University of Alberta

Computer algebra and a generalized m^{th} root ecological Finsler metric

by

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Aron Murphy

Contents

1 Introduction

Differential geometry involves the lengths of paths in space, which are given by a metric. Finsler geomery involves the study of Finsler metrics and their associated properties: induced metric tensor, connections, etc. Unlike Riemannian metrics, Finsler metrics depend on $\frac{dx}{dt}$ as well as the position x; they provide a more detailed analysis of the system under consideration.

When used in applications, a Finsler manifold may be interpreted as the configuration space of the system in question, and the metric as the energy cost to go from one state to another in the system. An example of such an interpretation is found in [3], in which the authors used a Finsler m^{th} root metric, $F(x,y) = e^{\alpha_1 x^1 + \alpha_2 x^2}((y^1)^m + (y^2)^m)^{\frac{1}{m}}$, and its associated tensors and scalars to model the predator prey interactions between the crown of thorns starfish, A. planci, and the Great Barrier Reef in Australia.

There is an expanded version of the metric above:

$$F(x,y) = e^{\alpha_1 x^1 + \alpha_2 x^2 + Ltan^{-1}\frac{y^1}{y^2}}((y^1)^m + (y^2)^m)^{\frac{1}{m}},$$

where $Ltan^{-1}\frac{y^1}{y^2}$ is a measure of diversity. Due to the complexity of the calculations, this expanded metric has not yet been studied. Recently, however, a Finsler computing package, based on Maple, has been developed, and the purpose of this thesis is to investigate the expanded m^{th} root metric using the Finsler package, to obtain values for such things as the Gauss-Berwald curvature, the Berwald connection coefficients, and the Douglas tensor. The focus here is primarily on differential geometry rather than on applications.

The main result is an extension of a theorem found in [3], which states that the Douglas tensor associated with the expanded metric is 0 if and only if $m = 2$. From an applications point of view, this means that the starfish/coral interaction is intrinsically social unless $m = 2$.

The first section covers the background information necessary to cover the geometry involved. The second is a discussion of a special case of the Antonelli m^{th} root metric. Finally, the third section covers calculations done for the more general case of the metric mentioned above, a comparison of the known value of the Gauss-Berwald curvature with the value obtained using the Finsler program is made, and the new theorem is stated.

1.1 Background Information

Definition 1.1.1 An **atlas** is a collection of coordinate charts $\{U_\lambda, h_\lambda\}_{\lambda \in \Lambda}$ on a manifold satisfying the following conditions:

- $h_\lambda : U_\lambda \to V_\lambda \subset \mathbb{R}^n$ is a homeomorphism, where V_λ is an open set,
- for all $\alpha, \beta \in \Lambda, h_\alpha \circ h_\beta^{-1} \in C^\infty$ when restricted to $h_\beta(U_\alpha \bigcap U_\beta)$.

A **maximal** atlas is an atlas not contained in any other atlas.

An **n-dimensional differentiable manifold,** M^n, is a separable Hausdorff space with a maximal atlas.

The torus, the sphere S^n and real projective n-space $\mathbb{R}P^n$ are examples of differentiable manifolds.

Notation 1.1.2 A coordinate chart λ is assumed and coordinates are imposed from V_λ. Given x $\in U_\lambda \subset M$, identified with $h_\lambda(x)$, the Euclidean coordinates of $\mathbb{R}^n \mid_{V_\lambda}$ are interpreted as coordinates on M^n. $\{\frac{\partial}{\partial x^i}\}$ is the induced coordinate basis on the tangent space of M^n; vectors in this tangent space can be written $y = \sum_{i=1}^n y^i \frac{\partial}{\partial x^i}$. Coordinates in TM^n are given by $(x^1, ..., x^n, y^1, ..., y^n)$, which is abbreviated (x, y). Functions F defined on TM can therefore locally be expressed as $F(x^1, ..., x^n, y^1, ..., y^n)$.

Definition 1.1.3 A **real vector bundle** $\xi = (E, B, \pi, +, \cdot)$ is a 5-tuple, where:

- E and B are topological spaces called the **total space** and the **base space**, respectively.
- $\pi : E \to B$ is called the **projection map**, it is continuous and surjective.
- $+ : E \oplus E := \{(e, e') \in E \times E \mid \pi(e) = \pi(e')\} \longmapsto E$, (adds elements in the same fiber).
- $\cdot : \mathbb{R} \times E \to E, : (t, e) \longmapsto t \cdot e$

$+$ and $\cdot$ are continuous and the restrictions to $\pi^{-1}(b)$, $b \in B$, make $\pi^{-1}(b)$ into a real vector space; $\pi^{-1}(b)$ is called the **fiber** over b.
A vector bundle is a special case of a **fibre bundle**, which is a map $f : E \to B$ such that every point in the base space, $b \in B$, has an open neighborhood U such that $f^{-1}(U)$ is homeomorphic to $U \times F$. Namely, if [9]

$$h : f^{-1}(U) \to U \times F$$

is the homeomorphism, then

$$\pi_U \circ h = f\mid_{f^{-1}(U)}$$

The homeomorphisms which commute with projection are called **local trivializations** of the fibre bundle f; E looks locally like the product $B \times F$.

Example 1.1.4 The Mobius strip, (M^2, S^1, π), is an example of a nontrivial (not a global product) fibre bundle. It has the circle for a base space and the fibers are intervals in $\mathbb{R}$, of the form (-1,1).

Definition 1.1.5 Given M^n, a manifold and x a point in M^n, we define a **tangent vector** ξ at x to be an assignment of an n-tuple of numbers for every $\lambda \epsilon \Lambda$, denoted ξ^i_λ, for $i \epsilon \{1, ..., n\}$, such that it obeys the relation:

$$\xi_\beta = D(h_\beta \circ h_\alpha^{-1}) \mid_{h_\alpha(x)} \cdot \xi_\alpha,$$

where D is the derivative.

The **tangent space at x** ϵ M^n, denoted $T_x M^n$, is a real vector space of tangent vectors at x. The **tangent bundle of** M^n, $(TM^n, M^n, \pi, +, \cdot)$, as a set, is the union of all $T_x M^n$. It has a natural projection $\pi : TM \to M$, mapping $T_x M$ onto x. The topology on the tangent bundle is defined by the pre-image of the projection of all the open sets $U \subset M$, namely $\pi^{-1}(U)$. In this way, the projection mapping is C^∞. The tangent space can also be shown to be a 2n-dimensional, C^∞ manifold [4].

The **slit tangent bundle**, $\widetilde{TM}$, is an open region of TM^n that:

- does not contain (x,0),
- contains (x,λy), $\lambda > 0$, if it contains (x,y). This is known as a *positive cone.*

Definition 1.1.6 A **section** of a fibre bundle gives an element of the fibre over every point in B. It is a C^∞ map $\sigma : B \to E$ such that $\pi \circ \sigma$ is the identity on B.

Example 1.1.7 The **zero section** of a vector bundle consists of all the zero vectors; those with length zero and therefore with all components equal to zero.

We can see that the slit tangent bundle is simply the tangent bundle with *at least the zero section removed.*

Notation 1.1.8 Throughout, **Einstein summation** is used to simplify notation. This means that repeated upper and lower indices are summed over $\{1, ..., n\}$. For example, given two tensors R_{ijk} and y^i, then $R_{ijk}y^i = \sum_{i=1}^{n} R_{ijk}y^i$.

Definition 1.1.9 **Tensors** are generalizations of scalars, vectors, and matrices. While these objects have 0, 1 or 2 indices, respectively, a tensor can have any finite number of indices.

Furthermore, a covariant tensor of rank 1, T_r, associated with a point P, transforms according to the equation

$$T'_r = T_s \frac{\partial x^s}{\partial x'^r}$$

about the point P.

Similarly, a contravariant tensor of rank 1 transforms according to the equation

$$T'^r = T^s \frac{\partial x'^r}{\partial x^s}$$

Following the same pattern, a mixed tensor of rank 3, for example, transforms according to

$$T'^{r}_{st} = T^{m}_{np}\frac{\partial x'^{r}}{\partial x^{m}}\frac{\partial x^{n}}{\partial x'^{s}}\frac{\partial x^{p}}{\partial x'^{t}}$$

See [15] for more information on tensors.

Notation 1.1.10 By $\partial_i F$, we mean $\frac{\partial F}{\partial x^i}$
By $\dot{\partial}_i F$, we mean $\frac{\partial F}{\partial \dot{x}^i}$ or $\frac{\partial F}{\partial y^i}$, (which one is meant will be clear from the context).

Definition 1.1.11 A C^∞, real-valued function f on $\mathbb{R}^n$ is **positively homogeneous of degree** d if $f(\lambda y) = \lambda^d f(y)$ for any $\lambda > 0$.

Example 1.1.12 $F(y) = \{(y^1)^m + (y^2)^m\}^{\frac{1}{m}}$ is p-homogeneous of degree 1 and
$F(y) = \frac{(y^1)^n}{(y^2)^n}$ is p-homogeneous of degree zero in y.

Definition 1.1.13 If $F(x^1, ..., x^n, y^1, ...y^n)$, where $y^i = \frac{dx^i}{dt}$, is a Finsler metric, then F has the following properties:

- F : $\widetilde{TM} \to \mathbb{R}$ is C^∞ on $\widetilde{TM}$ and is continuous an all of TM.
- $F(x, y) > 0$ for $y \neq 0$.
- F(x,λy) = λF(x,y), for all $\lambda > 0$.

$F(x, y)$ is also called the **fundamental function** of the Finsler space.

Definition 1.1.14 A Finsler space is called a **Berwald space** if the connection coefficients G^i_{jk} (see page 21) are functions of x^i alone, in some coordinate system. If this is true in one coordinate chart of the atlas, then it is true for all charts of the atlas.

