IMPACTS OF RECOVERY RATES AND TERMS OF TRADE ON STRANGE ATTRACTORS AND PREDICTABILITY IN SUSTAINABLE AGRICULTURE

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Executive Summary

Sustaining agricultural systems requires the ability to predict approaching extinction. Thus a suitable model needs to generate predictability as an intrinsic attribute. Such a model should provide several features for managers. It should incorporate learning about the minimum sustainable ecospheric threshold. The model requires a capability to cope with uneven system coevolution. Solutions should reveal how to maneuver parameters to achieve 'favourable' system dynamics.

Models based on assumptions of linearity and randomness are not able to explain sharp changes in system behaviour and do not result in predictability. Fortunately, theories for complex nonlinear dynamical systems are emerging from mathematics and physics in applications such as ecology, economics and immunology.

This paper addresses the relationships between the structural parameters of a complex three-dimensional system and the predictability of wealth and sustainability. The subsystems are agriculture, the ecosphere and industry. Their interaction is modelled by a dynamical system based on the predator prey paradigm. The ecosphere is considered as a living interactive system that can regenerate, reproduce and become extinct. The model explores the dynamics of the whole system as the structural properties of its parts coevolve over time. We demonstrate that the structural parameters may pass through bifurcation values, which not only result in new equilibria and periodic trajectories, but account for the presence of strange attractors.

Most of our attention is placed on exploring the conditions under which strange attractors appear and disappear in coevolution. The presence of strange attractors connotes great uncertainty and severely limits predictability The policy problem for sustainable agriculture is to prevent strange attractors from appearing.

The results are that agricultural terms of trade and ecospheric recovery rates are partially substitutable in sustaining agriculture. Strange attractors may be avoided and replaced by predictable periodic trajectories or stable node type equilibria by changing the rate of ecospheric recovery, or terms of trade or the productivity of the ecosphere in agricultural uses. Portraits of the trajectories are provided to make it easier to understand the dynamics.

The results suggest that sustainability is sensitive to learning processes which address these tradeoffs, the approach to minimum thresholds of persistence for the ecosphere and the mutualism in economic predation. It may be noted that farmers historically learned about these things through artisan apprenticeship. Just as this learning method has been replaced by social, biological and physical sciences to achieve remarkable productivity gains, so resources need to be shifted to address the co-requisite ecospheric recovery processes.

Keywords: system dynamics, sustainable agriculture, prediction, strange attractors, chaos.

1 Problem Statement

Agriculture faces problems of sustainability. The evidence lies in the extensive and expensive subsidies to agriculture in the industrialized world, and extreme poverty and ecospheric degradation in much of the less developed world. There is growing recognition that the irregularity in agriculture may be attributed to inherent nonlinearity in economic relationships, rent-seeking, and degradation which underlie the disorder that subsidies seek to reverse, yet often contribute to.

This paper is concerned with the problems of predictability surrounding sustainability. These problems are rooted in prevailing nonlinearity in economic and ecospheric relationships. Nonlinear types of order produce conditions in which inputs are not usually matched by outputs. Small causes can have disproportionately large effects different under changing circumstances. This type of behaviour is related to uneven changes to the equilibrium properties and dynamics inherent only in nonlinear systems. This type of order violates the stochastic and linearity assumptions common to many economic models of sustainability. Single equation models are similarly incapable of capturing the dynamics of nonlinear processes. Fortunately, new methods are available in mathematics and physics, the same basic sciences upon which economics was founded.

In this paper we model a complex system of agriculture, industry and the ecosphere in interaction. This system is modelled by a three-dimensional nonlinear dynamical system of predator prey type. This model has two purposes. The first is to understand the behaviour of the ecosystem over time. The second is to identify structural parameters and their ranges of value for which the system dynamics is sensitive. Attainment of these two purposes provides the predictability of the model.

There are certain sets of values for the structural parameters for which system behaviour is unpredictable. This is the case when the omega-limit set, destination of a system at time equal infinity, is a strange attractor. We explore the idea that the public policy problem for sustainable economic activity is to keep the economic systems away from 'strangeness' by maneuvering the structural parameters.

The model is relatively unrestricted by assumptions. There are three behavioural and two simplifying assumptions. The first behavioural assumption is that humans are conscious or may become conscious of a need to rehabilitate their ecosphere. The second is that human systems are predatory. The third assumption is that humans cannot migrate costlessly among systems to escape predation. In this sense, the model is about global systems. By the principle of holonomy, the parallelism of properties between parts of systems and their wholes, however, we expect that the model may be applied to micro systems (Gabor, 1949).

The first simplifying assumption in keeping with the rural nature of the system is that no industrial activity disturbs the ecosphere. The second is that the parameters are constants. In reality, some of them are probably subject to long period oscillations.

A unique feature of the model is that the ecosphere is neither considered as a feedstock nor as quasi-capital. Instead, the ecosphere is modelled as a living interacting system able to regenerate, reproduce and become extinct. This characterization is modelled on systems theory of nonlinear dynamical systems and their properties. The advantage of this approach is that four qualities may be considered, namely: a maximum condition, a minimum threshold below which the economy collapses, a rate of degradation and a rate of rehabilitation. The latter two rates combine in proportion to a composite ecospheric recovery rate.

The results stem from a model of three active systems, agriculture, industry and the ecosphere, and their relationships. These results are an extension of earlier work with a three-dimensional model, simplified by keeping the ecosphere in a stationary or passive state (Apedaile et al, 1994). This investigation explains how equilibria for wealth shift and change their stability properties, when the ecosphere is actively changing. The signatures of consequent trajectories in three dimensions as they search for these equilibria provide insights for predictability. Several kinds of dynamics leading to different flows of trajectories are examined through numerical experiments.

The solution to the model provides a precise definition of sustainable development. An economy beginning at any point on or above a minimum threshold of the ecosphere, will follow a trajectory leading to higher equilibrium points on the agriculture/industry plane, at the ecospheric threshold, as long as the rehabilitation rate for the ecosphere relative to the degradation rate exceeds a certain value. The ratio of these two rates is proportional to the ecospheric recovery rate. We understand these two rates to be determined by the learning processes of the system oikos (Anderson-Medellin et al, 1994).

The second outcome is that stronger agricultural terms of trade may go some way as a substitute for better recovery rates. This result is interpreted to mean that industry has to reduce its predation on agriculture, to the extent that ecospheric recovery rates stagnate. Both these factors lie within a social consciousness which may be heightened by improved predictability.

The third outcome is that the omega-limit sets may become strange attractors, stable periodic trajectories, stable foci or stable nodes. Of these outcomes the strange attractor generates uncertainty which can reduce predictability and economic (allocative) efficiency. This uncertainty may be manipulated by the State to enhance or mediate predation. The context of mutualism defined by the oikos of the systems may be altered with social and environmental consequences for agriculture and the ecosphere. Mutualism governs the extent and form of predation so as to maintain the productivity of a prey population.

The fourth result is that in the case of stable periodic or near periodic trajectories, the economy moves relatively rapidly through the highest attainable wealth coordinates starting with agriculture, moving on to industry and then accelerates to low values for both systems. The systems remain closer to the origin roughly ten times longer than they remain in regions of prosperity. This additional time is needed to rehabilitate the ecosphere.

Agriculture may be interpreted to represent renewable resource extraction in general. The model reveals several interesting tradeoffs to modify equilibria, their stability properties and the corresponding trajectories. The results indicate that research, normally devoted to productivity and the economic efficiency of agriculture, is also needed for enhancing recovery rates for the ecosphere, agriculture and industry.

In this paper we consider a model with initial conditions that are at or above a minimum for sustainability. Modelling the regeneration of the ecosphere when the extinction of agriculture is imminent is a subject for future study.

2 Notes on method

This work is a theoretical exploration of relationships among three systems. Predatory relationships are privileged over competition and cooperation. One of the continuing preoccupations of this research is to insert actual ranges of observed values for the parameters and initial conditions representing a real economic system.

Currently all values, for variables and parameters alike, are scaled around nominal values. For example the minimum threshold for ecospheric wealth is defined as 1.0. The concept of ecospheric wealth itself has yet to be resolved. The ecosphere is multifaceted and relatively unyielding to efforts to summarize it in one measure. Nevertheless, relative changes, though scaled, provide considerable insight into the performance of the model and the implications for structural change.

The procedure leading to the results reported below is to derive the various multiple equilibria in the solution for the system of equations. Values for the parameters are substituted into the equations to establish a number of cases defined as interesting to the strategic objectives of the research.

These objectives, in strategic order, are to learn the properties of the model, explore the consequences of the introduction of the ecosphere equation for the behaviour of the system, determine which parameters are instrumental in shifting the equilibria to improved levels of wealth, and to calculate the effects on the properties of the equilibria upon changing the values of parameters associated with the ecosphere.

At each step, for selected parameters, trajectories are simulated using PHASER software (Kocak, 1989). The trajectories provide information as to the dynamic behaviour of the three systems, agriculture, industry and the ecosphere, in interaction.

Two cases are used. Each is described by a set of parameter values which result in rich dynamic properties for the solutions (Table 1). The cases differ only in that the rates of recovery for the agriculture and industry systems are higher for Case II.

3 Basic model of agriculture, ecosphere and industry

3.1 the dynamical system

We consider the following three-dimensional dynamical system. The dynamical variables, A, I, and E represent the agricultural, industrial and ecospheric wealth respectively.

$$\frac{d}{dt}A = \mu\left\{\alpha_0 \frac{E}{e+E} A - \beta A^2 + (\gamma - \delta)\frac{A}{(a+A)}\frac{I}{(b+I)}\right\} \quad (1a)$$

$$\frac{d}{dt}I = -\xi I - \eta I^2 + \delta \frac{A}{(a+A)}\frac{I}{(b+I)} \quad (1b)$$

$$\frac{d}{dt}E = u(E - E_0) - vA(E - E_0) - w(E - E_0)^2. \quad (1c)$$

The system contains fourteen parameters. Four of these, u, v, w and E_0, define the rate of change of ecospheric wealth. Parameter u is the rehabilitation rate of the ecosphere, and v, the rate of degradation of the ecosphere. These two rates are not independent. The ecosphere recovery rate, e^{-1}, is proportional to the ratio u/v. The relation between u and w defines the maximal, or saturation level, of ecospheric wealth (Appendix 1). E_0 is the minimum level of ecospheric wealth to sustain economic activity. Any system or subsystems for which $E < E_0$, may be considered as experiencing ecospheric disaster leading in most cases to extinction. See Figure 3.

All other parameters were introduced in the paper by Apedaile et al (1994) and are summarized in Table 1.

3.2 equilibria

Let us describe possible equilibria. These are defined where

$$\frac{d}{dt}A = \frac{d}{dt}I = \frac{d}{dt}E = 0. \qquad (2)$$

In the plane $E = E_0$, the system reduces to the two dimensional one considered in Apedaile and Freedman et al, (1994). This reduced two dimensional system always possesses at least two equilibria $F_0(0,0)$ and $F_1\left(\frac{\alpha}{\beta},0\right)$ where $\frac{\alpha}{\beta} = \frac{\alpha_0}{\beta} \cdot \frac{E_0}{e+E_0}$.

