Some problems from convex geometry and geometric tomography

by

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Abstract

We address several related problems from convex geometry and geometric tomography, which separate along two main themes. The content of this thesis is based on five papers, either published or submitted.

The first theme concerns the origin-symmetry and unique determination of convex bodies. A convex body K is a compact and convex subset in n-dimensional Euclidean space with non-empty interior. We say K is origin-symmetric if it is equal to its reflection through the origin, that is $K = -K$. Makai, Martini, and Ódor have shown that a convex body is necessarily origin-symmetric if every hyperplane section through the origin has maximal $(n-1)$-dimensional volume amongst all parallel sections. We prove a stability version of their result.

Recently, Meyer and Reisner associated with every convex body K a new set, which they call the convex intersection body of K. It follows from previously known results that two origin-symmetric convex bodies coincide whenever their convex intersection bodies coincide. Removing the assumption of origin-symmetry, we show that Meyer and Reisner's convex intersection body does not uniquely determine a convex body up to congruency.

A convex polytope P is a convex body which is the convex hull of finitely many points. We show that P must be origin-symmetric if every hyperplane section through the origin has maximal $(n-2)$-dimensional surface area amongst all parallel sections. This gives partial confirmation to a conjecture made by Makai, Martini, and Ódor.

Our second theme concerns extensions of Grünbaum's inequality, which gives a sharp lower bound for the volume of each half of a convex body that is split by a hyperplane through its centroid. In particular, we generalize this inequality to the orthogonal projections of a convex body onto subspaces, and the intersections of a convex body with subspaces through its centroid.

Preface

Chapter 3 is an original work by M. Stephen and V. Yaskin which was published under the title "Stability results for sections of convex bodies" in Transactions of the American Mathematical Society, pages 6239–6261 in volume 369 of 2017. I was involved with mathematical proof and manuscript composition. V. Yaskin was involved with concept formation, mathematical proof, and manuscript editing.

Chapter 4 is an original work by M. Stephen and has been published under the title "On convex intersection bodies and unique determination problems for convex bodies" in the Journal of Mathematical Analysis and Applications, pages 295–312 in volume 443 of 2016.

Chapter 5 is an original work by M. Stephen and is based on the preprint "Maximal perimeters of polytope sections and origin-symmetry", available at arXiv:1803.01506 [math.MG].

Chapter 6 is an original work by M. Stephen and N. Zhang which was published under the title "Grünbaum's inequality for projections" in the Journal of Functional Analysis, pages 2628–2640 in volume 272 of 2017. I was involved with concept formation, mathematical proof, and manuscript composition. N. Zhang was involved with mathematical proof and manuscript composition.

Chapter 7 is an original work by S. Myroshnychenko, M. Stephen, and N. Zhang which published under the title "Grünbaum's inequality for sections" in the Journal of Functional Analysis in 2018, DOI: 10.1016/j.jfa.2018.04.001. N. Zhang and I were both involved with concept formation, mathematical proof, and manuscript composition. S. Myroshnychenko was involved with mathematical proof, preparation of figures, and manuscript editing.

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Chapter 1

Introduction

This thesis considers several problems from the related fields of convex geometry and geometric tomography. The setting for both of these fields is typically Euclidean space $\mathbb{R}^n$. In convex geometry, the properties of convex sets and convex/concave functions are investigated. Classic references for convex geometry are [2] and [49]. An excellent recent reference is [4]. In geometric tomography, a set $S \subset \mathbb{R}^n$ is studied via "lower-dimensional" data. This data could be the volume of the intersection of S with a subspace, the volume of its projection onto a subspace, etc. The standard reference for geometric tomography is [14].

Commonly studied sets are the star bodies and convex bodies. Call a set S star-shaped if, for every $x \in S$, S contains the line segment connecting x to the origin. A *star body* is a compact and star-shaped set S whose *radial function*, defined by

$$\rho_S(x) := \max\{a > 0 : ax \in S\} \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{o\},$$

is positive and continuous. Evaluated at a unit vector $\xi \in S^{n-1}$, $\rho_S(\xi)$ gives the distance from the origin to the boundary of S in the direction ξ.

A *convex body* $K \subset \mathbb{R}^n$ is a convex and compact set with non-empty interior. For every $\xi \in S^{n-1}$, there is an $x \in K$ so that the translated convex body $K - x$ lies within $\xi^- := \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq 0\}$; in this case, $\{x + \xi^\perp\}$ is called the *supporting hyperplane* of K with outer unit normal ξ. The *support* *function* of K is defined by

$$h_K(x) := \max_{y \in K} \langle x, y \rangle \quad \text{for} \quad x \in \mathbb{R}^n.$$

Evaluated at a unit vector $\xi \in S^{n-1}$, $h_K(\xi)$ gives the signed distance from the origin to the supporting hyperplane of K with outer unit normal ξ. Importantly, $h_K : \mathbb{R}^n \to \mathbb{R}$ is a positively homogeneous and convex function.

Now, assume the convex body K includes the origin as interior point. The *Minkowski functional* of K is then defined by

$$\|x\|_K := \min\{a > 0 : \, x \in aK\} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Note that $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ is a positively homogeneous and convex function, with $\|x\|_K^{-1} = \rho_K(x)$ for $x \in \mathbb{R}^n \backslash \{o\}$. Convex bodies which contain the origin as an interior point are also, in particular, star bodies.

My thesis is based upon the papers [52, 50, 51, 53, 43], which separate along two main themes. The first theme concerns origin-symmetry and the unique determination of star/convex bodies, and includes [52, 50, 51]. The second theme concerns extensions of Grünbaum's inequality for convex bodies, and includes [53, 43]. I properly introduce these topics in the following two subsections.

1.1 Origin-Symmetry and Uniqueness

In convex geometry and geometric tomography, we are frequently interested in subsets of $\mathbb{R}^n$ which have some type of symmetry. Of course, star/convex bodies with symmetry have more structure, and the additional structure is often useful in proofs. Origin-symmetry is particularly important. A set $S \subset \mathbb{R}^n$ is called *origin-symmetric* if it is equal to its reflection through the origin, i.e. $S = -S$. Convex bodies are origin-symmetric exactly when their support functions are even. Star bodies are origin-symmetric exactly when their radial functions are even. More generally, S is called *centrally symmetric* if there is an $x \in \mathbb{R}^n$ so that the translated set $S - x$ is origin-symmetric.

Now, consider a convex body $K \subset \mathbb{R}^n$. The *parallel section function* of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}.$$

Here, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$ is the hyperplane passing through the origin and orthogonal to the vector ξ, and $\{t\xi + \xi^\perp\}$ denotes the translate of $\xi^\perp$ containing $t\xi$. Brunn's theorem asserts that $A_{K,\xi}$ raised to the power $\frac{1}{n-1}$ is concave on its support. Therefore, the origin-symmetry of a convex body K implies

$$\mathrm{vol}_{n-1}\Big(K \cap \xi^\perp\Big) = \max_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap \{t\xi + \xi^\perp\}\Big) \quad \forall\, \xi \in S^{n-1}. \tag{1.1}$$

Using a particular integral transform, Makai, Martini, and Ódor [33] proved the converse statement: a convex body K which contains the origin in its interior and satisfies (1.1) must be origin-symmetric. In fact, they proved a more general statement for star bodies, but it is not as relevant to our current discussion. See [47] for an alternative proof using Fourier analytic methods.

The result of Makai et al. can be restated in terms of intersection bodies and cross-section bodies. The *intersection body* of a star body $L \subset \mathbb{R}^n$ is the star body $IL \subset \mathbb{R}^n$ with radial function

$$\rho_{IL}(\xi) = \mathrm{vol}_{n-1}\left(L \cap \xi^\perp\right), \quad \xi \in S^{n-1}.$$

Intersection bodies were first introduced by Lutwak in [29] in connection with the Busemann-Petty problem [15]. See [25, 39, 40, 60] for more information about intersection bodies and related concepts. The *cross-section body* of a convex body $K \subset \mathbb{R}^n$ is the star body $CK \subset \mathbb{R}^n$ with radial function

$$\rho_{CK}(\xi) = \max_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\left(K \cap \{\xi^\perp + t\xi\}\right), \quad \xi \in S^{n-1}.$$

Cross-section bodies were introduced by Martini [34]. See [10, 11, 30, 31] for more information about cross-section bodies and related concepts. It is immediately clear that $IK \subset CK$ for any convex body K containing the origin in the interior. By Brunn's Theorem, the intersection body and the cross-section body of an origin-symmetric convex body must coincide. The result of Makai et al. is equivalent to the following theorem.

Theorem 1.1 (Makai, Martini, and Ódor)**.** *If $K \subset \mathbb{R}^n$ is a convex body containing the origin in its interior and such that $IK = CK$, then K is origin-symmetric.*

Chapter 3 comes from my paper [52] with V. Yaskin. In [52], we proved a stability version of Theorem 1.1. For star bodies $K, L \subset \mathbb{R}^n$, the *radial metric* is defined as

$$\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

The notation $B_2^n(r)$ is used for the Euclidean ball in $\mathbb{R}^n$ with radius $r > 0$ centred at the origin. We proved the following: if $K \subset \mathbb{R}^n$ is a convex body such that $B_2^n(r) \subset K \subset B_2^n(R)$ and $\rho(CK, IK) \leq \varepsilon$ for small enough $\varepsilon > 0$, then $\rho(K, -K) \leq C \cdot \varepsilon^q$ for some constants $C > 0$, $0 < q < 1$. The upper bound for ε depends on R, $r > 0$, the constant C depends on n, R, and r, and q depend on n. See Theorem 3.1 in Chapter 3, or Theorem 2 in [52], for the precise statement.

Yaskin and I also established a stability version of a unique determination result due to Koldobsky and Shane [26]. Many classic problems in geometric tomography are concerned with the unique determination, possibly up to congruency, of a star/convex body within some collection. Two subsets of $\mathbb{R}^n$ are called *congruent* if one is the image of the other under an isometry. One well-known positive result is the Funk Section Theorem (e.g. Theorem 7.2.6 in [14]): whenever $K, L \subset \mathbb{R}^n$ are origin-symmetric star bodies such that

$$\mathrm{vol}_{n-1}\Big(K \cap \xi^\perp\Big) = \mathrm{vol}_{n-1}\Big(L \cap \xi^\perp\Big) \quad \text{for all} \quad \xi \in S^{n-1}, \tag{1.2}$$

or equivalently $IK = IL$, then necessarily $K = L$. Without origin-symmetry, more information is needed besides (1.2) to guarantee uniqueness, even up to congruency, and even in the case of convex bodies containing the origin as an interior point. For examples, see [3, 9].

Koldobsky and Shane have shown that if $A_{K,\xi}(0)$, $A_{L,\xi}(0)$ in (1.2) are replaced by fractional derivatives of non-integer order of the functions $A_{K,\xi}(t)$, $A_{L,\xi}(0)$ at $t = 0$, then this information does determine a convex body uniquely. Recall that fractional derivatives are an analytic extension of classical differentiation; see Chapter 2 for the formal definition. We call a star/convex body *m-smooth* or C^m if its surface is an m-smooth manifold.

Theorem 1.2 (Koldobsky and Shane)**.** *Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. Let $-1 < p < n-1$ be a non-integer, and m be an integer greater than p. If K and L are m-smooth with*

$$A^{(p)}_{K,\xi}(0) = A^{(p)}_{L,\xi}(0) \quad \forall\, \xi \in S^{n-1},$$

then $K = L$.

For p and m as in Theorem 1.2, our stability result is the following: if $K, L \subset \mathbb{R}^n$ are m-smooth convex bodies which contain $B_2^n(r)$, are contained in $B_2^n(R)$, and are such that

$$\sup_{\xi \in S^{n-1}} \left| A^{(p)}_{K,\xi}(0) - A^{(p)}_{L,\xi}(0) \right| \leq \varepsilon \quad \text{for some} \quad 0 < \varepsilon < 1,$$

then $\rho(K, L) \leq C \cdot \varepsilon^q$ for some constants $C > 0$, $0 < q < 1$. The constant C depends on n, p, R, and r, and q depend on n and p. See Theorem 3.4 in Chapter 3, or Theorem 5 in [52], for the precise statement.

Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in the interior. Considering Theorem 1.1 and the Funk Section Theorem, it is natural to ask whether $CK = CL$ implies that K and L are congruent, without the assumption of origin-symmetry. This question was first posed by Klee in [24]. It was only recently proven in the negative in [16], where an explicit counter-example was constructed. Subsequently, [44] and [45] gave alternative constructions. In fact, a stronger statement was proven in [45]: the cross-section body does not uniquely determine the Euclidean ball up to congruency amongst all convex bodies.

Chapter 4 is based on my paper [50]. Following the spirit of Klee's problem, I investigate in [50] whether Meyer and Reisner's convex intersection body [37] uniquely determines a convex body. Before I present their definition, let me provide their motivation. In general, intersection and cross-section bodies are not convex bodies. Busemann's Theorem implies that for a convex body K containing the origin, IK is convex when K is origin-symmetric. If K is not origin-symmetric, then IK is not necessarily convex, nor is it necessary that CK is so when $n > 3$; see [7, 41, 5]. Meyer and Reisner were interested in associating with every convex body a new function which would be the radial function of a necessarily convex body.

Let $K \subset \mathbb{R}^n$ be a convex body. Let $g = g(K) \in \operatorname{int}(K)$ be the centroid of K, and let K^{*y} denote the *polar body* of K with respect to the point $y \in \operatorname{int}(K)$; that is,

$$K^{*y} = \{x \in \mathbb{R}^n : \langle x - y, z - y\rangle \leq 1 \ \forall \ z \in K\}.$$

The *convex intersection body* of K is the (a priori) star body $CI(K) \subset \mathbb{R}^n$ with

$$\rho_{CI(K)}(\xi) = \min\left\{\operatorname{vol}_{n-1}\left[\left(K^{*g}\middle|\xi^{\perp}\right)^{*y}\right] : y \in \operatorname{relint}\left(K^{*g}\middle|\xi^{\perp}\right)\right\}, \quad \xi \in S^{n-1}.$$

Here, $\cdot\,|\xi^{\perp}$ is the orthogonal projection onto the hyperplane perpendicular to ξ, y is taken from the relative interior of $K^{*g}\big|\xi^{\perp}$, and the polar body of $K^{*g}\big|\xi^{\perp}$ with respect to y is taken within $\xi^{\perp}$.

The main result in [37] is that $CI(K)$ is always a convex body. However, they also demonstrate that the relationship between $CI(K)$ and IK parallels the relationship between IK and CK, when the centroid of K is at the origin. Indeed, if $g(K) = o$, then $CI(K) \subset IK$ and $CI(K) = IK$ if and only if K is origin-symmetric. Convex intersection bodies were also studied in [12], where it was shown that $CI(K)$ is "close" to the Euclidean ball when K is in isotropic position.

Is a convex body uniquely determined, up to congruency, by its convex intersection body? If the convex body is centrally-symmetric, then the previously discussed properties of convex intersection bodies imply an affirmative answer. A star body L is a *body of rotation* if its radial function is *rotationally* *symmetric* about some axis; i.e. there is an $\omega \in S^{n-1}$ for which

$$\rho_L(\xi) = \rho_L(\eta) \quad \text{whenever} \quad \xi, \eta \in S^{n-1} \quad \text{and} \quad \langle \xi, \omega \rangle = \langle \eta, \omega \rangle.$$

A convex body is a body of rotation if one of its translates is a star body of rotation. I proved the following:

Theorem 1.3 (appears as Theorem 2 in [50])**.** *Let $n \in \mathbb{N}$, $n \geq 2$. There are infinitely smooth convex bodies of rotation K, $L \subset \mathbb{R}^n$ such that K is not centrally-symmetric, L is origin-symmetric, and $CI(K) = CI(L)$.*

The convex bodies in the above theorem are necessarily non-congruent, so Meyer and Reisner's convex intersection bodies do not determine a convex body up to congruency. I adapt the construction from [16] to prove Theorem 1.3; see Chapter 4.

Brunn's Theorem and Theorem 1.1 together give nice conditions which are equivalent to origin symmetry. Several other characterizations of origin-symmetry are known. For example, Falconer [8] showed that a convex body $K \subset \mathbb{R}^n$ is origin-symmetric if and only if every hyperplane through the origin splits K into two halves of equal n-dimensional volume. Ryabogin and Yaskin [47] used the volume of conical sections to determine when star bodies are origin-symmetric.

Makai et al. [33] conjectured a further characterization of origin-symmetry for convex bodies in terms of the quermassintegrals of sections. The *quermassintegrals* $W_l(K)$ of a convex body $K \subset \mathbb{R}^n$ arise as coefficients in the expansion

$$\operatorname{vol}_n\Big(K + tB_2^n\Big) = \sum_{l=0}^{n} \binom{n}{l} W_l(K)\, t^l, \quad t \geq 0.$$

The addition of sets here is the well-known Minkowski addition

$$K + tB_2^n := \big\{x + ty : \ x \in K,\ y \in B_2^n\big\},$$

and B_2^n is the Euclidean ball with unit radius centred at the origin. Refer to [49] for a thorough overview of mixed volumes and quermassintegrals. For any

$0 \leq l \leq n-2$ and $\xi \in S^{n-1}$, consider the quermassintegral $W_l\big((K-t\xi)\cap\xi^\perp\big)$ of the $(n-1)$-dimensional convex body $(K-t\xi)\cap\xi^\perp$ in $\xi^\perp$. If K is origin-symmetric, then the monotonicity and positive multilinearity of mixed volumes together with the Alexandroff-Fenchel inequality imply

$$W_l\Big(K\cap\xi^\perp\Big) = \max_{t\in\mathbb{R}} W_l\Big((K-t\xi)\cap\xi^\perp\Big) \quad \text{for all} \quad \xi\in S^{n-1}. \tag{1.3}$$

For $l=0$, (1.3) is equivalent to (1.1), as $W_0\big((K-t\xi)\cap\xi^\perp\big)$ is the $(n-1)$-dimensional volume of $(K-t\xi)\cap\xi^\perp$. Makai et al. conjectured that if K contains the origin in its interior and satisfies (1.3) for any $1\leq l\leq n-2$, it must be origin-symmetric. Makai and Martini [32] proved a local variant of the conjecture for smooth perturbations of the Euclidean ball.

The content of Chapter 5 comes from my paper [51], where I consider the case of convex polytopes which satisfy (1.3) for $l=1$. A *convex polytope* $P\subset\mathbb{R}^n$ is a convex body which is the convex hull of finitely many points. It is common practice to restrict unsolved problems for general convex bodies to the class of polytopes (e.g. [42, 56, 57, 59, 62]), because polytopes have additional structure. Up to a constant depending on the dimension, $W_1\big((P-t\xi)\cap\xi^\perp\big)$ is the $(n-2)$-dimensional surface area of the $(n-1)$-dimensional polytope $(P-t\xi)\cap\xi^\perp$ in $\xi^\perp$. Letting $\mathrm{vol}_{n-2}\big(\mathrm{relbd}(P\cap\{t\xi+\xi^\perp\})\big)$ denote the $(n-2)$-dimensional volume of the relative boundary of $P\cap\{t\xi+\xi^\perp\}$, I proved the following:

Theorem 1.4 (appears as Theorem 1 in [51])**.** *Let $P\subset\mathbb{R}^n$ $(n\geq 3)$ be a convex polytope containing the origin in its interior, and such that*

$$vol_{n-2}\Big(relbd\big(P\cap\xi^\perp\big)\Big) = \max_{t\in\mathbb{R}} vol_{n-2}\Big(relbd\big(P\cap\{t\xi+\xi^\perp\}\big)\Big) \tag{1.4}$$

for all $\xi\in S^{n-1}$. Then $P=-P$.

