University of Alberta

Quantum Aspects of Black Hole Interiors

By

Patrick R. Brady ©

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Open your eyes and see how beautiful the world is,
and how lucky we are who are alive!

KARL POPPER

to

Mam and Dad.

Abstract

Classical models of generic black hole interiors have made progress in unravelling the nature of the internal geometry up to the onset of singular behavior at the inner (Cauchy) horizon. At this lightlike hypersurface, which corresponds to infinite external advanced time, the "Coulomb component" $|\Psi_2|$ of the Weyl curvature diverges exponentially with advanced time.

As classical curvatures rise, quantum effects become significant. The way in which the evolution is influenced by quantum effects is investigated. The geometry near the Cauchy horizon singularity is nearly conformally flat, allowing a detailed analysis of semiclassical effects using a linearised stress-energy tensor for conformally coupled fields. The results suggest that vacuum polarization effects initially reinforce the classical growth of curvature. At sufficiently late times, however, the results predict damping of the classical approach to a singularity. Unfortunately this damping occurs too late to be of any practical importance. Thus it seems that quantum gravity will be needed to completely understand the end stages of gravitational collapse.

Preface

The results in this thesis were obtained during a number of collaborative projects at the University of Alberta between 1991 and 1993. The presentation of course suffers from my personal bias, and as such, some of the opinions expressed may not be shared by all those who were involved in the work.

Chapters 4 and 5 are loosely based on

> Warren G. Anderson, Patrick R. Brady, Werner Israel and Sharon Morsink, "*Quantum effects in black hole interiors*", Phys. Rev. Letters **70**, 1041-1044 (1993).

While the thesis was being prepared reference [69] appeared on the gravitation/quantum cosmology electronic bulletin board (e-mail: gr-qc@xxx.lanl.gov). In this article Campos and Verdaguer have independently derived some of the results in chapter 4.

The material in chapter 6 is based on

> Warren G. Anderson, Patrick R. Brady and Roberto Camporesi, "*Vacuum polarisation and the black hole singularity*", Class. Quantum Grav. **10**, 497-503 (1993).

with some alterations.

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The kayaking fanatics (Eric D., Eric S., Loretta, Rob and Roger) have provided some great days out during the past year. Thanks guys, I hope we can do it again sometime.

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Most important of all I thank my parents, Kay and Paddy Brady. Over the years they have encouraged me to follow my dreams, and have made me realise the infinite possibilities which life offers. This thesis would have been impossible without their unfailing support, their good advice, and their belief in me; once again thank you both.

Contents

List of Figures

Conventions

We follow the sign conventions of Misner, Thorne and Wheeler [21] throughout the thesis. Specifically the metric signature is $- + ++$, the curvature tensors are defined as

$$\begin{aligned} R^{\mu}{}_{\nu\alpha\beta} &= \partial_{\alpha}\Gamma^{\mu}_{\nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \\ R_{\nu\beta} &= R^{\alpha}{}_{\nu\alpha\beta} \end{aligned}$$

where $\Gamma^{\mu}_{\sigma\alpha}$ is the Christoffel symbol.

We will also use units in which $G = c = \hbar = 1$ unless it is necessary to restore one or more of them for clarity (e.g. chapters 5 and 6).

CHAPTER 1

Introduction

When Einstein [1] published his vacuum field equations in 1915 he had already derived from them the advance of the perihelion of Mercury using a difficult perturbative approach. The following year Karl Schwarzschild [2] discovered the exact solution to the field equations representing the external gravitational field of a static, spherically symmetric body. This solution allows a very simple derivation of the perihelion advance. In a letter to Schwarzschild, who was then serving in the German army, Einstein wrote [3]:

> I have read your paper with the utmost interest. I had not expected that one could formulate the exact solution of the problem in such a simple way. I liked very much your mathematical treatment of the subject. Next Thursday I shall present the work to the Academy with a few words of explanation.

Thus black hole physics was born only months after General Relativity itself. The Schwarzschild solution has since provided us with a great deal of information about black holes.

During the following forty years advances in the relativistic theory of massive stars were rather slow. In 1931 it was noted, by Chandrasekhar [4] and independently by Landau [5], that there was an upper limit to the mass which a cold star could have and continue to support itself against Newtonian gravity. Both Eddington and Landau realized that any star above the Chandrasekhar limit would, upon using all of its nuclear fuel, contract dramatically producing an object from which light could not even escape [5, 6]. However, there was great reluctance to accept that this could occur in nature. Einstein himself was vehemently opposed to the idea of stellar collapse, and in 1939 he constructed model solutions to his field equations which showed that *stationary* configurations of matter with mass M must have a radius $R > 2GM/c^2$ [7]. That same year however, Oppenheimer and Volkoff [8] completed calculations showing that there was an upper limit to the mass of a star in General Relativity too. In subsequent work with Hartland Snyder [9], Oppenheimer went on to examine the gravitational collapse of a massive star which produces a black hole. This work now serves as the paradigm of gravitational collapse.

It was during the 1960's and early 1970's that most of our present knowledge of black holes was obtained. Advances in the understanding of classical processes involved in gravitational collapse were dramatic. A series of theorems (see chapter 12 of [10] for a review) established the uniqueness of stationary black holes, showing that their only external attributes were mass, angular momentum and electric charge; Wheeler used the graphic phrase "black holes have no hair" to summarize these results. New global methods were employed to prove the singularity theorems [11, 12] and to derive other properties of black holes [12].

Then, in 1974, Hawking announced the dramatic result that black holes were not so black after all [13]. They emit radiation with a thermal spectrum by quantum mechanical processes. Suddenly there was a glimpse of a new era in black hole theory and a future unification of gravitation and quantum mechanics. Since then the investigation of quantum effects in the strong gravitational fields around black holes has been pursued with great vigour (see [14] and references therein); however our understanding of the cryptic clues that these investigations supply about quantum gravity is sadly lacking.

The inevitability of a singularity inside a classical black hole was proved by Penrose under certain reasonable physical conditions [11]. In the last decade we have learned a great deal about the type of singularity to be expected in generic black hole interiors. With the early observation of the instability of the Cauchy horizon, in charged and rotating black holes, by Penrose [15], and the subsequent work on linear perturbations in the fixed background geometries [16], it became clear that known exact black hole solutions are not valid deep inside the black hole core. Hiscock [17], Poisson and Israel [18], Ori [19, 20] and others have now brought us a better understanding of the classical end-state of gravitational collapse. The type of singularity discovered by Poisson and Israel [18] has been called a mass-inflation singularity. This thesis summarizes some further investigations of the black hole core, with an emphasis on quantum effects obtaining in the vicinity of mass-inflation singularities.

This chapter reviews some background in black hole physics which is needed to put the remaining chapters in perspective. The picture of black hole formation due to gravitational collapse is outlined in the next section by reviewing the Oppenheimer-Snyder collapse scenario and then proceeding to discuss the more general situation. The classical picture breaks down near singularities and on long time scales where quantum effects become important. Some comments are made on the importance of such effects in black hole theory. The chapter ends with an outline of the thesis.

1.1 Black hole formation

Spherical collapse

The gravitational collapse of a spherically symmetric star, which produces a black hole, was first considered by Oppenheimer and Snyder [9] in 1939. Their work was poorly understood, however, going unnoticed for a long time.

For simplicity they examined the collapse of a spherical ball of dust. That is, the internal pressures and internal physics of the star were neglected. This assumption, which may appear unjustified at first sight, leads to a qualitatively correct picture of gravitational collapse (see [21] section 32.7) although this was not realized until long after their original work.

Since there is no pressure, the matter particles in the star follow geodesics of the spacetime; they are subject only to the gravitational force. Making one further assumption, that the radiation outside of the star is negligible, the external gravitational field is uniquely determined, by virtue of Birkhoff's theorem [21], to be the Schwarzschild solution:

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \qquad\qquad f = 1 - \frac{2m}{r} . \tag{1.1}$$

m denotes the total mass of the star and $d\Omega^2$ is the line element on the unit sphere.

To understand the process of collapse it is only necessary to study the motion of the stellar surface located at $r = R(\tau)$ where τ is the proper time along the timelike geodesic it follows. The equation of motion of this surface is simply

$$\dot{R}^2 = -(1 - \frac{2m}{R}) + E^2 , \tag{1.2}$$

where E is a constant and a dot represents differentiation with respect to proper time. Integrating this equation shows that a star can collapse to $R = 2m$, the gravitational radius of the system, in a finite proper time. In fact, once it reaches this radius the surface of the star must continue to $R = 0$. The time of free-fall, as measured by an observer on the surface of the star, from the gravitational radius to the origin is approximately one day for a star of $\sim 10^{10} M_{\odot}$.

Thus, the star collapses to zero radius in a finite proper time. For an external observer things appear different, however. A long way from the black hole t is the proper time along an observer's path. The motion of the stellar surface parameterized by this time

$$\left(\frac{dR}{dt}\right)^2 = \left(1 - \frac{2m}{R}\right)\left[E^2 - 1 + \frac{2m}{R}\right] . \tag{1.3}$$

reveals an interesting fact. As the star's surface approaches the gravitational radius $dR/dt \rightarrow 0$. In fact it takes an infinite amount of time, t, for the star to reach $R = 2m$. An external observer therefore sees the star approach asymptotically to this surface.

This can be better understood in terms of light signals emitted from the star (see Fig. 1.1). As the star collapses, light emitted from the surface experiences an increasing gravitational field which tends to focus it. The closer the star is to $r = 2m$ the longer it takes for the emitted light to reach a distant observer O. In particular the light which is emitted at the gravitational radius $r = 2m$ takes an infinite amount of time to reach O. All subsequent light signals are focussed so strongly that they actually move towards $r = 0$. Therefore the external observer can have no knowledge of what happens to the star after it passes $r = 2m$ this surface is therefore referred to as the black hole *event horizon*.

It is important to realize that the event horizon has no invariant local significance. It is a global construct, being the inner boundary of the causal past of an observer who remains outside of the black hole for an infinite time.

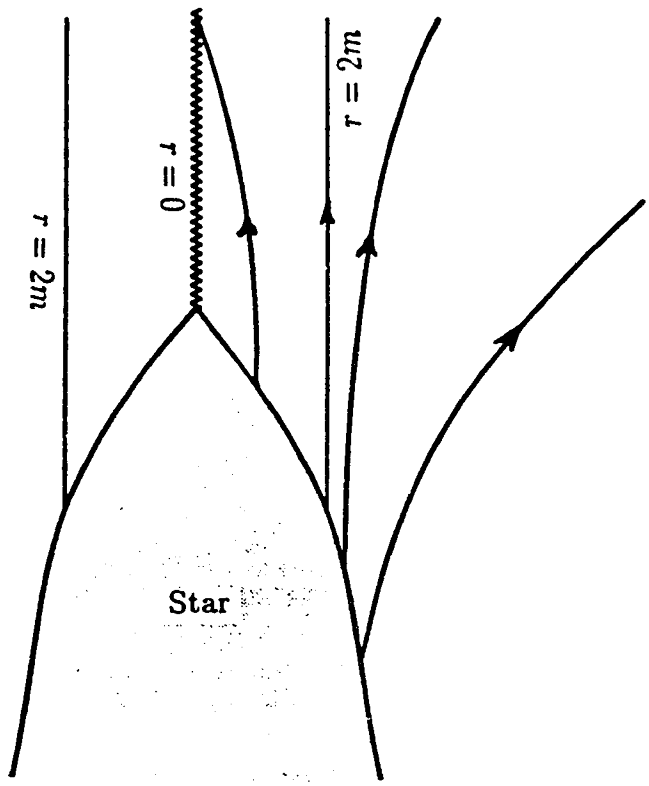

Figure 1.1: Gravitational collapse of a spherical star to black hole. The event horizon is labelled $r = 2m$ in this diagram. Outgoing lightrays are shown leaving the surface of the star and reaching the external observer O. Those emitted after the star passes through the event horizon actually hit $r = 0$

Thus we have two complementary views of gravitational collapse: for an external observer, the star appears asymptotically to approach the surface $r = 2m$ and never collapses further. Note that light from the star will be exponentially redshifted as its surface approaches $r = 2m$, making it quickly invisible to the external observer using optical devices. However the star actually passes, completely uninhibited, through the event horizon and reaches $r = 0$ where the curvature diverges. (In the Schwarzschild solution the Kretschmann invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48m^2/r^6 \rightarrow \infty$ as $r \rightarrow 0$.)

This picture remains qualitatively unchanged when the internal pressures of the star are non-zero (page 854 of [21]). Of course, no astrophysical collapse is expected to be spherically symmetric, so does the picture survive the inclusion of asphericities?

Spherical collapse with perturbations

During the 1960's many people argued that since black holes were exactly spherical they would not form in nature, however it gradually became clear that this was not true. An understanding of this fact was first achieved by numerical integration of the equations governing small perturbations of a spherical black hole [22, 23].

As the star collapses it radiates, by gravitational and other means, but certainly not fast enough to remove all the asphericities before it crosses the event horizon. Therefore one might think that an external observer should see a dirty black hole which is not exactly spherical. This is not the case. The method by which the black hole sheds this excess baggage was first elucidated in the work of Price [23].

In general the gravitational potential outside a black hole will scatter radiation which is present. Price noted that long wavelength radiation emitted from close to the black hole horizon ($2m < r < 3m$) is completely backscattered by the gravitational field and disappears down the black hole. Moreover, outgoing radiation emitted from close to the event horizon is redshifted by a very large factor as it moves to larger radii. Therefore information about the dirt on the black hole horizon which propagates outwards will become infinitely redshifted and is therefore completely backscattered down the black hole. It turns out that this backscattered radiation also interferes destructively with outgoing radiation, leading to a cleansing of the horizon and a Schwarzschild exterior at late times.

It might seem that the radiation that escaped from the collapsing star at earlier times (before it passed $r = 3m$) would simply escape to infinity. This is not the case though. It also gets scattered by the gravitational potential, leading to a diffusion of the radiation by a sequence of scatterings in the exterior of the black hole (see Gundlach *et al* [24] for a lucid exposition of this fact). Thus one finds a radiative tail of gravitational waves which decays with an inverse power law in external time. Along the event horizon of the black hole the stress-energy of the perturbations behaves like

$$\mathcal{F} \sim \sum_{l=2}^{\infty} P_l(\theta, \phi) v^{-4l-4} , \tag{1.4}$$

in terms of a multipole expansion where v is external advanced time. This fact is important for the mass inflation scenario [18] (see chapter 3).

General asphericities

Powerful global techniques [12] have been applied to prove that singularities cannot be avoided once a trapped surface* forms in collapsing matter [11]. Unfortunately these results tell us nothing about the nature of the singularities which are encountered, nor do they tell us whether an event horizon will seal them off from external observers (i.e. if a black hole forms or not).

Penrose [25] has conjectured that any spacetime singularity will be surrounded by an event horizon – calling this phenomenon *cosmic censorship*. The proof of this conjecture remains elusive today. Indeed there is a mounting body of counter-examples [26] which severely constrain attempts to formulate the conjecture as a mathematical theorem. Most of these *naked singularities* are regarded as unrealistic, however they do suggest that only some quite limited form of cosmic censorship may be true (e.g. Thorne's hoop conjecture). Due to the great

*A trapped surface T is a closed, spacelike two surface with the property that the two systems of null geodesics which meet T orthogonally converge locally in the future directions difficulty of this problem recent work has tended to focus either on the search for counter-examples, or on the derivation of conditions for the existence of trapped surfaces [27].

An interesting departure from these approaches has been given by Israel [28]. He suggests that we may try to avoid the issues of singularities (initially) and ask when a trapped surface will necessarily lead to the formation of an event horizon. In fact he managed to prove a very beautiful theorem which states (roughly) that " a trapped 2-surface can be extended to a 3-cylinder that is and remains spacelike at least as long as it remains regular, thus it acts as a permanent one-way membrane for causal effects". Such results bolster our belief in cosmic censorship, and it is generally taken as a working hypothesis.

Collapse with rotation and/or charge

The presence of rotation and/or charge in a collapsing star has a dramatic effect on the internal geometry of the black holes which form. The "no hair" theorems [10] prove that the only stationary electrovac black holes must belong to the Kerr-Newmann family. In view of our understanding of spherical collapse with perturbations we expect that the external gravitational field of a rotating black hole will settle down to a member of this family at late times. These solutions have a timelike singularity and suggest the possibility of travelling through the black hole into other universes. Since it is so difficult to analyse collapse with rotation, the Reissner-Nordström solution (which is the unique spherically symmetric electrovac solution of General Relativity [29]) is sometimes used as a model in which to test certain hypotheses, since these solutions have a similar causal structure to that of Kerr.

Gravitational collapse of charged dust was first considered by Novikov [30] and by de la Cruz and Israel [31]. Their work shows that the gravitational collapse proceeds, in its external aspects, as it does in the uncharged case. The surface of the collapsing material reaches its gravitational radius in a finite proper time and an event horizon forms which seals off the subsequent evolution from external observers. The difference is that the contraction of the matter halts at some minimum radius (inside the event horizon), and then the dust begins to re-expand. In fact the evolution continues and the body emerges into another asymptotic region which is identical to, but distinct from, the one in which the collapse began. In so doing the matter has passed into a region of unpredictability. The future boundary of the domain of dependence for Cauchy data posed on the surface Σ is a null hypersurface — the Cauchy horizon — beyond which the evolution can no longer be uniquely determined. In particular, information can come out of the timelike singularity at $r = 0$ and affect the evolution in a completely unknown manner. Even to construct the analytically continued spacetime in Fig. 3.1 requires the assumption that nothing escapes from the singularity, and that the spacetime is vacuum (except for the electric field) in this region.

The Cauchy horizon is a highly pathological surface; small time-dependent perturbations originating outside the black hole undergo an infinite gravitational blueshift as they evolve towards the Cauchy horizon. This blueshift of infalling radiation gave the first indications that these solutions, which so well describe the exterior geometry at late times, may not describe the generic internal structure. Penrose [15] pointed this out some twenty five years ago, and since then linear perturbations have been analysed in detail [16]. The divergence of the measured energy density of the perturbations at the Cauchy horizon led to the conjecture that a scalar curvature singularity would form once back-reaction was accounted for.

Since a generic collapse will almost certainly involve some rotation, an interesting problem is to understand what will be the structure inside the event horizon of such a black hole. This is the primary motivation for current investigations of black hole internal structure [17]-[20],[32]-[34]. So far most of the analysis has been restricted to spherical symmetry where perturbations are modelled by lightlike dust or a scalar field. A first attempt by Hiscock to understand the internal geometry of a charged black hole with an influx of lightlike particles showed that an observer-dependent singularity was present along the Cauchy horizon. Poisson and Israel [18] included an outgoing flux of material across the Cauchy horizon and constructed a solution in which there is a null singularity, characterized by the divergence of the mass-function, along the Cauchy horizon. An outline of this work, which provides the launchpad for my contributions, is presented in chapter 3. Poisson and Israel also argued that the physics behind their analysis is sufficiently general to believe that similar results should hold for generic collapse. Further detailed calculations seem to support this speculation that the singularity inside a generic black hole is null [20, 35].

1.2 Quantum gravitational collapse

The discussion up to this point has been entirely classical, ignoring the fact that matter (and ultimately the gravitational field) is governed by quantum mechanics. This idealization should indeed be justified during the early stages of gravitational collapse of a macroscopic star (with mass $\sim M_\odot$ or greater). However it will break down near classical singularities and on timescales of order $10^{71}(M/M_\odot)^3\, s$ where M is the mass of the collapsed body. Although there is no complete theory of quantum gravity as yet, a great deal of useful information has been obtained by studying the behavior of quantum matter in classical curved backgrounds.

In the absence of external fields the Minkowski vacuum is stable, in the sense that virtual particles are constantly being created and annihilated yet this process never produces real particles. In a sufficiently strong or rapidly varying gravitational field this is no longer true; real particles can be created out of the "vacuum". Thus, the notion of a no particle state is ambiguous in a curved background.

In the late sixties, Parker and others [36] realized the significance of the instability of the vacuum and began to discuss particle creation in the early universe. Around the same time Markov and Frolov [37] studied Schwinger pair production near charged black holes, while Zeldovich and later Starobinsky [38] began to study quantum effects around rotating black holes. They found that ingoing modes of the quantum field get amplified by the energy of rotation of the black hole and the scattered modes having a larger amplitude contain an increased number of quanta in the field. In fact they also predicted that this would occur even if there were no incoming field quanta. This process (which is the quantum mechanical analogue of superradiance) leads to a decrease in the angular momentum of the black hole. The event horizon area continues to increase during this process in accordance with Hawking's area theorem [12].

Hawking subsequently made the revolutionary discovery that quantum particle creation can also occur in static black hole spacetimes such as Schwarzschild [13]. His calculation showed that a black hole creates and emits particles as though it were a black body with temperature

$$T = \frac{\hbar\kappa}{2\pi ck}. \tag{1.5}$$

where $\kappa = c^4/4GM$ is the surface gravity of the black hole, and k is Boltzmann's constant. This remarkable discovery means that spatial inhomogeneities of the gravitational field can be converted into real quantum particles and may thus decrease in size. In particular when the particle production is allowed to affect the geometry, this leads to a picture of a black hole which decreases its mass by quantum emission of particles.

Particle production is not the only manifestation of the effects of quantum fields in curved spacetime. The presence of the gravitational field may also polarize the vacuum. Thus, even in the absence of particle creation, there may be contributions to physical observables (e.g. $\langle T_{\mu\nu} \rangle$ for the quantum field) which depend on the properties of the gravitational field.

The problem of the effect of vacuum polarization and particle production on the gravitational field is generally termed (semi-classical) backreaction. We will discuss this and the main steps in elevating flat space quantum field theory to a curved manifold in chapter 2. The main goal of this thesis is to summarize some investigations of quantum effects in regions where, classically, curvatures approach arbitrarily large values, the hope being that vacuum polarization and particle production might damp the classical growth of curvature.

1.3 Outline of thesis

The material in chapters 4 through 7 relies heavily on previous work both in quantum field theory in curved spaces and in black hole theory. It therefore seemed necessary to include some introductory chapters to provide sufficient background for the reader.

Quantum fields in curved spacetime

The quantization of a scalar field in an arbitrary background is discussed, paying particular attention to the ambiguity in the choice of vacuum state for the field. Unlike Minkowski space, where we usually single out the Poincaré invariant vacuum state since all inertial observers will agree on it, there is no general physical principle in a spacetime without symmetry to decide on one particular vacuum over another. Following this we outline the general problem of regularization of the stress-energy tensor for the quantum field. The classical stress-energy tensor of a conformally invariant field is traceless, however an important feature of regularization is the appearance of an anomalous trace for such fields. We outline an argument (due to Duff [39]) which suggests the inevitability of the anomalous trace. Using a two-dimensional example we then discuss some important aspects of quantum fields in black hole backgrounds. The chapter ends with a discussion of semi-classical gravity, outlining some problems which are usually encountered and describing a resolution which has been suggested by Simon [40].

Mass inflation inside charged black holes

The global structure of Reissner-Nordström black holes is discussed. The inner (Cauchy) horizon is unstable due to an infinite gravitational blueshift of infalling radiation. Such radiation will always be present due to the radiative tail of the gravitational collapse to form the black hole. A preliminary attempt to understand how perturbations might alter the internal structure of the black hole was undertaken by Hiscock [17]. We summarize his results here. The remainder of the chapter outlines the important aspects of the Poisson-Israel [18] analysis of the Cauchy horizon singularity which results when the Hiscock model is generalized to include an outflux of material across the Cauchy horizon.

Quantum stress-energy tensor in linearized gravity

In order to investigate the effects of quantum matter on the geometry near to mass-inflation singularities we need to calculate the renormalized stress-energy tensor. Due to a fortunate collection of circumstances the spacetime inside the black hole may be treated as nearly conformally flat. In this chapter we therefore calculate the regularized stress-energy tensor for a scalar field in a nearly conformally flat spacetime. The results we obtain are in agreement with previous work by Horowitz [41], who used general arguments to derive the stress-energy tensor to linear order in perturbations from flat space. Campos and Verdaguer [?] have independently obtained the result for nearly conformally flat geometries which is presented in section 4.4.

Semi-classical effects in mass inflation

Ori [19, 20] has constructed an exact mass-inflation solution in which quantum effects can be examined in detail. We first present his solution and then show that it can be cast (approximately) in a conformally flat form, thus allowing us to use the results of chapter 4. The stress-energy tensor involves an integral over the past lightcone of the point of evaluation. This integral gives the leading behavior in the stress tensor as the mass-inflation singularity is approached. Once this is evaluated we obtain semi-classical corrections to the spacetime. Interpretation of the results is somewhat hampered by the appearance of the regularization scale in a logarithm, which changes the sign of the leading corrections at some stage during the evolution. Arguments are advanced which suggest that an intensification of the classical growth of curvature occurs up to the time when curvatures approach Planck values. However the sign change hints that quantum corrections may ultimately act to keep curvatures bounded as the Cauchy horizon is approached.

Homogeneous mass inflation

Continuing the investigation of the black hole core we re-examine a classical model of the black hole interior which was first proposed by Page and Ori [42]. The model continues to assume a decaying influx of classical radiation (modelled by a lightlike influx of particles) and also includes outgoing lightlike dust, however we restrict the gravitational field to be homogeneous as $r \to 0$. An approximate solution is presented and it is argued that such solutions may be representative of behavior at small radii. We then go on to examine quantum effects in the neighborhood of the mass-inflation singularity (which is spacelike in this model). Unambiguous predictions of intensification of the classical growth of curvature are obtained for the particular case which is investigated. It is further argued that our model should capture the leading effects even though the general homogeneous solutions are somewhat different.

Non-spherical considerations

This chapter provides a progress report on one approach to the extension of current ideas on mass inflation to the case of non-spherical collapse. Very near to the Cauchy horizon of the black hole it is possible to obtain an asymptotic solution which is based only on the assumption that the singularity starts from finite "radius". This solution indicates that the Cauchy horizon singularity can remain null in the presence of non-spherical perturbations, with shear remaining bounded until the radius goes to zero. The leading divergences also suggest that the Cauchy horizon has a shock-wave type of structure. There are some technical points which must however be checked before this work is complete.

Conclusions

This short chapter attempts to give a coherent summary of the new results which have been presented in this thesis. The overall picture which emerges from the investigations of semi-classical effects is that quantum gravity will be needed in order to understand whether and how singularities are avoided inside black holes. Initial indications in chapter 5 do however suggest that vacuum polarization may tend to keep curvatures bounded in the neighborhood of the Cauchy horizon. Unfortunately, this is just at the boundary of validity of the approximations used. Finally we comment on some open problems associated with black hole interiors.

CHAPTER 2

Quantum fields in curved spacetimes

The exact methods used to investigate particle creation and vacuum polarization in curved spacetimes vary greatly, however they all fall under one general title: *quantum field theory in curved spacetime*. In this chapter a brief outline of the standard generalization of flat space quantum field theory to an arbitrary background is given. This is in no way a complete account of the subject. More detail can be found in the excellent reviews of Gibbons [43] and DeWitt [44], while more pedagogical treatments are given in Birrell and Davies [45] and Fulling [46].

Gibbons lists the following constituents of a field theory on an arbitrary background: (1) A Hilbert space, $\mathcal{H}$; (2) A classical spacetime; (3) A set of field operators $\{\Phi\}$; (4) Wave equations satisfied by the fields (linear for non-interacting fields, which is all we consider here); (5) Commutation relations satisfied by the field operators; (6) Rules for constructing a Fock basis for $\mathcal{H}$; (7) Regularization schemes which render formally divergent non-linear expressions in the fields finite.

In this thesis we always assume that the background spacetime is a continuous manifold. Explicit calculations in this thesis are only performed for the scalar field case (when the conclusions continue to hold for fields of other spins it is indicated explicitly).

2.1 Scalar field quantization

The action for a scalar field in n-dimensions is written as

$$S = -\frac{1}{2}\left\langle g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2 + \xi R\phi^2\right\rangle \tag{2.1}$$

where $\langle\ldots\rangle = \int d^n x\sqrt{-g}(\ldots)$, ∇ is the covariant derivative with respect to $g_{\mu\nu}$ and m is the mass of the field. There is also a coupling to the curvature where ξ is a dimensionless constant and R is the Ricci scalar. Variation of this action yields the wave equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - m^2\phi - \xi R\phi = 0\,. \tag{2.2}$$

One particular value of the coupling constant is of great interest to us, $\xi_c = (n-2)/n(n-1)$, for which the wave equation is invariant under rescaling of the metric provided $m = 0$ and the scalar field transforms in a very simple way. Specifically, under the transformation $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$, the wave equation becomes

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\phi} - \xi_c\bar{R}\bar{\phi} = 0\,, \tag{2.3}$$

provided $\bar{\phi}(x) = \Omega^{(2-n)/2}(x)\phi(x)$.

Before proceeding to the field quantization it is worthwhile to point out another important property of classical fields which are conformally invariant. The stress-energy tensor of the field is given by

$$T^{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}} \tag{2.4}$$

and in particular for a massless conformally coupled field the trace of this quantity is zero. This is important because it turns out that the regularization of the stress-energy tensor in curved space breaks the conformal invariance and leads to the appearance of an anomalous trace. (We will learn more about this later).