Definition 1.1.15 A Finsler space (M, F^n) is called **locally Minkowski** if there exists a coordinate system in which the fundamental function F depends on y^i alone. This coordinate system is called **adapted** in F^n.
Locally Minkowski spaces are also Berwald spaces.

Example 1.1.16 $F = e^{\alpha_i x^i}[(y^1)^2 + (y^2)^2]^{\frac{1}{2}}$ is locally Minkowski.

Definition 1.1.17 Given a Finsler metric F, the **metric tensor** associated with F is given by

$$g_{ij} = \tfrac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$$

This induced metric tensor is symmetric and is required to be nondegenerate, ensuring g_{ij} has an inverse. It is also homogeneous of degree zero in y.
The inverse metric tensor g^{ij} is defined by the relationship $g^{ik}g_{jk} = \delta^i_j$, where δ^i_j is the Kronecker delta:

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

If g_{ij} is interpreted as a matrix, then g^{ij} is the matrix inverse.

Notation 1.1.18 The indices of a tensor can be raised or lowered via the metric tensor. Given tensors A^{ijk} and R_{ijk}, then $A_l^{jk} = g_{il}A^{ijk}$ and $R_{jk}^m = g^{im}R_{ijk}$

Theorem 1.1.19 (Euler's Theorem) A C^∞, real-valued function f is p-homogeneous of degree d if and only if

$$\tfrac{\partial}{\partial y^i} f(y) \cdot y^i = d \cdot f(y)$$

Proof: From [5], suppose f is p-homogeneous of degree d, then it satisfies

$$f(\lambda y) = \lambda^d f(y) \text{ for all } \lambda > 0.$$

Differentiating with respect to the parameter λ, we get

$$(\tfrac{\partial}{\partial y^i} f(y))y^i = d\lambda^{d-1} f(y).$$

Setting $\lambda = 1$ in the above gives the required result.

Conversely, assume the above holds. If we evaluate f at λy, we get

$$df(\lambda y) = \partial f_{y^i} \mid_{\lambda y} \lambda y^i = \lambda \partial f(\lambda y) \partial \lambda.$$

For fixed y, let $g(\lambda) = f(\lambda y) \Rightarrow \frac{dg(\lambda)}{\lambda} = \partial g(\lambda)\partial\lambda$ which is separable. From this, we get $g(\lambda) = \lambda^d g(1) \Rightarrow f(\lambda y) = \lambda^d f(y)$, and f is p-homogeneous of degree d.

1.2 Sprays

Definition 1.2.1 A **vector field** on a differentiable manifold M is a C^∞ cross-section $\xi : M \to TM$ of the tangent bundle. That is, ξ is C^∞, lies in T_xM for each $x \epsilon M$, and $\pi \circ \xi = id$.

Definition 1.2.2 A vector field S on TM is a **second order differential equation** (or SODE) if the Jacobian map

$$D\pi : TTM \to TM$$

has the property

$$D\pi \circ S(\xi) = \xi \text{ for all } \xi \epsilon TM.$$

A SODE is a section of TTM.

Definition 1.2.3 [10] If $f : X \to X'$ is a C^∞ diffeomorphism, then we define $T(f) : T(X) \to T(X')$ to be $T_x(f)$ on each fiber $T_x(X)$.

Locally, we may assume that X and X' are open in vector spaces E and E'. $T_xf = f'(x)$ is the derivative. Then Tf is given by

$$Tf(x, \xi) = (f(x), f'(x)\xi) \text{ for } x \epsilon X, \xi \epsilon E.$$

Below, we write $f_\star$ instead of Tf for the **induced map**, which is also called the **tangent map**.

Definition 1.2.4 A **spray** is a SODE satisfying certain properties. Namely, given a positive real number λ and the total space of a C^{∞}-vector bundle, define $\lambda : E \to E$ by scalar multiplication on each fiber.

The induced map on TE, $\lambda_{\star} : TM \to TM$, satisfies $\lambda_{\star}(\xi) = \lambda\xi$.

Let $E = TM$. If a given SODE satisfies $S(\lambda\xi) = \lambda_{\star}\lambda S(\xi)$, for $\lambda > 0$, then S is called a *spray.*

Definition 1.2.5 Let $S_n \subset \mathbb{R}^n$ be an open, connected region (a submanifold of dimension n) with a local coordinate system. A **parameterized curve** in S_n is identified with a set of equations

$x^i = f^i(t)$,

where the f^i are C^{∞}, not all constant. These are the so-called *finite equations of a curve.*

Example 1.2.6

- From [6], the curve $\alpha = (cos\ t, sin\ t, t)$ is a parameterized curve whose trace in $\mathbb{R}^3$ is a helix on the cylinder $x^2 + y^2 = 1$.
- The curve $\alpha = (t, |t|)$ is not a parameterized curve since $|t|$ is not differentiable at $t = 0$.

Definition 1.2.7 By a **system of paths** in S_n, we mean any system of curves with finite equations

$$x^i = f^i(t, a), \ \ i = 1, ..., n$$

of C^{∞} curves with parameter t. The letter a denotes a set of $2n-2$ parameters which vary from curve to curve. The following conditions must also hold:

- There is a unique curve in the system passing through any two given points in S_n, sufficiently close.
- There is a unique curve through any point $x \,\epsilon\, S_n$, with $\frac{dx^i}{dt}$, $i = 1, ..., n$, a direction at x, arbitrary.

Theorem 1.2.8 (Douglas) A system of paths is a local spray; conversely, a local spray is a system of paths.

Every Finsler metric induces a spray, namely, the geodesic spray. On the other hand, not all sprays are of the geodesic type.

A global spray gives a family of smooth curves through each point of M^n, one in each direction. Also, for any two points $p, q \,\epsilon\, M$, sufficiently close, there is a unique spray curve joining them. In 1928, Douglas showed that local sprays are systems of paths in the real analytic case, and in 1937, T.Y. Thomas showed the same for the C^{∞} case.

1.3 Finsler Connections

Given a fundamental function F, we can define the notion of a "Finsler connection" . A "connection", roughly, is a first order directional operator acting on vector fields; it is a path dependent map from one tangent space of a manifold to another.

There are many connections, but the Finsler notion was first introduced by L. Berwald. Just as different topologies on a space yield different types of information about that space, so do different connections on the tangent space of a manifold.

The well known connections are the Berwald connection, the Levi-Civita connection of Riemannian geometry, and the Cartan connection. Only the Berwald connection can be extended to sprays, which are not necessarily Finsler geodesics; we will mainly discuss the Berwald connection here. One is referred to [1], [13] for a deeper discussion of the various connections.

Definition 1.3.1 Given a smooth manifold M^n and the tangent space T_xM, a **frame** z at x is a basis of T_xM. It is a set of n linearly independent tangent vectors, z^i, at x.

Let L be the set of all frames at all points of M^n and define the mapping $\pi_L : L \to M$, called the projection, such that $\pi_L(z) = x$. x is called the **origin** of z.

The set of all frames over x is written $\pi^{-1}(x)$; it is called the **fiber** over x and it forms the group $GL(n)$, the set of all invertible real square matrices.

Definition 1.3.2 Given a local coordinate system $\{U, x^i\}$ in a neighbourhood U of M^n, every tangent vector z_a of a frame $z = (z_a)$ at $x = (x^i) \epsilon U$ is written $z^i_a(\partial_{x^i})_x$ and we get a local coordinate system $\{\pi_L^{-1}(U), (x^i, z^i_a)\}$ on

L. This is the **canonical coordinate system** of L; we can regard L as a smooth manifold [1].

Definition 1.3.3 $L(M^n) = \{L, \pi_L, M^n\}$ is called the **frame bundle** of the manifold M^n, where L is the total space and M is the base space.

Definition 1.3.4 Given a frame bundle, we can consider the tangent space at a point z of L (note that this 'point' z is actually a frame, i.e. a set of vectors). We define the **vertical subspace** of L, denoted L_z^v, to be the kernel of the projection π_L' from the tangent space of L onto L at a point z,

$$L_z^v = \{X \in L_z : \pi_L'(X) = 0\}.$$

Definition 1.3.5 On a smooth manifold M, a C^∞ map $D : x \in M \to V_x \subset T_xM$ is called a **distribution** in M, where V_x is a subspace of the tangent space T_xM.

The *horizontal subspace* of L is not well-defined, a priori. It can be considered a complement of the vertical subspace and can be defined via a connection.

Definition 1.3.6 For a fixed $g \in GL(n)$, we have the mapping

$$\beta_g : z \in L \to zg \in L ,$$

called **right translation** of L by g. This means that

$$\pi_L^{-1}(x) = \{zg, g \in GL(n)\}$$

for a point z in $\pi_L^{-1}(x)$.

Definition 1.3.7 A distribution $\Gamma : z \epsilon L \to \Gamma_z \subset L_z$ in the total space L of the frame bundle $L(M^n)$ is called a **linear connection** in L, or on M^n, if the following two conditions are satisfied:

- $L_z = \Gamma_z \oplus L_z^v$
- $D\beta_g(\Gamma_z) = \Gamma_{zg}, g \,\epsilon\, GL(n)$

The space $\Gamma_z \subset L_z$ is called the horizontal subspace. From this it follows that a linear connection is the complement of the vertical distribution and is invariant under right translations.

Definition 1.3.8 A **spray connection** N in the total space T of the tangent bundle $T(M^n)$ is a distribution $y \epsilon T \to N_y \subset T_y$ which satisfies the direct sum splitting $T_y = N_y \oplus T_y^v$. This is also called a *Nonlinear Connection.*

Definition 1.3.9 Given $\pi : E \to B$, a bundle, and a continuous map $f : B' \to B$, then the **pullback**, (or **induced bundle**) is given by

$$f^\star(\pi) : E' \to B'$$

where

$$E' = \{(e, b) \,\epsilon\, E \times B' : f(b) = \pi(e)\}$$

and $f^\star(\pi)$ is the restriction of the projection map to B′.