Most of Chapter 5 is devoted to proving Theorem 1.4 using techniques similar to those developed in [56, 57, 59]. At this time, no other progress has been made towards solving the conjecture of Makai et al. [33], other than the local result of Makai and Martini [32] and my Theorem 1.4. However, in the last section of Chapter 5, I describe how to characterize the origin-symmetry of C^1 convex bodies using the dual quermassintegrals of sections; this is a dual version of the conjecture for quermassintegrals.

1.2 Extensions of Grünbaum's Inequality

An elegant inequality of Grünbaum [20] gives a lower bound for the volume of that portion of a convex body lying in a halfspace which slices the convex body through its centroid. The *centroid* of a convex body $K \subset \mathbb{R}^n$ is the affine covariant point

$$g(K) := \frac{1}{\mathrm{vol}_n(K)} \int_K x\, dx \in \mathrm{int}(K).$$

It is perhaps surprising that there are still many natural and unanswered questions about the centroid; see [22] for one recent and interesting result. For convenience, we assume in this section that the centroid of K is at the origin. Given a unit vector $\theta \in S^{n-1}$, we define $\theta^+ := \{x \in \mathbb{R}^n : \langle x, \theta \rangle \geq 0\}$. Specifically, Grünbaum's inequality states that

$$\mathrm{vol}_n(K \cap \theta^+) \geq \left(\frac{n}{n+1}\right)^n \mathrm{vol}_n(K) \quad \forall\, \theta \in S^{n-1}. \tag{1.5}$$

There is equality for a given $\theta \in S^{n-1}$ when, for example, K is the cone

$$\mathrm{conv}\left(\frac{-1}{n+1}\theta + B_2^{n-1},\ \frac{n}{n+1}\theta\right)$$

and B_2^{n-1} is the unit Euclidean ball in $\theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = 0\}$ centred at the origin. This volume inequality was independently proven in [41].

From Grünbaum's inequality, we can derive an integral inequality for log-concave functions. A function $f : \mathbb{R}^n \to [0, \infty)$ is called *log-concave* if its support $\mathrm{supp}(f)$ is a convex set, and $\log f$ is concave on $\mathrm{supp}(f)$. Consider any integrable and log-concave function f whose support has non-empty interior, and is such that $\int_{\mathbb{R}^n} x f(x)\, dx = o$. For each $\varepsilon > 0$, the set

$$K_{m,\varepsilon} = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \, f(x) \geq \varepsilon, \ |y|_2 \leq 1 + \frac{\log f(x)}{m} \right\}$$

is a convex body in $\mathbb{R}^{n+m}$ for large enough integers m. Notice that for any $x \in \mathbb{R}^n$ with $f(x) \geq \varepsilon$ and large enough m, we have

$$\mathrm{vol}_m \Big(K_{m,\varepsilon} \cap \{ (x,y) : \, y \in \mathbb{R}^m \} \Big) = \left(1 + \frac{\log f(x)}{m} \right)^m \kappa_m,$$

where κ_m is the m-dimensional volume of the unit Euclidean ball in $\mathbb{R}^m$. Applying Grünbaum's inequality to the shifted convex bodies $K_{m,\varepsilon} - g\big(K_{m,\varepsilon}\big)$ with centroids at the origin, and taking the limit as m goes to infinity and ε goes to zero, we obtain

$$\int_{\theta^+} f(x)\, dx \geq \frac{1}{e} \int_{\mathbb{R}^n} f(x)\, dx \quad \forall\, \theta \in S^{n-1}.$$

We refer to this as "Grünbaum's inequality for log-concave functions". Refer to Lemma 2.2.6 in [4] for an alternative proof.

Similar in spirit to Grünbaum's inequality is an inequality, attributed to Minkowski for $n = 2, 3$ and Radon for general n, which bounds the distance from $g(K)$ to a supporting hyperplane of K. See pages 57–58 of [2], Section 6.1 of [21], and the references therein. For $g(K) = o$, Minkowski and Radon's inequality states that

$$h_K(\theta) \geq \left(\frac{1}{n+1} \right) \big(h_K(-\theta) + h_K(\theta) \big) \quad \forall\, \theta \in S^{n-1}, \tag{1.6}$$

or alternatively

$$\rho_K(\theta) \geq \left(\frac{1}{n+1} \right) \big(\rho_K(-\theta) + \rho_K(\theta) \big) \quad \forall\, \theta \in S^{n-1}. \tag{1.7}$$

The correspondence between (1.6) and (1.7) follows from their equivalence to the containment $-K \subset nK$. There is equality in (1.6) and (1.7) for a given

$\theta \in S^{n-1}$ when, for example, K is the cone

$$\operatorname{conv}\left(\frac{-n}{n+1}\theta,\ \frac{1}{n+1}\theta + B_2^{n-1}\right).$$

The sum $h_K(-\theta) + h_K(\theta)$ gives the width of K in the direction $\theta \in S^{n-1}$, and $\rho_K(-\theta) + \rho_K(\theta)$ gives the length of the chord of K which passes through $g(K) = o$ and is parallel to θ. With this in mind, rewriting (1.6) as

$$\operatorname{vol}_1\big((K|E) \cap \theta^+\big) \geq \left(\frac{1}{n+1}\right)^1 \operatorname{vol}_1\big(K|E\big) \quad \forall\ E \in G(n,1),\ \theta \in S^{n-1} \cap E$$

and rewriting (1.7) as

$$\operatorname{vol}_1\big((K \cap E) \cap \theta^+\big) \geq \left(\frac{1}{n+1}\right)^1 \operatorname{vol}_1\big(K \cap E\big) \quad \forall\ E \in G(n,1),\ \theta \in S^{n-1} \cap E$$

emphasizes the connection with Grünbaum's inequality. We always let $G(n,k)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$, with $\cdot|E$ giving the orthogonal projection onto $E \in G(n,k)$.

In the past few years, there has been significant progress in extending the aforementioned inequalities. The first of these recent extensions was made by Fradelizi, Meyer, and Yaskin [12]. They considered the following problem: what is the largest constant $c_1 := c_1(n,k) > 0$, depending only on integers $1 \leq k \leq n$, so that

$$\operatorname{vol}_k\big(K \cap E \cap \theta^+\big) \geq c_1 \cdot \operatorname{vol}_k\big(K \cap E\big) \quad \forall\ E \in G(n,k),\ \theta \in S^{n-1} \cap E \qquad (1.8)$$

for every convex body $K \subset \mathbb{R}^n$ with $g(K) = o$? Let us emphasize that the value of c_1 cannot be obtained from Grünbaum's inequality because the centroid of $K \cap E$ is in general different from the centroid of K. Fradelizi et al. showed there is an absolute constant $c > 0$ so that

$$c_1 \geq c_2 := \frac{c}{(n-k+1)^2}\left(\frac{k}{n+1}\right)^{k-2},$$

but they did not prove $c_1 = c_2$. We refer to (1.8) as "Grünbaum's inequality for sections".

Following the result of Fradelizi et al., N. Zhang and I found an extension of Grünbaum's inequality to the (orthogonal) projections of a convex body [53]. This paper is the content of Chapter 6 in my thesis. Consider a concave function $\psi : K \to [0, \infty)$ which is not identically zero, and is supported on a convex body $K \subset \mathbb{R}^n$. Let $p > 0$. The *centroid* of the function $\psi^p : K \to [0, \infty)$ is the point

$$g(\psi^p, K) := \frac{\int_K x\, \psi^p\, dx}{\int_K \psi^p\, dx} \in \operatorname{int}(K).$$

We proved in [53] that

$$\operatorname{vol}_n\big(K \cap \theta^+\big) \geq \left(\frac{n}{n+1+p}\right)^n \operatorname{vol}_n(K) \quad \forall\, \theta \in S^{n-1} \tag{1.9}$$

when $g(\psi^p, K)$ is at the origin; see Theorem 6.6 in Chapter 6, or Theorem 8 in [53].

As a particular case of inequality (1.9), we get the following. For any integers $1 \leq k \leq n$ and convex body $K \subset \mathbb{R}^n$ with $g(K) = o$, then

$$\operatorname{vol}_k\Big((K|E) \cap \theta^+\Big) \geq \left(\frac{k}{n+1}\right)^k \operatorname{vol}_k\big(K|E\big) \quad \forall\, E \in G(n,k),\ \theta \in S^{n-1} \cap E. \tag{1.10}$$

There is equality when, for example,

$$K = \operatorname{conv}\left(-\left(1 - \frac{k}{n+1}\right)\theta + B_2^{k-1},\ \frac{k}{n+1}\theta + B_2^{n-k}\right), \tag{1.11}$$

$\theta \in E \cap S^{n-1}$, B_2^{k-1} is the unit ball in $E \cap \theta^\perp$, and B_2^{n-k} is the unit ball in $E^\perp$. Observe that (1.10), which we refer to as "Grünbaum's inequality for projections", provides a link between inequalities (1.5) and (1.6). Again, Grünbaum's inequality does not imply the result for projections because the centroid of $K|E$ is in general different from the centroid of K. See Corollary 6.7 in Chapter 6, or Corollary 9 in [53], for the proof of (1.10) and the complete characterization of the equality conditions. See Figure 6.1 for an illustration of the minimizing shape for K.

We conjectured in [53] that the best constant in Grünbaum's inequality for sections would be $\left(\frac{k}{n+1}\right)^k$, the same as in Grünbaum's inequality for projections. With this constant, Grünbaum's inequality for sections would link inequalities (1.5) and (1.7). Neither we, nor any of our colleagues to my knowledge, have yet been able to prove inequality (1.8) as a consequence of inequality (1.10) for $1 < k < n$.

Subsequent to our work, it was shown by Meyer, Nazarov, Ryabogin, and Yaskin in [36] that

$$\int_0^\infty f(s\theta)\,ds \geq \frac{1}{e^n}\int_{-\infty}^\infty f(s\theta)\,ds \quad \forall\, \theta \in S^{n-1} \tag{1.12}$$

for every log-concave $f : \mathbb{R}^n \to [0,\infty)$ with a finite and positive integral, and $\int_{\mathbb{R}^n} xf(x)\,dx = o$. This result can be seen as "Grünbaum's inequality for one dimensional sections of log-concave functions".

With S. Myroshnychenko and N. Zhang [43], I adapted the methods of [36] to prove a generalization of inequality (1.12) for γ-concave functions. Our resulting paper provides the content of Chapter 7 in my thesis. We say a function $f : \mathbb{R}^n \to [0,\infty)$ is *γ-concave* for $\gamma > 0$ if f^γ is concave on convex support. We prove

$$\int_0^\infty f(s\theta)\,ds \geq \left(\frac{\gamma+1}{\gamma n+\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\int_{-\infty}^\infty f(s\theta)\,ds \quad \forall\, \theta \in S^{n-1} \tag{1.13}$$

for every γ-concave function $f : \mathbb{R}^n \to [0,\infty)$, $\gamma > 0$, with $0 < \int_{\mathbb{R}^n} f(x)\,dx < \infty$ and $\int_{\mathbb{R}^n} xf(x)\,dx = o$. This result can be seen as "Grünbaum's inequality for one dimensional sections of γ-concave functions". See Theorem 7.1 in Chapter 7, or Theorem 1 in [43], for the precise statement and the characterization of the equality conditions.

We prove two more important inequalities in our paper [43], using inequality (1.13). First, we establish "Grünbaum's inequality for k-dimensional sections of γ-concave functions":

$$\int_{E\cap\theta^+} f(x)dx \geq \left(\frac{k\gamma+1}{(n+1)\gamma+1}\right)^{\frac{k\gamma+1}{\gamma}} \int_E f(x)dx \ \ \forall\ E\in G(n,k),\ \theta\in S^{n-1}\cap E$$

for every γ-concave $f:\mathbb{R}^n\to[0,\infty)$ with $\gamma>0$, $0<\int_{\mathbb{R}^n} f(x)\,dx<\infty$, and $\int_{\mathbb{R}^n} xf(x)\,dx = o$. See Corollary 7.7 in Chapter 7, or Corollary 7 in [43]. Second, we verify that $c_1=\left(\frac{k}{n+1}\right)^k$ is the best constant in Grünbaum's inequality for sections with equality when, for example, K has the form in (1.11). The complete characterization of the equality conditions is given in Corollary 7.8 in Chapter 7, and in Corollary 8 of [43].

Let me conclude this section with two observations. First, Grünbaum's inequality for projections can be proven using Grünbuam's inequality for sections. See the final remark in Chapter 7 for more explanation. My paper [53] with Zhang and the original proof of Grünbaum's inequality for projections are still included in this thesis as the methods used are different and interesting in their own right. Second, note that inequality (1.9) can be stated in terms of γ-concave functions. Indeed if $\psi:K\to[0,\infty)$ is concave and $p>0$, then ψ^p is p^{-1}-concave. We did not use the terminology of γ-concave functions in [53], nor is it used in Chapter 6.

Chapter 2

Common Preliminaries

In the following sections, I present those notations and background materials which are shared by several subsequent chapters.

2.1 Notation

The origin in $\mathbb{R}^n$ is given by o or 0. We use $|\cdot|$ and $|\cdot|_2$ for the Euclidean norm, and $\langle\cdot,\cdot\rangle$ for the dot product. The Euclidean ball in $\mathbb{R}^n$ with radius $r > 0$ and centred at $x \in \mathbb{R}^n$ will be denoted by $B_2^n(x,r)$. As shorthand, we write $B_2^n(r) := B_2^n(o,r)$ or $B^n(r) := B_2^n(o,r)$, and $B_2^n := B_2^n(o,1)$ or $B^n := B_2^n(o,1)$. The affine hull and linear span of a set $A \subset \mathbb{R}^n$ are respectively denoted by $\mathrm{aff}(A)$ and $\mathrm{span}(A)$. We let $\mathbb{R}x := \mathrm{span}(x)$ be the line through $x \in \mathbb{R}^n\backslash\{0\}$ and the origin. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. For $\xi \in S^{n-1}$, we define $\xi^+ := \{x \in \mathbb{R}^n : \langle x,\xi\rangle \geq 0\}$ and $\xi^- := \{x \in \mathbb{R}^n : \langle x,\xi\rangle \leq 0\}$.

Whenever we integrate over a k-dimensional subspace of $\mathbb{R}^n$, we are integrating with respect to the appropriately scaled k-dimensional Hausdorff measure on $\mathbb{R}^n$. Interpret $\mathrm{vol}_k(A)$ as the k-dimensional Hausdorff volume of the subset $A \subset \mathbb{R}^n$. The constants

$$\kappa_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \qquad \text{and} \qquad \omega_n := n\cdot\kappa_n$$

give the volume and surface area of the unit Euclidean ball $B_2^n \subset \mathbb{R}^n$, where

Γ denotes the Gamma function. Whenever we integrate over Borel subsets of the sphere S^{n-1}, we are using non-normalized spherical measure; that is, the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, scaled so that the measure of S^{n-1} is ω_n.

Recall the definition of the polar body K^{*y} of K with respect to the point $y \in \text{int}(K)$; that is,

$$K^{*y} = \{x \in \mathbb{R}^n : \langle x - y, z - y \rangle \leq 1 \ \forall \ z \in K\}.$$

If $o \in \text{int}(K)$, then we may variously write $K^{*o} = K^* = K^\circ$; in this case, h_K will be the Minkowski functional of the polar body K^*.

Let $\mathbb{Z}^n_{\geq 0}$ denote the collection of n - tuples of non-negative integers. For $\alpha \in \mathbb{Z}^n_{\geq 0}$, define

$$|\alpha| = [\alpha] := \sum_{j=1}^{n} \alpha_j$$

and the differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{[\alpha]}}{\partial x^\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

When differentiating a function $f = f(x, y)$, $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, the addition of a subscript $D_y^\alpha f$ indicates we are differentiating with respect to $y \in \mathbb{R}^n$. The Laplacian operator iterated k - times is represented by Δ^k. When needed, the addition of a subscript Δ_z will indicate with respect to what variables the Laplacian is taken.

2.2 Even, Odd, and Homogeneous Functions

A function $f : \mathbb{R}^n \to \mathbb{C}$ (respectively, $f : S^{n-1} \to \mathbb{C}$) is called *even* if $f(x) = f(-x)$ for all $x \in \mathbb{R}^n$ (respectively, $x \in S^{n-1}$). Similarly, f is *odd* if $f(-x) = -f(x)$ for all x in the domain.

We say $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ is (positively) *homogeneous* of degree $p \in \mathbb{C}$ if

$$f(x) = |x|^p f\left(\frac{x}{|x|}\right) \qquad \forall\, x \in \mathbb{R}^n \backslash \{0\}.$$

For any $f \in C(S^{n-1})$ and $p \in \mathbb{C}$, the *homogeneous extension* of f of degree $-n+p$ is given by

$$f_p(x) := |x|^{-n+p} f\left(\frac{x}{|x|}\right) \qquad \forall\, x \in \mathbb{R}^n \backslash \{0\}.$$

As usual, $C^k(S^{n-1})$ is the space of complex-valued functions on the sphere which are k - times continuously differentiable. It is easy to see that $f \in C^k(S^{n-1})$ if and only if $\tilde{f} \in C^k(\mathbb{R}^n \backslash \{0\})$, and

$$\lim_{m\to\infty} \|f_m - f\|_{C^k(S^{n-1})} = 0$$

for a sequence $\{f_m\} \subset C^k(S^{n-1})$ if

$$\lim_{m\to\infty} \sup_{\xi \in S^{n-1}} \left| D^\alpha \tilde{f}_m(\xi) - D^\alpha \tilde{f}(\xi) \right| = 0 \quad \forall\, \alpha \in \mathbb{Z}^n_{\geq 0} \text{ with } |\alpha| \leq k.$$

Here, $\tilde{f}_m$, $\tilde{f} : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ are the homogeneous extensions of f_m, f of degree zero, and the previous statements remain true if $\tilde{f}_m$, $\tilde{f}$ are replaced with the homogeneous extensions of degree $p \in \mathbb{R}$.

The *spherical gradient* of $f \in C(S^{n-1})$ is the restriction of $\nabla f\left(\frac{x}{|x|}\right)$ to S^{n-1}. It is denoted by $\nabla_o f$. The *spherical Laplacian* of $f \in C^2(S^{n-1})$ is the restriction of $\Delta f\left(\frac{x}{|x|}\right)$ to S^{n-1}. It is denoted by $\Delta_o f$. For a homogeneous function $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ of degree $m \in \mathbb{C}$, there is the well-known relation

$$(\Delta f)(\xi) = (\Delta_o f)(\xi) + m(m+n-2) f(\xi), \qquad \xi \in S^{n-1}; \tag{2.1}$$

this appears as equation (1.2.9) in [19].