Quantization

The field is decomposed with respect to a complete orthonormal set of modes $u_i(x)$ as

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x) . \tag{2.5}$$

The quantization of the theory proceeds as in flat space by adopting the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \quad [a_i, a_j] = 0 \quad [a_i^\dagger, a_j^\dagger] = 0 . \tag{2.6}$$

Now the construction of the Fock space can go ahead as usual. We first construct the state which is annihilated by all a_i, denoted $|0\rangle$:

$$a_i\,|0\rangle = 0 . \tag{2.7}$$

The entire Fock basis is then built up by the action on $|0\rangle$ with $a_i^\dagger$ according to the rule exemplified by

$$(n_i!\, n_j!)^{-1/2}(a_i^\dagger)^{n_i}(a_j^\dagger)^{n_j}\,|0\rangle = |n_i, n_j\rangle . \tag{2.8}$$

In particular the number of "particles" associated with a given mode u_j in a given state $|n_1, \ldots, n_j, \ldots\rangle$ is the expectation value of the operator $a_j^\dagger a_j$:

$$\langle n_1, \ldots, n_j, \ldots|\, a_j^\dagger a_j\, |n_1, \ldots, n_j, \ldots\rangle = n_j . \tag{2.9}$$

In Minkowski space there is a unique vacuum state singled out for *all* inertial observers. This is done by requiring that the state should be invariant under the action of the Poincaré group. Thus since this is *the* empty or no particle state in one inertial frame, it will also be empty in any other inertial frame. All inertial observers agree on the vacuum $|0\rangle$.

In an arbitrary spacetime there may be no such state. Not having any general guiding principle to choose a particular set of modes we can just as well decompose

Using the identities

$$\frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int d^n x\sqrt{-g}\,C \;=\; (n-4)\,(C+\frac{2}{3}\Box R)\,, \tag{2.26}$$
$$\frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int d^n x\sqrt{-g}\,\mathcal{G} \;=\; (n-4)\,\mathcal{G}\,, \tag{2.27}$$

which were obtained by Duff [39], we find that

$$\langle T\rangle|_{n=4} = A\,\mathcal{G} + B\,\left(C+\frac{2}{3}\Box R\right)\,. \tag{2.28}$$

This is exactly the ***trace anomaly*** which is so well known for conformally invariant fields in curved backgrounds. The precise values of A and B have been determined for fields of different spins [45]; it should be noted that for spin-1 fields dimensional regularization gives different values of these coefficients than other regularization schemes (e.g. zeta function, or point splitting).

Conformal anomaly in two dimensions

Although the work in the following chapters considers only four dimensional problems, it is worth pointing out the result analogous to (2.28) in two dimensions. For two-dimensional fields the effective action when analytically continued to n dimensions has a pole at $n = 2$. Arguments similar to those outlined above imply that the only possible counterterm which is compatible with conformal invariance is

$$\Gamma_{div} = \frac{a\,\mu^{n-2}}{n-2}\int d^n x\sqrt{-g}\;R\,. \tag{2.29}$$

Before regularization the stress-energy tensor is traceless. Duff [39] once again notes that

$$\frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int d^n x\sqrt{-g}R = (n-2)R\,, \tag{2.30}$$

and so the trace of the two dimensional stress-energy tensor also acquires an anomalous contribution:

$$\langle T\rangle \;=\; \frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\Gamma_m+\Gamma_{div}\right)\,, \tag{2.31}$$
$$\;=\; aR\,. \tag{2.32}$$

The value of the constant a is determined by explicit calculation; for a scalar field $a = (24\pi)^{-1}$.

2.2 Stress-energy tensor for quantum fields

In this section we briefly outline some of the important features associated with the regularization of the stress-energy tensor for a quantum field in a curved background. The discussion is heuristic and ignores many technical issues (the reader is referred to [45, 46] for the details). Attention is mainly focussed on conformally invariant fields.

The stress-energy tensor is bilinear in the quantized field and as such contains divergences. Many different ways have been proposed to remove these [45, 46], so one might wonder which of these procedures is correct. Wald [47] has suggested a set of axioms which the regularized stress tensor should satisfy. These axioms are widely accepted as reasonable properties to demand, thus $\langle T_{\mu\nu}\rangle_{reg}$ should be:

1. Covariantly conserved;
2. Causal;
3. Give the standard results for "off-diagonal" matrix elements;
4. Reproduce standard results in flat spacetime.

Indeed using these axioms Wald proved that $\langle T_{\mu\nu}\rangle_{reg}$ is unique up to the addition of a local conserved tensor. Thus any regularization procedure which produces a result satisfying these conditions may be regarded as correct.

In this thesis we generally consider the calculation of $\langle T_{\mu\nu}\rangle_{reg}$ as part of the wider dynamical problem of semi-classical gravity. This approximation to quantum gravity treats the gravitational field as unquantized (specifically one assumes that a continuous background manifold exists) allowing the curvature to respond to the quantum matter through the generalized field equations:

$$G_{\mu\nu} = 8\pi \left(T_{\mu\nu}^{\rm class} + \langle T_{\mu\nu} \rangle_{reg} \right) . \tag{2.15}$$

One approach to obtaining the regularized stress-energy tensor is to look for the effective action Γ_m which generates it according to

$$\frac{2}{\sqrt{-g}} \frac{\delta \Gamma_m}{\delta g_{\mu\nu}} = \langle T^{\mu\nu} \rangle_{reg} . \tag{2.16}$$

Consider the generating functional $Z[J]$ defined by

$$\langle out | \, in \rangle_J := e^{iZ[J]} = \int \mathcal{D}[\phi] \exp[i(S + \langle J\phi \rangle)] , \tag{2.17}$$

where the path integral is over all fields which are negative frequency in the past when $J = 0$. More generally setting $J = 0$ and varying $Z[0]$ with respect to $g_{\mu\nu}$ we find

$$\frac{2}{\sqrt{-g}} \frac{\delta Z[0]}{\delta g_{\mu\nu}} = \frac{\langle out | \, T^{\mu\nu} \, | in \rangle}{\langle out | \, in \rangle} . \tag{2.18}$$

The vacuum persistence amplitude $\langle out | \, in \rangle_J$ is a standard object in flat space quantum field theory. Its calculation in curved space can proceed via the usual perturbative expansion in powers of $\hbar$. The terms in this series will generally contain the metric and its derivatives through the curvature. We will always truncate the calculation at first order – the so called one-loop approximation*.

We will now discuss dimensional regularization of the one loop [terms only of $o(\hbar)$] effective action for conformally invariant fields. The presentation closely follows that of Duff [39].

Conformally invariant fields in four dimensions

The technique of dimensional regularization can be successfully applied to the effective action in curved spacetime [45, 46]. Starting from a conformally invariant

*For free fields this is all there is.

field theory in four dimensions, we analytically continue it and the spacetime to n dimensions. The one-loop effective action is generally a non-local functional of the spacetime curvature. It is also a function of the dimensionality of the background spacetime containing a simple pole at $n = 4$. Write

$$\Gamma_m = (n-4)^{-1} \int d^n x \sqrt{-g} G(n) \,. \tag{2.19}$$

where $G(n)$ contains the metric, its derivatives (in combinations like R^2, $R(\ln\Box)R$ etc.) and is a function of n. Introducing a counterterm Γ_{div} which also has a pole at $n = 4$ with residue such that it cancels the divergences in Γ_m we obtain the regularized effective action

$$\Gamma_{reg} = \Gamma_m + \Gamma_{div} \,. \tag{2.20}$$

Since the original field theory was conformally invariant Γ_{div} should only contain quantities which are conformally invariant or total divergences in four dimensions [39]. Duff therefore argues that the only possibility (on dimensional grounds) is

$$\Gamma_{div} = \frac{\mu^{n-4}}{n-4} \int d^n x \sqrt{-g} \, (A\, \mathcal{G} + B\, C) \,. \tag{2.21}$$

$$\mathcal{G} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \,. \tag{2.22}$$

$$C = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \,. \tag{2.23}$$

Notice that μ is a parameter with the dimensions of mass (and is related to the ultraviolet cutoff in the usual perturbation theory); A and B are constants which must be determined by explicit calculation; $\mathcal{G}$ is the Gauss-Bonnet combination which is a total divergence in four dimensions, and C is the square of the Weyl tensor in four dimensions.

The trace of the regularized stress-energy tensor obtained by varying (2.20) is

$$\langle T\rangle = \frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta\Gamma_{reg}}{\delta g_{\mu\nu}} = \frac{2g_{\mu\nu}}{\sqrt{-g}}\left(\frac{\delta\Gamma_{m}}{\delta g_{\mu\nu}} + \frac{\delta\Gamma_{div}}{\delta g_{\mu\nu}}\right) . \tag{2.24}$$

This quantity is finite at $n = 4$ (once the constants A and B have been appropriately chosen). Specifically we already know from (2.4) that

$$\left.\frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta\Gamma_{m}}{\delta g_{\mu\nu}}\right|_{n=4} = 0 . \tag{2.25}$$

Using the identities

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$$\frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int d^n x\sqrt{-g}\, \mathcal{G} \;=\; (n-4)\,\mathcal{G}\,. \tag{2.27}$$

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Before regularization the stress-energy tensor is traceless. Duff [39] once again notes that

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and so the trace of the two dimensional stress-energy tensor also acquires an anomalous contribution:

$$\begin{aligned} \langle T\rangle &= \frac{2g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\Gamma_m + \Gamma_{div}\right)\,, && (2.31)\\ &= aR\,. && (2.32)\end{aligned}$$

The value of the constant a is determined by explicit calculation; for a scalar field $a = (24\pi)^{-1}$.

In two dimensions this result, in combination with covariant conservation, is sufficient to completely determine the entire stress-energy tensor [48] (unfortunately the same is not true in four dimensions except in circumstances of high symmetry). We will now discuss quantum effects in two-dimensional black hole spacetimes.

2.3 Quantum effects in black hole spacetimes

Consider a two-dimensional spacetime

$$ds^2 = -e^{2\rho}dUdV\,. \tag{2.33}$$

It has Ricci scalar $R = 8e^{-2\rho}\partial_V\partial_U\rho$ and hence the trace anomaly is $\langle T\rangle = 8ae^{-2\rho}\partial_V\partial_U\rho$ where $a = (24\pi)^{-1}$ for a scalar field. Demanding that the regularized stress-energy tensor is conserved and has the trace (2.32) implies that

$$\begin{aligned} \langle T_{UU}\rangle &= 2a\left(\partial_U^2\rho - (\partial_U\rho)^2\right) + g_-(U)\,, && (2.34)\\ \langle T_{VV}\rangle &= 2a\left(\partial_V^2\rho - (\partial_V\rho)^2\right) + g_+(V)\,, && (2.35)\end{aligned}$$

while

$$\langle T_{UV}\rangle = -2a\partial_V\partial_U\rho\,. \tag{2.36}$$

The two arbitrary functions $g_+(V)$ and $g_-(U)$ reflect the freedom to choose boundary conditions.

To make the discussion more concrete let us consider the two dimensional section of the Schwarzschild spacetime given by

$$ds^2 = \frac{4m^2 f}{UV}dU\,dV\,, \tag{2.37}$$

where

$$f = 1 - 2m/r, \qquad UV = e^{r/2m}(1 - r/2m)\,. \tag{2.38}$$

The coordinates are shown in figure 2.3; they cover the entire spacetime, specifically $V = 0$ on the past black hole horizon and $U = 0$ on the future sheet. Comparing (2.37) and (2.33) we find that the stress-energy tensor is

$$\langle T_{UU}\rangle = 8m^2 a V^2 e^{-r/2m}\left(\frac{1}{4} + \frac{m}{r} + \frac{3m^2}{r^2}\right) + g_-(U)\,, \tag{2.39}$$

$$\langle T_{VV}\rangle = 8m^2 a U^2 e^{-r/2m}\left(\frac{1}{4} + \frac{m}{r} + \frac{3m^2}{r^2}\right) + g_+(V)\,, \tag{2.40}$$

$$\langle T_{UV}\rangle = \frac{32am^4}{r^4}e^{-r/2m}\,. \tag{2.41}$$

As mentioned above there are three important boundary conditions which correspond to taking the expectation value of $T_{\mu\nu}$ with respect to different states.

The Boulware vacuum

Placing the restriction on the stress-energy that it should be zero in the asymptotic region as $r \to \infty$ (both for large U and for large V), we find that the free functions in (2.39) and (2.40) are given by

$$g_+(V) = -\frac{a}{2V^2}\,, \qquad g_-(U) = -\frac{a}{2U^2}\,. \tag{2.42}$$

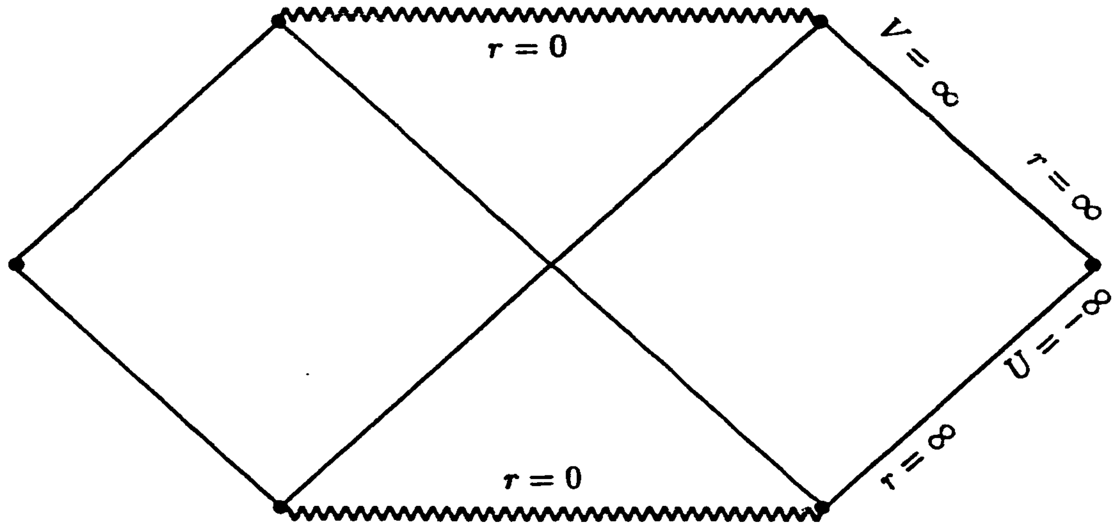

Figure 2.1: A two-dimensional Schwarzschild black hole. The coordinates are shown in order to make the discussion in the text clear

This is the vacuum state which an observer at fixed radius would experience. Thus we have chosen positive frequency with respect to the *static* time.

It is, however, a rather pathological state for black holes (as might be expected since an observer can only stay at a fixed radius outside of the black hole). Consider the value of the stress-energy tensor on the future event horizon of the black hole,

$$\langle T_{UU} \rangle \sim -\frac{a}{2U^2} \to -\infty \,, \tag{2.43}$$

as $U \to 0$. In fact $\langle T_{VV} \rangle$ behaves in an identical way on the past horizon $V = 0$. Thus this state is appropriate to describe physics outside a static spherical star.

Hartle-Hawking state

This is the state which is appropriate to a free-falling observer in this spacetime. The positive frequency states have been chosen with respect to the coordinates U and V (which are regular on the event horizon). The Hartle-Hawking state is defined by setting $g_+ = 0 = g_-$. It is straighforward to check that that the expectation value of the stress-energy tensor in this state is finite everywhere in the spacetime except at $r = 0$.

At infinity it is no longer zero however. Considering for example $V \to \infty$ we find $\langle T_{UU} \rangle = a/2U^2$, which corresponds to a thermal outflux from the black hole with temperature $(4m.2\pi k)^{-1}$. In order to see this we consider the energy density of the outflux as measured by an observer at infinity:

$$\langle T_{\mu\nu} \rangle \, u^\mu u^\nu |_{V=\infty} = \frac{a}{2(4m)^2}. \qquad (2.44)$$

A similar result holds when $U \to -\infty$, giving a thermal influx at the same temperature. Thus the Hartle-Hawking state may be interpreted as a black hole in thermal equilibrium with a heat bath.

Unruh vacuum

The boundary conditions are that the state should correspond to the vacuum in the asymptotic region when $U = -\infty$, and that the stress-energy tensor be regular on the future event horizon ($U = 0$) of the black hole. These two requirements fix the functions

$$g_+(V) = -\frac{a}{2V^2}, \qquad g_-(U) = 0\,. \qquad (2.45)$$

Clearly this state will also lead to a divergence on the past horizon ($V = 0$) of the black hole as the Boulware vacuum did. However this state is appropriate to describe gravitational collapse since the past horizon does not exist in this case (see

Fig. 1.1).

Examining the stress tensor at large r once again reveals the thermal outflux of Hawking radiation to future infinity. This is the quantum state in which we would like to estimate the stress-energy tensor in the later chapters.

This discussion can easily be generalized to charged black holes for which the same three states exist.

2.4 Backreaction and semi-classical gravity

A successful quantum theory of pure gravity (in four dimensions) may be expected to have the effect [49], at moderate curvatures, of modifying the Einstein-Hilbert Lagrangian by terms quadratic in curvature,

$$16\pi L_G = l_p^{-2} R + \alpha_1 C^2_{\alpha\beta\gamma\delta} + \beta_1 R^2 \,, \tag{2.46}$$

By including the matter action we can obtain a semiclassical theory of gravity with effective field equations

$$G_{\mu\nu} + \hbar(\alpha A_{\mu\nu} + \beta I_{\mu\nu}) = T_{\mu\nu} + \hbar \langle T_{\mu\nu} \rangle + O(\hbar^2) \tag{2.47}$$

which is correct to order $\hbar$. These equations are obtained perturbatively by the loop expansion method (asymptotic expansion in powers of $\hbar$). The tensors $A_{\mu\nu}$ and $I_{\mu\nu}$ are given in equations (4.81) and (4.82) respectively, $T_{\mu\nu}$ is the stress-energy tensor of the classical matter and $\hbar \langle T_{\mu\nu} \rangle$ is the one loop contribution from the quantum matter fields which are present. The quantum stress-energy tensor will contain powers and derivatives of curvature, it will also contain state dependent (usually non-local) terms with their explicit form depending on the background geometry. It is these effective field equations which we will use later to investigate semi-classical effects inside black holes.

In general, as can be seen by looking at the tensors $A_{\mu\nu}$ and $I_{\mu\nu}$, these equations contain second derivatives of curvature meaning that they involve fourth derivatives of the metric. Thus although the terms of $O(\hbar)$ are supposed only to induce small corrections to the classical Einstein equations, they alter the structure of the equations dramatically by the introduction of higher derivative terms. Consequently there will be an accompanying increase in the number of solutions which arise, many of the new ones being unphysical since they will not be perturbatively expandable in powers of $\hbar$ [40, 51]. A particularly important example of this problem was elucidated in the work of Horowitz, and Hartle [41, 50]. They considered solutions to the effective equations (2.47) for linearized perturbations about flat space. Some of the solutions strongly suggest that Minkowski space is unstable to vacuum fluctuations, however, these are exactly the ones which are non-analytic in $\hbar$.

To make this discussion more concrete we consider a simple example which demonstrates the problem. Schematically the effective equations can be written as

$$g'' = -\omega^2 g + \hbar g'''' + O(\hbar^2) , \tag{2.48}$$

where a prime represents a derivative with respect to some parameter t, say, with the dimensions of length, and ω has the dimensions of inverse length. Then the exact solution to the classical equations ($\hbar = 0$) is

$$g = c_1 e^{\imath\omega t} + c_2 e^{-\imath\omega t} , \tag{2.49}$$

where c_1 and c_2 are constants. Clearly this solution is oscillatory and remains regular for all t. Now let us consider a sample solution of the fourth order equations

$$g = c_3 e^{A_+ t} + c_4 e^{A_- t}, \quad \text{where} \;\; A_\pm = \frac{1 \pm \sqrt{1 + 4\hbar}}{2\hbar} \, . \tag{2.50}$$

It is easy to see that this solution blows up as $t \to \pm\infty$, furthermore it does not reduce to the classical solution (2.49) as we let $\hbar \to 0$. Thus the small correction term in the equations has completely changed the nature of the solution.

So how can one avoid this problem? One method is to obtain all the solutions to the effective field equations and then to exclude (by hand) those which are not expandable in powers of $\hbar$. On the other hand Simon [10] has advocated the *self-consistent* method of solving the effective equations which removes all the spurious solutions which are introduced by the higher derivatives. The application of his method is easily demonstrated using equation (2.48). Firstly multiply (2.48) by $\hbar$ to obtain what we call a perturbative constraint

$$\hbar g'' = -\hbar\omega^2 g + O(\hbar^2) \, . \tag{2.51}$$

This clearly implies that

$$\hbar g'''' = -\hbar\omega^2 g'' + O(\hbar^2) \, . \tag{2.52}$$

Substituting this into equation (2.48) we obtain the *reduced* equation

$$g'' = (-\omega^2 + \hbar\omega^4) g + O(\hbar^2) \, . \tag{2.53}$$

This method eliminates the higher derivatives and leads to a reduced equation with solutions which are manifestly perturbative in $\hbar$. The generalization of the method to the effective field equations (2.47) is straightforward.

Analogously, multiplying (2.47) by $\hbar$ we obtain a first order perturbative constraint

$$\hbar G_{\mu\nu} = 8\pi\hbar \left(\mathcal{T}_{\mu\nu} + O(\hbar^2) \right) \tag{2.54}$$

from which it follows that

$$\hbar R = -8\pi\hbar T + O(\hbar^2)\,. \tag{2.55}$$

and that

$$\hbar R_{\mu\nu} = 8\pi\hbar\left(T_{\mu\nu} - \frac{1}{2}T\right) + O(\hbar^2)\,. \tag{2.56}$$

Thus we have expressed the curvature in terms of the stress-energy of the classical matter (which is usually constructed only from the metric without derivatives). Substituting these expressions into the order $\hbar$ terms in (2.47) yields a set of equations which involve only second derivatives of the metric. In this way one obtains second order differential equations for the semiclassical theory in which solutions are automatically analytic in $\hbar$. The self-consistent method is adopted throughout the thesis.

With these preliminaries out of the way we can proceed to the discussion of black hole interiors.

CHAPTER 3

Mass inflation inside charged black holes

The instability of the Cauchy horizon inside Reissner-Nordström black holes has been investigated by many authors [15, 16]. In this chapter we summarise what is currently known about this instability and discuss the work of Hiscock [17], Poisson and Israel [18] and Ori [19] on the problem of the backreaction of perturbations near the Cauchy horizon.

The structure of the Reissner-Nordström solution is discussed in section 3.1, paying particular attention to the behaviour of timelike geodesics inside the event horizon. It is shown that an observer can pass through the black hole without encountering a singularity along his path.

A typical perturbing field Ψ decays in advanced time (which is infinite at the Cauchy horizon) according to an inverse power law, $\Psi \sim v^{-n}$ where n is a positive integer depending on the multipole order of the perturbation [23]. However, a measurement of the field's rate of variation by a free-falling observer crossing the Cauchy horizon yields the infinite result

$$\Psi_{,\alpha} u^{\alpha} \simeq \Psi_{,v} \dot{v} \sim v^{-(n+1)} e^{\kappa_i v} \tag{3.1}$$

where u^{α} is the four velocity of the observer (the dot denotes differentiation with respect to proper time) and $\kappa_i = (m^2 - e^2)^{1/2}/r_i^2$ denotes the surface gravity of the inner horizon, m is the mass of the black hole, e its charge and $r_i = m - (m^2 - e^2)^{1/2}$ is the inner horizon radius. The same observer measures a flux of energy given essentially by the square of equation (3.1) which is even more divergent. Using a test field of lightlike dust we demonstrate this instability in section 3.2.

The mechanism of this instability is the large blueshift occurring near to the Cauchy horizon. For reasonable observers residing outside the hole, $\dot{v} \simeq 1$, so that they require an infinite proper time to reach future null infinity ($v = \infty$). Internal observers require only a finite proper time to reach the Cauchy horizon, which implies that $\dot{v}$ will diverge as $v \to \infty$. As equation (3.1) shows, this divergence wins over the time decay of the perturbations and, as a result, the Cauchy horizon is said to be unstable.

The presence of this divergent flux also suggests that the backreaction of perturbations on the geometry will generate unbounded curvature along the Cauchy horizon. A preliminary investigation of the backreaction by Hiscock [17] showed that an observer-dependent singularity did form. The Hiscock model makes use of an exact solution of the Einstein field equations with null dust as a source. This model is presented in section 3.3 and it is shown that the singularity which forms along the Cauchy horizon is characterized by the divergence of curvature in a parallel propagated orthonormal frame. This type of non-scalar curvature singularity is generally believed to be unstable, in the sense that slight perturbations will produce a scalar curvature singularity in its place. For charged, spherical black holes Poisson and Israel [18] have shown that a scalar curvature singularity forms at the Cauchy horizon when the above influx is accompanied by an outflux emitted from the collapsing star. Since they found the singularity to be characterized by a diverging mass function, they called it a *mass-inflation* singularity. We give a detailed discussion of the Poisson-Israel [18] scenario in section 3.4.

An exact solution, modelling the outflux as a thin shell of lightlike matter, was used by Ori to examine the nature of the mass-inflation singularity in some detail [19]. He showed that an observer, who falls into the black hole, experiences finite tidal distortion at this singularity since the curvature is an integrable function of the observer's proper time.

For completeness the Ori model [19] limit of the presented solution is therefore taken in section 3.5. It is then easy to obtain the relationship between proper time, along a timelike geodesic, and advanced time near to the singularity. Using this it is shown that an extended object undergoes only finite distortion up to the moment it crosses the Cauchy horizon. For this reason Ori suggested that the spacetime may be continued beyond the mass-inflation singularity. This is difficult to believe since there seems to be no (classical) mechanism which can drive curvatures back down to finite values beyond the Cauchy horizon. Herman and Hiscock [33] have argued that the question is not really relevant since anything approaching the Cauchy horizon will probably be "fried" by the ingoing blueshifted radiation.

3.1 The Reissner-Nordström solution

The unique spherically symmetric, charged black hole solution of the Einstein field equations is the Reissner-Nordström solution. Using an advanced time coordinate v it has the line element

$$ds^2 = dv(2dr - f dv) + r^2 d\Omega^2 \qquad f = 1 - 2m/r + e^2/r^2 \,. \tag{3.2}$$

It is a static solution with timelike Killing vector $\xi^\alpha = \partial x^\alpha / \partial v$ outside the black hole. The global structure is different to that of Schwarzschild due to the presence of the charge. The Killing vector becomes null on two stationary lightlike hypersurfaces: the black hole event horizon $r = r_e$ and the inner (Cauchy) horizon $r = r_i$ determined by solving the quadratic $f = 0$, where the roots satisfy $0 < r_i < r_e$. The surface gravity is constant on each of the horizons being determined by the slope of f at that horizon, it is

$$\kappa_n = \frac{1}{2} |\partial_r f|_{r=r_n} \qquad n \in \{i, e\} . \tag{3.3}$$

Timelike geodesics

An analysis of timelike geodesics in this spacetime gives us some idea of the intrigueing global structure of this spacetime. The equation of a radial timelike geodesic can be reduced to

$$\dot{r}^2 = E^2 - f . \tag{3.4}$$

Let us suppose the energy satisfies $|E| > 1$, although a similar analysis can be done for the other cases. The right hand side of (3.4) can become zero at a finite radial value

$$r_b = \frac{-m + \sqrt{m^2 + (E^2 - 1)e^2}}{(E^2 - 1)} < r_i . \tag{3.5}$$

Therefore an observer who falls into the black hole decelerates until he reaches the finite radius r_b at which he comes to rest. Subsequently the same observer moves in the direction of increasing r and can return to arbitrarily large radii in a finite proper time. Thus it seems that it is possible to fall into a charged black hole, to avoid hitting the curvature singularity at $r = 0$ and to return to large radii again (see Fig. 3.1). Clearly this is very different to the behavior of geodesics in Sewharzschild spacetime (see section 1.1 in the Introduction).

Analytic continuation past the Killing Horizons

The system of coordinates (v, r) is singular on the ingoing sheet of the inner horizon: both r and v are constant there (v is actually infinite). They are also singular on the past event horizon. It is possible, however, to construct coordinates which are regular on each of these horizons in turn. As an example we do this for the inner black hole horizon. Introduce the coordinate

$$u = 2\int dr/f - v\,, \tag{3.6}$$

which is infinite on the outgoing sheet of the inner horizon. Moreover the coordinates (u, r) cover the region between the event and inner horizons (see Fig. 3.1). In terms of new coordinates (U, V) defined by

$$V = -e^{-\kappa_i v}\,, \tag{3.7}$$

$$U = -e^{-\kappa_i u}\,. \tag{3.8}$$

the line element (3.2) takes the form

$$ds^2 = \frac{f}{\kappa_i^2 UV}\,dU\,dV + r^2 d\Omega^2\,. \tag{3.9}$$

In order to see that the metric is indeed regular on the inner horizon combine (3.6), (3.7) and (3.8) to get

$$UV = (r - r_i)G(r)\,, \tag{3.10}$$

where $G(r)$ is a function which has a non-zero value at $r = r_i$. Since we can write

$$f = \frac{(r - r_e)(r - r_i)}{r^2} \tag{3.11}$$

it is easy to see that the metric component $g_{UV} \to (\text{constant}) \neq 0$ as $r \to r_i$. To analytically continue through the horizon, simply allow U and V to assume positive values. These coordinates then cover the entire black hole interior from the event horizon, $r = r_e$, down to the singularity at $r = 0$ - they are Kruskal coordinates for the inner horizon. One may analytically extend the spacetime through the other horizons in a similar manner, thus exhibiting the global structure of the manifold [see Fig. 3.1].

