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A Simple Approach to Valuing Risky Debt with Constant Elasticity of Variance Effects

by

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ABSTRACT

In this paper, we develop a simple approach to valuing risky debt that has realistic credit spreads and that incorporates agency effects, in the sense that we allow the volatility of the value of the firm to increase as the firm approaches default. We do this by assuming that the value of the firm follows a stochastic process that is based on the Constant Elasticity of Variance (CEV) process. We then provide a closed form expression for the value of risky debt when the elasticity of variance is in a certain range. Next, we consider the econometrics of the problem and find that we must generalize the model. Finally, we use Monte Carlo simulations to estimate the probability of default in our more general model.

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Chapter 1: The Model

1.1 Introduction

In this chapter, we present the fundamental ideas behind our model for risky debt. The main contribution of this chapter is that we consider a stochastic process for the value of the firm that allows the volatility of the value of the firm to increase as it approaches default. With this process, known as the Constant Elasticity of Variance process, we are able to derive a closed-form solution for the price of risky debt. When we proceed to test our model, in chapter 2, we find that our model requires some modification, as some firm's parameters do not satisfy the restrictions required by our analytic solution. Our modified model is a non-standard one, and so we proceed in that chapter with a discussion of the details of the econometric issues. We show that our modified model fits the data quite well for virtually all the firms in our sample whose parameters do not already satisfy the standard model. Next, in chapter 3, we discuss the problem of valuing risky debt for our non-standard model, for which no analytic solution exists. Although it is a difficult model to evaluate, a simple Monte Carlo procedure shows that our generalized model is quite effective in explaining a firm's credit rating.

The remainder of this chapter proceeds as follows. In section 1.2 we discuss the literature on risky debt. In section 1.3 we look at the assumptions used in our model, and we present our analytic solution for the price of risky debt in section 1.4. In section 1.5 we do some numerical work, to show how the model behaves as parameters are changed. The chapter closes with two appendices. Appendix 1 contains the proof of our analytic result, and also derives some more general results on first-passage times. Finally, Appendix 2 contains an alternative model with a different assumption regarding the payoff in case of default.

1.2 Literature Review

Risky debt was first modeled as a contingent claim by Black and Scholes (1973) and Merton (1974). In their models, the firm's debt goes into default if the value of the firm is below the face value of the debt at maturity. The recovery rate, i.e., the fraction of the face value of debt that bond holders receive in the event of default, is endogenously described as the ratio of the value of the firm to the face value of debt. One drawback to this model is that one must assume the firm has a very simple capital structure with a single issue of debt outstanding. Another undesirable assumption is that default can only occur at the time of maturity of the debt. This is unrealistic, as "early" default can occur if the firm is unable to meet any payment on any of the debt issues outstanding, or if it fails to meet some criterion stipulated in the covenants that may be associated with these debt contracts. Each of these events may happen at any instant. Empirically, it has been shown by Jones, Mason, and Rosenfeld (1984) and Franks and Torous (1989), that this assumption of no early default implies credit spreads that are much smaller than actual credit spreads. One major goal of this paper, therefore, is to provide a model that has realistic credit spreads.

Several recent papers, such as for example, Artzner and Delbaen (1993), Duffie, Schroder and Skiadas (1994), Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull

(1994), Lando (1994), and Madan and Unal (1993) model the "default process" itself as some sort of random process, in which case, the problem becomes one of modeling the intensity of this process. The default process is defined to be a random process that equals zero when there is no default, and is equal to some random value at the default time, which is assumed to be a (random) stopping time. Because the default process is usually given by a pure jump process with a continuous compensator, the time of default is a totally inaccessible stopping time, meaning that it is, in a sense, unpredictable. The compensator of this process, which determines the probability of default, is allowed to change over time, but its specification seems somewhat ad hoc. Jarrow, Lando and Turnbull (1994) on the other hand model the default process by means of a finite state Markov process where the states correspond to the firm's credit ratings. This seems to be a natural way to model the problem, however one disadvantage to their model is that no information is used beyond the firm's credit rating.

Jarrow and Turnbull (1995) also deal with credit risk in a model that is compatible with Heath, Jarrow and Morton (1992), but their main concern is with the pricing of options subject to credit risk. In their continuous trading economy, by assuming that the bankruptcy process is a simple Poisson process that is independent from the default-free interest rate process under the martingale probabilities, they are able to derive formulas for derivatives on financial securities subject to credit risk. They are not primarily concerned with the probability of default, though, which is a central part of our paper.

Our model is more in the tradition of Black and Cox (1976). Black and Cox develop a model in which default is allowed to occur at the first time at which the value of the firm's assets reaches a lower threshold; i.e., mathematically, the problem is a first-passage problem. This framework is consistent with either net-worth or cash-flow based insolvency. Their model generates credit spreads that are more consistent with empirical findings. This model is generalized by Nielsen, Saá-Requejo, and Santa-Clara (1994) and Longstaff and Schwartz (1995). One feature of these models is that the recovery rate is exogenously specified. An advantage to this approach is that one need not assume that strict absolute priority rules hold, which is consistent with recent empirical findings, such as Franks and Torous (1989,1994), Eberhart, Moore and Roenfeldt (1990), LoPucki and Whitford (1990), Weiss (1990), and Betker (1991, 1992). However, the papers that use the first-passage problem approach each assume that the underlying value of the firm follows a stochastic process with constant volatility. In our paper, we would like to model the value of the firm in such a way that the firm's volatility increases as the firm approaches default. This should capture agency costs, in the sense that, as a firm nears default, it is generally in the interests of management (acting on behalf of shareholders) to take bigger risks, thus increasing the volatility of the firm's value.

One disadvantage to the Longstaff and Schwartz (also Nielsen et al., 1994) paper is that they use the Vasicek (1977) model to model stochastic interest rates. Under the Vasicek model, interest rates may become negative, which is particularly a problem for long-term debt. Furthermore, there is some recent evidence that yield spreads are relatively insensitive to the volatility of interest rates (Kim, Ramaswamy and Sundaresan, 1993)

1.3 The Valuation Framework

In this section, we state the assumptions behind our model for risky debt. These assumptions will then be used in later sections to derive closed-form expressions for the valuation of risky corporate debt. The basic assumptions for this framework parallel those of Black and Scholes (1973), Merton (1974), Black and Cox (1976) and Longstaff and Schwartz (1995). We begin with a general assumption that is commonly made in the literature.

Assumption 1. *Markets are perfect, frictionless and securities trade in continuous time.*

This assumption allows us to apply stochastic calculus to the problem.

Assumption 2. *There is a threshold value K for the firm at which financial distress occurs. As long as the value of the firm, V, is greater than K. the firm continues to be able to meet its contractual obligations. If V reaches K. however, the firm immediately enters financial distress, defaults on all of its obligations, and some form of corporate restructuring takes place.*

Assumption 2 is, of course, the assumption that allows for early default, and as is shown by Longstaff and Schwartz, also allows for violation of the absolute priority rule. This assumption is used in Black and Cox (1976), Longstaff and Schwartz (1995), and Nielsen, Saá-Requejo, and Santa-Clara (1994). In general, both V and K may depend on time. Mathematically, then, the problem is known as a first-passage problem. In some cases, explicit solutions can be obtained. Black and Cox, for example, assume that V is lognormally distributed, and the threshold value is of the form Ke^{-cT}. Longstaff and Schwartz extend the Black and Cox model by allowing for a stochastic short-term riskless interest rate based on the Vasicek (1977) model. This allows them to investigate the effect on the value of risky debt of the correlation between the short-term interest rate and the value of the firm. In addition, Longstaff and Schwartz explain how the model could be extended to allow for time-varying K, in which case, numerical solutions are required. Nielsen, et al., also using numerical techniques, allow K to be quite general, even stochastic.

As Longstaff and Schwartz point out, this definition of financial distress is consistent with two types of insolvency, that is, the value of the firm's assets, V, being equal to K, may indicate one of two things. First, it may indicate that the firm is unable to meet its current obligations; second, it may indicate that the firm has violated some minimum-net-worth or working-capital requirements. See, e.g., Wruck (1990).

Violations of absolute priority rules play an important role in this assumption. Suppose the absolute priority rules were never violated, and the firm were forced to restructure when $V = K$, either through a Chapter 7 liquidation, a Chapter 11 reorganization a Chapter 11 liquidation, or a private debt restructuring. The reorganization of bankruptcy is simply a mechanism by which total assets of K are allocated to the various classes of corporate claimants. If this reallocation of assets occurs immediately when $V = K$, then if the absolute priority rule holds, if K equals the face value of the debt, and if there are no bankruptcy costs (as the Modigliani-Miller Theorem assumes), then the bondholders would receive the full value of their bonds, and so the bonds would be effectively riskless. Thus, violations of absolute priority rules are one reason that default-prone debt is actually risky. Several studies have in fact shown that absolute priority rules are frequently violated in corporate restructurings. For example, Franks and Torous (1989) find that absolute priority is violated in 78 percent of the bankruptcies in their sample, while similar percentages are found by Eberhart, Moore, and Roenfeldt (1990) and Weiss (1990). This bargaining game between various corporate claimants during restructuring has been modeled and incorporated into models for risky debt prices by, for example, Anderson and Sundaresan (1996), Mella and Perraudin (1993), and Leland (1994). Our model focuses more on the probability of default rather than the bargaining game that occurs after default. As in the Longstaff and Schwartz paper then, we take the allocation of the firm's assets as exogenously given.

Assumption 3. *If a reorganization occurs during the life of a security, the security holder receives a proportion* $1 - w$ *of the face value of the security at maturity.*

Thus, for example, if $w = 0$, there is no writedown of the security, while if $w = 1$, the security holder receives nothing in a restructuring. Altman (1992) finds that the average writedown, w, for secured, senior, senior subordinated, cash-pay subordinated, and non-cash-pay subordinated debt for a sample of defaulted bond issues during the 1985-1991 period is .395, .477, .693, .720, and .805, respectively. Franks and Torous (1994) also estimate w and find that the average writedown, w, for secured debt, bank debt, senior debt, and junior debt for a sample of firms that reorganized under Chapter 11 during the 1983-1990 period is .199, .136, .530, and .711, respectively. Betker (1992) obtains similar results. There is one constraint on the value of the w's: the total settlement on all the classes of claims cannot exceed K.

Although it is convenient to think of w as being a constant, some extensions are possible. The writedown w could easily be made a (deterministic) function of the time to maturity, or it could even be stochastic, provided the risk of w is nonsystematic. In the latter case, one simply replaces w with its expected value in the valuation expressions. For an alternative model, in which w is a function of time, see *Appendix* 2.

The following assumption is the key difference between our paper and the Black and Cox paper.

Assumption 4. *Let X represent the ratio $(V-K)/K$, the total value of the assets of the firm less the threshold value at which bankruptcy occurs divided by the threshold level. The dynamics of X, are given by*

(1) $$dX = \mu X dt + \sigma X^{\beta/2} dZ,$$

where μ, σ and β are constants, $0 < \beta < 2$, and Z is a standard Wiener process. In addition, we assume that the short-term riskless interest rate, r, is constant.

Note that an equivalent model would be one in which X were set equal to $V-K$, the only difference being that the parameter σ must be recalibrated. In comparing our volatilities to those used in Longstaff and Schwartz, for example, suppose that $X = 0.5$ (i.e., the firm is worth 50% more than its threshold level). If the other parameters are $r = 0.04$, $w = 0.5$, $\beta = 1$, then a volatility of $\sigma = 0.2$ in the Longstaff and Schwartz model corresponds to a volatility of, roughly, $\sigma = 0.42$ in our model. To see this, let $dZ = 0.01$ say, and note that the corresponding change to the value of the firm, dV, is the same for both models.

In testing our model, one of the questions will be how to define the default level. K. Anderson and Sundaresan (1996), using a game-theoretic model are able to endogenize the default value, but that value is not easily described. Leland (1994) on the other hand, by assuming that all debt service must be met by issuing new equity endogenizes the firm's bankruptcy point as the point where the value of equity is zero. Similarly, Nielsen et al. (1994) argue that the bankruptcy value, K, should reflect the market value of the firm's outstanding debt. If the firm consists of debt and equity, then $V = S + B$. If default occurs when $V = B$, then this is consistent with Leland's conclusion that bankruptcy occurs when the value of equity is zero. For simplicity, we shall assume that this is true, and so we use equity values, S, in order to estimate the relevant parameters. We will return to this point in Section 2.2.

The process described by equation (1) is called the constant elasticity of variance (CEV) model and it has been used in a variety of contexts in the financial literature. See for example, Cox (1975), Cox and Ross (1976), Emanuel and MacBeth (1982) and Schroder (1989) for discussions of the use of the CEV model for option pricing, and Choi and Longstaff (1985) for pricing options on agricultural futures. For tests of the CEV model in various contexts, see Ang and Peterson (1984), Rubinstein (1985), Tucker and Scott (1987), and Tucker, Peterson and Scott (1988).

