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UNIVERSITY OF ALBERTA

# SEMIPRIMITIVE COMPLETE p-ADIC GROUP ALGEBRAS

*by*

ALLEN HERMAN

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PERMANENT ADDRESS:

Department of Mathematics

University of Alberta

Edmonton, Alberta T6G 2G1

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SEMIPRIMITIVE COMPLETE $p$-ADIC GROUP ALGEBRAS submitted by ALLEN HERMAN in partial fulfillment of the degree of Master of Science.

M. Shirvani
M. Shirvani
Supervisor

B.N. Allison
B.N. Allison

A.H. Rhemtulla
A.H. Rhemtulla

D. Wiens
D. Wiens

Date November 29, 1991

## ABSTRACT

Let $p$ be a prime integer. Let $k$ be a complete $p$-adic field, that is, a subfield of $\Omega_p$, the completion of the algebraic closure of the field of $p$-adic numbers $\mathbb{Q}_p$, which is itself complete with respect to the $p$-adic absolute value $|\cdot|_p$. The aim of this thesis is to find all groups $G$ such that the ring of formal sums

$$\ell_1(k, G) = \left\{ \sum_{g \in G} \alpha_g g : \ \alpha_g \in k, |\alpha_g|_p \to 0 \right\}$$

is semiprimitive, i.e. its Jacobson radical is the zero ideal. These rings are the subject of an open question of S.K. Sehgal in his book *Topics in Group Rings*, where it is asked whether or not $\ell_1(\mathbb{Q}_p,\ G)$ is semiprime.

It is well-known that the corresponding complex Banach algebra $\ell_1(\mathbb{C}, G)$ with norm $\sum_{g \in G} |\alpha_g| < \infty$ is always semiprimitive. However, it has been shown that $\ell_1(\Omega_p,\ C_{p^\infty})$ has non-zero nilpotent elements, where $C_{p^\infty}$ is the divisible abelian $p$-group of rank one. It is this discrepancy which becomes the motivation for this study.

Appearing in the thesis are the previously published results that $\ell_1(k, G)$ is semiprimitive whenever the group $G$ is:

i) a residually finite group,

ii) a unique product group,

iii) an abelian group with no $C_{p^\infty}$-subgroups,

or

iv) a solvable group which has a normal series consisting of abelian $p'$-groups.

Also, a translation is presented of the proof that $\ell_1(k,\ C_{p^\infty})$ is semiprimitive if and only if the maximal algebraic extension of $\mathbb{Q}_p$ contained in $k$ is not of different zero over $\mathbb{Q}_p$. Of further interest are the previously unpublished results presented here, such as an example of a non-zero nilpotent element of $\ell_1(\Omega_p,\ C_{p^\infty})$ (Theorem 3.9), a proof of the statement $k \subset \Omega_p \Rightarrow J(\ell_1(k,G)) \subseteq J(\ell_1(\Omega_p,\ G))$ (Theorem 2.5), and proof of semiprimitivity of $\ell_1(k,G)$ for all nilpotent $p$-groups $G$ which have central, residually finite commutator subgroups (Theorem 4.12). Finally, the description of complete crossed products in Chapter IV and the characterization of the Jacobson radical of $\ell_1(\Omega_p,\ G)$ for locally finite abelian groups $G$ presented in the conclusion may also be of interest.

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# CHAPTER I

# INTRODUCTION TO $p$-ADIC $\ell_1$-GROUP ALGEBRAS

Define the Jacobson radical of a ring to be the intersection of all of its maximal right ideals. A ring is called semiprimitive if its Jacobson radical is the zero ideal.

Let $p$ be a fixed prime number.

The topic of this thesis is the study of the Jacobson radical of $\ell_1$-group algebras over complete $p$-adic fields, in the hope of finding conditions for which these rings are semiprimitive. This study is motivated by the fact that $\ell_1$-group algebras over the field of complex numbers are always semiprimitive [6], and yet there are cases for which the $\ell_1$-group algebra of a complete $p$-adic field is not semiprimitive [1]. This thesis presents an overview of the progress on this problem up to the present date.

The purpose of the first chapter is to introduce $p$-adic fields and $\ell_1$-group algebras in order to provide the background necessary for the rest of the thesis.

## A. The $p$-adic integers and the field of $p$-adic numbers.

One way of creating the ring of $p$-adic integers is to form the projective limit of the rings $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$, $n \in \mathbb{N}$. For a fixed $n \in \mathbb{N}$, any element $z \in \frac{\mathbb{Z}}{p^n\mathbb{Z}}$ can be uniquely expressed as $z = a_0 + a_1 p + \cdots + a_{n-1}p^{n-1}$, with $a_i \in \{0, 1, \ldots, p-1\}$ for $i = 0, 1, \ldots, n-1$. For any two integers $n \geq m \geq 1$, there is a natural projection $\varphi_{n,m} : \frac{\mathbb{Z}}{p^n\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{p^m\mathbb{Z}}$ given by $\varphi_{n,m}\left(\sum_{i=1}^{n-1} a_i p^i\right) = \sum_{i=0}^{m-1} a_i p^i$. Then it is easy to see that $\varphi_{n,n}$ is the identity map, and $\varphi_{n,\ell} = \varphi_{m,\ell}\varphi_{n,m}$ for any integers $n \geq m \geq \ell \geq 1$. The projective limit $\varprojlim \frac{\mathbf{Z}}{p^n\mathbf{Z}}$ is called the ring of $p$-adic integers, denoted $\mathbf{Z}_p$, and is defined to be

$$\left\{a \in \prod_{n=1}^{\infty} \frac{\mathbf{Z}}{p^n\mathbf{Z}} : (a)_m = \varphi_{n,m}(a)_n, \quad \text{for all} \quad n \geq m \geq 1\right\}. \quad [3, (18)]$$

Such an element looks like $(a_0, a_0 + a_1p, a_0 + a_1p + a_2p^2, \cdots)$ for some $a_i \in \{0, 1, \ldots, p-1\}$, for all $i = 0, 1, 2, \ldots$, and thus can be identified with the infinite series $\sum_{i=0}^{\infty} a_ip^i$. When all of the $a_i$ belong to $\{0, \ldots, p-1\}$, this infinite series is said to be in the standard form of an element of $\mathbf{Z}_p$. The addition and multiplication of the ring $\mathbf{Z}_p$ are comparable to the usual base$-p$ addition and multiplication in the integers. For all $\sum_{i=0}^{\infty} a_ip^i,\ \sum_{i=0}^{\infty} b_ip^i \in \mathbf{Z}$,

$$\sum_{i=0}^{\infty} a_ip^i + \sum_{i=0}^{\infty} b_ip^i = \sum_{i=0}^{\infty}(a_i + b_i)p^i$$

and

$$\Big(\sum_{i=0}^{\infty} a_ip^i\Big)\Big(\sum_{j=0}^{\infty} b_ip^i\Big) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_ib_jp^{i+j} = \sum_{\ell=0}^{\infty}\Big(\sum_{j=0}^{\ell} a_jb_{\ell-j}\Big)p^{\ell}.$$

The sums $a_i + b_i$ and $\sum_{j=0}^{\ell} a_jb_{\ell-j}$ take place in the integers, and so some rounding may be required to retain the standard form of the sum or product, as in the case of ordinary base$-p$ addition or multiplication.

It is clear that $\mathbf{Z}_p$ is a commutative ring with identity, the element $1 \cdot p^0 + 0p^1 + 0p^2 + \cdots = 1$ being the multiplicative identity. Also, $\mathbf{Z}_p$ has no zero divisors. For any non-zero $\sum_{i=0}^{\infty} a_ip^i,\ \sum_{j=0}^{\infty} b_jp^j \in \mathbf{Z}_p$, there are integers $m_1, m_2$ such that

$a_{m_1} \neq 0$, $b_{m_2} \neq 0$, and $a_{n_1} = 0$, $b_{n_2} = 0$ for all $0 \leq n_1 < m_1$, $0 \leq n_2 < m_2$. Thus $\left(\sum_{i=0}^{\infty} a_i p^i\right)\left(\sum_{j=0}^{\infty} b_j p^j\right) = \sum_{\ell=0}^{\infty}\left(\sum_{j=0}^{\ell} a_j b_{\ell-j}\right)p^\ell$ has $\sum_{j=0}^{\ell} a_j b_{\ell-j} = 0$ for all $0 \leq \ell < m_1 + m_2$, and $\sum_{j=0}^{m_1+m_2} a_j b_{\ell-j} = a_{m_1} b_{m_2} \not\equiv (\text{mod } p)$. Hence $\mathbb{Z}_p$ is an integral domain.

The field of fractions of $\mathbb{Z}_p$ is called the field of $p$-adic numbers. Let $\mathbb{Q}_p = \left\{\sum_{i=N}^{\infty} a_i p^i : a_i \in \{0, \ldots, p-1\}, \text{ for all } N \leq i, \text{ for some } N \in \mathbb{Z}\right\}$. (Negative integers $N$ are allowed.)

Define addition and multiplication on $\mathbb{Q}_p$ analogously to that of $\mathbb{Z}_p$. For all $\sum_{i=N}^{\infty} a_i p^i$, $\sum_{i=N}^{\infty} b_i p^i \in \mathbb{Q}_p$, with $N \leq M$ let $\sum_{i=N}^{\infty} a_i p^i + \sum_{i=M}^{\infty} b_i p^i = \sum_{i=N}^{\infty} (a_i + b_i) p^i$, where $b_i = 0$ for $N \leq i < M$, and $a_i + b_i$ is integer addition, for all $i \geq N$. Also, let $\sum_{i=N}^{\infty} a_i p^i \cdot \sum_{j=M}^{\infty} b_j p^j = \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} a_i b_j p^{i+j} = \sum_{\ell=N+M}^{\infty} \left(\sum_{i=N}^{\ell} a_i b_{\ell-i}\right) p^\ell$, with $\sum_{i=N}^{\ell} a_i b_{\ell-i}$ taking place in $\mathbb{Z}$, for all $\ell \geq N + M$. With these operations, $\mathbb{Q}_p$ is a commutative ring containing $\mathbb{Z}_p$ as a subring.

Let $\sum_{i=N}^{\infty} a_i p^i \in \mathbb{Q}_p$ be non-zero, and assume without loss of generality that $a_N \neq 0$. The element $\sum_{i=N}^{\infty} a_i p^i$ has an inverse in $\mathbb{Q}_p$ if the equation

$$1 = \left(\sum_{i=N}^{\infty} a_i p^i\right)\left(\sum_{i=-N}^{\infty} b_i p^i\right) = \sum_{\ell=0}^{\infty}\left(\sum_{j=N}^{\ell} a_j b_{\ell-j}\right)p^\ell$$

can be solved for some $\sum_{i=M}^{\infty} b_i p^i \in \mathbb{Q}_p$. Now, $1 \equiv a_N b_{-N} (\text{ mod } p)$ has a solution for some unique $b_{-N} \in \{1, \ldots, p-1\}$, and if $a_N b_{-N} = pk_{-N} + 1$, the linear equation $0 \equiv a_{N+1} b_{-N} + a_N b_{-N+1} + k_1 (\text{ mod } p)$ is linear with $b_{-N+1}$ as the unknown. Since $\{0, \ldots, p-1\}$ is a field when addition and multiplication are done modulo $p$, and $a_N \neq 0$, there is a unique solution for $b_{-N+1}$. Inductively,

$0 \equiv a_N b_{\ell-N} + k_\ell \pmod{p}$, for some integer $k_\ell$, and so as before there is a unique solution for $b_{\ell-N} \in \{0, \ldots, p-1\}$, for all $\ell \geq -N$. Thus $\sum_{i=N}^{\infty} a_i p^i$ has a unique inverse $\sum_{i=-N}^{\infty} b_i p^i \in \mathbb{Q}_p$, and so $\mathbb{Q}_p$ is a field. At least one of $\sum_{i=N}^{\infty} a_i p^i$, $\sum_{i=-N}^{\infty} b_i p^i$ must lie in $\mathbb{Z}_p$, so $\mathbb{Q}_p$ must be the field of fractions of $\mathbb{Z}_p$.

**B. The $p$-adic absolute value on $\mathbb{Q}_p$.**

For any field $F$, an absolute value on $F$ is a map $|\cdot| : F \to [0, \infty)$ such that for all $a,\ b \in F$,

(i) $|a| = 0 \Leftrightarrow a = 0$,

(ii) $|a \cdot b| = |a| \cdot |b|$,

(iii) $|a + b| \leq |a| + |b|$.

An absolute value on $F$ is called non-Archimedean if (iii) can be relaced by the stronger triangle inequality

(iii′) $|a + b| \leq \max\{|a|,\ |b|\}$.

A non-Archimedean absolute value $|\cdot|_p$ can be defined in a natural way on the field $\mathbb{Q}_p$. Set $|0|_p = 0$, and for a non-zero $\sum_{i=N}^{\infty} a_i p^i \in \mathbb{Q}_p$ with $a_N \neq 0$, let $|\sum_{i=N}^{\infty} a_i p^i|_p = p^{-N}$. $|\cdot|_p$ clearly satisfies property (i). For $\sum_{i=N}^{\infty} a_i p^i$, $\sum_{j=M}^{\infty} b_j p^j \in \mathbb{Q}_p$ with $a_N,\ b_M \neq 0$, $|\left(\sum_{i=N}^{\infty} a_i p^i\right)\left(\sum_{j=M}^{\infty} b_j p^j\right)|_p = |\sum_{\ell=N+M}^{\infty}\left(\sum_{i=N}^{\ell} a_i b_{\ell-i}\right)p^\ell|_p = p^{-(N+M)} = p^{-N} \cdot p^{-M} = |\sum_{i=N}^{\infty} a_i p^i|_p \cdot |\sum_{j=M}^{\infty} b_j p^j|_p$, so $|\cdot|_p$ also satisfies property (ii). Finally, $|\sum_{i=N}^{\infty} a_i p^i + \sum_{i=M}^{\infty} b_i p^i|_p = |\sum_{i=\min\{N,M\}}^{\infty} (a_i + b_i)p^i|_p \leq p^{-(\min\{N,M\})} = \max\left\{|\sum_{i=N}^{\infty} a_i p^i|_p,\ |\sum_{i=M}^{\infty} b_i p^i|_p\right\}$, so $|\cdot|_p$ also satisfies (iii′).

Proposition 1.1. *With respect to the topology induced by $|\cdot|_p$, (i) $\mathbb{Q}_p$ is complete, and (ii) $\mathbb{Z}_p$ is compact.*

Proof: (i): Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathbb{Q}_p$ which is Cauchy with respect to the topology induced by $|\cdot|_p$. Then for any $\varepsilon > 0$, and for all large enough $n$, $m \in \mathbb{N}$, $|a_n - a_m|_p < \varepsilon$. For all $n \in \mathbb{N}$, write $a_n = \sum_{i=-\infty}^{\infty} a_{n,i}p^i$ and let $N(n) \in \mathbb{Z}$ such that $a_{n,i} = 0$ for all $i < N(n)$. Then $|a_n - a_m|_p < \varepsilon \Longrightarrow |\sum_{i=-\infty}^{\infty} (a_{n,i} - a_{m,i})p^i|_p < \varepsilon \Rightarrow a_{n,i} = a_{m,i}$, for all $i$ such that $p^{-i} \geq \varepsilon$. Thus the sequences $\{a_{n,i}\}_{n=1}^{\infty}$ of elements of $\{0, \ldots, p-1\}$ must stabilize after some point. Hence for each $i \in \mathbb{Z}$, there is an $a_i \in \{0, \ldots, p-1\}$ such that $a_i = \lim_{n\to\infty} a_{n,i}$. Furthermore, the sequence $\{N(n)\}_{n=1}^{\infty}$ must also have a lower bound, else the sequence $\{a_m\}_{n=1}^{\infty}$ cannot be Cauchy. Hence there is an $N \in \mathbb{Z}$ such that $a_i = 0$ for all $i \leq N$. Thus it is clear that $a = \sum_{i=-\infty}^{\infty} a_i p^i$ is an element of $\mathbb{Q}_p$, and $|a_n - a|_p \to 0$ as $n \to \infty$, so $a = \lim_{n\to\infty} a_n$.

(ii): It suffices to show that any sequence of elements in $\mathbb{Z}_p$ has a subsequence which converges to an element of $\mathbb{Z}_p$ in the topology induced by $|\cdot|_p$.

Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Z}_p$, and write $a_n = \sum_{i=0}^{\infty} a_{n,i}p^i$ in standard form. The sequence $\{a_{n,i}\}_{i=1}^{\infty}$ is a sequence of elements from the finite set $\{0, \ldots, p-1\}$, and so it is possible to choose a subsequence $\{a_{n_0}^{(0)}\}_{n_0=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $a_{n_0,0}^{(0)} = b_0 \in \{0, \ldots, p-1\}$, for all $a_{n_0}^{(0)}$. Inductively, choose a subsequence $\{a_{n_i}^{(i)}\}_{n_i=1}^{\infty}$ of $\{a_{n_{i-1}}^{(i-1)}\}_{n_{i-1}=1}^{\infty}$ such that $a_{n_i,i}^{(i)} = b_i \in \{0, \ldots, p-1\}$ for all $a_{n_i}^{(i)}$. Then $\{a_{n_i}^{(i)}\}_{i=0}^{\infty}$ is a subsequence of $\{a_n\}$ which has the obvious limit $\sum_{i=0}^{\infty} b_i p^i \in \mathbb{Z}_p$.

□

**C. Algebraic extensions of $\mathbb{Q}_p$.**

The complete field $\mathbb{Q}_p$ is not algebraically closed; indeed, it is easy to see that the polynomial $x^2 - p \in \mathbb{Q}_p[x]$ is irreducible over $\mathbb{Q}_p$. (If $\alpha \in \mathbb{Q}_p$ with $|\alpha|_p = p^{-N}$, then $|\alpha^2| = p^{-2N}$, so $\alpha^2 \neq p$.) In this section, algebraic extension fields of $\mathbb{Q}_p$ are examined.

The first result shows that finite extensions of $\mathbb{Q}_p$ admit non-Archimedean absolute values that extend the $p$-adic absolute value of $\mathbb{Q}_p$. In order to show this result, the concept of a "norm" map is needed. The "trace" map also introduced here will be needed later on.

Let $K$ be a finite field extension of a field $F$. For all $\alpha \in K$, left multiplication by $\alpha$ is an $F$-linear map of $K$ into itself. Thus for any fixed basis of $K$ over $F$, left multiplication by $\alpha$ has a square matrix representation $M(\alpha)$ with entries from $F$. The trace and norm maps of the extension $K$ over $F$ are defined by $Tr_{K/F}(\alpha) = tr(M(\alpha))$, and $N_{K/F}(\alpha) = \det(M(\alpha))$. These are well-defined since the trace and determinant functions on an $n \times n$ matrix are constant on similar matrices. The next proposition outlines some of the basic properties of norms and traces.

Proposition 1.2. *Let $K$ be a finite extension of a field $F$. Then*

*(i) $Tr_{K/F}$ is an $F$-linear map from $K$ onto $F$.*

*(ii) $N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta)$, for all $\alpha, \beta \in K$,*

*(iii) $N_{K/F}(a\beta) = a^n N_{K/F}(\beta)$, for all $a \in F$, $\beta \in K$.*

PROOF: (i): Let $\alpha$, $\beta \in K$, and let $a \in F$. Note that for a fixed basis of $K$ over $F$, the matrix $M(a\alpha + \beta)$ that corresponds to left multiplication by $a\alpha + \beta$ is equal to $M(a) \cdot M(\alpha) + M(\beta)$. Since $M(a) = a \cdot I$, where $I$ is the identity matrix, $tr(M(a\alpha)) = a \cdot tr(M(\alpha))$. Thus $tr_{K/F}(a\alpha + \beta) = tr(M(a\alpha + \beta)) = tr(M(a\alpha)) + tr(\beta) = a\ tr(M(\alpha)) + tr(M(\beta)) = a\ Tr_{K/F}(\alpha) + Tr_{K/F}(\beta)$. Thus $Tr_{K/F}$ is $F$-linear.

(ii): Since $M(\alpha\beta) = M(\alpha)M(\beta)$ for all $\alpha$, $\beta \in K$, $N_{K/F}(\alpha\beta) = \det(M(\alpha\beta)) = \det(M(\alpha)M(\beta)) = \det(M(\alpha)) \cdot \det(M(\beta)) = N_{K/F}(\alpha) \cdot N_{K/F}(\beta)$.

(iii): For all $a \in F, N_{K/F}(a) = \det(M(a)) = \det(a \cdot I) = a^n$. Thus (iii) follows from (ii).

□

PROPOSITION 1.3. *let $F$ be a field of characteristic 0. Suppose $K = F(\alpha)$, where $\alpha$ is a root of the monic irreducible polynomial $f_\alpha(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in F[x]$. Then*

*(i) $Tr_{K/F}(\alpha) = -a_{n-1}$, $N_{K/F}(\alpha) = (-1)^n a_0$, and*

*(ii) if $f_\alpha(x)$ splits in $K$, and $\alpha_1, \ldots, \alpha_n$ are the distinct roots of $f_\alpha(x)$ in $K$, then $Tr_{K/F}(\alpha) = \sum_{i=1}^{n} \alpha_i$ and $N_{K/F}(\alpha) = (-1)^n \prod_{i=1}^{n} \alpha_i$.*

PROOF: (i): $K = F(\alpha)$ has the basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$ over $F$, and left multiplicaton by $\alpha$ has the matrix representation

$$M(\alpha) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & & 0 & -a_{n-2} \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}$$

in this basis. (Using $\alpha^n = -a_{n-1}\alpha^{n-1} - a_{n-2}\alpha^{n-2} - \cdots - a_0$).

Thus $tr(M(\alpha)) = -a_{n-1}$, and expanding the determinant along the first row yields $\det(M(\alpha)) = (-1)^n a_0$.

(ii): Since $F$ has characteristic zero, the extension $K$ is separable over $F$, and thus there are $n$ distinct roots $\alpha_1, \ldots, \alpha_n$ of $f_\alpha(x)$ in $K$. Furthermore, $f_\alpha(x) = \prod_{i=1}^{n}(x - \alpha_i)$. Thus $a_{n-1} = \sum_{i=1}^{n}(-\alpha_i)$ and $a_0 = \prod_{i=1}^{n}(-\alpha_i)$. (ii) follows then from (i).

□

PROPOSITION 1.4. *Let $F \subset K \subset L$ be fields with $\frac{L}{F}$ finite. Then for all $\alpha \in L$, $Tr_{L/F}(\alpha) = Tr_{K/F}(Tr_{L/K}(\alpha))$ and $N_{L/F}(\alpha) = N_{K/F}(N_{L/K}(\alpha))$.*

PROOF: [2, §7.4].

□

In general, a finite field extension of $\mathbb{Q}_p$ is a finite dimensional vector space over $\mathbb{Q}_p$. A norm ($K$-norm) on a vector space $V$ over a field $K$ with an absolute value $|\cdot|_K$ is a map $\|\cdot\| : V \to [0, \infty)$ such that

(i) $\|v\| = 0 \Leftrightarrow v = 0$,

(ii) $\|av\| = |a|_K \cdot \|v\|$, for all $a \in K$, $v \in V$,

and

(iii) $\|v + u\| \le \|v\| + \|u\|$, for all $v$, $u \in V$.

If $V$ is also an algebra over $K$, then $\|\cdot\|$ is called a $K$-algebra norm, if in addition to the above, the multiplication on $V$ satisfies

(iv) $\|v \cdot u\| \le \|v\| \cdot \|u\|$, for all $u$, $v \in V$.

Again, the norm $\|\cdot\|$ is non-Archimedean if (iii) can be replaced by

(iii′) $\|v+u\| \leq \max\{\|v\|, \|u\|\}$,

$\forall v,\ u \in V$. Two $K$-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $V$ over $K$ are said to be equivalent if there are positive constants $c_1,\ c_2 \in \mathbb{R}$ such that $c_1\|\cdot\|_1 \leq \|\cdot\|_2 \leq c_2\|\cdot\|_1$.

The following result is of vital importance in the study of non-Archimedean normed spaces.

PROPOSITION 1.5. *Let $\|\cdot\|$ be a non-Archimedean norm on a $K$-vector space $V$. Then if $v,\ u \in V$ with $\|v\| > \|u\|$, $\|v+u\| = \|v\|$.*

PROOF: If $\|v+u\| < \|v\|$, then $\|v\| = \|(v+u)-u\| \leq \max\{\|v+u\|, \|u\|\} < \|v\|$. This is a contradiction.

□

A well-known fact about norms of finite dimensional vector spaces over non-Archimedean fields is the following.

THEOREM 1.6. *If $V$ is a finite dimensional vector space over a non-Archimedean field $K$ then all $K$-norms on $V$ are equivalent.*

PROOF: Let $|\cdot|_K$ be the non-Archimedean absolute value on $K$, and let $\|\cdot\|_V$ be a $K$-norm on $V$.

If $V$ is one-dimensional over $K$, then the theorem is obvious.

Suppose $V$ is $n$-dimensional over $K$, and assume the theorem is true for all $K$-vector spaces $W$ of dimension less than $n$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$ over $K$, and define the maximum norm with respect to this basis to be

$$\|v\| = \max_{1\leq i\leq n} |\alpha_i|_K, \quad \text{where} \quad v = \sum_{i=1}^{n} \alpha_i v_i \in V,$$

with all $\alpha_i \in K$. It suffices to show that $\|\cdot\|_V$ is equivalent to $\|\cdot\|$.

For any $v = \sum_{i=1}^{n} \alpha_i v_i \in V$, $\|v\|_V \leq \max_{1\leq \ \leq n} |\alpha_i| \cdot \|v_i\|_V \leq M \cdot \|v\|$, where $M = \max_{1\leq i\leq n} \|v_i\|_V$.

Let $U = Kv_1 + \cdots + Kv_{n-1}$. By induction, there exists a $C > 0$ such that $C\|u\| \leq \|u\|_V$, for all $u \in U$. Consider $D = \|v_n\|_V^{-1} \cdot \inf_{v\in U} \{\|v_n - u\|_V\}$. Since $0 \neq v_n \notin U$, and $U$ is a closed subspace of $V$ ($U$ is complete), it is evident that $0 < D \leq 1$.

Set $N = \min\{C \cdot D, \ \|v_n\|_V \cdot D\}$.

Let $v \in V$, with $v = u + \alpha v_n$, for some $\alpha \in K$, $u \in U$. If $\alpha \neq 0$, then $\|v\|_V = |\alpha| \cdot \|\alpha^{-1}u + v_n\|_V \geq |\alpha| \cdot \|v_n\|_V \cdot D = \|\alpha v_n\|_V \cdot D$. Now, $\|v\|_V \geq D \cdot \|\alpha v_n\|_V$ implies $\|u\|_V = \|\alpha v_n + u - \alpha v_n\|_V \leq \max\{\|v\|_V, \ \|\alpha v_n\|_V\} \leq D^{-1}\|v\|_V$, so $\|v\|_V \geq D\|u\|_V$, also.

Hence

$$\begin{aligned}
\|v\|_V &\geq D \max\{\|u\|_V, \ \|\alpha v_n\|_V\} \\
&\geq D \max\{C\|u\|, \ |\alpha| \cdot \|v_n\|_V\} \\
&\geq N \max\{\|u\|, \ |\alpha|\} \\
&= N\|v\|.
\end{aligned}$$

Thus $N\|v\| \le \|v\|_V \le M\|v\|$, for all $v \in V$, proving the theorem.

□

COROLLARY 1.6.1. *If $K$ is a finite extension field of a non-Archimedean field $F$, then there is at most one absolute value on $K$ which extends that of $F$.*

PROOF: By Theorem 1.6, any two absolute values $|\cdot|_1$, $|\cdot|_2$ are equivalent. Thus there are positive constants $c_1, c_2 \in \mathbf{R}$ such that $c_1|\cdot|_1 \le |\cdot|_2 \le c_2|\cdot|_1$. If $\exists \alpha \in K$ such that $|\alpha|_1 < |\alpha|_2$ without loss of generality, then for a sufficiently large $n$, $c_2|\alpha^n|_1 \le |\alpha^n|_2$. This is a contradiction.

□

The norm map can be used to show that there is an absolute value on any finite extension field of $\mathbb{Q}_p$ that extends the $p$-adic absolute value on $\mathbb{Q}_p$.

THEOREM 1.7. *Let $K$ be a finite extension of $\mathbb{Q}_p$ of degree $n$. Then $|\alpha|_p = |N_{K/\mathbb{Q}_p}(\alpha)|_p^{1/n}$ is an absolute value on $K$ that extends the $p$-adic absolute value on $\mathbb{Q}_p$.*

PROOF: [4, Theorem 11].

□

Let $\widehat{\mathbb{Q}_p}$ denote the algebraic closure of $\mathbb{Q}_p$. Using Theorem 1.5, the $p$-adic absolute value of $\mathbb{Q}_p$ extends to $\widehat{\mathbb{Q}_p}$ by defining $|\alpha|_p = |N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}(\alpha)|^{1/[\mathbb{Q}_p(\alpha):\mathbb{Q}]}$ for all $\alpha \in \widehat{\mathbb{Q}_p}$. This makes sense since for all finite extensions $K$ of $\mathbb{Q}_p(\alpha)$ contained in $\widehat{\mathbb{Q}_p}$,

$$N_{K/\mathbb{Q}_p}(\alpha) = N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}\left(N_{K/\mathbb{Q}_p(\alpha)}(\alpha)\right) = N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}\left(\alpha^{[K:\mathbb{Q}_p(\alpha)]}\right) =$$
$$= \left(N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}(\alpha)\right)^{[K:\mathbb{Q}_p(\alpha)]}.$$

Recall that $\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\}$. Since $p\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a| < 1\}$ is clearly the unique maximal ideal of $\mathbb{Z}_p$, $\frac{\mathbb{Z}_p}{p\mathbb{Z}_p}$ is a field. In fact, it is obvious that $\frac{\mathbb{Z}_p}{p\mathbb{Z}_p}$ is just the field of $p$ elements $\mathbb{F}_p = \{0, \ldots, p-1\}$ with addition and multiplication modulo $p$. Analogously, for any extension $K$ of $\mathbb{Q}_p$ contained in $\widehat{\mathbb{Q}_p}$, define $\mathcal{O}_K = \{\alpha \in K : |\alpha|_p \leq 1\}$ to be the valuation ring of $K$, with unique maximal ideal $\mathcal{P}_K = \{\alpha \in K : |\alpha|_p < 1\}$. The field $\frac{\mathcal{O}_K}{\mathcal{P}_K}$ is called the residue class field of $K$, and is denoted $\widetilde{K}$.

PROPOSITION 1.8. *Let $K$ be an algebraic extension of $\mathbb{Q}_p$. Then $\widetilde{K}$ is a field extension of $\mathbb{F}_p$.*

PROOF: Since the absolute value $|\cdot|_p$ on $K$ extends the $p$-adic absolute value on $\mathbb{Q}_p$, $\mathcal{P}_K \cap \mathbb{Z}_p = p\mathbb{Z}_p$. Let $a$, $b \in \mathbb{Z}_p$. If $a + \mathcal{P}_K = b + \mathcal{P}_K$, then $a - b \in \mathcal{P}_K \cap \mathbb{Z}_p$. Thus the natural inclusion $a + p\mathbb{Z}_p \mapsto a + \mathcal{P}_K$ is an embedding of $\mathbb{F}_p$ into $\widetilde{K}$.