If the condition

$$\frac{\alpha}{\beta} > \frac{ab\xi}{\delta - b\xi} \qquad (3)$$

is satisfied, there exists at least one more equilibrium $\hat{F}_1(\hat{A},\hat{I})$ in the first quadrant. It was shown by Apedaile et al (1994), for the case of systems with high recovery rates $\frac{1}{a}$ and $\frac{1}{b}$, a,b, $<< 1$, that if parameter ξ increases beyond a certain value ξ_0, saddle-node bifurcation occurs. Two new equilibria $\hat{F}_2, \hat{F}_3$ appear, a saddle point and a stable node respectively, located at relatively high levels of A and I.

Similar new equilibria may be created by increasing α. Note that the parameter α in the two dimensional version of this model is a function of the level of performance of the ecosphere and of the ecospheric recovery rate in the three-dimensional model.

$$\alpha = \alpha_0 \frac{E}{e+E} \qquad (4)$$

Consequently in the plane $E = E_0$, α is a function on e. Thus the two new equilibria may also be attributed to increasing the ecospheric recovery rate, or reducing e.

For the specific set of parameter values used for Section 3, there are three equilibria in the plane $E = E_0$, for e = 11.25, and five equilibria in this plane for $e = 9.0$. For details refer to Table 1 and Figure A2.1 in Appendix 2.

In the generic case, there are also three equilibria beyond the plane $E = E_0$. Two of these, F_2^* and F_3^*, are located in the (A,E) plane where I = 0. Their coordinates are easily found from the system of equations (2):

$$F_2^*\left\{\frac{\alpha}{\beta}\frac{E_2^*}{e+E_2^*}, 0, E_2^*\right\}, \tag{5}$$

where

$$E_2^* = \frac{1}{2}\left\{e - E_0 - \frac{u}{w} + \frac{v\alpha}{w\beta}\right\} + \sqrt{\frac{1}{4}\left(e - E_0 - \frac{u}{w} + \frac{v\alpha}{w\beta}\right)^2 + \left(\frac{u}{v} + E_0 e\right)} \tag{6}$$

and

$$F_3^*\left(0, 0, \frac{u}{v} + E_0\right). \tag{7}$$

The fourth equilibrium, F_4^*, does not pertain to any of the coordinate planes. It is unique in the generic case as a point of intersection of three two-dimensional surfaces defined by the system of equations (2). Its coordinates are defined numerically. The forgoing apply for all non-negative A, I, and E.

3.3 stability properties

Local stability of the equilibrium of the dynamical system is defined by signs of the real parts of the eigenvalues, λ_i, i=1, 2, 3, of the linearized system. Positive values for any of the real parts of λ_i signify instability. Negative values of all real parts of λ_i signify stability (Hirsch and Smale, 1974). For the two cases considered below, these eigenvalues are evaluated numerically. The results are presented in the tables of Appendix 2. We emphasize the most important features inherent in the sets of equilibria.

a) Equilibria F_0 and F_1^* on the $E = E_0$ plane are saddle points such that F_0 is an attractor in the I-direction and a repeller in the A-direction. In contrast, F_1^* is stable in the

A-direction and unstable in the I-direction (Figure 1). This result and a detailed discussion in terms of predator prey modelling may be found in Freedman and Moson (1987) and Freedman and Waltman (1990).

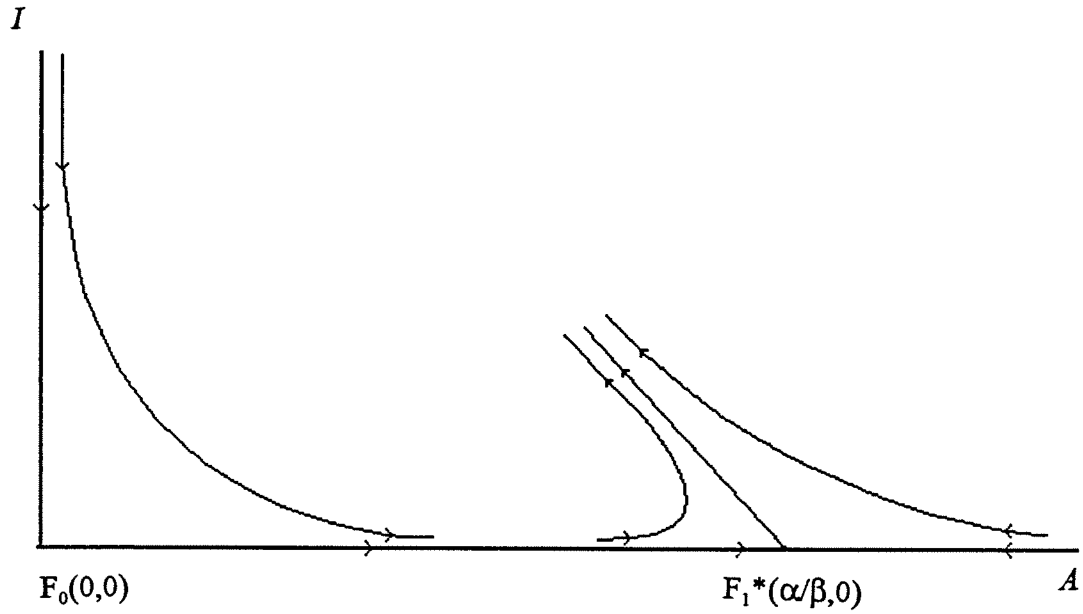

Figure 1. F_1^* is stable in the A-direction and unstable in the I-direction; F_0^* is unstable in the A-direction and stable in the I-direction.

b) For high recovery rates of agriculture and industry, a, b $< 1, \hat{F}_1$ is a stable focus located close to the point F_0. So the trajectory captured in the basin of attraction of $\hat{F}_1$ undergoes damped oscillation about this point, and due to the proximity of $\hat{F}_1$ to the origin;

i) the phase portrait of the oscillation is a very 'irregular' spiral (Figure 2).

ii) the speed of the trajectory passing between $\hat{F}_1$ and F_0, where the trajectories are 'dense', is low, much lower than the speed of the same trajectory far from F_0.

c) The equilibria lying on the $E = E_0$ plane for the trajectories starting at $E > E_0$ are repellers if the corresponding values of A and I are small. Otherwise the equilibria are attractors. This feature of their behaviour is inherent in equation (1c).

d) A saddle-node bifurcation leads to the appearance of a locally stable equilibrium, $\hat{F}_3$, which is also a global attractor for all the trajectories starting at $E \geq E_0$.

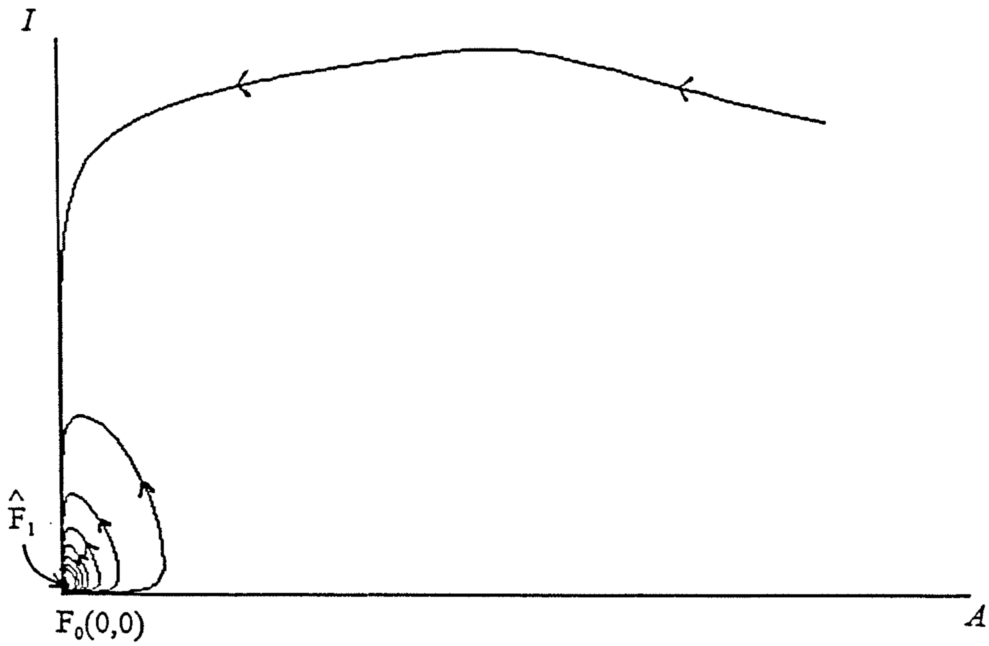

Figure 2. The phase portrait for high recovery rates of agriculture and industry, a, b << 1, illustrating the irregular spiral leading to the stable focus, $\hat{F}_1$, located close to the point F_0.

For mathematical completeness, we note that the trajectory starting at P_1 under the plane $E = E_0$, at low levels of A and I, tends to $-\infty$ along the E axis. Another trajectory starting at P_2 also under the plane $E = E_0$ but with greater original values of A and I, eventually comes to the equilibrium $\hat{F}_3$ in the $E = E_0$ plane (Figure 3).

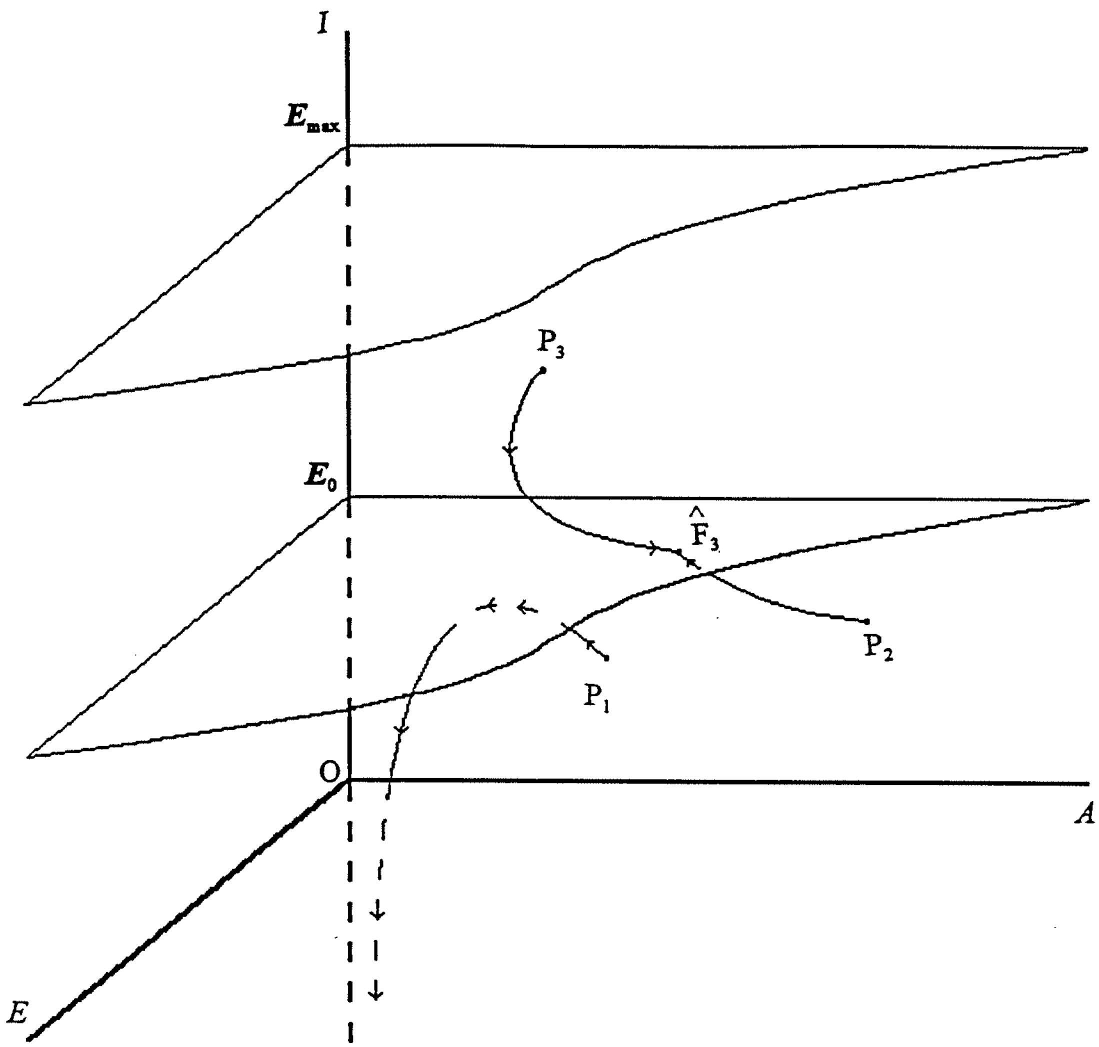

Figure 3. The trajectories starting at $E < E_0$ but at higher original levels of A and I are also attracted to $\hat{F}_3$. The others with $E < E_0$ and small original values of A and I tend to $-\infty$ along the negative E semi-axis.