This assumption generalizes *Assumption* 1 of Longstaff and Schwartz (1995). The main advantage of this model is that the volatility of the ratio, X, and hence, of the value of the firm, V, increases as the firm's value approaches K, that is, as the firm nears default. (The volatility of X is given by $\sigma X^{\beta/2}/X = \sigma X^{(\beta-2)/2}$, so for $\beta < 2$, this becomes large as X approaches zero.) This property captures the agency problem that Smith and Warner (1979) call asset substitution, where "a firm sells bonds for the stated purpose of engaging in low variance projects and the bonds are valued at prices commensurate with that low risk, the value of the stockholders' equity rises and the value of the bondholder's claim is reduced by substituting projects which increase the firm's variance rate." If any such change in variance occurs, our model should capture it.

This model has two other properties that should also be mentioned. First, X has an absorbing boundary at zero; that is, default is an absorbing state, which seems reasonable. Second, the expected return is given by $E[X_T] = X_t exp(\mu(T-t))$, that is, $E[(V-K)_T] = (V-K)_t exp(\mu(T-t))$. If the critical level, K, is constant, then this may seem unreasonable, as the firm would stop growing as the value of the firm neared K. However, the critical value may also increase. In fact, Black and Cox assume that the critical value takes the form $K_t = Kexp(ct)$. Thus, our local approximation may indeed be reasonable. Because any extension of our model (such as allowing interest rates to be random as in the Longstaff and Schwartz model) would require a numerical solution, the above model seems well worth investigating.

1.4 Valuing Risky Debt

In this section, we derive an expression for the value of a pure discount bond. If the bond does not go into default, the payoff of the bond is 1, while if the bond does default at some random default time, γ, the payoff to the bondholder is $1 - w$. This payoff function can be expressed as

$$1 - wI_{\gamma \le T},$$

Let $D(r, T)$ represent the price of a riskless discount bond, while $P(V, r, T)$ denotes the price of the risky bond.

Proposition 1: *The value of a risky discount bond is*

$$P(V, r, T) = \quad D(r, T) - wD(r, T)G(\tfrac{1}{2-\beta}, \xi),$$

where

$$\xi = \quad kX_t^{2-\beta} \exp\{r(2-\beta)(T-t)\},$$

$$k = \quad \frac{2r}{\sigma^2(2-\beta)[e^{r(2-\beta)(T-t)} - 1]},$$

$$X = \quad (V - K)/K,$$

and $G(m, v) = [\Gamma(m)]^{-1} \int_v^\infty e^{-u} u^{m-1} du$ is the standard complimentary gamma distribution.

Proof: By risk-neutral valuation, the value of the risky debt is the present value of $E^*[1 - wI_{\gamma \le T}] = 1 - wP^*\{\gamma \le T\}$, where P^* (E^*) represent the probability (expectation) in a risk-neutral world. Thus, it remains to show that $P^*\{\gamma \le T\} = G(\tfrac{1}{2-\beta}, \xi)$. Although this result is known, see *Appendix* 1 for a discussion and proof.

For an alternative model in which the payoff is a function of time, see *Appendix* 2.

1.5 Numerical Results

The numerical results are summarized in Figures 1 through 4 below. As mentioned above, the model must be recalibrated if we are to assume that the ratio $X = (V - K)/K$ satisfies the stochastic differential equation given in *Assumption* 4. In comparing our volatilities to those used in Longstaff and Schwartz, for example, suppose that $X = 0.5$ (i.e., the firm is worth 50 percent more than its threshold level). If the other parameters are $r = 0.04$, $w = 0.5$, $\beta = 1$, then a volatility of $\sigma = 0.2$ in the Longstaff and Schwartz model corresponds to a volatility of, roughly, $\sigma = 0.42$ in our model. To see this, let $dZ = 0.01$ say, and see the effect on the change in the value of the firm, dV. As Figures 1-4 illustrate, our results are comparable to the results of Longstaff and Schwartz in some cases, but give larger yield spreads in other cases. The critical difference between our models is, of course the exponent, β, which we interpret as a measure of the firm's agency problem. To test our results, the natural approach would be to estimate the firm's β and compare that estimate to the beta implied by our model for risky debt. In econometric terms, this is a model with multiplicative heteroskedasticity, which can be estimated using maximum likelihood.

Appendix 1

We begin with some general observations about first passage times. We shall present the moment generating functions for several first passage times. We then present a proof of the Proposition 1 using a different technique.

Consider the stochastic process $\{Y_t\}$ satisfying the stochastic differential equation

$$dY_t = (\alpha - \gamma Y_t)dt + \eta\sqrt{Y_t}\,dZ_t; \qquad Y_0 = y. \tag{A.1}$$

and consider the first passage time $\tau_L = \inf\{t: \; Y_t = k\}$, where $0 < y < k$. Here τ_L represents the first time that the process Y hits k from below.

Theorem A1: The moment generating function for τ_L is given by

(A.2)
$$E[exp(-s\tau_L)] = \frac{{}_1F_1(as,b;cy)}{{}_1F_1(as,b;ck)}$$

where

(A.3)
$$a = 1/\gamma,$$
$$b = 2\alpha/\eta^2,$$
$$c = 2\gamma/\eta^2,$$

Here ${}_1F_1(j, m; z)$ is the confluent hypergeometric function

(A.4)
$${}_1F_1(j, m; z) = \sum_{n=0}^{\infty} \frac{(j)_n z^n}{(m)_n n!},$$

where $(j)_n = j(j+1)\ldots(j+n)$. (Note that j and m need not be integers.)

Proof: Define the stochastic process $\{V(t)\}$ by

$$V(t) = e^{-st}{}_1F_1(as, b; cY_t).$$

From Ito's Lemma and the properties of the confluent hypergeometric function (see, e.g., Slater, 1960), it is not hard to show that this process is a uniformly integrable martingale. There are two relevant properties of the confluent hypergeometric function, ${}_1F_1(j, m; z)$. First, it satisfies Kummer's differential equation:

(A.5)
$$zf_{zz} - (m - z)f_z - jf = 0.$$

Second, it and its first two derivatives remain bounded for z on the interval $[0, k]$. Thus, if we consider the process stopped at time τ_L, then we have

$$E[V(\tau_L)] = V(0),$$

that is,

$$E[exp(-s\tau_L){}_1F_1(as, b; cY(\tau_L)] = E[exp(-s\tau_L){}_1F_1(as, b; ck)]$$

$$= {}_1F_1(as, b; cy),$$

giving us our result. •

Now let us consider the case where $0 < k < y$, and we have the stopping time τ_U again defined as $\tau_U = \inf\{t:\ Y_t = k\}$, although in this case, τ_U represents the first time that Y hits k from above. For the following, we must assume that $\alpha, \gamma, \eta > 0$.

Corollary A2: The moment generating function for τ_U is given by

$$E[exp(-s\tau_U)] = \frac{U(as,b;cy)}{U(as,b;ck)} \tag{A.6}$$

where a, b, and c are again given by (A.3) and

$$U(j, m; z) = \frac{\Gamma(1-m)}{\Gamma(1+j-m)}{}_1F_1(j,m;z) + \frac{\Gamma(m-1)}{\Gamma(j)}z^{1-m}{}_1F_1(1+j-m,2-m;z), \tag{A.7}$$

provided m is not an integer. The function $U(j, m; z)$ can be thought of as the second solution to Kummer's equation. There are various notations for this type of solution. (See Slater, 1960, p. 5.) This function is also related to the Whittaker function.

Proof: Here, the important properties of the function $U(j, m; z)$ are that it and its first two derivatives remain bounded as z approaches infinity. (See Slater, section 4.1.2) Otherwise, the proof is the same as the proof of the theorem. •

These results agree with the work of Kent (1978) in the following limiting case. Formally, for $\eta = 2$, as γ approaches zero, the process defined by equation (A.1) approaches a squared Bessel process. On the other hand, the moment-generating functions given by (A.2) and (A.6) approach (see Slater, 1960, p. 67) the moment-generating functions given by Theorem 3.1 in Kent's paper, as desired.

Now we consider a stochastic process of the type considered in *Assumption* 4. Let the stochastic process $\{X_t\}$ satisfy the stochastic differential equation

$$dX = \mu X dt + \sigma X^{\beta/2} dZ, \qquad X_0 = x, \tag{A.8}$$

where $0 < \beta < 2$. Suppose we again want to consider the first passage times of the form $\tau_L = \inf\{t\colon\ X_t = \kappa\}$, where $0 < x < \kappa$, and $\tau_U = \inf\{t\colon\ X_t = \kappa\}$, where $0 < \kappa < x$.

Now, we first note that if we set $Y = X^{2-\beta}$, then

$$\begin{aligned} dY_t &= [\,\tfrac{1}{2}(2-\beta)(1-\beta)\sigma^2 + (2-\beta)\mu Y]dt + \sigma(2-\beta)\sqrt{Y_t}\,dZ, \\ Y_0 &= x^{2-\beta}, \end{aligned} \tag{A.9}$$

so that in the notation of equation (A.1) we have the following substitutions:

$$\begin{aligned} \alpha &= \tfrac{1}{2}(2-\beta)(1-\beta)\sigma^2, \\ \gamma &= -(2-\beta)\mu, \\ \eta &= \sigma(2-\beta), \\ y &= x^{2-\beta}, \end{aligned} \tag{A.10}$$

and

$$k = \kappa^{2-\beta}.$$

In the notation of equation (A.3) then, we have

$$\begin{aligned} a &= \frac{1}{\gamma} = \frac{-1}{\mu(2-\beta)} \\ b &= \frac{2\alpha}{\eta^2} = \frac{1-\beta}{2-\beta}, \\ c &= \frac{2\gamma}{\eta^2} = \frac{-2\mu}{\sigma^2(2-\beta)}. \end{aligned} \tag{A.11}$$

Note that, because $\gamma < 0$, we have $a < 0$, $b > 0$, $c < 0$. Thus, we must modify the above theorem and corollary.

Corollary A3: The moment generating functions for τ_L and τ_U are

$$E[exp(-s\tau_L)] = \frac{{}_1F_1(as,b;cy)}{{}_1F_1(as,b;ck)}, \tag{A.12}$$

and

$$E[exp(-s\tau_U)] = \frac{e^{cy}U(b-as,b;-cy)}{e^{ck}U(b-as,b;-ck)}, \tag{A.13}$$

where a, b, c, y, and k are given by equations (A.10) and (A.11).

Proof: For τ_U, Slater's equation (1.3.3) shows that the function $e^zU(m-j, m, -z)$ is also a solution to Kummer's equation, (A.5). Its derivatives are given by Slater's equation (2.1.30) as

$$\frac{d^n}{dz^n}\{e^z\, U(m-j, m, -z)\} = (-1)^n e^z\, U(m-j, m+n, -z).$$

Finally, the proof that these functions are bounded as z approaches infinity is given in Slater's section 4.1.2. •

Note that the above moment-generating functions are very difficult to invert. It is for this reason that we simplify the problem by considering the limit as κ approaches zero; i.e., we consider the first passage time $\tau_U = \inf\{t\colon\ X_t = 0\}$. To do this, we need to find

$\lim\limits_{k\to 0} e^{ck}U(b-as, b, -ck)$, where the values a, b, c and k are given by equations (A.10) and (A.11). At this point, we must be careful about the behaviour of the values a, b, and c. For $0 < \beta < 2$, it is clear that $a < 0$, $-\infty < b < ½$, and $c < 0$. Next, we observe (see equation (A.4)) that ${}_1F_1(as, b; 0) = 1$. Now, it follows that

$$\lim_{k \to 0} e^{ck} U(b - as, b, -ck) =$$

$$\lim_{k \to 0} e^{ck} \left\{ \frac{\Gamma(1-b)}{\Gamma(1-as)} {}_1F_1(b - as, b; -ck) + \frac{\Gamma(b-1)}{\Gamma(b-as)} |ck|^{1-b} {}_1F_1(1 - as, 2 - b; -ck) \right\}$$

$$= \frac{\Gamma(1-b)}{\Gamma(1-as)}$$

since the exponent of $|ck|$ is positive. Thus, the moment generating function for this first passage time is

$$\text{(A.14)} \qquad E[exp(-s\tau_U)] = \frac{\Gamma(1-as)}{\Gamma(1-b)} exp(cx^{2-\beta}) U(b - as, b; -cx^{2-\beta}).$$

This can now be inverted to find the density for the first passage time.

While the above approach, using moment-generating functions, is fairly general, the final step, inverting the Laplace transform, is still rather difficult. As a result. we now present an alternative, more direct proof of the theorem.