□

PROPOSITION 1.9. *If $K$ is a finite extension of $\mathbb{Q}_p$ of degree $n$, then $\widetilde{K}$ is a finite extension of $\mathbb{F}_p$ of degree at most $n$.*

PROOF: Let $a_1 + \mathcal{P}_k, \ldots, a_{n+1} + \mathcal{P}_K \in \widetilde{K}$, for some $a_1, \ldots, a_{n+1} \in \mathcal{O}_K$. Then $a_1, \ldots, a_{n+1}$ are linearly dependent over $\mathbb{Q}_p$, so there exists $b_1, \ldots, b_{n+1} \in \mathbb{Q}_p$, not all 0, such that $a_1 b_1 + \cdots + a_{n+1} b_{n+1} = 0$. Normalize the $b_k$ so that $\max_{1 \leq i \leq n+1} |b_i|_p = 1$. Then

$$(a_1 + \mathcal{P}_K)(b_1 + \mathcal{P}_K) + \cdots + (a_{n+1} + \mathcal{P}_K)(b_{n+1} + \mathcal{P}_K) = 0$$

in $\widetilde{K}$, and not all of the $(b_k + \mathcal{P}_K)$ are zero. Thus $[\widetilde{K} : \mathbb{F}_p] \leq n$.

□

In general, if $F$ is an extension of $\mathbb{Q}_p$ with residue class field $\widetilde{F}$, and $K$ is a finite extension of $F$, then a similar proof to the above shows that $[\widetilde{K} : \widetilde{F}]$ is finite, and does not exceed $[K : F]$. The index $[\widetilde{K} : \widetilde{F}]$ is called the residue degree of the extension $\frac{K}{F}$, and is denoted $f\left(\frac{K}{F}\right)$.

Another important index for finite extensions of $\mathbb{Q}_p$ is the ramification index. For a field $K$ with an absolute value $|\cdot|$, $|K^*| = \{|a| : a \in K,\ a \neq 0\}$ is called the value group of $K$. The value group of a field $K$ is always a multiplicative subgroup of the positive real numbers. If $K$ is a finite extension of a valued field $F$, and $|\cdot|$ is an absolute value on $K$ which extends that of $F$, then $|K^*|$ contains $|F^*|$. Hence $|F^*|$ is a subgroup of $|K^*|$. The index $[|K^*| : |F^*|]$ is called the ramification index of the extension $\frac{K}{F}$, denoted $e\left(\frac{K}{F}\right)$.

Note that the value group $|\mathbb{Q}_p{}^*|_p = \{p^i : i \in \mathbb{Z}\} = \langle |p|_p \rangle$ is infinite cyclic. A field with an infinite cyclic value group is said to have a discrete valuation. An element $\pi$ in a field $K$ for which $\langle |\pi| \rangle = |K^*|$ is called a uniformizer. If the value group of a field is dense in the positive reals, then that field is said to have a dense valuation.

The next result relates the residue degree and the ramification index to the index $[K : F]$, where $K \supset F$ are finite extensions of $\mathbb{Q}_p$. The following important fact is needed:

PROPOSITION 1.10. *If $K$ is a finite extension of $\mathbb{Q}_p$, then $K$ is complete with respect to the topology defined by $|\cdot|_p$.*

PROOF: Let $\{a_i\}_{i=1}^{\infty} \subset K$ be a Cauchy sequence with respect to $|\cdot|_p$. Write $a_i = \alpha_{i,1}b_1 + \cdots + \alpha_{i,\ell}b_\ell$ where $\alpha_{n,i} \in \mathbb{Q}_p$, for $i = 1, \ldots, \ell$ and $b_1, \ldots, b_\ell$ form a basis of $K$ over $\mathbb{Q}_p$. Then for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n,\ m \geq N$ implies $|a_n - a_m|_p < \varepsilon$. Let $\|\cdot\|$ be the maximum norm with respect to the basis $\{b_1, \ldots, b_n\}$. By Theorem 1.6, $\|\cdot\|$ is equivalent to $|\cdot|_p$, so there exists a $c > 0$ such that $c\|\cdot\| \leq |\cdot|_p$. Thus if $n,\ m \geq N$, then $|(\alpha_{n,1}b_1 + \cdots + \alpha_{n,\ell}b_\ell) - (\alpha_{m,1}b_1 + \cdots + \alpha_{m,1}b_\ell)|_p < \varepsilon \implies |(\alpha_{n,1} - \alpha_{m,1})b_1 + \cdots + (\alpha_{n,\ell} - \alpha_{m,\ell})b_\ell|_p < \varepsilon \implies \max_{1\leq j\leq \ell} |\alpha_{n,j} - \alpha_{m,j}| < \frac{\varepsilon}{c}$. Thus each sequence $\{\alpha_{i,j}\}_{i=1}^{\infty}$ in $\mathbb{Q}_p$ is Cauchy with respect to $|\cdot|_p$, for each $j = 1, \ldots, \ell$. Since $\mathbb{Q}_p$ is complete, there exists $\alpha_j \in \mathbb{Q}_p$ such that $\alpha_j = \lim_{i\to\infty} \alpha_{i,j}$, for each $j = 1, \ldots, \ell$. It is clear that the element $a = \alpha_1 b_1 + \cdots + \alpha_\ell b_\ell \in K$ is the limit of the Cauchy sequence $\{a_i\}_{i=1}^{\infty}$ with respect to $|\cdot|_p$. Hence $K$ is complete.

□

Proposition 1.11. *Let $K$ be a finite extension of a complete field $F$ with a discrete valuation. Then* $e\left(\frac{K}{F}\right) \cdot f\left(\frac{K}{F}\right) = [K : F]$.

Proof: [3, Proposition 9.1 and Proposition 9.3].

□

A finite separable extension $K$ over a field $F$ with a discrete valuation is called unramified if $[K : F] = f\left(\frac{K}{F}\right)$, and totally ramified if $[K : F] = e\left(\frac{K}{F}\right)$. Furthermore, $K$ is called tamely ramified over $F$ if $e\left(\frac{K}{F}\right)$ is not divisible by $p$, and wildly ramified over $F$ if $e\left(\frac{K}{F}\right)$ is a power of $p$.

Primitive elements of ramified extensions always satisfy Eisenstein polynomials. A polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x]$ is called an Eisenstein polynomial if $a_i \in \mathcal{P}_F$, for each $i = 0, \ldots, n-1$, and $a_0 \notin \mathcal{P}_F^2$. Eisenstein polynomials are always irreducible [9, 3-3-1]. An example of a ramified extension over $\mathbb{Q}_p$ is $\mathbb{Q}_p(\zeta_n)$, where $\zeta_n$ is a primitive $p^n$-th root of unity. $\zeta_1$ is a root of the polynomial $x^p - 1 \in \mathbb{Q}_p[x]$, which has the factors $x - 1$ and $x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}_p[x]$. Changing the variable $x$ to $y + 1$ transforms the latter polynomial into $y^{p-1} + \binom{p}{p-1}y^{p-2} + \cdots + \binom{p}{1}y + p \in \mathbb{Q}_p[x]$, which is Eisenstein. Note that $\zeta_1 - 1$ is a root of the polynomial, and so $|\zeta_1 - 1|_p = |p|_p^{\frac{1}{p-1}}$ in $\mathbb{Q}_p(\zeta_1)$, and this extension is tamely ramified over $\mathbb{Q}_p$, with uniformizer $(\zeta_1 - 1)$.

Furthermore, if $n \geq 2$, then $\zeta_n$ has the minimal polynomial $x^p - \zeta_{n-1} \in \mathbb{Q}_p(\zeta_{n-1})[x]$, where $\zeta_n^p = \zeta_{n-1}$. A similar change of variable $x$ to $y + 1$ yields $(y+1)^p - \zeta_{n-1} = y^p + \binom{p}{p-1}y^{(p-1)} + \cdots + \binom{p}{1}y + (1 - \zeta_{n-1})$. By induction, since

$1-\zeta_{n-1}$ is a uniformizer of $\mathbb{Q}_p(\zeta_{n-1})$, this polynomial is Eisenstein, and the root $\zeta_n - 1$ is a uniformizer of $\mathbb{Q}_p(\zeta_n)$. (Note that, by induction, $|\zeta_n - 1|_p = |p|_p^{\frac{1}{p^{n-1}(p-1)}}$.)

The next theorem is a well-known result concerning finite extensions of discretely valued fields.

THEOREM 1.12. *Let $K$ be a finite separable extension of a field $F$. Then there exists a maximal unramified extension $W$ of $F$ contained in $K$, called the inertia field, such that $\frac{K}{W}$ is totally ramified. Furthermore, there exists a maximal tamely ramified extension $T$ of $F$ contained in $K$, called the ramification field, such that $\frac{T}{W}$ is totally and tamely ramified, and $\frac{K}{T}$ is totally and wildly ramified.*

PROOF: [9, Theorems 3.2.10 and 3.4.7].

□

**D. Complete extensions of $\mathbb{Q}_p$.**

It has been shown that any finite extension of $\mathbb{Q}_p$ is complete with respect to the extended "$p$-adic" absolute value. However, infinite algebraic extensions of $\mathbb{Q}_p$ are not complete in general, the best example being the algebraic closure of $\mathbb{Q}_p$.

PROPOSITION 1.13. *$\widehat{\mathbb{Q}_p}$ is not complete with respect to $|\cdot|_p$.*

EXAMPLE: [7, Theorem 16.6].

□

Since $\widehat{\mathbb{Q}_p}$ is a metric space under the topology induced by $|\cdot|_p$, it has a metric completion $\Omega_p$. The $p$-adic absolute value $|\cdot|_p$ on $\widehat{\mathbb{Q}_p}$ extends to $\Omega_p$ in the

$n \to \infty$

be $\lim_{n \to \infty} |\alpha_n|_p$. Any subfield of $\Omega_p$ containing $\mathbb{Q}_p$ is called a $p$-adic field, and is understood to possess the absolute value $|\cdot|_p$.

The main reason $\Omega_p$ is useful in mathematics is that it is both complete and algebraically closed, and so becomes the number theorist's $p$-adic analog for the complex numbers.

THEOREM 1.14. *$\Omega_p$ is algebraically closed.*

PROOF: [7, Theorem 17.1].

□

As with algebraic extensions of $\mathbb{Q}_p$, any $p$-adic field $k$ has a valuation ring $\mathcal{O}_k = \{\alpha \in k : |\alpha|_p \le 1\}$ with maximal ideal $\mathcal{P}_k = \{\alpha \in k : |\alpha|_p < 1\}$. Thus $k$ has a residue class field $\tilde{k} = \frac{\mathcal{O}_k}{\mathcal{P}_k}$, and as before, $\tilde{k}$ contains a copy of $\mathbb{F}_p$. Hence char $\tilde{k} = p$. Also, $k$ has a value group $|k^*|$ which is a subgroup of the positive real numbers.

PROPOSITION 1.15. *Suppose that $k$ is a subfield of $\Omega_p$ which is the completion of an algebraic extension $K$ over $\mathbb{Q}_p$. Then (i) $|k^*| = |K^*|$, and (ii) $\tilde{k} \cong \widetilde{K}$.*

PROOF: (i) Let $\alpha \in k$ be non-zero. Let $\{a_n\}_{n=1}^{\infty} \subset K$ be such that $\lim_{n \to \infty} a_n = \alpha$. Then for all large enough $n$, $|a_n - \alpha|_1 < \max\{|a_n|_p, |\alpha|_p\}$. Thus for all such $n$, $|a_n|_p = |\alpha|_p$, by Proposition 1.5.

coset $\beta + \mathcal{P}_k \in \tilde{k}$ can be respresented as $b + \mathcal{P}_k$, for some $b \in \mathcal{O}_K$. Hence the inclusion $b + \mathcal{P}_K \mapsto b + \mathcal{P}_k$ is the required isomorphism of $\widetilde{K}$ to $\tilde{k}$.

□

COROLLARY 1.15.1. *$|\Omega_p^*| = |\widehat{\mathbb{Q}_p}^*| = \{p^q : q \in \mathbb{Q}\}$ and $\widetilde{\Omega_p} = \widetilde{\widehat{\mathbb{Q}_p}}$ = the algebraic closure of $\mathbb{F}_p$.*

PROOF: [7, Theorem 16.2].

□

In fact, it can be shown that any complete subfield of $\Omega_p$ is the completion of an algebraic extension of $\mathbb{Q}_p$.

THEOREM 1.16. *Suppose that $K$ is an algebraic extension of $\mathbb{Q}_p$. Let $H$ be a subgroup of the galois group $Gal\left(\frac{\widehat{\mathbb{Q}_p}}{K}\right)$. Let $\overline{H}$ be the subgroup of $Gal\left(\frac{\Omega_p}{K}\right)$ obtained by extending the elements of $H$ by continuity to $\Omega_p$. Then the subfield of $\Omega_p$ fixed by all elements of $\overline{H}$ is the completion of $K$ in $\Omega_p$.*

PROOF: [8, Theorem 4].

□

COROLLARY 1.16.1. *Any complete subfield of $\Omega_p$ is the completion of an algebraic extension of $\mathbb{Q}_p$.*

PROOF: Let $k$ be a complete subfield of $\Omega_p$. Set $K = k \cap \widehat{\mathbb{Q}_p}$. Since $\frac{k}{K}$ and $\frac{\widehat{\mathbb{Q}_p}}{K}$ are linearly disjoint extensions, the galois group $\mathrm{Gal}\left(\frac{k\widehat{\mathbb{Q}_p}}{k}\right)$ is isomorphic to $\mathrm{Gal}\left(\frac{\widehat{\mathbb{Q}_p}}{K}\right)$

by the map which takes $\sigma \mapsto \sigma|_{\widehat{\mathbb{Q}_p}}$, for all $\sigma \in \mathrm{Gal}\left(\frac{k\widehat{\mathbb{Q}_p}}{k}\right)$. To see this, note that if $\{\beta_\lambda : \lambda \in \Lambda\}$ is a basis for $\widehat{\mathbb{Q}_p}$ over $K$, then since $k$, $\widehat{\mathbb{Q}_p}$ are linearly disjoint over $K$, $\{\beta_\lambda : \lambda \in \Lambda\}$ is also a basis for $k\widehat{\mathbb{Q}_p}$ over $k$. Thus for all $\sigma, \theta \in \mathrm{Gal}\left(\frac{k\widehat{\mathbb{Q}_p}}{k}\right)$, $\sigma = \theta \iff \sigma(\beta_\lambda) = \theta(\beta_\lambda)$, $\forall \lambda \in \Lambda \iff \sigma|_{\widehat{\mathbb{Q}_p}}(\beta_\lambda) = \theta|_{\widehat{\mathbb{Q}_p}}(\beta_\lambda), \forall \lambda \in \Lambda \Leftrightarrow \sigma|_{\widehat{\mathbb{Q}_p}} = \theta|_{\widehat{\mathbb{Q}_p}}$. Furthermore, any $\tau \in \mathrm{Gal}\left(\frac{\widehat{\mathbb{Q}_p}}{K}\right)$ can be lifted to an element $\tau' \in \mathrm{Gal}\left(\frac{k\widehat{\mathbb{Q}_p}}{k}\right)$ by defininING $\tau'$ to be linear on $k$. This lift is the inverse of the above restriction.

Now, let $\tau \in \mathrm{Gal}\left(\frac{\widehat{\mathbb{Q}_p}}{K}\right)$. Lift $\tau$ to $\tau' \in \mathrm{Gal}\left(\frac{k\widehat{\mathbb{Q}_p}}{k}\right)$ by letting $\tau'$ be linear on $k$. Extend $\tau'$ by continuity to an element $\overline{\tau'} \in \mathrm{Gal}\left(\frac{\Omega_p}{k}\right)$. Then $\overline{\tau'}$ is a continuous extension of $\tau \in \mathrm{Gal}\left(\frac{\widehat{\mathbb{Q}_p}}{K}\right)$ to $\mathrm{Gal}\left(\frac{\Omega_p}{K}\right)$. Since $\widehat{\mathbb{Q}_p}$ is dense in $\Omega_p$, this extension is unique. Thus $k$ must always lie in the field fixed by the subgroup of $\mathrm{Gal}\left(\frac{\Omega_p}{K}\right)$ obtained by extending the elements of $\mathrm{Gal}\left(\frac{\widehat{\mathbb{Q}_p}}{K}\right)$ by continuity to $\Omega_p$. Hence by Theorem 1.16, $k \subset \overline{K}$ (the closure of $K$ in $\Omega_p$), and so since $K \subset k$ and $k$ is closed, $k = \overline{K}$.

□

**E. $\ell_1$-Group algebras.**

Suppose that $R$ is an algebra over a valued field $K$, and suppose that a non-Archimedean $K$-norm $\|\cdot\|_R$ is defined on $R$. Suppose further that $R$ is complete with respect to the norm $\|\cdot\|_R$. Then the following holds:

LEMMA 1.17.1. *If $\{r_n\}_{n=1}^{\infty}$ is a sequence in $R$ such that $\lim_{n\to\infty} r_n = 0$, then $\sum_{n=1}^{\infty} r_n \in R$.*

PROOF: The sequence $\sum_{n=1}^{N} r_n$ is Cauchy with respect to $\|\cdot\|_R$, since $\|\sum_{n=1}^{N} r_n - \sum_{n=1}^{M} r_n\|_R = \|\sum_{n=M+1}^{N} r_n\| \leq \max_{M+1\leq n\leq N} \|r_n\|_R$, and $\lim_{n\to\infty} \|r_n\|_R = 0$.

□

Now, let $G$ be an arbitrary group. For any complete non-Archimedean $K$-algebra $R$, the $\ell_1$-group algebra of $G$ over $R$ is the ring of formal sums

$$\ell_1(R,G) = \Big\{ \sum_{g\in G} \alpha_g g : \alpha_g \in R,\ \forall g \in G, \quad \text{and} \quad \|\alpha_g\|_R \to 0 \Big\},$$

where $\|\alpha_g\|_R \to 0$ means that for any real number $\varepsilon > 0$, only finitely many $\alpha$ satisfy $\|\alpha_g\|_R \geq \varepsilon$.

For all $\sum_{g\in G} \alpha_g g$, let $\operatorname{supp}\Big(\sum_{g\in G} \alpha_g g\Big) = \{g \in G : \alpha_g \neq 0\}$ be the support of $\sum_{g\in G} \alpha_g g$. Then it is clear that the condition $\|\alpha_g\|_R \to 0$ implies that the support of any $\sum_{g\in G} \alpha_g g \in \ell_1(R,G)$ is countable.

The ring operations of $\ell_1(R,G)$ are analogous to those of the ordinary group algebra. The only difference lies in the fact that an element has countable, but not necessarily finite, support. Addition is given by

$$\sum_{g\in G} \alpha_g g + \sum_{g\in G} \beta_g g = \sum_{g\in G} (\alpha_g + \beta_g) g,$$

for all $\sum_{g\in G} \alpha_g g,\ \sum_{g\in G} \beta_g g \in \ell_1(R,G)$. Multiplication is defined by

$$\Big(\sum_{g\in G} \alpha_g g\Big)\Big(\sum_{h\in G} \beta_h h\Big) = \sum_{g\in G}\ \sum_{h\in G} \alpha_g \beta_h g h$$

$$= \sum_{x\in G} \Big(\sum_{g\in G} \alpha_g \beta_{g^{-1}x}\Big) x,$$

for all $\sum_{g\in G} \alpha_g g$, $\sum_{h\in G} \beta_h h \in \ell_1(R,G)$. It is easy to see that these definitions make sense. If $\sum_{g\in G} \alpha_g g$, $\sum_{g\in G} \beta_g g \in \ell_1(R,G)$, then the conditions $\|\alpha_g\|_R \to 0$, $\|\beta_g\|_R \to 0$ imply that $\|\alpha_g + \beta_g\|_R \leq \max\{\|\alpha_g\|_R, \|\beta_g\|_R\} \to 0$, so $\sum_{g\in G}(\alpha_g + \beta_g)g \in \ell_1(R,G)$. It follows that $\ell_1(R,G)$ is an abelian group under addition. If $\sum_{g\in G} \alpha_g g$, $\sum_{h\in G} \beta_h h \in \ell_1(R,G)$, then the conditions $\|\alpha_g\|_R \to 0$, $\|\beta_h\|_R \to 0$ imply that $\|\alpha_g\beta_{g^{-1}x}\| \to 0$ for all $g$, $x \in G$. Since $\operatorname{supp}\left(\sum_{g\in G} \alpha_g g\right)$, $\operatorname{supp}\left(\sum_{h\in G} \beta_h h\right)$ are countable, Lemma 1.17.1 implies that $\sum_{g\in G} \alpha_g \beta_{g^{-1}x} \in R$ for all $x \in G$. Furthermore, $\|\sum_{g\in G} \alpha_g\beta_{g^{-1}x}\|_R \leq \sup_{g\in G} \|\alpha_g\beta_{g^{-1}x}\|_R \leq \sup_{g\in G} \|\alpha_g\|_R \cdot \|\beta_{g^{-1}x}\|_R \to 0$, so multiplication is well-defined binary operation on $\ell_1(R,G)$ as well. It is clear that $\sum_{g\in G} 0g = 0$ is the zero element of $\ell_1(R,G)$ and if $1_G$ is the identity of the group $G$, then $1_K \cdot 1_G + 0$ is the multiplicative identity of $\ell_1(R,G)$. Finally, if $\sum_{g\in G} \alpha_g g$, $\sum_{g\in G} \beta_g g$, $\sum_{h\in G} \gamma_h h \in \ell_1(R,G)$, then

$$
\begin{aligned}
&\Big(\sum_{g\in G} \alpha_g g + \sum_{g\in G} \beta_g g\Big) \cdot \Big(\sum_{h\in G} \gamma_h h\Big) \\
&\qquad = \sum_{g\in G} \sum_{h\in G} (\alpha_g + \beta_g)\gamma_h gh \\
&\qquad = \sum_{g\in G} \sum_{h\in G} \alpha_g \gamma_h gh + \sum_{g\in G} \sum_{h\in G} \beta_g \gamma_h gh \\
&\qquad = \Big(\sum_{g\in G} \alpha_g g\Big)\Big(\sum_{h\in G} \gamma_h h\Big) + \Big(\sum_{g\in G} \beta_g g\Big)\Big(\sum_{h\in G} \gamma_h h\Big),
\end{aligned}
$$

so multiplication is right (and similarly left) distributive over addition. Thus the following has been shown.

PROPOSITION 1.17. *Let $R$ be a complete $K$-algebra with respect to a non-Archimedean $K$-norm $\|\cdot\|_R$. Let $G$ be any group. Then $\ell_1(R,G)$ is a ring.*

Define the supremum norm on the formal sums of $\ell_1(R,G)$ by $\|\sum_{g\in kG}\alpha_g g\| = \sup_{g\in G}\|\alpha_g\|_R$. It is evident that $\|\sum_{g\in G}\alpha_g g\| = 0 \iff \sum_{g\in G}\alpha_g g = 0$, and that $\|\sum_{g\in G}\alpha_g g + \sum_{g\in G}\beta_g g\| = \sup_{g\in G}\|\alpha_g+\beta_g\|_R \le \sup_{g\in G}\{\max\{\|\alpha_g\|_R, \|\beta_g\|_R\}\} = \max\{\|\sum_{g\in G}\alpha_g g\|, \|\sum_{g\in G}\beta_g g\|\}$. for all $\sum_{g\in G}\alpha_g g$, $\sum_{g\in G}\beta_g g \in \ell_1(R,G)$. Also, if $a \in K$, then since $\|\cdot\|_R$ is a $K$-algebra norm,

$$\|a\cdot\sum_{g\in G}\alpha_g g\| = \|\sum_{g\in G}a\alpha_g g\| = \sup_{g\in G}\|a\alpha_g\|_R = |a|_K\cdot\sup_{g\in G}\|\alpha_g\|_R$$
$$= |a|_K\cdot\|\Sigma\alpha_g g\|,\ \forall\sum_{g\in G}\alpha_g g \in \ell_1(R,G).$$

Thus $\|\cdot\|$ is a $K$-algebra norm on $\ell_1(R,G)$. Furthermore,

$$\|\Big(\sum_{g\in G}\alpha_g g\Big)\cdot\Big(\sum_{h\in G}\beta_h h\Big)\| = \|\sum_{x\in G}\Big(\sum_{g\in G}\alpha_g\beta_{g^{-1}x}\Big)x\|$$
$$= \sup_{x\in G}\|\sum_{g\in G}\alpha_g\beta_{g^{-1}x}\|_R \le \sup_{x\in G}\big(\sup_{g\in G}\|\alpha_g\beta_{g^{-1}x}\|_R\big)$$
$$\le \sup_{x\in G}\big(\sup_{g\in G}\|\alpha_g\|\cdot\sup_{g^{-1}x\in G}\|\beta_{g^{-1}x}\|\big)$$
$$= \|\sum_{g\in G}\alpha_g g\|\cdot\|\sum_{h\in G}\beta_h h\|,$$

for all $\sum_{g\in G}\alpha_g g$, $\sum_{h\in G}\beta_h h \in \ell_1(R,G)$. It follows that multiplication is continuous on $\ell_1(R,G)$ in the supremum norm. The following proposition is clear:

PROPOSITION 1.18. *$\ell_1(R,G)$ is a non-Archimedean $K$-algebra with $K$-norm $\|\cdot\|$.*

□

The following proposition justifies the use of the supremum norm, in the sense that $\ell_1(R,G)$ inherits the completeness of $R$. Furthermore, it shows that the oridnary group ring $RG = \left\{ \sum_{g\in G} \alpha_g g : \alpha_g \in R, \forall g \in G, \text{ and almost all } \alpha_g = 0 \right\}$ is dense in $\ell_1(R,G)$ with respect to the supremum norm.

PROPOSITION 1.19. *$\ell_1(R,G)$ is complete with respect to $\|\cdot\|$. Furthermore, the group ring $RG$ is dense in $\ell_1(R,G)$.*

PROOF: Let $\left\{f_n\right\}_{n=1}^{\infty} \subset \ell_1(R,G)$ be a Cauchy sequence with respect to $\|\cdot\|$. Let $f_n = \sum_{g\in G} \alpha_{n,g} g$, for each $n$. Then for all real numbers $\varepsilon > 0$, there is an $N$ such that for all $m,\ n \geq N$, $\|\sum_{g\in G} \alpha_{n,g} g - \sum_{g\in G} \alpha_{m,g} g\| < \varepsilon$. This implies that $\sup_{g\in G} \|\alpha_{n,g} - \alpha_{m,g}\|_R < \varepsilon$, for all $n,\ m \geq N$. Thus each sequence $\left\{\alpha_{n,g}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $R$, for all $g \in G$, and so there exists an $\alpha_g \in R$ such that $\lim_{n\to\infty} \alpha_{n,g} = \alpha_g$. Furthermore, for all $\varepsilon > 0$, there is an $N$ such that for all $n \geq N$, $\|\alpha_{n,g} - \alpha_g\|_R < \varepsilon$, for all $g \in G$, by the uniformity of the family of sequences $\left\{\alpha_{n,g}\right\}_{n=1}^{\infty}$. Thus for all $n \geq N$, $\sup_{g\in G} \|\alpha_{n,g} - \alpha_g\|_R < \varepsilon$, and so it can be concluded that $f = \sum_{g\in G} \alpha_g g$ satisfies $\|\alpha_g\|_R \to 0$ and that $f$ is the limit of the Cauchy sequence $\left\{f_n\right\}_{n=1}^{\infty}$. Therefore $\ell_1(R,G)$ is complete with respect to $\|\cdot\|$.

To prove the second statement, let $f = \sum_{g\in G} \alpha_g g \in \ell_1(R,G)$. For any $\varepsilon > 0$, let $f(\varepsilon) = \sum_{g\in G} \beta_g g$ where $\beta_g = \alpha_g$ when $\|\alpha_g\|_R \geq \varepsilon$, and $\beta_g = 0$ otherwise. Then only finitely many $\beta_g$ are non-zero, so $f(\varepsilon) \in RG$. It is clear that $\|f - f(\varepsilon)\| < \varepsilon$, and so it follows that the group ring $RG$ is dense in $\ell_1(R,G)$.

□

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# CHAPTER II

# THE NILPOTENT AND JACOBSON RADICALS OF $\ell_1(k, G)$

The purpose of this chapter is to collect general results pertaining to the primeness, semiprimeness, or semiprimitivity of $p$-adic $\ell_1$-group algebras.

Recall that a ring $R$ is

i) prime if the product of any two non-zero ideals of $R$ is also non-zero,

ii) semiprime if $R$ contains no non-zero ideal $A$ such that $A^2$ is zero, and

iii) semiprimitive if the intersection of maximal left or right ideals of $R$ is zero.

## A. Prime or semiprime conditions.

Let $k$ be a complete $p$-adic field, and let $G$ be a group. In this section, it is shown that $\ell_1(k, G)$ is prime if and only if $G$ has no non-trivial finite normal subgroups, and that $\ell_1(k, G)$ is semiprime if $G$ contains no finite normal subgroups of order a power of $p$. The proofs of these results use the fact that primeness or semiprimeness of $\ell_1(k, G)$ is related to that of the group ring $\tilde{k}G$, where $\tilde{k}$ is the residue class field of $k$.

Consider

$$\ell_1(\mathcal{O}_k, G) = \Big\{ \sum_{g\in G} \alpha_g\ g \in \ell_1(k,G) : |\alpha_g| \le 1, \quad \forall g \in G \Big\}.$$

Since the supremum norm that $\ell_1(\mathcal{O}_k, G)$ inherits as a subspace of $\ell_1(k,G)$ is non-Archimedean, it is obvious that $\ell_1(\mathcal{O}_k, G)$ is a subring of $\ell_1(k,G)$. If $I$ is an ideal of $\ell_1(k,G)$, then $I' = I \cap \ell_1(\mathcal{O}_k, G)$ is an ideal of $\ell_1(\mathcal{O}_k, G)$, since if $a,\ b \in I'$, then $\|a+b\| \le \max(\|a\|,\ \|b\|) \le 1$, and if $r \in \ell_1(\mathcal{O}_k, G)$, $\|ar\| \le \|a\| \cdot \|r\| \le 1$. Also, it is easy to see that if $I$ is a left ideal, then so is $I'$. Conversely, given an ideal $I'$ of $\ell_1(\mathcal{O}_k, G)$, there is a corresponding ideal $kI'$ of $\ell_1(k,G)$. If $\alpha a \in kI'$ and $f \in \ell_1(k,G)$, then by choosing $\beta \in k$ with $|\beta|^{-1} \le \|f\|$, then $\alpha a f = \alpha\beta a(\beta^{-1} f) \in kI'$. If $I'$ has elements of absolute value 1, then $kI' \cap \ell_1(\mathcal{O}_k, G)$ must be $I'$, evidently.

Another ideal of $\ell_1(\mathcal{O}_k, G)$ is $\ell_1(\mathcal{P}_k, G) = \Big\{ \sum_{g\in G} \alpha_g g \in \ell_1(k,G) :\ |\alpha_g| < 1,\ \forall g \in G \Big\}$. In an analogous manner to which $\frac{\mathcal{O}_k}{\mathcal{P}_k}$ is the residue class field $\tilde{k}$, $\frac{\ell_1(\mathcal{O}_k,G)}{\ell_1(\mathcal{P}_k,G)}$ is the group ring $\tilde{k}G$. To see this, note that $\ell_1(\mathcal{P}_k, G)$ is the set of elements of $\ell_1(k,G)$ whose coefficients belong to $\mathcal{P}_k$. Thus for distinct group elements $g,\ h \in G,\ g-h \notin \ell_1(\mathcal{P}_k, G)$. Hence the quotient homomorphism $\theta : \ell_1(\mathcal{O}_k, G) \longrightarrow \frac{\ell_1(\mathcal{O}_k,G)}{\ell_1(\mathcal{P}_k,G)}$ acts as an isomorphism on $G$. Also, $\theta|_{\mathcal{O}_k}$ is the natural projection of $\mathcal{O}_k$ onto $\tilde{k}$. Thus for any $f = \sum_{g\in G} \alpha_g\ g \in \ell_1(\mathcal{O}_k, G)$, since $f = \sum_{i=1}^{n} \alpha_{g_i} g_i + f'$, with $|\alpha_{g_i}| = 1$ for all $i = 1, \ldots, n$, and $f' \in \ell_1(\mathcal{P}_k, G)$, $\theta(f) = \sum_{i=1}^{n} \theta(\alpha_{g_i}) g_i$, where $\theta(\alpha_{g_i})$ is a non-zero element of $\tilde{k}$. In fact, it is clear that any finite sum $\sum_{i=1}^{n} a_i g_i \in \tilde{k}G$ is the image of some $\sum_{i=1}^{n} \alpha_{g_i} g_i + f' = f \in \ell_1(\mathcal{O}_k, G)$, just by arranging $\theta(\alpha_{g_i}) = a_i$, for each $i = 1, \ldots, n$. Thus $\frac{\ell(\mathcal{O}_k,G)}{\ell_1(\mathcal{P}_k,G)} \cong \tilde{k}G$.

