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THE SIGNALLING PROBLEM IN NONLINEAR HYPERBOLIC WAVE THEORY

by

YUANPING HE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled The Signalling Problem in Nonlinear Hyperbolic Wave Theory submitted by Yuanping He in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

This work is devoted to investigating weakly nonlinear hyperbolic waves arising from the action of small-amplitude, high-frequency boundary disturbances. By directly introducing a nonlinear phase variable corresponding to the leading wavefront and specifying a 'single-wave mode' boundary disturbance, we are able to construct an asymptotic solution and perform the requisite shock calculations. Furthermore, our result shows that, by properly arranging the relation of small-amplitude to high-frequency, a systematic procedure can be provided for constructing weakly nonlinear wave solutions with interior shocks and determining the shock initiation position (and time), when there is a local linear degeneracy at the leading wavefront.

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CHAPTER I

Introduction

Over the past decade, the study of asymptotic methods for the analysis of weakly nonlinear hyperbolic waves generated by small-amplitude high-frequency disturbances, has produced many new and important results. Based upon an earlier work of Choquet-Bruhat [1], who considered a single-wave mode in several space dimensions, Hunter and Keller [2] established a general nonresonant, multi-wave mode theory which they called the weakly nonlinear geometrical optics. Majda and Rosales [3], and Hunter *et al.* [4] later extended this theory to handle resonantly interacting, multi-wave mode problems. These studies, which concentrated on the small-amplitude high-frequency feature, provided a deeper understanding of nonlinear wave processes. Some earlier related references are to be found in [5]-[8].

In this paper, we shall develop a single-wave mode theory for the study of weakly nonlinear hyperbolic waves generated by small-amplitude, high-frequency boundary disturbances in one space dimension. We shall deal with quasi-linear, totally (or strictly) hyperbolic systems of the form

$$\boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u}, x)\boldsymbol{u}_x = \boldsymbol{b}(\boldsymbol{u}, x), \tag{1-1}$$

where $x \geq 0$ is the space variable, $t \geq 0$ is time, $\boldsymbol{u}$ is the vector of n state variables, $\boldsymbol{A}(\boldsymbol{u}, x)$ and $\boldsymbol{b}(\boldsymbol{u}, x)$ are smooth matrix and vector functions of their arguments, respectively, and subscripts refer to differentiation with respect to the corresponding variable.

As distinct from [1]-[4], in which initial value problems were studied, we consider a small-amplitude, high-frequency, single-wave mode boundary perturbation about the steady state $\boldsymbol{u} \equiv \mathbf{0}$ of the quasi-linear hyperbolic system (1-1). The perturbation takes the form

$$\boldsymbol{u}|_{x=0} = \epsilon\sigma_0(t/\delta)\boldsymbol{r}_0(0) + O(\epsilon^2), \quad t \geq 0, \tag{1-2}$$

where $\sigma_0(\cdot)$ is a smooth scalar function with compact support, $\boldsymbol{r}(\boldsymbol{u}, x)$ is the right eigenvector associated with a positive eigenvalue $\lambda = \lambda(\boldsymbol{u}, x)$ of $\boldsymbol{A}(\boldsymbol{u}, x)$, and

$\boldsymbol{r}(\mathbf{0}, x) \triangleq \boldsymbol{r}_0(x)$, $\lambda(\mathbf{0}, x) \triangleq \lambda_0(x)$. Also introduced in (1-2) are ϵ and δ, representing small but not independent positive parameters to describe this small-amplitude, high-frequency feature. We assume that

$$\delta = \epsilon^m, \tag{1-3}$$

where $m \geq 1$ is an integer. This is an extended version of previous results, since in [1]-[4] only the case $m = 1$ has been considered. Also to be noted in (1-2) is the $O(\epsilon^2)$ term which surfaces to ensure the well-posedness of the perturbation problem.

If instead of dealing with the linear phase variable as in [2], we introduce directly the nonlinear phase variable $\theta = \theta(x, t)$ corresponding to $\lambda = \lambda(\boldsymbol{u}, x)$

and define it as the solution of

$$\begin{cases} \theta_t + \lambda(\boldsymbol{u}, x)\theta_x = 0, \\ \theta|_{x=0} = t/\delta = t/\epsilon^m, \quad t \geq 0, \end{cases} \tag{1-4a,b}$$

then our asymptotic solution has the form

$$\boldsymbol{u} = \sum_{k=1}^{m} \frac{\epsilon^k}{k!}\sigma_0^k(\theta)\boldsymbol{r}_{k-1}(x) + O(\epsilon^{m+1}). \tag{1-5}$$

Also, the arrival time formula, which gives the time for phase θ to arrive at position x, is

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \sum_{k=1}^{m-1} \frac{\epsilon^k}{k!}\,\sigma_0^k(\theta)\int_0^x [\Lambda_k]_{\sigma_0\equiv 1}ds$$

$$+ \epsilon^m\{\theta + \frac{1}{m!}\sigma_0^m(\theta)\int_0^x [\Lambda_m]_{\sigma_0\equiv 1}ds\} \tag{1-6}$$

$$+ O(\epsilon^{m+1}),$$

where in the above, $\boldsymbol{r}_k(x)$ and $[\Lambda_k]_{\sigma_0\equiv 1}$ $(k = 1, 2, \ldots)$ are vector and scalar functions of x and depend only on the local behavior of the quasi-linear hyperbolic system about its steady state solution. These quantities are obtainable from the explicit solutions of linear algebraic systems and will be described later.

From (1-5), we find that the $O(\epsilon^2)$ term in (1-2) is specified when $m > 1$. In fact, as it will be shown in this paper, there exists a class of boundary disturbances in the form of (1-2) which admit the asymptotic solution (1-5). If we further require, as in the case of most applications, that one component of $\boldsymbol{u}$, say u_1, be specified on the boundary in the form

$$u_1|_{x=0} = \epsilon\sigma_0(t/\delta)r_{0,1}(0), \tag{1-7}$$

where $O(\epsilon^2)$ term vanishes (it is assumed without loss of generality that $r_{0,1}(0) \neq 0$, i.e., the first component of $\boldsymbol{r}_0(0)$ is nonzero), then the boundary disturbance, and hence the solution (1-5), is uniquely determined.

The freedom to choose m provides a great advantage. It turns out that the larger m is, corresponding to a steeper boundary pulse, the more accurate the asymptotic solution will be. As it is known that [9]

$$t_\theta = 0 \tag{1-8}$$

is a criterion for determining shock occurrence, (1-6) provides a way of detecting interior shocks and determining their precise location. Furthermore, it is possible to construct a weakly nonlinear solution with interior shocks. It turns out that a particular case of interest arises when

$$[\Lambda_k]_{\sigma_0\equiv 1} \equiv 0, \quad k = 1, 2, \ldots, q-1,$$

$$[\Lambda_q]_{\sigma_0\equiv 1} \not\equiv 0.$$

Then, by letting $m = q$, we find from (1-6) that

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \epsilon^q\{\theta + \frac{1}{q!}\,\sigma_0^q(\theta)\int_0^x [\Lambda_q]_{\sigma_0\equiv 1} ds\} \tag{1-9}$$

$$+ O(\epsilon^{q+1}),$$

so that (1-8) can be approximated by

$$1 + \frac{1}{(q-1)!}\,\sigma_0^{q-1}(\theta)\sigma_0'(\theta)\int_0^x [\Lambda_q]_{\sigma_0\equiv 1} ds = 0, \tag{1-10}$$

giving the location of shock initiation. When, for example, $q = 2$ this recovers an earlier result by Moodie and Swaters [9]. We shall include it as a particular application.

This paper is organized as follows: In Chapter II, after some preliminaries and notation are explained, the so-called signalling problem is formulated in §1. Then in §2 we introduce the nonlinear phase variable corresponding to the leading wavefront and carry out the eikonal transformation for the signalling problem. By considering the asymptotic expansion of the solution in Chapter III we arrive at the $O(1), O(\epsilon)$ and $O(\epsilon^k)$ problems. When these problems are solved in Chapter IV, § 1 it is found that for the solution to be approximated by a single-wave mode the boundary disturbance must take the form (1-2). Then, using the arrival time formula in the following section, conditions under which shocks will occur are discovered and expressions giving the position and time of the first shock occurrence are derived. The last chapter includes an example in which the recent results by Moodie and Swaters [9] for an inhomogeneous hyperelastic fluid filled tube problem are recovered.

CHAPTER II

Nonlinear Phase Variable and Eikonal Transformation

§1. Preliminaries and Notation

Throughout this paper, (1-1) is assumed to be totally (or strictly) hyperbolic, i.e., $\boldsymbol{A}(\boldsymbol{u},x)$ has n distinct and real eigenvalues $\{\lambda_i(\boldsymbol{u},x)\}_{i=1}^{n}$ with

$$\lambda_1(\boldsymbol{u},x) < \lambda_2(\boldsymbol{u},x) < \cdots < \lambda_n(\boldsymbol{u},x). \tag{2-1}$$

We further assume

$$\lambda_1(\boldsymbol{u},x) < \lambda_2(\boldsymbol{u},x) < \cdots < \lambda_p(\boldsymbol{u},x) \leq 0 < \lambda_{p+1}(\boldsymbol{u},x) < \cdots < \lambda_n(\boldsymbol{u},x), \tag{2-2}$$

when positive and negative eigenvalues are distinguished.