Proof of Proposition 1: Consider the process satisfying the following stochastic differential equation (in a risk-neutral world):

$$\text{(A.15)} \qquad dX_t = rX_t dt + \sigma X_t^{\beta/2} dZ_t,$$

From Rogers and Williamson (1987), Theorems (51.2), p. 295 and equation (52.2), p. 297, it can be shown that zero is an absorbing boundary for the above process; that is, if $X_t = 0$, then $X_s = 0$ for all $s \geq t$. Let τ_U represent the first time that $X = 0$. Now, $X_t = 0$ if and only if $\tau_U \leq t$. Thus, the probability of default before time T is given by

$$P\{\tau_U \leq T \mid X_t\} = P\{X_T = 0 \mid X_t\} = E[1_{\{X_T = 0\}} \mid X_t],$$

which is a martingale. If we denote this probability by $P\{\tau_U \leq T \mid X_t = x\} = f(x, t)$. then by Ito's lemma, this must satisfy the partial differential equation

$$rxf_x + \tfrac{1}{2}\sigma^2 x^\beta f_{xx} + f_t = 0,$$

with boundary conditions $f(x, 0) = 0$ for $x > 0$, and $f(0, t) = 1$, for $0 \le t \le T$. Using Mathematica for example (see Wolfram, 1993), it can be confirmed that the solution to this partial differential equation is $G(\frac{1}{2-\beta}, \xi)$, where

$$\xi = kX_t^{2-\beta}\exp\{r(2-\beta)(T-t)\},$$

$$k = \frac{2r}{\sigma^2(2-\beta)[e^{r(2-\beta)(T-t)} - 1]},$$

and $G(m, v) = [\Gamma(m)]^{-1}\int_v^\infty e^{-u}u^{m-1}du$ is the standard complimentary gamma distribution.

We can now confirm that this solution gives the moment generating function of equation (A.14). To do this, we first write the moment-generating function (i.e., the Laplace transform) in terms of the Whittaker function (Slater, 1960, equation (1.9.6) p. 13):

$$\frac{\Gamma(1-as)}{\Gamma(1-b)}e^{-y}U(b-as, b; y) = \frac{\Gamma(1-as)}{\Gamma(1-b)}e^{-y}e^{y/2}y^{-b/2}W_{-\frac{1}{2}b+as,\frac{1}{2}b-\frac{1}{2}}(y).$$

Here, $y = |cx^{2-\beta}|$, $a = -1/r(2-\beta)$, $b = (1-\beta)/(2-\beta)$, and $c = -2r/\sigma^2(2-\beta)$.

. Now, using the substitution $u = ye^{r(2-\beta)t}/(e^{r(2-\beta)t} - 1) = ye^{rt/(1-b)}/(e^{rt/(1-b)} - 1)$, we have,

$$P\{\tau_U \le T\} = [\Gamma(1-b)]^{-1}\int_\xi^\infty e^{-u}u^{-b}du$$

$$= \frac{ry^{1-b}}{(1-b)\Gamma(1-b)}\int_0^T e^{rt}(e^{rt/(1-b)} - 1)^{b-2}\exp\left\{\frac{-ye^{rt/(1-b)}}{(e^{rt/(1-b)} - 1)}\right\}dt,$$

which gives us the probability density of the random variable τ_U. Now, using the substitution $u = rt/(1-b)$, we find that the Laplace Transform of this density is

$$\frac{ry^{1-b}}{(1-b)\Gamma(1-b)}\int_0^\infty e^{-st}e^{rt}(e^{rt/(1-b)}-1)^{b-2}\exp\left\{\frac{-ye^{rt/(1-b)}}{(e^{rt/(1-b)}-1)}\right\}dt$$

$$=\frac{y^{1-b}}{\Gamma(1-b)}\int_0^\infty e^{-su(1-b)/r}e^{u(1-b)}(e^u-1)^{b-2}\exp\left\{\frac{-ye^u}{(e^u-1)}\right\}dt$$

$$=\frac{y^{1-b}e^{-y}}{\Gamma(1-b)}\int_0^\infty e^{-[s(1-b)/r-(1-b)]u}(e^u-1)^{b-2}\exp\left\{\frac{-y}{(e^u-1)}\right\}dt$$

$$=\frac{\Gamma(1-as)}{\Gamma(1-b)}e^{-y/2}|z|^{-b/2}W_{-\frac{1}{2}b+as,\frac{1}{2}b-\frac{1}{2}}(y),$$

as desired, where the last step follows from Erdayi et al., 1954, equation (41), p. 147.

Appendix 2

Another natural way to model the payoff at default is to assume that the weight w is a function of time. Suppose a default has occurred at the random time $\gamma = \inf\{t: V_t = K_t\}$. Let us assume that the payoff in case of default on a zero coupon bond with face value of $1 is $(1-w)e^{-r(T-\gamma)}$. In this case, if default occurs close to maturity, the bondholder receives approximately $1-w$, while if default occurs earlier, the bondholder receives the present value, as of the default date, of $1-w$. This seems to be a natural assumption for a zero coupon bond; otherwise, one may actually prefer to receive $1-w$ immediately rather than receive $1 at maturity. The value (at time zero) of this risky bond is then just

$$\begin{aligned} P(V,r,T) &= e^{-rT}E^*[(1-w)e^{-r(T-\gamma)}] \\ &= e^{-rT} - we^{-2rT}E^*[e^{r\gamma}] \\ &= D(r,T) - wD(r,T)^2E^*[e^{r\gamma}], \end{aligned}$$

where $D(r, T)$ represents the price of a riskless discount bond. It remains to evaluate $E^*[e^{r\gamma}]$.

Lemma: Assuming that $\beta < 2$,

$$E^*[e^{r\gamma}] = \frac{1}{\Gamma(\frac{1}{2-\beta})} \int_{\xi}^{\infty} u^{\frac{\beta}{2-\beta}}(u-c)^{\frac{-1}{2-\beta}} e^{-u} du\,,$$

where $\xi = f(T-t)$ is as in *Proposition* 1, and

$$c = \frac{2r}{\sigma^2(2-\beta)} X_t^{2-\beta} = \lim_{T\to\infty} f(T-t) < f(T-t).$$

(Note that, because $c < f(T-t) = \xi$, the integral is well defined.)

Proof: Let us re-write $\xi = f(T-t)$ as

$$\begin{aligned} f(T-t) &= \frac{2r}{\sigma^2(2-\beta)} \frac{e^{a(T-t)}}{[e^{a(T-t)}-1]} X_t^{2-\beta} \\ &= \frac{ce^{a(T-t)}}{[e^{a(T-t)}-1]}, \end{aligned}$$

where $a = r(2-\beta)$. Now, we know that the distribution (under the risk-neutral measure) of the random stopping time γ is given by

$$P^*\{\gamma \le T\} = \frac{1}{\Gamma(m)} \int_{\xi}^{\infty} e^{-u} u^{m-1},$$

where $m = \frac{1}{(2-\beta)}$. One interpretation of this is that $P^*\{\gamma \le T\} = P^*\{U \ge f(T-t)\}$, where $U = f(\gamma - t)$ is a random variable with the gamma distribution. Now consider the inverse transformation, $f^{-1}(u)$, which is given by

$$f^{-1}(u) = (1/a)\ln[(u/c-1)^{-1}+1].$$

It follows easily that

$$
\begin{aligned}
E^*[e^{r\gamma}] &= \frac{1}{\Gamma(m)} \int_{\xi}^{\infty} \exp\{r f^{-1}(u)\} e^{-u} u^{m-1} du \\
&= \frac{1}{\Gamma(m)} \int_{\xi}^{\infty} u^{r/a} e^{-u} u^{m-1} du \\
&= \frac{1}{\Gamma(\frac{1}{2-\beta})} \int_{\xi}^{\infty} u^{\frac{\beta}{1-\beta}} (u-c)^{\frac{-1}{1-\beta}} e^{-u} du,
\end{aligned}
$$

as desired.

Chapter 2: Econometrics

2.1 Introduction

In this chapter, we discuss the econometrics of our model. The basic model, described in Chapter 1, known as the Constant Elasticity of Variance model in continuous time, is known as multiplicative heteroskedasticity in discrete time. The maximum likelihood estimation procedure for multiplicative heteroskedasticity was first discussed by Harvey (1976). For an excellent summary, see also Greene (1993, pp. 405-407). Multiplicative heteroskedasticity is now widely used and many computer statistical packages (such as SHAZAM) do estimation for this model. However, after looking at the data, we found that the model of multiplicative heteroskedasticity did not adequately describe our data. For this reason, we develop two alternatives to the standard model. The first alternative to the model of multiplicative heteroskedasticity (Model I) is still consistent with our analytic result, which is an advantage. Unfortunately, the model has other problems--most notably there is a problem with multicollinearity--and so we, in the end were forced to abandon it. The second model (Model II) proved quite successful, although its use in some cases precludes the use of our analytic results of Chapter 1. As a consequence, we must use numerical procedures to find the value of risky debt. We proceed with the numerical work in Chapter 3. It should be noted that Model II has very important implications for a model of risky debt, as we discuss while presenting our results, at the end of this chapter.

Let us now briefly over-view the three models under consideration. The continuous-time analogue of multiplicative heteroskedasticity can be written as

$$dX_t = \mu X_t dt + \sigma X_t^{\beta/2} dZ_t,$$

where Z is a standard Wiener process. This process, also known as a Constant Elasticity of Variance (CEV) process, is the diffusion used in Chapter 1. One way of estimating the parameters μ, σ and β, given observations $X_1, X_2, \ldots, X_n$, is to regress $y_t = X_{t+1} - X_t$ on X_t; i.e., we consider the regression

$$y_t = b_0 + b_1 X_t + \varepsilon_t\,,$$

where the variance of the error term is assumed to be

$$\begin{aligned}\sigma_t^2 &= \sigma^2 X_t^{\,\beta} \\ &= \exp(\gamma_0 + \beta \ln(X_t)),\end{aligned}$$

where $\gamma_0 = \ln(\sigma^2)$. (Here, we include the intercept term, b_0, for completeness; later we shall ignore the intercept, and re-write the regression with returns, $\Delta S/S$, as the dependent variable rather than price differences, ΔS.) The estimation procedure for this model is, as mentioned above, well known.

We would now like to modify the model for multiplicative heteroskedasticity. We consider two alternatives to the standard model. The first can be written in continuous time as

Model I: $$dX_t = \mu X_t dt + \sigma (X_t + C)^{\beta/2} dZ_t,$$

where C is treated as another parameter to estimate. The second, again written in continuous time, is

Model II: $$dX_t = \mu X_t dt + \sigma X_t^{\beta/2} dZ_1 + \kappa dZ_2.$$

where, in this case, κ is the extra parameter that is to be estimated. Here Z_1 and Z_2 are independent standard Wiener processes. As far as the variance structure is concerned, Model II is equivalent to the process

$$dX_t = \mu X_t dt + \sqrt{\sigma^2 X_t^{\beta} + \kappa^2}\, dZ_t.$$

What each of these models have in common is that they attempt to measure what happens to the volatility of the process as X approaches zero. We find that the second model is quite effective in describing the data. This also has important implications for the study of risky debt.

The remainder of this chapter proceeds as follows. In section 2.2. we deal with some methodological preliminaries, focusing on the question of what it is we are trying to measure. In section 2.3, we present Model I, and in 2.4 we present Model II. We take the approach of deriving the scoring technique for estimating the models, as this seems to be an effective method of understanding them thoroughly. In section 2.5 we discuss data collection, and in section 2.6 we summarize our results.

2.2 Methodological Preliminaries

We would now like to briefly discuss what type of data will be used to test our model. In our model of risky debt, we consider the stochastic process $(V_t - K_t)/K_t$ or alternatively, simply $V_t - K_t$, where V_t is the value of the firm at time t, and K_t is an exogenously specified point (possibly a function of time, possibly even stochastic) at which the firm will go into default immediately if $V_t = K_t$. Two questions now are: what is this default level K, and how do we measure the value of the firm, V? As pointed out in Nielsen et al. (1994), "the critical level, $[K_t]$, should ideally reflect the market value at time t of all debt obligations faced by the firm." If we simply take K_t to be *equal* to the market value of debt, then for a firm with a relatively simple capital structure, $V_t - K_t$ should represent the market value of equity, which we denote by S_t. In other words, if we let B_t denote the market value of the firm's debt, then $V_t - K_t \approx V_t - B_t = S_t$. Thus, default occurs when the market value of the firm's equity is (approximately) equal to zero. This conclusion is consistent with Leland (1994) in which he endogenizes the firm's bankruptcy point and concludes that it is the point where the value of equity is zero. Leland works in a continuous time framework, as we do, but restricts his study to the case of perpetual debt. Nevertheless, using this as our bankruptcy point seems to be reasonable, and it considerably simplifies the problem of testing the model: we no longer have to measure V and K; we can simply observe the value of the firm's equity, S. It is for this reason that we use $X = V - K$ ($\approx S$) rather than $X = (V - K)/K$ as we did in Chapter 1. In Chapter 1, using the ratio $(V - K)/K$ allowed us to compare our results qualitatively with the work of Longstaff and Schwartz (1995). For the econometrics, however, using the difference, $V - K$, makes matters much simpler.

