# Invariants Associated to K-theoretic Methods and Complexity of Algebraic Cycle Groups

by

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**Abstract**

The Hodge conjecture can be formulated in terms of Algebraic Cycles Groups. That is, Hodge conjecture state that every Hodge class on X is algebraic. This tells us that is worthwhile to explore the complexity of the Algebraic Cycle Groups. On the other hand, it turns out that we can study algebraic Cycles from the point of view of K-theory and vice versa. The aim of thesis is to explore the complexity of Algebraic cycle groups and K-theoretic invariants and to build bridges between K-theory and Algebraic cycles. We will use the methods developed to capture arithmetic invariants.

To Mathematics and Science

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# 1 Preliminaries

## 1.1 Sheaves

Given a topological space X. There are several situations where the open subsets of X have some kind of algebraic structure. In fact, all manifolds admit such a structure. We will touch upon this later when we introduce locally ringed spaces. In such scenarios, we would like to capture invariants using the algebraic structure associated to the open sets; In a universal fashion, that is linking local data to global data. In this section we will introduce the notion of sheaves and examples, and use that to define a scheme.

To introduce sheaves, we need to introduce pre-sheaves. Intuitively, a pre-sheave is a gadget which associates to an open set an algebraic object, in a such way that such a structure forms a tower. The top of the tower is the lowest open set and as we go down we have bigger open sets. Formally we have the following definition. In all that follows below we shall fix a topological space X.

**Definition 1.1.** Fix a ring $K$. A presheaf $\mathbb{S}$ is given by the following datum:

- For each open set $U \subset X$ there is a K-module $\mathbb{S}(U)$
- If $V \subset U$, there exists $K - module$ morphism

$$p_{V,U} : \mathbb{S}(U) \to \mathbb{S}(V).$$

satisfying the following conditions:

– $p_{U,U} = id_{\mathbb{S}}$,

– If $W \subset V \subset U$, then $p_{W,U} = p_{W,V} \circ p_{V,U}$.

If we have two pre-sheaves $\mathbb{S}_1$ and $\mathbb{S}_2$, then morphisms of sheaves is just the most natural way of preserving both algebraic and topogical structures. That is, for each open sets $V \subset U \subset X$, the following diagram commutes:

$$\begin{tikzcd} \mathbb{S}_1(U) \arrow[r] \arrow[d, "p_{V,U}"'] & \mathbb{S}_2(U) \arrow[d, "r_{V,U}"] \\ \mathbb{S}_1(V) \arrow[r] & \mathbb{S}_2(V) \end{tikzcd}$$

In order for pre-sheaves to be useful we will need the notion of sheaves. Sheaves tells us how to pass from local to global data and vice versa.

**Definition 1.2.** A presheaf $\mathbb{S}$ is a sheaf if additionally it satsifies the following extra datum:

- (How to glue) If $s_i \in \mathbb{S}(U_i)$ and if $U_i \cap U_j \neq \emptyset$ we have that the following is satisfied:

  $$p_{U_i \cap U_j, U_i}(s_i) = p_{U_i \cap U_j, U_j}(s_j).$$

  for all i, then there exists an $s \in \mathbb{S}(U)$ such that $p^U_{U_i}(s) = s_i$.

- (Local morphism) If $s, t \in \mathbb{S}(U)$ and $p_{U_i,U}(s) = p_{U_i,U}(t)$ for all i, then $s = t$.

Morphisms of sheaves is the morphisms of the presheaf associated to them. We will work mainly with two specialized sheaves. Viz., locally free sheaves and coherent sheaves, as those will give us the machinery of homological algebra that we will need for the rest of the paper. The following defininition is taken from [43].

**Definition 1.3.** Let $\mathcal{R}$ be a sheaf of commutative rings over a topological space X.

- Define $\mathcal{R}^p$, for $p \geq 0$, by the presheaf

$$U \mapsto \mathcal{R}^p(U) = \mathcal{R}(U) \bigoplus \ldots \mathcal{R}(U).$$

  Here we take the direct sum p-times. $\mathcal{R}^p$, so defined, is clearly a sheaf of $\mathcal{R}$-modules and is called the direct sum of $\mathcal{R}$.

- If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules such that $\mathcal{M} \simeq \mathcal{R}^p$ for some $p \geq 0$, then $\mathcal{M}$ is said to be a free sheaf of modules.

- if $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules such that each $x \in X$ has a neighborhood U such that $\mathcal{M}|_U$ is free, then $\mathcal{M}$ is said to be locally free.

A natural question that might arise is that the definition of sheaves resembles the definition of vector bundles over a topological space X. It is true that the notions of locally free sheaves and vector bundles over connected topological space X are precisely the same. Formally, we have the following statement:

**Proposition 1.1.** Fix a connected topological space X. The category of locally free sheaves and the category of vector bundles over X are the same.

We will not prove the statement above, but the proof is very simple. One remark is in order though, the reason we require that the topological space X is connected is in order to have constant rank as we vary across sections.

Sometimes, we would like to work with sheaves that are a little more general than locally free sheaves. But, still be able to apply homological algebra machinery. For this, we need the notion of coherent sheaves.

**Definition 1.4.** Suppose we equip X with a sheaf of rings $\mathbb{O}$. A sheaf $F$ is coherent if it satisfies the following conditions:

- Each point $x \in X$ has a neighborhood $U_x$ such that there is a surjective sheaf morphism:

$$\mathbb{O}^n|_{U_x} \to F_{U_x}.$$

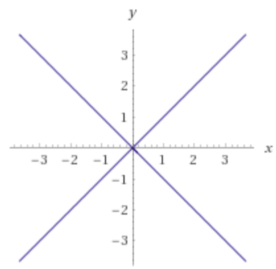
- The surjective map defined above has a finitely generated kernel.

## 1.2 Locally ringed spaces and Schemes

In analytic geometry we model things locally as Euclidean spaces. Such model only detects analytic structure. In order to detect an algebraic structure associated to the topological space we will define schemes. Before defining schemes, we will motivate things a little more to see why varieties fails to detect certain things and why we need this general notion of schemes.

One of the main things that varieties fails to capture is the notion of having more than one point associated to the fibers of an x shape. More precisely, if we look at the variety given by $V(x^2 - y^2)$ (figure given below). We

Figure 1




taken from Wolfram-Alpha

can see that if we project along the x-coordinate and look at the fibre above 0, then that is only 1 point above x = 0. In order to capture the geometry sharper, we would like to have the fibre above x = 0 come from two distinct points.

Another important reason is we want to define algebraic geometry without relying on being embeded in some space. If we recall the notion of manifolds, they are defined extrinically.

One final reason why we would want to work with schemes is because we would like to do geometry over rings such as $\mathbb{Z}$.

**Definition 1.5.** If X is given a sheaf of rings $\mathbb{O}$, then X is called a structure sheaf. X is called a locally ringed space if the stalks $\mathbb{O}_{X,x}$ over each $x \in X$ forms a local ring. That is, the stalk $O_{X,x} = lim_{x \in U} O_X(U)$ is a local ring. Here the direct limit it taken with respect to the maps $p_{U,V}$ and inclusions.

Let us see couple of simple examples of locally ringed spaces before we proceed:

- Suppose we consider $X = \mathbb{C}$. Define sheaf of rings $\mathbb{O}$ on X as follows: For each open set $U \subset X$, $\mathbb{O}(U)$ is the ring of complex continuous functions $\psi : U \to \mathbb{C}$.
- Suppose X is a general complex manifold, and $\mathbb{O}(U)$ is the ring of holomorphic functions $\psi : U \to \mathbb{C}$.

It is easy to see that the stalks at each $x \in X$ are local rings, where the maximal ideals at x is given by the functions which vanish at $x$.

Note that any manifold (Smooth, analytic, and complex) is a locally ringed space. The maximal ideal is given by the set of functions which vanish at x. From this perspective, we will see that schemes are generalizations of manifolds.

In order to define a scheme, we need something which plays the role of Euclidean sets from manifold theory. For this, we need to define the notion of affine schemes.

Suppose we fix a ring R. Let $X = Spec(R)$. Recall that the following set form a basis for the space:

$$D_f = \{P \in Spec(R) \ : \ f \notin P\}.$$

Construct a sheaf $\mathbb{O}$ on X by defining $\mathbb{O}(D_f) = R_f$. Recall, $R_f$ is the localization with respect to the multiplicative set $\{1, f, f^2, f^3, \ldots\}$. Moreover, for a point $x \in Spec(R)$, the stalk at x is given by $O_{X,x} = R_x$.

$(Spec(R), \mathbb{O})$ forms a locally ringed space, which is called affine scheme. Affine schemes are the atoms of schemes in the same way as Euclidean spaces are the atoms of abstract manifolds.

Let us see what we mean by locally isomorphic to affine schemes. First, let us define the notion of morphisms in the category of locally ringed spaces.

**Definition 1.6.** Suppose we have two locally ringed spaces $(X, \mathbb{O}_X)$ and $(Y, \mathbb{O}_Y)$. A morphism between them is something which respects set structure, topological structure, and algebraic structure. More precisely, it is given by the following datum:

- A continuous map $\psi : X \to Y$.

- For each open set $V \subset Y$, a homomorphism map

$$\psi_U : \mathbb{O}_Y(U) \to \mathbb{O}_{\mathbb{X}}(\psi^{-1}(U)).$$

  such that the map defined above is a sheaf mapping, i.e it commutes with the restriction mapping.

- For each $x \in X$, the map $\psi_x : \mathbb{O}_{(Y,\psi(x))} \to \mathbb{O}_{(X,x)}$ maps maximal ideal to maximal ideal.

Finally, we are ready to define schemes. Note, that the isomorphism below is in the category of locally ringed spaces.

**Definition 1.7.** A scheme $(X, \mathbb{O})$ is a locally ringed space, which is locally an affine scheme. That is, there exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of X such that each of the open set $U_\alpha$ are isomorphic to an affine scheme $(Spec(R_\alpha), \mathbb{O})$. We will assume that schemes are integral scheme of finite type over an algebraically closed field $\mathbb{K}$.

**Definition 1.8.** A morphism between schemes is just morphisms in the category of locally ringed spaces defined above.

# 2 Chow Group and algebraic cycles

We will follow similar expositions as [9] and [10]. Just as homology and homotopy theory is a functor from topological spaces to groups. It is possible to do the same thing for schemes in order to capture invariants for such objects and to do intersection theory. It turns out that such an invariant forms a homology theory for schemes called Chow groups.

In this section, we will discuss Chow groups, algebraic cycles, and properties of such objects. In particular functorial properties. In what follows we will assume that our schemes are Noetherian, i.e admits open covering by open affine subsets $Spec(R_i)$ such that $R_i$ is Noetherian ring. Varieties are reduced irreducible subschemes of X, and subvarieties are closed subschemes, which is itself a variety.

In ancient geometry people studied the shape of intersections of geometrical objects. Intersection theory studies intersections of schemes, i.e it studies the intersection of two closed subschemes of a scheme X. In order to be able to do intersection theory systematically we will work with algebraic cycles.

**Definition 2.1.** Suppose X is a scheme. We form a group of algebraic cycles on X by considering the free Abelian group of subvarieties of X. Such group is denoted by $Z(X)$. A cycle $\alpha \in Z(X)$ can be written as $\Sigma_i n_i Y_i$ where $Y_i$ is a subvariety of X and of course the sum above is finite.

Note, that to each subvariety $Y \subset X$ we can associate the cycle generated by it and is denoted by $(Y)$. Namely, if $Y \subset X$ is a subvariety, then we look at $Y_1, \dots, Y_s$ the irreducible components of the reduced scheme $Y_{red}$. Then, because the Noetherian condition each local ring $O_{Y,Y_i}$ has finite composition series. So, if we write $l_i$ for the length, then we can define the cycle associated to $Y$ as

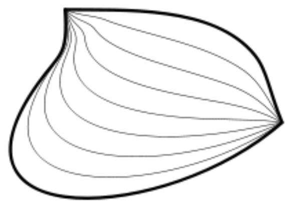
$$(Y) := \Sigma_i l_i Y_i.$$

Therefore, we can see that algebraic cycles are approximations of Schemes. We will later see how we can approximate Chow group using K-theory.

The group defined above is very large. Also, we don't have intersection theory. In order to fix those issues we will have to mod out by an appropriate equivalence relation. There is two ways to do this. One is clear for people who think more geometrically, and is conceptually clear. The other one is algebraic, we will present both definitions here.

Recall, in homotopy theory, we say two paths are equivalent if we can interpolate between them, i.e have the following picture:

Figure 2



taken from Wikipedia

We will define equivalence of cycles in similar fashion. The following definition is taken [9].

