University of Alberta

Regulators in Algebraic Geometry

by

Andrew Roberts ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Master of Science**.

in

Mathematics

Department of Mathematical and Statistical Sciences

Edmonton, Alberta

Fall 2002

0-612-81470-X

University of Alberta

Library Release Form

Name of Author: Andrew Roberts

Title of Thesis: Regulators in Algebraic Geometry

Degree: Master of Science

Year this Degree Granted: 2002

...Andrew Roberts...

Andrew Roberts
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Date: ..Sept. 5, 2002

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommended to the Faculty of Graduate Studies and Research for acceptance, the thesis entitled **Regulators in Algebraic Geometry** submitted by **Andrew Roberts** in partial fulfillment for the degree of **Master of Science**. in Mathematics.

....
James D. Lewis (Supervisor)

....
Arturo Pianzola (Chair)

....
Richard Sydora (Physics)

Date: 30/VIII/02

Abstract: The focus of this paper is to provide the reader with the necessary background to understand the study of algebraic cycles and the construction of regulator maps from certain cycle groups into cohomology. After covering the basics on varieties, cohomology (especially deRham cohomology), and Hodge Theory we proceed to build examples of regulators on the cycle groups $z^k(X,1)$, $K^M{}_2C(X)$ and $CH^2(X,2)$ for a general projective algebraic manifold in the first case and for a compact Riemann surface in the second. We go through detailed calculations using limit arguments and Stokes' Theorem to show that these regulators are well-defined maps. We end with some calculations suggesting that our regulator on $K^M{}_2C(X)$ is non-trivial.

Acknowledgment: I would like to thank Dr. James Lewis for his help and support in the completion of my thesis, committee members Arturo Pianzola and Richard Sydora for attending my defense, as well as the staff and students of the department for making my two years at the University of Alberta enjoyable ones.

Contents

Chapter 1

Introduction

This thesis will deal with the subject of cycle groups in Algebraic Geometry and with building so-called regulator maps from these cycle groups into cohomology. The main goals can be summarized as follows,

1. To learn the rudiments of Algebraic Geometry, particularly that area dealing with algebraic cycles.

2. To describe certain cycle groups on projective algebraic manifolds that are of current interest in Algebraic K-Theory and Algebraic Geometry.

3. To give examples of regulator maps from these cycle groups into cohomology.

4. To suggest ways of showing that these maps can be non-trivial.

The area of Algebraic Geometry explored in this thesis is chiefly concerned with groups of algebraic cycles. We will see examples of such groups such as $z^k(X,1)$, the Milnor K-Theory groups $K_{\bullet}^M\mathbf{C}(X)$, and the Chow groups $CH^k(X,m)$. The main idea is to find relationships between the cycle groups in Algebraic geometry (which are difficult to compute) and the more easily computable cohomology groups in Algebraic Topology. Such relationships can be established by defining maps from cycle groups into cohomology. These maps are what we call regulators. The hope is to find computable, non-trivial regulators which can tell us something of the cycle groups we are trying to study. The relationship of regulators to Algebraic Geomety and Algebraic Topology can be summarized in the following diagram,

Algebraic Topology		**Algebraic Geometry**
X a topological space		*X a projective algebraic manifold*
(Co)homology *(easily computable)* *(i.e.* $H^\bullet(X), H_\bullet(X)$*)*	*Regulators* $\longleftarrow$	*Cycle Groups* *(difficult to compute)* *(i.e.* $z^k(X,1), K_\bullet^M \mathbf{C}(X)$, $CH^k(X,m)$*)*

The thesis will be structured in the following way. Chapters 2 and 3 will cover the necessary background in varieties and cohomology with basics of commutative algebra found in the appendix. Chapter 4 will consist of our first example of a regulator, in this case on the cycle group $z^k(X,1)$. The cycle groups $K_\bullet^M \mathbf{C}(X)$ from Milnor K-Theory as well as the Tame symbol map will be introduced in Chapter 5. The ideas developed in Chapter 5 will then be used to define a second regulator map, this time on the cycle group $K_2^M \mathbf{C}(X)$ for the case where X is a compact Riemann surface. We will end with some motivational calculations as well as an indication of where these ideas lead in the subject.

Chapter 2

Varieties

In order to get an intuitive idea of what varieties are we give a brief introduction to varieties in complex space $\mathbf{C}^n$. We will then look at the more generalized scenario in affine and then projective space where the interaction between algebra and geometry will be more apparent. The main sources for this chapter are Mumford [Mum] and Dummit & Foote [D-F].

2.1 Varieties in $\mathbf{C}^n$

Consider a finite set of polynomials $f_1, \ldots, f_m$ in the polynomial ring $\mathbf{C}[Z_1, \ldots, Z_n]$. We let,

$$V(f_1, \ldots, f_m)$$

be the set of zeros of the f_i's.

__Definition__: A subset $X \subset \mathbf{C}^n$ is said to be a closed algebraic subset of $\mathbf{C}^n$ if $X = V(f_1, \ldots, f_m)$ for some finite set of polynomials $f_1, \ldots, f_m \in \mathbf{C}[Z_1, \ldots, Z_n]$.

Say we have $X = V(f_1, \ldots, f_m)$ a closed algebraic subset. It is obvious that if $m = (f_1, \ldots, f_m)$ is the ideal generated by the f_i's then $V(f_1, \ldots, f_m) = V((m))$. Also, since $\mathbf{C}[Z_1, \ldots, Z_n]$ is Noetherian [for more on Noetherian domains see Appendix], we know that this ideal is finitely generated. Thus we can represent any algebraic subset X as $V((m))$ for some ideal m in $\mathbf{C}[Z_1, \ldots, Z_n]$. This will be a more useful way to define algebraic subsets since we will be able to exploit all the properties of rings and ideals in order to examine varieties and algebraic subsets. In general we will denote $V((m))$ by $V(m)$ to simplify notation.

We make note of the following properties of closed algebraic subsets,

1. $V(0) = \mathbf{C}^n$ and $V(1) = \emptyset$

2. If m_1 and m_2 are ideals then $V(m_1) \cup V(m_2) = V(m_1 \cap m_2)$. Notice that $V(m_1 \cap m_2)$ is again a closed algebraic subset since $m_1 \cap m_2$ is an ideal.

3. If $\{m_j \ : \ j \in J\}$ is an arbitrary set of ideals then,

$$\bigcap_{j \in J} V(m_j) = V(\sum_{j \in J} m_j)$$

Notice that $V(\sum_{j \in J} m_j)$ is again a closed algebraic subset since $\sum_{j \in J} m_j$ is an ideal.

But we see that the three properties above are exactly the axioms for the closed sets of a topology. So we see that the closed algebraic subsets define the closed sets in a topology on $\mathbf{C}^n$. This is called the Zariski topology.

If m is an ideal of the ring $\mathbf{C}[Z_1, \ldots, Z_n]$ then the radical of m is defined to be

$$r(m) = \{x \in \mathbf{C}[Z_1, \ldots, Z_n] : x^s \in m \ for \ some \ s \in \mathbf{N}\}$$

Thus it is apparent that,

$$V(r(m)) = V(m)$$

Now, since $\mathbf{C}[Z_1, \ldots, Z_n]$ is Noetherian, we know that any radical ideal has a unique prime decomposition. In other words, for any ideal m we can write,

$$r(m) = m_1 \cap \cdots \cap m_p$$

in a unique way where the m_i are prime ideals. Thus we have that,

$$\begin{aligned} V(m) &= V(r(m)) \\ &= V(m_1 \cap \cdots \cap m_p) \\ &= V(m_1) \cup \cdots \cup V(m_p) \end{aligned}$$

So closed algebraic subsets of the form $V(m)$ where m is a prime ideal seem as though they are especially important. Indeed, we give them a special title,

Definition: If m is a prime ideal then the closed algebraic subset $V(m)$ is called an affine variety.

Note: The affine varieties are the irreducible Zariski closed sets.

In particular we know that if $f \in \mathbf{C}[Z_1, \ldots, Z_n]$ is an irreducible polynomial then (f) is a prime ideal. In this case $V((f))$ is an affine variety called a hypersurface. From the above discussion we arrive immediately at the following result,

Proposition: Any closed algebraic subset in $\mathbf{C}^n$ can be expressed uniquely as the union of finitely many affine varieties.

Now that we have gotten a feel for what affine varieties look like in $\mathbf{C}^n$ we will move to the more abstract environment of affine n-space. Here we will take full advantage of the algebraic structure of varieties in order to get a better handle on their geometry. All the results in the next section can be reduced to the particular case where we let our field k be the complex numbers. From time to time it will be helpful and instructive to do so.

2.2 Varieties in Affine n-space

2.2.1 Preliminary Definitions

Now, instead of just considering the field $\mathbf{C}$, we let k be any field and denote the space of n-tuples of elements of k by A^n. We call A^n affine n-space. An element f of the polynomial ring $k[Z_1, \ldots, Z_n]$ can be considered as a function $f : A^n \longrightarrow k$ by mapping,

$$(a_1, \ldots a_n) \in A^n \longmapsto f(a_1, \ldots a_n) \in k$$

In this sense we denote $k[Z_1, \ldots, Z_n]$ by $k[A^n]$ and call it the coordinate ring of A^n. For $f_1, \ldots, f_m \in k[A^n]$ we define,

$$V(f_1, \ldots, f_m) = \{a \in A^n : f_1(a) = \ldots = f_m(a) = 0\}$$

and call $X \subset A^n$ a closed algebraic subset if $X = V(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in k[A^n]$. As before we notice that the set of zeros of a collection of functions is equal to the set of zeros of the elements of the ideal generated by those functions in $k[A^n]$. This, along with the fact that $k[A^n]$ is Noetherian for any field k, ensure that the closed algebraic subsets of $k[A^n]$ are exactly the subsets $V(m)$ where m is an ideal in $k[A^n]$.

Definition: If m is a prime ideal then we call the closed algebraic set $V(m)$ an affine variety.

Using the same argument as in the previous section, we note that any closed algebraic set has a unique decomposition as a union of finitely many affine varieties. Also, it is easy to see that the closed algebraic subsets still define the closed sets for the Zariski topology on A^n.

2.2.2 The Algebra and Geometry of Affine Varieties

We now begin to examine the correspondence between the algebraic and geometric properties of varieties. In this section we will restrict ourselves to algebraically closed fields k (i.e. where every polynomial in $k[A^n]$ has a root in k). In particular, we recall that the Fundamental Theorem of Algebra states that

$\mathbf{C}$ is an algebraically closed field. Since this is really the field we want to work with this restriction will not be much of a sacrifice.

We let $X = V(m)$ be any closed algebraic subset in $k[A^n]$ and define the set,

$$I(X) = \{f \in k[A^n] : f = 0 \ on \ X\}$$

We can consider V and I as maps,

$$V : \{ideals \ in \ k[A^n]\} \longrightarrow \{closed \ algebraic \ subsets \ of \ A^n\}$$

$$I : \{closed \ algebraic \ subsets \ of \ A^n\} \longrightarrow \{ideals \ in \ k[A^n]\}$$

The question arises, "Are the maps I and V in bijective correspondence?" It is fairly easy to see that for any closed algebraic subset X we get $V(I(X)) = X$ but unfortunately it is not the case that $I(V(m)) = m$ for any ideal m. However, as a result of a famous theorem of Hilbert we get that they are in bijection on a restriction of their domains.

Theorem: (Hilbert's Nullstellensatz) Let k be any algebraically closed field. For any ideal m of $k[A^n]$, $I(V(m)) = r(m)$.

Corollary: If m is a prime ideal then $I(V(m)) = m$.

So if we restrict the the domains of the maps V and I we get the following one-to-one correspondence,

$$\{affine \ varieties \ in \ A^n\} \longleftrightarrow \{prime \ ideals \ in \ k[A^n]\}$$

So we see that varieties are simply the irreducible algebraic subsets.

It is well-known that any maximal ideal is prime and so Hilbert's Nullstellensatz puts maximal ideals in one-to-one correspondence with some subset of the affine varieties in A^n. Can we give a description of this subset? Say we have m a maximal ideal in $k[A^n]$. Then we know that $m \neq (1)$ but then the Nullstellensatz implies that $V(m) \neq \emptyset$. So we can pick $a = (a_1, \ldots, a_n) \in V(m)$. This implies that $m \subset (K_1 - a_1, \ldots, K_n - a_n)$. But this along with m being maximal implies that,

$$m = (K_1 - a_1, \ldots, K_n - a_n)$$

So any maximal ideal is of the form $(K - a)$ for some point $a \in A^n$. Thus we have a correspondence between any maximal ideal and the point $(a_1, \ldots, a_n) \in A^n$, i.e. we have,

$$\{points \ in \ A^n\} \longleftrightarrow \{maximal \ ideals \ in \ k[A^n]\}$$

We now look a little more closely at the polynomials in $k[A^n]$ and consider their restrictions to an algebraic subset $X = V(m) \subset A^n$. We have the following,

Proposition: For any algebraic subset $X = V(m) \subset A^n$ with $m = r(m)$ and any algebraically closed k, the ring $k[A^n]/m$ is canonically isomorphic to the ring of functions $X \longrightarrow k$ that are restrictions of polynomials $A^n \longrightarrow k$.

Proof: This is immediate considering that $f = g$ on X means that $f - g = 0$ on X. But this holds if and only if $f - g \in m$ and thus the images of f and g in $k[A^n]/m$ are equal. Thus the restricted polynomials are in one-to-one correspondence with the elements of the ring $k[A^n]/m$. Q.E.D.

Definition: We denote the ring $k[A^n]/m$ by $k[X]$ and call it the affine coordinate ring of X.

Now, instead of looking at $k[A^n]$, let us restict our attention to the coordinate ring $k[X]$ for some arbitrary algebraic subset $X = V(m)$. If $\pi : k[A^n] \longrightarrow k[X]$ is just the quotient map then it is well known that if β is a prime ideal in $k[X]$ then $\tilde{\beta} = \pi^{-1}(\beta)$ is a prime ideal in $k[A^n]$. Notice that it is always true that $\tilde{\beta} \supset m$ and so we have that $V(\tilde{\beta}) \subset V(m) = X$, i.e $V(\tilde{\beta})$ is a subvariety of X. So we get the correspondence,

$$\{subvarieties\ of\ X\} \longleftrightarrow \{prime\ ideals\ in\ k[X]\}$$

We also note that the coordinate ring of the subvariety $V(\beta)$ is given equivalently by,

$$k[V(\beta)] = k[A^n]/\tilde{\beta}$$

or,

$$k[V(\beta)] = k[X]/\beta$$

depending on whether we consider $V(\beta)$ as a variety in A^n or as a subvariety of X.

If we map maximal ideals of $k[X]$ back into $k[A^n]$ via π^{-1} we can use a similar argument to that used above to give the association,

$$\{points\ in\ X\} \longleftrightarrow \{maximal\ ideals\ in\ k[X]\}$$

As a result of Hilbert's Nullstellensatz we have seen that there exists an amazing correspondence between the geometry of varieties and algebraic sets and the algebra of polynomial and coordinate rings and their ideals. We summarize the results in the following chart,

Geometry		Algebra
Affine n-space A^n	$\longleftrightarrow$	Coordinate Ring $k[A^n]$
Points in A^n	$\longleftrightarrow$	Maximal Ideals in $k[A^n]$
Affine Varieties in A^n	$\longleftrightarrow$	Prime Ideals in $k[A_n]$
Affine Algebraic Subset X	$\longleftrightarrow$	Coordinate Ring $k[X]$
Points in X	$\longleftrightarrow$	Maximal Ideals in $k[X]$
Subvarieties in X	$\longleftrightarrow$	Prime Ideals in $k[X]$

2.2.3 The Local Rings $\mathcal{O}_{x,X}$

Now that we understand a bit about varieties and their associated polynomial rings we would like to talk about localizations of these rings about points in the variety. To do this we introduce a local ring $\mathcal{O}_{x,X}$ for each point x in our variety X. We will see that the coordinate ring $k[X]$ and it's localizations are very inter-related and that the local rings can be used to find information about particular points on the variety. We will also see that the local rings and their maximal ideals fit very well into the structure given by the Nullstellensatz in the previous section.

Definition: Let $X = V(m) \subset A^n$ be an affine variety and let x be any point in X. We define,

$$\mathcal{O}_{x,X} = \{f/g : f, g \in k[X],\ g(x) \neq 0\}$$

and call $\mathcal{O}_{x,X}$ the ring of rational functions at the point $x \in X$.

It is fairly obvious that $\mathcal{O}_{x,X}$ is a ring, but less obvious is that $\mathcal{O}_{x,X}$ is in fact a local ring. To see this consider the ideal $m_x = \{f/g \in \mathcal{O}_{x,X} : f(x) = 0\}$. Then $m \neq (1)$ and we notice that any $f/g \notin m_x$ is a unit (since $f(x) = 0$ implies $g/f \in \mathcal{O}_{x,X}$ and thus $(f/g)^{-1} = g/f$). So, by our Test for Locality in Appendix A, we have that $\mathcal{O}_{x,X}$ is a local ring with maximal ideal m_x.

An alternate way to formulate $\mathcal{O}_{x,X}$ is as a localization of the coordinate ring $k[X]$. Consider the multiplicatively closed set $D_x = \{g \in k[X] : g(x) \neq 0\}$. Then we know that $D_x^{-1}k[X]$ is defined as the smallest ring in which all elements of D_x become units. Clearly we must have $1/g \in D_x^{-1}k[X]$ and so without much work we can see that,

$$D_x^{-1}k[X] = \{f/g : f, g \in k[X], g(x) \neq 0\}$$

which is exactly $\mathcal{O}_{x,X}$. So an alternate definition of $\mathcal{O}_{x,X}$ is as the localization of $k[X]$ at D_x.

So it seems that the rings $k[X]$ and $\mathcal{O}_{x,X}$ are inter-related. Another instance of this is that, for any $x \in X$ these two rings have the same quotient field. This is quite clear since if $f/g \in Quot(k[X])$ then we write,

$$f/g = \frac{f/1}{g/1} \in Quot(\mathcal{O}_{x,X})$$

Conversely, if

$$\frac{f_1/g_1}{f_2/g_2} \in Quot(\mathcal{O}_{x,X})$$

then we write

$$\frac{f_1/g_1}{f_2/g_2} = \frac{f_1 g_2}{f_2 g_1} \in Quot(k[X])$$

We denote this quotient field as $k(X)$ and call it the rational function field of X.

We now show an important relationship between the coordinate ring of $k[X]$ and the local rings $\mathcal{O}_{x,X}$; that the coordinate ring of a variety is completely determined by the local rings of the variety.

Proposition: Let $x = V(m)$ be an affine variety and x an arbitrary point in X. The by taking intersections in the rational function field $k(X)$ we get,

$$k[X] = \bigcap_{x \in X} \mathcal{O}_{x,X}$$

Proof:

- "$\subset$" Let $f \in k[X]$. Then $f = f/1 \in \mathcal{O}_{x,X}$ and this holds for any $x \in X$. Thus $f \in \cap_{x \in X} \mathcal{O}_{x,X}$ and so $k[X] \subset \mathcal{O}_{x,X}$.
- "$\supset$" Let $f \in \cap_{x \in X} \mathcal{O}_{x,X}$. Then, for each $x \in X$, we have $f \in \mathcal{O}_{x,X}$ and thus we can write $f = h_x/g_x$ where $h_x, g_x \in k[X]$ and $g_x(x) \neq 0$. Consider the ideal,

  $$a = \{g \in k[X] : gf \in k[X]\}$$

  Then we see that $g_x \in a$ since $g_x f = g_x(h_x/g_x) = h_x \in k[X]$. Since $g_x(x) \neq 0$ we have that $x \notin V(\tilde{a})$ where $\tilde{a} = \{f \in k[A^n] : f \bmod m \in a\}$, and this holds for all $x \in X$. But notice that $\tilde{a} \supset m$ so that $V(\tilde{a}) \subset V(m)$ and so $V(\tilde{a}) = \emptyset$. But then, by the Nullstellensatz,

  $$1 \in I(V(\tilde{a})) = r(\tilde{a})$$

  and hence $1 \in \tilde{a}$. Thus $1 \in a$ and so by the definition of a we have that $f \in k[x]$. Thus $k[X] \supset \mathcal{O}_{x,X}$.

Therefore $k[X] = \cap_{x \in X} \mathcal{O}_{x,X}$. Q.E.D.

We've seen that we can build a local ring $\mathcal{O}_{x,X}$ at each point of a variety $X = V(m)$. So we can associate each point $x \in X$ with the single maximal ideal m_x sitting in $\mathcal{O}_{x,X}$. But recall that the Nullstellensatz associated each point $x \in X$ with the maximal ideal $(Z - x)$ in $k[X]$. What is the relationship here? Since we have that $k[X] = \cap_{x \in X} \mathcal{O}_{x,X}$, for any $x \in X$ we have an inclusion map,

$$i_x : k[X] \hookrightarrow \mathcal{O}_{x,X}$$

If we look more closely at the ideal $(Z - x)$ we see that,

$$\begin{aligned} (Z-x) &= \{f \in k[X] : f = (Z-x) \cdot (\tilde{f}_1, \ldots, \tilde{f}_n) \ for\ some\ \tilde{f}_1, \ldots, \tilde{f}_n \in k[X]\} \\ &= \{f \in k[X] : f(x) = 0\} \end{aligned}$$

and so we can think of $(Z - x)$ as the set $\{f/g \in m_x : g = 1\}$. Thus we can treat $(Z - x)$ as a subset of m_x. So we have,

$$\begin{array}{ccc} k[X] & \hookrightarrow & \mathcal{O}_{x,X} \\ \cup & & \cup \\ (K-x) & \hookrightarrow & m_x \end{array}$$

and there is no problem in associating these two maximal ideals with the same point $x \in X$ since $(Z - x)$ is simply m_x restricted to $k[X] \subset \mathcal{O}_{x,X}$. This indicates that the structure of the local rings of rational functions fits well into the structure of associations developed above as a result of the Nullstellensatz. Now we will proceed to use the local rings $\mathcal{O}_{x,X}$ to gain information about our variety X.

2.2.4 Tangent Spaces, Smoothness & Dimension

Now that we are able to examine a variety locally we can begin to talk about it's local characteristics. Two important examples of this are tangent spaces and smoothness of the variety at a point. We begin with some definitions,

<u>Definition</u>: Let $X = V(m)$ be an affine variety and let $p = (p_1, \ldots, p_n)$ be any point in X. If $\{f_1, \ldots, f_m\}$ is a generating set of the ideal m then we define,

$$T_p(X) = \{K = (K_1, \ldots, K_n) \in A^n : \vec{\nabla} f_j \cdot (K_1 - p_1, \ldots, K_n - p_n) = 0, \forall j\}$$

and call $T_p(X)$ the tangent space of X at p (It can be shown that this definition is independent of the choice of generators).

By using some basic linear algebra we can see that the dimension of the vector space $T_p(X)$ is given by,

$$dimT_p(X) = n - rk\left(\frac{\partial f_i}{\partial K_j}(p)\right)_{1\leq i\leq m, 1\leq j\leq n}$$

Using this we define,

<u>**Definition**</u>: Let X be a variety in A^n. We define the dimension of X with respect to the field k to be,

$$dim_k X = \min_{p\in X}\{dimT_p(X)\}$$

It is not too hard to see that it will always be the case that $0 \leq dimT_p(X) \leq n$ and thereby $0 \leq dim_k X \leq n$. This makes sense since we wouldn't want n-dimensional affine space to contain varieties of a higher dimension. We now define what it means for a point in X to be smooth.

<u>**Definition**</u>: A point p in a variety X is said to be smooth if $dim_k X = dimT_p(X)$ and is said to be singular if $dim_k X < dimT_p(X)$.

We introduce the notation X_{smooth} and X_{sing} for the set of smooth and singular points in X, and note that any we can write any variety as $X = X_{smooth} \cup X_{sing}$.

<u>**Definition**</u>: We say that X is a smooth variety if every point in X is smooth (i.e. $X = X_{smooth}$ and $X_{sing} = \emptyset$).

Let us consider the following simple example,

<u>**Example**</u>: Consider the variety $X = V(Z_1^2 - Z_2^3) \subset C^2$. Then,

$$\begin{aligned} dimT_{(Z_1,Z_2)}(X) &= 2 - rk\left(\frac{\partial}{\partial Z_1}(Z_1^2 - Z_2^3) \;\; \frac{\partial}{\partial Z_2}(Z_1^2 - Z_2^3)\right) \\ &= 2 - rk\left(2Z_1 \;\; -3Z_2^2\right) \end{aligned}$$

Thus we have,

$$dimT_{(Z_1,Z_2)}(X) = \begin{cases} 2 & if\ (Z_1, Z_2) = (0,0) \\ 1 & if\ (Z_1, Z_2) \neq (0,0) \end{cases}$$

So by definition we have that $dim_C X = 1$ and that all points with the exception of $(0,0)$ are smooth in X. This is suggested by the graph of the real zeroes of the function $f(Z_1, Z_2) = Z_1^2 - Z_2^3$, which is just the locus of the points $(a^2, a^3) \in C^2$ restricted to R^2.

By the definition of $T_p(X)$ we can see that,

$$T_{(0,0)}(X) = C^2$$

and that for (a^2, a^3) with $a \neq 0$ we have,

$$\begin{aligned}
T_{(a^2,a^3)}(X) &= \{(Z_1, Z_2) \in C^2 : \sum_{j=1}^{2} \frac{\partial}{\partial Z_j}(Z_1^2 - Z_2^3)|_{(a^2,a^3)} \cdot (Z_j - a_j) = 0\} \\
&= \{(Z_1, Z_2) \in C^2 : 2a^2(Z_1 - a^2) - 3a^3(Z_2 - a^3) = 0\} \\
&= \{(Z_1, Z_2) \in C^2 : Z_1 = \frac{3a}{2} Z_2 + \left(a^2 - \frac{3a^4}{2}\right)\}
\end{aligned}$$

which defines a line in C^2. So we see that the dimensions of $T_{(Z_1,Z_2)}(X)$ make intuitive sense.

In the Appendix we define the Krull dimension of a ring to be the maximal length (minus 1) of strictly increasing chains of prime ideals in the ring. Is it true that the dimension of a variety is the same as the Krull dimension of it's coordinate ring? The answer is "yes". The following result gives three diffent ways to think of dimension when speaking of varieties and asserts that they all agree. The proof is rather long and will be omitted but can found at [A-M, 124-125].

Proposition: Let X be a variety in affine n-space over an algebraically closed field k . Then the following integers are equal,

1. $dim_k X = \min_{p \in X}\{dim T_p X\}$
2. $tr.d._k k(X)$ = transcendence degree of $k(X)$ over k
3. $dim_K k[X]$ = Krull dimension of the coordinate ring $k[X]$

So, in the tradition of the associations derived from the Nullstellensatz, we see a very nice correspondence between the algebraic and the geometric in the notion of the dimension of a variety.

2.3 Complex Varieties & Complex Manifolds

We now return to the complex environment for a while and examine the relationship between complex varieties and complex manifolds. First we recall the basic definition,

Definition: A complex manifold M of dimension n is a topological space together with coordinate charts $\{(U_i, h_i)\}_{i \in I}$ such that the following hold,

1. the collection $\{U_i\}_{i \in I}$ is an open cover of M,

2. $h_i : U_i \longrightarrow V_i$ is a homeomorphism onto an open set $V_i \subseteq \mathbf{C}^n$,

3. the so-called "transition functions" $h_i \circ h_j^{-1} : V_j \longrightarrow V_i$ are holomorphic on the overlap $V_i \cap V_j$.

Now, in general, a complex variety is not a complex manifold. The problem is caused by the singular points which are not allowed in a manifold. What if we restrict ourselves to only smooth points? We will attempt to build coordinate charts in neighbourhoods of smooth points in a complex variety. We begin by using the following result [Mum, 7],

Proposition: Let $X = V(m)$ be a variety in $\mathbf{C}^{n+r}$ with $dim_{\mathbf{C}} X = n$ and let $p \in X_{smooth}$. Then there exist functions $f_1, \ldots, f_r \in m$ such that,

$$V(f_1, \ldots, f_r) = X \cap Y$$

where Y is a closed algebraic set with $p \notin Y$.

So we consider the setting above and define a function,

$$F : \mathbf{C}^{n+r} \longrightarrow \mathbf{C}^r$$

by the formula,

$$F(Z) = (f_1(Z), \ldots, f_r(Z))$$

Notice that $F(p) = 0$ since $p \in V(f_1, \ldots, f_r)$. We will write,

$$\mathbf{C}^{n+r} = \mathbf{C}^n \times \mathbf{C}^r$$

and will represent any $Z \in \mathbf{C}^{n+r}$ as,

$$Z = (z_1, \ldots, z_n, w_1, \ldots, w_r) = (z, w)$$

and, in particular, will write,

$$p = (a, b)$$

where $a \in \mathbf{C}^n$ and $b \in \mathbf{C}^r$. Now, since $p \in X_{smooth}$ we know that the $r \times (n + r)$ matrix,

$$DF(p) = \left(\frac{\partial f_i}{\partial z_j}(p) \mid \frac{\partial f_i}{\partial w_l}(p) \right)_{0 \leq i \leq r, 0 \leq j \leq n, 0 \leq l \leq r}$$

has rank r. So by permuting the variables (if necessary) we can assume that we have,

$$det \left(\frac{\partial f_i}{\partial w_l}(p) \right) \neq 0$$

But now the Implicit Function Theorem tells us that there exists an open neighbourhood U_p of p in X_{smooth} of the form,

$$U_p = U_{p,\mathbf{C}^n} \times U_{p,\mathbf{C}^r}$$

and a unique holomorphic function,

$$g : U_{p,\mathbf{C}^n} \longrightarrow U_{p,\mathbf{C}^r}$$

with $g(a) = b$ and $F(z, w) = 0$ if and only if $w = g(z)$. Now, since $V(f_1, \ldots, f_r) = \{F = 0\}$ we can define the set,

$$V_p = \{F = 0\} \cap U_p = \{(z, g(z)) : z \in U_{p,\mathbf{C}^n}\}$$

which lies in X_{smooth}. So we have a holomorphic function,

$$\phi_p : U_{p,\mathbf{C}^n} \longrightarrow V_p$$

given by,

$$\phi_p(z) = (z, g(z))$$

which has a holomorphic inverse,

$$\phi_p^{-1} : V_p \longrightarrow U_{p,\mathbf{C}^n}$$

given by projection,

$$\phi_p^{-1}(z, w) = z$$

For convenience of notation we let $h_p = \phi_p^{-1}$ and notice that now we have coordinate charts $\{V_p, h_p\}_{p \in X_{smooth}}$ such that,

1. $\{V_p\}_{p \in X_{smooth}}$ is an open cover of X_{smooth}
2. $h_p : V_p \longrightarrow U_{p,\mathbf{C}^n}$ are homeomorphisms from open sets in X_{smooth} onto open sets in $\mathbf{C}^n$
3. the transition functions $h_p \circ h_q^{-1}$ are holomorphic wherever defined

This shows that X_{smooth} is a complex manifold of dimension n. So we have the result,

Proposition: Let X be a variety in $\mathbf{C}^n$ with $dim_{\mathbf{C}} X = r$. Then X_{smooth} is a complex manifold of dimension r.