2.3 Fourier Transforms of Homogeneous Functions

We follow the conventions of [25] regarding Fourier transforms and distributions. Refer to [17, 46] for more information.

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ denote the space of *Schwartz test functions*; that is, functions in $C^\infty(\mathbb{R}^n)$ for which all derivatives decay faster than any rational function. The *Fourier transform* of $\phi \in \mathscr{S}$ is a test function $\mathscr{F}\phi$ defined by

$$\mathscr{F}\phi(x) = \widehat{\phi}(x) = \int_{\mathbb{R}^n} \phi(y) e^{-i\langle x,y\rangle}\, dy, \qquad x \in \mathbb{R}^n.$$

The continuous dual of $\mathscr{S}$ is denoted as $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$, and elements of $\mathscr{S}'$ are referred to as *distributions*. The action of $f \in \mathscr{S}'$ on a test function ϕ is denoted as $\langle f, \phi\rangle$. The Fourier transform of f is a distribution $\widehat{f}$ defined by

$$\langle \widehat{f}, \phi\rangle = \langle f, \widehat{\phi}\rangle, \qquad \phi \in \mathscr{S};$$

$\widehat{f}$ is well-defined as a distribution because $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is a continuous and linear bijection.

For $f \in C(S^{n-1})$ and $\mathrm{Re}(p) > 0$,

$$f_p(x) := |x|^{-n+p} f\left(\frac{x}{|x|}\right)$$

is locally integrable on $\mathbb{R}^n$ with at most polynomial growth at infinity. In this case, f_p is a distribution on $\mathscr{S}$ acting by integration, and we may consider its Fourier transform. Goodey, Yaskin, and Yaskina show in [18] that, for $f \in C^\infty(S^{n-1})$, the additional restriction $\mathrm{Re}(p) < n$ ensures the action of $\widehat{f_p}$ is also by integration, with $\widehat{f_p} \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

The *spherical Radon transform* $\mathcal{R} : C(S^{n-1}) \to C(S^{n-1})$ is defined by

$$\mathcal{R}f(u) = \int_{S^{n-1}\cap u^\perp} f(\xi)\, d\xi, \qquad u \in S^{n-1}, \qquad f \in C(S^{n-1}).$$

The following connection between the spherical Radon and Fourier transforms is well-known; for example, it appears as Lemma 3.7 in [25].

Lemma 2.1. *Let $f \in C(\mathbb{R}^n \backslash \{0\})$ be an even homogeneous function of degree $-n+1$. Then $\widehat{f} \in C(\mathbb{R}^n \backslash \{0\})$ is an even homogeneous function of degree -1 such that*

$$\mathcal{R}f(\xi) = \int_{S^{n-1} \cap \xi^{\perp}} f(\eta)\, d\eta = \frac{1}{\pi}\widehat{f}(\xi) \qquad \forall\ \xi \in S^{n-1}.$$

The restriction $\mathcal{R} : C_e^{\infty}(S^{n-1}) \to C_e^{\infty}(S^{n-1})$ to the even and infinitely smooth functions on the sphere gives a bijection, and we may consider the inverse transform $\mathcal{R}^{-1}$ on this domain; see Theorem C.2.5 in [14].

2.4 Spherical Harmonics

A detailed discussion on spherical harmonics is given in [19]. A *spherical harmonic* Q of dimension n is a real-valued harmonic and homogeneous polynomial in n variables whose domain is restricted to S^{n-1}. We say Q is of degree m if the corresponding polynomial has degree m. Any two spherical harmonics of the same dimension and different degrees are orthogonal. The collection $\mathcal{H}_m^n$ of all spherical harmonics with dimension n and degree m is a finite dimensional Hilbert space with respect to the inner product for $L^2(S^{n-1})$. If $\mathcal{B}_m$ is an orthonormal basis for $\mathcal{H}_m^n$ for each non-negative integer m, then the union of all $\mathcal{B}_m$ is an orthonormal basis for $L^2(S^{n-1})$. Given $f \in L^2(S^{n-1})$, and defining

$$\sum_{Q \in \mathcal{B}_m} \langle f, Q \rangle\, Q =: Q_m \in \mathcal{H}_m^n,$$

we call $\sum_{m=0}^{\infty} Q_m$ the *condensed harmonic expansion* for f. The condensed harmonic expansion does not depend on the particular orthonormal bases chosen for each $\mathcal{H}_m^n$.

We make extensive use in Chapter 3 of the mapping $I_p : C^{\infty}(S^{n-1}) \to C^{\infty}(S^{n-1})$ defined in [18], which sends a function f to the restriction of $\widehat{f_p}$ to S^{n-1}. For $0 < \mathrm{Re}(p) < n$ and any non-negative integer m, Goodey, Yaskin and Yaskina show I_p has an eigenvalue $\lambda_m(n,p)$ whose eigenspace includes all spherical harmonics of degree m and dimension n. These eigenvalues are given explicitly in the following lemma; refer to [18] for the proof.

Lemma 2.2. *If $0 < Re(p) < n$, then the eigenvalues $\lambda_m(n,p)$ are given by*

$$\lambda_m(n,p) = \frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{m}{2}} \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m+n-p}{2}\right)} \quad \textit{if } m \textit{ is even,}$$

and

$$\lambda_m(n,p) = i\, \frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{m-1}{2}} \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m+n-p}{2}\right)} \quad \textit{if } m \textit{ is odd.}$$

2.5 Fractional Derivatives

Let $m \in \mathbb{N} \cup \{0\}$, and let $h : \mathbb{R} \to \mathbb{C}$ be an integrable function which is m-smooth in a neighbourhood of the origin. For $p \in \mathbb{C} \backslash \mathbb{Z}$ such that $-1 < \mathrm{Re}(p) < m$, we define the *fractional derivative* for h of order p at zero as

$$\begin{aligned} h^{(p)}(0) =& \frac{1}{\Gamma(-p)} \int_0^1 t^{-1-p} \left(h(-t) - \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k!} t^k \right) dt \\ &+ \frac{1}{\Gamma(-p)} \int_1^\infty t^{-1-p} h(-t)\, dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k!(k-p)}. \end{aligned}$$

Given the simple poles of the Gamma function, the fractional derivatives of h at zero may be analytically extended to the integer values $0, \ldots, m-1$, and they will agree with the classical derivatives.

Let K be an infinitely smooth convex body. By Lemma 2.4 in [25], the parallel section function

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}\Big(K \cap \{t\xi + \xi^\perp\}\Big)$$

is infinitely smooth in a neighbourhood of $t = 0$ which is uniform with respect to $\xi \in S^{n-1}$. With the exception of a sign difference, the equality

$$\begin{aligned} A^{(p)}_{K,\xi}(0) =& \frac{\cos\left(\frac{p\pi}{2}\right)}{2\pi(n-1-p)} \left(\|x\|_K^{-n+1+p} + \|-x\|_K^{-n+1+p}\right)^{\wedge}(\xi) \\ &+ i\,\frac{\sin\left(\frac{p\pi}{2}\right)}{2\pi(n-1-p)} \left(\|x\|_K^{-n+1+p} - \|-x\|_K^{-n+1+p}\right)^{\wedge}(\xi), \end{aligned} \tag{2.2}$$

was proven by Ryabogin and Yaskin in [47] for all $\xi \in S^{n-1}$ and $p \in \mathbb{C}$ such that $-1 < \mathrm{Re}(p) < n-1$. The sign difference results from their use of $h(x)$ rather than $h(-x)$ in the definition of fractional derivatives.

Chapter 3

Stability Results for Sections of Convex Bodies

The content of this chapter comes from my paper with V. Yaskin [52]. All convex bodies in this chapter are assumed to contain the origin in their interiors. Our main result is the following:

Theorem 3.1. *Let K be a convex body in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. If there exists $0 < \varepsilon < \min\left\{\left(\frac{\sqrt{3}\,r}{6\sqrt{3}\,\pi r + 32\pi}\right)^2, \frac{r^2}{16}\right\}$ so that

$$\rho(CK, IK) \le \varepsilon,$$

then

$$\rho(K, -K) \le C(n,r,R)\,\varepsilon^q \quad \textit{where} \quad q = \begin{cases} \frac{1}{2} & \textit{if } n = 2, \\ \frac{1}{2(n+1)} & \textit{if } n = 3,4, \\ \frac{1}{(n-2)(n+1)} & \textit{if } n \ge 5. \end{cases}$$

Here, $C(n,r,R) > 0$ are constants depending on the dimension, r, and R.

Remark 3.2. In the proof of Theorem 3.1, we give the explicit dependency of $C(n, r, R)$ on r and R.

Our Theorem 3.1 is a stability version of the result of Makai et al. [33], which we stated in Theorem 1.1 in terms of intersection and cross-section bodies. The following corollary is a straightforward consequence of the Lipschitz property of the parallel section function (Lemma 3.8) and Theorem 3.1. Roughly speaking, if for every direction $\xi \in S^{n-1}$, the convex body K has a maximal section perpendicular to ξ that is close to the origin, then K is close to being origin-symmetric.

Corollary 3.3. *Let K be a convex body in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. Let $L = L(n)$ be the constant given in Lemma 3.8. If there exists

$$0 < \varepsilon < \min \left\{ \frac{r}{2}, \frac{3r^3}{LR^{n-1}\left(6\sqrt{3}\pi r + 32\pi\right)^2}, \frac{r^3}{16LR^{n-1}} \right\}$$

so that, for each direction $\xi \in S^{n-1}$, $A_{K,\xi}$ attains its maximum at some $t = t(\xi)$ with $|t(\xi)| \leq \varepsilon$, then

$$\rho(K, -K) \leq \widetilde{C}(n, r, R)\, \varepsilon^q.$$

Here, $\widetilde{C}(n, r, R) > 0$ are constants depending on the dimension, r, and R, and $q = q(n)$ is the same as in Theorem 3.1.

The proof of Theorem 3.1 is given in Section 3.3 and consists of a sequence of lemmas from Section 3.2. The main idea is the following. If K is of class C^∞, then we use Brunn's theorem and an integral formula from [6] to show that $\rho(CK, IK)$ being small implies that $\int_{S^{n-1}} \left|A'_{K,\xi}(0)\right|^2 d\xi$ is also small. If K is not smooth, we approximate it by smooth bodies, for which the above integral is small. Then we use the Fourier transform techniques from [47] and the tools of spherical harmonics similar to those from [18] to finish the proof.

As we will see below, the same methods can be used to obtain a stability version of the unique determination result of Koldobsky and Shane [26], which we stated in Theorem 1.2. The following is our stability result:

Theorem 3.4. *Let K and L be convex bodies in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R) \quad \textit{and} \quad B_2^n(r) \subset L \subset B_2^n(R)$$

for some $r, R > 0$. Let $-1 < p < n-1$ be a non-integer, and m be an integer greater than p. If K and L are m-smooth and

$$\sup_{\xi \in S^{n-1}} \left| A_{K,\xi}^{(p)}(0) - A_{L,\xi}^{(p)}(0) \right| \leq \varepsilon$$

for some $0 < \varepsilon < 1$, then

$$\rho(K, L) \leq C(n, p, r, R)\, \varepsilon^q \quad \textit{where} \quad q = \begin{cases} \frac{2}{n+1} & \textit{if } n \leq 2p+2, \\ \frac{4}{(n-2p)(n+1)} & \textit{if } n > 2p+2. \end{cases}$$

Here, $C(n, p, r, R) > 0$ are constants depending on the dimension, p, r, and R.

Remark 3.5. In the proof of Theorem 3.4, we give the explicit dependency of $C(n, p, r, R)$ on r and R.

3.1 Preliminaries

Let K be a convex body in $\mathbb{R}^n$ containing the origin in its interior. The *maximal section function* of K is defined by

$$m_K(\xi) = \max_{t \in \mathbb{R}} \mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}) = \max_{t \in \mathbb{R}} A_{K,\xi}(t), \qquad \xi \in S^{n-1}.$$

Note that m_K is simply the radial function for the cross-section body CK. For each $\xi \in S^{n-1}$, we let $t_K(\xi) \in \mathbb{R}$ be the closest to zero number such that

$$A_{K,\xi}(t_K(\xi)) = m_K(\xi).$$

Towards the proof of our first stability result, we use the formula

$$\begin{aligned} f_K(t) &:= \frac{1}{\omega_n} \int_{S^{n-1}} A_{K,\xi}(t)\, d\xi \\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{n-1}{2}\right)} \int_{K\cap\{|x|\geq|t|\}} \frac{1}{|x|} \left(1 - \frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} dx; \end{aligned} \tag{3.1}$$

refer to Lemma 1.2 in [6] or Lemma 1 in [1] for the proof.

Given another convex body L in $\mathbb{R}^n$, define

$$\delta_2(K, L) = \left(\int_{S^{n-1}} |h_K(\xi) - h_L(\xi)|^2\, d\xi\right)^{\frac{1}{2}}$$

and

$$\delta_\infty(K, L) = \sup_{\xi\in S^{n-1}} |h_K(\xi) - h_L(\xi)|\,.$$

These functions are, respectively, the L^2 and *Hausdorff metrics* for convex bodies in $\mathbb{R}^n$. The following theorem, due to Vitale [54], relates these metrics; refer to Proposition 2.3.1 in [19] for the proof.

Theorem 3.6. *Let K and L be convex bodies in $\mathbb{R}^n$, and let D denote the diameter of $K \cup L$. Then*

$$\frac{2\kappa_{n-1} D^{1-n}}{n(n+1)}\, \delta_\infty(K,\, L)^{n+1} \leq \delta_2(K,\, L)^2 \leq \omega_n\, \delta_\infty(K, L)^2.$$

3.2 Auxiliary Results

We first prove some auxiliary lemmas.

Lemma 3.7. *Let m be a non-negative integer. Let K be an m-smooth convex body in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. There exists a family $\{K_\delta\}_{0<\delta<1}$ of infinitely smooth convex *bodies in $\mathbb{R}^n$ which approximate K in the radial metric as δ approaches zero, with*

$$B_2^n\big((1+\delta)^{-1}r\big) \subset K_\delta \subset B_2^n\big((1-\delta)^{-1}R\big).$$

Furthermore,

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \sup_{|t|\le \frac{r}{4}} \big|A_{K,\xi}(t) - A_{K_\delta,\xi}(t)\big| = 0,$$

and

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \left|A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{K,\xi}(0)\right| = 0$$

for every $p \in \mathbb{R}$, $-1 < p \le m$.

Proof. For each $0 < \delta < 1$, let $\phi_\delta : [0,\infty) \to [0,\infty)$ be a C^∞ function with support contained in $[\delta/2, \delta]$, and

$$\int_{\mathbb{R}^n} \phi_\delta\big(|z|\big)\, dz = 1.$$

It follows from Theorem 3.4.1 in [49] that there is a family $\{K_\delta\}_{0<\delta<1}$ of C^∞ convex bodies in $\mathbb{R}^n$ such that

$$\|x\|_{K_\delta} = \int_{\mathbb{R}^n} \big\|x + |x|z\big\|_K \phi_\delta\big(|z|\big)\, dz,$$

and

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \big|\|\xi\|_{K_\delta} - \|\xi\|_K\big| = 0.$$

For each $\xi \in S^{n-1}$ and $z \in \mathbb{R}^n$ with $|z| \le \delta$, we have

$$\big\|\xi + |\xi|z\big\|_K = \|\xi + z\|_K = \|\lambda\eta\|_K = \lambda\|\eta\|_K$$

for some $\eta \in S^{n-1}$ and $0 < 1-\delta \le \lambda \le 1+\delta$. It then follows from the support of ϕ_δ and the inequality $R^{-1} \le \|\eta\|_K \le r^{-1}$ that

$$\|\xi\|_{K_\delta} = \int_{\mathbb{R}^n} \|\xi + z\|_K \phi_\delta\big(|z|\big)\, dz \le (1+\delta) r^{-1}$$

and

$$\|\xi\|_{K_\delta} = \int_{\mathbb{R}^n} \|\xi + z\|_K \phi_\delta\big(|z|\big)\, dz \ge (1-\delta) R^{-1},$$

which gives

$$B_2^n\big((1+\delta)^{-1} r\big) \subset K_\delta \subset B_2^n\big((1-\delta)^{-1} R\big).$$

This containment, with the limit of the difference of Minkowski functionals above, implies

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \big|\rho_{K_\delta}(\xi) - \rho_K(\xi)\big| = 0. \tag{3.2}$$

Therefore, $\{K_\delta\}_{0<\delta<1}$ approximate K with respect to the radial metric.

Furthermore, the radial functions $\{\rho_{K_\delta}\}_{0<\delta<1}$ approximate ρ_K in $C^m(S^{n-1})$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be any n-tuple of non-negative integers such that $1 \le [\alpha] \le m$, and consider the function

$$f(y,z) := \frac{\partial^{[\alpha]}}{\partial x^\alpha} \big\|x + |x| z\big\|_K \Big|_{x=y}.$$

Observe that f is uniformly continuous on

$$\big\{y \in \mathbb{R}^n, 2^{-1} \le |y| \le 2\big\} \times \big\{z \in \mathbb{R}^n, |z| \le 2^{-1}\big\}$$

since K is m-smooth. Therefore, we have

$$\frac{\partial^{[\alpha]}}{\partial x^\alpha} \big(\|x\|_{K_\delta} - \|x\|_K\big)\Big|_{x=\xi} = \int_{\mathbb{R}^n} \frac{\partial^{[\alpha]}}{\partial x^\alpha} \Big(\big\|x + |x| z\big\|_K - \|x\|_K\Big)\Big|_{x=\xi} \phi_\delta\big(|z|\big)\, dz$$

for all $\xi \in S^{n-1}$ and $\delta < 1/2$, which implies

$$\sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]}}{\partial x^{\alpha}} \big(\|x\|_{K_\delta} - \|x\|_K \big) \Big|_{x=\xi} \right| \le \sup_{\xi \in S^{n-1}} \sup_{|z|<\delta} \big| f(\xi, z) - f(\xi, 0) \big|.$$

Noting that $\big|(\xi, z) - (\xi, 0)\big| = |z| < \delta$, the uniform continuity of f then implies

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]}}{\partial x^{\alpha}} \big(\|x\|_{K_\delta} - \|x\|_K \big) \Big|_{x=\xi} \right| = 0. \tag{3.3}$$

It follows from the relation $\rho_K(x) = \|x\|_K^{-1}$ that $\frac{\partial^{[\alpha]}}{\partial x^{\alpha}} \rho_K \big|_{x=\xi}$ may be expressed as a finite linear combination of terms of the form

$$\rho_K^{d+1}(\xi) \prod_{j=0}^{d} \frac{\partial^{[\beta_j]}}{\partial x^{\beta_j}} \|x\|_K \Big|_{x=\xi},$$

where $d \in \mathbb{Z}^{\geq 0}$, and each β_j is an n-tuple of non-negative integers such that $[\beta_j] \geq 1$ and $[\alpha] = \sum_{j=0}^{d} [\beta_j]$. Of course, $\frac{\partial^{[\alpha]}}{\partial x^{\alpha}} \rho_{K_\delta} \big|_{x=\xi}$ may be expressed similarly. Equations (3.2) and (3.3) then imply

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]}}{\partial x^{\alpha}} \big(\rho_{K_\delta} - \rho_K \big) \Big|_{x=\xi} \right| = 0, \tag{3.4}$$

once we note that ρ_K and the partial derivatives of $\|x\|_K$, up to order m, are bounded on S^{n-1}.