**Definition 2.2.** Two cycles $\alpha_1$ and $\alpha_2$ are rationally equivalent if there is family of cycles interpolating between them. That is, if we have a cycle $\psi$ in $\mathbb{P}^1 \times X$ whose restriction on the fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are $\alpha_1$ and $\alpha_2$. More precisely, Let $R(X) \subset Z(X)$ be the subgroup generated by the following formal difference

$$(\psi \cap (\{t_0\} \times X)) - (\psi \cap (\{t_1\} \times X)).$$

Where $t_0, t_1 \in \mathbb{P}^1$ and $\psi$ is a subvariety of $\mathbb{P}^1 \times X$not contained in any fiber $\{t\} \times X$. Then two cycles $\alpha_1$ and $\alpha_2$ are rationally equivalence if their difference is in $Rat(X)$, and two subschemes are rationally equivalent if their associated cycles are equivalent.

**Definition 2.3.** The Chow group $A(X)$ of $X$ is the following quotient:

$$A(X) = Z(X)/Rat(X).$$

It is easy to see that $A(X)$ is graded by dimension. That is, we have the following proposition:

**Proposition 2.1.** If X is a scheme then the Chow group of X is graded by dimension. Formally we have:

$$A(X) = \bigoplus_{k \geq 1} A_k(X).$$

Now, that we have defined the geometrical definition of Chow group. Let us define the algebraic version:

Let us first recall the definition of divisor function and rational equivalence to zero.

**Definition 2.4.** Suppose that $V$ is a n + 1 subvariety of X and f is a rational function on V. That is, $\psi \in R(V)^{\times}$. Then the divisor associated to f is defined as:

$$[\psi] = \Sigma_i ord_{U_i}(\psi)[U].$$

Where the sum is taken over all subvarieties of V of codimesion 1. Recall that $ord_U(\psi)$ is the order of the function f along the subvariety U, defined by the local ring $\mathbb{O}_{V,W}$

**Definition 2.5.** An algebraic cycle $\alpha$ on X is rationally equivalent to zero, if we have subvarieties $U_1, \ldots, U_n$ of X and rational functions $\psi_i$ for each $U_i$ such that

$$\alpha = \Sigma_{i=1}^{k}[\psi_i].$$

Then we can define the Chow group in similar fashion, but here we get the grading for free.

**Definition 2.6.** The Chow group of k-cycles on X is defined as the quotient of algebraic k-cycles by algebraic k cycles that is equivalent to zero. Then we have

$$A_*(X) = \bigoplus_{k\geq 1} A_k(X).$$

Note, it is easy to see the equivalence between the two notions defining rational equivalence. Given a rational function $\psi : V \to \mathbb{P}^1$, then if we look at the graph of $\psi$, it is a cycle in $V \times \mathbb{P}^1$; moreover, the difference over the fibers of 0 and $\infty$ is $ord(\psi)$. Thus, we get that one inclusion.

The other inclusion is has simliar line of reasoning. Given a cycle $\overline{W} \subset X \times \mathbb{P}^1$ whose fibers are $\alpha_0 - \alpha_\infty$, then if we project on the X axis. That will determine a rational function $\psi$ on $W$ such that $ord(\psi) = [\alpha_0] - [\alpha_\infty]$.

## 2.1 Examples of Chow group and rational equivalence

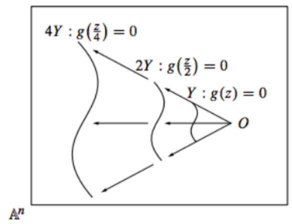
First of all, let us try to digest what it means to be rationally equivalent. Rational equivalence can be thought of as a very rigid homotopy between algebraic cycles. Using this intuition can get us little far in computing simple examples of Chow group. Recall, that any two path in $\mathbb{R}^n$ are homotopic to each. It is natural to ask the question what about two points in $\mathbb{A}^n$. Given any two points

$p_1$ and $p_2 \in \mathbb{A}^n$, then we can connect them by a line, so any two points are rationally equivalent.

A hyper surface is defined by the zeroes of a polynomial function. That is, we can think of it as a map $\phi : \mathbb{A}^n \to \mathbb{K}$. In fact, we can think of $\phi$ as a map $\phi : \mathbb{A}^n \to \mathbb{P}^1$. We can see that the fiber over $\{\infty\}$ is empty. Therefore, any hyper surface is actually rationally equivalent to the empty set. We can think of this as throwing the hyper surface away to infinity.

The first real example of Chow group is that of affine space. We have the following proposition which we shall prove. The idea is that we will show that any irreducible subvariety can be thrown away to infinity. That is, there is only one generator. We will make this more formal in the proof. The way we do this intuitively is that we project along arrows until we meet the irreducible subvariety. See figure below:

Figure 3




taken from [8]

**Proposition 2.2.** $A(\mathbb{A}^n) \cong \mathbb{Z}[\mathbb{A}^n]$

**Proof**: We will show that any proper subvariety $V \subset \mathbb{A}^n$ is rationally equivalent to the empty set. Since $V$ is a proper subvariety, so we may choose specific coordinate $z_1, \ldots, z_n$ such that $0 \notin V$. Consider the following family $\overline{V} \subset \mathbb{A}^n \times (\mathbb{A}^1 - \{0\})$ defined as

$$\overline{V} = \{(z, t) : \frac{z}{t} \in W\} = V(\{f(\frac{z}{t}) \ : \ f(z) \text{ vanishes on V}\}).$$

Geometrically, the fiber above t is just the stretching of V by a factor of $\frac{1}{t}$. If we take the closure of $\overline{V}$, then we will get a family in $\mathbb{A}^n \times \mathbb{P}^1$. If we look at the fiber above $t = 1$, then we will see that is precisely V. Since $0 \notin V$, there exists a polynomial function $g(z)$ that vanishes on V and has a non-zero constant term. Then, setting $F(t, z) = g(\frac{z}{t})$, which is a function on $(\mathbb{A}^1 - \{0\}) \times \mathbb{A}^n$ that can be extended to regular function on $(\mathbb{P}^1 - \{0\}) \times \mathbb{A}^n$.

Hence we can see that the fiber above $\infty$ is the empty set. Thus, we proved that any proper subvariety $V$ is rationally equivalent to $\emptyset$. $\square$

In fact, we can prove that for any open non-empty subset $U \subset \mathbb{A}^n$ we have $A(U) \cong Z[U]$. To do that we will need the following proposition:

**Proposition 2.3.** Let X be a scheme.

- If $X_1$,$X_2$ are closed subschemes of X, then there is a right exact sequence

$$A(X_1 \cap X_2) \to A(X_1) \bigoplus A(X_2) \to A(X_1 \cup X_2) \to 0.$$

- If $Y \subset X$ is a closed subscheme and $U = X - Y$ is its complement, then the inclusion and restriction maps of cycles gives a right exact sequence:

$$A(Y) \to A(X) \to A(U) \to 0.$$

If X is smooth, then the map $A(X) \to A(U)$ is a ring homomorphism.

Now, we are ready to prove that for open non-empty subset $U \subset \mathbb{A}^n$ we have $A(U) \cong \mathbb{Z}[U]$. Setting $Y = X - U$. Then, we have the following exact sequence from the proposition above:

$$A(Y) \to A(\mathbb{A}^n) \to A(U) \to 0.$$

Thus, the map $A(\mathbb{A}^n) \to A(U)$ is surjective. Therefore, we get the proposition. That is, we have $A(U) \cong \mathbb{Z}[U]$.

## 2.2 Push forwards and Pull-backs

Recall, in homotopy and homology theory for topology. The fundamental group and homology are both functorial. Given a map between topological spaces:

$$f : X \to Y.$$

This will induces a map:

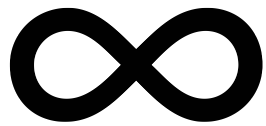
$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x)).$$

It is natural to ask the question if the same holds for Chow groups. Unfortunately, the situation for varieties is a little more restrictive. We will have to assume certain properties in order to get functorial properties. First, we will discuss push forwards, then present a way of integrating 0-cycles using push forwards. We will also see why we require the maps to be proper (to be defined below).

In a topological category, we know that continuous maps sends compact sets to compact sets. It is not necessarily true that inverse image of a compact set is a compact set under a continuous map. If a continuous map holds such a property, then it is said to be proper. The analogue in the category of schemes is proper morphism of scheme. In fact, we will see at the end of this subsection why we require proper morphism of schemes.

Let $\phi : X \to Y$ be a proper morphism of schemes. Given a subvariety $U$, $\phi(U)$ is a subvariety of Y, whose dimension is less than Y, as one would expect. We could define the pushforward to be just $\phi(U)$, but in order to take care of multiplicity we will have to add an extra component. If we look at the figure 8 (figure below), then the self-intersection should be associated to two points not one.

Figure 4



Drawn picture

**Definition 2.7.** Suppose that X and Y are schemes and $\phi : X \to Y$ is a proper morphism between them. Let $U$ be a subvariety, then define

$$\phi_*(U) = \begin{cases} mult(\phi_V)[\phi(V)] \text{ if } dim(\phi(V)) = dim(V), \\ 0 \text{ otherwise.} \end{cases}$$

where $mult(\phi_V) = [k(V) : k(\phi(V))]$.

Since an algebraic cycle is linear combination of subvarieties, so we can extend $\phi_*$ to a morphism between algebraic cycles on X and on Y. That is, we get the following extension $\phi_* : Z(X) \to Z(Y)$. Does this descend on the level of Chow group ? The answer is yes as the following theorem states:

**Theorem 2.1.** Let $\phi : X \to Y$ be a proper morphism of schemes. If $\beta$ is a cycle in X rationally equivalent to zero, then $\phi(\beta)$ is a cycle in Y, which is rationally equivalent to zero.

From the theorem above, we get an induced morphism on the level of Chow group that we will also denote by $\phi_*$. We have the following morphism:

$$\phi_* : A_k(X) \to A_k(Y).$$

Let us recall, a scheme over Y is just a scheme X equipped with a fixed proper morphism $\psi : X \to Y$. We call such $\psi$ the structure map. Suppose that we take $Y = Spec(k)$. Also, recall that $A_0(S) = \mathbb{Z}[S] \cong \mathbb{Z}$. Suppose that X is a scheme over Y and $\psi : X \to Y$ is the structure map. Then, from the theorem above we get the following morphism:

$$\psi_* : A_0(X) \to A_0(S) \cong \mathbb{Z}.$$

We can extend this morphism to the whole Chow group group by setting $\psi_*(A_k) = 0 \quad \forall \quad k \geq 0$. By abuse of notation we will call such an extension $\psi_*$. Thus, we get the following map:

$$\int_X := \psi_* : A_*X \to \mathbb{Z}.$$

We can ask the question if properness is necessary in order to define push-forwards. Suppose we consider $\mathbb{A}^1$, i.e the affine line. Then, we have seen earlier that any point in the affine is rationally equivalent to zero, however its push forward is not rationally equivalent to zero. Therefore, we get that properness is a necessary condition for push-forwards.

## 2.3 Pull-backs of algebraic cycles and Affine bundles

Recall that given any continuous map $f : X \to Y$. Any differential form on Y can be pulled back to a differential form on X. We would like to have similar property for algebraic cycles. In this section, we will discuss the similar notion in algebraic geometry; moreoever, we shall use the ideas that we will develop in this subsection to give another way of computing the Chow group of affine space in the next chapter.

First, let us recall what we mean by a morphism having relative dimension n. Suppose that $\psi : X \to Y$ is a flat morphism of schemes. Then, $\psi$ has relative dimension n, if for any subvariety $V$ $dim(\psi^{-1}(V)) = dim(V) + n$. In order to get nice functorial properties we shall assume that any morphism of schemes $\psi : X \to Y$ is flat. Then, for any closed subvariety $V \subset Y$ of dimension m, we have:

$$\psi^*(V) = (\psi^{-1}(V)).$$

Recall, here that $\psi^{-1}(V)$ is inverse image of scheme and $(\psi^{-1}(V))$ is the associated algebraic cycle associated to subscheme $\psi^{-1}(V)$.

The above map extends linearly to all algebraic cycles, i.e we get the following map which we will also denote by $\psi^*$ on the level of algebraic cycles.

$$\psi^* : Z_d(Y) \to Z_{d+n}(X).$$

We would like a theorem that tells us that the map above descends on the level of Chow group. As this would give us the required pull-back map as in differential forms.

In fact, we get that $\phi^*$ indeed does descend on the level of Chow group.

**Theorem 2.2.** Suppose that $\phi : X \to Y$ is a flat morphism of relative dimension n. If $\alpha_1$ is rationally equivalent to zero, then $\phi^*(\alpha_1)$ is rationally equivalent to

0.

From the theorem above we get that given a flat morphism of relative dimension n $\phi : X \to Y$. Then, we get an induced map on the level of Chow group. That is, we get the following map on the level of Chow group:

$$\phi^* : A_i(Y) \to A_{i+n}(X).$$

Is there any other functorial properties that are satisfied by pull-backs of algebraic cycle. Another important functorial property tells us how pull-back and push forwards interact with each other.