3.4 dynamics and limit sets

The system is substantially nonlinear. Thus the local linear analysis does not exhaust all the options for system behaviour. Nonlinearity may result in such phenomena as the appearance of periodic orbits, 'chaos' and other behaviour both desirable and undesirable from the economic point of view.

Before we begin our discussion of the nonlinear properties of the present model, we specify the sets of parameter values for the two cases to be considered. We specify these parameters to make the analysis tractable. The values chosen are representative and lead to a robust dynamical behaviour of solutions (Apedaile et al, 1994). We denote these cases as I and II respectively. The differences between them are in the recovery rates for agriculture and industry. For Case I, a = b = 0.1 corresponding to high rates of recovery. The values are a = b = 0.25 for Case II judged to be moderate recovery rates. The basic set of values for the other parameters is set out in Table 1.

Table 1. Values for parameters common to Cases I and II.

parameter	value	parameter	value
relative productivity of A and I	$\mu = 1$	min threshold for the ecosphere	$E_0 = 1.0$
base capacity use of E for A	$\alpha_0 = 20$	linear depreciation rate of industrial capital	$\xi = 1.0$
coeff of diminishing returns in agriculture	$\beta = 1.0$	coeff of diminishing rate of depreciation of capital	$\eta = 0$
ecospheric recovery rate	$e = 9.0$	rate of restoration or rehabilitation of the ecosphere	$u = 0.5$
industrial price index	$\delta = 2.0$	rate of degradation of the ecosphere	$v = 2.0$
		position of the E system relative to max condition of the 'natural' ecosphere	$w = 0.1$
agricultural price index (a)	$\gamma = 0.33$	agricultural price index (b)	$\gamma = 0.78$

When doing numerical experiments, we alter e slightly from the basic value. These small changes, as we shall see, exert great influence on the system behaviour, emphasizing the importance of the recovery rate for the ecosphere in the normal functioning of the economic system.

3.4.1 Hopf bifurcation and limit cycle

Apedaile et al (1994) demonstrate that the two-dimensional version of our system may pass through a Hopf bifurcation (Marsden and McCracken, 1976) under certain configurations of the parameter values. In that version, μ plays the role of a bifurcation parameter. Bifurcation, given the values of the other parameters, occurs when μ surpasses a certain value, $\mu = \mu_0$ that lies between 1.0 and 2.0. For Case I, $\mu_0 \approx 1.62$ (Apedaile et al, 1994). When $\mu < \mu_0$, $\hat{F}_1$ is a stable focus. When $\mu > \mu_0$, $\hat{F}_1$ is an unstable focus surrounded by a stable periodic trajectory called a limit cycle. The radius of this cycle grows with μ in proportion to $\sqrt{\mu - \mu_0}$.

In the three-dimensional model, this limit cycle on the $E = E_0$ plane plays the role of a global attractor if the equilibrium $\hat{F}_3$ is absent. In this case, all the trajectories starting at $E \geq E_0$ are attracted to the limit cycle. Figure 4 represents such a limit cycle for Case I with $\mu = 2$.

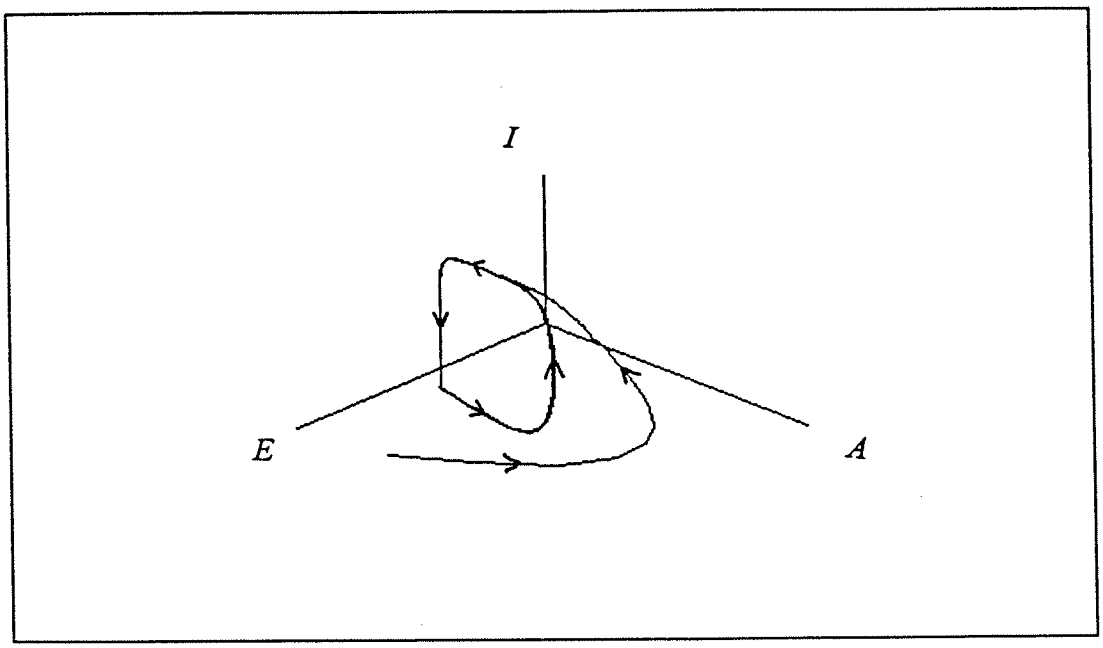

Figure 4. A limit cycle for Case I with $\mu = 2$.

3.4.2 Near-homoclinic periodic trajectories and strange attractors.

Stability properties (b) and (c) from Section 2.2 above imply the existence of a homoclinic trajectory, biasymptotic to the saddle-focus, $\hat{F}_1$. For an elaboration of the terminology, see Guckenheimer and Holmes (1983). That means there exists a closed trajectory that tends to $\hat{F}_1$ both for time extending to $+\infty$ and $-\infty$. A numerical approximate portrait of the homoclinic trajectory for Case II is presented in Figure 5.

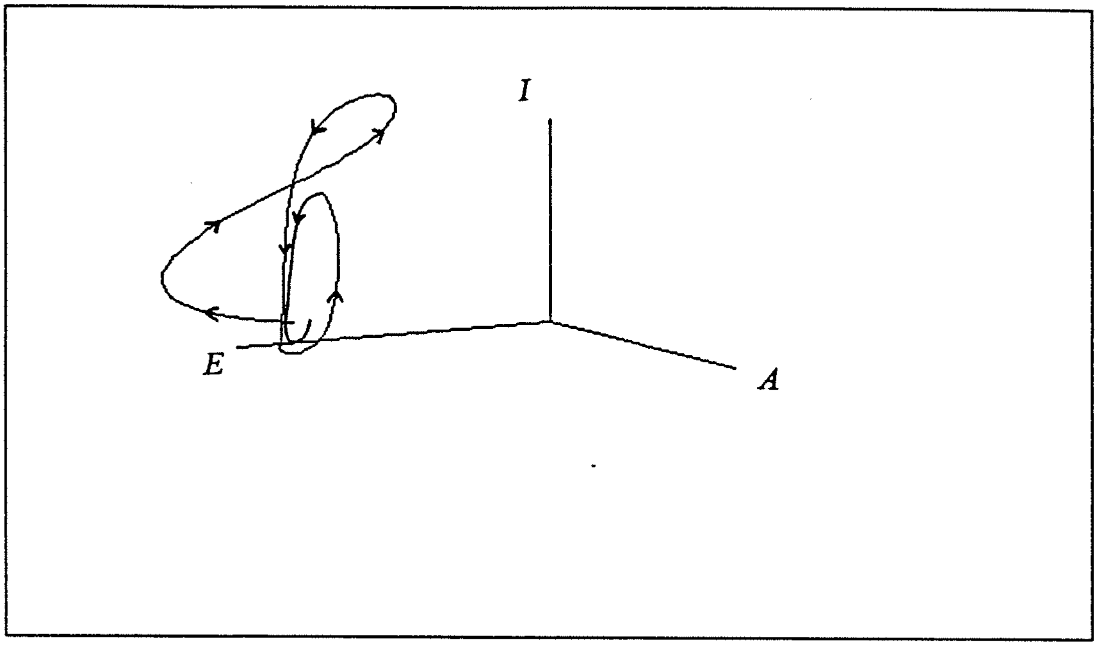

Figure 5. The approximate phase portrait of the homoclinic trajectory for Case II.

Shil'nikov (1965) proved that a system with a homoclinic trajectory, biasymptotic to a saddle focus, may possess under certain conditions a countable set of periodic near-homoclinic trajectories. This property is responsible for the apparent stochastic behaviour of the system. Trajectories are not attracted to some limit set, such as an equilibrium point or a periodic trajectory, but fill some region in the vicinity of the homoclinic trajectory. This kind of behaviour was discussed first by Lorenz (1963) for meteorological systems and was called a strange attractor. The main property of strange attractors is the great sensitivity to initial conditions and unpredictability of the trajectory coordinates for any point in time.

Our system displays a strange attractor in Case II when $e = 9.0$ and $\gamma = 0.33$ (Figure 6).

Strange attractors are interesting in economic systems because of the uncertainty generated by being unable to predict the future course of a trajectory. Systems might be able to manage uncertainty better by understanding the circumstances under which strange attractors may be replaced with more predictable attractors. Avoiding strange attractors diminishes the dependence on initial conditions, which characterizes fatalism in societies.

We have discovered four circumstances stemming from treatment of the ecosphere in the model which eliminates strange attractors.