We denote

$$\boldsymbol{\ell}^{(i)} = \boldsymbol{\ell}^{(i)}(\boldsymbol{u},x), \quad \boldsymbol{r}^{(i)} = \boldsymbol{r}^{(i)}(\boldsymbol{u},x)$$

as the left and right eigenvectors, respectively, associated with $\lambda_i = \lambda_i(\boldsymbol{u},x)$. They satisfy

$$\boldsymbol{\ell}^{(i)}\boldsymbol{A} = \lambda_i\boldsymbol{\ell}^{(i)}, \quad \boldsymbol{A}\boldsymbol{r}^{(i)} = \lambda_i\boldsymbol{r}^{(i)}, \quad i = 1,2,\ldots,n, \tag{2-3}$$

and the orthonormality condition

$$\boldsymbol{\ell}^{(i)}\boldsymbol{r}^{(j)} = \delta_{ij}, \quad i,j = 1,2,\ldots,n, \tag{2-4}$$

where δ_{ij} is the Kronecker function. In addition, we shall call each $r_0^{(i)}(x) \triangleq r^{(i)}(0, x)$ a *wave mode* of the system (1-1).

As defined by Lax [10], the i-th characteristic field is genuinely nonlinear if

$$(\mathrm{grad}_{\boldsymbol{u}} \lambda_i) \boldsymbol{r}^{(i)} \neq 0 \tag{2-5}$$

or, on the other hand, linearly degenerate, if

$$(\mathrm{grad}_{\boldsymbol{u}} \lambda_i) \boldsymbol{r}^{(i)} \equiv 0. \tag{2-6}$$

These are global conditions. For the convenience of our local asymptotic discussions, a modification must be given. Let $\boldsymbol{u} = \boldsymbol{u}_0(x, t)$ be a solution of (1-1), then the i-th characteristic field is called *locally linearly degenerate* about $\boldsymbol{u} = \boldsymbol{u}_0$ if

$$(\mathrm{grad}_{\boldsymbol{u}} \lambda_i) \boldsymbol{r}^{(i)} \equiv 0, \tag{2-7}$$

when evaluated at $\boldsymbol{u} = \boldsymbol{u}_0$.

Now, assuming a small amplitude, high-frequency disturbance is excited on the boundary $x = 0$, we may expect a weakly nonlinear wave to propagate into the steady state region. When our discussion is restricted to the region

$x \geq 0,\ t \geq 0,$ the so-called 'signalling problem' is formed, i.e.,

$$\boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u}, x)\boldsymbol{u}_x = \boldsymbol{b}(\boldsymbol{u}, x), \tag{1-1}$$

$$\boldsymbol{u} \equiv \boldsymbol{0}, \qquad x \geq 0, \quad t = 0, \tag{2-8}$$

$$\boldsymbol{u} = \epsilon \boldsymbol{g}_\epsilon(t/\delta), \quad x = 0, \quad t \geq 0, \tag{2-9}$$

where $\boldsymbol{g}_\epsilon(\cdot)$ is a smooth vector function with compact support and, in particular,

$$\boldsymbol{g}_\epsilon(0) = \boldsymbol{g}'_\epsilon(0) = \boldsymbol{0}, \tag{2-10}$$

are assumed. In addition, $\boldsymbol{g}_\epsilon(\cdot)$ is supposed Taylor expandable about the small parameter ϵ,

$$\boldsymbol{g}_\epsilon(\cdot) = \boldsymbol{g}^{(0)}(\cdot) + \epsilon \boldsymbol{g}^{(1)}(\cdot) + \frac{\epsilon^2}{2!}\boldsymbol{g}^{(2)}(\cdot) + \cdots . \tag{2-11}$$

As it is known, the well-posedness of the signalling problem relies on $\boldsymbol{g}_\epsilon(\cdot)$, which can not be arbitrarily given. The readers may refer to [11] (Chapter 4, Theorem 3.1) for a well established sufficient condition which ensures the existence and uniqueness of a smooth solution for the signalling problem in a finite time layer. However, in this paper, $\boldsymbol{g}_\epsilon(\cdot)$ is not given but will be determined in the course of the solution.

If $\boldsymbol{g}_\epsilon(\cdot)$ is a wave-mode, i.e., $\boldsymbol{g}_\epsilon(\cdot)$ is parallel to $\boldsymbol{r}_0^{(i)}(0)$ for some i, we shall call $\boldsymbol{g}_\epsilon(\cdot)$ a 'single-wave mode'. Since this will not happen in general, as we shall later see, we adopt the convention that $\boldsymbol{g}_\epsilon(\cdot)$ is a 'single-wave mode' if its leading term $\boldsymbol{g}^{(0)}(\cdot)$ is.

§2. Nonlinear Phase Variable and Eikonal Transformation

Let us pick a positive eigenvalue from $\lambda_{p+1}(\boldsymbol{u},x),\dots,\lambda_n(\boldsymbol{u},x)$, and for simplicity, denote it by

$$\lambda = \lambda(\boldsymbol{u},x),$$

as before, and $\boldsymbol{\ell} = \boldsymbol{\ell}(\boldsymbol{u},x)$, $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{u},x)$ as the corresponding left and right eigenvectors.

Now suppose $t = T(x)$, which is defined by

$$\begin{cases} \dfrac{dt}{dx} &= \dfrac{1}{\lambda_0(x)}, \\ t(0) &= 0, \end{cases} \tag{2-12}$$

corresponds to the leading wavefront, i.e., the time for the first disturbance to arrive at position x.

We introduce the nonlinear phase variable $\theta = \theta(x,t)$ associated with $\lambda = \lambda(\boldsymbol{u},x)$ and define it as the solution of

$$\theta_t + \lambda(\boldsymbol{u},x)\theta_x = 0, \tag{1-4a}$$

$$\theta|_{x=0} = t/\delta(\epsilon), \quad t \geq 0. \tag{2-13}$$

The existence and smoothness of $\theta = \theta(x,t)$ is guaranteed by the existence and smoothness of $\boldsymbol{u} = \boldsymbol{u}(x,t)$. The inverse function of $\theta = \theta(x,t)$ gives the so-called arrival time formula, i.e., the time for wave of phase θ to arrive at position x, which we write as

$$t = T(x, \theta; \epsilon). \tag{2-14}$$

In particular, $\theta = 0$ corresponds to the leading wavefront.

We regard θ as an independent variable and transform the signalling problem (1-1), (2-8), (2-9) into (x, θ) coordinates

$$(x,t) \mapsto (x, \theta), \tag{2-15}$$

by

$$\begin{cases} \theta &= \theta(x,t), \\ x &= x. \end{cases} \tag{2-16a,b}$$

Rewriting

$$\boldsymbol{u}(x,t) = \boldsymbol{U}(x, \theta; \epsilon), \tag{2-17}$$

then

$$\begin{cases} \partial_x &\mapsto \partial_x - \dfrac{\theta_t}{\lambda}\,\partial_\theta, \\ \\ \partial_t &\mapsto \theta_t \partial_\theta, \end{cases} \tag{2-18a,b}$$

and the signalling problem is transformed to

$$\theta_t\{I - \frac{A(U,x)}{\lambda(U,x)}\}U_\theta = b(U,x) - A(U,x)U_x, \tag{2-19}$$

$$U \equiv 0, \quad \theta = 0, \quad x \geq 0, \tag{2-20}$$

$$U = \epsilon g_\epsilon(\theta), \quad x = 0, \quad \theta \geq 0, \tag{2-21}$$

where ahead of the leading wavefront $u \equiv 0$ is noted.

Meanwhile, the relation that

$$t \equiv T(x, \theta(x,t); \epsilon), \tag{2-22}$$

gives

$$\begin{cases} \theta_t &= T_\theta^{-1}, \\ \theta_x &= -T_x/T_\theta, \end{cases} \tag{2-23a,b}$$

when (2-22) is differentiated with respect to t and x respectively. Then a substitution of (2-23) into (1-4a) and (2-13) results in

$$\begin{cases} T_x &= \dfrac{1}{\lambda(U,x)}, \\ T|_{x=0} &= \delta(\epsilon)\theta. \end{cases} \tag{2-24a,b}$$

The validity of the eikonal transformation (2-15) depends on the condition that the Jacobian

$$\frac{\partial(x,\theta)}{\partial(x,t)} = \theta_t \neq 0, \tag{2-25}$$

and

$$T_\theta = \theta_t^{-1} \neq 0. \tag{2-26}$$

It is easy to check that the transformation is valid in the neighborhood of $(x, \theta) = (0, 0)$ since

$$0 < \theta_t|_{x=0} = \frac{1}{\delta(\epsilon)} < +\infty. \tag{2-27}$$

One of the most important features of nonlinear hyperbolic systems is that the solution, however smooth it is at first, may develop shocks. A criterion for detecting such shocks is that

$$T_\theta = 0, \tag{2-28}$$

which in turn represents the breakdown of the transformation (2-15).