The above isomorphism is useful since non-zero ideals $I$ of $\ell_1(k,G)$ must always have elements of absolute value 1, and hence these $\theta(I \cap \ell_1(\mathcal{O}_k, G))$ are exactly the non-zero ideals of $\tilde{k}G$. Hence primeness and semiprimeness of $\tilde{k}G$ is nicely related to that of $\ell_1(k,G)$.

THEOREM 2.1. *([4]) Let $k$ be a complete p-adic field, and let $G$ be a group. The following are equivalent:*

*a) $\ell_1(\mathcal{O}_k, G)$ is prime;*

*b) $\ell_1(k,G)$ is prime;*

*c) $G$ has no finite normal subgroups other than $\langle 1 \rangle$.*

PROOF: (a) $\Rightarrow$ (b): Suppose $I, J$ are non-zero ideals of $\ell_1(k,G)$ such that $IJ = 0$. Then $I \cap \ell_1(\mathcal{O}_k, G) = I'$ and $J \cap \ell_1(\mathcal{O}_k, G) = J'$ are non-zero ideals of $\ell_1(O_k, G)$ such that $I'J' = 0$. Thus $\ell_1(\mathcal{O}_k, G)$ is not prime if $\ell_1(k,G)$ is not prime. (a) $\Rightarrow$ (b) follows from the contrapositive.

(b) $\Rightarrow$ (c): Let $H \neq \langle 1 \rangle$ be a finite normal subgroup of $G$. Let $T$ be a transversal of $H$ in $G$ containing 1, and let $I$ be the ideal of $\ell_1(k,G)$ generated by $\{h - 1 : h \in H\}$. If $f \in$I, then $f$ can be written as

$$\sum_{t \in T} \Big( \sum_{h \in H} \alpha_{ht} h \Big) t \qquad \text{with} \qquad \sum_{h \in H} \alpha_{ht} h = \sum_{h \in H} \beta_{ht}(h-1),$$

for some $\beta_{ht} \in k$, $\forall h \in H$, $\forall t \in T$. Thus $\sum_{h\in H} \alpha_{ht} = 0$, for all $t \in T$. Also, let $J$ denote the principal ideal generated by $\sum_{x\in H} x$.

Then, if

$$f = \sum_{t\in T}\Big(\sum_{h\in H}\alpha_{ht}h\Big)t \in I, \quad \text{and} \quad s = r\Big(\sum_{x\in H} x\Big) \in J, \quad \text{for some } r \in \ell_1(k,G),$$

$$s \cdot f = r\Big(\sum_{x\in H} x\Big)\Big(\sum_{t\in T}\Big(\sum_{h\in H}\alpha_{ht}h\Big)t\Big) = r \cdot \sum_{x\in H}\Big(\sum_{t\in T}\Big(\sum_{h\in H}\alpha_{ht}xh)t\Big)$$

$$= r \cdot \sum_{t\in T}\Big(\sum_{x\in H}\sum_{h\in H}\alpha_{ht}xh\Big)t \qquad \text{(by continuity and since } |\alpha_{ht}| \to 0)$$

$$= r \cdot \sum_{t\in T}\Big(\sum_{y\in H}\Big(\sum_{h\in H}\alpha_{ht}\Big)y\Big)t = 0, \quad \text{since } f \in I.$$

Therefore $JI = 0$, and so $\ell_1(k,G)$ is not prime. Again, the contrapositive of this result is (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): Let $G$ be a group that has no non-trivial finite normal subgroups. Suppose $I, J$ are ideals of $\ell_1(\mathcal{O}_k, G)$ such that $IJ = 0$.

If $I, J$ are non-zero, then there are non-zero elements $a \in I$, $b \in J$. Let $\lambda \in k$ be such that $|\lambda| = \min\{\|a\|, \|b\|\}$. Let $I' = \{f \in \ell_1(\mathcal{O}_k, G) : \lambda f \in I\}$ and $J' = \{f \in \ell_1(\mathcal{O}_k, G) : \lambda f \in J\}$. Then it is clear that $I'$ and $J'$ are non-zero ideals of $\ell_1(\mathcal{O}_k, G)$, and for all $r \in I'$, $s \in J'$, $rs = \lambda^{-1}(\lambda r)(\lambda s)\lambda^{-1} = \lambda^{-1}(0)\lambda^{-1} = 0$. Thus $I'J' = 0$. Furthermore, $I'$ and $J'$ must contain elements of absolute value 1 (some $k$-multiples of $a$ and $b$). Therefore, $\theta(I')$, $\theta(J')$ are non-zero ideals of $\tilde{k}G$ such that $\theta(I')\theta(J') = 0$.

However, $\tilde{k}G$ is prime if $G$ has no non-trivial finite normal subgroups [5, Theorem 4.2.10], so this is a contradiction. Thus, $I = J = 0$, and so $\ell_1(\mathcal{O}_k, G)$ is prime. □

THEOREM 2.2. *If $\tilde{k}G$ is semiprime, then $\ell_1(k, G)$ is semiprime.*

PROOF: Suppose $\ell_1(k, G)$ is not semiprime. Then if $I$ is a non-zero ideal of $\ell_1(k, G)$ such that $I^2 = 0$, $I' = I \cap \ell_1(\mathcal{O}_k, G)$ is a non-zero ideal of $\ell_1(\mathcal{O}_k, G)$ such that $(I')^2 = 0$. $I'$ must have elements of norm 1, thus $\theta(I')$ is a non-zero ideal of $\tilde{k}G$. But $(I')^2 = 0 \Rightarrow (\theta(I'))^2 = 0$, so $\tilde{k}G$ is not semiprime. This proves the contrapositive of the theorem.

□

COROLLARY 2.2.1. *If $G$ has no finite normal subgroup of order divisible by $p$, then $\ell_1(k, G)$ is semiprime.*

PROOF: The above condition is equivalent to the group ring $\tilde{k}G$ being semiprime, when $\tilde{k}$ is a field of characteristic $p$ [5, Theorem 4.2.13].

**B. The Jacobson radical of complete non-Archimedean normed rings.**

For any ring $R$, the Jacobson radical of $R$, denoted $J(R)$, is the intersection of all maximal right ideals of $R$. Thus $R$ is semiprimitive if and only if $J(R) = 0$.

If $\theta : R \to S$ is a surjective ring homomorphism, then it should be noted that $\theta(J(R)) \subseteq J(S)$, since maximal ideals of $S$ are the images of maximal ideals of $R$.

The main results of this section characterize the Jacobson radical for a $p$-adic $\ell_1$-group algebra. A few of these results extend to the more general class of complete non-Archimedean rings. Whenever possible, an effort is made to prove the main results in this broader class of rings.

The first result of this section allows one to conclude that the Jacobson radical of $\ell_1(k, G)$ is always a closed ideal.

PROPOSITION 2.3. *Let $R$ be a complete non-Archimedean normed ring. Then the maximal (right or left) ideals of $R$ are closed.*

PROOF: Suppose $M$ is a maximal right ideal of $R$. If $M$ is not closed, then $1 \in \overline{M}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of elements of $M$ which converges to 1. Then for any $f_n$ with $\|1 - f_n\| < 1$, $f_n = 1 - (1 - f_n)$ is a unit, with inverse $\sum_{i=0}^{\infty}(1 - f_n)^i \in R$. (Since $R$ is complete, $\sum_{i=0}^{\infty}(1 - f_n)^i \in R$.) This is impossible since $M$ is a proper ideal of $R$. Thus $\overline{M} = M$, so $M$ is closed.

□

COROLLARY 2.3.1. *If $R$ is as above, then $J(R)$ is closed. In particular, $J(\ell_1(k, G))$ is a closed ideal of $\ell_1(k, G)$.* □

PROOF: $J(R)$ is the intersection of all maximal (right) ideals of $R$, all of which are closed. So $J(R)$ is closed.

□

The next theorem concerning dense subrings of complete non-Archimedean rings has important corollaries in the $\ell_1(k, G)$ situation. An element $r$ of a ring $R$ with unity is called left (or right) quasi-regular if $(1-r)$ has a left (or right) inverse in $R$. The Jacobson radical of $R$ can be characterized as the set of all elements $r$ of $R$ for which $sr$ is left (or right) quasi-regular for all $s \in R$ [3, Theorem 4.1].

THEOREM 2.5. *Let $R$ be a complete, non-Archimedean normed ring. If $S$ is a dense subring of $R$, then $J(S) \subseteq J(R)$.*

PROOF: Let $w \in J(S)$ such that $w \neq 0$.

Consider the map $\theta : S \to S$ given by $\theta(s) = (1-sw)^{-1}$. This is well-defined since $1 - sw$ is a unit for all $s \in S$.

$\theta$ is also continuous. To see this, let $s_1,\ s_2 \in S$, and consider

$$1-(s_1+s_2)w = (1-s_1w)\left[1-(1-s_1w)^{-1}s_2w\right] = (1-s_1w)\left[1-\theta(s_1)s_2w\right].$$

Taking inverses, one obtains

$$\theta(s_1+s_2) = \left[1-\theta(s_1)s_2w\right]^{-1} \cdot \theta(s_1).$$

So if $\|s_2\|_R < \dfrac{1}{\|\theta(s_1)w\|_R}$, then $(1-\theta(s_1)s_2w)^{-1} = \sum_{i=0}^{\infty}(\theta(s_1)s_2w)^i$, so $\|(1-\theta(s_1)s_2w)^{-1}\|_R = 1$, and so $\|\theta(s_1+s_2)\|_R \leq \|\theta(s_1)\|_R$, $\forall s_2 \in S$ such that $\|s_2\|_R < \dfrac{1}{\|\theta(s_1)w\|_R}$.

Therefore,

$$\theta(s_1+s_2)-\theta(s_1) = \theta(s_1+s_2)\big[(1-s_1w)-(1-(s_1+s_2)w)\big]\theta(s_1)$$
$$= \theta(s_1+s_2)s_2w\theta(s_1)$$

and, in particular, for all $s_2$ with $\|s_2\|_R$ sufficiently small,

$$\|\theta(s_1+s_2)-\theta(s_1)\|_R \leq \|\theta(s_1)\|_R^2 \cdot \|w\| \cdot \|s_2\|_R.$$

This shows that $\theta$ is continuous.

Since $S$ is dense in $R$, $\theta$ has a continuous extension $\bar{\theta} : R \to R$. Let $r \in R$, and let $s_n$ be a sequence in $S$ converging to $r$. For each $n$, $1 = (1-s_nw)\theta(s_n) = (1-s_nw)\bar{\theta}(s_n)$. Letting $n \to \infty$, one obtains $1 = (1-rw)\bar{\theta}(r)$, for any $r \in R$. This shows that $rw$ is right quasi-regular, for all $r \in R$, so $w \in J(R)$.

□

This theorem has an important corollary which answers affirmatively an open question of Hare and Shirvani [1].

Corollary 2.5.1. *If $k$ is a complete $p$-adic field contained in $\Omega_p$, then $J(\ell_1(k,G)) \subseteq J(\ell_1(\Omega_p, G))$.*

Proof: Let $K$ be the algebraic closure of $k$ in $\Omega_p$. By a theorem due to Amitsur [4, Theorem 7.2.13], $J(K \otimes_k \ell_1(k,G)) = K \otimes_k J(\ell_1(k,G))$. But $K \otimes_k \ell_1(k,G)$ is a dense subring of $\ell_1(\Omega_p, G)$, since $K$ is dense in $\Omega_p$. Thus by Theorem 2.5, $J(\ell_1(k,G)) \subseteq J(K \otimes_k \ell_1(k,G)) \subseteq J(\ell_1(\Omega_p, G))$.

□

For any complete $p$-adic field $k$ contained in $\Omega_p$, denote by $\mathcal{X}_k$ the class of all groups $G$ for which $J(\ell_1(k,G)) = 0$. Then Corollary 2.5.1 implies

Corollary 2.5.2. *If $k$ is complete $p$-adic field, then $\mathcal{X}_k \supseteq \mathcal{X}_{\Omega_p}$.*

The last result of this section shows that it suffices to consider the Jacobson radical of $\ell_1(k,G)$ only for countable groups $G$.

Proposition 2.4. *([1]) $J(\ell_1(k,G)) \subseteq \bigcup_{H \in C} J(\ell_1(k,H))$, where $C$ is the set of all countable subgroups of $G$.*

Proof: Supppose $f \in J(\ell_1(k,G))$. Since supp$(f)$ is countable, $H = \langle \text{supp}(f) \rangle$ is a countable subgroup of $G$. For any $s \in \ell_1(k,H)$, there exists a $u \in \ell_1(k,G)$ such that $u(1 - sf) = 1$. Let $T$ be a left transversal of $H$ in $G$ containing 1, and write $u = \sum_{t \in T} t u_t$, where $u_t \in \ell_1(k, H)$, for all $t \in T$. Then $1 = \sum_{t \in T} t u_t(1 - sf) \Rightarrow 1 = u_1(1 - sf)$. Thus $sf$ is left quasi-regular in $\ell_1(k, H)$, $\forall s \in \ell_1(k, H)$.

Therefore $f \in J(\ell_1(k, H))$, for some countable subgroup $H$ of $G$.

□

**C. Groups for which $\ell_1(\Omega_p, G)$ is semiprimitive.**

In this section, certain classes of groups are shown to be in $\mathcal{X}_{\Omega_p}$. Note that by Corollary 2.5.2, a group class that is contained in $\mathcal{X}_{\Omega_p}$ must also be in $\mathcal{X}_k$, for any complete $p$-adic field $k$. It is shown that if $G$ is a residually finite group, a unique product group, or a torsion-free solvable group, then $J(\ell_1(\Omega_p, G)) = 0$. The results of this section are due to Hare and Shirvani [1].

Let $G$ be a group. Let $\{N_\lambda : \lambda \in \Lambda\}$ be a family of normal subgroups of $G$, indexed by some set $\Lambda$. Then $\{N_\lambda : \lambda \in \Lambda\}$ is a directed system of normal subgroups of $G$ if $\bigcap_{\lambda \in \Lambda} N_\lambda = \langle 1 \rangle$ and $N_{\lambda_1} \cap N_{\lambda_2} \supseteq N_{\lambda_3}$ for some $\lambda_3 \in \Lambda$, for any two distinct $\lambda_1$, $\lambda_2 \in \Lambda$. A group $G$ is residually finite if the set of normal subgroups of finite index in $G$ forms such a directed system.

THEOREM 2.6. *Let $G$ be a group. Let $\{N_\lambda : \lambda \in \Lambda\}$ be a directed system of normal subgroups of $G$ such that $G/N_\lambda \in \mathcal{X}_{\Omega_p}$, for all $\lambda \in \Lambda$. Then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: Suppose $f = \sum_{g \in G} \alpha_g \, g \in \ell_1(\Omega_p, G)$ is non-zero. Write $f = \sum_{i=1}^{n} \alpha_{g_i} g_i + f'$, where $|\alpha_{g_i}| = \|f\|$, for all $i = 1, \ldots, n$, and $\|f'\| < \|f\|$. Consider $\{g_i g_j^{-1} : 1 \leq i \neq j \leq n\}$. Since $\{N_\lambda : \lambda \in \Lambda\}$ is a directed system of normal subgroups of $G$, there is a $\lambda_0 \in \Lambda$ such that $g_i g_j^{-1} \notin N_{\lambda_0}$ for all $1 \leq i \neq j \leq n$.

Let $\pi : G \longrightarrow \dfrac{G}{N_{\lambda_0}}$ be the canonical group homomorphism. Then $\pi$ extends $k$-linearly to a homomorphism of group rings $\bar{\pi} : kG \twoheadrightarrow k \, \frac{G}{N_{\lambda_0}}$, and since this resulting homomorphism is continuous in the supremum norm it extends by continuity to a ring homomorphism $\hat{\pi} : \ell_1(k, G) \twoheadrightarrow \ell_1\left(k, \frac{G}{N_{\lambda_0}}\right)$. Since $\hat{\pi}\left(\sum_{g \in G} \alpha_g \, g\right) = \Sigma \alpha_g \pi(g)$, this homomorphism has the property that $\|\hat{\pi}(r)\| \leq \|r\|$, for all $r \in \ell_1(k, G)$.

Now, $\hat{\pi}(f) = \sum_{i=1}^{n} \alpha_{g_i} \pi(g_i) + \hat{\pi}(f')$. Since $g_i g_j^{-1} \notin N_{\lambda_0}$, for all $1 \leq i \neq j \leq n$, $\pi(g_i) \neq \pi(g_j)$ where $i \neq j$. Thus the elements $\pi(g_i)$ are distinct in $\frac{G}{N_{\lambda_0}}$ for $i = 1, \ldots, n$. Also, $\|\hat{\pi}(f')\| \leq \|f'\| < \|f\|$, so $\|\hat{\pi}(f)\| = \|f\|$. Hence $\hat{\pi}(f) \neq 0$, and so since $\ell_1\left(\Omega_p, \frac{G}{N_{\lambda_0}}\right)$ is semiprimitive, $\hat{\pi}(f) \notin J\left(\ell_1(\Omega_p, \frac{G}{N_{\lambda_0}})\right)$. Therefore, $f \notin J(\ell_1(\Omega_p, G))$, and so $J(\ell_1(\Omega_p, G)) = 0$.

□

COROLLARY 2.6.1. *If $G$ is residually finite, then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: If $N$ is a normal subgroup of finite index in $G$, then $\ell_1\left(\Omega_p, \frac{G}{N}\right)$ is just the group ring $\Omega_p G/N$, which is semiprimitive by Maschke's Theorem [5, Theorem 2.4.2]. Since the set of such $N$ forms a directed system of normal subgroups of $G$, Theorem 2.6 implies that $J(\ell_1(\Omega_p, G)) = 0$.

□

The next result can be used to place some classes of groups in $\mathcal{X}_{\Omega_p}$.

THEOREM 2.7. *If the group ring* $\widetilde{\Omega_p}G$ *is semiprimitive and has no zero divisors then* $J(\ell_1(\Omega_p, G)) = 0$.

PROOF: Suppose $s$ is a non-zero element of $J(\ell_1(k, G))$. Without loss of generality, we may suppose that $\|s\| = 1$. Let $f \in \ell_1(k, G)$ with $\|f\| = 1$.

Consider $\|1 - sf\|$. If $\|1 - sf\| < 1$, then $sf = 1 - (1 - sf)$ would be invertible, with inverse $\sum_{i=0}^{\infty}(1 - sf)^i$. This is impossible since $sf \in J(\ell_1(k, G))$. Therefore $\|1 - sf\| = 1$.

Since $sf$ is quasi-regular, there is a $r \in \ell_1(k, G)$ such that $r(1 - sf) = 1$. Note that $1 = \|r(1 - sf)\| \leq \|r\| \cdot \|1 - sf\| = \|r\|$.

Next, let $\theta : \ell_1(\mathcal{O}_{\Omega_p}, G) \longrightarrow \widetilde{\Omega_p}G$ be the quotient homomorphism with kernel $\ell_1(\mathcal{P}_{\Omega_p}, G)$. If $\|r\| > 1$, choose $\alpha \in \Omega_p$ with $|\alpha^{-1}| = \|r\|$. Then $(\alpha r)(1 - sf) = \alpha$, and so $\theta(\alpha r) \cdot \theta(1 - sf) = \theta(\alpha) = 0$. Since $1 = \|\alpha r\| = \|1 - sf\|$, and $\widetilde{\Omega_p}G$ has no zero divisors, this is impossible. Therefore $\|r\| = 1$. But then $1 = \theta(r(1 - sf)) =$

$\theta(r)(1-\theta(s)\theta(f))$, for all $\theta(f) \in \widetilde{\Omega_p}G$. Thus $\theta(s) \in J(\widetilde{\Omega_p}G) = 0$. This contradicts the assumption that $\|s\| = 1$. Therefore $J(\ell_1(\Omega_p, G)) = 0$.

□

There are some important classes of groups which satisfy the conditions of Theorem 2.5. Let $G$ be a group. $G$ is a unique product (u.p.) group if for any two non-empty finite subsets $A$, $B$ of $G$, there is at least one element $x \in G$ that can be uniquely expressed as $x = ab$, for some $a \in A$, $b \in B$. $G$ is called a two unique products (t.u.p.) group if for any two non-empty finite subsets $A$, $B$ of $G$ satisfying $\text{card}(A) + \text{card}(B) > 2$, then there are at least two elements $x$, $y \in G$ that can be uniquely expressed as $x = ab$, $y = cd$, for some $a$, $c \in A$, $b$, $d \in B$. Surprisingly, Strojnowski [6] has shown that these two definitions are equivalent. On the other hand, it is known that for t.u.p. groups, $\widetilde{\Omega_p}G$ is semisimple and has no zero divisors [5, Chapter 13]. An important subclass of the class of unique product groups is the class of ordered groups. A group $G$ is said to be ordered if there is a partial order relation "$<$" on $G$ such that for all $x$, $y$, $z \in G$, $x < y \Rightarrow xz < yz$ and $zx < zy$. For example, any additive subgroup of the real numbers is ordered using the usual "$<$" relation.

The next corollary is thus obvious.

COROLLARY 2.7.1. *(i) If $G$ is a unique product group, then $J(\ell_1(\Omega_p, G)) = 0$.*

*(ii) If $G$ is an ordered group, then $J(\ell_1(\Omega_p, G)) = 0$.*

A group $G$ is torsion-free group if $G$ has no elements of finite order except the unit element.

COROLLARY 2.7.2. *If $G$ is a torsion-free solvable group, then $J(\ell_1(\Omega_p, G)) = 0$.*

PROOF: For such groups $G$, it has been shown that $\widetilde{\Omega_p}G$ is semiprimitive [5, Theorem 7.4.6], and recently that $\widetilde{\Omega_p}G$ has no zero divisors [2, Theorem 1.4]. Thus Theorem 2.7 applies.

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# CHAPTER III

# SEMIPRIMITIVITY OF COMMUTATIVE $p$-ADIC $\ell_1$-GROUP ALGEBRAS

A $p$-adic $\ell_1$-group algebra $\ell_1(k, G)$ is commutative only when the group $G$ is abelian. The purpose of this chapter is to find all those abelian groups which are in the class $\mathcal{X}_{\Omega_p}$. It is found that the abelian groups contained in $\mathcal{X}_{\Omega_p}$ are exactly those without $C_{p^\infty}$-subgroups, where $C_{p^\infty}$ is the group isomorphic to the multiplicative subgroup of the complex numbers consisting of all $p^{\text{th}}$-power roots of 1. Furthermore, necessary and sufficient conditions are found for a complete $p$-adic field $k$ to have $C_{p^\infty} \in \mathcal{X}_k$. However, it remains an open problem whether or not the direct sum of more than one copy of $C_{p^\infty}$ lies in $\mathcal{X}_k$, for any $p$-adic field $k$. Thus the classification of abelian groups in $\mathcal{X}_k$ remains unfinished for certain complete $p$-adic fields $k$, namely those for which $C_{p^\infty} \in \mathcal{X}_k$.

## A. Continuous $k$-algebra homomorphisms of $\ell_1(k, G)$ into $\Omega_p$.

Let $G$ be any group, not necessarily abelian, and let $k$ be a complete $p$-adic field. Let $\widehat{G_k}$ denote the set of all continuous $k$-algebra homomorphisms from $\ell_1(k, G)$ into $\Omega_p$. The first proposition of this section is a simple characterization of the elements of $\widehat{G_k}$.

**PROPOSITION 3.1.** *Let $\phi : \ell_1(k,G) \to \Omega_p$ be a $k$-algebra homomorphism. Then $\phi$ is continuous $\Leftrightarrow |\phi(f)| \leq \|f\|, \forall f \in \ell_1(k,G)$.*

PROOF: ($\Leftarrow$): Let $\varepsilon > 0$. If $\|r - s\| < \varepsilon$, then $|\phi(r) - \phi(s)| = |\phi(r - s)| < \varepsilon$ under the assumption. Thus $\phi$ is continuous.

($\Rightarrow$): Suppose $\phi$ is continuous. Suppose there is a $g \in G$ such that $|\phi(g)| \neq 1$. Without loss of generality, assume $|\phi(g)| > 1$. Then for all $\alpha \in k$ such that $0 < \alpha < 1$, $|\phi(g)| < |\alpha^{-n}|$, for some $n \in \mathbb{N}$. Consider $f = \sum_{i=0}^{\infty} \alpha^{ni} g^i \in \ell_1(k,G)$. If $f_N = \sum_{i=0}^{N} \alpha^{ni} g^i$, then $\|f_N - f\| \to 0$ while $|\phi(f_N) - \phi(f)| = |\alpha^{nN} \phi(g)^N| > 1$, $\forall N \in \mathbb{N}$. This contradicts the continuity of $\phi$.

□

A $p$-adic character of an abelian group $G$ is a homomorphism $\phi : G \to \Omega_p$ such that $|\phi(g)| = 1$, $\forall g \in G$. It is easy to see that the set of characters is a ring with the operations $(\phi + \theta)(g) = \phi(g)\theta(g)$, $(\phi \cdot \theta)(g) = \phi(\theta(g))$, $\forall g \in G$, and for all characters $\phi$, $\theta$. It follows from Proposition 3.1 that the $k$-linear extension of a $p$-adic character $\phi$ to $\ell_1(k,G)$ given by $\phi(\sum_{g\in G} \alpha_g g) = \sum_{g \in G} \alpha_g \phi(g)$, $\forall \sum_{g \in G} \alpha_g g \in \ell_1(k,G)$ is a continuous $k$-algebra homomorphism of $\ell_1(k,G)$ into $\Omega_p$. Conversely, the restriction of any element $\phi \in \widehat{G_k}$ to $G$ is a $p$-adic character of $G$. Thus when $G$ is abelian it is convenient to call the elements of $\widehat{G_k}$ "characters".

**B. Semiprimitivity of $\ell_1(\Omega_p, G)$ for abelian $p'$-groups.**

A $p'$-group is a group without elements of order a power of $p$. The main result of this section shows that when $G$ is an abelian $p'$-group, then $\ell_1(k, G)$ is semiprimitive for all complete $p$-adic fields $k$. The proof uses the following important result of $p$-adic character theory.

THEOREM 3.2. *Let $G$ be an abelian group. Let $\phi \in \widehat{G_k}$ such that $\phi(g)$ is a root of unity in $\Omega_p$, for all $g \in G$. Then $\phi(\ell_1(k, G))$ is a complete subfield of $\Omega_p$.*

PROOF: Since $\ell_1(k, G)$ is complete and $\phi$ is continuous, $\phi(\ell_1(k, G))$ is a complete subring of $\Omega_p$.

Let $\sum_{g \in G} \alpha_g g$ be an element of the group ring $kG$. Then $\phi(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g \phi(g)$ is an element of the algebraic field extension $K$ of $k$ obtained by adjoining $\{\phi(g) : g \in G\}$ to $k$. Conversely, any element of $K$ is a finite sum $\sum_{g \in G} \alpha_g \phi(g)$, and so is the image under $\phi$ of an element of $kG$. Thus $\phi(kG) = K$. Since $kG$ is dense $\ell_1(k, G)$, $K$ is dense in $\phi(\ell_1(k, G))$. Hence $\phi(\ell_1(k, G)) = \overline{K}$.

Let $\alpha \in \overline{K}$ be non-zero. Then $\alpha$ is the limit of a Cauchy sequence of non-zero elements $\alpha_n \in K$. Consider the sequence $\{\alpha_n^{-1}\}$ in $K$. For any $\varepsilon > 0$, and for all sufficiently large integers $m, n$,

$$\begin{aligned}|\alpha_m^{-1} - \alpha_n^{-1}| &= |\alpha_m^{-1}| \cdot |\alpha_n^{-1}| \cdot |\alpha_n - \alpha_m| \\ &= |\alpha_m|^{-1} \cdot |\alpha_n|^{-1} \cdot \varepsilon \\ &= |\alpha|^2 \cdot \varepsilon.\end{aligned}$$

Thus $\{\alpha_n^{-1}\}$ is a Cauchy sequence in $K$. Let $\beta \in \overline{K}$ such that $\beta = \lim_{n\to\infty} \alpha_n^{-1}$. Since multiplication is continuous on $\overline{K} \subset \Omega_p$, $1 = \lim_{n\to\infty} \alpha_n \cdot \alpha_n^{-1} = (\lim_{n\to\infty} \alpha_n) \cdot (\lim_{n\to\infty} \alpha_n^{-1}) = \alpha\beta$, so $\beta = \alpha^{-1}$. Hence the closure of $K$ is a field.

□

COROLLARY 3.2.1. *If $G$ is an abelian group and $\phi \in \widehat{G_k}$ such that $\phi(g)$ is a root of unity in $\Omega_p$, for all $g \in G$, then $\ker\phi$ is a maximal ideal of $\ell_1(k, G)$.*

□

The proof of the main result of this section requires the following lemma.

LEMMA 3.3.1. *Let $G$ be an abelian group, and let $H \triangleleft G$. Then any element $\theta$ of $\widehat{H_k}$ extends to an element $\varphi$ of $\widehat{G_k}$. Furthermore, if $\theta(h)$ is a root of unity, for all $h \in H$, then $\varphi$ can be chosen so that $\varphi(g)$ is a root of unity, for all $g \in G$.*

PROOF: Let $\theta \in \hat{H}_k$.

Let $S$ be the set of all pairs $(A, \varphi)$ such that $H \lhd A \lhd G$, $\varphi \in \hat{A}_k$, and $\varphi$ extends $\theta$ to $A$. Define a partial order $\leq$ on $S$ by defining

$$(A_1, \varphi_1) \leq (A_2, \varphi_2) \Leftrightarrow A_1 \subseteq A_2 \quad \text{and } \varphi_2 \text{ extends}$$

$$\varphi_1 \quad \text{to} \quad A_2.$$

Since $(H, \theta) \in S$, $S$ is not empty. Suppose $\{(A_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ is a chain in $S$, for some index set $\Lambda$. Let $A = \bigcup_{\lambda \in \Lambda} A_\lambda$, and $\forall a \in A$, define $\varphi(a) = \varphi_\lambda(a)$, for all $\lambda \in \Lambda$ such that $a \in A_\lambda$. Then it is easy to see that $(A, \varphi) \in S$, and so $\leq$ defines an inductive partial order on $S$. Hence by Zorn's Lemma, there exists a maximal element $(A, \varphi) \in S$ with respect to the partial order $\leq$.

Suppose $A \neq G$. Let $g \in G$ such that $g \notin A$. If $g$ has infinite order modulo $A$, then $A' = \langle A, g \rangle = A \times \langle g \rangle$. Extend $\varphi$ to a homomorphism $\varphi' : A' \to \Omega_p$ by defining $\varphi'(g) = 1$. If $g$ has finite order modulo $A$, let $n$ be the least positive integer such that $g^n \in A$. Since $\Omega_p$ is algebraically closed, there exists an $\alpha \in \Omega_p$ such that $\alpha^n = \varphi(g^n)$. Define $\varphi' : A' \to \Omega_p$ by letting $\varphi'(hg^i) = \varphi(h)\alpha^i$. It is easy to see that $\varphi'$ is a continuous $k$-linear homomorphism extending $\varphi$ to $A'$. In either case, the maximality of the element $(A, \varphi)$ with respect to $\leq$ is contradicted. Therefore, $A = G$. $\square$

the set of all pairs $(A, \varphi)$ such that $H \lhd A \lhd G$, $\varphi \in \hat{A}_k$ extends $\theta$ on $h$, and $\varphi(a)$ is a root of unity, for all $a \in A$. Zorn's Lemma again applies to this set $S$, and the same proof as above shows that a maximal element of $S$ must be of the form $(G, \varphi)$, $\exists \varphi \in \widehat{G_k}$.