So, in the complex case, this gives us another way to consider dimension of varieties which meshes with the proposition from the last section. This result also indicates that varieties are a sort of generalization of manifolds where we now allow singular points.

2.4 Divisors

We will use the following two results from Mumford [Mum, 15-17] as motivation for this section,

Proposition: If $X \subset \mathbf{C}^n$ is a variety and x is a smooth point then the local ring $\mathcal{O}_{x,X}$ is a unique factorization domain.

Proposition: Let $X \subset \mathbf{C}^n$ be an r-dimensional variety with x a smooth point and $f \in \mathbf{C}[X]$. Let,

$$V(f) = Z_1 \cup \cdots \cup Z_k$$

be the decomposition of $V(f)$ into it's irreducible components (i.e. where the Z_i are subvarieties). Then all Z_i passing through x are of dimension $r-1$ and are in one-to-one correspondence with the irreducible factors of f in $\mathcal{O}_{x,X}$ (which is a U.F.D. since x is a smooth point).

Notice that an immediate consequence is that if X is a smooth variety of dimension r then, for any $f \in \mathbf{C}[X]$, the irreducible components of $V(f)$ are all of dimension $r-1$. The question arises, "Is $\mathbf{C}[X]$ a unique factorization domain?" If so, then we would end up in a situation where we have a one-to-one correspondence between the irreducible components of the polynomials in the coordinate ring $\mathbf{C}[X]$ and the subvarieties of codimension 1 in X. The answer to this question is in general "No". So we will look more closely at the this relationship.

We consider a smooth variety $X \subset \mathbf{C}^n$ of dimension r and define the group of (Weil) divisors of X, denoted $Div(X)$, to the free abelian group generated by subvarieties $Z \subset X$ of dimension $r-1$, i.e.,

$$Div(X) = \{\sum_{i=1}^{k} n_i \cdot Z_i : n_i \text{ } an \text{ } integer, \; Z_i \subset X \text{ } subvariety \text{ } of \text{ } dim \text{ } r-1\}$$

For any polynomial $f \in \mathbf{C}[X]$ we know that,

$$V(f) = Z_1 \cup \cdots \cup Z_k$$

where the Z_i are subvarieties of X of codimension 1. We define the order of vanishing of f along Z_i in the following way: first choose any $x \in Z_i$ and, considering f as an element of $\mathcal{O}_{x,X}$, we write,

$$f = \frac{g}{h} f_i^{r_i}$$

where $g, h \in \mathbf{C}[X]$, g, h not identically 0 on Z_i and $f_i \equiv 0$ on Z_i (one can check that r_i is independent of the choice of x or f_i). Then we define,

$$ord_{Z_i} f = r_i$$

We see that $r_i > 0$ since otherwise f_i couldn't be 0 on Z_i.

We define the divisor of f, denoted by (f), as an element in $Div(X)$ by,

$$(f) = \sum_{i=1}^{k} ord_{Z_i} f \cdot Z_i$$

If we use the convention that $ord_Z f = 0$ when Z is not a component of $V(f)$ then we can also write,

$$(f) = \sum_{codim_X Z=1} ord_Z f \cdot Z_i$$

It is immediate from the definition that $(\bullet) : \mathbf{C}[X] \longrightarrow Div(X)$ is a homomorphism since $(fg) = (f) + (g)$. So we extend the definition for $f \in \mathbf{C}(X) - 0$ as follows: if $f = g/h$ then we set,

$$ord_Z f = ord_Z g - ord_Z h$$

$$(f) = (g) - (h)$$

We say Z is a pole of f if $ord_Z f < 0$ and a zero of f if $ord_Z f > 0$. The image of the homomorphism $(\bullet) : \mathbf{C}(X) - 0 \longrightarrow Div(X)$ defines what is called the subgroup of principal divisors in $Div(X)$,

$$PrDiv(X) = \{D \in Div(X) : D = (f) \ for \ some \ f \in \mathbf{C}(X) - 0\}$$

The principal divisors are used to define the important Picard group,

$$Pic(X) = Div(X)/PrDiv(X)$$

Now we recall that we were interested in when the coordinate ring $\mathbf{C}[X]$ is a U.F.D. and that this happened when the irreducible components of the polynomials in $\mathbf{C}[X]$ were in one-to-one correspondence with the subvarieties of codimension 1. In light of the definitions we have just seen this condition is equivalent to every divisor of X being principal. Thus we get the result,

Proposition: For a smooth variety $X \subset \mathbf{C}^n$, $\mathbf{C}[X]$ is a U.F.D. $\Longleftrightarrow Pic(X) = 0$.

As a final remark we notice that we can recognize the local rings $\mathcal{O}_{x,X}$ inside $\mathbf{C}(X)$ as the set of f where $ord_Z f \geq 0$ for all codimension 1 subvarieties $Z \subset X$ which pass through the point x. It then follows that the coordinate ring $\mathbf{C}[X]$ is the set of f where $ord_Z f \geq 0$ for all codimension 1 subvarieties $Z \subset X$.

2.5 Complex Projective Varieties

We will now consider varieties in projective space rather than affine space. We will see that complex projective space $\mathbf{P}^n$ can be thought of as a compactification of regular complex (affine) space $\mathbf{C}^n$. The characteristic of compactness will simplify the integration we want to perform on these varieties. We begin by defining projective space,

2.5.1 Complex Projective Space

<u>**Definition**</u>: Complex projective space $\mathbf{P}^n$ can be defined in any of three equivalent ways,

1. as the point set $\{1 - \textit{dimensional subspaces contained in } \mathbf{C}^{n+1}\}$
2. as the unit sphere S^{2n-1} along with the equivalence relation $z \sim e^{it}z$ where $t \in R$
3. as $\mathbf{C}^{n+1} - \{0\}$ with the equivalence relation $z \sim \lambda z$ where $\lambda \in \mathbf{C} - \{0\}$

Each one of these definitions is useful for different reasons. In particular, the second tells us that $\mathbf{P}^n$ is a compact space (here we are using the quotient topology) and the third is the most useful for doing calculations. It is the third definition that we will work with most of the time.

We use the third definition to place a coordinate system on projective space. If $(z_0, \ldots, z_n)$ are the coordinates of a point in $\mathbf{C}^{n+1} - \{0\}$ then we will denote the corresponding point in $\mathbf{P}^n$ by the coordinates $[z_0, \ldots, z_n]$ where $[z_0, \ldots, z_n] = [\lambda z_0, \ldots, \lambda z_n]$, $\lambda \neq 0$. We call these the homogeneous coordinates of $\mathbf{P}^n$.

Let us consider the sets,

$$U_j = \{[z_0, \ldots, z_n] \in \mathbf{P}^n : z_j \neq 0\}$$

for $j = 0, \ldots, n$. The corresponding sets in $\mathbf{C}^{n+1}$ are certainly open and so under the quotient topology we have that the sets U_j are open in $\mathbf{P}^n$. Since projective space does not contain the point where all z_j's are 0, we see that,

$$\bigcup_{j=0}^{n} U_j = \mathbf{P}^n$$

so that the U_j's form an open cover of $\mathbf{P}^n$. In general, points in projective space do not have a unique representation in homogeneous coordinates. However, on the set U_j for any point $[z_0, \ldots, z_n]$,

$$[z_0, \ldots, z_n] = [z_0/z_j, \ldots, z_j/z_j = 1, \ldots, z_n]$$

is the unique representation with 1 in the j^{th} position. So we can define a homeomorphism,

$$h_j : U_j \longrightarrow \mathbf{C}^n$$

by the formula,

$$h([z_0, \dots, z_n]) = (z_0/z_j, \dots, z_{j-1}/z_j, z_{j+1}/z_j, \dots, z_n)$$

This map is really just a projection and we can see that the transition maps $h_j \circ h_i^{-1}$ will be holomorphic. So we have constructed coordinate charts $\{(U_j, h_j)\}_{0 \le j \le n}$, and we arrive at the result,

Proposition: $\mathbf{P}^n$ is a complex manifold of dimension n.

Using the open sets U_j we can derive a decomposition for $\mathbf{P}^n$ as follows,

Proposition: $\mathbf{P}^n \simeq \mathbf{C}^n \sqcup \mathbf{C}^{n-1} \sqcup \cdots \sqcup \mathbf{C} \sqcup \{\infty\}$ (where $\sqcup$ denotes the disjoint union).

Proof: First we look more closely at $\mathbf{P}^1 \subset \mathbf{C}^2$ which has homogeneous coordinates of the form $[z_0, z_1]$. Notice that,

$$\mathbf{P}^1 = U_1 \sqcup \{[z_0, 0] \in \mathbf{P}^1\}$$

But we know that $U_0 \simeq \mathbf{C}$. Also, all points in the set $\{[z_1, 0] \in \mathbf{P}^1\}$ are equal to $[1, 0]$ under the equivalence relation, so the set consists of just this point. So we have,

$$\mathbf{P}^1 \simeq \mathbf{C} \sqcup \{\infty\}$$

where we denote $\{\infty\} = [1, 0]$ and call it the "point at infinity". Now looking at P^n we can write,

$$\begin{aligned} \mathbf{P}^n &= U_n \sqcup \{[z_0, \dots, z_{n-1}, 0] \in \mathbf{P}^n\} \\ &\simeq \mathbf{C}^n \sqcup \{[z_0, \dots, z_{n-1}] \in \mathbf{P}^{n-1}\} \\ &\simeq \mathbf{C}^n \sqcup \mathbf{P}^{n-1} \end{aligned}$$

Iterating this for $\mathbf{P}^{n-1}, \mathbf{P}^{n-2}, \dots$ gives,

$$\mathbf{P}^n \simeq \mathbf{C}^n \sqcup \mathbf{C}^{n-1} \sqcup \cdots \sqcup \mathbf{P}^1$$

Now applying our result from earlier finishes off the proof,

$$\mathbf{P}^n \simeq \mathbf{C}^n \sqcup \mathbf{C}^{n-1} \sqcup \cdots \sqcup \mathbf{C} \sqcup \{\infty\}$$

Q.E.D.

Now that we have constucted complex projective space and demonstrated a couple of it's properties we will start to look at varieties in projective space.

Much of the next section will consist in restating for complex projective space the definitions and results developed for the affine case. The major difference will be that for projective spaces we must restrict ourselves to homogeneous functions and ideals.

2.5.2 Varieties in $\mathbf{P}^n$

First we need the idea of homogeneity.

Definition: A polynomial $f \in \mathbf{C}[Z_0, \ldots, Z_n]$ is said to be homogeneous if the condition,

$$f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n)$$

holds for all λ and some integer d.

This means that all terms in the polynomial have the same degree.

Now, since complex projective space is a quotient of complex affine space, we'd like to define closed algebraic sets in projective space as a sort of quotient of the closed algebraic sets in affine space. So the projective closed algebraic sets will be exactly the affine closed algebraic sets that factor through to projective space. But what do these look like? We start off with the following short example.

Example: Consider the closed algebraic set $V(X^2 - Y - Z) \subset \mathbf{C}^3$. We notice immediately that the polynomial is not homogeneous. Now, the affine point $(1,1,0) \in \mathbf{C}^3$ is clearly in $V(X^2 - Y - Z)$. However, considered as a point in $\mathbf{P}^2$, we have that $[1,1,0] = 2[1,1,0] = [2,2,0]$. But notice that the affine point $(2,2,0)$ is not in the set $V(X^2 - Y - Z)$. So we see that we run into problems when the polynomials generating the closed algebraic set are not homogeneous.

Now let us consider a closed algebraic set $V(f) \subset \mathbf{C}^n$ where f is a homogeneous polynomial. Thus if we have $f(z_0, \ldots, z_n) = 0$ then we know,

$$f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n) = \lambda \cdot 0 = 0$$

and so the condition $f = 0$ will factor through to homogeneous coordinates in projective space. So we make the following definition,

Definition: Let $f_1, \ldots, f_m \in \mathbf{C}[Z_0, \ldots, Z_n]$ be homogeneous polynomials. Then sets of the form,

$$V(f_1, \ldots, f_m) = \{[z_0, \ldots, z_n] \in \mathbf{P}^n : f_1(z_0, \ldots, z_n) = \cdots = f_1(z_0, \ldots, z_n) = 0\}$$

are called the closed algebraic subsets of $\mathbf{P}^n$.

We now recall the definition,

Definition: Consider a polynomial $g \in \mathbf{C}[Z_0, \dots, Z_n]$. The sum of all the terms of g of degree k is said to be the homogeneous component of g of degree k. An ideal in $\mathbf{C}[Z_0, \dots, Z_n]$ is said to be homogeneous if, for each polynomial g in the ideal, every homogeneous component of g is also in the ideal.

Proposition: Let m be an ideal in the polynomial ring $\mathbf{C}[Z_0, \dots, Z_n]$. Then m is a homogeneous ideal $\Longleftrightarrow$ m can be generated by homogeneous polynomials.

Proof:

- "$\Longrightarrow$" Assume that m is a homogeneous ideal with generating set $\{f_1, \dots, f_m\}$. Since m is homogeneous all the homogeneous components of the f_i's are also in m and clearly they will generate m. Thus the ideal m can be generated by homogeneous polynomials.

- "$\Longleftarrow$" Assume that m is an ideal generated by homogeneous polynomials $f_1, \dots, f_m$. We consider a polynomial $g \in m$ and use induction on the degree of g to show that all homogeneous components of g are also in m.

  First, let $deg(g) = 0$. Then $g = k$ for some $k \in \mathbf{C}$ (i.e. g is constant) and the only homogeneous component of g is g itself. Thus all homogeneous components of g belong to m.

  We make the assumption that if $deg(g) \leq n$ then all homogeneous components of g belong to m.

  Now assume we have $g \in m$ with $deg(g) = n + 1$. Since m is generated by polynomials $f_1, \dots, f_m$ we can write,

$$g = g_1 f_1 + \cdots + g_m f_m$$

  Without loss of generality we can assume that $deg(f_i) > 0 \ \forall i$ since otherwise we would have $m = (1)$ in which case our result is trivial. But this implies that $deg(g_i) < deg(g)$ and thus $deg(g_i) \leq n \ \forall i$. Thus all the homogeneous comoponents of the g_i belong to m. Now notice that the homogeneous components of g are just given by the homogeneous components of the g_i's times the f_i's which are all in m. Thus all homogeneous components of g belong to m.

  Therefore m is a homogeneous ideal. Q.E.D.

From this follows immediately that the closed algebraic sets in projective space are given exactly by $V(m)$ where m is a homogeneous ideal. So the closed algebraic sets in projective space can be considered as a subset of those in affine space. Namely, those that factor through the quotient map $\mathbf{C}^{n+1} - \{0\} \longrightarrow \mathbf{P}^n$.

As with the affine case we make the following definition,

Definition: A closed algebraic set $V(m)$ of $\mathbf{P}^n$ is called a projective variety exactly when m is a homogeneous prime ideal.

We recall from the section on affine varieties that any radical ideal can be expressed uniquely as the intersection of finitely many prime ideals. So for a closed algebraic set generated by the homogeneous ideal m we can write,

$$r(m) = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$$

for a unique collection of prime ideals $\mathcal{P}_i$. The question is "Are the $\mathcal{P}_i$ also homogeneous ideals?"

Proposition: If m is a homogeneous ideal whose radical is written uniquely as,

$$r(m) = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$$

for prime ideals $\mathcal{P}_i$, then each $\mathcal{P}_i$ is also a homogeneous ideal.

Proof: Define $m^\lambda = \{f(\lambda z_0, \ldots, \lambda z_n) : f \in m\}$ where $\lambda \in \mathbf{C} - \{0\}$. Then we see that,

$$m \textit{ homogeneous} \Longleftrightarrow m = m^\lambda \Longleftrightarrow r(m) = r(m)^\lambda$$

but then we have,

$$\begin{aligned} r(m) &= r(m)^\lambda \\ &= \{f(\lambda z_0, \ldots, \lambda z_n) : f \in r(m)\} \\ &= \{f(\lambda z_0, \ldots, \lambda z_n) : f \in \mathcal{P}_i \forall i\} \\ &= \cap_{i=1}^r \{f(\lambda z_0, \ldots, \lambda z_n) : f \in \mathcal{P}_i\} \\ &= \mathcal{P}_1^\lambda \cap \cdots \cap \mathcal{P}_r^\lambda \end{aligned}$$

Notice that since the $\mathcal{P}_i$'s are prime the $\mathcal{P}_i^\lambda$'s will also be prime. Thus by the uniqueness of the prime representation of $r(m)$ we get that for each i,

$$\mathcal{P}_i = \mathcal{P}_j^\lambda$$

for some j. But rewriting $\mathcal{P}_j$ in the same way and then iterating this we will get (since there are only finitely many $\mathcal{P}_i$),

$$\mathcal{P}_i = \mathcal{P}_i^\mu$$

where μ is some power of λ. But this implies that the prime ideals $\mathcal{P}_i$ are also homogeneous. Q.E.D.

So for any closed algebraic set $V(m)$ in $\mathbf{P}^n$ we can write,

$$\begin{aligned} V(m) &= V(r(m)) \\ &= V(\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r) \\ &= V(\mathcal{P}_1) \cup \cdots \cup V(\mathcal{P}_r) \end{aligned}$$

where the $\mathcal{P}_i$ are homogeneous prime ideals. Further, this representation is unique. So we arrive immediately at the following,

Proposition: Any closed algebraic set in $\mathbf{P}^n$ can be written uniquely as a union of projective varieties.

We now discuss some properties of projective varieties.

2.5.3 The Zariski Topology in $\mathbf{P}^n$

Many of the properties that we developed for affine varieties transfer directly to the projective case. In particular, if we consider closed algebraic sets $V(m_j)$ in $\mathbf{P}^n$ we have,

1. $m_1 \subseteq m_2$ then $V(m_1) \supseteq V(m_2)$
2. $V(m_1 \cap m_2) = V(m_1) \cup V(m_2)$
3. $\cap_{j\in J} V(m_j) = V(\sum_{j\in J} m_j)$

So, as in the affine case, we can see that the closed algebraic sets in $\mathbf{P}^n$ form a basis for the closed sets of a topology which we call the Zariski topology.

Using the Zariski topology we will see the following result,

Proposition: Every projective variety has a finite open covering by affine varieties.

Proof: Let $m \subset \mathbf{C}[Z_0, \ldots, Z_n]$ be a homogeneous prime ideal and $X = V(m)$ be the corresponding variety in $\mathbf{P}^n$. Recall from earlier the set $U_j = \{[Z_0, \ldots, Z_n] \in \mathbf{P}^n : Z_j \neq 0\}$. It is clear that the U_j are Zariski open since they are just the complements of the zero set of the homogeneous polynomials $f(Z_0, \ldots, Z_n) = Z_j$. We associate U_j with $\mathbf{C}^n$ as before by the coordinates,

$$Y_{j,i} = \frac{Z_i}{Z_j},\ for\ 0 \leq i \leq n,\ i \neq j$$

and notice that $X \cap U_j$ is an affine variety in $U_j \simeq \mathbf{C}^n$. We can see this by letting $\tilde{m}$ be the ideal,

$$\tilde{m} = \{f(Y_{j,0}, \ldots, Y_{j,j} = 1, \ldots, Y_{j,n}) : f \in m\}$$

and noticing that $X \cap U_j = V(\tilde{m})$. Thus we see that X has a finite open cover given by the affine varieties $X \cap U_0, \ldots, X \cap U_n$. Q.E.D.

This representation of projective varieties as being locally affine allows us to transfer our local definitions (i.e. local rings, smoothness, etc...) directly from the affine to the projective case.

2.5.4 Local Rings, Smoothness & Dimension for Projective Varieties

Proposition: Let $X = V(m)$ be a variety in $\mathbf{P}^n$. We define $\mathbf{C}[Z_0, \ldots, Z_n]/m$ to be the homogeneous coordinate ring of X and denote it by $\mathbf{C}[X]$.

For affine varieties we defined a local rings about a point by taking quotients fo polynomials in the coordinate ring whose denominator was non-zero at the point in question. The fact that projective varieties are locally affine allows us to make an analagous definition for local rings by switching from homogeneous coordinates to suitable affine coordinates.

Definition: Let $X = V(m) \subseteq \mathbf{P}^n$ be a projective variety and let $x \in U_j \subseteq X$ be a point in X with non-zero j^{th} component. As a point in $\mathbf{P}^n$, x can be written in homogeneous coordinates,

$$x_h = [x_0, \ldots x_n]$$

with $x_j \neq 0$. As a point in U_j, x can be written in affine coordinates,

$$x_a = \left(\frac{x_0}{x_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_n}{x_j}\right)$$

We define the local ring $\mathcal{O}_{x_h,X}$ of X about the point x by setting,

$$\mathcal{O}_{x_h,X} = \mathcal{O}_{x_a,X\cap U_j}$$

where $\mathcal{O}_{x_a,X\cap U_j}$ is the local ring of the affine variety $X \cap U_j$ as defined earlier.

Notice that it is equivalent to define $\mathcal{O}_{x_h,X}$ in terms of the coordinate ring of X as follows,

$$\mathcal{O}_{x_h,X} = \{f/g : f, g \in C[X],\ f, g\ homogeneous\ of\ same\ degree\ with\ g(x_h) \neq 0\}$$

In the same way we can define the other local attributes of a projective variety. We let x_h and x_a be as above and define,

Definition: The tangent space to a projective variety X at a point x is defined to be the vector space,

$$T_{x_h}(X) = T_{x_a}(X \cap U_j)$$

where j is chosen such that $x_j \neq 0$.

As with affine varieties we define the dimension of a projective variety X to be,

$$dim(X) = min_{x \in X}\{dimT_{x_h}(X)\}$$

and say that a point x in a projective variety X is smooth if $dim(T_{x_h}(X)) = dim(X)$ and singular if $dim(T_{x_h}(X)) > dim(X)$.

Notice that a projective variety X is smooth if and only if each of the affine varieties $X \cap U_j$ are smooth. We proved in an earlier section that a smooth affine variety in $\mathbf{C}^n$ is a complex manifold. So we see that a smooth projective manifold can be seen as a bunch of complex manifolds glued together in a smooth way. Thus a smooth projective variety is also a complex manifold. These have a special name,

Definition: A smooth projective variety is called a projective algebraic manifold.

Projective algebraic manifolds are important because their smoothness and compactness make them relatively manageable to work on.

2.5.5 Divisors for Projective Varieties

The concept of divisors can again be readily transferred from affine to projective space.

Definition: Let X be a projective variety. Then we define,

- the group of divisors of X to be the free abelian group generated by subvarieties $Z \subset X$ of codimension 1, i.e.,

$$Div(X) = \{\sum_i n_i \cdot Z_i : n_i\ an\ integer,\ Z_i \subset X\ a\ subvariety\ of\ codimension\ 1\}$$

- for any $f \in \mathbf{C}(X) - 0$ and any $Z \subset X$ of codimension 1 we define $ord_Z f = ord_{Z \cap U_i} f$ where $Z \cap U_i$ is one of the affine varities covering Z and where $ord_{Z \cap U_i} f$ is taken exactly as in the affine case outlined earlier.
- for any $f \in \mathbf{C}(X) - 0$ the divisor of f is given by,

$$(f) = \sum_{codim_X Z = 1} ord_Z f \cdot Z$$

  which is clearly an element of $Div(X)$. Elements of $Div(X)$ of the form (f) are called the principal divisors of X denoted $PrDiv(X)$.
- the Picard group of X is given by $Pic(X) = Div(X)/PrDiv(X)$.

Chapter 3

Cohomology

We begin this chapter by looking at the general structure of sheaf cohomology over a space X. We then look specifically at deRham and Dolbeault cohomologies involving differential forms on projective algebraic manifolds. The major sources for this chapter are Lewis [Lew1] and Warner [Warn].

3.1 Sheaf Cohomology

3.1.1 Introduction and Definitions

We let X be a topological space and k any PID (in particular we will be intersted in $k = \mathbf{R}$ and $k = \mathbf{C}$). We recall,

Definition: A sheaf $\mathcal{S}$ of k-modules over the space X is a topological space S together with a map $\pi : S \longrightarrow X$ which satisfies,

1. π is a local homeomorphism,
2. $\pi^{-1}(x)$ is a k-module for every $x \in X$,
3. the k-module operations on S are continuous in the topology of S.

We denote $\pi^{-1}(x) = \mathcal{S}_x$ and call it the stalk of $\mathcal{S}$ over $x \in X$.

Definition: If $\mathcal{S}$ and $\mathcal{S}'$ are sheaves of k-modules over the same space X then we define a sheaf homomorphism to be a map $\phi : \mathcal{S} \longrightarrow \mathcal{S}'$ such that,

1. $\pi' \circ \phi = \pi$,
2. ϕ is homomorphism on each stalk in $\mathcal{S}$.

Notice that the above implies that sheaf homomorphisms map stalks to stalks. This allows us to define a a sequence of sheaves and sheaf homomorphisms,

$$\ldots \longrightarrow \mathcal{S}_{i-1} \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_{i+1} \longrightarrow \ldots$$

to be exact if the induced sequence of k-modules,

$$\ldots \longrightarrow (\mathcal{S}_{i-1})_x \longrightarrow (\mathcal{S}_i)_x \longrightarrow (\mathcal{S}_{i+1})_x \longrightarrow \ldots$$

is exact for every $x \in X$.

Definition: If $\mathcal{S}$ is a sheaf of k-modules over X then a (global) section of $\mathcal{S}$ is given by a continuous map $f : X \longrightarrow \mathcal{S}$ such that $\pi \circ f = id$.

We denote the collection of global sections of $\mathcal{S}$ by $\Gamma(X, \mathcal{S})$ and notice that it can be given the structure of a k-module by defining,

1. $(f + g)(x) = f(x) + g(x)$ for all $f, g \in \Gamma(X, \mathcal{S})$,
2. $(cf)(x) = c(f(x))$ for all $f \in \Gamma(X, \mathcal{S})$ and $c \in k$.

We have the following important types of sheaves,

Definition: A sheaf $\mathcal{S}$ is said to be fine if, for each locally finite open cover $\{U_i\}_{i\in I}$ of X, there exists for each $i \in I$ an endomorphism ℓ_i of $\mathcal{S}$ such that,

1. $supp(\ell_i) \subset U_i$,
2. $\sum_i \ell_i = id$.

We call $\{\ell_i\}_{i\in I}$ a partition of unity for $\mathcal{S}$ subordinate to the cover $\{U_i\}_{i\in I}$ of X.

Now we are able to give a definition for a sheaf cohomology theory,

Definition: A sheaf cohomology theory $\mathcal{H} = \{H^q(X, -)\}$ is given by a covariant functor for each integer q,

$$H^q(X, -) : \{\text{sheaves of } k\text{-modules over } X\} \longrightarrow \{k\text{-modules}\}$$

which satisfy the following properties,

1. $H^0(X, \mathcal{S}) = \Gamma(X, \mathcal{S})$ and $H^q(X, \mathcal{S}) = 0$ for all $q < 0$,
2. if $\mathcal{S}$ is a fine sheaf then $H^q(X, \mathcal{S}) = 0$ for all $q \geq 1$ (i.e. $\mathcal{S}$ is acyclic),
3. given a short exact sequence (SES) of sheaves $0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{S} \longrightarrow \mathcal{T} \longrightarrow 0$ there is a long exact sequence (LES) of k-modules in cohomology,

$$0 \longrightarrow \Gamma(X, \mathcal{R}) \longrightarrow \Gamma(X, \mathcal{S}) \longrightarrow \Gamma(X, \mathcal{T}) \longrightarrow H^1(X, \mathcal{R}) \longrightarrow H^1(X, \mathcal{S})$$
$$\longrightarrow H^1(X, \mathcal{T}) \longrightarrow \ldots \longrightarrow H^i(X, \mathcal{R}) \longrightarrow H^i(X, \mathcal{S}) \longrightarrow H^i(X, \mathcal{T})$$
$$\longrightarrow H^{i+1}(X, \mathcal{R}) \longrightarrow H^{i+1}(X, \mathcal{S}) \longrightarrow \ldots$$

3.1.2 Existence of Sheaf Cohomology Theories

It is not immediately apparent that sheaf cohomology theories exist at all. However, it is possible to build an example.

We call an exact sequence of sheaves over X,

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow \dots \tag{3.1}$$

a resolution of the sheaf $\mathcal{A}$. We say the resolution is fine if each of the sheaves $\mathcal{C}_i$ are fine. To each resolution of $\mathcal{A}$ and each sheaf $\mathcal{S}$ we can associate the cochain complex,

$$\dots \longrightarrow 0 \longrightarrow \Gamma(X, \mathcal{C}_0 \otimes \mathcal{S}) \longrightarrow \Gamma(X, \mathcal{C}_1 \otimes \mathcal{S}) \longrightarrow \Gamma(X, \mathcal{C}_2 \otimes \mathcal{S}) \longrightarrow \dots \tag{3.2}$$

which we will denote by $\Gamma(X, \mathcal{C}_\bullet \otimes \mathcal{S})$. The homomorphisms in the resolution, when tensored with the identity map, give the homomorphisms for a cochain complex,

$$\dots \longrightarrow \mathcal{C}_0 \otimes \mathcal{S} \longrightarrow \mathcal{C}_1 \otimes \mathcal{S} \longrightarrow \mathcal{C}_2 \otimes \mathcal{S} \longrightarrow \dots \tag{3.3}$$

which in turn induce the homomorphisms for our cochain complex $\Gamma(X, \mathcal{C}_\bullet \otimes \mathcal{S})$. The exactness of the original resolution ensures that (3.3) and consequently $\Gamma(X, \mathcal{C}_\bullet \otimes \mathcal{S})$ are indeed cochain complexes.