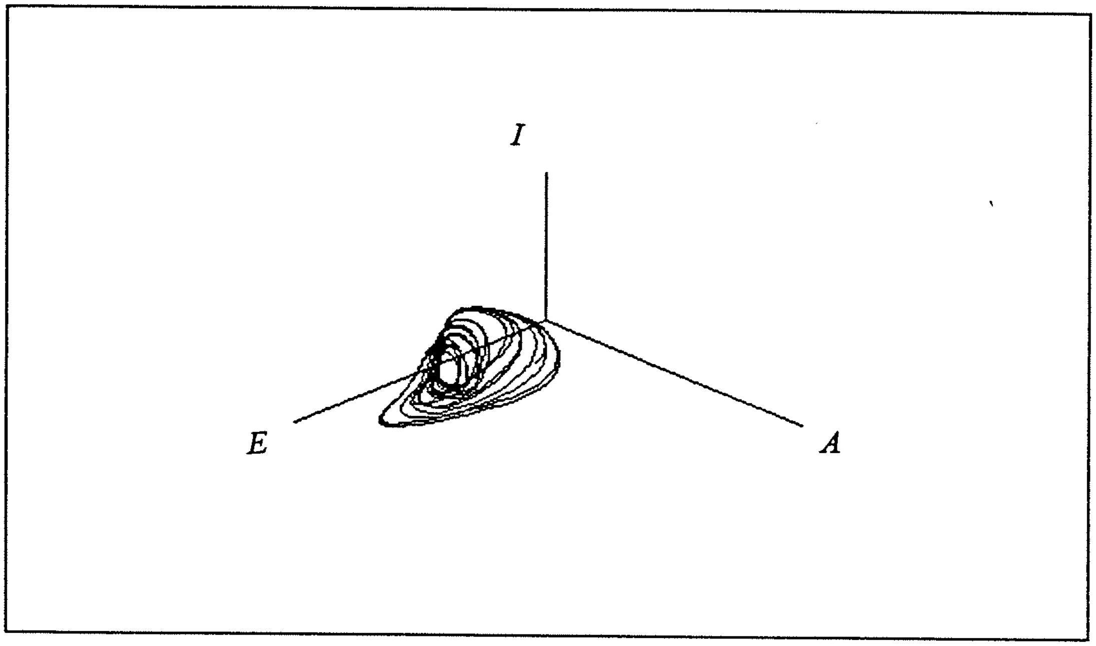

Figure 6. Strange attractor in Case II when $e = 9.0$ and $\gamma = 0.33$

a) Agricultural terms of trade:

The strange attractor disappears when γ increases to 0.78 from 0.33. A global attractor $\hat{F}_3$, in the form of a stable node, is created at a high level of agricultural wealth in the plane $E = E_0$ (Figure 7a). See details in Appendix 2, especially Figure A2.2. All the trajectories are attracted to $\hat{F}_3$. We conclude therefore that a modest shift to more favourable terms of trade for agriculture substantially removes uncertainty in the system behaviour, other things remaining equal.

b) Ecospheric wealth:

The strange attractor is replaced by a stable periodic trajectory as the ecospheric system approaches its 'natural' maximum condition (Figure 7b). Proximity to the highest natural level of ecological wealth is modelled by increasing the value of w to 1.0 from 0.1. Refer to Appendix 1.

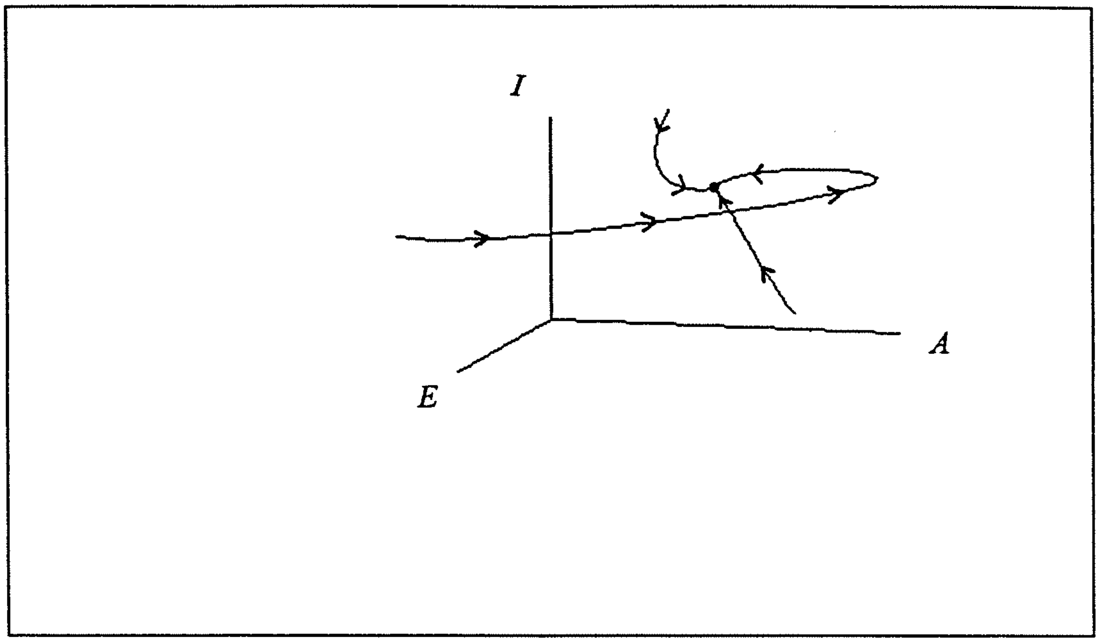

Figure 7a. A modest shift to more favourable terms of trade for agriculture makes the strange attractor disappear, replaced by a global attractor in the form of a stable node, substantially removing the uncertainty in the system's behaviour.

c) Agricultural and industrial recovery rates:

The strange attractor also vanishes if agricultural and industrial recovery rates are increased sharply to 10.0 from 4.0, modelled by $a = b = 0.1$, changed from $a = b = 0.25$. A stable periodic trajectory in the near-homoclinic region appears (Figure 7c).

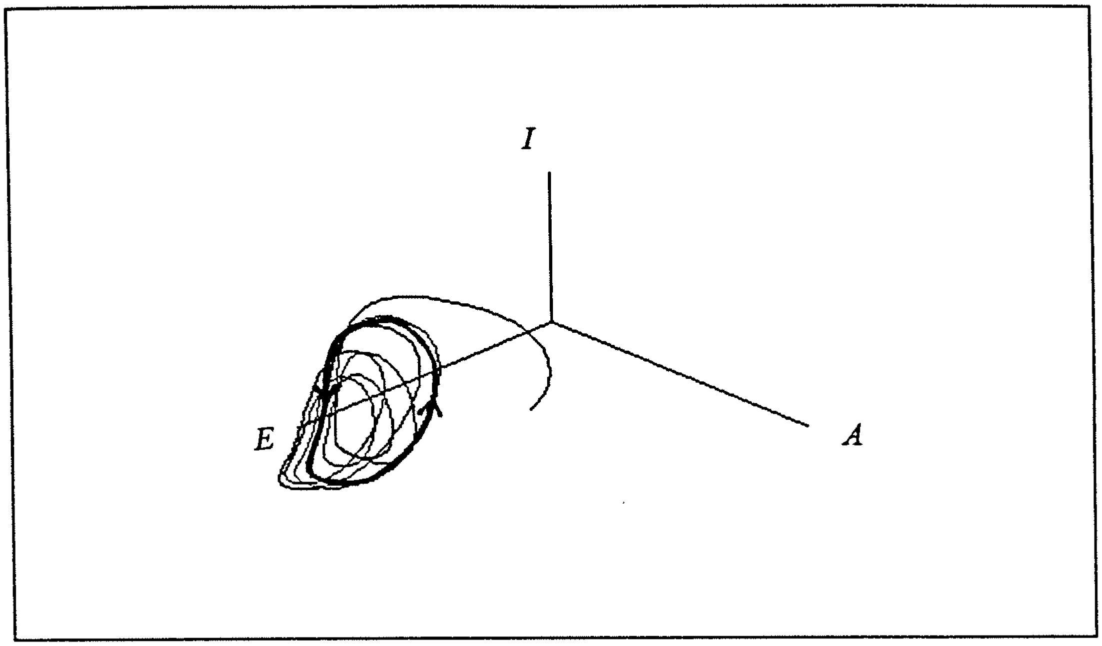

Figure 7b. The strange attractor is replaced by a stable periodic trajectory (shown in bold) as the ecospheric system approaches its 'natural' maximum condition.

d) Ecosphere recovery rate:

The small increase in the ecospheric recovery rate to 1/8.2 from 1/9 leads to the creation of the stable node $\bar{F}_3$. This stable node plays the role of a stable attractor for the system. The strange attractor disappears, and with it the unpredictability.

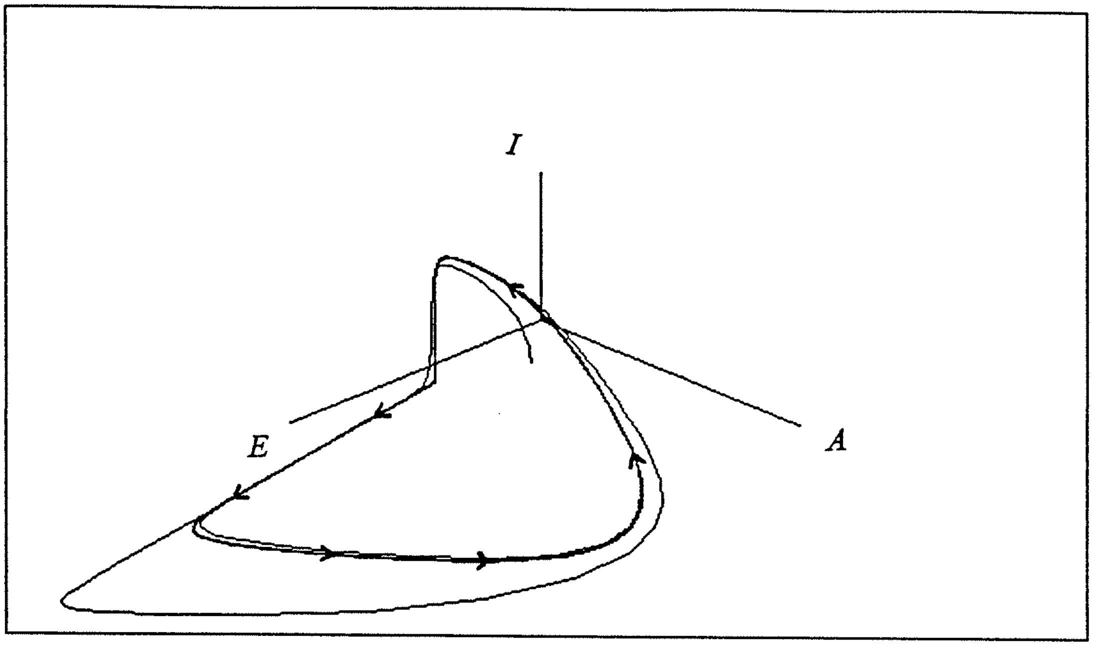

Figure 7c. Increased agricultural and industrial recovery rates remove the strange attractor and a stable periodic trajectory constitutes the omega-limit set of the system.

