree T obtained from T_1 and T_2 by adding the edge $v_1^* v_2^*$ is graceful. If, in addition, θ_1 in the above theorem has strength k', then θ_2 has strength $k' + k + 1$.

Proof. We define a labeling θ on $V(T)$ as follows. For a vertex $v \in V(T)$, let θ be determined by

$$\theta(v) = \begin{cases} \overline{\theta}_1(v) + k + 1 & \text{if } v \in V(T_1), \\ \hat{\theta}_2(v) & \text{if } v \in V(T_2) \text{ and } \theta_2(v) \leq k, \\ \hat{\theta}_2(v) + |V(T_1)| & \text{if } v \in V(T_2) \text{ and } \theta_2(v) > k. \end{cases}$$

θ is trivially a labeling. Edges of T_1 keep labels $1, 2, \ldots, |V(T_1)| - 1$. Also,

$$\begin{aligned} \theta(v_1^* v_2^*) &= (\overline{\theta}_1(v_1^*) + k + 1) - \hat{\theta}_2(v) \\ &= |V(T_1)| - 1 + k + 1 - k \\ &= |V(T_1)|. \end{aligned}$$

[7]The term comes from [7].

[8]This result is mentioned in [7], but we have not seen it stated explicitly in any of the early papers.

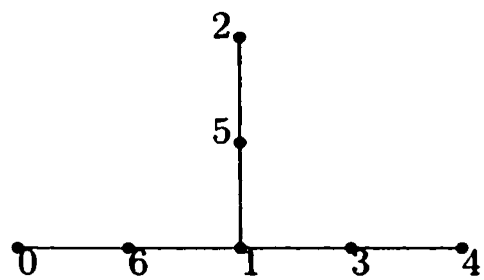
Finally, we increased labels of all the edges of T_2 by $|V(T_1)|$. So, θ is graceful.

The case where θ_1 has strength k' can be handled as in Theorem 2.4.

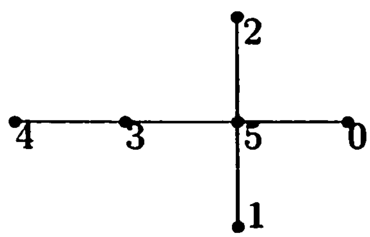
■

Example: Consider the trees T_1 and T_2 with labelings as follows:

(T_1, θ_1) :

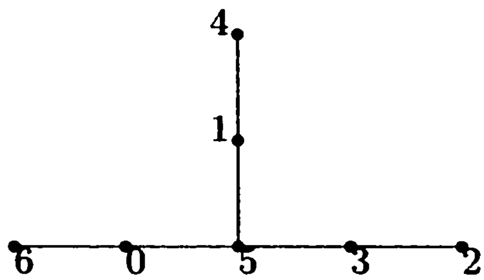


(T_2, θ_2) :

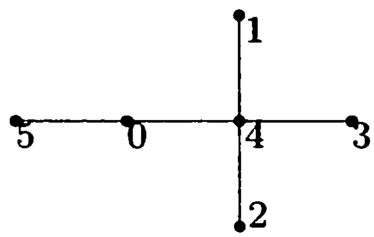


The complementary labeling for θ_1 and the reverse labeling for θ_2 are as follows:

$(T_1, \bar{\theta}_1)$:



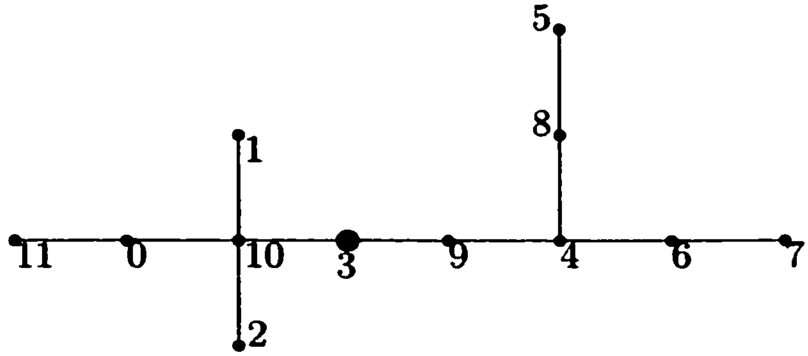
$(T_2, \hat{\theta}_2)$:



This is strongly graceful with $k = 3$
(recall Lemma 2.3)

From Theorem 2.4 we get:

(T, θ) :



From Theorem 2.5 we get:

(T, θ) :

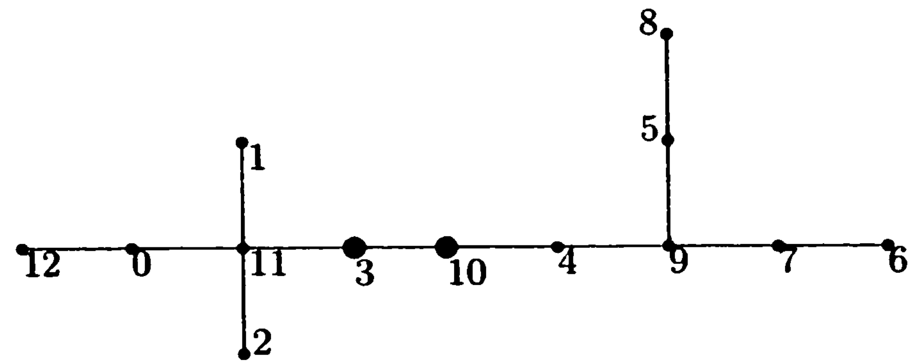


Figure 1.2.1: Examples of amalgamation-like constructions.

The next result is not used anywhere else in this section, but its usefulness will be justified in the next one, where it will be used to prove that all "banana trees" are graceful.

Theorem 2.6. (see [8]) Let T_1 and T_2 be graceful trees and let θ_1 and θ_2 be their respective graceful labelings. Suppose that θ_2 has strength k. Consider the disjoint union of T_1 and T_2 and define a labeling θ on this union as follows:

$$\theta(v) = \begin{cases} \theta_1(v) + k + 1 & \text{if } v \in V(T_1), \\ \theta_2(v) & \text{if } v \in V(T_2) \text{ and } \theta_2(v) \leq k, \\ \theta_2(v) + |V(T_1)| & \text{if } v \in V(T_2) \text{ and } \theta_2(v) > k. \end{cases}$$

Then the tree T obtained from the union of T_1 and T_2 by adding an edge between any vertices $u \in V(T_1)$ and $v \in V(T_2)$ with $|\theta(u) - \theta(v)| = |V(T_1)|$ is graceful. Suppose in addition that θ_1 has strength k'. If $\theta_2(v) \leq k$ and $\theta_1(u) > k'$, or $\theta_2(v) > k$ and $\theta_1(u) \leq k'$, then the resulting labeling has strength $k' + k + 1$.

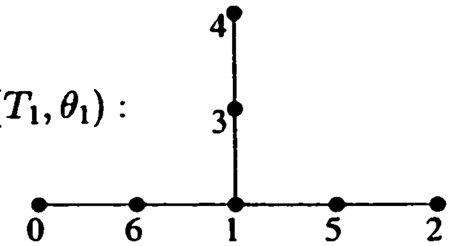
Proof. The argument is similar to that in the proof of Theorem 2.5. We omit the details.

■

Remark: This theorem generalizes Theorem 2.5. It produces more graceful trees.

Example: Consider the trees T_1 and T_2 with labelings as follows:

(T_1, θ_1) :



(T_2, θ_2) :

From the Theorem 2.6. we get:

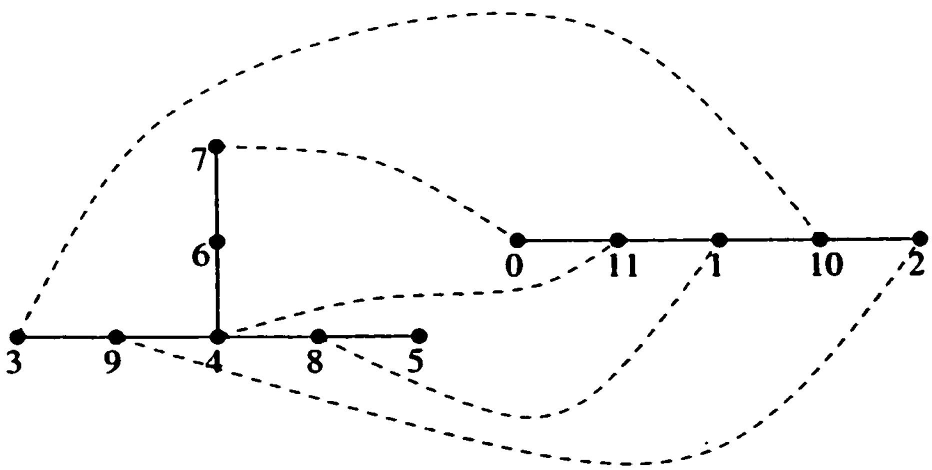


Figure 1.2.2: Another amalgamation-like construction.

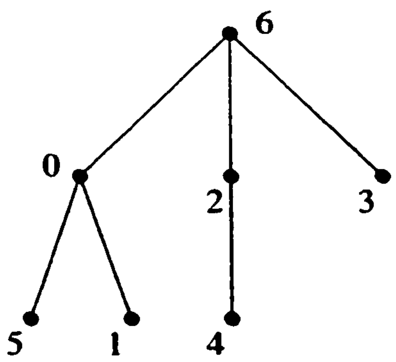
Pick any (one) dashed connection to get a gracefully labeled tree. We see that one of those connections has the center vertex of T_2 (which is a path on 5 vertices) as its end-point. This connection cannot be obtained via Theorem 2.5. In the next section, we will prove that the center vertex of the path on 5 vertices cannot be assigned the label 0 in any strongly graceful labeling.

1.2.2 The "Garland" construction.

The following construction[9] is due to K.M. Koh, D.G. Rogers and T.Tan.

Assume that T is a graceful tree on n vertices and θ is its graceful labeling. Let $w \in V(T)$ be such that $\theta(w) = n - 1$. For any positive integer p, we will use p isomorphic copies of T to construct a larger graceful tree T^*. Denote the copies of T by $T_1, T_2, \ldots, T_p$. Also, let w_i be the vertex corresponding to w in T_i, where i is from 1 to p. We construct T^* by taking a new vertex, call it w_0, and making it adjacent to all of $w_1, w_2, \ldots, w_p$.

Example: Let $n = 7$ and $p = 4$. Let the graceful tree T be as follows:



The following is the corresponding tree T^* together with its graceful labeling, which is obtained according to the proof of Theorem 2.7.

[9]This and the next constructions were not given names in the original sources. The names here are given by us for convenience.

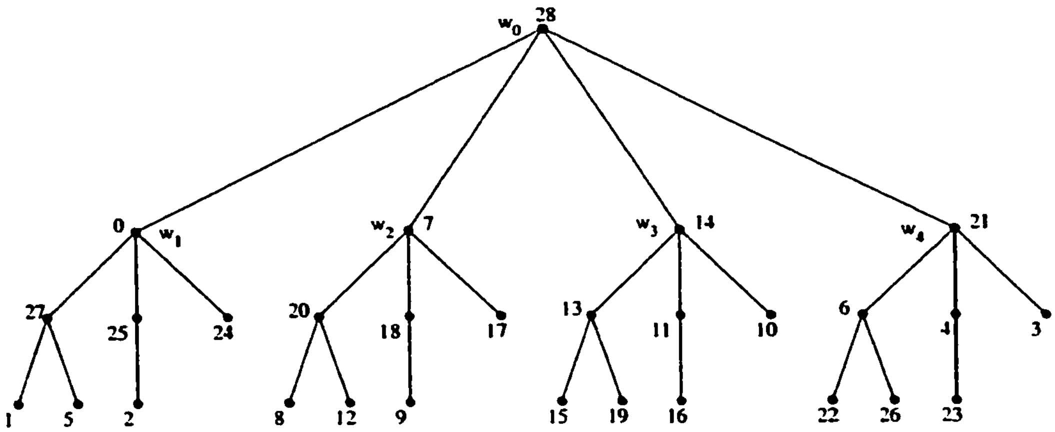


Figure 1.2.3: Example of the garland construction.

Theorem 2.7.(see [25]) Let T be a tree on n vertices and let θ be a graceful labeling of T with $\theta(w) = n-1$, where $w \in V(T)$. Then for each integer $p \geq 1$ there exists a graceful labeling θ^* for the tree T^* as described above, such that $\theta^*(w_0) = pn$.

Proof. Let p be given. We define θ^* as follows. Let $\theta^*(w_0) = pn$. Further, let v be a vertex of T and let v_i[10] be its image in the copy T_i of T, where $1 \leq i \leq p$. Set

$$\theta^*(v_i) = \begin{cases} in - 1 - \theta(v) & \text{if } dist(v,w) \text{ is even,} \\ (p+1-i)n - 1 - \theta(v) & \text{if } dist(v,w) \text{ is odd.} \end{cases}$$

We first verify that θ^* is a labeling. The range is easily seen to be from 0 to pn. Assume to the contrary that $\theta^*(u_i) = \theta^*(v_j)$, where either $u \neq v$ or $i \neq j$ (or both). The definition of θ^* implies that $\theta^*(u_i) \equiv \theta(u) (\text{mod } n)$ and $\theta^*(v_i) \equiv \theta(v) (\text{mod } n)$. Therefore $u \neq v$ is ruled out. However, $u = v$ leads us to conclusion that either both $dist(u,w)$ and $dist(v,w)$ are even or both of them are odd, which easily implies $i = j$. So, θ^* is shown to be a labeling.

We now verify that θ^* is graceful. We see that edges $w_0 w_i$ get labels that are multiples of n.

[10]Unless specified otherwise, v_i will have this meaning throughout the proof.

Consider the copies T_i and T_j of T, where $i + j = p + 1$ and $2i < p - 1$. Let uv be an edge in T such that $dist(u, w)$ is even. We claim that either

$$\theta^*(u_i v_i) = (p+i-2i)n+\theta(uv) \text{ and } \theta^*(u_j v_j) = (p+i-2i)n-\theta(uv),$$

or

$$\theta^*(u_i v_i) = (p+i-2i)n-\theta(uv) \text{ and } \theta^*(u_j v_j) = (p+i-2i)n+\theta(uv).$$

We have:

$$\begin{aligned} \theta^*(u_i v_i) &= |\theta^*(u_i) - \theta^*(v_i)| \\ &= |[in - 1 - \theta(u)] - [(p + 1 - i)n - 1 - \theta(v)]| \\ &= |(2i - p - 1)n + \theta(v) - \theta(u)| \\ &= |(p - 2i + 1)n + \theta(u) - \theta(v)| \end{aligned}$$

and

$$\begin{aligned} \theta^*(u_j v_j) &= |(2(p - i + 1) - p - 1)n + \theta(v) - \theta(u)| \\ &= |(p - 2i + 1)n + \theta(v) - \theta(u)|. \end{aligned}$$

Our claim follows from the two possible cases:

$$\theta(v) > \theta(u) \text{ and } \theta(u) > \theta(v).$$

Hence, the edges of T_i and T_j get labels

$$(p - 2i + 1)n \pm 1, (p - 2i + 1)n \pm 2, \ldots, (p - 2i + 1)n \pm (n - 1),$$

which makes it easy to see that θ^* is graceful. The proof is complete.

■

1.2.3 The "Attachment" construction.

The same authors are responsible for the following construction. Again, let p be a positive integer and $T_1, T_2, \ldots, T_p$ be isomorphic copies of a graceful tree T on n vertices. Let θ be a graceful labeling of T and let w be a vertex of T with $\theta(w) = n - 1$. Let $w_1, w_2, \ldots, w_p$ be the vertices corresponding to w in respective copies of T. We construct the tree T^* by simply identifying vertices $w_1, w_2, \ldots, w_p$, thus making them one vertex (call it w_0) in T^*. By the following theorem, K.M. Koh, D.G. Rogers, and T.Tan assert that the resulting tree T^* is graceful if the original tree and its graceful labeling satisfy a special condition.

Theorem 2.8.(see [26]) Let T be a graceful tree and θ be its graceful labeling. Let w be a vertex in T with $\theta(w) = n - 1$. If $\{\theta(v)|v \in N(w)\} \subset \{0\} \cup \{(n-1) - \theta(v)|v \in N(w)\}$ (where $N(w)$ is the neighborhood of w in T), then there exists a graceful labeling θ^* on T^*.

Proof. We define a labeling θ^* on T^* as follows. First set $\theta^*(w_0) = p(n-1)$. Again, let v be a vertex of T and let v_i be its image in the copy T_i of T, where $1 \leq i \leq p$. Set

$$\theta^*(v_i) = \begin{cases} (p-i)(n-1) + \theta(v) & \text{if } dist(v,w) \text{ is even,} \\ (i-1)(n-1) + \theta(v) & \text{if } dist(v,w) \text{ is odd.} \end{cases}$$

The technique used in the proof of Theorem 2.7 can be applied again to show that θ^* is a labeling.

We form two sets: I - the set of labels of the edges of T^* incident upon w_0, J - the set of labels of all edges of T^* whose endpoints are distinct from w_0.

The labels in I are in the same residue class modulo $n - 1$ as the corresponding labels of T. Also, the labels in J are calculated as in the proof of Theorem 2.7. It follows that if the prerequisite of the theorem is satisfied then θ^* is graceful.

■

Note that the last theorem properly generalizes the construction of Theorem 2.7. To demonstrate that, we will use the last theorem to build the tree as we built in the example for

Theorem 2.7. Let us take the tree T from that example and consider its complementary labeling. We then append an edge to the vertex numbered 0 using Theorem 2.5. The resulting tree is as follows:

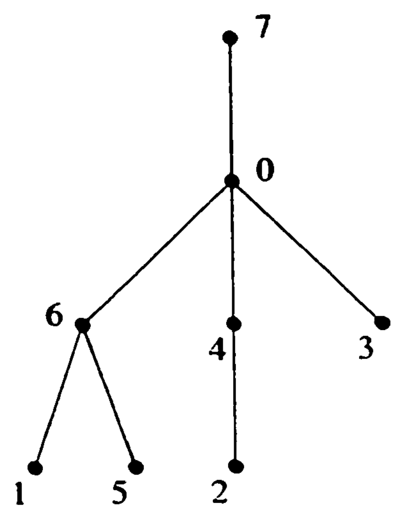


Figure 1.2.4: A sample tree to see that the attachment construction generalizes the garland construction.

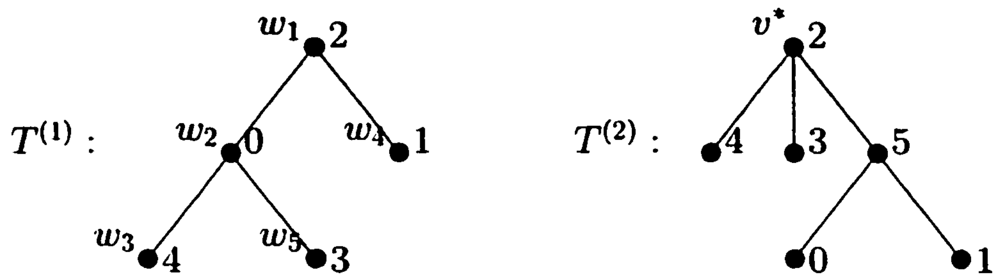
We use the new tree as the initial tree T for the last theorem, since this tree trivially satisfies the condition of theorem (the only vertex adjacent to the vertex marked $n-1$ is labeled 0). So, we can take 4 copies of our tree and glue them at w. Thus we get the exact construction of the example for Theorem 2.7.

1.2.4 The Δ-construction.

This is yet another construction of K.M. Koh, D.G. Rogers, and T.Tan. Unlike the two described earlier, this construction uses two graceful trees to construct a larger graceful tree. The construction goes as follows. Let $T^{(1)}$ and $T^{(2)}$ be two graceful trees with n_1 and n_2 vertices respectively. Choose arbitrarily a vertex v^* of $T^{(2)}$. Let $T_1, T_2, \ldots, T_{n_1}$ be isomorphic copies of $T^{(2)}$.

Let $\{w_1, w_2, \ldots, w_{n_1}\}$ be the set of vertices of $T^{(1)}$. We construct the new tree denoted by $T^{(1)} \Delta T^{(2)}$ as follows. Take $T^{(1)}$ and to each w_i attach T_i by identifying the vertex corresponding to v^* in T_i with w_i.

As an example of such a construction consider the trees $T^{(1)}$ and $T^{(2)}$ as follows:



Our next theorem guarantees that the following tree is graceful (the tree is drawn with the labeling provided in the proof of the theorem).

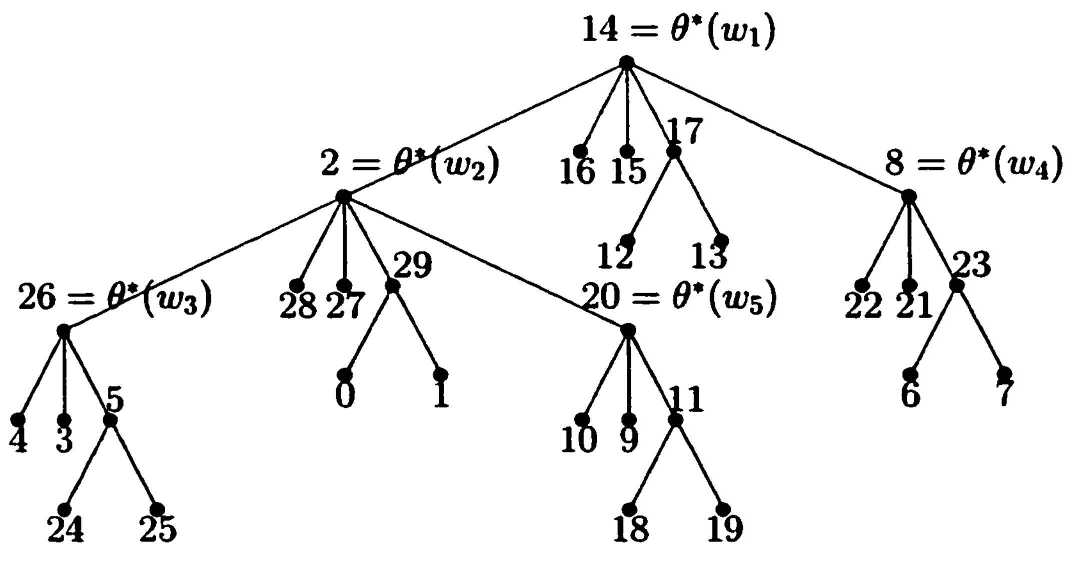


Figure 1.2.5: Example of the Δ-construction.

Theorem 2.9. (see [26]) Let $T^{(1)}$ and $T^{(2)}$ be graceful trees with n_1 and n_2 vertices respectively; also, let θ_1 and θ_2 be their respective graceful labelings. Then there exists a graceful labeling θ^* of the tree $T^{(1)}\Delta T^{(2)}$ as described above.

Proof. We define θ^* as follows. Let v_i be the image of some vertex $v \in V(T^{(2)})$ in T_i, where i ranges from 1 to n_1. Then we set

$$\theta^*(v_i) = \begin{cases} \theta_1(w_i)n_2 + \theta_2(v) & \text{if } dist(v, v^*) \text{ is even} \\ (n_1 - \theta_1(w_i) - 1)n_2 + \theta_2(v) & \text{if } dist(v, v^*) \text{ is odd} \end{cases}$$

Once this labeling is found, all we have to do is to verify that it works. Namely, the labelling θ_1 is used to label w_i's so that $\theta^*(w_i) \equiv \theta_2(v^*)(\text{mod } n_2)$ for all i. All the edge labels from 1 to $n_1n_2 - 1$ are split into groups of consecutive labels and those groups are assigned to the edges of the copies of $T^{(2)}$ in the manner that we are familiar with from Theorem 2.7.

■

Corollary 2.10. In the setting of the theorem, suppose that w_i and w_j are two distinct vertices of $T^{(1)}$ that form an edge. Let v be any vertex of $T^{(2)}$ and let v_i and v_j be the images of v in the corresponding copies of $T^{(2)}$. Then it is straightforward to check that $|\theta^*(v_i) - \theta^*(v_j)| = |\theta^*(w_i) - \theta^*(w_j)|$.

Corollary 2.11. If w_i and w_j are two distinct vertices of $T^{(1)}$ that form an edge, then $|\theta^*(w_i) - \theta^*(w_j)| = n_2|\theta_1(w_i) - \theta_1(w_j)|$.

1.2.5 The generalized Δ-construction.

This construction, as well as the next one, is a logical continuation of the constructions described above. It is due to M. Burzio and G. Ferrarese (see [6]). Using Corollary 2.10, we modify the Δ-construction by allowing the copies of $T^{(2)}$ corresponding to adjacent vertices of $T^{(1)}$ to be connected by any (one) edge whose end points are copies of the same vertex of $T^{(2)}$.

1.2.6 The Δ_{+1}-construction.

Unlike the two previous constructions, where all the vertices of one tree had to be substituted by a copy of the other tree, the construction being presented allows one special vertex of $T^{(1)}$ to remain itself.

We have two graceful trees $T^{(1)}$ and $T^{(2)}$ with graceful labelings θ_1 and θ_2 respectively. Let v^* be a fixed vertex in $T^{(2)}$ and $V(T^{(1)}) = \{w_1, w_2, \dots, w_{n_1}\}$ be the vertex set of $T^{(1)}$ (also, $T^{(2)}$ has n_2 vertices). Let $w \in V(T^{(1)})$ with $\theta_1(w) = n_1 - 1$ and consider the disjoint union of trees $T^{(1)} - w$.[11]

The authors of [6] came up with the generalized Δ-construction for this case, which looks as follows. Let $\overset{<}{v}$ and $\overset{>}{v}$ be vertices of $T^{(2)}$ with $\theta_2(\overset{<}{v}) = 0$ and $\theta_2(\overset{>}{v}) = n_2 - 1$. We define a labeling $\tilde{\theta}$ on $T^{(2)}$ as follows: for $v \in V(T^{(2)})$ we set

$$\tilde{\theta}(v) = \begin{cases} \theta_2(v) & \text{if } dist(\overset{<}{v}, v^*) \text{ is even,} \\ \bar{\theta}_2(v) & \text{if } dist(\overset{<}{v}, v^*) \text{ is odd.} \end{cases}$$

Since $\overset{<}{v}\overset{>}{v}$ is necessarily an edge in $T^{(2)}$ (recall Lemma 1.4), $dist(\overset{<}{v}, v^*)$ and $dist(\overset{>}{v}, v^*)$ always have different parity. Also, the above labeling is graceful for $T^{(2)}$ by Lemma 2.1.

Although the Δ-construction is formally defined for two trees, we can apply it in the obvious way to $T^{(1)} - w$ and $T^{(2)}$ to obtain a disjoint union of trees with $(n_1 - 1)n_2$ vertices: $G = (T^{(1)} - w)\Delta T^{(2)}$. Recall that by Theorem 2.9 a labeling of this union is defined by

$$\theta^*(v_i) = \begin{cases} \theta_1(w_i)n_2 + \tilde{\theta}_2(v) & \text{if } dist(v, v^*) \text{ is even,} \\ (n_1 - \theta_1(w_i) - 2)n_2 + \tilde{\theta}_2(v) & \text{if } dist(v, v^*) \text{ is odd,} \end{cases}$$

where v_i is the copy of $v \in V(T^{(2)})$ in the i^{th} copy of $T^{(2)}$.

[11]This short and intuitive notation means that we consider all the vertices in $V(T^{(1)})\backslash\{w\}$ and all the edges $\{uv : uv \in E(T^{(1)})$ and $u, v \in V(T^{(1)})\backslash\{w\}$.

It follows from Theorem 2.9 and from Corollary 2.11 that with this labeling the edges of G get all the different labels from 1 to $n_2(n_1-1)-1$ except the labels $n_2|\theta_1(w)-\theta_1(w_{k_j})|$, where w_{k_j} are the neighbors of w in $T^{(1)}$. Therefore, we build G into a tree as follows. Let $\overset{<}{v}_k$ and $\overset{>}{v}_k$ be the copies of $\overset{<}{v}$ and $\overset{>}{v}$ respectively in the k^{th} copy of $T^{(2)}$. We add a vertex u to the vertex set of G and make this new vertex adjacent to $\overset{<}{v}_k$ if $\tilde{\theta} = \theta_2$ and we make this vertex adjacent to $\overset{>}{v}_k$ otherwise. The resulting graph is clearly a tree. We denote this tree by $T^{(1)}\Delta_{+1}T^{(2)}$.

Theorem 2.12. (see [6]) The mapping θ_{+1} defined by $\theta_{+1}(v) = \theta^*(v)$ for each v in G and $\theta_{+1}(u) = (n_1 - 1)n_2$ is a graceful labeling for $T^{(1)}\Delta_{+1}T^{(2)}$.