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle f^{\star}(\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ B' & \underset{f}{\longrightarrow} & B \end{array}$$

$f^{\star}(\pi)$ is said to be the *pullback* of π by f.

Definition 1.3.10 Given the following diagram

$$\begin{array}{ccc} F(M) & \overset{\pi_T^{\star}}{\longrightarrow} & L(M) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_L} \\ T(M) & \underset{\pi_T}{\longrightarrow} & M \end{array}$$

the **spray bundle** $\pi_1(F(M))$ is the pullback of π_L by π_T. It is denoted $F(M^n) = \{F, \pi_1, T\}$. The total space of the spray bundle is given by

$$F = \{(y, z) \in T \times L : \pi_T(y) = \pi_L(z)\}.$$

In other words, F is the set of all pairs (y, z) such that y is a tangent vector at a point of x and z is a frame at the same point x.

The projection π_1 is given by

$$u = (y, z) \in F \to y \in T.$$

π_1 is the restriction of the projection map to T. This is analogous to the definition of pullback.

Just as in the definitions of linear and spray connections, we get the direct sum relation for the spray bundle

$$F_u = \Gamma_u \oplus F_u^v, \text{ where } u = (y, z),$$

splitting the bundle into horizontal and vertical subbundles.

The horizontal subspace Γ_u can further be split into subspaces, giving the relation,

$$\Gamma_u = \Gamma_u^v \oplus \Gamma_u^h.$$

Γ_u^h and Γ_u^v are called *horizontally* and *vertically horizontal subspaces* respectively (or *h-horizontal* and *v-horizontal* subspaces).

Definition 1.3.11 [4] A **pre-Finsler connection** $F\Gamma$ in the total space F of the spray bundle $F(M^n)$ is a pair (Γ, N) of a linear connection Γ in F, and a spray connection N in the total space T of $T(M^n)$.

Locally, a pre-Finsler connection is denoted by the triad $(F_{ik}^j, N_i^j, V_{ki}^j)$. These functions are called the *connection coefficients of the pre-Finsler connection.* Loosely speaking, F can be viewed as a horizontal connection, V as a vertical connection, and N is a nonlinear (or spray) connection.

Returning to the discussion of sprays, assume a local spray is given. We express this as a second order differential equation of the form [4]

$$\frac{d^2u^i}{dt^2} + 2G^i = 0, \text{ for } i = 1, ..., n$$

where the G^i are C^∞ in x and $y = \dot{x}$ and p-homogeneous of degree 2 in $y = \dot{x}$. Now define the nonlinear connection

$$G^i_j = \frac{\partial G^i}{\partial y^j}.$$

where each pair of i,j run from 1 to n!. This is called the **spray connection**. Taking one more derivative, define [4]

$$G^i_{jk}(x, y) = \frac{\partial G^i_j}{\partial y^k}.$$

From Euler's theorem, we have that G^i_j is p-homogeneous of degree 1 in y and G^i_{jk} are homogeneous of degree 0 in y. Therefore the Gs are allowed to depend on ratios of y^i (see 1.3.19 below).

Definition 1.3.12 Given the spray connection, G^i_j, we define **Berwald's nonlinear operator** by

$$\delta_i = \partial_i - G^i_j \dot{\partial}_j$$

Hence, if $f(x, y)$ is C^∞, $\delta_i f$ *transforms as a covariant vector.* This is the Finsler notion of the gradient operator.

Given the notion of a connnection, we can define the h- (or short bar) and v- (or long bar) covariant derivatives.

Definition 1.3.13 The **h-covariant derivative,** $\nabla^h T$, of a (1,1) type tensor T^i_j, with respect to a pre-Finsler connection, is given locally by

$$T^i_{j|k} = \partial_k T^i_j - (\dot{\partial}_r T^i_j) N^r_k + T^r_j F^i_{rk} - T^i_r F^r_{jk}$$

$$= \delta_k T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk}$$

Definition 1.3.14 The **v-covariant derivative,** $\nabla^v T$, of a (1,1) type tensor T^i_j with respect to a pre-Finsler connection, is given locally by

$$T^i_j|_k = \dot{\partial}_k T^i_j + T^r_j V^i_{rk} - T^i_r V^r_{jk}$$

Similar equations hold for the covariant derivative of a tensor of arbitrary rank, see [1].

Definition 1.3.15 The components of the **deflection tensor field D** of a pre-Finsler connection are given by

$$D^i_j = y^r F^i_{rj} - N^i_j$$

Notation 1.3.16 The symbol $(j|k)$ means that all terms appearing before it are to be added with the j and k reversed.
For example, $\Gamma^i_{jk} + (j|k) = \Gamma^i_{jk} + \Gamma^i_{kj}$

Given a linear connection Γ, we obtain *five torsion tensor fields* and *three curvature tensor fields for the pre-Finsler connection* $F\Gamma$ [4]:

- h-torsion T: $T^i_{jk} = F^i_{jk} - (j|k)$
- h-torsion R^1 : $R^i_{jk} = \delta_k N^i_j - (j|k)$
- hv-torsion V : V^i_{jk}

- hv-torsion P^1 : $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}$
- v-torsion S^1 : $S^i_{jk} = V^i_{jk} - (j|k)$
- h-curvature R^2 : $R^i_{hjk} = K^i_{hjk} + V^i_{hr} R^r_{jk}$

where $K^i_{hjk} = \delta_k F^i_{hj} + F^r_{hj} F^i_{rk} - (j|k)$

- hv-curvature P^2 : $P^i_{hjk} = F^i_{hjk} - V^i_{hk|j} + V^i_{hr} P^r_{jk}$

where $F^i_{hjk} = \dot{\partial}_k F^i_{hj}$

- v-curvature S^2 : $S^i_{hjk} = \dot{\partial}_k V^i_{hj} + V^r_{hj} V^i_{rk} - (j|k)$

Given a Finsler space with fundamental function $F(x, y)$, we can introduce a pre-Finsler connection based on F. This is called a Finsler connection.

Definition 1.3.17 Given a Finsler metric and hence an induced metric tensor, the **Levi-Civita symbol of the first kind** is given by:

$$\gamma_{ijk} = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})$$

It is p-homogeneous of degree 0.

The **Levi-Civita symbol of the second kind** is given by

$$\gamma^i_{jk} = g^{is}\gamma_{sjk}$$

It is also homogeneous of degree 0.

Definition 1.3.18 According to physical analogies, the **geodesics** of a space are the paths followed by a non-accelerating particle. In the plane, they are the straight lines, on the sphere, the great circles. The geodesics of a space depend on the metric of that space. Given the Levi-Civita symbols, the geodesic equations for F can be written:

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

γ^i_{jk} is not a connection in Finsler space, though it is in Riemannian geometry. For a Finsler space (M, F), there is a canonical spray with local coefficients G^i, given by [2]

$$2G^i = \frac{1}{2}g^{ij}\left(\frac{\partial^2 F^2}{\partial y^j \partial x^m}y^m - \frac{\partial F^2}{\partial x^j}\right).$$

This term was mentioned in the discussion of local sprays (see page 17). Equivalently, we can define G^i as follows

$$G^i = \frac{1}{2}\gamma^i_{jk}y^j yk.$$

The geodesic equations can therefore be written:

$$\frac{d^2x^i}{ds^2} + 2G^i = 0.$$

The G^i term is called the **spray function**.

Definition 1.3.19 From the canonical spray, we get the **local spray connection**, or **local nonlinear connection**, N^i_j by

$$N^i_j = \dot{\partial}_j G^i.$$

It is p-homogeneous of degree 1 and it will be labelled G^i_j. The **spray connection coefficients** are given by

$$G^i_{jk} = \dot{\partial}_k G^i_j.$$

These are also called the **Berwald connection coefficients.** They are p-homogeneous of degree 0.

The Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is determined from the fundamental function F by the five axioms of S. Okada [4]:

- ∇^hF= 0 (this is known as "F-metrical")
- $T = 0$ (h-symmetric, i.e. no h-torsion)
- $D = 0$
- $P^1 = 0$
- $V = 0$

Also, since $V^i_{jk} = 0$ for $B\Gamma$, the P^2 curvature reduces to $G^i_{hjk} = \dot{\partial}_h G^i_{jk}$. This is known as the *Douglas tensor*, $D^i_{hjk} = \dot{\partial}_h G^i_{jk}$.

Note that $D = 0$ if and only if the connection coefficients are independent of y (*affine linear*). D is known as the *spray curvature.*

Now the geodesic equations can be written

$$\frac{d^2x^i}{ds^2} + G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

Where $G^k = \frac{1}{2}\gamma^k_{rs}y^r y^s$ implies that

$$\dot{\partial}_i\dot{\partial}_j(2G^k) = \dot{\partial}_i\dot{\partial}_j(\gamma^h_{rs}y^r y^s)$$

which, upon differentiation, yields

$$G^h_{ij} = \gamma^h_{ij} + \dot{\partial}_i\gamma^h_{rj}y^r + \dot{\partial}_j\gamma^h_{is}y^s + \tfrac{1}{2}\dot{\partial}_i\dot{\partial}_j\gamma^h_{rs}y^r y^s.$$

Since γ^h_{is} is p-homogeneous of degree 0 in y, we have by Euler's theorem that $\dot{\partial}_i\gamma^h_{rj}y^r = \dot{\partial}_j\gamma^h_{is}y^s = 0$. Hence

$$G^h_{ij} = \gamma^h_{ij} + \tfrac{1}{2}\dot{\partial}_i\dot{\partial}_j\gamma^h_{rs}y^r y^s.$$

Note that if the γ^h_{ij} are independent of y, then $G^h_{ij} = \gamma^h_{ij}$.

Theorem 1.3.20 A Finsler space is locally Minkowski if and only if the spray curvature and the h-curvature are equal to 0 in $B\Gamma$.