THEOREM 3.3. *[4] Let $k$ be a complete $p$-adic field, and let $G$ be an abelian $p'$-group. If $f \in kG$, then there exists a $\phi \in \widehat{G_k}$ such that $|\phi(f)| = \|f\|$, and $\phi(g)$ is a root of unity, $\forall g \in G$.*

PROOF: Let $f = \sum_{i=1}^{n} \alpha_i g_i$, with $\alpha_i \in k$, $g_i \in G$, for $i = 1, \ldots, n$. Assume without loss of generality that $\|f\| = 1$. The proof proceeds by induction on $n$.

If $n = 1$, then let $\phi(g) = 1$, for $g \in G$. Then $|\phi(f)| = |\phi(\alpha_1 g_1)| = |\alpha_1| = 1 = \|\alpha_1 g_1\|$.

Assume the statement is true for $n-1$ but false for $n$. Then suppose $f = \sum_{i=1}^{n} \alpha_i g_i$ is such that $|\phi(f)| < 1$, $\forall \phi \in \widehat{G_k}$. Suppose $|\alpha_n| = 1$. Since $g_1^{-1} g_n \neq 1$, there is a $p'$-root of unity $\lambda \in \Omega_p$ of the same order as $g_1^{-1} g_n$. (If the order of $g_1^{-1} g_n$ is infinite, then let $\lambda$ by any $p'$-root of unity in $\Omega_p$).

Define a continuous $\Omega_p$-algebra homomorphic $\phi : \ell_1(\Omega_p, \langle g_1^{-1} g_n \rangle) \longrightarrow \Omega_p$ by $\phi(g_1^{-1} g_n) = \lambda$. Extend this to an element $\hat{\phi} \in \widehat{G_k}$ using Lemma 3.3.1. Then

By the inductive hypothesis, there exists a $\mu \in \widehat{\overline{G_k}}$ such that $|\sum_{i=1}^{n} \alpha_i(\phi(g_1) - \phi(g_i))\mu(g_i)| = 1$. The homomorphism $\phi + \mu$ (defined by $(\phi + \mu)\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} \alpha_g \phi(g)\mu(g)\right)$ is also continuous, and so $|\sum_{i=1}^{n} \alpha_i \phi(g_i)\mu(g_i)| < 1$, by the assumption. Also, $|\sum_{i=1}^{n} \alpha_i \phi(g_1)\mu(g_i)| = |\phi(g_1)| \cdot |\Sigma \alpha_i \mu(g_i)| < 1$. Therefore, $|\sum_{i=1}^{n} \alpha_i(\phi(g_1) - \phi(g_i))\mu(g_i)| < 1$, a contradiction. This proves the theorem.

□

COROLLARY 3.3.2. *If $G$ is an abelian $p'$-group, then $\ell_1(k, G)$ is semiprimitive, for all complete $p$-adic fields $k$.*

PROOF: Let $f$ be a non-zero element of $\ell_1(k, G)$. Let $\{f_n\}$ be a Cauchy sequence of elements of $kG$ which converges to $f$. Suppose that $n$ is large enough so that $\|f - f_n\| < \|f\|$. Then $\|f_n\| = \|f\|$. By Theorem 3.3, there exists a $\phi \in \widehat{G_k}$ such that $|\phi(f_n)| = \|f_n\|$, and $\phi(g)$ is a root of unity, $\forall g \in G$.

Now, $|\phi(f)| = |\phi(f_n) + \phi(f - f_n)| = |\phi(f_n)|$ by Proposition 1.5, since $|\phi(f - f_n)| \leq \|f - f_n\| < \|f\| \leq \|f_n\| = |\phi(f_n)|$. Thus $\phi(f)$ is non-zero in $\phi(\ell_1(k, G))$. By Theorem 3.2, $J(\phi(\ell_1(k, G)) = 0$, so $f \notin J(\ell_1(k, G))$, and the result follows. □

Let $G$ be an abelian group, all of whose elements are torsion. Then for any $p$-adic character $\phi$ of $G$, and for any $g \in G$, there is an integer $n$ such that $1 = \phi(g^n) = (\phi(g))^n$. Hence $\phi(g)$ is a root of unity in $\Omega_p$, for all $g \in G$. Theorem 3.2 thus implies that $\ker \phi$ is a maximal ideal of $\ell_1(k, G)$. In fact, for such a group $G$, all maximal ideals have this form.

LEMMA 3.4.1. *[2] Let $k$ be a complete $p$-adic field. Let $K$ be an algebraic field extension of $k$. Suppose $\|\cdot\|$ is a $k$-norm on $K$, as a vector space over $k$. Then for all $\alpha \in K$, $\|\alpha\| \geq |\alpha|_p$, where $|\cdot|_p$ is the $p$-adic absolute value on $K$.*

PROOF: Suppose that $K$ is a finite extension of $k$. Then by Theorem 1.6, $\|\cdot\|$ is equivalent to $|\cdot|_p$, so there is a real number $c > 0$ such that $c|\alpha|_p \leq \|\alpha\|$, for all $\alpha \in K$. Suppose there is a $\beta \in K$ such that $\|\beta\| < |\beta|_p$. Let $d < 1$ be a real number such that $\|\beta\| = d|\beta|_p$. Then for all positive integers $n$, $\|\beta^n\| \leq \|\beta\|^n = d^n|\beta^n|_p$. For any integer $n$ such that $d^n < c$, this is a contradiction.

For a general algebraic extension $K$ of $k$, any $\alpha \in K$ generates a finite extension $k(\alpha)$ of $k$, and so the above proof shows that $\|\alpha\| \geq |\alpha|_p$, as required.

□

THEOREM 3.4. *Let $G$ be an abelian group such that all elements of $G$ have finite order. Let $k$ be a complete $p$-adic field. Then any maximal ideal of $\ell_1(k,G)$ is the kernel of a $p$-adic character in $\widehat{G_k}$.*

PROOF: Let $M$ be a maximal ideal of $\ell_1(k,G)$. Let $\varphi_M : \ell_1(k,G) \longrightarrow \frac{\ell_1(k,G)}{M}$ be the canonical homomorphism. $\varphi_M$ is continuous when $\frac{\ell_1(k,G)}{M}$ is bestowed with the quotient topology $\|\varphi_M(f)\|_q = \inf_{e\in M} \|f+e\|$.

Let $\alpha \in k$. Consider $\|\varphi_M(\alpha)\|_q$. If $\|\varphi_M(\alpha)\|_q < |\alpha|_p$, then there exists a $\sum_{g\in G} \beta_g g \in M$ such that $\|(\alpha+\beta_1) + \sum_{g\neq 1} \beta_g g\| < |\alpha|$ and so $|\beta_1|_p = |\alpha|_p > |\beta_g|$, for all $g \neq 1$. Thus $\beta_1^{-1}\Big(\sum_{g\in G} \beta_g g\Big) = 1 + \sum_{g\neq 1} \beta_1^{-1}\beta_g g$ satisfies $\|\sum_{g\neq 1} \beta_1^{-1}\beta_g g\| < 1$, and hence is a unit in $\ell_1(k,G)$. This contradicts the assumption that $\sum_{g\in G} \beta_g g \in M$. Therefore, $\|\varphi_M(\alpha)\| = |\alpha|_p$, for all $\alpha \in k$, so $\varphi_M(k)$ is an isometric copy of $k$ in $\frac{\ell_1(k,G)}{M}$.

Since every $g \in G$ has finite order, $\varphi_M(g)$ is a root of unity in the field $\frac{\ell_1(k,G)}{M}$, for all $g \in G$. Thus the subfield of $\frac{\ell_1(k,G)}{M}$ obtained by adjoining the roots of unity in $\varphi_M(G)$ to $k$ is algebraic over $k$. Let $\mu$ be a $k$-linear field isomorphism which identifies $k(\varphi(G))$ with a subfield $K$ of $\Omega_p$. Note that this isomorphism induces a $p$-adic absolute value on $k(\varphi(G))$ by $|\alpha| = |\mu(\alpha)|_p$, for all $\alpha \in k(\varphi_M(G))$. By the previous lemma, $|\alpha| \leq \|\alpha\|$, $\forall \alpha \in k(\varphi(G))$. Thus the isomorphism $\mu : k(\varphi_M(G)) \rightarrow K$ is continuous. Since $k(\varphi_M(G))$ is dense in $\frac{\ell_1(k,G)}{M}$ (with respect to the quotient topology), $\mu$ extends by continuity to an isomorphism $\bar{\mu}$ of $\frac{\ell_1(k,G)}{M}$ with a subfield of $\overline{K}$.

Now, the map $\bar{\mu} \circ \varphi_M|_G$ is a homomorphism of $G$ into the roots of unity of $\overline{K}$, and hence is a $p$-adic character of $G$. Since $\varphi_M$ is continuous and $\varphi_M(k) \cong k$, $\bar{\mu} \circ \varphi_M\left(\sum_{g\in G} \alpha_g g\right) = \bar{\mu}\left(\sum_{g\in G} \alpha_g(\phi_M(g)\right)$ for all $\sum_{g\in G} \alpha_g g$, so $\ker(\bar{\mu} \circ \varphi_M) = \ker \varphi_M = M$.

□

The next corollary is obvious.

Corollary 3.4.2. *Let $k$ be a complete $p$-adic field. If $G$ is an abelian torsion group, then $f \in J(\ell_1(k,G)) \Leftrightarrow \phi(f) = 0$, for all $f \in \widehat{G_k}$.*

Corollary 3.4.3. *Let $G$ be an abelian torsion group. Then for all maximal ideals $M$ of $\ell_1(\Omega_p, G)$, $\frac{\ell_1(\Omega_p,G)}{M}$ is isometrically isomorphic to $\Omega_p$.*

Proof: Let $f \in \ell_1(\Omega_p, G)$. Write $f = \sum_{g\in G} \alpha_g g$. Let $\varphi_M$ be the $p$-adic character in $\widehat{G_k}$ such that $M = \ker \varphi_M$. Then $\varphi_M(f) = \sum_{g\in G} \alpha_g \varphi_M(g) \in \Omega_p$. Let $\alpha = \sum_{g\in G} \alpha_g \varphi_M(g)$. Then $\varphi_M(f-\alpha) = \varphi_M(f) - \varphi_M(\alpha) = \alpha - \alpha = 0$, so $f - \alpha \in M$.

This shows that for all $f + M \in \frac{\ell_1(\Omega_p,G)}{M}$, $f + M = \alpha + M$, $\exists \alpha \in \Omega_p$. So since $\frac{\ell_1(\Omega_p,G)}{M}$ contains an isometric copy of $\Omega_p$, $\frac{\ell_1(\Omega_p,G)}{M}$ must be isometrically isomorphic to $\Omega_p$.

**D. Abelian groups with no $C_{p^\infty}$-subgroups.**

The problem of determining those groups which are of the class $\mathcal{X}_k$ for any complete $p$-adic field $k$ has already been reduced via Corollary 2.5.2 and Theorem 2.6 to finding the countable group members of $\mathcal{X}_{\Omega_p}$. For abelian groups, the next proposition further reduces the problem to the case of $p$-groups.

PROPOSITION 3.5. *[4] Suppose that $H$ is a subgroup of an abelian group $G$. If $H \in \mathcal{X}_{\Omega_p}$ and $\frac{G}{H} \in \mathcal{X}_{\Omega_p}$, then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: Suppose $f = \sum_{g \in G} \alpha_g g \in J(\ell_1(\Omega_p, G))$, and let $T$ be a transversal of $H$ in $G$.

Let $\phi \in \widehat{H_{\Omega_p}}$, and let $\theta \in \widehat{G_{\Omega_p}}$ be the extension of $\phi$ given by Lemma 3.3.1. For any maximal ideal $M$ of $\ell_1\left(\Omega_p, \frac{G}{H}\right)$, let $\mu_M$ be the natural projection of $\ell_1\left(\Omega_p, \frac{G}{H}\right)$ onto $\frac{\ell_1\left(\Omega_p, \frac{G}{H}\right)}{M}$. As in the proof of Theorem 3.4, $\alpha \mapsto \mu_M(\alpha)$ is an isometric isomorphism of $\Omega_p$ onto $\mu_M(\Omega_p)$, so $\mu_M$ may be taken to be an $\Omega_p$-linear map.

Let $g \mapsto \bar{g}$ be the canonical homomorphism of $G$ onto $\frac{G}{H}$. Define $\psi_M : \ell_1(\Omega_p, G) \longrightarrow \frac{\ell_1\left(\Omega_p, \frac{G}{H}\right)}{M}$ to be the extension of the map $g \mapsto \theta(g)\mu_M(\bar{g})$ by linearity and continuity to the whole space. For any $\sum_{t \in T} \beta_t \bar{\mu}_M(\bar{t}) \in \frac{\ell_1\left(\Omega_p, \frac{G}{H}\right)}{M}$, it is clear that $\sum_{t \in T} \beta_t t \in \ell_1(\Omega_p, G)$ and $\psi_M\left(\sum_{t \in T} \beta_t t\right) = \sum_{t \in T} \beta_t \mu_M(\bar{t})$, so $\psi_M$ is certainly onto. So since $f \in J(\ell_1(\Omega_p, G))$ and $\frac{\ell_1\left(\Omega_p, \frac{G}{H}\right)}{M}$ is a field, $\psi_M(f)$ must be 0.

Therefore, $0 = \psi_M(f) = \sum_{t \in T}\left(\sum_{x \in H} \alpha_{xt}\phi(x)\right)\mu_M(\bar{t})$, for all maximal ideals $M$ of $\ell_1\left(\Omega_p, \frac{G}{H}\right)$. Since $\ell_1\left(\Omega_p, \frac{G}{H}\right)$ is semiprimitive, $\sum_{x \in H} \alpha_{xt}\phi(x) = 0$, for all $t \in T$. Since $\phi$ was arbitrary, $\sum_{\alpha \in H} \alpha_{xt} x \in J(\ell_1(\Omega_p, H))$ by Corollary 3.4.2. So since $H \in \mathcal{X}_{\Omega_p}$, then $\alpha_{xt} = 0$, for all $x \in H$, and for all $t \in T$. Thus $f = 0$, and so $G \in \mathcal{X}_{\Omega_p}$.

□

COROLLARY 3.5.1. *If $G$ is an abelian group with maximal $p$-subgroup $P$, then $G \in \mathcal{X}_{\Omega_p}$ if $P \in \mathcal{X}_{\Omega_p}$.*

PROOF: $\frac{G}{P}$ is an abelian $p'$-group, so by Corollary 3.3.2, $\frac{G}{P} \in \mathcal{X}_{\Omega_p}$. Proposition 3.5 implies the result.

□

For an abelian $p$-group $G$, let $G^{p^m} = \{g^{p^m} : g \in G\}$. Then $G^{p^m}$ is a subgroup of $G$, and if $G^{p^m} = \langle 1 \rangle$, then $G$ is said to have finite exponent $p^m$. If a countable abelian $p$-group $G$ satisfies $\bigcap_{m=1}^{\infty} G^{p^m} = \langle 1 \rangle$, then $G$ is said to have no elements of infinite height, i.e. no non-trivial elements $g$ such that the equations $g = x^{p^n}$ have solutions in $G$ for all integers $n \in \mathbb{N}$. The Ulm sequence [3] of an infinite abelian $p$-group is defined transfinitely by $G^{(1)} = \bigcap_{n=1}^{\infty} G^{p^m}$, $G^{(\sigma+1)} = \left(G^{(\sigma)}\right)^{(1)}$ if $\sigma$ is not a limit ordinal, and $G^{(\sigma)} = \bigcap_{\lambda<\sigma} G^{(\lambda)}$ if $\sigma$ is a limit ordinal. Infinite reduced abelian $p$-groups (the class of infinite abelian $p$-groups not containing $C_{p^\infty}$-subgroups) have the property that $G^{(\tau)} = \langle 1 \rangle$ for some ordinal $\tau$ [3]. The next theorem uses this fact to show that these groups belong to $\mathcal{X}_{\Omega_p}$.

THEOREM 3.6. *[4] Let $G$ be a countable abelian p-group with no $C_{p^\infty}$-subgroups. Then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: **Case 1:** Suppose $G$ has finite exponent $p^m$. If $m = 1$, then $G$ is a countable vector space over the field $\mathsf{F}_p$, and is thus residually finite. Hence $G \in \mathcal{X}_{\Omega_p}$ by Corollary 2.6.1. By induction, if $m > 1$, then $\frac{G}{G^{p^{m-1}}}$ has exponent $p$, and $G^{p^{m-1}} \in \mathcal{X}_{\Omega_p}$, so $G \in \mathcal{X}_{\Omega_p}$ follows from the $m = 1$ case and Proposition 3.5.

**Case 2:** Suppose $G$ has no element of infinite height. Since $\bigcap_{m=1}^{\infty} G^{p^m} = \langle 1 \rangle$, $\{G^p, G^{p^2}, \dots\}$ is a directed system of subgroups of $G$, and so since each $\frac{G}{G^{p^m}} \in \mathcal{X}_{\Omega_p}$ by Case 1, $G \in \mathcal{X}_{\Omega_p}$ by Theorem 2.6.

**Case 3:** Let $G$ be a countable reduced abelian $p$-group. Then there is an ordinal $\tau$ such that $G^{(\tau)} = \langle 1 \rangle$. The group $\frac{G}{G^{(1)}}$ has no elements of infinite height, and so $\frac{G}{G^{(1)}} \in \mathcal{X}_{\Omega_p}$ by Case 2. If $\sigma$ is not a limit ordindal, then $\frac{G^{(\sigma-1)}}{G^{(\sigma)}} \in \mathcal{X}_{\Omega_p}$ by Case 2, and $\frac{G}{G^{(\sigma-1)}} \in \mathcal{X}_{\Omega_p}$ by the inductive hypothesis, so $G \in \mathcal{X}_{\Omega_p}$ by Proposition 3.5. If $\sigma$ is a limit ordinal, then $\{\frac{G^{(\lambda)}}{G^{(\sigma)}} : \lambda < \sigma\}$ is a directed system of subgroups of $\frac{G}{G^{(\sigma)}}$, and so since each $\frac{G}{G^{(\lambda)}} \in \mathcal{X}_{\Omega_p}$ by the inductive hypothesis, $\frac{G}{G^{(\sigma)}} \in \mathcal{X}_{\Omega_p}$ by Theorem 3.6. Hence $G \cong \frac{G}{G^{(\tau)}} \in \mathcal{X}_{\Omega_p}$ by transfinite induction, as required.

□

**E. Nilpotent elements of $\ell_1(\Omega_p, C_{p^\infty})$.**

For all $n \geq 0$, let $\Gamma_n$ be the set of all roots of unity $\zeta \in \Omega_p$ such that $\zeta^{p^n} = 1$. Set $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. It is clear that $\Gamma$ is a multiplicative subgroup of $\Omega_p$, and that $\Gamma \cong C_{p^\infty}$. Furthermore, for all $n \geq 0$, choose primitive roots of unity $\zeta_n \in \Gamma_n$ such that $\zeta_n^p = \zeta_{n-1}$, and $\zeta_n^m \neq \zeta_{n-1}$ for $1 \leq m < p$. Set $K_n = \mathbb{Q}_p(\zeta_n)$, for all $n \geq 0$, and let $K_\infty = \mathbb{Q}_p(\Gamma)$. The main result of this section is that if $k$ is a complete $p$-adic field containing $K_\infty$ then $\ell_1(k,\ C_{p^\infty})$ contains non-zero nilpotent elements. This result is due originally to Fresnel and de Mathan [2]. However, the proof offered here is of a more algebraic nature than the original one.

For notational purposes, set $G = C_{p^\infty}$, and let $G_n$ be the unique cyclic subgroup of order $p^n$ in $G$, for each $n \geq 0$. For any $f = \sum_{g \in G} \alpha_g g \in \ell_1(\Omega_p, G)$, $f =$

$\lim_{n\to\infty} f_n$, where $f_n = \sum_{g\in G_n} \alpha_g g \in \Omega_p G_n$. However, for a fixed $n$, $\sum_{g\in G_n} \alpha_g g$ is only one representation of $f_n$ in the group ring $\Omega_p G_n$. As it turns out, this representation is not the most useful here. The representation described in the next proposition is better suited to the application of characters of $G$.

PROPOSITION 3.7. *Let $H$ be a finite abelian group. Let $\hat{H}$ be the set of p-adic characters of $H$. Then*

*(i)* $e(\theta) = \frac{1}{|H|} \sum_{h\in H} \theta(h)h$ *is an idempotent of* $\Omega_p H$*, for all* $\theta \in \hat{H}$.

*(ii)* $\{e(\theta)\}_{\theta\in\hat{H}}$ *is a full set of primitive, orthogonal idempotents of* $\Omega_p H$.

*(iii) Every* $f \in \Omega_p H$ *can be uniquely expressed as* $\sum_{\theta\in\hat{H}} a(\theta)e(\theta)$*, where* $a(\theta) \in \Omega_p$, $\forall\theta \in \hat{H}$.

PROOF: (i) $e(\theta)^2 = \left(\frac{1}{|H|}\right)^2 \sum_{x\in H} \sum_{y\in H} \theta(x)\theta(y)xy$

$$= \left(\frac{1}{|H|}\right)^2 \sum_{h\in H} \Big(\sum_{x\in H} \theta(x)\theta(x^{-1}h)\Big)h = \left(\frac{1}{|H|}\right)^2 \sum_{h\in H} (|H|\cdot\theta(h))h = e(\theta).$$

(ii) This is a well-known result concerning the representation theory of finite groups [1, Theorem 33.8].

(iii) Since $H$ is finite, $\Omega_p H$ is semiprimitive Artinian so by [1, (25.7)] if $\theta_1, \ldots, \theta_m$ are the distinct elements of $\hat{H}$, then

$$\Omega_p H = e(\theta_1) \cdot \Omega_p H \oplus \cdots \oplus e(\theta_m) \cdot \Omega_p H,$$

and $1 = \sum_{i=1}^{m} e(\theta_i)$. Thus for any $h \in H$, $h = \sum_{i=1}^{m} he(\theta_i) = \sum_{i=1}^{m} \frac{1}{|H|} \sum_{x \in H} \theta_i(x) hx = \sum_{i=1}^{m} \frac{1}{|H|} \sum_{y \in H} \theta_i(h^{-1}y)y = \sum_{i=1}^{m} \theta_i(h^{-1})e(\theta_i)$. For a general $f = \sum_{h \in H} \alpha_h h \in \Omega_p H$, where $\alpha_h \in \Omega_p$, for all $h$, it is clear that $f = \sum_{h \in H} \alpha_h \Big( \sum_{\theta \in H} \theta h^{-1}) e(\theta) \Big) = \sum_{\theta \in \hat{H}} \Big( \sum_{h \in H} \alpha_h (h^{-1} \Big) e(\theta)$ is the unique representation for $f$ with $a(\theta) = \sum_{h \in H} \alpha_h \theta(h^{-1}) \in \Omega_p$, $\forall \theta \in \hat{H}$.

□

Now, for each cyclic subgroup $G_n$ of order $p^n$ in $G$, for $n \geq 0$, let $\{e_n(\theta)\}_{\theta \in \widehat{G}_n}$ be the set of primitive orthogonal idempotents of $\Omega_p G_n$. By the proposition, it is clear that the given $f \in \ell_1(\Omega_p, G)$ is the limit of elements of the form $\sum_{\theta \in \widehat{G}_n} a_n(\theta) e_n(\theta) = f_n$, with $f_n \in \Omega_p G_n$ as before, for all $n \geq 0$. It is this representation of $f$ that is needed.

One drawback for representing $f \in \ell_1(\Omega_p, G)$ as $\lim_{n \to \infty} \sum_{\theta \in \widehat{G}_n} a_n(\theta) e_n(\theta)$ is that $\|f\|$ becomes difficult to determine, since for a given $\varphi \in \widehat{G}$, $\|e_n(\varphi|_{G_n})\|$ diverges as $n \to \infty$. Thus the condition $|a_n(\varphi|_{G_n})| \to 0$ is not enough to ensure that a given sequence is Cauchy with respect to $\| \cdot \|$. The next result gives a sufficient condition for such a sequence to be Cauchy. The following lemma is useful:

LEMMA 3.8.1. *Let $n \geq 0$. For all $\theta \in \widehat{G}_n$, $e_n(\theta) = \sum_{\substack{\varphi \in \widehat{G}_{n+1} \\ \varphi|_{G_n} = \theta}} e_{n+1}(\varphi)$.*

PROOF: $e_n(\theta)$ is an idempotent of $\Omega_p G_{n+1}$, so consider the ideal generated by $e_n(\theta)$ in $\Omega_p G_{n+1}$. Since $\Omega_p G_{n+1}$ is the direct sum of the minimal ideals $e_{n+1}(\varphi)(\Omega_p G_{n+1})$, $\varphi \in \widehat{G}_{n+1}$, $e_n(\theta)(\Omega_p G_{n+1})$ is the sum of certain of these ideals.

Now, let $\varphi \in \widehat{G}_{n+1}$, and consider

$$\begin{aligned} e_{n+1}(\varphi)e_n(\theta) &= \frac{1}{p^{n+1}}\Big(\sum_{x \in G_{n+1}} \varphi(x)x\Big) \cdot \frac{1}{p^n}\Big(\sum_{y \in G_n} \theta(y)y\Big) \\ &= \frac{1}{p^{2n+1}} \sum_{x \in G_{n+1}} \sum_{y \in G_n} \varphi(x)\theta(y)xy \\ &= \frac{1}{p^{2n+1}} \sum_{g \in G_{n+1}} \Big(\sum_{y \in G_n} \varphi(gy^{-1})\theta(y)\Big)g \\ &= \frac{1}{p^{2n+1}} \sum_{g \in G_{n+1}} \varphi(g)\Big(\sum_{y \in G_n} \varphi(y^{-1}\theta(y)\Big)g. \end{aligned}$$

Since

$$\sum_{y \in G_n} \varphi|_{G_n}^{-1}(y)\theta(y) = \begin{cases} |G_n|, & \text{if } \varphi|_{G_n} = \theta \\ 0, & \text{if } \varphi|_{G_n} \neq \theta \end{cases}$$

(by the orthogonality relations for finite groups [1, Lemma 31.17]), this implies that

$$e_{n+1}(\varphi)e_n(\theta) = \begin{cases} e_{n+1}(\varphi), & \text{if } \varphi|_{G_n} = \theta \\ 0, & \text{if } \varphi|_{G_n} \neq \theta. \end{cases}$$

Thus $e_n(\theta) \cdot (\Omega_p G_{n+1}) = \bigoplus_{\substack{\varphi \in \widehat{G}_{n+1} \\ \varphi|_{G_n} = \theta}} e_{n+1}(\varphi)(\Omega_p G_{n+1})$, and since $e_n(\theta)$ is an idempotent, $e_n(\theta) = \sum_{\substack{\varphi \in \widehat{G}_{n+1} \\ \varphi|_{G_n} = \theta}} e_{n+1}(\varphi)$. □

PROPOSITION 3.8. *Let $f_n$ be a sequence of elements of $\Omega_p G$ such that supp $f_n \subseteq G_n$, for all $n$. Write $f_n = \sum_{\theta \in G_n} a_n(\theta) e_n(\theta)$, where $a_n(\theta) \in \Omega_p$ for all $\theta \in \widehat{G}_n$. Then $\{f_n\}$ is a Cauchy sequence with respect to $\|\cdot\|$ if, for all $\varphi \in \widehat{G}$, $p^{n+1}|a_{n+1}(\varphi|_{G_{n+1}}) - a_n(\varphi|_{G_n})| \to 0$ as $n \to \infty$.*

PROOF: $\{f_n\}$ is Cauchy with respect to $\|\cdot\| \Leftrightarrow \|f_{n+1} - f_n| \to 0$ as $n \to \infty$

$$\Leftrightarrow \Big\| \sum_{\varphi \in \widehat{G}_{n+1}} a_{n+1}(\varphi) e_{n+1}(\varphi) - \sum_{\theta \in \widehat{G}_n} a_n(\theta) e_n(\theta) \Big\| \to 0 \quad \text{as} \quad n \to \infty$$

$$\Leftrightarrow \Big\| \sum_{\varphi \in \widehat{G}_{n+1}} a_{n+1}(\varphi) e_{n+1}(\varphi) - \sum_{\theta \in \widehat{G}_n} a_n(\theta) \Big( \sum_{\substack{\varphi \in \widehat{G}_{n+1} \\ \varphi|_{G_n} = \theta}} e_{n+1}(\varphi) \Big) \Big\| \to 0 \quad \text{as} \quad n \to \infty$$

$$\Leftrightarrow \Big\| \sum_{\varphi \in \widehat{G}_{n+1}} \big(a_{n+1}(\varphi) - a_n(\varphi|_{G_n})\big) e_{n+1}(\varphi) \Big\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since all $\varphi \in \widehat{G}_n$ are the restrictions of some $\phi \in \widehat{G}$ to $G_n$, a sufficient condition for this is that for all $\varphi \in \widehat{G}$, $\|(a_{n+1}(\varphi|_{G_{n+1}}) - a_n(\varphi|_{G_n})) e_{n+1}(\varphi)\| \to 0$ as $n \to \infty$. Since $\|e_n(\varphi)\| = p^n$, for all $n \geq 0$, this condition is equivalent to $p^{n+1}|a_{n+1}(\varphi|_{G_{n+1}}) - a_n(\varphi|_{G_n})| \to 0$ as $n \to \infty$ for all $\varphi \in \widehat{G}$.

□

The main result of this section now can be shown.

THEOREM 3.9. *Let $G \cong C_{p^\infty}$. Then $\ell_1(\Omega_p, G)$ posseses non-zero nilpotent elements. More precisely, there exists an $f = \sum_{g \in G} \alpha_g g \in \ell_1(\Omega_p, G)$ such that $|\alpha_1| = 1$ and $f^2 = 0$.*

PROOF: The index of $\widehat{G}_n$ in $\widehat{G}_{n+1}$ is $p$, since for all finite abelian groups $H$, $\hat{H} \cong H$.

So write $f = \lim_{n \to \infty} f_n$, where $f_n = \sum_{\theta \in \widehat{G}_n} a_n(\theta) e_n(\theta) \in \Omega_p G_n$, and the $a_n(\theta)$ are determined by the following.

Enumerate $\widehat{G}_1$ as $\{\theta_0, \ldots, \theta_{p-1}\}$. For $n > 1$, let $\theta_0, \ldots, \theta_{p^n-1}$ be an enumeration of $\widehat{G}_n$ such that $\theta_i$ extends $\theta_j \in \widehat{G}_{n-1}$ if and only if $i \equiv j \mod p^{n-1}$. Define $a_n(\theta_i)$ according to the following rules:

$$a_n(\theta_i) = \begin{cases} p^n, & \text{if } \; i \equiv 0 \mod p \quad \text{and if} \quad n > 1, \; a_m(\,\theta_i|_{G_m}\,) \neq 1 \quad \text{for all} \quad m < n. \\ p^{2n}, & \text{otherwise.} \end{cases}$$

Thus for any $\varphi \in \widehat{G}$, $a_n(\,\varphi|_{G_n}\,) = p^n$ for at most one $n \in \mathbb{N}$, and $a_n(\,\varphi|_{G_n}\,) = p^{2n}$ for all other $n$. To finish the proof, it is left to show:

(i) $f_n = \sum_{\theta \in G_n} a_n(\theta) e_n(\theta)$ is a Cauchy sequence with respect to $\|\cdot\|$.

(ii) If $f = \lim_{n \to \infty} f_n = \sum_{g \in G} \alpha_g g \in \ell_1(\Omega_p, G)$, then $|\alpha_1| = 1$.

(iii) $f^2 = \lim_{n \to \infty} f_n^2 = 0$.