2.3 Model I

Our first alternative to the standard model of multiplicative heteroskedasticity may be thought of as multiplicative heteroskedasticity with a shift. The continuous-time model is

$$dS_t = \mu S_t dt + \sigma(S_t + C)^{\beta/2} dZ_t,$$

where S_t represents the market value of the firm's equity at time t. In discrete time then, we consider the regression

$$\Delta S_t = b_0 + b_1 S_t + \varepsilon_t \, ,$$

(where $\Delta S_t = S_{t+1} - S_t$) which may also be written as

$$y_t = \boldsymbol{x}_t^{\mathrm{T}} \boldsymbol{b} + \varepsilon_t,$$

where $y_t = \Delta S_t$, $\boldsymbol{x}_t^{\mathrm{T}} = (1, S_t)$ and $\boldsymbol{b}^{\mathrm{T}} = (b_0, b_1)$. (As mentioned above, we include the intercept term for completeness; later we shall ignore the intercept, and re-write the regression with returns, $\Delta S/S$, as the dependent variable rather than price differences, ΔS.) The variance of the error term is thus assumed to be

$$\begin{aligned} \sigma_t^2 &= \sigma^2 (S_t + C)^{\gamma_1} \\ &= \exp(\gamma_0 + \gamma_1 \ln(S_t + C)) \\ &= \exp(\gamma^T z_t), \end{aligned}$$

where $\gamma_0 = \ln\sigma^2$, $\gamma_1 \equiv \beta$, $\gamma^T = (\gamma_0, \gamma_1)$ and $z_t^{\mathrm{T}} = (1, \ln(S_t + C))$. Here, C is another parameter that must be estimated. In the standard model for multiplicative heteroskedasticity, $C = 0$. The log likelihood is

$$\begin{aligned} \ln L &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_t \ln\sigma_t^2 - \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\sigma_t^2} \\ &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_t \gamma^T z_t - \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^T z_t)} \\ &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\gamma_0 - \frac{1}{2}\sum_t \gamma_1 \ln(S_t + C) - \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp\{\gamma_0 + \gamma_1 \ln(S_t + C)\}}. \end{aligned}$$

where $\varepsilon_t = \Delta S_t - b_0 - b_1 S_{t^-} = y_t - \boldsymbol{x}_t^{\mathrm{T}} \boldsymbol{b}$. The likelihood equations are

$$\frac{\partial \ln L}{\partial b_0} = \sum_t \frac{\varepsilon_t}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} = 0$$

$$\frac{\partial \ln L}{\partial b_1} = \sum_t S_t \frac{\varepsilon_t}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} = 0$$

$$\frac{\partial \ln L}{\partial \gamma_0} = -\frac{n}{2} + \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)}$$

$$= \frac{1}{2}\sum_t \left(\frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} - 1\right) = 0$$

$$\frac{\partial \ln L}{\partial \gamma_1} = -\frac{1}{2}\sum_t \ln(S_t + C) + \frac{1}{2}\sum_t \ln(S_t + C) \frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)}$$

$$= \frac{1}{2}\sum_t \ln(S_t + C)\left(\frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} - 1\right) = 0$$

$$\frac{\partial \ln L}{\partial C} = -\frac{1}{2}\sum_t \frac{\gamma_1}{(S_t + C)} + \frac{1}{2}\sum_t \frac{\gamma_1}{(S_t + C)} \frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)}$$

$$= \frac{1}{2}\sum_t \frac{\gamma_1}{(S_t + C)}\left(\frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} - 1\right) = 0.$$

If we let $\boldsymbol{\theta}^{\mathrm{T}} = (\gamma_0, \gamma_1, C) = (\ln\sigma^2, \beta, C)$ and $\boldsymbol{w}_t^{\mathrm{T}} = (1, \ln(S_t + C), \gamma_1/(S_t + C))$, we can write the likelihood equations in matrix form:

$$\frac{\partial \ln L}{\partial \boldsymbol{b}} = \sum_t \boldsymbol{x}_t \frac{\varepsilon_t}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} = \mathbf{0} = \mathbf{X}^{\mathrm{T}}\Omega^{-1}\varepsilon.$$

$$\frac{\partial \ln L}{\partial \boldsymbol{\theta}} = \frac{1}{2}\sum_t \boldsymbol{w}_t\left(\frac{\varepsilon_t^2}{\exp(\boldsymbol{\gamma}^{\mathrm{T}} z_t)} - 1\right) = \mathbf{0}.$$

Here, $\mathbf{X}$ is the matrix whose t^{th} row is x_t^{T}, and Ω is the variance-covariance matrix for the error terms, $\varepsilon_t = y_t - x_t^{\mathrm{T}}\boldsymbol{b}$; i.e., $\Omega = E[\varepsilon\varepsilon^{\mathrm{T}}] = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$.

We now consider the method of scoring. Although there are more efficient methods of estimating this model, an understanding of the scoring method serves two purposes. First, it will allow us to estimate the asymptotic variances. Second, it will reveal any multicollinearity that is inherent in the problem. We shall see that there is a multicollinearity problem when we try to estimate the variance parameters. If we choose to use a different method to find the maximum likelihood estimates, it must be a method that can bypass the problem of multicollinearity.

The first few terms in the Hessian can easily be written in matrix notation:

$$\frac{\partial^2 \ln L}{\partial b \partial b^{\mathrm{T}}} = -\sum_t \frac{1}{\exp(\gamma^{\mathrm{T}} z_t)} x_t x_t^{\mathrm{T}} = -\mathbf{X}^{\mathrm{T}} \Omega^{-1} \mathbf{X},$$

$$\frac{\partial^2 \ln L}{\partial b \partial \theta^{\mathrm{T}}} = -\sum_t \frac{\varepsilon_t}{\exp(\gamma^{\mathrm{T}} z_t)} x_t w_t^{\mathrm{T}}.$$

The remaining terms, however, do not easily lend themselves to matrix notation. In scalar notation then, we have

$$\frac{\partial^2 \ln L}{\partial \gamma_0^2} = -\frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^{\mathrm{T}} z_t)},$$

$$\frac{\partial^2 \ln L}{\partial \gamma_0 \partial \gamma_1} = -\frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^{\mathrm{T}} z_t)} \ln(S_t + C),$$

$$\frac{\partial^2 \ln L}{\partial \gamma_0 \partial C} = -\frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^{\mathrm{T}} z_t)} \frac{\gamma_1}{(S_t + C)},$$

$$\frac{\partial^2 \ln L}{\partial \gamma_1^2} = -\frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^{\mathrm{T}} z_t)} [\ln(S_t + C)]^2,$$

$$\frac{\partial^2 \ln L}{\partial \gamma_1 \partial C} = \frac{1}{2}\sum_t \frac{1}{(S_t + C)}\left(\frac{\varepsilon_t^2}{\exp(\gamma^{\mathrm{T}} z_t)} - 1\right) - \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^{\mathrm{T}} z_t)} \frac{\gamma_1}{(S_t + C)} \ln(S_t + C),$$

$$\frac{\partial^2 \ln L}{\partial C^2} = -\frac{1}{2}\sum_t \frac{\gamma_1}{(S_t+C)^2}\left(\frac{\varepsilon_t^2}{\exp(\gamma^T z_t)} - 1\right) - \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\exp(\gamma^T z_t)}\frac{\gamma_1^2}{(S_t+C)^2}.$$

Taking expectations gives

$$E[\frac{\partial^2 \ln L}{\partial b \partial b^T}] = -\mathbf{X}^T\Omega^{-1}\mathbf{X},$$

$$E[\frac{\partial^2 \ln L}{\partial b \partial \theta^T}] = \mathbf{0},$$

$$E[\frac{\partial^2 \ln L}{\partial \gamma_0^2}] = -\frac{n}{2},$$

$$E[\frac{\partial^2 \ln L}{\partial \gamma_0 \partial \gamma_1}] = -\frac{1}{2}\sum_t \ln(S_t + C),$$

$$E[\frac{\partial^2 \ln L}{\partial \gamma_0 \partial C}] = -\frac{1}{2}\sum_t \frac{\gamma_1}{(S_t+C)},$$

$$E[\frac{\partial^2 \ln L}{\partial \gamma_1^2}] = -\frac{1}{2}\sum_t [\ln(S_t + C)]^2,$$

$$E[\frac{\partial^2 \ln L}{\partial \gamma_1 \partial C}] = -\frac{1}{2}\sum_t \frac{\gamma_1}{(S_t+C)}\ln(S_t + C),$$

$$E[\frac{\partial^2 \ln L}{\partial C^2}] = -\frac{1}{2}\sum_t \frac{\gamma_1^2}{(S_t+C)^2}.$$

Now, if we let $\delta^T = (b_0, b_1, \gamma_0, \gamma_1, C) = (b_0, b_1, \ln\sigma^2, \beta, C)$, the information matrix can be written in matrix notation as

$$-E[\frac{\partial^2 \ln L}{\partial \delta \partial \delta^T}] = \begin{bmatrix} \mathbf{X}^T\Omega^{-1}\mathbf{X} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{W}^T\mathbf{W} \end{bmatrix} \equiv -\mathbf{H}.$$

Here $\mathbf{W}$ is the matrix such that the t^{th} row equals the vector w_t^T. The scoring method is then

$$\hat{\delta}_{i+1} = \hat{\delta}_i - \mathbf{H}_i^{-1}g_i,$$

where $\hat{\delta}_i$ is the estimate at iteration i and g_i is the vector $\left.\frac{\partial \ln L}{\partial \delta}\right|_{\delta=\hat{\delta}_i}$. Since $\mathbf{H}_i$ is block diagonal, the iteration can be written as separate equations. The first set of equations is

$$\begin{aligned}\hat{b}_{i+1} &= \hat{b}_i + (\mathbf{X}^T\Omega_i^{-1}\mathbf{X})^{-1}\mathbf{X}^T\Omega_i^{-1}\varepsilon_i \\ &= \hat{b}_i + (\mathbf{X}^T\Omega_i^{-1}\mathbf{X})^{-1}\mathbf{X}^T\Omega_i^{-1}(y - \mathbf{X}\hat{b}_i) \\ &= (\mathbf{X}^T\Omega_i^{-1}\mathbf{X})^{-1}\mathbf{X}^T\Omega_i^{-1}y.\end{aligned}$$

Thus, as in the standard model for multiplicative heteroskedasticity, the updated coefficient vector, $\hat{b}_{i+1}$ is computed by feasible generalized least squares (FGLS) using the previously computed estimate of θ to compute Ω. The approach for θ is also similar to the procedure used in the standard model for multiplicative heteroskedasticity:

$$\hat{\theta}_{i+1} = \hat{\theta}_i + (\mathbf{W}^T\mathbf{W})^{-1}\sum_t w_t\left(\frac{\varepsilon_t^2}{\exp(\hat{\gamma}_i^T z_t)} - 1\right).$$

Thus, to find the updated value of θ we first regress $\left(\frac{\varepsilon_t^2}{\exp(\gamma^T z_t)} - 1\right)$ on w_t and then add the regression coefficients to θ_i. This procedure is iterated to convergence. Finally, the asymptotic covariance matrix is simply $-\mathbf{H}^{-1}$, which is block diagonal with blocks

$$\begin{aligned}\text{Asy.Var}(\hat{b}_{ML}) &= (\mathbf{X}^T\Omega^{-1}\mathbf{X})^{-1}, \\ \text{Asy.Var}(\hat{\theta}_{ML}) &= 2(\mathbf{W}^T\mathbf{W})^{-1}.\end{aligned}$$

Unfortunately, this iterative procedure is not very effective due to multicollinearity between $\ln(S_t + C)$ and $\gamma_1/(S_t + C)$. It is still possible to solve the maximum likelihood problem, though. An effective procedure is Generalized Reduced

Gradients (GRG), which is the method used by EXCEL's Solver. Once the parameters are estimated, one can, in principle, use the above procedure to arrive at estimates of the asymptotic variance. Although this seems to be the most natural econometric model given our mathematical model, convergence proved to be very slow and the results did not seem very reliable. Furthermore, using S as an independent variable also creates problems, as S does not have the appropriate convergence properties needed to allow one to apply the usual asymptotic results. Using returns, $\Delta S/S$, rather than price changes, ΔS, as the dependent variable does ameliorate the situation somewhat (as we report in section 2.6 below), but the maximum likelihood estimation procedures are still rather slow to converge. For these reasons, we shall use a different econometric model.