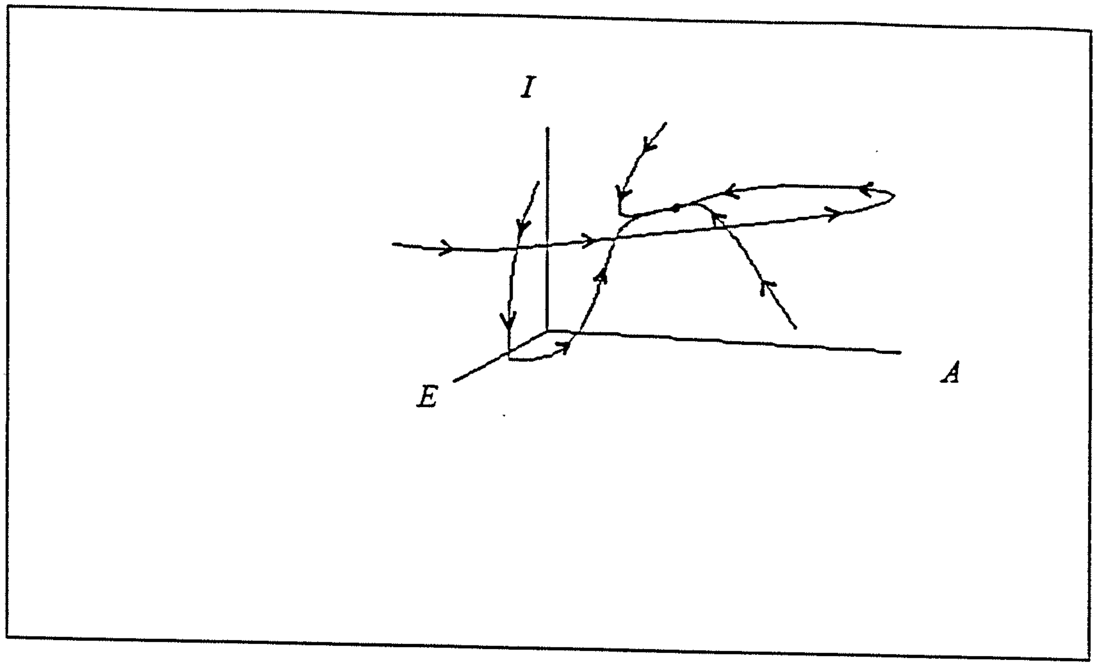

Figure 7d. The strange attractor is replaced by a stable node when the ecospheric recovery rate e^{-1} is slightly increased to 1/8.2 from 1/9 keeping $\gamma = 0.33$

4 Numerical experiments

4.1 basic solutions

4.1.1 Case I: Strong recovery rates for agriculture and industry

This case produces a stable periodic trajectory bounded by a minimum threshold for ecospheric wealth and an upper asymptote for maximum possible ecospheric wealth in the absence of predation by agriculture. This maximum is an artificial form of the ecosphere in which humans are economically inactive. The values of the parameters are contained in Table 1. The trajectory is pictured in Figure 8.

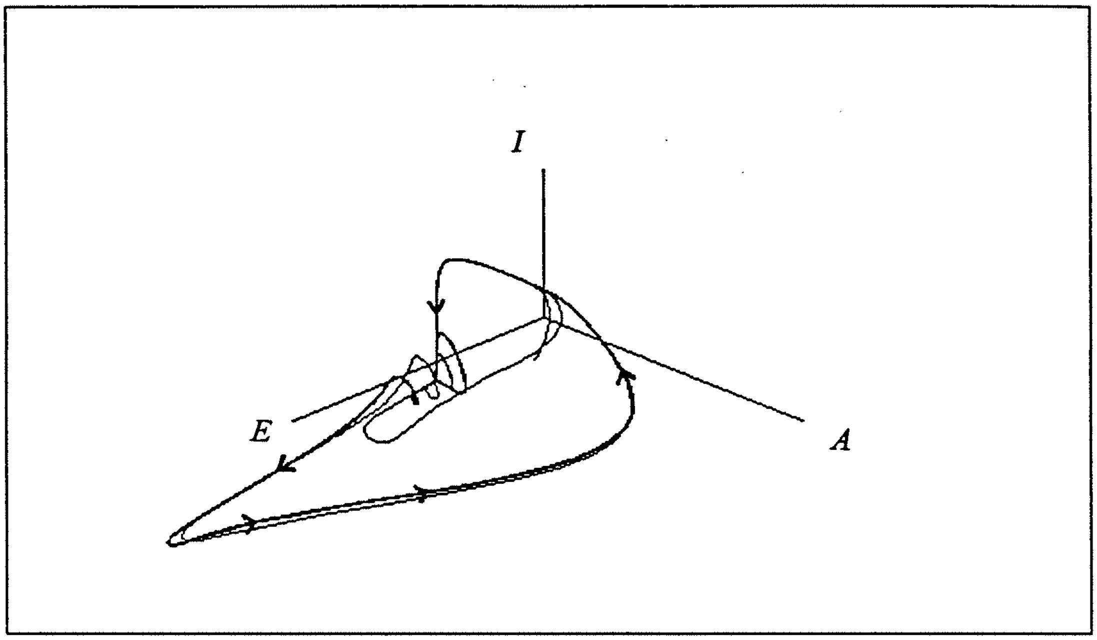

Figure 8. Case I is characterized by the stable periodic trajectory.

When this equilibrium approaches the maximum condition for the ecosphere, as would be the case of extensive agriculture, the omega-limit set is a stable focus instead of a periodic trajectory. This focus occurs close to the lowest levels of agriculture and industry wealth on the trajectory.

This case is not particularly attractive from a sustainability point of view. The economic systems spend most of their histories close to $F_0(0, 0, E_0)$. The ecospheric recovery rates are low relative to the recovery rates for agriculture and industry. This circumstance corresponds to economic systems which have not learned to rehabilitate degraded aspects of the ecosphere.

4.1.2 Case II: Weaker recovery rates for agriculture and industry

This case is the same as Case I with the exception of lower rates of recovery for agriculture and industry. The corresponding omega-limit set is a strange attractor. The region occupied by this attractor is smaller than the one embraced by the stable periodic trajectory of Case I (Figure 9). This result demonstrates the robust nonlinear capabilities of the model, and reveals an economic sensitivity to recovery rates. The strange attractor disappears when the systems are allowed to operate closer to the maximum level of environmental wealth. This intuitive result means that the economy is more predictable when the condition of the ecosphere is nearer its natural maximum.

The advantages of this case is that the low part of the stable periodic trajectory is not as close to the origin as it is in the first case. Furthermore, the systems spend less time, relative to the whole period of the oscillation of the trajectory, restoring the ecosphere at low levels of economic wealth. Of course in both cases the periodic trajectory nevertheless acts as a trap for the economy. Sustainable development therefore requires that the trajectories somehow break away from this stable attractor. This is the context for our investigation of the ways in which equilibria shift and change their stability properties.

4.1.3 Case I and II compared

Case I is characterized by the stable periodic trajectory represented in Figure 8. This trajectory has a much higher range of A and I values compared to Case II. The difference is that the recovery rates of agriculture and industry are larger than for Case II.

Figure 12 depicts the projections of trajectories in the $E = E_0$ plane for both Cases I and II. Both orbits start at the same initial conditions. The strange attractor of Case II fills the interior shaded domain encompassed by the stable periodic trajectory of Case I. Higher levels of agricultural and industrial wealth are attainable for Case I at the expense of larger swings in the fortunes of these two economic systems over time.

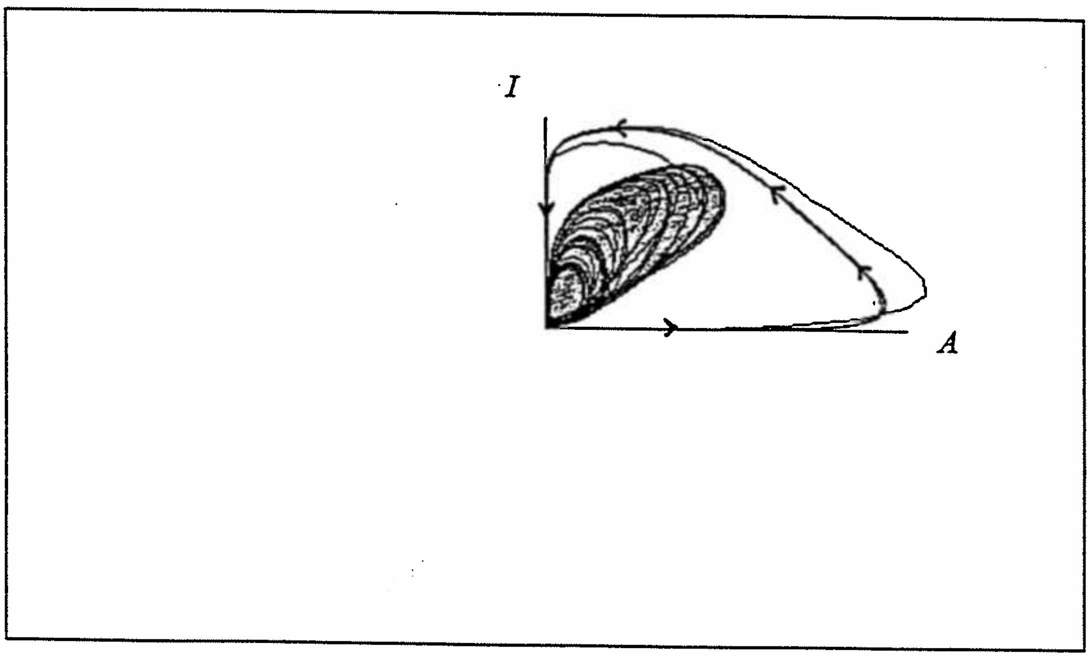

Figure 9. The strange attractor of Case II fills the interior shaded domain encompassed by the stable periodic trajectory of Case I.

4.2 Recovery rate for the ecosphere

A series of experiments are carried out to simulate the equilibrium conditions as the ecospheric recovery rate improves. The ecospheric recovery rate is linked to the average rate of use of the ecosphere's capacity by agriculture. As the recovery rate improves, the economy takes advantage of the improvement to increase the wealth of agriculture and industry.

Ecospheric recovery rates were simulated two ways: The first by increasing the rehabilitation rate u relative to a constant rate of ecospheric degradation v. The second by decreasing the rate of degradation v while maintaining the same rehabilitation rate u. The number of equilibria, their stability properties and their locations on the $E = E_0$ plane are identical either way. Only the results for dynamics, though qualitatively the same, are quantitatively slightly different. Both figures represent the higher levels of agricultural and industrial wealth that are possible as the economic systems learn to increase the recovery rate of the ecosphere.

The trajectories in Figures 6, 7d and 10 represent the changes in the omega-limit sets according to the two approaches to raising the recovery rate of the ecosphere. At the lowest and next lowest recovery rates, there are stable periodic trajectories (Figure 10). For the mid-level recovery rate (e = 9.0), all starting points for the trajectory end around a strange attractor (Figure 6). The higher recovery rates produce stable nodes (Figure 7d). The coordinates of the stable node are higher, the higher the recovery rate.

Replacing the strange attractor by a stable periodic trajectory by reducing e^{-1} to 1/10 from 1/9 keeping $\gamma = 0.33$ is not a particularly favourable outcome. The amplitude of the trajectory is much smaller than that achieved by increasing the agricultural and industrial recovery rates. Reducing the highest possible levels of A and I is a relatively high cost to pay for improved predictability (Figure 10).

High recovery rates and stable nodes have their own drawbacks. Stable nodes connote inflexibility to system response to population growth and demand for higher incomes. All the costs of achieving the high rates of recovery are not captured by the present version of the model.

The origin of the trajectories represents the starting point for planning and development of the economic and ecospheric systems. The presence of at least two basins of attraction on the plane representing the minimum ecospheric threshold results in at least two categories of trajectory. One category originating at generally lower levels of agricultural wealth, irrespective of industrial wealth, may be expected to end in omega-limit sets closer to the origin of the three-dimensional system. Those that are associated with higher recovery rates beyond the bifurcation levels of e, and which originate with higher levels of agricultural wealth, are attracted to omega-limit sets which may preserve or even enhance these levels. Each starting point represents an actual system, such as regional, national, global or household (micro) economy.