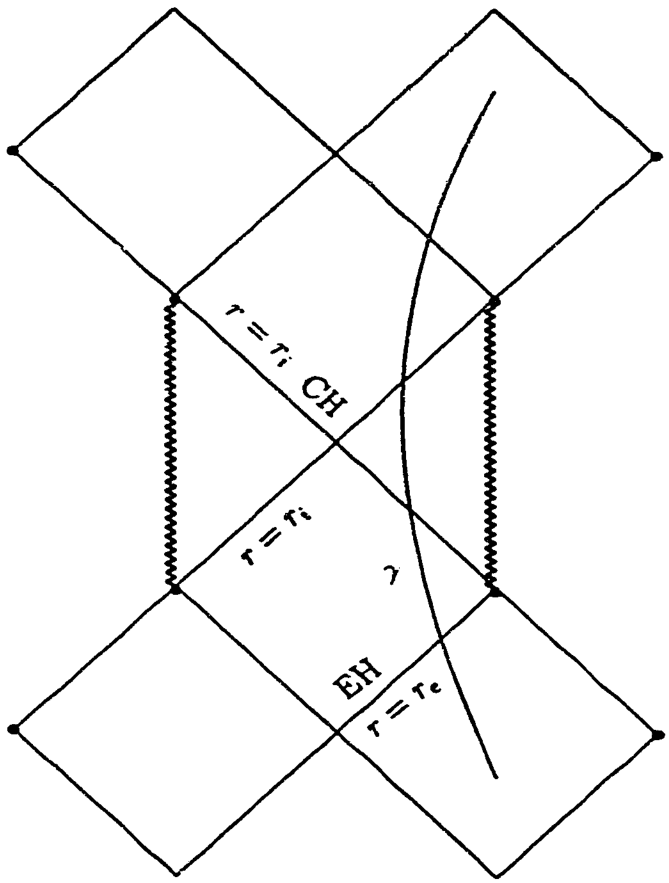

Figure 3.1: Analytically extended Reissner-Nordström spacetime. EH is the event horizon $r = r_e$ and CH is the Cauchy horizon. A timelike observer follows the trajectory γ originating in our universe at O and continueing through the black hole to another asymptotic region.

3.2 Cauchy horizon instability

The instability of the Cauchy horizon of a Reissner-Nordström black hole is a direct consequence of the global structure of the spacetime manifold. The ingoing sheet of the inner horizon is the "continuation" of future null infinity ($\mathcal{I}^+$) inside of the black hole. An external observer requires an infinite amount of time to reach $\mathcal{I}^+$, however an observer who falls into the black hole requires only a finite amount of time to reach the Cauchy horizon. This observer will therefore see the entire history of the external universe flash quickly by. In particular the ratio of the proper time for an external observer to that of an internal observer will diverge as $v \to \infty$ implying that ingoing radiation will undergo an infinite gravitational blueshift at the Cauchy horizon. In a realistic situation an influx of radiation will always be present in the form of the radiative tail of the gravitational collapse which formed the black hole. We now examine a simple model of the gravitational perturbations[17, 18].

Superposed on the Reissner-Nordström background, we consider an influx of test radiation described by the stress-energy tensor

$$T_{\alpha\beta} = \frac{L(v)}{4\pi r^2} l_\alpha l_\beta \tag{3.12}$$

and characterized by the luminosity function $L(v)$: the vector $l_\alpha = -\partial_\alpha v$ is tangent to ingoing radial null geodesics. We imagine radiation abutting the $v = \infty$ surface (see Fig. 3.2) and place restrictions on $L(v)$ by requiring that a radially free-falling observer in the exterior of the black hole measures the decaying radiative influx predicted by Price [23]. Thus we write the luminosity function (as $v \to \infty$)

$$L(v) = \gamma(\kappa_i v)^{-p} \tag{3.13}$$

where $p \geq 12$ is an integer and γ is a dimensionless constant.

To see that the Cauchy horizon is unstable, consider a radially free-falling observer in the vicinity of the Cauchy horizon who measures the energy density $\rho_{obs} = T_{\alpha\beta}u^\alpha u^\beta = T_{vv}\dot{v}^2$ for the influx of radiation. Normalizing the observer's four velocity so that $u^\alpha u_\alpha = -1$ we can write, using (3.4),

$$\dot{v} \simeq \frac{-2|E|}{f}\,, \tag{3.14}$$

as $v \to \infty$ and $f \to 0$. It can also be shown that near to the Cauchy horizon

$$f \simeq -2e^{-\kappa_i v} \tag{3.15}$$

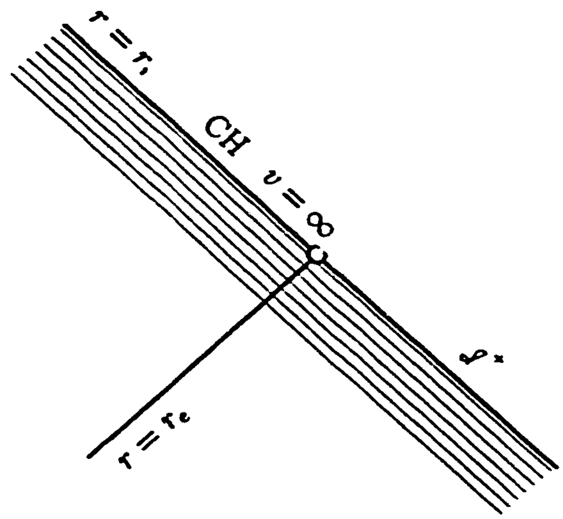

Figure 3.2: A portion of the Reissner-Nordström spacetime, with infalling null radiation abutting the $v = \infty$ surface. The system of coordinates (v, r) covers the region below the $v = \infty$ line, and becomes singular on that line. Shown are the singularity ($r = 0$), the Cauchy horizon ($r = r_i$), the black hole horizon ($r = r_e$), and the asymptotic future null infinity.

along the observer's path. Thus $\dot{v} \simeq |E|\, e^{\kappa_i v}$ diverges at the Cauchy horizon, indicating that the observer requires only a finite proper time to reach $v = \infty$. Consequently, the observed energy density is

$$\rho_{\text{obs}} \simeq \frac{\gamma E^2}{4\pi r_i^2}(\kappa_i v)^{-p} e^{2\kappa_i v} \tag{3.16}$$

from which we conclude that the energy density of the influx, as measured by a free-falling observer crossing the Cauchy horizon, will diverge as $v \to \infty$ indicating an instability of that horizon as Penrose [15] pointed out.

3.3 Charged Vaidya solutions

Since an instability of the Cauchy horizon has been established, what is the effect of the divergent influx on the geometry? There exists an exact solution of the

Einstein field equations with a stress-energy of the form (3.12). Hiscock [17] made a preliminary study of the backreaction of perturbations on the geometry using this solution. His results are summarized below.

The presence of the null dust simply makes the mass in (3.2) a function of advanced time v, so that it satisfies

$$\frac{dm}{dv} = L(v). \tag{3.17}$$

Therefore, enforcing the the inverse power law decay (3.13) on the influx the mass of the black hole

$$m(v) = m_1 - \frac{\gamma}{(p-1)\kappa_i}(\kappa_i v)^{-p+1} \tag{3.18}$$

increases to the value m_1 as $v \to \infty$. At first sight it seems that the radiation has not changed the geometry in any significant way. There continue to be two apparent horizons in the spacetime. The outgoing sheets of these horizons are no longer null, however the Cauchy horizon persists being located at $v = \infty$ and $r_i = m_1 - \sqrt{m_1^2 - e^2}$. Of course it is the curvature, not the metric, which is important.

Since this solution includes material which streams along the Cauchy horizon, one expects that the curvature as measured by a free-falling observer will diverge there. Consider an orthonormal frame

$$\begin{aligned}
e^{\alpha}_{(0)} &= (\dot{v}, \dot{r}, 0, 0) = u^{\alpha}, &(3.19)\\
e^{\alpha}_{(1)} &= (\dot{v}, f\dot{v} - \dot{r}, 0, 0), &(3.20)\\
e^{\alpha}_{(2)} &= (0, 0, r^{-1}, 0), &(3.21)\\
e^{\alpha}_{(3)} &= (0, 0, 0, (r\sin\theta)^{-1}), &(3.22)
\end{aligned}$$

where u^{α} is the observer's 4-velocity, and a dot represents differentiation with respect to proper time. It is straightforward to calculate the projected components of curvature in this frame. The three non-trivial components are

$$R_{(a)(B)(c)(D)} = \delta_{BD} \left(\eta_{ac} \left[\frac{1}{4} \partial_r^2 f + \frac{1}{2} \right] + 4\pi I_{ac} \rho_{obs} \right) \tag{3.23}$$

where $a, c \in \{0, 1\}$, $B, D \in \{2, 3\}$, $\eta_{ab} = diag(-1, 1)$ and I_{ab} is the two-dimensional matrix with all its entries equal to unity. As expected they diverge like the measured energy density (3.16) as the Cauchy horizon is approached. Despite this fact, it is possible to check that all the second order algebraic curvature scalars are bounded on the Cauchy horizon.

Can *any* curvature scalar diverge at the Cauchy horizon of this solution? For an influx given by (3.13), it seems not. In the coordinate system (v, r) the components of the metric and its inverse are bounded above and below (by zero) at the Cauchy horizon. In fact these components are C^∞ at the Cauchy horizon. Since an arbitrary curvature scalar is constructed from the metric and its derivatives, it is clear that all scalars must be bounded there also.

A singularity at which projected components of curvature diverge while scalars remain bounded has been called an *intermediate*, or *whimper* singularity by Ellis and King [52]. Detailed studies indicate that such singularities are generally unstable and exist only in circumstances of high symmetry [53]. It is therefore expected that a slight perturbation will induce a catastrophic plunge into a scalar curvature singularity. In this spherically symmetric case we now show that the inclusion of some outgoing null dust (proposed by Poisson and Israel [18]) is sufficient to completely destabilize the Cauchy horizon.

3.4 Backreaction of Spherical perturbations

Our aim, in this section, is to construct an approximate solution which includes backreaction of perturbations (modelled by a lightlike influx and outflux) on the geometry and in this way to examine the Cauchy horizon singularity in detail. The general argument follows Poisson and Israel [18].

It is convenient to use null coordinates on the "radial" two spaces so that the spherical line element is

$$ds^2 = -\frac{2F}{r} du dv + r^2 d\Omega^2 \,. \tag{3.24}$$

where $F = F(u, v)$ and $r = r(u, v)$, and the coordinates are such that u is a retarded time and v an advanced time. The stress-energy tensor for a radial electromagnetic field is

$$E_\mu{}^\nu = e^2/4\pi r^4 \,\mathrm{diag}(-1, -1, 1, 1) \tag{3.25}$$

where e is the charge on the black hole. Poisson and Israel used crossflowing null dust to model the perturbations of the geometry, arguing that the large blueshift near to the Cauchy horizon should make the Isaacson [54] effective stress energy desciption valid for the ingoing radiation. They also pointed out that the nature of the outflux is not important, its only purpose is to initiate the contraction of the Cauchy horizon. The stress-energy tensor is

$$T_{\mu\nu} = \rho_{\text{in}} l_\mu l_\nu + \rho_{\text{out}} n_\mu n_\nu \,. \tag{3.26}$$

where $l_\mu = -\partial_\mu v$ and $n_\mu = -\partial_\mu u$ are radial null vectors pointing inwards and outwards respectively, and, ρ_{in} and ρ_{out} represent the energy densities of the inward and outward fluxes. Each term of (3.26) is independently conserved so that

$$\rho_{\text{in}} = \frac{L_{\text{in}}(v)}{4\pi r^2} \,, \qquad \rho_{\text{out}} = \frac{L_{\text{out}}(u)}{4\pi r^2} \,. \tag{3.27}$$

The functions $L_{\rm in}(v)$ and $L_{\rm out}(u)$ are to be determined by the boundary conditions, however, it is important to note that they have no direct operational meaning since they depend on the parametrization of the null coordinates.

The field equations can now be written as a pair of non-linear hyperbolic wave equations for r^2 and $\ln F$

$$(r^2)_{,uv} = F(e^2/r^2 - 1)/r\,, \tag{3.28}$$

$$(\ln F)_{,uv} = F(1 - 3e^2/r^2)/(2r^3)\,, \tag{3.29}$$

subject to the constraints

$$(r^2)_{,vv} - \frac{F_{,v}}{F}(r^2)_{,v} = -2L_{\rm in}(v)\,, \tag{3.30}$$

$$(r^2)_{,uu} - \frac{F_{,u}}{F}(r^2)_{,u} = -2L_{\rm out}(u)\,. \tag{3.31}$$

A comma denotes partial differentiation. Characteristic initial data for these equations is supplied along null rays $u = u_0$ and $v = v_0$.

Imagine that the inflow is turned on at a finite advanced time $v = v_0$ and the outflow is turned on at $u = u_0$. In the pure inflow (outflow) regime the solution is an ingoing (outgoing) Vaidya-Reissner-Nordström spacetime with mass function $m(v)[m(u)]$. The structure of the spacetime with cross-flow is shown in the Penrose diagram of Fig. 3.4.

We choose v to be standard advanced time far from the black hole, thus $v = \infty$ at the Cauchy horizon. We also use a convenient parametrization of the null coordinate u such that

$$F(u, v_0) = r_i, \tag{3.32}$$

where r_i is the radius of the static portion of the Cauchy horizon for $u < u_0$. It is now easy to obtain the behaviour of the radius along $u = u_0$ by solving the equations

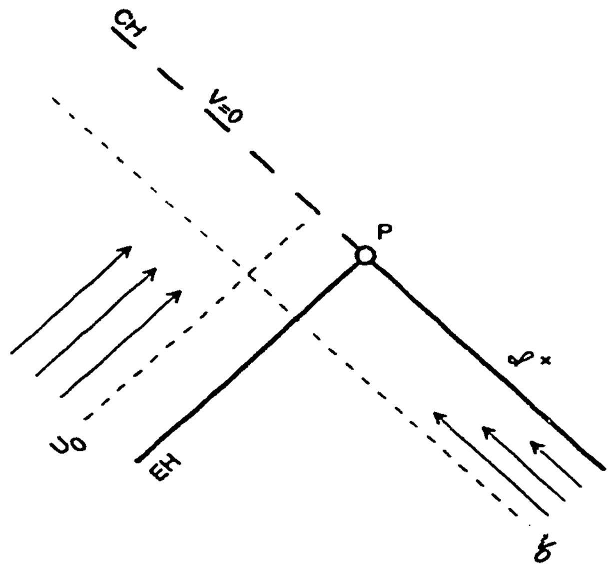

Figure 3.3: The structure of the spacetime, with crossflowing null dust. EH is the event horizon and CH is the Cauchy horizon. The lines $u = u_0$ and $v = v_0$ at which the fluxes are switched on are also shown.

for a radial null geodesic in the Vaidya spacetime. A first integral of the geodesic equations is

$$\frac{dr}{dv} = \frac{f}{2} = \frac{1}{2}\left(1 - \frac{2m(v)}{r} + \frac{e^2}{r^2}\right) \tag{3.33}$$

where $m(v)$ is given by (3.18). This has the asymptotic solution

$$r|_{u=u_0} = r_i + \frac{\gamma}{\kappa_i^2 r_i (p-1)}(\kappa_i v)^{-p+1}\left[1 + (p-1)(\kappa_i v)^{-1} + \ldots\right] , \tag{3.34}$$

as $v \longrightarrow \infty$. Now to determine $F(u_0, v)$ we substitute (3.34) into the constraint (3.30) to obtain (as $v \longrightarrow \infty$)

$$(\ln F)_{,v} \simeq -\kappa_i , \tag{3.35}$$

which implies that

$$F(u_0, v) \simeq r_i e^{-\kappa_i v} . \tag{3.36}$$

Along the characteristic line $v = v_0$, the only remaining free datum is the radius of the two-sphere which must satisfy

$$(r^2)_{,uu} = -2L_{\text{out}}(u) . \tag{3.37}$$

The behaviour of $L_{\rm out}(u)$ is not important so we simply assume that it is analytic in a neighbourhood of4 u_0.

For a given $L_{\rm out}(u)$ equations (3.32) and (3.34)-(3.37) complete the specification of the characteristic initial data. We can now proceed to find an approximate solution to the field equations.

It is crucial to obtain a serviceable approximation to the function $F(u, v)$ near to the Cauchy horizon, to this end we formally integrate (3.29) to

$$F = r_i\, g_1(u) g_2(v) \exp\left[\frac{1}{2}\int_{u_0}^{u}\!\!\int_{v_0}^{v} du' dv' \frac{F'}{(r')^5}\left\{(r')^2 - 3e^2\right\}\right] . \qquad (3.38)$$

The functions $g_1(u)$ and $g_2(v)$ are determined by the initial data along the null rays $v = v_0$ and $u = u_0$. With the parametrization of the null coordinates defined by (3.32) we set $g_1(u) = 1$ and $g_2(v) = e^{-\kappa_i v}$.

In order to proceed, we must estimate the behavior of the integral in (3.38). It is expected that the major contribution to this integral should come from near to the Cauchy horizon since $v = \infty$ there. However, at least initially, we expect $r(u, v)$ to be a well behaved function with the *slow* contraction of ingoing null rays governed primarily by the outflux from the collapsing star. Thus we conclude that a good approximation to this metric coefficient is

$$F \simeq r_i\, e^{-\kappa_i v} e^{\mathcal{F}(u)} , \qquad (3.39)$$

where

$$\mathcal{F}(u) = e^{-\kappa_i v_0}\left[\frac{r_i^2 - 3e^2}{2\kappa_i r_i^4}\right](u - u_0) , \qquad (3.40)$$

near to $v = \infty$.

With this ansatz we can rewrite the constraints (3.30) and (3.31) as

$$(r^2)_{,uu} - \mathcal{F}_{,u}(r^2)_{,u} \;\simeq\; -2L_{\rm out}(u) , \qquad (3.41)$$

$$(r^2)_{,vv} + \kappa_i (r^2)_{,v} \;\simeq\; -2L_{\rm in}(v) . \qquad (3.42)$$

According to (3.39) we see that $F(u, v)$ goes to zero very rapidly near to the Cauchy horizon. Equation (3.28) can therefore be approximated by

$$(r^2)_{,uv} \simeq 0 \tag{3.43}$$

which gives the solution $r^2 \simeq R_{in}(v) + R_{out}(u)$. Substituting this into (3.41) and (3.42), and using the luminosity function defined in (3.13) we obtain the approximate solution

$$r^2 \simeq r_i^2 + \frac{2\gamma}{\kappa_i^2(p-1)}(\kappa_i v)^{-p+1} - 2\int_{u_0}^{u} e^{\mathcal{F}'} \left[\int_{u_0}^{u'} e^{-\mathcal{F}''} L_{\text{out}}(u'')du''\right] du' + \dots \tag{3.44}$$

Provided we limit our analysis to small $(u - u_0)$ the above approximation for $F(u, v)$ should be sufficient. It is worthwhile to notice that it can be separated into a function of u times a function of v. In particular by rescaling the coordinates we could make F constant on the Cauchy horizon.

It should also be noted that

$$\int_{u_0}^{u} e^{\mathcal{F}'} \left[\int_{u_0}^{u'} e^{-\mathcal{F}''} L_{\text{out}}(u'')du''\right] du' \simeq \int_{u_0}^{u} \left[\int_{u_0}^{u'} L_{\text{out}}(u'')du''\right] du' + \dots \tag{3.45}$$

when $u - u_0$ is small. Thus as stated above the contraction of the ingoing lightrays near the Cauchy horizon is governed primarily by the outflux from the star.

3.4.1 The mass function

For a spherically symmetric system it is possible to introduce a geometrically defined mass function $m(x^a)$ via the gradient of the area of the two spheres:

$$g^{\alpha\beta}\nabla_\alpha r \nabla_\beta r = 1 - \frac{2m}{r} + \frac{e^2}{r^2}\,. \tag{3.46}$$

where ∇ indicates the four dimensional covariant derivative. At infinity it corresponds to the ADM mass (see for example page 293 of [10]). Furthermore, it is equivalent to Hawking's quasi-local mass [55]. It also acquires operational meaning in the spherically symmetric case since it determines the Weyl curvature $|\Psi_2| = (m - e^2/r)/r^3$.

Using (3.28), (3.31) and (3.46) it is easy to show that the mass function $m(x^a)$ satisfies

$$m_{,v} = -\frac{L_{in}(v)(r^2)_{,u}}{2F} \simeq \frac{\gamma(\kappa_i v)^{-p} e^{\kappa_i v}}{r_i} \int L_{out}(u) e^{-\mathcal{F}} du \, . \tag{3.47}$$

Thus the mass in the crossflow region ($u > u_0$ and $v > v_0$)

$$m(u,v) \simeq \frac{\gamma}{\kappa_i}(r_i \kappa_i v)^{-p} e^{\kappa_i v} \int L_{out}(u) e^{-\mathcal{F}} du \, , \tag{3.48}$$

inflates to infinity exponentially in advanced time v. This is the result obtained by Poisson and Israel (equation (4.17) in [18]).

3.5 The strength of the mass inflation singularity

The strength of the mass-inflation singularity is best discussed in terms of a simple model of a black hole interior considered by Ori [19]. He replaced the continuous outflux by a delta function source at $u = u_0$. This has the advantage that the spacetime continues to be described by an ingoing Vaidya solution for $u > u_0$. In this region the mass function does not approach a finite asymptotic value, rather it inflates to infinity on the Cauchy horizon.

Writing $L_{out}(u) = a\,\delta(u - u_0)$ in (3.44) and (3.48) the line element of Ori's solution is

$$ds^2 \simeq -\frac{2r_i e^{\mathcal{F}} e^{-\kappa_i v}}{r} du dv + r^2 d\Omega^2 \, , \tag{3.49}$$

where

$$r^2 \simeq r_i^2 + \frac{2\gamma}{\kappa_i^2(p-1)}(\kappa_i v)^{-p+1} - 2a(u-u_0)\,. \tag{3.50}$$

The mass function (3.48) is

$$m(v) \simeq \frac{a\gamma}{r_i\kappa_i}(\kappa_i v)^{-p}e^{\kappa_i v} + \ldots\,. \tag{3.51}$$

for $u > u_0$, in complete agreement with [19].

Although there is a scalar curvature singularity in this model at the Cauchy horizon, Ori suggested that the tidal distortion of extended objects approaching the singularity is physically more relevant as a means of determining the strength of the singularity. Thus a singularity is weak (according to Ori) if an extended object undergoes only finite tidal distortion all the way up to the singularity.

The tidal forces experienced by an observer are proportional to the projected components of curvature in an orthonormal frame $\{e^\alpha_{(a)}\}$ parallel propagated along his path (see for example [21] page 860). Choosing $u^\alpha = e^\alpha_{(0)} = (\dot u, \dot v, 0, 0)$ as the timelike vector, we have

$$\frac{2F}{r}\dot u \dot v = 1\,. \tag{3.52}$$

where a dot represents diferentiation with respect to the observers proper time τ. The geodesic equations and this first integral imply that

$$\ddot v \simeq \kappa(\dot v)^2 \qquad \text{as} \quad v \to \infty\,, \tag{3.53}$$

which is easily integrated. If we choose $\tau = 0$ on the Cauchy horizon the relation between it and the advanced time is

$$\tau \simeq \text{const} \times e^{-\kappa_i v} \qquad \text{as} \quad v \to \infty\,. \tag{3.54}$$

Assuming that internal stresses of the body can be neglected to a first approximation, the tidal distortion is given by twice integrating the projected curvature. From equation (3.23) it is easy to see that the leading terms in the tidal forces are

$$R_{\dots} \sim \frac{L_m(v)}{r^2}(\dot{r})^2 \sim |\ln \kappa_i \tau|^{-p}(\kappa_i \tau)^{-2} . \tag{3.55}$$

Integrating this function twice with respect to τ gives a bounded quantity as $\tau \to 0$, indicating that the singularity is weak according to the definition above.

In fact using τ as a coordinate the metric can be recast in the form

$$ds^2 \simeq -\frac{2r_i e^{\mathcal{F}}}{r} du d\tau + r^2 d\Omega^2 , \tag{3.56}$$

where

$$r^2 \simeq r_i^2 + \frac{2\gamma}{\kappa_i^2(p-1)}|\ln(\kappa_i \tau)|^{-p+1} - 2a(u - u_0) . \tag{3.57}$$

Thus the metric and its inverse are bounded at the Cauchy horizon and $\sqrt{-g}$ is non-zero there. It is this fact that led Ori to suggest that spacetime might be continued beyond the mass-inflation singularity [19].

3.6 Conclusion

In this chapter we have presented a detailed analysis of the Cauchy horizon instability for Reissner-Nordström black holes. In the test field approximation, where a null dust was used to model perturbations on the fixed background, it was shown that the Cauchy horizon is unstable due to an infinite gravitational blueshift of infalling radiation. This was first pointed out by Penrose [15] some twenty five years ago, and linearized perturbations have been analysed by many authors [16].

An approximate solution to the Einstein field equations coupled to cross-flowing null dust was also presented. As pointed out by Poisson and Israel [18] it indicates that the mass function diverges exponentially in external advanced time, which is infinite on the Cauchy horizon. Unlike the Hiscock model [17], where no scalar curvature diverges, the presence of a continuous outflux $L_{\rm out}(U)$ means that the Kretschmann invariant diverges like the locally measured energy density

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim 16 e^{-2\kappa_i v} e^{\mathcal{F}(u)} \frac{L_{\rm out}(u)L_{\rm in}(v)}{r_i^4} \tag{3.58}$$

As pointed out by Ori [19], the presence of the outflux from the star does not enhance the singularity enough to cause infinite tidal distortion of an observer reaching the Cauchy horizon. Ori's conclusion that spacetime can perhaps be continued beyond the mass-inflation singularity has stirred some debate [33, 32]. In any case it is difficult to see how the infinite curvature might be contained in a thin layer at the Cauchy horizon followed by a region of bounded curvature.

The discussion in this chapter has been limited to spherical symmetry. Poisson and Israel [18] argued that the presence of asphericities should not significantly change the mass-inflation scenario. Indeed, asymptotic and perturbative analyses [19] of the singularity inside more realistic black holes suggest that this should be the case. Further investigation of aspherical models of black hole interiors is currently under way [56], and a progress report on this work is presented in chapter 7.

CHAPTER 4

Quantum stress-energy tensor in linearised gravity

The general problem of calculating the stress-energy tensor for quantum fields in a curved background is immensely difficult. Unless the spacetime contains a high degree of symmetry it is usually impossible to obtain exact results [45]. In this chapter we present a derivation of an approximate stress-energy tensor for a scalar field in a nearly flat background and in any conformally related spacetime. We will use these results in the next chapter to investigate quantum effects near mass-inflation singularities.

Much work has been done in the past along these lines [57]. In fact the result obtained here has already been derived by Horowitz [41] using a very different approach. He starts from an axiomatic standpoint, listing a number of basic properties which any candidate for the stress-energy tensor of a quantum field should satisfy. The axioms are those suggested by Wald [47]. For weak gravitational fields he then invokes some general properties of Poincaré invariant distributions in Minkowski space which single out a unique candidate for the linearised stress-energy tensor of an arbitrary quantum field. Later Hartle and Horowitz [50] calculated the leading quantum corrections to the gravitational action coupled to N scalar fields, and argued that it was a candidate for a semi-classical theory to gravity.

In the cosmological setting, it is now well known that the stress-energy tensor for a conformally coupled field can be obtained from the conservation law and the trace anomaly alone in a conformally flat geometry [45]. However the early universe probably was not exactly isotropic and homogeneous. Therefore many authors have attempted to invoke quantum effects to damp out anisotropies and inhomogeneities [61]. Most of this work is limited to the case of small anisotropies [58], although an exception is the work of Parker and Hu [60], who used a combination of analytic and numerical methods to analyse more general anisotropies. The techniques used to perform these calculations vary greatly, from adiabatic approximations [60] to perturbatively evaluating the modes [62] to the use of functional methods [59]. The most efficient of the approaches seems to be that of Hartle and Hu [59], based on the effective action and background field method. Unfortunately, there is a shortcoming. The usual effective action generates matrix elements between the *in* and *out* states, and thus gives complex results in general. Even more frustrating is that this approach generates results which are acausal. To evaluate a matrix element at a given spacetime point requires performing an integral over the causal future and past of the point of interest. This is particularly disturbing when one wishes to consider the backreaction of the quantum fields on the geometry, meaning that in-out matrix elements do not lend themselves to straightforward interpretation. It would therefore seem preferable to use a formalism with the advantages of the standard effective action, but which would generate expectation values which are physically relevant.

Such a formalism does in fact exist. It was developed by Schwinger [63], Keldysh [64] and others. Its adaptation to curved spacetime was undertaken by Jordan [65] who showed that it generates real and causal results to two loop order.

Further use has been made of it by Calzetta and Hu [66] to study anisotropy dissipation in the early universe. Their results are in qualitative agreement with the earlier investigations. The curved spacetime formalism has been further developed by Paz [67].

Since this method is not in widespread use at the present time, section 4.1 is devoted to an exposition for flat space field theories. The presentation closely follows that of Calzetta and Hu [66]. In section 4.2 the formalism is applied to calculate the in-in effective action for a scalar field in the presence of a weak gravitational field. Divergences which arise are controlled using the standard methods of dimensional regularization [68], and the renormalization procedure is discussed in section 4.3. Finally we obtain the in-in expectation value of the renormalized stress-energy tensor for a scalar field. The in-state is the vacuum associated with the flat space background field theory. The result for conformally related spacetimes is then easily obtained. Campos and Verdaguer [69] have independently derived the same result and applied it to discuss quantum effects in the spacetime of a cosmic string. A brief discussion of the range of validity of the results is given in the conclusion.