2.4 Model II

For our second model, one modification that we make is to work with returns, dS/S, rather than simply changes, dS. The other modification we now make is to write the model, in continuous-time, as (ignoring the time subscripts)

$$dS = \mu S dt + \sigma S^{\beta/2} dZ_1 + \kappa dZ_2,$$

or, using returns, as

$$dS/S = \mu dt + \sigma S^{(\beta/2)-1} dZ_1 + \kappa/S dZ_2,$$

where Z_1 and Z_2 are independent standard Wiener processes. (We assume this process is stopped at $S = 0$, i.e., if $S_t = 0$, then $S_u = 0$ for all $u \geq t$.) Note that this would imply an incomplete market. We shall ignore any problems caused by market incompleteness, such as the fact that prices are no longer unique in an incomplete model. Any variation in prices should be minor, however. An alternative way of writing the model would be:

$$dS = \mu S dt + \sqrt{\sigma^2 S^\beta + \kappa^2}\, dZ,$$

or once again, using returns:

$$dS/S = \mu dt + \sqrt{\sigma^2 S^{\beta-2} + (\kappa / S)^2}\, dZ,$$

In either case, when using returns the square of the volatility becomes:

$$\sigma_t^2 = \sigma^2 S_t^{\beta-2} + (\kappa/S_t)^2 = [\sigma^2 S_t^\beta + \kappa^2]/S_t^2,$$

so that the log likelihood is

$$\begin{aligned}
\ln L &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_t \ln\sigma_t^2 - \frac{1}{2}\sum_t \frac{\varepsilon_t^2}{\sigma_t^2} \\
&= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_t \ln(\sigma^2 S_t^{\beta-2} + (\kappa/S_t)^2) - \frac{1}{2}\sum_t \frac{S_t^2\varepsilon_t^2}{\sigma^2 S_t^\beta + \kappa^2} \\
&= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_t [\ln(\sigma^2 S_t^\beta + \kappa^2) - \ln(S_t^2)] - \frac{1}{2}\sum_t \frac{S_t^2\varepsilon_t^2}{\sigma^2 S_t^\beta + \kappa^2}.
\end{aligned}$$

Here we write $\varepsilon_t = \Delta S_t/S_t - \mu\Delta t = y_t - a$, say, where $y_t = \Delta S_t/S_t$ and $a = \mu\Delta t$, in the discrete-time notation. The likelihood equations are

$$\frac{\partial \ln L}{\partial a} = \sum_t \frac{S_t^2\varepsilon_t}{\sigma^2 S_t^\beta + \kappa^2} = 0$$

$$\begin{aligned}
\frac{\partial \ln L}{\partial(\kappa^2)} &= -\frac{1}{2}\sum_t \frac{1}{\sigma^2 S_t^\beta + \kappa^2} + \frac{1}{2}\sum_t \frac{S_t^2\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^2} \\
&= \frac{1}{2}\sum_t \frac{1}{\sigma^2 S_t^\beta + \kappa^2}\left(\frac{S_t^2\varepsilon_t^2}{\sigma^2 S_t^\beta + \kappa^2} - 1\right) = 0
\end{aligned}$$

$$\frac{\partial \ln L}{\partial(\sigma^2)} = -\frac{1}{2}\sum_t \frac{S_t^\beta}{\sigma^2 S_t^\beta + \kappa^2} + \frac{1}{2}\sum_t \frac{S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^2}$$

$$= \frac{1}{2}\sum_t \frac{S_t^\beta}{\sigma^2 S_t^\beta + \kappa^2}\left(\frac{S_t^2\varepsilon_t^2}{\sigma^2 S_t^\beta + \kappa^2} - 1\right) = 0$$

$$\frac{\partial \ln L}{\partial \beta} = -\frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t) S_t^\beta}{\sigma^2 S_t^\beta + \kappa^2} + \frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t) S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^2}$$

$$= \frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t) S_t^\beta}{\sigma^2 S_t^\beta + \kappa^2}\left(\frac{S_t^2\varepsilon_t^2}{\sigma^2 S_t^\beta + \kappa^2} - 1\right) = 0.$$

Let **1** denote a column vector of ones. If we now write $\theta^{\mathrm{T}} = (\kappa^2, \sigma^2, \beta)$ and

$$w_t^{\mathrm{T}} = \left(\frac{1}{\sigma^2 S_t^\beta + \kappa^2}, \frac{S_t^\beta}{\sigma^2 S_t^\beta + \kappa^2}, \frac{\sigma^2 \ln(S_t) S_t^\beta}{\sigma^2 S_t^\beta + \kappa^2}\right),$$

then we can write the likelihood equations as

$$\frac{\partial \ln L}{\partial a} = \mathbf{1}^{\mathrm{T}}\Omega^{-1}\varepsilon,$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{1}{2}\sum_t w_t\left(\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right).$$

The first terms in the Hessian can be written

$$\frac{\partial^2 \ln L}{\partial a^2} = -\mathbf{1}^{\mathrm{T}}\Omega^{-1}\mathbf{1},$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial a} = -\sum_t w_t\left(\frac{\varepsilon_t}{\sigma_t^2}\right),$$

and the other terms are

$$\frac{\partial^2 \ln L}{\partial(\kappa^2)^2} = \frac{1}{2}\sum_t \frac{1}{(\sigma^2 S_t^\beta + \kappa^2)^2} - \sum_t \frac{S_t^2\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^3},$$

$$\frac{\partial^2 \ln L}{\partial(\sigma^2)^2} = \frac{1}{2}\sum_t \frac{S_t^{2\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2} - \sum_t \frac{S_t^{2\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^3},$$

$$\frac{\partial^2 \ln L}{\partial(\sigma^2)\partial(\kappa^2)} = \frac{1}{2}\sum_t \frac{S_t^{\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2} - \sum_t \frac{S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^3},$$

$$\frac{\partial^2 \ln L}{\partial\beta^2} = -\frac{1}{2}\sum_t \frac{\sigma^2\kappa^2 \ln(S_t)^2 S_t^{\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2} + \frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t)^2 S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^2}$$

$$-\sum_t \frac{\sigma^4 \ln(S_t)^2 S_t^{2\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^3},$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial(\kappa^2)} = \frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t) S_t^{\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2} - \sum_t \frac{\sigma^2 \ln(S_t) S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^3},$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial(\sigma^2)} = -\frac{1}{2}\sum_t \frac{\kappa^2 \ln(S_t) S_t^{\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2} + \sum_t \frac{\kappa^2 \ln(S_t) S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^3} - \frac{1}{2}\sum_t \frac{\ln(S_t) S_t^{\beta+2}\varepsilon_t^2}{(\sigma^2 S_t^\beta + \kappa^2)^2}.$$

Taking expectations,

$$E[\frac{\partial^2 \ln L}{\partial a^2}] = -\mathbf{1}^{\mathrm{T}}\Omega^{-1}\mathbf{1},$$

$$E[\frac{\partial^2 \ln L}{\partial\theta\partial a}] = 0,$$

$$E[\frac{\partial^2 \ln L}{\partial(\kappa^2)^2}] = -\frac{1}{2}\sum_t \frac{1}{(\sigma^2 S_t^\beta + \kappa^2)^2},$$

$$E[\frac{\partial^2 \ln L}{\partial(\sigma^2)^2}] = -\frac{1}{2}\sum_t \frac{S_t^{2\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2},$$

$$E[\frac{\partial^2 \ln L}{\partial(\sigma^2)\partial(\kappa^2)}] = -\frac{1}{2}\sum_t \frac{S_t^{\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2},$$

$$E[\frac{\partial^2 \ln L}{\partial\beta^2}] = -\frac{1}{2}\sum_t \frac{\sigma^4 \ln(S_t)^2 S_t^{2\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2}.$$

$$E[\frac{\partial^2 \ln L}{\partial\beta\partial(\kappa^2)}] = -\frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t) S_t^\beta}{(\sigma^2 S_t^\beta + \kappa^2)^2},$$

$$E[\frac{\partial^2 \ln L}{\partial\beta\partial(\sigma^2)}] = -\frac{1}{2}\sum_t \frac{\sigma^2 \ln(S_t) S_t^{2\beta}}{(\sigma^2 S_t^\beta + \kappa^2)^2},$$

Finally, we can write

$$-E[\frac{\partial^2 \ln L}{\partial\delta\partial\delta^{\mathrm{T}}}] = \begin{bmatrix} \mathbf{1}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{W}^{\mathrm{T}}\mathbf{W} \end{bmatrix} = -\mathbf{H},$$

where $\delta^{\mathrm{T}} = (a, \kappa^2, \sigma^2, \beta)$ and $\mathbf{W}$ is the matrix such that the t^{th} row equals w_t. So, the scoring procedure is much the same as before, except that now we iterate between first using feasible generalized least squares to regress returns versus a constant vector, and then regressing $(\frac{\varepsilon_t^2}{\sigma_t^2} - 1)$ on $w = (1/(\sigma^2 S^\beta + \kappa^2), S^\beta/(\sigma^2 S^\beta + \kappa^2), \sigma^2 \ln(S) S^\beta/(\sigma^2 S^\beta + \kappa^2))$. Note that there is no intercept term here. Because the asymptotic covariance matrix is block diagonal, the asymptotic variances are

$$\text{Asy.Var}(\hat{a}) = (\mathbf{1}^{\mathrm{T}}\Omega^{-1}\mathbf{1})^{-1}$$

$$\text{Asy.Var}(\hat{\theta}) = 2(\mathbf{W}^{\mathrm{T}}\mathbf{W})^{-1}.$$

The two advantages to this approach are first that we use returns rather than price changes, and second, that the multicollinearity in w may not be as severe as it was in the previous model. As we shall see in the next section, Model II seems to fit the data well.

2.5 Data and Methodology

Data were collected for a total of 27 firms on a daily basis from January 1975 through January 1981 where possible. The firms selected were the same firms used in

Jones, Mason, and Rosenfeld (1984). As they discuss, the firms were selected according to a number of criteria:

1. Simple Capital Structures (i.e., one class of stock, no convertible bonds, small number of debt issues, no preferred stock).
2. Small proportion of private debt to total capital.
3. Small proportion of short term notes payable or capitalized leases to total capital.
4. All publicly traded debt is rated.

The market value of equity is just the price per share multiplied by the numbers of shares outstanding. The number of shares outstanding varied considerably, for most firms, over the period studied. There was no attempt made to keep the capital structure fixed.

2.6 Results and Conclusions

Table 1a contains a summary of the results for Model I; Table 2, for Model II. For each model we first estimated the restricted model ($C = 0$ for Model I, $\kappa = 0$ for Model II) --which corresponds to the standard model for multiplicative heteroskedasticity--and then we estimated the unrestricted model. For each firm, the first row corresponds to the restricted model, the second to the unrestricted model. The method used to obtain the maximum likelihood estimators was Generalized Reduced Gradients (GRG), which is the method used by EXCEL's Solver. No matrix inversion is required by this method. so the problem of multicollinearity noted above was overcome.

One can see substantial differences between the two models, even when they were attempting to estimate the same (restricted) parameters. Model I seems to be the one that is behaving badly. For one thing, the estimates seemed to be quite sensitive to changes in the sample chosen. Also, some of the values observed as estimates of C seem excessive. Furthermore, a few attempts were made to use the parameters to simulate data, and these few simulations quickly diverged.

Whether Model I fails because of the multicollinearity between $\ln(S_t + C)$ and $\gamma_1/(S_t + C)$, or because we regressed on stock prices, is not clear, initially. In order to better understand this point, it seems to be worth modifying this model to apply it to returns, dS/S, rather than to price changes, dS. When we use returns as the dependent variable, the variance of the error term becomes $\sigma_t^2 = \sigma^2(S_t + C)^{\gamma_2} S_t^{-2}$. The results of these regressions are reported in Table 1b. One finds that the two approaches give similar results in the restricted model ($C = 0$), but using price changes gives anomalous results in the unrestricted model. On the other hand, whether regressing using price changes or returns, the same firms seem to show a significant difference when we add the constant C to the model; it is only the estimates of C (and the corresponding changes in β and σ) that differ with the two approaches. Even when using returns, however, the maximum likelihood estimation procedures for this model were slow to converge (sometimes over 400 iterations were required). So, although the model is a natural one given our earlier work, and is mathematically fairly straightforward, it does not seem to be as practical as Model II.

Model II seems to work well. For fifteen of the twenty-seven firms studied, the parameter κ^2 was significant. A simple method to test for the significance of this parameter is the likelihood ratio test. The test statistic is $2(\ln L_U - \ln L_R)$, where L_U is the likelihood function evaluated at the unrestricted estimates, while L_R is the likelihood function evaluated at the restricted estimates. As we use one restriction, $\kappa = 0$, this test statistic has a χ^2 distribution with one degree of freedom. The p-values for these tests are provided (for each model).

Despite the apparent problems with Model I, in most cases, if C was significant in Model I, then κ was significant in Model II, and vice versa.