Proof

($\Rightarrow$)

Locally Minkowski $\Rightarrow F = F(y)$

$$\Rightarrow G^i = 0 \text{ since } 2G^i = \tfrac{1}{2}\left(\frac{\partial^2 F^2}{\partial y^j\,\partial x^m} - \frac{\partial F^2}{\partial x^j}\right)$$

$$\Rightarrow G^i_{jk} = 0$$

$$\Rightarrow D^i_{hjk} = R^2 = 0$$

($\Leftarrow$)

$$D^i_{hjk} = 0$$

$$\Rightarrow G^i_{jk} = G^i_{jk}(x)$$

$\Rightarrow$ We have a Berwald space

$$\Rightarrow R^i_{jkl} = G^r_{hj}G^i_{rk} - G^r_{hk}G^i_{rj} = 0$$

$$\Rightarrow \text{In some coordinate system, } G^i_{jk} = 0$$
$$\Rightarrow G^i_j = 0 \text{ by Euler's Theorem}$$

For a connection coefficient, if $T = 0$, and the N^i_j are given, we have $G_{ijk} = \frac{1}{2}(\delta_k g_{ij} + \delta_i g_{kj} - \delta_j g_{ik})$. In a Berwald space, the δs reduce to ∂s, and since $G^i_{jk} = 0$,

$$0 = \tfrac{1}{2}(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik})$$
$$\Rightarrow g_{ij} = g_{ij}(y).$$

Since $g_{ij}y^i y^j = 2F$, $F = F(y)$, and the proof is complete.

Definition 1.3.21 The **Cartan Torsion Tensor** is defined as

$$C_{ijk} = \tfrac{1}{2}\dot{\partial}_k g_{ij}$$

If this tensor is equal to 0, then we have Riemannian geometry. In other words: *y-dependence of the metric tensor is the difference between Finsler and Riemannian geometry.*

Definition 1.3.22 The **Cartan Connection Coefficients** are given by

$$\Gamma^{*i}_{jk} = g^{il}(\gamma_{ljk} - C_{ljm}G^m_k - C_{jkm}G^m_l + C_{lkm}G^m_j)$$

or equivalently,

$$\Gamma^{*i}_{jk} = \tfrac{1}{2}g^{il}(\delta_j g_{lk} + \delta_k g_{lj} - \delta_l g_{jk})$$

which is similar to the definition of the Christoffel symbols, except we use δ instead of ∂.

The Cartan connection $C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ is determined from the fundamental function F by the five axioms of M. Matsumoto [4]:

- $\nabla^h g = 0$ (this is known as "h-metrical")
- $\nabla^v g = 0$ ("v-metrical")
- $D = 0$
- $P^1 = 0$
- $T = 0$ (h-symmetric, i.e. symmetric in the lower indices)
- $S^1 = 0$ (v-symmetric)

The first two conditions together give us that $C\Gamma$ is "metrical".

Since, for $C\Gamma$, $S^1 = 0$, we have that C^i_{jk} is symmetric in its lower indices. From this, we have that the S-curvature reduces to $C^r_{hj}C^i_{rk} - C^r_{hk}C^i_{rj}$. Furthermore, if the Finsler space is 2-dimensional, then all indices are either 1 or 2, and the S-curvature reduces to 0 identically.

Berwald's pre-Finsler connection is not metrical, in general, but it is metrical in Berwald spaces [1].

The Berwald connection is important, however, since it is intrinsic to spray theory and to the study of the geometry of arbitrary SODEs given locally by $\frac{d^2x^i}{ds^2} + f^i(x, \dot{x}, s) = 0$. The geometry of such a system is known as KCC theory, after Kosambi, Cartan, and Chern [1].

1.4 Curvatures

Historically, the study of the curvature of curves in two- and three-space led to the Frenet formulas, which describe a curve in terms of its curvature, torsion, and initial starting point and direction [9]. The Frenet equations can be viewed as "structure equations" for 1-dimensional submanifolds of $\mathbb{R}^3$. They determine one and only one curve up to its position in space [14]. Likewise, the study of surfaces in three-space led to the notions Gauss-Berwald curvature K and the consequent structure equations, known as the Gauss-Weingarten equations and the Gauss-Codazzi equations [14]. These equations determine one and only one surface, up to its position in space.

The Gauss-Weingarten and the Gauss-Codazzi equations are the Riemannian versions of the Finsler structure equations, which, in the case of a linear connection, define the torsion and curvature tensor fields mentioned earlier. For a more in depth discussion of the various structure equations, see [1], [4], [6], [14].

Most important here is the Gauss-Berwald curvature K, which measures the rate of change of the direction of the normal vector, N, at a point $p \in M^n$. Intuitively, K measures how N pulls away from $N(p)$ in a neighbourhood of p.

The Gauss-Berwald curvature is important because it is invariant under isometries, which are distance preserving, bijective maps between metric spaces. It can therefore help to classify various structures up to isometry.

Definition 1.4.1 Given a linear connection (for example, the Berwald connection), we define the **Riemann curvature tensor** to be

$$R^i_{kjl} = \delta_l G^i_{jk} - \delta_k G^i_{jl} + G^r_{jk} G^i_{rl} - G^r_{jl} G^i_{rk}$$

This is the R^2 tensor field, given on page 19, for the Berwald connection, where $F^i_{jk} = G^i_{jk}$ and $V^i_{jk} = 0$.

Definition 1.4.2 In a two dimensional Finsler space, the *Gauss-Berwald curvature* is given by [4]

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}}$$

where the denominator is the determinant of the metric tensor g_{ij}.

Examples of surfaces with constant Gauss-Berwald curvature include the sphere, the cone, the plane, and the pseudosphere. Of these, the sphere has curvature $K = \frac{1}{r^2}$, r the radius, the pseudosphere has $K = -1$, and the cone and the plane have 0 Gauss-Berwald curvature.

An example of a Finsler space with constant Gauss-Berwald curvature is given by the metric [5]

$$F = \sqrt{(y^1)^2 + \sinh^2(x^1)(y^2)^2} + y^1 \tanh(x^1)$$

This space has curvature $K = -\frac{1}{4}$.

2 m^{th} Root Metrics

Definition 2.0.3 A Finsler metric F(a^α) is called a **one-form metric** if F(a^α) is a p-homogeneous of degree one function of n arguments $a^\alpha(x, y)$.

Definition 2.0.4 We follow [4], section 5.4. The function L, given by,

$$L = \{(y^1)^m + (y^2)^m + ... + (y^n)^m\}^{\frac{1}{m}}, \; m \geq 3, \text{ m an integer},$$

is known as the m^{th} **root metric.**

Definition 2.0.5 Given an n-dimensional Finsler space $F^n = (M^n, F)$, with metric function

$$F = e^{\phi} L, \text{ where } \phi = \alpha_i \, x^i \text{ with } \alpha_i \text{ constant, usually} > 0,$$

the metric F is known as **Antonelli's m^{th} root ecological metric** [4]. The coordinates $(x^1, ..., x^n, y^1, ..., y^n)$ are called *adapted* [1].

The m^{th} root metric has been used in the study of coral reef ecology on the Great Barrier Reef, to specify certain morphological traits in a certain coral genus, see [3].

F can be written:

$$F = \{(a^1)^m + (a^2)^m + ... + (a^n)^m\}^{1/m}$$

where

$$a^i = e^{\phi} y^i, \text{ for } i = 1, ..., n.$$

We can see that Antonelli's metric is a special one-form metric, it is also locally Minkowski for $m = 2$, since $K = 0$. For $m \geq 3$, however, it is not locally Minkowski since the Gauss-Berwald curvature K is not equal to 0 identically (see p. 28).

The motivation here is to take an expanded version of the m^{th} root metric which has a diversity term, $L\,tan^{-1}(\frac{y^1}{y^2})$, in the exponential, and to use computer software to obtain the spray functions G^i, the Berwald connection coefficients G^i_{jk}, and the Gauss-Berwald curvature K.

3 Results

3.1 The FINSLER package

The FINSLER package written on Maple (see [8]) by Solange F. Rutz and Renato Portugal (see [1], [12]), allows tensorial manipulation and componentwise calculations. There are many built-in tensors and commands in the package, some of which are already defined in the RIEMANN package (see [11]), upon which the FINSLER package was developed. The commands and tensors which were important in my calculations are defined here [1]:

- The command **show** evaluates, assigns and prints the values of the components of a tensor. If the indices given are numerical or have component names, the value of a specific component will be returned.
- **coordinates** defines the coordinate names. The number of arguments must be equal to the value of the variable **dimension**, a positive integer which must already have been specified.
- **Dcoordinates** defines the names of the directional coordinates, or the y^i.

- **metricfunction** derives the components of the metric tensor using as input the square of the metric function.

- The spray coefficients **G[i]** are defined as $G^i = \frac{1}{2}\Gamma^i_{jk} y^j y^k$, where Γ^i_{jk} are the Christoffel symbols of the second kind, corresponding to the metric function, when defined.

- **N[i,-j]**, the *nonlinear connection*, is defined as $N^i_j = \frac{\partial G^i}{\partial y^j}$.

- **G[i,-j,-k]**, the *spray connection*, is defined as $G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$.

- **B[i,-j]**, the *deviation tensor*, is defined as $B^i_j = 2\partial_j G^i + 2G^r G^i_{jr} - y^r \partial_r G^i_j - G^i_r G^r_j$.

  Note that $B^i_j = R^i_{jkl} y^k y^l$, see page 24.

- **K(v1,v2)**, the Gauss-Berwald curvature, is defined as

$$K(v^i, v^j) = \frac{B_{ij} V^i V^j}{(g_{ij} g_{kl} - g_{il} g_{jk}) V^i V^j y^k y^l}.$$

Also important when dealing with such large expressions are Maple's simplification routines. The important ones that used are listed below [7].