Our next step is to uniformly approximate the parallel section function $A_{K,\xi}$. Fix $\xi \in S^{n-1}$, and define the hyperplane

$$H_t = \xi^{\perp} + t\xi$$

for any $t \in \mathbb{R}$ such that $|t| < r$. Let S^{n-2} denote the Euclidean sphere in H_t centred at $t\xi$, and let $\rho_{K \cap H_t}$ denote the radial function for $K \cap H_t$ with respect to $t\xi$ on S^{n-2}. Then, for $|t| < r$,

$$A_{K,\xi}(t) = \frac{1}{n-1} \int_{S^{n-2}} \rho_{K \cap H_t}^{n-1}(\theta)\, d\theta. \tag{3.5}$$

For $|t| < r/2$ and $0 < \delta < 1$, $A_{K_\delta,\xi}(t)$ may be expressed similarly. Fixing $\theta \in S^{n-2}$, and with angles α and β as in Figure 3.1, we have

$$\left|\rho_{K\cap H_t}(\theta) - \rho_{K_\delta\cap H_t}(\theta)\right| \leq \frac{\sin\beta}{\sin\alpha}\left|\rho_K(\eta_1) - \rho_{K_\delta}(\eta_1)\right|.$$

By restricting to $|t| \leq r/4$, α may be bounded away from zero and π. Indeed, if $\alpha < \pi/2$, then

$$\tan\alpha \geq \frac{r/2 - |t|}{R} \geq \frac{r}{4R},$$

and if $\alpha > \pi/2$, then

$$\tan(\pi - \alpha) \geq \frac{r/2 + |t|}{R} \geq \frac{r}{2R}.$$

Therefore

$$0 < \arctan\left(\frac{r}{4R}\right) \leq \alpha \leq \pi - \arctan\left(\frac{r}{4R}\right) < \pi.$$

We now have

$$\left|\rho_{K\cap H_t}(\theta) - \rho_{K_\delta\cap H_t}(\theta)\right| \leq \frac{1}{\sin\left(\arctan\left(\frac{r}{4R}\right)\right)} \sup_{\eta\in S^{n-1}} \left|\rho_K(\eta) - \rho_{K_\delta}(\eta)\right|, \quad (3.6)$$

where the upper bound is independent of $\xi \in S^{n-1}$, t with $|t| \leq r/4$, and $\theta \in S^{n-2}$. This inequality, the integral expression (3.5), and equation (3.2) imply

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \sup_{|t|\leq \frac{r}{4}} \left|A_{K,\xi}(t) - A_{K_\delta,\xi}(t)\right| = 0.$$

Lemma (2.4) in [25] establishes the existence of a small neighbourhood of $t = 0$, independent of $\xi \in S^{n-1}$, on which $A_{K,\xi}$ is m-smooth. The following is an elaboration of Koldobsky's proof, so that we may uniformly approximate the derivatives of $A_{K,\xi}$. Again fix $\xi \in S^{n-1}$, and fix $\theta \in S^{n-2} \subset H_t$. Let $\rho_{K,\theta}$ denote the m-smooth restriction of ρ_K to the two dimensional plane spanned by ξ and θ, and consider $\rho_{K,\theta}$ as a function on $[0, 2\pi]$, where the angle is

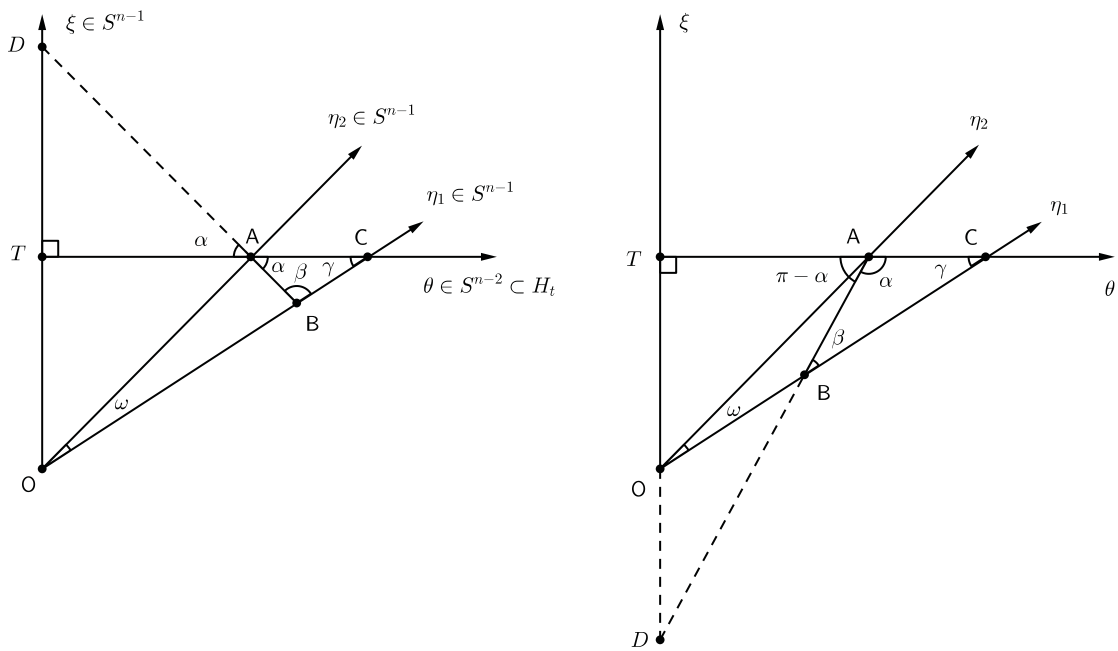

Figure 3.1: The diagrams represent two extremes: when the angle α is small ($\alpha < \pi/2$), and when it is large ($\alpha > \pi/2$). The point O represents the origin in $\mathbb{R}^n$, and $\left|\overline{OT}\right| = t$ where $0 \leq t \leq r/4$. The points A and C are the boundary points for K and K_δ in the direction θ, with two obvious possibilities: either $\left|\overline{TA}\right| = \rho_{K \cap H_t}(\theta)$ and $\left|\overline{TC}\right| = \rho_{K_\delta \cap H_t}(\theta)$, or the opposite. The point B is a boundary point for the same convex body as A, but in the direction η_1. The point D lies outside of the convex body for which A and B are boundary points.

measured from the positive θ-axis. A right triangle then gives the equation

$$\rho_{K\cap H_t}^2(\theta) + t^2 = \rho_{K,\theta}^2\left(\arctan\left(\frac{t}{\rho_{K\cap H_t}(\theta)}\right)\right),$$

which we can use to implicitly differentiate $y(t) := \rho_{K\cap H_t}(\theta)$ as a function of t. Indeed,

$$F(t,y) := y^2 + t^2 - \rho_{K,\theta}^2\left(\arctan\left(\frac{t}{y}\right)\right)$$

is differentiable away from $y = 0$, with

$$F_y(t,y) = 2y + \frac{2t}{y^2+t^2}\rho_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\rho_{K,\theta}'\left(\arctan\left(\frac{t}{y}\right)\right).$$

The containment $B_2^n(r) \subset K \subset B_2^n(R)$ implies $\rho_{K,\theta}$ is bounded above on S^{n-1} by R, and

$$\rho_{K\cap H_t}(\theta) \geq \frac{\sqrt{15}\,r}{4}$$

for $|t| \leq r/4$. If

$$M = 1 + \sup_{\xi\in S^{n-1}} \left|\nabla_o \rho_K(\xi)\right| < \infty,$$

and $\lambda \in \mathbb{R}$ is a constant such that

$$0 < \lambda < \min\left\{\frac{15\sqrt{15}\,r^3}{128RM}, \frac{r}{4}\right\},$$

then

$$\left|F_y\big(t, \rho_{K\cap H_t}(\theta)\big)\right| > \frac{\sqrt{15}\,r}{4}$$

for $|t| \leq \lambda$. Therefore, by the Implicit Function Theorem, $y(t) = \rho_{K\cap H_t}(\theta)$ is differentiable on $(-\lambda, \lambda)$, with

$$y'(t) = \frac{\rho_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\rho'_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\left(y^2+t^2\right)^{-1}y - t}{y + t\rho_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\rho'_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\left(y^2+t^2\right)^{-1}}.$$

Recursion shows that $\rho_{K\cap H_t}(\theta)$ is m-smooth on $(-\lambda, \lambda)$, independent of $\xi \in S^{n-1}$ and $\theta \in S^{n-2}$. It follows from the integral expression (3.5) that $A_{K,\xi}$ is m-smooth on $(-\lambda, \lambda)$ for every $\xi \in S^{n-1}$. This argument also shows that $A_{K_\delta,\xi}$ is m-smooth on the same interval, for $\delta > 0$ small enough. Using the resulting expressions for the derivatives of $A_{K,\xi}$ and $A_{K_\delta,\xi}$, and applying equations (3.2), (3.4), and the inequality (3.6), we have

$$\lim_{\delta\to 0}\sup_{\xi\in S^{n-1}}\sup_{|t|\leq\lambda}\left|A^{(k)}_{K,\xi}(t) - A^{(k)}_{K_\delta,\xi}(t)\right| = 0$$

for $k = 1, \ldots, m$.

Finally, for any $p \in \mathbb{R}$ such that $-1 < p < m$ and $p \neq 0, 1, \ldots, m-1$, we will uniformly approximate $A^{(p)}_{K,\xi}(0)$. With $\lambda > 0$ as chosen above, we have

$$\begin{aligned}A^{(p)}_{K,\xi}(0) =& \frac{1}{\Gamma(-p)}\int_0^\lambda t^{-1-p}\left(A_{K,\xi}(-t) - \sum_{k=0}^{m-1}\frac{(-1)^k A^{(k)}_{K,\xi}(0)}{k!}t^k\right)dt \\ &+ \frac{1}{\Gamma(-p)}\int_\lambda^\infty t^{-1-p}A_{K,\xi}(-t)\,dt + \frac{1}{\Gamma(-p)}\sum_{k=0}^{m-1}\frac{(-1)^k\lambda^{k-p}A^{(k)}_{K,\xi}(0)}{k!(k-p)}.\end{aligned}$$

The first integral in this equation can be rewritten as

$$\int_0^\lambda t^{-1-p}\int_0^t \frac{A^{(m)}_{K,\xi}(-z)}{(m-1)!}(t-z)^{m-1}\,dz\,dt,$$

using the integral form of the remainder in Taylor's Theorem. We also have

$$\begin{aligned}
&\int_\lambda^\infty t^{-1-p} A_{K,\xi}(-t)\,dt \\
&= \int_{K\cap\{\langle x,-\xi\rangle\geq\lambda\}} \langle x,-\xi\rangle^{-1-p}\,dx \\
&= \int_{B_K(\xi)} \langle \eta,-\xi\rangle^{-1-p} \int_{\lambda\langle\eta,-\xi\rangle^{-1}}^{\rho_K(\eta)} r^{n-2-p}\,dr\,d\eta \\
&= \frac{1}{n-1-p}\int_{B_K(\xi)} \Big(\langle\eta,-\xi\rangle^{-1-p}\rho_K^{n-1-p}(\eta) - \lambda^{n-1-p}\langle\eta,-\xi\rangle^{-n}\Big)\,d\eta,
\end{aligned}$$

where

$$B_K(\xi) = \Big\{\eta\in S^{n-1}\Big|\langle\eta,\xi\rangle<0 \text{ and } \rho_K(\eta)\geq\lambda\langle\eta,-\xi\rangle^{-1}\Big\}.$$

Therefore, with the set $B_{K_\delta}(\xi)$ defined similarly, we have

$$\begin{aligned}
&\left|A_{K,\xi}^{(p)}(0) - A_{K_\delta,\xi}^{(p)}(0)\right|\cdot\left|\Gamma(-p)\right| \\
&\leq \frac{1}{(m-1)!}\left(\sup_{|z|\leq\lambda}\left|A_{K,\xi}^{(m)}(z) - A_{K_\delta,\xi}^{(m)}(z)\right|\right)\int_0^\lambda\int_0^t t^{-1-p}(t-z)^{m-1}\,dz\,dt \qquad (3.7)\\
&\quad+\left(\sup_{\eta\in S^{n-1}}\left|\rho_K^{n-1-p}(\eta) - \rho_{K_\delta}^{n-1-p}(\eta)\right|\right)\int_{B_K(\xi)\cap B_{K_\delta}(\xi)} \frac{\langle\eta,-\xi\rangle^{-1-p}}{|n-1-p|}\,d\eta \qquad (3.8)\\
&\quad+\int_{B_K(\xi)\setminus B_{K_\delta}(\xi)}\left|\frac{\langle\eta,-\xi\rangle^{-1-p}\rho_K^{n-1-p}(\eta) - \lambda^{n-1-p}\langle\eta,-\xi\rangle^{-n}}{n-1-p}\right|\,d\eta \qquad (3.9)\\
&\quad+\int_{B_{K_\delta}(\xi)\setminus B_K(\xi)}\left|\frac{\langle\eta,-\xi\rangle^{-1-p}\rho_{K_\delta}^{n-1-p}(\eta) - \lambda^{n-1-p}\langle\eta,-\xi\rangle^{-n}}{n-1-p}\right|\,d\eta \qquad (3.10)\\
&\quad+\sum_{k=0}^{m-1}\frac{\lambda^{k-p}}{k!|k-p|}\left|A_{K,\xi}^{(k)}(0) - A_{K_\delta,\xi}^{(k)}(0)\right|,
\end{aligned}$$

for $\delta>0$ small enough. The integrals in expressions (3.7) and (3.8) are finite, with

$$\int_0^\lambda\int_0^t t^{-1-p}(t-z)^{m-1}\,dz\,dt = \frac{\lambda^{m-p}}{m(m-p)},$$

since p is a non-integer less than m, and

$$\int_{B_K(\xi)\cap B_{K_\delta}(\xi)} \langle \eta, -\xi\rangle^{-1-p}\, d\eta \le \left(\frac{R}{\lambda}\right)^{1+p} \omega_n.$$

Furthermore, the integrands in expression (3.9) and (3.10) are bounded above by

$$\left(\frac{2R}{\lambda}\right)^{1+p} (2R)^{n-1-p} + \lambda^{n-1-p}\left(\frac{2R}{\lambda}\right)^{n} \qquad \text{if } p < n-1,$$

and

$$\left(\frac{2R}{\lambda}\right)^{1+p} \left(\frac{r}{2}\right)^{n-1-p} + \lambda^{n-1-p}\left(\frac{2R}{\lambda}\right)^{n} \qquad \text{if } p > n-1,$$

noting that $B_2^n(r/2) \subset K_\delta \subset B_2^n(2R)$ for $\delta < 1/2$.

It is now sufficient to prove

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \int_{S^{n-1}} \chi_{B(\xi,\delta)}\, d\eta = 0,$$

where

$$\begin{aligned} &B(\xi,\delta) = B_K(\xi)\Delta B_{K_\delta}(\xi) \\ &= \left\{ \eta\in S^{n-1} \,\middle|\, \rho_K(\eta) \ge \frac{\lambda}{\langle \eta,-\xi\rangle} > \rho_{K_\delta}(\eta) \text{ or } \rho_{K_\delta}(\eta) \ge \frac{\lambda}{\langle \eta,-\xi\rangle} > \rho_K(\eta) \right\}. \end{aligned}$$

We will prove the equivalent statement

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \int_{S^{n-1}} \chi_{B(-\xi,\delta)}\, d\eta = 0,$$

where the sign of ξ has changed, so that we may use Figure 3.1.

Towards this end, fix any $\theta \in S^{n-2}$, and consider Figure 3.1 specifically when $t = \lambda$. In this case,

$$\left|\overline{OA}\right| = \rho_K(\eta_2) = \lambda\langle \eta_2, \xi\rangle^{-1} \text{ and } \left|\overline{OC}\right| = \rho_{K_\delta}(\eta_1) = \lambda\langle \eta_1, \xi\rangle^{-1}$$

or

$$\left|\overline{OC}\right| = \rho_K(\eta_2) = \lambda\langle\eta_2,\xi\rangle^{-1} \text{ and } \left|\overline{OA}\right| = \rho_{K_\delta}(\eta_1) = \lambda\langle\eta_1,\xi\rangle^{-1}.$$

Any $\eta \in B(-\xi,\delta)$ lying in the right half-plane spanned by ξ and θ will lie between η_1 and η_2. Furthermore, the angle ω converges to zero as δ approaches zero, uniformly with respect to $\xi \in S^{n-1}$ and $\theta \in S^{n-2}$. Indeed, we have

$$0 \le \sin\omega \le \frac{2\,\sin\beta\,\sin\gamma}{r\,\sin\alpha}\left|\rho_K(\eta_1) - \rho_{K_\delta}(\eta_1)\right|,$$

using the fact that both K and K_δ contain a ball of radius $r/2$, and with $\sin\alpha$ uniformly bounded away from zero as before. It follows that the spherical measure of $B(-\xi,\delta)$ converges to zero as δ approaches zero, uniformly with respect to $\xi \in S^{n-1}$. □

Lemma 3.8. *Let K be a convex body in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. If

$$L(n) = 8(n-1)\pi^{\frac{n-1}{2}}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^{-1},$$

then

$$|A_{K,\xi}(t) - A_{K,\xi}(s)| \le L(n)\,R^{n-1}\,r^{-1}\,|t-s|$$

for all $s,t \in [-r/2, r/2]$ and $\xi \in S^{n-1}$.

Proof. For $\xi \in S^{n-1}$, Brunn's Theorem implies $f := A_{K,\xi}^{\frac{1}{n-1}}$ is concave on its support, which includes the interval $[-r,r]$. Let

$$L_0 = \max\left\{\left|\frac{f\left(\frac{-3r}{4}\right) - f(-r)}{\frac{-3r}{4} - (-r)}\right|, \left|\frac{f(r) - f\left(\frac{3r}{4}\right)}{r - \frac{3r}{4}}\right|\right\},$$

and suppose $s, t \in [-r/2, r/2]$ are such that $s < t$. If

$$\frac{f(t) - f(s)}{t - s} > 0,$$

then

$$\frac{f\left(\frac{-3r}{4}\right) - f(-r)}{\frac{-3r}{4} - (-r)} \geq \frac{f(s) - f\left(\frac{-3r}{4}\right)}{s - \left(\frac{-3r}{4}\right)} \geq \frac{f(t) - f(s)}{t - s} > 0\,;$$

otherwise, we will obtain a contradiction of the concavity of f. Similarly, if

$$\frac{f(t) - f(s)}{t - s} < 0,$$

then

$$\frac{f(r) - f\left(\frac{3r}{4}\right)}{r - \frac{3r}{4}} \leq \frac{f\left(\frac{3r}{4}\right) - f(t)}{\frac{3r}{4} - t} \leq \frac{f(t) - f(s)}{t - s} < 0.$$

Therefore,

$$\left| A_{K,\xi}^{\frac{1}{n-1}}(t) - A_{K,\xi}^{\frac{1}{n-1}}(s) \right| \leq L_0 \, |t - s|$$

for all $s, t \in [-r/2, r/2]$. Now, we have

$$|A_{K,\xi}(t) - A_{K,\xi}(s)| \leq (n-1) \left(\max_{t_0 \in \mathbb{R}} A_{K,\xi}(t_0) \right)^{\frac{n-2}{n-1}} \left| A_{K,\xi}^{\frac{1}{n-1}}(t) - A_{K,\xi}^{\frac{1}{n-1}}(s) \right|$$

by the Mean Value Theorem, and

$$L_0 \leq \frac{4}{r} \cdot 2 \left(\max_{t_0 \in \mathbb{R}} A_{K,\xi}(t_0) \right)^{\frac{1}{n-1}} = \frac{8}{r} A_{K,\xi}^{\frac{1}{n-1}}\big(t_K(\xi)\big).$$

Finally, since K is contained in a ball of radius R, we have

$$A_{K,\xi}\big(t_K(\xi)\big) \leq \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} R^{n-1}.$$

Combining these inequalities gives

$$|A_{K,\xi}(t) - A_{K,\xi}(s)| \leq L(n)\, R^{n-1}\, r^{-1}\, |t-s|$$

for all $s, t \in [-r/2, r/2]$ and $\xi \in S^{n-1}$. $\square$

We now prove two lemmas that will be the core of the proof of Theorem 3.1.