Proof. We merely compute that $\theta_{+1}(u\,\overset{<}{v}_k) = n_2|\theta_1(w) - \theta_1(w_k)|$ in the case $\tilde{\theta} = \theta_2$ and $\theta_{+1}(u\,\overset{>}{v}_k) = n_2|\theta_1(w) - \theta_1(w_k)|$ otherwise, which completes the proof.

■

The following special case is remarkable from two points of view. First, it applies the construction from two graceful trees to modify a given graceful tree. Second, it is one of the very few results that allows a given graceful tree to grow while preserving gracefulness.

Theorem 2.13. (see [6]) The subdivision graph of a graceful tree is graceful.

Proof. Let $T^{(1)}$ be the original graceful tree and let $T^{(2)}$ be the tree consisting of two vertices and the edge between them. Naturally (recall the construction above), one of the vertices of $T^{(2)}$ is named $\overset{<}{v}$, while the other one is named $\overset{>}{v}$. Fix $v^* = \overset{<}{v}$ and obtain the tree $T^{(1)}\Delta_{+1}T^{(2)}$, which represents the subdivision graph of $T^{(1)}$ and, hence, of the original tree, and which is graceful by Theorem 2.12 and the generalized Δ-construction.

■

1.2.7 A construction using the matrix representation of graceful and strongly graceful trees.

The construction that we are going to present in this section is due to Thom Grace (see [18]). This construction, in some sense, fills a blank space. Namely, one of the most commonly used and most efficient tools of describing any kind of a graph (and a tree, in particular) is the adjacency matrix. Yet, there are very few (if any except this one) results in the area of graceful graphs that use the adjacency matrix for their construction or to prove results about gracefulness.

We will start with an observation, which is obvious from the definitions, but is very useful nevertheless.

Theorem 2.14. Let T be a tree on n vertices. T is graceful if and only if there is a labeling θ of vertices of T such that each diagonal[12] (except the principal one) of the adjacency matrix A of T under this labeling contains exactly one 1. Also, if such labeling θ has strength k, then the adjacency matrix A under θ can be represented as follows:

$$A = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix},$$

where X is a $(k+1) \times (n-k-1)$ submatrix.

It is now easy to describe the construction by Thom Grace. Let T be a graceful tree and let θ be a graceful labeling for T. Let $V(T) = \{v_0, v_1, \dots, v_{n-1}\}$ be the set of vertices of T

[12]Here and everywhere further in this section, when we say "diagonal," we mean "down diagonal."

and assume for simplicity that we assign names to vertices so that $\theta(v_i) = i$. Let $T_0, T_1, \ldots, T_{n-1}$ be strongly graceful trees with same number of vertices (but not necessarily the number of vertices in T). Furthermore, let $\theta_0, \theta_1, \ldots, \theta_{n-1}$ be the respective labelings for the trees above with the same strength, say k. We construct a tree T^* by attaching to the vertex v_i of T the vertex u of T_i with $\theta_i(u) = |V(T_i)| - 1$, where i ranges from 1 to n.

Theorem 2.15. (see [18]) The tree T^* built as constructed above is graceful.

Proof. Consider the following permutation of numbers from 1 to n[13]:

$$\pi(i) = \begin{cases} (n-1) - \lfloor \frac{i-1}{2} \rfloor & \text{if } i \text{ is odd,} \\ \frac{i}{2} - 1 & \text{if } i \text{ is even} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Let A_i be the adjacency matrix for T_i. Let X_i be the submatrix for A_i as described in Theorem 2.14. We set

$$Y_i = \begin{cases} X_{\pi(i)} & \text{if i is odd} \\ X^T_{\pi(i)} & \text{if i is even,} \end{cases}$$

where i runs from 1 to n. We aim to construct an adjacency matrix A^* of T^* which contains exactly one 1 on each diagonal except the principle one. That would be sufficient by Theorem 2.14.

We start out by putting Y_i's into M starting from the upper right corner.

We consider an example.

[13]This very important point is not pronounced in the original paper. Therefore, We consider it important to emphasize it here.

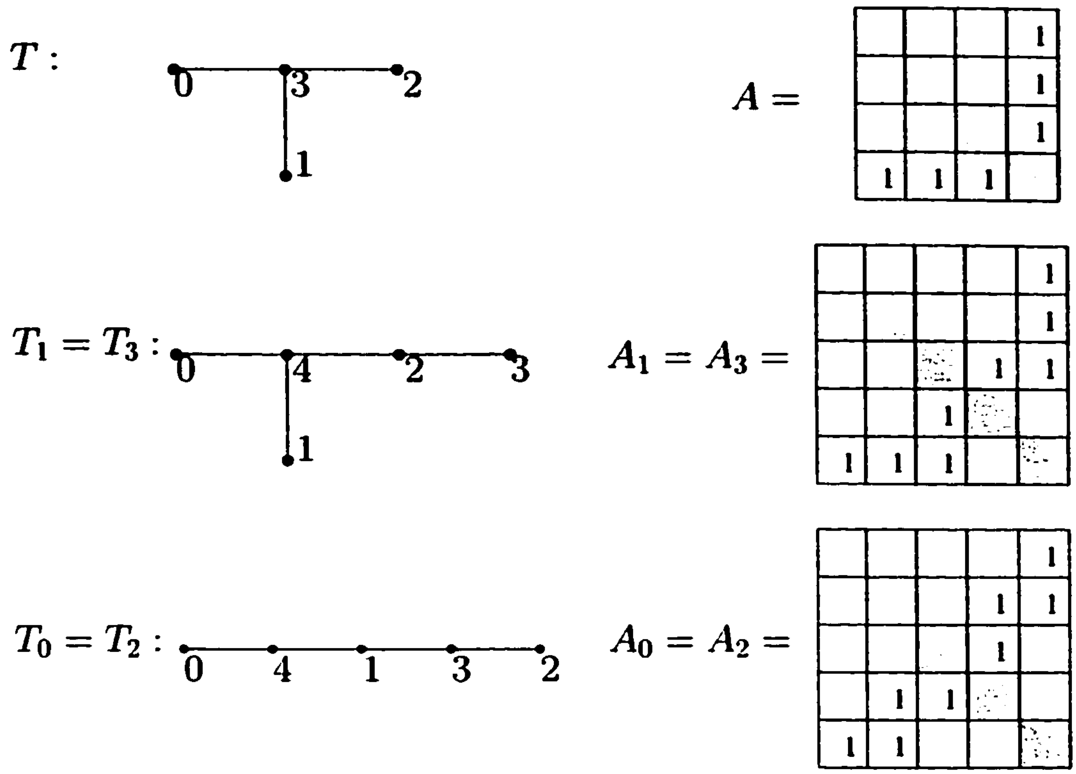


Figure 1.2.6: Trees to demonstrate the construction that uses the matrix representation.

In our example, $k = 2$ (recall Lemma 2.3) and $n = 4$. Therefore, we have: $Y_1 = X_3$, $Y_2 = X_0^T$, $Y_3 = X_2$, $Y_4 = X_1^T$. This takes care for the right upper part of the adjacency matrix of the resulting tree. Also, $Y_8 = X_3^T$, $Y_7 = X_0$, $Y_6 = X_2^T$, $Y_5 = X_1$

So, after we put all the Y_i's, we get the following matrix:

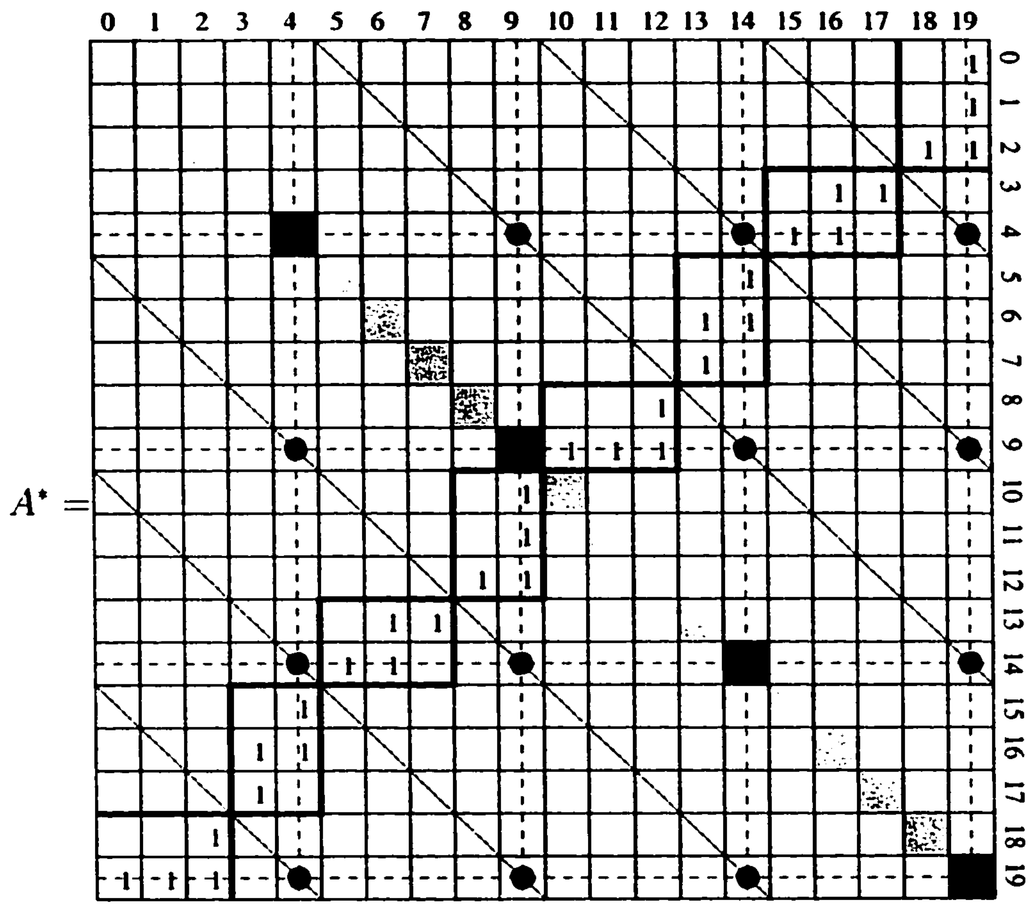


Figure 1.2.7: Obtaining the adjacency matrix of a bigger graceful tree.

It is clear that every diagonal except the main one and the ones marked by lines contain exactly one 1. Thus we only have to put a 1 to each of the marked diagonals. We have to do it in a manner that preserves the edges of T.

So, we look at the points of intersection of columns containing the last columns of each Y_i for odd i and rows containing the last rows of each Y_i for even i. Those rows and columns are drawn as dashed lines on the above picture. The points of intersection are drawn as filled circles and squares. Now, if we look at these points as correspondent to the positions in A, then, by putting 1's to repeat the pattern in A, we complete A^*. For our example, we put 1's corresponding to the pairs: $(4, 19)$, $(9, 19)$ and $(14, 19)$.

Here is the resulting tree for our example:

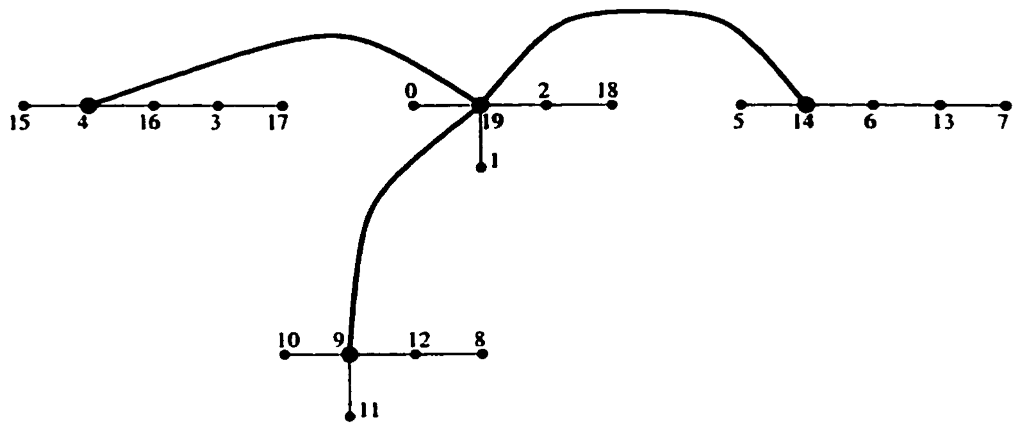


Figure 1.2.8: A graceful tree obtained with the construction that uses the adjacency matrix.

The labeling that we have got is a graceful labeling of the announced tree T^*, thus the proof is complete.

■

1.2.8 A further example.

Consider the following tree on $n = 19$ vertices.

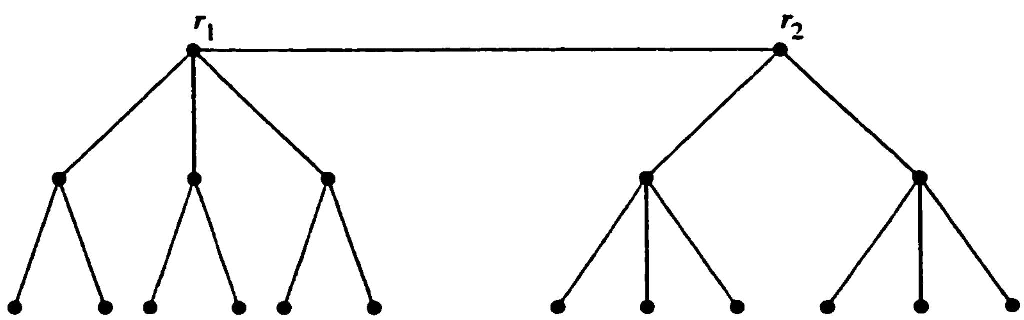


Figure 1.2.9: A tree of diameter 5.

We would like to obtain a graceful labeling of this tree. We note that the tree can be regarded as two trees joined by an edge (the long horizontal edge on the picture). We will refer to these subtrees as left and right. The left one has $n_1 = 10$ vertices and the right one has $n_2 = 9$ vertices. We also fix their roots: r_1 and r_2 respectively.

The right subtree is strongly graceful as a caterpillar by Theorem 1.10. The left subtree has diameter 4 and it is not a caterpillar; hence it is not strongly graceful by Theorem 1.12.

We will show that we cannot get a graceful labeling of our tree by Theorem 2.6 using its left and right subtrees.

We claim that the root of the left subtree cannot be labeled 3 in any graceful labeling of that subtree.

Assume to the contrary that we can label r_1 with 3 and complete a graceful labeling. Then since we have to get edge-labels 8 and 9, the placement of the vertex-labels 0, 1 and 9 is forced as follows.

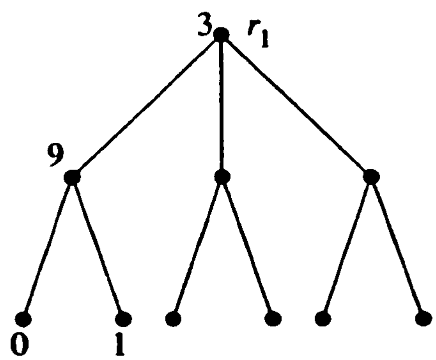


Figure 1.2.10: Illustration of the proof of a property of all graceful labelings of the left subtree of the tree in Figure 1.2.9.

However, there is no way we can get the edge-label 7 now. Thus the contradiction is obtained and the statement of our claim is verified.

As an exhaustive computer search showed[14], the only graceful labelings of the right subtree are as follows.

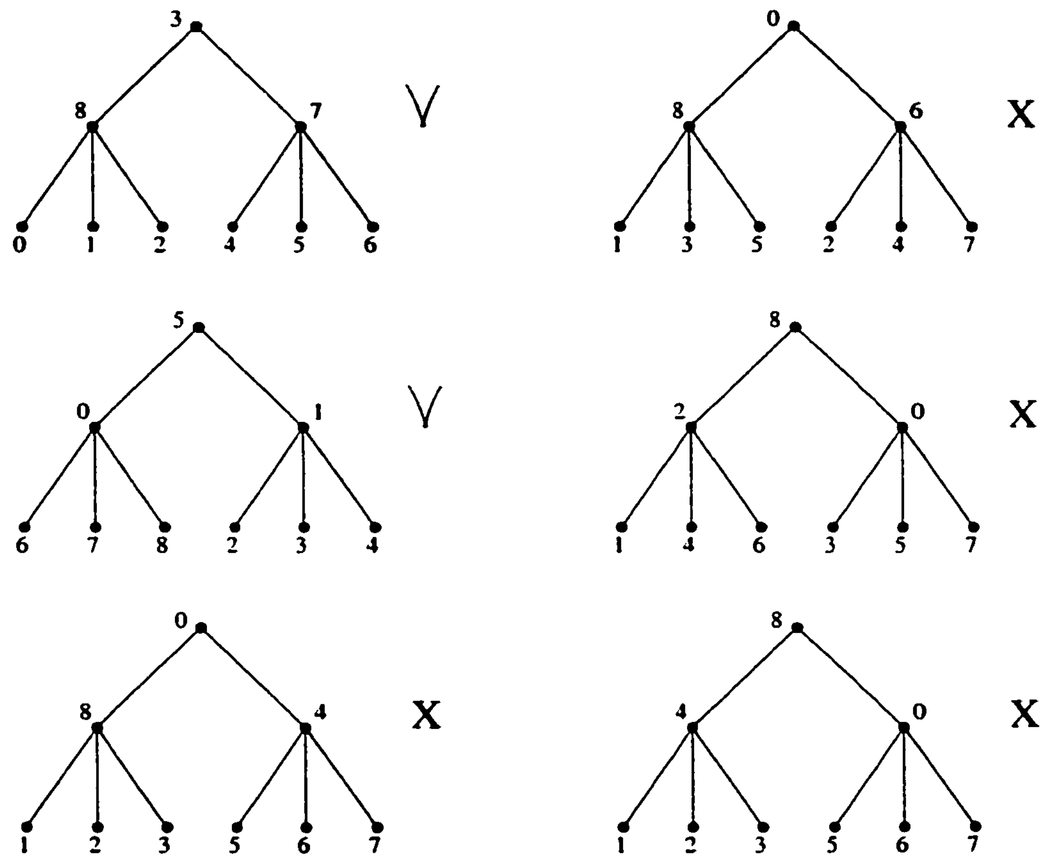


Figure 1.2.11: Graceful labelings of the right subtree of the tree in Figure 1.2.9.

The labelings marked with "X" are not strongly graceful. Thus we can have r_2 labeled either 3 or 5 in a strongly graceful labeling. The corresponding strengths (recall Lemma 2.3) are $k = 6$ and $k = 1$.

Now, if we try to build our big tree using Theorem 2.6, then the vertex r_2 will be labeled in the resulting tree with either 3 or $5 + n_1$, that is, it will be labeled either 3 or 15. We need to connect the two subtrees with the edge labeled n_1, that is,

[14]The code is available from www.cs.ualberta.ca/~goldenbe

with the edge labeled 10. Therefore, r_1 has to be labeled either 13 or 5. Now, let the original label of r_1 in the left subtree be $\theta_1(r_1)$. Then either $13 = \theta_1(r_1) + k + 1$, where $k = 6$, or $5 = \theta_1(r_1) + k + 1$, where $k = 1$. One leads to $\theta_1(r_1) = 6$; the other leads to $\theta_1(r_1) = 3$. Both are impossible.

This finishes the proof that we cannot get our tree by Theorem 2.6.

Using a heuristic computer search, the following labeling of our tree on 19 vertices was obtained.

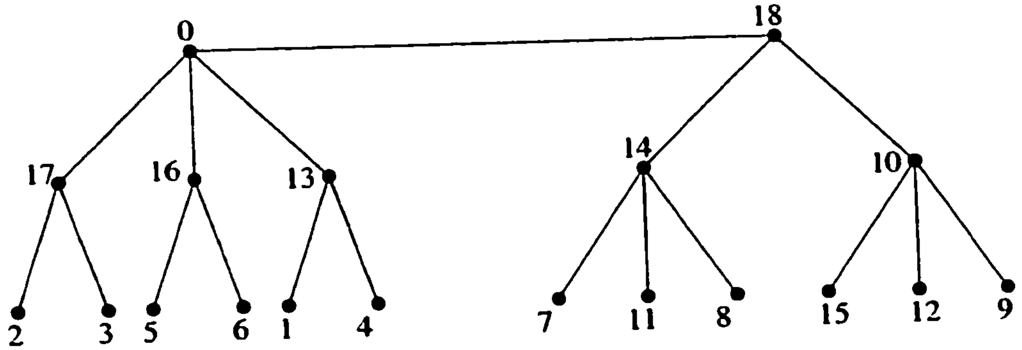


Figure 1.2.12: A graceful labeling for the tree in Figure 1.2.9.

Besides gracefulness, the above labeling has two nice properties:

(i) The "center edge" has the maximal edge-label. Such a labeling is called *m-edge-centered* (the term is introduced in [7]).

(ii) The edges of the right subtree have all the labels from 1 to 8; the edges of the left subtree have all the labels from 9 to 17.

It would be interesting to find conditions for two graceful trees such that some construction would give us a labeling with property (ii). Such a construction may lead to some new results.

1.3 Assigning particular labels and some well characterized classes of graceful trees.

In this section, we will survey some attempts to answer the question: "Given a tree, where can we assign a particular label so that it is possible to extend it into a graceful labeling?" We will show that answering this question can help us establish results on gracefulness. Also, some well known results about specific classes of graceful trees are described.

A remarkable result on this topic is due to Aleksander Rosa.

Theorem 3.1. (see [30]) Let T be a path on n vertices. Let v be any fixed vertex of T. If $n \neq 5$, then there exists a strongly graceful labeling θ of T with $\theta(v) = 0$.

Proof.

Let $m = n-1$ be the number of edges in T. Also, let us denote the vertices of T by $v_1, v_2, \ldots, v_n$, so $v_i v_{i+1}$ are the edges, where i ranges from 1 to m.

We first show that if θ is a labeling of T with strength k, then the following two statements hold:

(i) If m is even (say, $m = 2q$), then either $\theta(v_1) \leq q$ and $\theta(v_n) \leq q$, or $\theta(v_1) > q$ and $\theta(v_n) > q$. In this case, either $\theta(v_1) + \theta(v_n) = q$ and $q = k$, or $\theta(v_1) + \theta(v_n) = 3q$ and $q = k-1$, respectively.

(ii) If m is odd (say, $m = 2q-1$), then $\theta(v_1) \leq q$ and $\theta(v_n) > q$ or visa versa and $k = q-1$; in this case, $\theta(v_n) - \theta(v_1) = q$ or $\theta(v_1) - \theta(v_n) = q$ respectively.

Assume for now that $\theta(v_1)$ and $\theta(v_n)$ are less than k. Then let us consider the set X of all vertices of T such that for every $x \in X$ we have $f(x) > q$ and the set Y containing all the rest vertices except v_1 and v_n.

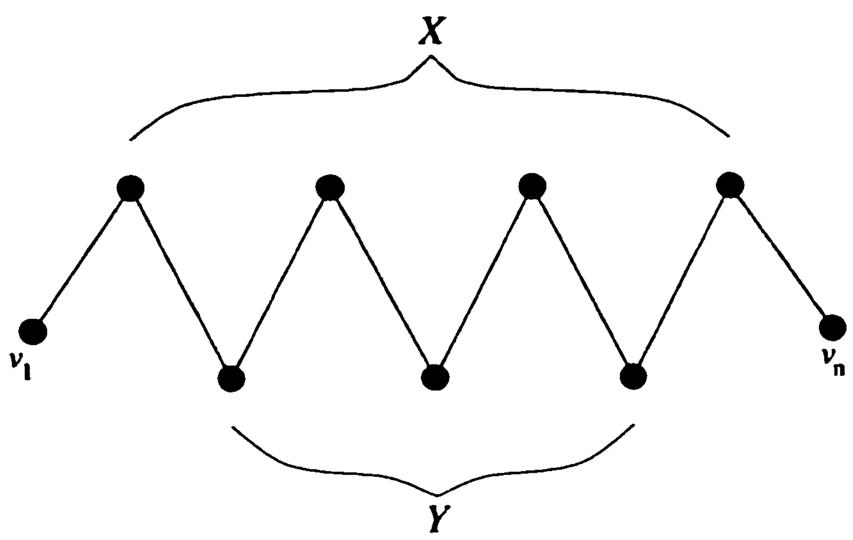


Figure 1.3.1: Making use of independent sets of a path.

Then by the definition of graceful labeling we have the following system of equations:

$$\begin{cases} \sum_{x\in X}\theta(x)+\sum_{y\in Y}\theta(y)+\theta(v_1)+\theta(v_n)=0+1+2+\ldots+m \\ 2\sum_{x\in X}\theta(x)-2\sum_{y\in Y}\theta(y)-\theta(v_1)-\theta(v_n)=1+2+\ldots+m \end{cases}$$

After multiplying the first equation by 2 and adding the resulting equation to the second one, we have:

$$\theta(v_1)+\theta(v_n)=3\sum_{i=1}^{m} i-4\sum_{x\in X}\theta(x)$$

Since $\theta(X) = \{q+1, q+2, \ldots, 2q\}$, we get that $\theta(v_1)+\theta(v_n)=q$, which is exactly what we wanted to show.

If $\theta(v_1)$ and $\theta(v_n)$ are greater than or equal to k, the argument is essentially the same. This completes the proof of (i). The proof of (ii) is similar.

We now prove the theorem. Denote the edge v_iv_{i+1} by e_i. We denote a strongly graceful labeling of T by θ_m^i if $\theta_m^i(e_i)=n-1$.

Assume for now that m is odd, so $m = 2q-1$. Then taking in consideration the symmetry of a path with respect to its center edge and the complementary labeling introduced in Section 1.2, it is enough to show the following statement.

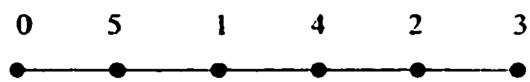
(iii) Strongly graceful labelings $\theta_m^1, \theta_m^3, \ldots, \theta_m^{q-1}$ exist if q is even and that $\theta_m^1, \theta_m^3, \ldots, \theta_m^{q-2}, \theta_m^{q-1}$ exist if q is odd.

We will use simultaneous induction on q to prove this and the following statement.

(iv) For every pair of positive integers whose difference is q there exists a strongly graceful labeling of T such that the end vertices of T get those two numbers as labels.

We first assert that (iii) and (iv) hold true for $q = 1, 2, 3$. We will only consider $q = 3$.

For (iii), the following diagram exhibits the labelings θ_5^1.



For (iv), 0 and 3 are the labels of the end-vertices of θ_5^1. By Lemma 2.1, the complementary labeling puts 2 and 5 on the end-vertices: Finally, 1 and 4 are the labels of the end-vertices in the following diagram:

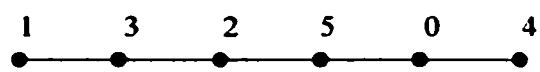


Figure 1.3.2: Justification of existence of certain graceful labelings for P_6.

Now, we perform the induction step. So, assume that the two statements hold for $q-1$. Rosa gives the following explicit way to obtain θ_{2q-1}^{q-1}:

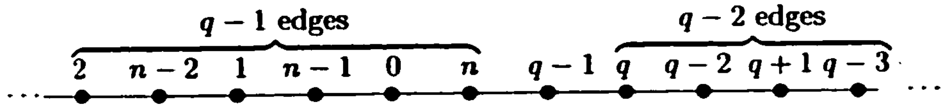


Figure 1.3.3: A strongly graceful labeling of P_{2q} with $(q-1)^{st}$ edge labeled m, for even q greater than 3.

Note that strength of the labeling in Figure 1.3.3 is $q-1$ (recall Lemma 2.3). The reverse labeling leads to θ^q_m.

Taking this into account, we only have to prove that there exist strongly graceful labelings $\theta^1_{2q-1}, \theta^3_{2q-1}, \dots, \theta^{2\lfloor\frac{q-1}{2}\rfloor-1}_{2q-1}$. So, we choose an odd number s between 1 and $\lfloor\frac{q-1}{2}\rfloor-1$ (say, $s=2t-1$) and find θ^s_{2q-1} corresponding to this number. By (iv), we know that for a path on $2q-2t$ vertices (call them $w_1, w_2, \dots, w_{2q-2t}$; the edges are w_iw_{i+1}) there exists a strongly graceful labeling (call it θ_1) with $\theta_1(w_1)=q-1$, $\theta_1(w_{2q-2t})=t-1$, and strength $q-t$. We easily obtain the desired strongly graceful labeling of the path on $2q-1$ edges by Theorem 2.6:

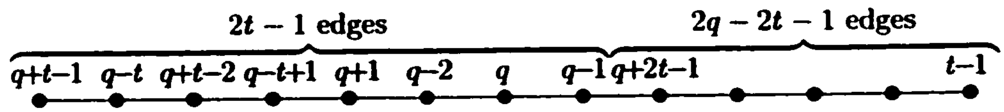


Figure 1.3.4: A strongly graceful labeling of P_{2q} with the desired (but not $(q-1)^{st}$) odd-positioned edge labeled m, for q greater than 3.