4.1 Closed-time-path functional formalism

As mentioned in the introduction there exists a functional formalism which can be used to calculate in-in expectation values. Since it is not, as yet, a standard technique we will summarize it in this section. The presentation follows closely Calzetta and Hu [66] with some help from Ramond [70].

Consider an interacting scalar field theory in flat space described by the action

$$S = -\left\langle \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi)\right\rangle . \quad (4.1)$$

where we have used the notation $\langle\ldots\rangle := \int d^4x$. Let us suppose that the interaction is adiabatically turned off in the asymptotic past and future, so that the field theory may be regarded as free in these regions. The vacuum to vacuum persistence amplitude is defined as

$$\langle out|in\rangle_J := e^{iZ[J]} = \int \mathcal{D}[\phi]\exp\left[i(S_\epsilon + \langle J\phi\rangle)\right] , \quad (4.2)$$

where $J(x)$ is an arbitrary c-number source. The boundary conditions in the path integral have been made explicit by the inclusion of an $i\epsilon$ term in the action, as indicated by the subscript ϵ. The out vacuum, $|out\rangle$, and the in vacuum, $|in\rangle$, coincide with the free field vacua in the asymptotic regions where the interactions are turned off. In particular the Heisenberg states evolve in time according to

$$|in\rangle_{J(t)} = T\exp\left[i\int_{-\infty}^{t} dt' \int d^3\mathbf{x}\,\mathbf{J}(\mathbf{x},\mathbf{t}')\,\phi(\mathbf{x},\mathbf{t}')\right]|in\rangle \quad (4.3)$$

where T means temporal ordering. The vacuum persistence amplitude therefore generates time-ordered matrix elements between the in and out states:

$$\langle out|\,T\left[\phi(x_1)\ldots\phi(x_n)\right]|in\rangle = (-i)^n \frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_1)} e^{iZ[J]}\Big|_{J=0} . \quad (4.4)$$

Now consider the classical field

$$\phi_{cl}(x) = \frac{\delta Z[J]}{\delta J} . \quad (4.5)$$

The in-out effective action is the Legendre transform of $Z[J]$

$$\Gamma[\phi_{cl}] = Z[J] - \langle J\phi\rangle , \quad (4.6)$$

where we have now assumed that (4.5) is invertible as

$$J(x) = -\frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}} . \quad (4.7)$$

$\Gamma[\phi_{cl}]$ is an action which gives (4.7) as the effective equations governing the evolution of ϕ_{cl}. It entails all the quantum corrections to the field theory (4.1).

We wish to construct the functional which generates expectation values with respect to $|in\rangle$ rather than matrix elements. This formalism has been developed by Keldysh [63] and Schwinger [64]. Introduce two sources $J^+(x)$ and $J^-(x)$ and consider the quantity

$$e^{iZ[J^+,J^-]} :=_{J^-}\langle in|\, in\rangle_{J^+} \ . \tag{4.8}$$

Instead of letting the *in* vacuum evolve only in the presence of a single source and comparing the result with the *out* vacuum in the future, one considers the *in* vacuum evolving independently in the presence of two different sources and compares the results some time in the future. we may rewrite (4.8) as

$$\begin{aligned} e^{iZ[J^+,J^-]} &= \int \mathcal{D}[\varphi] \, \langle in|\, \tilde{T} \exp\left[-i\int_{-\infty}^{t_1} dt \int d^3\mathbf{x}\, \mathbf{J}^-\phi\right] |\varphi\rangle \\ &\quad \times \langle \varphi|\, T \exp\left[i\int_{-\infty}^{t_1} dt \int d^3\mathbf{x}\, \mathbf{J}^+\phi\right] |in\rangle \ , \end{aligned} \tag{4.9}$$

where $\tilde{T}$ is anti-temporal ordering, and $|\varphi\rangle$ is a complete orthonormal basis at t_1. It is now easy to see that $Z[J^+,J^-]$ generates expectation values such as,

$$\begin{aligned} (-i)^{n-m} &\frac{\delta^{n+m} e^{iZ[J^+,J^-]}}{\delta J^-(x_1)\ldots\delta J^-(x_m)\delta J^+(x'_1)\ldots\delta J^+(x'_n)} \\ &= \langle in|\, \tilde{T}\left[\phi(x_1)\ldots\phi(x_m)\right] T\left[\phi(x'_1)\ldots\phi(x'_n)\right] |in\rangle \end{aligned} \tag{4.10}$$

This generating functional also has a path integral representation

$$e^{iZ[J^+,J^-]} = \int \mathcal{D}[\phi^+]\mathcal{D}[\phi^-] \exp\left[i(S_\epsilon[\phi^+] + \left\langle J^+\phi^+\right\rangle_x - S_\epsilon^*[\phi^-] - \left\langle J^-\phi^-\right\rangle_{x'})\right] \ . \tag{4.11}$$

The boundary conditions for this path integral are manifest by the explicit identification of $S_\epsilon[\phi^+]$ with (4.1) and

$$S_\epsilon^*[\phi^-] = -\left\langle \frac{1}{2}\eta^{\mu\nu}\partial'_\mu\phi_-\partial'_\nu\phi_- + \frac{1}{2}(m^2 + i\epsilon)(\phi^-)^2 + V(\phi^-)\right\rangle \ . \tag{4.12}$$

That is the path integral is over all fields $\phi^{\pm}$ which are $\mp$-frequency in the infinite past $(t \to -\infty)$ and which coincide at $t = t_1$.

We can now construct the in-in effective action by analogy with (4.6). Classical fields are defined by

$$\phi_{\rm cl}^{+}(x) = \frac{\delta Z[J^+, J^-]}{\delta J^+}, \qquad \phi_{\rm cl}^{-}(x) = -\frac{\delta Z[J^+, J^-]}{\delta J^-}. \tag{4.13}$$

If $J^+ = J^- = J$ then $\phi_{\rm cl}^{+} = \phi_{\rm cl}^{-} = \phi_{\rm cl}$ is the expectation value of the Heisenberg field with respect to the state which evolved from the *in* state in the presence of the source J. Assuming that (4.13) is invertible define the in-in effective action by

$$\Gamma[\phi_{\rm cl}^{+}, \phi_{\rm cl}^{-}] = Z[J^+, J^-] - \left\langle J^+ \phi_{\rm cl}^{+} \right\rangle + \left\langle J^- \phi_{\rm cl}^{-} \right\rangle . \tag{4.14}$$

The equations of motion for $\phi_{\rm cl}^{\pm}$ are therefore

$$\frac{\delta \Gamma[\phi_{\rm cl}^{+}, \phi_{\rm cl}^{-}]}{\delta \phi_{\rm cl}^{\pm}} = \mp J^{\pm} . \tag{4.15}$$

Jordan [65] has shown that these equations are both real and causal to two loop order. In particular we recover the equation of motion for the expectation value

$$\left. \frac{\delta Z[J^+, J^-]}{\delta J^+} \right|_{J^+ = J^- = 0} = \langle in |\, \phi(x) \,| in \rangle = \phi_{\rm cl}(x) \tag{4.16}$$

when $J^+ = J^- = 0$ in (4.15).

As usual exact calculations of the effective action are generally impossible, so it is important that the usual perturbative and background field methods apply here. It must simply be kept in mind that the fields ϕ^+ and ϕ^- are linked by common boundary conditions on $t = t_1$.

For a free field theory [$V = 0$ in (4.1)] the path integral becomes Gaussian and can be performed using the standard techniques [70]:

$$e^{iZ[J^+, J^-]} = \exp\left[-\frac{1}{2} \left\langle \mathbf{J}\, \mathbf{G}\, \mathbf{J}^T \right\rangle_{xx'} \right] . \tag{4.17}$$

Here we have introduced the notation $\mathbf{J} = (J^+, -J^-)$, $(\ldots)^T$ means transpose and $\mathbf{G}(x, x')$ is a matrix-valued kernel with entries $G_{++}(x, x')$ and $G_{--}(x, x')$ which are symmetric and $G_{+-}(x, x') = G_{-+}(x', x)$. This kernel is determined by the boundary conditions on the classical fields (4.13) which were stated following equation (4.12). Notice that $\mathbf{\Phi}_{\rm cl} = (\phi^+_{\rm cl}, \phi^-_{\rm cl})$ satisfies the equation of motion

$$\mathbf{L}\mathbf{\Phi} = \mathbf{J} \tag{4.18}$$

where $\mathbf{L}$ is the diagonal operator

$$\mathbf{L} = \begin{bmatrix} \Box_f - m^2 + i\epsilon & 0 \\ 0 & -(\Box_f - m^2 - i\epsilon) \end{bmatrix} . \tag{4.19}$$

$\Box_f$ is the flat space D'Alembertian. The solution which satisfies the boundary conditions - that $\phi^+_{\rm cl}$ ($\phi^-_{\rm cl}$) contains only negative (positive) frequencies as $t \to -\infty$ and that $\phi^+_{\rm cl} \to \phi^-_{\rm cl}$ and $\partial_t\phi^+_{\rm cl} \to \partial_t\phi^-_{\rm cl}$ as $t \to t_1$ is simply

$$\mathbf{\Phi}^T_{\rm cl} = \left\langle \mathbf{G}\,\mathbf{J}^T \right\rangle_{x'} \tag{4.20}$$

where

$$\mathbf{G}(x, x') = \begin{bmatrix} \Delta_F & -\Delta^+ \\ \Delta^- & -\Delta_D \end{bmatrix} . \tag{4.21}$$

The Feynman, Dyson and positive and negative frequency Wightman functions are

$$\Delta_F = -\int \frac{d^n p}{(2\pi)^n} e^{ipx} (p^2 + m^2 - i\epsilon)^{-1} \tag{4.22}$$

$$\Delta_D = -\int \frac{d^n p}{(2\pi)^n} e^{ipx} (p^2 + m^2 + i\epsilon)^{-1} \tag{4.23}$$

$$\Delta^+ = 2\pi i \int \frac{d^n p}{(2\pi)^n} e^{ipx} \delta(p^2 + m^2)\theta(-p^0) \tag{4.24}$$

$$\Delta^- = -2\pi i \int \frac{d^n p}{(2\pi)^n} e^{ipx} \delta(p^2 + m^2)\theta(p^0) \tag{4.25}$$

For interacting fields we can adopt the background field method. In particular the one-loop effective action can be obtained by expanding about some classical field configuration and using the saddle point approximation to obtain

$$\Gamma[\phi^+, \phi^-] = S[\phi_{\rm cl}^+] - S[\phi_{\rm cl}^-] - \frac{i}{2} \ln \det \mathbf{L}^{-1} . \tag{4.26}$$

The operator $\mathbf{L}$ is given by

$$L_{ab} = \frac{\delta^2 (S_\epsilon[\phi^+] - S_\epsilon^*[\phi^-])}{\delta\Phi^a \delta\Phi^b} , \tag{4.27}$$

where $a, b \in \{+, -\}$. In general, care must be taken to correctly incorporate the boundary conditions following (4.19).

The generalization to curved backgrounds follows that of the in-out approach. The effective action is a functional of the metric $g_{\mu\nu}$ and the matter fields ϕ. In this case we consider the gravitational and matter fields to evolve in the presence of two independent sources as in the flat space case, thus $\Gamma = \Gamma[g^+, \phi^+; g^-, \phi^-]$. The boundary conditions on the background fields are the same as in flat space, although one must deal with the decomposition into positive and negative frequency modes with great care. In general, there may exist no unique decomposition of this kind [45]. The in-in expectation value of the stress tensor is obtained by varying with respect to $g_{\mu\nu}^+$ and then setting $g_{\mu\nu}^+ = g_{\mu\nu}^-$ and $\phi^+ = \phi^-$:

$$\langle in | T^{\mu\nu} | in \rangle = 2 \frac{\delta \Gamma[g^+, \phi^+; g^-, \phi^-]}{\delta g_{\mu\nu}^+} \Bigg|_{\substack{g_{\mu\nu}^+ = g_{\mu\nu}^- = g_{\mu\nu} \\ \phi^+ = \phi^- = \phi}} \tag{4.28}$$

This technique will now be applied to a scalar field in both nearly flat, and nearly conformally flat spacetime. Both of these cases allow unambiguous imposition of the boundary conditions since a global momentum representation of the fields exists on the background (conformally) flat spacetimes. This defines the usual flat space vacuum as the in state. It also singles out the conformal vacuum in the cosmological case [45, 46].

4.2 Effective action

In this section we use the CTP formalism to obtain an expression for the expectation value of the renormalised effective action in a nearly flat spacetime.

We begin with a spacetime which can be naturally separated into a flat piece and something which we will assume is small:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = (\eta_{\alpha\beta} + \gamma_{\alpha\beta})dx^\alpha dx^\beta. \tag{4.29}$$

Ultimately we want to calculate the renormalised stress-energy tensor to linear order in $\gamma_{\alpha\beta}$ for a scalar field on this background. The effective action must therefore be calculated to second order, since a variation (necessary to obtain the stress tensor) will reduce the order by one.

The action for a massless scalar field is written

$$S_f = -\frac{1}{2}\langle g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \xi R\phi^2\rangle, \tag{4.30}$$

where the notation is now $\langle\ldots\rangle = \int d^4x\sqrt{g}(\ldots)$. When there is more than one variable we will indicate this by adding subscripts to the closing bracket. Integrating by parts, and assuming that surface terms can be discarded, this can be rewritten as

$$S_f = \frac{1}{2}\langle\phi\Box\phi - \xi R\phi^2\rangle, \tag{4.31}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. Now using the fact that

$$\sqrt{g} = 1 + \frac{1}{2}\gamma + \frac{1}{8}\gamma^2 - \frac{1}{4}\gamma_{\alpha\beta}\gamma^{\alpha\beta} + \ldots \tag{4.32}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \gamma^{\alpha\beta} + \gamma^\alpha{}_\sigma\gamma^{\sigma\beta} + \ldots \tag{4.33}$$

we can write

$$\begin{aligned}\Box\phi = {} & \Box_f\phi - \left(\gamma^{\alpha\beta}\partial_\alpha\partial_\beta + \gamma^\beta\partial_\beta\right)\phi + \left(\partial_\alpha\left[\gamma^\alpha{}_\sigma\gamma^{\sigma\beta}\partial_\beta\right]\right. \\ & \left. -\frac{1}{4}\left[\partial_\alpha(\gamma_{\mu\nu}\gamma^{\mu\nu}) + \frac{1}{2}\partial_\alpha(\gamma^2)\right]\eta^{\alpha\beta}\partial_\beta - \frac{1}{2}(\partial_\alpha\gamma)\gamma^{\alpha\beta}\partial_\beta\right)\phi + \ldots.\end{aligned} \tag{4.34}$$

Here $\Box_f = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, $\gamma = \gamma^\alpha{}_\alpha$ and a vector $\gamma^\beta = \partial_\alpha\gamma^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\gamma$ has been introduced.

Varying the action (4.31) with respect to ϕ gives the wave equation for ϕ

$$\Box\phi - \xi R\phi = 0, \tag{4.35}$$

which can easily be expanded in powers of $\gamma_{\alpha\beta}$. The result is

$$\Box_f\phi + (V^{(1)} + V^{(2)} + \ldots)\phi = 0, \tag{4.36}$$

where

$$V^{(n)} = \Box^{(n)} - \xi R^{(n)}. \tag{4.37}$$

$\Box^{(n)}$ is the contribution (4.34) with 'n' powers of γ and $R^{(n)}$ is the corresponding term in the expansion of the Ricci scalar.

The one loop in-in effective action is given formally by (4.26) as

$$\Gamma = -\frac{i}{2}\ln\det(\mathbf{L}^{-1}) \tag{4.38}$$

where

$$\mathbf{L} = \begin{bmatrix} \Box_{(+)} - \xi R_{(+)} + i\epsilon & 0 \\ 0 & -(\Box_{(-)} - \xi R_{(-)} - i\epsilon) \end{bmatrix}. \tag{4.39}$$

In particular denoting $\mathbf{L}^{-1}$ by $\mathbf{G}$ we have

$$\mathbf{LG} = \delta(x - x')\mathbf{I} \tag{4.40}$$

where $\mathbf{I}$ is the 2×2 unit matrix. In the absence of any gravitational perturbation $\mathbf{G} = \mathbf{G}^0$ given by (4.21)-(4.25).

The idea is to perturbatively evaluate $\mathbf{G}$ for the operator $\mathbf{L}$ in (4.39) using the flat space counterpart as a zeroth order approximation. In this way we also have a clear handle on the quantum state for which we are evaluating expectation values. Write

$$\mathbf{L} = \mathbf{L}^{(0)} + \mathbf{V}^{(1)} + \mathbf{V}^{(2)} + \dots, \tag{4.41}$$

using notation such that the diagonal matrices $\mathbf{V}^{(n)}$ contain all terms of order n in $\gamma_\pm$, and $\mathbf{L}^{(0)}$ is the corresponding flat space operator (4.19) with $m = 0$. This allows us to formally write the solution to (4.40) as

$$\begin{aligned} \mathbf{G} &= \mathbf{G}^0 - \mathbf{G}^0(\mathbf{V}^{(1)} + \mathbf{V}^{(2)} + \dots)\mathbf{G} \\ &= \mathbf{G}^0 - \mathbf{G}^0(\mathbf{V}^{(1)} + \mathbf{V}^{(2)} + \dots)\mathbf{G}^0 + \mathbf{G}^0\mathbf{V}^{(1)}\mathbf{G}^0\mathbf{V}^{(1)}\mathbf{G}^0 + \dots . \end{aligned} \tag{4.42}$$

Using the operator identity $\det \mathbf{M} = \exp[\mathrm{Tr} \ln \mathbf{M}]$ we can rewrite (4.38) as

$$\Gamma = -\frac{i}{2} \mathrm{Tr} \ln \mathbf{G} . \tag{4.43}$$

Inserting (4.42) and further expanding the result gives

$$\begin{aligned} \Gamma &= \frac{i}{2}\mathrm{Tr}[\mathbf{V}_+^{(1)}\Delta_F + \mathbf{V}_+^{(2)}\Delta_F] - \frac{i}{4}\mathrm{Tr}[\mathbf{V}_+^{(1)}\Delta_F\mathbf{V}_+^{(1)}\Delta_F] \\ &\quad - \frac{i}{2}\mathrm{Tr}[\mathbf{V}_+^{(1)}\Delta^+\mathbf{V}_-^{(1)}\Delta^-] + (\text{ terms which will not contribute to } \delta/\delta g_{\mu\nu}^+) \end{aligned} \tag{4.44}$$

It is important to remember that γ_+ and γ_- should not be put equal until after the calculation is complete and the variation has been taken.

Expression (4.44) contains divergent terms which must be regularized. In this chapter we use dimensional regularisation to carry this out. First the geometry and field theory are continued to n dimensions where the above expressions are finite. Then expanding the results about four dimensions we find a pole at $n = 4$. To cancel this divergence we add a counterterm comprising an integral of a local polynomial of curvature. The coefficients in this polynomial also have poles at $n = 4$ with residues that exactly cancel the divergences.

Beginning with the terms which are linear in γ, insert the flat space Feynmann Green function $\triangle_F$ (4.22) into $(i/2)\mathrm{Tr}[\mathbf{V}_+^{(1)}\triangle_F + \mathbf{V}_+^{(2)}\triangle_F]$. We then obtain expressions proportional to $\int d^np\, p_\mu p_\nu(p^2 - i\epsilon)^{-1}$, $\int d^np\, p_\mu(p^2 - i\epsilon)^{-1}$ and $\int d^np\, (p^2 - i\epsilon)^{-1}$. These quantities vanish identically when they are regularised [68] so that the effective action contains no contributions from these tadpole graphs. We will now consider each of the remaining terms in turn.

The ++ graph

The first non-trivial term $-(i/4)\mathrm{Tr}[\mathbf{V}_+^{(1)}\triangle_F\mathbf{V}_+^{(1)}\triangle_F]$ is the same as for the in-out formalism. The evaluation proceeds along the lines described by Hartle and Hu [59].

Denoting $I = -(i/4)\mathrm{Tr}[\mathbf{V}_+^{(1)}\triangle_F\mathbf{V}_+^{(1)}\triangle_F]$ we first expand this to nine terms which can be treated separately. Thus

$$I = -\frac{i}{4}\sum_{j=1}^{9} T_j \tag{4.45}$$

where

$$T_1 = \langle\, \gamma_+^{\mu\nu}(x)\partial_\mu\partial_\nu\, \triangle_F(x,x')\, \gamma_+^{\alpha\beta}(x')\partial'_\alpha\partial'_\beta\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.46}$$

$$T_2 = \langle\, \gamma_+^{\mu}(x)\partial_\mu\, \triangle_F(x,x')\, \gamma_+^{\alpha}(x')\partial'_\alpha\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.47}$$

$$T_3 = \xi^2\langle\, R_+^{(1)}(x)\, \triangle_F(x,x')\, R_+^{(1)}(x')\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.48}$$

$$T_4 = \langle\, \gamma_+^{\mu\nu}(x)\partial_\mu\partial_\nu\, \triangle_F(x,x')\, \gamma_+^{\alpha}(x')\partial'_\alpha\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.49}$$

$$T_5 = \xi\langle\, \gamma_+^{\mu\nu}(x)\partial_\mu\partial_\nu\, \triangle_F(x,x')\, R_+^{(1)}(x')\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.50}$$

$$T_6 = \langle\, \gamma_+^{\mu}(x)\partial_\mu\, \triangle_F(x,x')\, \gamma_+^{\alpha\beta}(x')\partial'_\alpha\partial'_\beta\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.51}$$

$$T_7 = \xi\langle\, \gamma_+^{\mu}(x)\partial_\mu\, \triangle_F(x,x')\, R_+^{(1)}(x')\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.52}$$

$$T_8 = \xi\langle\, R_+^{(1)}(x)\, \triangle_F(x,x')\, \gamma_+^{\alpha\beta}(x')\partial'_\alpha\partial'_\beta\, \triangle_F(x',x)\,\rangle_{x'x}, \tag{4.53}$$

$$T_9 = \xi\langle\, R_+^{(1)}(x)\, \triangle_F(x,x')\, \gamma_+^{\alpha}(x')\partial'_\alpha\, \triangle_F(x',x)\,\rangle_{x'x}. \tag{4.54}$$

Since all of these integrals are evaluated in the same manner, I will simply demonstrate the method for T_2 and state the result at the end. Using the flat space expression for $\triangle_F(x, x')$ we have

$$T_2 = \langle \gamma_+^\mu(x)\, \hat{T}^{(2)}_{\nu\beta}(x, x')\, \gamma_+^\beta(x') \rangle_{x' x}, \tag{4.55}$$

$$\hat{T}^{(2)}_{\nu\beta}(x, x') = -\int \frac{d^n p}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n}\, e^{i(p-q).(x-x')} \left\{ p_\nu q_\beta \left[(p^2 - i\epsilon)(q^2 - i\epsilon)\right]^{-1} \right\} \tag{4.56}$$

Substituting $\hat{q} = p - q$ and relabelling as q we get

$$\hat{T}^{(2)}_{\nu\beta}(x, x') = -\int \frac{d^n q}{(2\pi)^n} e^{iq.(x-x')} \int \frac{d^n p}{(2\pi)^n} \{p_\nu p_\beta - p_\nu q_\beta\} \left[(p^2 - i\epsilon)([p-q]^2 - i\epsilon)\right]^{-1} . \tag{4.57}$$

The integrals over p can now be done (using the standard results [68] which are summarized in appendix B) and reducing (4.57) to

$$\hat{T}^{(2)}_{\nu\beta}(x, x') = -\int \frac{d^n q}{(2\pi)^n} e^{iq.(x-x')} \left\{ -\eta_{\nu\beta} p^2 \left(\frac{1}{4(n-1)}\right) + q_\nu q_\beta \left(\frac{n}{4(n-1)} - \frac{1}{2}\right) \right\} I_1 . \tag{4.58}$$

It is of crucial importance that one first expands all the quantities about $n = 4$ before taking the limit, as will now become apparent. About $n = 4$

$$I_1 \overset{n=4}{=} -2(4\pi)^{-2} \left[\frac{1}{n-4} + \frac{1}{2}\psi(1) + \frac{1}{2}\ln(q^2 - i\epsilon) + O(n-4)\right] \tag{4.59}$$

where $\psi(x) = d/dx(\ln\Gamma(x))$. Thus

$$\begin{aligned}\hat{T}^{(2)}_{\nu\beta}(x, x') = {} & \frac{i}{3(4\pi)^2} \left\{\frac{1}{2}\eta_{\nu\beta}\Box_f + \partial_\nu\partial_\beta\right\} \left(\frac{1}{n-4}\delta(x - x') + H^+(x - x')\right) \\ & + \frac{i}{18(4\pi)^2} \{-\eta_{\nu\beta}\Box_f + \partial_\nu\partial_\beta\}\, \delta(x - x')\end{aligned} \tag{4.60}$$

where

$$H^+(x - x') = \frac{1}{2} \int \frac{d^n q}{(2\pi)^n} e^{iq.(x-x')} \left\{\psi(1) + \ln(q^2 - i\epsilon)\right\} . \tag{4.61}$$

The operator acting on the second Dirac delta function in (4.60) arose from a term of $O(n-4)$ multiplying a pole term in I_1. It is such contributions to I which conspire to give the trace anomaly for conformally coupled massless fields. Finally we can combine these results as

$$T_2 = i(3(4\pi)^2)^{-1}\left\langle \gamma_+^{\nu}(x)\gamma_+^{\beta}(x')\left\{\left[\frac{1}{2}\eta_{\nu\beta}\Box_f + \partial_\nu\partial_\beta\right]\left(\frac{1}{n-4}\delta(x-x') + H^+(x-x')\right)\right.\right.$$
$$\left.\left. + \frac{1}{6}\left[-\eta_{\nu\beta}\Box_f + \partial_\nu\partial_\beta\right]\delta(x-x')\right\}\right\rangle_{x\,x'} . \tag{4.62}$$

At this stage the pole term, proportional to $(n-4)^{-1}$, is already apparent.

The same procedure can be used to isolate the divergences in the other terms. After a rather long calculation one obtains

$$I = -a\left\langle \left[\frac{1}{2}\left(\gamma_+^{\alpha\rho}(x)\gamma^+_{\alpha\rho}(x')\Box_f\Box_f + \gamma_+^{\alpha\beta}(x)\gamma_+^{\nu\rho}(x')\partial_\alpha\partial_\beta\partial_\rho\partial_\nu - 2\gamma_+^{\rho\mu}(x)\gamma_\mu^{+\alpha}(x')\partial_\rho\partial_\alpha\Box_f\right)\right.\right.$$
$$\left.\left. - \frac{1}{6}D(x,x')\right]\left\{\left(\frac{1}{n-4} - \frac{8}{45}\right)\delta(x-x') + H^+(x-x')\right\}\right\rangle_{x\,x'}$$
$$+ \frac{2}{16\pi^2}\left(\frac{\xi}{3} - \xi^2 - \frac{1}{36}\right)\left\langle D(x,x')\left\{\frac{1}{n-4}\delta(x-x') + H^+(x-x')\right\}\right\rangle_{x\,x'}$$
$$-(2880\pi^2)^{-1}\frac{(11-60\xi)}{12}\left\langle D(x,x')\,\delta(x-x')\right\rangle_{x\,x'} \tag{4.63}$$

where $a = (1920\pi^2)^{-1}$, all the operators (e.g. $\Box_f$, ∂_α, ...) act only on the variable x and

$$D(x,x') := \left(\gamma_+^{\sigma\rho}(x) - \eta^{\sigma\rho}\gamma_+(x)\right)\left(\gamma_+^{\alpha\beta}(x') - \eta^{\alpha\beta}\gamma_+(x')\right)\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma . \tag{4.64}$$

This unwieldy expression can be made a little more manageable by inserting the appropriate curvatures corresponding to the operators in (4.63). Using the results in appendix B we have

$$I \doteq -a\left\langle C^+_{\alpha\beta\gamma\delta}(x)\left[\left(\frac{1}{n-4} - \frac{8}{45}\right)\delta(x-x') + H^+(x-x')\right]C_+^{\alpha\beta\gamma\delta}(x')\right\rangle_{x\,x'}$$
$$+ \left\langle R_+^2(x)\right\rangle_x\left(\frac{2}{16(n-4)\pi^2}\left\{\frac{\xi}{3} - \xi^2 - \frac{1}{36}\right\} - \frac{(11-60\xi)}{12(2880\pi^2)}\right)$$
$$+ \frac{2}{16\pi^2}\left\{\frac{\xi}{3} - \xi^2 - \frac{1}{36}\right\}\left\langle R_+(x)H^+(x-x')R_+(x')\right\rangle_{x\,x'} . \tag{4.65}$$

where the symbol $\doteq$ indicates that the result only holds to second order in $\gamma_{\alpha\beta}$. For a conformally coupled massless field the non-local contribution involving the Ricci scalar vanishes, since $\xi/3 - \xi^2 = 1/36$ in this case. The residue of the pole at $(n-4)^{-1}$ also becomes proportional simply to the square of the Weyl tensor as shown in [45, 39]. It is also noteworthy that there is no divergence proportional to the Gauss-Bonnet Lagrangian, as would generally be expected [39], because it vanishes to second order in curvature.