Now, by Proposition 3.8, (i) follows if for all $\varphi \in \widehat{G}$, $p^{n+1}|a_{n+1}(\varphi|_{G_{n+1}}) - a_n(\varphi|_{G_n})| \to 0$ as $n \to \infty$. So let $\varphi \in \widehat{G}$. For all large enough $n$, $a_n(\varphi|_{G_n}) = p^{2n}$, so $\lim_{n\to\infty} p^{n+1}|a_{n+1}(\varphi|_{G_{n+1}}) - a_n(\varphi|_{G_n})| = \lim_{n\to\infty} p^{n+1}|p^{2n+2} - p^{2n}| = \lim_{n\to\infty} p^{n+1} \cdot p^{-2n} = \lim_{n\to\infty} p^{-n+1} = 0$. Thus (i) is shown. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence by Proposition 3.8, there exists an $f \in \ell_1(\Omega_p, G)$ such that $f = \lim_{n\to\infty} f_n$.

Write $f = \sum_{g\in G} \alpha_g g$, and write $f_n = \sum_{g\in G_n} \beta_{g,n} g$. Since $f = \lim_{n\to\infty} f_n$, $\alpha_1 = \lim_{n\to\infty} \beta_{1,n}$. Now, it is evident that $\beta_{1,n} = \sum_{\theta\in\widehat{G}_n} \frac{1}{p^n} \cdot a_n(\theta)$. For each $n \in \mathbb{N}$, note that for each $n \geq 2$, $a_n(\theta) = p^n$ for exactly $p^{n-1} - 1$ choices of $\theta \in \widehat{G}_n$. Thus $|\sum_{\theta\in\widehat{G}_n} \frac{1}{p^n} a_n(\theta)| = p^n|(p^{n-1}-1)p^{2n+1} + (p^n - p^{n-1}+1)p^{2n}| = p^n \cdot |p^{3n} - p^{3n-1} + p^{2n} + p^{2n+1} - p^n| = p^n \cdot |p^n| = p^n \cdot p^{-n} = 1$, for all $n \geq 2$. Hence $|\alpha_1| = \lim_{n\to\infty} |\beta_{1,n}| = 1$, proving (ii). Thus $f \neq 0$.

The orthogonality of the idempotents $e_n(\theta)$ implies that $f_n^2 = \sum_{\theta\in\widehat{G}_n} (a_n(\theta))^2 e_n(\theta)$.

So

$$\begin{aligned}\|f_n^2\| &\leq \max_{\theta\in\widehat{G}_n} \|(a_n(\theta))^2 e_n(\theta)\| = \max_{\theta\in\widehat{G}_n} p^n |a_n(\theta)|^2 \\ &= p^n \cdot p^{-2n} = p^{-n}.\end{aligned}$$

Thus $\lim_{n\to\infty} \|f_n^2\| = 0$, and so $\lim_{n\to\infty} f_n^2 = 0$. Therefore $f^2 = 0$.

□

COROLLARY 3.9.1. *Let $K_\infty = \mathbb{Q}_p(\Gamma)$, and let $\overline{K}_\infty$ be the closure of $K_\infty$ in $\Omega_p$. Then if $G \cong C_{p^\infty}$, $\ell_1(\overline{K}_\infty, G)$ contains non-zero nilpotent elements.*

PROOF: In the proof of Theorem 3.9, it is clear that the coefficients of each $f_n$ lie in $K_\infty$, for all $n$, since each $a_n(\theta) \in K_n$. Hence the coefficients of $f$ lie in $\overline{K}_\infty$.

COROLLARY 3.9.2. *$\ell_1(\Omega_p,\ C_{p^\infty})$ is not semiprime, and hence $C_{p^\infty} \notin \mathcal{X}_{\Omega_p}$.*

PROOF: The non-zero nilpotent element $f$ generates a non-zero nilpotent ideal, and this ideal is contained inside $J(\ell_1(\Omega_p, G))$.

**F. The different of finite extensions.**

The characterization of complete $p$-adic fields $k$ for which $C_{p^\infty} \in \mathcal{X}_k$ is based on the different of finite $p$-adic field extensions. The purpose of this section is to introduce the different of a finite extension and provide background which will be needed for the proofs to come.

Let $K$ be an algebraic extension of $\mathbb{Q}_p$. Let $F$ be a finite extension of $K$. Let the codifferent of $F$ over $K$ be defined by $\mathcal{D}^{-1}_{F/K} = \{\alpha \in F : Tr_{F/K}(\alpha \mathcal{O}_F) \subseteq \mathcal{O}_k\}$. Then since $|Tr_{F/K}(\alpha)| \leq |\alpha|$, for all $\alpha \in F$, it is clear that $\mathcal{D}^{-1}_{F/K}$ is an $\mathcal{O}_F$-module contained in $F$, and $\mathcal{O}_F \subseteq \mathcal{D}^{-1}_{F/K}$. Define the different of $F$ over $K$ to for all $\alpha \in \mathcal{D}_{F/K}$. Furthermore, $\mathcal{D}_{F/K}$ is an ideal of $\mathcal{O}_F$ contained in $F$, since $\alpha \cdot \mathcal{D}_{F/K}^{-1} \subset \mathcal{O}_F \Rightarrow (\alpha\mathcal{O}_F)\mathcal{D}_{F/K}^{-1} \subseteq \alpha\mathcal{D}_{F/K}^{-1} \subseteq \mathcal{O}_F$.

A simple but important property of the different is transitivity.

PROPOSITION 3.10. *Suppose $K, F$, and $L$ are algebraic extensions of $\mathbb{Q}_p$ such that $K \subset F \subset L$ and $\frac{L}{K}$ is finite. Then $\mathcal{D}_{L/K} = \mathcal{D}_{L/F} \cdot \mathcal{D}_{F/K}$.*

PROOF: If $\alpha \in L$, then

$$\begin{aligned}
\alpha \in \mathcal{D}_{L/K}^{-1} &\Leftrightarrow Tr_{L/K}(\alpha\mathcal{O}_L) \subset \mathcal{O}_K \\
&\Leftrightarrow Tr_{F/K}(Tr_{L/F}(\alpha\mathcal{O}_L)) \subset \mathcal{O}_K \\
&\Leftrightarrow Tr_{L/F}(\alpha\mathcal{O}_L)) \subset \mathcal{D}_{F/K}^{-1} \\
&\Leftrightarrow (Tr_{L/F}(\alpha\mathcal{O}_L))\mathcal{D}_{F/K} \subset \mathcal{O}_F \\
&\Leftrightarrow Tr_{L/F}(\alpha\mathcal{D}_{F/K}\mathcal{O}_L) \subset \mathcal{O}_F \\
&\Leftrightarrow \alpha\mathcal{D}_{F/K} \subset \mathcal{D}_{L/F}^{-1} \\
&\Leftrightarrow \alpha\mathcal{D}_{F/K}\mathcal{O}_L \subset \mathcal{D}_{L/F}^{-1} \\
&\Leftrightarrow \alpha \in \mathcal{D}_{L/F}^{-1}\mathcal{D}_{F/K}^{-1}.
\end{aligned}$$

Thus $\mathcal{D}_{L/K}^{-1} = \mathcal{D}_{L/F}^{-1}\mathcal{D}_{F/K}^{-1}$, and so $\mathcal{D}_{L/F}\mathcal{D}_{F/K} = \mathcal{D}_{L/K}$. □

In some circumstances, $\mathcal{D}_{F/K}$ can be easily determined via a simple calculation.

PROPOSITION 3.11. *Suppose $F$ is a finite extension of a $p$-adic field $K$ of degree $n$. Suppose $\alpha \in F$ such that*

$$F = K(\alpha) \quad \text{and} \quad \mathcal{O}_F = \bigoplus_{i=0}^{n-1} \alpha^i \mathcal{O}_k.$$

*If $f(X) \in K[X]$ is the minimum polynomial of $\alpha$ over $K$, then $\mathcal{D}_{F/K} = f'(\alpha) \cdot \mathcal{O}_F$.*

PROOF: [6, Proposition 3.7.21].

□

Now, for any finite extension $F$ of a $p$-adic field $K$, set $|\mathcal{D}_{F/K}| = \sup_{\beta \in \mathcal{D}_{F/K}} |\beta|$. If $\alpha \in F$ satisfies the conditions of proposition, then it is obvious that $|\mathcal{D}_{F/K}| = |f'(\alpha)|$, where $f(X)$ is the minimum polynomial of $\alpha$ over $K$.

This section concludes with a calculation of $|\mathcal{D}_{K_N/K_s}|$, for any $N \geq s \geq 1$ where $K_N = K(\zeta_N)$. Let $\zeta_s$ be a primmitive $p^s$-th root of unity in $K_s$, and let $\zeta_N$ be a primitive $p^N$-th root of unity in $K_N$ such that $(\zeta_N)^{p^{N-s}} = \zeta_s$. Then $\zeta_N$ is a root of the irreducible polynomial $X^{p^{N-s}} - \zeta_s \in K_s[X]$. Furthermore, $\mathcal{O}_{K_N} = \bigoplus_{i=1}^{p^{N-s}-1} \zeta_N^i \mathcal{O}_{K_s}$ since $|\zeta_N| = 1$ and $\{1, \zeta_N, \ldots, \zeta_N^{p^{N-s}-1}\}$ is a basis of $K_N$ over $K_s$. Thus $|\mathcal{D}_{K_N/K_s}| = |p^{N-s}(\zeta_N)^{p^{N-s}-1}| = p^{s-N}$.

**G. Semiprimitivity and $\ell_1(k,\ C_{p^\infty})$, for arbitrary complete $p$-adic fields $k$.**

Let $L$ be an algebraic extension of a $p$-adic field $K$. Then $L$ is of different zero over $K$ if $|\mathcal{D}_{L/K}| = \inf_{F\in\mathcal{L}} |\mathcal{D}_{F/K}| = 0$, where $\mathcal{L}$ denotes the set of all finite extensions of $K$ contained in $L$. In [2], it was shown that $\ell_1(k,\ C_{p^\infty})$ is semiprimitive if and only if $k \cap \mathbb{Q}_p$ is not of different zero over $\mathbb{Q}_p$. This section presents the proof of this result in detail.

PROPOSITION 3.12. *Suppose that $L$ is of different non-zero over $\mathbb{Q}_p$. Suppose $(LK_1) \cap K_\infty = K_n$. Then for all integers $s \geq n$, the map $T_s : LK_\infty \to LK_s$ defined by*

$$T_s = \liminj_{N\geq s} \frac{1}{p^{N-s}} Tr_{LK_N/LK_s}$$

*is a continuous $LK_s$-linear map.*

PROOF: To see that $T_s$ is well-defined, let $\alpha \in LK_N$, for some $N \geq s$. For all $m \geq N$,

$$\begin{aligned} p^{\frac{1}{m-s}} Tr_{LK_m/LK_s}(\alpha) &= \frac{1}{p^{m-s}} Tr_{LK_N/LK_s}(Tr_{LK_m/LK_N}(\alpha)) \\ &= \frac{1}{p^{m-s}} Tr_{LK_n/LK_s}(p^{m-N}\alpha) \\ &= \frac{1}{p^{N-s}} Tr_{LK_N/LK_s}(\alpha). \end{aligned}$$

(Note that $n \geq 1$ is needed so that $[LK_m : LK_N]$ is a power of $p$. This is because $[K_1 : \mathbb{Q}_p] = p - 1$, for all prime $p$.) Thus $T_s = \liminj_{N \geq s} Tr_{LK_N/LK_s}$ is well-defined. Linearity is easy, since for all $\alpha,\ \beta \in LK_\infty,\ \gamma \in LK_s$,

$$\begin{aligned}
T_s(\gamma\alpha + \beta) &= \liminj_{N \geq s} \frac{1}{p^{N-s}} Tr_{LK_N/LK_s}(\gamma\alpha + \beta) \\
&= \frac{1}{p^{N-s}} Tr_{LK_N/LK_s}(\gamma\alpha + \beta) \quad \text{(for all large enough } N\text{)} \\
&= \gamma \cdot \frac{1}{p^{N-s}} Tr_{LK_N/LK_s}(\alpha) + \frac{1}{p^{N-s}} Tr_{LK_N/LK_s}(\beta) \\
&= \gamma T_s(\alpha) + T_s(\beta).
\end{aligned}$$

To prove the continuity of $T_s$, the fact that $|\mathcal{D}_{L/\mathbb{Q}_p}| \neq 0$ is needed. First, note that every finite extension $F$ of $K_s$ contained in $LK_s$ is contained in an extension of the form $MK_s$, where $M$ is a finite extension of $\mathbb{Q}_p$. Furthermore, $K_s \subset F \subset MK_s$, implies $|\mathcal{D}_{MK_s/K_s}| \leq |\mathcal{D}_{F/K_s}|$ from the definition. Also, $|\mathcal{D}_{L/\mathbb{Q}_p}| \neq 0$ implies that $|\mathcal{D}_{LK_s/K_s}| = |\mathcal{D}_{LK_s/L}| \cdot |\mathcal{D}_{L/\mathbb{Q}_p}| \cdot |\mathcal{D}_{K_s/\mathbb{Q}_p}|^{-1} \neq 0$. Thus there is a $d \in \mathbb{N}$ such that $p^{-d} \leq |\mathcal{D}_{LK_s/K_s}| \leq |\mathcal{D}_{F/K_s}|$, for all finite extensions $F$ of $K_s$ contained in $LK_s$.

Now, let $F$ be a finite extension of $K_s$ contained in $LK_s$. By transitivity of differents, for all $N \geq s$, $|\mathcal{D}_{FK_N/F}| \cdot |\mathcal{D}_{F/K_s}| = |\mathcal{D}_{FK_N/K_N}| \cdot |\mathcal{D}_{K_N/K_s}|$. Since $|\mathcal{D}_{K_N/K_s}| = p^{s-N}$, $|\mathcal{D}_{FK_N/K_N}| \leq 1$, and $|\mathcal{D}_{F/K_s}| \geq p^{-d}$, this equality implies that $|\mathcal{D}_{FK_N/F}| \leq p^{s-N+d}$.

To show the continuity of $T_s$, suppose $\alpha \in LK_N$ with $|\alpha| \leq 1$. Then $\alpha \in FK_N$, for some finite extension $F$ of $K_s$ contained in $LK_s$. By the above, $|\left(\frac{\alpha}{p^{N-s-d}}\right)^{-1}| = |\alpha|^{-1} \cdot |p^{N-s-d}| \geq p^{s-N+d} \geq |\mathcal{D}_{FK_n/F}|$; Hence $|\frac{\alpha}{p^{N-s-d}}| \leq |\mathcal{D}^{-1}_{FK_N/F}|$, and since $\mathcal{D}^{-1}_{FK_N/F}$ is a fractional ideal and $FK_N$ is a discretely valued field, this implies that $\frac{\alpha}{p^{N-s-d}} \in \mathcal{D}^{-1}_{FK_N/F}$. Then $|Tr_{FK_N/F}(\frac{\alpha}{p^{N-s-d}})| \leq 1 \Rightarrow p^{N-s-d} \cdot |Tr_{FK_N/F}(\alpha)| \leq 1 \Rightarrow |Tr_{FK_N/F}(\alpha)| \leq p^{s-N+d}$, whenever $|\alpha| \leq 1$.

Finally, note that $Tr_{LK_N/LK_s}(\alpha) = Tr_{FK_N/FK_s}(\alpha) = Tr_{FK_N/F}(\alpha)$. Hence

$$\begin{aligned}
|T_s(\alpha)| &= |\frac{1}{p^{N-s}} Tr_{LK_N/LK_s}(\alpha)| \quad \text{(for all large enough } N\text{)} \\
&= p^{N-s} \cdot |Tr_{MK_N/M}(\alpha)| \quad \text{(for some extension } M\text{)} \\
&\leq p^{N-s} \cdot p^{s-N+d} = p^d.
\end{aligned}$$

Thus $T_s$ is continuous.

□

COROLLARY 3.12.1. *The continuous extension of $T_s$ to $\overline{LK_\infty}$ is an $\overline{LK_s}$ linear map $\overline{T_s} : \overline{LK_\infty} \to \overline{LK_s}$ such that $\overline{T_s}(\zeta) = 0$ for all $\zeta \in \Gamma$ for which $\zeta^{p^s} \neq 1$, and $\overline{T_s}(\zeta) = \zeta$ for all $\zeta \in \Gamma$ for which $\zeta^{p^s} = 1$.*

PROOF: Since $T_s$ is continuous, the continuous extension $\overline{T_s}$ is well-defined and $LK_s$-linear. If $\zeta \in \Gamma$ and $\zeta \in K_N \backslash K_s$ for some $N > s$, then $Tr_{K_N/K_s}(\zeta) = 0$, so $\overline{T_s}(\zeta) = 0$; If $\zeta \in \Gamma$ and $\zeta \in K_s$, then since $\overline{T_s}$ is $LK_s$-linear $\overline{T_s}(\zeta) = \zeta$. □

THEOREM 3.13. *Let $k$ be a complete $p$-adic field, such that $k \cap \widehat{\mathbb{Q}_p} = L$ is not of different zero over $\mathbb{Q}_p$. Then $C_{p^\infty} \in \mathcal{X}_k$.*

PROOF: Let $G \cong C_{p^\infty}$, and let $G_n$ be the unique cyclic subgroup of order $p^n$ in $G$, for all $n \geq 0$. Suppose $f = \sum_{g \in G} \alpha_g g \in J(\ell_1(k, G))$. By Theorem 3.4, $\theta(f) = 0$, for all $\theta \in \widehat{G}_k$.

Let $\theta \in \widehat{G}$ such that $\theta(G) \neq \{1\}$. Suppose $\ker\theta = G_t$, for some $t \geq 0$, and suppose $LK_1 \cap K_\infty = K_n$ for some integer $n \geq 1$. Then for all $s \geq n$,

$$0 = \theta(f) \Rightarrow 0 = \overline{T_s}(\theta(f)) = \sum_{g \in G} \alpha_g \overline{T_s}(\theta(g)) = \sum_{g \in G_{s+t}} \alpha_g \theta(g).$$

Hence for all $t \geq 0$, for all $\theta \in \widehat{G}$ such that $\theta(G) \neq \{1\}$ and $\ker\theta \subseteq G_t$, and for all integers $N \geq n + 1$,

$$(*) \qquad \sum_{g \in G_N} \alpha_g \theta(g) = 0.$$

Now, each element $f \in \ell_1(k, G)$ can be considered as a map of $\widehat{G}$ into $\Omega_p$. Let $\mathcal{F} : \ell_1(k, G) \to \mathrm{Hom}(\widehat{G}, \Omega_p)$ be the map defined by $\mathcal{F}(f)(\theta) = \theta_k(f)$, where $\theta_k$ is the element corresponding to $\theta$ in $\widehat{G}_k$. $\mathcal{F}$ is clearly a well-defined ring homomorphism, using the fact that each $\theta_k$ is a ring homomorphism. Furthermore, $\mathcal{F}(f)$ is the zero map if and only if $f \in J(\ell_1(k, G))$, by Corollary 3.4.2. Another fact is that since the group ring $kG$ is semiprimitive, the restriction of $\mathcal{F}$ to $kG$ is injective.

Now, let $e_m$ be the idempotents $\frac{1}{p^m} \sum_{x \in G_m} x$, for all $m \geq 1$, and let $f_N = \sum_{g \in G_N} \alpha_g g$, for all $N \geq n$. Let $\phi \in \widehat{G}_k$. If $\phi(G) \neq \{1\}$, the identity (*) implies that $\phi(f_N) = 0$, for all $N \geq n$, and so $\phi(f_N e_m) = \phi(f_N)\phi(e_m) = 0$, for all $m \geq 1$, $N \geq n$. If $\Delta \in \widehat{G}_k$ with $\Delta(G) = \{1\}$, then since $\Delta(e_m) = 1$, for all $m \geq 1$, it follows that $\Delta(f_N e_m) = \Delta(f_N)\Delta(e_m) = \Delta(f_N)$. Therefore, $\phi(f_N) = \phi(f_N e_m)$, for all $\phi \in \widehat{G}_k$. Injectivity of the map $\mathcal{F}$ on $kG$ implies that $f_N = f_N e_m$, for all $N \geq n$, and for all $m \geq 1$.

Thus $\sum_{g \in G_N} \alpha_g g = \left( \sum_{g \in G_N} \alpha_g g \right) \left( \frac{1}{p^m} \sum_{x \in G_m} x \right) = \sum_{g \in G_n} \left( \frac{1}{p^m} \sum_{x \in G_m} \alpha_{x^{-1} g} \right) g$, so $\alpha_g = \frac{1}{p^m} \sum_{x \in G_m} \alpha_{x^{-1} g}$, for all $g \in G_m$. But then $f = \sum_{g \in G} \alpha_g g$ is constant on all cosets of $G_n$, for every $n \geq 1$. The condition $|\alpha_g| \to 0$ forces $f$ to be 0.

□

The converse to Theorem 3.13 is a consequence of the following characterizing property of fields of different zero over $\mathbb{Q}_p$.

PROPOSITION 3.14. *Let $L$ be a field of different zero over $\mathbb{Q}_p$, with $K_\infty \not\subset L$, such that $L \cap K_\infty = K_n$, for some $n \in \mathbb{N}$. Then for all $N > n$, $|\mathcal{D}_{LK_{N+1}/LK_N}| = 1$.*

PROOF: **Case 1:** Suppose $n = 0$. Then $LK_1$ is tamely ramified over $L$, and thus it is defined by the polynomial $X^e - a$, where $e, p$ are relatively prime and $|a| < 1$

[6, 3-4-3]. Let $0 < c < 1$. Since the valuation on $L$ must be dense, there exists a $\rho \in L$ such that $c < |\rho^e a| < 1$. The polynomial $x^e - p^e a$ defines the extension $LK_1$ of $L$, and if $\alpha$ is a root, then $|\mathcal{D}_{LK_1/L}| = |e\alpha^{e-1}| = |\alpha^{e-1}| = |\rho^e a|^{\frac{e-1}{e}} > c^{\frac{e-1}{e}}$. Hence $|\mathcal{D}_{LK_1/L}| = 1$.

**Case 2**: Suppose $n \geq 1$, so $LK_{N+1}$ is wildly ramified of degree $p$ over $LK_N$. The proof in this case is a consequence of the fact that if $F$ is a finite extension of $K_N$ not containing $K_{N+1}$ with $|\mathcal{D}_{F/K_N}| \leq p^{-2m}$, for some $m \in \mathbb{N}$, then $|\mathcal{D}_{FK_{N+1}/F}| \geq p^{-\frac{1}{p}m-1}$. To see this, consider the following statements:

a) Let $\zeta_N$ be a primitive $p^N$-th root of unity. For all $\alpha \in F$, $|\alpha^p - \zeta_N| \geq p^{-\frac{p}{p-1}}$.

Proof of a): It suffices to show a) holds for all $\alpha \in F$ with $|\alpha| = 1$.

Since $f(X) = X^p - \zeta_N$ is is irreducible over $F$, so is $f(\alpha + Y) = (\alpha + Y)^p + \zeta_N = Y^p + \sum_{i=1}^{p-1} \binom{p}{i}\alpha^i Y^{p-i} + (\alpha^p + \zeta_N)$. If $\beta$ is a root of $f(\alpha + Y)$ in $FK_{N+1}$, then $|\beta| = |\alpha^p + \zeta_N|^{\frac{1}{p}}$. If $|\beta| = 1$, then $|\alpha^p - \zeta_N| = 1$, so a) is trivial. If $|\beta| < 1$, then since $|p\alpha^{p-1}\beta| = p^{-1}|\beta| > |\binom{p}{i}\alpha^i\beta^{p-i}|$ for $i = 1, \ldots, p-2$, and since $\beta$ is a root of $f(\alpha + Y)$, it follows that $|p\alpha^{p-1}\beta| \leq |\beta^p| = |\alpha^p - \zeta_N|$. Thus $p^{-1}|\beta| \leq |\beta|^p \Rightarrow p^{-1} \leq |\beta|^{p-1} = |\alpha^p - \zeta_N|^{\frac{p-1}{p}} \Rightarrow p^{-\frac{p}{p-1}} \leq |\alpha^p - \zeta_N|$, proving a).

b) For any integer $m \geq 2$, if there exists an $\alpha \in F$ such that $|\alpha^p - \zeta_N| \leq p^{-\left(\frac{p}{p-1} - \frac{1}{p^{m-1}(p-1)}\right)}$, then $|\mathcal{D}_{FK_{N+1}/F}| \geq p^{-\frac{1}{p^{m-1}}}$.

PROOF OF b): Set $\gamma = \frac{1}{p} \cdot (\alpha^p - \zeta_N)$, so $\gamma \in F$ and $|\gamma| = p \cdot |\alpha^p - \zeta_N|$. Since $f(X) = X^p - \zeta_N$ is irreducible over $F$, the monic polynomial $f_0(Y) = \gamma^{-p}((\gamma Y + \alpha)^p - \zeta_N)$ is also irreducible, with coefficients in $\mathcal{O}_F$. Thus $|\mathcal{D}_{FK_{N+1}/F}| \geq f_0'(\beta)$, where $\beta$ is a root of $f_0(Y)$ in $FK_{N+1}$. Now, $f_0'(Y) = \gamma^{-p} \cdot p(\gamma Y + \alpha)^{p-1} \cdot \gamma = p\gamma^{-(p-1)}(\gamma Y + \alpha)^{p-1}$, so $|f_0'(\beta)| = |p\gamma^{-(p-1)}|$, since $|\gamma\beta + \alpha| = |\zeta_N|^{1/p} = 1$. Therefore, $|f_0'(\beta)| = p^{-1} \cdot p^{-(p-1)}|\alpha^p - \zeta_N|^{-(p-1)} = p^{-p} \cdot |\alpha^p - \zeta_N|^{-(p-1)} \geq p^{-p} p^{\left[-\left(\frac{p}{p-1} - \frac{1}{p^{m-1}(p-1)}\right)\right] \cdot [-(p-1)]} = p^{-p} \cdot p^{p - \frac{1}{p^{m-1}}} = p^{-\frac{1}{p^{m-1}}}$. This proves b).

Thus the proof in Case 2 is reduced to finding an $\alpha \in L$ satisfying the conditions of b). To see this, let $F$ be any finite extension of $K_n$ contained in $L$. Let $M$ be the maximal unramified extension of $F$ over $\mathbb{Q}_p$, and $M' = MK_N$ the maximal unramified extension of $F$ over $K_N$. Let $\zeta_n$ be a primitive $p^n$-th root of unity in $F$, so that $\pi_n = \zeta_n - 1$ is a uniformizer of $M'$, and let $\mu$ be a uniformizer of $F$.

Then $\mu$ is a root of an Eisenstein polynomial over $M'$ of the form

$$U(X) = X^t + \pi_n(\eta_{t-1}X^{t-1} + \cdots + \eta_1 X + \eta_0),$$

where $|\eta_i| \leq 1$ for $i = 0, \ldots, t-1$ and $|\eta_0| = 1$. Also, $|\mathcal{D}_{F/K_N}| = |\mathcal{D}_{F/M'}| = |U'(\mu)|$, by Proposition 3.11. Now, $U'(\mu) = t\mu^{t-1} + \pi_n \cdot \sum_{i=1}^{t-2} i\eta_i\mu^{i-1}$. Consider $|U'(\mu)|$. Since

$|i|$, $|\eta_i|$ are integer powers of $|\pi_n|$, for each $i = 1, \ldots, t-1$, $\pi_n$ is a uniformizer of $M$ and $|\mu|^t = |\pi_n|$, it is clear that $|i\pi_n\eta_i\mu^{t-i}|$ are all distinct for $i = 1, \ldots, t-1$. Similarly, $|t\mu^{t-1}|$ is distinct from the others.

Now, suppose $|\mathcal{D}_{F/K_N}| \leq p^{-2m}$, for some $m \in \mathbb{N}$. Then $|U'(\mu)| \leq p^{-2m}$, so for all $i = 1, \ldots, t-1$, $|i\pi_n\eta_i\mu^{t-i}| \leq p^{-2m}$, and $|t\mu^{t-1}| \leq p^{-2m}$. So since $|p| \leq |\pi_n| < 1$, it follows that $|\eta_i| \leq p^{-m}$ for $i \not\equiv 0 \mod p^m$ and $t \equiv 0 \mod p^m$. Thus $U(X)$ must be of the form

$$U(X) = X^{rp^m} + \pi_n\left(\delta_{r-1}X^{(r-1)p^m} + \cdots + \delta_1 X^{p^m} + \delta_0\right) + p^m R(X)$$

where $\delta_j \in \mathcal{O}_{M'}$, for $j = 0, \ldots, r-1$, $|\delta_0| = 1$, and $R(X) \in \mathcal{O}_{M'}[X]$

Similarly, $\mu$ is also a root of an Eisenstein polynomial over $M$ of the form

$$V(X) = X^{sp^m} + p\left(v_{s-1}X^{(s-1)p^m} + \cdots + v_1 X^{p^m} + v_0\right) + p^m S(X)$$

where $v_\ell \in \mathcal{O}_M$, for $\ell = 0, \ldots, s-1$, $|v_0| = 1$, and $S(X) \in \mathcal{O}_M[X]$.

If $\alpha$, $\alpha' \in \mathcal{O}_F$, then define $\alpha \equiv \alpha' \mod p^m$ if $\alpha - \alpha' \in p^m\mathcal{O}_F$. Consider the following claims:

**Claim 1**: There exists polynomials $P(X)$, $Q(X) \in \mathcal{O}_M[X]$ such that

i) $p \equiv \mu^{sp^a} P(\mu^{p^m}) \mod p^m$, and

ii) $\pi_n \equiv \mu^{rp^\alpha} Q(\mu^{p^m}) \mod p^m$.

PROOF OF CLAIM 1: : The form of the polynomial $V(X)$ shows that $p(\nu_{s-1}\mu^{(s-1)p^m} + \cdots + \nu_1\mu^{p^m} + \nu_0) \equiv -\mu^{sp^m} \mod p^m$. Since $|\nu_0| = 1$, $\nu_{s-1}\mu^{(s-1)p^m} + \cdots + \nu_1\mu^{p^m} + \nu_0$ has an inverse in $\mathcal{O}_{M(\mu^{p^m})}$. Thus there exists a $P(X) \in \mathcal{O}_M[X]$ such that $(\nu_{s-1}\mu^{(s-1)p^m} + \cdots + \nu_1\mu^{p^m} + \nu_0)^{-1} = -P(\mu^{p^m})$. Hence $p \equiv \mu^{sp^m} P(\mu^{p^m}) \mod p^m$, proving i).

As above, the form of the polynomial $U(X)$ shows that there exists a $Q_0(X) \in \mathcal{O}_{M'}[X]$ such that $\pi_n \equiv \mu^{rp^m} Q_0(\mu^{p^m}) \mod p^m$. The polynomial $Q_0(X)$ can be written as $Q_0(X) = Q_{1,0}(X) + \pi_n Q_{2,0}(X)$, where $Q_1(X) \in \mathcal{O}_M[X]$ and $Q_2(X) \in \mathcal{O}_{M'}[X]$. Hence $\pi_n \equiv \mu^{rp^m}(Q_{1,0}(\mu^{p^m}) + \pi_n Q_{2,0}\mu^{p^m})) \mod p^m$.

Proceeding by induction, assume

$$\pi_n \equiv \mu^{p^m}(Q_{1,v}(\mu^{p^m}) + \pi_n \mu^{vrp^m} Q_{2,v}(\mu^{p^m})) \mod p^m.$$

Then

$$\pi_n \equiv \mu^{p^m}(Q_{1,v}(\mu^{p^m}) + \mu^{(v+1)rp^m}(Q_{1,v}(\mu^{p^m}) + \pi_n \mu^{vrp^m} Q_{2,v}(\mu^{p^m}) Q_{2,v}(\mu^{p^m}) \mod p^m.$$

Write $Q_{2,v}(X) = Q'_{2,v}(X) + Q''_{2,v}(X)$, where $Q'_{2,v} \in \mathcal{O}_M[X]$ and $Q''_{2,v}(X) \in \mathcal{O}_{M'}[X]$.