CHAPTER III

Weakly Nonlinear Waves and the Transport Equations

We construct the asymptotic solution for (2-19)-(2-21) in the form

$$\boldsymbol{U}(x,\theta;\epsilon)=\epsilon\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\boldsymbol{U}^{(k)}(x,\theta), \tag{3-1}$$

where

$$\boldsymbol{U}^{(k)}(x,\theta),\quad \boldsymbol{U}_x^{(k)}(x,\theta),\quad \boldsymbol{U}_\theta^{(k)}(x,\theta)=O(1),\quad k=0,1,2,\dots, \tag{3-2}$$

are required, i.e., $\boldsymbol{U}^{(k)}(x,\theta)\ (k=0,1,2,\dots)$ are independent of ϵ when θ is viewed as an independent variable.

To find the asymptotic expansion solution, first we rewrite (2-19) (use (2-23a)) as

$$(\boldsymbol{I}-\boldsymbol{A}/\lambda)\boldsymbol{U}_\theta=(\boldsymbol{b}-\boldsymbol{A}\boldsymbol{U}_x)T_\theta, \tag{3-3}$$

and then form the transport equation by applying $\boldsymbol{\ell}$ to both sides of (3-3) obtaining

$$\boldsymbol{\ell b}-\boldsymbol{\ell A U}_x=0. \tag{3-4}$$

Also we note that the integration of (2-24) would give

$$\begin{aligned} t&=T(x,\theta;\epsilon)\\ &=\delta(\epsilon)\theta+\int_0^x\frac{ds}{\lambda(\boldsymbol{U}(s,\theta;\epsilon),s)}\,. \end{aligned} \tag{3-5}$$

Based on (3-3)–(3-5), we are able to construct a weakly nonlinear wave solution of (2-19)–(2-21) in the form of the asymptotic expansion (3-1), subject to (3-2), when $g_\epsilon(\cdot)$ is of a single-wave mode.

Before proceeding, we need to Taylor expand the following (matrix, vector and scalar) functions about $\boldsymbol{u} \equiv \boldsymbol{0}$:

$$\frac{1}{\lambda(\boldsymbol{U},x)} = \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \Lambda^{(k)}(\overbrace{\boldsymbol{U},\ldots,\boldsymbol{U}}^{k})/k!, \tag{3-6}$$

$$\boldsymbol{A}(\boldsymbol{U},x) = \sum_{k=0}^{\infty} \boldsymbol{A}^{(k)}(\overbrace{\boldsymbol{U},\ldots,\boldsymbol{U}}^{k})/k!, \tag{3-7}$$

$$\boldsymbol{b}(\boldsymbol{U},x) = \sum_{k=1}^{\infty} \boldsymbol{b}^{(k)}(\overbrace{\boldsymbol{U},\ldots,\boldsymbol{U}}^{k})/k!, \tag{3-8}$$

$$(\lambda\boldsymbol{\ell})(\boldsymbol{U},x) = \sum_{k=0}^{\infty} \boldsymbol{a}^{(k)}(\overbrace{\boldsymbol{U},\ldots,\boldsymbol{U}}^{k})/k!, \tag{3-9}$$

$$(\boldsymbol{\ell b})(\boldsymbol{U},x) \triangleq c(\boldsymbol{U},x) = \sum_{k=1}^{\infty} c^{(k)}(\overbrace{\boldsymbol{U},\ldots,\boldsymbol{U}}^{k})/k!, \tag{3-10}$$

$$(\frac{1}{\lambda}\boldsymbol{A})(\boldsymbol{U},x) \triangleq \boldsymbol{D}(\boldsymbol{U},x) = \sum_{k=0}^{\infty} \boldsymbol{D}^{(k)}(\overbrace{\boldsymbol{U},\ldots,\boldsymbol{U}}^{k})/k!, \tag{3-11}$$

where

$$\boldsymbol{A}^{(0)} = \big(a_{ij}(\boldsymbol{0},x)\big)_{n\times n},$$

and

$$A^{(k)}(\overbrace{U,\dots,U}^{k}) = \Big(\sum_{k_1+\cdots+k_n=k} (\partial^k a_{ij}/\partial u_1^{k_1}\dots\partial u_n^{k_n})_0 U_1^{k_1}\dots U_n^{k_n} \Big)_{n\times n},$$

$$b^{(k)}(\overbrace{U,\dots,U}^{k}) = \Big(\sum_{k_1+\cdots+k_n=k} (\partial^k b_i/\partial u_1^{k_1}\dots\partial u_n^{k_n})_0 U_1^{k_1}\dots U_n^{k_n} \Big)_{n\times 1},$$

$$a^{(k)}(\overbrace{U,\dots,U}^{k}) = \Big(\sum_{k_1+\cdots+k_n=k} (\partial^k (\lambda\ell_i)/\partial u_1^{k_1}\dots\partial u_n^{k_n})_0 U_1^{k_1}\dots U_n^{k_n} \Big)_{1\times n},$$

$$c^{(k)}(\overbrace{U,\dots,U}^{k}) = \sum_{k_1+\cdots+k_n=k} (\partial^k c/\partial u_1^{k_1}\dots\partial u_n^{k_n})_0 U_1^{k_1}\dots U_n^{k_n},$$

are matrix, vector and scalar valued k-linear forms, respectively.

Now substituting (3-1) into (3-6)–(3-11) and rewriting (3-6)–(3-11) as

$$\frac{1}{\lambda(U,x)} = \frac{1}{\lambda_0(x)} + \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} \Lambda_j(x,\theta), \tag{3-12}$$

$$A(U,x) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} A_j(x,\theta), \tag{3-13}$$

$$b(U,x) = \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} b_j(x,\theta), \tag{3-14}$$

$$(\lambda\ell)(U,x) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} a_j(x,\theta), \tag{3-15}$$

$$c(\boldsymbol{U}, x) = \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} c_j(x, \theta), \tag{3-16}$$

$$\boldsymbol{D}(\boldsymbol{U}, x) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} \boldsymbol{D}_j(x, \theta), \tag{3-17}$$

we have the following lemma:

LEMMA. *For* $j \geq 1$,

$$\Lambda_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} \Lambda^{(k)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_k)}), \tag{3-18}$$

$$\boldsymbol{A}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} \boldsymbol{A}^{(k)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_k)}), \tag{3-19}$$

$$\boldsymbol{b}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} \boldsymbol{b}^{(k)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_k)}), \tag{3-20}$$

$$\boldsymbol{a}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} \boldsymbol{a}^{(k)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_k)}), \tag{3-21}$$

$$c_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} c^{(k)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_k)}), \tag{3-22}$$

$$\boldsymbol{D}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} \boldsymbol{D}^{(k)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_k)}), \tag{3-23}$$

where $\binom{j}{k\, i_1 \ldots i_k} = \frac{j!}{k! i_1! \ldots i_k!}$ *subject to* $k + i_1 + \cdots + i_k = j$.

Proof: We need only prove (3-18). Employing (3-1) in (3-6) to obtain

$$\frac{1}{\lambda(U,x)} = \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \Lambda^{(k)}(U,\ldots,U)/k!$$

$$= \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \frac{1}{k!} \Lambda^{(k)}(\epsilon \sum_{i_1=0}^{\infty} \frac{\epsilon^{i_1}}{i_1!} U^{(i_1)}, \ldots, \epsilon \sum_{i_k=0}^{\infty} \frac{\epsilon^{i_k}}{i_k!} U^{(i_k)})$$

$$= \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \sum_{i_1,i_2,\ldots,i_k=0}^{\infty} \frac{\epsilon^{k+i_1+\cdots+i_k}}{k! i_1! \ldots i_k!} \Lambda^{(k)}(U^{(i_1)}, \ldots, U^{(i_k)})$$

$$= \frac{1}{\lambda_0(x)} + \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} \sum_{\substack{k+i_1+\cdots+i_k=j \\ k\geq 1}} \binom{j}{k\, i_1 \ldots i_k} \Lambda^{(k)}(U^{(i_1)}, \ldots, U^{(i_k)}),$$

we have therefore that

$$\Lambda_j = \sum_{\substack{k+i_1+\cdots+i_k=j \\ k\geq 1}} \binom{j}{k\, i_1 \ldots i_k} \Lambda^{(k)}(U^{(i_1)}, \ldots, U^{(i_k)})$$

$$= \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\, i_1 \ldots i_k} \Lambda^{(k)}(U^{(i_1)}, \ldots, U^{(i_k)}).$$

This completes the proof.

Now we solve for (3-1) by first substituting (3-12) into (3-5), and then inserting the derived equation together with (3-1), (3-13)–(3-17) into (3-3), (3-4) to form the $O(1), O(\epsilon)$ and $O(\epsilon^k)$ problems.