Now we consider a fine resolution of the constant sheaf $\mathcal{K} = X \times k$,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow \dots$$

which are known to exist from classical cohomology theory. We define a sheaf cohomology theory from this resolution as follows: for each sheaf $\mathcal{S}$ and integer q define,

$$H^q(X, \mathcal{S}) = H^q(\Gamma(X, \mathcal{C}_\bullet \otimes \mathcal{S})$$

then we see that,

1. labelling the homomorphisms in (3.2) as follows,

$$\dots \longrightarrow 0 \xrightarrow{\phi} \Gamma(X, \mathcal{C}_0 \otimes \mathcal{S}) \xrightarrow{\psi} \Gamma(X, \mathcal{C}_1 \otimes \mathcal{S}) \longrightarrow \dots$$

we get,

$$
\begin{aligned}
H^0(X,\mathcal{S}) &= H^0(\Gamma(X,\mathcal{C}_\bullet\otimes\mathcal{S}) \\
&= ker(\psi)/im(\phi) \\
&= \Gamma(X,0\otimes\mathcal{S})/0 \\
&\simeq \Gamma(X,\mathcal{S})
\end{aligned}
$$

as required. Clearly we also have $H^q(X,\mathcal{S}) = H^q(\Gamma(X,\mathcal{C}_\bullet\otimes\mathcal{S})) = 0$ whenever $q<0$.

2. say $\mathcal{S}$ is a fine sheaf with a partition of unity $\{\ell_i\}_{i\in I}$ subordinate to a cover $\{U_i\}_{i\in I}$ of X. Then we can see that $\mathcal{C}_i\otimes\mathcal{S}$ is also a fine sheaf for any $i\in I$ since $\{id_i\otimes\ell_i\}_{i\in I}$ is a partition of unity for $\mathcal{C}_i\otimes\mathcal{S}$ (where id_i denotes the identity map on the sheaf $\mathcal{C}_i$). We label maps in the resolution (3.1) and the cochain complex (3.2) as follows,

$$
\mathcal{C}_{q-1} \overset{\tilde{\phi}}{\longrightarrow} \mathcal{C}_q \overset{\tilde{\psi}}{\longrightarrow} \mathcal{C}_{q+1}
$$

$$
\Gamma(X,\mathcal{C}_{q-1}\otimes\mathcal{S}) \overset{\phi}{\longrightarrow} \Gamma(X,\mathcal{C}_q\otimes\mathcal{S}) \overset{\psi}{\longrightarrow} \Gamma(X,\mathcal{C}_{q+1}\otimes\mathcal{S})
$$

where we are assuming that $q\geq 1$. Now, any global section $f\in\Gamma(X,\mathcal{C}_{q-1}\otimes\mathcal{S})$ is given by a collection of maps $\{f_i\otimes g_i\}_{i\in I}$ where,

$$
\begin{aligned}
f_i &: U_i \longrightarrow \mathcal{C}_{q-1} \\
g_i &: U_i \longrightarrow \mathcal{S}
\end{aligned}
$$

and thus,

$$
f_i\otimes g_i : U_i \longrightarrow \mathcal{C}_{q-1}\otimes\mathcal{S}
$$

and which satisfy,

$$
f = \sum_i (f_i\otimes g_i)\circ(id_{q-1}\otimes\ell_i)
$$

So we define the homomorphism $\phi:\Gamma(X,\mathcal{C}_{q-1}\otimes\mathcal{S})\longrightarrow\Gamma(X,\mathcal{C}_q\otimes\mathcal{S})$ as follows,

$$
\begin{aligned}
\phi(f) &= \phi(\sum_i (f_i\otimes g_i)\circ(id_{q-1}\otimes\ell_i)) \\
&= \sum_i ((\tilde{\phi}\circ f_i)\otimes g_i)\circ(id_q\otimes\ell_i) \in \Gamma(X,\mathcal{C}_q\otimes\mathcal{S})
\end{aligned}
$$

and we define ψ analagously. But now we notice that,

$$ker(\psi) = \Gamma(X, ker(\tilde{\psi}) \otimes \mathcal{S})$$

$$im(\phi) = \Gamma(X, im(\tilde{\phi}) \otimes \mathcal{S})$$

But the exactness of the resolution (1) tells us that $ker(\psi) = im(\phi)$. Thus we have,

$$\begin{aligned} H^q(X, \mathcal{S}) &= H^q(\Gamma(X, \mathcal{C}_\bullet \otimes \mathcal{S})) \\ &= ker(\psi)/im(\phi) \\ &= 0 \end{aligned}$$

for all $q \geq 1$, as required.

3. The third property follows from the first and the standard result from cohomology theory that any SES,

$$0 \longrightarrow M_\bullet \longrightarrow N_\bullet \longrightarrow P_\bullet \longrightarrow 0$$

of chain complexes indues a LES in cohomology,

$$\ldots \longrightarrow H^{q-1}(P_\bullet) \longrightarrow H^q(M_\bullet) \longrightarrow H^q(N_\bullet) \longrightarrow H^q(P_\bullet) \longrightarrow \ldots$$

So we have defined a sheaf cohomology theory and thus shown existence. We will see other examples later on.

3.1.3 Uniqueness of Sheaf Cohomology Theories

Say we have two sheaf cohomology theories $\mathcal{H} = \{H^q(X, -)\}$ and $\tilde{\mathcal{H}} = \{\tilde{H}^q(X, -)\}$ over a topological space X. It is immediate that $H^0(X, \mathcal{S}) \simeq \tilde{H}^0(X, \mathcal{S})$ since the definition of sheaf cohomology theory states they must both be equal to $\Gamma(X, \mathcal{S})$. Now we want to show that $H^q(X, \mathcal{S}) \simeq \tilde{H}^q(X, \mathcal{S})$ for any $q \geq 1$.

Given any sheaf $\mathcal{S}$ we consider the sheaf of germs of discontinuous functions (which we will call $\mathcal{S}_0$). Any point m in a stalk $\mathcal{S}_x$ is the value of some section of $\mathcal{S}$ at the point $x \in X$. Thus $\mathcal{S}$ can be imbedded into $\mathcal{S}_0$ by sending m to the germ of the section at x. It is well known that the sheaf of germs of discontinous sections in fine. Thus we have a SES of sheaves,

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \longrightarrow \mathcal{S}_0/\mathcal{S} \longrightarrow 0$$

where $\mathcal{S}_0$ is a fine sheaf. This SES induces a LES in each of our cohomology theories $\mathcal{H}$ and $\tilde{\mathcal{H}}$ giving us the following diagram (NOTE: since the topological space will always be understood to be X we drop it in our notation):

$$\begin{array}{ccccccccccccc}
0 & \rightarrow & \Gamma(\mathcal{S}) & \rightarrow & \Gamma(\mathcal{S}_0) & \overset{\alpha}{\rightarrow} & \Gamma(\mathcal{S}_0/\mathcal{S}) & \overset{\delta}{\rightarrow} & H^1(\mathcal{S}) & \rightarrow & H^1(\mathcal{S}_0) & \rightarrow & \cdots \\
 & & \downarrow & & \downarrow \mu & & \downarrow \lambda & & \downarrow \phi & & \| \; 0 \; \| & & \\
0 & \rightarrow & \Gamma(\mathcal{S}) & \rightarrow & \Gamma(\mathcal{S}_0) & \overset{\beta}{\rightarrow} & \Gamma(\mathcal{S}_0/\mathcal{S}) & \overset{\tilde{\delta}}{\rightarrow} & \tilde{H}^1(\mathcal{S}) & \rightarrow & \tilde{H}^1(\mathcal{S}_0) & \rightarrow & \cdots
\end{array}$$

where the maps μ and λ are simply the identity maps. How can we define the homomorphism $\phi : H^1(\mathcal{S}) \longrightarrow \tilde{H}^1(\mathcal{S})$? We define ϕ by requiring that,

$$\phi \circ \delta = \tilde{\delta} \circ \lambda$$

It is clear that ϕ is then well-defined as long as the following property holds,

- "If $x, y \in \Gamma(\mathcal{S}_0/\mathcal{S})$ are such that $\delta(x) = \delta(y)$ then $(\tilde{\delta} \circ \lambda)(x) = (\tilde{\delta} \circ \lambda)(y)$."

We show this holds as follows: Let $x, y \in \Gamma(\mathcal{S}_0/\mathcal{S})$ be such that $\delta(x) = \delta(y)$. Then $x - y \in ker(\delta) = im(\alpha)$ and so there exists some $z \in \Gamma(\mathcal{S}_0)$ such that,

$$x - y = \alpha(z)$$

Rewriting we get,

$$x = \alpha(z) + y$$

and so,

$$\begin{aligned}
\lambda(x) &= \lambda(\alpha(z)) + \lambda(y) \\
&= (\lambda \circ \alpha)(z) + \lambda(y)
\end{aligned}$$

But we know the second square commutes so that $\lambda \circ \alpha = \beta \circ \mu$. Thus,

$$\lambda(x) = (\beta \circ \mu)(z) + \lambda(y)$$

Further, we get,

$$\begin{aligned}
\tilde{\delta}(\lambda(x)) &= \tilde{\delta}((\beta \circ \mu)(z)) + \tilde{\delta}(\lambda(y)) \\
&= (\tilde{\delta} \circ \beta)(\mu(z)) + \tilde{\delta}(\lambda(y))
\end{aligned}$$

But by the exactness of the second LES we know $\tilde{\delta} \circ \beta \equiv 0$ so that,

$$(\tilde{\delta} \circ \lambda)(x) = (\tilde{\delta} \circ \lambda)(y)$$

as required. In fact, since λ is an isomorphism we can say ϕ is also. Thus we have that,

$$H^1(X,\mathcal{S}) \simeq \tilde{H}^1(X,\mathcal{S})$$

Since this result holds for any sheaf $\mathcal{S}$ we know in particular that it holds for $\mathcal{S}_0/\mathcal{S}$. Thus we know $H^1(X,\mathcal{S}_0/\mathcal{S}) \simeq \tilde{H}^1(X,\mathcal{S}_0/\mathcal{S})$. So, from our LES's, we have the diagram,

$$\begin{array}{ccccccccccc}
\cdots & \to & H^1(\mathcal{S}_0) & \to & H^1(\mathcal{S}_0/\mathcal{S}) & \overset{\simeq}{\to} & H^2(\mathcal{S}) & \to & H^2(\mathcal{S}_0) & \to & \cdots \\
 & & \| & & & & & & \| & & \\
 & & 0 & & \downarrow\simeq & & \downarrow\psi & & 0 & & \\
 & & \| & & & & & & \| & & \\
\cdots & \to & \tilde{H}^1(\mathcal{S}_0) & \to & \tilde{H}^1(\mathcal{S}_0/\mathcal{S}) & \overset{\simeq}{\to} & \tilde{H}^2(\mathcal{S}) & \to & \tilde{H}^2(\mathcal{S}_0) & \to & \cdots
\end{array}$$

We define ψ in the obvious way by following the three isomorphisms around the other sides of the square. This means that ψ is also an isomorphism and so,

$$H^2(X,\mathcal{S}) \simeq \tilde{H}^2(X,\mathcal{S})$$

Continuing this process inductively along the LES's then defines isomorphisms,

$$H^q(X,\mathcal{S}) \simeq \tilde{H}^q(X,\mathcal{S})$$

for all $q \geq 2$. Thus we have that $\mathcal{H} \simeq \tilde{\mathcal{H}}$.

So we see that given any topological space X, sheaf cohomology theories over X are unique up to isomorphism.

3.1.4 The Abstract DeRham Isomorphism Theorem

In later sections we will use the following important theorem concerning the existence of sheaf cohomology theories,

Theorem: (Abstract DeRham Isomorphism Theorem) Given any fine resolution,

$$0 \to \mathcal{S} \to \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \cdots$$

of a sheaf $\mathcal{S}$ we have a sheaf cohomology theory defined by,

$$H^q(X,\mathcal{S}) \simeq H^q(\Gamma(\mathcal{S}_\bullet))$$

3.2 deRham Cohomology

For the whole of this section we will let X be a projective algebraic manifold. We can consider X in two ways,

1. as a complex manifold with $dim_{\mathbf{C}} X = n$ and local coordinates $z = (z_1, \ldots, z_n)$,
2. as a real manifold with $dim_{\mathbf{R}} X = 2n$ and local coordinates $x = (x_1, \ldots, x_{2n})$.

3.2.1 Notation for k-Forms on X

We will denote k-forms on X as follows:

1. Real-Valued Forms: Let $U \subset X$ have local coordinates $(x_1, \ldots, x_{2n})$. A real-valued k-form ω on X can be locally represented as,

$$\omega = \sum_I h_I dx_I$$

where $h_I : U \subset X \to \mathbf{R}$ is C^∞ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, 2n\}$.

The vectorspace of real-valued k-forms on X will be denoted by $E^k_{\mathbf{R}}(X)$.

2. Complex-Valued Forms: Let $U \subset X$ have local coordinates $(z_1, \ldots, z_n)$. A complex-valued (p, q)-form ω on X can be locally represented as,

$$\omega = \sum_{I,J} h_{IJ} dz_I \wedge d\bar{z}_J$$

where $h_{IJ} : U \subset X \to \mathbf{C}$ is C^∞ and $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ for $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ and $J = \{j_1, \ldots, j_q\} \subset \{1, \ldots, n\}$.

The vectorspace of complex-valued (p, q)-forms on X will be denoted by $E^{p,q}_{\mathbf{C}}(X)$ with $\overline{E^{p,q}_{\mathbf{C}}(X)} = E^{q,p}_{\mathbf{C}}(X)$.

We define the vectorspace of complex-valued k-forms on X by setting,

$$E^k_{\mathbf{C}}(X) = \oplus_{p+q=k} E^{p,q}_{\mathbf{C}}(X)$$

Thus, by using $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ as a local coordinate for the underlying real structure of X, we notice that the real-valued forms are just the complex-valued forms where the functions h_{IJ} are real-valued. Thus we have that,

$$E^k_{\mathbf{R}}(X) \subset E^k_{\mathbf{C}}(X)$$

We will show later that in fact $E^k_{\mathbf{R}}(X) \otimes \mathbf{C} \simeq E^k_{\mathbf{C}}(X)$.

3.2.2 Operators on Forms

We define an operator $d : E^k_{\mathbf{R}}(X) \longrightarrow E^{k+1}_{\mathbf{R}}(X)$ inductively as follows,

1. "$k = 0$": Let $f \in E^0_{\mathbf{R}}(X)$. Then we define,

$$df = \sum_{i=1}^{2n} \frac{\partial f}{\partial x_i} dx_i$$

   Thus df is just the differential of the function f.

2. "$k \geq 1$": Let $\omega = \sum_I h_I dx_I \in E^k_{\mathbf{R}}(X)$ where $h_I \in E^0_{\mathbf{R}}(X)$. Then we define,

$$d\omega = \sum_I dh_I \wedge dx_I$$

This definition can easily be extended to complex-valued forms. Say we have $\omega = \sum_{I,J} h_{IJ} dz_I \wedge d\bar{z}_J \in E^{p,q}_{\mathbf{C}}(X)$ (where we let $k = p+q$). Then we define the operators,

1. $\partial : E^{p,q}_{\mathbf{C}}(X) \longrightarrow E^{p+1,q}_{\mathbf{C}}(X)$
2. $\bar{\partial} : E^{p,q}_{\mathbf{C}}(X) \longrightarrow E^{p,q+1}_{\mathbf{C}}(X)$

by,

1. $\partial\omega = \sum_{I,J} \partial h_{IJ} \wedge dz_I \wedge d\bar{z}_J$, where $\partial h_{IJ} = \sum_{i=1}^n \frac{\partial h_{IJ}}{\partial z_i} dz_i$
2. $\bar{\partial}\omega = \sum_{I,J} \bar{\partial} h_{IJ} \wedge dz_I \wedge d\bar{z}_J$, where $\bar{\partial} h_{IJ} = \sum_{i=1}^n \frac{\partial h_{IJ}}{\partial \bar{z}_i} d\bar{z}_i$

We can then define the operator $\tilde{d} : E^k_{\mathbf{C}}(X) \longrightarrow E^{k+1}_{\mathbf{C}}(X)$ by the formula,

$$\tilde{d} = \partial + \bar{\partial}$$

Claim: The map $\tilde{d}$ is an extension of the map d.

Proof: Now, if $\omega = \sum_{I,J} h_{IJ} dz_I \wedge d\bar{z}_J \in E^k_{\mathbf{R}}(X) \subset E^k_{\mathbf{C}}(X)$ then by using $(x_1, \ldots, x_{2n}) = (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ as a local coordinate on the underlying real structure of X we have the following calculation,

$$
\begin{aligned}
\tilde{d}\omega &= (\partial + \bar{\partial})\omega \\
&= \partial\omega + \bar{\partial}\omega \\
&= \sum_{I,J} \partial h_{IJ} \wedge dz_I \wedge d\bar{z}_J + \sum_{I,J} \bar{\partial} h_{IJ} \wedge dz_I \wedge d\bar{z}_J \\
&= \sum_{I,J} (\partial h_{IJ} + \bar{\partial} h_{IJ}) \wedge dz_I \wedge d\bar{z}_J \\
&= \sum_{I,J} (\sum_{i=1}^{n} \frac{\partial h_{IJ}}{\partial z_i} dz_i + \sum_{i=1}^{n} \frac{\partial h_{IJ}}{\partial \bar{z}_i} d\bar{z}_i) \wedge dz_I \wedge d\bar{z}_J \\
&= \sum_{I,J} (\sum_{i=1}^{2n} \frac{\partial h_{IJ}}{\partial x_i} dx_i) \wedge dz_I \wedge d\bar{z}_J \\
&= \sum_{I,J} dh_{IJ} \wedge dz_I \wedge d\bar{z}_J \\
&= d\omega
\end{aligned}
$$

Thus $\tilde{d}$ and d agree on $E^k_{\mathbf{R}}(X)$. Q.E.D.

In particular this means we have the commutative diagram,

$$
\begin{array}{ccc}
 & \tilde{d} & \\
E^k_{\mathbf{C}}(X) & \longrightarrow & E^{k+1}_{\mathbf{C}}(X) \\
i \uparrow & & i \uparrow \\
 & d & \\
E^k_{\mathbf{R}}(X) & \longrightarrow & E^{k+1}_{\mathbf{R}}(X)
\end{array}
$$

We make a couple of final remarks concerning these operators,

1. In §3.2.5 we will show that $E^k_{\mathbf{C}}(X) \simeq E^k_{\mathbf{R}}(X) \otimes \mathbf{C}$. In this sense we can write,

$$\tilde{d} = d \otimes 1 : E^k_{\mathbf{R}}(X) \otimes \mathbf{C} \to E^{k+1}_{\mathbf{R}}(X) \otimes \mathbf{C}$$

   and this operator is global.

2. If we define the projection $Pr_{p,q} : E^k_{\mathbf{C}}(X) \to E^{p,q}_{\mathbf{C}}(X)$ then we have the identities,

   (a) $\partial = Pr_{p+1,q} \circ \tilde{d}$

   (b) $\bar{\partial} = Pr_{p,q+1} \circ \tilde{d}$

3. It is well known that $d^2 = \tilde{d}^2 = 0$. For the map $\tilde{d}$ we have,

$$\tilde{d}^2 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

If we let $\omega \in E^{p,q}_{\mathbf{C}}(X)$ then,

$$0 = \tilde{d}^2\omega = \partial^2\omega + \partial\bar{\partial}\omega + \bar{\partial}\partial\omega + \bar{\partial}^2\omega$$

where we know,

(a) $\partial^2\omega \in E^{p+2,q}_{\mathbf{C}}(X)$

(b) $\partial\bar{\partial}\omega, \bar{\partial}\partial\omega \in E^{p+1,q+1}_{\mathbf{C}}(X)$

(c) $\bar{\partial}^2\omega \in E^{p,q+2}_{\mathbf{C}}(X)$

By Hodge type considerations it must hold that,

$$\partial^2\omega = \bar{\partial}^2\omega = (\partial\bar{\partial}\omega + \bar{\partial}\partial\omega) = 0$$

and so we get the identities,

$$\partial^2 = 0,\ \bar{\partial}^2 = 0,\ \partial\bar{\partial} = -\bar{\partial}\partial$$

Note: From now on we will often use the notation d to represent both d and $\tilde{d}$. It will be clear from the context which map is begin used.

3.2.3 Real deRham Cohomology

Since we know $d^2 = 0$ we have a cochain complex,

$$0 \xrightarrow{i} E^0_{\mathbf{R}}(X) \xrightarrow{d} E^1_{\mathbf{R}}(X) \rightarrow \cdots \rightarrow E^k_{\mathbf{R}}(X) \xrightarrow{d} E^{k+1}_{\mathbf{R}}(X) \rightarrow \cdots$$

which we will denote $E^{\bullet}_{\mathbf{R}}(X)$. From this, we define the q^{th} real deRham cohomology group of X to be,

$$H^q_{dR}(X, \mathbf{R}) = H^q(E^{\bullet}_{\mathbf{R}}(X))$$

This agrees with the classical definition,

$$H^q_{dR}(X, \mathbf{R}) = \{closed\ q-forms\}/\{exact\ q-forms\}$$

where a q-form $\omega \in E^q_{\mathbf{R}}(X)$ is said to be,

1. closed if $d\omega = 0$,
2. exact if there exists $\gamma \in E^{q-1}_{\mathbf{R}}(X)$ such that $d\gamma = \omega$.

(Notice that $d^2 \equiv 0$ means that every exact q-form is also closed, and so the above is well-defined).

3.2.4 Complex deRham Cohomology

We can extend our definition to the complex case by using our map $\tilde{d}$. Since $\tilde{d} = 0$ we have the cochain complex,

$$0 \overset{i}{\rightarrow} E^0_{\mathbf{C}}(X) \overset{\tilde{d}}{\rightarrow} E^1_{\mathbf{C}}(X) \rightarrow \cdots \rightarrow E^k_{\mathbf{C}}(X) \overset{\tilde{d}}{\rightarrow} E^{k+1}_{\mathbf{C}}(X) \rightarrow \cdots$$

which we will denote by $E^{\bullet}_{\mathbf{C}}(X)$. We define the q^{th} complex deRham cohomology group of X to be,

$$H^q_{dR}(X, \mathbf{C}) = H^q(E^{\bullet}_{\mathbf{C}}(X))$$

3.2.5 Relationship between $H^q_{dR}(X, \mathbf{R})$ and $H^q_{dR}(X, \mathbf{C})$

Say we have $\omega \in E^k_{\mathbf{C}}(X)$. Then ω has a local representation,

$$\omega = \sum_{I,J} h_{IJ} dz_I \wedge d\bar{z}_J$$

Since the z_i are complex coordinates we can write,

$$z_i = x_i + \sqrt{-1}y_i, \; z_i = x_i - \sqrt{-1}y_i$$

where $\{x_i, y_i\}$ form a coordinate system for the underlying real structure of X. Thus it follows that,

$$dz_i = d(x_i + \sqrt{-1}y_i) = dx_i + \sqrt{-1}dy_i$$

$$d\bar{z}_i = d(x_i - \sqrt{-1}y_i) = dx_i - \sqrt{-1}dy_i$$

So, if $|I| = p$ and $|J| = q$ with $p + q = k$, then we have,

$$dz_I = (dx_{i_1} + \sqrt{-1}dy_{i_1}) \wedge \cdots \wedge (dx_{i_p} + \sqrt{-1}dy_{i_p})$$

$$d\bar{z}_J = (dx_{j_1} - \sqrt{-1}dy_{j_1}) \wedge \cdots \wedge (dx_{j_q} - \sqrt{-1}dy_{j_q})$$

So what does $dz_I \wedge d\bar{z}_J$ look like?

It is not too difficult to see that if we expand $dz_I \wedge d\bar{z}_J$ we will end up with a sum where each term is a wedge product of $p + q = k$ differentials from the set,

$$\{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n\}$$

each multiplied by ± 1 or $\pm\sqrt{-1}$. Thus we have that $\omega = \sum_{I,J} h_{IJ} dz_I \wedge d\bar{z}_J$ is a sum of real k-forms each multiplied by a complex number. Thus we can say that $\omega \in E^k_{\mathbf{R}}(X) \otimes \mathbf{C}$. So we see that,

$$E^k_{\mathbf{C}}(X) \simeq E^k_{\mathbf{R}}(X) \otimes \mathbf{C}$$

But then we have that,

$$\begin{aligned} H^q_{dR}(X, \mathbf{C}) &\simeq H^q(E^\bullet_{\mathbf{C}}(X)) \\ &\simeq H^q(E^k_{\mathbf{R}}(X) \otimes \mathbf{C}) \\ &\simeq H^q(E^k_{\mathbf{R}}(X)) \otimes \mathbf{C} \\ &\simeq H^q_{dR}(X, \mathbf{R}) \otimes \mathbf{C} \end{aligned}$$

Thus, the relationship we were looking for is,

$$H^q_{dR}(X, \mathbf{C}) \simeq H^q_{dR}(X, \mathbf{R}) \otimes \mathbf{C}$$

3.2.6 deRham Cohomology and Sheaf Cohomology

We can ask the question, "Can the classical deRham cohomology as defined above be realized as a sheaf cohomology theory?" The answer is "Yes", and we will proceed to demonstrate this fact.

We first define the sheaf of germs of real (resp. complex) k-forms on X as follows: take all k-forms at a point $x \in X$ and consider two such forms equivalent if they agree on some open neighbourhood of x, the resulting equivalence classes form the elements of the stalk of our sheaf at the point x. We will denote this sheaf as $\mathcal{E}^k_{\mathbf{R}}(X)$ (resp. $\mathcal{E}^k_{\mathbf{C}}(X)$).

We will now consider the real case (the complex case will follow by a similar argument). We notice that the map $d : E^k_{\mathbf{R}}(X) \longrightarrow E^{k+1}_{\mathbf{R}}(X)$ induces a sheaf homomorphism,

$$d : \mathcal{E}^k_{\mathbf{R}}(X) \longrightarrow \mathcal{E}^{k+1}_{\mathbf{R}}(X)$$

(We keep d as our notation since the context will be clear.) Consider the constant sheaf $\mathcal{R} = X \times \mathbf{R}$. We can imbed $\mathcal{R}$ into $\mathcal{E}^0_{\mathbf{R}}(X)$ by sending the value $a \in (\mathcal{R})_x$ to the germ at $x \in X$ of the function with constant value a. Thus we have a sequence of sheaf homomorphisms,

$$0 \rightarrow \mathcal{R} \xrightarrow{i} \mathcal{E}^0_{\mathbf{R}}(X) \xrightarrow{d} \mathcal{E}^1_{\mathbf{R}}(X) \xrightarrow{d} \mathcal{E}^2_{\mathbf{R}}(X) \rightarrow \cdots$$

It turns out that this is a fine resolution of the constant sheaf $\mathcal{R}$. Applying the Abstract deRham Isomorphism Theorem we get a sheaf cohomology theory defined by,

$$H^q(X, \mathcal{R}) = H^q(\Gamma(\mathcal{E}^\bullet_{\mathbf{R}}(X)))$$

But, for any k, $\Gamma(\mathcal{E}^k_{\mathbf{R}}(X))$ is isomorphic to $E^k_{\mathbf{R}}(X)$ in the obvious way. Thus we have,

$$H^q(X, \mathcal{R}) \simeq H^q(E^\bullet_{\mathbf{R}}(X))$$

which is exactly our definition of real deRham cohomology. Thus we see that deRham cohomology can also be represented as an axiomatic sheaf cohomology theory as follows,

$$H^q_{dR}(X, \mathbf{R}) \simeq H^q(X, \mathcal{R})$$

where $\mathcal{R} = X \times \mathbf{R}$. The argument is similar for the complex case so that we have,

$$H^q_{dR}(X, \mathbf{C}) \simeq H^q(X, \mathcal{C})$$

where $\mathcal{C} = X \times \mathbf{C}$.

Thus we see that both real and complex deRham cohomology can be interpreted as sheaf cohomology theories.

3.3 Hodge Theory

3.3.1 Building an Inner Product on Exterior Algebras

Let $\{V, \langle , \rangle\}$ be an inner product space of finite dimension with orthonormal basis $\{e_1, \ldots, e_n\}$ and choice of orientation $e_1, \ldots, e_n > 0$. We extend the inner product to $\bigwedge(V) = \sum_{k \geq 0} \bigwedge^k(V)$ (the exterior algebra of V) as follows,

1. $\langle \bigwedge^q(V), \bigwedge^k(V) \rangle = 0$ if $q \neq k$
2. for $v_1 \wedge \cdots \wedge v_k,\ w_1 \wedge \cdots \wedge w_k \in \bigwedge^k(V)$ we set $\langle v_1 \wedge \cdots \wedge v_k,\ w_1 \wedge \cdots \wedge w_k \rangle = det(\langle v_i, w_j \rangle)$
3. extend by linearity over all of $\bigwedge^*(V)$

We also define the so-called star operator $* : \bigwedge^*(V) \to \bigwedge^*(V)$ as follows,

1. $*(e_1 \wedge \cdots \wedge e_n) = 1$, $*(1) = e_1 \wedge \cdots \wedge e_n$
2. $*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_n$
3. extend by linearity over all of $\bigwedge^*(V)$

From the definition of $*$ we get the following,

Proposition: On $\bigwedge^p(V)$, $** = (-1)^{p(n-p)}$.