Set $Q_{1,v+1}(X) = Q_{1,v}(X) + X^{(v+1)r}Q_{1,v}(X)Q'_{2,v}(X) \in \mathcal{O}_M[X]$ and $Q_{2,v+1}(X) = X^{vr}Q_{2,v}(X) + Q_{2,v}(X)Q''_{2,v}(X) \in \mathcal{O}_{M'}[X]$. Substituting and evaluating at $\mu^{p^m}$ yields

$$\pi_n \equiv \mu^{rp^m}(Q_{1,v+1}(\mu^{p^m}) + \pi_n \mu^{(v+1)rp^m} Q_{2,v+1}(\mu^{p^m})) \mod p^m.$$

Now, letting $v$ be large enough so that $\mu^{krp^m} \in p^m\mathcal{O}_{M'}$, then $\pi_n \equiv \mu^{rp^m}(Q_{1,v}(\mu^{p^m}))$ mod $p^m$, with $Q_{1,v}(X) \in \mathcal{O}_M[X]$, proving ii).

**Claim 2:** For any $W(X) \in \mathcal{O}_M[X]$, there exists a polynomial $\widetilde{W}(X) \in \mathcal{O}_M[X]$, such that

$$(\widetilde{W}(X))^p \equiv W(X^p) \mod p\mathcal{O}_M[X].$$

(This congruence occurs if $(\widetilde{W}(X))^p - W(X^p) \in p\mathcal{O}_M[X]$.)

Proof of Claim 2: The residue field $\widetilde{M}$ is a perfect field [5, Theorem 4.6], so there is a unique solution to the polynomial $X^p - \bar{\omega}$, for all $\omega \in \mathcal{O}_M$, where $\omega \mapsto \bar{\omega}$ is the canonical map of $\mathcal{O}_M$ onto $\widetilde{M}$. Thus any pre-image $\tilde{\omega}$ of such a solution must satisfy $\tilde{\omega}^p - \omega \in p\mathcal{O}_M$.

So if $W(X) = \omega_m X^m + \cdots + \omega_1 X + \omega_0$, choosing $\tilde{\omega}_0, \ldots, \tilde{\omega}_m$ as above yields

$$(\widetilde{W}(X))^p \equiv W(X^p) \mod p\mathcal{O}_M[X],$$

which proves claim.

To finish the proof of Case 2, construct a finite sequence $\alpha_1, \ldots, \alpha_m \in L$ satisfying

i) $\alpha_i = A_i(\mu^{p^{m-i}})$, and

ii) $\alpha_i^p \equiv \zeta_N + \mu^{S(p^m+p^{m-1}+\cdots+p^{m-i+1})} B_i(\mu^{p^{m-i}}) \mod p^m$, where $A_i(X)$, $B_i(X) \in \mathcal{O}_M[X]$.

Let $Q(X) \in \mathcal{O}_M[X]$ be the polynomial satisfying $\pi_n \equiv \mu^{rp^m} Q(\mu^{p^m}) \mod p^m$, as in Claim 1. Set $A_1(X) = 1 + X^r \widetilde{Q}(X)$, where $\widetilde{Q}(X)$ is obtained from $Q(X)$. Set $\alpha_1 = A_1(\mu^{p^{m-1}})$. Then $\alpha_1^p = 1 + \mu^{rp^m} Q(\mu^{p^m}) + pT(\mu^{p^{m-1}})$, with $T(X) \in \mathcal{O}_M[X]$, by Claim 2. Thus

$$\alpha_1^p \equiv \zeta_n + \mu^{sp^m} P(\mu^{p^m}) T(\mu^{p^{m-1}}) \mod p^m,$$

by Claim 1. Set $B_1(X) = P(X^p)T(X) \in \mathcal{O}_M[X]$.

By induction, suppose $\alpha_i \in F$ satisfying i) and ii). Set $\alpha_{i+1} = \alpha_i - \mu^{s(p^m+\cdots+p^{m-i})} \widetilde{B}_i(\mu^{p^{m-i-1}})$. Thus $\alpha_{i+1}$ is of the form $A_{i+1}(\mu^{p^{m-\alpha-1}})$, for some $A_{i+1}(X) \in \mathcal{O}_M[X]$. Also, $\alpha_{i+1}^p = \alpha_i^p - \mu^{s(p^{\alpha-1}+\cdots+p^{\alpha-i+1})} B_i(\mu^{p^{m-i}}) + p\mu^{s(p^{m-1}+\cdots+p^{m-i})} T_i(\mu^{p^{m-i}})$ for some $T_i(X) \in \mathcal{O}_M[X]$. Hence by induction,

$$\alpha_{i+1}^p \equiv \zeta_n + p\mu^{s(p^{m-1}+\cdots+p^{m-i})} T_i(\mu^{p^{m-i-1}}) \mod p^m.$$

Thus by Claim 1,

$$\alpha_{i+1}^p \equiv \zeta_n + \mu^{s(p^m+\cdots+p^{m-i})} P(\mu^{p^m}) T_i(\mu^{p^{m-i-1}}) \mod p^m.$$

Finally, the element $\alpha_m \in F$ satisfies

$$|\alpha_m^p - \zeta_n| \leq |\mu^{s(p^m+\cdots+p)}| = p^{-\frac{s(p^m+\cdots+p)}{sp^m}} = p^{-\left(\frac{p}{p-1} - \frac{1}{p^{m-1}(p-1)}\right)},$$

and so this proves Case 2 and Proposition 3.14.

□

COROLLARY 3.14.1. *For all $N \geq n$, $|\mathcal{D}^{-1}_{LK_{N+1}/LK_N}| = 1$.*

PROOF: Obvious, since $|\mathcal{D}_{LK_{N+1}/LK_N}| = 1$, and $\mathcal{D}^{-1}_{LK_{N+1}/LK_N}\mathcal{D}_{LK_{N+1}/LK_N} \subseteq \mathcal{O}_{LK_{N+1}}$.

□

Now, let $\varepsilon$ be a positive real number. Since the valuation on $L$ is dense, there is a $\eta_n \in L$ such that $1 < |\eta_n| < 1 + \varepsilon$. Similarly, there is a $\beta_{n+1} \in LK_{n+1}$ with $1 < |\beta_{n+1}| < \frac{1+\varepsilon}{|\eta_n|}$. The above property of the codifferent of the extension $LK_{n+1}$ over $L$ implies that $|Tr_{LK_{n+1}/L}(\beta_{n+1})| > 1$. Let $Tr_{LK_{n+1}/2}(\beta_{n+1}) = \rho_r \in L$. Let $\gamma_n \in L$ such that $\gamma_n\rho_n = \eta_n$. Note that $Tr_{LK_{n+1}/L}(\gamma_n\beta_{n+1}) = \eta_n$, and $|\gamma_n\overline{\beta_{n+1}}| < |\gamma_n| \cdot \frac{1+\varepsilon}{|\eta_n|} = |\rho_n|^{-1} \cdot (1+\varepsilon) < 1 + \varepsilon$. Set $\eta_{n+1} = \gamma_n\beta_{n+1}$. Inductively, construct a sequence of elements $\{\eta_N\}_{N=n}^{\infty}$ of $LK_\infty$ such that $|\eta_N| < 1 + \varepsilon$, and $Tr_{LK_{N+1}/LK_N}(\eta_{N+1}) = \eta_N$, for all $N \geq n$.

PROPOSITION 3.15. *Let $t_\eta : LK_\infty \to L$ be defined by $t_\eta(\alpha) = \varinjlim_{N \geq n} Tr_{LK_N/L}(\eta_N \alpha)$. Then $t_\eta$ is a well-defined $L$-linear continuous projection of $LK_\infty$ onto $L$.*

PROOF: Let $\alpha \in LK_N$, for some $N \geq n$. For any $m \geq N$,

$$\begin{aligned} Tr_{LK_m/L}(\eta_m \alpha) &= Tr_{LK_N/L}(Tr_{LK_m/LK_N}(\eta_m \alpha)) \\ &= Tr_{LK_N/L}(\alpha \cdot Tr_{LK_m/LK_N}(\eta_m)) \\ &= Tr_{LK_N/L}(\alpha \cdot \eta_N). \end{aligned}$$

Thus $t_\eta$ is well-defined.

It is clear that $t_\eta$ is $L$-linear. If $\alpha \in LK_N$, then $|Tr_{LK_N/L}(\eta_N \alpha)| \leq |\eta_N \alpha| < |\alpha|(1+\varepsilon)$, so it is also clear that $t_\eta$ is continuous on $LK_\infty$.

□

Let $\bar{t}_\eta$ be the continuous extension of the map $t_\eta$ to $\overline{LK}_\infty$, extended naturally to a map of $\ell_1(\overline{LK}_\infty, G) \longrightarrow \ell_1(k, G)$. The following lemma shows that $t_\eta$ maps $J(\ell_1(\overline{LK}_\infty, G))$ into $J(\ell_1(k, G))$.

LEMMA 3.16.1. *Define $\|f\|_C = \sup_{\theta \in \widehat{G}_{LK_\infty}} |\theta(f)|$, for all $f \in \ell_1(\overline{LK}_\infty, G)$. Then $\|\bar{t}_\eta(f)\|_C \leq \|f\|_C \cdot (1+\varepsilon)$.*

PROOF: By continuity of $\bar{t}_\eta$, it suffices to show this result for all $f = \sum_{g \in G_m} \alpha_g g \in (LK_N)G_N$, for any $N \geq n$.

For any $\sigma \in \text{Gal}\left(\frac{LK_N}{L}\right)$, $\sigma$ is an isometry, so for all $\theta \in \widehat{G}_{\overline{LK_\infty}}$, $|\sum_{g \in G_N} \sigma(\alpha_g)\sigma(\theta(g))| \leq \|f\|_C$.

Let $\psi \in \widehat{G}$ be fixed with $\ker \psi = \langle 1 \rangle$. Then for all $\sigma \in \text{Gal}\left(\frac{LK_N}{L}\right)$, there exists an isomorphism, $\theta_\sigma : \Gamma \to \Gamma$ such that $(\sigma \circ \theta_\sigma \circ \psi)(g) = \psi(g)$, for all $g \in G_N$. Since $\ker \theta_\sigma = \langle 1 \rangle$, any element $\tau \in \widehat{G}$ can be written as $\tau = \theta_\sigma \circ \phi$, for some $\phi \in \widehat{G}$. Hence $|\sum_{g \in G_N} \sigma(\alpha_g)\phi(g)| \leq \|f\|_C$, for all $\phi \in \widehat{G}_{\overline{LK_\infty}}$. Therefore, $|\sum_{g \in G_N} \sigma(\eta_N)\sigma(\alpha_g)\phi(g)| \leq \|f\|_C \cdot (1+\varepsilon)$, for all $\phi \in \widehat{G}_{\overline{LK_\infty}}$, and all $\sigma \in \text{Gal}\left(\frac{LK_N}{L}\right)$. Hence

$$|\sum_{g \in G_N} \sum_{\sigma \in \text{Gal}(LK_N/L)} \sigma(\eta_N \alpha_g)\phi(g)| \leq \|f\|_C(1+\varepsilon), \quad \forall \phi \in \widehat{G}_k,$$

and thus $|\sum_{g \in G_N} t_\eta(\alpha_g)\phi(g)| \leq \|f|_C(1+\varepsilon)$.

□

The converse to Theorem 3.13 now can be shown.

THEOREM 3.16. *Suppose $k$ is a complete $p$-adic field such that $k \cap \widehat{\mathbb{Q}_p} = L$ is a field of different zero over $\mathbb{Q}_p$. Then $\ell_1(k, G)$ is a field of different zero over $\mathbb{Q}_p$. Then $\ell_1(k, G)$ is not semiprimitive.*

PROOF: By Theorem 3.9, there eixsts an $f = \sum_{g \in G} \alpha_g g \in J(\ell_1(\overline{LK_\infty}, G))$ with $\alpha_1 = 1$. By the lemma, $\bar{t}_\eta(f) \in J(\ell_1(k, G))$ with $\bar{t}_\eta(\alpha_1) = 1$, so $\bar{t}_\eta(f) \neq 0$.

□

Thus for complete $p$-adic fields $k$ with $k \cap \widehat{\mathbb{Q}_p}$ of different zero over $\mathbb{Q}_p$, the abelian groups containeed in the class $\mathcal{X}_k$ are exactly those with no $C_{p^\infty}$-subgroups. However, it is sill an open question [4] as to whether $G \cong \bigoplus_1^N C_{p^\infty} \in \mathcal{X}_k$ for $N > 1$ when $k \cap \widehat{\mathbb{Q}_p}$ is not of different zero over $\mathbb{Q}_p$. If so, $\mathcal{X}_k$ would contain the entire class of abelian groups, by Corollary 3.5.1.

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# CHAPTER IV

# SEMIPRIMITIVE $p$-ADIC $\ell_1$-GROUP ALGEBRAS OVER SOLVABLE GROUPS

If a group $G$ is solvable, then it has a non-trivial normal subgroup $H$. Suppose that $f = \sum_{g\in G} \alpha_g g \in \ell_1(\Omega_p, G)$. Then if $T$ is a transversal of $H$ in $G$, $f = \sum_{t\in T} \left( \sum_{h\in H} \alpha_{ht} h \right) t$. Let $\varepsilon > 0$. Since $\|\alpha_g\| \geq \varepsilon$ for only finitely many $g \in G$, $\| \sum_{h\in H} \alpha_{ht} h \| \geq \varepsilon$ for only finitely many $t \in T$. Furthermore, for any fixed $t \in T$, $\|\alpha_{ht}\| \geq \varepsilon$ for only finitely many $h \in H$. Thus $\sum_{h\in H} \alpha_{ht} h \in \ell_1(\Omega_p, H)$ for every $t \in T$. Thus the sub-algebra of $\ell_1(\Omega_p, G)$ consisting of finite sums $\sum_{t\in T} a_t t$, with $a_t \in \ell_1(\Omega_p, H)$, $\forall t \in T$, is dense in $\ell_1(\Omega_p, G)$. The first section of this chapter defines $\ell_1(\Omega_p, G)$ as the completion of this crossed product of $\ell_1(\Omega_p, H)$ by $\frac{G}{H}$. The remainder of the chapter is devoted to finding conditions for $H \in \mathcal{X}_{\Omega_p}$ to imply $G \in \mathcal{X}_{\Omega_p}$.

## A. Complete crossed products.

Let $R$ be a normed ring. Suppose that $A$ is a complete subring of $R$ with the same 1, and that $G$ is a subgroup of the group of units of $R$ with $\|g\|_R = 1$, $\forall g \in G$. Suppose also that $G$ acts on $A$ by conjugation, and that for any $g \in G$, the map $\phi_g : A \to A$ given by $a \mapsto g^{-1}ag$ is an isometry. Thus $H \leq A \cap G$ is clearly a normal subgroup of $G$. Let $T$ be a transversal of $H$ in $G$ containing 1. Suppose finally that $R = \bigoplus_{t\in T} At$, i.e. that every element of $R$ can be uniquely written as $\sum_{t\in T} a_t t$, with $a_t \in A$, $\forall t \in T$ and almost all $a_t = 0$. Then $R$ is called the crossed product of $A$ by $\frac{G}{H}$, denoted $R = A[G]$. If $R$ is non-Archimedean, then the maximum norm given by $\|\sum_{t\in T} a_t t\|_R = \max_{t\in T} \|a_t\|_R$ is one possible norm for $R$.

Suppose that $R = A[G]$ is a crossed product with the maximum norm. Let $A\{G\} = \left\{\sum_{t\in T} a_t t : a_t \in A,\ \forall t \in T, \text{ and } \|a_t\|_R \to 0\right\}$, and let $\|\cdot\|_{A\{G\}}$ be the supremum norm $\|\sum_{t\in T} a_t t\|_{A\{G\}} = \sup_{t\in T} \|a_t\|_R$ on these formal sums. The following proposition holds:

PROPOSITION 4.1.

*(i)* *$A\{G\}$ is a ring, and*

*(ii)* *$A\{G\}$ is the completion of $A[G]$ when $A[G]$ has the maximum norm.*

PROOF: (i): The addition in $A\{G\}$ is componentwise, with zero element having all $a_t = 0$. Multiplication in $A\{G\}$ is defined using a property of the transversal $T$. Let $t \in T$. For any $u \in T$, there are elements $h_{t,u} \in H$, and $v_{t,u} \in T$ such that $uv_{t,u} = h_{t,u}t$. Since $T$ is a transversal, $\{v_{t,u}\}_{u\in T} = T$. Hence if $\sum_{u\in T} a_u u$, $\sum_{v\in T} b_v v \in A\{G\}$, then define their product by

$$\left(\sum_{u\in T} a_u u\right)\cdot\left(\sum_{v\in T} b_v v\right) = \sum_{u\in T}\sum_{v\in T}(a_u u)(b_v v) = \sum_{u\in T}\sum_{v\in T} a_u(ubu^{-1})uv$$

$$= \sum_{t\in T}\left(\sum_{u\in T} a_u(ub_{v_{t,u}}u^{-1})h_{t,u}\right)t.$$

For any $\varepsilon > 0$, it is clear that $\| \sum_{u \in T} a_u (u b_{v_{t,u}} u^{-1}) h_{t,u} \|_{A\{G\}} \le \sup_{u \in T} \|a_u b_{v_{t,u}}\|_R < \varepsilon$ for almost all $a_u b_{v_{t,u}}$, for all $t \in T$. The identity element has $a_1 = 1$, and all other $a_t = 0$. Distributivity laws are part of the definition. This multiplication is continuous in $A\{G\}$ since $\forall f_1, f_2 \in A\{G\}$, $\|f_1 \cdot f_2\|_{A\{G\}} \le \|f_1\|_{A\{G\}} \|f_2\|_{A\{G\}}$. Thus $A\{G\}$ is a normed ring.

(ii): It is clear that $A[G]$ is a subring of $A\{G\}$, and that the subspace topology on $A[G]$ is the one given by the maximum norm. Also any infinite sum $\sum_{t \in T} a_t t$ can be approximated within any $\varepsilon > 0$ by a finite sum $\sum_{t \in T} b_t t \in A[G]$ by letting $b_t = a_t$ whenever $\|a_t\|_R \ge \varepsilon$ and $b_t = 0$ otherwise. Thus $A[G]$ is dense in $A\{G\}$.

Now, let $\varepsilon > 0$. Any Cauchy sequence $\left\{ \sum_{t \in T} a_{n,t} t \right\}_{n=1}^{\infty} \subset A\{G\}$ has the property that for $n, m$ large enough,

$$\|\sum_{t \in T} a_{n,t} t - \sum_{t \in T} a_{m,t} t\|_{A\{G\}} \le \varepsilon, \quad \text{so} \quad \sup_{t \in T} \|a_{n,t} - a_{m,t}\|_R \le \varepsilon.$$

Thus $\{a_{n,t}\}_{n=1}^{\infty}$ is a Cauchy sequence in $A$, for all $t \in T$. Since $A$ is complete, $\lim_{n \to \infty} a_{n,t} = a_t \in A$, and $\|a_t\|_R \to 0$, so $\sum_{t \in T} a_t t \in A\{G\}$ is the limit of $\left\{ \sum_{t \in T} a_{n,t} t \right\}_{n=1}^{\infty}$. Thus $A\{G\}$ is complete.

□

Ideals in $A$ are related to ideals of $A\{G\}$. If $J$ is an ideal of $A\{G\}$, then $J \cap A$ is an ideal of $A$. Conversely, if $I$ is a closed ideal of $A$ such that $g^{-1} I g = I$, $\forall g \in G$, then let $I\{G\} = \left\{\sum_{t\in T} \alpha_t t \in A\{G\} : \alpha_t \in I,\ \forall t \in T\right\}$. It is clear that $I\{G\}$ is additively closed, and if $\sum_{u\in T} \alpha_u u \in I$, then for any $\sum_{r\in T} b_v v \in A\{G\}$, the product $\left(\sum_{u\in T} \alpha_u u\right)\left(\sum_{v\in T} b_v v\right) = \sum_{u\in T}\sum_{v\in T} \alpha_u (u b_v u^{-1}) uv = \sum_{t\in T}\left(\sum_{u\in T} \alpha_u (u b_{v_{t,u}} u^{-1}) h_{t,u}\right) t$. Each $\sum_{u\in T} \alpha_u (u b_{v_{t,u}} u^{-1}) h_{t,u}$ is a convergent sum of elements from $I$, so since $I$ is closed, the sum must be in $I$. Thus $I\{G\}$ is an ideal of $A\{G\}$. Furthermore, since $I$ is closed, $I\{G\}$ must also be closed.

Consider the quotient ring $\frac{A\{G\}}{I\{G\}}$. Topologically $\frac{A\{G\}}{I\{G\}}$ is the quotient of a complete space modulo a closed subset, hence must be complete. Also, consider $I[G] = \left\{\sum_{t\in T} \alpha_t t \in A[G] : \alpha_t \in I,\ \forall t \in T\right\}$. In a similar fashion to the above, $I[G]$ is an ideal of $A[G]$. By a well-known result of crossed products [3, Lemma 1.4], $\frac{A[G]}{I[G]} \cong \frac{A}{I}\left[\frac{G}{H_I}\right]$, where $H_I = \{h \in H : h = 1 + \alpha,\ \exists \alpha \in I\} = H \cap (1 + I)$.

Furthermore, if $\frac{A[G]}{I[G]}$ has the quotient topology given by $\|f + I[G]\|_q = \inf_{s\in I[G]} \|f + s\|_{A[G]}$, $\forall f \in A[G]$, and $\frac{A}{I}\left[\frac{G}{H_I}\right]$ has the maximum topology on its formal sums, then this isomorphism is an isometry. If $f \in A[G]$, then

$$
\begin{aligned}
\|f + I[G]\|_q &= \inf_{\Sigma \alpha_t t \in I[G]} \|\sum_{t\in T} (a_t + \alpha_t) t\|_{A[G]} \\
&= \inf_{\Sigma \alpha_t t \in I[G]} \left(\max_{t\in T} \|a_t + \alpha_t\|_{A[G]}\right) \\
&= \max_{t\in T} \left(\inf_{\alpha_t \in I} \|a_t + \alpha_t\|_{A[G]}\right) \\
&= \max_{t\in T} \left(\|a_t + I\|_q\right) \\
&= \|\pi(f)\|_{\frac{A}{I}\left[\frac{G}{H_I}\right]},
\end{aligned}
$$

where $\pi$ is the isomorphism $\frac{A[G]}{I[G]} \longrightarrow \frac{A}{I}\left[\frac{G}{H_I}\right]$. Considering $\frac{A\{G\}}{I\{G\}}$ as a normed ring using the quotient norm $\|f + I\{G\}\|_q = \inf_{j \in I\{G\}} \|f + j\|_{A\{G\}}$, it is clear that since $A[G]$ is dense in $A\{G\}$, $\frac{A[G]}{I[G]}$ is dense in $\frac{A\{G\}}{I\{G\}}$. Thus $\frac{A\{G\}}{I\{G\}}$ is isometric to the completion of $\frac{A}{I}\left[\frac{G}{H_I}\right]$, which is just $\frac{A}{I}\left\{\frac{G}{H_I}\right\}$, by Proposition 4.1.

**B. Extension theorems.**

Suppose $H$ is a group, and that $H \in \mathcal{X}_{\Omega_p}$. Suppose $G$ is a group extension of $H$ such that $H \triangleleft G$. Under what conditions does $G$ belong to $\mathcal{X}_{\Omega_p}$? Properties of complete crossed products are used to show that if $\frac{G}{H}$ is a finite group, or an infinite cyclic group then semiprimitivity of $\ell_1(\Omega_p, G)$ follows from that of $\ell_1(\Omega_p, H)$. Also, investigations in character theory yield a similar result when $\frac{G}{H}$ is an abelian $p'$-group.

The first result is a generalized form of Maschke's Theorem for group rings in the larger realm of crossed products.

THEOREM 4.2. *(Maschke's Theorem for crossed products).*

*Let $R = A[G]$ be a crossed product, with $H = A \cap G$. Suppose that if $[G : H] = n$, then $n^{-1} \in A$.*

*Let $V, W$ be right $R$-modules with $W \subseteq V$. Then if $V = W \oplus U$, for some $A$-submodule $U$ of $V$, then $V = W \oplus U'$, for some $R$-submodule $U'$ of $V$. (In* *other words, if $V$ is completely reducible as an $A$-module, then $V$ is completely reducible as an $R$-module.)*

PROOF: Suppose that $T$ is a transversal of $H$ in $G$, containing 1. Let $\mu : V \to W$ be the $A$-module projection given by the decomposition $V = W \oplus U$. Define

$$\pi : V \to W \quad \text{by} \quad \pi(v) = n^{-1}\Big(\sum_{t\in T}\mu(vt)t^{-1}\Big), \quad \forall v \in V.$$

$\pi$ is clearly well-defined, and is additive since $\forall v_1,\ v_2 \in V$,

$$\begin{aligned}\pi(v_1+v_2) &= n^{-1}\Big(\sum_{t\in T}\mu((v_1+v_2)t)t^{-1}\Big) = n^{-1}\Big(\sum_{t\in T}\mu(v_1t+v_2t)t^{-1}\Big)\\ &= n^{-1}\Big(\sum_{t\in T}(\mu(v_1t)+\mu(v_2t))t^{-1}\Big) = n^{-1}\Big(\sum_{t\in T}(\mu(v_1t)t^{-1}+\mu(v_2t)t^{-1})\Big)\\ &= n^{-1}\Big(\sum_{t\in T}\mu(v_1t)t^{-1}+\sum_{t\in T}\mu(v_2t)t^{-1}\Big) = \pi(v_1)+\pi(v_2).\end{aligned}$$

Note that for any $g \in G$, and for all $t \in T$, $gt = u_{gt}h_{gt}$, for some $u_{gt} \in T$, $h_{gt} \in H$. Thus

$$\begin{aligned}\pi(vg) &= n^{-1}\Big(\sum_{t\in T}\mu(vgt)t^{-1}\Big)\\ &= n^{-1}\Big(\sum_{t\in T}\mu(vu_{gt}h_{gt})t^{-1}\Big) = n^{-1}\Big(\sum_{t\in T}\mu(vu_{gt})h_{gt}t^{-1}\Big)\\ &= n^{-1}\Big(\sum_{t\in T}\mu(vu_{gt})u_{gt}^{-1}g\Big) = \pi(v)\cdot g.\end{aligned}$$

Also, $\forall a \in A$,

$$\begin{aligned}\pi(va) &= n^{-1}\Big(\sum_{t\in T}\mu(vat)t^{-1}\Big) = n^{-1}\Big(\sum_{t\in T}\mu(vt(t^{-1}at))t^{-1}\Big)\\ &= n^{-1}\Big(\sum_{t\in T}\mu(vt)(t^{-1}at)t^{-1}\Big) = n^{-1}\Big(\sum_{t\in T}\mu(vt)t^{-1}a\Big) = \pi(v)\cdot a.\end{aligned}$$

Thus since $\pi$ is additive, $\pi(vr) = \pi(v)r$ for all $r \in R$, $v \in V$. Hence $\pi$ is an $R$-module homomorphism of $V$ into $W$. Furthermore, if $w \in W$, $\pi(w) = n^{-1}\left(\sum_{t\in T}\mu(wt)t^{-1}\right) = n^{-1}\left(\sum_{t\in T} w\right) = w$. Hence $\pi(V) = W$, and for all $v \in V$, $\pi - \pi(v) \in \ker\pi$. Thus since $\ker\pi$ is an $R$-submodule of $V$, $V = W \oplus \ker\pi$ is a direct sum of $R$-submodules.

□

The next result shows that the Jacobson radical of a crossed product behaves well under finite extensions of the group $G$.

THEOREM 4.3. *Let $R = A[G]$ be a crossed product. Suppose $|\frac{G}{H}| = n$, and that $n^{-1} \in A$. Then $J(R) = J(A) \cdot R$.*

PROOF: $\underline{J(A) \subset J(R)}$: Let $V$ be an irreducible $R$-module. Then $V = vR$, for any non-zero $v \in V$, and so $V = v \cdot \left(\bigoplus_{t\in T} tA\right) = \bigoplus_{t\in T} vtA$. So $V$ is a finitely generated $A$-module. Thus by Nakayama's Lemma [2, Lemma 7.4.2], $VJ(A) \neq V$, since $V \neq 0$. If $t \in T$, then since $t$ is a unit of $R$, and conjugation of $A$ by $t$ is a ring isomorphism of $A$ into itself, $(VJ(A))t = Vt(t^{-1}J(A)t) = VJ(A)$. Thus $VJ(A)$ is an $R$-submodule of $V$. Since $V$ is irreducible and $VJ(A) \neq V$, $VJ(A) = 0$. Thus $VJ(A) = 0$ for all irreducible $R$-modules $V$, hence $J(A) \subseteq J(R)$.

$\underline{J(R) \subset J(A) \cdot R}$: Let $W$ be an irreducible $A$-module. Let $V = W \otimes_A R = W \otimes_A \left(\bigoplus_{t\in T} At\right) = \bigoplus_{t\in T}(W \otimes_A t)$. Each $W \otimes_A t$ is also an irreducible $A$-module, so $V$

is the direct sum of irreducible $A$-modules. Therefore, $V$ is completely reducible as an $A$-module. Hence by Theorem 4.2, $V$ is completely reducible as an $R$-module. Therefore $VJ(R) = 0$.

Suppose $\sum_{t\in T} a_t t \in J(R)$. Then for all $w \in W$, $0 = (w\otimes 1)\Big(\sum_{t\in T} a_t t\Big) = \sum_{t\in T}(wa_t\otimes t)$. Thus $wa_t = 0$, $\forall t \in T$. Since $W$ is an arbitrary irreducible $A$-module, each $a_t \in J(A)$, and so $J(R) \subseteq \sum_{t\in T} J(A)t = J(A)R$.

□

Theorem 4.3 implies the following corollaries.

COROLLARY 4.3.1. *Let $G$ be a group with $H \triangleleft G$ and $\frac{G}{H}$ finite. Then if $\ell_1(k,H)$ is semiprimitive, so is $\ell_1(k,G)$.*

COROLLARY 4.3.2. *If $|G| = n$ and $A$ is any semiprimitive algebra with $n^{-1} \in A$, then the group ring $AG$ is semiprimitive.*

The next result shows that the Jacobson radical of a complete crossed product also behaves well under infinite cyclic group extensions. The proof requires the next two lemmas.

LEMMA 4.4.1. *Let $R = A\{G\}$ be a complete crossed product. Then $J(R) \cap A \subseteq J(A)$.*

PROOF: Suppose $a \in J(R) \cap A$. Then for any $b \in A$, $ab$ is quasi-regular in $R$, so there is an $f \in R$ such that $(1-ab)f = 1$. Now, $f = \sum_{t \in T} f_t t$, $f_t \in A$, $\forall t \in T$, and so since $T$ contains 1, $(1-ab)f_1 = 1$ and $(1-ab)f_t = 0$, for $t \neq 1$. Therefore, $f_1$ is a quasi-inverse for $(1-ab)$ in $A$, so since $b$ was arbitrary, $a \in J(A)$.