The picture above clearly gives us the labeling where the s^{th} edge is formed by vertices with labels q and $q - 1$. Hence it suffices to consider the edge complementary labeling to finish the proof for (iii) (assuming that (iv) holds). So, we are only left to finish the inductive proof for (iv). However, (iv) follows from $(q + t - 1) - (t - 1) = q$. Since s was arbitrary, $q+t-1$ and $t - 1$ represent all conceivable pairs of numbers whose difference is q.

If m is even, the argument is essentially the same.

■

Corollary 3.2. The only graceful labeling (up to symmetry) of the path on 5 vertices with the center vertex labeled 0 was shown in the proof of Lemma 1.9. Therefore, any vertex of any path may be assigned the label 0 under some graceful labeling.

Remark. For the sake of being complete here, we have to say that the only other paper that we found dealing with graceful labelings of paths is [13]. The authors of that paper came up with a method to enumerate all the graceful labelings of a given path, such that the end vertex is labeled 1. They also came up with a simple recursion for the number of such labelings.

We are in the position now to demonstrate how knowledge about labeling of particular vertices is applicable to resolve the question of gracefulness for some classes of trees.

The gracefulness of trees with maximum vertex degree 3 is still an open question. Following the spirit of the proof of the above theorem, one might try to prove the result by induction. Hence let us perform the base step of the induction by proving that all trees with exactly one vertex of degree 3 are graceful. This is equivalent to proving of the following theorem:

Theorem 3.3. (see [20]) Any of the leaves can be labeled 0 in some graceful labeling of a tree with at most 3 leaves.
Proof. The only non-trivial case is when a tree has exactly 3 leaf-vertices. Such a tree looks as follows:

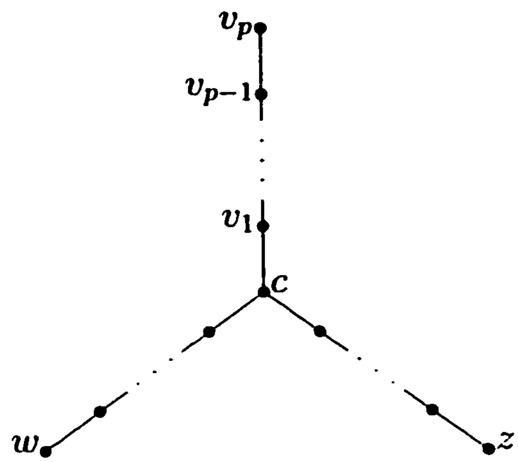


Figure 1.3.5: Canonical representation of a tree with at most three leaves.

Let us utilize the result by Aleksander Rosa proved above[15]. We apply the theorem to the path connecting the leaf-vertices w and z. By the theorem, this path can be assigned a graceful labeling, call it θ', such that the "center" vertex c of our tree gets the label 0, that is, $\theta'(c) = 0$. Let the remaining tail have p vertices: $v_1, v_2, \dots, v_p$ and suppose that we want to label v_p with the label 0. We do it by labeling the path $v_1, v_2, \dots, v_p$ strongly gracefully with strength k and v_1 labeled $k+1$ (we can achieve that by labeling v_p with either 0 or $p-1$ depending on the parity of p). We complete the proof using Theorem 2.6 and possibly Lemma 2.1.

■

[15] After We came up with this application, we found out that it was published in [20].

Corollary 3.4. A tree T with 3 end-vertices is strongly graceful, unless the distance between every pair of leaves in this tree is 5 (i.e. unless all the "tails" have length 2) .
Proof. The statement follows from theorems 3.1 and 3.3., and from Corollary 3.2.

■

Let us follow the authors of [20] and explore the next logical step – proving that all the trees with at most two vertices of degree 3 are graceful. In fact, a stronger result is proved in the paper.

Theorem 3.5. (see [20]) All trees with at most 4 end-vertices are graceful.
Proof. Let T be a tree with exactly 4 end-vertices. There are two possible cases.

(i) T has a vertex of degree 4. Then if at least one of the tails of T has length not equal to 2, then we easily complete the proof using the results that have already been discussed in this section and Theorem 2.4. If all the tails have length 2, then we provide a graceful labeling as follows:

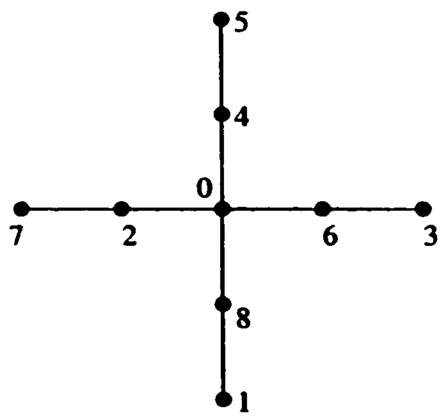


Figure 1.3.6: Graceful labeling of an exceptional tree with at most four leaves.

(ii) T has exactly 2 vertices of degree 3. The tree looks as follows:

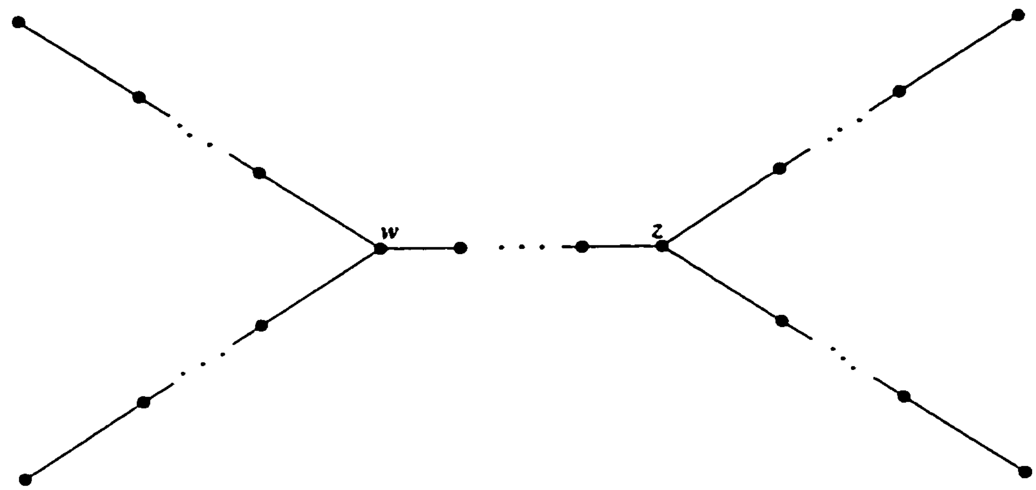


Figure 1.3.7: Canonical representation of a tree with two vertices of degree 3.

Now, if there is a tail (either incident upon w or incident upon z) with length not equal to 2, then we can complete the proof using the results that have been previously discussed in this section and Theorem 2.4.

Otherwise, the matters are more complicated. The authors of [20] came up with labelings depending on the equivalence class modulo 4 to which m (the number of edges of T) belongs. We give the pictorial representation for theses labelings below. From left to right, k is a positive integer such that $m = 4k$, $m = 4k + 1$, $m = 4k + 2$ or $m = 4k + 3$.

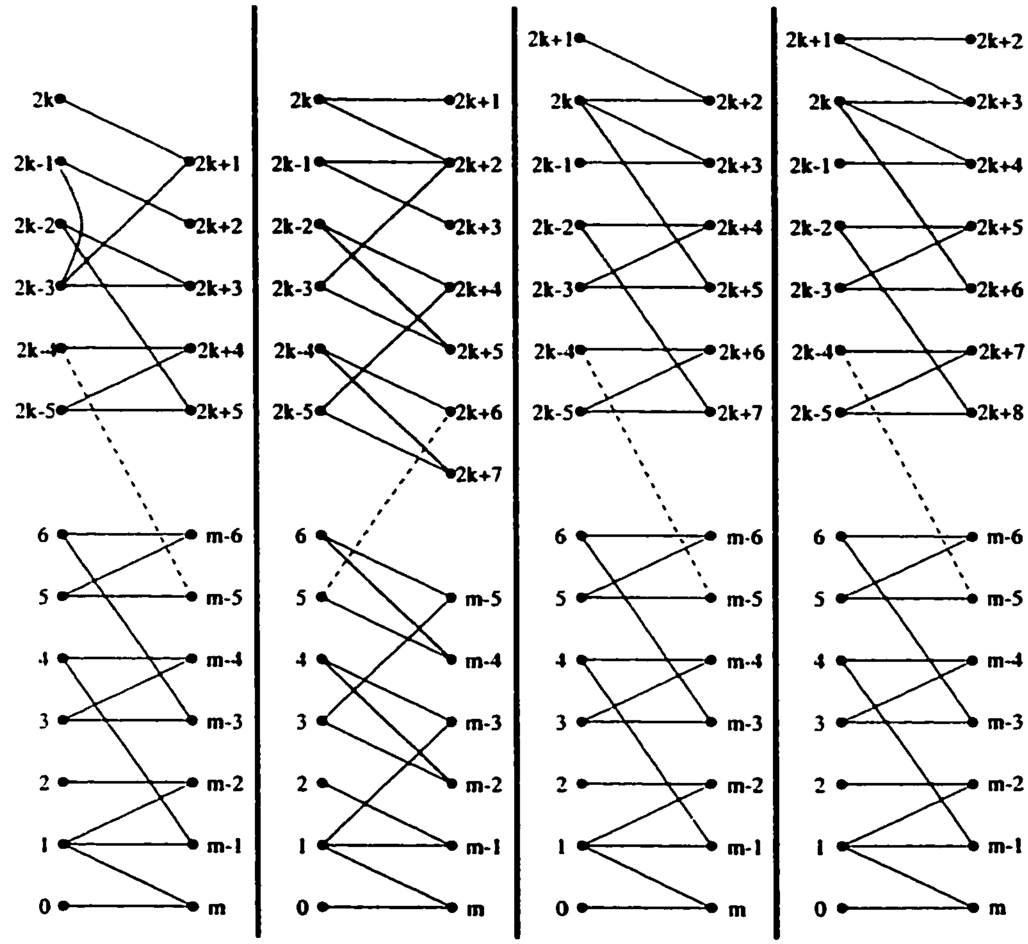


Figure 1.3.8: Graceful labelings for the exceptional cases for trees with at most four leaves.

■

This survey of important ideas in the area of the Graceful Tree Conjecture would not be complete without showing some classes of trees that were only recently proved to be graceful. One of the most important and recent papers establishing such results was written by Wen-Chin Chen, Hsueh-I Li and Yeong-Nan Yeh.

Theorem 2.6 of Section 2.1 will be our main tool for the rest of this section.

Definition 3.6. Consider a tree T consisting of a *distinguished vertex* that is joined by an edge to a leaf of each of the star trees $S_1, S_2, \ldots, S_l$ for some fixed positive integer l, where an i^{th} star S_i has exactly $i-1$ edges. Then T is said to be a *banana tree.*

This example shows a banana tree (with $l = 5$) labeled gracefully with strength 4. The labeling is obtained as in the proof of the next theorem.

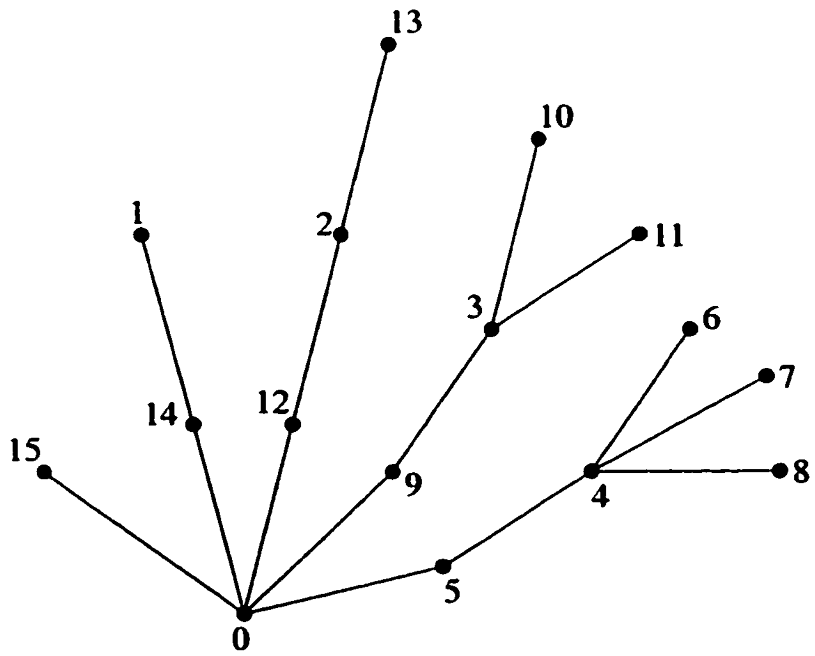


Figure 1.3.9: Graceful labeling for the banana tree with 5 branches.

Theorem 3.7. (see [8]) All banana trees are strongly graceful.

Proof. We proceed by induction on l. On the i^{th} step, we will obtain a strongly graceful labeling for the case $l = i$, with strength $i - 1$, and the distinguished vertex labeled 0. For $l = 1$ the result is evident. Assume that the desired labeling is obtained for $l = i - 1$.

We use Theorem 2.6 to join the obtained strongly graceful tree to the leaf of a gracefully labeled star tree with $i - 1$ edges (with the center vertex labeled 0) that is labeled 1. In the resulting tree, that vertex gets the label $1+(i-2)+1 = i$. So, the edge connecting the distinguished vertex with the mentioned leaf gets the label i, which is exactly the number of vertices in our star. Since the labeling we used for the star tree was strongly graceful, by the last remark to Theorem 2.6, the obtained labeling of the resulting tree is strongly graceful with strength $0 + (i - 2) + 1 = (i - 1)$, which completes the proof.

■

A related class of graceful trees (see [28]) is the class of *olive trees*.

Definition 3.8. An *olive tree* consists of a distinguished vertex joined by an edge to an end vertex of each of the l paths $P_1, P_2, \dots, P_l$, where the i^{th} path has exactly $i - 1$ edges.

Here is a class of trees, in which not every member is known to be graceful.

Definition 3.9. A tree T is called a *lobster* if removal of all its leaves leaves a caterpillar.

We already know that all paths are graceful. If we append leaves to a given path, then the resulting tree is a caterpillar and it is also graceful.

This already looks like an induction approach. The next step would be to add leaves to the leaves of a given caterpillar, resulting in a lobster. If we can prove that all lobsters are graceful, it would serve as a basis for an inductive argument towards proving the GTC.

For some reason, the conjecture that all lobsters are graceful stands out (this is surprising because this conjecture was not in place before the GTC was proposed). The conjecture was made by Bermond in 1979 (see [3]) and was not proved or disproved yet. Therefore, partial results are of interest.

As an example of a graceful subclass of lobsters, we give a proof of gracefulness of the class of *firecrackers*, which was defined in [8].

Definition 3.10. Consider a tree T comprised of a *distinguished path*, each of whose vertices is adjacent to the center vertex of exactly one star tree. Such a tree is called a *firecracker*.

Theorem 3.11. (see [8]) All firecrackers are graceful.

Proof. Let T be a firecracker on n vertices. Let the path component of T consist of $l-1$ edges $w_i w_{i+1}$, where i ranges from 1 to $l-1$. We consider 2 cases.

Case 1. l is even. We prove that there is a strongly graceful labeling θ of T with strength k, such that $\theta(w_1) = 1$ and $\theta(w_l) = k+2$.

We proceed by induction on n. The following picture gives the desired labeling for $l = 2$.

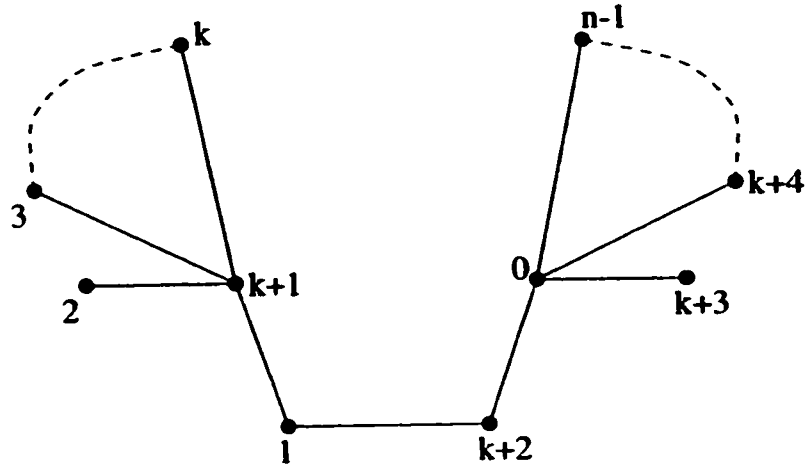


Figure 1.3.10: Graceful labeling for a firecracker with the distinguished path of length 2.

Assume that we have the desired labeling for the case $l = i-2$. We obtain the labeling for the case $l = i$ with the distinguished path $w_1 w_2 \ldots w_i$. To do that, we split this firecracker into two, one of which get vertices $w_1, w_2, \ldots, w_{i-2}$ (and the attached star trees), while the other one gets w_{i-1} and w_i. By induction, we know that the first firecracker admits a strongly graceful labeling with w_1 labeled 1, and w_{i-2} labeled $k_1 + 2$, where k_1 is the strength of the labeling. Also, the second firecracker admits a strongly graceful labeling with w_{i-1} labeled 1, and w_i labeled $k_2 + 2$, where k_2 is the strength of the labeling. We complete the proof by joining the vertices v_{i-2} and v_{i-1} using Theorem 2.6.
Case 2. l is odd. We prove the gracefulness of our firecracker. The case $l = 1$ is trivial. If l is greater than 1, then we split the firecracker into two, such that one of the resulting firecrackers has the parameter $l = 1$ (the only vertex of its path is labeled 1 and the center vertex of its star component is labeled 0). Then we completer the proof by applying the result of the first case and Theorem 2.6.

■

As an example, the following firecracker was gracefully labeled using the construction in the proof.

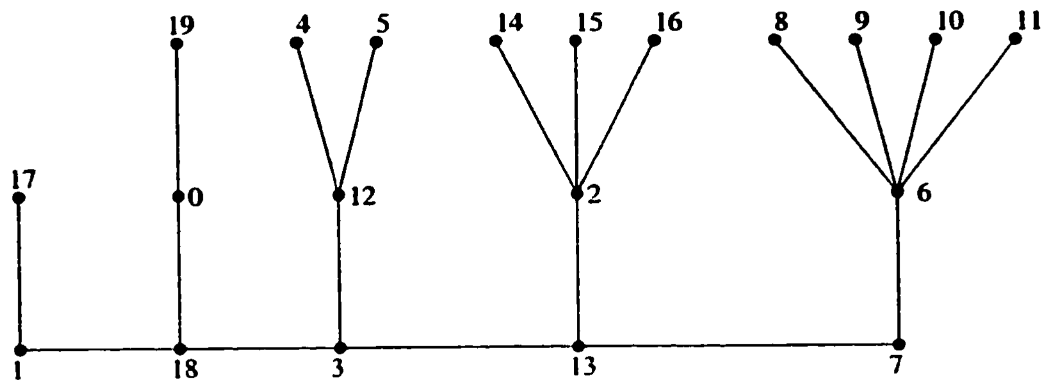


Figure 1.3.11: A graceful firecracker on 20 vertices.

Although the next result that we cite here does not span a class with infinite number of trees, it is still very important, since it empirically justifies the GTC. By computation, the authors of [1] were able to show that all trees on up to 27 vertices are graceful.

1.4 The Transplant Lemma and related results.

In this section, we will examine a very powerful way to transform a given tree with a graceful labeling into another tree while preserving gracefulness.

1.4.1 The Transplant Lemma.

Lemma 4.1. (*the Transplant Lemma*, see [21]) Let T be a tree and θ be a graceful labeling for T. Also, let x, y, u, v be vertices of T such that u, v are leaves and ux, vx are edges, where u and v are not necessarily distinct. If $\theta(ux) = |\theta(v) - \theta(y)|$ and

$\theta(vx) = |\theta(u) - \theta(y)|$, then we can obtain a new graceful tree T' by replacing the edges ux, vx in T with the edges uy, vy. Then θ is also a graceful labeling for T'.

Proof. the set of vertices of T' is same as the set of vertices of T. So, θ is a labeling for T'. Also, the two constrains on θ tell us that the set of edge labels will also be preserved by the transformation. That is, uy will get the labels of vx and vy will receive the label of ux in T. Thus θ is a graceful labeling for T'.

■

Below is an example where $u \neq v$.

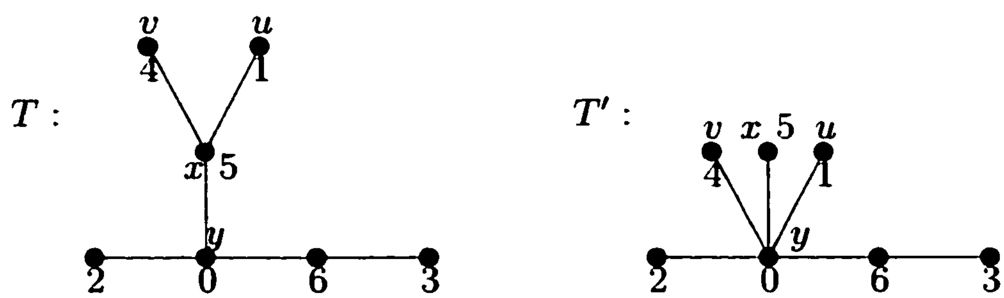


Figure 1.4.1: Moving a pair of leaves using the Transplant Lemma.

Below is an example where $u = v$.

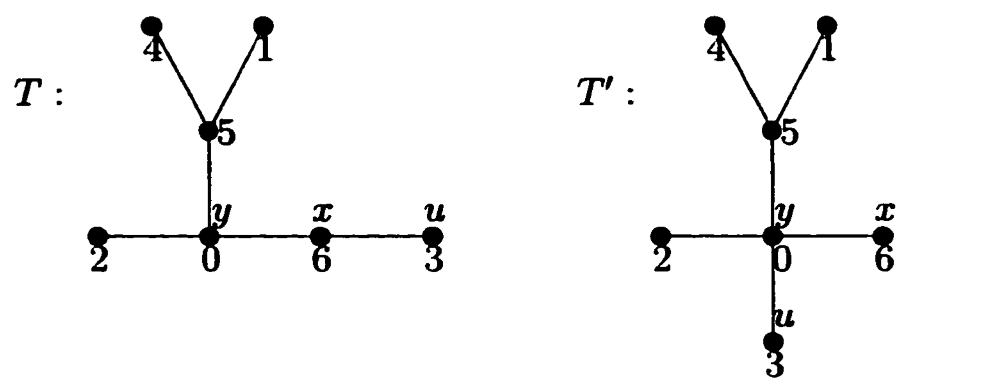


Figure 1.4.2: Moving a single leaf using the Transplant Lemma.

Remark. Note that we have either $\theta(x) < \theta(u)$ and $\theta(v) < \theta(y)$, or $\theta(y) < \theta(u)$ and $\theta(v) < \theta(x)$.

A convenient way to check the hypothesis is given by the following result.

Lemma 4.2. In the setting of Lemma 4.1.,

$$\theta(x) + \theta(y) = \theta(u) + \theta(v).$$

Proof. Consider the case when $\theta(x) < \theta(u)$, $\theta(v) < \theta(y)$ (the other case is similar). Then $\theta(ux) = |\theta(v) - \theta(y)|$ means that

$$\theta(u) - \theta(x) = \theta(y) - \theta(v) \ (*).$$

Similarly, from $\theta(vx) = |\theta(u) - \theta(y)|$ means that

$$\theta(v) - \theta(x) = \theta(y) - \theta(u) \ (**).$$

By adding (*) and (**), we easily get the result. In particular, if $u = v$, then we have $\theta(x) + \theta(y) = 2\theta(u)$.

■

In the following subsections we illustrate the power of the Transplant Lemma.

1.4.2 Trees of diameter 4.

This subsection is devoted to a proof of the following result.

Theorem 4.3. (see [21]) Trees of diameter 4 are graceful[16].

Proof. As in the proof of Theorem 1.12, a tree of diameter 4 is represented as follows.

[16] That all trees of diameter 4 are graceful was conjectured by A.Rosa in 1982 (see [29]). C. Huang, A. Kotzig and A. Rosa (see [20]) proposed that settling this was a keystone in proving the GTC. In 1989, the conjecture was finally proved by Zhao (see [36]). He gave an explicit labeling for any given tree of diameter 4. The proof is entirely technical. The proof presented here is due to Dejun Jin, Fanhong Meng and Jingong Wang (in Chinese, see [21]).

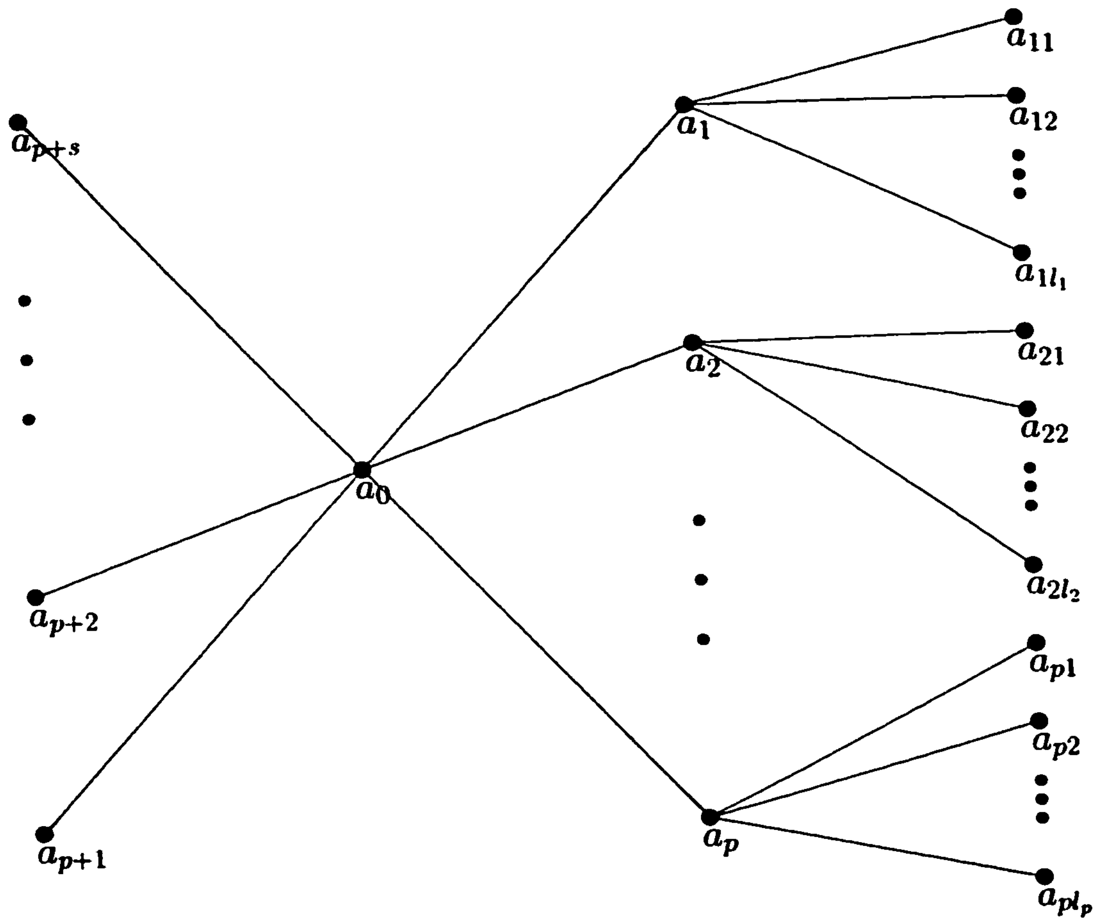


Figure 1.4.3: Canonical representation of a tree of diameter 4.

Here s is the number of leaf neighbors of the center vertex a_0, and p is the number of its non-leaf neighbors of a_0. The degree of a_0 is $p + s$.

Suppose we are given a tree T as above. We use the Transplant Lemma to prove that T is graceful.

The general idea of the proof is to start from the n-star (where $n = |V(T)|$, $m = n - 1 = |E(T)|$). It has a trivial graceful labeling θ with the center vertex labeled 0. We then apply the

Transplant Lemma several times until the given tree T is obtained. The center vertex will serve as a_0 of T. The leaf labeled $n-1$ will serve as a_1.

Whenever we move a vertex, the label goes with it. Instead of saying "move the vertex with label ℓ," we simply say "move label ℓ.'

Assume for now that the degree of a_0 is odd, i.e. that $p+s$ is odd. The reason for this will become clear later. Let $p+s=2\lambda+1$. We consider three cases.

Let the degrees of $a_1, a_2, \ldots, a_p$ be $l_1+1, l_2+1, \ldots, l_p+1$, respectively.

Case 1. $l_1, l_2, \ldots, l_p$ are odd.