Lemma 3.9. *Let K be a convex body in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. Let $\{K_\delta\}_{0<\delta<1}$ be as in Lemma 3.7. If there exists $0 < \varepsilon < \frac{r^2}{16}$ so that

$$\rho(CK, IK) \leq \varepsilon,$$

then, for $\delta > 0$ small enough,

$$\int_{S^1} \left|A'_{K_\delta,\xi}(0)\right| d\xi \leq \left(6\pi + \frac{32\,\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon} \qquad \textit{when } n = 2,$$
$$\int_{S^{n-1}} \left|A'_{K_\delta,\xi}(0)\right|^2 d\xi \leq C(n)\left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}}\right)\sqrt{\varepsilon} \qquad \textit{when } n \geq 3.$$

Here, $C(n) > 0$ are constants depending only on the dimension.

Proof. By Lemma 3.7, we may choose $0 < \alpha < 1/2$ small enough so that for every $0 < \delta < \alpha$,

$$\sup_{\xi \in S^{n-1}} \sup_{|t| \leq r/4} \left|A_{K,\xi}(t) - A_{K_\delta,\xi}(t)\right| \leq \varepsilon.$$

We first show that for each $0 < \delta < \alpha$ and $\xi \in S^{n-1}$, there exists a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \leq \sqrt{\varepsilon}$ for which

$$\left|A'_{K_\delta,\xi}\big(c_\delta(\xi)\big)\right| \leq 3\sqrt{\varepsilon}.$$

Indeed, if $\xi \in S^{n-1}$ is such that $|t_{K_\delta}(\xi)| \le \sqrt{\varepsilon}$, then

$$A'_{K_\delta,\xi}\big(t_{K_\delta}(\xi)\big) = 0,$$

and we may take $c_\delta(\xi) = t_{K_\delta}(\xi)$.

Assume $\xi \in S^{n-1}$ is such that $|t_{K_\delta}(\xi)| > \sqrt{\varepsilon}$. Letting s denote the sign of $t_{K_\delta}(\xi)$, we have

$$\begin{aligned}
&\big|A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K_\delta,\xi}(0)\big| = A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K_\delta,\xi}(0)\\
&= \Big(A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0)\Big) + \Big(A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(s\sqrt{\varepsilon})\Big)\\
&\qquad\qquad\qquad\qquad + \Big(A_{K,\xi}(0) - A_{K_\delta,\xi}(0)\Big)\\
&\le \sup_{\xi \in S^{n-1}} \left|\max_{t\in\mathbb{R}} A_{K,\xi}(t) - A_{K,\xi}(0)\right| + 2 \sup_{\xi\in S^{n-1}} \sup_{|t|\le r/4} \big|A_{K,\xi}(t) - A_{K_\delta,\xi}(t)\big|\\
&\le 3\varepsilon.
\end{aligned}$$

It then follows from the Mean Value Theorem that there is a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \le \sqrt{\varepsilon}$ for which

$$\big|A'_{K_\delta,\xi}\big(c_\delta(\xi)\big)\big| = \left|\frac{A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K_\delta,\xi}(0)}{\sqrt{\varepsilon} - 0}\right| \le 3\sqrt{\varepsilon}.$$

With the numbers $c_\delta(\xi)$ as above, for the case $n = 2$ we have

$$\begin{aligned}
&\int_{S^1} \big|A'_{K_\delta,\xi}(0)\big|\, d\xi\\
&\le \int_{S^1} \left(\big|A'_{K_\delta,\xi}\big(c_\delta(\xi)\big)\big| + \left|\int_{c_\delta(\xi)}^0 A''_{K_\delta,\xi}(t)\, dt\right|\right) d\xi\\
&\le 6\pi\sqrt{\varepsilon} + \int_{S^1} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \big|A''_{K_\delta,\xi}(t)\big|\, dt\, d\xi.
\end{aligned} \tag{3.11}$$

When $0 < \delta < 1/2$, K_δ is contained in a ball of radius $2R$, and contains a ball of radius $r/2$. Lemma 3.8 then implies

$$\sup_{\xi\in S^{n-1}} \sup_{t\in(-\sqrt{\varepsilon},\sqrt{\varepsilon})} \big|A'_{K_\delta,\xi}(t)\big| \le \frac{2L(n)\,(2R)^{n-1}}{r}.$$

So, when $n \geq 3$,

$$
\begin{aligned}
&\int_{S^{n-1}} \left|A'_{K_\delta,\xi}(0)\right|^2 d\xi \\
&\leq \int_{S^{n-1}} \left(\left|A'_{K_\delta,\xi}\big(c_\delta(\xi)\big)\right|^2 + \left| \int_{c_\delta(\xi)}^0 2A''_{K_\delta,\xi}(t) A'_{K_\delta,\xi}(t)\, dt \right| \right) d\xi \\
&\leq 9\,\omega_n\,\varepsilon + \frac{4L(n)\,(2R)^{n-1}}{r} \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left|A''_{K_\delta,\xi}(t)\right| dt\, d\xi \qquad (3.12)
\end{aligned}
$$

Considering inequalities (3.11) and (3.12), we still need to bound

$$
\int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left|A''_{K_\delta,\xi}(t)\right| dt\, d\xi
$$

for arbitrary n. Rearranging the equation

$$
\begin{aligned}
\frac{d^2}{dt^2} A_{K_\delta,\xi}^{\frac{1}{n-1}}(t) &= \frac{d}{dt}\left(\frac{1}{n-1} A_{K_\delta,\xi}^{\frac{2-n}{n-1}}(t)\, A'_{K_\delta,\xi}(t) \right) \\
&= \frac{2-n}{(n-1)^2} A_{K_\delta,\xi}^{\frac{3-2n}{n-1}}(t)\, \left(A'_{K_\delta,\xi}(t)\right)^2 + \frac{1}{n-1} A_{K_\delta,\xi}^{\frac{2-n}{n-1}}(t)\, A''_{K_\delta,\xi}(t)
\end{aligned}
$$

gives

$$
A''_{K_\delta,\xi}(t) = (n-1) A_{K_\delta,\xi}^{\frac{n-2}{n-1}}(t) \frac{d^2}{dt^2} A_{K_\delta,\xi}^{\frac{1}{n-1}}(t) + \frac{n-2}{n-1} \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)}.
$$

Brunn's Theorem implies that the second derivative of $A_{K_\delta,\xi}^{\frac{1}{n-1}}$ is non-positive for $|t| < r$, so

$$
\begin{aligned}
\left|A''_{K_\delta,\xi}(t)\right| &\leq (1-n) A_{K_\delta,\xi}^{\frac{n-2}{n-1}}(t) \frac{d^2}{dt^2} A_{K_\delta,\xi}^{\frac{1}{n-1}}(t) + \frac{n-2}{n-1} \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)} \\
&= -A''_{K_\delta,\xi}(t) + 2\left(\frac{n-2}{n-1}\right) \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)}.
\end{aligned}
$$

Because K_δ contains a ball of radius $r/2$ centred at the origin, we have

$$A_{K_\delta,\xi}(t) \geq \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \left(\frac{3\pi r^2}{16}\right)^{\frac{n-1}{2}}$$

for $|t| \leq r/4$, and so

$$\begin{aligned} \frac{n-2}{n-1} \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)} &\leq \frac{n-2}{n-1} \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2L(n)\,(2R)^{n-1}}{r}\right)^2 \left(\frac{16}{3\pi r^2}\right)^{\frac{n-1}{2}} \\ &= \frac{\widetilde{L}(n)\, R^{2n-2}}{r^{n+1}} \end{aligned}$$

for all $|t| \leq \sqrt{\varepsilon}$, where $\widetilde{L}(n)$ is a constant depending only on n. Therefore,

$$\begin{aligned} &\int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left|A''_{K_\delta,\xi}(t)\right| dt\, d\xi \\ &\leq \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \Big(-A''_{K_\delta,\xi}(t)\Big)\, dt\, d\xi + \frac{4\,\omega_n\, \widetilde{L}(n)\, R^{2n-2}}{r^{n+1}} \sqrt{\varepsilon}. \end{aligned} \tag{3.13}$$

We will bound the first term on the final line above using formula (3.1). Letting

$$\widetilde{C}(n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)},$$

formula (3.1) becomes

$$\begin{aligned} f_{K_\delta}(t) &= \widetilde{C}(n) \int_{S^{n-1}} \int_{|t|}^{\rho_{K_\delta}(\xi)} \frac{1}{r} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-3}{2}} r^{n-1}\, dr\, d\xi \\ &= \widetilde{C}(n) \int_{S^{n-1}} \int_{|t|}^{\rho_{K_\delta}(\xi)} r \left(r^2 - t^2\right)^{\frac{n-3}{2}} dr\, d\xi \\ &= \frac{\widetilde{C}(n)}{(n-1)} \int_{S^{n-1}} \left(\rho^2_{K_\delta}(\xi) - t^2\right)^{\frac{n-1}{2}} d\xi. \end{aligned}$$

The derivatives of $A_{K_\delta,\xi}$ and $\left(\rho^2_{K_\delta}(\xi) - t^2\right)^{\frac{n-1}{2}}$ are bounded on $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ uniformly with respect to $\xi \in S^{n-1}$, so

$$f'_{K_\delta}(t) = \frac{1}{\omega_n} \int_{S^{n-1}} A'_{K_\delta,\xi}(t)\, d\xi = -\widetilde{C}(n)\, t \int_{S^{n-1}} \left(\rho^2_{K_\delta}(\xi) - t^2\right)^{\frac{n-3}{2}}\, d\xi.$$

Observing $\widetilde{C}(2) = \pi^{-1}$, and using that $0 < \varepsilon < r^2/16$ and $r/2 \le \rho_{K_\delta} \le 2R$ for $\delta < 1/2$, we have

$$\begin{aligned}
&\left| \int_{S^{n-1}} A'_{K_\delta,\xi}(\pm\sqrt{\varepsilon})\, d\xi \right| \\
&= \omega_n \left| f'_{K_\delta}(\pm\sqrt{\varepsilon}) \right| = \widetilde{C}(n)\, \omega_n \sqrt{\varepsilon} \int_{S^{n-1}} \left(\rho^2_{K_\delta}(\xi) - \varepsilon\right)^{\frac{n-3}{2}}\, d\xi \\
&\le \begin{cases} 16\, \pi \left(\sqrt{3}\, r\right)^{-1} \sqrt{\varepsilon} & \text{if } n = 2, \\ \widetilde{C}(n)\, \omega_n^2\, (2R)^{n-3} \sqrt{\varepsilon} & \text{if } n \ge 3. \end{cases}
\end{aligned}$$

This implies

$$\begin{aligned}
\left| \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} -A''_{K_\delta,\xi}(t)\, dt\, d\xi \right| &= \left| \int_{S^{n-1}} \left(A'_{K_\delta,\xi}(-\sqrt{\varepsilon}) - A'_{K_\delta,\xi}(\sqrt{\varepsilon}) \right) d\xi \right| \\
&\le \begin{cases} 32\, \pi \left(\sqrt{3}\, r\right)^{-1} \sqrt{\varepsilon} & \text{if } n = 2, \\ 2\, \widetilde{C}(n)\, \omega_n^2\, (2R)^{n-3} \sqrt{\varepsilon} & \text{if } n \ge 3. \end{cases} \qquad (3.14)
\end{aligned}$$

Noting that $\widetilde{L}(2) = 0$, inequalities (3.11), (3.13), and (3.14) give

$$\int_{S^1} \left| A'_{K_\delta,\xi}(0) \right| d\xi \le \left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon}$$

when $n = 2$. For $n \ge 3$, inequalities (3.12), (3.13), and (3.14) give

$$\int_{S^{n-1}} \left| A'_{K_\delta,\xi}(0) \right|^2 d\xi \le C(n) \left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \sqrt{\varepsilon},$$

where $C(n)$ is a constant depending on n. □

Lemma 3.10. *Let K and L be infinitely smooth convex bodies in $\mathbb{R}^n$ such that*

$$B_2^n(r) \subset K \subset B_2^n(R) \quad \textit{and} \quad B_2^n(r) \subset L \subset B_2^n(R)$$

for some $r, R > 0$. Let $p \in (0, n)$. If $\varepsilon > 0$ is such that

$$\left\| I_p \left(\|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p} \right) \right\|_2 \leq \varepsilon,$$

then when $n \leq 2p$,

$$\rho(K, L) \leq C(n, p)\, R^2 r^{\frac{-3n-1+2p}{n+1}} \varepsilon^{\frac{2}{n+1}},$$

and when $n > 2p$,

$$\rho(K, L) \leq C(n, p)\, R^2 r^{\frac{-3n-1+2p}{n+1}} \left(\varepsilon^2 + \frac{R^{2(n+1-p)}}{r^2} \right)^{\frac{n-2p}{(n+2-2p)(n+1)}} \varepsilon^{\frac{4}{(n+2-2p)(n+1)}}.$$

Here, $\| \cdot \|_2$ denotes the norm on $L^2(S^{n-1})$, and $C(n, p) > 0$ are constants depending on the dimension and p.

Proof. Define the function

$$f(\xi) := \|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}$$

on S^{n-1}. Towards bounding the radial distance between K and L by $\|f\|_2$, the $L^2(S^{n-1})$ norm of f, note that the identity

$$\rho_K(\xi) - \rho_L(\xi) = \rho_K(\xi)\rho_L(\xi)\big(\|\xi\|_L - \|\xi\|_K\big)$$

implies

$$\big|\rho_K(\xi) - \rho_L(\xi)\big| \leq R^2 \big|\|\xi\|_K - \|\xi\|_L\big|.$$

By Theorem 3.6, we have

$$\delta_\infty(K^\circ, L^\circ) \leq C(n) D^{\frac{n-1}{n+1}} \big(\delta_2(K^\circ, L^\circ)\big)^{\frac{2}{n+1}},$$

where $C(n) > 0$ is a constant depending on n, and D is the diameter of $K^\circ \cup L^\circ$. Both K° and L° are contained in a ball of radius r^{-1} centred at the origin.

We then have $D \leq 2r^{-1}$, and

$$\sup_{\xi \in S^{n-1}} \left| \|\xi\|_K - \|\xi\|_L \right| \leq C(n) r^{\frac{1-n}{n+1}} \left(\int_{S^{n-1}} \left(\|\xi\|_K - \|\xi\|_L \right)^2 d\xi \right)^{\frac{1}{n+1}}$$

for some new constant $C(n)$. There exists a function $g : S^{n-1} \to \mathbb{R}$ such that

$$\big(\|\xi\|_K - \|\xi\|_L \big) g(\xi) = \|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}.$$

If $\xi \in S^{n-1}$ is such that $\|\xi\|_K \neq \|\xi\|_L$, then an application of the Mean Value Theorem to the function t^{-n+p} on the interval bounded by $\|\xi\|_K$ and $\|\xi\|_L$ gives

$$|g(\xi)| \geq (n-p) \left(\max \left\{ \|\xi\|_K, \|\xi\|_L \right\} \right)^{-n-1+p} \geq (n-p) r^{n+1-p}.$$

Therefore,

$$\left| \|\xi\|_K - \|\xi\|_L \right| \leq (n-p)^{-1} r^{-n-1+p} |f(\xi)|.$$

Combining the above inequalities, we get

$$\sup_{\xi \in S^{n-1}} \left| \rho_K(\xi) - \rho_L(\xi) \right| \leq C(n,p) R^2 r^{\frac{-3n-1+2p}{n+1}} \|f\|_2^{\frac{2}{n+1}}, \tag{3.15}$$

for some constant $C(n,p)$.

We now compare the L^2 norm of f to that of $I_p(f)$ by considering two separate cases based on the dimension n, as in the proof of Theorem 3.6 in [18]. In both cases, we let $\sum_{m=0}^{\infty} Q_m$ be the condensed harmonic expansion for f, and let $\lambda_m(n,p)$ be the eigenvalues from Lemma 2.2. As in [18], the condensed harmonic expansion for $I_p f$ is then given by $\sum_{m=0}^{\infty} \lambda_m(n,p) Q_m$.

Assume $n \leq 2p$. An application of Stirling's formula to the equations given in Lemma 2.2 shows that $\lambda_m(n,p)$ diverges to infinity as m approaches infinity. The eigenvalues are also non-zero, so there is a constant $C(n,p)$ such that $C(n,p)|\lambda_m(n,p)|^2$ is greater than one for all m. Therefore,

$$\begin{aligned}\|f\|_2^2 &= \sum_{m=0}^{\infty} \|Q_m\|_2^2 \\ &\le C(n,p) \sum_{m=0}^{\infty} \left|\lambda_m(n,p)\right|^2 \|Q_m\|_2^2 = C(n,p)\|I_p(f)\|_2^2 \le C(n,p)\varepsilon^2.\end{aligned}$$

Combining this inequality with (3.15) gives the first estimate in the theorem.