**Proposition 2.4.** Suppose that $\psi$ is flat and $\theta$ is a proper map of schemes. Then, if the square below

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\theta'"'] & X \arrow[d, "\theta"] \\ Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutes, then we have that $\theta'$ and $\psi'$ inherits the properties of $\theta$ and $\psi$, that is $\psi'$ is flat and $\theta'$ is proper. Also, we have that flat pullback commutes with proper push forwards on the level of cycles. That is, we have the following condition on the level of cycles:

$\phi'_* {\psi'}^* \alpha = \psi^* \phi_* \alpha$ where $\alpha \in Z_* Y'$.

**Proof**:

We will give the main idea of the proof the details are given in reference [9] and [10]. Some of the details are missing in [9] and [10], which we will explain here. First it suffices to consider the case where $X$ and $Y$ are varieties and $\theta$ is surjective. The reason it suffices, is that we can base change in order to work with surjective map and another base change to work with variety. After that, this turns into calculation in a local ring. The details are given in reference [9] and [10]. □

Another proposition which tells us how pull-back and push forward interact with respect to exact sequence is the following.

**Proposition 2.5.** Let Y be a closed subscheme of a scheme X, and let $U = X - Y$. Let $i : Y \to X$, $j : U \to X$ be the inclusions. Then, the sequence

$$A_kY \xrightarrow{i_*} A_kX \xrightarrow{j^*} A_kU \to 0,$$

is exact for all k.

**Proof**: It is easy to check that on the level of cycles we get that the following sequence is exact.

$$Z_kY \xrightarrow{i_*} Z_kX \xrightarrow{j^*} Z_kU \to 0.$$

Therefore, if we descend on the level of Chow group, then we get that the composition of two arrows is zero;moreover, we get exactness on the right. Finally, now we should prove exactness in the middle. Suppose that $\beta$ is a cycle in $Z_kX$ that is inside of the kernel of $j^*$. That is, $j^*(\beta) = 0$. Here, it makes sense to use the more algebraic version of rational equivalence. By definition, we then have that $\beta$ can be represented as $\beta = \Sigma_i[\psi_i]$.

Recall each $\psi_i$ is a rational function on $V_i$ subvarieties of U. Therefore, we get that $\psi_i$ we can think of it as a rational function in the closure of each $V_i$ in X. Denote that function by $\overline{\psi_i}$ for each i. Then, we get what we want that is, we get the following equality:

$$j^*(\beta - \Sigma_i[\overline{\psi_i}]) = 0.$$

Note that the composition of two arrows is zero gives the other inclusion. Therefore we are done.

□

## 2.4 Affine bundles

In this final section of the introduction to Chow group, we will talk about affine bundles which is similar to vector bundles in the algebraic category see section 3.4. Affine bundles is a non-linear version of vector bundles see section 3.4.

**Definition 2.8.** An affine $(E, X, \pi)$ is given by the following data:

- A scheme E.
- A scheme X.
- A morphism $\pi : E \to X$ that is of rank n over X.

Such that if X can be covered by open sets $U_\alpha$ then

$$\pi^{-1}(U) \cong U_\alpha \times \mathbb{A}^n.$$

Moreover, if we restrict ourself to a fiber of $\pi^{-1}(U_\alpha)$, then that correspond to projection from $U_\alpha \times \mathbb{A}^n$ to $U_\alpha$.

One might expect that the pull-back on the level of Chow group is surjective. That is indeed the case as the following proposition states.

**Proposition 2.6.** Let $p : E \to X$ be an affine bundle of rank n. Then, the flat pull-back

$$p^* : A_k(X) \to A_{k+n}(X),$$

is surjective for all k.

**Proof**:

see [8] and [9]

□

# 3 K-theory

## 3.1 Algebraic K-theory

Classical K-theory was developed in order to generalize linear algebra for projective R-modules. We know that every vector space has a basis, so we would like to have the same thing for projective R-modules and vector bundles. K-theory will be interesting, because we could do intersection theory at the level of K-theory, and we won't require moving lemma. Another reason is that K-theory is related to algebraic cycles, so we could do calculations at the level of algebraic cycles and then pass to K-theory. This provides a dictionary between K-theory and algebraic cycles. A good introduction to K-theory is [17],[18],[20],and [38].

## 3.2 Algebraic K-theory

In this section we shall develop K-theory for Abelian categories. Suppose $\mathcal{C}$ is an Abelian category. Let $I(\mathcal{C})$ denote free Abelian group generated by the isomorphism classes of $\mathcal{C}$. We quotient out by the relation $[A_1] = [A_0] + [A_2]$ if there is a short exact sequence:

$$0 \to A_0 \to A_1 \to A_2 \to 0.$$

We denote that group by $K_0(X)$. If we look at the category of coherent sheaves over a scheme X. That will give an invariant for X. It turns out that this invariant is related to algebraic cycles. We would like to also define higher K-theory. Let us recall a few basic properties from homotopy theory. Recall, that $[X, Y]$ denotes the homotopy classes of continuous maps from X to Y.

**Definition 3.1.** Suppose $n \geq 0$ and we are working with a pointed space $(X, x)$, then define

$$\pi_n(X, x) := [(\mathcal{S}^n, \infty), (X, x)].$$

Recall, that if X is path connected, then the isomorphism classes of $\pi_n(X, x)$ is indepedent of $x \in X$.

Recall, that a map $\phi : (X, x) \to (Y, y)$ between two pointed topological spaces induce a map on the level of homotopy. That is, we have the following map:

$$\bar{\phi}_n : \pi_n(X, x) \to \pi_n(Y, y).$$

A natural question that might arise is that if we have isomorphism for all n, then is it true that f is a homotopy equivalence. In general, the answer is no. Albeit, for special kind of spaces that are built combinatorially, then it is true.

**Theorem 3.1.** If $\phi : X \to Y$ is a continuous of CW complexes such that $\bar{\phi} : \pi_n(X, x) \to \pi_n(Y, \phi(x))$ is isomorphism for all $n \geq 1$, then $\phi$ is a homotopy equivalence.

One last thing we will need from homotopy theory is that a long exact sequence of homotopy maps is induced from fiberation map. That is, if we have a fiberation map

$$p : E \to B.$$

Then, that will induce a long exact sequence of homotopy groups: That is, if we choose a base point $x_0 \in B$. Let F be the fiber over $x_0$, and i being the inclusion mapping $i : F \to E$. Then, we have the following long exact sequence:

$$\ldots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots$$

Given an Abelian category, there is a construction due to Quillen that takes an Abelian category and changes into a simplicial category. Then, from that we can apply Milnor's geometric realization functor which constructs a topological space, and we define the K-theory being the homotopy groups of that topological space. First, we will define higher K-theory, then after that we will explain the construction, and maybe give an insight for how it was constructed. That insight is just speculative from the point of view of the writer of the thesis.

**Definition 3.2.** Suppose $\mathcal{C}$ is an Abelian category and let QP be the category obtained from $\mathcal{C}$ by applying the $Q-$ construction, then

$$K_i(\mathcal{C}) := \pi_{i+1}(BQP).$$

Where BQP is Milnor's geometric realization functor.

First, let us recall simplicial sets. Recall, in algebraic topology an elementary way of constructing homology is to build it out of maps from $\Delta_n$ into the space X. Sometimes it is useful to consider a set with some kind of ordering based on simplices. This is achieved through simplicial sets (see [11] for more explanation) .

**Definition 3.3.** Suppose we consider the category of standard simplices for which we denote by $\Delta$. This has objects given by $[n] = [0, 1, \ldots, n]$ and morphisms given by family of morphisms, that is we have the following family:

$$Hom([n], [m]) := \{increasing\ maps\ [0, \ldots, n] \to [0, \ldots, m]\}.$$

Recall, as we have in algebraic Topology we have the obvious maps:

$$\partial_i : [n-1] \to [n],$$

$$\tau_j : [n+1] \to [n].$$

A simplicial set is just a functor from

$$\Delta^{op} \to Sets.$$

Given a simplicial set it is possible to realize that geometrically by building a topological space out of the simplicial sets. Conceptually, the way we do this is by gluing the standard simplices through the simplicial sets.

Formally, we have the following definition which is due to Milnor.

**Definition 3.4.** Suppose that we have a simplicial set $S$, then construct the following set

$$[S] = \amalg_{n \geq 0} S_n \times \Delta^n / \sim$$

The equivalence relation is the one which glues together the simplices. We put the quotient topology on this final object [S].

If $\mathcal{C}$ is a category. We can define a simplicial set out of this category. Before introducing this, let us first look at an example which might provide us with some insight about how this is constructed.

Suppose X is a polyhedron. We would like to create some model on this X, where somehow gives us the barycentric division of X. More generally, given a category $\mathcal{C}$ we would like to construct a category that subdivides and linearize $\mathcal{C}$ such that we can associate a topological structure to such a construction.

This could be what motivated Quillen's construction, he wanted to take the homotopy group of a linearization of an Abelian category.

**Definition 3.5.** Suppose that $\mathcal{C}$ is a category. We define the nerve of the category denoted by $\mathcal{NC}$ a simplicial set. In order to define this simplicial set we need to know what it's n-simplices looks like. The n-simplices of $\mathcal{NC}$ are the set of n-tuples of composable morphisms in $\mathcal{C}$. That is,

$$\mathcal{NC}_n = \{C_n \to C_{n-1} \to \ldots \to C_0\},$$

Where $\partial_i$ and $\tau_i$ are defined in the obvious way. That is, define $\partial_i$ : $\mathcal{NC}_{n-1} \to \mathcal{NC}_n$ and $\tau_i : \mathcal{NC}_n \to \mathcal{NC}_{n-1}$ by composing two consequent maps for $0 < i < n$. If $i = 0$ or $i = n$ then the last map or first map. We define $\tau_i$ by repeating $C_i$ and inserting an identity.

We shall end this subsection by Quillen's Q- construction and important theorem relating Quillen's construction to classical K-theory. The construction is by Quillen himself and more detailed discussion can be found in reference [16]. In our case we will make things simpler by considering Abelian categories instead of exact ones. The idea is that we start with an Abelian category for which we collapse the morphisms by a specific equivalence relation. This goes back to the idea of linearizing the category.

**Definition 3.6.** Suppose $\mathcal{C}$ is an Abelian category. We shall define a new category $\mathcal{QC}$. The objects of $\mathcal{QC}$ are the same objects of $\mathcal{C}$. The morphisms are defined as

$$Hom_{\mathcal{QC}}(P, Q) = \{P \leftarrow X \rightarrow Q\}/\sim,$$

Such that the map $P \leftarrow X$ is epimorphism and the morphism $X \rightarrow Q$ is a monomorphism. Two morphisms $P \leftarrow X \rightarrow Q$ and $P \leftarrow X' \rightarrow Q$ are related if they fit into the following commutative diagram:

$$\begin{tikzcd}
P \arrow[d, "="'] & X \arrow[l, "p"] \arrow[r, "i"] \arrow[d, "\psi"] & Q \arrow[d, "="] \\
P & X' \arrow[l, "p"] \arrow[r, "i"] & Q
\end{tikzcd}$$

For smooth schemes X. It turns out that we can study K-theory of coherent sheaves using vector bundles. That is, if we consider vector bundles over a smooth scheme X, then K-theory of that will be the same as K-theory of coherent sheaves on X.

## 3.3 Introduction to Vector bundles

In this section we will study vector bundles and classify them. For our purpose we can study K-theory of coherent sheaves by studying K-theory of vector bundles. Therefore, it is worthwhile classifying vector bundles. We will head in this direction and classify vector bundles over a paracompact space X.

In finite dimensional linear algebra it is easy to classify things. There is only one isomorphism class per dimension. If we do linear algebra locally, then we get more interesting objects. This turns out to be the right recipe for doing geometry. The key ingredient for doing linear algebra locally is vector bundles. Essentially, the problem of classifying all vector bundles is still an open problem. We will classify vector bundles over a paracompact space X.

**Definition 3.7.** A $\mathbb{C}$-vector bundle $(\pi, E, \mathbb{C})$ of rank r over an $\mathbb{C}$-manifold X is given by the following datum:

- A holomorphic surjective map $\pi : E \to X$.
- Each fiber with respect to a point is a $\mathbb{C}$ vector space. That is,

  $$E_x := \pi^{-1}(x),$$

  is a vector space.
- Each fiber over an open set is locally a trivial bundle. That is, if $x \in X$ is a point. Then, there exists a neighorhood $U_x$ and a homeomorphism

  $$h_x : \pi^{-1}(U_x) \to U_x \times \mathbb{C}^r,$$

  such that $h_x(E_x) \subset \{x\} \times \mathbb{C}^r$. Moreover, we have that the composition below is an isomorphism:

  $$h_x{}^p : E_x \to \{p\} \times \mathbb{C}^r \to \mathbb{C}^r.$$

An important class is that of the pull-back bundle.