3.3.2 Extension to Manifolds

Let X be a real compact oriented Riemannian manifold of dimension n with local coordinates $\{x_1, \ldots, x_n\}$ and choice of orientation $dx_1 \wedge \cdots \wedge dx_n > 0$. We define,

1. $T(X)$ = real tangent bundle with local basis given by $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$
2. $T^*(X)$ = real cotangent bundle with local basis given by $\{dx_1, \ldots, dx_n\}$

We can define a local metric on $T(X)$ as follows: let $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ be a partition of unity on X (it is well known that such partitions of unity exist). If $\pi : T(X) \to X$ is the usual projection then we know,

$$\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbf{R}^n$$

and so we can define a metric $\langle\ ,\ \rangle_\alpha$ on U_α by,

$$\langle (p, \vec{v}), (q, \vec{w}) \rangle_\alpha = \begin{cases} 0 & if\ p \neq q \\ \vec{v} \cdot \vec{w} & if\ p = q \end{cases}$$

Now we can define a global metric $\langle\ ,\ \rangle$ on $T(X)$ by "glueing together" the $\langle\ ,\ \rangle_\alpha$ using our partition of unity. In particular we let,

$$\langle \xi, \xi' \rangle = \sum_{\alpha \in A} \langle \phi_\alpha(\xi), \phi_\alpha(\xi') \rangle_\alpha$$

By using the diffeomorphisms,

$$\psi : T_p(X) \to T_p^*(X)$$

defined by,

$$\psi(\xi_p) = \langle \xi_p, - \rangle$$

we see that our metric $\langle\ ,\ \rangle$ on $T(X)$ induces a metric $\langle\ ,\ \rangle^*$ on $T^*(X)$. In particular, for $\gamma_1, \gamma_2 \in T^*(X)$, $\langle\ ,\ \rangle^*$ is defined by,

$$\langle \gamma_1, \gamma_2 \rangle^* = \langle \psi^{-1}(\gamma_1), \psi^{-1}(\gamma_2) \rangle$$

In the same way as before, we can extend this inner product to the exterior algebra $\bigwedge^* T^*(X)$. By noticing that $\bigwedge^k T^*(X) \simeq E^k_{\mathbf{R}}(X)$ we see that we have really defined a metric on the space of real-valued forms on the manifold X.

In general the $dx_1, \ldots, dx_n$ will not be orthonormal. However, once we have our metric we know that we can always find such an oriented orthonormal frame for $T^*(X)$ which we will denote $\varphi_1, \ldots, \varphi_n$.

Now we may define an extension of the star operator to forms on X,

$$* : E^*_{\mathbf{R}}(X) \to E^*_{\mathbf{R}}(X)$$

where, for $\omega = \sum_I f_I dx_I \in E^k_{\mathbf{R}}(X)$, we have,

$$*(\omega) = \sum_I f_I * (dx_I)$$

and where,

1. $*(\varphi_1 \wedge \cdots \wedge \varphi_n) = 1$, $*(1) = \varphi_1 \wedge \cdots \wedge \varphi_n$
2. $*(\varphi_1 \wedge \cdots \wedge \varphi_p) = \varphi_{p+1} \wedge \cdots \wedge \varphi_n$ and extend linearly

as before.

3.3.3 The Hodge Inner Product on $E^*_R(X)$

Notice that the element of volume for our manifold is given locally by,

$$dV = \varphi_1 \wedge \cdots \wedge \varphi_n > 0$$

and that, since X is compact,

$$Vol(X) = \int_X dV < \infty$$

This implies that the following is a well-defined inner product on $E^*_{\mathbf{R}}(X) = \sum_{k \geq 0} E^k_{\mathbf{R}}(X)$,

Definition: For $\alpha \in E^k_{\mathbf{R}}(X)$ and $\beta \in E^q_{\mathbf{R}}(X)$ the Hodge inner product $\langle\ ,\ \rangle_X$ is given by,

$$\langle \alpha, \beta \rangle_X = \begin{cases} 0 & if\ p \neq q \\ \int_X \alpha \wedge *\beta & if\ p = q \end{cases}$$

We will use this inner product to examine the space $E^*_{\mathbf{R}}(X)$ more closely.

3.3.4 The Euclidean Laplacian

We first intoduce the operator,

$$\delta : E^k_{\mathbf{R}}(X) \to E^{k-1}_{\mathbf{R}}(X)$$

defined by,

$$\delta = (-1)^{n(k+1)+1} * d*$$

Proposition: δ is the adjoint of d with respect to the Hodge inner product.

Proof: We need to show that for all $\alpha, \beta \in E^*_{\mathbf{R}}(X)$ it holds that $\langle d\alpha, \beta \rangle_X = \langle \alpha, \delta\beta \rangle_X$. We first notice that we can simplify to the case $\alpha \in E^{p-1}_{\mathbf{R}}(X)$, $\beta \in E^p_{\mathbf{R}}(X)$ since otherwise we have,

$$\langle d\alpha, \beta \rangle_X = 0 = \langle \alpha, \delta\beta \rangle_X$$

by the definition of the Hodge inner product and so the result obviously holds. We go on to calculate,

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta \tag{3.4}$$

Consider the form $d{*}\beta$. Since $\beta \in E^p_{\mathbf{R}}(X)$ we know that $d{*}\beta \in E^{n-p+1}_{\mathbf{R}}(X)$. Thus we have,

$$* * (d * \beta) = (-1)^{(n-p+1)(p-1)} d * \beta \tag{3.5}$$

and also,

$$* * (d * \beta) = *(*d * \beta) = *(-1)^{n(p+1)+1}\delta\beta = (-1)^{n(p+1)+1} * \delta\beta \tag{3.6}$$

by the definitions of $**$ and δ. Combining (3.5) and (3.6) we can write,

$$d * \beta = (-1)^{(n-p+1)(p-1)}(-1)^{n(p+1)+1}\alpha \wedge *\delta\beta \tag{3.7}$$

and substituting (3.7) back into (3.4) yields,

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{(n-p+1)(p-1)}(-1)^{n(p+1)+1}(-1)^{p-1}\alpha \wedge *\delta\beta$$

Let us look more closely at the sign of the second term. We work *mod*2 to get,

$$\begin{aligned} & (n-p+1)(p-1) + n(p+1) + 1 + (p-1) \\ \equiv\ & (n-p)(p-1) + n(p+1) + 1 \\ \equiv\ & np - n - p^2 + p + np + n + 1 \\ \equiv\ & 2np - p^2 + p + 1 \\ \equiv\ & 1 + p - p^2 \\ \equiv\ & 1 + p(1-p) \\ \equiv\ & 1, \ \textit{since } p(1-p) \textit{ is always even} \end{aligned}$$

Thus the sign is negative and so we have,

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta - \alpha \wedge *\delta\beta$$

Now, $\partial X = \emptyset$ so,

$$\int_{\partial X} \alpha \wedge *\beta = 0$$

Thus, by applying Stokes' Theorem we get,

$$\begin{aligned}
0 &= \int_X d(\alpha \wedge *\beta) \\
&= \int_X d\alpha \wedge *\beta - \int_X \alpha \wedge *\delta\beta \\
&= \langle d\alpha, \beta\rangle_X - \langle \alpha, \delta\beta\rangle_X
\end{aligned}$$

Thus $\langle d\alpha, \beta\rangle_X = \langle \alpha, \delta\beta\rangle_X$ as required. Q.E.D.

We now introduce another operator,

Definition: The Euclidean Laplacian $\Delta_d : E^k_{\mathbf{R}}(X) \to E^k_{\mathbf{R}}(X)$ is defined by,

$$\Delta_d = d\delta + \delta d$$

Proposition:

1. Δ_d is self-adjoint,
2. Δ_d commutes with the d, δ and $*$ operators,
3. $\Delta_d \alpha = 0 \iff d\alpha = \delta\alpha = 0$.

Proof:

1.

$$\begin{aligned}
\langle \Delta_d \alpha, \beta\rangle_X &= \langle d\delta\alpha + \delta d\alpha, \beta\rangle_X \\
&= \langle d\delta\alpha, \beta\rangle_X + \langle \delta d\alpha, \beta\rangle_X \\
&= \langle \delta\alpha, \delta\beta\rangle_X + \langle d\alpha, d\beta\rangle_X \\
&= \langle \alpha, d\delta\beta\rangle_X + \langle \alpha, \delta d\beta\rangle_X \\
&= \langle \alpha, d\delta + \delta d\beta\rangle_X \\
&= \langle \alpha, \Delta_d \beta\rangle_X
\end{aligned}$$

2. We prove only $d\Delta_d = \Delta_d d$ since the others follow by similar proofs.

$$\begin{aligned}
d\Delta_d &= d(d\delta + \delta d) \\
&= d^2\delta + d\delta d \\
&= d\delta d \\
&= d\delta d + \delta d^2 \\
&= (d\delta + \delta d)d \\
&= \Delta_d d
\end{aligned}$$

3. If $d\alpha = \delta\alpha = 0$ the $\Delta_d\alpha = 0$ clearly follows. From proof of 1. we had,

$$\langle \Delta_d\alpha, \beta\rangle_X = \langle \delta\alpha, \delta\beta\rangle_X + \langle d\alpha, d\beta\rangle_X$$

Thus we have,

$$\begin{aligned} \langle \Delta_d\alpha, \alpha\rangle_X &= \langle \delta\alpha, \delta\alpha\rangle_X + \langle d\alpha, d\alpha\rangle_X \\ &= \|\delta\alpha\|^2 + \|d\alpha\|^2 \end{aligned}$$

and so, if $\Delta_d\alpha = 0$ then it must be that $d\alpha = \delta\alpha = 0$ as well. Q.E.D.

<u>**Definition**</u>: If $\Delta_d\alpha = 0$ then we say α is harmonic.

3.3.5 The Complex Laplacian

Now instead of a Riemannian manifold, suppose we have a projective algebraic manifold X with Hermitian metric.

Recall that $E^*_{\mathbf{C}}(X) \simeq E^*_{\mathbf{R}}(X) \otimes \mathbf{C}$. This fact allows us to extend our star operator,

$$* : E^k_{\mathbf{R}}(X) \to E^{n-k}_{\mathbf{R}}(X)$$

to an operator on complex forms,

$$\tilde{*} : E^k_{\mathbf{C}}(X) \to E^{2n-k}_{\mathbf{C}}(X)$$

by defining,

$$\tilde{*} = * \otimes 1$$

(We will often use the notation $*$ to represent both $*$ and $\tilde{*}$.)

We define the complexified tangent and cotangent bundles to be,

1. $T(X)_{\mathbf{C}} = T(X) \otimes \mathbf{C}$ with local basis $\{\partial/\partial z_1, \partial/\partial \bar{z}_1, \ldots, \partial/\partial z_n, \partial/\partial \bar{z}_n\}$ or $\{\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n\}$
2. $T^*(X)_{\mathbf{C}} = T^*(X) \otimes \mathbf{C}$ with local basis $\{dz_1, d\bar{z}_1, \ldots, dz_n, d\bar{z}_n\}$ or $\{dx_1, dy_1, \ldots, dx_n, dy_n\}$

We use out Hermitian product on $T(X)_{\mathbf{C}}$ to define a product on $T^*(X)_{\mathbf{C}}$ in a similar way to the real case, and extend this product to the exterior algebra $\bigwedge^* T^*(X) \simeq E^*_{\mathbf{C}}(X)$. Thus we get an inner product on $E^*_{\mathbf{C}}(X)$.

<u>**Definition**</u>: Let $\alpha \in E^k_{\mathbf{C}}(X)$ and $\beta \in E^q_{\mathbf{C}}(X)$. The Hodge inner product on $E^*_{\mathbf{C}}(X)$ is given by,

$$\langle \alpha, \beta \rangle_X = \begin{cases} 0 & if\ p \neq q \\ \int_X \alpha \wedge *\bar{\beta} & if\ p = q \end{cases}$$

Note: If β is a real-valued form then $\bar{\beta} = \beta$ and so this inner product is an extension of that defined for the real case.

Proposition: The decomposition $E^*_{\mathbf{C}}(X) \simeq \oplus_{p,q} E^{p,q}_{\mathbf{C}}(X)$ is orthogonal with respect to the Hodge inner product.

Proof: Say $\alpha \in E^{p,q}_{\mathbf{C}}(X)$ and $\beta \in E^{s,t}_{\mathbf{C}}(X)$. If $p + q \neq s + t$ then it is obvious from the definition that $\langle \alpha, \beta \rangle_X = 0$. However, what if $p + q = s + t$ but $(p, q) \neq (s, t)$? In this case any local representation of the forms α and $*\bar{\beta}$ will necessarily have differentials in common and so,

$$\alpha \wedge *\bar{\beta} = 0$$

and thus,

$$\langle \alpha, \beta \rangle_X = \int_X \alpha \wedge *\bar{\beta} = \int_X 0 = 0$$

So, if $(p, q) \neq (s, t)$ then $\langle E^{p,q}_{\mathbf{C}}(X), E^{s,t}_{\mathbf{C}}(X) \rangle_X = 0$. Q.E.D.

We now introduce the operators $\bar{*} : E^*_{\mathbf{C}}(X) \to E^*_{\mathbf{C}}(X)$ defined by,

$$\bar{*}(\omega) = *(\bar{\omega})$$

and $\bar{\partial}^* : E^{p,q}_{\mathbf{C}}(X) \to E^{p,q-1}_{\mathbf{C}}(X)$ defined by,

$$\bar{\partial}^* = -\bar{*}\bar{\partial}\bar{*}$$

Proposition: $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ with respect to the Hodge inner product.

Proof: Similar to the proof that δ is the adjoint of d.

Using these operators we define,

Definition: The complex Laplacian $\Delta_{\bar{\partial}} : E^{p,q}_{\mathbf{C}}(X) \to E^{p,q}_{\mathbf{C}}(X)$ is defined by,

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

As in the real case we have the following,

Proposition:

1. $\Delta_{\bar{\partial}}$ is self-adjoint
2. $\Delta_{\bar{\partial}}$ commutes with the operators $\bar{\partial}$, $\bar{\partial}^*$, and $\bar{*}$

3. $\Delta_{\bar{\partial}}(\alpha) = 0 \iff \bar{\partial}(\alpha) = \bar{\partial}^*(\alpha) = 0$

Definition: If $\Delta_{\bar{\partial}}(\alpha) = 0$ then we say that α is harmonic.

3.3.6 Motivation for the Hodge Theorem

We now want to examine forms on the level of cohomology. The following proposition allows us to characterize equivalence classes in $H^k_{dR}(X, \mathbf{R})$.

Proposition: A d-closed k-form η is of minimal norm in $\eta + dE^{k-1}_{\mathbf{R}}(X)$ if and only if $\delta\eta = 0$.

Proof:

1. "$\Longleftarrow$" Say $\delta\eta = 0$ and take and $(\eta + d\nu) \in \eta + dE^{k-1}_{\mathbf{R}}(X)$. Then,

$$\begin{aligned}
\|\eta + d\nu\|^2 &= \langle \eta + d\nu, \eta + d\nu \rangle_X \\
&= \langle \eta, \eta \rangle_X + 2\langle \eta, d\nu \rangle_X + \langle d\nu, d\nu \rangle_X \\
&= \|\eta\|^2 + 2\langle \eta, d\nu \rangle_X + \|d\nu\|^2 \\
&= \|\eta\|^2 + \|d\nu\|^2 + 2\langle \delta\eta, \nu \rangle_X \\
&= \|\eta\|^2 + \|d\nu\|^2 + 2\langle 0, \nu \rangle_X \\
&= \|\eta\|^2 + \|d\nu\|^2 \\
&\geq \|\eta\|^2
\end{aligned}$$

Thus η is of minimum norm in $\eta + dE^{k-1}_{\mathbf{R}}(X)$.

2. "$\Longrightarrow$" Suppose η has minimal norm in $\eta + dE^{k-1}_{\mathbf{R}}(X)$. Then,

$$\frac{d}{dt}\left(\|\eta + td\nu\|^2\right)|_{t=0} = 0$$

for all $\nu \in E^{k-1}_{\mathbf{R}}(X)$. But this implies that,

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\|\eta + td\nu\|^2\right)|_{t=0} \\
&= \frac{d}{dt}\left(\|\eta\|^2 + 2\langle \eta, td\nu \rangle_X + \|td\nu\|^2\right)|_{t=0} \\
&= \frac{d}{dt}\left(\|\eta\|^2 + 2t\langle \eta, d\nu \rangle_X + t^2\|d\nu\|^2\right)|_{t=0} \\
&= \left(2\langle \eta, d\nu \rangle_X + 2t\|d\nu\|^2\right)|_{t=0} \\
&= 2\langle \eta, d\nu \rangle_X \\
&= 2\langle \delta\eta, \nu \rangle_X
\end{aligned}$$

Thus, $\langle \delta\eta, \nu \rangle_X = 0$ for all ν which implies that $\delta\eta = 0$, as required. Q.E.D.

This implies that forms of minimal norm exist and thus that each equivalence class $\{\eta\} \in H^k_{dR}(X, \mathbf{R})$ can be represented by such a form. This occurs when,

1. η is d-closed (i.e. $d\eta = 0$)
2. $\delta\eta = 0$

Thus the representative η is a harmonic form (since $d\eta = \delta\eta = 0$ implies $\Delta_d \eta = 0$).

We introduce the following notation,

$$\mathcal{H}^k(X) \;=\; space\ of\ harmonic\ k-forms\ on\ X \;=\; ker(\Delta_d)$$

The above result also holds in the complex case so that every equivalence class in $H^{p,q}(X)$ has a unique harmonic representative. So we also have,

$$\mathcal{H}^{p,q}(X) \;=\; space\ of\ harmonic\ (p,q)-forms\ on\ X \;=\; ker(\Delta_{\bar{\partial}})$$

3.3.7 The Hodge Theorem

We state both the real and complex versions of the Hodge Theorem:

Theorem: (The Hodge Theorem, Real Version) Let X be a compact oriented Remannian manifold. Then the following hold,

1. $dim_{\mathbf{R}} \mathcal{H}^k(X) < \infty$ (therefore $\mathcal{H}^k(X)$ is a closed subspace of $E^k_{\mathbf{R}}(X)$ and thus we can write $E^k_{\mathbf{R}}(X) = \mathcal{H}^k(X) \oplus (\mathcal{H}^k(X))^{\perp}$ with repect to $\langle\ ,\ \rangle_X$),
2. the orthogonal projection $Pr_{\mathcal{H}} : E^k_{\mathbf{R}}(X) \to \mathcal{H}^k(X)$ is well-defined,
3. there is a unique operator $G : E^k_{\mathbf{R}}(X) \to (\mathcal{H}^k(X))^{\perp}$ such that $G(\mathcal{H}^k(X)) = 0$, G commutes with d, δ, and $*$, and $Id = Pr_{\mathcal{H}} + \Delta_d G$ on $E^k_{\mathbf{R}}(X)$.

The above theorem immediately extends to the complex case by considering the space $\mathcal{H}^{p,q}(X)$,

Theorem: (The Hodge Theorem, Complex Version) Let X be a compact complex manifold with Hermitian metric. Then the following hold,

1. $dim_{\mathbf{C}} \mathcal{H}^{p,q}(X) < \infty$ (therefore $\mathcal{H}^{p,q}(X)$ is a closed subspace of $E^{p,q}_{\mathbf{C}}(X)$ and thus we can write $E^{p,q}_{\mathbf{C}}(X) = \mathcal{H}^{p,q}(X) \oplus (\mathcal{H}^{p,q}(X))^{\perp}$ with repect to $\langle\ ,\ \rangle_X$),

2. the orthogonal projection $Pr_{\mathcal{H}} : E^{p,q}_{\mathbf{C}}(X) \to \mathcal{H}^{p,q}(X)$ is well-defined,

3. there is a unique operator $G : E^{p,q}_{\mathbf{C}}(X) \to (\mathcal{H}^{p,q}(X))^{\perp}$ such that $G(\mathcal{H}^{p,q}(X)) = 0$, G commutes with $\bar{\partial}$, $\bar{\partial}^*$, and $\bar{*}$, and $Id = Pr_{\mathcal{H}} + \Delta_{\bar{\partial}} G$ on $E^{p,q}_{\mathbf{C}}(X)$.

We get the following important corollaries,

Corollary: (Hodge Decomposition Theorem) We have the following orthogonal decompositions (with respect to the Hodge inner product),

1. $E^k_{\mathbf{R}}(X) \simeq \mathcal{H}^k(X) \oplus dE^{k-1}_{\mathbf{R}}(X) \oplus \delta E^{k+1}_{\mathbf{R}}(X)$
2. $E^k_{\mathbf{C}}(X) \simeq \mathcal{H}^k_{\mathbf{C}}(X) \oplus dE^{k-1}_{\mathbf{C}}(X) \oplus \delta E^{k+1}_{\mathbf{C}}(X)$
3. $E^{p,q}_{\mathbf{C}}(X) \simeq \mathcal{H}^{p,q}(X) \oplus \bar{\partial} E^{p,q-1}_{\mathbf{C}}(X) \oplus \bar{\partial}^* E^{p,q+1}_{\mathbf{C}}(X)$
4. By defining the operator $\partial^* : E^{p,q}_{\mathbf{C}}(X) \to E^{p-1,q}_{\mathbf{C}}(X)$ by the formula $\partial^* = - * \partial *$ we get an analagous relationship for ∂ and ∂^* as for $\bar{\partial}$ and $\bar{\partial}^*$ (i.e. ∂^* is the adjoint of ∂). Thus we also have the decomposition,

$$E^{p,q}_{\mathbf{C}}(X) \simeq \mathcal{H}^{p,q}(X) \oplus \partial E^{p-1,q}_{\mathbf{C}}(X) \oplus \partial^* E^{p+1,q}_{\mathbf{C}}(X)$$

Corollary: We have the isomorphisms,

1. $\mathcal{H}^k(X) \simeq H^k_{dR}(X, \mathbf{R})$
2. $\mathcal{H}^{p,q}(X) \simeq H^{p,q}(X)$

Corollary:

1. (Poincaré Duality) The bilinear pairing,

$$H^k_{dR}(X, \mathbf{R}) \times H^{n-k}_{dR}(X, \mathbf{R}) \to \mathbf{R}$$

given by,

$$(\{\eta\}, \{\nu\}) \mapsto \int_X \eta \wedge \nu$$

is non-singular. Thus $H^k_{dR}(X, \mathbf{R}) \simeq (H^{n-k}_{dR}(X, \mathbf{R}))^{\vee}$.

2. (Kodaira-Serre Duality) The bilinear pairing,

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbf{C}$$

given by,

$$(\{\eta\}, \{\nu\}) \mapsto \int_X \eta \wedge \nu$$

is non-singular. Thus $H^{p,q}(X) \simeq (H^{n-p,n-q}(X))^{\vee}$.

We also have the following important relation among the Laplacians $\Delta_d = d\delta + \delta d$, $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $\Delta_\partial = \partial\partial^* + \partial^*\partial$,

Proposition: If X is a projective algebraic manifold then,

$$\frac{\Delta_d}{2} = \Delta_{\bar\partial} = \Delta_\partial$$

3.4 Dolbeault Cohomology

3.4.1 Definitions

We can also define a sheaf cohomology theory on complex-valued forms of type (p,q). This is called Dolbeault cohomology.

First we define,

1. $\Omega^p_{\mathbf{C}}(X)$ = sheaf of germs of holomorphic p-forms on X
2. $\mathcal{E}^{p,q}_{\mathbf{C}}(X)$ = sheaf of germs of C^∞ (p,q)-forms on X

Notice that any form in $\Omega^p_{\mathbf{C}}(X)$ is of type $(p,0)$ so that we have the inclusion map,

$$i : \Omega^p_{\mathbf{C}}(X) \longrightarrow \mathcal{E}^{p,0}_{\mathbf{C}}(X)$$

Using our operator $\bar\partial$ we then have, for each $p \geq 0$, the following sequence of sheaf homomorphisms,

$$0 \rightarrow \Omega^p_{\mathbf{C}}(X) \xrightarrow{i} \mathcal{E}^{p,0}_{\mathbf{C}}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,1}_{\mathbf{C}}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,2}_{\mathbf{C}}(X) \rightarrow \cdots$$

Which turns out to be a fine resolution of the sheaf $\Omega^p_{\mathbf{C}}(X)$. Applying the Abstract deRham Isomorphism Theorem we get a sheaf cohomology theory defined by,

$$H^q(X, \Omega^p_{\mathbf{C}}(X)) = H^q(\Gamma(\mathcal{E}^{p,\bullet}_{\mathbf{C}}(X)))$$

Thus we can define,

Definition: The $(p,q)^{th}$ Dolbeault cohomology group of X is given by,

$$H^{p,q}(X) = H^q(X, \Omega^p_{\mathbf{C}}(X))$$

Note:

1. In this way we see that Dolbeault cohomology is a sheaf cohomology theory.

2. Since $\Gamma(\mathcal{E}^{p,\bullet}_{\mathbf{C}}(X)) \simeq E^{p,\bullet}_{\mathbf{C}}(X)$, we can see that an alternate definition of Dolbeault cohomology is,

$$H^{p,q}(X) = H^q(E^{p,\bullet}_{\mathbf{C}}(X))$$

where $E^{p,\bullet}_{\mathbf{C}}(X)$ denotes the cochain complex,

$$0 \rightarrow E^{p,0}_{\mathbf{C}}(X) \xrightarrow{\bar{\partial}} E^{p,1}_{\mathbf{C}}(X) \xrightarrow{\bar{\partial}} E^{p,2}_{\mathbf{C}}(X) \rightarrow \cdots$$

Thus we can also write,

$$H^{p,q}(X) = \frac{\{ker\bar{\partial} : E^{p,q}_{\mathbf{C}}(X) \rightarrow E^{p,q+1}_{\mathbf{C}}(X)\}}{\{\bar{\partial}E^{p,q-1}_{\mathbf{C}}(X)\}}$$

or, in more compact notation,

$$H^{p,q}(X) = \frac{E^{p,q}_{\mathbf{C},\bar{\partial}-closed}(X)}{\bar{\partial}E^{p,q-1}_{\mathbf{C}}(X)}$$

3.4.2 Relationship of Dolbeault and deRham Cohomology

We recall that on any projective algebraic manifold X we have that,

$$\Delta_{\bar{\partial}} = \frac{\Delta_d}{2}$$

Say we have $\omega \in E^k_{\mathbf{C}}(X)$ with $\Delta_{\bar{\partial}}(\omega) = 0$. Since we know that,

$$E^k_{\mathbf{C}}(X) \simeq \oplus_{p+q=k} E^{p,q}_{\mathbf{C}}(X)$$

we can write,

$$\omega = \oplus_{p+q=k}\omega^{p,q}$$

But then,

$$\begin{aligned} 0 &= \Delta_{\bar{\partial}}(\omega) \\ &= \Delta_{\bar{\partial}}(\oplus_{p+q=k}\omega^{p,q}) \\ &= \oplus_{p+q=k}\Delta_{\bar{\partial}}(\omega^{p,q}) \\ &= \oplus_{p+q=k}\frac{\Delta_d}{2}(\omega^{p,q}) \\ &= \oplus_{p+q=k}\frac{1}{2}\Delta_d(\omega^{p,q}) \end{aligned}$$

which implies that,

$$\Delta_d(\omega^{p,q}) = 0,\ \forall\ p+q=k$$

Thus each component of a harmonic form is harmonic and we get,

$$\mathcal{H}^k(X) \otimes \mathbf{C} \simeq \oplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

Now, by our corollary to the Hodge Theorem, we get that this descends to cohomology. This gives the following relationship between Dolbeault and deRham cohomology,

Theorem: $H^k_{dR}(X, \mathbf{C}) \simeq H^k_{dR}(X, \mathbf{R}) \otimes \mathbf{C} \simeq \oplus_{p+q=k} H^{p,q}(X)$

3.4.3 An Alternate Definition for Dolbeault Cohomology

There is an alternate way to define Dolbeault cohomology that will be more useful in our later calculations. We begin by showing the following proposition,

Proposition: If $\eta \in E^{p,q}_{\mathbf{C},d-closed}(X)$ is a coboundary (i.e. $\eta = d\omega$ for some $\omega \in E^{p+q-1}_{\mathbf{C}}(X)$) then $\eta = \partial\bar{\partial}\sigma$ for some $\sigma \in E^{p-1,q-1}_{\mathbf{C}}(X)$.