The substitution of (3-12) into (3-5) gives

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \delta(\epsilon)\theta + \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} \int_0^x \Lambda_j(s,\theta)ds$$

$$\triangleq \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} T^{(j)}(x,\theta). \tag{3-24}$$

To match the order of $\delta(\epsilon)$ with ϵ, we require that

$$\delta(\epsilon) = \epsilon^m, \tag{1-3}$$

where $m \geq 1$ is an integer. m is chosen to be an integer in order to balance the two sides of (3-3) and $m \geq 1$ is required so that the as yet to be derived $O(1)$ problem is homogeneous. As we explained, the freedom to choose m is a particular advantage. Now in (3-24) we have

$$T^{(0)}(x,\theta) = \int_0^x \frac{ds}{\lambda_0(s)}, \tag{3-25a}$$

$$T^{(j)}(x,\theta) = \int_0^x \Lambda_j(s,\theta)ds, \quad j \neq m, \tag{3-25b}$$

$$T^{(m)}(x,\theta) = m!\theta + \int_0^x \Lambda_m(s,\theta)ds. \tag{3-25c}$$

After replacing (3-1), (3-12)–(3-17), (3-24) in (3-3), (3-4), equating like powers of ϵ we obtain through somewhat tedious but straightforward calculation:

$O(1)$ *problem*

$$(\boldsymbol{I}-\frac{\boldsymbol{A}_0}{\lambda_0})\,\boldsymbol{U}_\theta^{(0)}=\boldsymbol{0}, \tag{3-26}$$

$$c^{(1)}(\boldsymbol{U}^{(0)})-\boldsymbol{a}_0\boldsymbol{U}_x^{(0)}=0, \tag{3-27}$$

$O(\epsilon)$ *problem*

$$(\boldsymbol{I}-\frac{\boldsymbol{A}_0}{\lambda_0})\boldsymbol{U}_\theta^{(1)}=\boldsymbol{M}_1(x,\theta), \tag{3-28}$$

$$c^{(1)}(\boldsymbol{U}^{(1)})-\boldsymbol{a}_0\boldsymbol{U}_x^{(1)}=N_1(x,\theta), \tag{3-29}$$

where

$$\boldsymbol{M}_1(x,\theta)=\boldsymbol{D}_1\boldsymbol{U}_\theta^{(0)}+T_\theta^{(1)}(\boldsymbol{b}_1-\boldsymbol{A}_0\boldsymbol{U}_x^{(0)}), \tag{3-30}$$

$$N_1(x,\theta)=\boldsymbol{a}_1\boldsymbol{U}_x^{(0)}-c^{(2)}(\boldsymbol{U}^{(0)},\boldsymbol{U}^{(0)}), \tag{3-31}$$

$O(\epsilon^k)$ *problem*

$$(\boldsymbol{I}-\frac{\boldsymbol{A}_0}{\lambda_0})\boldsymbol{U}_\theta^{(k)}=\boldsymbol{M}_k(x,\theta), \tag{3-32}$$

$$c^{(1)}(\boldsymbol{U}^{(k)})-\boldsymbol{a}_0\boldsymbol{U}_x^{(k)}=N_k(x,\theta), \tag{3-33}$$

where

$$\boldsymbol{M}_k(x,\theta) = \sum_{\substack{k_1+k_2=k\\k_1>0}} \binom{k}{k_1\ k_2} \boldsymbol{D}_{k_1} \boldsymbol{U}_\theta^{(k_2)}$$

$$- \sum_{\substack{k_1+k_2+k_3=k\\k_1>0}} \binom{k}{k_1\ k_2\ k_3} T_\theta^{(k_1)} \boldsymbol{A}_{k_2} \boldsymbol{U}_x^{(k_3)}$$

$$+ \frac{1}{k+1} \sum_{\substack{k_1+k_2=k+1\\k_1,k_2>0}} \binom{k+1}{k_1\ k_2} \boldsymbol{b}_{k_1} T_\theta^{(k_2)}, \tag{3-34}$$

$$N_k(x,\theta) = \sum_{\substack{k_1+k_2=k\\k_1>0}} \binom{k}{k_1\ k_2} \boldsymbol{a}_{k_1} \boldsymbol{U}_x^{(k_2)}$$

$$- \frac{1}{k+1} \sum_{j=2}^{k+1} \sum_{i_1+\cdots+i_j=k+1-j} \binom{k+1}{j\ i_1 \ldots i_j} c^{(j)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_j)}). \tag{3-35}$$

CHAPTER IV

Weakly Nonlinear Wave Solutions and Shock Calculation

§1. Solutions to the $\mathbf{O(1), O(\epsilon)}$ and $\mathbf{O(\epsilon^k)}$ $(\mathbf{k < m})$ Problems

THE $O(1)$ PROBLEM

First we solve the $O(1)$ problem. Equation (3-26) shows that $\boldsymbol{U}_\theta^{(0)}$ is parallel to $\boldsymbol{r}_0(x)$ so that $\boldsymbol{U}^{(0)}$ can be written as

$$\boldsymbol{U}^{(0)}(x,\theta) = \sigma(x,\theta)\boldsymbol{r}_0(x), \tag{4-1}$$

when the condition

$$\boldsymbol{U}^{(k)}(x,\theta)|_{\theta=0} = \mathbf{0}, \quad k = 0,1,2,\ldots, \tag{4-2}$$

due to

$$\boldsymbol{U}(x,\theta;\epsilon)|_{\theta=0} = \mathbf{0}, \tag{4-3}$$

is noted and applied.

Denoting $\sigma(x,\theta)|_{x=0} = \sigma_0(\theta)$, then the boundary condition (2-21) at $x = 0$ leads to

$$\boldsymbol{g}^{(0)}(\theta) = \sigma_0(\theta)\boldsymbol{r}_0(0),$$

or

$$\boldsymbol{g}^{(0)}(\cdot) = \sigma_0(\cdot)\boldsymbol{r}_0(0). \tag{4-4}$$

Hence we come to the conclusion that only a single-wave mode perturbation such as (4-4) on the boundary would admit the asymptotic expansion (3-1) which is subject to (3-2). If the leading term of the boundary perturbation is not of a single-wave mode, then (3-1), (3-2) would fail and wave interaction may come into play. In other words, (3-1), (3-2) allow us to distinguish a class of (nonresonant) nonlinear waves.

$\sigma(x,\theta)$ can be determined through the transport equation (3-27). Applying (4-1) to (3-27) we obtain

$$\sigma_x - \Gamma_0(x)\sigma = 0, \tag{4-5}$$

where

$$\Gamma_0(x) = \frac{c^{(1)}(r_0) - a_0 r_0'}{a_0 r_0}\,. \tag{4-6}$$

Integrate to get

$$\sigma(x,\theta) = \sigma_0(\theta)\exp\{\int_0^x \Gamma_0(s)ds\}. \tag{4-7}$$

For the sake of clarity in the following discussions, we may 'normalize' r_0 at this stage by requiring that r_0 satisfy

$$\Gamma_0(x) \equiv 0, \tag{4-8}$$

so that

$$\sigma(x,\theta) = \sigma_0(\theta). \tag{4-9}$$

In fact, this procedure can be done simply by replacing $\boldsymbol{r}_0$ with

$$\tilde{\boldsymbol{r}}_0 = \boldsymbol{r}_0 \exp\{\int_0^x \Gamma_0(s)ds\}, \tag{4-10}$$

and it is easy to check that

$$\tilde{\Gamma}_0 = \frac{c^{(1)}(\tilde{\boldsymbol{r}}_0) - \boldsymbol{a}_0\tilde{\boldsymbol{r}}_0'}{\boldsymbol{a}_0\tilde{\boldsymbol{r}}_0} \equiv 0. \tag{4-11}$$

We shall always assume that $\boldsymbol{r}_0$ is 'normalized' unless otherwise stated. Therefore we have

$$\boldsymbol{U}^{(0)}(x,\theta) = \sigma_0(\theta)\boldsymbol{r}_0(x). \tag{4-12}$$

THE $O(\epsilon)$ PROBLEM

We consider two cases: $m > 1$ and $m = 1$.

CASE 1. $m > 1$.

Substituting (4-12) into (3-30), (3-31) to obtain

$$\boldsymbol{M}_1(x,\theta) = \boldsymbol{D}^{(1)}(\boldsymbol{U}^{(0)})\boldsymbol{U}_\theta^{(0)} + (\boldsymbol{b}^{(1)}(\boldsymbol{U}^{(0)}) - \boldsymbol{A}_0\boldsymbol{U}_x^{(0)}) \int_0^x \Lambda^{(1)}(\boldsymbol{U}_\theta^{(0)})ds$$

$$= \{\boldsymbol{D}^{(1)}(\boldsymbol{r}_0)\boldsymbol{r}_0 + (\boldsymbol{b}^{(1)}(\boldsymbol{r}_0) - \boldsymbol{A}_0\boldsymbol{r}_0') \int_0^x \Lambda^{(1)}(\boldsymbol{r}_0)ds\}\sigma_0(\theta)\sigma_0'(\theta), \tag{4-13}$$

$$\triangleq \boldsymbol{p}_1(x)\sigma_0(\theta)\sigma_0'(\theta),$$

$$
\begin{aligned}
N_1(x,\theta) &= \boldsymbol{a}_1 \boldsymbol{U}_x^{(0)} - c^{(2)}(\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}) \\
&= \{\boldsymbol{a}^{(1)}(\boldsymbol{r}_0)\boldsymbol{r}_0' - c^{(2)}(\boldsymbol{r}_0, \boldsymbol{r}_0)\}\sigma_0^2(\theta) \qquad (4\text{-}14) \\
&\triangleq q_1(x)\sigma_0^2(\theta),
\end{aligned}
$$

i.e., the separation of variables holds, and the $O(\epsilon)$ problem now takes the form

$$(\boldsymbol{I} - \frac{\boldsymbol{A}_0}{\lambda_0})\boldsymbol{U}_\theta^{(1)} = \boldsymbol{p}_1(x)\sigma_0(\theta)\sigma_0'(\theta), \qquad (4\text{-}15)$$

$$c^{(1)}(\boldsymbol{U}^{(1)}) - \boldsymbol{a}_0\boldsymbol{U}_x^{(1)} = q_1(x)\sigma_0^2(\theta). \qquad (4\text{-}16)$$