Let $\pi : E \to X$ be a vector bundle. Let $\phi : X' \to X$ be a continuous map. Construct the pull-back bundle as follows:

$$\phi^\star E = \{(x', e) \in X' \times E : \phi(x') = \pi(e)\} \subset X' \times E.$$

Equip $\phi^\star E$ with the subspace topology, consider the projection map onto the first factor and call that map $\pi'$. Thus, we have constructed a new vector bundle $\phi^\star E$.

Moreover, by construction the following diagram commutes:

$$\begin{array}{ccc} \phi^\star E & \xrightarrow{\pi_2} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\phi} & X \end{array}$$

**example 3.1.** A good visual example of the pull-back bundle is achieved with the Möbius band. Let E be the Möbius considered as a real band over the circle and let $\phi$ from the figure eight to circle achieved by imagining the figure eight as the wedge of two circles. The induced bundle $\phi^\star E$ is then two Möbius bands glued together at a single fiber.

The best visual way to view vector bundles is as smoothly varying vector spaces. For each point $x \in X$ in the base space, there is an associated vector space and as we move from vector space to vector space, the family varies smoothly. This is precisely what condition 3 above says.

Given the data of an $\mathbb{S}$ vector bundle $(\pi, E, \mathbb{C})$ of rank r. This defines gluing data. That is, if $x \in U_\alpha \cap U_\beta$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
U_\alpha \cap U_\beta \times \mathbb{C}^r \arrow[r, "h_\alpha^{-1}"] \arrow[dr, "\pi_1"'] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "h_\beta"] \arrow[d, "\pi"'] & U_\alpha \cap U_\beta \times \mathbb{C}^r \arrow[dl, "\pi_1"] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

From The commutative diagram above we get a smooth map $g_{\alpha\beta}$ as follows:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl_r(\mathbb{C}).$$

These maps satisfies the following compatibility conditions:

- $g_{\alpha\beta} g_{\beta\eta} g_{\eta\alpha} = I_r$ on $U_\alpha \cap U_\beta \cap U_\eta$,
- $g_{\alpha\alpha} = I_r$ on $U_\alpha$,
- $g_{\alpha\beta} g_{\beta\alpha} = I_r$.

The first condition is called Čech cocycle condition. Those maps are called transition maps. Geometrically, what is happening here is that the vector bundles are locally trivial. Globally things are twisted and the way they are twisted is precisely through these transition maps $g_{\alpha\beta}$.

Vector bundles are built from trivial bundles along with gluing data coming from transition maps. Given a vector bundle we get transition maps, also the converse is true. Suppose that to each non-empty intersection $U_\alpha \cap U_\beta$ we assign an $\mathcal{S}$-function:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl_r(\mathbb{C}).$$

Those $g_{\alpha\beta}$ satisfies the compatibility conditions above. Then we can construct a $\mathbb{S}$ vector bundle $(\pi, E, \mathbb{C})$ having these as the transition maps. The way we do this as as follows:

$$\bar{E} = \bigcup_{\alpha} U_\alpha \times \mathbb{C}^r.$$

Equip $\bar{E}$ with the product topology and regular manifold structure. Then, we define an equivalence relation on $\bar{E}$ as follows:

$$(x, v) \sim (u, s) \Longleftrightarrow x = u \textit{ and } s = g_{\alpha\beta}(x)v.$$

Here $(x, v) \in U_\beta \times \mathbb{C}^r$ and $(u, s) \in U_\alpha \times \mathbb{C}^r$.

This is well-defined equivalence relation because of the compatibility conditions. Finally, define $E = \bar{E}/\sim$ equipped with the quotient topology and let $\pi : E \to X$ be the natural map. Then, it is easy to show that E carries an $\mathbb{S}$ structure and is a $\mathbb{S}$ vector bundle.

We will define the Universal bundle. In order to define a topology on it we need to define Stiefel manifold.

**Definition 3.8.** Define the Stiefel manifold $V_n(\mathbb{C}^k)$ to be the space of orthonormal n-frames in $\mathbb{C}^k$. This is a subspace of the product of n copies of the unit sphere $\mathbb{S}^{k-1}$. This is a closed subspace as one can readily verify. It is also compact since the product of n copies of the unit sphere $\mathbb{S}^{k-1}$ is compact.

Now we are ready to define the Universal bundle and put a topology on it.

**Definition 3.9.** Define the Grassmannian manifold $G_n(\mathbb{C}^k)$ for nonnegative integers $n \leq k$ as the collection of all n-dimensional vector subspaces of $\mathbb{C}^k$. Given $w \in G_n(\mathbb{C}^k)$, then that means it is an n-dimensional vector spaces passing through the origin in $\mathbb{C}^k$.

There is a natural surjection $\pi : V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ sending an n-frame to the subspace it spans. We can equip $G_n(\mathbb{C}^k)$ with the quotient topology with respect to this surjection.

Recall, we can define $\mathbb{C}^\infty = \bigcup_{k\geq 1} \mathbb{C}^k$. Thus, doing the same thing for Grassmannians, we can construct the infinite Grassmannian space as $G_n(\mathbb{C}^\infty) = \bigcup_{k\geq 1} G_n(\mathbb{C}^k)$. The topology we equip $G_n(\mathbb{C}^\infty)$ is the topology induced from the inclusion maps. There is a canonical vector bundles over $G_n(\mathbb{C}^k)$ and $G_n(\mathbb{C}^\infty)$. Define $E_n$ as follows:

$$U_n(\mathbb{C}^k) = \{(l, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k : v \in l\}.$$

As, earlier define $U_n(\mathbb{C}^\infty) = \bigcup_k U_n(\mathbb{C}^k)$. Equip $U_n(\mathbb{C}^\infty)$ with the topology induced from the inclusion maps.

**Lemma 3.2.** The projection map $\pi : U_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ given by $(l, v) \mapsto l$, is a vector bundle for finite and infinite k.

To classify vector bundles we will deal with the case when k is infinite. In this case simplify the notation by writing $U_n(\mathbb{C}^\infty)$ and $G_n(\mathbb{C}^\infty)$ as $U_n$ and $G_n$ respectively. It turns out that we can probe the geometry of vector bundles through homotopic methods. This gives us a topological classification for vector bundles. If we would like to also classify holomorphic vector bundles, then there is extra work that has to be done, as we don't have partition of unity. Please refer to [12] for the case of classifying holomorphic vector bundles.

**Theorem 3.3.** For paracompact space X, the map $[X, G_n] \to Vect^n(X)$, $[f] \mapsto f^\star U_n$ is a bijection. Moreover, $G_n$ is called the classifying space.

We will provide a heuristic intuition for why it is actually $G_n$, which is used as the classifying space. Let us say that we know that $[X, Y]$ is in bijection with $Vect^n(X)$ and we want to find out what that space $Y$ might be. We can think of vector bundle as assigning a vector space to each point. Therefore we want Y to be the space of all n-dimensional vector spaces.

We shall give a sketch of the proof for this theorem. Because of the theorem above, the bundle $\pi : U_n \to G_n$ is called the universal bundle, as it can be used to classify all vector bundles over paracompact base. Most topological spaces ocuring in nature are paracompact. We have to try try hard to construct a non paracompact topological space.

**Sketch**: First, we need to show that the above map is well-defined. That is, we want to show the following proposition:

**Proposition 3.1.** If $f, g : X \to Y$ are homotopic and $E \to Y$ is a vector bundle over $Y$, then $f^\star E$ and $g^\star E$ are isomorphic.

Suppose we have a homotopy $h_t$ interpolating between $f$, consider the homotopy $h_t^\star E$. This contains an interpolation between $f^\star E$ and $g^\star E$. We can check then that they are isomorphic by building the isomorphism using partition of unity.

The whole idea of the proof of the theorem depends on the following idea:

For an n-dimensional vector bundle $\pi : E \to X$, an isomorphism $E \cong f^\star(E_n)$ is equivalent to a map $g : E \to \mathbb{R}^\infty$ that is linear injection on each fiber.

In order to verify the above, we just consider the diagram and in one direction if we trace through the diagram we get the required injection on each fiber. In the other direction, we see that linear injection on each fiber defines a map by pulling it back along the fiber.

The surjectivity of the map above comes from that we can use the criterion above along with partition of unity in order to build the required vector bundle.

For injectivity we use the criterion above and building the homotopy by mimking the straight line homotopy. After that we pull it back and that will give us that the original maps are homotopic. Thus, invoking proposition 3.1

we get that their pull-back must be isomorphic. □

Before, moving on to invariants that we can capture using Chow groups. First we will note the following important relationship between cycles and K-theory.

**Theorem 3.4.** There is an isomorphism between the zeroeth K-theory of X and algebraic cycles. More explicitly, we have the following isomorphism:

$$ch : K_0(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}.$$

Having understood algebraic cycles using K-theoretic methods. A natural question is if there is cycle theoretic interpretation of higher K-theory. This led to higher Chow groups and isomorphism between higher K-theory.

# 4 Invariants

In this section, we will see how we can detect invariants using algebraic cycles, in order to detect those invariants. First, we need some filtration on the level of Chow groups. In the next few sections we will develop enough machinery to show that such filtration exists. We will restrict to the case of $X = X/\mathbb{C}$ projective algebraic manifold of dimension d for simplicity. In the next section we will develop complexes of sheaves, sheafification, and sheaf cohomology.

## 4.1 More sheaf theory

From a presheaf $\mathcal{F}$ we can create a sheaf associated to it. This is called sheafification. There are many ways to do this. From personal bias the best way to do the sheafification construction is using the notion of Etale spaces. That is, one can think of a sheaf as a special topological space where the algebraic data are living over the fibers. We will follow Etale space theory presented in [42].

**Definition 4.1.**

- An Etale space over a topological space X is a topological space Y together with a continuous surjective mapping $\pi : Y \to X$ such that $\pi$ is a local homeomorphism.
- a section of an Etale space $Y \to X$ over an open set $U \subset X$ is a continuous map $f : U \to Y$ such that $\pi \circ f = 1_U$. The set of sections over U is denoted by $\Gamma(U, Y)$.

From the above definition we can see that sections of Etale spaces form a subsheaf of continuous sections from X to Y. We will use Etale spaces in order to straighten out a presheaf into a sheaf. Suppose we are given a sheaf $\mathcal{F}$ over a topological space X, then we can define the stalks over each $x \in X$ as follows:

$$\mathcal{F}_x := \varinjlim_{x \in U} F(U).$$

The direct limit is taken with respect to the restriction map. It is easy to check that using point wise operation that we will have an induced algebraic structure on the stalks. There is a natural map from the sections of the presheaf into the stalks defined as follows:

$$r_x^U : \mathcal{F}(U) \to \mathcal{F}_x,$$

$$s \mapsto s_x := [s].$$

If we set $\overline{\mathcal{F}} = \bigcup_{x \in X} \mathcal{F}_x$. There is a natural way to put a topology on $\overline{\mathcal{F}}$ such that the natural $\pi : \overline{\mathcal{F}} \to \mathcal{F}$ is an Etale space. For the sake of completeness we will show how it can be done. First we define a function $\overline{s}$ for each $s \in \mathcal{F}(U)$:

$$\overline{s} : U \to \overline{\mathcal{F}},$$

$$\overline{s}(x) = s_x \ for\ each\ x \in U.$$

It is easy to see that $\pi \circ \overline{s} = 1_U$ where $\pi : \overline{\mathcal{F}} \to X$ defined by mapping $\mathcal{F}_x \mapsto x$. Finally, define the basis for the topology of $\overline{\mathcal{F}}$ as:

$$\{\overline{s}(U)\} \text{ where U is open in X, s } \in \mathcal{F}(U).$$

It is easy to check that $\pi : \overline{\mathcal{F}} \to X$ is a Etale space over X. If we start with a sheaf $\mathcal{F}$ and take its sheafification, then that will be a sheaf isomorphic to $\mathcal{F}$. We have the following result:

**Theorem 4.1.** If $\mathcal{F}$ is a sheaf, then

$$\tau : \mathcal{F} \to \overline{\mathcal{F}},$$

is a sheaf isomorphism.

**Definition 4.2.** Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then one can form the quotient sheaf $\mathcal{Q}$ defined to be the sheaf associated(using sheafification above) to the presheaf:

$$U \mapsto \frac{\mathcal{F}(U)}{\mathcal{G}(U)}.$$

Given a sheaf $\mathcal{F}$, as we have algebraic data on the stalks, we can talk about the notion of exact sequence of sheaves.

**Definition 4.3.** If $\mathcal{A}$,$\mathcal{B}$, and $\mathcal{C}$ are sheaves over X, then morphism of sheaves

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C},$$

is said to be short exact sequence of sheaves iff the induced maps on stalks is exact for each $x \in X$. That is,

$$0 \to \mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x \to 0,$$

is exact for each $x \in X$. Similarly, we can define exactness at each position of the morphism; 0 denotes the (constant) zero sheaf.