Proof: Using the Hodge Decomposition Theorem we can write,

$$\omega = h_1 + \partial x_1 + \partial^* y_1 \tag{3.8}$$
$$\omega = h_2 + \bar{\partial} x_2 + \bar{\partial}^* y_2 \tag{3.9}$$

where the h_i are harmonic and where the asterix denotes the adjoint with respect to the Hodge inner product. Since the h_i are harmonic we have $\partial h_i = \bar{\partial} h_i = 0$. Thus from (1) we get,

$$\bar{\partial}\omega = \bar{\partial}\partial x_1 + \bar{\partial}\partial^* y_1$$

and from (2) we get,

$$\partial\omega = \partial\bar{\partial} x_2 + \partial\bar{\partial}^* y_2$$

Since $\eta = d\omega = \partial\omega + \bar{\partial}\omega$ we have,

$$\eta = \bar{\partial}\partial x_1 + \bar{\partial}\partial^* y_1 + \partial\bar{\partial} x_2 + \partial\bar{\partial}^* y_2 \tag{3.10}$$

Now, since $d\eta = 0$ and η is of type (p,q) we can say that,

$$\partial\eta = (Pr_{p+1,q} \circ d)(\eta) = 0$$

$$\bar{\partial}\eta = (Pr_{p,q+1} \circ d)(\eta) = 0$$

Thus, using (3) and the fact that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$, we have,

$$0 = \partial\eta = \partial\bar{\partial}\partial x_1 + \partial\bar{\partial}\partial^* y_1 + \partial\partial\bar{\partial}x_2 + \partial\partial\bar{\partial}^* y_2 = \partial\bar{\partial}\partial^* y_1$$

$$0 = \bar{\partial}\eta = \bar{\partial}\bar{\partial}\partial x_1 + \bar{\partial}\bar{\partial}\partial^* y_1 + \bar{\partial}\partial\bar{\partial}x_2 + \bar{\partial}\partial\bar{\partial}^* y_2 = \bar{\partial}\partial\bar{\partial}^* y_2$$

Now we calculate,

1.

$$\begin{aligned}
\|\bar{\partial}\partial^* y_1\|^2 &= \langle\bar{\partial}\partial^* y_1, \bar{\partial}\partial^* y_1\rangle_X \\
&= -\langle\bar{\partial}\partial^* y_1, \partial^{ast}\bar{\partial}y_1\rangle_X \\
&= -\langle\partial\bar{\partial}\partial^* y_1, \bar{\partial}y_1\rangle_X \\
&= -\langle\partial\eta, \bar{\partial}y_1\rangle_X \\
&= -\langle 0, \bar{\partial}y_1\rangle_X \\
&= 0
\end{aligned}$$

Thus we have $\|\bar{\partial}\partial^* y_1\| = 0$ and so,

$$\bar{\partial}\partial^* y_1 = 0 \tag{3.11}$$

2.

$$\begin{aligned}
\|\partial\bar{\partial}^* y_2\|^2 &= \langle\partial\bar{\partial}^* y_2, \partial\bar{\partial}^* y_2\rangle_X \\
&= -\langle\partial\bar{\partial}^* y_2, \bar{\partial}^*\partial y_2\rangle_X \\
&= -\langle\bar{\partial}\partial\bar{\partial}^* y_2, \partial y_2\rangle_X \\
&= \langle\bar{\partial}\eta, \partial y_2\rangle_X \\
&= \langle 0, \partial y_2\rangle_X \\
&= 0
\end{aligned}$$

Thus we have $\|\partial\bar{\partial}^* y_2\| = 0$ and so,

$$\partial\bar{\partial}^* y_2 = 0 \tag{3.12}$$

Substituting (4) and (5) into (3) gives,

$$\begin{aligned}
\eta &= \bar{\partial}\partial x_1 + \partial\bar{\partial}x_2 \\
&= -\partial\bar{\partial}x_1 + \partial\bar{\partial}x_2 \\
&= \partial\bar{\partial}(x_2 - x_1)
\end{aligned}$$

where $(x_2 - x_1) = \sigma \in E_{\mathbf{C}}^{p-1,q-1}(X)$ as required. Q.E.D.

Now recall our definition for Dolbeault cohomology,

$$H^{p,q}(X) = \frac{E^{p,q}_{\mathbf{C},\bar{\partial}-closed}(X)}{\bar{\partial}E^{p,q-1}_{\mathbf{C}}(X)}$$

Since we know that the harmonic spaces defined by Δ_d and $\Delta_{\bar{\partial}}$ are the same, we can also say that the conditions d-closed and $\bar{\partial}$-closed are equivalent. Thus, applying the results of the above proposition we get the following,

Proposition: An equivalent definition for Dolbeault cohomology is given by,

$$H^{p,q}(X) = \frac{E^{p,q}_{\mathbf{C},d-closed}(X)}{\partial\bar{\partial}E^{p-1,q-1}_{\mathbf{C}}(X)}$$

This definition will turn out to be more useful for our calculations in later sections.

Chapter 4

A Real Regulator on $z^k(X,1)$

Now that we have covered the necessary background we would like to define a first example of a regulator map. The following calculation is motivated by [Lew3, 143-144].

Let X be a projective algebraic manifold with $dim_{\mathbf{C}} X = n$. We define $z^k(X)$ to be the free abelian group generated by irreducible subvarieties of codimension k in X. In other words we have the cycle group,

$$z^k(X) = \{\sum_j n_j \cdot Z_j : codim_X Z_j = k, n_j \textit{ an integer}\}$$

(Note that in this section we will not indicate ranges on sums but will assume that the index is always finite.)

In particular notice that $z^0(X) \simeq X$ and $z^1(X) = Div(X)$. We extend this idea by defining,

$$\tilde{z}^k(X,1) = \{\sum_j (f_j, Z_j) : codim_X Z_j = k, f_j \in \mathbf{C}(Z_j)^\times\}$$

$$z^k(X,1) = \{\sum_j (f_j, Z_j) \in \tilde{z}^k(X,1) : \sum_j div_{Z_j} f_j = 0\}$$

where we recall,

$$div_{Z_j} f_j = \sum_{codim_{Z_j} D=1} ord_D f_j \cdot D$$

and say that D is a zero of f_j if $ord_D f_j > 0$ and a pole of f_j if $ord_D f_j < 0$. We can write $div_{Z_j} f_j$ equivalently as follows: say f_j has pole set and zero set given by $\{P_j^i\}$ and $\{Q_j^i\}$ respectively with $ord_{P_j^i} f_j = m_j^i$ and $ord_{Q_j^i} f_j = n_j^i$, then we can write,

$$div_{Z_j} f_j = \sum_i (n_j^i P_j^i + m_j^i Q_j^i)$$

We now define a regulator map,

$$r : z^k(X,1) \longrightarrow H^{k,k}(X,\mathbf{R})$$

as follows: for any $\xi = \sum_j (f_j, Z_j) \in z^k(X,1)$ we define the function $r(\xi) \in H^{n-k,n-k}(X,\mathbf{R})^\vee$ by the formula,

$$r(\xi)(\{\omega\}) = \sum_j \int_{Z_j} log|f_j| \wedge \omega \in \mathbf{R}$$

where $\{\omega\}$ is an equivalence class of $(n-k,n-k)$-forms in $H^{n-k,n-k}(X,\mathbf{R})$. We then use Kodaira-Serre Duality which gives the isomorphism,

$$H^{n-k,n-k}(X,\mathbf{R})^\vee \simeq H^{k,k}(X,\mathbf{R})$$

to associate $r(\xi)$ with an element in $H^{k,k}(X,\mathbf{R})$. So this defines our map r.

We notice a couple of things,

1. The form ω is of type $(n-k, n-k)$ which matches the dimension of Z_j which is $n-k$. Thus the integral is not trivial.

2. The map r is well-defined on forms $\omega \in E_{\mathbf{R}}^{n-k,n-k}(X)$ but not necessarily on equivalence classes of forms in cohomology. We recall from Hodge Theory that,

$$H^{p,q}(X,\mathbf{R}) \simeq \frac{E^{p,q}_{\mathbf{R},d-closed}(X)}{\partial\bar{\partial}E_{\mathbf{R}}^{p-1,q-1}(X)}$$

and so in our case we are interested in,

$$H^{n-k,n-k}(X,\mathbf{R}) \simeq \frac{E^{n-k,n-k}_{\mathbf{R},d-closed}(X)}{\partial\bar{\partial}E_{\mathbf{R}}^{n-k-1,n-k-1}(X)}$$

Thus, to show our regulator is well-defined on cohomology we must show that if $\omega \in \partial\bar{\partial}E_{\mathbf{R}}^{n-k-1,n-k-1}(X)$ then $r(\xi)(\{\omega\}) = 0$ for any choice of ξ.

Proposition: Let $\{\omega\} \in H^{n-k,n-k}(X,\mathbf{R})$ be such that $\omega = \partial\bar{\partial}\eta$ for some $\eta \in E_{\mathbf{R}}^{n-k-1,n-k-1}(X)$. Then for any $\xi \in z^k(X,1)$ we have $r(\xi)(\{\omega\}) = 0$.

Proof: Let $\xi = \sum_j (f_j, Z_j) \in z^k(X,1)$ and $\omega = \partial\bar{\partial}\eta$ for some $\eta \in E_{\mathbf{R}}^{n-k-1,n-k-1}(X)$. Then we have,

$$
\begin{aligned}
r(\xi)(\omega) &= r(\xi)(\partial\bar{\partial}\eta) \\
&= \sum_j \int_{Z_j} log|f_j| \wedge \partial\bar{\partial}\eta
\end{aligned}
$$

but notice that $d(\bar{\partial}\eta) = (\partial + \bar{\partial})(\bar{\partial}\eta) = \partial\bar{\partial}\eta + \bar{\partial}\bar{\partial}\eta = \partial\bar{\partial}\eta$, so that we can write,

$$
= \sum_j \int_{Z_j} log|f_j| \wedge d(\bar{\partial}\eta)
$$

Now notice that, $d(log|f_j| \wedge \bar{\partial}\eta) = d(log|f_j|) \wedge \bar{\partial}\eta + log|f_j| \wedge d(\bar{\partial}\eta)$ which implies that $log|f_j| \wedge d(\bar{\partial}\eta) = d(log|f_j| \wedge \bar{\partial}\eta) - d(log|f_j|) \wedge \bar{\partial}\eta$. So we can split our integral,

$$
= \sum_j \int_{Z_j} d(log|f_j| \wedge \bar{\partial}\eta) - \sum_j \int_{Z_j} d(log|f_j|) \wedge \bar{\partial}\eta
$$

Now we will look at the first integral seperately. We know that $f_j \in \mathbb{C}(Z_j)$ is meromorphic so we will denote it's zero and pole sets by $\{P_j^i\}$ and $\{Q_j^i\}$ respectively. Then the pole set of $log|f_j|$ is the union of the P_j^i and Q_j^i, relabel these poles R_j^i. Notice that the R_j^i are divisors of $log|f_j|$. We define the following,

- $T_\epsilon^i(Z_j) = Z_j$ – tubular ϵ-nbhd of R_j^i
- $C_\epsilon^i(Z_j) = \partial T_\epsilon^i(Z_j)$ (i.e. boundary of $T_\epsilon^i(Z_j)$)
- $T_\epsilon(Z_j) = \cap_i T_\epsilon^i(Z_j)$

Thus we can re-express the first integral as,

$$
\sum_j \int_{Z_j} d(log|f_j| \wedge \bar{\partial}\eta) = \sum_j \lim_{\epsilon \to 0} \int_{T_\epsilon(Z_j)} d(log|f_j| \wedge \bar{\partial}\eta)
$$

Now, since $log|f_j| \wedge \bar{\partial}\eta$ is C^∞ on the surface $T_\epsilon(Z_j)$ we can apply Stokes' Theorem to get,

$$
= \sum_j \lim_{\epsilon \to 0} \int_{\partial T_\epsilon(Z_j)} log|f_j| \wedge \bar{\partial}\eta
$$

$$
= \sum_j \sum_i \lim_{\epsilon \to 0} \int_{C_\epsilon^i(Z_j)} log|f_j| \wedge \bar{\partial}\eta
$$

We notice a couple of things:

1. Without loss of generality, we can assume the divisor R_j^i is a zero of f_j since poles of f_j are simply zeroes of $1/f_j$ and so we could make a similar argument for this case.

2. WLOG, we can assume that R_j^i is a simple pole since $log|f_j|^r = rlog|f_j|$.

3. As a result of the above, f_j can be used as a coordinate about the divisor R_j^i (i.e. we have a biholomorphic relationship between the cordinate systems $(z_1, \ldots, z_n) \longleftrightarrow (f_j = z_1, \ldots, z_n)$ where R_j^i is defined precisely by $|z_1| = 0$).

It is equivalent to describe the tube $C_\epsilon^i(Z_j)$ by $|f_j| = \epsilon$ (i.e. $|z_1| = \epsilon$) so we have,

$$\begin{aligned} \left| \int_{C_\epsilon^i(Z_j)} log|f_j| \wedge \bar{\partial}\eta \right| &= \left| \int_{|z_1|=\epsilon} log|z_1| \wedge \bar{\partial}\eta \right| \\ &= \left| \int_{|z_1|=\epsilon} log(\epsilon) \wedge \bar{\partial}\eta \right| \\ &= log(\epsilon) \left| \int_{|z_1|=\epsilon} \bar{\partial}\eta \right| \end{aligned}$$

But this integral represents the volume of the ϵ-tube about the divisor R_j^i where the differential dz_1 moves around the curve $C_\epsilon^i(Z_j)$ and the differentials $dz_2, \ldots, dz_n$ move laterally along the ϵ-tube. Since we are working in compact space we know we can bound the length of the ϵ-tube by some constant M. Thus the volume is bounded by $V \leq (2\pi\epsilon)(M)$. Thus we get,

$$\begin{aligned} log(\epsilon) \left| \int_{|z_1|=\epsilon} \bar{\partial}\eta \right| &\leq log(\epsilon)(2\pi\epsilon)(M) \\ &= 2\pi M(\epsilon log\epsilon) \longrightarrow 0, \ as \ \epsilon \to 0 \end{aligned}$$

Thus $\lim_{\epsilon\to 0} \int_{C_\epsilon^i} log|f_j| \wedge \bar{\partial}\eta = 0 \ \forall i, j$ and so,

$$\sum_j \int_{Z_j} d(log|f_j| \wedge \bar{\partial}\eta) = 0$$

Thus the first integral is zero and our main calculation is reduced to,

$$= -\sum_j \int_{Z_j} d(log|f_j|) \wedge \bar{\partial}\eta$$

Now we notice that,

$$\begin{aligned} d(log|f_j|^2) \wedge \bar{\partial}\eta &= d(log|f_j| + log|f_j|) \wedge \bar{\partial}\eta \\ &= d(log|f_j|) \wedge \bar{\partial}\eta + d(log|f_j|) \wedge \bar{\partial}\eta \\ &= 2d(log|f_j|) \wedge \bar{\partial}\eta \end{aligned}$$

Thus $d(log|f_j|) \wedge \bar{\partial}\eta = 1/2d(log|f_j|^2) \wedge \bar{\partial}\eta$ and so we can re-express our integral as,

$$\begin{aligned} &= -\frac{1}{2}\sum_j \int_{Z_j} d(log|f_j|^2) \wedge \bar{\partial}\eta \\ &= -\frac{1}{2}\sum_j \int_{Z_j} d(log(f_j\bar{f}_j) \wedge \bar{\partial}\eta \\ &= -\frac{1}{2}\sum_j \int_{Z_j} d(log(f_j) + log(\bar{f}_j)) \wedge \bar{\partial}\eta \\ &= -\frac{1}{2}\sum_j \int_{Z_j} (dlog(f_j) \wedge \bar{\partial}\eta + dlog(\bar{f}_j) \wedge \bar{\partial}\eta) \\ &= -\frac{1}{2}\sum_j \left(\int_{Z_j} dlog(f_j) \wedge \bar{\partial}\eta + \int_{Z_j} dlog(\bar{f}_j) \wedge \bar{\partial}\eta\right) \\ &= -\frac{1}{2}\sum_j \left(\int_{Z_j} \frac{df_j}{f_j} \wedge \bar{\partial}\eta + \int_{Z_j} \frac{d\bar{f}_j}{\bar{f}_j} \wedge \bar{\partial}\eta\right) \end{aligned}$$

Let us look now at the second integral in the sum above. We notice that,

$$\int_{Z_j} \frac{d\bar{f}_j}{\bar{f}_j} \wedge \bar{\partial}\eta = \int_{Z_j} \overline{\frac{df_j}{f_j} \wedge \partial\eta}$$

and so we examine the integral $\int_{Z_j} \frac{df_j}{f_j} \wedge \partial\eta$. We know a couple of things,

1. $\eta \in E_{\mathbf{R}}^{n-k-1,n-k-1}(X)$ which implies that $\partial\eta \in E_{\mathbf{R}}^{n-k,n-k-1}(X)$.
2. $\frac{df_j}{f_j} = d(log(f_j)) = \partial log(f_j) + \bar{\partial} log(f_j) = \partial log(f_j) = \sum_i \frac{\partial log(f_j)}{\partial z_1} dz_1$ Thus we get, $\frac{df_j}{f_j} \wedge \partial\eta = \sum_i \frac{\partial log(f_j)}{\partial z_1} dz_1 \wedge \partial\eta \in E_{\mathbf{R}}^{n-k+1,n-k-1}(X)$.

But notice that $dim(Z_j) = n - k < n - k + 1$. So the degree of the form exceeds the dimension of the region over which we are integrating. This implies immediately that,

$$\int_{Z_j} \frac{df_j}{f_j} \wedge \partial\eta = 0$$

and thus that,

$$\int_{Z_j} \frac{d\bar{f}_j}{\bar{f}_j} \wedge \bar{\partial}\eta = 0$$

So we have again reduces our main integral,

$$= -\frac{1}{2}\sum_j \left(\int_{Z_j} \frac{df_j}{f_j} \wedge \bar{\partial}\eta \right)$$

We know from above that $\int_{Z_j} \frac{df_j}{f_j} \wedge \partial\eta = 0$ so we can write,

$$\begin{aligned}
&= -\frac{1}{2}\sum_j \left(\int_{Z_j} \frac{df_j}{f_j} \wedge \partial\eta + \int_{Z_j} \frac{df_j}{f_j} \wedge \bar{\partial}\eta \right) \\
&= -\frac{1}{2}\sum_j \left(\int_{Z_j} \frac{df_j}{f_j} \wedge (\partial\eta + \bar{\partial}\eta) \right) \\
&= -\frac{1}{2}\sum_j \left(\int_{Z_j} \frac{df_j}{f_j} \wedge (\partial + \bar{\partial})\eta \right) \\
&= -\frac{1}{2}\sum_j \left(\int_{Z_j} \frac{df_j}{f_j} \wedge d\eta \right)
\end{aligned}$$

Now we notice that $d\left(\frac{df_j}{f_j} \wedge \eta\right) = \frac{d^2 f_j}{f_j} \wedge \eta \pm \frac{df_j}{f_j} \wedge d\eta$ and thus we have $\frac{df_j}{f_j} \wedge d\eta = \pm d\left(\frac{df_j}{f_j} \wedge \eta\right)$ and so we again rewrite our integral,

$$= \pm\frac{1}{2}\sum_j \int_{Z_j} d\left(\frac{df_j}{f_j} \wedge \eta\right)$$

Here we would like to apply Stokes' Theorem as we did earlier and so using the same notaion as above we write,

$$= \pm\frac{1}{2}\sum_j \lim_{\epsilon\to 0} \int_{T_\epsilon(Z_j)} d\left(\frac{df_j}{f_j} \wedge \eta\right)$$

Now we have that $\frac{df_j}{f_j} \wedge \eta$ is C^∞ on $T_\epsilon(Z_j)$ and so applying Stoke's Theorem we get,

$$= \pm\frac{1}{2}\sum_j \lim_{\epsilon\to 0} \int_{\partial T_\epsilon(Z_j)} \frac{df_j}{f_j} \wedge \eta$$

We have from residue theory that,

$$\lim_{\epsilon\to 0} \int_{\partial T_\epsilon(Z_j)} \frac{df_j}{f_j} \wedge \eta = 2\pi i \int_{div_{Z_j} f_j} \eta$$

so our integral becomes,

$$
\begin{aligned}
&= \pm\frac{1}{2}\sum_j 2\pi i \int_{div_{Z_j} f_j} \eta \\
&= \pm\pi i \sum_j \int_{div_{Z_j} f_j} \eta \\
&= \pm\pi i \int_{\sum_j div_{Z_j} f_j} \eta \\
&= \pm\pi i \int_0 \eta \\
&= 0
\end{aligned}
$$

Therefore if $\{\omega\} \in H^{n-k,n-k}(X, \mathbf{R})$ is of the form $\omega = \partial\bar{\partial}\eta$ for some $\eta \in E_{\mathbf{R}}^{n-k-1,n-k-1}(X)$ and ξ is any element in $z^k(X,1)$ then we have that $r(\xi)(\{\omega\}) = 0$. Q.E.D.

Therefore our regulator map $r : z^k(X,1) \longrightarrow H^{k,k}(X, \mathbf{R})$ factors through cohomology and is thus well-defined.

Chapter 5

Milnor K-Theory and the Tame Symbol

In this chapter we wish to define certain cycle groups from algebraic K-Theory on which we will build a regulator into cohomology in the next chapter. The main sources for this chapter are Bass & Tate [B-T] and Lewis [Lew3].

5.1 Milnor K-Theory

5.1.1 Introduction & Definitions

Let k be any field with multiplicative group $k^{\times}$. Since any group can be considered as a $\mathbf{Z}$-module (i.e. for $n \in \mathbf{Z}$, $nx = x^n$) we can define the tensor product of $k^{\times}$ to be,

$$T(k^{\times}) = \coprod_{n \geq 0} T^n(k^{\times})$$

where $T^n(k^{\times})$ is all finite linear combinations (with scalars in $\mathbf{Z}$) of n-tensors of elements of $k^{\times}$. (In particular notice that we have the association $k^{\times} \longleftrightarrow T^1(k^{\times})$ given by $a \longleftrightarrow [a]$.)

Now, for any $a \neq 0$ *or* 1 in $k^{\times}$ we set $r_a = [a] \otimes [1-a] \in T^2(k^{\times})$. We set $\mathcal{R}$ to be the two-sided ideal in $T(k^{\times})$ generated by all such r_a's. Notice that $\mathcal{R}$ is a graded ideal generated by elements of degree 2 and so $\mathcal{R}_0 = \mathcal{R}_1 = \{0\}$, and for $n \geq 2$,

$$\mathcal{R}_n = \coprod_{P+Q=n-2} T^P(k^{\times})\mathcal{R}_2 T^Q(k^{\times})$$

Now we define the Milnor K-groups;

Definition: For any $n \geq 0$ let $K_n^M k = T^n(k^\times)/\mathcal{R}_n$.

Definition: We define,

$$K_\bullet^M k = T(k^\times)/\mathcal{R}$$

(or equivalently, $= \coprod_{n\geq 0} T^n(k^\times)/\mathcal{R}_n = \coprod_{n\geq 0} K_n^M k$).

In particular, notice that the above definitions imply that $K_0^M k = \mathbf{Z}$ and $K_1^M k = k^\times$. We will give a characterization of $K_n^M k$ for $n \geq 2$ later on.

5.1.2 Ring Structure of $K_\bullet^M k$

We have the problem that $K_\bullet^M k$ has two notions of multiplication: internal multiplication in $k^\times$ and the tensor product in $T(k^\times)$. We solve this difficulty by noticing that,

$$K_1^M k = T^1(k^\times)/\mathcal{R}_1 \simeq k^\times/\{0\} \simeq k^\times$$

and defining the map $\ell : k^\times \longrightarrow K_1^M k$ to be the canonical isomorphism $a \longmapsto \ell(a)$. By imposing the identities,

$(K1)$ $\ell(ab) = \ell(a) + \ell(b)$

$(K2)$ if $a + b = 1$ or 0 with $a, b \in k^\times$ then $\ell(a)\ell(b) = 0$

we see that $K_\bullet^M k$ is generated by the set $\{\ell(a) : a \in k^\times\}$ under the operations,

1. "multiplication": which is consistent with the tensor product in $T(k^\times)$ modded out by $\mathcal{R}$ since (K2) implies that $0 = \ell(a)\ell(1-a) \simeq [a] \otimes [1-a] \in \mathcal{R}$.
2. "addition": which is consistent with multiplication within $k^\times$. Using the logarithm structure given by (K1) we associate $\ell(a) + \ell(b) \in K_1^M k = T^1(k^\times)$ with $ab \in k^\times$.

Claim: $K_\bullet^M k$ assumes a ring structure when given the operations defined above.

Proof:

- Using (K1) and commutivity of multiplication in $k^\times$ we have,

$$\ell(a) + \ell(b) = \ell(ab) = \ell(ba) = \ell(b) + \ell(a)$$

Therefore addition is commutative.

- Using (K1) and asscoiativity of multiplication in $k^\times$ we have,

  $$(\ell(a)+\ell(b))+\ell(c)=\ell((ab)c)=\ell(a(bc))=\ell(a)+(\ell(b)+\ell(c))$$

  Therefore addition is associative.

- Using (K1) and the identity element 1 in $k^\times$ we have,

  $$\ell(1)+\ell(a)=\ell(1a)=\ell(a)$$

  Therefore $\ell(1)=0$ (i.e. is the additive identity).

- Using (K1) and the existence of inverses in $k^\times$ we have,

  $$\ell(a)+\ell(a^{-1})=\ell(aa^{-1})=\ell(1)=0$$

  Therefore $\ell(a^{-1})=-\ell(a)$ (i.e. is the additive inverse).

- Using the associativity of tensor products we have,

  $$\ell(a)(\ell(b)\ell(c))\simeq a\otimes(b\otimes c)=(a\otimes b)\otimes c\simeq(\ell(a)\ell(b))\ell(c)$$

  Therefore multiplication is associative.

- Using the linearity of tensor products we have,

  $$\ell(a)(\ell(b)+\ell(c))\simeq a\otimes(b+c)=a\otimes b+a\otimes c\simeq\ell(a)\ell(b)+\ell(a)\ell(c)$$

  $$(\ell(a)+\ell(b))\ell(c)\simeq(a+b)\otimes c=a\otimes c+b\otimes c\simeq\ell(a)\ell(c)+\ell(b)\ell(c)$$

  Therefore the distributive laws hold.

Extending these properties linearly over all elements we see that $K_\bullet^M k$ is a ring. Q.E.D.

5.1.3 Some Identities in $K_\bullet^M k$

First notice that,

$$\frac{1-a}{1-a^{-1}}=\left(\frac{1-a}{1-a^{-1}}\right)\frac{a}{a}=\frac{a-a^2}{a-1}=\frac{-a(a-1)}{a-1}=-a$$

Thus,

$$\begin{aligned}
\ell(a)\ell(-a) &= \ell(a)\ell\left(\frac{1-a}{1-a^{-1}}\right)\\
&= \ell(a)\ell((1-a)(1-a^{-1})^{-1})\\
&= \ell(a)(\ell(1-a)+\ell((1-a^{-1})^{-1}))\\
&= \ell(a)\ell(1-a)+\ell(a)\ell((1-a^{-1})^{-1})\\
&= \ell(a)(-\ell(1-a^{-1}))\\
&= (-\ell(a))\ell(1-a^{-1})\\
&= \ell(a^{-1})\ell(1-a^{-1})\\
&= 0
\end{aligned}$$

So we have the identity,

$(K3)\quad \ell(a)\ell(-a) = 0$

Using (K3) we can establish the following,

$$\begin{aligned}
\ell(a)\ell(b) + \ell(b)\ell(a) &= \ell(a)\ell(b) + \ell(a)\ell(-a) + \ell(b)\ell(a) + \ell(b)\ell(-b) \\
&= \ell(a)(\ell(b) + \ell(-a)) + \ell(b)(\ell(a) + \ell(-b)) \\
&= \ell(a)\ell(b(-a)) + \ell(b)\ell(a(-b)) \\
&= \ell(a)\ell(-ab) + \ell(b)\ell(-ab) \\
&= (\ell(a) + \ell(b))\ell(-ab) \\
&= \ell(ab)\ell(-ab) \\
&= 0
\end{aligned}$$

So we have the identity,

$(K4)\quad \ell(a)\ell(b) = -\ell(b)\ell(a)$

Thus $K_\bullet^M k$ is anti-commutative.

5.1.4 A characterization of $K_2^M k$

By definition we have that,

$$K_2^M k = T^2(k^\times)/\mathcal{R}_2$$

and we know that $T^2(k^\times)$ is generated by the set $\{a \otimes b : a, b \in k^\times\}$ and $\mathcal{R}_2$ is generated by the set $\{a \otimes (1-a) : a \in k^\times\}$. We use the symbol $\{a, b\}$ to represent the coset in $K_2^M k$ to which $a \otimes b$ belongs. Thus we can characterize $K_2^M k$ as the abelian group generated by the set,

$$\{\{a, b\} : a, b \in k^\times\}$$

and subject to the identities (called the Steinberg relations),

1. $\{aa', b\} = \{a, b\}\{a', b\}$
2. $\{a, 1-a\} = 1$
3. $\{a, b\} = \{b, a\}^{-1}$
4. $\{a, -a\} = 1$

where 1. comes from the linearity of the tensor product, 2. from identity (K2), 3. from the anti-commutivity of $K_\bullet^M k$, and 4. from identity (K3).

5.1.5 Generalization for $K_n^M k$

We say a little about $K_n^M k$ in general.

Proposition: If $a_1 + \cdots + a_n = 0$ *or* 1 for $a_1, \ldots, a_n \in k^\times$ then $\ell(a_1)\cdots\ell(a_n) = 0$.

Proof: (by induction) We know from identity ($K2$) that the proposition holds for $n = 2$. As our induction hypothesis we assume the proposition holds for any $k < n$.

Say $a_1 + \cdots + a_n = 1$. If $a_n = 1$ then notice that $a_1 + \cdots + a_{n-1} = 0$ and so by our ind. hyp. we get,

$$\ell(a_1)\cdots\ell(a_{n-1}) = 0$$

and consequently,

$$\ell(a_1)\cdots\ell(a_n) = 0$$

so that the proposition holds. So we consider the case where $a_n \neq 1$. We can write,

$$a_1 + \cdots + a_{n-1} = 1 - a_n$$

Since $1 - a_n \neq 0$ it has an inverse and so we get,

$$a_1(1-a_n)^{-1} + \cdots + a_{n-1}(1-a_n)^{-1} = 1$$

By our ind. hyp. we now have that,

$$\begin{aligned} 0 &= \ell(a_1(1-a_n)^{-1}) + \cdots + \ell(a_{n-1}(1-a_n)^{-1}) \\ &= (\ell(a_1) - \ell(1-a_n)) + \cdots + (\ell(a_{n-1}) - \ell(1-a_n)) \\ &= \ell(a_1)\cdots\ell(a_{n-1}) + \text{ other terms} \end{aligned}$$

where each of the "other terms" has at least one occurance of $\ell(1-a_n)$. We know that $\ell(a_n)\ell(1-a_n) = 0$ by identity ($K2$). Thus, multiplying the above equation by $\ell(a_n)$ makes all the "other terms" drop away to 0 giving,

$$\ell(a_1)\cdots\ell(a_n) = 0$$

as required.

A similar proof can be used to show that if $a_1 + \cdots + a_n = 1$ then $\ell(a_1)\cdots\ell(a_n) = 0$. Thus the proposition is proved by induction. Q.E.D.

Thus we have the generalized identity,

$(K1)_n$ if $a_1 + \cdots + a_n = 0$ or 1 with $a_1, \ldots, a_n \in k^\times$ then $\ell(a_1) \cdots \ell(a_n) = 0$.

As a direct result of identity (K2) we have,

$(K2)_n$ if $a_i + a_{i+1} = 1$ for some $i < n$ then $\ell(a_1) \cdots \ell(a_n) = 0$.