One interesting result that can be seen from Table I is that, with only one exception (Seagram) if κ was significant, then β was greater than 2, and vice versa. This has very important implications to a model of risky debt. This implies that volatility does *not* approach zero as the value of equity approaches zero. In fact, the volatility of returns approaches infinity as the price approaches zero, much as in the case where $\beta < 2$. (Actually, if κ is significant, it is as if the stock has a $\beta = 0$ component.) With only one exception then, each firm has either $\beta < 2$, or $\kappa > 0$, which implies that zero is an accessible boundary for the equity process. When κ is ignored, on the other hand, there are seven firms for which the estimate of β is greater than 2, which would ordinarily imply that zero is inaccessible; i.e., the best estimate of the probability of default for these seven firms is zero, which in turn would imply that their debt was riskless. So, adding κ to the model strengthens it considerably.

Another observation that is perhaps worth mentioning is that if κ is significant for a firm, then our estimate of β increases (relative to the restricted model) while our estimate of σ decreases. It is clear that if β increases, then σ must decrease, in order get the same "local" volatility, but why β should increase when κ is significant is not clear. However, this does have implications for our modelling, due to the fact that large β's are difficult to deal with, as we shall see in Chapter 3.

We are now ready to apply our econometric results to the problem of risky debt. Note that there are two cases that we must consider: either $\beta < 2$ and $\kappa = 0$, in which case we have an analytic result for the value of risky debt, or $\beta > 2$ and $\kappa > 0$, in which case, numerical procedures are required. We consider such procedures in the following chapter.

Chapter 3: Numerical Procedures

3.1 Introduction

In this chapter, we use the parameters estimated in Chapter 2 and apply them to the problem of evaluating risky debt. In Chapter 2, however, we found that the model used in Chapter 1, for which we have an analytic solution, only applied to some of our firms. The remaining firms follow a stochastic process that satisfies a stochastic differential equation of the form

$$dS = \mu S dt + \sigma S^{\beta/2} dZ_1 + \kappa dZ_2,$$

or alternatively,

$$dS = \mu S dt + \sqrt{\sigma^2 S^\beta + \kappa^2}\, dZ,$$

where $\beta > 2$. In light of the results of the econometric work, we must now proceed with some numerical procedures in order to find the value of risky debt for those firms whose equity follows the above stochastic process. Ultimately, we shall use Monte Carlo methods to do this. However, a brief discussion of some other methods may be in order. In particular, we will show that the conventional recombining trinomial tree is not feasible.

This chapter proceeds as follows. In section 3.2 we look at some general issues regarding constructing lattices. In section 3.3 we consider a rather natural trinomial model and show that it is not feasible. In section 3.4 we discuss the Monte Carlo techniques we then used, and in section 3.5 we present our results.

3.2 Lattice Procedures

One of the first lattice procedures in option pricing was the seminal paper, Cox, Ross, and Rubinstein (CRR; 1979) in which they construct a binomial process of the stock price that converges in distribution to a geometric Brownian motion, while the European option in this binomial model converges to the value obtained by the Black-Scholes formula as the time step shrinks to zero ($\Delta t \rightarrow 0$). The binomial model is very useful in valuing contingent claims in cases where no closed-form solution has been obtained. Subsequently, Cox and Rubinstein (1985) and Brennan and Schwartz (1978) both showed that such lattice models are equivalent to some numerical solution to the partial differential equation that the value of an option satisfies.

However, the CRR model, with its limit of geometric Brownian motion, does not allow for the various types of diffusion processes that are often observed in finance. Interest rate models, in particular, tend to involve either mean-reversion or heteroskedasticity (or both). Such models are difficult to approximate for two reasons. First, path independence or recombining is no longer straightforward as in the CRR model. Second, convergence from the discrete- to continuous-time processes is no longer easily guaranteed.

Four papers that deal with these issues are Hull and White (1990), Nelson and Ramaswamy (1990), Tian (1992), and Barone-Adesi, Dinenis, and Sorwar (1997). (See also Tian (1994) for a discussion of the convergence properties of these models as well as for an excellent summary of the first three papers, to which I am indebted.) Each of these papers uses a transformation which we shall now describe.

Consider a diffusion process θ that follows the stochastic differential equation

$$d\theta(t) = \mu(\theta, t)dt + \sigma(\theta, t)dZ_t.$$

The process θ may represent a stock price, the interest rate, or some other economic variable. We assume that the drift rate is adjusted to a risk-neutral world. Let $\phi = \phi(\theta, t)$ be a transformation of θ. Then, by Ito's Lemma, ϕ satisfies

$$d\phi = q(\theta, t)dt + \frac{\partial \phi}{\partial \theta}\sigma dZ_t,$$

where

$$q(\theta, t) = \frac{\partial \phi}{\partial t} + \mu\frac{\partial \phi}{\partial \theta} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial \theta^2}.$$

If we can find a transformation θ such that

$$\sigma\frac{\partial \phi}{\partial \theta} = v,$$

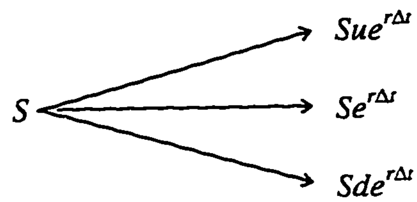
for some positive constant v, then the transformed process ϕ will be homoskedastic. Furthermore, we require that the function $\phi(\theta, t)$ be deterministic, and at least twice differentiable in ϕ and once differentiable in t. Applying this condition to our model, we require

$$\phi = \int \frac{1}{\sigma(S)}dS = \int \frac{1}{\sqrt{\sigma^2 S^\beta + \kappa^2}}dS,$$

which unfortunately does not have an analytic solution in general. Thus, we cannot use this transformation, and so it seems reasonable to attempt to construct a tree directly. based on the non-transformed model. One possible, simple trinomial tree is discussed in the following section.

3.3 A Simple Trinomial Tree

In this section, we will demonstrate that a conventional recombining trinomial tree fails to model the type of heteroskedasticity observed in our sample. A simple trinomial tree would look like this:



This arrangement, in which the middle "no change" branch actually corresponds to growth at the risk-free rate simplifies the mathematics substantially. In addition, a tree with this type of symmetry ought to encounter fewer problems with negative probabilities. Note that we are also assuming that the jump size, u, is proportional rather than additive; i.e., we are constructing a tree with geometric growth rather than an evenly spaced grid. The transition probabilities are denoted

$$P\{S_{t+\Delta t} = S_t u e^{r\Delta t} \mid S_t\} = p_u(t, S_t) = p_u,$$

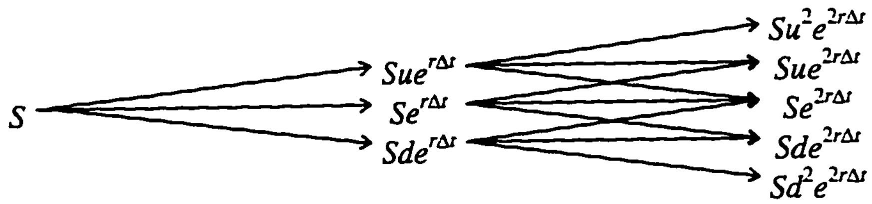
$$P\{S_{t+\Delta t} = S_t e^{r\Delta t} \mid S_t\} = p_m(t, S_t) = p_m,$$

$$P\{S_{t+\Delta t} = S_t d e^{r\Delta t} \mid S_t\} = p_d(t, S_t) = p_d.$$

Note that we suppress the dependence of these probabilities on t and S_t.

Although a trinomial tree is incomplete (unlike a binomial tree) the reason for applying this method to a heteroskedastic model is that one can vary the volatility at each node by changing the probabilities, p_u, p_m and p_d: for maximum volatility, set $p_m = 0$; for zero volatility, set $p_u = p_d = 0$. The idea behind this model is that one chooses the maximum volatility at the outset; then one can simply vary the probabilities throughout the tree in order to capture the heteroskedasticity.

Now consider a two-step tree:



This tree illustrates two assumptions that are fairly standard. First, in order that the tree recombine, we assume that $u = 1/d$. Second, we assume that u is a constant, independent of t or S_t, again, so that the tree recombines. We now show that this model fails.

The probabilities are determined by the first and second moments. For the first moment, we require

$$E[S_{t+\Delta t} | S_t] = S_t e^{r\Delta t}.$$

From the trinomial model on the other hand,

$$E[S_{t+\Delta t} | S_t] = S_t u e^{r\Delta t} p_u + S_t e^{r\Delta t} p_m + S_t d e^{r\Delta t} p_d$$

Solving these two equations, we find that

$$u p_u + p_m + d p_d = 1,$$

and since

$$p_u + p_m + p_d = 1,$$

it follows that

$$(u-1)\, p_u + (d-1)\, p_d = 0. \tag{3.1}$$

Now, we consider the second moment. For small Δt, we must have

$$Var(S_{t+\Delta t} | S_t) = S_t^2 \sigma_t^2 \Delta t.$$

where

$$S_t^2\sigma_t^2 = \sigma^2 S_t^{\,\beta} + \kappa^2.$$

From the trinomial model,

$$Var(S_{t+\Delta t} \mid S_t) = (S_t u e^{r\Delta t} - S_t e^{r\Delta t})^2 p_u + 0 \cdot p_m + (S_t d e^{r\Delta} - S_t e^{r\Delta})^2 p_d$$

which, after simplifying leads to the equation

$$(u-1)^2 p_u + (d-1)^2 p_d = \sigma_t^2 \Delta t e^{-2r\Delta t}. \tag{3.2}$$

Next, using the fact that $u = 1/d$, and again simplifying, we find from equation (3.1) that

$$(d-1)^2 p_d = (u-1)(1 - 1/u)p_u,$$

so that, from equation (3.2)

$$(u-1)(u - 1/u)p_u = \sigma_t^2 \Delta t e^{-2r\Delta t}. \tag{3.3}$$

For $\beta > 2$, (the case that particularly presents problems in terms of finding an analytic solution) we are most concerned about large prices. (For simplicity, we may assume $\kappa = 0$ for the moment.) The maximum price in the tree would be $S_0 u^n e^{rT}$, where n is the number of steps in the tree. At this price, we would want the maximum volatility which corresponds to $p_m = 0$, so that $p_d = 1 - p_u$. From equation (3.1) then, and the fact that $u = 1/d$,

$$(u-1)p_u + (1/u - 1)(1 - p_u) = 0,$$

or,

$$(u - 1/u)p_u = 1 - 1/u.$$

Substituting this into equation (3.3) gives

$$(u-1)(1 - 1/u) = (\sigma_t^*)^2 \Delta t e^{-2r\Delta t},$$

where $(\sigma_t^*)^2$ is the volatility at the maximum price; i.e., we have

$$(u-1)(1-1/u) = [\sigma^2 S_0^{(\beta-2)} u^{n(\beta-2)} e^{rT(\beta-2)} + \kappa^2 S_0^{-2} u^{-2n} e^{-2rT}]\Delta t e^{-2r\Delta t}. \quad (3.4)$$

This equation can now be solved for u.

Using the parameters estimated for the firms in our study having $\beta > 2$, a quick check using Solver in EXCEL revealed that in no case was there a feasible solution for $u > 1$. Thus, the above trinomial model is not feasible in this case. The underlying reason for this is that the volatility is unbounded. Intuitively, what happens is that when we choose a maximum stock price, say, for the tree, that price is associated with a maximum volatility which defines u. Unfortunately, the implied u leads to a maximum stock price that is larger than the one originally chosen, so that we have a vicious circle. This seems significant. Whether or not a multinomial tree is feasible is difficult to answer, but we can outline what the multinomial model would involve.

The basic difference between a multinomial tree and a simple trinomial tree is that in the multinomial case, at each node of the tree, the maximum upward jump, say, is given by u^J, rather than simply u; that is, at step t, the possible jumps are of the form

$$S_{t+\Delta t} = \begin{cases} S_t u^{J(t)} e^{r\Delta t} \\ S_t e^{r\Delta t} \\ S_t d^{J(t)} e^{r\Delta t} \end{cases}$$

We now describe how J is chosen at each step of the tree. At $t = 0$, we may set u so that

$$S_0^2(u-1)(1-1/u) = [\sigma^2 S_0^{\beta} + \kappa^2]\Delta t e^{-2r\Delta t}.$$

For example, using the relevant parameters for the firm Allied Chemical, we find that $u = 1.22$. Next, as an example of an intermediate calculation, suppose that the value of equity has increased to $S_t = S_0 u^k e^{rk\Delta t}$. (In the Allied Chemical example, for $k = 50$, this corresponds to $S_t = 5\times10^{10}$.) At this point in the tree, $J(t)$ is chosen as the minimum integer satisfying

$$S_t^2(u^{J(t)} - 1)(1 - 1/u^{J(t)}) = [\sigma^2 S_t^\beta + \kappa^2]\Delta t e^{-2r\Delta t}.$$

In our Allied example, again for $k = 50$, this gives $J(t) = 126$, which is quite large. It seems that the multinomial model quickly becomes rather unwieldy. It is for this reason that we proceed with a Monte Carlo simulation.