A typical example is the exponential sequence:

**example 4.1.** Let X be a connected complex manifold. Let $\mathcal{O}$ the sheaf of holomorphic functions on X and $\mathcal{O}^*$ be sheaf of nowhere holomorphic functions on X. Then we have the following sequence:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0.$$

The maps defined are the ones that we would expect. It is easy to check that the above is a short exact sequence of sheaves. This is denoted by:

$$0 \to \mathcal{F} \to \mathcal{F}^\star.$$

It can be checked that the category of sheaves forms an Abelian category. Therefore, we can speak of the notion of graded sheaf. That is a graded sheaf $\mathcal{F}^*$ is a family of sheaves indexed by $\mathbb{Z}$:

$$\mathcal{F}^* = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^n.$$

A complex of sheaves is given by $(\mathcal{F}^*, d)$ such that $d^2 = 0$. Finally, a resolution of a sheaf $\mathcal{F}$ is completing $\mathcal{F}$ to an long exact sequence. That is, it is an exact sequence of sheaves of the form:

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

Recall from commutative algebra that taking hom functor is left exact. Similar situations occur for sheaves. Suppose we have the following short exact sequences of sheaves over X:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0.$$

If we take global section then we will get the following induced sequence:

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X).$$

That will be exact at every position, but not necessarily at $\mathcal{C}(X)$. We would like to use homological algebra in order to extend from local picture to global one. For that we will develop sheaf cohomology.

**Definition 4.4.** A sheaf $\mathcal{F}$ over a space X is soft if for any closed subset $S \subset X$ the restriction mapping

$$\mathcal{F}(X) \to F(S),$$

is surjective; That is, any local section can be extended to a global section.

**Definition 4.5.** Let X be a topological space. We say a presheaf of sets $\mathcal{S}$ is flabby or flasque if $\mathcal{S}(X) \to \mathcal{S}(U)$ is surjective for all open sets U in X. It can be shown that a flasque sheaf is soft.

Given a sheaf $\mathcal{F}$ we can construct a resolution of $\mathcal{F}$ where the resolution of the sheaf $\mathcal{F}$ is made up of soft sheaves. Let $\mathcal{S}$ be the sheaf $\mathcal{F}$ given in the form of an Etale space. That is, $(\mathcal{S}, \pi : \mathcal{S} \to X)$ is the Etale space associated to $\mathcal{F}$, which we recall is isomorphic to $\mathcal{F}$. Let $\mathcal{C}^0$ be the sheaf if sections of $\mathcal{S}$. That is given $U \subset X$ open $\mathcal{C}^0(U)$ to be the sheaf of discontinuous sections of $\mathcal{S}$

over X, that is:

$$\mathcal{C}^0(U) = \{f : U \to \mathcal{S} : \pi \circ f = 1_U\}.$$

Then by taking quotients and splicing the short exact sequence up we get the following long exact sequence:

$$0 \to \mathcal{S} \to C^0(\mathcal{S}) \to C^1(\mathcal{S}) \to \dots$$

Taking global sections of the above sequence and taking the cohomology of that we arrive at sheaf cohomology denoted by $H_{SH}(X, \mathbb{S})$. We will need the following concepts for Leray spectral sequence. Essentially given a map of topological spaces, we would like to have some way of pulling back a sheaf on Y to one on X, and pushing forward a sheaf on X on Y.

**Definition 4.6.** Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_*\mathcal{F}$ on Y $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subset Y$. For any sheaf $\mathcal{G}$ on Y, we define the inverse image sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$, where U is any open set in X, and the limit is taken over all open sets V of y containing $f(U)$.

## 4.2 Introduction to spectral sequences

In this section, we shall give an introduction to spectral sequence. Spectral sequence provides ways to calculate cohomology using smaller pieces of it. It is a discrete Riemann sum. Although it provides an elegant proof for many things in (co)homology theory, it is hard to conceptualize because it handles a lot of data. We will treat spectral sequence in the most general setting.

A complex $(K^{\star}, d) = \{K^0 \to K^1 \to K_2 \to \ldots\}$ is a sequence of objects in an Abelian category $\mathcal{C}$ with differentials

$$d : K^p \to K^{p+1},$$

such that $d^2 = 0$. Given a complex we can form a graded cohomology

$$H^{\star}(K^{\star}) = \bigoplus_{p \geq 0} H^p(K^{\star}),$$

where $H^p(K^{\star})$ is just cochains divided by coboundaries. That is,

$$H^p(K^{\star}) = ker(\{d : K^p \to K^{p+1}\})/dK^{p-1} = Z^p/B^p.$$

It is known that given a subcomplex $(J^{\star}, d) \subset (K^{\star}, d)$, then we get an induced sequence in the cohomology. We would like to generalize the notion of subcomplex and that long exact sequence in cohomology to something three-dimensional, which gives pieces of information about the cohomology in a discrete fashion. For simplicity, we assume that for $p$ big enough we have that the $K^m = 0$ for all, $m \geq p$. A filtered complex $(F^pK^{\star}, d)$ is a decreasing sequence of subcomplexes:

$$K^{\star} = F^0K^{\star} \supseteq F^1K^{\star} \supseteq \ldots \supseteq F^NK^{\star} = \{0\}.$$

Given a complex, we can induce a graded complex, which is something as smaller version of the original cohomology. That we have the following induced graded complex:

$$GrK^{\star} = \bigoplus_{p \geq 0} Gr^p K^{\star},$$

$$Gr^p K^{\star} := F^p K^{\star} / F^{p+1} K^{\star}.$$

The filtration above also induces a filtration on the cohomology $F^p H^{\star}(K^{\star})$ defined as:

$$F^p H^q(K^{\star}) = F^p Z^q / F^p B^q.$$

Therefore, we have the following associated cohomology:

$$GrH^{\star}(K^{\star}) = \bigoplus_{p,q} Gr^p H^q(K^{\star}),$$

$$Gr^p H^q(K^{\star}) = F^p H^q(K^{\star}) / F^{p+1} H^q(K^{\star}).$$

Finally, we are ready to define cohomological spectral sequence.

**Definition 4.7.** A cohomological spectral sequence is a sequence $\{E_r, d_r\}(r \geq 0)$ of bigraded objects in an Abelian category:

$$E_r = \bigoplus_{p,q \geq 0} E_r^{p,q},$$

together with differentials:

$$d_r : E_r^{p,q} \to E_r^{p+r,q+1-r} \qquad d_r^2 = 0,$$

such that

$$H^\star(E_r) = E_{r+1}.$$

Essentially with most spectral sequence as r gets big enough, then the spectral sequence converges. That is $E_r = E_{r+1} = E_{r+2} = \ldots$ for $r \geq r_0$ and we call the convergent limit $E_\infty$.

We can think of spectral sequence as three-dimensional grid, where for each r we have a plane of cohomological data. The planes are related to each other from the fact that we build them in an inductive fashion.

A natural question if given a cohomology, does there exist a spectral sequence that converges to it ? The answer is given by the following proposition:

**Proposition 4.1.** Let $K^\star$ be a filtered complex. Then there exists a spectral sequence $\{E_r\}$ that converges to the cohomology as r gets big enough. More precisely:

$$E_0^{p,q} = \frac{F^pK^{p+q}}{F^{p+1}K^{p+q}},$$

$$E_1^{p,q} = H^{p+q}(Gr^pK^\star),$$

$$E_\infty^{p,q} = Gr^p(H^{p+q}(K^\star)).$$

**Proof idea**: The whole idea is based on the fact that spectral sequence is algebraic discrete Riemann sum. Since we have already the initial term being defined, we can define the second term by using the cohomology of the first term which is precisely $H^{p+q}(Gr^pK^\star)$. We keep doing this process inductively, because the complex is bounded eventually everything collapses to $E_\infty^{p,q} = Gr^p(H^{p+q}(K^\star))$. $\square$

The above spectral sequence approximate regular cohomology. A natural question if there exists a spectral sequence that approximate generalized cohomology theory. The answer is answered by the following proposition:

**Proposition 4.2.** Given a generalized cohomology theory $K^{\star}$. Suppose that X is a finite CW-complex. Then we can approximate the generalized cohomology theory of X with respect to ordinary cohomology theory. That is, there is spectral sequence $\{E_r\}$ taking values in ordinary cohomology theory such that:

$$E_2^{p,q} = H^p(X; E^q(pt)) \Longrightarrow E^{p+q}(X).$$

**Proof idea**:

The proof is very similar to proposition 3.7 given the fact that we have an induced filtration coming from CW complex structure on X. However, a slick way to prove this from generalized spectral sequences satisfies axioms stated in reference [7] page 56. □

**example 4.2.** A very easy example to show the use of Atiyah-Hirzebruch spectral sequence is the calculation of $K^{\star}(\mathbb{CP}^n)$. Recall, from elementary algebraic topology we have:

$$H^p(\mathbb{CP}^k; V) = \begin{cases} V \; if \; p \; is \; even, with \; 0 \leq p \leq 2k \\ 0 \; otherwise. \end{cases}$$

Therefore by hypothesis of the spectral sequence we have

$$E_2^{p,q} = \begin{cases} \mathbb{Z} \ p, q \ even, 0 \leq p \leq 2k, \\ \\ 0 \ otherwise. \end{cases}$$

Therefore,

$$E_2^{p,q} \cong E_\infty^{p,q}.$$

Thus $K^\star(\mathbb{CP}^n)$ is isomorphic to its grading.

## 4.3 Čech cohomology

Čech cohomology provides us a way of looking at the cohomology complex of sheaves in way to capture topological datum associated to the space. This arises very naturally when we consider certain invariants associated to a space. One way to study a given space is to transport pieces of it and look at how we transported the space. We could associate an invariant to such transportation. It turns out that invariant is related to Chow groups and this is the direction that we will be heading towards. Hypercohomology allows us to do that. First we need the notion of Čech cohomology with coefficients in a sheaf.

Suppose $(X, \mathcal{F})$ is a sheaf on the complex manifold X. Here $\mathcal{F}$ is coherent sheaf on X. Suppose we have an open covering $\mathcal{U} = \{U_\alpha\}$ of X. We can define a cohomology theory on X which depends only on the topological structure of

X. That is a q simplex is an ordered collection of $q+1$ of the covering $\mathcal{U}$.

$$\sigma = (U_0, \dots, U_q).$$

Moreover, we have that $\bigcap_{i=0}^{q} U_i \neq \emptyset =: |\sigma|$. A q-cochain of $\mathcal{U}$ is a mapping $f$ from q-simplex $\sigma$ to $\mathcal{F}(|\sigma|)$. Set of cochains is denoted by $C^q(\mathcal{U}, \mathcal{F})$. There is a natural boundary operators $\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$. We can use this boundary operator to define Čech cohomology. More explicitly, define the coboundary operator :

$$\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}),$$

$$f\sigma \mapsto \Sigma_{i=0}^{q+1}(-1)^i p_{|\sigma|,\sigma_i} f(\sigma_i),$$

$$\sigma_i = (U_0, \dots, U_{i-1}, U_{i+1}, \dots, U_{q+1}).$$

It is easy to check that $\delta^2 = 0$. Moreover, we have the following cochain complex:

$$C^*(\mathcal{U}, \mathcal{S}) := \mathcal{C}^0(\mathcal{U}, \mathcal{S}) \to \dots \to \mathcal{C}^q(\mathcal{U}, \mathcal{S}) \to \dots$$

Then Čech cohomology is the cohomology of the cochain complex above. One of the main properties of Čech cohomology that we will use is the following. Given $\mathcal{B}$ refinement of $\mathcal{U}$. There is a natural group homomorphism:

$$\psi_{\mathcal{B}}^{\mathcal{U}} : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{B}, \mathcal{F}),$$

$$H^q(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F}).$$

The left-hand side is sheaf cohomology defined above.

## 4.4 Double complexes

**Definition 4.8.** Double complex is given by the following $(K^{\star,\star}; d, \delta)$: that is, we are given the following datum:

- $K^{\star,\star} := \bigoplus_{p,q \geq 0} K^{p,q}$,
- $d : K^{p,\star} \to K^{p+1,\star}$,
- $\delta : K^{\star,q} \to K^{\star,q+1}$,
- $d^2 = \delta^2 = 0$,
- $d\delta + \delta d = 0$.

From the last identity $d\delta + \delta d = 0$, we get the associated single complex $(sK^{\star}, D = d + \delta)$. There is a filtration on $(sK^{\star}, D)$ defined as:

$$F_1{}^{\theta} sK^n = \bigoplus_{p+q=n, p \geq \theta} K^{p,q},$$

$$F_2{}^{\theta} sK^n = \bigoplus_{p+q=n, q \geq \theta} K^{p,q}.$$

By earlier work, we see that there is two spectral sequences $\{A^r\}$ and $\{B^r\}$ that converges to the single complex $(sK^{\star}, D)$.