We exclude the case of σ_0 or $\sigma_0' \equiv 0$ to avoid trivial solutions. Let $\boldsymbol{r}_1(x)$ be a particular solution of

$$(\boldsymbol{I} - \frac{\boldsymbol{A}_0}{\lambda_0})\boldsymbol{r}_1 = \boldsymbol{p}_1(x), \qquad (4\text{-}17)$$

which vanishes when $\boldsymbol{p}_1(x)$ is identically zero and therefore

$$\boldsymbol{U}_\theta^{(1)} = \sigma_0(\theta)\sigma_0'(\theta)\{\sigma_1(x,\theta)\boldsymbol{r}_0(x) + \boldsymbol{r}_1(x)\}, \qquad (4\text{-}18)$$

where $\sigma_1(x,\theta)$ is to be determined by the transport equation (4-16). Differentiate (4-16) with respect to θ and then apply (4-18) to obtain

$$(\sigma_1)_x - \Gamma_0(x)\sigma_1 = K_1(x), \qquad (4\text{-}19)$$

where

$$K_1(x) = \frac{c^{(1)}(\boldsymbol{r}_1) - \boldsymbol{a}_0\boldsymbol{r}_1' - 2q_1}{\boldsymbol{a}_0\boldsymbol{r}_0}\,. \tag{4-20}$$

Since $\boldsymbol{r}_0$ is 'normalized,' $\Gamma_0 \equiv 0$, and hence

$$(\sigma_1)_x = K_1(x),$$

or

$$\begin{aligned} \sigma_1(x,\theta) &= \sigma_1(0,\theta) + \int_0^x K_1(s)ds \\ &\triangleq \sigma_1(\theta) + \int_0^x K_1(s)ds. \end{aligned} \tag{4-21}$$

Now we apply (1-7) to determine $\sigma_1(\theta)$. From (1-7) it follows that

$$U_1^{(k)}|_{x=0} = 0, \qquad k = 1, 2, \ldots,$$

and hence

$$U_{1,\theta}^{(k)}|_{x=0} = 0, \qquad k = 1, 2, \ldots\,. \tag{4-22}$$

Applying the case $k = 1$ to (4-18), (4-21) gives

$$\sigma_1(\theta) r_{0,1}(0) + r_{1,1}(0) \equiv 0$$

or, since $r_{0,1}(0) \neq 0$,

$$\sigma_1(\theta) \equiv -\frac{r_{1,1}(0)}{r_{0,1}(0)}\,, \tag{4-23}$$

i.e., $\sigma_1(\theta)$ is a constant, where $r_{0,1}(x)$ and $r_{1,1}(x)$ are the first components of $\boldsymbol{r}_0(x)$ and $\boldsymbol{r}_1(x)$, respectively.

Now, (4-18) becomes

$$\boldsymbol{U}_\theta^{(1)} = \sigma_0(\theta)\sigma_0'(\theta)\{\boldsymbol{r}_0(x)[-\frac{r_{1,1}(0)}{r_{0,1}(0)} + \int_0^x K_1(s)ds] + \boldsymbol{r}_1(x)\}. \tag{4-24}$$

We may 'normalize' $\boldsymbol{r}_1(x)$ in the sense that

$$\begin{cases} K_1(x) \equiv 0, \\ r_{1,1}(0) \equiv 0. \end{cases} \tag{4-25a,b}$$

This procedure may be carried out simply by replacing $\boldsymbol{r}_1(x)$ with

$$\tilde{\boldsymbol{r}}(x) = \boldsymbol{r}_0(x)[-\frac{r_{1,1}(0)}{r_{0,1}(0)} + \int_0^x K_1(s)ds] + \boldsymbol{r}_1(x). \tag{4-26}$$

We always assume that $\boldsymbol{r}_1(x)$ is 'normalized' unless otherwise stated, and hence that

$$\boldsymbol{U}_\theta^{(1)} = \sigma_0(\theta)\sigma_0'(\theta)\boldsymbol{r}_1(x). \tag{4-27}$$

Integrate (using (4-2)) to get

$$\boldsymbol{U}^{(1)}(x,\theta) = \frac{1}{2}\,\sigma_0^2(\theta)\boldsymbol{r}_1(x). \tag{4-28}$$

NOTE. (i). As we shall see, in order to proceed to the resolution of higher order problems, we need $\boldsymbol{U}_\theta^{(1)}$ to take the form (4-27), which in turn requires only that $\sigma_1(\theta) \equiv$ constant. While the condition (4-22) guarantees $\sigma_1(\theta)$ to be a constant, other similar conditions, such as the j-th component of $\boldsymbol{U}^{(k)}$ vanishing at $x = 0$ for all higher order $(k \geq 1)$ terms are also available, especially when $r_{0,1}(0) = 0$. In general, all choices of $\sigma_1(\theta) =$ constant, together with $\sigma_k(\theta) =$ constant, as we shall soon see, consist of the class of boundary disturbances to admit solutions of the form (1-5).

(ii) In the case of $\sigma_1(\theta)$ remaining as a function of θ, we may rewrite (4-18), (4-21) and integrate to get

$$\boldsymbol{U}^{(1)} = \frac{1}{2}\,\sigma_0^2(\theta)\boldsymbol{r}_1(x) + \sigma^*(\theta)\boldsymbol{r}_0(x).$$

This corresponds to the interaction of two boundary disturbances of the same wave mode, one being of greater amplitude than the other, that is,

$$\boldsymbol{U} = \boldsymbol{U}^{[1]} + \boldsymbol{U}^{[2]}, \quad x = 0$$

where

$$\boldsymbol{U}^{[1]}|_{x=0} = \epsilon\sigma_0(\theta)\boldsymbol{r}_0(0) + \frac{1}{2}\,\epsilon^2\sigma_0^2(\theta)\boldsymbol{r}_1(0),$$

$$\boldsymbol{U}^{[2]}|_{x=0} = \epsilon^2\sigma^*(\theta)\boldsymbol{r}_0(0).$$

While their contribution to $\boldsymbol{U}^{(1)}$ is linear, the nonlinear effect will surface in higher order problems. The above discussion remains valid for $O(\epsilon^k)$ $(k < m)$ problems.

CASE 2. $m = 1$.

Now (3-25c) gives

$$T_\theta^{(1)} = 1 + \sigma'(\theta) \int_0^x \Lambda^{(1)}(\mathbf{r}_0(s))ds,$$

and $\boldsymbol{M}_1(x, \theta)$ no longer has the form of separation of variables. Although $\boldsymbol{U}^{(1)}(x, \theta)$ can be solved by a similar step as in CASE 1, the further successive solutions of the $O(\epsilon^k)$ problem will be too complicated to handle.

However, now we have

$$\boldsymbol{U}(x, \theta; \epsilon) = \epsilon\sigma_0(\theta)\mathbf{r}_0(x) + O(\epsilon^2), \tag{4-29}$$

and

$$t = \int_0^x \frac{ds}{\lambda_0(s)}$$

$$+ \epsilon\{\theta + \sigma_0(\theta) \int_0^x \Lambda^{(1)}(\mathbf{r}_0(s))ds\} \tag{4-30}$$

$$+ O(\epsilon^2).$$

Shock occurrence and shock initiation position can also be calculated by solving

$$t_\theta = T_\theta = 0,$$

or

$$1+\sigma_0'(\theta)\int_0^x \Lambda^{(1)}(r_0(s))ds=0, \tag{4-31}$$

approximately. We state the result as

SHOCK CONDITION. $(m=1)$

If there exists (x,θ), $x>0$, $\theta>0$ such that (4.31) holds then a shock will occur at (x_s,t_s) behind the leading wavefront $t=\int_0^x \frac{ds}{\lambda_0(s)}$, where

$$x_s=\min\{x>0:\ 1+\sigma_0'(\theta)\int_0^x \Lambda^{(1)}(r_0(s))ds=0\}, \tag{4-32}$$

$$t_s=\int_0^{x_s}\frac{ds}{\lambda_0(s)}+\epsilon\{\theta_s+\sigma_0(\theta_s)\int_0^{x_s}\Lambda^{(1)}(r_0(s))ds\}+O(\epsilon^2). \tag{4-33}$$

One immediate observation of this shock condition gives

(i) If $\Lambda^{(1)}(r_0)\not\equiv 0$, a function $\sigma_0(\cdot)$ is always constructable such that a shock will occur behind the leading wavefront.