3.4 Monte Carlo

The Monte Carlo simulation is fairly straightforward. First, we generate a sequence of i.i.d. standard normal random variables, $\{z_t\}$. In EXCEL, the command RAND() generates i.i.d. uniform(0,1) random variables, while the command NORMSINV converts these to standard normal random variables. We then construct our (risk-neutral) sequence of equity prices, based on Model II of Chapter 3, as follows:

$$S_{t+1} = S_t + (r - \delta)S_t\Delta t + \sqrt{(\sigma^2 S_t^\beta + \kappa^2)\Delta t}\, z_t,$$

and we stop the random process at the first time that the price hits zero; i.e., we set $S_{t+1} = 0$ if $S_t \leq 0$. Here, r represents the risk-free interest rate, which was assumed to be a constant rate of 8 percent per year (continuously compounded) for each firm, while δ represents the firm's dividend yield, which was estimated from the data by comparing the total returns to the returns calculated from the changes in prices. The time increment is given by $\Delta t = T/n$, where T was taken to be 20 years, and n, the number of steps in the simulation was 50 for each firm. For our starting point, S_0, we used the final capitalization value in our sample (an out of sample approach) which seems to be sound methodologically.

We ran 1000 simulations for each of the 15 firms for which κ was significant (and $\beta > 2$) and used the results to estimate the probability of default, on a 20 year bond, for each of these firms. For the remaining firms, we calculated the probability of default by using ours analytic results of Chapter 1. These results are tabulated in Table 3.

One could also easily calculate the price of a 20-year zero coupon bond from these simulations, although zero coupon bonds are not very realistic. To evaluate coupon bonds, one would have to use the simulations to estimate the probability of default at each of the coupon payment dates. As an aside, note that one could also easily estimate prices of 20- year zero coupon bonds based on our alternative model given in Appendix 2 of Chapter 1, in which the payoff at default is equal to $(1 - w)e^{-r(T-\gamma)}$, where γ is the random default time. At any rate, these calculations require knowledge of the value of w. Rather than trying to estimate this value, we simply use the probability of default in order to try to explain the firm's credit rating. Our results are given in the next section.

3.5 Results and Conclusions

The probabilities of default that we estimated for each firm, together with the firms' credit ratings, are given in Table 3. If we assign a "1" to credit rating AAA, a "2" to credit rating AA, etc., we can calculate the correlation between credit ratings and the probability of default. If the firm had multiple credit ratings, we used the lowest rating, which should give the best indication of the firm's likeliness to default. Presumably, if a firm has bonds with multiple credit ratings, the higher rated bonds must give the creditors some sort of protection in the event of a default, rather than somehow providing a lower likelihood of default.

Note that some firms are reported twice. We calculated the analytic result whenever possible, so that we could test its effectiveness. Some firms had $\beta < 2$ when κ was ignored, but when Model II was estimated, we found $\kappa > 0$ and $\beta > 2$. These firms are repeated in the tables. Otherwise, firms with $\beta < 2$ are listed in the analytic section and firms with $\beta > 2$ are listed under the numerical results.

The resulting correlation, using all 27 firms in our sample, between the estimated probability of default and the credit rating was calculated to be equal to 0.67 ($r^2 = 0.45$, t-ratio = 2.52, one-tailed p-value = 0.009), which seems quite promising for such a simple, one-factor model. If we were to ignore the parameter κ, i.e., if we were to use the standard model of multiplicative heteroskedasticity, then only 19 of the firms would have $\beta < 2$. The correlation between the probability of default and the credit rating for these firms is only 0.37 ($r^2 = 0.13$, t-ratio = 0.56, one-tailed p-value = 29%). On the other hand, if we use "Model I revisited" for the few firms for which kappa was not significant (Model II) while C was significant in Model I revisited (i.e., Model I using returns instead of price changes) we find $r = 0.69$ ($r^2 = 0.48$, t-ratio = 2.75, one-tailed p-value = 0.5%), a slight improvement over simply using Model II alone. Curiously, for the 15 firms with $\beta > 2$, the correlation was 0.84 ($r^2 = 0.71$). Thus, Model II does seem to be successful in explaining firms' credit ratings, and it is a marked improvement over the standard model of multiplicative heteroskedasticity.

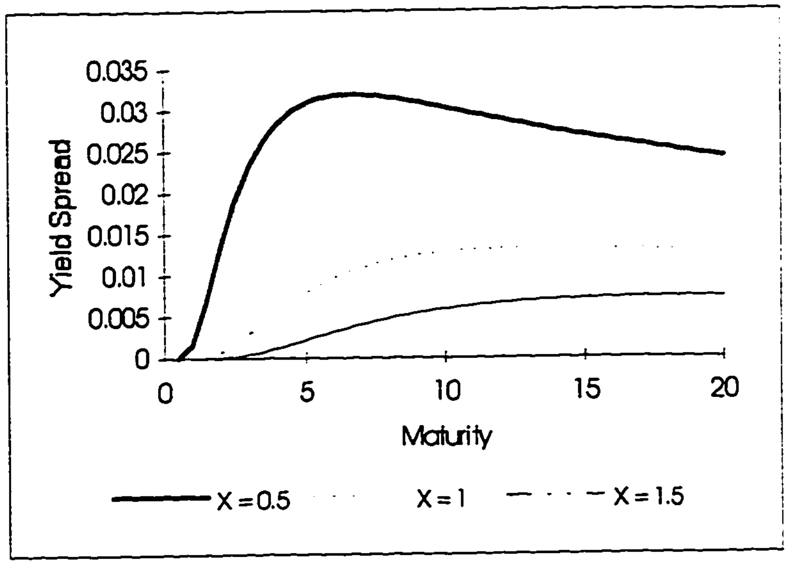


Fig 1. Credit spreads for an 8% bond for different values of X. The parameter values used are $r = 0.04$, $w = 0.5$, $\sigma = 0.4$, and $\beta = 1$.

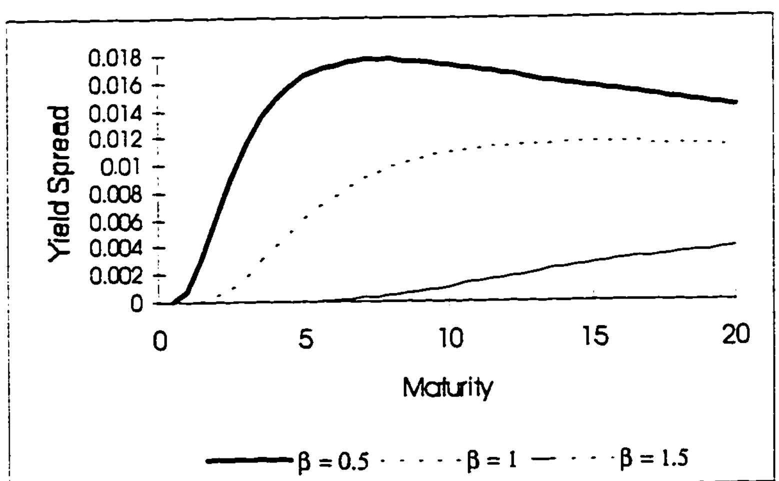


Fig 2. Credit spreads for an 8% bond for different values of β. The parameter values used are $r = 0.04$, $w = 0.5$, $\sigma = 0.4$, and $X = 1$.

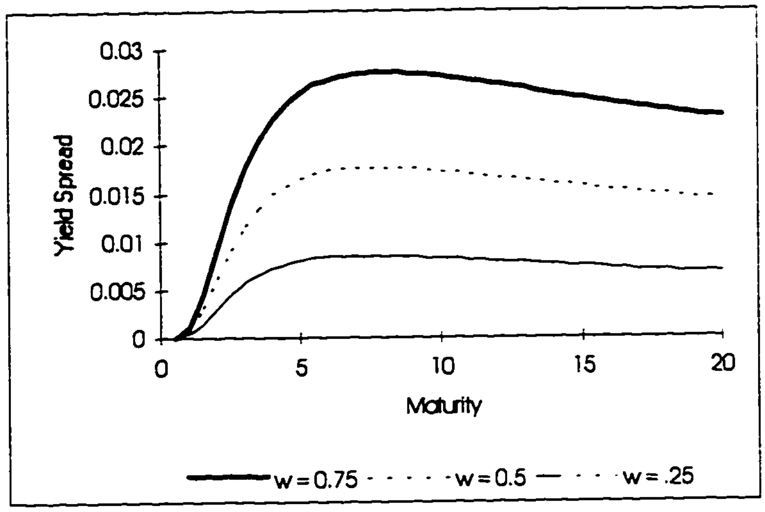


Fig 3. Credit spreads for an 8% bond for different values of w. The parameter values used are $r = 0.04$, $\beta = 1$, $\sigma = 0.4$, and $X = 1$.

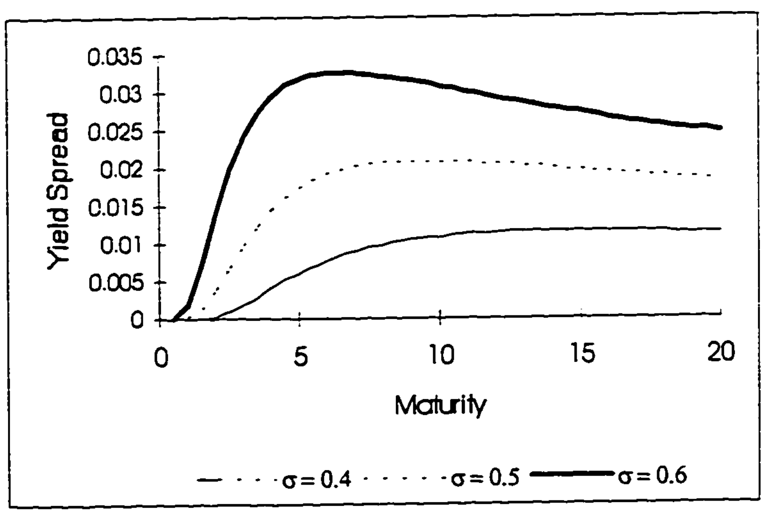


Fig 4. Credit spreads for an 8% bond for different values of σ. The parameter values used are $r = 0.04$, $\beta = 1$, $w = 0.5$, and $X = 1$.

Table 1a: The parameters for each firm are estimated using maximum likelihood estimation (with p-values based on the likelihood ratio test) from the following system of equations:

$$\Delta S_t = b_0 + b_1 S_t + \varepsilon_t,$$
$$E[\varepsilon_t] = 0,\ E[\varepsilon_t^2 \mid S_t] = \sigma^2 (S_t + C)^\beta.$$

Firm	b_1	b_0	β	σ	C	p-value for C
Allied	-0.00335	4505.00	2.6808	0.0029	0	5.54E-04
	-0.00312	4237.05	42.5241	4.02E-149	18290724	
Anheuser Busch	-0.00320	4130.56	3.1199	0.0001	0	0.0512
	-0.00409	5102.77	8.8227	5.50E-23	2237278	
Braniff	-0.00387	743.66	0.9123	396.1854	0	0.0002
	-0.00302	586.34	16.5681	1.31E-48	3000565	
Brown Group Inc.	-0.00377	718.69	0.5316	2188.933	0	0.2749
	-0.00376	716.03	0.9842	111.6338	111789	
Bucyrus	-0.00982	4321.21	0.7465	1371.675	0	0.3406
	-0.00983	4327.36	0.5767	4448.488	-97136	
Champion Spark Plug	-0.01649	6800.89	1.0467	181.2166	0	0.3899
	-0.01664	6864.34	1.2138	55.7014	72664	
Cities Service Co.	0.00133	-784.13	3.3039	2.58E-04	0	4.32E-09
	0.00130	-776.45	14.1106	7.70E-43	6487426	
CPC International Inc.	-0.00527	6746.50	2.1978	0.0603	0	0.0038
	-0.00545	6936.88	14.2552	2.11E-43	6616299	
Crane Co.	-0.00601	2007.40	1.3676	19.8023	0	0.4854
	-0.00600	2008.94	1.3317	25.1855	-6176	
Food Fair Inc.	-0.01964	783.41	0.8638	199.0463	0	0.4395
	-0.02007	800.40	0.7281	436.7890	-6692	
Fuqua	-0.00217	351.13	1.5661	7.4005	0	0.0327
	-0.00236	372.53	1.1878	77.6641	-21710	
General Cigar	-0.00254	124.49	0.8491	181.3345	0	0.0095
	-0.00254	124.80	0.4759	1505.224	-16311	
Kane Miller Corp.	-0.01182	381.94	1.3106	17.0321	0	0.3605
	-0.01103	356.56	1.4923	5.9122	5277	
MGM Grand Hotels Inc.	-0.00272	1130.31	2.2295	0.1288	0	0
	-0.00305	1085.34	10.9422	2.04E-28	1229210	