As noted above, the present model is not designed for systems achieving below the $E = E_0$ plane. Nevertheless an interesting observation may be made. Trajectories for systems with low ecospheric recovery rates, starting below this plane, experience a rapid descent along the E-axis. This is the ecological disaster scenario. Exceptional circumstances could prevent this unfortunate outcome. If the ecospheric recovery rate is high enough so that the stable equilibrium $\hat{F}_3$ in the $E = E_0$ plane exists, trajectories starting below this threshold, but with high original values of A and I, will converge to $\hat{F}_3$ (Figure 3).

This exception provides an apparent rationale for development assistance among systems. Injections of agricultural and industrial wealth to less developed systems from wealthy systems should reverse the doomsday scenario, when accompanied by improvements to the ecospheric recovery rate and appropriate values for the other parameters. However, this latter condition is usually missing in aid programs. Furthermore, our model suggests that wealth transfers corresponding to 'Big Push' theories of development are probably not a substitute for major restructuring of the relationships among the parameters of system behaviour, especially the predation parameters.

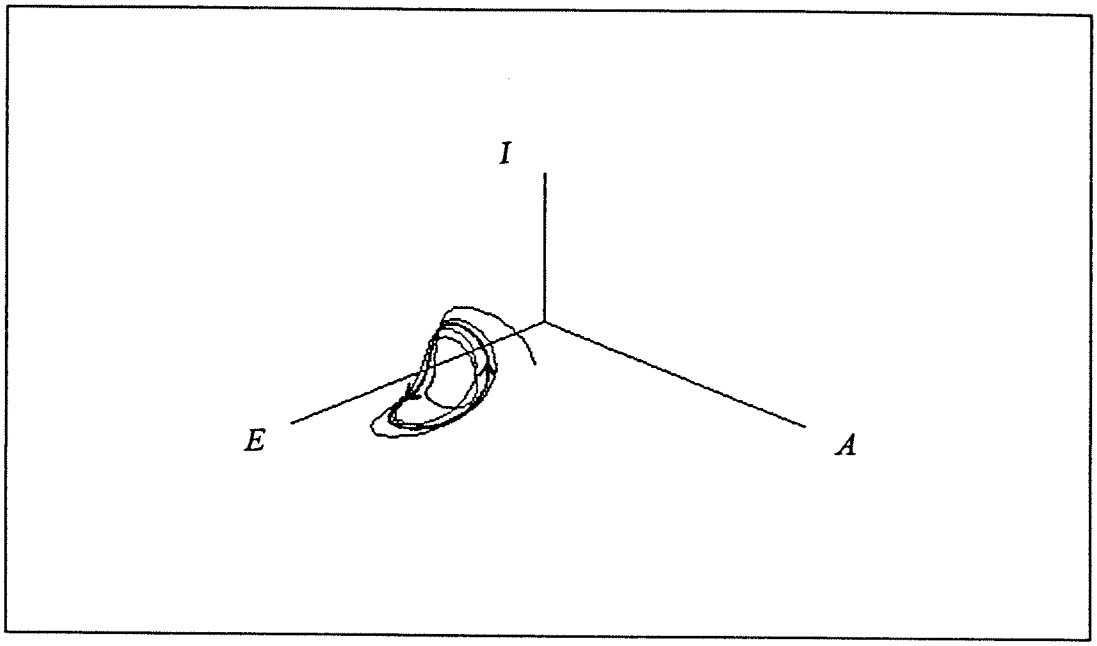

Figure 10. Replacing the strange attractor by a stable periodic trajectory, by reducing e^{-1} to 1/10 from 1/9 keeping $\gamma = 0.33$, is not a particularly favourable outcome.

4.3 Upper achievement bound for the ecosphere

The second and third elements in the equation (1c), describing the rate of change in ecospheric wealth, are unrelated to predation. They describe a logistic relationship between the condition of the ecosphere and time. The coefficient w in the last term allows us to examine the effect on economic development as the systems operate further away from the minimum threshold approaching the maximum or best possible condition of the environment. The approach of E to maximum E is modelled using the parameter w to rescale the logistic, changing its shape, its relation to time and the position of the ecospheric system relative to the maximum. See Appendix 1 for details. When w is closer to 1.0 compared to 0.1 in Cases I and II, the whole system is operating nearer the natural ceiling for ecospheric wealth.

The effect of w is to change the nature of the omega-limit set for the system in each of the two cases. Generally for all initial conditions above the minimum threshold, the effect of increasing the value of w is to suppress the appearance of a strange attractor. In the first case, a stable node occurs. In the second case, a stable periodic trajectory emerges. Thus when the ecospheric system operates close to its natural ceiling, the economic systems stabilize at the highest possible levels of A and I. These results confirm again that a healthy environment introduces predictability into the economy.

4.4 minimum threshold for the ecosphere

The existence of an $E = E_0$ plane appears to be of conceptual importance to modelling persistence of economic systems. To help understand this point, return to Figure 3. This three-dimensional diagram represents the scope for sustainable development defined on the agriculture/industry axes sandwiched between the maximum and minimum thresholds (planes) for ecospheric wealth. The diagram contains trajectories converging from any starting point in this sandwich to a stable node on the minimum threshold of the ecosphere.

Figure 3 illustrates how important it is for humans to learn about the recovery rate of the ecosphere and the terms of trade between the two economic systems. The $E = E_0$ plane may be understood as the upper boundary of a 'risk' zone characterized by great uncertainty and a need for fast learning. An extreme 'over-fishing' problem would be represented figuratively by the trajectories moving 'through' the E_0 plane into the risk zone. In actuality, our model establishes the plane as a separatrix through which trajectories cannot pass. Nevertheless, the economic consequences of approaching the plane are severe as illustrated by Figure 7c in which the two economic systems defer agricultural and industrial wealth for lengthy periods to enable ecospheric recovery.

4.5 terms of trade for agriculture

The terms of trade for agriculture interact with the ecospheric recovery rate. An experiment was carried out to explore this interaction. Adverse but improving agricultural terms of trade were used. Improvement stabilized the equilibria for the five different recovery rates used in the experiment. The only strange attractor which emerged was for the most unfavourable recovery rate. In the other cases, the global attractors are stable nodes. Thus the effect of reducing predation upon agriculture by industry is to reduce uncertainty for the trajectories for all the systems.

The second result of improving terms of trade for agriculture is the achievement of higher levels of wealth as equilibrium $\hat{F}_3$ moves outward (Figure A2.2 in Appendix 2). Agricultural wealth is the first to respond as might be expected. However, industrial wealth soon responds and the economy improves at lower ecospheric recovery rates. Thus favourable agricultural terms of trade appear to be a partial substitute for favourable ecospheric recovery rates.

Substitutability of favourable agricultural terms of trade for ecospheric recovery rate appears to be limited, however. If the ecospheric recovery rate is too low, unpredictability returns, even with relatively favourable agricultural terms of trade. For example, the strange attractor described in Section 3.4.2 disappears when γ increases to 0.78 from 0.33 (Figure 7a), but another strange attractor appears when e^{-1} deteriorates to 1/11.25 from 1/9 (Figure 11). The scope of this substitution merits further research.

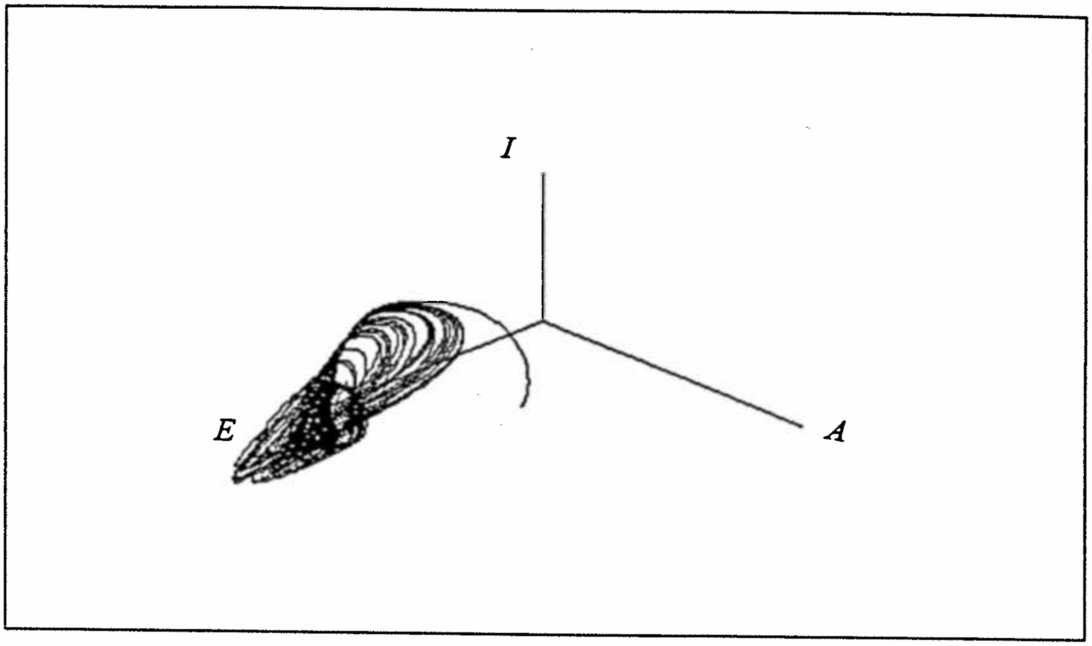

Figure 11. Favourable agricultural terms of trade are offset by a deterioration in the recovery rate for the ecosphere and a new strange attractor appears to destroy predictability.

5 Conclusions

The model represents predatory behaviour by one ecospheric and two economic systems in relation to each other. The model is applicable when humans are conscious of their predation on the ecosphere. It is restricted to circumstances when the minimum ecospheric threshold, E_0 , is greater than zero. Thus it applies to any economy for which the ecosphere is not irretrievably damaged. Behaviour of the system with initial conditions near or below this minimum ecospheric threshold is being explored by further research.

Our interest lies in identifying structural parameters whose change can achieve the following favourable outcomes for the economy and the ecosphere. These desirable outcomes are; increased predictability, smaller radii for oscillations, ways to escape limit cycles, shorter time intervals in economic recession, greater levels of agricultural and industrial wealth, and reduced sensitivity to initial conditions. The structural parameters may be viewed as policy 'variables'.

The first conclusion is that recognition of possible omega-limit sets seem to determine the scope for public policy to manage uncertainty for agricultural economic systems. The relative values for agricultural terms of trade, economic and ecospheric recovery rates and the agricultural capacity utilization rate for the ecosphere determine the presence or absence of strange attractors. Strange attractors are associated with 'chaos' and uncertainty about the future path of the economy. Historical patterns based on time-series analysis are of little help for prediction. So-called stabilization policies may be misdirected in their reliance on periodicity when strange attractors may be responsible for income and price uncertainty.

Predictability is afforded only by avoiding the circumstances that result in strange attractors. Consequently the policy problem could be defined as preventing strange attractors in economic system behaviour. This paper has begun the definition of these conditions for predictability. Further research could address the empirical estimation of the dynamical behaviour of structural parameters and ways to fingerprint the distinctive signatures of economic systems as they search for their equilibria.

The second conclusion is that the system has to determine correctly the minimum threshold for the ecosphere, E_0. The predator prey nature of the model ensures that mutualism brings the solutions to this minimum level. Misunderstanding the mutualism which sets this threshold for the prey system is a recipe for economic disaster. Getting E_0 right appears to be as important as getting agricultural and industrial prices right.