From the n-star we apply the Transplant Lemma to move from a_0 to a_1 the labels

$$\lambda+1, n-(\lambda+2), \lambda+2, n-(\lambda+3), \ldots, \lambda+k, n-(\lambda+k+1),$$

where k is the largest value for which $\lambda+k \leq n-(\lambda+k+1)$.

Thus a_0 is left with exactly $2\lambda+1=p+s$ neighbors with labels

$$n-1, 1, n-2, 2, n-3, \ldots, \lambda, n-(\lambda+1).$$

These vertices will serve as $a_1, a_2, \ldots, a_{p+s}$, respectively.

Next we move from a_1 to a_2 the middle block of consecutive labels:

$$\lambda+\frac{l_1+1}{2}+1, \lambda+\frac{l_1+1}{2}+2, \ldots, n-(\lambda+\frac{l_1+1}{2}+1).$$

It is easy to see that the sum of the first and the last labels in this block is n, which is equal to $1+(n-1)$. The same can be said about any two labels taken symmetric about the middle of the block. Thus the hypothesis of the Lemma 4.2 is satisfied. Also, we had $n-(2\lambda+1)$ leaves with a_1. After moving $n-2(\lambda+\frac{l_1+1}{2})$ of them, we have exactly l_1 left with a_1.

Similar reasoning justifies moving the following labels from a_2 to a_3:

$$\lambda + \frac{l_1 + l_2}{2} + 1, \lambda + \frac{l_1 + l_2}{2} + 2, \ldots, n - 1 - (\lambda + \frac{l_1 + l_2}{2} + 1).$$

In general, we move from a_k to a_{k+1} the labels

$$\lambda + \frac{\sum_{i=1}^{k} l_i + 1}{2} + 1, \lambda + \frac{\sum_{i=1}^{k} l_i + 1}{2} + 2, \ldots, n - (\lambda + \frac{\sum_{i=1}^{k} l_i + 1}{2} + 1)$$

for odd k. If k is even, we move instead

$$\lambda + \frac{\sum_{i=1}^{k} l_i}{2} + 1, \lambda + \frac{\sum_{i=1}^{k} l_i}{2} + 2, \ldots, n - 1 - (\lambda + \frac{\sum_{i=1}^{k} l_i}{2} + 1).$$

Now, let us see why this works. Note that $a_1, a_2, \ldots, a_p$ are labeled so that $\theta(a_i) + \theta(a_{i+1})$ are alternately n and $n-1$. So, when we move leaves from a_2 to a_3, the parity of the sum of the moved pairs of labels is different from that of the pairs moved from a_1 to a_2 and so on.

Suppose that n is even. Then since $p+s$ is odd, the number of leaves moved from a_0 to a_1 is even. Also, a_1 is labeled $(n-1)$ and a_2 is labeled 1; so, the sum of the labels of a_1 and a_2 is even. Hence we can move a consecutive block of an odd number of leaves from a_1 to a_2 and leave the desired odd number of leaves with a_1. Now, the sum of the labels of a_2 and a_3 is odd. So, we can move an even number of leaves from a_2 to a_3 and leave the desired number of leaves with a_2, and so on. The situation is similar if n is odd.

Case 2. $l_1, l_2, \ldots, l_p$ are even.

We proceed as in Case 1. Start with the same n-star. and move from a_0 to a_1 the labels

$$\lambda + 1, n - (\lambda + 2), \lambda + 2, n - (\lambda + 3), \ldots, \lambda + k, n - (\lambda + k + 1)$$

where k is the largest value for which $\lambda + k \leq n - (\lambda + k + 1)$.

Thus a_0 is left with exactly $2\lambda + 1 = p + s$ neighbors with labels

$$n-1, 1, n-2, 2, n-3, \dots, \lambda, n-(\lambda+1).$$

These vertices will serve as $a_1, a_2, \dots, a_{p+s}$, respectively.

Now, we move leaves from a_1 to a_2. Note that now an even number of leaves remain with a_1. Move the labels

$$\lambda+\frac{l_1}{2}+1, \lambda+\frac{l_1}{2}+2, \dots, \frac{n}{2}-1, \frac{n}{2}+1, \dots, n-(\lambda+\frac{l_1+1}{2}+1).$$

The labels that we move are not consecutive. Namely, we leave the leaf with the label $\frac{n}{2}$ (the middle one) with a_1. It is easy to see that the hypothesis of Lemma 1.2 is satisfied. Also, as in the first case, exactly l_1 leaves remain with a_1.

Now, we move the following labels from a_2 to a_3:

$$\lambda+\frac{l_1+l_2}{2}, \lambda+\frac{l_1+l_2}{2}+1, \dots, \frac{n}{2}-2, \frac{n}{2}+1, \dots, n-1-(\lambda+\frac{l_1+l_2}{2}).$$

In general, we move from a_k to a_{k+1} the labels

$$\lambda+\frac{\sum_{i=1}^{k} l_i}{2}+1, \lambda+\frac{\sum_{i=1}^{k} l_i}{2}+2, \dots, \frac{n}{2}-\frac{k+1}{2},$$

$$\frac{n}{2}+\frac{k+1}{2}, \dots, n-1-(\lambda+\frac{\sum_{i=1}^{k} l_i}{2})$$

for odd k. If k is even, we move instead

$$\lambda+\frac{\sum_{i=1}^{k} l_i}{2}, \lambda+\frac{\sum_{i=1}^{k} l_i}{2}+1, \dots, \frac{n}{2}-\frac{k}{2}-1,$$

$$\frac{n}{2}+\frac{k}{2}, \dots, n-1-(\lambda+\frac{\sum_{i=1}^{k} l_i}{2}).$$

Let us see why this works.

Now, let us see why this works. Note that $a_1, a_2, \dots, a_p$ are labeled so that $\theta(a_i) + \theta(a_{i+1})$ are alternately n and $n-1$. So, when we move leaves from a_2 to a_3, the parity of the sum of the moved pairs of labels is different from that of the pairs moved from a_1 to a_2 and so on.

We know that n is even. Since $\theta(a_0) + \theta(a_1) = n - 1$, we can move the desired block with even number of labels from a_0 to a_1. Let these labels be $\alpha, \alpha + 1, \ldots, \alpha + 2\beta - 1$. We know that $\theta(a_1) + \theta(a_2) = n$. Since $\alpha + (\alpha + 2\beta - 1) = n - 1$, we cannot move α to a_2. Therefore, we leave α and $\alpha + \beta$ with a_1. From the rest, we move an even number of labels starting from the pair $(\alpha + \beta - 1, alpha + \beta + 1)$ to a_2. We have a similar situation when we move labels from a_2 to a_3. The only difference is that we leave behind the largest label in the block of labels that we moved to a_2. In general, we necessarily leave behind one label from the boundary of the just moved block (smallest and largest boundary alternate) and the label that is closest to the middle in the rest of the block.

Case 3. $l_1, l_2, \ldots, l_w$ are odd and $l_{w+1}, l_{w+2}, \ldots, l_p$ are even.

We start as in Case 1 and proceed according to the scheme of the first case until we reach a_{w+1}. So, we have a consecutive block of an even number of labels (namely $l_{w+1} + l_{w+2} + \ldots + l_p$ labels) at a_{w+1}. So, we can switch to the scheme of Case 2 and finish the construction of T.

This completes the proof for the case when $p + s$ is odd.

Now, assume that $p + s$ is even.

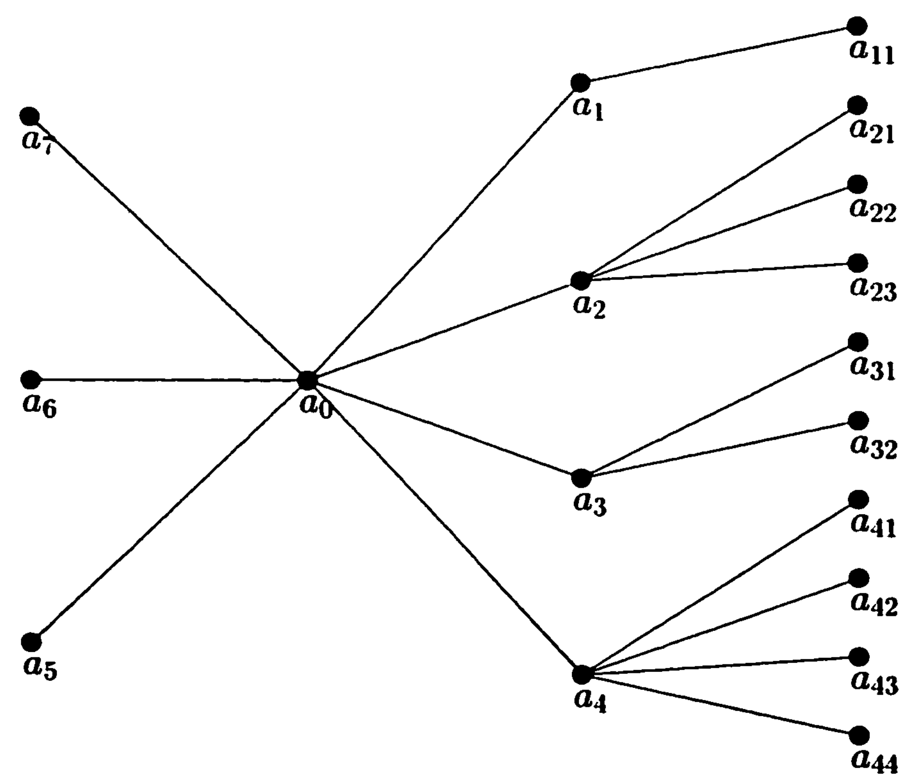
Suppose that $s \neq 0$. Set aside one vertex from the first column in our representation of T, say a_{p+s}. We apply the described scheme to what has remained of T. Note that according to our scheme a_0 gets label 0 and we get all edges labeled from 1 to $(n-2)$. So, if we give a_{p+s} the label $(n-1)$, then we will get a graceful labeling of T.

Now suppose that $s = 0$. Set aside one of the branches from the right part of our picture of T. For instance, let us set aside vertices $a_p, a_{p1}, a_{p2}, \ldots, a_{pl_p}$. We apply the scheme to the remaining vertices (there are $n - l_p - 1$ of them). Now, give the label $n - l_p - 1$ to a_p and labels $-1, -2, \ldots, -p$ to the vertices $a_{p1}, a_{p2}, \ldots, a_{pl_p}$. After we raise all labels of the vertices of T by p we get a graceful labeling.

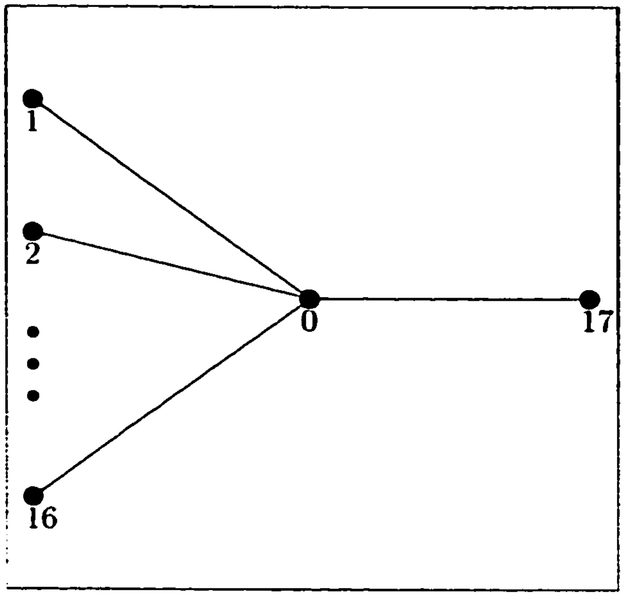
This completes the proof.

■

Example. Consider the tree T below.



We get a graceful labeling in the following way.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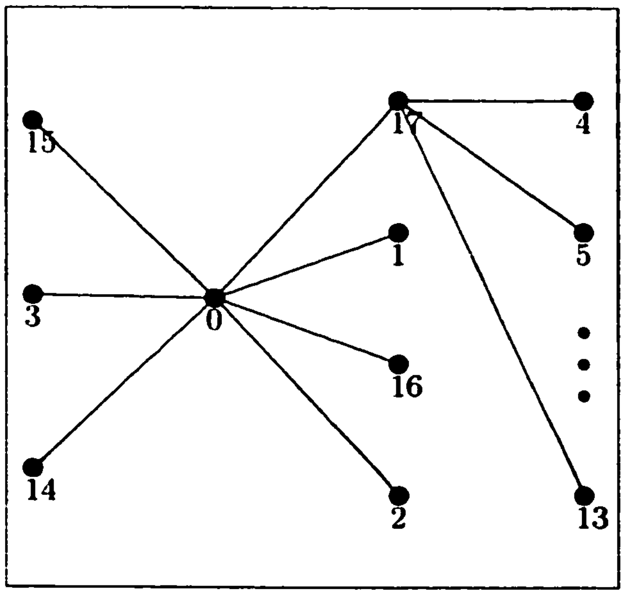
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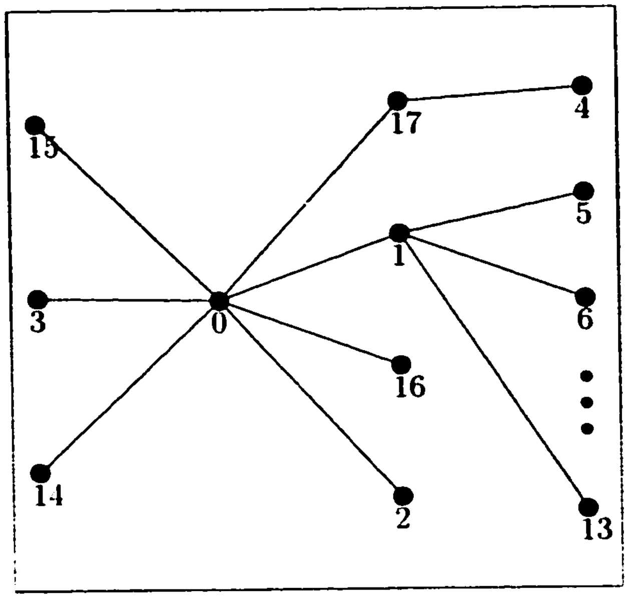

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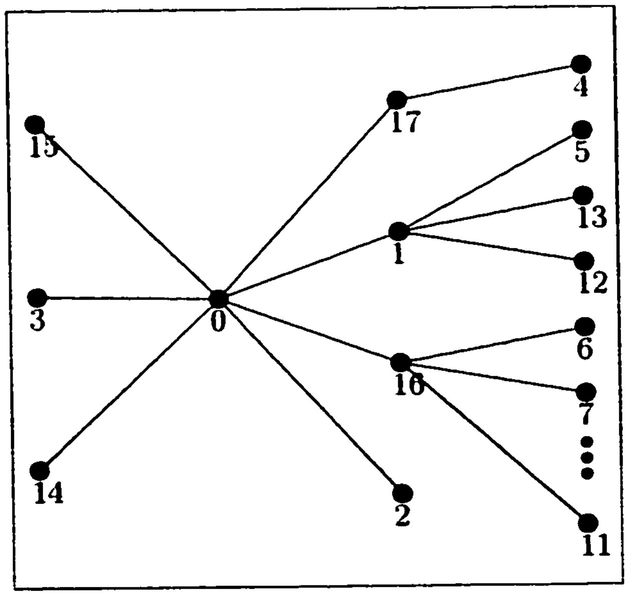

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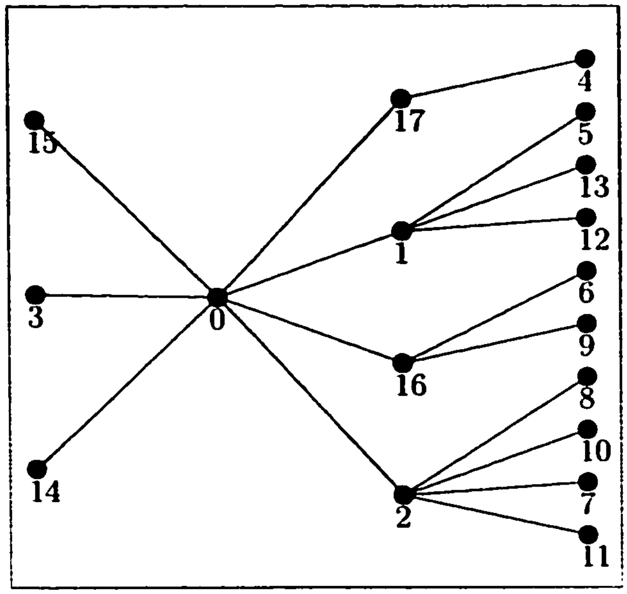


Figure 1.4.4: An example of a graceful labeling of a tree of diameter 4.

1.4.3 Trees of diameter 5.

A natural question is: "Can we generalize the result of the above theorem?" Although we were not able to prove gracefulness of all trees of diameter 5, the approach used in proving Theorem 2.3 can be used for settling some particular cases.

Let the canonical representation of trees of diameter 5 be similar to that used for trees of diameter 4 with an extra level. Note that this can only happen within one branch, say the one headed a_1. We order the vertices $a_{11}, a_{12}, \dots, a_{1l_1}$ so that the vertices of this set with even degree have smaller indices. So, our tree looks as follows:

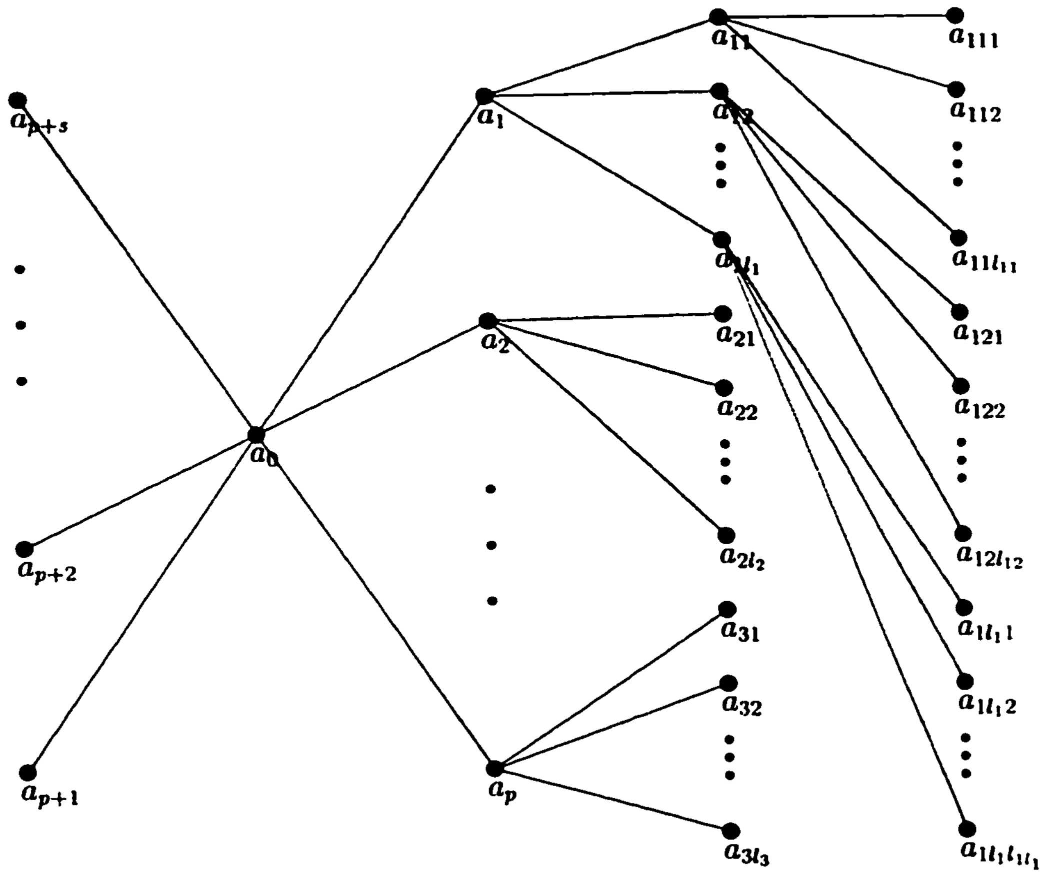


Figure 1.4.5: Canonical representation of a tree of diameter 5.

Theorem 4.4. Let T be a tree of diameter 5 in the canonical representation. If all l_i's are odd, then T is graceful.

Proof. Assume for now that $s = 0$ and that p is odd. Also, assume without loss of generality that $a_{11}, a_{12}, \dots, a_{1l_1}$ are ordered from the top down so that the vertices with odd number of leaf-neighbors go first.

We proceed using the scheme described in the proof of the main theorem (we ignore the leaves with triple indices so far). Since p is odd, a_p gets the label $(n-1) - \lfloor \frac{p}{2} \rfloor = n - \frac{p+1}{2}$. Also, $a_{11}, a_{12}, \dots, a_{1l_1}$ get labels

$$\lambda + 1, \lambda + 2, \dots, \lambda + \frac{l_1 + 1}{2}, (n-1) - \lambda - 1,$$

$$(n-1)-\lambda-2, \dots, (n-1)-\lambda-\frac{l_1-1}{2},$$

where $\lambda = \frac{p+1}{2}$.

We assign labels to $a_{11}, a_{12}, \dots, a_{1l_1}$ in the following order:

$$\lambda+1, (n-1)-(\lambda+1), \lambda+2, (n-1)-(\lambda+2), \dots, \lambda+\frac{l_1+1}{2}-1,$$

$$(n-1)-(\lambda+\frac{l_1+1}{2}-1), \lambda+\frac{l_1+1}{2}.$$

After we have moved leaves from a_{p-1} to a_p, we are going to move leaves from a_p to a_{11}. Note that the sum of labels of a_p and a_{11} is $(n-\frac{p+1}{2})+\lambda+1 = (n-\frac{p+1}{2})+\frac{p-1}{2}+1 = n$. Since p is odd and l_i are odd, $l_1+1_2+\ldots+l_p+1$ is even.

Now, we are going to move exactly $n-(l_1+l_2+\ldots+l_p+p+1)$ leaves to a_{11}. Hence if n is even, then we move an odd number of leaves, which is both what we want to do and what we can do by the Transplant Lemma. Also, we move a consecutive block of labels.

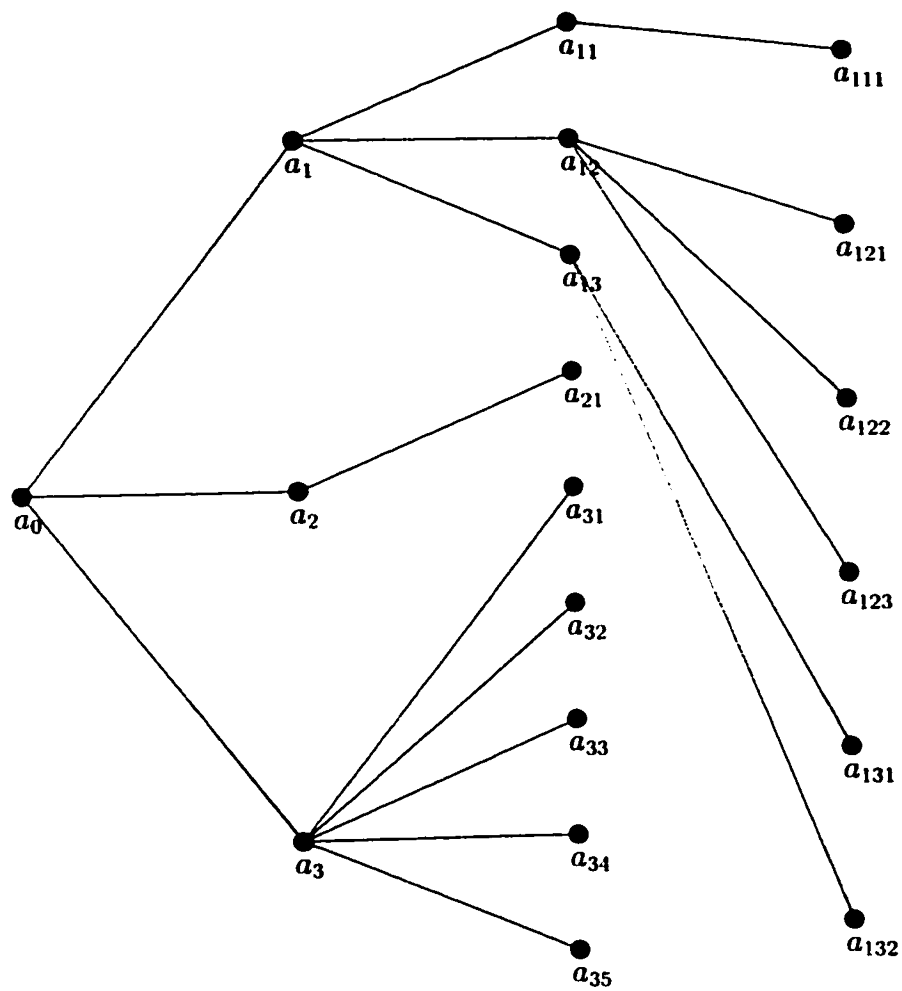
Note that $\theta(a_{1i})+\theta(a_{1(i+1)})$ are alternately n and $n-1$. Hence we can complete the labeling as in Theorem 4.3.

If $s \neq 0$, then we label the leaf-neighbors of a_0 with the largest labels.

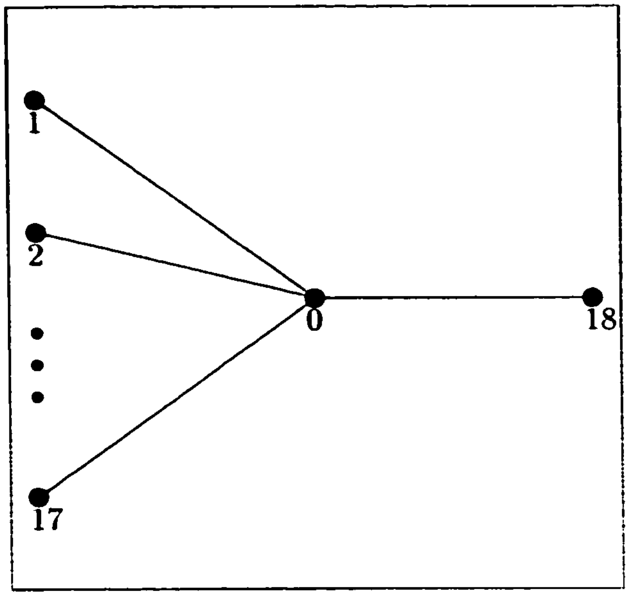
Suppose now that p is even. We can deal with this case just as we did in Theorem 4.3.