□

LEMMA 4.4.2. *Let $R = A\{G\}$ be a crossed product with $A$ an algebra over a $p$-adic field $k$. Suppose $\frac{G}{H}$ is infinite cyclic with transversal $\{x^i\}_{i \in \mathbf{Z}}$. Let $\beta \in k$ with $|\beta| = 1$. Then the map $\varphi_\beta(x) = \beta x$ extends $A$-linearly to an isometric ring automorphism of $R$ defined by*

$$\varphi_\beta\Big(\sum_{i \in \mathbf{Z}} a_i x^i\Big) = \sum_{i \in \mathbf{Z}} a_i \beta^i x^i.$$

PROOF: Since

$$|\beta| = 1, \ \|\varphi_\beta\Big(\sum_{i \in \mathbf{Z}} a_i x^i\Big)\| = \|\sum_{i \in \mathbf{Z}} a_i x^i\|.$$

Thus $\varphi_\beta$ is an isometry. Furthermore, $\varphi_{\beta^{-1}} = (\varphi_\beta)^{-1}$ is obvious, so $\varphi_\beta$ is a bijection. $\varphi_\beta$ is also clearly additive. To see that $\varphi_\beta$ is multiplicative, let $\sum_{i \in \mathbf{Z}} b_i x^i$, $\sum_{j \in \mathbf{Z}} c_j x^j \in R$ with $b_i, c_j \in A$, $\forall i, j$. Consider

$$\varphi_\beta\Big[\Big(\sum_{i \in \mathbf{Z}} b_i x^i\Big)\Big(\sum_{j \in \mathbf{Z}} c_j x^j\Big)\Big] = \varphi_\beta\Big[\Big(\sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} b_i x^i c_j x^j\Big)\Big]$$

$$= \varphi_\beta\Big[\sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} b_i (x^i c_j x^{-i}) x^{i+j}\Big] = \sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} b_i (x^i c_j x^{-i}) \beta^{i+j} x^{i+j}$$

(since $x^i c_j x^{-i} \in A$, $\forall i, j$, and $\varphi_\beta$ is $A$-linear).

$$= \sum_{i\in\mathbf{Z}} \sum_{i\in\mathbf{Z}} (b_i\beta^i x^i)(c_j\beta^j x^j) \quad \text{(since } \beta \in \Omega_p,\ \beta \text{ is central in } R\text{)}$$

$$= \Big(\sum_{i\in\mathbf{Z}} b_i\beta^i x^i\Big)\Big(\sum_{i\in\mathbf{Z}} c_j\beta^j x^j\Big) = \varphi_\beta\Big(\sum_{i\in\mathbf{Z}} b_i x^j\Big)\varphi_\beta\Big(\sum_{i\in\mathbf{Z}} c_j\beta^j x^j\Big).$$

Thus $\varphi_\beta$ is a ring automorphism of $R$ onto itself which is also an isometry.

□

THEOREM 4.4. *Let $R = A\{G\}$ be a complete crossed product, with $A$ an algebra over a p-adic field $k$ with infinite residue field $\tilde{k}$. Then if $\frac{G}{H}$ is an infinite cyclic group, $J(R) \subseteq J(A)\{G\}$. In particular, if $J(A) = 0$, then $J(R) = 0$.*

PROOF: Since $\frac{G}{H}$ is an infinite cyclic group, there is an $x \in G$ such that $\{x^i\}_{i\in\mathbf{Z}}$ is a transversal of $H$ in $G$. Let $f \in J(R)$. If $f = \sum_{i\in\mathbf{Z}} a_i x^i$, with $a_i \in A$, $\forall i \in \mathbf{Z}$, then it suffices to show that $a_0 \in J(A)$. (The same proof using $fx^{-i}$ will show $a_i \in J(A)$, for any $i \in \mathbf{Z}$.)

For a fixed $m \in \mathbf{Z}$, $f = \sum_{i=-m}^{m} a_i x^i + \varepsilon_m$, with $\varepsilon_m = \sum_{i=-\infty}^{-m-1} a_i x^i + \sum_{m+1}^{\infty} a_i x^i \in R$. Since $f \in R$, $\|\varepsilon_m\| \to 0$ as $m \to \infty$. Let $\beta \in k$ with $|\beta| = 1$, and apply $\varphi_\beta$ to $f - \varepsilon_m$, to obtain

$$\sum_{i=-m}^{m} a_i\beta^i x^i = f_\beta + \varepsilon_{m,\beta}, \quad \text{where} \quad f_\beta = \varphi_\beta(f), \quad \text{and} \quad \varepsilon_{m,\beta} = \varphi_\beta(\varepsilon_m). \qquad (*)$$

Since $\varphi_\beta$ is an isometric ring automorphism, $f_\beta \in J(R)$ and $\|\varepsilon_{m,\beta}\|_R = \|\varepsilon_m\|_R$.

Since $\bar{k}$ is infinite, it is possible to choose $2m+1$ distinct $\beta_{-m}, \cdots, \beta_0, \cdots, \beta_m$ with $|\beta_i - \beta_j| = 1$, for $-m \le i \ne j \le m$. The system of equations in matrix form obtained by writing down $(*)$ for all of these choices of $\beta$ is

$$(a_{-m}x^{-m}, \ldots, a_0, \ldots, a_{-m}x^m)M = (s_{-m}, \ldots, s_0, \ldots, s_m), \qquad (**)$$

where

$$M = \begin{pmatrix} \beta_{-m}^{-m} & \cdots & \beta_0^{-m} & \cdots & \beta_m^{-n} \\ \vdots & & \vdots & & \vdots \\ 1 & & 1 & & 1 \\ \vdots & & \vdots & & \vdots \\ \beta_{-m}^{m} & \cdots & \beta_0^{m} & \cdots & \beta_m^{m} \end{pmatrix},$$

(The $(i,j)^{\text{th}}$ entry of $M$ is $\beta_j^i$, for $-m \le i, j \le m$.)

and $s_i = f_{\beta_i} + \varepsilon_{m,\beta_i}$, for each $i = -m, \ldots, m$. Thus

$$\det(M) = \Big(\prod_{i=-m}^{m} \beta_i^{-m}\Big) \cdot \det \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \beta_{-m} & \cdots & \beta_0 & \cdots & \beta_m \\ \vdots & & \vdots & & \vdots \\ \beta_{-m}^{2m} & \cdots & \beta_0^{2m} & \cdots & \beta_m^{2m} \end{pmatrix}$$

$$= \Big(\prod_{i=-m}^{m} \beta_i^{-m}\Big) d, \qquad \text{where} \quad d = \prod_{-m \le i < j \le m} (\beta_j - \beta_i).$$

(The determinant of the above matrix is the standard van der Monde determinant.) So since $|\beta_j - \beta_i| = 1$ when $j > i$, and $|\beta_i^{-m}| = 1$ for all $i = -m, \ldots, m$, $M$ is invertible, and $|\det M| = 1$. Furthermore, since $M^{-1} = \left(\frac{1}{\det(M)}\right)$ (matrix of cofactors of $M$), the entries of $M^{-1}$ are polynomials in the $\beta_i$ with coefficients being rational integers multiplied by $\frac{1}{\det(M)}$. Thus the entries of $M^{-1}$ have absolute value not exceeding 1.

Now, multiply (**) by $M^{-1}$ to obtain

$$(a_{-m}x^m, \ldots, a_0, \ldots, a_m x^m) = (s_{-m}, \ldots, s_0, \ldots, s_m)M^{-1} \qquad (***)$$
$$= (f_{\beta_{-m}}, \ldots, f_{\beta_0}, \ldots, f_{\beta_m})M^{-1} + (\varepsilon_{m,\beta_{-m}}, \ldots, \varepsilon_{m,\beta_0}, \ldots, \varepsilon_{n,\beta_m})M^{-1}.$$

The entries of the first vector are all elements of $J(R)$, and the entries of the second vector have norm at most $\max_{-m\leq i\leq m} \|\varepsilon_{m,\beta_i}\| = \|\varepsilon_m\|$ (since entries of $M^{-1}$ have absolute value $\leq 1$).

Thus $a_0 = y_m + \delta_m$ with $y_m$ and $\delta_m$ being the $0^{\text{th}}$- component of the first and second vectors in (* * *), respectively. Note that $y_m \in J(R)$, and $\|\delta_m\| \leq \|\varepsilon_m\|$. Therefore, $a_0$ is the limit of the sequence $\{y_1, y_2, \ldots\} \subset J(R)$ since $\|\delta_m\| \to 0$ as $m \to \infty$. Hence $a_0 \in J(R)$. So by Lemma 4.5.1, $a_0 \in J(A)$.

□

Theorem 4.4 has the following implications concerning the semiprimitivity of $\ell_1(k, G)$, where $k$ is a $p$-adic field.

COROLLARY 4.4.3. *Let $k$ be a complete p-adic field such that $\tilde{k}$ is infinite. Let $G$ be a group with $H \triangleleft G$ such that $\frac{G}{H}$ is infinite cyclic and $H \in \mathcal{X}_k$. Then $G \in \mathcal{X}_k$.*

COROLLARY 4.4.4. *Let $G, H$ be as in the previous corollary. If $k$ is any complete p-adic field, and $H \in \mathcal{X}_{\Omega_p}$, then $G \in \mathcal{X}_k$.*

PROOF: Theorem 4.4 shows that $G \in \mathcal{X}_{\Omega_p}$, and then Corollary 2.5.2 implies that $G \in \mathcal{X}_k$. □

The final result of this section shows that if $H$ is a normal subgroup of a group $G$ such that $\frac{G}{H}$ is an abelian $p'$-group, then the Jacobson radical of $\ell_1(k, G)$ is again nicely related to that of $\ell_1(k, H)$. The proof requires the following lemma:

LEMMA 4.5.1. *[See 2, Theorem 4.3.3]. Let $G$ be an abelian $p'$-group, and $\{g_j\}_{j=1}^{\infty} \subset G$ with $g_{j_1} \neq g_{j_2}$ for $j_1 \neq j_2$. Then there are $p$-adic characters $\phi_i \in \widehat{G_{\Omega_p}}$, $i = 1, 2, 3, \ldots$ such that if $A_n = (\phi_i(g_j))_{i,j=1}^{n}$, then*

*(i) $|\det A_n| = 1$, for all $n$, and*

*(ii) if $A_n^{-1} = (b_{i,j}^{(n)})_{i,j=1}^{n}$, then $|b_{i,j}^{(n)}| \leq 1$ for $i$, $j = 1, \ldots, n$.*

PROOF: The proof proceeds by induction on $n$, the case $n = 1$ being trivial using $\phi_1 = 1$.

Assume that the result holds for the case $n - 1$. Let $\phi_1, \ldots, \phi_{n-1} \in \widehat{G_{\Omega_p}}$ satisfy properties (i) and (ii). Let $\mu \in \widehat{G_{\Omega_p}}$, and consider the $n \times n$ matrix

$$M(\mu) = (a_{i,j}) \quad \text{where} \quad a_{i,j} = \begin{cases} \phi_i(g_j), & \text{for } i = 1 \leq i \leq n-1, \quad 1 \leq j \leq n \\ \mu(g_j), & \text{for } i = n, \quad 1 \leq j \leq n. \end{cases}$$

Expanding the determinant along the $n^{\text{th}}$-row yields $\det(M(\mu)) = \sum_{i=1}^{n} \mu(g_i) c_i$, where $|c_i| \leq 1$ for $1 \leq i \leq n-1$ and $c_n = (-1)^{n-1} \cdot \det A_{n-1}$. The induction hypothesis implies that $|c_n| = 1$.

By Lemma 3.3.3, there is a $\phi_n \in \widehat{G_{\Omega_p}}$ such that $|\sum_{i=1}^{n} c_i \phi_n(x_i)| = 1$. So let $A_n = M(\phi_n)$, and (i) is proved. (ii) follows immediately from the cofactor formula for inverses, since the entries of $A_n = M(\phi_n)$ all have absolute value not exceeding 1, and $|\det A_n| = 1$.

□

THEOREM 4.5. *[1]. Let $G$ be a group with $H \triangleleft G$ such that $\frac{G}{H}$ is an abelian $p'$-group. Then $J(\ell_1(\Omega_p, G)) \subseteq J(\ell_1(\Omega_p, H))\{G\}$.*

PROOF: Let $T$ be a transversal of $H$ in $G$, and write $f = \sum_{t \in T} a_t t$ where each $a_t = \sum_{h \in H} \alpha_{h t} h \in \ell_1(k, H)$ and $\|a_t\| \to 0$. Let $\pi : G \to \frac{G}{H}$ be the canonical map.

Apply Lemma 4.5.1 to $\frac{G}{H}$ by using $\{t_j H : |a_{t_j}\| > 0$ in the expansion of $f\}$ (a countable set). This yields $\{\phi_i\}_{i \in \mathbb{N}} \subset \widehat{(\frac{G}{H})}_{\Omega_p}$ satisfying properties (i) and (ii). Extend each $\phi_i$ to a $\Omega_p$-linear homomorphism $\widehat{\phi_i} : \ell_1(\Omega_p, G) \to \Omega_p$ by defining $\widehat{\phi_i}\Big(\sum_{g \in G} \alpha_g g\Big) = \sum_{g \in G} \alpha_g \phi_i(\pi(g))$. Since $\widehat{\phi_i} = \phi_i \circ \pi$ is the composition of a continuous $\Omega_p$-linear homomorphism $\widehat{\phi_i}$ is also a continuous $\Omega_p$-linear homomorphism.

Furthermore, the map $\psi_i : G \longrightarrow \ell_1(\Omega_p, G)$ given by $\psi_i(g) = \widehat{\phi_i}(g) g$ extends linearly to a continuous $\Omega_p$-algebra automorphism of $\ell_1(\Omega_p, G)$ onto itself.

Now,

$$\psi_i(f) = \psi_i\Big(\sum_{j=1}^{\infty} a_{t_j} t_j\Big) = \psi_i\Big(\sum_{j=1}^{\infty} \sum_{h\in H} \alpha_{ht_j} ht_j\Big) = \sum_{j=1}^{\infty} \sum_{h\in H} \alpha_{ht_j} \widehat{\phi}_i(ht_j) ht_j$$

$$= \sum_{j=1}^{\infty} \sum_{h\in H} \alpha_{ht_j} \phi_i(\pi(ht_j)) ht_j = \sum_{j=1}^{\infty} \sum_{h\in H} \alpha_{ht_j} \phi_i(\pi(t_j)) ht_j$$

$$= \sum_{j=1}^{\infty} a_{t_j} \widehat{\phi}_i(t_j) t_j.$$

Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\|a_{t_n}\| < \varepsilon$ for $n > N$. Let $B_N = (\widehat{\phi}_i(t_{N+j}))$, $1 \le i \le N$, $j = 1, 2, \ldots$ (an $N \times \infty$ matrix). Then $\big(\psi_i(f)\big)_{i=1,\ldots,N} = A_N\big(a_{t_i} t_i\big)_{i=1,\ldots,N} + B_N\big(a_{t_i} t_i\big)_{i=N+1,N+2,\ldots}$, and so using Lemma 4.5.1,

$$A_N^{-1}\big(\psi_i(f)\big)_{i=1,\ldots,N} = (a_{t_i} t_i)_{i=1,\ldots,N} + A_N^{-1} B_N\big(a_{t_i} t_i\big)_{i=N+1,N+2,\ldots}. \qquad (*)$$

Since $\psi_i$ is an automorphism, $\psi_i(f) \in J(\ell_1(\Omega_p, G))$, for all $i \in \mathbb{N}$. Let $\big(y_i^{(N)}\big)_{i=1,\ldots,N} = A_N^{-1}\big(\psi_i(f)\big)_{i=1,\ldots,N}$. Then each $y_i^{(N)}$ is a linear combination of $\psi_1(f), \ldots, \psi_N(f)$, so $y_i^{(N)} \in J(\ell_1(\Omega_p, G))$, for each $i = 1, \ldots, N$, for all $N$.

Let $\big(z_i^{(N)}\big)_{i=1,\ldots,N} = A_N^{-1} B_N\big(a_{t_j} t_j\big)_{j=N+1,N+2,\ldots}$. By Lemma 4.4.1, the entries of $A_N^{-1}$ and $B_N$ have absolute value at most 1. So since $\|a_{t_j}\| < \varepsilon$ for all $j > N$, $\|z_i^{(N)}\| < \varepsilon$ for all $i = 1, \ldots, N$.

Also, $(*)$ shows that $a_{t_i} t_i = y_i^{(N)} + z_i^{(N)}$, for each $i = 1, \ldots, N$, and for all $N$.

Finally, for all $i \in \mathbb{N}$, $\lim_{N\to\infty} \|z_i^{(N)}\| = 0$, since $\varepsilon > 0$ was arbitrarily and $\|a_{t_i}\| \to 0$ as $i \to \infty$. Hence $a_{t_i}t_i = \lim_{N\to\infty} y_i^{(N)}$, and so since $J(\ell_1(\Omega_p, G))$ is closed, $a_{t_i}t_i \in J(\ell_1(\Omega_p, G))$. Thus $a_{t_i} = (a_{t_i}t_i)t_i^{-1} \in J(\ell_1(\Omega_p, G)) \cap \ell_1(\Omega_p, H)$. By Lemma 4.3.1, $a_{t_i} \in J(\ell_1(\Omega_p, H))$, for all $i \in \mathbb{N}$. This proves the result.

□

COROLLARY 4.5.2. *If $H \in \mathcal{X}_{\Omega_p}$ and $H \triangleleft G$ with $\frac{G}{H}$ an abelian $p'$-group, then $G \in \mathcal{X}_{\Omega_p}$.*

**C. Generalized solvable groups for which $p$-adic $\ell_1$-group algebras are semiprimitive.**

Let $G$ be a solvable group with subnormal series $\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ such that $\frac{G_i}{G_{i-1}}$ is abelian for $i = 1, \ldots, n$. Then this series is a $p'$-subnormal series for $G$ if each $\frac{G_i}{G_{i-1}}$ is an abelian $p'$-group. Let $\mathcal{S}_{p'}$ denote the class of solvable groups $G$ wich have a $p'$-subnormal series.

The results in this section show that when a solvable group $G$ has a subnormal series for which none of the abelian factors are infinitely generated $p$-groups, then $G \in \mathcal{X}_{\Omega_p}$. The first theorem shows that $\mathcal{X}_{\Omega_p}$ is closed under normal $\mathcal{S}_{p'}$-extensions.

THEOREM 4.6. *[1] Supppose $H \triangleleft G$ with $H \in \mathcal{X}_{\Omega_p}$ and $\frac{G}{H} \in \mathcal{S}_{p'}$. Then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: Let $\pi : G \to \frac{G}{H}$ be the canonical homomorphism. Suppose $\langle 1 \rangle = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_n = \frac{G}{H}$ is a subnormal series for $\frac{G}{H}$ such that $\frac{K_i}{K_{i-1}}$ is an abelian $p'$-group for $i = 1, \ldots, n$. Let $\pi^{-1}(K_i) = G_i$ for $i = 0, \ldots, n$. Then each $G_{i-1} \triangleleft G_i$, and $\frac{G_i}{G_{i-1}} \cong \frac{K_i}{K_{i-1}}$, for each $i = 1, \ldots, n$. Thus $H = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ is a series of subgroups of $G$ such that $\frac{G_i}{G_{i-1}}$ is an abelian $p'$-group, for $i = 1, \ldots, n$. Thus, since $H \in \mathcal{X}_{\Omega_p}$, a simple induction using Corollary 4.5.2 shows that $G \in \mathcal{X}_{\Omega_p}$.

□

COROLLARY 4.6.1. *Suppose that a group $G$ has a directed system $\{N_\lambda : \lambda \in \Lambda\}$ of normal subgroups such that each factor group $\frac{G}{N_\lambda} \in \mathcal{S}_{p'}$. Then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: Theorem 4.6 shows that if $G \in \mathcal{S}_{p'}$, then $G \in \mathcal{X}_{\Omega_p}$. Thus Theorem 2.6 implies the result.

□

In general, if $G$ is solvable, and $G$ has a subnormal series $\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with each abelian factor $\frac{G_i}{G_{i-1}}$ a $p'$-group or a finite $p$-group, then it can be shown by induction that $G \in \mathcal{X}_{\Omega_p}$. Since $G_1$ is abelian with no $C_{p^\infty}$-subgroup, $G_1 \in \mathcal{X}_{\Omega_p}$ by Theorem 3.6. If $\frac{G_2}{G_1}$ is an abelian $p'$-group, then by Theorem 4.6, $G_2 \in \mathcal{X}_{\Omega_p}$. If $\frac{G_2}{G_1}$ is a finite $p$-group, then Corollary 4.3.1 implies that $G_2 \in \mathcal{X}_{\Omega_p}$. Repetition of this process yields $G \in \mathcal{X}_{\Omega_p}$. Hence the main result on solvable groups is:

THEOREM 4.7. *Let $G$ be a solvable group that has a subnormal series none of whose abelian factor groups are infinitely generated $p$-groups. Then $G \in \mathcal{X}_{\Omega_p}$.*

**D. Semiprimitivity of $\ell_1(\Omega_p, G)$ and nilpotent groups.**

The results obtained so far reduce the problem of semiprimitivity of $\ell_1(\Omega_p, G)$ for nilpotent groups to the case where $G$ is an infinitely generated nilpotent $p$-group, with $[G : Z(G)]$ infinite. ($Z(G)$ is the center of $G$.) When $G$ is nilpotent, then the subgroup $G_p$ of $G$ generated by all elements of order a power of $p$ is normal in $G$ [4]. Since the factor group $\frac{G}{G_p}$ must be in $\mathcal{S}_{p'}$, $G \in \mathcal{X}_{\Omega_p}$ follows from $G_p \in \mathcal{X}_{\Omega_p}$. If $G$ is a finitely generated nilpotent group, then $G$ is residually finite [4], so $G \in \mathcal{X}_{\Omega_p}$. When $[G : Z(G)]$ is finite, then $G \in \mathcal{X}_{\Omega_p}$ if $Z(G) \in \mathcal{X}_{\Omega_p}$. (Of course, since $Z(G)$ is abelian, $Z(G) \in \mathcal{X}_{\Omega_p}$ only when $Z(G)$ does not contain a $C_{p^\infty}$-subgroup.)

For a nilpotent group $G$ with center $Z(G)$, it is natural to identify $\ell_1(\Omega_p, G)$ with the complete crossed product $\ell_1(\Omega_p,\ Z(G))\{G\}$ and then use the fact that $\ell_1(\Omega_p, Z(G))$ is central. The following proposition gives a sufficient condition for semiprimitivity of $\ell_1(\Omega_p, G)$ in this situation.

PROPOSITION 4.8. *Let $R = A\{G\}$ be a complete crossed product with $G \cap A = H$. Suppose that $A$ is semiprimitive and central in $R$. Then if $\frac{A}{M}\left\{\frac{G}{H_M}\right\}$ is* *semiprimitive for all maximal ideals $M$ of $A$, then $R$ is also semiprimitive. (Recall that $H_M = H \cap (1+M)$ for any ideal $M$ of $A$.)*

PROOF: Let $T$ be a transversal of $G \cap A = H$ in $G$ containing 1, and let $f \in R$ be non-zero. Write $f = \sum_{t \in T} a_t t$, with $a_t \in A$, $\forall t \in T$, and $\|a_t\|_R \to 0$. Without loss of generality, assume that $a_1 \neq 0$.

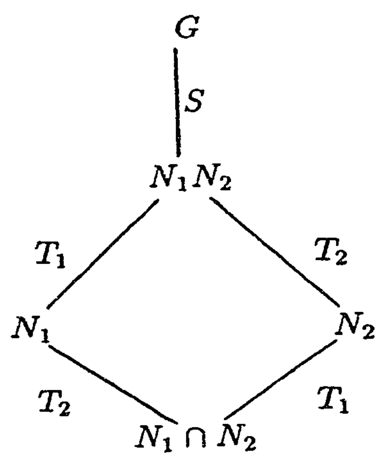
Since $J(A) = 0$, there is a maximal ideal $M$ of $A$ with $a_1 \notin M$. Let $\pi : A\{G\} \longrightarrow \frac{A\{G\}}{M\{G\}} \cong \frac{A}{M}\left\{\frac{G}{H_M}\right\}$ be the canonical homomorphism. Then $\pi(f) = \pi\Big(\sum_{t \in T} a_t t\Big) = \sum_{t \in T} \pi(a_t)\pi(t)$. Now $\pi|_G : G \to \frac{G}{H_M}$ is the quotient group homomorphism, and $H_M \subseteq H$, so if $T$ is a transversal of $H$ in $G$, $\{\pi(t)\}_{t \in T}$ is a transversal of $\frac{H}{H_M}$ in $\frac{G}{H_M}$. Hence the expansion $\sum_{t \in T} \pi(a_t)\pi(t)$ is in the standard unique form of $\frac{A}{M}\left\{\frac{G}{H_M}\right\}$. Thus since $\pi(a_1) \neq 0$, $\pi(f) \neq 0$, and so $\pi(f) \notin J\left(\frac{A}{M}\left\{\frac{G}{H_M}\right\}\right) \Rightarrow f \notin J(R)$.

□

The above result can be used to show the semiprimitivity of $\ell_1(\Omega_p, G)$ for certain countable, infinitely generated $p$-groups of nilpotency index 2, namely those with commutator subgroup $G'$ central and finite. (The commutator subgroup $G'$ is $\langle\{[x,y] : x,\ y \in G\}\rangle$, where $[x,y] = x^{-1}y^{-1}xy$, $\forall x,\ y \in G$.) The following lemmas are vital for the proof:

LEMMA 4.9.1. *Let $k\{G\}$ be a complete crossed product with $k$ a complete p-adic field which is central in $kG$. Let $I \triangleleft k\{G\}$. Suppose that $N_1, N_2$ are normal subgroups of $G$ such that $I = (I \cap k\{N_1\})\{G\} = (I \cap k\{N_2\})\{G\}$. Then $I = (I \cap k\{N_1 \cap N_2\})\{G\}$.*

PROOF: Choose transversals $S, T_1$, and $T_2$ as shown in the diagram:




Let $f \in I \cap k\{N_2\}$. Since $k\{N_2\} = (k\{N_1 \cap N_2\})\{N_2\}$, $f$ can be written as $\sum_{t \in T_1} a_t t$, with $a_t \in k\{N_1 \cap N_2\}$, $\forall t \in T_1$. Since $I = (I \cap k\{N_1\})\{G\}$, $f$ can also be written as $\sum_{s \in S} \left( \sum_{t \in T_1} b_{t,s} t \right) s$, with $b_{ts} \in I \cap k\{N_1\}$, $\forall t \in T_1$, $\forall s \in S$. Since $k\{N_2\} \subset k\{N_1 N_2\}$, every $b_{ts}$ with $s \neq 1$ must be zero. (Since $k\{G\} = (k\{N_1 N_2\})\{G\}$ is a complete crossed product, these representations are unique.) Thus $f = \sum_{t \in T_1} b_{t,1} t$. Again, uniqueness implies that $b_{t,1} = a_t$, $\forall t \in T_1$, and so $a_t \in I \cap k\{N_1\} \cap k\{N_1 \cap N_2\} = I \cap k\{N_1 \cap N_2\}$. Thus $f \in \big(I \cap k\{N_1 \cap N_2\}\big)\{N_2\}$. This implies that $I \cap k\{N_2\} = \big(I \cap k\{N_1 \cap N_2\}\big)\{N_2\}$, and hence that $I = (I \cap k\{N_2\})\{G\} = (I \cap k\{N_1 \cap N_2\})\{G\}$. □

LEMMA 4.9.2. *Let $k\{G\}$ be a complete crossed product with $k$ a complete $p$-adic field which is central in $k\{G\}$, and $G$ a countable group. Suppose that $\{N_\lambda : \lambda \in \Lambda\}$ is a family of normal subgroups of $G$, such that $\bigcap_{\lambda\in\Lambda} N_\lambda = H = k \cap G$. Then if $0 \neq I \triangleleft k\{G\}$ such that $I = \big(I \cap k\{N_\lambda\}\big)\{G\}$, $\forall \lambda \in \Lambda$, then $I = k\{G\}$.*

PROOF: Since $G$ is countable, $H$ has countably many cosets $H,\ g_1H,\ g_2H, \cdots$ in $G$. For each $i \geq 1$, since $\bigcap_{\lambda\in\Lambda} N_\lambda = H$, there is a $\lambda_i \in \Lambda$ such that $g_iH \cap N_{\lambda_i} = \emptyset$.

Put $M_0 = G$, and $M_j = N_{\lambda_1} \cap \cdots \cap N_{\lambda_j}$ for all $j \in \mathbb{N}$. Then $\bigcap_{j=0}^{\infty} M_j$ contains $H$, and has empty intersection with all other cosets of $H$ in $G$, hence $\bigcap_{j=0}^{\infty} M_j = H$.

Let $T_j$ be a transversal of $M_{j+1}$ in $M_j$, for all $j \geq 0$. Define $Y_j = T_0T_1 \cdot \ldots \cdot T_{j-1} = \big\{t_0 \cdot t_1 \cdot \ldots \cdot t_{j-1} : t_i \in T_i, \text{for } i = 0, \ldots, j-1\big\}$ and $X_j = T_j \cdot T_{j+1} \cdot T_{j+2} \cdots = \big\{t_j \cdot t_{j+1} \cdot t_{j+2} \cdot \ldots \ : t_i \in T_i, \text{ for } i \geq j \text{ with almost all } t_i = 1\big\}$. Then $Y_j$ is a transversal of $M_j$ in $G$, and $X_j$ is a transversal of $H$ in $M_j$, for all $j \geq 0$.

Let $f = \sum_{t\in X_0} \alpha_t t \in I$. Fix $n \in \mathbb{N}$. Since each $t \in X_0$ can be written $t = y_n(t)x_n(t)$, with $y_n(t) \in Y_n$, $x_n(t) \in X_n$, $\forall t \in X_0$, $f$ can be written as

$$f = \sum_{y\in Y_n} y \cdot \Big( \sum_{y_n(t)=y} \alpha_t x_n(t)\Big).$$

By the previous lemma, $I = (I \cap k\{N_\lambda\})\{G\}$, $\forall \lambda \in \Lambda$ implies that $I = (I \cap k\{M_i\})\{G\}$, for all $i \in \mathbb{N}$. Thus $f \in I = (I \cap k\{M_n\})\{G\} \Longrightarrow$

$$\sum_{y_n(t)=y} \alpha_t x_n(t) \in I \cap k\{M_n\},\ \forall y \in Y_n, \quad \text{for all } n \geq 0. \tag{*}$$

Assume that $\|f\|_R = 1 = |\alpha_1|$. If $t_1, \dots, t_r$ are the other elements of $X_0$ with $|\alpha_{t_i}| = 1$, for $i = 1, \dots, r$, then each of them can be excluded from some $X_n$, since $\bigcap_{n=0}^{\infty} X_n = \langle 1 \rangle$. Thus for all sufficiently large $n$, $y_n(t_i) \neq 1$ for $i = 1, \dots, r$. Hence for $y = 1$, and for all large enough $n$, $(*)$ reads $\alpha_1 + \sum_{y_n(t)=1} \alpha_t x_n(t) \in I \cap k\{M_n\}$, with $|\alpha_i| < 1$ for all of the terms appearing in the sum. Since $\alpha_1 \in k$, $\alpha_1$ is a unit in $k\{G\}$, and thus so is $\alpha_1 + \sum_{y_n(t)=1} \alpha_t x_n(t) \in I$, hence $I$ must be all of $k\{G\}$.

□

For all $g \in G$, the centralizer of $g$ in $G$ is $C_G(g) = \{x \in G : gx = xg\}$. Thus $\bigcap_{g \in G} C_G(g) = Z(G)$.

THEOREM 4.9. *Let $R = \Omega_p\{G\}$ be a complete crossed product with $G$ a countable nilpotent $p$-group of class 2 such that $Z(G) = \Omega_p \cap G$. Suppose that $G' \cong C_{p^n}$, for some $n \geq 1$. Then $\Omega_p\{G\}$ is a simple ring.*

PROOF: In view of the preceding lemma, it suffices to show that if $I$ is a non-zero ideal of $R$, $I = (I \cap \Omega_p\{C_G(g)\})\{G\}$, for all $g \in G$.

Let $g \in G$. If $g \in Z(G)$, then $C_G(g) = G$, so there is nothing to show.

So suppose that $g \notin Z(G)$. Assume that the subgroup $[g, G] = \{[g, x] : x \in G\}$ has order $p^m$, for some $m \leq n$. Let $\zeta$ be a $p^m$-th root of unity in $\Omega_p$. Let $X$ be a transversal of $C_G(g)$ in $G$, and let $X_i = \{x \in X : [g, x^{-1}] = \zeta^i\}$ for all $i = 1, \dots, p^m$.