Getting E_0 right is a quality-of-information problem. Information about the ecosphere will always be incomplete and even erroneous. Consequently learning processes appear to be critical to knowing about the minimum threshold for the ecosphere. Unlike the maximum level, the processes of determining E_0 are as much social, political and economic as they are ecological and biological. A pluri-science approach to learning appears to be indicated.

The third conclusion is that the dynamics of the system and the properties of the omega-limit sets determine whether the two economic systems can avoid extreme poverty for long periods of time. Long times are involved when, failing knowledge about the system dynamics, the ecosphere must be restored by deferred predation alone. Enhanced ecosphere recovery rates invoked at higher levels of agricultural and industrial wealth may shorten the recovery time.

The central feature of this third conclusion is that sustained development of agriculture and industry requires two conditions to be fulfilled. The first is that the agricultural system can surpass the levels of wealth associated with the low level equilibrium points. The parameters, specifically, γ, α and e are involved. Secondly, bifurcations seem essential to break the trapping of economies within limit cycles and to changing the dynamics of the system so that the omega-limit set occurs at high levels of A and I.

Together these conditions enable different initial conditions for wealth to result in different forms of development, a property of the model which conforms to evidence in both industrialized and less industrialized countries. Prescriptions for development probably need to be more case specific than is acknowledged by growth and development economics.

The fourth conclusion is that there exist reasonable values and relationships among the parameters for the equilibria to shift outwards in the agriculture/industry plane. One of these favourable circumstances is a combination of improved ecospheric recovery rates and favourable agricultural terms of trade. These offer the possibility of control over the form and extent of mutualism to manage the predation among the systems so as to sustain development. Work is needed on better understanding the process of equilibrium shifts.

Further research is also needed to understand the influence of economies of size upon agricultural and industrial predation. It would be useful to understand how competition and cooperation can alter the predictability of outcomes and the scope for achievement of higher levels of wealth. A third area for research is to incorporate human demographics. Continuing work is required to reconcile the scaling, needed to make the models mathematically tractable, with commonly understood measures of parameters and variables.

6 References

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Appendix 1. **Parameter *w* and saturation level of ecospheric wealth.**

Let us consider equation (1c), responsible for the growth (or depreciation) of ecological wealth, and the meaning of the parameter w.

In the absence of any economic activity ($I=0,\ A=0$) this equation would look like:

$$dE/dt = u(E-E_0) - w(E-E_0)^2. \qquad \text{(A1.1)}$$

The number of parameters in this equation may be reduced to one by rescaling

$$\tilde{E} = E\sqrt{w};\quad \tilde{E}_0 = E_0\sqrt{w};\quad \tau = t\sqrt{w} \qquad \text{(A1.2)}$$

and defining

$$\Gamma \equiv u/\sqrt{w} \qquad \text{(A1.3)}$$

Now equation (A1.1) may be rewritten as

$$d\tilde{E}/d\tau = (\tilde{E} - \tilde{E}_0)[\Gamma - (\tilde{E} - \tilde{E}_0)] \qquad \text{(A1.4)}$$

the solution of this logistic equation with the initial condition

$$\tilde{E}(0) = \tilde{E}_0 + E^* \qquad \text{(A1.5)}$$

under the assumption

$$E^* < \Gamma \qquad \text{(A1.6)}$$

is

$$\tilde{E} = \tilde{E}_0 + \Gamma/(1 + (\Gamma/E^* - 1)e^{-\Gamma\tau}) \qquad \text{(A1.7)}$$

It is seen from this solution that, for $\tau \to \infty$, $E \to \tilde{E}_0 + \tilde{\Gamma}$. Thus the constant Γ defines the saturation level of ecological wealth of the system.

The graph of the function (A1.7) is depicted on the Figure A1.1.

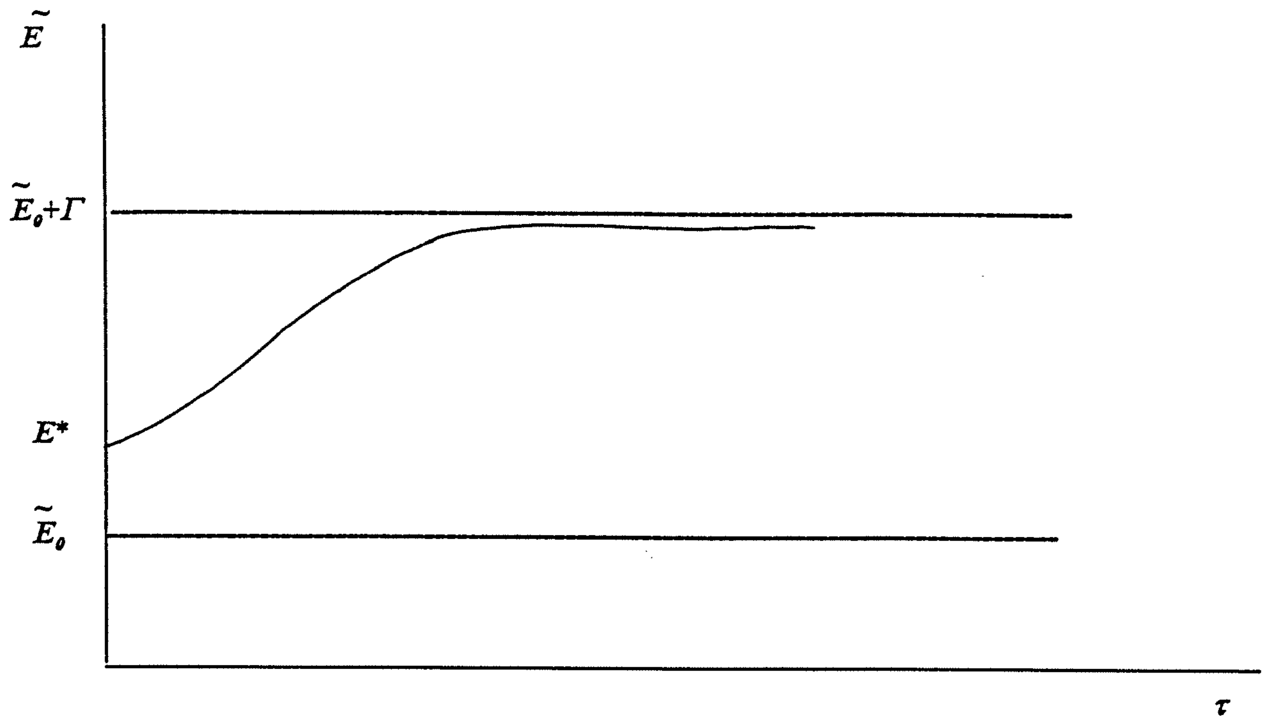

Figure A1.1. As time tends to ∞ the logistic curve approaches the saturation level

Now let us compare two systems S_1 and S_2 at the same instant in time with level $E(t)$ of ecological wealth and equal rehabilitation rates $u_1 = u_2 = u$ and different values w_1 and w_2 of the parameter w. We assume for convenience $w_1 < w_2$.

The latter leads, in accordance with formulae (A1.2, A1.3) to

$$\left.\begin{array}{l}\tilde{E}_1 < \tilde{E}_2 \\ \\ \Gamma_1 > \Gamma_2\end{array}\right\} \qquad \text{(A1.6)}$$

This may be interpreted as follows: The system S_2 is closer to its natural ceiling (saturation level) than the system S_1.

The situation is presented visually in Figure A1.2.

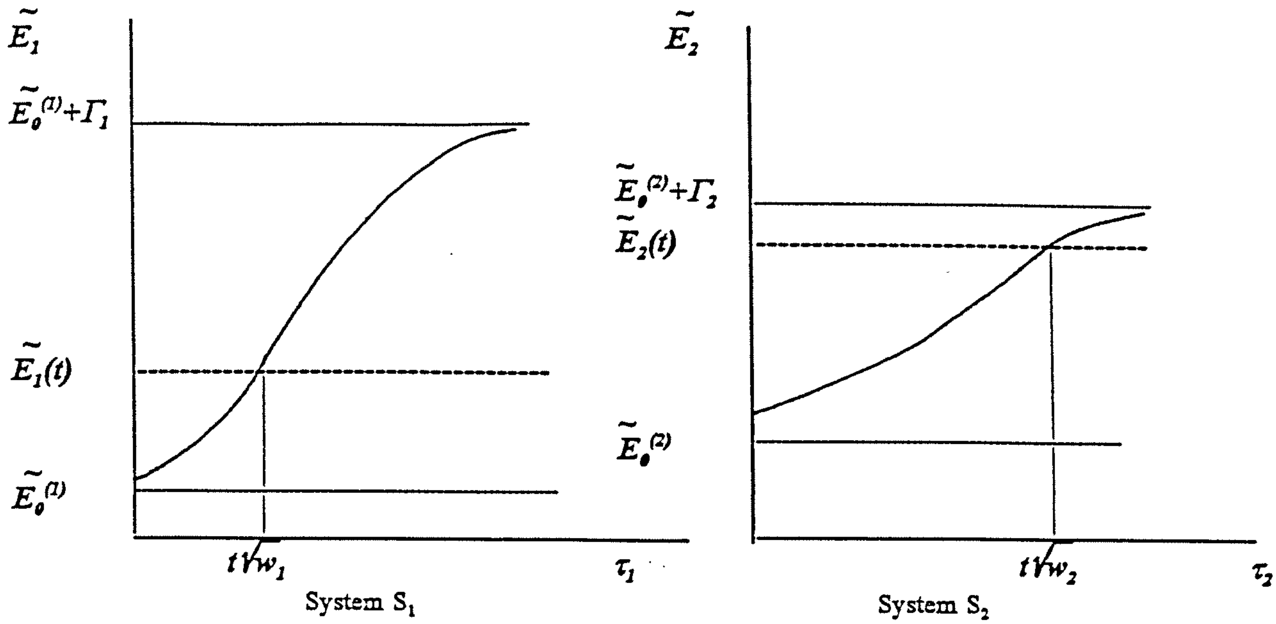

Figure A1.2. Two systems with different w: $w_2 = 4w_1$

For this example given in Figure A1.2 $w_2 = 4w_1$. It is seen that at a given moment t system S_2 almost attaines its saturation level $\tilde{E}_0^{(2)}+\Gamma_2$, and system S_1 is still on its way, far from the corresponding "ceiling" $\tilde{E}_0^{(1)}+\Gamma_1$.

Thus, by taking w large we can model the closeness of an ecosystem to its highest possible natural level of ecospheric wealth.

Appendix 2. Equilibria: Their Number, Locations and Corresponding Eigenvalues.

A2.1) Number of equilibria on the plane $E = E_0$: dependence on ecological recovery rate e^{-1} and agricultural terms of trade γ/δ.

As demonstrated in the paper by Apedaile et al., 1994, the two-dimensional reduction of the system (1a-c) obtained by assumption $E = E_0$ surpasses a saddle-node bifurcation when parameter ξ exceeds a certain value ξ_0. The bifurcation results in the creation of two new equilibria--a saddle point and a stable node. The last is important because as one can see in section 4.2 the stable node may become a global attractor for the 3-dimensional system.

In this Appendix we demonstrate visually the creation of the new equilibria as a result of a change in the recovery rate of the ecosphere e^{-1} and agricultural terms of trade γ/δ. The latter can be modified by changing the agricultural price level γ.