As a generalization of (K1) we can write,

$(K3)_n$ the map $L_n : k^\times \times \cdots \times k^\times \longrightarrow K_n k$ given by $(a_1, \ldots, a_n) \longmapsto \ell(a_1) \cdots \ell(a_n)$ is multilinear and anti-commutative so that, for any $i < n$, $L_n(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = -L_n(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$.

So we have a complete characterization of the groups $K_n^M k$:

- $K_0^M k \simeq \mathbf{Z}$
- $K_1^M k \simeq k^\times$
- for $n \geq 2$, $K_n^M k$ is an abelian group generated by the symbols $\{a_1, \ldots, a_n\} \simeq L_n(a_1, \ldots, a_n)$ and subject to the identities $(K1)_n$, $(K2)_n$, and $(K3)_n$.

5.2 κ-Algebras

We define a graded ring $\kappa = \coprod_{n \geq 0} \kappa_n$ by,

$$\kappa = \frac{\mathbf{Z}[t]}{2t\mathbf{Z}[t]}$$

Then if we let ϵ be the image of the indeterminate t in κ then we can write,

$$\kappa = \mathbf{Z}[\epsilon]$$

and we can see that,

$$\kappa_0 = \mathbf{Z}, \ \kappa_n = \mathbf{Z}_2 \epsilon^n \ for \ n \geq 1$$

Thus κ is the ring of polynomials in the variable ϵ with constant term in $\mathbf{Z}$ and higher degree terms with coefficients in $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$.

Now we recall the definition of an algebra. Say A and B are rings and let $f : A \longrightarrow B$ be a ring homomorphism. For any $a \in A$ and $b \in B$ we define,

$$ab = f(a)b$$

This definition of scalar multiplication gives B the structure of an A-module as well as a ring. These structures are compatible in the usual ways (i.e. associativity, commutivity, etc.). The ring B along with this A-module structure is said to be an A-algebra. So we see that an A-algebra is fully defined by a ring

B together with a ring homomorphism $f : A \longrightarrow B$.

In this way, we define a graded κ-algebra to be a graded ring $A = \coprod_{n \geq 0} A_n$ along with a graded ring homomorphism $\kappa \longrightarrow A$ which will be defined by,

$$\epsilon \longmapsto \epsilon_A \in A_1$$

where $\epsilon_A \in center(A)$. (Note that since we have a homomorphism this fully defines the map.) Further, we will say A is a κ-Algebra if it also satisfies the property,

$$a^2 = \epsilon_A a$$

for all $a \in A_1$.

<u>**Example**</u>: We consider the following important example of a κ-Algebra. Let k be any field. Consider the graded ring $K_\bullet^M k$ from Milnor K-Theory along with the ring homomorphism $\kappa \longrightarrow K_\bullet^M k$ defined by,

$$\epsilon \longmapsto \ell(-1) \in K_1^M k$$

We recall that we had $\ell(1) = 0$ so that,

$$2\ell(-1) = \ell((-1)^2) = \ell(1) = 0$$

Thus we have that for any $\ell(a) \in K_1^M k$,

$$\begin{aligned} & 2\ell(-1)\ell(a) = 0 \\ \Longrightarrow \quad & (\ell(-1) + \ell(-1))\ell(a) = 0 \\ \Longrightarrow \quad & \ell(-1)\ell(a) + \ell(-1)\ell(a) = 0 \\ \Longrightarrow \quad & \ell(-1)\ell(a) = -\ell(-1)\ell(a) \\ \Longrightarrow \quad & \ell(-1)\ell(a) = \ell(a)\ell(-1) \end{aligned}$$

Thus by extending over all of $K_\bullet^M k$ we have that $\ell(-1) \in center(K_\bullet^M k)$.

We also recall identity (K3) from our section on Milnor K-Theory which stated that $\ell(a)\ell(-a) = 0$ for any $\ell(a) \in K_1^M k$. Thus we have,

$$\begin{aligned} 0 &= \ell(a)\ell(-a) \\ &= \ell(a)\ell(-1 \cdot a) \\ &= \ell(a)(\ell(-1) + \ell(a)) \\ &= \ell(a)\ell(-1) + \ell(a)^2 \end{aligned}$$

$$\Longrightarrow \quad \ell(a)^2 = -\ell(a)\ell(-1)$$

$$\Longrightarrow \quad \ell(a)^2 = \ell(-1)\ell(a)$$

Thus, the graded ring $K_\bullet^M k$ along with the ring homomorphism defined by $\epsilon \longmapsto \ell(-1)$ is a κ-Algebra.

5.3 The Definition of the Tame Symbol

Say we are given a field k and let,

$$\nu : k^\times \longrightarrow \mathbf{Z}$$

be any discrete valuation on k [for more on discrete valuations see Appendix]. Given such a map we have a corresponding discrete valuation ring (DVR) defined by,

$$\mathcal{O} = \{a \in k^\times \ : \ \nu(a) \geq 0\}$$

We know that any DVR is a local ring and thus has a unique maximal ideal m. For the above DVR this maximal ideal is given by,

$$m = \{a \in k^\times \ : \ \nu(a) > 0\}$$

We know a couple of things about m,

1. any $\pi \in \mathcal{O}$ with $\nu(\pi) = 1$ generates m
2. any other non-zero ideal is of the form (π^r) for some $r \geq 0$

We pick any such $\pi \in \mathcal{O}$ (we know such a π exists since the map ν is always onto) and so $m = (\pi)$. We then have the residue field k_ν defined by,

$$k_\nu = \mathcal{O}/(\pi)$$

Now we can define a map,

$$d_\pi : k^\times \longrightarrow (K_\bullet^M k_\nu)(\ell(\pi))$$

in the following way: any $a \in k^\times$ can be written as $a = a_0\pi^i$ where $a_0 \in k^\times$ contains no powers of π and so we let,

$$d_\pi(a) = \ell(\bar{a}_0) + i\ell(\pi)$$

(where $\bar{a}_0 \in k_\nu$ is the corresponding element in the residue field). By extending d_π over K-Theory we get a map,

$$\partial_\pi : K_\bullet^M k \longrightarrow (K_\bullet^M k_\nu)(\ell(\pi))$$

We define the maps,

$$\partial_\pi^0, \partial_\nu : K_\bullet^M k \longrightarrow K_\bullet^M k_\nu$$

by,

$$\partial_\pi(x) = \partial_\pi^0(x) + \partial_\nu(x)\ell(\pi)$$

It happens that these maps are independent of the choice of generator π and that in general we have,

$$\partial_\nu : K_\bullet^M k \longrightarrow K_{\bullet-1}^M k_\nu$$

In particular we want to consider the group $K_2^M k$ which we know to be generated by the symbols,

$$\{\{a,b\} \ : \ a,b \in k^\times\}$$

subject to the Steinberg relations. Say we have $a = a_0\pi^i, b = b_0\pi^j \in k^\times$ (notice that $\nu(a) = i$ and $\nu(b) = j$). By extension of d_π to K-Theory we have,

$$\begin{aligned} \partial_\pi(\{a,b\}) &= (\ell(\bar{a}_0) + i\ell(\pi))(\ell(\bar{b}_0) + j\ell(\pi)) \\ &= \ell(\bar{a}_0)\ell(\bar{b}_0) + j\ell(\bar{a}_0)\ell(\pi) + i\ell(\pi)\ell(\bar{b}_0) + ij\ell(\pi)^2 \end{aligned}$$

We recall that $K_\bullet^M k$ is a κ-Algebra and so $\ell(\pi)^2 = \ell(-1)\ell(\pi)$. Using this and our identities from K-Theory we get,

$$\begin{aligned} \partial_\pi(\{a,b\}) &= \ell(\bar{a}_0)\ell(\bar{b}_0) + \ell(\bar{a}_0^j)\ell(\pi) + \ell(\bar{b}_0^i)\ell(\pi) + \ell((-1)^{ij})\ell(\pi) \\ &= \ell(\bar{a}_0)\ell(\bar{b}_0) + (\ell(\bar{a}_0^j) + \ell(\bar{b}_0^i) + \ell((-1)^{ij}))\ell(\pi) \end{aligned}$$

In particular we see that,

$$\begin{aligned} \partial_\nu(\{a,b\}) &= \ell(\bar{a}_0^j) + \ell(\bar{b}_0^i) + \ell((-1)^{ij}) \\ &= \ell\left((-1)^{ij}\frac{\bar{a}_0^j}{\bar{b}_0^i}\right) \\ &= \ell\left(\overline{(-1)^{ij}\frac{a_0^j}{b_0^i}}\right) \\ &= \ell\left(\overline{(-1)^{\nu(a)\nu(b)}\frac{a^\nu(b)}{b^{\nu(a)}}}\right) \end{aligned}$$

This obliges us to define the Tame symbol map,

$$T := \partial_\nu : K_2^M k \longrightarrow K_1^M k_\nu$$

by,

$$T(\{a,b\}) = (-1)^{\nu(a)\nu(b)}\left(\frac{a^{\nu(b)}}{b^{\nu(a)}}\right)$$

Chapter 6

A Real Regulator on $K_2^M \mathbf{C}(X)$

We will now define a regulator on the cycle group $K_2^M \mathbf{C}(X)$ in the case where X is a compact Reimann surface. The motivation for the definition of this regulator and the subsequent calculations can be found in [Lew3].

6.1 Definition of the Tame Symbol on $K_2^M \mathbf{C}(X)$

Let X be a projective algebraic manifold with $dim_{\mathbf{C}} X = 1$ (i.e. x is a compact Riemann surface). We consider the field $\mathbf{C}(X)$ of rational functions on X. Let D be any subvariety of X with $codim_X D = 1$. For each such subvariety we can define the discrete valuation,

$$\nu_D : \mathbf{C}(X)^{\times} \longrightarrow \mathbf{Z}$$

by,

$$\nu_D(f) = order\ of\ vanishing\ of\ f\ along\ the\ subvariety\ D$$

This information defines a Tame symbol map,

$$T_D : K_2^M \mathbf{C}(X) \longrightarrow K_1^M k_{\nu_D}$$

where k_{ν_D} is the proper residue field. What exactly is this residue field? The discrete valuation ν_D has the corresponding DVR,

$$\mathcal{O}_D = \{f \in \mathbf{C}(X)\ :\ \nu_D(f) \geq 0\}$$

with unique maximal ideal $m_D = \{f \in \mathbf{C}(X)\ :\ \nu_D(f) > 0\}$. Say we have that $D = V(f_D)$ where $f_D \in \mathbf{C}(X)$. Then it follows that $\nu_D(f_D) = 1$ and so f_D generates m_D. Thus we have that the residue field is simply,

$$k_{\nu_D} \simeq \mathcal{O}_D/(f_D) \simeq \mathbf{C}(D)$$

Thus our Tame symbol map becomes,

$$T_D : K_2^M \mathbf{C}(X) \longrightarrow K_1^M \mathbf{C}(D)$$

and is defined by,

$$T_D(\{f,g\}) = (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\right)_D$$

Combining these, we can define the Tame symbol map,

$$T : K_2^M \mathbf{C}(X) \longrightarrow \oplus_{codim_X D=1} K_1^M \mathbf{C}(D)$$

by,

$$T(\{f,g\}) = \sum_{codim_X D=1} T_D(\{f,g\})$$

Finally we notice a couple of things,

1. from K-Theory we know that $K_1^M \mathbf{C}(D) \simeq \mathbf{C}(D)^\times$
2. $codim_X D = 1$ implies that $dim_{\mathbf{C}} D = 0$ and so the D are just points in X, thus we get that $\mathbf{C}(D)^\times \simeq \mathbf{C}^\times$

So our final definition for the Tame symbol map on $K_2^M \mathbf{C}(X)$ is,

$$T : K_2^M \mathbf{C}(X) \longrightarrow \oplus_{codim_X D=1} \mathbf{C}^\times$$

and defined by,

$$T(\{f,g\}) = \sum_{codim_X D=1} (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\right)_D$$

6.2 Definition of the Regulator

We now define a regulator map from the kernel of the Tame symbol map to the first real deRham cohomology group of X. We will in fact consider r as a map on both $K_2^M \mathbf{C}(X)$ as well as the restricted domain $ker(T)$. The reason for considering r on this retricted domain will be clarified in the conclusion. We define this regulator,

$$r : ker(T) \subset K_2^M \mathbf{C}(X) \longrightarrow H^1_{dR}(X, \mathbf{R})$$

in the following manner. Let $\{f,g\} \in ker(T)$ and consider a class $\{\omega\} \in H^1_{dR}(X, \mathbf{R})$. We define,

$$r(\{f,g\}) : H^1_{dR}(X,\mathbf{R}) \longrightarrow \mathbf{R}$$

to be the map given by,

$$r(\{f,g\})(\omega) = \int_X (log|g|dlog|f| - log|f|dlog|g|) \wedge \omega$$

Thus $r(\{f,g\}) \in H^1_{dR}(X,\mathbf{R})^\vee$. Since $dim_{\mathbf{C}}X = 1$, Poincaré duality implies that $H^1_{dR}(X,\mathbf{R}) \simeq H^1_{dR}(X,\mathbf{R})^\vee$ and so $r(\{f,g\})$ is canonically associated with an element in $H^1_{dR}(X,\mathbf{R})$. This gives an implicit definition for the map r. The question now is whether r is a well-defined map.

6.3 Is r a well-defined map?

In order to show that r is well-defined we must show two things. We must show that the maps $r(\{f,g\}) : H^1_{dR}(X,\mathbf{R}) \longrightarrow \mathbf{R}$ factor through both cohomology and Milnor K-Theory.

First we will show that $r(\{f,g\})$ factors through cohomology. Recall that from Hodge Theory we have that $H^1_{dR}(X,\mathbf{R}) \simeq H^{1,0}(X) \oplus H^{0,1}(X)$ where,

$$H^{1,0}(X) = \frac{E^{1,0}_{\mathbf{R},d-closed}(X)}{\partial\bar{\partial}E^{0,-1}_{\mathbf{R}}(X)} = \frac{E^{1,0}_{\mathbf{R},d-closed}(X)}{\{0\}} = E^{1,0}_{\mathbf{R},d-closed}(X)$$

$$H^{0,1}(X) = \frac{E^{0,1}_{\mathbf{R},d-closed}(X)}{\partial\bar{\partial}E^{-1,0}_{\mathbf{R}}(X)} = \frac{E^{0,1}_{\mathbf{R},d-closed}(X)}{\{0\}} = E^{0,1}_{\mathbf{R},d-closed}(X)$$

since both $E^{0,-1}_{\mathbf{R}}(X)$ and $E^{-1,0}_{\mathbf{R}}(X)$ are empty since there are no (-1)-forms. Thus each d-closed 1-form $\omega \in E^1_{\mathbf{R}}(X)$ forms it's own class in $H^1_{dR}(X,\mathbf{R})$ and so it is trivial that r factors through cohomology.

It is more difficult to show that r factors through Milnor K-Theory. Let $f \in \mathbf{C}(X)$ with $f \neq 0, 1$. In this case recall that $\{f, 1-f\} = 0$ in $K^M_2\mathbf{C}(X)$. So for r to factor through K-Theory we must show that, for any class $\{\omega\} \in H^1_{dR}(X,\mathbf{R})$, we have $r(\{f,1-f\})(\omega) = 0$. We begin by noticing that since $f \in \mathbf{C}(X)$ we have that f is meromorphic. Recall that if $\mathbf{P}^1$ is one-dimensional complex projective space then $\mathbf{P}^1 \simeq \mathbf{C} \cup \{\infty\}$. Thus we can consider f as a holomorphic function $X \longrightarrow \mathbf{P}^1$ by associating poles with the point at infinity. If t is a local coordinate on $\mathbf{P}^1$ then we have $f = f^*(t)$ where $f^*(t)$ denotes the "pullback" of t.

$$\begin{array}{ccccc} & f & & t & \\ X & \longrightarrow & \mathbf{P}^1 & \longrightarrow & \mathbf{C} \\ \| & & & & \| \\ & & f = f^*(t) & & \\ X & & \longrightarrow & & \mathbf{C} \end{array}$$

So we can write,

$$log|1-f|dlog|f| - log|f|dlog|1-f| = f^*(log|1-t|dlog|t| - log|t|dlog|1-t|)$$

If we let,

$$\nu = log|1-f|dlog|f| - log|f|dlog|1-f|$$
$$\eta = log|1-t|dlog|t| - log|t|dlog|1-t|$$

then we write this as $\nu = f^*(\eta)$. We notice that $\nu \in E^1_{\mathbf{R},d-closed}(X)$ and $\eta \in E^1_{\mathbf{R},d-closed}(\mathbf{P}_1)$ and define a map,

$$G_f := \int_X \nu \wedge (\bullet) : E^1_{\mathbf{R},d-closed}(X) \longrightarrow \mathbf{R}$$

where $\nu = f^*(\eta)$ as above. Further, we define another map,

$$G := \int_{P^1} (\bullet) \wedge \mu : E^1_{\mathbf{R},d-closed}(\mathbf{P}_1) \longrightarrow \mathbf{R}$$

where μ is any 1-form on $\mathbf{P}^1$. Then we have,

$$\begin{aligned} r(\{f, 1-f\})(\omega) &= \int_X \nu \wedge \omega \\ &= G_f(\omega) \\ &= (f^*G)(\omega) \\ &= G(f_*(\omega)),\ \textit{via the projection formula} \end{aligned}$$

where $f_*(\omega)$ denotes the "pushforward" of ω. (It is worth noting that in this particular case the pushforward is also called the trace.) What exactly is $f_*(\omega)$? First we give a representation of ω. We know from Hodge theory that $H^{0,1}(X) = \overline{H^{1,0}(X)}$. Combining this with the earlier decomposition of $H^1_{dR}(X, \mathbf{R})$ we have,

$$H^1_{dR}(X, \mathbf{R}) \simeq H^{1,0}(X) \oplus \overline{H^{1,0}(X)} \simeq E^{1,0}_{\mathbf{R},d-closed}(X) \oplus \overline{E^{0,1}_{\mathbf{R},d-closed}(X)}$$

so that we can express $\omega \in H^1_{dR}(X, \mathbf{R})$ as $\omega = \omega^{1,0} + \overline{\omega^{1,0}}$ where $\omega^{1,0} \in H^{1,0}(X)$ and $\overline{\omega^{1,0}} \in \overline{H^{1,0}(X)}$. Let $\omega^{1,0} = f(z)dz$ for some local coordinate z. Then $\omega^{1,0}$ d-closed implies,

$$0 = d\omega^{1,0} = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz$$

and so $\partial f / \partial \bar{z} = 0$. This implies that f is holomorphic and so $\omega^{1,0}$ is a holomorphic 1-form on X. Similarly, $\overline{\omega^{1,0}} = \bar{f}(z) d\bar{z}$ d-closed implies,

$$0 = d\overline{\omega^{1,0}} = \frac{\partial \bar{f}}{\partial z} dz \wedge d\bar{z} + \frac{\partial \bar{f}}{\partial \bar{z}} d\bar{z} \wedge d\bar{z}$$

and so $\partial \bar{f} / \partial z = 0$. This implies that $\bar{f}$ is an antiholomorphic 1-form on X and so $\overline{\omega^{1,0}}$ is an antiholomorphic 1-form on X. Thus,

$$H^1_{dR}(X, \mathbf{R}) = \{holomorphic\ 1{-}forms\ on\ X\} \oplus \{antiholomorphic\ 1{-}forms\ on\ X\}$$

Now, it is well known that a pushforward always takes holomorphic forms to holomorphic forms. Thus,

$$\begin{aligned} f_*(\omega) &= f_*(\omega^{1,0} + \omega^{0,1}) \\ &= f_*(\omega^{0,1}) + f_*(\omega^{0,1}) \\ &= f_*(\omega^{0,1}) + f_*(\overline{\omega^{1,0}}) \\ &= f_*(\omega^{0,1}) + \overline{f_*(\omega^{1,0})} \end{aligned}$$

where $f_*(\omega^{0,1}) \in$ {holomorphic 1-forms on $\mathbf{P}^1$} and $\overline{f_*(\omega^{1,0})} \in$ {antiholomorphic 1-forms on $\mathbf{P}^1$}. But since $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$, $\mathbf{P}^1$ has a CW-cell structure composed of a 0-cell $e^0 = \{\infty\}$ and a 2-cell $e^2 = \mathbf{C}$. Thus, from basic cohomology theory we know,

$$H^n(\mathbf{P}^1, \mathbf{R}) = \begin{cases} \mathbf{R} & for\ n = 0, 2 \\ 0 & otherwise \end{cases}$$

In particular, $H^1(\mathbf{P}^1, \mathbf{R}) = 0$ which implies that there are no 1-forms on $\mathbf{P}^1$. Thus it must be that $f_*(\omega) = 0$. But this implies that $G(f_*(\omega)) = 0$ and thus $r(\{f, 1-f\})(\omega) = 0$ from our earlier calculation. Thus the maps $r(\{f, g\})$ factor through Milnor K-Theory.

Finally, we conclude that the regulator r is well-defined.

6.4 A Property of the Regulator r

Now that we have a well defined regulator we would like to see whether this map can tell us anything about the element in it's domain $ker(T) \subset K_2^M \mathbf{C}(X)$. It turns out that for $f, g \in \mathbf{C}(X)$ with either g or f constant then $r(\{f, g\}) = 0$. So we will see that our regulator can detect constancy in the functions determining the symbols in $ker(T) \subset K_2^M \mathbf{C}(X)$. We show this below,

Claim: Let $f, g \in \mathbf{C}(X)$ with either g or f constant. Then $r(\{f,g\}) = 0$.

Proof: We will first assume that g is a constant. Let $\{\omega\}$ be any class in $H^1_{dR}(X, \mathbf{R})$ and recall that we defined the regulator map $r : ker(T) \longrightarrow H^1_{dR}(X, \mathbf{R})$ by the formula,

$$r(\{f,g\})(\omega) = \int_X (log|g|dlog|f| - log|f|dlog|g|) \wedge \omega$$

Notice that since g is constant we have $dlog|g| = 0$ and so $log|g|dlog|f| - log|f|dlog|g| = log|g|dlog|f|$. But we also have that $d(log|g|log|f|) = log|g|dlog|f| + log|f|dlog|g|$ so that if g is constant we can say $log|g|dlog|f| - log|f|dlog|g| = d(log|g|log|f|)$. Thus we can rewrite our integral,

$$\begin{aligned} r(\{f,g\})(\omega) &= \int_X (log|g|dlog|f| - log|f|dlog|g|) \wedge \omega \\ &= \int_X d(log|g|log|f|) \wedge \omega \end{aligned}$$

Now notice that since $\omega \in H^1_{dR}(X, \mathbf{R}) \simeq E^{1,0}_{\mathbf{R},d-closed}(X) \oplus E^{0,1}_{\mathbf{R},d-closed}(X)$ we can write $\omega = \omega^{1,0} + \omega^{0,1}$ and $d\omega = d\omega^{1,0} + d\omega^{0,1} = 0$. But then we have $d(log|g|log|f| \wedge \omega) = d(log|g|log|f|) \wedge \omega + (log|g|log|f|) \wedge d\omega = d(log|g|log|f|) \wedge \omega$. Thus our integral becomes,

$$= \int_X d(log|g|log|f| \wedge \omega)$$

Now let Z_f be the set of zeroes of f and P_f be the set of poles of f. Then we let $\Sigma = Z_f \cup P_f$. The set Σ is exactly the set of poles of the function we are integrating and we define $T_\epsilon(\Sigma)$ to be the surface obtained by removing an ϵ-neighbourhood of Σ from X. Now use the facts that $X = \lim_{\epsilon \to 0} T_\epsilon(\Sigma)$ and that the surface $T_\epsilon(\Sigma)$ is smooth to apply Stoke's Theorem and rewrite our integral as,

$$\begin{aligned} &= \lim_{\epsilon \longrightarrow 0} \int_{T_\epsilon(\Sigma)} d(log|g|log|f| \wedge \omega) \\ &= \lim_{\epsilon \longrightarrow 0} \int_{\partial T_\epsilon(\Sigma)} log|g|log|f| \wedge \omega \end{aligned}$$

Finally, factoring the constant $log|g|$ out of our integral gives,

$$= log|g| \lim_{\epsilon \longrightarrow 0} \int_{\partial T_\epsilon(\Sigma)} log|f| \wedge \omega$$

Now we look a little more closely at the set $\partial T_\epsilon(\Sigma)$. Since X is one-dimensional, we know that the zeroes and poles of f are isolated points in

X so that we can say $\partial T_\epsilon(\Sigma) = \sum_{p \in \partial T_\epsilon(\Sigma)} \partial T_\epsilon(p)$ where $\partial T_\epsilon(p)$ is simply the boundary of an ϵ-neighbourhood around the point p in X. So we can write the integral as,

$$= log|g| \sum_{p \in \partial T_\epsilon(\Sigma)} \lim_{\epsilon \to 0} \int_{\partial T_\epsilon(p)} log|f| \wedge \omega$$

Locally about each point p we can use the local coordinate $z_p = f$ and thus give a local representation of the 1-form ω as $hdz_p + \bar{h}d\bar{z}_p$ where h is some holomorphic function. Thus we can rewrite our integral as,

$$\begin{aligned}
&= log|g| \sum_{p \in \partial T_\epsilon(\Sigma)} \lim_{\epsilon \to 0} \int_{|z_p|=\epsilon} log|z_p| \wedge (hdz_p + \bar{h}d\bar{z}_p) \\
&= log|g| \sum_{p \in \partial T_\epsilon(\Sigma)} \lim_{\epsilon \to 0} \int_{|z_p|=\epsilon} log(\epsilon) \wedge (hdz_p + \bar{h}d\bar{z}_p) \\
&= log|g| \sum_{p \in \partial T_\epsilon(\Sigma)} \lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} (hdz_p + \bar{h}d\bar{z}_p) \\
&= log|g| \sum_{p \in \partial T_\epsilon(\Sigma)} \left(\lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} hdz_p \right) + \\
&\qquad \left(\lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} \bar{h}d\bar{z}_p \right)
\end{aligned}$$

Now since X is a projective algebraic complex manifold we know that X is compact. But then, since h is a holomorphic function on X, we know that there must exist some upper bound $M \in R$ such that $|h| < M$. So looking at the moduli of the integrals in the above equation we see that,

$$\begin{aligned}
\lim_{\epsilon \to 0} log(\epsilon) \left| \int_{|z_p|=\epsilon} hdz_p \right| &\leq \lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} |h||dz_p| \\
&\leq \lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} M|dz_p| \\
&= \lim_{\epsilon \to 0} log(\epsilon) M \int_{|z_p|=\epsilon} |dz_p| \\
&= \lim_{\epsilon \to 0} log(\epsilon) M (2\pi i \epsilon) \\
&= 2\pi i M \lim_{\epsilon \to 0} \epsilon log(\epsilon) \\
&= 2\pi i M \lim_{\epsilon \to 0} \frac{log(\epsilon)}{1/\epsilon} \\
&= 2\pi i M \lim_{\epsilon \to 0} \frac{1/\epsilon}{-1/\epsilon^2}, \textit{ by Hopitâl's rule}
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi i M \lim_{\epsilon \to 0}(-\epsilon) \\
&= 2\pi i M(0) \\
&= 0
\end{aligned}
$$

A similar calculation will show that $\lim_{\epsilon\to 0} log(\epsilon)\left|\int_{|z_p|=\epsilon} \bar{h} d\bar{z}_p\right| \leq 0$. Thus we get that the integrals

$$\lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} h dz_p$$

and

$$\lim_{\epsilon \to 0} log(\epsilon) \int_{|z_p|=\epsilon} \bar{h} d\bar{z}_p$$

are both zero. So our integral becomes,

$$
\begin{aligned}
&= log|g| \sum_{p \in \partial T_\epsilon(\Sigma)} (0+0) \\
&= 0
\end{aligned}
$$

Thus we have shown that if g is constant then $r(\{f,g\}) = 0$. A similar calculation shows that the same holds if f is constant. Q.E.D.

So, after a rather long calculation, we have seen that our regulator r is capable of detecting constancy in the functions $f, g \in \mathbf{C}(X)^\times$. This suggests that the regulator is sensitive enough to detect certain properties of the elements in it's domain.

6.5 A Method for Building forms in $H^1_{dR}(X, \mathbf{R})$

In order to actually work with our regulator $r : K_2^M \mathbf{C}(X) \to H^1_{dR}(X, \mathbf{R})$ we need to be able to find explicit elements in cohomology. In this section we develop a method of finding such forms.

We begin by recalling a bit about residues. Say we have a holomorphic function $f(z) : U \subset \mathbf{C} \to \mathbf{C}$ where U is some open subset of $\mathbf{C}$. We recall the Cauchy Integral Formula from single-variable complex analysis which states, for any fixed $p \in U$ and $\epsilon > 0$,

$$\int_{|z-p|=\epsilon} \frac{f(z)}{(z-p)} dz = 2\pi\sqrt{-1} f(p)$$

Since we know $dz = d(z-p)$ we can re-express the above formula as,

$$lim_{\epsilon \to 0} \int_{|z-p|=\epsilon} f(z) \frac{d(z-p)}{(z-p)} = 2\pi\sqrt{-1} f(z) \mid_{z-p=0}$$

From this we define the idea of a residue,

Definition: The residue of the meromorphic 1-form $\eta = f(z)\frac{d(z-p)}{(z-p)}$ at the point $z = p$ is given by,

$$Res_{z=p}\eta = f(z) \mid_{z-p=0} = f(p)$$

We'd also like to extend this idea in order to define residues of meromorphic 2-forms.