Table 1a (continued):

Firm	b_1	b_0	β	σ	C	p-value for C
National Tea Co.	-0.03214	1276.96	9.2740	2.60E-16	0	0
	-0.03062	1207.80	97.0447	7.50E-267	359923	
NVF Co.	-0.00049	221.04	2.3243	0.0958	0	7.38E-07
	0.00091	99.70	3.7322	8.12E-05	87313	
Procter & Gamble Co.	-0.00403	26902.00	1.4068	20.6674	0	0.3813
	-0.00409	27312.23	1.3986	21.0712	453613	
Pullman Inc.	-0.01114	4209.00	1.1337	111.8972	0	0.0106
	-0.01152	4340.87	9.1696	4.90E-24	2365083	
Rapid American Corp.	-0.00110	143.07	1.2865	30.6656	0	3.11E-07
	-0.00051	108.67	9.5601	2.35E-22	-420342	
Raytheon Co.	-0.00037	2113.34	2.3553	0.0291	0	1.96E-11
	-0.00036	1889.83	5.7690	6.84E-13	1945702	
Republic Steel Corp.	-0.00495	2238.30	1.9901	0.3160	0	0.1957
	-0.00497	2246.57	1.6840	2.7364	-79422	
Seagram Ltd.	0.00021	294.95	3.5325	8.33E-05	0	2.69E-06
	0.00021	298.84	1.9054	1.2396	-474591	
Sunbeam Corp.	-0.01429	4343.39	0.6880	1551.715	0	0.2967
	-0.01426	4332.39	1.1137	82.4294	169372	
Tandy Corp.	-0.00199	2193.52	1.9309	0.8354	0	5.67E-13
	-0.00178	1888.43	5.1760	1.93E-10	1181879	
United Brands	-0.00508	598.96	0.7503	775.2869	0	0.0018
	-0.00458	547.75	9.3635	2.83E-23	1011247	
Upjohn Co.	-0.00358	4854.20	1.2784	52.8822	0	0.1015
	-0.00327	4483.78	0.5980	7916.478	-656142	
Whittaker Corp.	-0.00057	352.80	1.4209	19.0215	0	0.1373
	-0.00057	348.96	1.4163	19.0217	4680	

Table 1b: The parameters for each firm are estimated using maximum likelihood estimation (with p-values based on the likelihood ratio test) from the following system of equations:

$$\Delta S_t / S_t = \mu + \varepsilon_t,$$
$$E[\varepsilon_t] = 0,\ E[\varepsilon_t^2 \mid S_t] = \sigma^2 (S_t + C)^{\beta} S_t^{-2}.$$

Firm	μ	β	σ	C	p-value for C
Allied	0.0609	2.6914	0.0023	0	0.0071
	0.1474	3.4568	6.52293E-05	383714.5	
Anheuser Busch	0.1474	3.1199	0.0001	0	0.2756
	0.1376	3.2605	3.54160E-04	67788.5	
Braniff	-0.0250	0.9124	328.0089	0	0.0011
	-0.0220	2.8434	0.0006	359270.9	
Brown Group Inc.	0.0978	0.5772	1378.6042	0	0.3086
	0.0925	0.8758	190.2176	86688.8	
Bucyrus	0.0239	0.7433	1165.2569	0	0.3066
	0.0247	0.3925	12757.4098	-189817.0	
Champion Spark Plug	0.0016	1.0785	122.5933	0	0.2747
	-0.0151	1.8386	0.5348	312679.7	
Cities Service Co.	-0.0151	3.3039	2.13332E-04	0	0.0015
	-0.0117	3.6275	1.65037E-05	219915.0	
CPC International Inc.	0.1210	2.1613	0.0646	0	0.1378
	0.1424	2.4315	0.0081	190376.3	
Crane Co.	0.1581	1.3409	19.4411	0	0.4825
	0.1501	1.3634	16.6419	5728.3	
Food Fair Inc.	-0.0159	0.6419	541.6659	0	0.4949
	-0.0159	0.5959	706.7472	-2868.8	
Fuqua	0.2903	1.5617	6.2900	0	0.0602
	0.3160	1.2309	49.2808	-19151.6	
General Cigar	0.0883	0.8461	152.6290	0	0.0121
	0.0891	0.4826	1200.8033	-15952.3	
Kane Miller Corp.	0.0272	1.2746	17.0316	0	0.1743
	0.0223	2.6913	0.0039	37299.2	

Table 1b (continued):

Firm	μ	β	σ	C	p-value for C
MGM Grand Hotels Inc.	0.3627	2.2045	0.1254	0	8.88178E
	0.3870	4.3914	2.62656E-07	288715.0	
National Tea Co.	0.0358	0.3845	2383.7588	0	0.1416
	0.0381	1.9990	0.0974	148161.5	
NVF Co.	0.5492	2.3195	0.0816	0	4.30277E
	0.4808	3.6398	1.24908E-04	81726.0	
Procter & Gamble Co.	-0.0245	1.4072	17.0543	0	0.1459
	-0.0274	2.4290	0.0029	4866198	
Pullman Inc.	0.1174	1.1050	111.5242	0	0.0371
	0.1346	2.2622	0.0304	371367.6	
Rapid American Corp.	0.2386	1.2850	25.6123	0	3.30261E
	0.2749	3.5985	1.01981E-04	117597.5	
Raytheon Co.	0.4353	2.3430	0.0263	0	3.27427E
	0.4142	4.1828	1.70127E-07	1024160.9	
Republic Steel Corp.	0.0381	1.9823	0.2756	0	0.0248
	0.0455	0.6922	1776.8379	-287278.5	
Seagram Ltd.	0.1463	3.5262	7.19652E-05	0	0.0122
	0.1462	3.3175	3.65723E-04	-103058.5	
Sunbeam Corp.	0.0409	0.6078	2139.4493	0	0.2098
	0.0226	1.6120	1.8253	444497.3	
Tandy Corp.	0.5267	1.9033	0.8324	0	2.79776E
	0.4738	3.7744	7.71865E-06	638972.5	
United Brands	0.1885	0.7317	715.5021	0	0.0043
	0.2204	2.0497	0.1371	154975.3	
Upjohn Co.	0.0489	1.2792	43.5997	0	0.4129
	0.0491	1.2599	51.0344	-41586.4	
Whittaker Corp.	0.4739	1.4111	16.7112	0	9.92092E
	0.4538	2.4007	0.0239	96479.1	

Table 2: The parameters for each firm are estimated using maximum likelihood estimation (with p-values based on the likelihood ratio test) from the following system of equations:

$$\Delta S_t / S_t = \mu + \varepsilon_t,$$
$$E[\varepsilon_t] = 0,\ E[\varepsilon_t^2 \mid S_t] = \sigma^2 S_t^{\beta-2} + \kappa^2 S_t^{-2}.$$

	Final Equity	Dividend rate	β	σ	κ	p-value for κ
Allied	1657706	0.0203	2.6836	0.0024	0	1.58E-06
	1657706	0.0203	4.8426	4.47E-09	213601	
Anheuser Busch	1483837	0.0425	1.4167	16.0724	0	3.11E-14
	1483837	0.0425	3.9739	2.29E-06	140594	
Braniff	82578.4	0.0326	0.9102	401.6678	0	0
	82578.4	0.0326	4.0223	8.96E-06	69557	
Brown Group Inc.	226858.0	0.0965	0.5840	1325.312	0	0.4651
	226858.0	0.0965	0.5898	1269.065	5678	
Bucyrus	408280.0	0.0119	0.7288	1277.488	0	0.5000
	408280.0	0.0119	0.7351	1225.929	3830	
Champion Spark Plug	339850.4	0.0865	1.0371	160.1781	0	0.4376
	339850.4	0.0865	1.0375	157.1328	23686	
Cities Service Co.	3685494	0.0684	3.3093	2.05E-04	0	7.50E-09
	3685494	0.0684	3.7596	6.98E-06	148183	
CPC International Inc.	1429211	0.0712	2.0704	0.1219	0	2.79E-04
	1429211	0.0712	4.2632	1.78E-07	162736	
Crane Co.	416225.3	0.0870	1.3513	18.2211	0	0.4733
	416225.3	0.0870	1.3548	17.7527	8173	
Food Fair Inc.	34030.8	0.0559	0.6339	565.0817	0	0.5000
	34030.8	0.0559	0.6393	547.6059	1180	
Fuqua	170607.0	-0.0243	1.5592	6.3848	0	0.4708
	170607.0	-0.0243	1.5620	6.2827	153	
General Cigar	49460.5	-0.0135	0.8460	152.7035	0	0.5000
	49460.5	-0.0135	0.8509	148.7976	0	
Kane Miller Corp.	25463.8	0.0932	1.2869	15.9858	0	0.4087
	25463.8	0.0932	1.2883	15.5700	2405	
MGM Grand Hotels Inc.	305510.5	0.1661	2.2053	0.1247	0	0.0000
	305510.5	0.1661	3.4619	3.07E-04	61766	

Table 2 (continued):

	Final Equity	Dividend rate	β	σ	κ	p-value for κ
National Tea Co.	50000.0	-0.0675	0.4096	2087.701	0	0.4336
	50000.0	-0.0675	0.4175	1940.269	4505	
NVF Co.	270874.5	-0.0313	2.3207	0.0810	0	6.59E-08
	270874.5	-0.0313	2.8281	0.0033	23185	
Procter & Gamble Co.	5542307	0.0486	1.4162	15.8920	0	0.0161
	5542307	0.0486	3.2421	6.11E-05	802953	
Pullman Inc.	505840.5	0.0522	1.1239	98.8447	0	0.0001
	505840.5	0.0522	3.6116	6.41E-05	108244	
Rapid American Corp.	132284.8	0.0788	1.2850	25.6052	0	2.83E-09
	132284.8	0.0788	3.1806	0.0004	21538	
Raytheon Co.	2879666	-0.0477	2.3409	0.0267	0	1.11E-15
	2879666	-0.0477	3.4833	6.25E-05	171171	
Republic Steel Corp.	410618.3	0.0982	1.9818	0.2764	0	0.5000
	410618.3	0.0982	1.9852	0.2704	6	
Seagram Ltd.	1981851	0.0433	3.5279	7.11E-05	0	0.5000
	1981851	0.0433	3.5897	4.64E-05	2419	
Sunbeam Corp.	271523.5	0.0671	0.6077	2140.078	0	0.0975
	271523.5	0.0671	2.7242	0.0018	83771	
Tandy Corp.	1833488	0.1487	1.9057	0.8194	0	3.33E-15
	1833488	0.1487	2.8544	0.0011	137762	
United Brands	171709.0	0.0089	0.7317	715.4970	0	9.60E-04
	171709.0	0.0089	2.9656	0.0010	38905	
Upjohn Co.	1910396	0.0392	1.2221	65.0227	0	0.5000
	1910396	0.0392	1.2225	64.8545	2236	
Whittaker Corp.	476542.3	0.0897	1.4080	17.0116	0	2.39E-04
	476542.3	0.0897	1.9495	0.5624	32643	

Table 3

Analytic results: Firm	Prob(default)	Credit rating	Numerical Rating
Anheuser Busch	0.02554	A	3
Braniff	0.08389	BBB/CC	3/8
Brown	0.25105	A	3
Bucyrus	0.35899	A	3
Champion Spark Plug	0.43963	AA	2
Crane	0.10902	BBB	4
Food Fair	0.59497	BB/B	5/6
Fuqua	0.22354	B	6
General Cigar	0.20732	BB/B	5/6
Kane	0.52782	B	6
National Tea	0.30553	B	6
Procter and Gamble	0.00011	AAA	1
Pullman	0.27110	BBB	4
Rapid American	0.39529	B/CC	6/8
Republic Steel	0.00000	A	3
Seagram	0.00000	A	3
Sunbeam	0.48205	A	3
United Brands	0.34604	B	6
Upjohn	0.05097	AAA	1

Numerical results: Firm	Prob(default)	Credit rating	Numerical Rating
Allied Chemical	0.521	AA/A	2/3
Anheuser Busch	0.276	A	3
Braniff	0.818	BBB/CC	4/8
Cities Service	0.26	A	3
CPC	0.148	AA/A	2/3
MGM	0.792	BBB/B	4/6
NVF	0.611	B	6
Procter and Gamble	0.063	AAA	1
Pullman	0.516	BBB	4
Rapid American	0.681	B/CC	6/8
Raytheon	0.37	AA/A	2/3
Sunbeam	0.459	A	3
Tandy	0.723	BBB/B	4/6
United Brands	0.512	B	6
Whittaker	0.429	BB/B	5/6

Model I Revisited Firm	Prob(default)	Credit rating	Numerical Rating
Fuqua	0.46674	B	6
General Cigar	0.30234	BB/B	5/6
Republic Steel	0.57294	A	3

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