We will derive two spectral sequences that will be useful for us:

$$E_0^{p,q} = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}} \cong K^{p,q}.$$

Therefore, from the above computation we have:

$$H^{p+q}(F^p(sK^\star)) \cong H^{p+q}(K^{p,\star}) \cong H^q_\delta(K^{p,\star}).$$

Given $A_1$ as above, we have a complex on it given by quotient of the operator $D$. Since $\delta = 0$ on $E_1$ we have that:

$$E_2^{p,q} \cong H^p_d(H^q_\delta(K^{p,\star})).$$

Conclusion, given a double complex $(K^{\star,\star}; d, \delta)$: we have two spectral sequences $\{A^r\}$ and $\{B^r\}$ :

$$A_2^{p,q} = H^p_d(H^q_\delta(K^{p,\star})),$$

$$B_2^{p,q} = H^p_\delta(H^q_p(K^{\star,q})).$$

**example 4.3.** Let $\varepsilon^k_X$ be the sheaf of germs of $C^\infty$ complex-valued forms (see chapter 5) on $X$, and $E^k_X := H^0(X, E^k_X)$. One has a Hodge decomposition

$$\varepsilon^k_X = \bigoplus_{p+q=k} \mathcal{E}^{p,q}_X, \quad E^k_X = \bigoplus_{p+q=k} E^{p,q}_X.$$

The complex $(E^\bullet_X, d)$ is filtered by subcomplexes $(F^pE^\bullet_X, d)$, $p \geq 0$, where

$$F^pE^k_X = \bigoplus_{i+j=k, i\geq p} E^{i,j}_X, \ D := d = \partial + \bar{\partial}.$$

$[d^2 = 0$, hence by type, $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0.]$ Explicitly,

$$\begin{array}{c} F^pE^\bullet_X : 0 \to \cdots \to 0 \to F^pE^p_X \xrightarrow{d} \cdots \xrightarrow{d} F^pE^{2d}_X \\ \cap \\ E^\bullet_X : E^0_X \xrightarrow{d} \cdots \xrightarrow{d} E^{p-1}_X \xrightarrow{d} E^p_X \xrightarrow{d} \cdots \xrightarrow{d} E^{2d}_X \end{array}$$

The Hodge to de Rham spectral sequence is given by

$$E^{p,q}_1 := H^{p+q}(Gr^p_F E^\bullet_X) \Rightarrow H^{p+q}_{\mathrm{DR}}(X).$$

But

$$H^{p+q}(Gr^p_F E^\bullet_X) = H^q_{\bar{\partial}}(E^{p,\bullet}_X) =: H^{p,q}(X).$$

Degeneration at $E_1$ is a result of the equivalence of Laplacians ($\frac{\Delta_d}{2} = \Delta_{\overline{\partial}} = \Delta_\partial$).

## 4.5 Hypercohomology

Essentially we we apply what we constructed so far to sheaves. Suppose we have a bounded complex of sheaves $(\mathcal{S}^{\geq 0}, d)$. From this we have the Čech double complex $(C^\star(U, \mathcal{S}^\star), d, \delta)$. This induces in the same way a single complex $(sC^\star, D = d \pm \delta)$. More explicitly,

$$(sC^\star, D = d \pm \delta) = (M^\star := \bigoplus_{i+j=\star} C^i(\mathcal{U}, \mathcal{S}^j), D = d \pm \delta).$$

**Definition 4.9.** The k-th hypercohomology is defined to be the cohomology of the single complex

$$(M^\star := \bigoplus_{i+j=\star} C^i(\mathcal{U}, \mathcal{S}^j)).$$

More explicitly,

$$\mathbb{H}^k(\mathcal{S}^\star) = \varinjlim_{\mathcal{U}} H^k(M^\star).$$

**example 4.4.** Leray spectral sequence. This will be useful for us when we discuss Bloch-Beilinson filtration. Essentially the whole idea is that we will use Leray spectral sequence in order glue fibers and pass from local to global data. More on this in the last section of the thesis.

Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{F}$ a sheaf on $X$.

Recall we defined $f_*\mathcal{F}$. We will define now a spectral sequence associated to $f_*\mathcal{F}$.

Let us resolve $\mathcal{F}$ with a flasque resolution, of $\mathcal{A}^\bullet$,

$$0 \to \mathcal{F} \to \mathcal{A}^\bullet.$$

*Remark.* Note that $f_*\mathcal{A}$ is flasque for any flasque sheaf $\mathcal{A}$ on $X$.

Furthermore because of flasqueness,

$$\mathbb{H}^i(f_*\mathcal{A}^\bullet) = H^i_{SH}\big(\Gamma(Y, f_*\mathcal{A}^\bullet)\big) = H^i_{SH}\big(\Gamma(X, \mathcal{A}^\bullet)\big) \simeq H^i_{SH}(X, \mathcal{F}).$$

See section 5.1 which explains more the above equalities. The $E_2$-term of one of the Grothendieck spectral sequences associated to $\mathbb{H}^i(f_*\mathcal{A}^\bullet)$ is again, via flasqueness:

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(f_*\mathcal{A}^\bullet)) = H^p(X, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Keep in mind that $R^q f_*\mathcal{F}$, called the Leray cohomology sheaf, is really the sheaf associated to the presheaf:

$$U \subset Y \text{ open } \mapsto H^q(f^{-1}(U), \mathcal{F}).$$

• Suppose $f : X \to S$ is a smooth and proper morphism of smooth quasiprojective varieties over $\mathbb{C}$. Then it is well known, by Deligne (see [26]), that the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_*\mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q}),$$

degenerates at $E_2$. In the case where $X = S \times Y$, this yields the Künneth formula:

$$H^i(S \times Y, \mathbb{Q}) \simeq \bigoplus_{p+q=i} H^p(S, \mathbb{Q}) \otimes H^q(Y, \mathbb{Q}).$$

## 4.6 Arithmetic invariants and it's relationship to Chow groups

Given a space X of some kind we would like an algebraic way to do calculus on the space. One natural thing that we might do is move the space by transporting small chunks of it around and capturing invariants associated to such movements. This is made precise through the notion of Kähler differentials and Arithmetic De Rham cohomology.

We will do this algebraically first then after that we will surgically do it for smooth algebraic variety by using sheafification.

**Definition 4.10.** Let A be a ring, $\alpha : A \to B$ an A-algebra. An A-derivation is an A-linear map $B \to U$ that satisfies Leibniz rule:

$$d(fg) = fd(g) + gd(f) \text{ for all f,g } \in B.$$

We way we define differential forms below is really in the same way we do it for geometry. That is locally we have smooth functions with respect to scalar field $\mathbb{R}$. This will be precise in the definition below. The definition below is the abstract one from the explicit construction which satisfies the universal property below, then from that our comment will make sense.

**Definition 4.11.** A relative differential form is given by $(\Omega^1_{B/A}, d : B \to \Omega^1_{B/A})$.

That is it is given by the following datum:

- A B-module $\Omega^1_{B/A}$,
- A-derivation $d : B \to \Omega^1_{B/A}$.

such that the following universal property is satisfied for any B-module M with an derivation $d' : B \to M$ there is a unique B-module homomorphism $f : \Omega^1_{B/A} \to M$ such that $d' = f \circ d$.

We can construct relative differential form as follows. Let $\Omega^1_{B/A}$ be the elements generated by the symbols $\{db : b \in B\}$ quotient by the submodule generated by expression of the form:

- $d(b + b') - d(b) - d(b')$,
- $d(bb') - d(b)b' - bd(b')$,
- $da$.

Define $\Omega^p_{B/A} = \bigwedge^p \Omega_{B/A}$ and $\Omega^0_{B/A} := B$.

Given a smooth projective space X we can define arithmetic De Rham cohomology by gluing things using sheafification.

**Definition 4.12.** Let X be a smooth projective space over a field K. let $k \subset K$ be a subfield. Define the presheaf:

$$U \subset X \text{ Zariski open } \mapsto \Omega^p_{X(K)/k}(U) = \Omega^p_{\mathcal{O}_X(U)/k^*}(U).$$

Define Arithmetic De Rham cohomology to be $H^k_{DR}(X(K)/k) = \mathbb{H}^k(\Omega^*_{X(K)/k})$.

Now are ready to define Arithmetic Gauss Manin connection. Again here X is smooth projective defined over $K$. Define

$$Filt^m \Omega^p_{X(K)/k} := Im(\Omega^m_{K/k} \otimes_K \Omega^{p-m}_{X(K)/k} \to \Omega^p_{X(K)/k}).$$

Therefore we have:

$$Gr^m \Omega^p_{X(K)/k} \cong \Omega^m_{K/k} \otimes_K \Omega^{p-m}_{X(K)/k}.$$

We can easily see that we have the following short exact sequence:

$$0 \to \Omega^{m+1}_{K/k} \otimes \Omega^*_{X(K)/K} \to Gr^{m+1,m-1}\Omega^*_{X(K)/k} \to \Omega^m_{K/k} \otimes \Omega^*_{X(K)/K} \to 0$$

Passing that to hyper cohomology we get the following connecting map

$$\nabla_{X(K)/k} : \Omega^m_{K/k} \otimes H^i_{DR}(X(K)/K) \to \Omega^{m+1}_{K/k} \otimes H^i_{DR}(X(K)/K).$$

*Remark.* We have a filtration on $\Omega^*_{X(K)/k}$ given by $Filt^m \Omega_{X(K)/K}$. Computing the $E_1$ term of $H^*_{DR}(X(K)/K)$ we get

$$E_1^{p,q} = \Omega^p_{K/k} \otimes \mathbb{H}^q(\Omega^*_{X(K)/K}),$$

with $d_1 = \nabla_{X(K)/k}$. By Deligne ( See [25]) this degenerate at $E_2 \Longrightarrow \nabla^2 = 0$, therefore we get flat connection. By repeating the above argument we get

$$\nabla(\Omega^m_{K/k} \otimes F^p H^i_{DR}(X(K)/K)) \subset \Omega^{m+1}_{K/k} \otimes F^{p-1} H^i_{DR}(X/K).$$

This can be understood as the arithmetic Griffiths transversality. We will get to this important point later when we introduce Hodge theory. We will explain more about Griffiths transversality.

## 4.7 Deligne Cohomology

Given a cycle $[\eta] \in A^r(X)$. We would like to have a topological approximation of $[\eta]$. It turns out that given $[\eta]$ as specified. The fundamental class gives us an element in $H^{2r}(X, \mathbb{Z}(r))$. It is possible that this topological approximation fails. In that case $[\eta] = 0 \in H^{2r}(X, \mathbb{Z}(r))$, but $[\eta] \neq 0$ in $A^r(X)$. In that case we could try to detect the cycle using a secondary cycle class map known as the Abel-Jacobi map.

$$AJ([\eta]) \in J(H^{2r-1}(X, \mathbb{Z}(r)).$$

$J(H^{2r-1}(X, \mathbb{Z}(r))$ is the complex torus. Deligne cohomology combines both cycle class maps into a single object $H^{2r}_{\mathcal{D}}(X, \mathbb{A}(r))$. It's definition requires all of the concepts related to spectral sequence that we have studied so far. We will be interested in a specific filtration on Chow groups, which we will use in order to detect arithmetic invariants. In order to have this filtration we will need the notion of Beilinson Absolute Hodge cohomology, which is a natural extension of Deligne cohomology. The idea is that there will be a cycle class map between Chow group and Beilinson absolute Hodge cohomology. We will define Deligne cohomology below, which can be extended to Beilinson's absolute Hodge cohomology( see [26]).

**Definition 4.13.** Let $\mathbb{A} \subset \mathbb{R}$ be a subring and $r \geq 0$ an integer. We recall the Tate twist $\mathbb{A}(r) = (2\pi \mathrm{i})^r \cdot \mathbb{A}$, and declare $\mathbb{A}(r)$ a pure $\mathbb{A}$-Hodge structure of weight $-2r$ and of (pure) Hodge type $(-r, -r)$. We introduce the Deligne complex $\mathbb{A}_{\mathcal{D}}(r)$:

$$\mathbb{A}(r) \to \underbrace{\mathcal{O}_X \to \Omega_X \to \cdots \to \Omega_X^{r-1}}_{=:\Omega_X^{\bullet < r}}.$$

**Definition 4.14.** Deligne cohomology is given by the hypercohomology:

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) := \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(r)).$$

**example 4.5.** When $\mathbb{A} = \mathbb{Z}$, we have a quasi-isomorphism

$$\mathbb{Z}_{\mathcal{D}}(1) \approx \mathcal{O}_X^{\times}[-1],$$

hence

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq H^1(X, \mathcal{O}_X^{\times}) =: \mathrm{Pic}(X) \simeq \mathrm{A}^1(X).$$

There is a different way of defining Deligne cohomology, which is useful for our purposes.