Now say we have a holomorphic function $f(x, y) : U \subset \mathbf{C}^2 \to \mathbf{C}$ where U is some open subset of $\mathbf{C}^2$. Then we get that,

$$\eta = \frac{dx \wedge dy}{f}$$

is a meromorphic 2-form on U. In order to rewrite the 2-form η we notice a few things. Firstly we know,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Using this and the fact that $dx \wedge dx = dy \wedge dy = 0$ we can write,

$$dx \wedge df = \frac{\partial f}{\partial y} dx \wedge dy$$

and,

$$-dy \wedge df = df \wedge dy = \frac{\partial f}{\partial x} dx \wedge dy$$

Thus we can write $dx \wedge dy$ in two ways,

$$dx \wedge dy = \frac{dx \wedge df}{\partial f / \partial y} = -\frac{dy \wedge df}{\partial f / \partial x}$$

From this we see we can express η in two ways,

$$\eta = \frac{dx}{\partial f / \partial y} \wedge \frac{df}{f} = -\frac{dy}{\partial f / \partial x} \wedge \frac{df}{f}$$

We now can make the definition,

Definition: The residue of the meromorphic 2-form $\eta = \frac{dx \wedge dy}{f}$ along $f = 0$ is given by,

$$Res_{f=0}\eta = \frac{dx}{\partial f/\partial y}\mid_{f=0} = -\frac{dy}{\partial f/\partial x}\mid_{f=0}$$

Notice that $Res_{f=0}\eta$ is a 1-form defined on the curve $f = 0$. In fact, we have the following,

Proposition: The 1-form $Res_{f=0}\eta$ is holomorphic on the curve $f = 0$ provided that $\nabla f(p) \neq (0,0)$ whenever $f(p) = 0$.

Proof: Say we have $\partial f/\partial y \neq 0$. Then,

$$\begin{aligned}
\bar{\partial} Res_{f=0}\eta &= \bar{\partial}\left(\frac{dx}{\partial f/\partial y}\mid_{f=0}\right) \\
&= \frac{\partial}{\partial \bar{x}}\left(\frac{1}{\partial f/\partial y}\right) dx \wedge d\bar{x} + \frac{\partial}{\partial \bar{y}}\left(\frac{1}{\partial f/\partial y}\right) dx \wedge d\bar{y} \\
&= \left(\frac{-1}{(\partial f/\partial y)^2}\right)\left(\frac{\partial^2 f}{\partial \bar{x}\partial y}\right) dx \wedge d\bar{x} + \left(\frac{-1}{(\partial f/\partial y)^2}\right)\left(\frac{\partial^2 f}{\partial \bar{y}\partial y}\right) dx \wedge d\bar{y} \\
&= \left(\frac{-1}{(\partial f/\partial y)^2}\right)\left[dx \wedge \left(\frac{\partial}{\partial \bar{x}}\left(\frac{\partial f}{\partial y}\right) d\bar{x} + \frac{\partial}{\partial \bar{y}}\left(\frac{\partial f}{\partial y}\right) d\bar{y}\right)\right] \\
&= \left(\frac{-1}{(\partial f/\partial y)^2}\right)\left[dx \wedge \bar{\partial}\left(\frac{\partial f}{\partial y}\right)\right]
\end{aligned}$$

But since f is holomorphic we know that $\bar{\partial}\left(\frac{\partial f}{\partial y}\right) = 0$. Thus we have,

$$\bar{\partial} Res_{f=0}\eta = \left(\frac{-1}{(\partial f/\partial y)^2}\right)[dx \wedge 0] = 0$$

Thus $Res_{f=0}\eta$ is holomorphic on the curve $f = 0$. If we instead have $\partial f/\partial x \neq 0$ then we use the expression $Res_{f=0}\eta = \frac{-dy}{\partial f/\partial x}$ and the above proof holds similarly. Q.E.D.

We are now ready to begin building forms in $H^1_{dR}(X, \mathbf{R})$. We begin with an illustrative example and then generalize this to get our method.

Example: Consider the homogeneous polynomial $F(z_0, z_1, z_2) = z_0 z_2^2 - z_1^3 - z_0^2 z_1 - z_0^3$ and let,

$$X = \{[z_0, z_1, z_2] \in \mathbf{P}^2 \ : \ F(z_0, z_1, z_2) = 0\}$$

Notice that X is smooth and $dim_{\mathbf{C}} X = 1$ so that we have that X is a compact Reimann curve (i.e. projective algebraic manifold of dimension 1).

We have local coordinates on X (when $z_0 \neq 0$) given by,

$$x = \frac{z_1}{z_0},\ y = \frac{z_2}{z_0}$$

and a local affinization $f(x,y) : \mathbf{C}^2 \to \mathbf{C}$ of F given by,

$$\begin{aligned} f(x,y) &= F/z_0^3 \\ &= \left(\frac{z_2}{z_0}\right)^2 - \left(\frac{z_1}{z_0}\right)^3 - \left(\frac{z_1}{z_0}\right) - 1 \\ &= y^2 - x^3 - x - 1 \end{aligned}$$

We see that f is certainly holomorphic. We have the meromorphic 2-form on X,

$$\eta = \frac{dx \wedge dy}{f}$$

and so,

$$\omega = Res_{f=0}\eta = \frac{dx}{2y} = \frac{dy}{3x^3+1}$$

is a holomorphic 1-form on the curve $f = 0$, which is exactly X. We notice a couple of things about ω,

1. $\bar{\partial}\omega = 0$ since ω is holomorphic
2. $\partial\omega = 0$ since $\partial\omega \in E^{2,0}_{\mathbf{C}}(X)$ and we know that $E^{2,0}_{\mathbf{C}}(X) = 0$ since $dim_{\mathbf{C}}X = 1$

Thus we have that $d\omega = 0$ (i.e. ω is d-closed) and so,

$$\omega \in H^{1,0}(X) \simeq E^{1,0}_{\mathbf{C},d-closed}(X)$$

Now, if we let $\nu = \omega + \overline{\omega}$ then we have,

$$\nu \in H^1_{dR}(X, \mathbf{C})$$

since we know that $H^1_{dR}(X,\mathbf{C}) \simeq H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$. Further, we notice that,

$$\begin{aligned} \overline{\nu} &= \overline{(\omega + \overline{\omega})} \\ &= \overline{\omega} + \omega \\ &= \omega + \overline{\omega} \\ &= \nu \end{aligned}$$

Thus ν is closed under conjugation which implies that ν is in fact a real-valued from. Thus we have,

$$\nu \in H^1_{dR}(X, \mathbf{R})$$

as we wanted.

This method generalizes as follows: Say we are given a compact Riemann curve X. We can always find a homogeneous polynomial $F(z_0, z_1, z_2)$ of degree $d \geq 3$ such that,

$$X = \{[z_0, z_1, z_2] \in \mathbf{P}^2 \ : \ F(z_0, z_1, z_2) = 0\}$$

We let $f(x, y) = F/z_0^d$ and $\eta = \frac{dx \wedge dy}{f}$. Then $\omega = Res_{f=0}\eta$ is a holomorphic 1-form on the curve X and,

$$\omega + \overline{\omega} \in H^1_{dR}(X, \mathbf{R})$$

This gives us an explicit way to build elements of $H^1_{dR}(X, \mathbf{R})$ for any compact Riemann curve X.

6.6 A Further Property of the Regulator r

In this section we wish to show that we can find explicit instances where our regulator r is non-zero. This will indicate that in general our regulator is non-trivial and is capable of giving us useful information about the elements in it's domain.

Let X be a compact Riemann surface defined by,

$$X = \{[z_0, z_1, z_2] \in \mathbf{P}^2 \ : \ F(z_0, z_1, z_2) = 0\}$$

where $F : \mathbf{C}^3 \longrightarrow \mathbf{C}$ is some homogeneous polynomial of degree $d \geq 3$ which can be represented in the following form: we fix distinct $t_1, \ldots t_{d-1} \in \mathbf{R}^\times$ and $\lambda \in \mathbf{C}$ and write,

$$F_{\vec{t},\lambda}(z_0, z_1, z_2) = \left(\prod_{j=1}^{d-1} (z_0 - t_j z_2) \right) (z_1 + z_2 + z_0\sqrt{-1}) + \lambda(z_0^d + z_1^d + z_2^d)$$

where $\vec{t} = (t_1, \ldots, t_{d-1})$. We also define,

$$F_{\vec{t}} = F_{\vec{t},0} = \left(\prod_{j=1}^{d-1} (z_0 - t_j z_2) \right) (z_1 + z_2 + z_0\sqrt{-1})$$

We can consider local coordinates on X (when $z_0 \neq 0$),

$$x = \frac{z_1}{z_0}, \ y = \frac{z_2}{z_0}$$

and can then define,

$$h_{\vec{t},\lambda}(x,y) = F_{\vec{t},\lambda}/z_0^d$$

and,

$$h_{\vec{t}}(x,y) = h_{\vec{t},0}(x,y) = F_{\vec{t}}/z_0^d = \left(\prod_{j=1}^{d-1}(1-t_j y)\right)(x+y+\sqrt{-1})$$

Now, using our method from the previous section we have the form,

$$\omega_{\vec{t},\lambda} = \left(\frac{dx}{\partial h_{\vec{t},\lambda}/\partial y} + \overline{\frac{dx}{\partial h_{\vec{t},\lambda}/\partial y}}\right) \in H^1_{dR}(X,\mathbf{R})$$

with alternate discription,

$$\omega_{\vec{t},\lambda} = -\left(\frac{dy}{\partial h_{\vec{t},\lambda}/\partial x} + \overline{\frac{dy}{\partial h_{\vec{t},\lambda}/\partial x}}\right)$$

Notice that since x and y are local coordinates on X we know $x, y \in \mathbf{C}(X)^\times$.

We will be interested in the calculation,

$$r(\{x,y\})(\omega_{\vec{t},\lambda}) = \int_X (log|y|dlog|x| - log|x|dlog|y|) \wedge \omega_{\vec{t},\lambda}$$

which we will show is non-zero. To simplify our calculations we will first let $\lambda \longrightarrow 0$ and show that,

$$r(\{x,y\})(\omega_{\vec{t}}) \neq 0$$

(where $\omega_{\vec{t}} = \omega_{\vec{t},0}$) and then go on to argue that the result still holds for a general $\lambda \neq 0$.

Using the fact that $log|x| = log(\sqrt{x\bar{x}})$ and $log|y| = log(\sqrt{y\bar{y}})$ we see that,

1. $2dlog|x| = \frac{dx}{x} + \frac{d\bar{x}}{\bar{x}}$
2. $2dlog|y| = \frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}$

This allows us to re-express our regulator calculation as,

$$r(\{x,y\})(\omega_{\vec{t}}) = \frac{1}{2}\int_X log|y|\left(\frac{dx}{x} + \frac{d\bar{x}}{\bar{x}}\right) \wedge \omega_{\vec{t}} - \frac{1}{2}\int_X log|y|\left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \omega_{\vec{t}}$$

We will need the following,

<u>**Proposition**</u>: Let X, x, y, and $\omega_{\vec{t},\lambda}$ be as above. Then,

$$\int_X d(log|y|log|x|\omega_{\vec{t},\lambda}) = 0$$

for any choice of $\vec{t}$ and λ.

__Proof__: Let $T_\epsilon(X)$ be the surface obtained by removing ϵ-tubes from X about all the zeroes and poles of x and y. Then $log|y|log|x|\omega_{\vec{t},\lambda}$ is a smooth form on $T_\epsilon(X)$ and so applying Stokes' gives,

$$\begin{aligned}\int_X d(log|y|log|x|\omega_{\vec{t},\lambda}) &= lim_{\epsilon\to 0}\int_{T_\epsilon(X)} d(log|y|log|x|\omega_{\vec{t},\lambda}) \\ &= lim_{\epsilon\to 0}\int_{\partial T_\epsilon(X)} log|y|log|x|\omega_{\vec{t},\lambda}\end{aligned}$$

Notice that $\partial T_\epsilon(X)$ is composed of the boudaries of the ϵ-tubes about each of the zeroes/poles of x and y. Take a particular zero of y and let ∂Z_ϵ denote the boundary of the ϵ-tube about this zero. Since $y \in \mathbf{C}(X)^\times$, y can be used as a local coordinate with ∂Z_ϵ defined by $|y| = \epsilon$. Thus we have,

$$\begin{aligned}\left|\int_{\partial Z_\epsilon} log|y|log|x|\omega_{\vec{t},\lambda}\right| &= \left|\int_{|y|=\epsilon} log(\epsilon)log|x|\omega_{\vec{t},\lambda}\right| \\ &= \left|log(\epsilon)\int_{|y|=\epsilon} log|x|\omega_{\vec{t},\lambda}\right|\end{aligned}$$

We know that, for small enough ϵ, $log|x|$ is bounded on $|y| = \epsilon$ (i.e. $log|x| \leq M$ for some $0 < M < \infty$). Thus,

$$\begin{aligned}&\leq log(\epsilon)\left|\int_{|y|=\epsilon} M\omega_{\vec{t},\lambda}\right| \\ &= log(\epsilon)M\left|\int_{|y|=\epsilon} \omega_{\vec{t},\lambda}\right|\end{aligned}$$

This integral represents a volume of the ϵ-tube Z_ϵ which is bounded by $(2\pi\epsilon)(N)$ for some $N > 0$. Thus,

$$\begin{aligned}&\leq log(\epsilon)M(2\pi\epsilon)(N) \\ &= 2\pi MN\epsilon log(\epsilon) \to 0, \ as\ \epsilon \to 0\end{aligned}$$

Similar calculations show the same result would hold if we had chosen a pole of y or a zero or pole of x. This implies that,

$$lim_{\epsilon \to 0} \int_{T_\epsilon(X)} d(log|y|log|x|\omega_{\vec{t},\lambda}) = 0$$

and so our result follows as required. Q.E.D.

Using the proposition above we see (for the case $\lambda = 0$),

$$\begin{aligned}
0 &= \int_X d(log|y|log|x|\omega_{\vec{t}}) \\
&= \int_X d(log|y|log|x|) \wedge \omega_{\vec{t}} + \int_X (log|y|log|x|) \wedge d\omega_{\vec{t}} \\
&= \int_X (log|y|dlog|x| + log|x|dlog|y|) \wedge \omega_{\vec{t}} + \int_X (log|y|log|x|) \wedge 0 \\
&= \int_X log|y| \left(\frac{dx}{x} + \frac{d\bar{x}}{\bar{x}}\right) \wedge \omega_{\vec{t}} + \int_X log|x| \left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \omega_{\vec{t}}
\end{aligned}$$

Thus,

$$\int_X log|y| \left(\frac{dx}{x} + \frac{d\bar{x}}{\bar{x}}\right) \wedge \omega_{\vec{t}} = -\int_X log|x| \left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \omega_{\vec{t}}$$

This allows us to express our regulator calculation as,

$$r(\{x,y\})(\omega_{\vec{t}}) = \int_X log|y| \left(\frac{dx}{x} + \frac{d\bar{x}}{\bar{x}}\right) \wedge \omega_{\vec{t}} \tag{6.1}$$

or equivalently as,

$$r(\{x,y\})(\omega_{\vec{t}}) = -\int_X log|x| \left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \omega_{\vec{t}} \tag{6.2}$$

Now we look more closely at the surface X. Using the properties of of projective algebraic subsets and the definition of $h_{\vec{t}}$ we get,

$$X = V(h_{\vec{t}}) = \left[\bigcup_{j=1}^{d-1} V(1 - t_j y)\right] \cup V(x + y + \sqrt{-1})$$

So integration over X splits into a sum of integrals over these varieties. For any $j = 1, \ldots, d-1$, we see that along $V(1 - t_j y)$ we have $1 - t_j y = 0$ and thus $y = t_j^{-1} \in \mathbf{R}^\times$. So along any of the $V(1 - t_j y)$ we have $dy = 0 = d\bar{y}$. But then, substituting $V(1 - t_j y)$ for X in equation (6.2) we get,

$$-\int_{V(1-t_j y)} log|x| \left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \omega_{\vec{t}} = 0$$

Thus our regulator calculation is reduced to integrating over the variety,

$$V(x+y+\sqrt{-1}) = \{(x,y) \in \mathbf{C} : x+y+\sqrt{-1} = 0\}$$

which defines a line in $\mathbf{C}^2$ which implies that $V(x+y+\sqrt{-1}) \simeq \mathbf{C}$. Thus our regulator calculation now becomes,

$$r(\{x,y\})(\omega_{\vec{t}}) = -\int_{\mathbf{C}} log|x| \left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \omega_{\vec{t}}$$

over $x+y+\sqrt{-1} = 0$.

We now need a more explicit equation for the form $\omega_{\vec{t}}$. First we calculate,

$$\frac{\partial h_{\vec{t}}}{\partial y} = \prod_{j=1}^{d-1}(1-t_j y) - \left[\sum_{i=1}^{d-1} t_i \left(\prod_{j=1, j\neq i}^{d-1} (1-t_j y)\right)\right](x+y+\sqrt{-1})$$

But notice that on $V(x+y+\sqrt{-1})$ this simplifies to,

$$\frac{\partial h_{\vec{t}}}{\partial y} = \prod_{j=1}^{d-1}(1-t_j y)$$

Using this and the fact that along $V(x+y+\sqrt{-1})$ we have $dx = -dy$ we can write,

$$\begin{aligned}
\omega_{\vec{t}} &= \left(\frac{dx}{\prod_{j=1}^{d-1}(1-t_j y)} + \overline{\frac{dx}{\prod_{j=1}^{d-1}(1-t_j y)}}\right) \\
&= \left(\frac{-dy}{\prod_{j=1}^{d-1}(1-t_j y)} + \overline{\frac{-dy}{\prod_{j=1}^{d-1}(1-t_j y)}}\right) \\
&= \left(\frac{-dy}{\prod_{j=1}^{d-1}(1-t_j y)} + \frac{-d\bar{y}}{\prod_{j=1}^{d-1}(1-t_j \bar{y})}\right)
\end{aligned}$$

along $V(x+y+\sqrt{-1})$. Thus we are interested in the following calculation over $x+y+\sqrt{-1} = 0$ (we will drop the idexing on the products as $j = 1, \ldots, d-1$ will always be implied),

$$\begin{aligned}
& r(\{x,y\})(\omega_{\vec{t}}) \\
= & -\int_{\mathbf{C}} log|x| \left(\frac{dy}{y} + \frac{d\bar{y}}{\bar{y}}\right) \wedge \left(\frac{-dy}{\prod(1-t_j y)} + \frac{-d\bar{y}}{\prod(1-t_j \bar{y})}\right) \\
= & \left[\int_{\mathbf{C}} \frac{log|x|}{y\prod(1-t_j\bar{y})} dy \wedge d\bar{y}\right] + \left[\int_{\mathbf{C}} \frac{log|x|}{\bar{y}\prod(1-t_j y)} d\bar{y} \wedge dy\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left[\int_{\mathbf{C}} \frac{log|y+\sqrt{-1}|}{y\prod(1-t_j\bar{y})} dy \wedge d\bar{y}\right] - \left[\int_{\mathbf{C}} \frac{log|y+\sqrt{-1}|}{\bar{y}\prod(1-t_j y)} dy \wedge d\bar{y}\right] \\
&= \int_{\mathbf{C}} log|y+\sqrt{-1}| \left(\frac{1}{y\prod(1-t_j\bar{y})} - \frac{1}{\bar{y}\prod(1-t_j y)}\right) dy \wedge d\bar{y} \\
&= \int_{\mathbf{C}} log|y+\sqrt{-1}| \left(\frac{1}{y\prod(1-t_j\bar{y})} - \overline{\frac{1}{y\prod(1-t_j\bar{y})}}\right) dy \wedge d\bar{y}
\end{aligned}
$$

But notice that for any $z = Re(z) + Im(z)\sqrt{-1} \in \mathbf{C}$ we have $z - \bar{z} = (Re(z) + Im(z)\sqrt{-1}) - (Re(z) - Im(z)\sqrt{-1}) = 2\sqrt{-1}Im(z)$ so that we can write,

$$
= 2\sqrt{-1}\int_{\mathbf{C}} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) dy \wedge d\bar{y}
$$

Now notice that if we denote $y = z_1 + z_2\sqrt{-1}$ then $dy = dz_1 + dz_2\sqrt{-1}$ and so $dy \wedge d\bar{y} = 2\sqrt{-1}dz_1 \wedge dz_2$. So we can integrate over the real Euclidean variables z_1 and z_2.

$$
\begin{aligned}
&= 2\sqrt{-1}\int_{\mathbf{C}} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) 2\sqrt{-1}dz_1 \wedge dz_2 \\
&= -4\int_{\mathbf{C}} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) dz_1 \wedge dz_2
\end{aligned}
$$

We denote the half planes,

$$
\mathbf{H}^+ = \{z \in \mathbf{C} : Im(z) > 0\}
$$

$$
\mathbf{H}^- = \{z \in \mathbf{C} : Im(z) < 0\}
$$

and so we can write $\mathbf{C} = \mathbf{H}^+ \cup \mathbf{H}^- \cup \mathbf{R}$. But integrating over $\mathbf{R}$ gives zero so we can re-express our regulator calculation,

$$
\begin{aligned}
= & -4\int_{\mathbf{H}^+} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) dz_1 \wedge dz_2 \\
& -4\int_{\mathbf{H}^-} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) dz_1 \wedge dz_2
\end{aligned}
$$

But notice that $\mathbf{H}^- = \overline{\mathbf{H}^+}$ so we have,

$$
\begin{aligned}
= & -4\int_{\mathbf{H}^+} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) dz_1 \wedge dz_2 \\
& -4\int_{\mathbf{H}^+} log|\bar{y}+\sqrt{-1}|Im\left(\overline{\frac{1}{y\prod(1-t_j\bar{y})}}\right) dz_1 \wedge dz_2
\end{aligned}
$$

$$
\begin{aligned}
&= -4\int_{\mathbf{H}^+} log|y+\sqrt{-1}|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)dz_1\wedge dz_2 \\
&\qquad -4\int_{\mathbf{H}^+} log|\bar{y}+\sqrt{-1}|\left[-Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)\right]dz_1\wedge dz_2 \\
&= -4\int_{\mathbf{H}^+}\left[log|y+\sqrt{-1}|-log|\bar{y}+\sqrt{-1}|\right]Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)dz_1\wedge dz_2 \\
&= -4\int_{\mathbf{H}^+} log\left|\frac{y+\sqrt{-1}}{\bar{y}+\sqrt{-1}}\right|Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)dz_1\wedge dz_2
\end{aligned}
$$

We notice the following interesting fact,

$$
\begin{aligned}
log\left|\frac{y+\sqrt{-1}}{\bar{y}+\sqrt{-1}}\right|>0 &\iff \left|\frac{y+\sqrt{-1}}{\bar{y}+\sqrt{-1}}\right|>1 \\
&\iff |y+\sqrt{-1}|^2>|\bar{y}+\sqrt{-1}|^2 \\
&\iff (y+\sqrt{-1})\overline{(y+\sqrt{-1})}>(\bar{y}+\sqrt{-1})\overline{(\bar{y}+\sqrt{-1})} \\
&\iff (y+\sqrt{-1})(\bar{y}-\sqrt{-1})>(\bar{y}+\sqrt{-1})(y-\sqrt{-1}) \\
&\iff y\bar{y}+(\bar{y}-y)\sqrt{-1}+1>y\bar{y}+(y-\bar{y})\sqrt{-1}+1 \\
&\iff -(y-\bar{y})\sqrt{-1}>(y-\bar{y})\sqrt{-1} \\
&\iff -2\sqrt{-1}(y-\bar{y})>0 \\
&\iff -2\sqrt{-1}(2\sqrt{-1}Im(y))>0 \\
&\iff 4Im(y)>0 \\
&\iff Im(y)>0 \\
&\iff y\in\mathbf{H}^+
\end{aligned}
$$

So we see that on $\mathbf{H}^+$ we have that $log\left|\frac{y+\sqrt{-1}}{\bar{y}+\sqrt{-1}}\right|>0$.

Now we examine $Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)$. By letting $\vec{t}\to 0$ we see that,

$$
\begin{aligned}
\lim_{\vec{t}\to 0}\left[Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)\right] &= Im\left(\frac{1}{y}\right) \\
&= Im\left(\frac{\bar{y}}{y\bar{y}}\right) \\
&= Im\left(\frac{\bar{y}}{|y|^2}\right) \\
&= \frac{Im(\bar{y})}{|y|^2} \\
&= \frac{-Im(y)}{|y|^2} \\
&< 0
\end{aligned}
$$

on $\mathbf{H}^+$.

Combining these last two results tells us that,

$$-4\int_{\mathbf{H}^+} lim_{\vec{t}\to 0}\left[log\left|\frac{y+\sqrt{-1}}{\bar{y}+\sqrt{-1}}\right| Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right)\right] dz_1\wedge dz_2 > 0$$

since our integrand is strictly negative on all of $\mathbf{H}^+$. But since we can see that the integrand varies continuously with $\vec{t}$ we can say that,

$$lim_{\vec{t}\to 0}\left[-4\int_{\mathbf{H}^+} log\left|\frac{y+\sqrt{-1}}{\bar{y}+\sqrt{-1}}\right| Im\left(\frac{1}{y\prod(1-t_j\bar{y})}\right) dz_1\wedge dz_2\right] > 0$$

which immediately implies that,

$$lim_{\vec{t}\to 0}\left[r(\{x,y\})(\omega_{\vec{t}})\right] > 0$$

In particular we can conclude that $r(\{x,y\})(\omega_{\vec{t}}) \neq 0$ when $\vec{t}$ is close to 0. But since the map $\vec{t} \longmapsto r(\{x,y\})(\omega_{\vec{t}})$ is rational, one can argue that this implies that $r(\{x,y\})(\omega_{\vec{t}})$ is non-zero almost-everywhere by using a continuity argument.

Using the fact that $\omega_{\vec{t},\lambda}$ varies continuously with λ, a similar argument will show that $r(\{x,y\})(\omega_{\vec{t},\lambda}) \neq 0$ for $\vec{t}$ and λ close to zero. One may in fact argue that this result holds true for all $\lambda \in \mathbf{C}$ away from a set of discrete points since otherwise the regulator would have to be zero for all $\lambda \in \mathbf{C}$. So we can reasonabley expect the regulator to be non-zero for "almost-all" λ.

This demonstrates that the regulator $r : K_2^M\mathbf{C}(X) \to H^1(X,\mathbf{R})$ is non-trivial. We will see in the conclusion a way of showing that r is in fact non-trivial on the restricted domain $ker(T)$ as well.

Chapter 7

Conclusion

We would like to conclude with a few remarks on how our results fit into the larger picture of the study of algebraic cycles.

For any projective algebraic manifold X and $k \geq 0$, the following sequence of sheaves is known to be exact,

$$\begin{array}{ccccccc}
\mathcal{K}^M_{k,X} & \longrightarrow & K^M_k\mathbf{C}(X) & \longrightarrow & \oplus_{cd_X Z=1} K^M_{k-1}\mathbf{C}(Z) & \longrightarrow \\
& & & \tilde{T} & & T \\
\oplus_{cd_X Z=2} K^M_{k-2}\mathbf{C}(Z) & \longrightarrow & \cdots & \longrightarrow & \oplus_{cd_X Z=k-2} K^M_2\mathbf{C}(Z) & \longrightarrow \\
& div & & i & & \\
\oplus_{cd_X Z=k-1} K^M_1\mathbf{C}(Z) & \longrightarrow & \oplus_{cd_X Z=k} K^M_0\mathbf{C}(Z) & \longrightarrow & 0 & \\
& & \| & & & \\
& & z^k(X) & & &
\end{array}$$

where div is the divisor map, T is the Tame symbol map, $\tilde{T}$ is a higher Tame symbol map (that we did not define), and $\mathcal{K}^M_{k,X}$ is the Milnor sheaf [Lew2, 277-278]. By applying global sections this exact sequence becomes a chain complex. This induces a sheaf cohomology theory,

$$H^{k-m}_{Zar}(X, \mathcal{K}^M_{k,X}) := H^{k-m}_{Zar}(\Gamma(exact\ seq.))$$

where the cohomology groups are called the Chow groups and are denoted by $CH^k(X,m)$. For $0 \leq m \leq 2$ we have the definitions,

- $CH^k(X) = CH^k(X,0) = \frac{ker(i)}{im(div)} \simeq \frac{z^k(X)}{im(div)}$
- $CH^k(X,1) = \frac{ker(div)}{im(T)}$
- $CH^k(X,2) = \frac{ker(T)}{im(\tilde{T})}$

As a point of interest it happens that, for $0 \leq m \leq 2$, the Chow groups of X are related to the K-Theory of X,

$$K_m(X) \longleftrightarrow CH^k(X, m)$$

by the famous Riemann-Roch Theorem which states,

$$K_m(X) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \oplus_{k \geq 0} CH^k(X, m) \otimes_{\mathbf{Z}} \mathbf{Q}$$

when $0 \leq m \leq 2$.

In particular we have been interested in the case where X is a compact Riemann surface (i.e. $dim_{\mathbf{C}} X = 1$) and where $k = 2$. Then our exact sequence becomes,

$$\begin{array}{ccccccccc}
 & \tilde{T} & & T & & div & & & \\
0 & \rightarrow & K_2^M \mathbf{C}(X) & \rightarrow & \oplus_{cd_X Z=1} K_1^M \mathbf{C}(Z) & \rightarrow & \oplus_{cd_X Z=2} K_0^M \mathbf{C}(Z) & \rightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & & \simeq & & & & & \\
0 & \rightarrow & K_2^M \mathbf{C}(X) & \rightarrow & \oplus_{cd_X Z=1} \mathbf{C}^\times & \rightarrow & 0 & \rightarrow & 0
\end{array}$$

So in this particular case the Chow groups are given by,

- $CH^2(X) = 0$
- $CH^2(X, 1) = 0$
- $CH^2(X, 2) = ker(T)$

This means that our regulator from Chapter 6, when restricted to $ker(T) \subset K_2^M \mathbf{C}(X)$, can be written as,

$$r : CH^2(X, 2) \longrightarrow H^1_{dR}(X, \mathbf{R})$$

So this regulator can be considered as working on two different cycle groups: firstly, on $K_2^M \mathbf{C}(X)$; secondly, on the restricted domain $CH^2(X, 2)$.