Assume $n > 2p$. Hölder's inequality gives

$$\begin{aligned}\|f\|_2^2 &= \sum_{m=0}^{\infty} \|Q_m\|_2^2 \\ &= \sum_{m=0}^{\infty} \left(\left|\lambda_m(n,p)\right|^{\frac{4}{n+2-2p}} \|Q_m\|_2^{\frac{4}{n+2-2p}} \right) \cdot \left(\left|\lambda_m(n,p)\right|^{\frac{-4}{n+2-2p}} \|Q_m\|_2^{\frac{2n-4p}{n+2-2p}} \right) \\ &\le \left(\sum_{m=0}^{\infty} \left|\lambda_m(n,p)\right|^2 \|Q_m\|_2^2 \right)^{\frac{2}{n+2-2p}} \left(\sum_{m=0}^{\infty} \left|\lambda_m(n,p)\right|^{\frac{-4}{n-2p}} \|Q_m\|_2^2 \right)^{\frac{n-2p}{n+2-2p}},\end{aligned}$$

where we again note that the eigenvalues are all non-zero. It follows from Lemma 2.2 and Stirling's formula that there is a constant $C(n,p)$ such that

$$\left|\lambda_m(n,p)\right|^{\frac{-4}{n-2p}} \le C(n,p) m^2$$

for all $m \ge 1$, and

$$\left|\lambda_0(n,p)\right|^{\frac{-4}{n-2p}} \le C(n,p).$$

Using the identity

$$\|\nabla_o f\|_2^2 = \sum_{m=1}^{\infty} m(m+n-2)\|Q_m\|_2^2 \tag{3.16}$$

given by Corollary 3.2.12 in [19], we then have

$$\|f\|_2^2 \le C(n,p) \left(\|I_p(f)\|_2^2\right)^{\frac{2}{n+2-2p}} \left(\|Q_0\|_2^2 + \|\nabla_o f\|_2^2\right)^{\frac{n-2p}{n+2-2p}}.$$

The Minkowski functional of a convex body is the support function of the corresponding polar body, so

$$\nabla_o \|\xi\|_K^{-n+p} = (-n+p)\|\xi\|_K^{-n-1+p}\, \nabla_o h_{K^\circ}(\xi).$$

Because K° is contained in a ball of radius r^{-1}, it follows from Lemma 2.2.1 in [19] that

$$|\nabla_o h_{K^\circ}(\xi)| \le 2r^{-1}$$

for all $\xi \in S^{n-1}$. We now have

$$\left\|\nabla_o \|\xi\|_K^{-n+p}\right\|_2^2 \le 4(n-p)^2 R^{2(n+1-p)} r^{-2} \omega_n.$$

This constant bounds the squared L^2 norm of $\nabla_o \|\xi\|_L^{-n+p}$ as well, so

$$\left\|\nabla_o f\right\|_2^2 \le 16(n-p)^2 R^{2(n+1-p)} r^{-2} \omega_n.$$

Therefore,

$$\|f\|_2^2 \le C(n,p)\varepsilon^{\frac{4}{n+2-2p}} \left(\varepsilon^2 + R^{2(n+1-p)} r^{-2}\right)^{\frac{n-2p}{n+2-2p}},$$

where the constant $C(n,p) > 0$ is different from before. This inequality with (3.15) gives the second estimate in the theorem. $\square$

3.3 Proofs of Stability Results

We are now ready to prove our stability results.

Proof of Theorem 3.1. Let $\{K_\delta\}_{0<\delta<1}$ be the family of smooth convex bodies from Lemma 3.7. We will show that $\rho(K_\delta, -K_\delta)$ is small for $0 < \delta < \alpha$, where α is the constant from the proof of Lemma 3.9. The bounds in the theorem will then follow from

$$\rho(K,-K) \le \lim_{\delta\to 0} \big(2\rho(K,K_\delta) + \rho(K_\delta,-K_\delta)\big) = \lim_{\delta\to 0} \rho(K_\delta,-K_\delta).$$

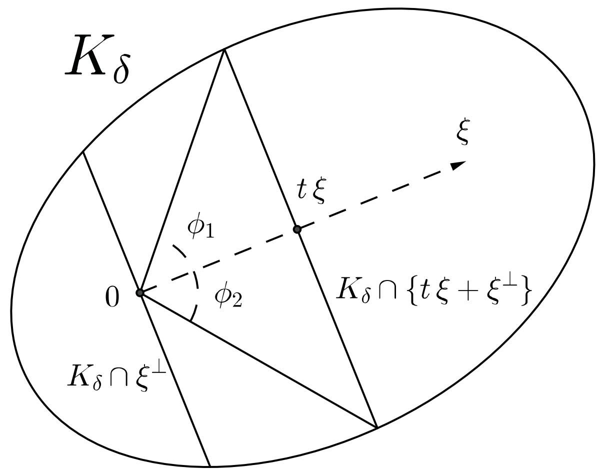

Figure 3.2: K_δ is a convex body in $\mathbb{R}^2$, and $\xi \in S^1$.

We begin by separately considering the case $n = 2$. Let the radial function ρ_{K_δ} be a function of the angle measured counter-clockwise from the positive horizontal axis. For any $\xi \in S^1$, let the angles ϕ_1 and ϕ_2 be functions of $t \in (-r, r)$ as indicated in Figure 3.2. If ξ corresponds to the angle θ, then the parallel section function for K_δ may be written as

$$A_{K_\delta,\theta}(t) = \rho_{K_\delta}(\theta + \phi_1)\,\sin\phi_1 + \rho_{K_\delta}(\theta - \phi_2)\,\sin\phi_2.$$

Implicit differentiation of

$$\cos\phi_j = \frac{t}{\rho_{K_\delta}\left(\theta - (-1)^j \phi_j\right)} \quad (j = 1, 2)$$

gives

$$\left.\frac{d\phi_j}{dt}\right|_{t=0} = \frac{(-1)}{\rho_{K_\delta}\left(\theta - (-1)^j \frac{\pi}{2}\right)},$$

so

$$A'_{K_\delta,\theta}(0) = -\frac{\rho'_{K_\delta}\left(\theta + \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta + \frac{\pi}{2}\right)} + \frac{\rho'_{K_\delta}\left(\theta - \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta - \frac{\pi}{2}\right)}.$$

Since $f(\phi) := \rho_{K_\delta}(\phi + \pi/2) - \rho_{K_\delta}(\phi - \pi/2)$ is a continuous function on $[0, \pi]$

with

$$f(0) = \rho_{K_\delta}(\pi/2) - \rho_{K_\delta}(-\pi/2) = -\big(\rho_{K_\delta}(-\pi/2) - \rho_{K_\delta}(\pi/2)\big) = -f(\pi),$$

there exists an angle $\theta_0 \in [0, \pi]$ such that $\rho_{K_\delta}(\theta_0 + \pi/2) = \rho_{K_\delta}(\theta_0 - \pi/2)$. With this θ_0, we get the inequality

$$\left| \int_{\theta_0}^{\theta} \left(-\frac{\rho'_{K_\delta}\left(\phi + \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\phi + \frac{\pi}{2}\right)} + \frac{\rho'_{K_\delta}\left(\phi - \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\phi - \frac{\pi}{2}\right)} \right) d\phi \right| \leq \int_0^{2\pi} \left| A'_{K_\delta,\phi}(0) \right| d\phi.$$

Integrating the left side of this inequality, and applying Lemma 3.9 to the right side, gives

$$\left| \log \left(\frac{\rho_{K_\delta}\left(\theta - \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta + \frac{\pi}{2}\right)} \right) \right| \leq \left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon}.$$

This implies

$$\begin{aligned} 1 - \exp\left[\left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon} \right] &\leq \exp\left[-\left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon} \right] - 1 \\ &\leq \frac{\rho_{K_\delta}\left(\theta - \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta + \frac{\pi}{2}\right)} - 1 \\ &\leq \exp\left[\left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon} \right] - 1. \end{aligned}$$

It follows that

$$\begin{aligned} -2 \left(\exp\left[\left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon} \right] - 1 \right) R &\leq \rho_{K_\delta}\left(\theta - \frac{\pi}{2}\right) - \rho_{K_\delta}\left(\theta + \frac{\pi}{2}\right) \\ &\leq 2 \left(\exp\left[\left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon} \right] - 1 \right) R, \end{aligned}$$

since K_δ is contained in a ball of radius $2R$. Viewing ρ_{K_δ} again as a function of vectors, we have

$$\sup_{\xi \in S^1} |\rho_{K_\delta}(\xi) - \rho_{K_\delta}(-\xi)| \leq 2 \left(\exp\left[\left(6\pi + \frac{32\pi}{\sqrt{3}\, r} \right) \sqrt{\varepsilon} \right] - 1 \right) R.$$

The inequality $e^t - 1 \leq 2t$ is valid when $0 < t < 1$; therefore, if

$$\varepsilon < \left(\frac{\sqrt{3}\, r}{6\sqrt{3}\pi r + 32\pi} \right)^2,$$

then

$$\sup_{\xi \in S^1} |\rho_{K_\delta}(\xi) - \rho_{K_\delta}(-\xi)| \leq \left(24\pi + \frac{128\pi}{\sqrt{3}\, r} \right) R\sqrt{\varepsilon}.$$

Consider the case when $n > 2$. For K_δ with $p = 1$, Equation (2.2) becomes

$$I_2 \left(\|x\|_{K_\delta}^{-n+2} - \|-x\|_{K_\delta}^{-n+2} \right)(\xi) = -2\pi i\,(n-2) A'_{K_\delta,\xi}(0),$$

so

$$\begin{aligned} &\left\| I_2 \left(\|x\|_{K_\delta}^{-n+2} - \|x\|_{-K_\delta}^{-n+2} \right) \right\|_2 = 2\pi(n-2) \left(\int_{S^{n-1}} \left| A'_{K_\delta,\xi}(0) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \widetilde{C}(n) \left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right)^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} \end{aligned}$$

by Lemma 3.9. Finally, by Lemma 3.10,

$$\rho(K_\delta, -K_\delta) \leq C(n) \frac{R^2}{r^{\frac{3n-3}{n+1}}} \left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right)^{\frac{1}{n+1}} \varepsilon^{\frac{1}{2(n+1)}}$$

when $n = 3$ or 4, and

$$\begin{aligned} \rho(K_\delta, -K_\delta) \leq C(n) &\left[\left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \sqrt{\varepsilon} + \frac{R^{2(n-1)}}{r^2} \right]^{\frac{n-4}{(n-2)(n+1)}} \\ &\cdot \left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right)^{\frac{2}{(n-2)(n+1)}} \frac{R^2 \varepsilon^{\frac{1}{(n-2)(n+1)}}}{r^{\frac{3n-3}{n+1}}} \end{aligned}$$

when $n \geq 5$, where $C(n) > 0$ are constants depending on the dimension. $\square$

We now present the proof of our second stability result.

Proof of Theorem 3.4. Apply Lemma 3.7 to K and L; let $\{K_\delta\}_{0<\delta<1}$ and

$\{L_\delta\}_{0<\delta<1}$ be the resulting families of smooth convex bodies. For each δ, define the constant

$$\varepsilon_\delta := \sup_{\xi \in S^{n-1}} \left| A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{K,\xi}(0) \right| + \sup_{\xi \in S^{n-1}} \left| A^{(p)}_{L_\delta,\xi}(0) - A^{(p)}_{L,\xi}(0) \right| + \varepsilon.$$

Defining the auxiliary function

$$f_\delta(\xi) := \|\xi\|_{K_\delta}^{-n+1+p} - \|\xi\|_{L_\delta}^{-n+1+p},$$

we have

$$\begin{aligned}
&\cos\left(\frac{p\pi}{2}\right) I_{1+p}\big(f_\delta(x) + f_\delta(-x)\big)(\xi) + i \sin\left(\frac{p\pi}{2}\right) I_{1+p}\big(f_\delta(x) - f_\delta(-x)\big)(\xi) \\
&= 2\pi(n-1-p)\Big(A^{(p)}_{K_\delta,\,\xi}(0) - A^{(p)}_{L_\delta,\,\xi}(0)\Big)
\end{aligned}$$

from Equation (2.2). The function of ξ on the left side of this equality is split into its even and odd parts, because I_{1+p} preserves even and odd symmetry. Therefore,

$$\begin{aligned}
&\frac{\cos\left(\frac{p\pi}{2}\right)}{\pi(n-1-p)} I_{1+p}\big(f_\delta(x) + f_\delta(-x)\big)(\xi) \\
&\qquad = \Big(A^{(p)}_{K_\delta,\,\xi}(0) - A^{(p)}_{L_\delta,\,\xi}(0)\Big) + \Big(A^{(p)}_{K_\delta,\,-\xi}(0) - A^{(p)}_{L_\delta,\,-\xi}(0)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{i\sin\left(\frac{p\pi}{2}\right)}{\pi(n-1-p)} I_{1+p}\big(f_\delta(x) - f_\delta(-x)\big)(\xi) \\
&\qquad = \Big(A^{(p)}_{K_\delta,\,\xi}(0) - A^{(p)}_{L_\delta,\,\xi}(0)\Big) - \Big(A^{(p)}_{K_\delta,\,-\xi}(0) - A^{(p)}_{L_\delta,\,-\xi}(0)\Big)
\end{aligned}$$

By the definition of ε_δ,

$$\begin{aligned}
\Big|I_{1+p}\big(2f_\delta\big)(\xi)\Big| &\leq \Big|I_{1+p}\Big(f_\delta(x) + f_\delta(-x)\Big)(\xi)\Big| + \Big|I_{1+p}\Big(f_\delta(x) - f_\delta(-x)\Big)(\xi)\Big| \\
&\leq \frac{2\pi(n-1-p)}{\cos\left({}^{p\pi}/_{2}\right)}\,\varepsilon_\delta + \frac{2\pi(n-1-p)}{\sin\left({}^{p\pi}/_{2}\right)}\,\varepsilon_\delta,
\end{aligned}$$

which implies

$$\|I_{1+p}(f_\delta)\|_2 \le \pi\sqrt{\omega_n}\,(n-1-p)\Big(\big|\sec\left({}^{p\pi}\!/_2\right)\big| + \big|\csc\left({}^{p\pi}\!/_2\right)\big|\Big)\varepsilon_\delta.$$

Both K_δ and L_δ are contained in a ball of radius $2R$ when $0<\delta<1/2$, and contain a ball of radius $r/2$. It now follows from Lemma 3.10 that

$$\rho(K_\delta, L_\delta) \le C(n,p)\, R^2 r^{\frac{-3n+1+2p}{n+1}}\, \varepsilon_\delta^{\frac{2}{n+1}}$$

when $n \le 2p+2$, and

$$\rho(K_\delta, L_\delta) \le C(n,p)\, R^2 r^{\frac{-3n+1+2p}{n+1}} \left(\varepsilon_\delta^2 + \frac{R^{2(n-p)}}{r^2}\right)^{\frac{n-2-2p}{(n-2p)(n+1)}} \varepsilon_\delta^{\frac{4}{(n-2p)(n+1)}}$$

when $n > 2p+2$, where $C(n,p)>0$ are constants depending on the dimension and p. Finally, the bounds in the theorem statement follow from the observations

$$\rho(K,L) \le \lim_{\delta\to 0}\Big(\rho(K,K_\delta) + \rho(L,L_\delta) + \rho(K_\delta,L_\delta)\Big) = \lim_{\delta\to 0}\rho(K_\delta, L_\delta),$$

and $\lim_{\delta\to 0}\varepsilon_\delta = \varepsilon$. $\square$

Chapter 4

On Convex Intersection Bodies and Unique Determination Problems for Convex Bodies

The content of this chapter comes from my paper [50]. The main goal is to prove that Meyer and Reisner's convex intersection body does not uniquely determine a convex body up to congruency. I do this by constructing two convex bodies, one which is not centrally symmetric and one which is origin-symmetric, whose convex intersection bodies coincide; recall Theorem 1.3.

First, in Section 4.2, I adapt the method of construction used in [16] so that it generates counter-examples to a more general question of unique determination for convex bodies. Roughly, if we "smoothly" associate every convex body $K \subset \mathbb{R}^n$ with an origin-symmetric star body $\widetilde{K} \subset \mathbb{R}^n$ such that $\widetilde{K} = IK$ whenever K is origin-symmetric, then $\widetilde{K}$ does not uniquely determine K; for the precise statement, see Theorem 4.2. In Section 4.3, I prove Theorem 1.3 by showing convex intersection bodies satisfy the hypotheses of Theorem 4.2. Finally, I give some concluding remarks in Section 4.4.

4.1 Preliminaries and Additional Notation

In this section, we collect those notations and background materials specific to this chapter.

The Legendre polynomial of dimension $n \in \mathbb{N}$, $n > 1$, and degree three is given by

$$P_3^n(t) = \left(\frac{n+2}{n-1}\right) t^3 - \left(\frac{3}{n-1}\right) t;$$

equation (3.3.18) in [19] gives the general formula for Legendre polynomials.

For $f : S^{n-1} \to \mathbb{C}$, we will let $\tilde{f}$ denote its homogeneous extension to $\mathbb{R}^n \backslash \{0\}$ of degree zero. We will say $f \in C^k(S^{m-1} \times S^{n-1})$ if

$$f\left(\frac{x}{|x|}, \frac{y}{|y|}\right) \in C^k\big(\mathbb{R}^m \backslash \{0\} \times \mathbb{R}^n \backslash \{0\}\big).$$

Let $r = r(\phi) \in C^2(S^1)$ be a planar curve in polar coordinates. Its curvature is then given by the well-known formula

$$\frac{2(r')^2 - r \cdot r'' + r^2}{\big((r')^2 + r^2\big)^{\frac{3}{2}}}; \tag{4.1}$$

see, for example, formula (0.41) in [14].

The *Santaló point* $s = s(K) \in \mathbb{R}^n$ of a convex body $K \subset \mathbb{R}^n$ is the unique point such that

$$\mathrm{vol}_n\left(K^{*s}\right) = \min\left\{\mathrm{vol}_n\left(K^{*y}\right) \middle| y \in \mathrm{int}(K)\right\};$$

see [48] or [49].

In this chapter, $\mathcal{S}^n$ is the collection of star bodies in $\mathbb{R}^n$, $\mathcal{K}^n$ is the collection of convex bodies in $\mathbb{R}^n$, and $\mathcal{K}_o^n$ is the collection of convex bodies with the origin in the interior. The *Hausdorff metric* is defined on $\mathcal{K}^n$ by

$$d_H(K, L) = \max_{\xi \in S^{n-1}} \big|h_K(\xi) - h_L(\xi)\big|, \qquad K,\, L \in \mathcal{K}^n.$$

The following lemma appears as Proposition 1 in [23].

Lemma 4.1. *The Santaló map* $s : (\mathcal{K}^n, d_H) \to \mathbb{R}^n$ *is continuous. Furthermore, for every convex body* K*, there exist positive constants* $C = C(K)$ *and* $\delta = \delta(K)$ *so that*

$$\big|s(K) - s(L)\big| \le C\, d_H(K, L)$$

whenever $L \in \mathcal{K}^n$ *and* $d_H(K, L) \le \delta$.

Using the Santaló point, the radial function of $CI(K)$ may be rewritten as

$$\rho_{CI(K)}(\xi) = \mathrm{vol}_{n-1}\left[\left(K^{*g}\big|\xi^{\perp}\right)^{*s(\xi)}\right],$$

where $s(\xi) = s\left(K^{*g}\big|\xi^{\perp}\right)$. For a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior and a $y \in \mathrm{int}(K)$, the volume of K^{*y} is given by

$$\mathrm{vol}_n(K^{*y}) = \int_{K^*} \frac{1}{\big(1 - \langle y, x\rangle\big)^{n+1}}\, dx;$$

see, for example, Lemma 3 in [38]. If the centroid of K is at the origin, then

$$\begin{aligned} \rho_{CI(K)}(\xi) = \mathrm{vol}_{n-1}\left[\left(K^{*}\big|\xi^{\perp}\right)^{*s(\xi)}\right] &= \int_{(K^*|\xi^{\perp})^*} \frac{1}{\big(1 - \langle s(\xi), x\rangle\big)^{n}}\, dx \\ &= \int_{K\cap\xi^{\perp}} \frac{1}{\big(1 - \langle s(\xi), x\rangle\big)^{n}}\, dx, \end{aligned} \tag{4.2}$$

using the fact that $(K^*|\xi^{\perp})^* = K \cap \xi^{\perp}$ (e.g. equation 0.38 in [14]).