**Definition 4.15.** Let $h : (A^\bullet, d) \to (B^\bullet, d)$ be a morphism of complexes. We define

$$\mathrm{Cone}\big(A^\bullet \xrightarrow{h} B^\bullet\big),$$

by the formula

$$\big[\mathrm{Cone}\big(A^\bullet \xrightarrow{h} B^\bullet\big)\big]^q := A^{q+1} \oplus B^q, \quad \delta(a,b) = (-da, h(a) + db).$$

$\mathrm{Cone}\big(\mathbb{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega^\bullet\big)[-1]$ is given by:

$$\mathbb{A}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-2} \xrightarrow{(0,d)} \big(\Omega_X^r \oplus \Omega_X^{r-1}\big)$$

$$\xrightarrow{\delta} (\Omega_X^{r+1} \oplus \Omega_X^r\big) \xrightarrow{\delta} \cdots \xrightarrow{\delta} (\Omega_X^d \oplus \Omega_X^{d-1}\big) \to \Omega_X^d$$

Using the holomorphic Poincaré lemma, one can show that the natural map

$$\mathbb{A}_{\mathcal{D}}(r) \to \mathrm{Cone}\big(\mathbb{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet\big)[-1],$$

is a quasi-isomorphism.

Thus

$$H_{\mathcal{D}}^k(X, \mathbb{A}(r)) \simeq \mathbb{H}^r\big(\mathrm{Cone}\big(\mathbb{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet\big)[-1]\big).$$

There is a cycle class map

$$cl_{r,0} : A^r(X) \to H_D^{2r}(X, \mathbb{Z}(r)).$$

The details can be found in [26]. These ideas can be extended to Beilinson Absolute Hodge cohomology [6]. Essentially the main idea of the filtration relies on the fact that we can spread our variety over a family. After that, we will capture pieces of the family using a cycle class map into Beilinson Absolute Hodge cohomology. Finally, glue everything together using spectral sequence.

## 4.8 Arithmetic cycle class map

We would like to have a connection between arithmetic De Rham cohomology and Chow groups. In order to have that we will need to define arithmetic cycle class map. Normally, we expect to have a connection because what we did above geometrically is that we have transported small chunks of the space using a connection. Given such a transportation we can associate a invariant. Chow groups provide a lot of information about the variety. We expect to also capture this invariant inside using a commutative diagram into the Chow group. We will build the theory required to give the idea of the proof.

Let $K \subset \mathbb{C}$ be a subfield containing $\overline{\mathbb{Q}}$, and consider our smooth projective variety X defined over $\mathbb{C}$. Let

$$\mathcal{K}^M_{r,X} = \mathcal{O}^*_X \otimes \ldots \otimes \mathcal{O}^*_X / < \tau_1 \otimes \ldots \otimes \tau_r : \tau_1 + \tau_j = 1, i \neq j >,$$

be the Milnor sheaf over X (this is discussed in detail in [25]). Then we have the following result (also discussed in [25]),

$$A^r(X) \simeq H^r_{Zar}(X, K^M_{r,X}) = \mathbb{H}^r(\mathcal{K}^M_{r,X} \to 0 \to 0 \ldots). \tag{1}$$

where, $A^r(X)$ is the Chow group taken with respect to co-dimension. There is the following cycle class map:

$$c_{r,K} : A^r(X/K) \to \mathbb{H}^{2r}(\Omega^{*\geq r}_{X(K)/\overline{\mathbb{Q}}}).$$

The way this is determined is as follows, there is a natural map

$$\mathcal{K}^M_{r,X} \to \Omega^{*\geq r}_{X(K)/\overline{\mathbb{Q}}}[r],$$

$$\{f_1, \ldots, f_r\} \mapsto \bigwedge_j dlog f_j,\ \ f_j \in \mathbb{O}^*_X.$$

The above map factors through a morphism of complexes:

$$(\mathcal{K}^M_{r,X} \to 0 \to 0 \to \ldots) \to \Omega^{*\geq r}_{X(K)/\overline{\mathbb{Q}}}[r].$$

Using the (1), we have the following cycle class map:

$$c_{r,K} : \mathrm{CH}^r(X/_K) \to \mathbb{H}^{2r}(\Omega^{\bullet \geq r}_{X(K)/\overline{\mathbb{Q}}}),$$

Recall, we have the connection:

$$\nabla_{X(K)/k} : \Omega^m_{K/k} \otimes H^i_{DR}(X(K)/K) \to \Omega^{m+1}_{K/k} \otimes H^i_{DR}(X(K)/K),$$

If we look at the Leray spectral sequence of $H^*_{DR}(X(K)/K)$ this will have $E_1$ term:

$$E^{p,q}_1 = \Omega^p_{K/k} \otimes \mathbb{H}^q(\Omega^*_{X(K)/K}).$$

so that $d_1 = \nabla := \nabla_{X(K)/k}$, which degenerate to $E_2$ by Deligne.

Similarly, there is a Leray spectral sequence associated to:

$$F^r H^{2r}_{DR}(X(K)/\overline{\mathbb{Q}}) := \mathbb{H}^{2r}(\Omega^{*\geq r}_{X(K)/\overline{\mathbb{Q}}}).$$

That will degenerate to $E_2$. Therefore, that will define a Leray filtration:

$$F^v_1 \mathbb{H}^{2r}(\Omega^{*\geq r}_{X(K)/\overline{\mathbb{Q}}}).$$

Now we are ready to define one of the invariants we will be interested in. Take $K = \mathbb{C}$. The invariants are defined as:

$$
\begin{multline}
\Omega^{v-1}_{K/\overline{\mathbb{Q}}} \otimes F^{r-v+1} H^{2r-v}_{DR}(X(K)/\overline{\mathbb{Q}}) \to \\
\Omega^{v}_{K/\overline{\mathbb{Q}}} \otimes F^{r-v} H^{2r-v}_{DR}(X(K)/\overline{\mathbb{Q}}) \\
\to \Omega^{v+1}_{K/\overline{\mathbb{Q}}} \otimes F^{r-v-1} H^{2r-v}_{DR}(X(K)/\overline{\mathbb{Q}}) \quad (2)
\end{multline}
$$

We also have the following sequence, which is the same as above; however, we don't consider filtration:

$$
\begin{multline}
\Omega^{v-1}_{K/\overline{\mathbb{Q}}} \otimes H^{2r-v}_{DR}(X(K)/\overline{\mathbb{Q}}) \to \\
\Omega^{v}_{K/\overline{\mathbb{Q}}} \otimes H^{2r-v}_{DR}(X(K)/\overline{\mathbb{Q}}) \\
\to \Omega^{v+1}_{K/\overline{\mathbb{Q}}} \otimes H^{2r-v}_{DR}(X(K)/\overline{\mathbb{Q}}) \quad (3)
\end{multline}
$$

We have two invariants coming from the sequence above. Intuitively, those invariants measure the rigidness of X. The full story is something like this; first, we move the space arithmetically as we developed in section 4.5, then after that taking cohomology in 1 and 2 measure the rigidness of X.

**Definition 4.16.** $\nabla DR^{r,v}(X(\mathbb{C}/\mathbb{Q})$, which is the cohomology of (3) is called the space of de Rham invariants. The space of Mumford-Griffiths invariants, $\nabla J^{r,v}(X(\mathbb{C})/\mathbb{Q})$ is given by the cohomology of (2).

Using a filtration of the Chow group $\{F^v(A^r(X/\mathbb{C})\otimes\mathbb{Q})\}_{v\geq 0}$ we will show that we can capture those invariants inside of the Chow group. We will prove that such a filtration exists. That is, we will have the following commutative diagrams:

$$\begin{tikzcd}
\left(Gr_F^v(A^r(X))\otimes\mathbb{Q}\right) \arrow[r] \arrow[d] & \nabla J^{r,v}(X(\mathbb{C})/\mathbb{Q}) \arrow[dl] \\
\nabla DR^{r,v}(X(\mathbb{C}/\mathbb{Q}) &
\end{tikzcd}$$

*Remark.* There is a natural map, which doesn't have to be injective:

$$\nabla J^{r,v}(X(\mathbb{C})/\mathbb{Q}) \to \nabla DR^{r,v}(X(\mathbb{C}/\mathbb{Q}).$$

The goal of the next chapter is to develop the machinery required to prove the filtration on the Chow group. Moreover, we will give the idea of the map from the Chow group to the space of de Rham invariants.

# 5 Hodge theory

## 5.1 Classical Hodge theory

In this section we will start with classical Hodge theory. After that, we will go into modern Hodge theory. We will explain a few things we didn't explain in details before. Let X be a projective algebraic manifold of dimension d over $\mathbb{C}$. Recall differential forms:

$$E_X^k := \mathbb{C} \ valued \ C^\infty \ k - forms \ on \ X.$$

Since X is a complex manifold. This differential forms breaks into (p,q) decomposition. That is, we can decompose $E_X$ as:

$$E_x^k = \bigoplus_{p+q=k} E_X^{p,q}, \quad \overline{E_x^{p,q}} = E_X^{q,p}, \{\ldots\} = \ complex \ conjugation$$

Locally, this can be represented as:

$$\Sigma_{|I| \ = \ p, |J| \ = \ q} f_{IJ} dz_I \wedge dz_J,$$

$$f_{IJ} - \mathbb{C} \ \textbf{valued} \ C^\infty \ \textbf{functions},$$

$$I = 1 \leq i_1 \leq \ldots \leq i_p \leq d,$$

$$J = 1 \leq j_1 \leq \ldots \leq j_q \leq d.$$

We have the following operator

$$d : E_X^k \to E_X^{k+1}.$$

Thus we can define a cohomology with respect to this operator called De Rham cohomology:

$$H_{DR}^k(X, \mathbb{C}) = \frac{ker\ d : E_X^k \to E_X^{k+1}}{dE_X^{k-1}}.$$

A natural question is if the (p,q) decomposition descends in cohomology level. After answering this, let us divert into De Rham theorem. We can prove it trivially using the machinery we developed. This theorem relates De Rham cohomology and singular cohomology of X. We will give the idea of the proof.

**Theorem 5.1** (De Rham)**.**

$$H^i(X, \mathbb{C}) \simeq H^i_{DR}(X, \mathbb{C}).$$

**Proof Idea:** A very fast way to prove this is using sheaf theory. We developed sheaf theory in details in the past couple of sections. It is known that whenever X is paracompact and Hausdorff, then sheaf cohomology can be calculated using global sections. More explicitly,

$$H^i_{SH}(X, \mathcal{F}) = \frac{ker(\Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{F}^{i+1}))}{Im(\Gamma(X, \mathcal{F}^{i-1}) \to \Gamma(X, \mathcal{F}^i))}.$$

Here $\mathcal{F}$ is a sheaf on X and $\mathcal{F} \to \mathcal{F}^*$ is a resolution by fine sheaf. Using Poincaré lemma we have the following exact sequence of sheaves:

$$0 \to \mathbb{C}_X \to E^0_X \to E^1_X \to \ldots \cong \mathbb{C}_X \to 0 \to 0 \to \ldots$$

The understood isomorphism above is quasi-isomorphism. Recall, $\mathbb{C}_X$ is the sheaf of locally constant functions on X. Since we have a resolution by fine sheaves, we get an isomorphism between De Rham cohomology and sheaf cohomology. Now, we can use a different resolution to get an isomorphism between sheaf cohomology and singular cohomology, concluding the proof.□

**Theorem 5.2** (Hodge decomposition)**.**

$$H^i_{sing}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^i_{DR}(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X).$$

**Proof Idea:** We choose a Riemannian metric on X, after that we define the Laplacian:

$$\Delta = dd^* + d^*d,$$

$d^*$ is the adjoint with respect to the Riemannian metric. By Hodge and Weyl any cohomology class has a unique representive that is harmonic; moreover, it lies in the kernel of $\Delta$. After that we show that a smooth projective variety X has a Kähler metric, because X can be embedded in projective space which has the natural Fubini-study metric, which is Kähler. Recall that a metric m is Kähler if it is d-closed

Any Kähler metric satisfies the Kähler identity:

$$\Delta = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = 2(\partial\partial^* + \partial^*\partial) = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}.$$

Therefore, a form is harmonic iff its associated (p,q) forms is harmonic. Combining all that with extra details we get Hodge decomposition (see [13]) .□

## 5.2 Abstract Hodge theory

**Definition 5.1.** Let $\mathbb{A} \subset \mathbb{R}$ be a subring. A $\mathbb{A}$-Hodge structure (HS) of weight $N \in \mathbb{Z}$ is given by the following datum:

- A finitely generated $\mathbb{A}$-module V, and either of the following two equivalent conditions:

- A decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=N} V^{p,q}, \quad \overline{V^{p,q}} = V^{q,p},$$

  where the line above $V^{p,q}$ is regular complex conjugation induced from the second factor on the complexified space.