■

Example: Let T be a tree of diameter 5 as follows:



We get a graceful labeling of T in the following way: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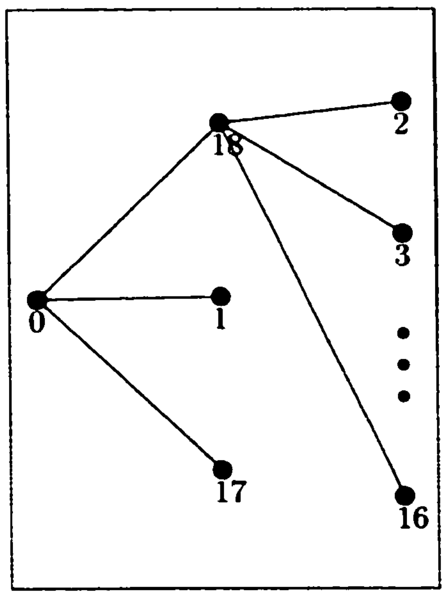
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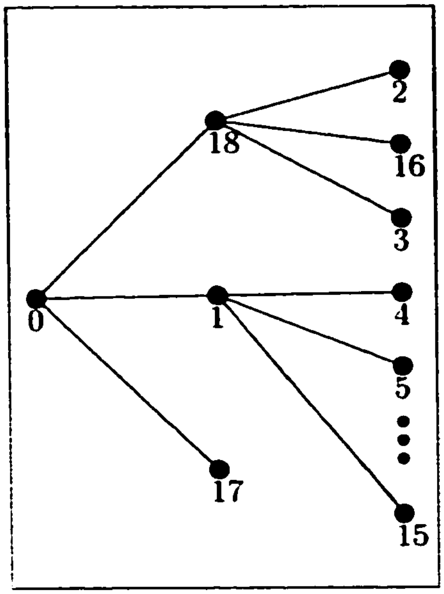
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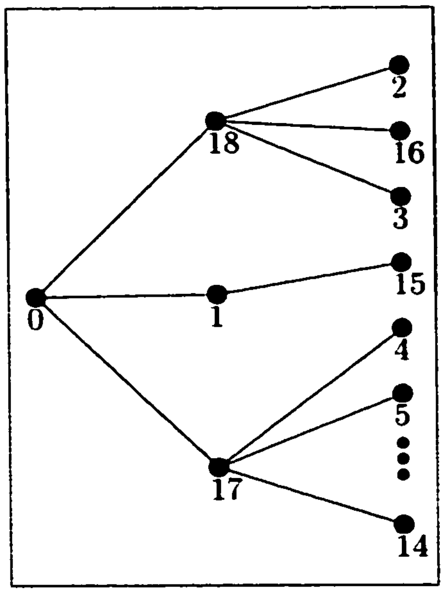
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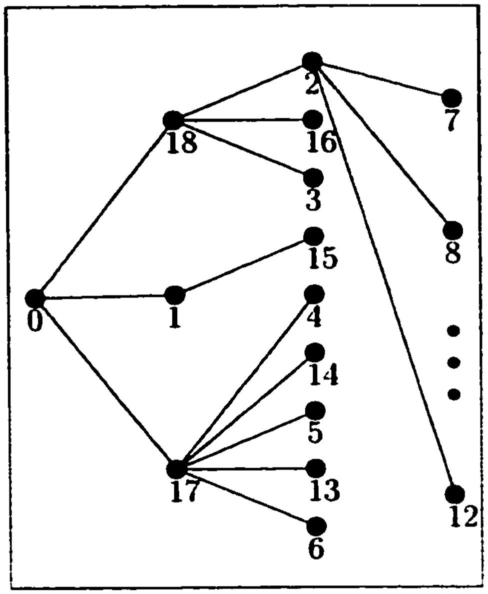
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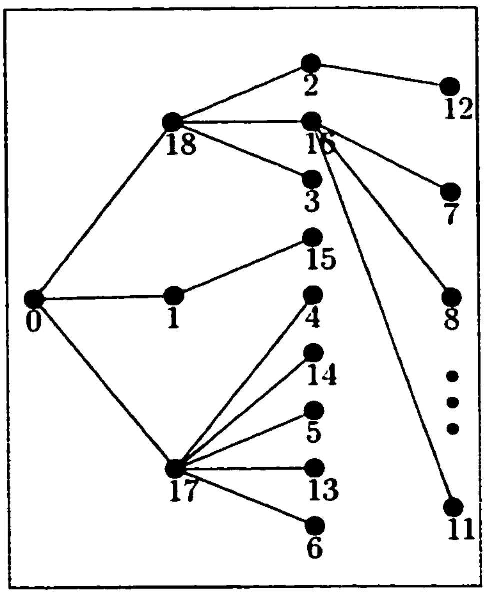
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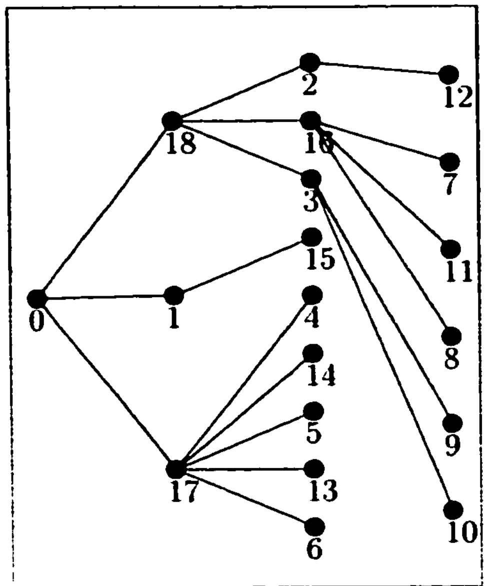


Figure 1.4.6: A class of trees of diameter 5 is graceful.

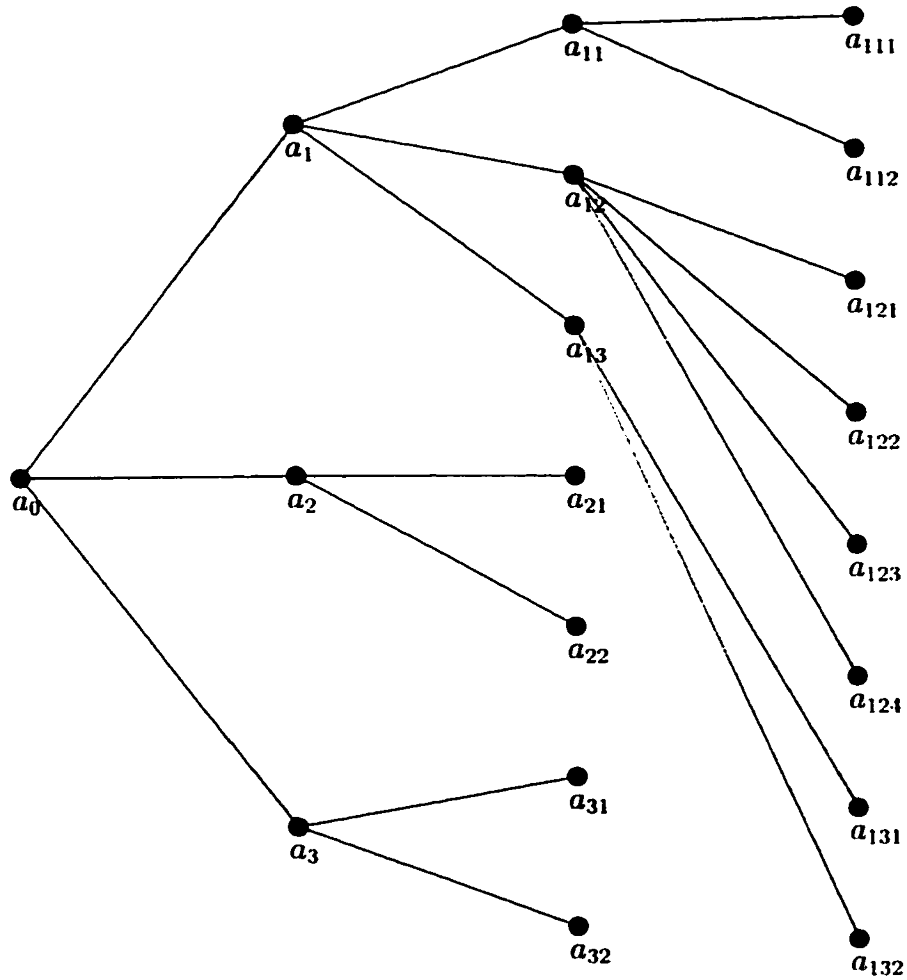
Theorem 4.5. Let T be a tree of diameter 5. If, in the canonical representation, l_1 is odd and all the l_{1j}'s are even, then T is graceful.

Proof. The argument is similar to that of Theorem 2.4. We omit the details.

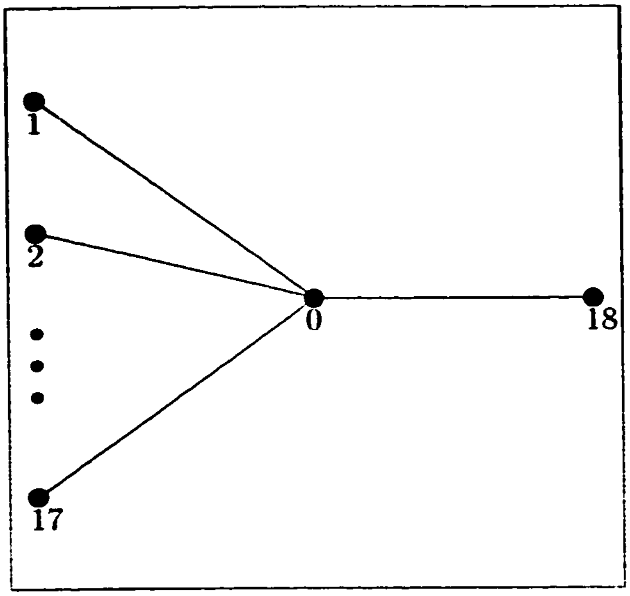
■

Note that there are no restrictions on $l_2, l_3, \ldots, l_p$.

Let us demonstrate this on the following example:



We construct a graceful labeling in the following way: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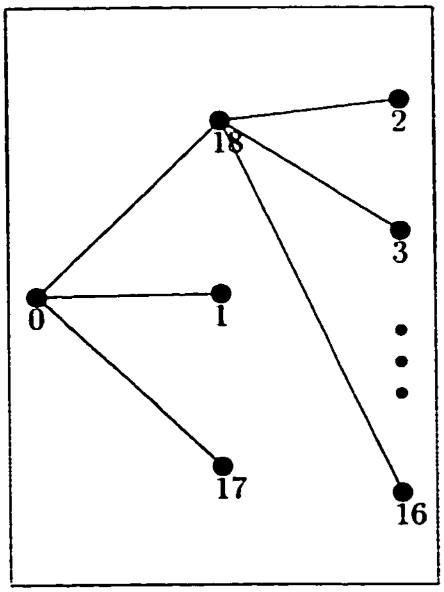
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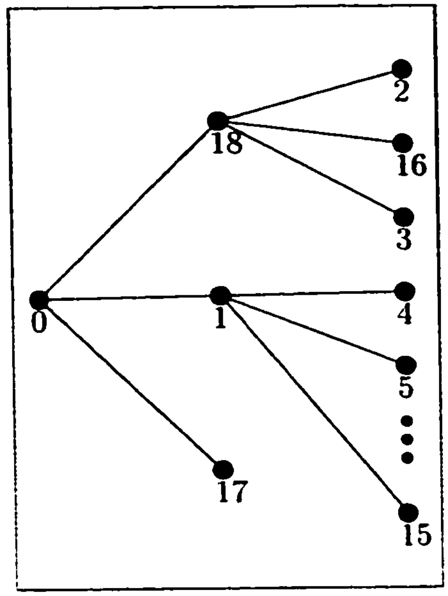
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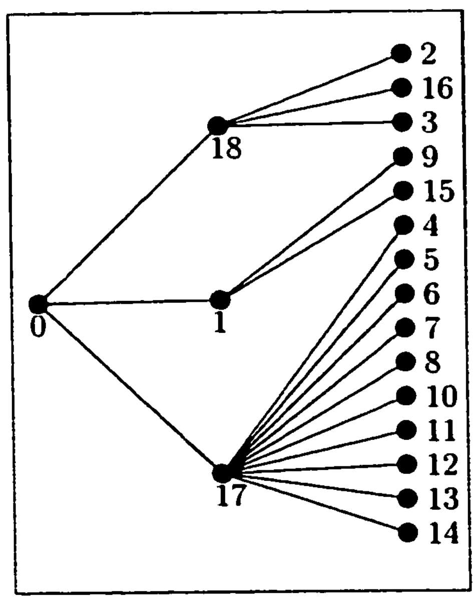
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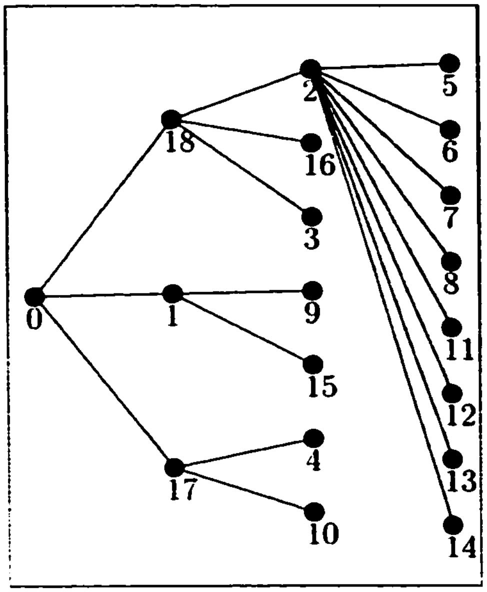
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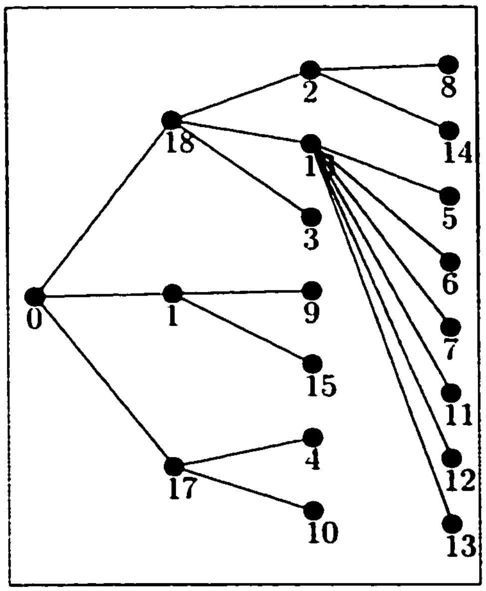
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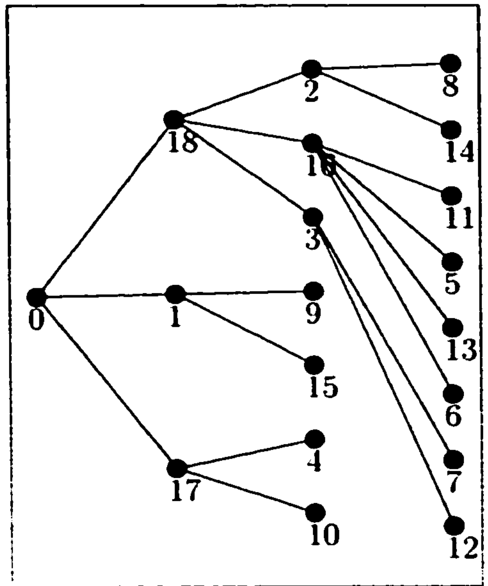


Figure 1.4.7: Another class of graceful trees of diameter 5.

1.4.4 Experimental demonstration of the power of the Transplant Lemma

The "Transplant Lemma Game" starts with an n-star T_0, along with one of its labelings θ_0. The ordered pair (T_0, θ_0) is the only element in level 0. The ordered pair $(T_0, \overline{\theta_0})$ is placed in a reserve list.

To construct level l for $l \geq 1$, apply the Transplant Lemma in every possible way to each (T, θ) in level $l-1$. Let (T', θ') be obtained.

(i) If (T', θ') has already appeared in an earlier level or the reserve list, discard it.

(ii) If neither (T', θ') nor $(T', \overline{\theta'})$ has appeared anywhere, put (T', θ') in level l and append $(T', \overline{\theta'})$ to the reserve list.

The game ends when we have an empty level. This means that we have generated all trees and their graceful labelings obtainable via the Transplant Lemma from the n-star.

The following table summarizes results of the Game for $n = 4, 5, 6, 7, 8, 9, 10, 11$. The number in the third column is the sum of the numbers of graceful labelings obtained this way (up to complementary labeling).

Number of vertices	Number of trees	Number of graceful labelings	Trees obtained by the Transplant Lemma Game
4	2	2	2
5	3	6	3
6	6	18	6
7	11	76	11
8	23	299	23
9	47	1550	47
10	106	8816	106
11	235	57833	235

Table 1: Empirical results about the Transplant Lemma.

Please see the Appendix A for discussion of the algorithmic and implementation issues.

Remark: The first attempt in the same direction was made by Anton Kotzig in 1984 (see [27]). He considered taking a gracefully labelled tree and replacing one edge with another while preserving gracefulness. Kotzig was able to prove that this transformation involving an edge with label 1 is always possible. He also conjectured that all graceful labelings of a given tree can be obtained from any other graceful labeling of the same tree by applying a series of such transformations. He was able to show by hand that his conjecture is true for trees on up to 6 vertices.

1.5 Concluding Thoughts.

In this section, we will give a brief overview of the approaches surveyed in this chapter and try to assess usefulness of each of the approaches as far as proving (or disproving) of the Graceful Tree Conjecture is concerned.

At the best of our knowledge, all trees on up to 27 vertices are graceful(see [1]). On the other hand, in order to disprove the conjecture, we have to:

(i) find (or simply guess) a candidate tree;
(ii) show that the candidate tree does not admit any graceful labeling.

It is very improbable that a tree with a nice structural property will do as a candidate; this lacking of structure excludes a mathematical proof in (ii) above with high certainty. Furthermore, since our candidate tree must have at least 28 vertices and because all the algorithms for exhaustive search of graceful labelings so far run in exponentional time, a proof of the negative result by computation does not seem possible unless our algorithm is improved by new results on assigning particular labels (such a result could dramatically reduce the search space). Thus if the GTC is not true, we have very scarce chances to find that out for sure. Therefore, let us hope for the best and continue looking for positive results.

The approach of the second section is interesting because it results in constructing trees with large number of vertices, large diameter, large maximal vertex degree, and sometimes with all of those properties simultaneously. This gives us yet more ground for believing in the GTC, despite the fact that gracefulness of trees on 28 vertices is still a conjecture. The gracefulness of trees of diameter 5 is not proved, nor is the gracefulness of trees with maximal vertex degree 3. Also, some of the constructions proved to be useful for an approach discussed in the third section. Although we were not able to come up with new constructions, we showed a labeling with properties that suggested that there were other constructions to look for.

Section 3 deals with two approaches.

As we mentioned before, finding conditions that restrict the usage of particular labels for certain vertices may help us to speed up exhaustive searches for graceful labelings. The approach was shown to be a very useful tool for establishing some classes of graceful trees.

Proving gracefulness of infinite families of trees may eventually lead to settling the GTC. A systematic way for doing that was started by Rosa when he proved gracefulness of paths and caterpillars. However, the next step (lobsters) is difficult. The union of classes of trees known to be graceful so far do not nearly approach the class of all trees. Gracefulness of caterpillars remains one of the most general results in this area.

The approach of Section 4 looks most promising. The Transplant Lemma connects all the graceful trees by an invisible thread of the simple transformation. It has already led to a relatively simple proof of gracefulness of trees of diameter 4, which was an open question for decades.

Our experiment suggests that the number of graceful labelings that we can get by applying the lemma is much larger than the number of trees, which makes it highly probable that we can get a graceful labeling for every tree.

Undoubtedly, the next logical step is to prove gracefulness of trees of diameter 5. We showed that the proof of gracefulness for trees of diameter 4 extends naturally to prove gracefulness of a family of trees of diameter 5.

Another approach is to come up with a rule for generating gracefully labeled trees with the Transplant Lemma (probably, a restricted version of the Transplant Lemma Game) and then identify the class of trees obtained by applying such a rule.

Chapter 2

Polyomino Rectification.

2.1 Introduction

In this chapter, we explore the rectification problem, that is, determining which rectangular boards can be tiled by a copy of a polyomino.

The notion of *polyomino* has a long but unclear history. However, its defining moment was the paper by Solomon W. Golomb in 1954 (see [15]). We repeat its definition.

Definition 1.1. A *polyomino* is a finite set of squares of an infinite checkerboard, connected "edge to edge." If a polyomino consists of n squares, then it is said to be an *n-omino*.

For small $n = 1, 2, 3, 4, 5$, the n-ominoes are called *monominoes*, *dominoes*, *trominoes*, *tetrominoes* and *pentominoes*[1] respectively.

The *Rectification Problem* can be formulated as follows:

Problem 1.2. Given a polyomino, find the dimensions of all finite rectangular boards that can be packed (i.e. covered completely without overlap) with copies of this polyomino.

To approach this problem, let us make some easy observations first.

[1] Pentomino is a registered trademark of Solomon W. Golomb.

Lemma 1.3. Suppose that an n-omino β is given. Let B be any board packable by β. Then

(i) The number of squares of B is divisible by n. In particular, if n is prime, then B has dimensions of the form $kn \times m$ (i.e. the length of one of the sides of B is divisible by n).

(ii) Any board B' that can be packed by B can be packed by β as well.

In the terminology of NDD, we say that β divides B if B is packable by β.

Definition 1.4. A board B packable by a polyomino β is called *prime* with respect to β if B is not packable by any combination of smaller boards which are packable by β. Otherwise, B is called *composite* with respect to β.

To solve Problem 1.2 for a particular polyomino, it is sufficient to find all the prime boards for this polyomino.

In the remaining sections of this chapter, we solve Problem 1.2 for all the n-ominoes, $n = 1, 2, \ldots, 5$.

2.2 Packing rectangular boards with monominoes, dominoes and trominoes.

There is one monomino and one domino. They are 1×1 and 1×2 rectangles respectively. Trivially, the monomino can pack any rectangle. Also, the domino can pack any rectangle with even number of squares, that is, any $2k \times m$ rectangle.

There are only two trominoes as shown on the following picture. They are called the I-tromino and the V-tromino respectively.

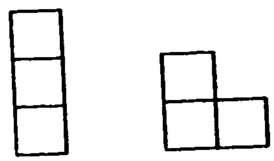


Figure 2.2.1: The two trominoes.

The I-tromino is itself its only prime board. It can pack an $n \times m$ board if and only if either $3|n$ or $3|m$.

The V-tromino has two prime boards as shown below (see [24]).

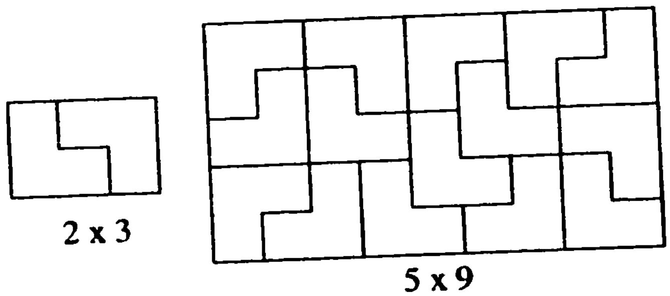


Figure 2.2.2: Packing prime boards for the V-tromino.

We have to show that there are no other primes. Suppose that a board $k \times m$ is packable and assume, without loss of generality, that k is divisible by 3.

If m is even, then our board is composite, since it can be packed by the copies of 2×3.

Assume that $m \geq 3$ is odd. If k is even, then the first three columns can be packed by the copies of 2×3, while the remaining part can be packed by the copies of 3×2. Now, suppose that k is odd. We certainly cannot get another prime for $k = 3$ or, symmetrically, for $m = 3$. (by a simple case study). For $k \geq 9$ and $m \geq 5$, we can place a copy of the 5×9 to one of the corners of our board and pack the rest with the copies of 2×3.

Hence 2×3 and 5×9 the only prime boards for the V-tromino. Therefore, a $k \times m$ board is packable by this tromino if and only if the following three conditions hold:

(i) 3 divides km;
(ii) both k and m are greater than 1;
(iii) $k \neq 3$ if m is odd and vice versa.

2.3 Packing rectangular boards with tetrominoes.

The only 5 tetrominoes are shown on the picture below. The letters are assigned for convenience and reflect the shape of the tetrominoes.

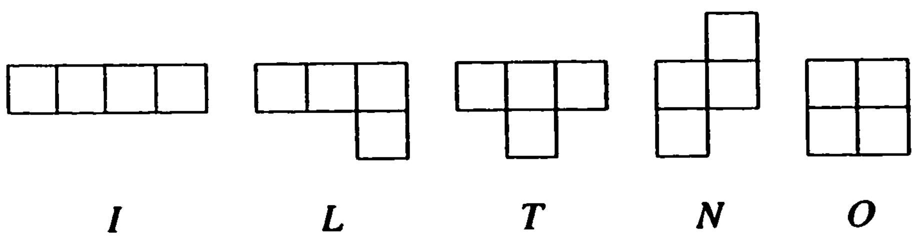


Figure 2.3.1: The five tetrominoes.

2.3.1 Packing with the N-tetromino

It is easy to see that the N-tetromino cannot pack any rectangular board. Note that we have to fill all the corners somehow. We can fill a corner only as shown below.

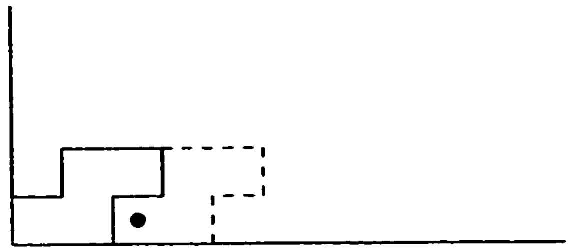


Figure 2.3.2: Packing with the N-tetromino.

However, we have only one choice for filling the square marked with the dot (this choice is shown by the dashed lines). It is now obvious that we cannot fill the bottom row of our board.

2.3.2 Packing with the O and the I-tetrominoes

The O-tetromino is the only (and trivial) prime board with respect to itself. Hence only boards $2k \times 2m$ can be packed with it.

The I-tetromino is the only (and trivial) prime board with respect to itself. Hence only boards $4k \times m$ can be packed with it.

However, we still need a justification that a $2k \times 2m$ board cannot be packed by the I-tetromino if both k and m are odd. This result is a special case of a theorem of de Bruijn (see [11]). We assign a number to each square of our board. A square on the i^{th} row and j^{th} column is assigned $i + j (\text{mod } 4)$. Then our rectangle can be split into four regions as in this example:

2	3	0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2	3	0
0	1	2	3	0	1	2	3	0	1
1	2	3	0	1	2	3	0	1	2
2	3	0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2	3	0

Figure 2.3.3: A $2k \times 2m$ board cannot be packed by the I-tetromino if both k and m are odd.

Since one of the sides of each of the regions except the right bottom one is divisible by 4, we have an equal number of zeros, ones, threes and fours in each of these three regions. This equality does not hold in the last region. The justification we sought follows from the fact that each I-tetromino fills exactly one square with each of 0, 1, 2, and 3.

2.3.3 Packing with the L-tetromino

There are two prime boards for the L-tetromino, which can be packed as shown below.

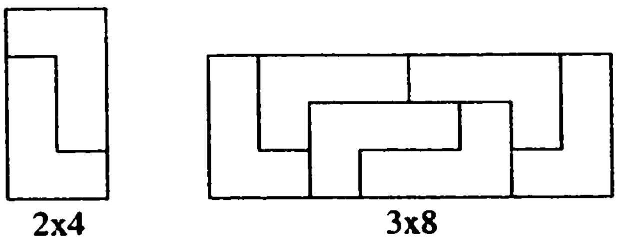


Figure 2.3.4: Prime boards for the L-tetromino.

We show that these are indeed the only two primes. The proof is due to Klarner (see [22]).

Suppose that a $k \times m$ board is packable by the L-tetromino. We claim that km is divisible by 8 (that this product is divisible by 4 is obvious). Assume, without loss of generality, that m is even. We color the columns black and white alternatively starting from white. The number of squares of each color is surely even. Since each copy of our tetromino fills an odd number of squares of each color, we have that the total number of pieces must be even, which completes the proof of our claim, so km is divisible by 8.

Assume, since $km \equiv 0 (\text{mod } 4)$, that m is divisible by 4. If k is odd, then m is divisible by 8 and we are certain to pack our board with our to prime boards if $k \geq 3$ (for $k = 1$, the board is unpackable). If k is even, then our board is certainly composite. Thus we have completed the proof and 2×4 and 3×8 are the only prime boards for the L-tetromino.

Hence a $k \times m$ board can be packed with the L-tetromino if and only if km is divisible by 8 with $k, m \geq 2$.

2.3.4 Packing with the T-tetromino

The only prime board for the T-tetromino is 4×4, which can be packed as shown below.

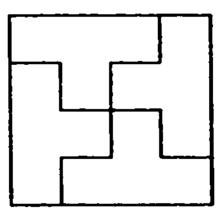


Figure 2.3.5: Packing 4×4 board with the T-tetromino.

Thus only boards $4k \times 4m$ are packable with this tetromino.

However, we still have to prove that there are no other prime boards. The argument, due to D. Walkup (see [33]), is quite difficult. We give a detailed exposition.

First we introduce some convenient notions.

We restrict our attention to lattice points (a, b) where $0 \leq a \leq k$ and $0 \leq b \leq m$.

A *segment* is a line of unit length that connects two of these lattice points. Except for points on the boundary of the $k \times m$ board, there are four segments leading from each of the lattice points; those leading up or to the right are called *outer segments* and those leading down or to the left are called *inner segments.*

By $[x, y]$ we will understand the unit square whose left bottom corner is the point (x, y).

We classify lattice points as follows. A lattice point (x, y) is

(i) an α-point if $x \equiv y \equiv 0 (\bmod 4)$;

(ii) a β-point if $x \equiv y \equiv 2 (\bmod 4)$.

In all the pictures in this subsection, a dot marks an α-point, while an empty circle marks a β-point.

Segments and lattice points on the boundaries of a T-tetromino are called *sides* and *corners* of that tetromino respectively.

Suppose that a $k \times m$ board is packable by the T-tetromino. We prove that both k and m must be divisible by 4 by proving the following three propositions for the solution (which we assumed to exist):

(i) Proposition $R(j)$: Every outer segment of every α-point on the line $x + y = 4j$ is a side of a copy of the T-tetromino;

(ii) Proposition $P(j)$: Every inner segment of every α-point on the line $x + y = 4j$ is a side of a copy of the T-tetromino;

(iii) Proposition $Q(j)$: No β-point on the line $x + y = 4j - 2$ is a corner of a copy of the T-tetromino.

Let us show how proving these three statements will help us prove our main claim. Assume that $m \not\equiv 0 (\bmod 4)$.