(ii) If $\Lambda^{(1)}(r_0)\equiv 0$, which is equivalent to $(\text{grad}_u\lambda)r\equiv 0$ about $u\equiv\mathbf{0}$, i.e., the leading wavefront is locally linearly degenerate about the steady state solution. In this case a shock will never occur behind the leading wavefront under the small-amplitude, high-frequency relation

$$m=1,\quad \delta=\epsilon,$$

within $0<x\ll\epsilon^{-1}$.

However, if we choose $m \geq 2$, a shock is still possible being formed behind the leading wavefront, as we will show later.

THE $O(\epsilon^k)$ PROBLEM $(k < m)$.

By the method of induction, we may assume

$$U^{(j)}(x,\theta) = \frac{1}{j+1}\sigma_0^{j+1}(\theta)\,\boldsymbol{r}_j(x), \quad j = 0, 1, \ldots, k-1. \tag{4-34}$$

Now considering $\boldsymbol{M}_k(x,\theta)$ and $N_k(x,\theta)$, we first show that they have the form of separation of variables. Since

$$\boldsymbol{D}_j = \sum_{s=1}^{j} \sum_{i_1+\cdots+i_s=j-s} \binom{j}{s\; i_1 \ldots i_s} \boldsymbol{D}^{(s)}(\boldsymbol{U}^{(i_1)}, \ldots, \boldsymbol{U}^{(i_s)})$$

$$= \sum_{s=1}^{j} \sum_{i_1+\cdots+i_s=j-s} \sigma_0^j(\theta) \binom{j}{s\; i_1 \ldots i_s} \boldsymbol{D}^{(s)}\Big(\frac{1}{i_1+1}\boldsymbol{r}_{i_1}, \ldots, \frac{1}{i_j+1}\boldsymbol{r}_{i_s}\Big)$$

$$= \sigma_0^j(\theta)[\boldsymbol{D}_j]_{\sigma_0 \equiv 1}, \tag{4-35}$$

with the same argument giving

$$\boldsymbol{A}_j = \sigma_0^j(\theta)[\boldsymbol{A}_j]_{\sigma_0 \equiv 1},$$

$$\boldsymbol{b}_j = \sigma_0^j(\theta)[\boldsymbol{b}_j]_{\sigma_0 \equiv 1},$$

$$\boldsymbol{a}_j = \sigma_0^j(\theta)[\boldsymbol{a}_j]_{\sigma_0 \equiv 1},$$

$$T^{(j)} = \sigma_0^j(\theta)[T^{(j)}]_{\sigma_0 \equiv 1}, \quad (j < m)$$

$$\Lambda_j = \sigma_0^j(\theta)[\Lambda_j]_{\sigma_0 \equiv 1},$$

we have therefore that

$$\boldsymbol{M}_k(x,\theta) = \Bigg\{ \sum_{\substack{k_1+k_2=k \\ k_1>0}} \binom{k}{k_1\ k_2} [\boldsymbol{D}_{k_1}]_{\sigma_0 \equiv 1} \boldsymbol{r}_{k_2}$$

$$- \sum_{\substack{k_1+k_2+k_3=k \\ k_1>0}} \binom{k}{k_1\ k_2\ k_3} k_1 [T^{(k_1)}]_{\sigma_0 \equiv 1} [\boldsymbol{A}_{k_2}]_{\sigma_0 \equiv 1} \frac{1}{k_3+1} \boldsymbol{r}'_{k_3}$$

$$+ \frac{1}{k+1} \sum_{\substack{k_1+k_2=k+1 \\ k_1,k_2>0}} \binom{k+1}{k_1\ k_2} [\boldsymbol{b}_{k_1}]_{\sigma_0 \equiv 1} k_2 [T^{(k_2)}]_{\sigma \equiv 1} \Bigg\} \sigma_0^k \sigma_0'$$

$$\triangleq \boldsymbol{p}_k(x) \sigma_0^k(\theta) \sigma_0'(\theta), \tag{4-36}$$

$$N_k(x,\theta) = \Big\{ \sum_{k_1+k_2=k} \binom{k}{k_1\ k_2} [a_{k_1}]_{\sigma_0 \equiv 1} \frac{1}{k_2+1} r'_{k_2}$$

$$- \frac{1}{k+1} \sum_{j=2}^{k+1} \sum_{i_1+\cdots+i_j=k+1-j} \binom{k+1}{j\ i_1 \ldots i_j} c^{(j)} \left(\frac{1}{i_1+1} r_{i_1}, \ldots, \frac{1}{i_j+1} r_{i_j} \right) \Big\}$$

$$\times \sigma_0^{k+1}(\theta)$$

$$\triangleq q_k(x) \sigma_0^{k+1}(\theta), \tag{4-37}$$

and the $O(\epsilon^k)$ problem is simplified to

$$(I - \frac{A_0}{\lambda_0}) U_\theta^{(k)} = p_k(x) \sigma_0^k(\theta) \sigma'_0(\theta), \tag{4-38}$$

$$c^{(1)}(U^{(k)}) - a_0 U_x^{(k)} = q_k(x) \sigma_0^{k+1}(\theta). \tag{4-39}$$

Solving (4-38), (4-39) as before, we let $r_k(x)$ be a particular solution of

$$(I - \frac{A_0}{\lambda_0}) r_k = p_k(x), \tag{4-40}$$

so that $r_k(x)$ disappears when $p_k(x)$ is identically zero. Hence we have

$$U_\theta^{(k)} = \sigma_0^k(\theta) \sigma'_0(\theta) \{ \sigma_k(x,\theta) r_0(x) + r_k(x) \}. \tag{4-41}$$

Now we differentiate the transport equation (4-39) with respect to θ and apply (4-40) to obtain

$$(\sigma_k)_x - \Gamma_0(x)\sigma_k = K_k(x), \tag{4-42}$$

where

$$K_k(x) = \frac{c^{(1)}(\boldsymbol{r}_k) - \boldsymbol{a}_0\boldsymbol{r}'_k - (k+1)q_k}{\boldsymbol{a}_0\boldsymbol{r}_0}. \tag{4-43}$$

Again we note that $\Gamma_0(x) \equiv 0$, and hence

$$(\sigma_k)_x = K_k(x),$$

or, integrating, we get

$$\begin{aligned}\sigma_k(x,\theta) &= \sigma_k(0,\theta) + \int_0^x K_k(s)ds \\ &\triangleq \sigma_k(\theta) + \int_0^x K_k(s)ds.\end{aligned} \tag{4-44}$$

Now apply (4-22) to the first component of (4-41) (using (4-44)) to determine $\sigma_k(\theta)$, we have

$$\sigma_k(\theta)r_{0,1}(0) + r_{k,1}(0) = 0,$$

or

$$\sigma_k(\theta) = -\frac{r_{k,1}(0)}{r_{0,1}(0)}, \tag{4-45}$$

where $r_{k,1}(x)$ is the first component of $\boldsymbol{r}_k(x)$. Hence (4-41) becomes

$$\boldsymbol{U}_\theta^{(k)} = \sigma_0^k(\theta)\sigma_0'(\theta)\{\boldsymbol{r}_0(x)\,[-\frac{r_{k,1}(0)}{r_{0,1}(0)} + \int_0^x K_k(s)ds] + \boldsymbol{r}_k(x)\}. \tag{4-46}$$

Also we 'normalize' $\boldsymbol{r}_k(x)$ in the sense that

$$\begin{cases} K_k(x) \equiv 0, \\ r_{k,1}(0) = 0. \end{cases} \tag{4-47a,b}$$

This is accomplished by replacing $\boldsymbol{r}_k(x)$ by

$$\tilde{\boldsymbol{r}}_k(x) = \boldsymbol{r}_k(x) + \boldsymbol{r}_0(x)\,[-\frac{r_{k,1}(0)}{r_{0,1}(0)} + \int_0^x K_k(s)ds]. \tag{4-48}$$

As before, we assume $\boldsymbol{r}_k(x)$ is 'normalized' unless stated otherwise.

Now (4-41) becomes

$$\boldsymbol{U}_\theta^{(k)}(x,\theta) = \sigma_0^k(\theta)\sigma_0'(\theta)\boldsymbol{r}_k(x). \tag{4-49}$$

Integrate to get

$$\boldsymbol{U}^{(k)}(x,\theta) = \frac{1}{k+1}\sigma_0^{k+1}(\theta)\boldsymbol{r}_k(x). \tag{4-50}$$

This completes the process of induction. We shall summarize the above results in the next section.

§2. Weakly Nonlinear Wave Solution and Shock Calculation

By means of the method of induction, we have ascertained that the asymptotic expansion for the solution takes the form

$$U(x,\theta;\epsilon) = \sum_{k=1}^{m} \frac{\epsilon^k}{k!}\sigma_0^k(\theta)r_{k-1}(x) + O(\epsilon^{m+1}), \quad \theta \geq 0, \tag{1-5}$$

where $r_k(x)$ $(k = 1,2,\ldots,m-1)$ are all particular solutions of the algebraic systems

$$(I - \frac{A_0}{\lambda_0})r_k = p_k, \quad k = 1,2,\ldots,m-1. \tag{4-51}$$

These can be solved successively as we stated above.