- A finite descending filtration

$$V_{\mathbb{C}} \supset \ldots \supset F^r \supset F^{r+1} \supset \ldots \supset \{0\},$$

  satisfying

$$V_{\mathbb{C}} = F^r \bigoplus \overline{F^{N-r+1}}, \quad \forall r \in \mathbb{Z}$$

It is easy to see that both definitions are equivalent. We would like to equip Hodge structure $H_{\mathbb{Z}}$ with extra structure that allows us to move things around the cohomology. This will be useful for capturing the invariants.

**Definition 5.2.** A polarized Hodge structure of weight n consists of a Hodge structure $(H_{\mathbb{Z}}, Q)$ where Q is a quadratic form that satisfies Hodge-Riemann Bilinear relations. viz., we have that the following relations are satisfied:

- $Q(\theta, \phi) = (-1)^n Q(\phi, \theta)$,
- $Q(H^{pq}, H^{uv}) = 0$ if $(p, q) \neq (u, v)$,
- $i^{p-q} Q(\theta, \overline{\theta}) > 0$ for $\theta \in H^{p,q}, \theta \neq 0$.

Even though, the proof of Hodge decomposition relied on the fact of us choosing a Kähler metric. Deligne showed that such choice is irrelevant. First we need to build the category of polarizable Hodge structure. A morphism of Hodge structure is a morphism of the underlying space which respects the Hodge structure. More explicitly,

**Definition 5.3.** A morphism of Hodge structure $\theta : X \to Y$ of weight i is homomorphism of modules groups such that when taking the complexified space we have $\theta(H^{p,q}(X)) \subset H^{p,q}(Y)$.

**example 5.1.** $X/\mathbb{C}$ smooth projective. By Hodge decomposition $H^i(X,\mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of weight i.

The next example is very important, since cohomologies normally come with a twist.

**example 5.2.** $\mathbb{A}(r) := (2\pi\ i)^r\mathbb{A}$ is an $\mathbb{A}$-Hodge structure of weight -2r of pure Hodge type $(r,-r)$, called the Tate twist.

We would like to extend these ideas to singular varieties. Essentially the whole idea is that we might not have Hodge structure on the grading, since we have singularities. That doesn't allow Hodge structure by Betti numbers; however, once dividing out the grading we might remove the singularities and thus have a Hodge structure.

**Definition 5.4.** An $\mathbb{A}$-mixed Hodge structure ($\mathbb{A}-MHS$) is given by the following datum:

- A finitely generated $\mathbb{A}$-module $V_{\mathbb{A}}$,
- A finite descending "Hodge" filtration on $V_{\mathbb{C}} = V \otimes \mathbb{C}$,

$$V_{\mathbb{C}} \supset \ldots \supset F^r \supset F^{r+1} \supset \ldots \supset \{0\}.$$

- An increasing "weight" filtration on $V_{\mathbb{A}} \otimes \mathbb{Q} := V_{\mathbb{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$\{0\} \subset \ldots \subset W_{l-1} \subset W_l \subset \ldots \subset V_{\mathbb{A}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Moreover, the filtrations interact in that $\{F^r\}$ induces a pure Hodge structure of weight l on $Gr_l^W = \frac{W_l}{W_{l-1}}$.

We have the following result by Deligne.

**Theorem 5.3** (Deligne)**.** Let Y be a complex variety. Then $H^i(Y, \mathbb{Z})$ has a canonical and functorial $\mathbb{Z} - MHS$.

In order to build the category of mixed Hodge structure, we will need to know how the morphisms behave.

**Definition 5.5.** A morphism $\psi : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ of $\mathbb{A}$-MHS is an $\mathbb{A}$-linear map that respects both filtrations. That is, we have the following behaviour:

$$h(W_l V_{1,\mathbb{A}\otimes\mathbb{Q}}) \subset W_l V_{2,\mathbb{A}\otimes\mathbb{Q}},$$

$$h(F^r V_{1,\mathbb{C}}) \subset F^r V_{2,\mathbb{C}}, \forall r.$$

Deligne showed that the category of $\mathbb{A}$-MHS is an Abelian category. Conceptually, what we are doing using this MHS, is that we are linearizing the space in a bigraded fashion. We will use the following notation in the proof of the filtration on the Chow group.

**Definition 5.6.** Let V be an $\mathbb{A}$-MHS. Define

$$\Gamma_{\mathbb{A}} := hom_{\mathbb{A}-MHS}(\mathbb{A}(0), V),$$

$$J_{\mathbb{A}} = Ext^1_{\mathbb{A}-MHS}(\mathbb{A}(0), V).$$

Finally, we will need the following concept in order to make a connection to the invariants we constructed earlier. Those invariants intuitively measure the rigidness of the space. Since our purpose is capturing those invariants in the Chow group, we will have move small chunks of the Chow group in a similar way to those invariants. This is done using variation of Hodge structure. Recall, a local system $V$ on a space X is a locally constant sheaf of $\mathbb{Z}$ modules on X. We can equip $X$ with an integrable connection $\nabla$.

**Definition 5.7.** Let X be a complex manifold. A polarized variation of Hodge structure of weight n over X is given by the following datum:

- A local system $V_{\mathbb{Z}}$ over X of $\mathbb{Z}$ module of finite rank,
- A decreasing filtration $\mathcal{F}$ of $V = \mathcal{O}_X \otimes V_{\mathbb{Z}}$ by holomorphic subbundles,
- An integrable connection $\nabla$ on V,
- An existence of a flat bilinear form $S : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$.

Moreover, the data above interact with each other. $\nabla(\mathcal{F}^p) \subset F^{p-1}$ That is, we have that Griffiths transversality is satisfied. We have an induced pure Hodge structure of weight n on the stalks. That is for every $x \in X$, we have an induced

Hodge structure on $V_{\mathbb{Z},x}$.

The ideas presented above will give us a family of Hodge structures. We would like to extend these ideas to families with singularities. This extends to the theory of variations of mixed Hodge structures.

## 5.3 Bloch Beilinson filtration and invariants

In order to show that there is a commutative diagram into the invariants we constructed earlier, consider the following Bloch-Beilinson filtration theorem. In what follows we will assume that $K = \mathbb{C}$.

**Theorem 5.4.** Let X/K be a smooth projective variety of dimension d. Then for all r, there is a filtration, depending on $k \subset K$,

$$A^r(X_K, \mathbb{Q}) = F^0 \supset F_1 \supset \ldots \supset F^v \supset \ldots$$

which satisfies the following

- $F^1 = A^r_{hom}(X_k, \mathbb{Q})$
- $F^2 \subset KerAJ \otimes \mathbb{Q} : A^r_{hom}(X_K, \mathbb{Q}) \to J(H^{2r-1}(X_K(\mathbb{C}), \mathbb{Q}(r))$
- $F^{v_1}A^{r_1}(X, \mathbb{Q}) \bullet F^{v_2}A^{r_2}(X, \mathbb{Q}) \subset F^{v_1+v_2}A^{r_1+r_2}(X, \mathbb{Q})$ where $\bullet$ is the intersection product.
- $F^v$ is preserved under the action of correspondences between smooth projective varieties over $\mathbb{K}$.
- Let $Gr^v_F := \frac{F^v}{F^{v+1}}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \bigoplus_{p+q=2d}[\Delta_X(p,q)] \in H^{2d}(X \times X, \mathbb{Q}(d))$ are algebraic and defined over K. Then

$$\Delta_X(2d-2r+l, 2r-l)_*|_{Gr^v_F}A^r(X, \mathbb{Q}) = \delta_{l,v} identity$$

- Let $D^r(X) := \bigcap_v F^v$, and $k = \overline{\mathbb{Q}}$. If Bloch-Beilinson conjecture on the injectivity of the Abel-Jacobi map ($\otimes\mathbb{Q}$) holds for quasi-projective varieties defined over $\mathbb{Q}$, then $D^r(X) = 0$.

**Proof Idea:**

Essentially the main idea is as follows. First of all, we spread the variety over a family of varieties such that the fibers capture pieces of the original variety. After that, we glue all the pieces together using spectral sequence in order to pass from local to global data.

Before, proceeding let us write few an example of a spread.

**example 5.3.**

$$Y/\mathbb{C} = \mathrm{Spec}\left\{ \frac{\mathbb{C}[x,y]}{(\pi y^2 + (\sqrt{\pi}+4)x^3 + \mathrm{e}x)} \right\}.$$

$$\mathcal{S}/\mathbb{Q} = \mathrm{Spec}\left\{ \frac{\mathbb{Q}[u,v,w]}{(u-v^2)} \right\},$$

Set:

$$\mathcal{Y}_{\mathcal{S}} = \mathrm{Spec}\left\{ \frac{\mathbb{Q}[x,y,u,v,w]}{\big(uy^2 + (v+4)x^3 + wx, u - v^2\big)} \right\}$$

The inclusion

$$\frac{\mathbb{Q}[u,v,w]}{(u-v^2)} \subset \frac{\mathbb{Q}[x,y,u,v,w]}{\big(uy^2 + (v+4)x^3 + wx, u - v^2\big)},$$

defines a morphism $\mathcal{Y}_{\mathcal{S}} \to \mathcal{S}$, as varieties over $\mathbb{Q}$. Let $\eta \in \mathcal{S}$, be the generic point. Then

$$\mathbb{Q}(\eta) = \mathrm{Quot}\left( \frac{\mathbb{Q}[u,v,w]}{(u-v^2)} \right).$$

Note that the embedding

$$\mathbb{Q}(\eta) \hookrightarrow \mathbb{C}, \quad (u, v, w) \mapsto (\pi, \sqrt{\pi}, \mathrm{e}), \Rightarrow \mathcal{Y}_{\mathcal{S},\eta} \times \mathbb{C} = Y/\mathbb{C}.$$

The mathematical details of the proof idea is as follows. Consider the $\overline{\mathbb{Q}}$ spread $p : \mathcal{X} \to \mathcal{S}$ such that p is smooth and proper. Then $X$ is spread over the generic point $\eta$. That is $\mathcal{X}_\eta \simeq X$. Then there is a cycle class map ( refer to [6] for the details):

$$A^r(\mathcal{X}; \mathbb{Q}) \to \underline{H_{\mathbb{H}}}^{2r}(\mathcal{X}, \mathbb{Q}(r))$$

Recall that $E_\infty^{v,2r-v}(p)$ is the v piece associated to the Leray filtration of p. There is a filtration $\{F^v A^r(\mathcal{X}, \mathbb{Q})\}_{v \geq 0}$ such that

$$Gr_F^v A^r(\mathcal{X}; \mathbb{Q}) \hookrightarrow E_\infty^{v,2r-v}(p)$$

We have the following short exact sequence:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho) \to E_\infty^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) \to 0,$$

where

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) = \Gamma(H^\nu(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))),$$

$$\underline{E}_\infty^{\nu,2r-\nu}(\rho) = \frac{J(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}$$

$$\subset J(H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))).$$

[The inclusion is a result of the following short exact sequence:

$$W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)).]$$

By definition

$$F^\nu \mathrm{A}^r(X_K;\mathbb{Q}) = \lim_{\substack{\rightarrow \\ U\subset S/\overline{\mathbb{Q}}}} \mathcal{F}^\nu \mathrm{A}^r(\mathfrak{X}_U;\mathbb{Q}), \quad \mathfrak{X}_U := \rho^{-1}(U)$$

$$F^\nu \mathrm{A}^r(X_{\mathbb{C}};\mathbb{Q}) = \lim_{\substack{\rightarrow \\ K\subset\mathbb{C}}} F^\nu \mathrm{A}^r(X_K;\mathbb{Q})$$

Finally, we have what we want,

$$Gr_F^\nu \mathrm{A}^r(X_K;\mathbb{Q}) = \lim_{\substack{\rightarrow \\ U\subset S/\overline{\mathbb{Q}}}} Gr_{\mathcal{F}}^\nu \mathrm{A}^r(\mathfrak{X}_U;\mathbb{Q}),$$

$$Gr_F^\nu \mathrm{A}^r(X_{\mathbb{C}};\mathbb{Q}) = \lim_{\substack{\rightarrow \\ K\subset\mathbb{C}}} Gr_F^\nu \mathrm{A}^r(X_K;\mathbb{Q})$$

□

Finally, we could use the machinery we developed so for to prove the following theorem.

**Theorem 5.5.** The following commutative diagram exists:

$$\begin{tikzcd}
\left(Gr_F^v(A^r(X)) \otimes Q\right) \arrow[r, dashed] \arrow[d] & \nabla J^{r,v}(X(\mathbb{C})/\mathbb{Q}) \arrow[dl] \\
\nabla DR^{r,v}(X(\mathbb{C}/\mathbb{Q}) &
\end{tikzcd}$$

**Proof** See [27]. We get the invariants from $\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho)$ term of the spectral sequence.

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