If $m \equiv 1 (\bmod 4)$, then we consider the α-point $(m - 1, 0)$; its outer segments are sides of some T-tetrominoes. However, the only T-tetromino capable of filling the square $[m-1,0]$ (without creating a dead end) has the point $m-1,2$ as its corner. Now, $m-1,2$ is a β-point, which contradicts the Proposition $Q(j)$.

If $m \equiv 2 \pmod 4$, then we consider the β-point $(m,0)$; notably, the square [m-1,0] is a corner of our board and thus $(m,0)$ is a corner of some T-tetromino, which contradicts $Q(j)$.

If $m \equiv 3 \pmod 4$, then we consider the β-point $(m-1,0)$; by the proposition $Q(j)$, the tetromino that covers the square $[m-1,0]$ has to have its long boundary running from $(m-3,0)$ to $(n,0)$. But then we have only one choice for filling the square $[n-1,1]$, which again results in contradiction with $Q(j)$.

Thus we actually need only to prove the third proposition. However, we prove the first two propositions concurrently to help us prove what we want.

The proof goes by induction on j. The statement $R(0)$ is trivial. The statement $P(j)$ only makes sense for $j \geq 1$. We will show that $P(1)$ follows from $R(0)$. The statement $Q(1)$ is also not hard to see (just consider the possibilities to fill the bottom left corner of the board).

Claim 1: $R(j-1)$ implies $P(j)$ for all $j \geq 1$.
Assume to the contrary that $j \geq 1$ and A is an α-point on the line $x+y=4j$ whose inner segments are not segments of any tetromino in the solution. We can assume without loss of generality that A is an inner point of our board. We consider the points B and C from the statement of $R(j-1)$ and their outer segments (drawn bold on the following picture.

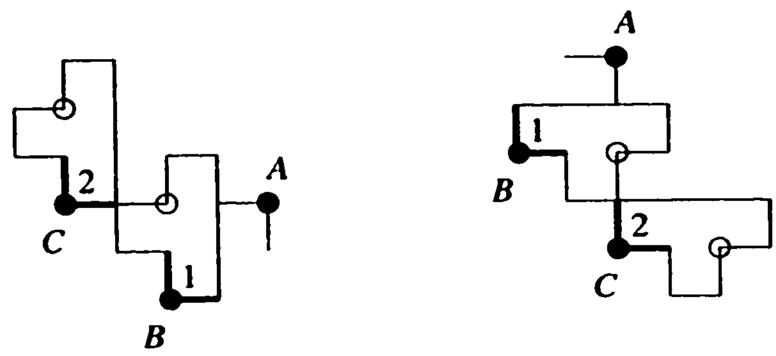


Figure 2.3.6: $R(j-1)$ implies $P(j)$ for all $j \geq 1$.

By our assumption, we have only two choices to fill the square labeled "1". For each of these choices, we have only one choice to fill the square labeled "2". This results to the picture above. We can continue putting the forced pieces until we hit the border of our board which is when we get the contradiction.

Claim 2: $Q(j-1)$, $R(j-1)$ and $P(j)$ together imply $Q(j)$ for all $j \geq 1$. Assume to the contrary that $j \geq 1$ and A is a β-point on the line $x + y = 4j - 2$ whose and A is a corner of some T-tetromino. We can easily get a contradiction if A is on the boundary of our board.

Suppose that A is an inner point. We consider the point B, for which $Q(j-1)$ holds and the pairs of points E, F and C, D for which $R(j-1)$ (and $P(j-1)$) and $P(j)$ hold respectively. Thus all the bold lines on the picture below must be segments of some T-tetrominoes. Suppose that we make A a corner of the T-tetromino as on the below picture (we actually do not have many choices since we have to take care about the bold lines afterwards). Then the only way to fill the square labeled "1" makes the point B a corner of a tetromino, which contradicts $Q(j-1)$ completes the proof of our claim.

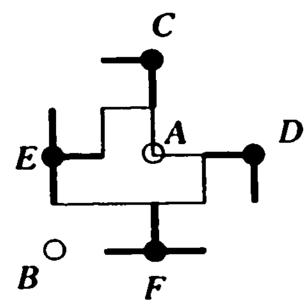


Figure 2.3.7: $Q(j-1)$, $R(j-1)$ and $P(j)$ together imply $Q(j)$ for all $j \geq 1$.

Claim 3: $P(j)$ and $Q(j)$ together imply $R(j)$ for all $j \geq 1$.

Again, let point A be the one to give contradiction to our claim.

If A is on the boundary of the board, then consider points B and C for which $P(j)$ holds and points D and E for which $Q(j)$ holds. Then we must put the tetrominoes marked "1" and "2", and then "3", "4", and "5" as on the picture below. Thus we obtain a contradiction similar to the one we had in the proof of the first claim.

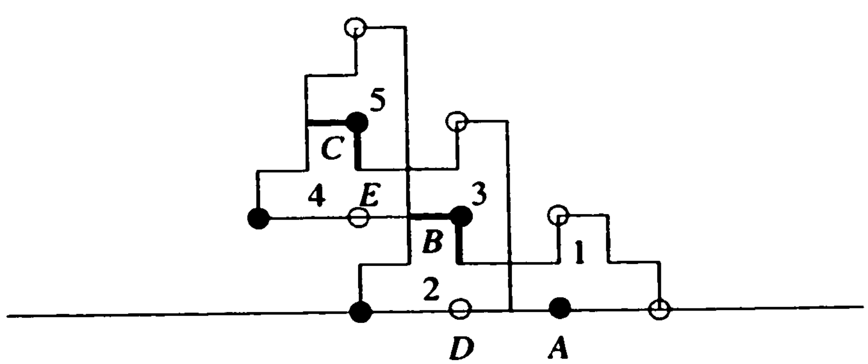


Figure 2.3.8: $P(j)$ and $Q(j)$ together imply $R(j)$ for all $j \geq 1$ (for a point on the boundary).

Suppose that A is an inner point. We know that $P(j)$ holds for A and B and that $Q(j)$ holds for the point C (see the picture below). Suppose for instance that the horizontal outer segment of A (drawn dashed) is not a side of any T-tetromino in our solution. To fill both of the squares marked "1" and not to get into the contradiction used in the proof of the first claim, we have to put the two tetrominoes as on the below picture. However, this leads us to the same contradiction.

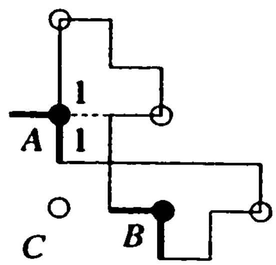


Figure 2.3.9: $P(j)$ and $Q(j)$ together imply $R(j)$ for all $j \geq 1$ (for an inner point).

2.4 Packing rectangular boards with pentominoes.

There are 12 pentominoes. They are shown on the picture below with the corresponding letters. As we will see, Problem 1.2. is only difficult for the Y-pentomino.

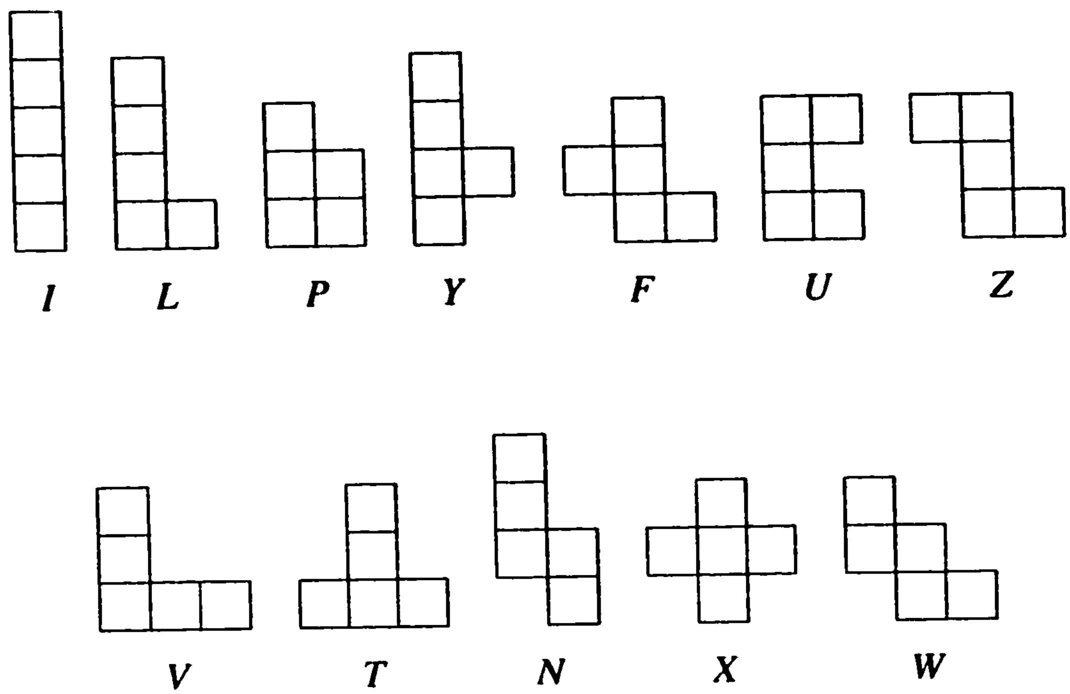


Figure 2.4.1: The twelve pentominoes.

2.4.1 Pentominoes that cannot pack any rectangular board.

In this subsection, we prove that 8 of the 12 pentominoes cannot pack any rectangular board. The following proof can also be found in [10].

We proceed in the order starting from the easiest case. The 8 "forbidden" pentominoes are X, T, F, Z, U, W, N, V.

It is easy to note that the X-pentomino cannot fill any corner and thus cannot pack any rectangle (see the picture below).

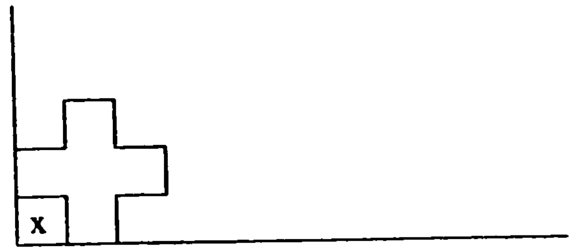


Figure 2.4.2: The X-pentomino cannot pack any board.

The T, F and Z-pentomino can fill a corner in a unique way, but, after that, the squares marked "x" cannot be filled (i.e. this square is *inaccessible*).

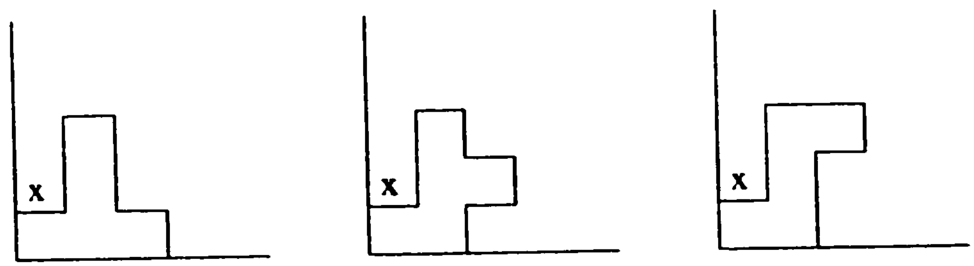


Figure 2.4.3: The T, F and Z-pentomino cannot pack any board.

The U-pentomino can fill a corner in two essentially different ways, each of which results in a dead end.

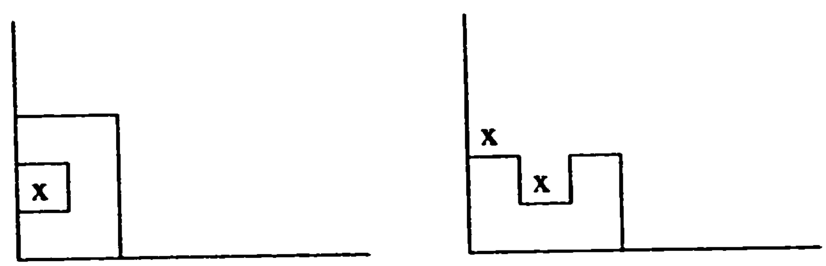


Figure 2.4.4: The U-pentomino cannot pack any board.

The case drawn on the left is obviously impossible. In the other case (on the right) it is not possible to fill both squares marked with the cross (i.e. we have an "*inaccessible pair*").

The W-pentomino can fill a corner in only one way (shown on the picture below.

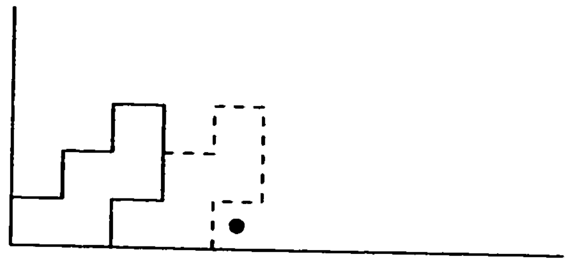


Figure 2.4.5: The W-pentomino cannot pack any board.

However, the position marked with the dot can now be filled in only one way, which makes it clear that we cannot pack the bottom edge of our board.

Similar (as for the W-pentomino) argument goes for the two possible ways the N-pentomino can fill a corner.

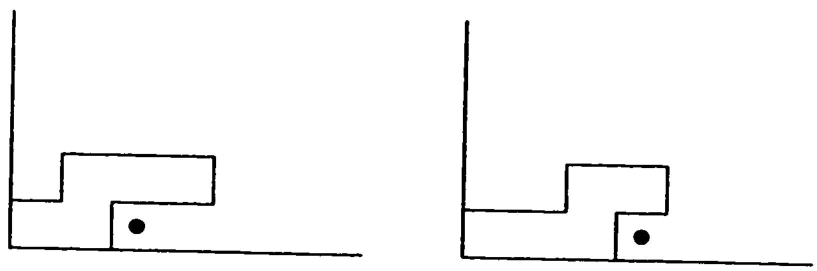


Figure 2.4.6: The N-pentomino cannot pack any board.

There are actually two possibilities to fill the squares marked with the dot, but then it is clear that we cannot pack the bottom edge of the board.

The V-pentomino can fill a corner in 2 different ways shown on the below diagram.

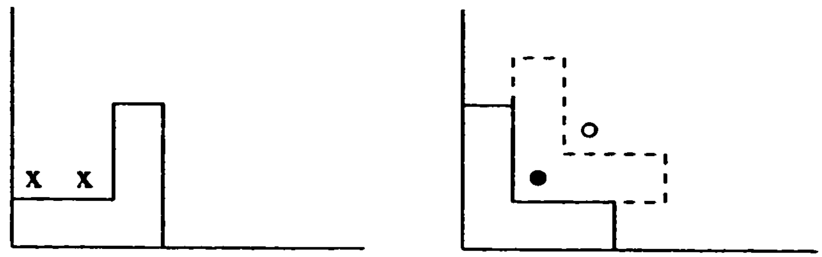


Figure 2.4.7: The V-pentomino can fill a corner in 2 different ways.

The case on the left results in an unaccessible pair. The case on the right leaves only one possibility to fill the square marked with the dot. After we use that possibility, there is again only one possibility to fill the the square marked with the empty circle. we end up with the following picture,

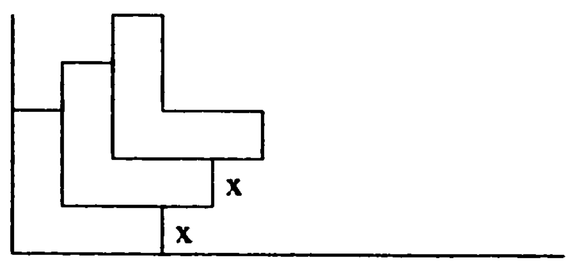


Figure 2.4.8: The V-pentomino cannot pack any board.

which leaves us with an unaccessible pair. Thus the V-pentomino cannot pack any rectangular board as well.

2.4.2 Packing with the I, L and P-pentominoes.

In this subsection, we solve Problem 1.2 for the 3 pentominoes listed in the title. We leave out the Y-pentomino for now, since that case is the most difficult and it certainly deserves a separate subsection. The solution that we describe in this subsection was published in [10].

Packing with the I-pentomino.

This case is trivial. The I-pentomino is the only prime board with respect to itself. Hence the only packable boards have dimensions $5k \times m$.

Packing with the L-pentomino.

The only two primes for this pentomino are the boards 2×5 and 7×15. These boards can be packed as follows (our solution for the 7×15 case is different from the very first solution published in [9]):

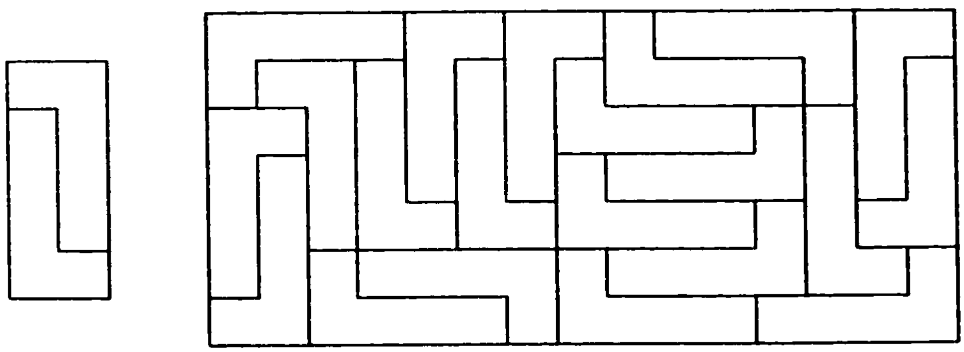


Figure 2.4.9: Prime boards for the L-pentomino.

However, we still have to prove that 7×15 is prime and that there are no other primes.

Definitely, we cannot pack 7×15 with copies of the 2×5 board (by a simple parity argument). By Lemma 1.3, we need not consider boards with the area not divisible by 5. It is clear that we do not have other primes with dimensions $1 \times m$ or $3 \times m$. Also, all the boards $2 \times m$ are either unpackable or composite (i.e. can be packed with copies of 2×5).

We show that $5 \times m$ boards are unpackable for odd m greater than 3. There are three ways to fill the left most column (note that four of the five squares in that column must be filled by one pentomino), which result in the cases depicted below.

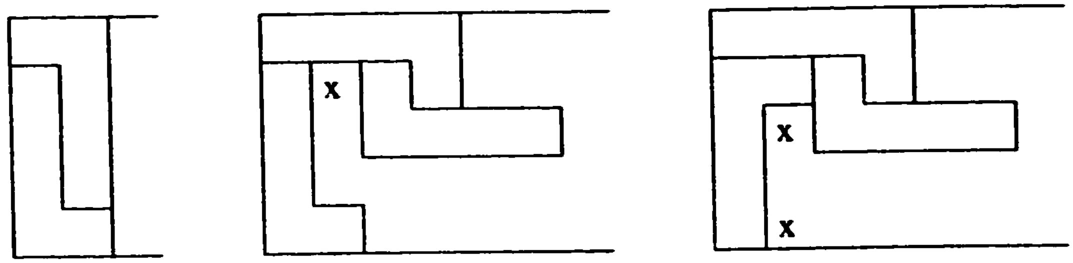


Figure 2.4.10: A $5 \times m$ board cannot be packed with copies of the L-pentomino for odd m greater than 3.

On the left most picture, we have filled a 5×2 rectangle; since m is odd, we have not accomplished anything. The other two pictures present us with either an inaccessible square or an inaccessible pair of squares denoted by crosses.

Since all the boards not considered so far can be packed by the two prime boards (e.g. the 13×15 board can be put together by placing on it 9 boards of size 2×5 so that the uncovered part is 7×15), we have completed the proof that these two boards are indeed the only prime ones for the L-pentomino.

Hence the packable boards are $2k \times 5m$ and $k \times 5m$ for all $m \geq 2$, odd $k \geq 5$.

Packing with the P-pentomino.

Interestingly enough, the only prime boards are 2×5 and 7×15 again. These two boards can be packed as follows (the solution for 7×15 was first published in [23]):

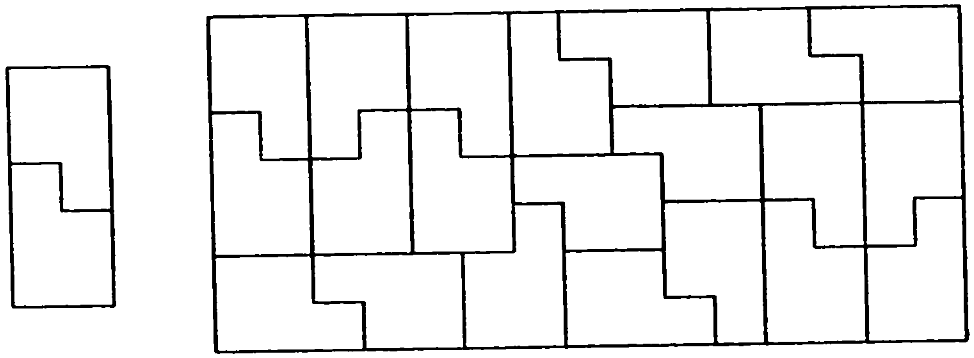


Figure 2.4.11: Prime boards for the P-pentomino.

By the same argument as for the L-pentominoes, it suffices to show that $5 \times n$ boards are not packable with the P-pentomino when n is odd. This is due to Cibulis and Liu (see [10]).

We color squares in the rows 1, 3 and 5 of our $5 \times n$ board black and the remaining squares white. We have a total of $3n$ black squares. Assume to the contrary that the board is packable. Then there are n copies of the P-pentomino in the solution. Each of this copies occupies at most 3 black squares. To fill all the black squares, we have to have each piece fill exactly 3 black squares. Hence each piece occupies exactly 2 white squares and they must both be in the same row. This shows that n must be even.

Thus the set of boards packable with the P-pentomino is the same as the set of boards packable with the L-pentomino.

2.4.3 Packing with the Y-pentomino.

The Y-pentomino is the most troublesome; it is even not clear if this pentomino can pack any rectangular board.

We first summarize the solution in the following table.

Rows and columns of the table correspond to the dimensions of the boards. A cross means that the board cannot be packed.

A circle means that the board is composite. A number in brackets means that the board is prime and gives the reference to the source where the solution[2] appeared for the first time. All the larger boards beyond this table except those pertaining to the first row of it are composite. A $5 \times m$ board is packable for m divisible by 10 only, 5×10 being the smallest packable board. The boards $5k \times m$ are not packable for $m = 1, 2, 3, 4, 6, 7, 8$. The diagrams for primes are given in Appendix B.

	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
5	×	[5]	×	×	×	×	×	×	×	×	×	○	×	×	×	×	×	×	×	×
10	×	○	×	×	×	[31]	○	[5]	×	×	○	○	○	×	[9]	○	○	○	[9]	○
15	×	○	×	×	×	[31]	[19]	[5]	[12]	×	[12]	○	[12]	[5]	[12]	○	○	○	○	○
20	[31]	○	[31]	×	[9]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
25	×	○	×	×	×	○	○	○	[12]	[12]	○	○	○	[5]	○	○	○	○	○	○
30	[31]	○	[9]	×	[9]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
35	×	○	[9]	×	[9]	○	○	○	○	[12]	○	○	○	○	○	○	○	○	○	○
40	○	○	○	×	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
45	[9]	○	[9]	×	[9]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
50	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
55	[9]	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
60	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
65	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
70	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
75	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
80	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
85	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
90	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
95	○	○	○	[4]	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○

Table 2: Summary about boards that the Y-pentomino packs.

[2]Some of the diagrams in Appendix B are different from those published in the referenced sources and were obtained by computer. Some details of implementation are explained in Appendix C.

In the reminder of this subsection, we show that our answer is correct.

Justification of unpackability.

Let us for convenience give names to the 8 possible orientations of the Y-pentomino depending on where the square that "sticks out" is.

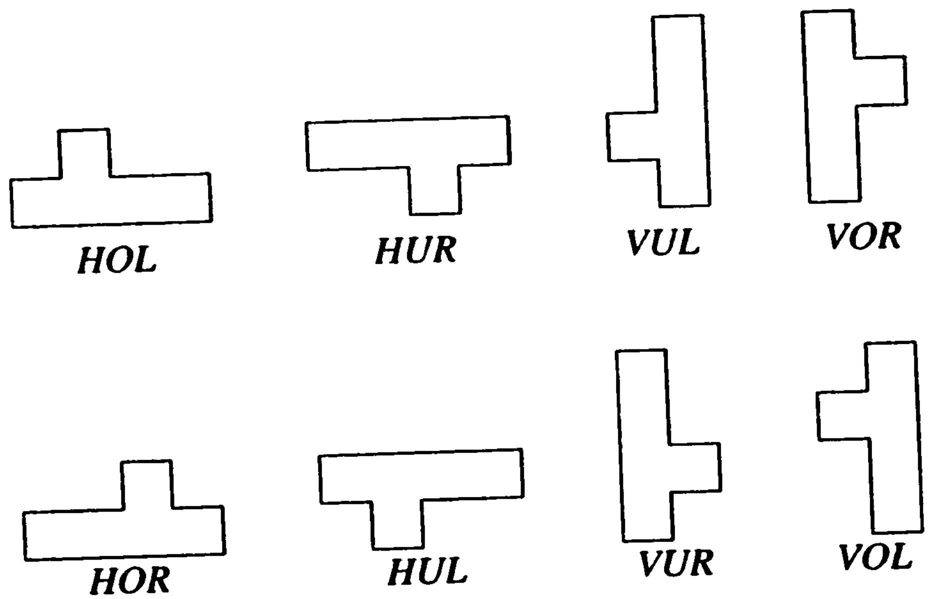


Figure 2.4.12: The eight orientations of the Y-pentomino.

We first show that the boards $5k \times m$ cannot be packed for $m = 1, 2, 3, 4, 6, 7, 8$ (i.e. that our table does not lack the first columns).

The claim is trivial for $m = 1$. For $m = 2, 3$ we get an unaccessible pair and an *unaccessible triple* as follows:

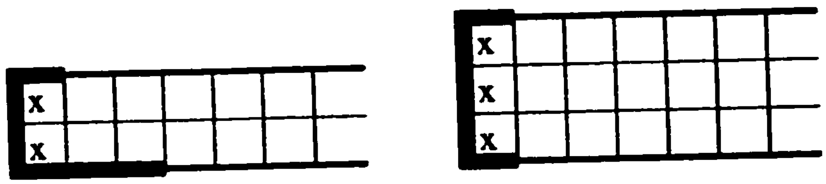


Figure 2.4.13: Canonical unaccessible pairs and triples.

We do not need very long "walls" to get inaccessible squares as on the pictures above. What we really need is shown by the bolder lines. These two situations will be referred to as the *canonical inaccessible pair* and the *canonical inaccessible triple* respectively.

For $m = 4$, there are two possible ways to fill the left most (i.e. the first) column.

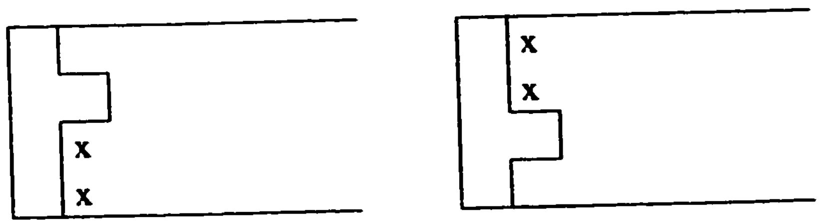


Figure 2.4.14: A board $5k \times 4$ is not packable by copies of the Y-pentomino.

Both ways leave us with canonical inaccessible pairs.

Similar argument goes for $m = 7$, since we again have to use exactly one copy of VOR or VUR to fill the first column.

For $m = 6$, we essentially have only one way not to get into trouble right away.

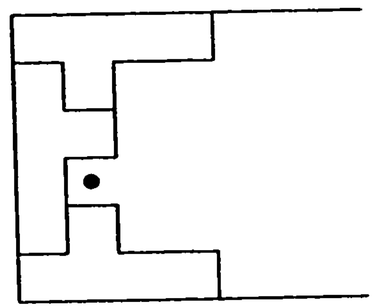


Figure 2.4.15: A board $5k \times 6$ is not packable by copies of the Y-pentomino (Step 1).

If we fill the square marked with the dot by an HUL, we will get a canonical inaccessible square. Similarly, trying to fill that square using VUL, then we will have to use VOR or VUR in the fourth column, which again results in our canonical "bad" situations. Thus the only way to proceed far enough is as shown on the picture below.

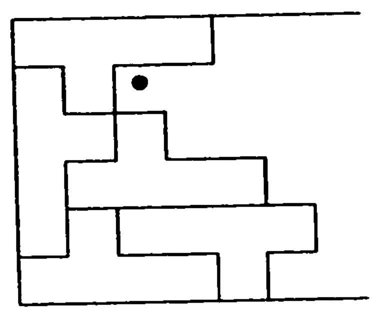


Figure 2.4.16: A board $5k \times 6$ is not packable by copies of the Y-pentomino (Step 2).

Then we must have one of the two outcomes on the picture below.