The + graph

The new term which is not present in the in-out approach is the + graph. It is this term which makes the equations real and causal [65]. Once again we will demonstrate the evaluation of this graph by considering only one contribution. The others follow in a similar manner. Write

$$\begin{aligned} J &= -\frac{i}{2}\mathrm{Tr}\left[V_+^{(1)}\Delta^+(x,x')V_-^{(1)}\Delta^-(x,x')\right] \\ &= -2i\pi^2\left\langle \int\frac{d^np}{(2\pi)^n}\int\frac{d^nq}{(2\pi)^n}e^{i(q-p)(x-x')}\left\{-\gamma_+^{\alpha\beta}(x)q_\alpha q_\beta + i\gamma_+^{\alpha}(x)q_\alpha + \xi R_{(+)}\right\}\right. \\ &\qquad \left. \times\left\{-\gamma_-^{\mu\nu}(x')p_\mu p_\nu + i\gamma_-^{\mu}(x')p_\mu + \xi R_{(-)}\right\}\mathcal{O}(p,q)\right\rangle_{x,x'} \qquad (4.66) \end{aligned}$$

where $\mathcal{O}(p,q) = \delta(q^2)\theta(-q^0)\delta(p^2)\theta(p^0)$. This expands to nine terms as in the ++ case.

As an example consider the sample term

$$J_1 = \xi^2\left\langle \int\frac{d^nq}{(2\pi)^n}\int\frac{d^np}{(2\pi)^n}\, e^{i(q-p)(x-x')}R_+(x)R_-(x')\mathcal{O}(p,q)\right\rangle_{x,x'}. \qquad (4.67)$$

Introducing the new variable $\acute{p} = q - p$ and then relabelling it as p

$$J_1 = \xi^2\left\langle \int\frac{d^np}{(2\pi)^n}\, e^{ip(x-x')}R_+(x)R_-(x')\int\frac{d^nq}{(2\pi)^n}\mathcal{O}(p-q,q)\right\rangle_{x,x'}. \qquad (4.68)$$

To carry out the integration over q it is convenient to consider two cases: (i) The vector p spacelike and (ii) p timelike. The result has a step discontinuity at p null so that this case need not be considered.

(i) p spacelike: Choose the axes in q-space such that $p^0 = 0$. The integrand then contains the product $\theta(-q^0)\theta(q^0)$ and therefore the integral over q is identically zero.

(ii) p-timelike: Choosing the rest frame of p to define the q^0 axis, we have $p = (p^0, 0, 0, 0)$. Now introducing hyperspherical coordinates on the spatial sections

$$\int \frac{d^n q}{(2\pi)^n} \mathcal{O}(q-p,q) \;=\; \int \frac{|\mathbf{q}|^{n-2} d\Omega^{n-1} d|\mathbf{q}| dq^0}{(2\pi)^n} \mathcal{O}(q-p,q) \tag{4.69}$$

Using the identity

$$\delta(|\mathbf{q}|^2 - (q^0)^2)\theta(q^0) = \frac{\delta(q^0 + |\mathbf{q}|)}{2|\mathbf{q}|}\,, \tag{4.70}$$

the integration over q^0 is easily performed to give

$$\int \frac{d\Omega^{n-1}}{2(2\pi)^n} \int d|\mathbf{q}||\mathbf{q}|^{n-3} \delta((p^0)^2 + 2p^0|\mathbf{q}|)\theta(-p^0 - |\mathbf{q}|)$$

$$= \frac{(\pi)^{n-1/2}}{\Gamma(n-1/2)(2\pi)^n} \frac{|p^0|^{n-4}}{2^{n-2}} \theta(-p^0)\,. \tag{4.71}$$

There are no poles when this expression is expanded about $n = 4$. Therefore

$$J_1 = \frac{\xi^2}{4(2\pi)^3} \int \frac{d^n p}{(2\pi)^n}\, e^{ip(x-x')} R_+(x) R_-(x')\theta(-p^2)\theta(-p^0)\,. \tag{4.72}$$

The other terms can be evaluated in a similar manner (using the results in appendix B) and always producing finite terms as $n \to 4$. The result is

$$J \doteq -a\left\langle C^+_{\alpha\beta\gamma\delta}(x) H^-(x-x') C_-^{\alpha\beta\gamma\delta}(x')\right\rangle_{xx'}$$

$$+ \frac{2}{16\pi^2}\left\{\frac{\xi}{3} - \xi^2 - \frac{1}{36}\right\}\left\langle R_+(x) H^-(x-x') R_-(x')\right\rangle_{xx'}\,, \tag{4.73}$$

where we have introduced the notation

$$H^-(x-x') = 2\pi i \int \frac{d^n p}{(2\pi)^n} e^{ip(x-x')}\theta(-p^2)\theta(-p^0)\,. \tag{4.74}$$

For a conformally coupled scalar field the term involving the Ricci scalar is not present, just as the corresponding term disappeared for the ++ graph.

4.3 Renormalization of the effective action

The effective action, valid to second order in the perturbation from flat space, is $\Gamma = I + J$ where I and J are given by (4.65) and (4.73) respectively. It contains a pole at $n = 4$. In order to remove this we introduce a counterterm which is constructed from a polynomial of curvature. It is a purely local quantity in which the coefficients are proportional to $(n-4)^{-1}$:

$$S_c = \frac{\lambda^{4-n}}{n-4}\left\langle A\left(R^2_{\alpha\beta\gamma\delta} - 4R^2_{\alpha\beta} + R^2\right) + B\left(C^2_{\alpha\beta\gamma\delta}\right) + CR^2\right\rangle_{x\,x'} \qquad (4.75)$$

where A, B and C are coefficients chosen to exactly cancel the divergences in I. The parameter λ is arbitrary and has the dimensions of length. Examining (4.65) we find that $B = (1920\pi^2)^{-1}$, as determined previously by many authors [57, 39, 59]. The coefficient $C = -(8\pi^2)^{-1}\{\xi/3 - \xi^2 - 1/36\}$ depends on the curvature coupling and vanishes for the conformal case.

As mentioned earlier the Gauss-Bonnet combination, which is multiplied by A above, vanishes to second order in the perturbation. Thus it is not possible to determine the value of A unless the perturbation expansion is carried to higher order. Such a calculation is not done here.

Adding the counterterm S_c to Γ we obtain the renormalised effective action which is free of divergences

$$\begin{aligned}\Gamma_{ren} \;\doteq\;& a\left\langle C^+_{\alpha\beta\gamma\delta}(x)\, H^+_{(\lambda)}(x-x')\, C^{\alpha\beta\gamma\delta}_+(x') - C^+_{\alpha\beta\gamma\delta}(x)\, H^-(x-x')\, C^{\alpha\beta\gamma\delta}_-(x')\right\rangle_{x\,x'} \\ &+ \frac{2}{16\pi^2}\left\{\frac{\xi}{3} - \xi^2 - \frac{1}{36}\right\}\left\langle R^+(x)\, H^+_{(\lambda)}(x-x')\, R^+(x') - R^+(x)\, H^-(x-x')\, R^+(x')\right\rangle_{x\,x'} \\ &- \frac{11 - 60\xi}{12(2880\pi^2)}\left\langle R^2_+(x)\right\rangle \end{aligned} \qquad (4.76)$$

where

$$H^+_{(\lambda)} = -\frac{1}{2}\int\frac{d^nq}{(2\pi)^n}\, e^{iq(x-x')}[\ln(q^2\lambda^2 - i\epsilon)]\,. \qquad (4.77)$$

Note that the arbitrariness of λ has been used to absorb some local terms in $C^2_{\alpha\beta\gamma\delta}$. The Ricci squared term must be treated with some care however, since the non-local contribution disappears in the $\xi \rightarrow 1/6$ conformal coupling limit. I have therefore chosen to keep this explicit in (4.76).

We can now proceed to evaluate the stress-energy tensor, remembering that we must not set $g^+_{\alpha\beta} = g^-_{\alpha\beta}$ until after the variation.

4.4 The stress-energy tensor

In this section we show how to derive an expression for the stress tensor for a massless scalar field which is valid to linear order in a perturbation from flat space. The result obtained is in agreement with that derived by Horowitz [41] using only the Wald axioms [47] and the Poincaré invariance of the flat background. Similar results have also been obtained by Barvinsky and Vilkovisky within the framework of their covariant perturbation formalism [71].

For a conformally coupled field we use a result due to Page [72] to obtain the stress tensor in a spacetime which is related to the nearly flat one by an arbitrary conformal transformation. This result has recently been derived independently by Campos and Verdaguer [69].

Perturbations from flat space

Before obtaining the stress-energy tensor from the effective action (4.76) I wish to clarify the method used to obtain the variation of the non-local terms. In general the exact nature of the distribution is required to determine how to vary a quantity like

$$\left\langle R^+(x)H(x,x')R^-(x')\right\rangle_{x\,x'} . \tag{4.78}$$

In particular, one needs to know the explicit dependence on the metric in the distribution $H(x,x')$ and also its density weight. These details are unnecessary in present circumstances since the result has only been derived to second order. Thus, upon variation, it produces a quantity valid only through linear order, while ambiguities arising from the density weight of the distribution are entirely second order effects. The linearized results of variations of the quantities of interest are given explicitly in appendix B.

The stress tensor is given by the variation

$$\langle 0|T^{\alpha\beta}|0\rangle = \frac{2}{\sqrt{-g^+}}\left.\frac{\delta\Gamma_f^{\rm in}}{\delta g^+_{\alpha\beta}}\right|_{g^+_{\alpha\beta}=g^-_{\alpha\beta}} . \tag{4.79}$$

explicitly

$$\begin{aligned}\langle 0|T^{\alpha\beta}|0\rangle &\doteq a\left\langle\left[A^{\alpha\beta}H_\lambda(x-x')\right]\right\rangle_{x'} - \frac{11-60\xi}{6(2880\pi^2)}I^{\alpha\beta}\\ &\quad -\frac{1}{8\pi^2}\left\{\frac{\xi}{3}-\xi^2-\frac{1}{36}\right\}\left\langle I^{\alpha\beta}H_\lambda(x-x')\right\rangle_{x'} .\end{aligned} \tag{4.80}$$

where $\doteq$ is now taken to mean equality to linear order. Here the tensors $A^{\alpha\beta}$ and $I^{\alpha\beta}$ are the linearised variations of $C^2_{\gamma\delta\epsilon\kappa}$ and R^2 respectively:

$$A_{\alpha\beta} \doteq \frac{2}{3}R_{,\alpha\beta} - 2\Box_f R_{\alpha\beta} + \frac{1}{3}\eta_{\alpha\beta}\Box_f R \tag{4.81}$$

$$I_{\alpha\beta} \doteq 2R_{,\alpha\beta} - 2\eta_{\alpha\beta}\Box_f R . \tag{4.82}$$

The distribution H_λ is the same as that given by Horowitz [41]:

$$\begin{aligned}H_\lambda &= -\int\frac{d^nq}{(2\pi)^n}e^{iq(x-x')}\left[\ln(q^2-i\epsilon)+\ln\lambda^2+2\pi i\theta(-p^2)\theta(-p^0)\right]\\ &= -\int\frac{d^nq}{(2\pi)^n}e^{iq(x-x')}\left[\ln|q^2\lambda^2|+i\pi\theta^-(q)\right]\end{aligned} \tag{4.83}$$

where $\theta^-(q) = -\theta(-q^2)\, sgn(q^0)$ and the second term is obtained using the identity $\ln(q^2\lambda^2 - i\epsilon) = \ln|q^2\lambda^2| - i\pi\theta(-q^2)$ [73].

Observe that choosing $\xi = 1/6$ (corresponding to conformal coupling) and taking the trace of (4.80) we obtain

$$\langle 0|T_\mu{}^\mu|0\rangle \doteq (2880\pi^2)^{-1}\Box_f R \tag{4.84}$$

which is the only part of the trace anomaly that survives to linear order [45].

Perturbations from conformally flat space

In the next chapter we will approximate a black hole interior by an inhomogeneous and anisotropic cosmological model to investigate quantum effects near to mass-inflation singularities. In fact, it turns out that we can consider the deviations from conformal flatness to be small. Therefore we now obtain the stress-energy tensor for conformally coupled fields in a spacetime with small deviations from conformal flatness.

Consider a spacetime of the form

$$ds^2 = \Omega^2\left(\left[\eta_{\alpha\beta} + \gamma_{\alpha\beta}\right] dx^\alpha dx^\beta\right) = \Omega^2 d\bar{s}^2\,. \tag{4.85}$$

Applying Page's result [72] for the conformal transformation of the stress-energy tensor (6.33) to (4.80) gives

$$\begin{aligned}\langle 0|T^\mu{}_\nu|0\rangle \;\doteq\; & a\Omega^{-4}\left\{\left\langle \overline{A}^\mu{}_\nu H_\lambda(x-x')\right\rangle_{x'} - 8\left(\overline{C}^{\alpha\mu\beta}{}_\nu \ln\Omega\right)_{,\alpha\beta}\right\} \\ & - \frac{1}{6}(2880\pi^2)^{-1} I^\mu{}_\nu\,. \end{aligned} \tag{4.86}$$

Only terms which contribute to linear order have been retained, and barred tensors are evaluated in the nearly flat metric $d\bar{s}^2$.

4.5 Conclusion

In this chapter the closed time path formalism[63]-[67] has been used to calculate an expression for the renormalized stress-energy tensor of massless scalar fields to linear order in perturbations from a flat background. This result is in agreement with that derived by Horowitz [41] using very different methods. In particular it is seen that the non-local term is given by an integral over a logarithmic form-factor. The coordinate representation is not derived here.

In the special case of conformal coupling we could use the result of Page [72] to obtain an expression for the renormalized stress-energy tensor in the case of perturbations form conformal flatness. This result was recently derived independently by Campos and Verdaguer [69].

Although the validity of these expressions is clearly limited by the requirement that the perturbations should be small, a reminder of the more precise meaning of this statement is important. In obtaining the results (4.80) and (4.86) all terms non-linear in γ have been neglected. When the general forms of these terms are investigated one finds that the requirement is more correctly stated as

$$\nabla\nabla\gamma \gg (\nabla\gamma)^2 \,. \tag{4.87}$$

This condition is satisfied by the black hole interior near to the Cauchy horizon as we will see forthwith.

CHAPTER 5

Semi-classical effects in mass inflation

Classical models of generic black hole interiors [17]-[20], [32]-[34] have made progress in unravelling the nature of the internal geometry up to the onset of singular behavior at the inner (Cauchy) horizon. At this lightlike hypersurface, which corresponds to infinite external advanced time, the Weyl curvature scalar $|\Psi_2|$ diverges exponentially with advanced time. However, Ori [19] has pointed out that the divergence is rather weak, in the sense that it is integrable. More precisely, this means that there exist coordinates in which the metric and its inverse are bounded (and non-zero) on the Cauchy horizon. The approximate solution (3.39), (3.40) and (3.44) given in chapter 3 can be used to show this quite clearly. Using that advanced time coordinate $V = -e^{-\kappa_0 v}$ (which is closely related to the proper time along timelike geodesics approaching the Cauchy horizon) the solution is

$$ds^2 \simeq \frac{\text{const}}{r} dU dV + r^2 d\Omega^2 , \tag{5.1}$$

$$r^2 \simeq r_0^2 - \gamma |\ln(-V)|^{-p+1} - b^2(U - U_0)^2 , \tag{5.2}$$

where $V \to 0^-$ on the Cauchy horizon, γ and b are constants and the outflux of stress-energy across the Cauchy horizon has been taken as constant when $U > U_0$ and zero before U_0. Consequently, Ori has also speculated that spacetime can be continued beyond the mass-inflation singularity [19, 20], although it is far from clear how the high (infinite) curvature would be confined to a thin layer at the Cauchy horizon. In any case, this question is purely academic since quantum effects become important as curvatures rise, completely changing the nature of the spacetime.

Although no satisfactory theory of quantum gravity is known yet, some questions can be addressed within the framework of semi-classical gravity. While curvatures are sub-Planckian it should be justifiable to treat spacetime as a continuous manifold and to ignore the fluctuations of the gravitational field itself. One therefore considers the effects of quantized matter fields by allowing the curvature to respond to the expectation value of the quantum stress-energy tensor. Since quantum matter may violate the energy conditions (necessary for the proof of the singularity theorems [12]) the hope is sometimes expressed that curvature singularities may be completely avoided in such a theory.

Deep inside a black hole, vacuum polarization and particle production may induce a tension along the cylinders r = constant and hence damp the classical rise of curvature. Thus the possibility of a self-regulatory spacetime with bounded curvature may exist [74, 75]. On the other hand, it is entirely possible that the semi-classical corrections will act to further destabilize the classical picture.

In this chapter we report on an attempt to estimate the influence of quantum effects on the mass inflation scenario; in particular, to examine whether vacuum polarization and pair creation will act so as to damp the classical rise of curvature and possibly limit it to sub-Planck values.

Different ways in which the evolution might be influenced by quantum effects at the semi-classical level have been considered by Balbinot and Poisson [76]. The essence of their analysis was to consider generalized Einstein equations derived from the action

$$\mathcal{L} = \sqrt{g}\left(R + \alpha C^2_{\mu\nu\rho\sigma}\right) + \mathcal{L}_{\rm matter}\,, \qquad (5.3)$$

where α is some phenomenological coupling constant. The Weyl squared contribution was used to mimic vacuum polarization. The conclusion of their work is that vacuum polarization can either damp or intensify the classical rise of curvature depending on the sign of α in (5.3). It is remarkable that the results of their investigations are so closely aligned with the more detailed approach of [77]. In fact the leading behavior is almost identical in both cases; in the latter analysis, however, the ambiguity derives from the regularization scale which enters the results in a crucial manner.

There continues to be some debate about the interpretation of semi-classical gravity [78]. Most of the trouble originates from the fact that the effective equations are fourth order, unlike Einstein's equations which are second order, so even in regimes where quantum corrections are expected to be small one can obtain unphysical solutions which indicate just the opposite. In order to avoid this problem we adopt a viewpoint which has been advocated by Simon [40]. He suggests that a self-consistent approach to semi-classical gravity only allows solutions which are perturbative in $\hbar$, as the derived equations are. He has also suggested a general approach to ensuring that this be true (see section 2.4). A discussion of the semi-classical equations, in the next section, shows that Simon's prescription [40] for solving them is equivalent to first estimating the renormalized stress-energy tensor on the classical background and then looking for perturbative corrections to the metric.

Section 5.2 therefore introduces a simple classical model of a spherical black hole interior, due to Ori [19], which appears to capture the essence of the physics behind mass inflation. Since this is an exact classical mass-inflation solution it provides an excellent vehicle for the investigation of quantum effects. One obtains qualitatively similar results using the Poisson-Israel scenario [18], which is outlined in chapter 3, however the added complexity simply clouds the important issues. The Ori model can be recast in a form which is conformal to a Kerr-Schild metric. In fact this spacetime is nearly conformally flat provided curvatures remain below the Planck scale, allowing us to use the result of chapter 4 to approximate the quantum stress-energy tensor in the Ori spacetime.

The stress-energy tensor for a conformally invariant massless scalar field contains a non-local term, an integral over the past light-cone of the point where the stress-energy is evaluated. This integral is evaluated in section 5.4. We then proceed to find semi-classical corrections to the spacetime near to the Cauchy horizon. Unfortunately the regularization scale plays a crucial role in the interpretation of the results. In the conclusion arguments are presented which suggest that this renormalization ambiguity is of little importance and that semi-classical effects tend to reinforce the classical singularity up to the time when quantum gravitational degrees of freedom become activated. Thus it seems that quantum gravity will be needed to fully understand the final stages of gravitational collapse.

5.1 The semi-classical equations

In section 2.4 semi-classical gravity was discussed in some detail. It was explained that this approximation to quantum gravity treats the spacetime as classical, that is as a continuous pseudo-Riemannian manifold. This metric satisfies a set of effective field equations

$$G_{\alpha\beta} = 8\pi \left[T^{\rm class}_{\alpha\beta} + \langle in|\, T_{\alpha\beta}\, |in\rangle\right] \ , \tag{5.4}$$

where $T^{\rm class}_{\alpha\beta}$ is the stress-energy of classical matter and $\langle in|\, T_{\alpha\beta}\, |in\rangle$ is the renormalized stress-energy tensor of the quantum matter fields in the spacetime. $\langle in|\, T_{\alpha\beta}\, |in\rangle$ includes contributions from all possible fields including the gravitons which are treated as linearized perturbations about the classical spacetime. In general $\langle in|\, T_{\alpha\beta}\, |in\rangle$

is calculated using the standard loop expansion of quantum field theory; one constructs an asymptotic series by expanding in powers of a small parameter $\hbar/L^2$ where L is the radius of curvature of the background. In this chapter we consider effective equations which are calculated only to first order in this parameter, the so called one-loop approximation. Thus we write

$$\langle in|\, T_{\alpha\beta}\,|in\rangle = \hbar\,\langle in|\, T_{\alpha\beta}\,|in\rangle + O(\hbar^2)\,. \tag{5.5}$$

Substituting this into (5.4) the effective equations are valid only through linear order in $\hbar$. Since these equations are constructed perturbatively (in powers of $\hbar$), physically relevant solutions should also be analytic in $\hbar$. To enforce this requirement we adopt the self-consistent procedure advocated by Simon [40] in what follows (see section2.4).

We write the spherically symmetric line element as

$$ds^2 = -fA^2dv^2 + 2Advdr + r^2d\Omega^2\,, \tag{5.6}$$

where $f = 1 - 2m(r,v)/r + e^2/r^2$, $A = A(r,v)$ and $d\Omega^2$ is the line element on the unit two sphere. The effective field equations (5.4) and (5.5) can now be written as a system of four coupled partial differential equations:

$$
\begin{aligned}
m' &= 4\pi r^2\,(T_r{}^r - T)\,, && (5.7)\\
\dot{m} &= 4\pi r^2\,(T_v{}^r)\,, && (5.8)\\
f\frac{A'}{A} &= 4\pi r\,(2T_r{}^r - T)\,. && (5.9)\\
-2A^{-1}\left[\left\{\frac{(A^2f)'}{2A}\right\}' + \left(\frac{A'}{A}\right)^{\dot{}}\right] &= \left(\frac{4m}{r^3} - \frac{6e^2}{r^4}\right) + 8\pi\,(T - 2P)\,. && (5.10)
\end{aligned}
$$

We use a dot ($\dot{}$) to represent differentiation with respect to v and a ($'$) with respect to r. Here $T_{\alpha\beta} = T^{\rm class}_{\alpha\beta} + \hbar\,\langle in|\,T_{\alpha\beta}\,|in\rangle$, $T = T_v{}^v + T_r{}^r$ and $P = T_\theta{}^\theta = T_\phi{}^\phi$. The stress-energy of a classical, spherically symmetric electromagnetic field has already been implicitly included in the above equations.

In this chapter we continue to model the radiative tail of the gravitational collapse using a lightlike influx of particles, therefore

$$T^{\text{class}}_{\mu\nu} = \frac{L(v)}{4\pi r^2} (\partial_\mu v)(\partial_\nu v) \ . \tag{5.11}$$

where $L(v)$ is given by the analysis of Price [23]. Multiplying (5.7) and (5.8) by $\hbar$ we have the perturbative constraints (see section 2.4 for a detailed discussion of this approach)

$$\hbar(m') = O(\hbar^2) \ , \tag{5.12}$$

$$\hbar(\dot{m}) = \hbar[L(v)A^{-1}] + O(\hbar^2) \ . \tag{5.13}$$

Bearing these two conditions in mind, (5.9) implies

$$\hbar\,(A') = O(\hbar^2). \tag{5.14}$$

In general $\hbar \langle in | T_{\alpha\beta} | in \rangle$ is constructed from the metric and its derivatives. Simon's prescription can now be applied to equations (5.7)-(5.10) in order to ensure that the solutions are analytic in $\hbar$. Whenever $\hbar(m')$ or $\hbar(A')$ appear in the one loop renormalized stress-energy tensor they should be set to zero for consistency. On the other hand $\hbar(\dot{m})$ is replaced according to the rule (5.13).

In the present instance this means that the investigation of quantum effects on mass-inflation can proceed by first obtaining a solution to the classical Einstein field equations with null dust, then calculating the renormalized stress-energy tensor on this background and finally inserting it as a source in the one-loop equations (5.7)-(5.10) to obtain semi-classical corrections to the metric. We now carry out this procedure.

5.2 The classical Ori model

Ori [19] has introduced a particularly simple model of a charged black hole interior which seems to capture the essence of the physics behind mass-inflation. The influx of gravitational waves is modelled by a radial stream of lightlike particles. The outflow is treated schematically as a thin, transparent lightlike shell Σ within and parallel to the event horizon (Fig. 5.1). This idealization of the outflux reduces the problem of finding a mass-inflation solution to matching two exact, charged Vaidya solutions along the null hypersurface Σ. A general approach to such problems has been worked out by Barrabès and Israel [79].

The metric in each of the domains $\mathcal{V}_-$ and $\mathcal{V}_+$ separated by Σ has the charged Vaidya form for pure inflow:

$$
\begin{aligned}
(ds^2)_\pm &= dv_\pm (2\,dr - f_\pm\,dv_\pm) + r^2\,d\Omega^2\,, \\
f_\pm &= 1 - \frac{2m_\pm}{r} + \frac{e^2}{r^2}\,.
\end{aligned}
\tag{5.15}
$$

The mass $[m_\pm(v_\pm)]$ in each region is a function of the advanced time only, and e is the constant charge in the black hole. We choose the coordinate v_- as the standard advanced time at large radii outside the black hole. In particular, $v_- \to \infty$ at future null infinity and on the Cauchy horizon (see Fig. 5.1). In $\mathcal{V}_-$ we take the mass function to approach the finite asymptotic value m_0. Thus the Cauchy horizon is static and located at a constant radius $r_0 = m_0 - \sqrt{m_0^2 - e^2}$ in this region. The advanced time parameters v_+ and v_- are unequal: they are related by noting that the area of two-spheres is continuous across the shell. Thus the equations of Σ with respect to the two abutting coordinate systems are

$$
f_+\,dv_+ = f_-\,dv_- = 2\,dr \qquad \text{along } \Sigma\,. \tag{5.16}
$$

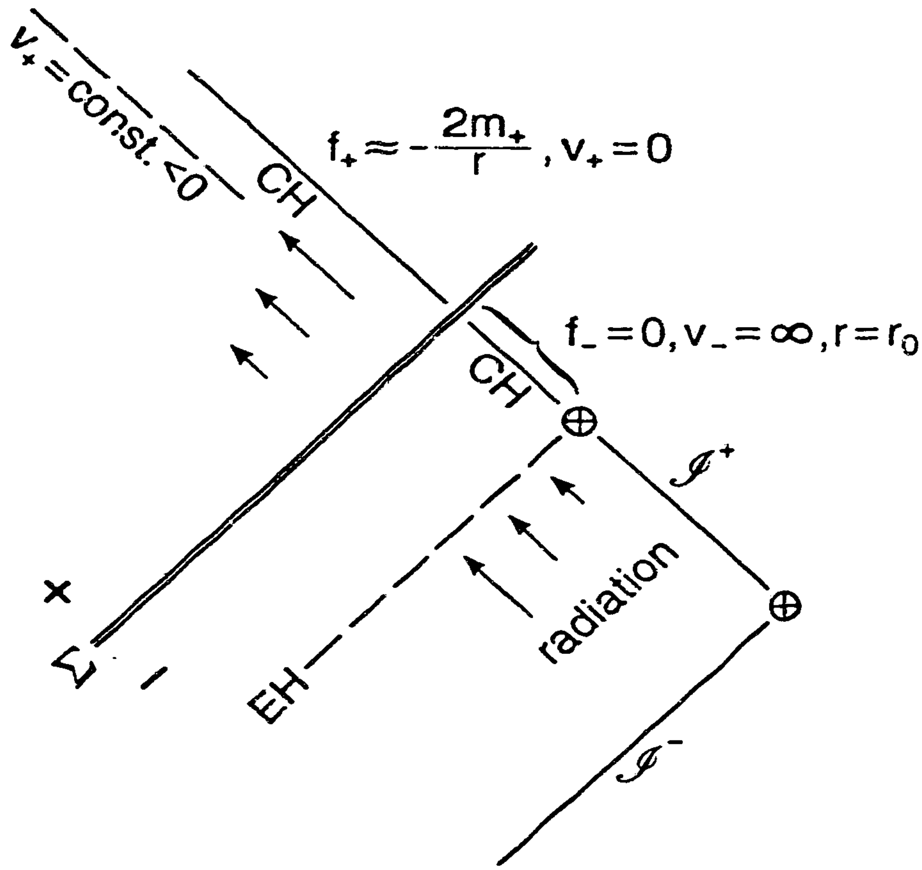

Figure 5.1: Ori model. Infalling radiation passes through a transparent, "outgoing" lightlike shell Σ inside a charged spherical hole. EH is the event horizon, and CH the Cauchy horizon.