- **normal(x,expanded)** simplifies rational functions. The argument x represents an expression, and the second argument, 'expanded', returns a result in which the numerator and denominator are expanded polynomials, simplified when possible.

- **simplify(x,symbolic)** allows symbolic manipulation of expressions, ignoring the issue of branches for multi-valued functions. This means that an expression like sqrt(x^2) simplifies to x under the symbolic option, ignoring the possible values of the sign of x.

The accepted value of K for the m^{th} root ecological metric is given by [4]:

$$K = \frac{m(m-2)}{4(m-1)^2}\left\{\frac{\alpha_1^2(z^{m-1}-\frac{\alpha_2}{\alpha_1})^2(z^m+1)^2}{z^{2m-1}}\right\}\frac{y^1y^2}{F^2}$$

where F is the m^{th} root ecological metric and $z = \frac{y^2}{y^1}$.

Using the FINSLER package, the Gauss-Berwald curvature for Antonelli's m^{th} root ecological metric was obtained; the result was compared to that given in [4], and they matched. The comparison was done by calculating the curvature using the package, and subtracting from it the known value in [4]. There was no difference between the known and the computed values. The sample calculation and comparison to the curvature in [1] are found in **Appendix A**.

Due to the intrinsic complexity in the simplification and manipulation of large expressions, a direct comparison between the obtained and known results was avoided. The issue of what is apparently "simple" is not a trivial one in symbolic computing. In the above expression, for instance, F actually stands for

$$F = e^{b_1x^1+b_2x^2}\{(y^1)^m + (y^2)^m\}^{\frac{1}{m}},$$

for which Maple would substitute once the metric function is defined. All tensorial expressions are derived from the metric function. This results in expressions given as functions of the positional and directional coordinates. Also, if terms like $(a+b)^2$ are not expanded, the command *Normal* is not as effective when dealing with long polynomial denominators.

Once an expression is simplified, it may be manipulated to satisfy such human demands as compactness, for example, with the commands *Collect, Combine, Convert, Factor,* etc. The computational cost of doing so should be considered, however, especially when an expression may be checked against known results by means of a subtraction which simplifies to 0, proving the correctness of the result. This was the case for the Gauss-Berwald curvature mentioned above.

Altering the metric to read:

$$F = e^{\alpha_1 x^1 + \alpha_2 x^2 + Ltan^{-1}(\frac{y1}{y2})}((y^1)^m + (y^2)^m)^{\frac{1}{m}},$$

the spray coefficients, G^i, Berwald's connection coefficients, and the Gauss-Berwald curvature were obtained (**see Appendix B**).

Since there are no accepted values for the curvature with $L \neq 0$, a comparison with known values was made by substituting $L = 0$ into the new expression for curvature, and comparing with the result given in [4]. This yielded a match, indicating that the new value for K is correct.

Also calculated were the components of the Douglas tensor, which led to a new theorem:

Theorem 3.1.1 (Antonelli/Murphy, 2004)

For the mixed (L, m) metric of this thesis, $D^i_{jkl} = 0$ if and only if $m = 2$.

This is an expanded version of a similar theorem in [4], which states that $D^i_{jkl} = 0$ if and only if $m = 2$ for the metric in which $L = 0$.

The proof can be found in **Appendix C**, which shows the calculations and comparisons of the various components of the tensor with the known values, but an outline is given here.

First $m = 2$ was substituted into the expanded metric F, and the Douglas tensor was calculated. It turned out to be 0.

Next, it was assumed that $m \neq 2$, and the D^i_{jkl} was calculated. Then, $L = 0$ was substituted into the components of the Douglas tensor, and each component reduced to the value given in [4]. This means that the D^i_{jkl} for the expanded metric contain those of the simpler metric. Since we already have that D^i_{jkl} for the metric in which $L = 0$ vanish only when $m = 2$, we know that the new values for the D tensor will vanish only when $m = 2$.

3.2 Conclusions

Just as any predator prey interaction is complicated, so too are the models created to describe it. With greater detail in the model comes greater difficulty in calculating its various characteristics; as such, there are many useful

Finsler metrics whose properties have not yet been investigated.

Computer calculations, although relatively new to the field, can be a very powerful tool in geometry. They provide an effective means to verify previous results and to expand upon them, as was the case with the ecological metric.

Specifically, the Finsler package was useful in confirming the known values for the Gauss-Berwald curvature and the Douglas tensor found in [3]. Furthermore, when applied to the more general m^{th} root metric, the program helped to expand upon a known theory, namely that the starfish/coral reef interaction is intrinsically social unless $m = 2$ (theorem 3.1.1).

4 Appendices

A

Here is the calculation of the Gauss-Berwald curvature for the m^{th} root ecological metric. Also obtained are the spray connections, the Berwald connections and the metric tensor.

```
> restart;
> libname := 'D:/finsler', libname:
> with(Finsler);
```

Warning, the protected name apply has been redefined and unprotected

Warning, the name init has been redefined

$$[Dcoordinates, Hdiff, K, connection, init, metricfunction, tddiff]$$

What follows is the simplification procedure which, once defined by the commcnd *Simpfcn*, is performed in every step of each calculation throughout the session. It was the most effective in simplifying the obtained expressions so that their difference from known expressions simplifies to zero.

```
> simp:=(x->simplify(normal(x,expanded),symbolic)); simpfcn(simp);
```

$$simp := x \to \mathrm{simplify}(\mathrm{normal}(x,\ expanded),\ symbolic)$$

```
> Dimension:= 2:
> coordinates(x1,x2):
```

The coordinates are :

$$X^{1} = x1$$

$$X^{2} = x2$$

```
> Dcoordinates(y1,y2):
```

'Y assigned to DCoordinateName'

$$The\ d - coordinates\ are:$$

$$Y^{1} = y1$$

$$Y^{2} = y2$$

```
> F:= exp(b1*x1+b2*x2)*(y1^m+y2^m)^(1/m):
> F2:=F^2:
> metricfunction(F2):
```

$$The\ components\ of\ the\ metric\ are:$$

$$g_{x1\ x1} = \frac{e^{(2\,b1\,x1+2\,b2\,x2)}\,(y1^m + y2^m)^{(\frac{2}{m})}\,(y1^{(2\,m)} + y1^m\,m\,y2^m - y1^m\,y2^m)}{y1^{(2+2\,m)} + 2\,y1^{(2+m)}\,y2^m + y1^2\,y2^{(2\,m)}}$$

$$g_{x1\ x2} = -\frac{e^{(2\,b1\,x1+2\,b2\,x2)}\,(y1^m + y2^m)^{(\frac{2}{m})}\,y1^m\,y2^m\,(-2+m)}{y1^{(1+2\,m)}\,y2 + 2\,y1^{(1+m)}\,y2^{(1+m)} + y1\,y2^{(1+2\,m)}}$$

$$g_{x2\ x2} = \frac{e^{(2\,b1\,x1+2\,b2\,x2)}\,(y1^m + y2^m)^{(\frac{2}{m})}\,(y2^{(2\,m)} + y1^m\,m\,y2^m - y1^m\,y2^m)}{y2^2\,y1^{(2\,m)} + 2\,y2^{(2+m)}\,y1^m + y2^{(2+2\,m)}}$$

```
> show(G[i]);
```

$$G^{x1} = -\frac{y1^{(2-m)}\, b1\, y2^m - y1^2\, b1\, m + y1^2\, b1 - y1\, y2\, b2\, m}{2\,(m-1)}$$

$$G^{x2} = -\frac{-y2\, y1\, b1\, m - y2^2\, b2\, m + y2^2\, b2 + y2^{(2-m)}\, b2\, y1^m}{2\,(m-1)}$$

```
> show(G[i,-j,-k]);
```

$$G^{x1}{}_{x1\ x1} = -\frac{y1^{(-m)}\, b1\, y2^m\, m}{2} + y1^{(-m)}\, b1\, y2^m + b1$$

$$G^{x2}{}_{x1\ x1} = -\frac{y2^{(2-m)}\, b2\, y1^{(-2+m)}\, m}{2}$$

$$G^{x1}{}_{x1\ x2} = \frac{m\,(-2\, y1^{(-m+1)}\, y2^m\, b1 + y1^{(-m+1)}\, y2^m\, b1\, m + y2\, b2)}{2\, y2\,(m-1)}$$

$$G^{x2}{}_{x1\ x2} = \frac{m\,(y1\, b1 - 2\, y2^{(-m+1)}\, b2\, y1^m + y2^{(-m+1)}\, y1^m\, b2\, m)}{2\, y1\,(m-1)}$$

$$G^{x1}{}_{x2\ x2} = -\frac{b1\, y1^{(2-m)}\, y2^{(-2+m)}\, m}{2}$$

$$G^{x2}{}_{x2\ x2} = -\frac{y2^{(-m)}\, b2\, y1^m\, m}{2} + b2 + y2^{(-m)}\, b2\, y1^m$$

This is the result Maple returned for the Gauss-Berwald curvature.

```
> Gauss:=simp(K(v1,v2));
```

$$Gauss := \frac{1}{4} e^{(-2\, b1\, x1 - 2\, b2\, x2)}\, (y1^m + y2^m)^{(-\frac{2}{m})}\, m(y1^{(2m)}\, y2^{(-2m+2)}\, m\, b2^2$$
$$-\, 2\, y1^{(2m)}\, y2^{(-2m+2)}\, y2^{(-2m+2)}\, m\, b2^2$$