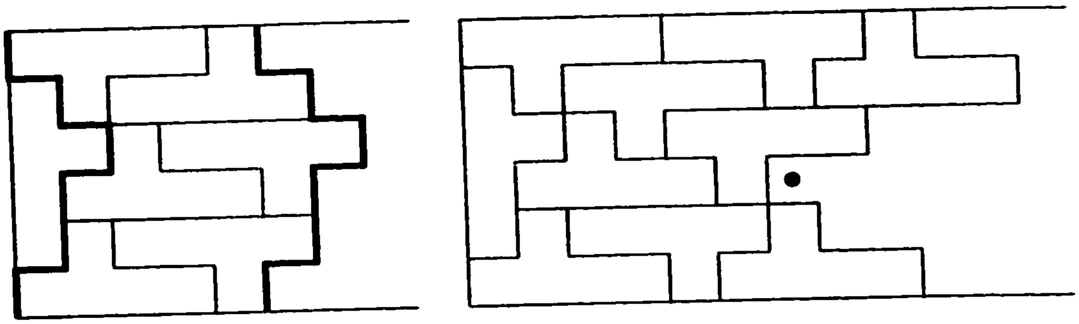


Figure 2.4.17: A board $5k \times 6$ is not packable by copies of the Y-pentomino (Step 3).

The bold lines on the left picture have the same shape. That means that we have not accomplished anything (we have to pack a rectangle eventually!). On the other hand, we can to either of the following two positions after we fill the square marked with the dot on the right picture (and put some other forced copies of the Y-pentomino).

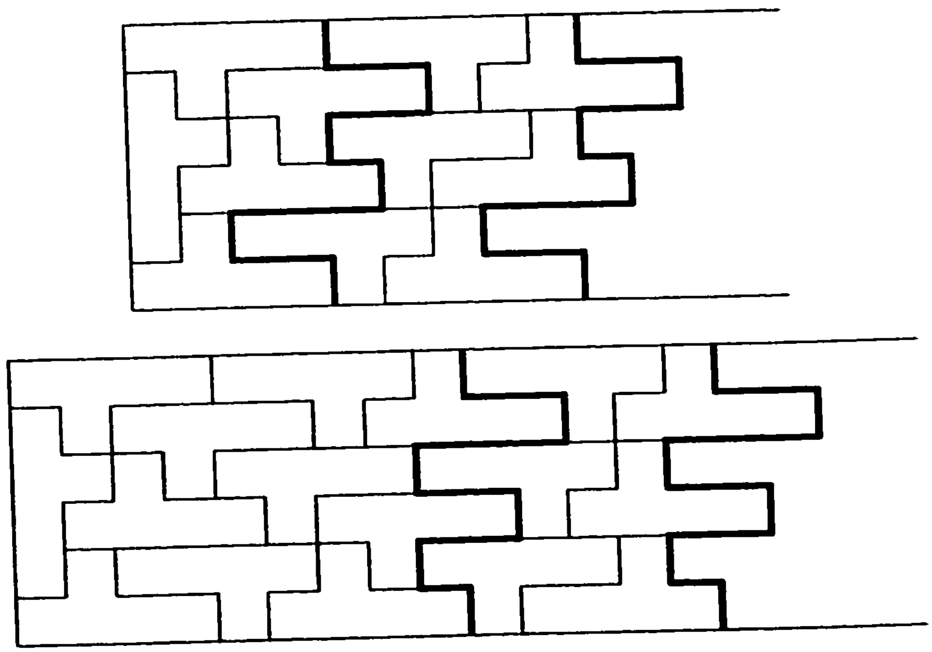


Figure 2.4.18: A board $5k \times 6$ is not packable by copies of the Y-pentomino (Step 4).

Both of the above pictures show that this attempt leads nowhere as well. Thus we are done with the case $5k \times m$, where $m = 6$.

For $m = 8$, we have to start as on the below picture not to get into trouble right away.

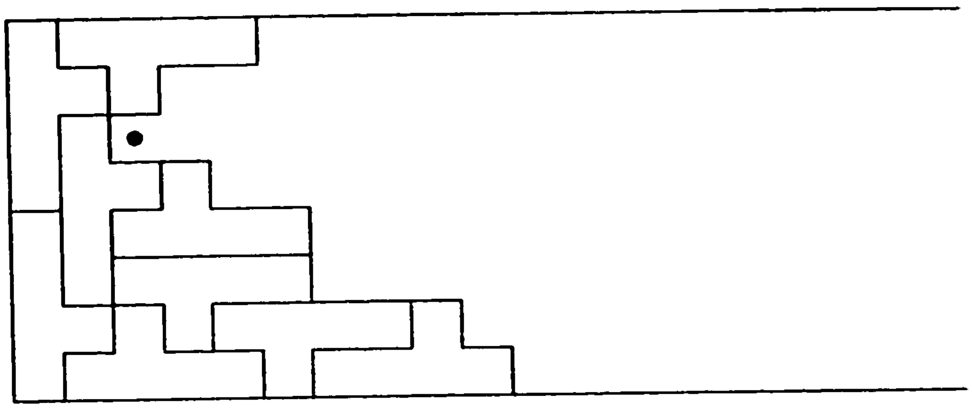


Figure 2.4.19: A board $5k \times 8$ is not packable by copies of the Y-pentomino (Step 1).

We have essentially two ways to fill the dotted square. After putting some forced pieces, we get the following two possible positions.

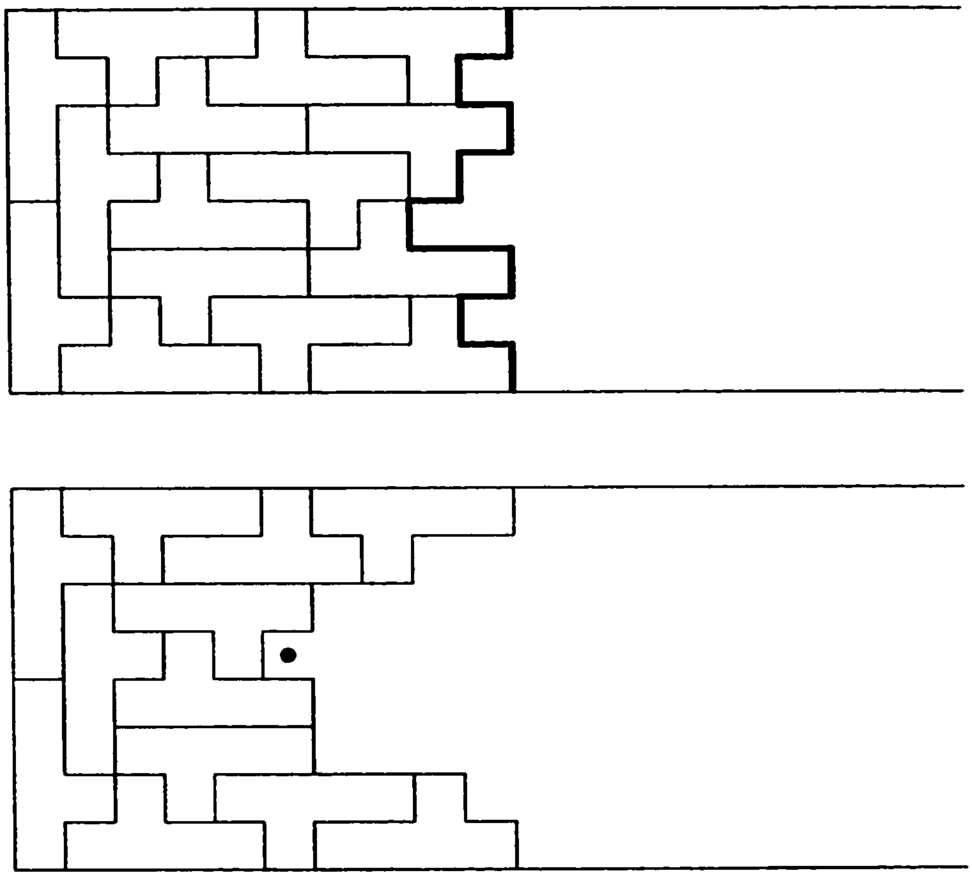


Figure 2.4.20: A board $5k \times 8$ is not packable by copies of the Y-pentomino (Step 2).

The border as in the position on the top will be dealt with in one of the cases for the position on the bottom (this will be shown by a matching bold line). There are two ways to fill the dotted square in the position on the bottom not to get into trouble right away. After putting also some forced pieces, we get the following two positions.

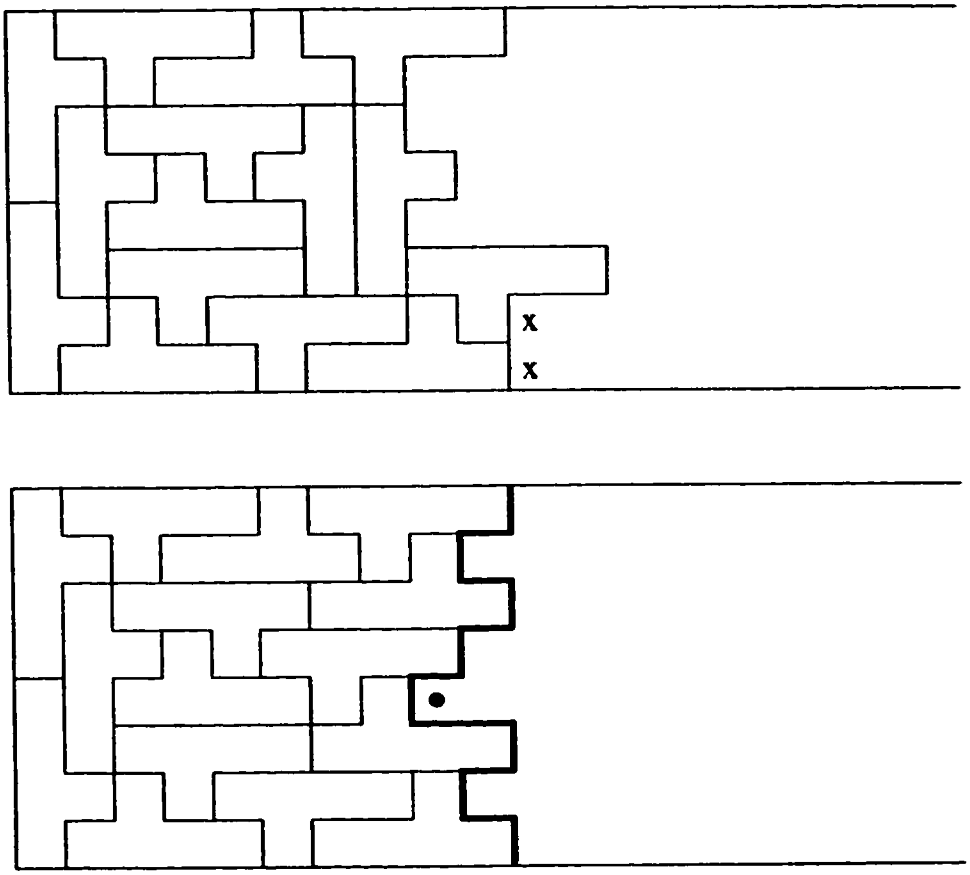
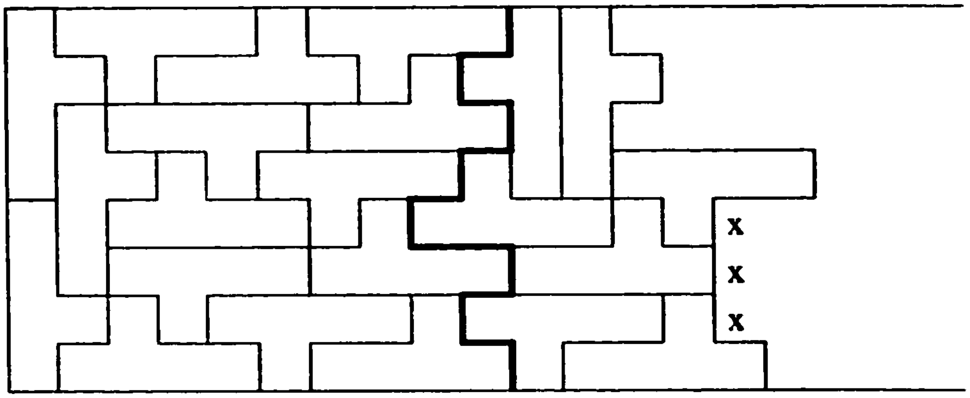


Figure 2.4.21: A board $5k \times 8$ is not packable by copies of the Y-pentomino (Step 3).

The position on the top presents us with a canonical inaccessible pair. The two ways to proceed from the position on the bottom lead us to the situations on the diagram below.



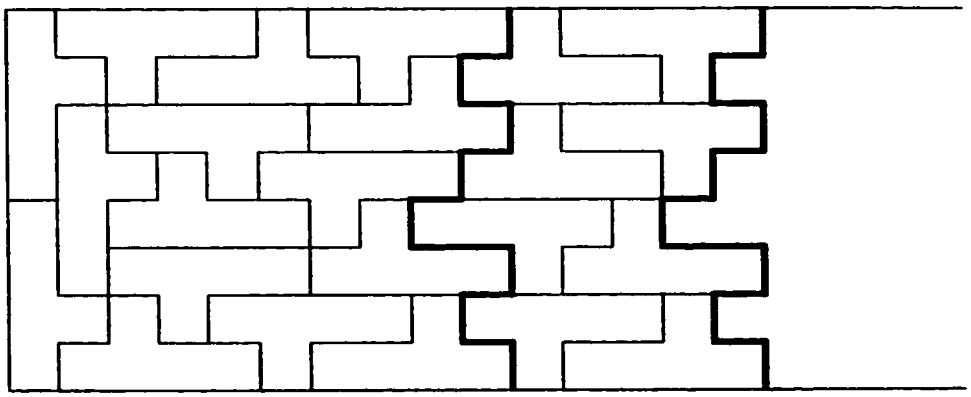


Figure 2.4.22: A board $5k \times 8$ is not packable by copies of the Y-pentomino (Step 4).

In the situation on the top, we are presented with the canonical inaccessible triple. In the situation on the bottom, we repeat the shape of the border, so we have not made any progress. This completes the proof for $m = 8$.

To finish with the infinite families of unpackable boards, we show that the $5 \times m$ boards are only packable for m divisible by 10. We have to start as on the picture below (the shaded HUL could have been the HUR, which would affect the next step and result in the same situation).

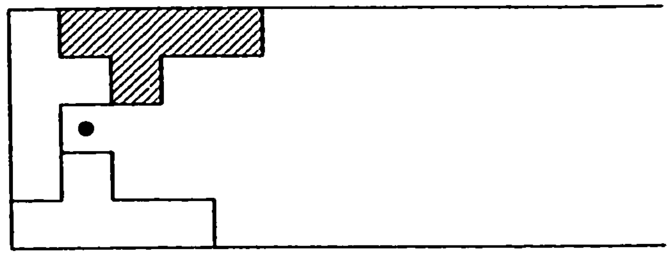


Figure 2.4.23: A board $5 \times m$ is not packable by copies of the Y-pentomino if m is not divisible by 10 (Step 1).

The two ways to fill the dotted square result in the following situations.

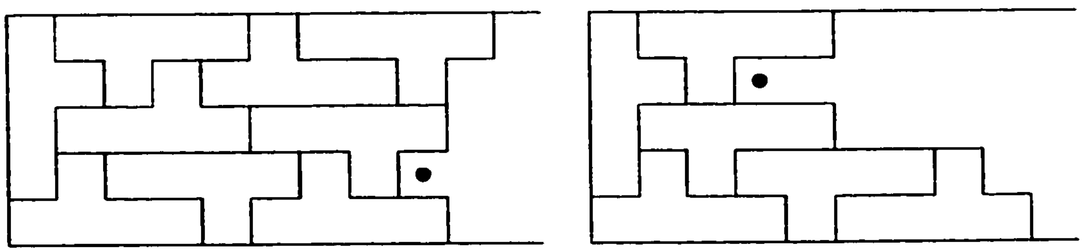


Figure 2.4.24: A board $5 \times m$ is not packable by copies of the Y-pentomino if m is not divisible by 10 (Step 2).

The picture on the left does not leave us any choices except for completing the 5×10 rectangle. The picture on the right provides us with two ways of filling the dotted square, which result in the following pictures.

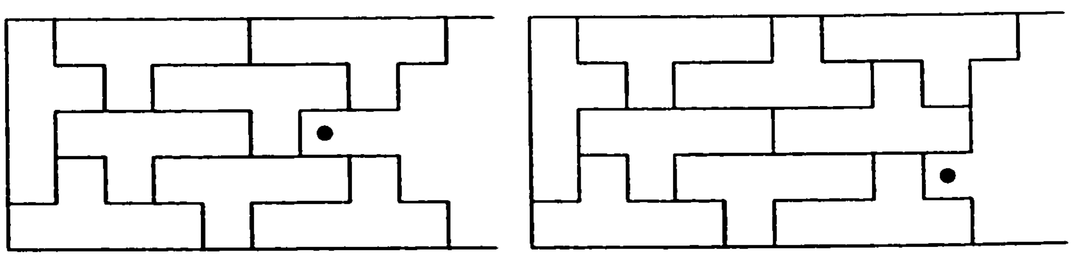


Figure 2.4.25: A board $5 \times m$ is not packable by copies of the Y-pentomino if m is not divisible by 10 (Step 3).

Any attempt to fill the dotted square on the left picture will result in a dead end. The picture on the right leads us to completing the 5×10 again. This finishes our proof for $m = 5$.

All the crosses in our "answer-table" starting from the second row were verified by exhaustive computer search.

Justification about the composite boards.

We have already shown that the 39 boards that were claimed to be prime are indeed prime (this clearly follows from the negative results proved above).

Now we show that all the circles in the table are correct and that all the boards larger than those in the table (except for the ones whose one side is 5) are composite as well. We move along the columns of our table.

For $5k \times 9$, we know that we can pack the rectangles for $k = 4, 6, 8, 9, 10, 11$. Now, $12 = 8+4$, $13 = 9+4$, ..., $23 = 10+13$. Then we can get all the bigger rectangles whose one side is 9 by appending the 20×9. By the very similar arguments, all the circles inside our table are correct. Also, all the boards larger than the ones in the table and pertaining to the columns of the table are composite.

We now move to the columns beyond the table.

A $10 \times m$ board is packable for $m = 5, 23, 24, 25, 26, 27$ (since $24 = 19 + 5$ and $26 = 21 + 5$). Thus such a board is packable for all $m \geq 23$. Similarly, a $15 \times m$ board is packable for $m = 10, 19, 20, \ldots, 28$. Thus such a board is packable for all $m \geq 19$. Now consider a $5k \times m$ board for $k \geq 2, m \geq 28$. We can certainly pack it by $10 \times m$ and $15 \times m$ boards. This completes the proof.

Bibliography

[1] R. E. L. Aldred and Brendan D. McKay. Graceful and harmonious labellings of trees. *Bulletin of the ICA*, 23:69–72, 1998.

[2] Ibrahim Cahit Arkut, Refik Cahit Arkut, and Nasir Ghani. Graceful label numbering in optical MPLS networks. *Proceedings of SPIE*, 4233, 2000.

[3] J.C. Bermond. Radio antennae and french windmills. *Graph Theory and Combinatorics*, pages 13–37, 1979.

[4] J. Bitner. Tiling $5n$ x 12 rectangles with Y-pentominoes. *J. Recreat. Math.*, 7:276–278, 1970.

[5] C.J. Bouwkamp and D.A. Klarner. Packing a box with Y-pentacubes. *J. Recreat. Math.*, 3:10–26, 1970.

[6] M. Burzio and F. Ferrarese. The subdivision graph of a graceful tree is graceful. *Discrete Mathematics*, 181:275–281, 1998.

[7] Frank Van Bussel. Towards the graceful tree conjecture. Master's thesis, University of Toronto, 2000.

[8] Wen-Chin Chen, Hsueh-I Lu, and Yeong-Nan Yeh. Operations of interlaced trees and graceful trees. *Southeast Asian Bulletin of Mathematics*, 21:337–348, 1997.

[9] A. Cibulis and I.Mizniks. Tiling rectangles with pentominoes. *Latv. Univ. Zinãt Raksti*, 612:57–61, 1988.

[10] Adris Cibulis and Andy Liu. Packing rectangles with the L and P pentominoes. *Math. Horizons*, to appear.

[11] N.G. de Bruijn. Filling boxes with bricks. *American Mathematical Monthly*, 76:37–40, 1969.

[12] Julian Fogel, Mark Goldenberg, and Andy Liu. Packing rectangles with Y-pentominoes. Submitted.

[13] Roberta W. Frucht and Luis C. Salinas. Graceful numbering of snakes with constraints on the first label. *Ars Combinatoria*, 20-B:143–157, 1985.

[14] J.A. Gallian. A dynamic survey of graph labelling. *The Electronic Journal of Combinatorics*, 2000.

[15] Solomon W. Golomb. Checkerboards and polyominoes. *American Mathematical Monthly*, 61(10):675–682, 1954.

[16] Solomon W. Golomb. How to number a graph. In R.C. Read, editor, *Graph Theory and Computing*, pages 23–37. Academic Press, N.Y., 1972.

[17] Solomon W. Golomb. Normed division domains. *American Mathematical Monthly*, 88(9):680–686, 1981.

[18] Thom Grace. More graceful trees. *Congressus Numerantum*, 39:433–439, 1983.

[19] L. Haselgrove. Packing a square with Y-pentominoes. *J. Recreat. Math.*, 7:229, 1974.

[20] C. Huang, A. Kotzig, and A. Rosa. Further results on tree labelling. *Utilitas Mathematica*, 21C:31–48, 1982.

[21] Dejun Jin, Fanhong Meng, and Jingong Wang. The gracefulness of the tree with a diameter 4. *Acta Scientiarum Naturalium Universitatis Jilinensis*, 1:17–22, 1993.

[22] D.A. Klarner. Covering a rectangle with L-tetrominoes. *American Mathematical Monthly*, 70:760–761, 1963.

[23] D.A. Klarner. Packing a rectangle with congruent n-ominoes. *J. Comb. Theory*, 3:107–115, 1969.

[24] D.A. Klarner and F. Gőbel. Packing boxes with congruent fugures. *Indag. Math.*, 31:465–472, 1969.

[25] K.M. Koh, D.G. Rogers, and T. Tan. On graceful trees. *Nanta Mathematica*, 10(2):207–211, 1977.

[26] K.M. Koh, D.G. Rogers, and T. Tan. Two theorems on graceful trees. *Discrete Mathematics*, 25:141–148, 1979.

[27] Anton Kotzig. Recent results and open problems in graceful graphs. *Congressus Numerantum*, 44:197–219, 1984.

[28] A.M. Pastel and H. Raynaud. Les oliviers sont gracieux. *Colloq. Grenoble*, Publications Universite de Grenoblé, 1978.

[29] A. Rosa. On certain valuations of the vertices of a graph. In *Theory of Graphs*, pages 349–355. Gordon and Breach, N.Y. and Dunod Paris, 1967.

[30] A. Rosa. Labeling snakes. *ARS Combinatoria*, 3:67–74, 1977.

[31] K. Scherer. Some new results on Y-pentominoes. *J. Recreat. Math.*, 12:201–204, 1979/80.

[32] D. Sheppard. The factorial representation of balanced labelled graphs. *Discrete Mathematics*, 15:379–388, 1976.

[33] D. Walkup. Covering a rectangle with T-tetrominoes. *American Mathematical Monthly*, 72:986–988, 1965.

[34] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 1996.

[35] Robert Alan Wright, Bruce Richmond, Andrew Odzylko, and Brendan D. McKay. Constant time generation of free trees. *SIAM J. Comput.*, 15(2):540–548, 1986.

[36] S.L. Zhao. All trees of diameter four are graceful. *Graph Theory and its Applications: East and West*, pages 700–706, 1986.

Appendix A

A note on the experiment concerning the Transplant Lemma.

This is a brief discussion of our implementation of the Transplant Lemma Game (see Subsection 1.4.4 for the rules)

The program has two parts:

(i) Constructing all levels of the game;

(ii) Checking that all trees appear at least once in some level.

Suppose that we have a candidate tree with a graceful labeling and we have to decide whether we have to add it to the level l. Namely, we have to check if this tree has appeared on the previous levels with this particular labeling. This is the place where we thank ourselves for the decision that our tree of trees is obtained in the breadth-first search order. The thing is that we do not have to compare our candidate tree against trees on level $l-3$ or above (since otherwise the tree from which we obtained our candidate tree would appear earlier than on the level $l-1$ because it would have been obtained from our candidate tree).

To check whether or not we really obtained graceful labelings of all trees on n vertices, we have to generate all such trees first. Then we have to look up our tree for all of the generated trees;

thus we need to solve the problem of tree isomorphism. The algorithms for solving these tasks were taken from [35].

Our implementation allowed to check that the Transplant Lemma gives graceful labelings to all trees on up to 11 vertices. The main reason for this upper bound is that the number of trees and labelings that we have to keep grows exponentially and for 11 vertices our program used almost all of the available memory. However, we can already think of improvements which would probably allow to raise that number of vertices to 13.

In our implementation, a graceful tree was represented in more ways than is necessary. This was done to simplify the program. For 12 vertices and assuming that the size of integer in the machine representation is 4 bytes, the structure that represented a single tree was 1976 bytes. Instead, we can keep only the canonical level sequences (see [35]) and graceful labelings of our trees, which would reduce the size per tree to 24 bytes. Of course, this would lead to some programming difficulties but they are solvable.

Appendix B

Packing prime boards for the Y-pentomino.

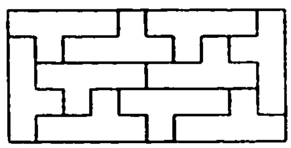


Figure B1: Packing the 5×10 board with copies of the Y-pentomino.

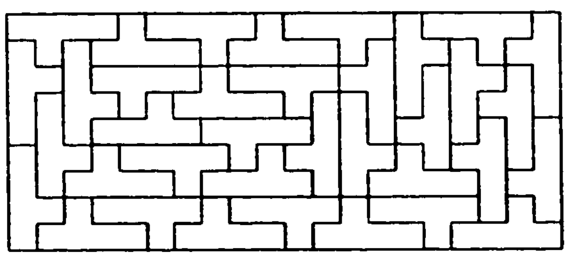


Figure B2: Packing the 9×20 board with copies of the Y-pentomino.

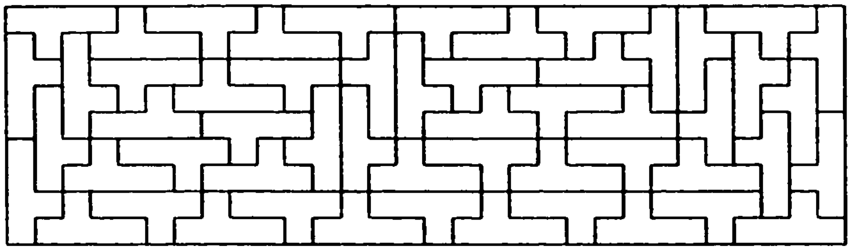


Figure B3: Packing the 9×30 board with copies of the Y-pentomino.

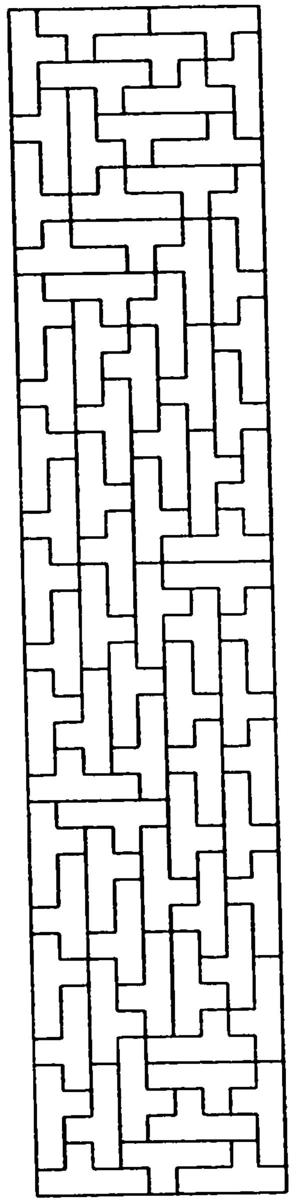


Figure B4: Packing the 9×45 board with copies of the Y-pentomino.