The stress-energy tensor of the null dust in each of these regions is

$$T^{(\pm)}_{\alpha\beta} = \frac{L^{(\pm)}(v_\pm)}{4\pi r^2}(\partial_\alpha v_\pm)(\partial_\beta v_\pm)\,. \tag{5.17}$$

Accordingly the only non-trivial Einstein equations are

$$\frac{dm_\pm}{dv_\pm} = L^{(\pm)}(v_\pm)\,. \tag{5.18}$$

The vector tangent to Σ

$$n^\alpha_\pm = \frac{dx^\alpha_\pm}{dr} = \left(\frac{2}{f_\pm}, 1, 0, 0\right)\,, \tag{5.19}$$

is lightlike (and is therefore also normal to the shell). The transparency of the shell finds mathematical expression as $[T_{\alpha\beta} n^\alpha n^\beta] = 0$ [79] where the square brackets indicate the jump across the shell, explicitly

$$\frac{L^+(r_+)}{f_+^2} = \frac{L^-(r_-)}{f_-^2} \,. \tag{5.20}$$

Combining (5.18), (5.16) and (5.20) we can write

$$\frac{dm_+}{dr_-} = \left(\frac{f_+}{f_-}\right)\frac{dm_-}{dr_-} \,. \tag{5.21}$$

This is an equation for the interior mass function m_+ once $m_-(r_-)$ is specified. The ansatz

$$m_-(r_-) = m_0 - \frac{\alpha}{(p-1)\kappa_0}(\kappa_0 r_-)^{-(p-1)} \tag{5.22}$$

reproduces the correct power-law decay $dm_-/dr_- \sim r_-^{-p}$ of the externally observed gravitational wave flux (p=12 for quadrupole waves [23]). Here $\kappa_0 = (m_0^2 - e^2)^{1/2}/r_0^2$ is the surface gravity of the static portion of the Cauchy horizon, and α is a dimensionless constant.

Integration of the foregoing equations to get an asymptotic ($r_- \to \infty$) solution is now straightforward. Equation (5.16) yields

$$r = r_0 + \frac{\alpha}{\kappa_0^2 r_0 (p-1)}(\kappa_0 r_-)^{-(p-1)}\left[1 + (p-1)(\kappa_0 r_-)^{-1} + \ldots\right] \quad \text{along } \Sigma \,. \tag{5.23}$$

Substituting this into equation (5.21) gives

$$\frac{dm_+}{dr_-} \simeq m_+ \kappa_0 \left[1 - p(\kappa_0 r_-)^{-1} + \ldots\right] , \tag{5.24}$$

which has the solution

$$m_+(r_-) \simeq \text{const} \times (\kappa_0 r_-)^{-p} e^{\kappa_0 r_-} \,. \tag{5.25}$$

Finally one uses (5.16) to obtain the relation between r_+ and r_- as

$$-r_+ = \text{const} \times \exp(-\kappa_0 r_-) \,. \tag{5.26}$$

In terms of r_+, the mass function therefore has the asymptotic form

$$m_+(r_+) \;=\; m_0 |\ln(-r_+/m_0)|^{-p}(-r_+/m_0)^{-1} \qquad (r_+/m_0 \to 0^-)\,. \tag{5.27}$$

where we have set a dimensionless prefactor in (5.27) (depending on the luminosity and initial deformation of the collapsing star) equal to unity.

We now have the exact mass-inflation solution in $\mathcal{V}_+$, given by the Vaidya geometry

$$\begin{aligned} ds^2 &= dr_+(2\,dr - f_+\,dr_+) + r^2\,d\Omega^2\,, \\ f_+ &= 1 - \frac{2m_+}{r} + \frac{e^2}{r^2}\,, \end{aligned} \tag{5.28}$$

where m_+ has the diverging form (5.27). It has a timelike singularity at $r = 0$ which is intersected by a null singularity along the Cauchy horizon at $r_+ = 0$. The null singularity is characterized by a divergence of the mass-function which is evident in the Weyl curvature scalar (5.30).

A Vaidya geometry with metric of the form (5.28) has the Ricci curvature

$$R_{\alpha\beta} = \frac{2}{r^2}\frac{dm}{dr_+}\,(\partial_\alpha r_+)(\partial_\beta r_+) + \frac{e^2}{r^4}\,E_{\alpha\beta} \tag{5.29}$$

where $E^\alpha{}_\beta = \mathrm{diag}(-1,-1,1,1)$ and with m_+ written simply as m from here on. The sole non-vanishing Newman-Penrose component of the Weyl curvature is

$$-\Psi_2 = \frac{1}{2}C^{\theta\phi}{}_{\theta\phi} = [m(r_+) - e^2/r]r^{-3}\,. \tag{5.30}$$

5.3 $\langle U|\,T_{\alpha\beta}\,|U\rangle$ in the Ori model

We aim to estimate the expectation value $\langle U|\,T_{\alpha\beta}\,|U\rangle$ (in the Unruh state) of the stress-energy for a conformally coupled massless field on the Vaidya background (5.28) when the mass function has the classically diverging form (5.27).

In general, finding $\langle U|T_{\alpha\beta}|U\rangle$ is a problem of notorious difficulty [45, 46]. It becomes tractable in the present instance because of a number of special circumstances:

1. We are only interested in the asymptotic form of $\langle U|T_{\alpha\beta}|U\rangle$ near the Cauchy horizon.

2. The singularity is relatively mild; the curvature is a diverging but integrable function of v_+ [19, 32].

3. The special form (5.27) of the mass function means that, of two terms containing the same total number of m-factors and v-derivatives, the term with the smaller number of m-factors is dominant; e.g. $\ddot{m} >> m\dot{m}/r^2$. (The ratio of the two sides is "merely" a logarithmic factor; however this factor does become infinite as $v \to 0^-$, and for $v = -10^{10}$ Planck times in a solar mass black hole, it has already grown to 10^{23} !)

This conjunction of circumstances permits us to treat the geometry as a linear perturbation of conformally flat space. The terms linear in derivatives of $m(v)$, which we retain, actually dominate the neglected non-linear terms up to the time when curvatures reach Planck values.

To make this discussion more concrete, consider the space $d\bar{s}^2$ which is conformally related to Ori's solution (5.28):

$$d\bar{s}^2 = \left(\frac{r_0}{r}\right)^2 ds^2 = 2\left(\frac{r_0}{r}\right)^2 dr dv_+ - \left(\frac{r_0}{r}\right)^2 f_+ dv_+^2 + r_0^2 d\Omega^2 \,. \tag{5.31}$$

We will sometimes refer to this (and equivalently (5.34) below) as the ***conformal*** metric, as opposed to the ***physical*** Vaidya metric given by (5.28). It generates a new Ricci curvature which is free of the strongly divergent terms in (5.29), while the Weyl curvature is only multiplied by a factor $(r/r_0)^2$ which is of order unity in the neighborhood of the shell Σ. In fact, noting that the curvature of the two spheres is small on Planck scales we can approximate the line element $r_0^2 d\Omega^2$ by a corresponding flat metric. This is made explicit via the coordinate transformation

$$x + iy := r_0 e^{i\phi} \sin\theta \tag{5.32}$$

so that $r_0^2 d\Omega^2 \approx dx^2 + dy^2$. In what follows it is also useful to rescale the coordinate v_+ so that it more nearly represents Planck scales. We set

$$v := \epsilon v_+ , \quad u := 2\epsilon r_0^2/r, \quad \epsilon := l_p/r_0 . \tag{5.33}$$

where l_p is the Planck length. The conformal metric now takes the Kerr-Schild (flat plus lightlike) form

$$d\tilde{s}^2 = -du\,dv + dx^2 + dy^2 + 2L(u,v)dv^2 . \tag{5.34}$$

$$L = \frac{1}{8}k\left[u^3 m(v) - \epsilon r_0^2 u^2 - e^2 u^4/4r_0^2\epsilon\right] . \tag{5.35}$$

$$k = (l_p r_0^3)^{-1} . \tag{5.36}$$

Since $m(v)$ is much larger than $|e|$ near the Cauchy horizon we we will approximate

$$L(u,v) \simeq \frac{ku^3 m}{8} . \tag{5.37}$$

The spacetime (5.34) is manifestly almost flat for $|\Psi_2| \ll l_p^{-2}$:

$$l_p^2|\Psi_2| \approx \frac{l_p^2 m}{8 r r_0^2} \approx \frac{r^2}{r_0^2} L(u,v) . \tag{5.38}$$

Thus the approximation to the stress-energy tensor in a nearly conformally flat spacetime should be valid also in the Ori background. This is indeed a surprising fact since we are dealing with a region where effects of curvature are large. Therefore it must be emphasized that the approximation scheme does break down as curvatures approach Planck values. This is not a very severe limitation since non-perturbative quantum gravitational effects will then be important, invalidating the whole semi-classical analysis.

We can now adopt the in-vacuum expression (4.86). This needs to be supplemented by a local, conserved tensor representing initial conditions appropriate to the Unruh state for an evaporating black hole. Inside the hole, this is just the lightlike influx of negative energy that accompanies the thermal outflux to infinity [45, 46]. However, this remains negligible up to the moment when the classical curvature becomes Planckian if the black hole is larger than 100kg [32], and it will therefore be ignored. It is necessary to remember that our approximation is only valid through linear order in $L(u, r)$ however. Therefore

$$\langle U|T^{\mu}{}_{\nu}(x)|U\rangle \;\simeq\; a\left(\frac{r_0}{r}\right)^4 \left\{\left\langle \bar{A}^{\mu}{}_{\nu} H_{\lambda}(x-x')\right\rangle_{x'} - 8\left(C^{\alpha\mu\beta}{}_{\nu}\ln\frac{r}{r_0}\right)_{;\alpha\beta}\right\} - \frac{1}{6(2880\pi^2)} I^{\mu}{}_{\nu}\,. \tag{5.39}$$

where

$$H_{\lambda}(x-x') = -\int\frac{d^4q}{(2\pi)^4}\, e^{iq(x-x')}\left[\ln|q^2\lambda^2| + i\pi\theta\ (q)\right]\,, \tag{5.40}$$

and the barred quantities are evaluated in the conformal spacetime (5.34). The tensors $A_{\mu\nu}$ and $I_{\mu\nu}$ are given by (4.81) and (4.82) respectively. For the Ori model the Ricci scalar $R \equiv 0$ so that the tensor $I_{\mu\nu}$ will give no contribution. Thus there are only two terms to be evaluated using the curvatures of the conformal metric.

5.4 Coordinate representation for $H_{\lambda}(x - x')$

For the problem at hand it is possible to reduce the non-local term in (5.39) to a line integral along an ingoing and an outgoing null ray. The observation which allows us to do this is that the curvature is "plane" symmetric with respect to x and y. Thus we can trivially integrate over these coordinates to produce only a two dimensional integral. Write

$$
\begin{aligned}
I &= \langle g(x')H_\lambda(x-x')\rangle_{x'} \\
&= -\int d^4x' g(x') \int \frac{d^4q}{(2\pi)^4}\, e^{iq(x-x')}\left[\ln|q^2\lambda^2| + i\pi\theta^-(q)\right] \qquad (5.41)
\end{aligned}
$$

where g is a scalar test function of u and v only, and the integral is defined in the background Minkowski space with coordinates (t,z,x,y) [here $t=(v+u)/2$ and $z=(v-u)/2$]. We integrate over x and y to produce δ-functions in q^x and q^y and then integrate out these momenta, thus reducing (5.41) to

$$
I = -\int d\mathbf{x}\, g(\mathbf{x}') \int \frac{d\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q}.(\mathbf{x}-\mathbf{x}')}\left[\ln|\lambda^2(\mathbf{q}.\mathbf{q})| + i\pi\theta^-(q)\right] . \qquad (5.42)
$$

where $\mathbf{x}=(t,z)$, $\mathbf{q}=(q^t,q^z)$ and $\mathbf{q}.\mathbf{x} = -q^t t + q^z z$. It is convenient to introduce coordinates with their origin at the point of evaluation $\mathbf{x}$. Define

$$
V = (t'-t)+(z'-z'), \quad U=(t'-t)-(z'-z)\,, \qquad (5.43)
$$

$$
q^u = -(q^t-q^z)/2, \qquad q^v = -(q^t+q^z)/2\,. \qquad (5.44)
$$

The distribution $\theta^-(q)$ can now be rewritten as

$$
\begin{aligned}
\theta^-(q) &= \theta(q^u)\theta(q^v) - \theta(-q^u)\theta(-q^v) \qquad (5.45)\\
&= \frac{1}{2}sgn(q^u) + \frac{1}{2}sgn(q^v)\,. \qquad (5.46)
\end{aligned}
$$

After a little manipulation the integral is separated into two pieces (which have identical forms):

$$
\begin{aligned}
I = -\frac{1}{2}\Bigg[&\int dU\; g(U,0)\int\frac{d\omega}{(2\pi)}\; e^{-i\omega U}\left(\ln|\lambda\omega| + \frac{i\pi}{2}sgn(\omega)\right) \\
&+ \int dV\; g(0,V)\int\frac{d\omega}{(2\pi)}\; e^{-i\omega V}\left(\ln|\lambda\omega| + \frac{i\pi}{2}sgn(\omega)\right)\Bigg] \qquad (5.47)
\end{aligned}
$$

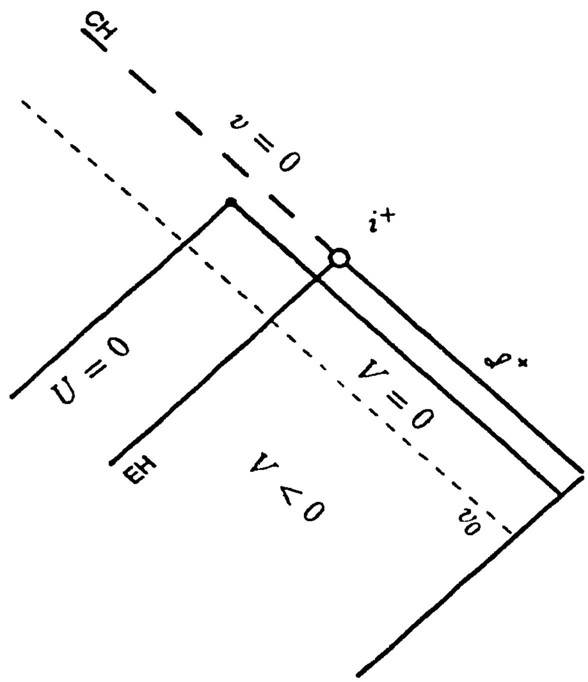

Figure 5.2: Lightcone coordinates. The relation between the coordinates U and V and the spacetime coordinates. EH is the event horizon, and CH the Cauchy horizon.

It is actually possible to eliminate ω completely from these integrals. Some care is necessary in this procedure since $H_\lambda(x - x')$ is a distribution, therefore the technical details appear in appendix C.

Using equation (C.7) we reduce (5.47) to

$$I = -\frac{1}{2}\left\{\int_{-\infty}^{\infty} g(U+u,0)\,\frac{\partial}{\partial U}\left[(\psi(1) - \ln|U/\lambda|)\,\theta(-U)\right]dU \right.$$
$$\left. + \int_{-\infty}^{\infty} g(0,V+v)\,\frac{\partial}{\partial V}\left[(\psi(1) - \ln|V/\lambda|)\,\theta(-V)\right]dV\right\} \quad (5.48)$$

Integrating by parts implies

$$I = \frac{1}{2}\left\{\int_{-\infty}^{0} \left.\frac{\partial g}{\partial U}\right|_{V=0} \ln|U/\lambda|\,dU + \int_{-\infty}^{0} \left.\frac{\partial g}{\partial V}\right|_{U=0} \ln|V/\lambda|\,dV\right\}, \quad (5.49)$$

where the arbitrariness of the length scale λ has been used to absorb the $v(1)$ term which is essentially a local contribution.

It is possible to formally integrate (5.49) to a form which will be useful later. Consider, for example, the term

$$\mathrm{Int} = \frac{1}{2}\int_{V_0}^{0} \frac{\partial g}{\partial V} \ln |V/\lambda| \, dV \, . \tag{5.50}$$

where I have changed the lower limit to V_0 for later convenience. The limit $V_0 \to -\infty$ can of course always be taken later. Making the coordinate transformation from $V = v' - v$ to $y = V/v + 1$ we can rewrite this as

$$\begin{aligned} \mathrm{Int} &= \frac{1}{2}\int_{y_0}^{1} \left[\frac{\partial g}{\partial y} \ln |v/\lambda| + \frac{\partial g}{\partial y} \ln |1-y| \right] dy \, . & (5.51) \\ &= \frac{1}{2} \ln |v/\lambda| \left[g(u,v) - g(u,v_0) \right] + \frac{1}{2}\int_{y_0}^{1} \frac{\partial g}{\partial y} \ln |1-y| dy \, . & (5.52) \end{aligned}$$

Finally integrating by parts

$$\begin{aligned} \mathrm{Int} &= \frac{1}{2} \ln |v/\lambda| \left[g(u,v) - g(u,v_0) \right] - \frac{1}{2}\sum_{n=1}^{\infty} \left[\left(\frac{\partial^n g(u,v)}{\partial v^n} \right) \Big|_{v=v_0} \frac{|v-v_0|^n}{n!} \right. \\ &\quad \left. \times \left\{ \ln |(v-v_0)/\lambda| - \ln |v/\lambda| + \sum_{j=1}^{n} \left(\frac{1}{j} \right) \right\} \right] . & (5.53) \end{aligned}$$

where $v_0 = V_0 + v$. This expression will be used in the next section.

5.5 Semi-classical corrections

With this machinery in hand we now examine the semi-classical corrections induced by the presence of the quantum matter. The first equations of interest are those for the mass function, since it characterizes the classical singularity along the Cauchy horizon. If quantum effects accelerate its growth then we can say that the singularity is strengthened in the semi-classical regime. Equations (5.7) and (5.8) read (to order $\hbar$)

$$\frac{\partial m_{sc}}{\partial v} = \frac{dm_+}{dv} - \frac{2\pi r^4 \hbar}{\epsilon r_0^2} \langle U | T_v{}^u | U \rangle \ . \tag{5.54}$$

$$\frac{\partial m_{sc}}{\partial u} = \frac{2\pi r^4 \hbar}{\epsilon r_0^2} \langle U | T^v{}_v | U \rangle \tag{5.55}$$

where r is regarded as a function of u given by $r = 2\epsilon r_0^2/u$, and the subscript "sc" indicates the semi-classical mass function. We have also reinstated the subscript on the classical mass $m_+(v)$ to indicate that it is the known function in (5.27). The components of the quantum stress tensor are given by (5.39) with $A^{\mu}{}_{\nu}$ and $C_{\mu\nu\alpha\beta}$ evaluated in the conformal spacetime. Thus

$$\langle U | T_v{}^u | U \rangle \simeq a\left(\frac{r_0}{r}\right)^4 \left\{ \int d^4x' \left[-8ku' \,\ddot{m}_+(v') H_\lambda(x - x')\right] + \frac{8}{3}\left(R \ln\left[\frac{r_0}{r}\right]\right)_{,vv} \right\} \tag{5.56}$$

$$\langle U | T_v{}^v | U \rangle \simeq a\left(\frac{r_0}{r}\right)^4 \left\{ \int d^4x' \left[8k \,\dot{m}_+(v') H_\lambda(x - x')\right] - \frac{8}{3}\left(R \ln\left[\frac{r_0}{r}\right]\right)_{,uv} \right\} \tag{5.57}$$

where $\bar{R} = 6kum$ is the Ricci scalar in the conformal metric. Remember that a is a positive constant. Before considering the evaluation of the non-local terms in these components of the stress-energy tensor a couple of comments are in order. Firstly we have considered the influx to be turned on at some finite advanced time v_0 as in Fig. 5.4. Thus $\dot{m}_+(v)$ and all other derivatives of the mass are identically zero for $v < v_0$ and the lower limit of integration is therefore $V_0 = v_0 - v$ in (5.49). Secondly the range of integration over U is $-u \leq U \leq 0$ since $u \to 0$ as $r \to \infty$. Therefore, combining (5.49) and (5.53) with these limits of integration gives

$$\int d^4x' \left[-8ku' \,\ddot{m}_+(v') H_\lambda(x - x')\right] \simeq -4ku\ddot{m} \left(\ln|v/\lambda| + \ln|u/\lambda|\right) + \ldots \tag{5.58}$$

and

$$\int d^4x' \left[8k \,\dot{m}_+(v') H_\lambda(x - x')\right] \simeq 4ku\dot{m} \left(\ln|v/\lambda| + \ln|u/\lambda|\right) + \ldots \ . \tag{5.59}$$

Only the leading terms are explicitly shown here. Inserting (5.56)-(5.59) in (5.54) and (5.55) we readily obtain the approximate solution

$$m_{sc} \simeq m_+ + \frac{8\pi r_0^2 a\hbar}{\epsilon} k u \dot{m}_+ \left\{ \ln|v/\lambda| + \ln|u/\lambda| - 4\ln|r/r_0| \right\} + \ldots \, . \tag{5.60}$$

At this point the reader should notice that the length scale λ enters the result in a crucial manner through the logarithm so the sign of the correction term will change at some stage during the evolution. Further discussion of this point is postponed to the end of this section. For now we proceed to equations (5.9) and (5.10) for the function $A(u, v)$.

The tangential pressures are evaluated in the same way as the other components of the stress-energy (5.56) and (5.57):

$$P \simeq a\left(\frac{r_0}{r}\right)^4 \left[-4k\dot{m} \left(\ln|v/\lambda| + \ln|u/\lambda| \right) - \frac{40}{3} \left(R \ln|r/r_0| \right)_{,uv} \right] + \ldots \tag{5.61}$$

Remembering that only terms of less than $O(\hbar^2)$ are kept for consistency, (5.9) becomes

$$A' \simeq 0 \, . \tag{5.62}$$

This result arises due to the truncation of the stress-energy tensor at linear order in $L(u, v)$ so that we are unable to obtain further information at this point. Equation (5.62) implies that (5.10) is trivially satisfied when m_{sc} has the form (5.60). Thus to linear order A is a function of v alone.

5.6 Discussion

We have now seen that the quantum corrections are of a particularly simple form with the mass function given by (5.60) and $A(u, v)$ effectively constant up to the time when curvatures become Planckian. Although these expressions give the quantum corrected solution, there is one more piece of information which is useful to understand the effects on the geometry. Classically the outflux from the star has merely the catalytic effect of initiating the contraction of the Cauchy horizon. The blueshifted influx and any outflux from the star produces a scalar curvature singularity along the Cauchy horizon. The quantum outflux, however, is potentially much stronger. It is straightforward to use (5.39) to estimate this flux of quantum material which crosses the horizon,

$$\hbar \langle in| T^{\alpha\beta} |in\rangle (\partial_\alpha r)(\partial_\beta r) \approx -32a \left(\frac{r_0}{r}\right)^4 \hbar (k u m \ln|r/r_0|)_{,uu} \sim -m/r_0^3 \,. \qquad (5.63)$$

which is of the order $|\Psi_2|$ times the classical outflux. It therefore remains small up to the time when our approximations break down and quantum gravitational effects become important. This has the important consequence that the classical picture of ingoing light rays contracting very slowly (on Planck scales) under irradiation by the star is not affected by the quantum corrections up to the time when curvatures become Planckian (cf [76]).

Do the semi-classical corrections oppose the classical plunge towards a singularity? Denoting the advanced time at which the Weyl scalar reaches Planck values by

$$|v_p| = \frac{m_0^2 l_p^2}{r_0^3} [\ln (r_0^3/m_0 l_p^2)]^{-p} \,. \qquad (5.64)$$

it is clear from equation (5.60) that there are now two essentially different possibilities:

- If $\lambda \gg |v_p|$, the logarithm in (5.60) is negative as v_4 approaches v_p, and (5.60) predicts damping of the classical growth of curvature due to quantum effects. Examining the behavior of the radial coordinate r along outgoing null rays near to the Cauchy horizon indicates that they are defocussed by the quantum matter. This suggests that quantum effects might lead to a core region with curvatures bounded from above as suggested in [74, 75].

- If, on the other hand, $\lambda \lesssim |e_p|$, quantum effects would further destabilize the classical plunge toward a curvature singularity. The combined effects of the quantum fluxes (5.63) and (5.56) would tend to focus both ingoing and outgoing light rays. However following the evolution all the way to the Cauchy horizon (which is, of course, way beyond our approximations) brings us back to the first possibility of a core with bounded curvature.

Unfortunately, this ambiguity cannot be resolved within the present theoretical framework, which provides no information about λ [49]. The quantum theory of massless fields propagating on a fixed classical background has no inherent length-scale. Something further can be said if one is willing to entertain an arguable hypothesis about the origin of this incompleteness of the semi-classical theory [49].

A successful quantum theory of the gravitational field is expected to have the effect, at moderate curvatures, of modifying the Einstein-Hilbert Lagrangian by terms quadratic in curvature,

$$16\pi L_G = l_p^{-2} R + \alpha_1 C^2_{\alpha\beta\gamma\delta} + \beta_1 R^2 \,. \tag{5.65}$$

where α_1 and β_1 are constants of order unity. It is precisely these coupling constants which get renormalised to remove the divergences which arise during the calculation of the quantum stress-energy tensor for the scalar field. Thus the effective equations including the quantum gravitational contributions are

$$G_{\mu\nu} + l_p^2(\alpha_1 A_{\mu\nu} + \beta_1 I_{\mu\nu}) = 8\pi \left\{ T^{\rm class}_{\mu\nu} + \hbar \left\langle in \right| T_{\mu\nu} \left| in \right\rangle \right\} \,. \tag{5.66}$$

This suggests that the incompleteness of the semi-classical theory is related to the neglect of quantum gravitational effects.

It is arguable that a quantum theory of massless fields which includes gravity would become a *complete* theory, with the net coefficients of $A_{\mu\nu}$ and $I_{\mu\nu}$ determined [49]. Suppose this is true. Then α_1 in (5.66) is expected to be of order unity, and the term $\alpha_1 A_{\mu\nu}$ may be interpreted as representing effects associated with gravitational vacuum polarization. Now, it is reasonable to expect that, once the quantum gravitational degrees of freedom are activated, gravitons will have effects not too disimilar from photons and other massless fields *, i.e. that

$$\langle in| T_{\mu\nu} |in\rangle \sim -\alpha_1 t_p^2 A_{\mu\nu} . \tag{5.67}$$

as curvatures approach Planck values. Comparison of this and (5.58) or (5.59) now shows that the logarithmic factor should also be of order unity, that is $\lambda \approx |v_p|$.

If this conclusion is correct, (5.60) should be interpreted as an intensification (rather than a damping) of the classical influences tending to produce a curvature singularity at the Cauchy horizon, at least up to the time when curvatures become Planckian.

Interestingly the sign change which occurs (due to the logarithm) would seem to be in agreement with general arguments advanced by Fradkin and Vilkovisky [80] which suggest that α_1 should be negative. One way or another, it seems clear that the resolution of the singularity problem lies at energy scales where perturbative calculations break down. The change of sign of the semi-classical corrections as this scale is approached may indicate a quantum damping of the classical growth of curvature, however these calculations do suggest that the ultimate, quantum stage

*Although this is a plausible assumption the non-linear and non-conformal nature of the gravitons may lead to important differences of evolution of the black hole is inaccessible to semi-classical considerations.

CHAPTER 6

Homogeneous Mass Inflation

To date, most of the work on black hole interiors has focussed on the singularity which forms at the Cauchy horizon. The detailed structure of the geometry in this region of spacetime was first elucidated by Poisson and Israel [18]. As discussed in chapter 3 they found a null singularity characterized by a divergence of the mass function of the spherical black hole. Radiative outflux from the collapsing star causes the contraction of the Cauchy horizon, which is initially a stationary null hypersurface (i.e. its lightlike generators have zero expansion). Thus the radius of the Cauchy horizon decreases with increasing retarded time, ultimately contracting to zero. In this chapter we consider what the structure of the singularity may be near to $r = 0$. In the Ori model spacetime the $r = 0$ singularity is timelike, being intersected by the null (mass-inflation) singularity at infinite advanced time. This is probably not representative however. In fact, indications from numerical integration [81] of the field equations for a charged black hole perturbed by a scalar field are that a rather large portion of the singularity is *spacelike*. The mass-inflation singularity becomes spacelike once the radius of the Cauchy horizon reaches zero.

In this chapter we consider a simple analytic model which may be representative of the spacetime structure in this region. The classical homogeneous mass inflation (HMI) model was first considered by Page and independently by Ori [42]. They used a two fluid stress-energy tensor to model perturbations of the interior of a charged black hole, and looked for solutions which were homogeneous in r. The advantage of the homogeneity requirement is that one obtains only ordinary differential equations to solve. Although the system is highly non-linear, it is possible to analyse the general solutions in great qualitative detail [42]. In their original work, Page and Ori considered an influx which did not decay in advanced time, and for this reason its conclusions seemed only to have tenuous connections to the internal physics of black holes. It is possible, however, to show that this model is compatible with arbitrary behaviour of the null fluids which leads us to to the interpretation discussed in section 6.2.

In section 6.1 a classical solution for the HMI model in the region $r^2 \ll e^2$ is presented. This particular solution did not explicitly appear in the work of Page and Ori [42], although it is implicit therein. A brief discussion of the properties of the solutions is then presented. In particular, it is suggested that homogeneous solutions may be generic as $r \to 0$.

We go on to consider quantum effects when $r \ll e^2$, calculating the renormalised stress-energy tensor for a conformally coupled massless scalar field on the classical HMI background [82]. The expectation value is taken with respect to a quantum state possessing the natural symmetries of the background. This is not the physically relevant state, which appeared empty in the infinite past when the matter which formed the black hole was dispersed to almost zero density. This means that particle creation effects have been neglected, however, we argue that the dominant terms in the stress-energy tensor should be insensitive to reasonable changes of initial state.