$-2\,y1^{(2m)}\,y2^{(-2m+2)}\,b2^2 - 2\,y1^{(1+m)}\,y2^{(-m+1)}\,b1\,b2\,m + 2\,y2^{(2-m)}\,y1^m\,m\,b2^2$
$+4\,y1^{(1+m)}\,y2^{(-m+1)}\,b1\,b2 - 4\,y2^{(2-m)}\,y1^m\,b2^2 + m\,y1^2\,b1^2 - 4\,b1\,y2\,y1\,b2\,m$
$+m\,y2^2\,b2^2 - 2\,y1^2\,b1^2 + 8\,b1\,y2\,y1\,b2 - 2\,y2^2\,b2^2 + 2\,y2^m\,y1^{(2-m)}\,b1^2\,m$
$-2\,y2^{(1+m)}\,y1^{(-m+1)}\,b1\,m\,b2 - 4\,y2^m\,y1^{(2-m)}\,b1^2 + 4\,y2^{(1+m)}\,y1^{(-m+1)}\,b1\,b2$
$+\,y2^{(2m)}\,y1^{(-2m+2)}\,b1^2\,m - 2\,y2^{(2m)}\,y1^{(-2m+2)}\,b1^2)/(m^2 - 2\,m + 1)$

Defining Gauss1 to be the curvature given in [4] and subtracting from the expression Gauss above, we obtain that the two expressions match for any value of m. Note that the F given in Gauss1 is the original m^{th} root metric.

```
> z := y2/y1;
```

$$z := \frac{y2}{y1}$$

```
> Gauss1 := (m*(m-2)*b1^2*(1+z^m)^2*(z^(m-1)-b2/b1)^2*(y1*y2))/
(4*(m-1)^2*(F)^2*z^ (2*m-1));
```

$$Gauss1 := \frac{1}{4}\,\frac{m\,(-2+m)\,b1^2\,(1+(\frac{y2}{y1})^m)^2\,((\frac{y2}{y1})^{(m-1)} - \frac{b2}{b1})^2\,y1\,y2}{(m-1)^2\,(e^{(b1\,x1+b2\,x2)})^2\,((y1^m + y2^m)^{(\frac{1}{m})})^2\,(\frac{y2}{y1})^{(-1+2\,m)}}$$

```
> zero:=simp(Gauss-Gauss1);
```

$$zero := 0$$

In using the *Simp* function above, defined before the step which determined the simplification procedure to be applied in the session by means of the *Simpfcn* command, we are applying the same simplification procedure to calculate the difference *zero*, which actually simplified to 0. Note that it may have been given in many more complex ways, such as $sin^2(x) + cos^2(x) - 1$.

B

This is the Maple session which derives the expression for the the Gauss-Berwald curvature for the extended m^{th} root metric, as well as the expressions for the spray coefficients. The Berwald connection coefficients were also obtained, but the length of their expressios made it impractical to include them here.

```
> restart;
> libname := 'D:/finsler', libname:
> with(Finsler):
Warning, the protected name apply has been redefined and unprotected

Warning, the name init has been redefined

> Dimension:= 2:
> coordinates(x1,x2):
```

The coordinates are :

$$X^{1} = x1$$

$$X^{2} = x2$$

```
> Dcoordinates(y1,y2):
'Y assigned to DCoordinateName'
```

The d – coordinates are :

$$Y^{1} = y1$$

$$Y^{2} = y2$$

This is the expanded metric function. It is the same as the previous m^{th} root metric except that now there is an $L\,arctan(\frac{y1}{y2})$ in the exponential.

```
> F:= exp(b1*x1+b2*x2+L*arctan(y1/y2))*(y1^m+y2^m)^(1/m):
> F2:=F^2:
> metricfunction(F2):
```

The components of the metric are :

$$g_{x1\ x1} = (e^{(b1\,x1+b2\,x2+L\arctan(\frac{y1}{y2}))})^2\,((y1^m+y2^m)^{(\frac{1}{m})})^2(2\,L^2\,y2^2\,y1^2\,(y1^m)^2+2\,L^2\,y2^2\,y1^2\,(y2^m)^2$$
$$+4\,L\,y2^3\,(y1^m)^2\,y1+2\,L\,y2\,(y1^m)^2\,y1^3-2\,L\,y2\,y1^3\,(y2^m)^2+y1^m\,m\,y2^4\,y2^m$$
$$+y1^m\,m\,y1^4\,y2^m-2\,y1^m\,y2^2\,y1^2\,y2^m+4\,L^2\,y2^2\,y1^2\,y1^m\,y2^m+4\,L\,y2^3\,y1^m\,y1\,y2^m$$
$$+2\,y1^m\,m\,y2^2\,y1^2\,y2^m-y1^m\,y2^4\,y2^m-y1^m\,y1^4\,y2^m+(y1^m)^2\,y2^4+(y1^m)^2\,y1^4$$
$$+2\,(y1^m)^2\,y2^2\,y1^2)\Big/((y2^2+y1^2)^2\,y1^2\,(y1^m+y2^m)^2)$$

$$g_{x1\ x2} = -(e^{(b1\,x1+b2\,x2+L\arctan(\frac{y1}{y2}))})^2\,((y1^m+y2^m)^{(\frac{1}{m})})^2(2\,L^2\,y2^2\,y1^2\,(y1^m)^2+2\,L^2\,y2^2\,y1^2\,(y2^m)^2$$
$$+3\,L\,y2^3\,(y1^m)^2\,y1+L\,y2\,(y1^m)^2\,y1^3-3\,L\,y2\,y1^3\,(y2^m)^2+y1^m\,m\,y2^4\,y2^m$$
$$+y1^m\,m\,y1^4\,y2^m-4\,y1^m\,y2^2\,y1^2\,y2^m+4\,L^2\,y2^2\,y1^2\,y1^m\,y2^m+2\,L\,y2^3\,y1^m\,y1\,y2^m$$
$$-2\,L\,y2\,y1^m\,y1^3\,y2^m+2\,y1^m\,m\,y2^2\,y1^2\,y2^m-2\,y1^m\,y2^4\,y2^m-2\,y1^m\,y1^4\,y2^m$$
$$-L\,y2^3\,(y2^m)^2\,y1)\Big/(y2\,(y2^2+y1^2)^2\,(y1^m+y2^m)^2\,y1)$$

$$g_{x2\ x2} = (e^{(b1\,x1+b2\,x2+L\arctan(\frac{y1}{y2}))})^2\,((y1^m+y2^m)^{(\frac{1}{m})})^2(2\,L^2\,y2^2\,y1^2\,(y1^m)^2+2\,L^2\,y2^2\,y1^2\,(y2^m)
+2\,L\,y2^3\,(y1^m)^2\,y1-4\,L\,y2\,y1^3\,(y2^m)^2+y1^m\,m\,y2^4\,y2^m+y1^m\,m\,y1^4\,y2^m
-2\,y1^m\,y2^2\,y1^2\,y2^m+4\,L^2\,y2^2\,y1^2\,y1^m\,y2^m-4\,L\,y2\,y1^m\,y1^3\,y2^m
+2\,y1^m\,m\,y2^2\,y1^2\,y2^m-y1^m\,y2^4\,y2^m-y1^m\,y1^4\,y2^m-2\,L\,y2^3\,(y2^m)^2\,y1+y2^4\,(y2^m)^2
+y1^4\,(y2^m)^2+2\,y2^2\,y1^2\,(y2^m)^2)\Big/(y2^2\,(y2^2+y1^2)^2\,(y1^m+y2^m)^2)$$

Following are the spray coefficients.

```
> show(G[i]);
```

$$G^{x1} = y1\,(b2\,y2^4\,L\,(y1^m)^2\,y1-b2\,y2^2\,L\,(y1^m)^2\,y1^3+2\,b1\,y1^2\,L\,y2^3\,(y1^m)^2
-b1\,y1\,y1^m\,y2^4\,y2^m+b2\,y2\,y1^m\,m\,y1^4\,y2^m+2\,b1\,y1^3\,y1^m\,m\,y2^2\,y2^m
+2\,b2\,y2^3\,y1^m\,m\,y1^2\,y2^m-2\,b2\,y2^2\,L\,y1^m\,y1^3\,y2^m+4\,b1\,y1^2\,L\,y2^3\,y1^m\,y2^m
+b2\,y2^5\,y1^m\,m\,y2^m-b1\,y1^5\,y1^m\,y2^m+2\,b2\,y2^4\,L\,y1^m\,y1\,y2^m-2\,b1\,y1^3\,y1^m\,y2^2\,y2^m
+b1\,y1^5\,y1^m\,m\,y2^m+b1\,y1\,y1^m\,m\,y2^4\,y2^m+2\,b1\,y1^2\,L\,y2^3\,(y2^m)^2
-b2\,y2^2\,L\,y1^3\,(y2^m)^2+b2\,y2^4\,L\,(y2^m)^2\,y1-b1\,y1^5\,(y2^m)^2-b1\,y1\,y2^4\,(y2^m)^2
-2\,b1\,y1^3\,y2^2\,(y2^m)^2)\Big/(2(2\,L\,y2^3\,(y1^m)^2\,y1+L^2\,y2^2\,y1^2\,(y1^m)^2-y1^m\,y2^4\,y2^m
-y1^m\,y1^4\,y2^m+2\,y1^m\,m\,y2^2\,y1^2\,y2^m-2\,y1^m\,y2^2\,y1^2\,y2^m-2\,L\,y2\,y1^m\,y1^3\,y2^m
+2\,L^2\,y2^2\,y1^2\,y1^m\,y2^m+y1^m\,m\,y2^4\,y2^m+2\,L\,y2^3\,y1^m\,y1\,y2^m+y1^m\,m\,y1^4\,y2^m
+L^2\,y2^2\,y1^2\,(y2^m)^2-2\,L\,y2\,y1^3\,(y2^m)^2))$$