Each equilibrium is a point of intersection of the isoclines corresponding to system (2) on the plane $E = E_0$. Two isoclines always exist as far as the system of two independent equations and of two variables (A and I) are concerned. One of these isoclines does not depend on the parameter e. This isocline is shown in bold in Figure A2.1. The other isocline does depend on e. Different representations of this isocline correspond to different values of e. Note here that Figure A2.1 corresponds to Case II, with $\gamma = 0.78$.

Parameter e is proportional to the ratio v/u. Therefore, there are two simple ways of changing e:

1) by changing the restoration rate, u, and keeping v, the degradation rate constant,
2) by changing v and keeping u constant.

We explore both these ways. As one can see from equations (1a), (1b) and (2), the sets of equilibria in the $E = E_0$ plane are the same independently of the way chosen. For both these cases, the results are qualitatively, and with high accuracy, quanitively the same. The accuracy demonstrates the robustness of the model.

As seen from the Figure A2.1, for $e > e_7 \approx 10.3$ the system possesses only one equilibrium F in the first quadrant. This equilibrium is a stable focus F_4 (Table A2.1). $e = e_7$ is a bifurcation value of the parameter e. At this value of e the two isoclines are tangent to each other. Thus, they acquire one more common point; a saddle-node is born! For e smaller than e_7 and larger than e_3 the system possesses three equilibria --a stable focus F_4, a saddle point F_5, and a stable node F_6. See the example in Table A2.2.

When e becomes less than $e_3 \approx 8.14$, another bifurcation value, the system loses two of its equilibria and preserves the third. The preserved equilibrium is a stable node F_4, located at a high level of agricultural wealth A (Table A2.3).

Similar consideration may be given to the parameter γ. The corresponding picture is presented on Figure A2.2. $\gamma_3 \approx 0.49$ and $\gamma_5 \approx 0.89$ are bifurcation values. Thus, by increasing γ, making terms of trade more favourable for agriculture, the system can be forced to have only one equilibrium with a high level of A.

A2.2) Locations of the equilibria; eigenvalues of the linearized system.

The results reported in this section are obtained numerically using an original program written in C language (see Tables A2.1-A2.3). The program solves the system (2) and at each equilibrium point calculates the variational matrix of the system (1a-c) and its eigenvalues λ_i, i=1,2,3.

Table A2.1

Parameters: e = 11.25; u = 0.40; v = 2.00; γ = 7/9; δ = 2.00; a = 0.25; b = 0.25; η = 0.00; ξ = 1.00; α_0 = 20.00; β = 1.00; w = 0.10; E_0 = 1.00; μ = 1.00

F_i=(A; I; E)	λ_0	λ_1	λ_2
F_0=(0.000; 0.000; 1.000)	1.633	0.4	-1.000
F_1=(1.633; 0.000; 1.000)	5.938	-1.632658	-2.865
F_2=(0.000; 0.000; 5.000)	6.154	-0.4	-1.000
F_3=(0.243; 0.000; 0.138)	2.944	0.789737	-0.947
F_4=(0.067; 0.171; 1.000)	-0.071 ± 0.790i		0.267
F_5=(0.181; 0.591; 1.371)	0.200 ± 0.477i		-0.484

Table A2.2

Parameters: e = 10.00; u = 0.45; v = 2.00; γ = 7/9; δ = 2.00; a = 0.25; b = 0.25; η = 0.00; ξ = 1.00; α_0 = 20.00; β = 1.00; w = 0.10; E_0 = 1.00; μ = 1.00

F_i=(A; I; E)	λ_0	λ_1	λ_2
F_0=(0.000; 0.000; 1.000)	1.818	0.450	-1.000
F_1=(1.818; 0.000; 1.000)	6.033	-1.818	-3.186
F_2=(0.000; 0.000; 5.500)	7.097	-0.450	-1.000
F_3=(0.268; 0.000; 0.136)	3.141	0.875	-1.057
F_4=(0.076; 0.217; 1.000)	-0.067 ± 0.746i		0.298
F_5=(0.511; 1.093; 1.000)	0.2948	-0.572	-0.742
F_6=(0.981; 1.344; 1.000)	-0.370	-0.787	-1.512
F_7=(0.215; 0.675; 1.193)	0.231 ± 0.344i		-0.540

Table A2.3

Parameters: e = 7.50; u = 0.60; v = 2.00; γ = 7/9; δ = 2.00; a = 0.25; b = 0.25; η = 0.00; ξ = 1.00; α_0 = 20.0; β = 1.00; w = 0.10; E_0 = 1.00; μ = 1.00

F_i=(A; I; E)	λ_0	λ_1	λ_2
F_0=(0.000; 0.000; 1.000)	2.353	0.600	-1.000
F_1=(2.353; 0.000; 1.000)	6.232	-2.353	-4.106
F_2=(0.000; 0.000; 7.000)	9.655	-0.600	-1.000
F_3=(0.343; 0.000; 0.131)	3.630	1.130	-1.387
F_4=(1.855; 1.512; 1.000)	-0.874	-1.400	-3.109
F_5=(0.308; 0.854; 0.834)	0.800	-0.370	-0.560

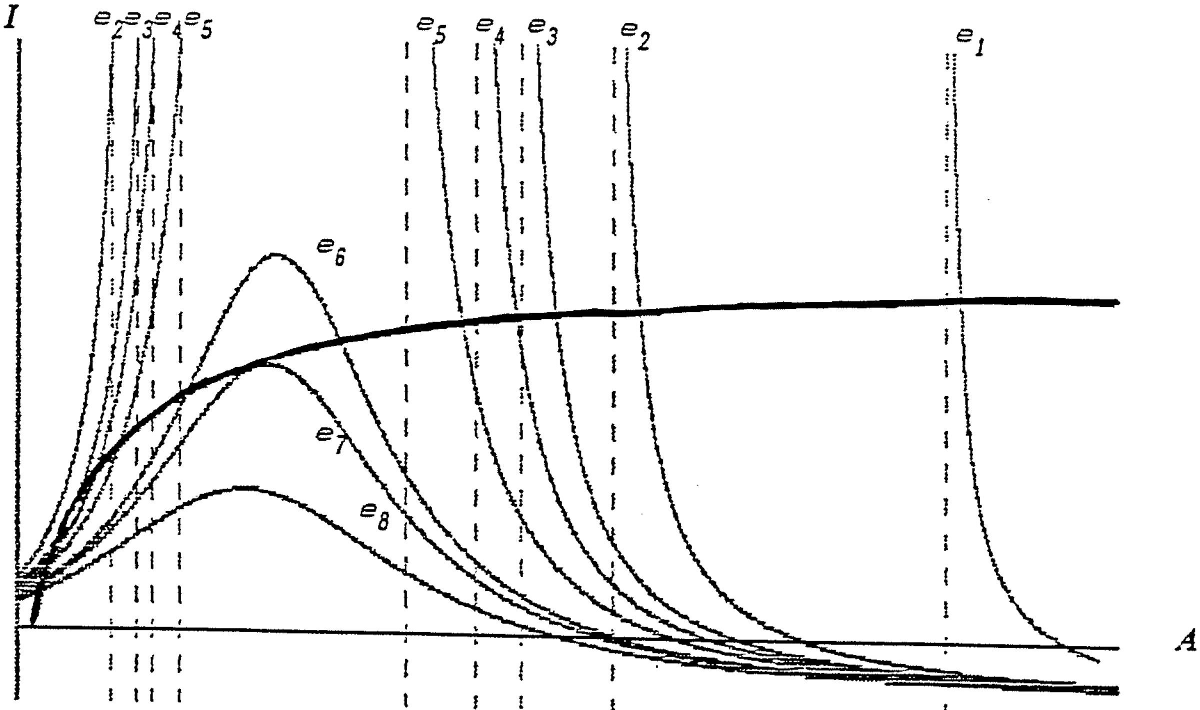

Figure A2.1. Isoclines corresponding to the different values of e: $e_1 = 2.0$; $e_2 = 7.5$; $e_3 = 8.14$; $e_4 = 8.18$; $e_5 = 9.0$; $e_6 = 10.0$; $e_7 = 10.31$; $e_8 = 11.25$. e_3 and e_7 are bifurcation values of e.

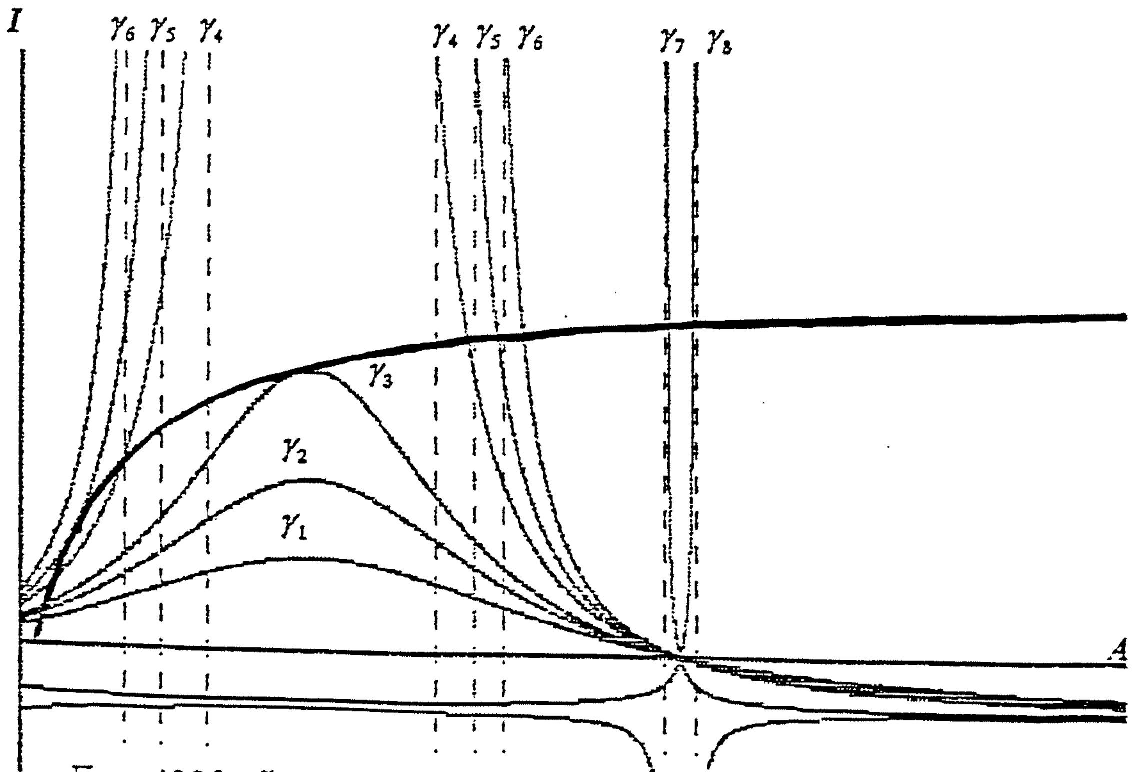

Figure A2.2 Isoclines corresponding to the different values of γ/δ: $\delta = 2.0$ for all the isoclines; $\gamma_1 = 0.0$; $\gamma_2 = 0.33$; $\gamma_3 = 0.49$; $\gamma_4 = 0.78$; $\gamma_5 = 0.89$; $\gamma_6 = 1.0$; $\gamma_7 = 1.9$; $\gamma_8 = 2.1$. γ_3 and γ_5 are bifurcation values of γ.