4.2 The General Method of Construction

We can consider intersection bodies, cross-section bodies, and convex intersection bodies as maps from $\mathcal{K}_o^n$ to $\mathcal{S}^n$:

$$K \mapsto IK, \quad K \mapsto CK, \quad K \mapsto CI(K) \quad \text{for } K \in \mathcal{K}_o^n.$$

The questions of unique determination described in the introduction are equivalent to asking whether these maps are injective. In [16], the constructed counter-example is specifically for Klee's problem, and the methods of construction are not stated in general terms. However, these methods can be applied to any map $K \mapsto \widetilde{K}$ from $\mathcal{K}_o^n$ to $\mathcal{S}^n$ which shares certain key properties with the maps above.

Suppose $K \mapsto \widetilde{K}$ has the following properties:

- $\widetilde{K}$ is always origin-symmetric.
- $IK = \widetilde{K}$ for all origin-symmetric $K \in \mathcal{K}_o^n$.
- There is a sequence $\{K_m\} \subset \mathcal{K}_o^n$ of non-centrally-symmetric convex bodies such that $\{\widetilde{K}_m\}$ are infinitely smooth, with

$$\lim_{m\to\infty} \left\|\rho_{\widetilde{K}_m} - a\right\|_{C^k(S^{n-1})} = 0 \quad \forall\, k \in \mathbb{N},$$

  where $a > 0$ is a constant independent of k.

We can think of this last property as a type of smoothness for the map $K \mapsto \widetilde{K}$.

The above three properties ensure $K \mapsto \widetilde{K}$ is not injective:

Theorem 4.2. *There exists a non-centrally-symmetric convex body K and an (infinitely smooth) origin-symmetric convex body L such that $\widetilde{K} = \widetilde{L}$. Namely, take $L = L_m$ defined by*

$$\rho_{L_m} = \left[(n-1)\mathcal{R}^{-1}\rho_{\widetilde{K}_m}\right]^{\frac{1}{n-1}},$$

and $K = K_m$, for large enough m.

At its core, Theorem 4.2 asserts ρ_{L_m} is positive with non-negative curvature for large enough m. The main idea behind the proof is that smooth convergence of functions on S^{n-1} implies smooth convergence of the distributional Fourier transforms of their homogeneous extensions. This intuition, clarified in the following lemma and corollary, was used in [16]. For convenience, we present proofs of these auxiliary results. Lemma 4.3 is a simple generalization of Lemma 3.1 in [55].

Lemma 4.3. *Let $f \in C_e^\infty(S^{n-1})$. For any $k \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{R}$ with $0 < q < 2k+1$, we have*

$$
\begin{aligned}
&|x|^{2k} \left(f\left(\frac{y}{|y|}\right) |y|^{-n+q} \right)^{\wedge}(x) \\
&= \begin{cases}
\frac{(-1)^{k+1}\pi}{2\Gamma(2k-q+1)\sin\left(\frac{\pi(2k-q)}{2}\right)} \\
\times \int_{S^{n-1}} |\langle x,\xi\rangle|^{2k-q} \Delta^k \left[f\left(\frac{y}{|y|}\right) |y|^{-n+q} \right](\xi)\, d\xi \text{ if } q \text{ is not even,} \\
\frac{(-1)^{\frac{q}{2}}}{(2k-q)!} \Bigg[-\int_{S^{n-1}} \langle x,\xi\rangle^{2k-q} \log\left|\left\langle \frac{x}{|x|},\xi\right\rangle\right| \Delta^k \left[f\left(\frac{y}{|y|}\right) |y|^{-n+q} \right](\xi)\, d\xi \\
+ \int_{S^{n-1}} \langle x,\xi\rangle^{2k-q} \Delta^k \left[f\left(\frac{y}{|y|}\right) |y|^{-n+q} \log|y| \right](\xi)\, d\xi \Bigg] \text{ if } q \text{ is even,}
\end{cases}
\end{aligned}
$$

for all $x \in \mathbb{R}^n \backslash \{0\}$.

Proof. The formula for when q is not even is given by Lemma 3.16 in [25].

Suppose q is even. We will use the first formula in the lemma statement to calculate

$$\lim_{p \to q} |x|^{2k} \left(f\left(\frac{y}{|y|}\right) |y|^{-n+p} \right)^{\wedge}(x). \tag{4.3}$$

Indeed, as p approaches q, both the numerator and denominator of

$$\frac{(-1)^{k+1}\pi \int_{S^{n-1}} |\langle x,\xi\rangle|^{2k-p} \Delta^k \left[f\left(\frac{y}{|y|}\right) |y|^{-n+p} \right](\xi)\, d\xi}{2\Gamma(2k-p+1)\sin\left(\frac{\pi(2k-p)}{2}\right)}$$

approach zero. This is clear for the denominator; we now prove this is also true for the numerator. That is, we need to show

$$\int_{S^{n-1}} g\Delta^k h\, d\xi = 0,$$

where $g(\xi) = \langle x,\xi\rangle^a$, with $a = 2k-q$, and h is the homogeneous extension of

f of degree $-n+q$. Equation (2.1) gives

$$\begin{aligned}
&\Delta^k h(\xi) = \Delta(\Delta^{k-1}h)(\xi) \\
&= \Delta_o(\Delta^{k-1}h)(\xi) + (-n-2k+q+2)(-2k+q)\Delta^{k-1}h(\xi), \quad \xi \in S^{n-1},
\end{aligned}$$

since $\Delta^{k-1}h$ is homogeneous of degree $-n-2k+q+2$. Recalling that the spherical Laplacian is self-adjoint, we then have

$$\begin{aligned}
&\int_{S^{n-1}} g\Delta^k h\,d\xi \\
&= \int_{S^{n-1}} (\Delta_o g)\Delta^{k-1}h\,d\xi + (-n-2k+q+2)(-2k+q)\int_{S^{n-1}} g\Delta^{k-1}h\,d\xi
\end{aligned}$$

But again by equation (2.1),

$$\Delta_o g(\xi) = \Delta g(\xi) - (2k-q)(n+2k-q-2)g(\xi), \quad \xi \in S^{n-1},$$

so

$$\int_{S^{n-1}} g\Delta^k h\,d\xi = \int_{S^{n-1}} (\Delta g)\Delta^{k-1}h\,d\xi.$$

It is clear that we can continue, repeatedly reducing the iterations of the Laplace transform on h by transferring them to g. Continuing $a/2$ times, we get

$$\begin{aligned}
\int_{S^{n-1}} g\Delta^k h\,d\xi &= \int_{S^{n-1}} (\Delta^{a/2+1}g)\,\Delta^{k-a/2-1}h\,d\xi \\
&= \int_{S^{n-1}} 0\cdot\Delta^{k-a/2-1}h\,d\xi = 0 \qquad (4.4)
\end{aligned}$$

because g is a polynomial in ξ of degree a.

Using l'Hospital's rule, we find that the limit (4.3) is equal to

$$\frac{(-1)^{\frac{q}{2}}}{(2k-q)!}\Bigg[-\int_{S^{n-1}} g(\xi)\log|\langle x,\xi\rangle|\Delta^k h(\xi)\,d\xi \\ +\int_{S^{n-1}} g(\xi)\Delta^k\Big[h(y)\log|y|\Big](\xi)\,d\xi\Bigg].$$

This expression is equal to the formula given in the lemma statement for even q; use the identity

$$\log|\langle x,\xi\rangle| = \log|\langle x/|x|,\xi\rangle| + \log|x|$$

and equation (4.4) to verify. Finally, we note that

$$|x|^{2k}\left(f\left(\frac{y}{|y|}\right)|y|^{-n+q}\right)^{\wedge}(x) = \lim_{p\to q}|x|^{2k}\left(f\left(\frac{y}{|y|}\right)|y|^{-n+p}\right)^{\wedge}(x);$$

this follows from Lemma 3.11 in [25]. □

As a consequence of Lemma 4.3, we have the following result; it is a generalization of Corollary 3.17 in [25].

Corollary 4.4. *Let $\{f_m\}\subset C_e^\infty(S^{n-1})$ be a sequence of functions converging to $f\in C_e^\infty(S^{n-1})$ with respect to $\|\cdot\|_{C^k(S^{n-1})}$, for every $k\in\mathbb{N}$. Then for every $q>0$,*

$$\lim_{m\to\infty}\left\|\left[f_m\left(\frac{x}{|x|}\right)|x|^{-n+q}\right]^{\wedge}-\left[f\left(\frac{x}{|x|}\right)|x|^{-n+q}\right]^{\wedge}\right\|_{C^k(S^{n-1})}=0\quad\forall\ k\in\mathbb{N}.$$

Proof. Observe that

$$\left|\left\langle x,\frac{y}{|y|}\right\rangle\right|^{2l-q}\quad\text{and}\quad\left|\left\langle x,\frac{y}{|y|}\right\rangle\right|^{2l-q}\log\left|\left\langle\frac{x}{|x|},\frac{y}{|y|}\right\rangle\right|$$

extend to k - smooth functions on $\mathbb{R}^n\backslash\{0\}\times\mathbb{R}^n\backslash\{0\}$, for large enough $l\in\mathbb{N}$.

For any $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq k$, we can then calculate

$$D^\alpha \left[f_m \left(\frac{x}{|x|} \right) |x|^{-n+q} \right]^\wedge \quad \text{and} \quad D^\alpha \left[f \left(\frac{x}{|x|} \right) |x|^{-n+q} \right]^\wedge$$

using the formulas from Lemma 4.3, where the derivatives will pass through the integrals. □

Proof of Theorem 4.2. We already know $\rho_{\widetilde{K}_m}$ is even and infinitely smooth, so $\mathcal{R}^{-1}\rho_{\widetilde{K}_m}$ is well-defined. Extending $\rho_{\widetilde{K}_m}$ to $\mathbb{R}^n \backslash \{0\}$ with its natural homogeneity of degree -1, it then follows from Lemma 2.1, Lemma 4.3, and the identity

$$(\rho_{\widetilde{K}_m})^{\wedge\wedge} = (2\pi)^n \rho_{\widetilde{K}_m}$$

that

$$\mathcal{R}^{-1}\rho_{\widetilde{K}_m}(\xi) = \frac{\pi}{(2\pi)^n} (\rho_{\widetilde{K}_m})^\wedge(\xi) \quad \forall\, \xi \in S^{n-1}.$$

Since

$$\lim_{m\to\infty} \|\rho_{\widetilde{K}_m} - a\|_{C^k(S^{n-1})} = 0$$

for every $k \in \mathbb{N}$, Corollary 4.4 implies

$$\lim_{m\to\infty} \left\| (\rho_{\widetilde{K}_m})^\wedge - (a|x|^{-1})^\wedge \right\|_{C^2(S^{n-1})} = 0.$$

In particular, this shows $(\rho_{\widetilde{K}_m})^\wedge$ converges uniformly to a positive constant on S^{n-1}. Therefore, ρ_{L_m} defines a star body for large enough m. We also see that the first and second order angular derivatives of $(\rho_{\widetilde{K}_m})^\wedge$ converge uniformly to zero; the same is true for ρ_{L_m}. It then easily follows from formula (4.1) that the restriction of ρ_{L_m} to any two-dimensional plane H has positive curvature, which means $L_m \cap H$ is convex; see the proof of Lemma 4.5 for a similar and more explicit argument. We can conclude L_m is an origin-symmetric convex body, for large enough m.

Finally, from the definition of L_m and the polar coordinate formula for volume, we have

$$\rho_{\widetilde{L}_m}(\xi) = \rho_{IL_m}(\xi) = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \rho_{L_m}^{n-1}(\eta)\, d\eta = \rho_{\widetilde{K}_m}(\xi) \quad \forall\, \xi \in S^{n-1}.$$

□

4.3 The Construction for Convex Intersection Bodies

Recall that $CI(K)$ is origin-symmetric for every $K \in \mathcal{K}_o^n$, with $CI(K) = IK$ whenever K is origin-symmetric. To apply Theorem 4.2 to the map $K \mapsto CI(K)$, we need to find a sequence $\{K_m\} \subset \mathcal{K}_o^n$ of non-centrally-symmetric convex bodies such that

$$\lim_{m\to\infty} \left\|\rho_{CI(K_m)} - a\right\|_{C^k(S^{n-1})} = 0 \quad \forall\, k \in \mathbb{N},$$

where $a > 0$ is a constant independent of k. We do this through the following series of lemmas.

Lemma 4.5. *Define the function*

$$\rho_{K_\varepsilon}(\xi) = \Big(1 + \varepsilon P(\langle \xi, e_1\rangle)\Big)^{\frac{1}{n+1}}$$

for $\xi \in S^{n-1}$, where $P = P_3^n$ and $\varepsilon > 0$. For sufficiently small $\varepsilon > 0$, ρ_{K_ε} is the radial function of an infinitely smooth convex body K_ε which is a body of rotation about the x_1 - axis, whose centroid is at the origin, and which is not centrally-symmetric.

Proof. Clearly, the homogeneous extension of ρ_{K_ε} of degree -1 is positive and infinitely smooth on $\mathbb{R}^n\backslash\{0\}$, once $\varepsilon > 0$ is smaller that the maximum value of $|P|$ on the interval $[-1, 1]$. So, ρ_{K_ε} defines a infinitely smooth star body, K_ε.

Given that $\langle\,\cdot\,, e_1\rangle$ is rotationally-symmetric about the x_1 - axis, K_ε is a star body of rotation about this axis. It is then necessary that the centroid of

K_ε lies on the x_1 - axis, with its x_1 coordinate given by

$$\begin{aligned}\frac{1}{\mathrm{vol}_n(K_\varepsilon)}\int_{K_\varepsilon} x_1\,dx &= \frac{1}{\mathrm{vol}_n(K_\varepsilon)}\int_{S^{n-1}}\int_0^{\rho_{K_\varepsilon}(\xi)} r^n\,\xi_1\,dr\,d\xi\\ &= \frac{1}{(n+1)\mathrm{vol}_n(K_\varepsilon)}\int_{S^{n-1}}\Big(1+\varepsilon P(\langle\xi,e_1\rangle)\Big)\,\xi_1\,d\xi\\ &= \frac{\varepsilon}{(n+1)\mathrm{vol}_n(K_\varepsilon)}\int_{S^{n-1}} P(\langle\xi,e_1\rangle)\,\langle\xi,e_1\rangle\,d\xi.\end{aligned}$$

This last integral is equal to zero, because $P(\langle\,\cdot\,,e_1\rangle)$ and $\langle\,\cdot\,,e_1\rangle$ are spherical harmonics of different degrees. Therefore, K_ε has its centroid at the origin.

If we show that the restriction of K_ε to the x_1, x_2 - plane is convex, then this will imply K_ε is convex, since it is a body of rotation. Letting θ be the angle with the positive x_1 - axis,

$$r(\theta) := \Big(1+\varepsilon P(\cos\theta)\Big)^{\frac{1}{n+1}}$$

gives the boundary of K_ε in the x_1, x_2 - plane in polar coordinates. Recall formula (4.1). The restriction of K_ε will be convex if the curvature of r is positive for all θ, so we need to show that the numerator of (4.1) is positive. This follows from the observations that r is bounded away from zero, every term of $r\cdot r''$ is multiplied by a factor of ε, and so

$$2(r')^2 - r\cdot r'' + r^2 \geq r^2 - r\cdot r'' > 0$$

for small enough $\varepsilon > 0$.

If K_ε has a center of symmetry, then it must lie on the x_1 - axis, and r must have the same curvature at $\theta = 0$ and $\theta = \pi$. Let

$$f(t) := \frac{(n-1)+(n+2)t}{(n-1)(1+t)^{\frac{n+2}{n+1}}}.$$

Using the facts $P(\pm 1) = \pm 1$ and $P'(\pm 1) = 3(n+1)/(n-1)$, it is easy to verify that the curvatures at $\theta = 0, \pi$ are, respectively, $f(\varepsilon)$ and $f(-\varepsilon)$. However, there is some open interval containing $t = 0$ on which f strictly increases, since $f'(0) > 0$. So $f(\varepsilon) \neq f(-\varepsilon)$ for small enough $\varepsilon > 0$. $\square$

For $\phi \in S^{n-1}$ lying in the x_1, x_2 - plane, let

$$v = v(\phi) \in S^{n-1} \tag{4.5}$$

be the unit vector obtained by rotating ϕ counter-clockwise about the origin in the x_1, x_2 - plane by $\pi/2$ radians. It is clear that $K_\varepsilon^{*g}\big|\phi^\perp = K_\varepsilon^*\big|\phi^\perp$ is a body of rotation about the axis in the direction v. The Santaló point of a convex body is affinely invariant, so we may uniquely define $s_\varepsilon = s_\varepsilon(\phi) \in \mathbb{R}$ so that

$$s\left(K_\varepsilon^*\big|\phi^\perp\right) = s_\varepsilon v. \tag{4.6}$$

We will show s_ε is an infinitely smooth function on S^1 with its derivatives absolutely bounded by $\varepsilon > 0$. This is done with the help of the following lemmas and corollary.

Lemma 4.6. *Let $f \in C^\infty\big(S^{n-1}\big)$, and let F be the homogeneous extension of $\mathcal{R}f$ to $\mathbb{R}^n \backslash \{0\}$ of degree zero. Then, for any $k \in \mathbb{N}$ and $y \in \mathbb{R}^n \backslash \{0\}$,*

$$F(y) = \frac{C}{|y|^{2k+1}} \int_{S^{n-1}} |\langle y, \xi \rangle|^{2k+1} \Delta^{k+1} \left[|z|^{-n+1} f\left(\frac{z}{|z|}\right)\right](\xi)\, d\xi;$$

$C > 0$ is a constant depending on k. Furthermore, F is infinitely smooth on its specified domain.

Proof. Let f^e denote the even part of f on S^{n-1}. For all $y \in \mathbb{R}^n \backslash \{0\}$, we have

$$\begin{aligned} F(y) = \mathcal{R}f\left(\frac{y}{|y|}\right) &= \int_{S^{n-1}\cap(y/|y|)^\perp} f^e(\xi)\, d\xi \\ &= \frac{1}{\pi}\left[|z|^{-n+1} f^e\left(\frac{z}{|z|}\right)\right]^\wedge \left(\frac{y}{|y|}\right) \\ &= \frac{|y|}{\pi}\left[|z|^{-n+1} f^e\left(\frac{z}{|z|}\right)\right]^\wedge (y) \end{aligned}$$

using Lemma 2.1, and

$$
\begin{aligned}
&\frac{|y|}{\pi}\left[|z|^{-n+1} f^e\left(\frac{z}{|z|}\right)\right]^{\wedge}(y) \\
&= \frac{C}{|y|^{2k+1}} \int_{S^{n-1}} |\langle y,\xi\rangle|^{2k+1} \Delta^{k+1}\left[|z|^{-n+1} f^e\left(\frac{z}{|z|}\right)\right](\xi)\, d\xi \\
&= \frac{C}{|y|^{2k+1}} \int_{S^{n-1}} |\langle y,\xi\rangle|^{2k+1} \Delta^{k+1}\left[|z|^{-n+1} f\left(\frac{z}{|z|}\right)\right](\xi)\, d\xi
\end{aligned}
$$

using Lemma 4.3.