Finally, the semiclassical backreaction equations are solved to find vacuum polarisation corrections to the classical background. The correction to the mass function diverges more strongly than in the classical case. There continues to be much controversy about the meaning of semi-classical gravity, and the correct way to solve the effective equations. We follow the self consistent approach advocated by Simon [40]. This method makes the perturbative nature of the calculations explicit, while it also avoids spurious solutions which indicate instabilities (of flat space for example). Of course this semi-classical analysis will become invalid once curvatures reach Planck values, so one may only speculate on the existence of a singularity in a complete theory of quantum gravity.

6.1 The HMI Background

It is most convenient to proceed from a double null formulation of the spherical field equations to the homogeneous problem for cross-flowing null dust. Therefore we begin with the line element

$$ds^2 = -\frac{2F}{r}dU\,dV + r^2 d\Omega^2, \tag{6.1}$$

where $d\Omega^2$ is the line element on a unit sphere. Along with the electromagnetic stress-energy tensor $E^{\mu}{}_{\nu} = e^2/8\pi r^4\,\mathrm{diag}(-1,-1,1,1)$ we include both ingoing and outgoing null dust with

$$T_{\alpha\beta} = \frac{L_{\rm in}(V)}{4\pi r^2}(\partial_\alpha V)(\partial_\beta V) + \frac{L_{\rm out}(U)}{4\pi r^2}(\partial_\alpha U)(\partial_\beta U)\,. \tag{6.2}$$

The Einstein field equations are

$$\begin{aligned} (r^2)_{,UV} &= F(e^2/r^2 - 1)/r\,, & (6.3)\\ (\ln F)_{,UV} &= F(1 - 3e^2/r^2)/(2r^3)\,, & (6.4)\end{aligned}$$

subject to the constraints

$$\begin{aligned} (r^2)_{,VV} - \frac{F_{,V}}{F}(r^2)_{,V} &= -2L_{\rm in}(V)\,, & (6.5)\\ (r^2)_{,UU} - \frac{F_{,U}}{F}(r^2)_{,U} &= -2L_{\rm out}(U)\,. & (6.6)\end{aligned}$$

It is very important to realise that the luminosity functions [$L_{\text{in}}(V)$ and $L_{\text{out}}(U)$] have no direct operational meaning. They depend on the parametrization of the null coordinates. We select coordinates such that $L_{\text{in}}(V) = 1/2 = L_{\text{out}}(U)$. So what is the relation between V and the standard advanced time coordinate v? Let us consider an influx which decays with advanced time according to an inverse power law along the event horizon [23, 24]: $L(v) = \text{const.} \times v^{-p}$. The relation between the two coordinates is simply

$$L_{\text{in}}(V)\left(\frac{dV}{dv}\right)^2 = \text{const.} \times v^{-p} \tag{6.7}$$

from the transformation law for (6.2). Thus as $v \to \infty$ we find $V \to 0$. This relation is important later when the structure of the homogeneous solutions is considered.

Following Page and Ori [42] we restrict r and F to be independent of $t = (V - U)/2$ and hence to be functions only of

$$\eta = (V + U)/2\,. \tag{6.8}$$

Introduce the functions

$$\sqrt{\frac{2F}{r^3}}\frac{dr}{d\eta} = -B^{1/2}(r) = -(2r\,m(r) - r^2 - e^2)^{1/2}\,, \tag{6.9}$$

$$\frac{2Fe^2}{r^3} = g(r)\,. \tag{6.10}$$

so that

$$ds^2 = \frac{2F}{r}(-d\eta^2 + dt^2) + r^2 d\Omega^2, \tag{6.11}$$

$$= \left(\frac{r}{e}\right)^2\left(-e^2B^{-1}dr^2 + g(r)dt^2 + e^2d\Omega^2\right). \tag{6.12}$$

In equation (6.9) we have introduced the local mass function $m(r)$. The field equations (6.3) and (6.4) become

$$(Bg)^{-1}(Bg)' = \frac{4}{B}\left(\frac{e^2 - mr}{r}\right) \tag{6.13}$$

$$\left(\frac{3\sqrt{Bg}}{r} + g'\sqrt{\frac{B}{g}}\right)' = \sqrt{\frac{g}{2B}}\left(1 - \frac{3e^2}{r^2}\right). \tag{6.14}$$

while the two constraints reduce to a single equation for the mass

$$m' = -e^2(gr^2)^{-1}, \tag{6.15}$$

where a prime (′) denotes differentiation with respect to r.

Using (6.13) and (6.15) we can derive a master equation for the mass function. Integrate (6.13) to

$$Bg = e^2 \exp\left(4\int \frac{e^2 - mr}{Br} dr\right) . \tag{6.16}$$

where a constant of integration has been set to unity. This corresponds to the freedom to rescale t. Substituting for $g(r)$ into (6.15) implies that

$$m' = -(e^2r^2)^{-1} B \exp\left(4\int \frac{e^2 - mr}{Br} dr\right) \tag{6.17}$$

$$= -\frac{r^2}{e^2B} \exp\left(\int \frac{4rm'}{B} dr\right) . \tag{6.18}$$

Now, multiply by $-e^2B/r^2$, take the logarithm and differentiate with respect to r to get

$$\left(\frac{Bm'}{r^2}\right)' = \frac{4(m')^2}{r}. \tag{6.19}$$

Finally, recasting this as an integral equation (for purely aesthetic reasons) we have

$$m' = \frac{r^2}{B}\int \frac{4(m')^2}{r} dr . \tag{6.20}$$

This will allow us to obtain an approximate solution to the homogeneous problem quite easily.

Consider the region where $r^2 \ll e^2$, so that $B(r) \approx 2m(r)\, r - e^2$. It is easily verified that

$$m(r) = \frac{e^2}{r}, \qquad g(r) = 1. \tag{6.21}$$

is a solution to the field equations in this region. Thus we have the HMI spacetime

$$ds^2 = \frac{r^2}{e^2}[-dr^2 + dt^2 + e^2 d\Omega^2], \tag{6.22}$$

which is of a particularly simple form.

Page and Ori [42] were able to obtain qualitative information about the general solutions to the homogeneous equations. Their results indicate that the mass function inflates to infinity like $1/r$, as occurs in (6.21). Generically they found that the mass function also contains an oscillatory part with an amplitude which diverges as $r \to 0$, we will comment further on the implication of this when we deal with quantum effects.

6.2 Properties of the homogeneous solutions

The spacetime (6.22) appears to present a very different picture of the black hole interior than the standard mass-inflation scenario [18]. So why has the truncation to dependence only on η been so brutal? Furthermore, are homogeneous solutions likely to be of any physical significance? The first of these questions cannot be answered satisfactorily at present, although some comments and observations can be made. Throughout the following discussion we will refer to Fig. 6.1 in which the black hole interior is schematically represented.

In this diagram there are essentially three different regions, about which we have various levels of knowledge:

1. The exterior gravitational field, including $\mathcal{I}^+$ and the event horizon, is quite well understood. When a slightly aspherical star collapses to form a black hole, the asymptotic field (near i^+) settles down quickly to the Reissner-Nordström

solution. The deviations from spherical symmetry remain small and in fact die away as an inverse power of advanced (or retarded on $\mathcal{I}^+$) time[23, 24]. Thus the Reissner-Nordström spacetime is a good asymptotic approximation to the external gravitational field.

2. Significant advances in our understanding of spacetime near to the Cauchy horizon have been made with the work of Poisson and Israel [18], Ori [19, 20] and others. This work indicates a finite "length" null singularity along the Cauchy horizon. However there is a (debatably) serious drawback in these analyses — the boundary conditions which are used always assume the existence of a static portion of the Cauchy horizon between i^+ and the retarded time when mass-inflation occurs. In all of these spherically symmetric models this essentially forces a null singularity to exist, since the contraction of ingoing null rays is governed only by the tranverse flux of perturbations across the horizon. To avoid this problem one would like to impose boundary conditions at the event horizon and evolve this data inwards (we currently have no reason to believe that this will change the mass-inflation picture).

3. Near to $r = 0$ (but away from the Cauchy horizon) our knowledge is greatly lacking. It is natural to assume that there will exist a spacelike singularity in this region; this is based on the intuition that the null singularity along the Cauchy horizon should connect to a spacelike singularity once the Cauchy horizon radius reaches zero. Some support for this has recently appeared in the work of Gnedin and Gnedin [81]. As $r \rightarrow 0$ it seems plausible that spatial derivatives will become unimportant and the solutions tend to those of equations (6.13)-(6.15) — the black hole core is *nearly* homogeneous for small radii. Although no formal proof of this fact yet exists, one may argue that since future light-cones encounter less of the spacelike singularity as $r \rightarrow 0$ the spacetime will, at least locally, appear homogeneous.

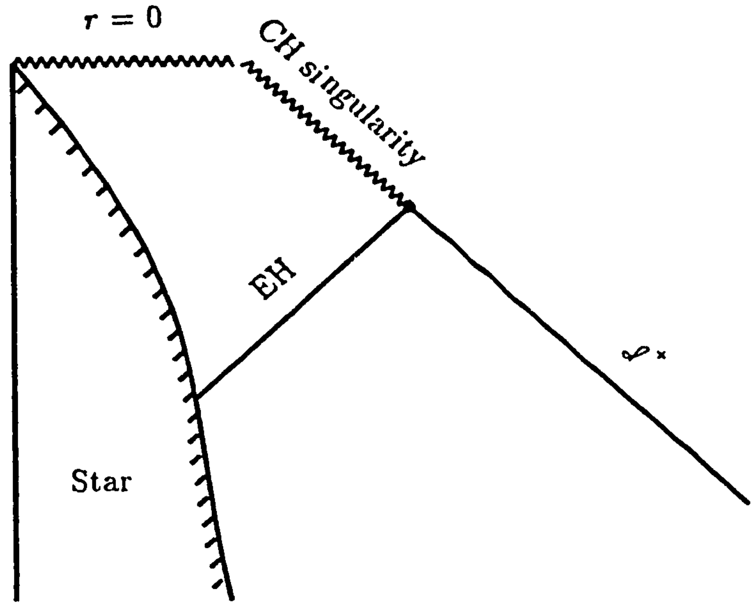

Figure 6.1: Schematic representation of the spacetime of a collapsing charged star

Although the spacetime (6.22) is at the unstable fixed point of these equations it provides an analytic solution which exhibits the properties which were discussed by Page [42]. It has a singularity only at $r = 0$. This singularity is indeed spacelike, and so it suggests that the solution may be indicative of the general situation in the shaded portion of the diagram. In the work of Page [42] it was shown that the solutions of (6.13)-(6.15) would generally have no Cauchy horizon; more precisely he showed that outgoing null geodesics would encounter an $r = 0$ singularity before the advanced time v became infinite. A crucial input into his result was that $L(v) =$ constant when v was the standard advanced time for an external observer. As was emphasised in section 1.1 it is always possible to find a coordinate system in which $L_{\text{in}}(V) =$ constant. For the usual power law decay of infalling perturbations the relation between the two coordinate systems is given in (6.7), where it shows that $V \to 0^-$ on the Cauchy horizon. Now along an outgoing null geodesic in the solution (6.22)

$$r = \text{const} - \frac{1}{\sqrt{2}} V \tag{6.23}$$

and generally as $V \to 0^-$ we find $r \to$ constant. Thus without the global solution to allow the determination of the constant in (6.23) we can say nothing about the presence (or not) of a Cauchy horizon.

Moreover Page [42] has shown that, for arbitrary solutions of (6.13)-(6.15), along outgoing null geodesics $r = 0$ is encountered at some finite value of $V = V_r$. However it cannot be determined whether this lies before or after the Cauchy horizon without knowing the global structure of the spacetime.

6.3 Vacuum Polarisation

Next, we wish to consider semiclassical corrections to the metric functions, due to the presence of a conformally coupled massless scalar field ϕ obeying

$$g^{\alpha\beta}\nabla_\alpha \nabla_\beta \phi - \frac{1}{6} R\phi = 0 . \qquad (6.24)$$

Later we will argue that our conclusions should be insensitive to the quantum state in which we evaluate the stress-energy tensor. For now we ignore this problem and present a derivation of the one loop stress-energy tensor in a state possessing the natural symmetries of the manifold: $R^2 \times S^2$. The method is to calculate the ζ-function and effective potential for $d\bar{s}^2 = -dr^2 + dl^2 + e^2 d\Omega^2$, and then to use Page's generalisation of a result due to Brown and Cassidy to obtain the stress energy tensor in the physical spacetime (6.22). The ζ-function is obtained from the heat kernel for this space (which is homogeneous) and is defined via the Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(0,0;t)dt. \qquad (6.25)$$

Here

$$K(0,0;t) = ((4\pi\, e)^2\, t)^{-1} \sum_{l=0}^{\infty} (2l+1)\exp[-t(12(l+1/2)^2 + 1)/12e^2] \qquad (6.26)$$

is the heat kernel for $d\bar{s}^2$ at $x = x' = 0$. After analytically continuing this ζ-function to a meromorphic function in the complex s-plane we find

$$\zeta(s) = \frac{2e^{2s-4}}{(4\pi)^2} \sum_{k=0}^{\infty} \frac{\Gamma(s+k-1)(-1)^k}{\Gamma(s)k!12^k} \zeta_R(2s+2k-3,\ 1/2) \tag{6.27}$$

where $\zeta_R(z, q)$ is the Riemann-Hurwitz zeta function [83]. The effective potential is formally given by $V = (-1/2)[\zeta'(0) + \log(\mu^2)\zeta(0)]$ [45] and is now easily evaluated using (6.27). It is

$$V = -\left(\frac{A}{16\pi e^4} + \frac{T}{2}\ \ln(e^2\mu^2)\right), \tag{6.28}$$

$$\begin{aligned} A &= \sum_{k=3}^{\infty} \frac{(-1)^k}{k(k-1)\,12^k} \zeta_R(2k-3, 1/2) \\ &\quad -2\zeta_R'(-3, 1/2) - \frac{1}{6}\zeta_R'(-1, 1/2) + \frac{7}{960} - \frac{1}{288}\psi(1/2), \end{aligned} \tag{6.29}$$

where μ is an arbitrary mass scale, $T = (1440\pi^2 e^4)^{-1}$ is the trace anomaly, $\zeta_R'(z, q) \equiv \partial\zeta_R(z, q)/\partial z$, and $\psi(x) = \Gamma'(x)/\Gamma(x)$. By performing the usual variation of the effective action we obtain the stress tensor in this static space

$$\langle \bar{T}^\alpha{}_\beta \rangle_{ren} = \frac{2\bar{g}^{\alpha\mu}}{\sqrt{\bar{g}}} \frac{\delta\sqrt{\bar{g}}V}{\delta\bar{g}^{\beta\mu}} = \text{diag}[-V, -V, V + \frac{T}{2}, V + \frac{T}{2}]. \tag{6.30}$$

In fact, one may evaluate the above stress-energy tensor for massless conformally coupled fields of other spins*. This simply changes the numerical values of the T and A above. For spin-1/2 (two-component) theory the trace anomaly is $T = (480\pi^2 e^4)^{-1}$ and

$$A = 4\zeta_R'(-3) + \frac{1}{60}, \tag{6.31}$$

where $\zeta_R(z)$ is the Riemann zeta function [83]. For the Maxwell field we find $T = (120\pi^2 e^4)^{-1}$ and

$$\begin{aligned} A &= \sum_{k=3}^{\infty} \frac{1}{k(k-1)\,2^{2k-1}} \zeta_R(2k-3, 3/2) \\ &\quad -4\zeta_R'(-3, 3/2) + \zeta_R'(-1, 3/2) + \frac{127}{480} - \frac{1}{16}\psi(3/2). \end{aligned} \tag{6.32}$$

*These results were communicated to me by Roberto Camporesi

We now invoke a result due to Page [72]. He finds that the renormalised stress-energy tensor for a conformally coupled massless fields transforms conformally as

$$
\begin{aligned}
T^{\mu}{}_{\nu} &= \Omega^{-4}\overline{T}^{\mu}{}_{\nu} - 8\alpha\Omega^{-4}[\overline{\nabla}_{\alpha}\overline{\nabla}^{\beta}(\overline{C}^{\alpha\mu}{}_{\beta\nu}\ln\Omega) + \frac{1}{2}\overline{R}_{\alpha}{}^{\beta}\overline{C}^{\alpha\mu}{}_{\beta\nu}\ln\Omega] \\
&\quad + \beta[(R_{\alpha}{}^{\beta}C^{\alpha\mu}{}_{\beta\nu} - 2H^{\mu}{}_{\nu}) - \Omega^{-4}(\overline{R}_{\alpha}{}^{\beta}\overline{C}^{\alpha\mu}{}_{\beta\nu} - 2\overline{H}^{\mu}{}_{\nu})] \\
&\quad - \frac{1}{6}\gamma[I^{\mu}{}_{\nu} - \Omega^{-4}\overline{I}^{\mu}{}_{\nu}], \qquad (6.33)
\end{aligned}
$$

where

$$
H_{\mu\nu} \equiv -R^{\alpha}{}_{\mu}R_{\alpha\nu} + \frac{2}{3}RR_{\mu\nu} + (\frac{1}{2}R^{\alpha}{}_{\beta}R^{\beta}{}_{\alpha} - \frac{1}{4}R^{2})g_{\mu\nu}, \qquad (6.34)
$$

$$
I_{\mu\nu} \equiv 2R_{|\mu\nu} - 2RR_{\mu\nu} + (\frac{1}{2}R^{2} - 2R_{|\alpha}{}^{\alpha})g_{\mu\nu}, \qquad (6.35)
$$

and, in the case of a scalar field, $\alpha = 12/(2^{9}45\pi^{2})$, $\beta = -4/(2^{9}45\pi^{2})$, $\gamma = 8/(2^{9}45\pi^{2})$. The barred tensors are evaluated in $d\overline{s}^{2} = \Omega^{-2}ds^{2}$, and in our case $\Omega^{2} = r^{2}/\epsilon^{2}$. It is now a straightforward matter to obtain $\langle T^{\alpha}{}_{\beta}\rangle_{ren}$ for the metric (6.22).

Before examining the backreaction, some comments are in order about the quantum state in which this stress-energy tensor is evaluated. The boundary conditions defining the state in which our system was prepared should appear in the derivation of $\overline{T}^{\mu}{}_{\nu}$. However, it is not an easy task to impose these in our case since we only have an asymptotic classical solution for $r^{2} \ll \epsilon^{2}$. Instead we suggest that the behaviour near the singularity will be insensitive to the initial conditions. In particular, since the singularity is spacelike, the quantum influx due to Hawking radiation can be ignored as it will be swamped by the classical radiation (modelled by the null dust). This certainly will *not* be true at late times when black hole evaporation occurs, however we will say nothing about this phase of the evolution.

Other non-local effects can be expected to contribute to the full stress-energy tensor as $\int \sqrt{-g}(curvature)^{2}dr$ [84] – behaving at most like r^{-3}. For these reasons we feel that the dominant behaviour is captured with our present model.

6.4 Backreaction

Simon [40] has advocated the self-consistent method (see section 2.4) of solving the semi-classical equations

$$G_{\mu\nu} = 8\pi T^{(0)}_{\mu\nu} + \hbar \langle T_{\mu\nu}\rangle_{ren} + O(\hbar^2), \tag{6.36}$$

where $T^{(0)}_{\mu\nu}$ is the stress-energy of classical matter and all the terms of $O(\hbar)$ are gathered into $\langle T_{\mu\nu}\rangle_{ren}$. This approach makes the perturbative nature of the solutions explicit. We therefore expand the metric in a power series in $\hbar$:

$$m(r) = m_{(0)}(r) + \hbar m_{(1)}(r) + O(\hbar^2), \tag{6.37}$$

$$g(r) = g_{(0)}(r) + \hbar g_{(1)}(r) + O(\hbar^2), \tag{6.38}$$

where $m_{(0)}$, $g_{(0)}$ are the classical metric functions given by (6.21). Substituting (6.37) and (6.38) into the generalised field equations and keeping only terms of less than $O(\hbar^2)$ we obtain the backreaction equations for the terms of $O(\hbar)$

$$r^2 g'_{(1)} + 2m'_{(1)} + 6m_{(1)} = -8\pi r^3(\langle T^t{}_t\rangle_{ren} + \langle T^r{}_r\rangle_{ren}), \tag{6.39}$$

$$-r^2 g_{(1)} + r^2 m'_{(1)} = -4\pi r^4 \langle T^t{}_t\rangle_{ren}. \tag{6.40}$$

where $\langle T_{\mu\nu}\rangle_{ren}$ is given by (6.33). As expected they are already linear in the unknown functions $m_{(1)}$ and $g_{(1)}$. Using (6.40) to eliminate $g_{(1)}$ from equation (6.39) we obtain

$$r^2 m''_{(1)} + 4r m'_{(1)} + 6m_{(1)} = -4\pi r^3\left[6\langle T^t{}_t\rangle_{ren} + 2\langle T^r{}_r\rangle_{ren} + r\frac{d}{dr}\langle T^t{}_t\rangle_{ren}\right] \tag{6.41}$$

This linear inhomogeneous ordinary differential equation for the correction to the mass function can be easily solved once the quantum stress tensor is inserted.

Combining the results of the previous section we obtain

$$\langle T^t{}_t \rangle_{ren} = \frac{4\alpha}{3r^4}\ln\left(\frac{r}{\lambda}\right) - \left[\frac{\beta(36e^2r^2 + 30e^4) + 6e^2\gamma r^2}{3r^8}\right], \tag{6.42}$$

$$\langle T^r{}_r \rangle_{ren} = \frac{4\alpha}{3r^4}\ln\left(\frac{r}{\lambda}\right) + \left[\frac{\beta(12e^2r^2 + 18e^4) + 6e^2\gamma r^2}{3r^8}\right], \tag{6.43}$$

(6.44)

where $\overline{T}_{\mu\nu}$ has been absorbed into the (essentially) non-local contribution containing the logarithm. We are interested only in the effects of this quantum material as $r \to 0$. To leading order equation (6.41) becomes

$$r^2 m''_{(1)} + 4rm'_{(1)} + 6m_{(1)} \simeq -\frac{128\pi\beta e^4}{r^5}. \tag{6.45}$$

The unique solution is

$$m_{(1)} \simeq [c\cos(\sqrt{15/4}\,\ln r) + d\sin(\sqrt{15/4}\,\ln r)]r^{-3/2} - \frac{8\pi\beta e^4}{r^5}, \tag{6.46}$$

where c and d are arbitrary constants. Recalling that β is negative, we see that semi-classical effects intensify the classical growth of the mass. The correction to g_{tt} is

$$g_{(1)} \simeq \text{constant} \times r^{-2} \quad \text{as } r \to 0. \tag{6.47}$$

It is interesting to notice that the solution suggests a growth in the anisotropy due to the quantum effects.

6.5 Conclusion

In this chapter we have presented a detailed discussion of a homogeneous model [42] of a charged black hole interior, which is perturbed by cross-flowing null dust. The main feature is the spacelike singularity which is present, in contrast to the (by now) standard mass-inflation scenario [18]. It was pointed out in section 1.2 that such homogeneous solutions are likely to be representative of solutions near to $r = 0$, within spherical symmetry. Having said this we must remember that (6.22) is not the general homogeneous solution to equations (6.13)-(6.15). In fact the HMI background is at an unstable fixed point of the equations [42]. The general solutions exhibit a highly anisotropic approach to the singularity, with violent oscillations in the metre as $r \to 0$. More generally, once the spherical symmetry is broken, the approach to the spacelike singularity inside a black hole will be described by the BKL [85] analysis.

It is widely believed that singularities in general relativity indicate regimes where a quantum treatment is necessary for a correct physical description. We have attempted a semi-classical analysis of the region near the mass inflation singularity of the HMI model. This is only a toy model which is useful as a tool to begin our investigations. When $r^2 \ll e^2$ our findings indicate that vacuum polarisation effects intensify the classical growth of curvature, rather than having a regulating effect as might be hoped. If this remains true in more realistic analyses, we will have to wait for a theory of quantum gravity to be able to resolve the singularity problem.

Without doubt the most serious drawback in this work is the imposition (or not) of boundary conditions. It may, in fact, be possible to obtain some of the non-local terms by examining the solution for $r^2 \gg e^2$ and matching the solutions near $r = |e|$. Nevertheless, it is difficult to see how this will alter our conclusions. One might also argue that the highly anisotropic approach to the singularity of the general solutions will lead to copious particle production, and so invalidate the above analysis. Particle production is a non-local effect, however, and should not change the leading results for the reasons mentioned at the end of section 1.3.

CHAPTER 7

Non-spherical considerations

Thus far all of our calculations have been limited to the spherically symmetric case. However, it has been argued by Poisson and Israel [18] that their analysis (as outlined in chapter 3) should be representative of the more general situation when asphericities are present.

A simple argument can be put forward to support this. When the collapsing star is not exactly symmetric it tends to radiate away its (quadrupole and higher) asymmetries. The external field settles down to a Reissner-Nordström black hole with a radiative tail which decays according to an inverse power law [23]. But what of the internal geometry?

The Cauchy horizon of the Reissner-Nordström spacetime is a null hypersurface that extends an infinite affine distance into the past. Between the event horizon and the Cauchy horizon the radiative tail of the collapse is scattered so that the outflux crossing the Cauchy horizon also decays with an inverse power law in retarded time u (which tends to negative infinity on the event horizon) [86, 20]. Even when the perturbations are allowed to interact with the geometry one therefore expects that the tail of the Cauchy horizon will be determined by the background Reissner-Nordström geometry.

This chapter outlines an investigation of the Cauchy horizon singularity when arbitrary asphericities are present [35]. The analysis is not as yet complete, however a qualitative picture has emerged which is in agreement with the work of Ori [20]. For simplicity we only consider aspherical perturbations of charged collapse, although the analysis also applies to Kerr black holes.

A key ingredient for mass inflation is that there exists an initially (when $u \to -\infty$) stationary Cauchy horizon in the spacetime. we therefore assume this to be true and look for an asymptotic solution as the singularity on the Cauchy horizon is approached. We continue to include crossflowing null dust in the spacetime so that it is easy to see how the analysis fits into the spherically symmetric case in the appropriate limit.

Potentially the major difference between the spherical and less symmetric situations is the presence of shear, which is known to dominate the dynamics as a spacelike singularity is approached [85]. If the shear became very large (or unbounded) as the Cauchy horizon was approached, it would presumably provoke a rapid contraction of ingoing light-rays and a spacelike singularity. As we will see the shear actually remains bounded on the Cauchy horizon.

In section 7.1 we present the field equations for this scenario. The line element contains only one less degree of freedom than the most general spacetime, and hence leads to rather complicated equations of motion. We therefore only give the leading terms which contribute to the asymptotic solution we obtain. Then we outline the assumptions which are made, providing some motivation from the spherical situation. Once this is done, it is straightforward to derive an asymptotic solution near to the Cauchy horizon. There is a null singularity present at which the Kretschmann invariant diverges. The leading divergences in the Weyl curvature suggest that the solution is asymptotically type N. Finally we mention the technical issues which must be addressed before this work is complete.

7.1 The field equations

The line element which acts as our starting point is

$$ds^2 = -\frac{2F}{r}dv(du + Adx + Bdy) + r^2(e^{2\beta}dx^2 + e^{-2\beta}dy^2)\,, \tag{7.1}$$

where F, A, B, r and β are functions of all four spacetime coordinates. This line element has one less degree of freedom than the most general spacetime. We feel, however, that it is sufficient to capture the leading behaviour near to the Cauchy horizon since inclusion of other terms appear only to modify sub-leading terms in the equations.

It is convenient at this stage to introduce a null tetrad and to always work in terms of the projected components of curvature:

$$e^{(1)}_\alpha = (\sqrt{F/r}, \sqrt{F/r}\,A, \sqrt{F/r}\,B, 0), \tag{7.2}$$

$$e^{(2)}_\alpha = (0, re^{\beta}/\sqrt{2}, ire^{-\beta}/\sqrt{2}, 0), \tag{7.3}$$

$$e^{(3)}_\alpha = (0, re^{\beta}/\sqrt{2}, -ire^{-\beta}/\sqrt{2}, 0), \tag{7.4}$$

$$e^{(4)}_\alpha = (0, 0, 0, \sqrt{F/r}). \tag{7.5}$$

In order to allow easy comparison with the Poisson-Israel analysis we introduce the stress-energy tensor of the cross-flowing null dust:

$$T_{\alpha\beta} = \rho_{in}e^{(4)}_\alpha e^{(4)}_\beta + \rho_{out}e^{(1)}_\alpha e^{(1)}_\beta\,. \tag{7.6}$$

It is then easy to recover the spherically symmetric results in the appropriate limit. Covariant conservation determines the functions

$$\rho_{in} = \frac{L_{in}(v, x, y)}{4\pi rF}, \qquad \rho_{out} = \frac{L_{out}(u, x, y)}{4\pi rF}. \tag{7.7}$$

Two further restrictions also arise from the conservation equations, namely that A and B should be independent of v. (We have something more to say about this later.)