Now we present the method for shock calculation. Using (3-25) we rewrite the arrival time formula as

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \sum_{k=1}^{m-1} \frac{\epsilon^k}{k!}\sigma_0^k(\theta)\int_0^x [\Lambda_k]_{\sigma_0\equiv 1}ds$$

$$+ \epsilon^m\{\theta + \frac{1}{m!}\sigma_0^m(\theta)\int_0^x [\Lambda_m]_{\sigma_0\equiv 1}ds\}$$

$$+ O(\epsilon^{m+1}). \tag{1-6}$$

where

$$\Lambda_k(x,\theta) = \sigma_0^k(\theta)[\Lambda_k]_{\sigma_0\equiv 1}, \quad k = 1,2,\ldots,m-1,m, \tag{4-52}$$

and

$$[\Lambda_k]_{\sigma_0\equiv 1} = \sum_{j=1}^{k} \sum_{i_1+\cdots+i_j=k-j} \binom{k}{j\ i_1 \ldots i_j} \Lambda^{(j)}\left(\frac{1}{i_1+1} r_{i_1}, \ldots, \frac{1}{i_j+1} r_{i_j}\right) \tag{4-53}$$

$$k = 1, 2, \ldots, m-1, m,$$

are noted. The method of shock calculation can then be summarized as follows.

SHOCK CONDITION $(m \geq 1)$. If there exists an integer $q \geq 1$ such that

$$[\Lambda_k]_{\sigma_0\equiv 1} \begin{cases} \equiv 0, & k = 1, 2, \ldots, q-1, \\ \not\equiv 0, & k = q, \end{cases}$$

then when $m = q$ is chosen, a shock will occur behind the leading wavefront if there exists $(x, \theta), x > 0, \theta > 0$ and

$$t_\theta = 0$$

or

$$1 + \frac{1}{(q-1)!} \sigma_0^{q-1}(\theta) \sigma_0'(\theta) \int_0^x [\Lambda_q]_{\sigma_0\equiv 1} ds = 0, \tag{1-9}$$

approximately. The shock initiates at (x_s, θ_s) in (x, θ) coordinates or (x_s, t_s) in (x, t) coordinates where

$$x_s = \min\{x > 0 : \ 1 + \frac{1}{(q-1)!} \sigma_0^{q-1}(\theta) \sigma_0'(\theta) \int_0^x [\Lambda_q]_{\sigma_0\equiv 1} ds = 0\},$$

and (x_s, θ_s) satisfies (1-9),

$$t_s = \int_0^{x_s} \frac{ds}{\lambda_0(s)} + \epsilon^q \{\theta_s + \frac{1}{q!}\sigma_0^q(\theta_s) \int_0^{x_s} [\Lambda_q]_{\sigma_0 \equiv 1} ds\}$$

$$+ O(\epsilon^{q+1}).$$

Since $[\Lambda_k]_{\sigma_0 \equiv 1}$ $(k = 1, 2, \ldots, m)$ are independent of $\sigma_0(\theta)$ and m, the above result, which includes the ***shock condition*** $(m = 1)$ in Chapter IV, §1 as a special case, suggests a systematic procedure to detect shocks behind the leading wavefront and also provides a method to calculate the shock initiation distance and time. The particular importance of this result is its validity when the leading wavefront is locally linearly degenerate about the steady state solution, which corresponds to

$$[\Lambda_1]_{\sigma_0 \equiv 1} = \Lambda^{(1)}(\boldsymbol{r}_0) \equiv 0.$$

In addition, (4-53) explicitly gives the expressions for $[\Lambda_k]_{\sigma_0 \equiv 1}$ in terms of

$\boldsymbol{r}_j \ (0 \le j \le k-1)$, and one may easily compute the first several $[\Lambda_k]_{\sigma_0 \equiv 1}$s' :

$$[\Lambda_1]_{\sigma_0 \equiv 1} = \Lambda^{(1)}(\boldsymbol{r}_0),$$

$$[\Lambda_2]_{\sigma_0 \equiv 1} = \Lambda^{(1)}(\boldsymbol{r}_1) + \Lambda^{(2)}(\boldsymbol{r}_0, \boldsymbol{r}_0),$$

$$[\Lambda_3]_{\sigma_0\equiv 1} = \Lambda^{(1)}(\boldsymbol{r}_2) + 3\Lambda^{(2)}(\boldsymbol{r}_0,\boldsymbol{r}_1)$$

$$+ \Lambda^{(3)}(\boldsymbol{r}_0,\boldsymbol{r}_0,\boldsymbol{r}_0),$$

$$[\Lambda_4]_{\sigma_0\equiv 1} = \Lambda^{(1)}(\boldsymbol{r}_3) + 3\Lambda^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_1) + 4\Lambda^{(2)}(\boldsymbol{r}_0,\boldsymbol{r}_2)$$

$$+ 6\Lambda^{(3)}(\boldsymbol{r}_0,\boldsymbol{r}_0,\boldsymbol{r}_0) + \Lambda^{(4)}(\boldsymbol{r}_0,\boldsymbol{r}_0,\boldsymbol{r}_0,\boldsymbol{r}_0).$$

NOTE. When $[\Lambda_k]_{\sigma_0\equiv 1}$ $(k = 1,2,\ldots,m)$ are bounded, which in turn requires only that $\boldsymbol{r}_0,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{m-1}$ be bounded when all the coefficients of the k-linear forms in (3-6)–(3-11) are assumed to be smooth and bounded, then the arrival time formula (1-6) is valid for $0 \le x \ll \epsilon^{-1}$.

In the next chapter we present an example application.

CHAPTER V

An Application: Weakly Nonlinear Waves in Fluid-Filled Hyperelastic Tubes

Consider the propagation of weakly nonlinear waves in fluid-filled hyperelastic tethered tubes subjected to axial strain. The one dimensional model developed in [9] is

$$A_t + (Au)_x = 0, \tag{5-1}$$

$$u_t + uu_x + p_x = 0. \tag{5-2}$$

This is a non-dimensional form, $x \geq 0$ is the axial variable, $t \geq 0$ is time, u is the fluid velocity in the axial direction and p is the transmural pressure. A is the cross sectional area.

The constitutional relation derived in [9] gives

$$\begin{aligned} A &= A(x,p) \\ &= A_0(x) + \varphi_0(x)p + \varphi_1(x)p^2 + \varphi_2(x)p^3 + O(p^4), \end{aligned} \tag{5-3}$$

where

$$\varphi_0(x) = A_0^{3/2}(x)/[2h_0(x)(W_1^0/(1+e)^2 + W_2^0)], \tag{5-4}$$

$$\varphi_1(x) = 3\varphi_0^2(x)/2A_0(x), \tag{5-5}$$

$$\varphi_2(x) = [\frac{5}{2} - \beta(x)]\varphi_0^3(x)/A_0^2(x), \tag{5-6}$$

$$\beta(x) = \frac{W_{11}^0/(1+e) + 2(1+e)W_{12}^0 + (1+e)^3 W_{22}^0}{W_1^0 + (1+e)^2 W_2^0}\,, \tag{5-7}$$

are all known functions involving the strain energy function W. In particular, $A_0(0) = 1$, $\varphi_0(0) = 1/2$.

The system admits a steady state solution: $p = u = 0$.

Now, if the boundary is perturbed by

$$p|_{x=0} = \epsilon g(\frac{t}{\delta}) + O(\epsilon^2), \quad t \geq 0, \tag{5-8}$$

we may consider the mixed initial and boundary problem prescribed by

$$p = u = 0, \quad t = 0, \quad x \geq 0, \tag{5-9}$$

$$p|_{x=0} = \epsilon g(\frac{t}{\delta}) + O(\epsilon^2), \quad t \geq 0. \tag{5-8}$$

As proved in [9], a shock is not possible on the leading wavefront but it is possible to construct a solution which leads to interior shocks and the shock initiation distance and time can be calculated. We shall apply the theory developed in this paper to recover these results.