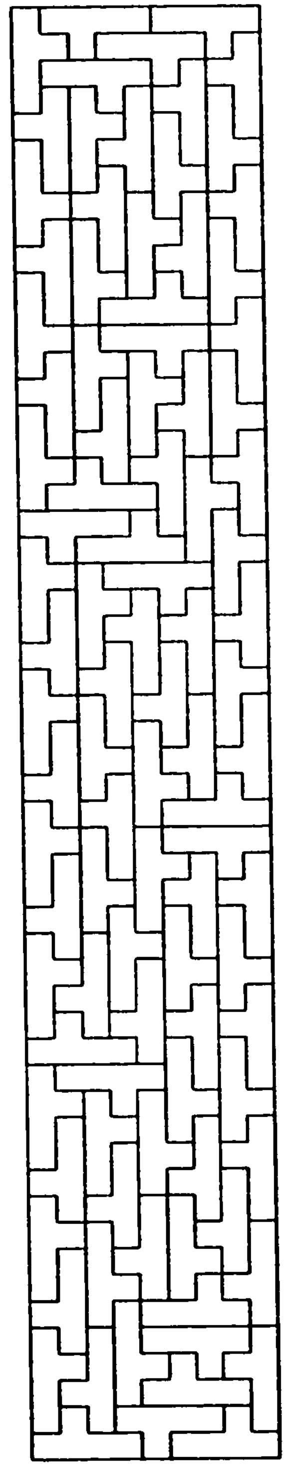


Figure B5: Packing the 9×55 board with copies of the Y-pentomino.

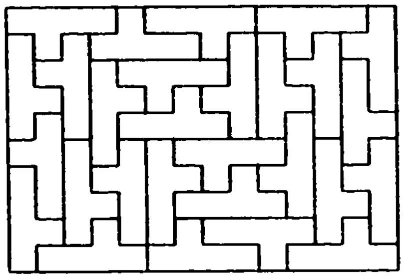


Figure B6: Packing the 10×14 board with copies of the Y-pentomino.

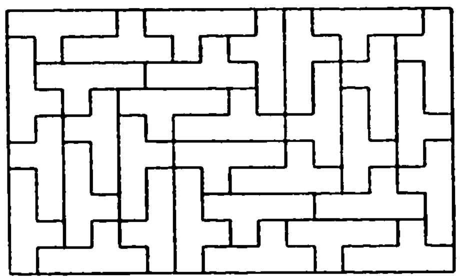


Figure B7: Packing the 10×16 board with copies of the Y-pentomino.

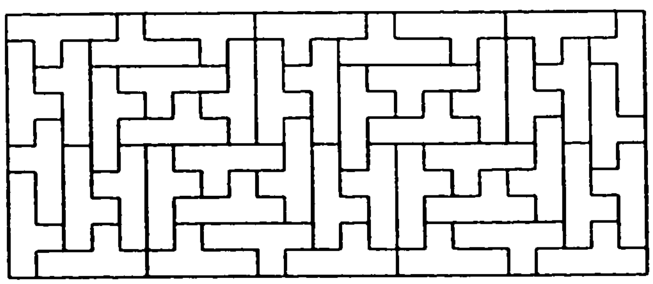


Figure B8: Packing the 10×23 board with copies of the Y-pentomino.

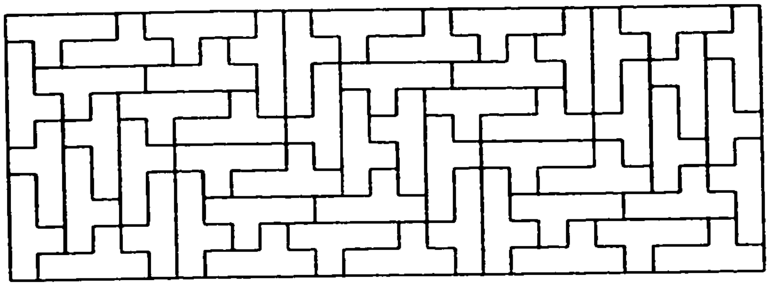


Figure B9: Packing the 10×27 board with copies of the Y-pentomino.

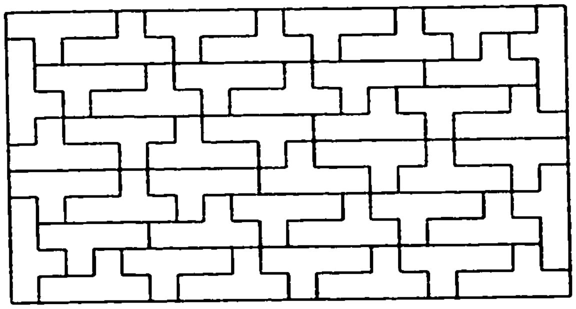


Figure B10: Packing the 11×20 board with copies of the Y-pentomino.

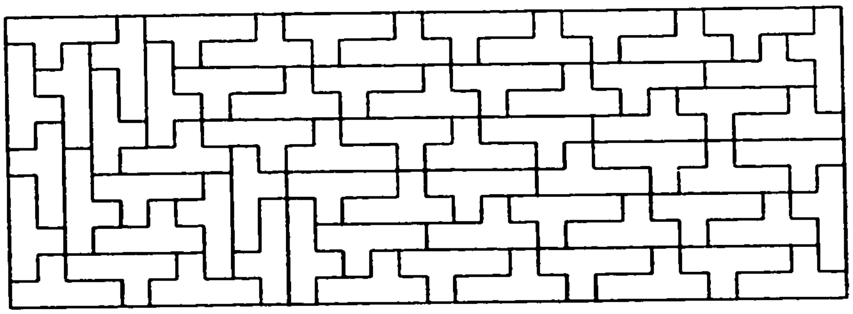


Figure B11: Packing the 11×30 board with copies of the Y-pentomino.

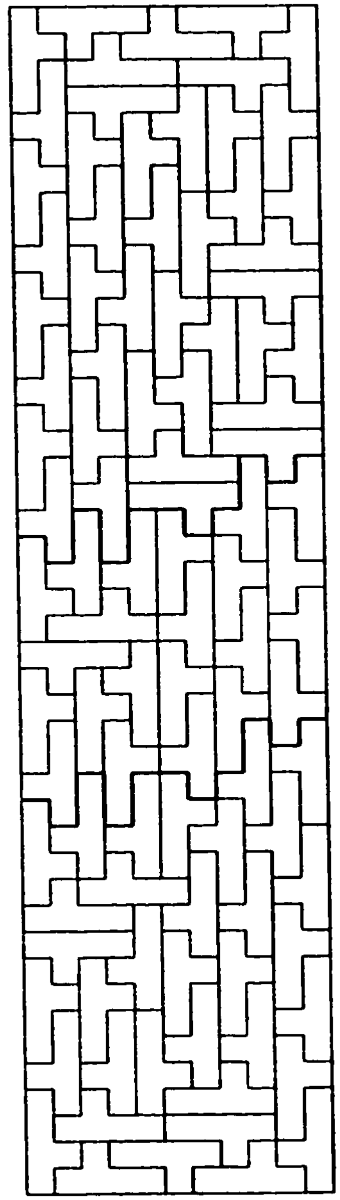


Figure B12: Packing the 11×45 and the 11×35 boards[1] with copies of the Y-pentomino.

[1] By removing the area between the bold lines or inserting more copies of this area, we can get solutions for all the listed boards. The diagram is for the first board in the list.

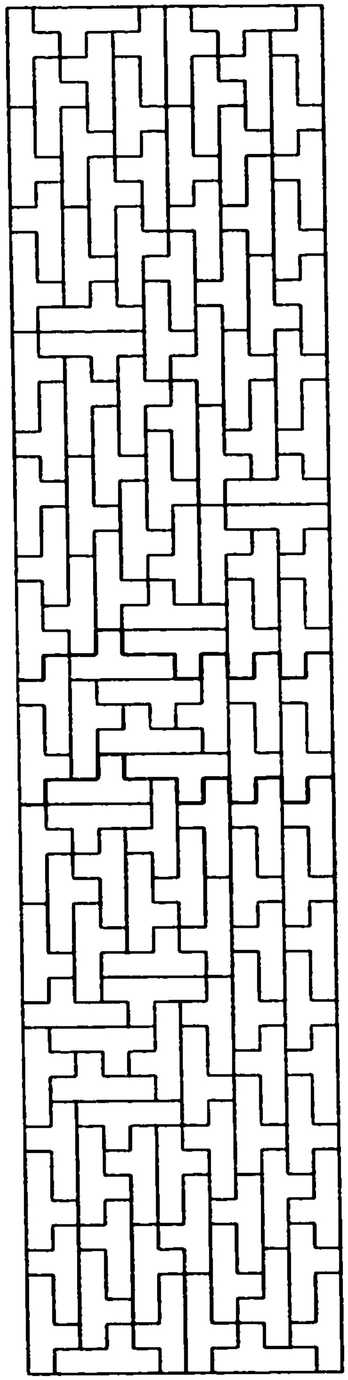


Figure B13: Packing the $12 \times m$ boards for $m = 55, 50, 60, 65, 70, 75, 80, 85, 90, 95^2$ with copies of the Y-pentomino.

[2]See the note for 11x45.

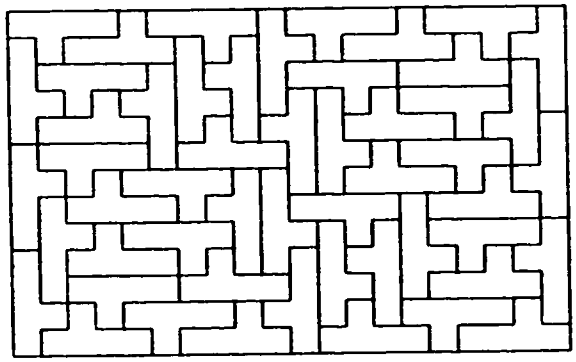


Figure B14: Packing the 13×20 board with copies of the Y-pentomino.

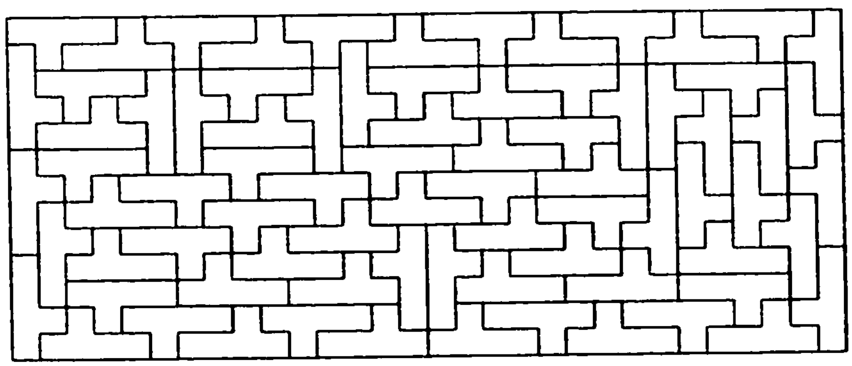


Figure B15: Packing the 13×30 board with copies of the Y-pentomino.

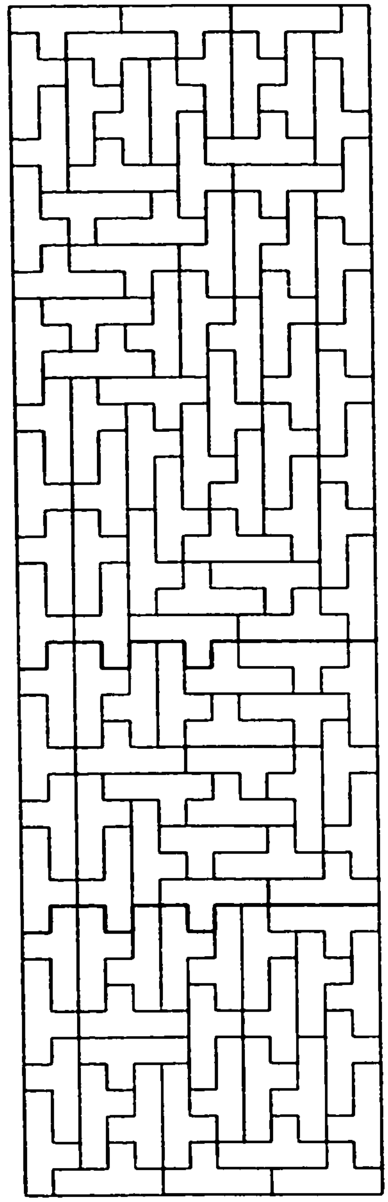


Figure B16: Packing the 13 × 45 and the 13 × 35 boards[3] with copies of the Y-pentomino.

[3]See the note for 11x45.

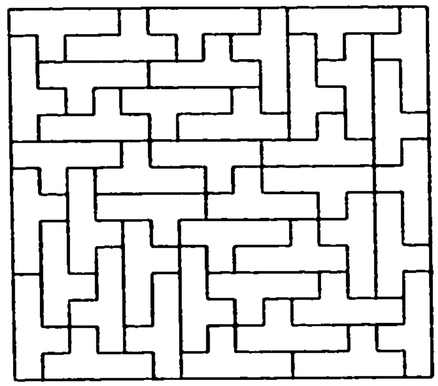


Figure B17: Packing the 14×15 board with copies of the Y-pentomino.

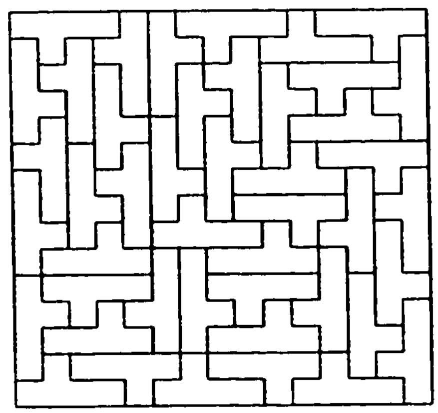


Figure B18: Packing the 15×15 board with copies of the Y-pentomino.

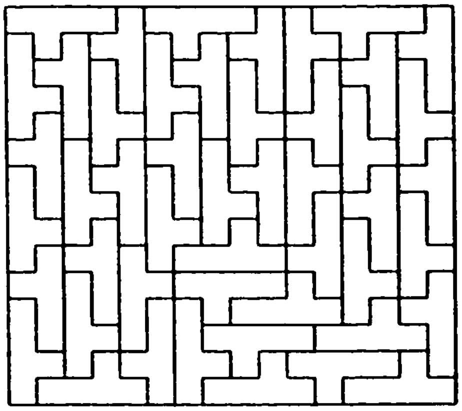


Figure B19: Packing the 15×16 board with copies of the Y-pentomino.

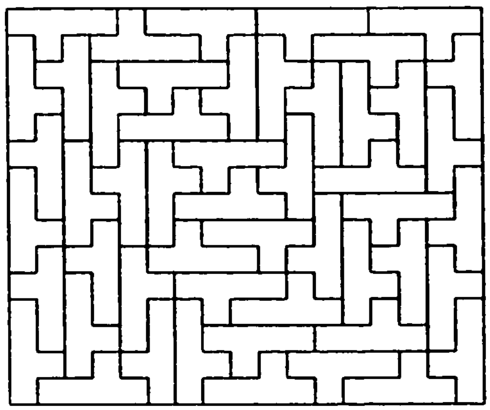


Figure B20: Packing the 15×17 board with copies of the Y-pentomino.

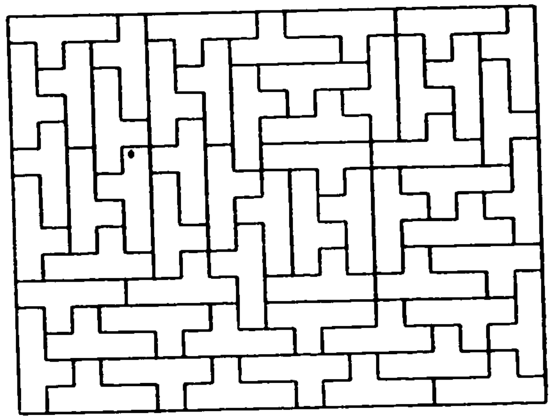


Figure B21: Packing the 15×19 board with copies of the Y-pentomino.

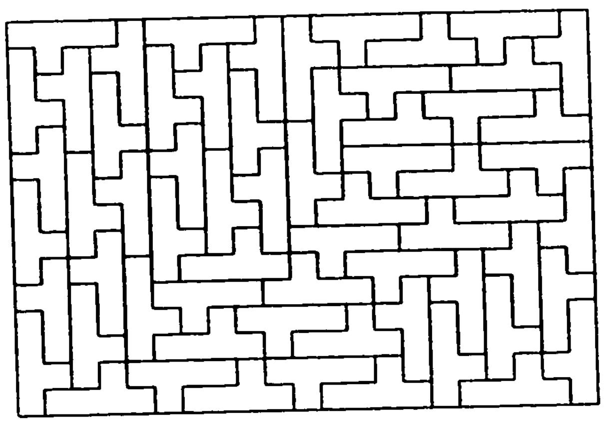


Figure B22: Packing the 15×21 board with copies of the Y-pentomino.

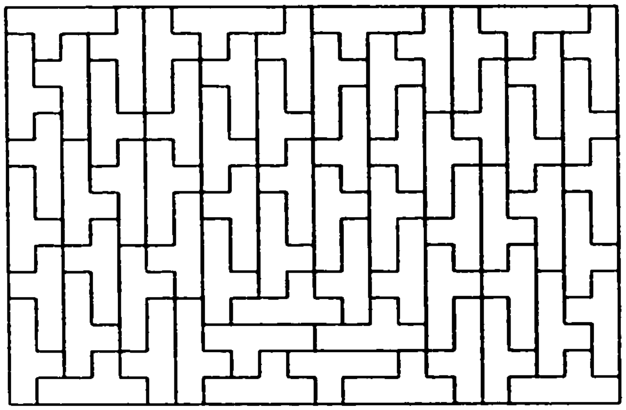


Figure B23: Packing the 15×22 board with copies of the Y-pentomino.

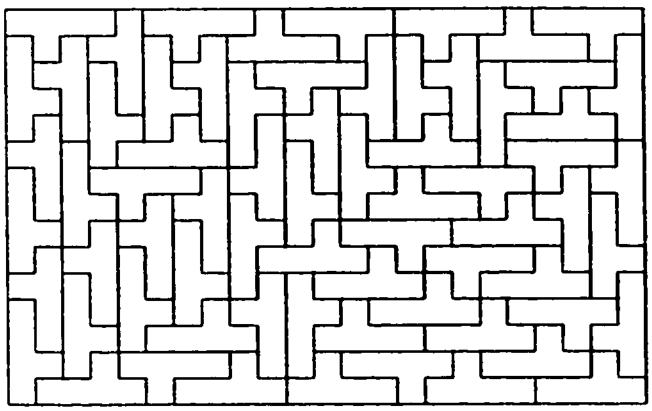


Figure B24: Packing the 15×23 board with copies of the Y-pentomino.

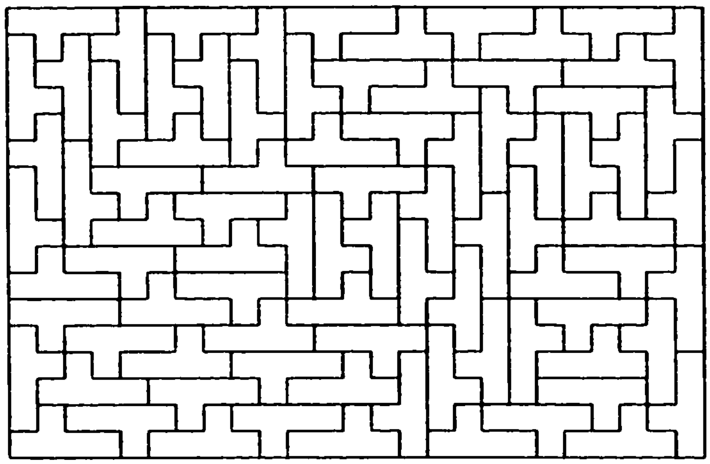


Figure B25: Packing the 17×25 board with copies of the Y-pentomino.

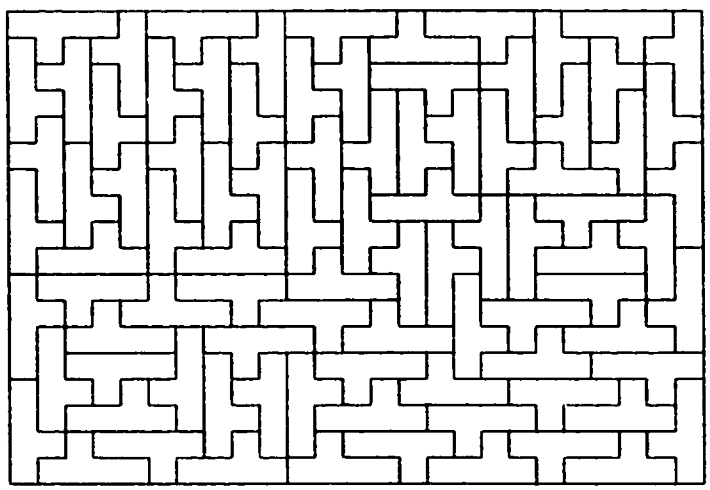


Figure B26: Packing the 18×25 board with copies of the Y-pentomino.

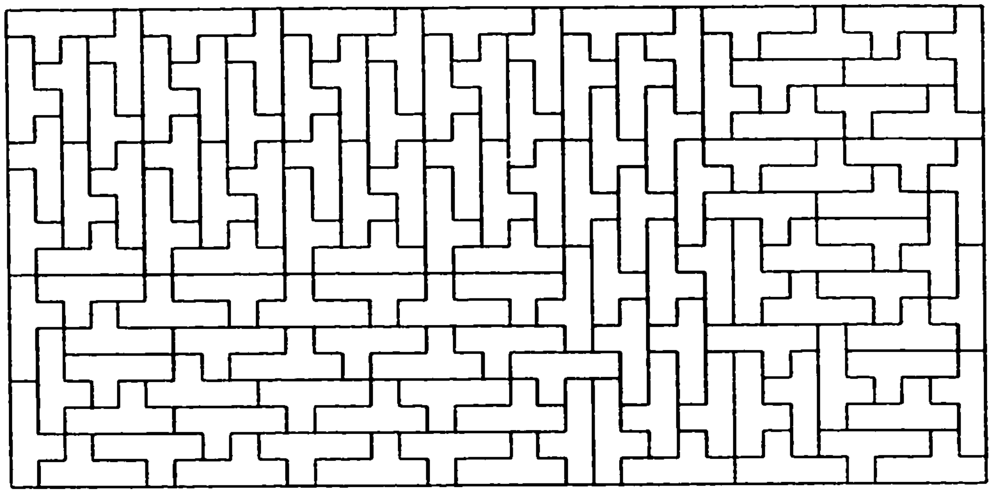


Figure B27: Packing the 18×35 board with copies of the Y-pentomino.

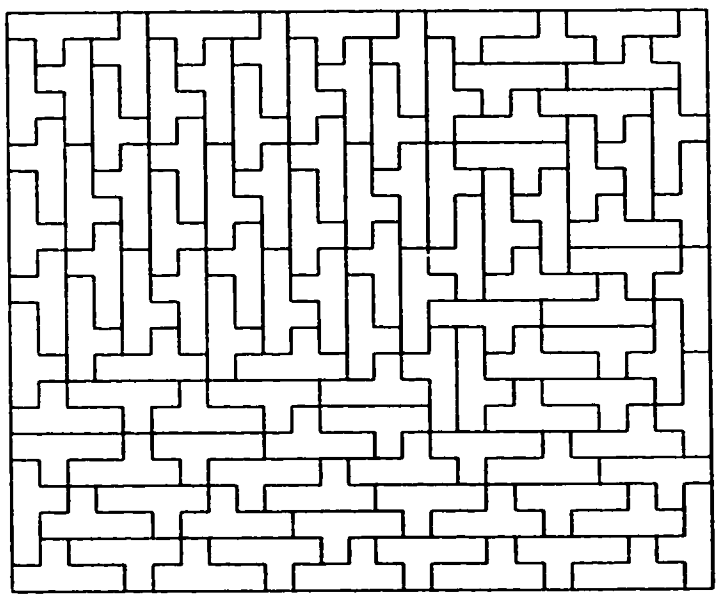


Figure B28: Packing the 22×25 board with copies of the Y-pentomino.

Appendix C

A note on generating solutions for prime boards for the Y-pentomino.

In this appendix, we give a brief description of the rather simple and almost standard algorithm whose implementation in C succeeded to get almost all the prime boards listed in Appendix B.

We are trying to fill the given rectangular board with copies of the Y-pentomino. The first idea that comes to mind is recursion. This is the one that we are going to use. The recursion is going to imitate the way one would try to fill the board should one do it by hand.

It is natural to place the first pentomino into the left upper corner. Where do we put the next piece? Well, two possible answers are: "right next to the first pentomino" or "that depends on the orientation of the first pentomino; we have to consider cases." The algorithm is based on the simplest answer to the above question. Namely, we again put the next pentomino to the left upper corner.

We define the left upper corner as a non-occupied square with the minimal sum of the row and the column coordinates. This is still not precise, since the mentioned sums are identical for squares on the same up-diagonals. Assume that we have positioned the board so that the number of columns is at least as large as the number of rows. Then we would like to "go faster" in the horizontal direction. Hence out of two candidate squares with identical sums of coordinates, we pick the one with the larger column-coordinate.

Thus we fill the board in the order shown on this picture.

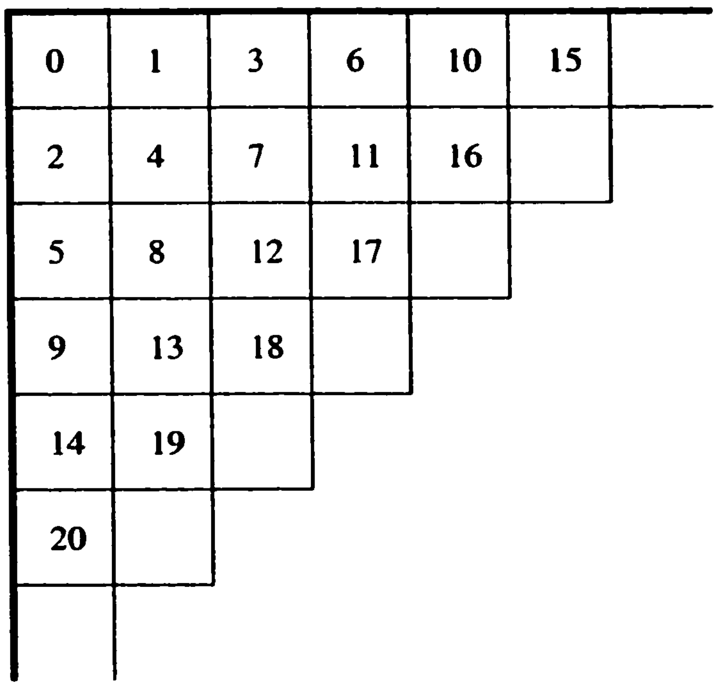


Figure C1: Ordering squares of a board for computer-aided packing with the Y-pentomino.

We claim that there is only one way to fill the corner on each step. To see that, let us also assign labels to the squares of the eight possible orientations of the Y-pentominoes:

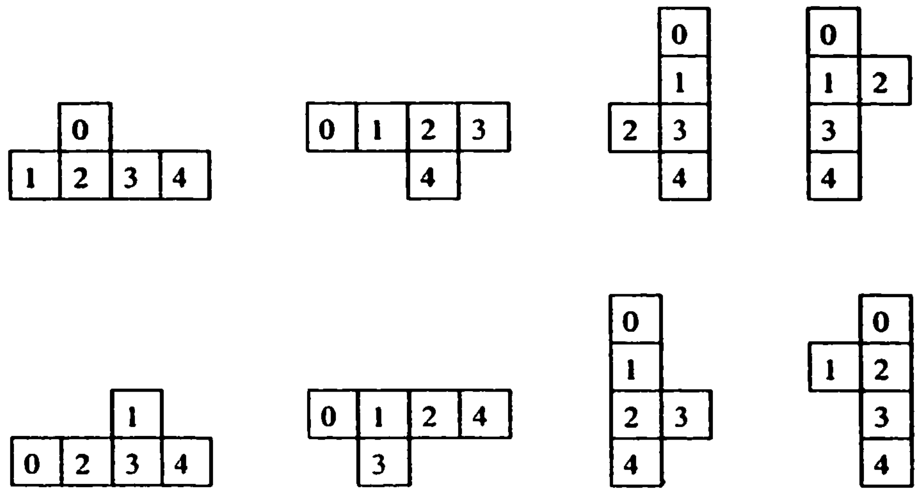


Figure C2: Ordering squares the Y-pentomino.

Then we can only fill the corner with the square of a copy of the Y-pentomino that is labeled 0. Note that if we could put such a copy to fill the corner in some other way, then the square labeled 0 of that copy would have gone to a square of the board that should have been occupied by this step.

It is easy to find the next corner efficiently. We just follow the labels of squares of our board in increasing order starting from the corner that was just filled. The first non-occupied square is the new corner.

Although the problem of covering boards with Y-pentominoes is completely solved, improving the described idea can be still of some interest (there are infinitely many unsolved tiling problems and the idea might be applicable to solving some of them).

The advantage of the diagonal approach as described above is that the impossible configurations are found pretty quickly. However, the process can be much improved by explicitly applying our knowledge about canonical unaccessible pairs and triples described in Subsection 2.5.6 of Chapter 2 and, possibly, throwing in some other heuristics.