We can now proceed to the Einstein field equations, of which we list only the ones which are needed to determine the leading behaviour of the solution:

$$(r^2)_{,uv} = F\left[S^{(1)}\right] \tag{7.8}$$

$$(\ln F)_{,uv} + 2\beta_{,u}\beta_{,v} = F\left[S^{(2)}\right] \tag{7.9}$$

$$2r^2\beta_{,uv} + (r^2)_{,v}\beta_{,u} + (r^2)_{,u}\beta_{,v} = F\left[S^{(3)}\right] \tag{7.10}$$

$$(\ln F)_{,v}(r^2)_{,v} - (r^2)_{,vv} - 2(\beta_{,v})^2 = 2L_{in}(v,x,y) + F\left[S^{(4)}\right] \tag{7.11}$$

$$(\ln F)_{,u}(r^2)_{,u} - (r^2)_{,uu} - 2(\beta_{,u})^2 = 2L_{out}(u,x,y)\ . \tag{7.12}$$

There are also two further equations which are needed to estimate the behaviour of A and B:

$$\begin{aligned}\left(\partial_v\mathcal{D}_3 - \frac{r_{,v}}{r}\mathcal{D}_3 + \beta_v\mathcal{D}_2\right)(\ln F - \ln r) &= \overline{U}_{,v} - 2\overline{V}_{,u} - \overline{U}\frac{r_{,v}}{r} - 4\lambda_{,u}\overline{V} \\ &- 2\mathcal{D}_3\left(\frac{r_{,v}}{r}\right) + \beta_{,v}U + 2\mathcal{D}_2\beta_{,v} + 4\beta_{,v}\left(r^{-1}\mathcal{D}_2 r - \mathcal{D}_3\beta\right) + F\left[S^{(5)}\right] \end{aligned} \tag{7.13}$$

$$\begin{aligned}\left(\beta_{,u}\mathcal{D}_3 - \frac{r_{,u}}{r}\mathcal{D}_2\right)(\ln F - \ln r) &= 2\mathcal{D}_3\beta_{,u} - 2\mathcal{D}_2\left(\frac{r_{,u}}{r}\right) + U\frac{r_{,u}}{r} \\ &+ 4\beta_{,u}\left(r^{-1}\mathcal{D}_3 r - \mathcal{D}_2\beta\right) - \beta_{,u}\overline{U} - 2\partial_u\mathcal{D}_2\lambda + U_{,u}\ . \end{aligned} \tag{7.14}$$

These equations have been simplified using the notation

$$\mathcal{D}_2 = \left[\frac{1}{r\sqrt{2}}\left(e^{-\beta}\partial_x - ie^{\beta}\partial_y\right) - \Sigma\partial_u\right] = \overline{\mathcal{D}}_3 \tag{7.15}$$

$$\Sigma = \frac{1}{r\sqrt{2}}\left(Ae^{-\beta} - iBe^{\beta}\right) \tag{7.16}$$

$$U = \frac{1}{r\sqrt{2}}\left(A_{,u}e^{-\beta} - iB_{,u}e^{\beta}\right) \tag{7.17}$$

$$V = \frac{1}{r\sqrt{2}}\left(A_{,v}e^{-\beta} - iB_{,v}e^{\beta}\right) \tag{7.18}$$

$$\sigma = \frac{e^{-\beta}}{r\sqrt{2}}(\mathcal{D}_2 A - \mathcal{D}_3 A) + i\frac{e^{\beta}}{r\sqrt{2}}(\mathcal{D}_2 B + \mathcal{D}_3 B) \tag{7.19}$$

Two points should be noted:

1. $S^{(1)}$-$S^{(5)}$ are complicated expressions involving the metric and its derivatives. The idea is that we can neglect these to first approximation provided $F \to 0$

on the Cauchy horizon, as it does in the spherical case. We will make this assumption and see that the solution does indeed satisfy this condition.

2. To recover the spherical case set $F = F(u, v)$, $r = r(u, v)$ and A and B equal to zero. The function $e^{2\beta} = (x^2 - 1)^{-1}$ where $x = \cos\theta$ and $y = \phi$.

7.2 Asymptotic analysis

Boundary conditions

It is convenient to fix the advanced time coordinate v to be the standard advanced time for an external observer far from the black hole. Thus $v \to \infty$ on the Cauchy horizon. The Price power-law tail of backscattered radiation [23] then fixes the influx to be

$$L_{in}(v, x, y) = C_0(x, y)v^{-p} + \dots \tag{7.20}$$

As the discussion in the introduction indicates we also assume that the function r goes to a non-zero value as $v \to \infty$. The physical picture is schematically represented in Fig. 7.2: initially the Cauchy horizon is a stationary null hypersurface described asymptotically by the Reissner-Nordström solution as $u \to -\infty$. We consider outgoing perturbations to be turned on at the null surface S and ask what happens to the future. Therefore we take

$$(r^2)_{,v}\Big|_S \overset{v\to\infty}{\simeq} v^{-p}\left[\gamma_0(x, y) + \gamma_1(x, y)v^{-1} + \dots\right] \tag{7.21}$$

where the $\gamma_i(x, y)$ are functions to be determined. This boundary condition clearly agrees with the spherical case if $\gamma_i =$ constant.

Close to the Cauchy horizon we also assume that the metric is analytic in u near to the outgoing null ray S.

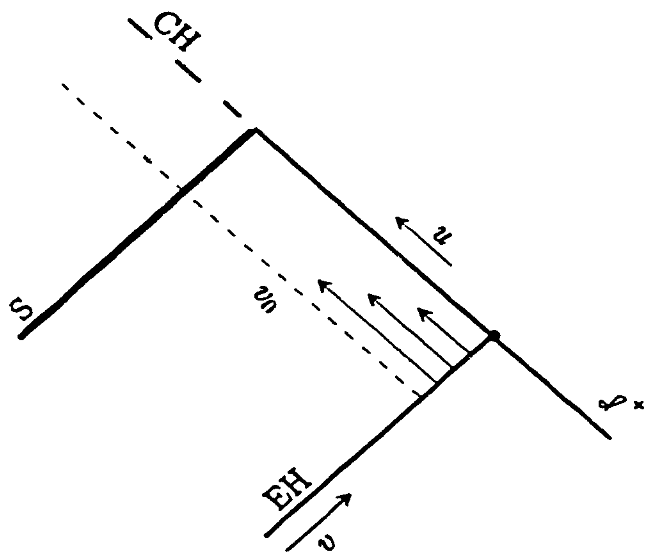

Figure 7.1: Schematic representation of non-spherical black hole interior. CH is the Cauchy horizon, S is the outgoing null ray along which boundary data is supplied and v_0 is an ingoing null ray

The solution

We can now proceed to solve the equations in the asymptotic region. To do so we assume that we can neglect the $S^{(i)}$ since $F \to 0$ on the Cauchy horizon. This assumption is valid along S provided we have an initially stationary Cauchy horizon. We will call the solution obtained in this approximation the *zeroth* order solution.

Let us begin with (7.8) for which the zeroth order solution is

$$r^2 \simeq R^+(v,x,y) + R^-(u,x,y)\,, \tag{7.22}$$

where R^+ is an arbitrary function of v, x and y, and similarly R^- an arbitrary function of u, x and y. These two functions are to be determined by the boundary conditions along S and v_0, respectively. The important one is R^+ which has its v-dependence given by (7.21). Specifically $r^2 \to r_i^2(x,y)$ as $v \to \infty$ along S, therefore write

$$r^2 = r_i^2 - v^{-p+1}\left[\frac{\gamma_0(x,y)}{p-1} + \frac{\gamma_1(x,y)}{p}v^{-1} + \ldots\right] + R^-(u,x,y) \tag{7.23}$$

where $R^- = 0$ along S.

Now introducing the two coordinates

$$t := R^+ + R^-, \qquad \chi := R^- - R^+ , \tag{7.24}$$

into (7.10) and using the approximate solution (7.22) we can rewrite the zeroth order equation for β as

$$\beta_{,t} + t\,(\beta_{,tt} - \beta_{,\chi\chi}) \simeq 0 . \tag{7.25}$$

Introducing the Fourier transform $\hat{\beta}(k,t)$ with respect to χ reduces this to a Bessel equation for the Fourier transform, with the general solution

$$\hat{\beta}(t,k) = C_1(k)J_0(kt) + C_2(k)Y_0(kt) \tag{7.26}$$

where J_0 and Y_0 are Bessel functions of the first and second kind respectively. J_0 is well behaved for all values of the argument, whereas Y_0 diverges logarithmically as its argument goes to zero. However, on the Cauchy horizon, using (7.23), we have that t is initially bounded away from zero and so β and its derivatives are also well behaved there. This allows us to proceed with the knowledge that there is no danger of β diverging as $v \to \infty$. Specifically we find that

$$\beta_{,v} \sim \beta v^{-p} , \tag{7.27}$$

as the Cauchy horizon is approached. The shear of the null generators of the Cauchy horizon (essentially $\beta_{,u}$) is therefore bounded initially.

We can now formally integrate (7.9) to get

$$F \simeq g_1(v,x,y)g_2(u,x,y)\exp\left[-\int du \int dv\,(2\beta_{,u}\beta_{,v})\right] . \tag{7.28}$$

The function $g_1(v,x,y)$ is what we wish to determine. It must go to zero as the Cauchy horizon is approached if our approximations are to be valid. Since we now know β and r^2, in principle, we can proceed to find $g_1(v, x, y)$. Inserting (7.22). (7.20), (7.21) and (7.27) into (7.11) we obtain

$$(\ln g_1)_{,v} \simeq \frac{C_0(x,y)}{2\gamma_0(x,y)} + \dots \tag{7.29}$$

The physical requirement that the energy density of the influx should be positive gives $C_0 > 0$, while $\gamma_0 < 0$ since the radius decreases towards the Cauchy horizon. Thus g_1 goes to zero exponentially as $v \to \infty$.

These are the main ingredients of the asymptotic solution and we will not go into more detail here. We simply mention that by further manipulations of (7.13) and (7.14) we can show that $C_0(x,y)/2\gamma_0(x,y)$ is actually independent of x and y, and also obtain some information about the functions A and B. Let us now proceed to show that there is actually a scalar curvature singularity at $v = \infty$.

7.3 The Cauchy horizon singularity

The solution which was outlined in the previous section is sufficient to give us the leading divergences, as we approach the Cauchy horizon, when asphericities are present. Once again we must emphasise that these are preliminary results, although they are in qualitative agreement with [20].

We estimate the leading divergence in the square of the Weyl tensor as

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \ . \tag{7.30}$$

This estimate suggests that the asymptotic form of the geometry in the nonspherical case is characterised by a diverging Weyl tensor of Petrov type-N, with the degenerate principal null vector aligned with the direction of the blueshifted energy flux.

The Kretschmann invariant is given by

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + 128\pi^2 T_{\alpha\beta}T^{\alpha\beta}\,, \tag{7.31}$$

since $R \equiv 0$ in the presence of only a null dust. In the spherical case the square of $T_{\alpha\beta}$ dominates this expression with the leading divergence proportional to the energy density of the blueshifted influx. In the aspherical situation however, we find contributions from each term which are of the same order of magnitude. In fact the diverging term in $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ could be interpreted as the gravitational energy streaming along the Cauchy horizon.

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto \Psi_4^2 \quad \text{as} \quad v \to \infty. \tag{7.32}$$

which is a stronger divergence. We finish this chapter with a brief discussion of the implication of these results, while pointing out some of the problems which must be addressed before the investigation is completed.

7.4 Discussion

Within spherical symmetry since the divergences are proportional to $v^{-p}e^{\kappa_0 v}$ the tidal distortion of an extended object, which reaches the Cauchy horizon, is finite. Do asphericities enhance the singularity so that tidal distortion diverges along an observer's path which approaches the singularity? It seems not. The tidal forces continue to be proportional to $(\kappa_i\tau)^{-2}|\ln\kappa_i\tau|^{-p}$ where τ is the observer's proper time which has been normalised to be zero at the Cauchy horizon.

During th discussion of the leading divergences in curvature we have ignored one of the problems which remains to be solved. We have assumed that $\beta_{,u}$ goes to a finite (this is fine) non-zero value (this is not clear) on the Cauchy horizon.

Physically this is not unreasonable since we would expect any gravitational perturbation which crosses that Cauchy horizon to lead to non-zero shear there, however, the information we have obtained about the solution so far is insufficient to determine this unambiguously. Further investigation is therefore needed to confirm this physical argument.

The second, and potentially more dangerous problem, is the choice of $e^{(1)}_{\alpha}$ as the outgoing vector along which the null dust moves. Having already chosen our coordinates to put the metric in the form (7.1), covariant conservation of the null dust restricts A and B to be independent of v. This means that the starting spacetime is no longer as general as we would wish. In principle one can simply remove the assumption that the outgoing null dust is present and impose, by hand, that the shear is non-zero to the future of S. This leads to a self-consistent picture which is as described above, however, it is not possible to obtain the spherically symmetric mass-inflation solution in the limit. This issue will be addressed in future work.

The qualitative picture which emerges from this analysis is in agreement with Ori [20]. Some work is needed to complete this analysis of non-spherical effects near to the Cauchy horizon, however the initial indications suggest that a null singularity exists even in the presence of aspherical collapse.

CHAPTER 8

Conclusions

Classical models of generic black hole interiors [17]-[34] have made progress in unravelling the nature of the internal geometry up to the onset of singular behavior at the inner (Cauchy) horizon. At this lightlike hypersurface, which corresponds to infinite external advanced time, the "Coulomb component" $|\Psi_2|$ of the Weyl curvature diverges exponentially with advanced time. (For spherical symmetry, $|\Psi_2| \approx m/r^3$ in terms of the Schwarzschild local mass function m.)

In chapter 5 we summarized an attempt to estimate the influence of quantum effects on this scenario; in particular, to examine whether vacuum polarization and pair creation will act so as to damp the classical rise of curvature and possibly limit it to sub-Planck values. It turns out that near to the Cauchy horizon the spacetime is well approximated by a nearly conformally flat geometry, so that it is possible to use a result obtained in chapter 4 as an estimate for the renormalized stress-energy tensor of a massless scalar field near to the Cauchy horizon. Solutions of the semi-classical equations indicate that vacuum polarization initially reinforces the classical rise of curvature. At sufficiently late times (see section 5.6 for the details) quantum effects will act to curb the growth of curvature near to the Cauchy horizon, however. An ambiguity enters the results through an arbitrary mass scale during the regularization of the stress-energy tensor, and the exact value of this qauntity determines when the damping occurs. Arguments were advanced which suggest that this damping occurs only as our approximations break down, that is as curvatures approach Planck scales. Thus, although the indications do suggest that quantum effects will ultimately act to maintain a bounded curvature near the Cauchy horizon, it seems a complete theory of quantum gravity may be needed to understand whether and how singularities are avoided inside black holes.

Inside spherically symmetric black holes, ingoing light rays inevitably contract to $r = 0$ (at least up to and including the Cauchy horizon). In chapter 6 we suggested that spatial variations of the metric are unlikely to be important as $r = 0$ is approached. Based on a model proposed by Page and Ori [42], we presented an approximate solution [82] which has a spacelike singularity at $r = 0$ at which the mass function of the black hole diverges. A brief discussion of the relevance of this, and other, homogeneous solutions was given. However, it was pointed out that appproximate solutions as $r \rightarrow 0$ do not contain enough information to tell if a Cauchy horizon persists: one needs to construct the global spacetime for this purpose. Vacuum polarization effects were also considered in the neighbourhood of the singularity of the approximate homogeneous mass inflation solution (6.22). A semi-classical analysis predicts (unambiguously) that quantum effects intensify the classical growth of curvature. It was also argued that this result should be fairly robust against reasonable changes of quantum state, and is probably indicative of semi-classical effects in the general solutions of the homogeneous equations.

Finally, in chapter 7 we sketched a generalization of the Poisson and Israel [18] analysis to a spacetime with less symmetry [35]. This is very much work in progress, however a qualitative picture has already emerged. The assumption which is essential is that the Cauchy horizon is initially a stationary null hypersurface (zero expansion of lighlike generators). Once this is imposed an asymptotic solution can be obtained which is qualitatively similar to the spherical situation [18]. The only major difference is that the leading divergence in the Newman-Penrose Weyl scalars is no longer Ψ_2 but rather Ψ_4, suggesting that the spacetime has a "shock-wave" structure as the Cauchy horizon is approached. Some technical difficulties remain to be resolved before this analysis is complete. The results are in qualitative agreement with [20], although there are some indications that the presence of shear does enhance the singularity enough to cause infinite tidal distortion of extended objects which approach the singularity.

Thus it seems that we are nearing a fairly complete picture of the internal structure of black holes, and the classical endstate of gravitational collapse. As for the quantum gravitational effects obtaining in the vicinity of the classical singularity, the investigations here suggest that a fully fledged theory of quantum gravity will be needed to completely resolve the singularity issue in real black holes.

Interestingly it has been suggested [87] that we are at a stage where string theory can provide an approach to this problem. Since strings do not couple to the metric which appears in the usual Einstein-Hilbert action, but rather to a conformally related one, it is entirely possible that solutions to string theory are not singular in the usual sense. However it is at present unclear what is a singularity in string theory: is it a scalar *curvature* singularity? If so, in which conformal frame are divergences important? Do singularities in the dilaton mean anything? These are open questions. Also despite some success at generating exact string solutions it seems that mathematical aspects of the theory are still too poorly understood to obtain exact realistic black-hole solutions.

Finally let me mention what I consider to be the most pressing open question in the study of black-hole interiors: how long is the null portion of the Cauchy horizon singularity? This problem has recently been brought up by the work of Gnedin and Gnedin [81]. In their article they suggest that the Cauchy horizon is completely destroyed once they consider the non-linear evolution of scalar field perturbations of the black-hole interior. This claim is, perhaps, a little misleading since the grid used in their numerical experiments does not actually reach the Cauchy horizon. This means that they could not *see* a null portion of the singularity even if it exists. It would, however, seem important to show that a null singularity does in fact survive the shift of boundary conditions, from a null ray crossing the Cauchy horizon, to the event horizon; that is turning the outflux on at (or before) the event horizon of the black hole. In my opinion, no satisfactory demonstration of this fact has yet been given. Further numerical work is unlikely to yield conclusive results due to the very fact that the Cauchy horizon is the future boundary of the evolution. This problem should be addressed and probably can be solved using some suitable analytic approximations [86].

Appendix A

Curvature and field equations in spherical symmetry

Most of this thesis has been concerned with spherical symmetry. It therefore seems useful to include a summary of geometric quantities in such spacetimes. The results here were obtained in this two dimensional covariant form by Poisson and Israel [18].

The spacetime metric is written as

$$ds^2 = g_{ab}dx^a dx^b + r^2 d\Omega^2 \tag{A.1}$$

where g_{ab} is the metric on the radial two sections, r^2 is a scalar function of x^a and $d\Omega^2$ is the line element on the unit two sphere. Following PI notation such that four dimensional quantities are indicated by a superscript 4, while two dimensional quantities have no superscript, the Riemann curvature is

$$^4R_{abcd} = R_{abcd} \tag{A.2}$$

$$^4R_{a\theta c\theta} = \sin^{-2}\theta \; ^4R_{a\phi c\phi} = -r r_{;ab} \tag{A.3}$$

$$^4R_{\theta\phi\theta\phi} = r^2 \sin^2\theta \, (1 - r^{,a} r_{,a}) \tag{A.4}$$

where (,) represents partial differentiation, and (;) is the covariant derivative associated with g_{ab}.

The Ricci tensor is

$$^4R_{ab} = R_{ab} - 2r_{;ab}/r \tag{A.5}$$

$$^4R_{\theta\theta} = \sin^{-2}\theta \; ^4R_{\phi\phi} = 1 - (r\Box r + r^{,a} r_{,a}) \, , \tag{A.6}$$

and the Ricci scalar

$$ {}^4R = R + 2(1 - 2r\Box r - r^{;a} r_{,a})/r^2 \,. \tag{A.7} $$

The Einstein field equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ can now be written down in a convenient form. This was used in a number of places throughout the text to derive the field equations in different coordinate systems. Introducing the scalars $m(x^a)$ and $\kappa(x^a)$

$$ g^{ab} r_{,a} r_{,b} \;:=\; f := 1 - 2m/r + e^2/r^2 \,. \tag{A.8} $$

$$ \kappa \;:=\; -\frac{1}{2}\partial_r f := -(m - e^2/r)/r^2 \,. \tag{A.9} $$

the field equations can be written as

$$ r_{;ab} + \kappa g_{ab} \;=\; -4\pi r(T_{ab} - g_{ab}T) \,. \tag{A.10} $$

$$ R - 2\partial_r \kappa \;=\; 8\pi(T - 2P) \,. \tag{A.11} $$

Here T_{ab} is the two dimensional submatrix of the stress-energy tensor $T_{\alpha\beta}$, $T = g^{ab}T_{ab}$ and $P = T^{\varphi}{}_{\varphi} = T^{\theta}{}_{\theta}$. Therefore $T_{\alpha\beta}$ is the stress-energy of all other forms of matter which are included as source. (The uncharged case can be obtained by setting $e = 0$ in the above equations.)

Appendix B

Integrals and variational derivatives

n-dimensional integrals

In chapter 4 we had cause to use the following integrals from dimensional regularization. These integrals are obtained from the review article of Liebbrant [68]. The integrals are evaluated in the Lorentzian sector. Defining

$$I_1 := -i\int\frac{d^nq}{(2\pi)^n}\left[q^2(p-q)^2\right]^{-1} = \frac{\Gamma(2-n/2)\Gamma(n/2-1)\Gamma(n/2-1)(p^2)^{n/2-2}}{(4\pi)^{n/2}\Gamma(n-2)}\,. \tag{B.1}$$

we can write the integrals we need as

$$-i\int\frac{d^nq}{(2\pi)^n}q_\mu\left[q^2(p-q)^2\right]^{-1} = \frac{1}{2}p_\mu I_1\,, \tag{B.2}$$

$$-i\int\frac{d^nq}{(2\pi)^n}q_\mu q_\nu\left[q^2(p-q)^2\right]^{-1} = \left[\frac{-p^2}{4(n-1)}\eta_{\mu\nu}+\frac{n}{4(n-1)}p_\mu p_\nu\right]I_1\,, \tag{B.3}$$

$$-i\int\frac{d^nq}{(2\pi)^n}q_\mu q_\nu q_\gamma\left[q^2(p-q)^2\right]^{-1} = \left[\frac{(n/2+1)}{4(n-1)}p_\mu p_\nu p_\gamma-\frac{p^2}{8(n-1)}E_{\mu\nu\gamma}\right]I_1\,. \tag{B.4}$$

$$-i\int\frac{d^nq}{(2\pi)^n}q_\mu q_\nu q_\gamma q_\sigma\left[q^2(p-q)^2\right]^{-1} = \left[\frac{(n/2+1)(n/2+2)}{4(n^2-1)}p_\mu p_\nu p_\gamma p_\sigma-\frac{(n/2+1)p^2}{8(n^2-1)}G_{\mu\nu\gamma\sigma}\right.$$
$$\left.+\frac{(p^2)^2}{16(n^2-1)}H_{\mu\nu\gamma\sigma}\right]I_1\,. \tag{B.5}$$

The tensors E, G and H are given by

$$E_{\mu\nu\gamma} := \eta_{\mu\nu}p_\gamma+\eta_{\nu\gamma}p_\mu+\eta_{\gamma\mu}p_\nu\,, \tag{B.6}$$

$$G_{\mu\nu\gamma\sigma} := \eta_{\mu\nu}p_\gamma p_\sigma+\eta_{\nu\gamma}p_\mu p_\sigma+\eta_{\nu\sigma}p_\gamma p_\mu+\eta_{\mu\gamma}p_\nu p_\sigma+\eta_{\mu\sigma}p_\gamma p_\nu+\eta_{\gamma\sigma}p_\mu p_\nu\,, \tag{B.7}$$

$$H_{\mu\nu\gamma\sigma} := \eta_{\mu\nu}\eta_{\gamma\sigma}+\eta_{\mu\sigma}\eta_{\nu\gamma}+\eta_{\nu\sigma}\eta_{\mu\gamma}\,. \tag{B.8}$$

Other momentum integrals

In the same chapter we also need the following integrals over the kernel $\mathcal{O}(p,q) = \delta(q^2)\theta(-q^0)\delta(p^2)\theta(p^0)$. Once again, defining

$$I_2 := \int \frac{d^nq}{(2\pi)^n}\mathcal{O}(p-q,q) = \frac{\theta(-p^2)\theta(-p^0)}{4(2\pi)^3}\,, \qquad (B.9)$$

we can write

$$\int \frac{d^nq}{(2\pi)^n}\, q_\mu\, \mathcal{O}(p-q,q) = \frac{p_\mu}{8}\frac{I_2}{(2\pi)^4}\,. \qquad (B.10)$$

$$\int \frac{d^nq}{(2\pi)^n}\, q_\mu q_\nu\, \mathcal{O}(p-q,q) = \left[\frac{p_\mu p_\nu}{12} - \frac{p^2\eta_{\mu\nu}}{48}\right]\frac{I_2}{(2\pi)^4}\,. \qquad (B.11)$$

$$\int \frac{d^nq}{(2\pi)^n}\, q_\mu q_\nu q_\rho\, \mathcal{O}(p-q,q) = \left[\frac{p_\mu p_\nu p_\rho}{16} - \frac{p^2}{6.16}E_{\mu\nu\rho}\right]\frac{I_2}{(2\pi)^4}\,. \qquad (B.12)$$

$$\int \frac{d^nq}{(2\pi)^n}\, q_\mu q_\nu q_\rho q_\sigma\, \mathcal{O}(p-q,q) = \left[\frac{p_\mu p_\nu p_\rho p_\sigma}{20} - \frac{p^2}{160}G_{\mu\nu\rho\sigma} + \frac{p^4}{48.20}H_{\mu\nu\rho\sigma}\right]\frac{I_2}{(2\pi)^4}\,. \qquad (B.13)$$

Linearized variational derivatives

For the symmetric kernels $H_\lambda^{(+)}$ and $H^{(-)}$ we find the following variational derivatives are needed in chapter 4:

$$\frac{1}{\sqrt{-g^+}}\frac{\delta}{\delta g^+_{\mu\nu}}\left\langle R^{(+)}(x)H(x-x')R^{(-)}(x')\right\rangle_{x\,x'}$$
$$\doteq \left\langle R^{(-)}(x')\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box_f\right)\, H(x-x')\right\rangle_{x'} \qquad (B.14)$$

$$\frac{1}{\sqrt{-g^+}}\frac{\delta}{\delta g^+_{\mu\nu}}\left\langle R^{(+)}_{\alpha\beta}(x)\, H(x-x')R^{\alpha\beta}_{(-)}(x')\right\rangle_{x\,x'}$$
$$\doteq \left\langle\left[R^{\mu\beta}_{(-)}(x')\left(\partial_\beta\partial^\nu - \frac{1}{2}\delta^\nu_\beta\Box_f\right) - \frac{1}{2}\eta^{\mu\nu}R^{\alpha\beta}_{(-)}\partial_\alpha\partial_\beta\right]\, H(x-x')\right\rangle_{x'} \qquad (B.15)$$

The fact that

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \equiv 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2\,. \qquad (B.16)$$

for variational purposes, in view of the Gauss-Bonnet identity, means that these variational derivatives are sufficient for the purpose of the chapter.

Appendix C

Non-local terms in linearized stress-energy tensor

In chapter 5 we showed how to reduce the non-local integral (5.41) to

$$I = -\frac{1}{2}\left[\int dU\ g(U,0)\int\frac{d\omega}{(2\pi)}\ e^{-i\omega U}\left(\ln|\lambda\omega| + \frac{i\pi}{2}sgn(\omega)\right)\right.$$
$$\left. + \int dV\ g(0,V)\int\frac{d\omega}{(2\pi)}\ e^{-i\omega V}\left(\ln|\lambda\omega| + \frac{i\pi}{2}sgn(\omega)\right)\right] \quad (C.1)$$

Here we demonstrate how to eliminate ω from the integrals. Noting the identity

$$\ln|\omega| + \frac{i\pi}{2}sgn(\omega) = \lim_{\sigma\rightarrow 0^+}\left[\ln(\sigma + i\omega)\right] \quad (C.2)$$

the integrals in (C.1) take the form

$$J = \int\frac{d\omega}{(2\pi)}\ e^{-(\sigma+i\omega)U}\ln(\sigma + i\omega)\ . \quad (C.3)$$

The limit as $\sigma \rightarrow 0^+$ is to be understood throughout the following discussion.

In order to evaluate this it is most convenient to work with a new integral

$$I_n = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-(\sigma+i\omega)U}(\sigma + i\omega)^n \quad n < 0 \quad (C.4)$$

and then to differentiate with respect to n to recover the logarithm. The integrand of (C.4) has a pole of order n at $\omega = i\sigma$, in the positive half plane. Therefore we integrate around a contour which closes along a semicircle at infinity in the upper half plane. The contribution from this part of the contour vanishes provided $U < 0$.

Making a coordinate transformation $s = -(\sigma + i\omega)U$ and using the residue theorem we find

$$I_n = \frac{U^{-n-1}}{\Gamma(-n)}\theta(-U) \quad n < 0 \,. \tag{C.5}$$

In particular, we differentiate with respect to n and then set $n = -1$ to get

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-(\sigma+i\omega)U} (\sigma + i\omega)^{-1} \ln|\sigma + i\omega| = \theta(-U)\left[\psi(1) - \ln U\right] \,. \tag{C.6}$$

Now it is clear that

$$\begin{aligned} I \;=\; -\frac{1}{2}\Bigg\{ & \int_{-\infty}^{\infty} g(U,0)\, \frac{\partial}{\partial U} \left[(\psi(1) - \ln|U/\lambda|)\; \theta(-U)\right] dU \\ & + \int_{-\infty}^{\infty} g(0,V)\, \frac{\partial}{\partial V} \left[(\psi(1) - \ln|V/\lambda|)\; \theta(-V)\right] dV \Bigg\} \,. \end{aligned} \tag{C.7}$$

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