Consider for a moment our result from §6.6, where we found $f, g \in \mathbf{C}(X)^\times$ (namely the local coordinates x and y) and a form $\omega \in H^1_{dR}(X, \mathbf{R})$ such that,

$$r(\{f, g\})(\omega) \neq 0$$

But here we only have $\{f, g\} \in K_2^M \mathbf{C}(X)$ which does not necessarily mean that $\{f, g\}$ will define a class in $CH^2(X, 2)$. For this we also need that $T(\{f, g\}) = 0$. However, if we could find $c_i \in \mathbf{C}$ and $h_i \in \mathbf{C}(X)^\times$ for $i = 1, \ldots, N$ such that,

$$T(\xi) = 0$$

where $\xi = \{f,g\}\prod_{i=1}^{N}\{c_i,h_i\}$, then applying our results from §6.4 and §6.6 would imply,

$$
\begin{aligned}
& r(\xi)(\omega) \\
= \; & r(\{f,g\}\prod_{i=1}^{N}\{c_i,h_i\})(\omega) \\
= \; & \int_X (log|g|dlog|f| - log|f|dlog|g|) \wedge \omega \\
& \qquad + \sum_{i=1}^{N}\left[\int_X (log|h_i|dlog|c_i| - log|c_i|dlog|h_i|) \wedge \omega\right] \\
= \; & \int_X (log|g|dlog|f| - log|f|dlog|g|) \wedge \omega + \sum_{i=1}^{N}[0] \\
= \; & r(\{f,g\})(\omega) \\
\neq \; & 0
\end{aligned}
$$

This would mean we now have a $\xi \in CH^2(X,2)$ for which our regulator is non-zero so that the regulator is non-trivial even on the restricted domain $ker(T) \simeq CH^2(X,2)$.

In fact, Lewis and Bloch [Blo2] have seperately found $\{c_i,h_i\}$'s such that ξ can be constructed in the case where $X \subset \mathbf{P}^3$ is a general cubic (=elliptic) curve.

Bibliography

[A-M] M.F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley (1969).

[B-T] H. Bass and J. Tate, *The Milnor ring of a global field*, in Algebraic K-Theory II, Lecture Notes in Mathematics, Springer-Verlag **342** (1972), 349-446.

[Blo1] S. Bloch, *Algebraic cycles and higher K-Theory*, Adv. Math. **61** (1986), 267-304.

[Blo2] ______, Lectures on Algebraic Cycles, Duke University Mathematics Series IV (1980).

[D-F] D.S. Dummit and R.M. Foote, Abstract Algebra, Second Edition, Wiley. (1999).

[G-H] P.A. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley (1978).

[Lew1] J.D. Lewis, A Survey of the Hodge Conjecture, Second Edition, CRM Monograph Series (American Math. Soc.), Vol. 10 (1999).

[Lew2] ______, *Real regulators on Milnor complexes*, K-Theory **25** (2002), 277-298.

[Lew3] ______, *Lectures on Algebraic Cycles*, Bol. Soc. Mat. Mexicana **3** Vol.7 (2001), 137-192.

[MS] S. Muller-Stach, Algebraic cycle complexes, Proceedings of the NATO Advanced Study Institute on the Arithmetic and Geometry of Algebraic Cycles, (Lewis, Yui, Gordon, Muller-Stach, S. Saito, eds.), Kluwer Academic Publishers, Dordrecht, The Netherlands, Vol. 548 (2000), 285-305.

[Mum] D. Mumford, Algebraic Geometry I, Springer-Verlag (1976).

[Warn] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer-Verlag (1983).

Appendix A

Rudiments of Commutative Algebra

The main sources for the material found in this appendix are Atiyah & MacDonald [A-M] and Dummit & Foote [D-F].

A.1 Rings & Ideals

We begin by recalling several important definitions which will be used throughout the following sections,

Definition: A commutative ring A is a set with two binary operations (called addition and multiplication) such that,

1. A is an abelian group with respect to addition
2. multiplication is associative and commutative
3. multiplication is distributive over addition

Henceforth, by a ring we will mean a commutative ring.

Definition: An ideal m of a ring A is an additive subgroup of A such that $mA \subseteq A$ (i.e. if $a \in A$ and $x \in m$ then $ax \in m$).

Definition: An ideal m of a ring A is said to be,

1. a principal ideal if it is of the form $m = \{ax \mid a \in A\}$ for some $x \in A$. In this case we write $m = (x)$. (Notice that $(1) = A$ and that if x is a unit of A then $(x) = (1)$).
2. prime if $m \neq (1)$ and if $xy \in m$ implies that either $x \in m$ or $y \in m$.

3. maximal if there does not exist an ideal a of A such that $m \subset a \subset (1)$ (notice the strict inclusions).

Definition: The annihilator of an ideal m of a ring A is the set $Ann(m) = \{x \in A \mid xm = 0\}$. It is apparent from the definition that $Ann(m)$ is an ideal of A. (If $m = (y)$ is a principal ideal we write $Ann(y)$ in place of $Ann((y))$.)

Definition: Let m and n be ideals of a ring A. We define,

1. the product of m and n, denoted mn, to be the set of all finite sums of elements of the form xy where $x \in m$ and $y \in n$.

2. the n^{th}-power of m, denoted by m^n, inductively as $m^1 = m$ and $m^n = mm^{n-1}$ for $n \geq 2$.

We note that both the product of ideals and the n^{th}-power of an ideal are again ideals of A.

Definition: Let m be an ideal of a ring A. We define the radical of m to be,

$$r(m) = \bigcap_{\mathcal{P} \supset m} \mathcal{P}$$

where $\mathcal{P}$ are prime ideals. Since the intersection of ideals is again an ideal, this implies that $r(m)$ is also an ideal. An alternate definition of $r(m)$ is as the set $\{x \in A : x^n \in m \text{ } for \text{ } some \text{ } integer \text{ } n \geq 0\}$.

Definition: Two ideals m and n are said to be coprime if $m + n = (1)$. One can see immediately from the definition that m and n are coprime if and only if $\exists$ $x \in m$ and $y \in n$ such that $x + y = 1$.

Now that we have these basic definitions we can proceed.

A.2 Local Rings

We let A be a ring and recall the following,

Definition: If a ring A has exactly one maximal ideal m, then A is called a local ring. The field $k = A/m$ is called the residue field of A.

How do we recognize a local ring and it's maximal ideal? We use the following proposition,

Proposition: (Test for Locality of a Ring) Let A be a ring and $m \neq (1)$ an ideal of A with the property that every $x \in A - m$ is a unit of A. Then A is a local ring with m it's only maximal ideal.

Proof: Any ideal $a \neq (1)$ contains only non-units (since if a contained a unit x then $x(x^{-1}) = 1 \in a$ and thus $a = (1)$) and so is contained in m. Thus m is the only maximal ideal of A. Q.E.D.

Local rings do in fact exist. If a ring A is also a field then it's only ideals are 0 and (1) and so any field is a local ring with maximal ideal 0. So we see that the residue field of a field is itself.

We can construct non-trivial examples of local rings. Let A be a ring and D any multiplicatively closed subset of A. It happens that one can construct a "smallest" ring denoted $D^{-1}A$ in which all elements of D become units. The ring $D^{-1}A$ is a generalization of the ring of fractions. We call $D^{-1}A$ the localization of A at D. We now define an important example of a localization.

Definition: Let P be a prime ideal of the ring A. Notice that $A-P$ is a multiplicatively closed subset of A by the definition of an ideal. We let $D = A - P$ and in this case denote $D^{-1}A$ by A_P. We call A_P the localization of A at the prime ideal P.

Example: If we let $A = \mathbf{Z}$ and $P = (p)$ for some prime $p \in \mathbf{Z}$ then,

$$\mathbf{Z}_{(p)} = \{a/b \in Q \mid p \textit{ does not divide } b\} \subseteq \mathbf{Q}$$

which is the "smallest" ring in which all elements of $A - P = \mathbf{Z} - (p)$ become units.

It turns out that A_P is a local ring with maximal ideal P' where P' is the extension of the prime ideal P over A_P.

A.3 Integral Dependence

We let B be and ring with identity and let A be a subring of B.

Definition: An element x in B is said to be integral over A if x is a root of a monic polynomial with coefficients in A, i.e. if x is a root of a polynomial,

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

where $a_i \in A$ for $i = 1, \ldots, n$ and $n \in \mathbf{Z}_+$.

In particular, every $x \in A$ is integral over A since it is a root of the monic polynomial $x + (-x) = 0$ where we know $-x \in A$ since A is a ring.

Proposition: An element x in B is integral over A if and only if $A[x]$ is a finitely generated A-module.

Proof:

- "If": We know that in general $A[x]$ is generated as an A-module by the infinite set $\{1, x, x^2, \ldots\}$. So if we assume that $A[x]$ is finitely generated we know that it will be generated by the set $\{1, x, \ldots, x^n\}$ for some positive integer n. Since $x^{n+1} \in A[x]$ we can write,

$$x^{n+1} = a_1 x^n + \cdots + a_n x + a_{n+1}$$

where $a_i \in A$ for $i = 1, \ldots, n+1$. Rewriting this we have,

$$x^{n+1} - a_1 x^n - \cdots - a_n x - a_{n+1} = 0$$

where we know that $-a_i \in A$ for $i = 1, \ldots, n+1$. Thus x is integral over A.

- "Only If": We assume $x \in B$ is integral over A so that for some $n \in \mathbf{Z}_+$,

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

which implies,

$$x^n = -(a_1 x^{n-1} + \cdots + a_n)$$

But then, for any $r \geq 0$,

$$x^{n+r} = -(a_1 x^{n+r-1} + \cdots + a_n x^r)$$

If we let $\mathcal{A}$ be the A-module generated by the set $\{1, x, \ldots, x^{n-1}\}$ then notice that, for $r = 0$, $x^{n+r} = x^n \in \mathcal{A}$. Now assume $x^{n+r} \in \mathcal{A}$. Then,

$$x^{n+r+1} = x x^{n+r} \in \mathcal{A}$$

since $x, x^{n+r} \in \mathcal{A}$. Thus, by induction on r, all positive powers of x and in $\mathcal{A}$ and so $A[x] \subseteq \mathcal{A}$. Since $\mathcal{A}$ is generated by a finite set $A[x]$ must be also. Therefore $A[x]$ is a finitely generated A-module. Q.E.D.

Corollary: If $x_1, \ldots, x_n$ are elements of B which are integral over A then the polynomial ring $A[x_1, \ldots, x_n]$ is a finitely generated A-module.

Proof: By induction on n. The base case is established in the previous proposition. Assume that $A[x_1, \ldots, x_{n-1}]$ is a finitely generated A-module. Denote $A_{n-1} = A[x_1, \ldots, x_{n-1}]$. Since x integral over A implies x is integral over

A_{n-1}, we see that $A[x_1, \ldots, x_n] = A_{n-1}[x_n]$ is finitely generated by applying the previous proposition. Q.E.D.

<u>**Proposition**</u>: The set C of elements of B which are integral over A is a subring of B containing A.

<u>**Proof**</u>: We know that every $a \in A$ is integral over A so we certainly have that $A \subseteq C$. This implies that $0, 1 \in C$. If $x, y \in C$ then we know that $A[x, y]$ is finitely generated. Since $A[x \pm y]$ and $A[xy]$ are both contained in $A[x, y]$, both must also be finitely generated. So both $x \pm y$ and xy must be integral over A, and thus are both in C. So C is closed under addition and multiplication and so it is a ring. Q.E.D.

So we have the inclusions $A \subseteq C \subseteq B$.

<u>**Definition**</u>: C is called the integral closure of A in B. If $C = A$ then A is said to be integrally closed in B. If $C = B$ then B is said to be integral over A.

<u>**Example**</u>: Let $B = \mathbf{Q}$ and $A = \mathbf{Z}$. Let $x/y \in \mathbf{Q}$ where $x, y \in \mathbf{Z}$ are relatively prime. Say x/y is integral over A. Then we have, for $a_1, \ldots, a_n \in \mathbf{Z}$,

$$\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{x}{y}\right) + a_n = 0$$

$$x^n + a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 0$$

$$x^n = -\left(a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n\right)$$

so we get that y divides x^n. But x and y were chosen to be relatively prime so it must be that $y = \pm 1$. Thus $x/y \in \mathbf{Z}$. So we see that $\mathbf{Z}$ is integrally closed in $\mathbf{Q}$.

<u>**Proposition**</u>: If $A \subseteq B \subseteq C$ are rings with B integral over A and C integral over B then C is integral over A (i.e. integral dependence is transitive).

<u>**Proof**</u>: Let $x \in C$. Since C is integral over B we have,

$$x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n = 0$$

where $b_1, \ldots, b_n \in B$. Let B' be the ring $A[b_1, \ldots, b_n]$ and notice that,

- B integral over $A \Longrightarrow B'$ is a finitely generated A-module.
- x integral over $B' \Longrightarrow B'[x]$ is a finitely generated B'-module.

Together, this implies that $B'[x]$ is a finitely generated A-module. Now, since $A \subseteq B'$, $A[x]$ is a sub-A-module of $B'[x]$ and so $A[x]$ is a finitely generated

A-module. Thus x is integral over A. Since $x \in C$ was arbitrary we have that C is integral over A. Q.E.D.

Definition: Let A be any ring with $x, y \in A$. If $xy = 0$ if and only if either $x = 0$ or $y = 0$ then A is said to be an integral domain.

Definition: The integral closure of an integral domain A in it's field of fractions K is called the normalization of A. If A is equal to it's normalization then we say that A is a normal ring.

Example: Recall that earlier in this section $\mathbf{Z}$ was shown to be integrally closed in it's field of fractions $\mathbf{Q}$. Thus $\mathbf{Z}$ is a normal ring.

A.4 Valuation Rings & Discrete Valuation Rings

A.4.1 Valuation Rings

Let B be an integral domain and K it's field of fractions.

Definition: B is said to be a valuation ring of K if, for each $x \neq 0$, either $x \in B$ or $x^{-1} \in B$ (or both).

Valuation rings are related to local rings and integral dependence in the following way,

Proposition: If B is a valuation ring of it's field of fractions K then,

1. B is a local ring.
2. if B' is a ring such that $B \subseteq B' \subseteq K$ then B' is also a valuation ring of K.
3. B is integrally closed in K.

Proof:

1. Let m be the set of non-units of B (i.e. if $x \in m$ then $x = 0$ or $x^{-1} \notin m$). Notice that if $a \in B$ and $x \in m$ then $ax \in m$ since otherwise $(ax)^{-1} \in B$ and thus $x^{-1} = a(ax)^{-1} \in B$. Now let x and y be non-zero elements of m. Then, since B is a valuation ring, either $xy^{-1} \in B$ or $(xy^{-1})^{-1} = x^{-1}y \in B$. Say $xy^{-1} \in B$ then $x + y = (1 + xy^{-1})y \in Bm \subseteq m$ (and similarly if $x^{-1}y \in B$). Thus m is an ideal and so B is a local ring with maximal ideal m.
2. Clear from the definition of a valuation ring.

3. Say $x \in K$ is integral over B. Then,

$$x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n = 0$$

where $b_1, \ldots, b_n \in B$. Rewriting this equation we get,

$$x + b_1 + b_2 x^{-1} + \cdots + b_n x^{1-n} = 0$$

$$x = -(b_1 + b_2 x^{-1} + \cdots + b_n x^{1-n})$$

Say $x \notin B$. Then, since B is a valuation ring, $x^{-1} \in B$. But then,

$$x = -(b_1 + b_2 x^{-1} + \cdots + b_n x^{1-n}) \in B$$

which is a contradiction. So only elements of B are integral over B. Q.E.D.

A.4.2 Discrete Valuation Rings

Now we let k be any field and $k^\times = k - \{0\}$ it's multiplicative group.

__Definition__: A discrete valuation on k is a map $v : k^\times \longrightarrow \mathbf{Z}$ such that,

1. $v(xy) = v(x) + v(y)$ (i.e. v is a homomorphism)
2. $v(x+y) \geq \min\{v(x), v(y)\}$

Let $V = \{0\} \cup \{x \in k^\times \mid v(x) \geq 0\}$. It is immediate from the definition of v that V is a ring. Since v is a homomorphism we have,

$$0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$$

which implies that $v(x^{-1}) = -v(x)$. Thus, for any $x \in k^\times$, we have $x \in V$ or $x^{-1} \in V$. Thus V is a valuation ring of the field k. V is called the valuation ring of v.

__Definition__: An integral domain A is a discrete valuation domain if there is a discrete valuation v of it's field of fractions k such that A is the valuation ring of v.

__Example__: Consider $\mathbf{Z}$ and and it's field of fractions $\mathbf{Q}$. We take $p \in \mathbf{Z}$ to be any fixed prime number. For any $\alpha/\beta \in \mathbf{Q}$ we can write,

$$\frac{\alpha}{\beta} = p^a \frac{\tilde{\alpha}}{\tilde{\beta}}$$

in a unique way with both $\tilde{\alpha}$ and $\tilde{\beta}$ relatively prime to p. We define the discrete valuation,

$$v_p\left(\frac{\alpha}{\beta}\right) = v_p\left(p^a\frac{\tilde{\alpha}}{\tilde{\beta}}\right) = a$$

The valuation ring of v_p is,

$$\{0\}\cup\{\,\alpha/\beta \in \mathbf{Q} \mid v_p(\alpha/\beta) \geq 0\} = \{0\}\cup\{\alpha/\beta \in \mathbf{Q} \mid p \textit{ does not divide } \beta\} = \mathbf{Z}_{(p)}$$

Thus the localization ring $\mathbf{Z}_{(p)}$ is a discrete valuation ring.

NOTE: Making a similar definition of a discrete valuation v_f on $k(x)$ for an irreducible $f \in k[x]$ we have that $k[x]_{(f)}$ is a discrete valuation ring.

A.5 Noetherian Domains & Dedekind Domains

A.5.1 Noetherian Domains

Let A be a ring. If any chain of ideals in A $m_1 \subseteq m_2 \subseteq m_3 \subseteq \cdots$ has the property that $\exists\ n \in \mathbf{Z}_+$ such that $m_k = m_n$ for all $k \geq n$ then A is said to satisfy the ascending chain condition (a.c.c.) on ideals.

Definition: A ring A is said to be Noetherian if it satisfies the a.c.c. on ideals.

Proposition: If m is an ideal of a Noetherian ring A then A/m is a Noetherian ring.

Proof: Any infinite chain of ideals $m_1 \subseteq m_2 \subseteq m_3 \subseteq \cdots$ in A/m corresponds to an infinite chain of ideals $m_1' \subseteq m_2' \subseteq m_3' \subseteq \cdots$ in A where m_k' is just the extension of m_k from A/m to A. Thus the chain in A/m must satisfy the a.c.c. since it's corresponding chain in A does. Thus A/m is Noetherian. Q.E.D.

Proposition: The following are equivalent,

1. A is a Noetherian ring.
2. every non-empty collection of ideals of A contains a maximal element.
3. every ideal of A is finitely generated.

Proof:

- "1. $\Rightarrow$ 2." Let A be a Noetherian ring and suppose Σ is a non-empty collection of ideals of A which does not contain a maximal element. Then we can construct inductively a chain $m_1 \subsetneq m_2 \subsetneq m_3 \subsetneq \cdots$ which violates the a.c.c. But this violates the assumption that A is Noetherian, thus Σ must contain a maximal element.

- "2. $\Rightarrow$ 3." Assume that 2. holds and let m be some ideal of A. Let Σ be the collection of finitely generated ideals contained in m. By 2. we have that Σ contains a maximal element, call in n. If $n \neq m$ then take $x \in m - n$. We have that n is finitely generated so the ideal $n + Ax$ is also finitely generated and is still contained in m. Thus $n + Ax \in \Sigma$. But notice that $n + Ax \supset m$ which violates the maximality of m. Thus $n = m$ and so every ideal of A is finitely generated.

- "3. $\Rightarrow$ 1." Assume that 3. holds and let $m_1 \subseteq m_2 \subseteq m_3 \subseteq \cdots$ be an ascending chain of ideals of A. Let,

$$m = \bigcup_{i=1}^{\infty} m_i$$

  Then m is again an ideal of A. Thus m is finitely generated, say by $x_1, \ldots, x_m \in A$. For each x_i we can pick $n_i \in \mathbf{Z}_+$ such that $x_i \in m_{n_i}$. If we let $n = \max\{n_1, \ldots, n_m\}$ then $x_1, \ldots, x_m \in m_r$ for all $r \geq n$. But since these elements generate all of m it must be that $m_r = m_n$ for all $r \geq m$. So A satisfies the a.c.c. and so A is Noetherian. Q.E.D.

(We should note that all the notions apply to A-modules as well. We simply replace "rings" by "modules" and "ideals" by "sub-modules" in the statements and proofs in this section.)

Proposition: (Hilbert's Basis Theorem) If A is a Noetherian ring then the polynomial ring $A[x]$ is also Noetherian.

Proof: Omitted, see [A-M, 81].

Corollary: If A is a Noetherian ring then the polynomial ring $A[x_1, \ldots, x_n]$ is also Noetherian.

Proof: By induction on n. The base case ($n = 1$) is established by Hilbert's Basis Theorem. Now assume A being Noetherian implies that $A[x_1, \ldots, x_{n-1}]$ is Noetherian. Notice that, if we denote $A_{n-1} = A[x_1, \ldots, x_{n-1}]$, then we have,

$$A[x_1, \ldots, x_n] = A_{n-1}[x_n]$$

and so $A[x_1, \ldots, x_n]$ is Noetherian by applying Hilbert's Basis Theorem. Q.E.D.

Though many rings (such as $\mathbf{Z}$ or the polynomial ring $A[x_1, \ldots, x_n]$) have the property that elements can be factored in a unique way, this is not true for rings in general. A simple example is the ring $\mathbf{Z}[i]$ where 10 can be factored as $2 \cdot 5$ or as $(3+1)(3-i)$. However, for Noetherian rings we can develop a notion of factorization of ideals, and for certain ideals this factorization is unique. We begin now with some defininitions and Propositions with this end in mind.

Definition: An ideal m in a ring A is said to be primary if $m \neq (1)$ and if $xy \in m$ implies that either $y \in m$ or $x^n \in m$ for some $n > 0$.

Where prime ideals are a generalization of prime numbers (indeed the prime ideals in $\mathbf{Z}$ are (p) for p a prime number), primary ideals are a sort of generalization of powers of prime numbers (again, in $\mathbf{Z}$ the primary ideals are (p^n) for p a prime number and $n > 0$). It turns out that in Noetherian domains every ideal can be factored as a finite intersection of primary ideals. This is called a primary decomposition. Further, this primary decomposition can be written in a minimal way and this leads us to define,

Definition A primary decomposition $m = \bigcap_{i=1}^{n} p_i$ of an ideal m is called minimal if it satisfies,

1. $p_i \not\supseteq \bigcap_{i \neq j} p_j$ for all $i \neq j$
2. $r(p_i) \neq r(p_j)$ for all $i \neq j$

The idea behind a minimal primary decomposition is simply that we get rid of all redundant information in order to give the most straight forward factorization of the ideal. Ideals in Noetherian domains always have minimal primary decompositions. Before we can prove this we will need a few things,

Definition: An ideal m is said to be irreducible if $m = a \cap b$ implies that either $m = a$ or $m = b$.

Proposition: Let A be a Noetherian ring. Then every ideal is a finite intersection of irreducible ideals.

Proof: Suppose the above were false. Then the set Σ of ideals violating the above condition is non-empty. Thus Σ contains a maximal element, call it a. Thus a is reducible, and so we can write $a = b \cap c$. But now $b \supset a$ and $c \supset a$ implies that $b, c \notin A$ since otherwise it would violate the maximality of a. Thus both b and c are finite intersections of irreducible ideals and so a is as well. Hence a contradiction. Q.E.D.

Proposition: If A is a Noetherian ring then every irreducible ideal of A is primary.

Proof: Let A be a Noetherian ring and m an irreducible ideal of A. Notice that m irreducible in A implies that the trivial ideal 0 is irreducible in A/m. Also notice that if 0 is primary in A/m then if $xy = 0$ in A/m then either $x^n = 0$ or $y = 0$. In other words, if $xy \in m$ then either $x^n \in m$ or $y \in m$ (i.e. m is a primary ideal of A). Thus it is enough to descend to the quotient ring and show that if 0 is irreducible in A/m then 0 is primary in A/m.

So assume 0 is irreducible in A/m and, for $x, y \in A/m$, let $xy = 0$ and $y \neq 0$. We have the descending chain of ideals in A/m,

$$(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots$$

and from this, since any element which annihilates (x^n) will also annihilate (x^r) for any $r > n$, we get the following ascending chain of ideals,

$$Ann(x) \subseteq Ann(x^2) \subseteq Ann(x^3) \subseteq \cdots$$

Since A Noetherian implies that A/m is Noetherian we have that $Ann(x^n) = Ann(x^{n+1}) = \cdots$ for some $n > 0$. Now notice that if $a \in (y)$ then $ax = 0$ since $xy = 0$. Also, if $a \in (x^n)$ then $a = bx^n$ for some $b \in A/m$. So if $a \in (x^n) \cap (y)$ then $ax = bx^{n+1} = 0$ and so $b \in Ann(x^{n+1}) = Ann(x^n)$. Thus $bx^n = 0$ which says that $a = 0$. Thus $(x^n) \cap (y) = 0$. Since we are assuming 0 is irreducible and $y \neq 0$, it must hold that $x^n = 0$. Thus 0 irreducible implies either $y = 0$ or $x^n = 0$ and so 0 is primary. Therefore, by our argument above, if m is irrducible in A then m is primary. Q.E.D.

Now we arrive at our desired result,

Proposition: (Primary Decomposition Theorem) Every ideal in a Noetherian ring has a minimal primary decomposition.

Proof The existence of a primary decomposition follows immediately from the propositions above. Now let the ideal m have a primary decompostion $m = \bigcap_{i=1}^{n} p_i$. If any one of the p_i's contain the intersection of the remaining p_j's then we simply remove this p_i from the decompostion. This doesn't change the intersection and so now we have a primary decompostion satisfying 1. in definition 7. If there is some finite set $\{1, \ldots, n\}$ such that $r(p_{i_1}) = r(p_{i_2}) = \cdots = r(p_{i_n})$ then we notice that $r(p_{i_1} \cap \cdots \cap p_{i_n}) = r(p_{i_j})$ and so we replace the ideals $p_{i_1}, \ldots, p_{i_n}$ in the decomposition by the single ideal $p_{i_1} \cap \cdots \cap p_{i_n}$. This doesn't change the intersection and now the decompostion satisfies 2. Thus m has a minimal primary decomposition. Q.E.D.

Proposition: Let an ideal m of a Noetherian ring A have a minimal primary decomposition $m = \bigcap_{i=1}^{n} p_i$. Then m is a radical if and only if the p_i are prime ideals for all $i = 1, \ldots, n$. In this case the decomposition is unique.

Proof (Outline) Let m be a radical with $m = \bigcap_{i=1}^{n} p_i$. By the definition of the radical we get that $m = \bigcap_{i=1}^{n} r(p_i)$. This is also a minimal primary decompostion and is uniqueness will follow from the irreducibility of the prime ideals $r(p_i)$. Thus it must be that $p_i = r(p_i)$ and so we have a unique minimal prime decomposition. Q.E.D.

So we have seen that many rings that we are interested in have the Noetherian property and that this is exactly enough to develop a notion of unique factorization of certain ideals, something which doesn't hold for rings in general. We now turn our attention to a subclass of Noetherian rings where this factorization is unique for any ideal.

A.5.2 Dedekind Domains

First we need to develop the notion of the dimension of a ring.

Definition: Let A be a ring. A strictly increasing sequence of prime ideals $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ is said to have length n. We define the Krull dimension of the ring A to be the supremum of the lengths of all such sequences in A, and write $dim_K A = n$.

We see that the dimension of a ring is some number in the set $\{0, 1, 2, 3, ...\} \cup \{\infty\}$. We give a few simple examples,

Example:

1. If A is a field then A has only two ideals 0 and (1) (since any other ideal will contain a unit and is thus equal to (1)). So the Krull dimension of a field is always 0.
2. A Noetherian ring always has finite Krull dimension since any strictly increasing sequence of prime ideals must be of finite length.
3. Consider the ring of integers Z. We know that the only prime ideals in Z are of the form (p) where p is a prime number. So the only strictly increasing sequences of prime ideals are of the form $(0) \subset (p)$ since it can never happen that $(p) \subset (q)$ for two distinct primes p and q. Thus we get that the $dim_K Z = 1$.

Now that we have the notion of dimension we can define,

Definition: Let A be a Noetherian domain of Krull dimension one which is integrally closed (i.e. integrally closed in it's field of fractions). Then we say that A is a Dedekind domain.

It turns out that be the Dedkind domains are restricted enough to have a generalized notion of "unique factorization".

Proposition: If A is a Dedekind domain then every primary ideal is a prime power.

Proof: Omitted, see [A-M, 95].

Proposition: Any non-zero ideal of a Dedekind domain can be uniquely expressed as a product of primary ideals.

Proof: Let A be a Dedekind domain and let m be any non-zero ideal of A. Since A is Noetherian we know that m has a primary decompostion,

$$m = \bigcap_{i=1}^{n} q_i$$

where the q_i are distinct primary ideals. We let $p_i = r(q_i)$. We know the p_i are prime ideals and, since A has dimension 1, each p_i must be a distinct maximal ideal. Since, for $i \neq j$, the ideal,

$$p_i + p_j = \{x + y \mid x \in p_i, y \in p_j\}$$

strictly contains p_i we know that $p_i + p_j = (1)$ (otherwise it would contradict the maximality of p_i). So the p_i are pairwise coprime. But since,

$$r(q_i + q_j) = r(r(q_i) + r(q_j)) = r(p_i + p_j) = r(1) = (1)$$

we see that $q_i + q_j = (1)$ for $i \neq j$ and so the q_i are also pairwise coprime. It holds for coprime ideals m_i that $\prod m_i = \bigcap m_i$ [see Atiyah pg. 6-7]. Thus we have that,

$$m = \bigcap q_i = \prod q_i$$

and so m can be expressed as a product of primary ideals. The uniqueness of this expression follows from the fact that each q_i is an "isolated primary component" [A-M, 54]. Q.E.D.

So we arrive finally at our generalization of unique factorization to rings and ideals,

Proposition: Let A be a Dedekind domain and m any non-zero ideal of A. Then m has a unique factorization as a product of prime ideals.

Proof: Follows immediately from the last two propositions above. Q.E.D.

So it is in Dedekind domains that we get a notion of "unique factorization as a product of primes" which is exactly analagous to prime factorization of numbers in **Z**. Fortunately Dedekind domains are common enough that this turns out to be useful.