Let $I$ be a proper ideal of $R$, and let $f \in I$. Since $\Omega_p\{G\} = (\Omega_p\{C_G(g)\})\{G\}$, it is evident that $f = \sum_{x \in X} a_x x$, with $a_x \in \Omega_p\{C_G(g)\}$, $\forall x \in X$. Let $f_i = \sum_{x \in X_i} a_x x$ for each $i = 1, \ldots, p^m$, so $f = \sum_{i=1}^{p^m} f_i$. For a fixed $i$, note that $x \in X_i \Rightarrow g^{-1}xg = [g, x^{-1}]x = \zeta^i x$, hence $g^{-1}f_i g = \sum_{x \in X_i} a_x g^{-1}xg = \sum_{x \in X_i} a_x \zeta^i x = \zeta^i f_i$. Thus $g^{-1}fg = \sum_{i=1}^{p^m} \zeta^i f_i$. More generally, $g^{-r}fg^r = \sum_{i=1}^{p^m} \zeta^{ir} f_i$. The system of equations obtained when $r = 1, \ldots, p^m$ is

$$\begin{pmatrix} f \\ g^{-1}fg \\ \vdots \\ g^{-p^m}fg^{p^m} \end{pmatrix} = \begin{pmatrix} 1 & & \ldots & 1 \\ 1 & \zeta & \ldots & \zeta^{p^m} \\ 1 & \zeta^2 & \ldots & \zeta^{2p^m} \\ \vdots & \vdots & & \vdots \\ 1 & \zeta^{p^n} & \ldots & \zeta^{p^{2m}} \end{pmatrix} \begin{pmatrix} f_0 \\ \\ \vdots \\ \\ f_{p^n} \end{pmatrix}$$

The coefficient matrix is in Van der Monde form, and thus is invertible. Hence since $f \in I$, this implies that each $f_i$ is a linear combination of elements of $I$, hence is also in $I$.

Next, fix $i$, and let $x_i \in X_i$. Then $g^{-1}f_i x_i^{-1} g = g^{-1}f_i g g^{-1} x_i^{-1} g = \zeta^i f_i [g, x_i] x_i^{-1} = \zeta^i f_i \zeta^{-i} x_i^{-1} = f_i x_i^{-1}$, so $f_i x_i^{-1}$ is centralized by $g$. Write $f_i x_i^{-1} = \sum_{t \in T} \alpha_t t$, where $T$ is a transversal of $Z(G)$ in $G$. Then $g^{-1} f_i x_i^{-1} g = f_i x_i^{-1} \Rightarrow \sum_{t \in T} \alpha_t g^{-1} t g = \sum_{t \in T} \alpha_t t \Rightarrow \sum_{t \in T} \alpha_t [g, t^{-1}] t = \sum_{t \in T} \alpha_t t \Rightarrow [g, t^{-1}] = 1$, for all $t$ such that $\alpha_t \neq 0$. Thus $f_i x_i^{-1} \in \Omega_p\{C_G(g)\}$, for each choice of $x_i \in X_i$, and for each $i = 1, \ldots, p^m$.

Thus $f = \sum_{i=1}^{p^m} f_i = \sum_{i=1}^{p^m} (f_i x_i^{-1}) x_i \in (I \cap \Omega_p\{C_G(g)\})\{G\}$, for all $g \in G$, proving the theorem.

□

The next result is needed for examining the case of nilpotent $p$-groups, but is interesting in its own right:

THEOREM 4.10. *Let $P$ be an abelian $p$-group such that $\bigcap_{i=1}^{\infty} P^{p^i} = \langle 1 \rangle$. If $\Omega_p\{P\}$ is any complete crossed product of $\Omega_p$ by $\frac{P}{\Omega_p \cap P}$ with $\Omega_p$ central, then $\Omega_p\{P\}$ is semiprimitive.*

PROOF: Since $\Omega_p$ is a field, and $P$ is a $p$-group without elements of infinite height, $P \cap \Omega_p \cong C_{p^m}$, for some $m \geq 0$. Let $Q$ be a maximal cyclic subgroup of $P$ containing $P \cap \Omega_p$. Then $P = Q \oplus Q'$, for some subgroup $Q'$ of $P$. The complete crossed product $\Omega_p\{Q\}$ is just the crossed product $\Omega_p[Q]$, which is the direct sum of $s = [Q : \Omega_p \cap P]$ copies of $\Omega_p$, and so $\Omega_p[Q] = \bigoplus_{i=1}^{s} {\Omega_p}^{(i)}$.

Finally, $\Omega_p\{P\} = \ell_1(\Omega_p\{Q\}, Q') \cong \ell_1\left(\bigoplus_{i=1}^{s} {\Omega_p}^{(i)}, Q'\right) \cong \bigoplus_{i=1}^{s} \ell_1({\Omega_p}^{(i)}, Q')$. Since $Q'$ has no $C_{p^\infty}$-subgroup, $\ell_1({\Omega_p}^{(i)}, Q')$ is semiprimitive, $\forall i = 1, \ldots, s$, by Theorem 3.6, and thus $\Omega_p\{P\}$ is semiprimitive.

□

The proof that countable nilpotent $p$-groups of nilpotency class 2 with finite commutator subgroups are members of the class $\mathcal{X}_{\Omega_p}$ now can be achieved.

THEOREM 4.11. *Let $G$ be a countable, nilpotent $p$-group with $G' \subseteq Z(G)$ and $G' \cong C_{p^n}$, $\exists n \geq 0$. Then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: Consider $\ell_1(\Omega_p, G)$ as $\ell_1(\Omega_p, Z(G))\{G\}$. Let $M$ be a maximal ideal of $\ell_1(\Omega_p, Z(G))$, and consider $\frac{\ell_1(\Omega_p,Z(G))\{G\}}{M\{G\}} \cong \frac{\ell_1(\Omega_p,Z(G))}{M}\left\{\frac{G}{G_M}\right\}$, where $G_M = (1+M)\cap Z(G)$. Since $Z(G)$ is abelian, $\frac{\ell_1(\Omega_p,Z(G))}{M} \cong \Omega_p$. For this complete crossed product $\Omega_p\left\{\frac{G}{G_M}\right\}$, $\Omega_p \cap \frac{G}{G_M} = \frac{Z(G)}{G_M}$. However, $\frac{Z(G)}{G_M}$ may not be the entire center of $\frac{G}{G_M}$. An element $gG_M \in Z\left(\frac{G}{G_M}\right) \iff [g,x] \in (1+M)$, $\forall x \in G \iff [g,G] \subseteq G' \cap (1+M)$. The complete crossed product $\Omega_p\left\{Z\left(\frac{G}{G_M}\right)\right\}$ is semiprimitive by Theorem 4.10.

Consider $\Omega_p\left\{\frac{G}{G_M}\right\}$ as $\left(\Omega_p\left\{Z\left(\frac{G}{G_M}\right)\right\}\right)\left\{\frac{G}{G_M}\right\}$. For any maximal ideal $I$ of $\Omega_p\left\{Z\left(\frac{G}{G_M}\right)\right\}$, $I \cap \Omega_p$ must be 0, otherwise $I$ contains a unit. Since $\left(\frac{G}{G_M}\right)' = \frac{G'}{G_M}G_M \subseteq \frac{\ell_1(\Omega_p,Z(G))}{M} \cong \Omega_p$, this means that $(1+I)\cap\left(\frac{G}{G_M}\right)' = \{1\}$. Therefore, letting $\frac{G}{G_M} = \overline{G}$,

$$Z\left(\frac{\overline{G}}{\overline{G}_I}\right) = \frac{\langle\{\bar{g} \in \overline{G} : [\bar{g}, \overline{G}] \subseteq \overline{G}' \cap (1+I)\}\rangle}{\overline{G}_I}$$
$$= \frac{\langle\{\bar{g} \in \overline{G} : [\bar{g}, \overline{G}] = \langle\bar{1}\rangle\}\rangle}{\overline{G}_I} = \frac{Z(\overline{G})}{\overline{G}_I}.$$

Thus

$$\frac{(\Omega_p\{Z(\overline{G})\}\{\overline{G}\}}{I\{\overline{G}\}} \cong \frac{\Omega_p\{Z(\overline{G})\}}{I}\left\{\frac{\overline{G}}{\overline{G}_I}\right\} \cong \Omega_p\left\{\frac{\overline{G}}{\overline{G}_I}\right\}$$

satisfies $\Omega_p \cap \frac{\overline{G}}{\overline{G}_I} = \frac{Z(\overline{G})}{\overline{G}_I} = Z\left(\frac{\overline{G}}{\overline{G}_I}\right)$. Since $G' \cong C_{p^n}$, $\left(\frac{\overline{G}}{\overline{G}_I}\right)' = \frac{\overline{G}'}{\overline{G}_I} \cong C_{p^m}$, for some $m \leq n$. Thus by Theorem 4.9, $\Omega_p\left\{\frac{\overline{G}}{\overline{G}_I}\right\}$ is simple, for all maximal ideals $I$ of $\Omega_p\{Z(\overline{G})\}$.

Therefore by Proposition 4.8, $\Omega_p\left\{\frac{G}{G_M}\right\}$ is semiprimitive for all maximal ideals $M$ of $\ell_1(\Omega_p, Z(G))$, and so by Proposition 4.8 again, $\ell_1(\Omega_p, G)$ is semiprimitive.

□

The proof for the case where $G'$ is a finite abelian group requires the following development:

Let $\mathcal{E}(k,G)$ be the subset of $\ell_1(k,G)$ whose elements sum to 0 on every right coset of every non-trivial subgroup of $G$. Then since $0 \in \mathcal{E}(k,G)$, this set is always a non-empty vector space over $k$. Define an action of $G$ on $\mathcal{E}(k,G)$ by $\Big(\sum_{x\in G} \alpha_x x\Big)^g = \sum_{x\in G} \alpha_{xg} x$, for all $\sum_{x\in G} \alpha_x x \in \mathcal{E}(k,G)$, and for all $g \in G$. For any non-trivial subgroup $H$ of $G$, the sum of $\Big(\sum_{x\in G} \alpha_x x\Big)^g$ on any right coset $Hy$ of $H$ is $\sum_{h\in H} \alpha_{hyg} = \sum_{x\in H(yg)} \alpha_x = 0$ when $\sum_{x\in G} \alpha_x x \in \mathcal{E}(k,G)$. Thus the action of $G$ on $\mathcal{E}(k,G)$ is well-defined.

Lemma 4.12.1. *(i) Suppose $G$ is a finite abelian $p$-group. No non-trivial subgroup of $G$ has a non-zero fixed point on $\mathcal{E}(k,G)$.*

*(ii) $G$ acts faithfully on every non-zero orbit $f^G = \{f^g : g \in G\}$, with $0 \neq f \in \mathcal{E}(k,G)$.*

*(iii) If $G$ is non-cyclic, then $\mathcal{E}(k,G) = 0$.*

PROOF: (i): Suppose that $H$ is a non-trivial subgroup of $G$. If $f = \sum_{x \in G} \alpha_x x \in \mathcal{E}(k,G)$ is a fixed point for $H$, then $f^h = f$, $\forall\ h \in H$. Thus $\sum_{x \in G} \alpha_{xh} x = \sum_{x \in G} \alpha_x x$, $\forall h \in H$. For all $y \in G$, $0 = \sum_{h \in H} \alpha_{hy} = \sum_{h \in H} \alpha_{yh} = \sum_{h \in H} \alpha_y = |H| \cdot \alpha_y$, since $\alpha_{yh} = \alpha_y$, $\forall h \in H$. But then $\alpha_y = 0$, $\forall y \in G$, so $f = 0$.

(ii) Let $f$ be a non-zero element of $\mathcal{E}(k,G)$. $G$ acts on the orbit $f^G = \{f^g : g \in G\}$. Thus $f$ is a fixed point for the subgroup $G_f = \{g \in G : f^g = f\}$, and so by (i), $G_f = \langle 1 \rangle$. Hence $G$ acts faithfully on $f^G$.

(iii) Suppose $\mathcal{E}(k,G) \neq 0$. By (ii), $G$ acts faithfully and regularly (in the sense that there are no non-trivial fixed points) on the $k$-vector space $\mathcal{E}(k,G)$. By a well-known result concerning Frobenius groups [1, Lemma 13.6], this implies that every subgroup of $G$ of order $p^2$ is cyclic. But since $G$ is a non-cyclic, abelian $p$-group, it certainly has non-cyclic subgroups of order $p^2$, a contradiction. Therefore, $\mathcal{E}(k,G) = 0$.

□

THEOREM 4.12. *Let $G$ be a nilpotent $p$-group with no central elements of infinite height. If $G'$ is central and finite in $G$, then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: In view of Theorem 4.11, it suffices to show this result when $G'$ is not cyclic.

Let $H_1, \ldots, H_n$ be the non-trivial proper subgroups of $G'$. Since $G'$ is not cyclic, $n \geq 3$. Note that since $G'$ is central, $H_i \triangleleft G$ for each $i = 1, \ldots, n$.

Let $\theta_i : \ell_i(\Omega_p, G) \longrightarrow \ell_1(\Omega_p, \frac{G}{H_i})$ be the natural $\Omega_p$-linear extension of the canonical homomorphism of $G$ onto $\frac{G}{H_i}$, for $i = 1 \ldots, n$. Set $J_i = \ker \theta_i$, for each $i = 1, \ldots, n$.

Suppose $f \in \bigcap_{i=1}^{n} J_i$. Fix $i$. If $T_i$ is a transversal of $H_i$ in $G$, then $f = \sum_{g \in G} \alpha_g g = \sum_{t \in T_i} \Big( \sum_{h \in H_i} \alpha_{ht} h \Big) t$. Thus $0 = \theta_i(f) = \sum_{t \in T_i} \Big( \sum_{h \in T_i} \alpha_{ht} \Big) \theta_i(t) = \sum_{h \in H_i} \alpha_{ht} = 0$, $\forall t \in T_i$. This means that $\sum_{h \in H_i} \alpha_{hy} = 0$ for all $y \in G$, and for all $i = 1, \ldots, n$. Hence $\sum_{g \in G'} \alpha_g g \in \mathcal{E}(\Omega_p, G')$. By Lemma 4.12.1, since $G'$ is not cyclic, $\alpha_g = 0$, for all $g \in G'$. (In particular, $\alpha_1 = 0$).

Furthermore, for any $x \in G$, $fx^{-1} \in \bigcap_{i=1}^{n} J_i$, so by the above, the coefficient of 1 for $\Sigma \alpha_g g x^{-1}$ is 0. Therefore, $\alpha_x = 0$, $\forall x \in G$. Hence $\bigcap_{i=}^{n} J_i = 0$.

For any non-zero $f \in \ell_1(\Omega_p, G)$, let $i_0$ be such that $\theta_{i_0}(f) \neq 0$. Then using induction on $n$, with Theorem 4.11 handling the case where $n = 1$, $\theta_{i_0}(f) \notin J(\ell_1(\Omega_p, \frac{G}{H_{i_0}}))$. Therefore, $f \notin J(\ell_1(\Omega_0, G))$, and so $G \in \mathcal{X}_{\Omega_p}$.

□

Corollary 4.12.2. *If $G$ is countable nilpotent $p$-group of nilpotency class 2 and $G'$ is residually finite, then $G \in \mathcal{X}_{\Omega_p}$.*

PROOF: Apply Corollary 2.6.1 to the previous theorem.

□

The following is an example of a nilpotent $p$-group which is not residually finite, yet has a finite central commutator subgroup. Thus this group satisfies the conditions of this section to belong to the class $\mathcal{X}_{\Omega_p}$.

Suppose that $p$ is an odd prime, and let

$$G_n = \langle x_n, y_n : x_n^p = y_n^p = [x_n, y_n]^p = 1, \quad z_n = [x_n, y_n] \quad \text{central in } G \rangle,$$

for each $n \in \mathbb{N}$. (If $p = 2$, set

$$G_n = \langle x_n, y_n : x_n^2 = y_n^2 = [x_n, y_n] = z_n,\ z_n^2 = 1,\ z_n \quad \text{central in } G \rangle.)$$

Form the central product

$$G = \bigotimes_{\substack{n=1 \\ z_1 = z_2 = z_3 = \cdots}}^{\infty} G_n,$$

where the commutators $z_n$ are identified to one element $z$. Then $G$ is easily seen to be nilpotent with center $\langle z \rangle = G'$. Thus since $G'$ is finite cyclic, Theorem 4.9 implies that $G \in \mathcal{X}_{\Omega_p}$. However, $G$ is not residually finite, since any normal subgroup of finite index in $G$ must contain some pair $x_n, y_n$ from the same $G_n$, and hence contains their commutator $z$.

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# CHAPTER V

# CONCLUSION

To this point the problem of determining those groups $G$ for which $\ell_1(k, G)$ is semiprimitive, for a given complete $p$-adic field $k$, remains unsolved. Some open questions have been outlined in Hare and Shirvani's article [1]. The most significant of the remaining problems is the following:

If $G$ is of the form $\bigoplus_1^n C_{p^\infty}$, for some $n \in \mathbb{N}$, then is $\ell_1(\mathbb{Q}_p, G)$ semiprimitive? If so, then it can be shown that $\ell_1(\mathbb{Q}_p, G)$ is semiprimitive for all abelian groups $G$. It is also difficult to procede in the solvable or nilpotent group case without a complete understanding of the abelian case. This thesis concludes by offering an alternate characterization of the Jacobson radical of $\ell_1(k, G)$, for arbitrary locally finite abelian groups $G$. In particular, $J\Big(\ell_1(\mathbb{Q}_p, \bigoplus_1^n C_{p^\infty})\Big)$ can be characterized in this fashion.

For a countable locally finite abelian group $G$, let $1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n \lhd \cdots$ be a chain of finite normal subgroups of $G$ for which $G = \bigcup_{n=0}^{\infty} G_n$. As in Proposition 3.7, for each finite subgroup $G_n$, and for each $p$-adic character $\theta \in \widehat{G_n}$, set

$$e_n(\theta) = \frac{1}{|G_n|} \sum_{x \in G} \theta(x) x,$$

to obtain a full set of primitive orthogonal idempotents of the group ring $\Omega_p G_n$. In Chapter III, it was shown that for all $m > n$, $e_n(\theta) = \sum_{\substack{\varphi \in \widehat{G_m} \\ \varphi|_{G_n} = \theta}} e_m(\varphi)$. Furthermore, an element $f \in \Omega_p G_n$, for some $n \in \mathbb{N}$, can be written uniquely as

$\sum_{\theta \in \widehat{G_m}} a_n(\theta)e_n(\theta)$, where $a_n(\theta) \in \Omega_p$, $\forall \theta \in \widehat{G_n}$. The above equality implies that the unique expression of $f$ as an element of $\Omega_p G_n$ is

$$\sum_{\theta \in \widehat{G_n}} a_n(\theta) \sum_{\substack{\varphi \in \widehat{G_m} \\ \varphi|_{G_n} = \theta}} e_m(\varphi) = \sum_{\varphi \in \widehat{G_n}} a_n(\,\varphi|_{G_n}\,) e_m(\varphi).$$

Thus it is possible to define a new norm on $\Omega_p G = \bigcup_{n=0}^{\infty} \Omega_p G_n$ by

$$\|f\|_* = \lim_{n \to \infty} \big( \max_{\theta \in \widehat{G_n}} |a_n(\theta)| \big).$$

It is trivial to see that $\|\cdot\|_*$ is a norm on $\Omega_p G$.

Compare $\|\cdot\|_*$ to the supremum norm $\|\cdot\|$.

PROPOSITION 5.1. *Let $f \in \Omega_{p_n}$, for some $n \in \mathbb{N}$. Write $f = \sum_{\theta \in \widehat{G_n}} a_n(\theta)e_n(\theta)$, $\exists a_n(\theta) \in \Omega_p$, $\forall \theta \in \widehat{G_n}$, in terms of the basis of primitive orthogonal idempotents, and $f = \sum_{x \in G_n} \alpha_x x$, $\exists \alpha_x \in \Omega_p$, $\forall x \in G_n$, in terms of the basis consisting of elements of $G$. Then*

*i)* $a_n(\theta) = \sum_{x \in G_n} \alpha_x \theta^{-1}(x)$, $\forall \theta \in \widehat{G_n}$.

*ii)* $\|f\|_* \leq \|f\|$, *for all* $f \in \Omega_p G$.

PROOF: (i): The orthogonality relations imply that

$$\Big( \sum_{x \in G_n} \alpha_x x \Big) e_n(\theta) = a_n(\theta) e_n(\theta), \ \forall \theta \in \widehat{G_n}.$$

Now,

$$\Big(\sum_{x\in G_n}\alpha_x x\Big)\Big(\frac{1}{|G_n|}\sum_{y\in G_n}\theta(y)y\Big)=\sum_{x\in G_n}\alpha_x\cdot\frac{1}{|G_n|}\sum_{y\in G_n}\theta(y)xy$$

$$=\sum_{x\in G_n}\alpha_x\frac{1}{|G_n|}\sum_{z\in G_n}\theta(x^{-1}z)z=\sum_{x\in G_n}\alpha_x\cdot\theta^{-1}(x)\Big(\frac{1}{|G_n|}\sum_{z\in G_n}\theta(z)z\Big)$$

$$=\Big(\sum_{x\in G_n}\alpha_x\theta^{-1}(x)\Big)e_n(\theta).$$

So $a_n(\theta)=\sum_{x\in G_n}\alpha_x\theta^{-1}(x),\ \forall\theta\in\widehat{G_n}$.

ii): If $f\in\Omega_pG$, then $f\in\Omega_pG_n$, for some $n\in\mathbb{N}$. Write $f=\sum_{\theta\in\widehat{G_n}}a_n(\theta)e_n(\theta)$ and $f=\Sigma\alpha_x x$, as above.

If $\|f\|_*=|a_n(\theta)|$, for some $\theta\in\widehat{G_n}$, then since $|\theta^{-1}(x)|=1,\ \forall x\in G_n$, the formula $a_n(\theta)=\sum_{x\in G_n}\alpha_x\theta^{-1}(x)$ implies that $|a_n(\theta)|\le\max_{x\in G_n}|\alpha_x|=\|f\|$.

□

The above shows that the identity map $\imath:(\Omega_pG,\ \|\cdot\|)\longrightarrow(\Omega_pG,\ \|\cdot\|_*)$ is a contraction, in the sense that $\|f\|_*\le\|f\|,\ \forall f\in\Omega_pG$.

Let $\overline{\Omega_pG}^*$ denote the completion of $\Omega_pG$ with respect to the $\|\cdot\|_*$ norm.

PROPOSITION 5.2. *The identity map $\imath:(\Omega_pG,\ \|\cdot\|)\longrightarrow(\Omega_pG,\ \|\cdot\|_*)$ extends to a continuous $\Omega_p$-algebra homomorphism $\varphi:\ell_1(\Omega_p,G)\longrightarrow\overline{\Omega_pG}^*$.*

PROOF: Since the identity map is a contraction, Cauchy sequences of $(\Omega_pG,\ \|\cdot\|)$ are mapped to Cauchy sequences of $(\Omega_pG,\ \|\cdot\|_*)$. Hence the extension of the identity map to the completion of $\ell_1(\Omega_p, G)$ is a well-defined map $\gamma$ into $\overline{\Omega_p G}^*$. It is clear that $\gamma$ is still a contraction, hence must be an $\Omega_p$-algebra homomorphism since it extends $\imath$.

□

This map $\gamma$ has the property that $J(\ell_1(\Omega_p, G)) = \ker\gamma$. The proof of this fact rests on character theory. Recall that $\widehat{G}$ is the set of $p$-adic characters, mapping $G$ into $\Omega_p$. As in the $\ell_1(k, G)$ situation, the elements of $\widehat{G}$ extend $\Omega_p$-linearly to continuous homomorphisms of $\Omega_p G$ into $\Omega_p$. These characters also extend to $\overline{\Omega_p G}^*$, by the next proposition.

PROPOSITION 5.3. *Let $\varphi \in \widehat{G}$, then for any $n \in \mathbb{N}$, and any $\theta \in \widehat{G_n}$,*

$$\varphi(e_n(\theta)) = \begin{cases} 1, & \textit{if} \quad \varphi|_{G_n} = \theta^{-1} \\ 0, & \textit{otherwise.} \end{cases}$$

*Furthermore, $\varphi$ extends to a continuous $\Omega_p$-algebra homomorphism $\varphi^* : \overline{\Omega_p G}^* \longrightarrow \Omega_p$, which is also a contraction.*

PROOF:

$$\varphi(e_n(\theta)) = \varphi|_{G_n}\left(\frac{1}{|G_n|}\sum_{x\in G_n}\theta(x)x\right) = \frac{1}{|G_n|}\sum_{x\in G_n}\theta(x)\,\varphi|_{G_n}(x).$$

Therefore, the orthogonality relations for characters of finite groups implies

$$\varphi(e_n(\theta)) = \begin{cases} 1, & \text{if} \quad \varphi|_{G_n} = \theta^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

If $f = \sum_{\theta \in \widehat{G_n}} a_n(\theta)e_n(\theta)$, $\exists a_n(\theta) \in \Omega_p$, $\forall \theta \in \widehat{G_n}$, then the above implies $\varphi(f) = \sum_{e \in \widehat{G_n}} a_n(\theta)\varphi(e_n(\theta)) = a_n(\varphi|_{G_n}^{-1})$. Hence $|\varphi(f)| \leq \|f\|_*$, so $\varphi$ is a contraction, and thus extends to the completion $\overline{\Omega_p G}^*$. Continuity of $\varphi$ on $\overline{\Omega_p G}^*$ implies that $\varphi$ is an $\Omega_p$-linear homomorphism.

□

$\overline{\Omega_p G}^*$ may be a more natural ring completion than $\ell_1(\Omega_p, G)$ in the case of abelian locally finite groups. For instance, $\overline{\Omega_p G}^*$ is always semiprimitive, as the next theorem shows.

THEOREM 5.4. $\bigcap_{\varphi \in \widehat{G}} \ker \varphi^* = \{0\}$. *In particular,* $J(\overline{\Omega_p G}^*) = \{0\}$.

PROOF: The set $\{e_n(\theta) \in \Omega_p G : \theta \in \widehat{G_n},\ n \in \mathbb{N}\}$ is a spanning set for $\Omega_p G$. Let $\mathcal{B}$ be a basis of $\Omega_p G$ as a vector space over $\Omega_p$ whose elements are chosen from this spanning set. Then

$$\Omega_p G = \Big\{ \sum_{e \in \mathcal{B}} a_e e : a_e \in \Omega_p, \quad \text{with almost all} \quad a_e = 0 \Big\}.$$

with the $\|\cdot\|_*$-norm on $\Omega_p G$ given by

$$\|\sum_{e \in \mathcal{B}} a_e e\|_* = \sup_{e \in \mathcal{B}} |a_e|.$$

Thus the completion $\overline{\Omega_p G}^*$ can be identified with set of formal sums $\sum_{e \in \mathcal{B}} a_e e$, $a_e \in \Omega_p$, $\forall e \in \mathcal{B}$, satisfying $|a_e| \to 0$, with the $\|\cdot\|_*$-norm given by $\|\sum_{e \in \mathcal{B}} a_e e\| = \sup_{e \in \mathcal{B}} |a_e|$.

Let $f \in \overline{\Omega_p G}^*$ be non-zero. Let $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ be any basis chosen from the set $\{e_n(\theta) : n \in \mathbb{N},\ \theta \in \widehat{G_n}\}$. $f$ can be expressed as $f = \sum_{i=1}^{m} a_i e_i + \sum_{i=m+1}^{\infty} b_i e_i$, for some $a_i,\ b_i \in \Omega_p$ such that $\|f\|_* = \|a_i\|$, for $i = 1, \ldots, m$ and $\|f\|_* > |b_i|$ for all $i > m$, after possibly renumerating the basis $\mathcal{B}$. The subset $\{e_i\}_{i=1}^{m}$ of $\mathcal{B}$ corresponds to some subset of $\{e_n(\theta) : 1 \leq n \leq N,\ \theta \in \widehat{G_n}\}$, for some large enough integer $N \in \mathbb{N}$. Put $f_N = \sum_{i=1}^{m} a_i e_i \in \Omega_p G_n$, and re-write $f_N$ as $\sum_{\theta \in \widehat{G_N}} a_N(\theta) e_N(\theta)$, with $a_n(\theta) \in \Omega_p$. In this expression, the $a_n(\theta)$ either satisfy $|a_n(\theta)| = \|f\|_* = \|f_N\|_*$ or $a_n(\theta) = 0$. Choose $\varphi \in \widehat{G}$ such that $\varphi|_{G_N} \in \widehat{G_N}$ has $a_N(\varphi|_{G_N}^{-1}) \neq 0$. Then

$$\begin{aligned}\varphi(f) &= \varphi(f_N) + \varphi\Big(\sum_{i=m+1}^{\infty} b_i e_i\Big)\\ &= \varphi\Big(\sum_{\theta \in \widehat{G_N}} a_N(\theta) e_N(\theta)\Big) + \varphi\Big(\sum_{i=m+1}^{\infty} b_i e_i\Big)\\ &= a_N\Big(\varphi|_{G_N}^{-1}\Big) + \varphi\Big(\sum_{i=m+1}^{\infty} b_i e_i\Big),\end{aligned}$$

and

$$|a_N(\varphi|_{G_N}^{-1})| = \|f\|_*, \quad \text{but} \quad |\varphi\Big(\sum_{i=m+1}^{\infty} b_i e_i\Big)| < \|f\|_*.$$

So $|\varphi(f)| = |a_N(\varphi|_{G_N}^{-1})| = \|f\|_*$. Thus for all such $\varphi$, $\varphi(f) \neq 0$.

□

The main result concerning $\overline{\Omega_p G}^*$ is the following:

THEOREM 5.5. $\ker\gamma = J(\ell_1(\Omega_p, G))$.

PROOF: Let $\varphi \in \widehat{G}$. Let $\varphi_*$ be the $\Omega_p$-linear extension of $\varphi$ to $\overline{\Omega_p G}^*$, and $\varphi_0$ the $\Omega_p$-linear extension of $\varphi$ to $\ell_1(\Omega_p, G)$. $\varphi_0$ and $\varphi_* \circ \gamma$ are both continuous $\Omega_p$-linear homomorphisms from $\ell_1(\Omega_p, G)$ to $\Omega_p$, and they agree on $\Omega_p G$, so $\varphi_0 = \varphi_* \circ \gamma$. Hence $\ker\varphi_0 = \ker(\varphi_* \circ \gamma) = \gamma^{-1}(\ker\varphi_*)$, and so $J(\ell_1(\Omega_p, G)) = \bigcap_{\varphi \in \widehat{G}_{\Omega_p}} \ker\varphi = \bigcap_{\varphi \in \widehat{G}} \gamma^{-1}(\ker\varphi_*) = \gamma^{-1}\Big(\bigcap_{\varphi \in \widehat{G}} \ker\varphi_*\Big) = \gamma^{-1}(\{0\})$, by Theorem 5.4. Therefore $J(\ell_1(\Omega_p, G)) = \ker\gamma$.

□

If $k$ is an arbitrary complete $p$-adic field, then in order that $\sum_{\theta \in \widehat{G_n}} a_n(\theta)e_n(\theta) \in kG_n$, for some $n \in \mathbb{N}$, certain restrictions apply to the $a_n(\theta)$ so that the coefficients of $a_n(\theta)e_n(\theta)$ lie in $k$, for each $\theta \in G_n$. However, the same characterization of the Jacobson radical of $\ell_1(k, G)$ remains. If $\varphi \in \widehat{G}$, and $\varphi_0$, $\varphi_*$ are the continuous $k$-linear extensions of $\varphi$ to $\ell_1(k, G)$, $\overline{kG}^*$ respectively, then $\varphi_0 = \varphi_* \circ \gamma$, as before, and $\ker\varphi_0 = \ker(\varphi_* \circ \gamma) = \gamma^{-1}\ker\varphi_*$ implies $J(\ell_1(k, G)) = \bigcap_{\varphi_0 \in \widehat{G_k}} \ker\varphi_0 = \gamma^{-1}\Big(\bigcap_{\varphi \in \widehat{G}} \ker\varphi_*\Big) = \gamma^{-1}(\{0\})$. It is an open question as to whether or not a similar characterization is possible for arbitrary locally finite non-abelian groups $G$.

## Bibliography for Chapter V

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