$$G^{x2} = y2(b1\, y1^2\, L\, y2^3\, (y1^m)^2 - 2\, b2\, y2^3\, (y1^m)^2\, y1^2 - b2\, y2\, (y1^m)^2\, y1^4$$
$$- 2\, b2\, y2^2\, L\, (y1^m)^2\, y1^3 - b2\, y2^5\, (y1^m)^2 - b1\, y1^4\, L\, y2\, (y1^m)^2 - b2\, y2\, y1^m\, y1^4\, y2^m$$
$$- 2\, b1\, y1^4\, L\, y2\, y1^m\, y2^m + b1\, y1^5\, y1^m\, m\, y2^m + b1\, y1\, y1^m\, m\, y2^4\, y2^m$$
$$+ 2\, b2\, y2^3\, y1^m\, m\, y1^2\, y2^m - b2\, y2^5\, y1^m\, y2^m + 2\, b1\, y1^3\, y1^m\, m\, y2^2\, y2^m$$
$$- 4\, b2\, y2^2\, L\, y1^m\, y1^3\, y2^m + 2\, b1\, y1^2\, L\, y2^3\, y1^m\, y2^m + b2\, y2^5\, y1^m\, m\, y2^m$$
$$- 2\, b2\, y2^3\, y1^m\, y1^2\, y2^m + b2\, y2\, y1^m\, m\, y1^4\, y2^m - b1\, y1^4\, L\, y2\, (y2^m)^2$$
$$+ b1\, y1^2\, L\, y2^3\, (y2^m)^2 - 2\, b2\, y2^2\, L\, y1^3\, (y2^m)^2) \Big/ (2(2\, L\, y2^3\, (y1^m)^2\, y1$$
$$+ L^2\, y2^2\, y1^2\, (y1^m)^2 - y1^m\, y2^4\, y2^m - y1^m\, y1^4\, y2^m + 2\, y1^m\, m\, y2^2\, y1^2\, y2^m$$
$$- 2\, y1^m\, y2^2\, y1^2\, y2^m - 2\, L\, y2\, y1^m\, y1^3\, y2^m + 2\, L^2\, y2^2\, y1^2\, y1^m\, y2^m + y1^m\, m\, y2^4\, y2^m$$
$$+ 2\, L\, y2^3\, y1^m\, y1\, y2^m + y1^m\, m\, y1^4\, y2^m + L^2\, y2^2\, y1^2\, (y2^m)^2 - 2\, L\, y2\, y1^3\, (y2^m)^2))$$

```
> Gauss := (K(v1,v2)):

> Gauss := normal(Gauss, expanded):

> factor(Gauss);
```

What follows is the Gauss-Berwald curvature for the metric. Given such a large expression for K, it was necessary to perform some sort of check. First L = 0 was substituted into K. Then a comparison was made with the value of K given in [4], which was obtained from the metric in which L = 0. They matched; this indicaties that the expression for the curvature here is correct.

$$\frac{1}{4}(y2^2+y1^2)^2\,(y1^m+y2^m)^2(12\,y2^6\,L^2\,y1^2\,(y1^m)^4+36\,y1^6\,(y1^m)^2\,(y2^m)^2\,y2^2\,L^2$$
$$-8\,y1^6\,(y1^m)^2\,(y2^m)^2\,y2^2\,m+16\,(y1^m)^3\,y2^m\,y2^7\,m\,L\,y1-22\,(y1^m)^3\,y2^m\,y2^6\,m\,L^2\,y1^2$$
$$+8\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^6\,L^2\,y1^2-8\,(y1^m)^2\,(y2^m)^2\,y2^6\,m\,y1^2+36\,(y1^m)^2\,(y2^m)^2\,y2^6\,L^2\,y1^2$$
$$-44\,(y1^m)^2\,(y2^m)^2\,y2^6\,m\,L^2\,y1^2-22\,y1^m\,(y2^m)^3\,y2^6\,m\,L^2\,y1^2$$
$$-24\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^7\,L\,y1-12\,(y1^m)^3\,y2^m\,m^2\,y2^7\,L\,y1$$
$$+8\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^7\,L\,y1+16\,(y1^m)^2\,(y2^m)^2\,y2^7\,m\,L\,y1-4\,(y1^m)^3\,y2^m\,m^3\,y2^7\,L\,y1$$
$$+12\,y1^7\,y1^m\,(y2^m)^3\,m^2\,y2\,L+4\,y1^7\,y1^m\,(y2^m)^3\,m^3\,y2\,L+y1^8\,(y1^m)^2\,(y2^m)^2\,m^4$$
$$-16\,y1^7\,y1^m\,y2\,(y2^m)^3\,m\,L+24\,y1^7\,(y1^m)^2\,(y2^m)^2\,m^2\,y2\,L+36\,y1^6\,y1^m\,(y2^m)^3\,y2^2\,L^2$$
$$+4\,(y1^m)^2\,(y2^m)^2\,m^4\,y2^6\,y1^2-16\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^6\,y1^2+20\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^6\,y1^2$$
$$-12\,(y1^m)^3\,y2^m\,m^2\,y2^6\,L^2\,y1^2+12\,y1^m\,(y2^m)^3\,m^2\,y2^6\,L^2\,y1^2$$
$$+36\,y1^5\,(y1^m)^3\,y2^m\,m^2\,y2^3\,L-4\,y1^4\,y1^m\,(y2^m)^3\,m^3\,y2^4\,L^2+4\,y1^4\,y1^m\,(y2^m)^3\,y2^4\,m\,L^2$$
$$+20\,y1^6\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^2-4\,y1^5\,(y1^m)^3\,y2^m\,m^3\,y2^3\,L$$
$$+112\,y1^3\,(y1^m)^2\,(y2^m)^2\,y2^5\,m\,L-24\,y1^3\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^5\,L$$
$$-8\,y1^3\,(y1^m)^3\,y2^m\,m^3\,y2^5\,L+32\,y1^3\,(y1^m)^3\,y2^m\,y2^5\,m\,L+24\,y1^3\,(y1^m)^3\,y2^m\,m^2\,y2^5\,L$$
$$-112\,y1^5\,(y1^m)^2\,(y2^m)^2\,y2^3\,m\,L+96\,y1^5\,(y1^m)^2\,(y2^m)^2\,y2^3\,L-4\,y1^8\,(y1^m)^2\,(y2^m)^2\,m^3$$
$$-24\,y1^5\,y1^m\,(y2^m)^3\,m^2\,y2^3\,L+8\,y1^5\,y1^m\,(y2^m)^3\,m^3\,y2^3\,L+48\,y1^5\,y1^m\,(y2^m)^3\,y2^3\,L$$
$$+5\,y1^8\,(y1^m)^2\,(y2^m)^2\,m^2+8\,y1^3\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^5\,L-48\,y1^3\,(y1^m)^3\,y2^5\,y2^m\,L$$
$$+12\,y1^6\,y2^2\,(y2^m)^4\,L^2-96\,y1^3\,(y1^m)^2\,y2^5\,(y2^m)^2\,L-32\,y1^5\,y1^m\,(y2^m)^3\,y2^3\,m\,L$$
$$-8\,y1^5\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^3\,L+24\,y1^5\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^3\,L$$
$$+48\,y1^5\,(y1^m)^3\,y2^m\,y2^3\,L-80\,y1^5\,(y1^m)^3\,y2^m\,y2^3\,m\,L-2\,y1^8\,(y1^m)^2\,(y2^m)^2\,m$$
$$+6\,y1^4\,(y1^m)^2,(y2^m)^2\,m^4\,y2^4+30\,y1^4\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^4-12\,y1^4\,(y1^m)^2\,(y2^m)^2\,y2^4\,m$$
$$+16\,y1^4\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^4\,L^2+8\,y1^4\,(y1^m)^2\,(y2^m)^2\,y2^4\,m\,L^2$$
$$+4\,y1^4\,(y1^m)^3\,y2^m\,y2^4\,m\,L^2-4\,y1^4\,(y1^m)^3\,y2^m\,m^3\,y2^4\,L^2-48\,y1^3\,y1^m\,y2^5\,(y2^m)^3\,L$$
$$+80\,y1^3\,y1^m\,(y2^m)^3\,y2^5\,m\,L-36\,y1^3\,y1^m\,(y2^m)^3\,m^2\,y2^5\,L+4\,y1^3\,y1^m\,(y2^m)^3\,m^3\,y2^5\,L$$
$$+12\,y1^6\,(y1^m)^3\,y2^m\,m^2\,y2^2\,L^2-2\,y1^6\,(y1^m)^3\,y2^m\,m^3\,y2^2\,L^2+36\,(y1^m)^3\,y2^m\,y2^6\,L^2\,y1^2$$
$$-24\,y1^4\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^4+4\,y1^6\,(y1^m)^2\,(y2^m)^2\,m^4\,y2^2+12\,y1^6\,(y1^m)^3\,y2^m\,y2^2\,L^2$$
$$+12\,y1^m\,(y2^m)^3\,y2^6\,L^2\,y1^2-22\,y1^6\,y1^m\,(y2^m)^3\,y2^2\,m\,L^2$$
$$-12\,y1^6\,y1^m\,(y2^m)^3\,m^2\,y2^2\,L^2-2\,(y1^m)^3\,y2^m\,m^3\,y2^6\,L^2\,y1^2$$
$$-8\,y1^7\,(y1^m)^2\,(y2^m)^2\,m^3\,y2\,L-16\,y1^7\,(y1^m)^2\,(y2^m)^2\,y2\,m\,L$$
$$-16\,y1^6\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^2-2\,y1^m\,(y2^m)^3\,m^3\,y2^6\,L^2\,y1^2$$
$$-2\,y1^6\,y1^m\,(y2^m)^3\,m^3\,y2^2\,L^2-44\,y1^6\,(y1^m)^2\,(y2^m)^2\,y2^2\,m\,L^2$$
$$+8\,y1^6\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^2\,L^2-22\,y1^6\,(y1^m)^3\,y2^m\,y2^2\,m\,L^2-2\,(y1^m)^2\,(y2^m)^2\,y2^8\,m$$
$$+5\,(y1^m)^2\,(y2^m)^2\,m^2\,y2^8+(y1^m)^2\,(y2^m)^2\,m^4\,y2^8-4\,(y1^m)^2\,(y2^m)^2\,m^3\,y2^8)($$
$$-y1^2\,y2\,b1\,y1^m\,L-y1^2\,y2\,b1\,L\,y2^m+y1^3\,b1\,y2^m+y1\,b1\,y2^m\,y2^2-y1\,y2^2\,y1^m\,L\,b2$$
$$-y1^2\,b2\,y1^m\,y2-b2\,y1^m\,y2^3-y1\,y2^2\,L\,y2^m\,b2)^2\Big/((e^{(b1\,x1+b2\,x2+L\arctan(\frac{y1}{y2}))})^2$$
$$((y1^m+y2^m)^{(\frac{1}{m})})^2(2\,L\,y2^3\,(y1^m)^2\,y1+L^2\,y2^2\,y1^2\,(y1^m)^2-y1^m\,y2^4\,y2^m-y1^m\,y1^4\,y2^m$$
$$+2\,y1^m\,m\,y2^2\,y1^2\,y2^m-2\,y1^m\,y2^2\,y1^2\,y2^m-2\,L\,y2\,y1^m\,y1^3\,y2^m$$
$$+2\,L^2\,y2^2\,y1^2\,y1^m\,y2^m+y1^m\,m\,y2^4\,y2^m+2\,L\,y2^3\,y1^m\,y1\,y2^m+y1^m\,m\,y1^4\,y2^m$$
$$+L^2\,y2^2\,y1^2\,(y2^m)^2-2\,L\,y2\,y1^3\,(y2^m)^2)^4)$$

```
> Gauss0 := subs(L=0,Gauss):

> z := y2/y1:

> Gauss1 := (m*(m-2)*b1^2*(1+z^m)^2*(z^(m-1)-b2/b1)^2*(y1*y2))/
(4*(m-1)^2*(F)^2*z^ (2*m-1)):

> Gauss2 := subs(L=0,Gauss1):

> zero := Gauss0 - Gauss2:

> zero := simplify(zero,symbolic);
```