To proceed, first we rewrite (5-1), (5-2) in a standard form

$$\begin{pmatrix} p \\ u \end{pmatrix}_t + \begin{pmatrix} u & AA_p^{-1} \\ 1 & u \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix}_x = \begin{pmatrix} -uA_xA_p^{-1} \\ 0 \end{pmatrix}\,, \tag{5-10}$$

and now

$$\boldsymbol{u} = \begin{pmatrix} p \\ u \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} u & AA_p^{-1} \\ 1 & u \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -uA_xA_p^{-1} \\ 0 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are

$$\lambda = u \pm (AA_p^{-1})^{1/2}. \tag{5-11}$$

Choose the one which is positive (about $\boldsymbol{u} \equiv \boldsymbol{0}$), i.e.,

$$\lambda = u + (AA_p^{-1})^{1/2}. \tag{5-12}$$

The left and right eigenvectors associated with λ are

$$\boldsymbol{\ell} = \frac{1}{2}((AA_p^{-1})^{-1/2}, 1), \quad \boldsymbol{r} = \begin{pmatrix} (AA_p^{-1})^{1/2} \\ 1 \end{pmatrix}, \tag{5-13}$$

and $\boldsymbol{\ell r} = 1$ is satisfied. In particular,

$$\boldsymbol{\ell}_0 = \boldsymbol{\ell}(\boldsymbol{0}, x) = \frac{1}{2}((\varphi_0/A_0)^{1/2}, 1), \tag{5-14}$$

$$\boldsymbol{r}_0 = \boldsymbol{r}(\boldsymbol{0}, x) = \begin{pmatrix} (A_0/\varphi_0)^{1/2} \\ 1 \end{pmatrix}. \tag{5-15}$$

Taylor expand $(\frac{1}{\lambda})$ about $\boldsymbol{u} \equiv \boldsymbol{0}$ obtaining

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \Lambda^{(1)}(\boldsymbol{u}) + \frac{1}{2}\Lambda^{(2)}(\boldsymbol{u}, \boldsymbol{u}) + O(\|\boldsymbol{u}\|^3), \tag{5-16}$$

where

$$\Lambda^{(1)}(\boldsymbol{u}) = (\varphi_0/A_0)((A_0/\varphi_0)^{-1/2}, -1)\begin{pmatrix} p \\ u \end{pmatrix}, \tag{5-17}$$

$$\Lambda^{(2)}(\boldsymbol{u},\boldsymbol{u}) = (p,u)\begin{pmatrix} 3(1-\beta)(\varphi_0/A_0)^{5/2} & -2(\varphi_0/A_0)^2 \\ -2(\varphi_0/A_0)^2 & 2(\varphi_0/A_0)^{3/2} \end{pmatrix}\begin{pmatrix} p \\ u \end{pmatrix}. \tag{5-18}$$

It is easy to check that

$$\begin{aligned} \Lambda^{(1)}(\boldsymbol{r}_0) &= (\varphi_0/A_0)((A_0/\varphi_0)^{-1/2}, -1)\begin{pmatrix} (A_0/\varphi_0)^{1/2} \\ 1 \end{pmatrix} \\ &\equiv 0, \end{aligned} \tag{5-19}$$

i.e., the leading wavefront is locally linearly degenerate (about $\boldsymbol{u} \equiv \boldsymbol{0}$). Therefore, to construct the weakly nonlinear wave solution which will build up interior shocks, we must choose $m \geq 2$. Let the relation of small-amplitude to high-frequency be $\delta = \epsilon^2$, then we have from (1-5), (1-6) that

$$\boldsymbol{u} = \epsilon\sigma_0(\theta)\boldsymbol{r}_0(x) + \frac{1}{2}\epsilon^2\sigma_0^2(\theta)\boldsymbol{r}_1(x) + O(\epsilon^3), \tag{5-20}$$

$$\begin{aligned} t = \int_0^x \frac{ds}{\lambda_0(s)} + \epsilon^2\{\theta + \frac{1}{2}\sigma_0^2(\theta)\int_0^x [\Lambda_2]_{\sigma_0\equiv 1} ds\} \\ + O(\epsilon^3), \end{aligned} \tag{5-21}$$

where $[\Lambda_2]_{\sigma_0\equiv 1} = \Lambda^{(2)}(\boldsymbol{r}_0,\boldsymbol{r}_0) + \Lambda^{(1)}(\boldsymbol{r}_1)$.

It remains to find $\boldsymbol{r}_1$, where $\boldsymbol{r}_1$ is a particular solution of

$$(\boldsymbol{I} - \frac{A_0}{\lambda_0})\,\boldsymbol{r}_1 = \frac{1}{\lambda_0}\boldsymbol{A}^{(1)}(\boldsymbol{r}_0)\boldsymbol{r}_0,$$

or

$$(\lambda_0 I - A_0)\, r_1 = A^{(1)}(r_0)\, r_0. \tag{5-22}$$

Now

$$A^{(1)}(r_0) r_0 = \begin{pmatrix} r_0^{(1)}(\partial_p u)_0 + r_0^{(2)}(\partial_u u)_0 & r_0^{(1)}(\partial_p(AA_p^{-1}))_0 + r_0^{(2)}(\partial_u(AA_p^{-1}))_0 \\ 0 & r_0^{(1)}(\partial_p u)_0 + r_0^{(2)}(\partial_u u)_0 \end{pmatrix} \begin{pmatrix} r_0^{(1)} \\ r_0^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} -(A_0/\varphi_0)^{1/2} \\ 1 \end{pmatrix}, \tag{5-23}$$

and (5-22) therefore becomes

$$\begin{pmatrix} (A_0/\varphi_0)^{1/2} & -(A_0/\varphi_0) \\ -1 & (A_0/\varphi_0)^{1/2} \end{pmatrix} \begin{pmatrix} r_1^{(1)} \\ r_1^{(2)} \end{pmatrix} = \begin{pmatrix} -(A_0/\varphi_0)^{1/2} \\ 1 \end{pmatrix}. \tag{5-24}$$

It is obvious that

$$r_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{5-25}$$

is a particular solution.

However, r_0 is yet to be 'normalized'. Now

$$\Gamma_0(x) = \frac{1}{\lambda_0} c^{(1)}(r_0) - \ell_0 r_0'$$

$$= -\frac{3}{4}\frac{A_0'}{A_0} + \frac{1}{4}\frac{\varphi_0'}{\varphi_0}, \tag{5-26}$$

which gives

$$\exp\{\int_0^x \Gamma_0(s)ds\} = [\frac{A_0}{A_0(0)}]^{-3/4} [\frac{\varphi_0}{\varphi_0(0)}]^{1/4}. \tag{5-27}$$

Hence after $\boldsymbol{r}_0$ is 'normalized',

$$\boldsymbol{r}_0 = \begin{pmatrix} (A_0/\varphi_0)^{1/2} \\ 1 \end{pmatrix} \left[\frac{A_0}{A_0(0)}\right]^{-3/4} \left[\frac{\varphi_0}{\varphi_0(0)}\right]^{1/4}, \tag{5-28}$$

and

$$\boldsymbol{r}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \left[\frac{A_0}{A_0(0)}\right]^{-3/2} \left[\frac{\varphi_0}{\varphi_0(0)}\right]^{1/2}. \tag{5-29}$$

Therefore,

$$\begin{aligned}[\Lambda_2]_{\sigma_0\equiv 1} &= \Lambda^{(2)}(\boldsymbol{r}_0,\boldsymbol{r}_0) + \Lambda^{(1)}(\boldsymbol{r}_1) \\ &= -3\beta\left(\frac{\varphi_0}{A_0}\right)^{3/2} \left[\frac{A_0}{A_0(0)}\right]^{-3/2} \left[\frac{\varphi_0}{\varphi_0(0)}\right]^{1/2}.\end{aligned} \tag{5-30}$$

This is a non-zero function. This fact confirms that our choice of $m = 2$ is appropriate.

Since we do not have to go further to find $\boldsymbol{r}_2, \boldsymbol{r}_1$ need not be 'normalized', and we just impose that $r_{1,1}(0) = 0$.

Now we note that $A_0(0) = 1$, $\varphi_0(0) = \frac{1}{2}$ and

$$\begin{aligned}\boldsymbol{u}|_{x=0} &= \epsilon g(\theta) \begin{pmatrix} 1 \\ * \end{pmatrix} + O(\epsilon^2) \\ &= \epsilon\sigma_0(\theta) \begin{pmatrix} 2^{1/2} \\ 1 \end{pmatrix} + O(\epsilon^2).\end{aligned} \tag{5-31}$$

which gives

$$\sigma_0(\theta) = 2^{-1/2} g(\theta). \tag{5-32}$$

Therefore, (5-20), (5-21) in turn, can be simplified to give

$$\boldsymbol{u} = \epsilon 2^{-\frac{1}{4}} g(\theta) A_0^{-\frac{3}{4}} \varphi_0^{\frac{1}{4}} \begin{pmatrix} (A_0/\varphi_0)^{1/2} \\ 1 \end{pmatrix}$$

$$+ \epsilon^2 2^{-\frac{3}{2}} g^2(\theta) A_0^{-\frac{3}{2}} \varphi_0^{\frac{1}{2}} \begin{pmatrix} 0 \\ (A_0/\varphi_0)^{-1/2} \end{pmatrix} \tag{5-33}$$

$$+ O(\epsilon^3),$$

$$t = \int_0^x \varphi_0(\eta) A_0^{-1}(\eta) d\eta$$

$$+ \epsilon^2 \{\theta - (3/2^{3/2}) g^2(\theta) \int_0^x \beta(\eta) \varphi_0^2(\eta) A_0^{-3}(\eta) d\eta\} \tag{5-34}$$

$$+ O(\epsilon^3).$$

The readers may find that (5-34) recovers the same result as in [9] (replacing ϵ by $\epsilon^{1/2}$).

It is also interesting to note that we have distinguished the boundary perturbation as

$$\boldsymbol{u}|_{x=0} = \epsilon 2^{-\frac{1}{2}} g(\frac{t}{\epsilon^2}) \begin{pmatrix} 2^{1/2} \\ 1 \end{pmatrix}$$

$$+ \frac{1}{4} \epsilon^2 g^2(\frac{t}{\epsilon^2}) \begin{pmatrix} 0 \\ 2^{-1/2} \end{pmatrix} \tag{5-35}$$

$$+ O(\epsilon^3).$$

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