$$zero := 0$$

C

This Maple session shows that, for the expanded m^{th} root metric, the Douglas tensor vanishes for $m = 2$. After defining the *Dimension, Coordinates, and Dcoordinates* as before, the metric function is defined for $m = 2$.

```
> F:= exp(b1*x1+b2*x2+L*arctan(y1/y2))*(y1^2+y2^2)^(1/2):
> F2:=F^2:
```

The metric tensor is given, and the Douglas tensor is calculated, note that it is equal to 0 for $m = 2$.

Douglas tensor is calculated.

```
> metricfunction(F2):
```

The components of the metric are :

$$g_{x1\ x1} = \frac{(2L^2 y2^2 + 2L\,y2\,y1 + y1^2 + y2^2)\,(e^{(b1\,x1 + b2\,x2 + L\arctan(\frac{y1}{y2}))})^2}{y1^2 + y2^2}$$

$$g_{x1\ x2} = -\frac{(2L\,y2\,y1 - y2^2 + y1^2)\,L\,(e^{(b1\,x1 + b2\,x2 + L\arctan(\frac{y1}{y2}))})^2}{y1^2 + y2^2}$$

$$g_{x2\ x2} = \frac{(2L^2 y1^2 - 2L\,y2\,y1 + y1^2 + y2^2)\,(e^{(b1\,x1 + b2\,x2 + L\arctan(\frac{y1}{y2}))})^2}{y1^2 + y2^2}$$

```
> show(G[i,-j,-k,-l]);
```

$$G^{i}{}_{j\ k\ l} = 0$$

Now, assuming that $m \neq 2$, the various components of the Douglas tensor were calculated. After substituting $L = 0$ into the expression, each component was compared to the known value given in [3], by obatining the difference, *diff*. The known value is labelled *D111known*, for example.

```
> F:= exp(b1*x1+b2*x2+L*arctan(y1/y2))*(y1^m+y2^m)^(1/m):

> D1111 := show(G[x1,-x1,-x1,-x1]):

> D1111 := subs(L=0,D1111):

> D1111known := b1/2*m*(m-2)*y2^m/(y1^m)*1/y1:

> diff1 := D1111 - D1111known:

> diff1 := normal(diff1,expanded);
```

$$diff1 := 0$$

```
> D1121 := show(G[x1,-x1,-x2,-x1]):

> D1121 := subs(L=0,D1121):

> D1121known := -b1/2*m*(m-2)*y2^(m-1)/(y1^(m-1))*1/y1:

> diff2 := D1121 - D1121known:

> diff2 := normal(diff2,expanded);
```

$$diff2 := 0$$

```
> D1221 := show(G[x1,-x2,-x2,-x1]):

> D1221 := subs(L=0,D1221):

> D1221known := b1/2*m*(m-2)*y2^(m-2)/(y1^(m-2))*1/y1:

> diff3 := D1221 - D1221known:

> diff3 := normal(diff3,expanded);
```

$$diff3 := 0$$

```
> D1112 := show(G[x1,-x1,-x1,-x2]):

> D1112 := subs(L=0,D1112):

> D1112known := -b1/2*m*(m-2)*y2^(m-1)/(y1^(m-1))*1/y1:

> diff4 := D1112 - D1112known:

> diff4 := normal(diff4,expanded);
```

$$diff4 := 0$$

```
> D1122 := show(G[x1,-x1,-x2,-x2]):

> D1122 := subs(L=0,D1122):

> D1122known := b1/2*m*(m-2)*y2^(m-2)/(y1^(m-2))*1/y1:

> diff5 := D1122 - D1122known:

> diff5 := normal(diff4,expanded);
```

$$diff5 := 0$$

```
> D1222 := show(G[x1,-x2,-x2,-x2]):
> D1222 := subs(L=0,D1222):
> D1222known := -b1/2*m*(m-2)*y2^(m-3)/(y1^(m-3))*1/y1:
> diff6 := D1222 - D1222known:
> diff6 := normal(diff4,expanded);
```

$$diff6 := 0$$

```
> D2111 := show(G[x2,-x1,-x1,-x1]):
> D2111 := subs(L=0,D2111):
> D2111known := -b2/2*m*(m-2)*y1^(m-3)/(y2^(m-3))*1/y2:
> diff7 := D2111 - D2111known:
> diff7 := normal(diff7,expanded);
```

$$diff7 := 0$$

```
> D2211 := show(G[x2,-x2,-x1,-x1]):
> D2211 := subs(L=0,D2211):
```

```
> D2211known := b2/2*m*(m-2)*y1^(m-2)/(y2^(m-2))*1/y2:
> diff8 := D2211 - D2211known:
> diff8 := normal(diff8,expanded);
```

$$diff8 := 0$$

```
> D2221 := show(G[x2,-x2,-x2,-x1]):
> D2221 := subs(L=0,D2221):
> D2221known := -b2/2*m*(m-2)*y1^(m-1)/(y2^(m-1))*1/y2:
> diff9 := D2221 - D2221known:
> diff9 := normal(diff9,expanded);
```

$$diff9 := 0$$

```
> D2112 := show(G[x2,-x1,-x1,-x2]):
> D2112 := subs(L=0,D2112):
> D2112known := b2/2*m*(m-2)*y1^(m-2)/(y2^(m-2))*1/y2:
> diff10 := D2112 - D2112known:
> diff10 := normal(diff10,expanded);
```

$$diff10 := 0$$

```
> D2212 := show(G[x2,-x2,-x1,-x2]):
> D2212 := subs(L=0,D2212):
> D2212known := -b2/2*m*(m-2)*y1^(m-1)/(y2^(m-1))*1/y2:
> diff11 := D2212 - D2212known:
> diff11 := normal(diff11,expanded);
> diff11 := normal(diff11,expanded);
```

$$\mathit{diff11} := 0$$

```
> D2222 := show(G[x2,-x2,-x2,-x2]):
> D2222 := subs(L=0,D2222):
> D2222known := b2/2*m*(m-2)*y1^m/(y2^m)*1/y2:
> diff12 := D2222 - D2222known:
> diff12 := normal(diff12,expanded);
```

$$\mathit{diff12} := 0$$

Notice that the difference between the calculated value for $\mathbb{D}$, and the known value is equal to 0 in each case. The Douglas tensor for the m^{th} root metric, with $L = 0$, vanishes if and only if $m = 2$ [3]. Since the Douglas tensor for the $L \neq 0$ case reduces to those of the $L = 0$ case upon substitution, we can say that the Douglas tensor for the expanded metric is 0 if and only if $m = 2$.

Bibliography

[1] P.L. Antonelli (ed.). *Handbook of Finsler Geometry*, Kluwer Academic Publishers, 2003.

[2] P.L. Antonelli, I. Bucataru, and S.F. Rutz. Computer algebra and two and three dimensional Finsler geometry. *Publ. Math. Debrecen*, 42:291, 2003.

[3] P.L. Antonelli, and R.H. Bradbury, *Volterra-Hamilton models in the ecology and evolution of colonial organisms*, volume 2 of the Series in Mathematical Biology and Medicine. World Scientific, 1996.

[4] P. Antonelli, R. Ingarden, and M. Matsumoto. *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, volume 58 of *Fundamental Theories of Physics*. Kluwer Academic Publishers, 1993.

[5] D. Bao, S.S. Chern, and Z. Shen. *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.

[6] M.P. DoCarmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[7] Maple 8, help.

[8] http://www.maplesoft.com

[9] http://www.mathworld.wolfram.com

[10] http://www.planetmath.org

[11] Portugal, R. and Sautu, S. L. Applications of Maple to General Relativity, Computer Physics Communications, **105**, 233 - 253, 1997.

[12] Rutz, S.F. and Portugal, R., *FINSLER: A Computer algebra package for Finsler geometry*, Nonlinear Analysis **47** (2001), 6121-6134.

[13] Z. Shen. *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[14] D. Struik. *Lectures on Classical Differential Geometry*, Addison-Wesley, 1950.

[15] J.L. Synge, and A. Schild. *Tensor Calculus*, Dover, 1978.