Combining these equations, we get the alternate definition of F in the lemma statement. Noting that $|\langle y,z\rangle|^{2k+1}$ is at least k - smooth on $\mathbb{R}^{2n}$, and

$$\Delta^{k+1}\left[|z|^{-n+1} f\left(\frac{z}{|z|}\right)\right]$$

is infinitely smooth on $\mathbb{R}^n\backslash\{0\}$, it follows that F is k - smooth on $\mathbb{R}^n\backslash\{0\}$. However, $k \in \mathbb{N}$ is arbitrary, so F is infinitely smooth. $\square$

We will use Lemma 4.6 in the form of the following immediate corollary.

Corollary 4.7. *Let $f \in C^\infty\big(S^{m-1} \times S^{n-1}\big)$, and define*

$$F(x,y) := \left[\mathcal{R} f\left(\frac{x}{|x|},\cdot\right)\right]\left(\frac{y}{|y|}\right) = \int_{S^{n-1}\cap(y/|y|)^\perp} f\left(\frac{x}{|x|},\xi\right) d\xi$$

for $(x,y) \in \mathbb{R}^m\backslash\{0\} \times \mathbb{R}^n\backslash\{0\}$. Then, for any $k \in \mathbb{N}$,

$$F(x,y) = \frac{C}{|y|^{2k+1}} \int_{S^{n-1}} |\langle y,\xi\rangle|^{2k+1} \Delta_z^{k+1}\left[|z|^{-n+1} f\left(\frac{x}{|x|},\frac{z}{|z|}\right)\right](\xi)\, d\xi;$$

$C > 0$ is a constant depending on k. Furthermore, F is infinitely smooth on its specified domain.

Lemma 4.8. *Let $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, let H be a k - dimensional subspace of $\mathbb{R}^n$, let $f : S^{n-1} \to \mathbb{R}$ be an odd function absolutely bounded by $M > 0$, and let $g : S^{n-1} \to \mathbb{R}$ be absolutely bounded by $\varepsilon > 0$. If K_ε is the convex body defined* *in Lemma 4.5, then*

$$\left| \int_{S^{n-1}\cap H} f(\xi) \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{m_1}}{(1-r\,g(\xi))^{m_2}}\, dr\, d\xi \right| \le C\varepsilon$$

for small enough $\varepsilon > 0$. The constant $C \ge 0$ depends on M, m_1, m_2, and n.

Proof. Because f is odd, the even part of

$$h(\xi) := \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{m_1}}{(1-r\,g(\xi))^{m_2}}\, dr$$

vanishes when integrating against f over $S^{n-1}\cap H$. Recall

$$\rho_{K_\varepsilon}(\xi) = \big(1+\varepsilon P(\langle \xi, e_n\rangle)\big)^{\frac{1}{n+1}},$$

where P is an odd polynomial. Let N be the maximum value of $|P|$ on the interval $[-1,1]$. It follows that

$$\begin{aligned}
&\left| \int_{S^{n-1}\cap H} f(\xi) h(\xi)\, d\xi \right| \le \frac{M}{2}\int_{S^{n-1}\cap H} \left|h(\xi)-h(-\xi)\right| d\xi \\
&\le \frac{M}{2}\int_{S^{n-1}\cap H} \left(\frac{(1+N\varepsilon)^{m_1+1}}{(m_1+1)(1-2\varepsilon)^{m_2}} - \frac{(1-N\varepsilon)^{m_1+1}}{(m_1+1)(1+2\varepsilon)^{m_2}} \right) d\xi \\
&\le g(\varepsilon) := \frac{M\omega_k}{2(m_1+1)} \left(\frac{(1+N\varepsilon)^{m_1+1}}{(1-2\varepsilon)^{m_2}} - \frac{(1-N\varepsilon)^{m_1+1}}{(1+2\varepsilon)^{m_2}} \right).
\end{aligned}$$

The function g is smooth in a neighbourhood of $0 \in \mathbb{R}$ with $g(0) = 0$, giving the desired result. $\square$

We are now ready to prove the smoothness result for s_ε.

Lemma 4.9. *Let s_ε be the function on S^1 defined by equation (4.6), where S^1 is the intersection of S^{n-1} with the x_1, x_2 - plane. For small enough $\varepsilon > 0$, $s_\varepsilon \in C^\infty(S^1)$. Furthermore, for any $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^2_{\ge 0}$ with $\alpha_1+\alpha_2 = k$,*

$$|D^\alpha \tilde{s}_\varepsilon(\xi)| \le C(k,n)\varepsilon \qquad \forall\, \xi \in S^1. \tag{4.7}$$

The constant $C(k,n) \ge 0$ depends only on k and the dimension n.

Proof. Again let N be the maximum value of $|P|$ on the interval $[-1,1]$. Define the function $g(t) = (1+Nt)^{\frac{1}{n+1}}$. We then have the containment $B^n(g(-\varepsilon)) \subset K_\varepsilon \subset B^n(g(\varepsilon))$; therefore

$$\begin{aligned} d_H\left(K_\varepsilon^*\middle|\phi^\perp, B^{n-1}(1)\right) &= \max_{\xi\in S^{n-1}\cap\phi^\perp}\left|h_{K_\varepsilon^*|\phi^\perp}(\xi)-1\right| \le \max_{\xi\in S^{n-1}}\left|h_{K_\varepsilon^*}(\xi)-1\right| \\ &\le C\max\{|g(\varepsilon)-g(0)|,\, |g(-\varepsilon)-g(0)|\} \\ &\le C\varepsilon \quad \forall\, \phi\in S^1, \end{aligned}$$

because g is smooth in a neighbourhood of $0\in\mathbb{R}$. With the observation that the Santaló point of a Euclidean ball is always at its center, Lemma 4.1 implies

$$|s_\varepsilon(\phi)| = \left|s\left(K_\varepsilon^*\middle|\phi^\perp\right) - s\left(B^{n-1}(1)\right)\right| \le C\varepsilon \quad \forall\, \phi\in S^1,$$

for small enough $\varepsilon>0$.

Next, we show s_ε is infinitely smooth. The centroid of K_ε is at the origin, so we may use equation (4.2) to get

$$\rho_{CI(K_\varepsilon)}(\phi) = \int_{K_\varepsilon\cap\phi^\perp}\frac{1}{(1-\langle s_\varepsilon v, x\rangle)^n}dx, \quad \phi\in S^1\subset S^{n-1}, \tag{4.8}$$

where $v=v(\phi)$ is the vector-valued function on S^1 defined by equation (4.5). Since K_ε is contained in $B^n(2)$ for small enough $\varepsilon>0$, the function

$$\begin{aligned} G(\phi,t) &= \int_{K_\varepsilon\cap\phi^\perp}\frac{1}{\left(1-\langle v,x\rangle t\right)^n}\,dx \\ &= \int_{S^{n-1}\cap\phi^\perp}\int_0^{\rho_{K_\varepsilon}(\xi)}\frac{r^{n-2}}{\left(1-\langle v, r\xi\rangle t\right)^n}\,dr\,d\xi \end{aligned}$$

is defined for $\phi\in S^1$ and $-1/2<t<1/2$. Because v is infinitely smooth on S^1 and ρ_{K_ε} is infinitely smooth on S^{n-1}, it follows from Corollary 4.7 that G is also so on its specified domain. For small enough $\varepsilon>0$, we know that K_ε^* contains $B^n(1/2)$, and $-1/2<s_\varepsilon(\phi)<1/2$ for all ϕ. The uniqueness of the Santaló point then implies s_ε is the unique real number with $-1/2<s_\varepsilon<1/2$

and

$$0 = \frac{\partial G}{\partial t}\bigg|_{t=s_\varepsilon} = \int_{K_\varepsilon \cap \phi^\perp} \frac{n\langle v, x\rangle}{(1-\langle v, x\rangle s_\varepsilon)^{n+1}} dx.$$

For all $\phi \in S^1$, we have

$$\frac{\partial^2 G}{\partial t^2}\bigg|_{t=s_\varepsilon} = \int_{K_\varepsilon \cap \phi^\perp} \frac{n(n+1)\langle v, x\rangle^2}{(1-\langle v, x\rangle s_\varepsilon)^{n+2}} dx > 0,$$

so the Implicit Function Theorem implies $s_\varepsilon \in C^\infty(S^1)$.

We will now prove inequality (4.7) for the first order partial derivatives of $\tilde{s}_\varepsilon$. From now on, suppose that $x \in \mathbb{R}^n \backslash \{0\}$ lies in the x_1, x_2 - plane. From our application of the Implicit Function Theorem, we have

$$\begin{aligned} F(x,x) &:= \int_{S^{n-1}\cap(x/|x|)^\perp} f\left(\frac{x}{|x|}, \xi\right) d\xi \\ &= \int_{K_\varepsilon \cap (x/|x|)^\perp} \frac{\langle \tilde{v}, y\rangle}{\left(1-\langle \tilde{v}, y\rangle\, \tilde{s}_\varepsilon\right)^{n+1}} dy = 0 \end{aligned} \tag{4.9}$$

where

$$f(\phi, \xi) = \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{n-1}\langle v, \xi\rangle}{\left(1 - r\langle v, \xi\rangle\, s_\varepsilon\right)^{n+1}} dr.$$

Here, $\tilde{v}$ and $\tilde{s}_\varepsilon$ are functions of $x \in \mathbb{R}^2$, while v and s_ε are functions of $\phi \in S^1$. This function $f : S^1 \times S^{n-1} \to \mathbb{C}$ is infinitely smooth. Using the expression from Corollary 4.7 to calculate the partial derivative of $F(x,x)$ with respect to x_j $(j = 1, 2)$, we find that it is equal to

$$\begin{aligned} &\int_{S^{n-1}\cap(x/|x|)^\perp} \frac{\partial}{\partial x_j}\left[f\left(\frac{x}{|x|}, \xi\right)\right] d\xi - \frac{(2k+1)x_j}{|x|^2}\int_{S^{n-1}\cap(x/|x|)^\perp} f\left(\frac{x}{|x|}, \xi\right) d\xi \\ &+ \frac{(2k+1)C}{|x|^{2k+1}} \int_{S^{n-1}} |\langle x, \xi\rangle|^{2k} \mathrm{sgn}(\langle x, \xi\rangle)\xi_j \Delta_z^{k+1}\left(|z|^{-n+1} f\left(\frac{x}{|x|}, \frac{z}{|z|}\right)\right)(\xi) d\xi. \end{aligned} \tag{4.10}$$

We will expand these integrals into several more terms; one term will be equal to $\partial \tilde{s}_\varepsilon / \partial x_j$ multiplied by a factor bounded away from zero, and the remaining terms will be bounded by an appropriate constant times $\tilde{s}_\varepsilon$ or ε. With equation (4.9) and inequality (4.7) for $\tilde{s}_\varepsilon$, this will imply inequality (4.7) for $\partial\tilde{s}_\varepsilon/\partial x_j$. Any constants mentioned will be independent of ε, when $\varepsilon > 0$ is small enough, and of the variables $x \in \mathbb{R}^n$ and $\xi \in S^{n-1}$ when present.

Consider the first integral from expression (4.10). We have

$$\begin{aligned}
&\int_{S^{n-1}\cap(x/|x|)^\perp} \frac{\partial}{\partial x_j}\left[f\left(\frac{x}{|x|},\xi\right)\right] d\xi \\
&= \int_{S^{n-1}\cap(x/|x|)^\perp} \left\langle \frac{\partial\tilde{v}}{\partial x_j},\xi\right\rangle \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{n-1}}{(1-r\langle\tilde{v},\xi\rangle\tilde{s}_\varepsilon)^{n+1}}\,dr\,d\xi \\
&+ (n+1)\int_{S^{n-1}\cap(x/|x|)^\perp} \left\langle \frac{\partial\tilde{v}}{\partial x_j},\xi\right\rangle \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{n}\langle\tilde{v},\xi\rangle}{(1-r\langle\tilde{v},\xi\rangle\tilde{s}_\varepsilon)^{n+2}}\,dr\,d\xi\ \tilde{s}_\varepsilon \\
&+ (n+1)\int_{S^{n-1}\cap(x/|x|)^\perp} \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{n}\langle\tilde{v},\xi\rangle^2}{(1-r\langle\tilde{v},\xi\rangle\tilde{s}_\varepsilon)^{n+2}}\,dr\,d\xi\ \frac{\partial\tilde{s}_\varepsilon}{\partial x_j}. \qquad (4.11)
\end{aligned}$$

Now, restrict x to S^1. By Lemma 4.8, the first term above is absolutely bounded by a constant times $\varepsilon > 0$. The second term consists of an absolutely bounded integral multiplied by s_ε. For the final term, we have $\partial\tilde{s}_\varepsilon/\partial x_j$ multiplied by

$$\begin{aligned}
&(n+1)\int_{S^{n-1}\cap(x/|x|)^\perp} \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{r^{n}\langle\tilde{v},\xi\rangle^2}{(1-r\langle\tilde{v},\xi\rangle\tilde{s}_\varepsilon)^{n+2}}\,dr\,d\xi \\
&\geq \frac{n+1}{2^{n+2}}\int_{S^{n-1}\cap(x/|x|)^\perp} \int_0^{\rho_{K_\varepsilon}(\xi)} r^n\langle\tilde{v},\xi\rangle^2\,dr\,d\xi \\
&= \frac{1}{2^{n+2}}\int_{S^{n-1}\cap(x/|x|)^\perp} \langle\tilde{v},\xi\rangle^2\rho_{K_\varepsilon}^{n+1}(\xi)\,d\xi \\
&\geq \frac{1}{2^{n+3}}\int_{S^{n-1}\cap(x/|x|)^\perp} \langle\tilde{v},\xi\rangle^2\,d\xi = \frac{\omega_{n-2}}{2^{n+3}}\int_{-1}^{1} t^2(1-t^2)^{\frac{n-4}{2}}\,dt > 0,
\end{aligned}$$

where the last equality comes from Lemma 1.3.1 in [19].

The second integral from expression (4.10),

$$-\frac{(2k+1)x_j}{|x|^2}\int_{S^{n-1}\cap(x/|x|)^\perp} f\left(\frac{x}{|x|},\xi\right) d\xi, \qquad (4.12)$$

is absolutely bounded by a constant times $\varepsilon > 0$ by Lemma 4.8.

Finally, consider the third integral from expression (4.10),

$$\frac{(2k+1)C}{|x|^{2k+1}} \int_{S^{n-1}} |\langle x,\xi\rangle|^{2k} \operatorname{sgn}(\langle x,\xi\rangle)\xi_j \Delta_z^{k+1}\left(|z|^{-n+1} f\left(\frac{x}{|x|}, \frac{z}{|z|}\right)\right)(\xi)\, d\xi. \tag{4.13}$$

Observe that $|\langle x,\xi\rangle|^{2k}\operatorname{sgn}(\langle x,\xi\rangle)\xi_j$ is even with respect to ξ; in fact, all further partial derivatives of $|\langle x,\xi\rangle|^{2k}\operatorname{sgn}(\langle x,\xi\rangle)\xi_j$ with respect to x_j ($j=1,2$) will be even with respect to ξ. Now, we need to determine the $(k+1)$-iterated Laplacian of

$$h(z) := |z|^{-n+1} f\left(\phi, \frac{z}{|z|}\right) = \frac{\langle v,z\rangle}{|z|^n} \int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n-1}}{\left(1 - r\langle v, z/|z|\rangle s_\varepsilon\right)^{n+1}}\, dr.$$

Letting $i = 1,2,\ldots,n$, we have

$$\begin{aligned}
\frac{\partial^2 h}{\partial z_i^2} &= \left(\frac{\partial^2}{\partial z_i^2}\left[\frac{\langle v,z\rangle}{|z|^n}\right]\right) \int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n-1}}{\left(1 - r\langle v, z/|z|\rangle s_\varepsilon\right)^{n+1}}\, dr \\
&\quad + 2\left(\frac{\partial}{\partial z_i}\left[\frac{\langle v,z\rangle}{|z|^n}\right]\right)\left(\frac{\partial}{\partial z_i}\int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n-1}}{\left(1 - r\langle v, z/|z|\rangle s_\varepsilon\right)^{n+1}}\, dr\right) \\
&\quad + \frac{\langle v,z\rangle}{|z|^n}\left(\frac{\partial^2}{\partial z_i^2}\int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n-1}}{\left(1 - r\langle v, z/|z|\rangle s_\varepsilon\right)^{n+1}}\, dr\right).
\end{aligned}$$

Note that partial derivatives of $\langle v,z\rangle/|z|^n$ of even orders are odd with respect to the variable z. Also,

$$\begin{aligned}
&\frac{\partial}{\partial z_i}\int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n-1}}{\left(1 - r\langle v, z/|z|\rangle s_\varepsilon\right)^{n+1}}\, dr \\
&= \frac{\tilde{\rho}_{K_\varepsilon}^{n-1}}{\left(1 - \tilde{\rho}_{K_\varepsilon}\langle v, z/|z|\rangle s_\varepsilon\right)^{n+1}} \frac{\partial \tilde{\rho}_{K_\varepsilon}}{\partial z_i} \\
&\quad + (n+1)\left(\int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n}}{\left(1 - r\langle v, z/|z|\rangle s_\varepsilon\right)^{n+2}}\, dr\right)\left(\frac{\partial}{\partial z_i}\left[\frac{\langle v,z\rangle}{|z|}\right]\right) s_\varepsilon.
\end{aligned} \tag{4.14}$$

Again, we know that s_ε and the derivatives of ρ_{K_ε} uniformly converge to zero with ε. Observe that all further derivatives of (4.14) (with respect to z) are similarly bounded. Given the even and odd symmetry mentioned above, it is clear that

$$\left| \int_{S^{n-1}} |\langle x, \xi \rangle|^{2k} \operatorname{sgn}(\langle x, \xi \rangle) \xi_j \Delta^{k+1} h(\xi)\, d\xi \right| \leq C\varepsilon.$$

We have finished bounding the integrals in expression (4.10); with equation (4.9), these bounds give inequality (4.7) for the first order partial derivatives of $\tilde{s}_\varepsilon$. Recursion gives inequality (4.7) for the partial derivatives of higher order; let us make some additional comments on this.

The first term in equation (4.11) is of a similar form as the right hand side of equation (4.9); differentiating this term with respect to x_j proceeds in a similar way. Further derivatives of the remainder of equation (4.11) result in appropriately bounded terms multiplied by derivatives of $\tilde{s}_\varepsilon$. Expression (4.12) is also of a similar form as the right hand side of equation (4.9); its derivatives are treated correspondingly. Derivatives of expression (4.13) may be bounded using the identity

$$D_x^\alpha \left[\Delta_z^{k+1} \left(|z|^{-n+1} f\left(\frac{x}{|x|}, \frac{z}{|z|} \right) \right) \right] = \Delta_z^{k+1} \left(|z|^{-n+1} D_x^\alpha \left[f\left(\frac{x}{|x|}, \frac{z}{|z|} \right) \right] \right),$$

where $\alpha \in \mathbb{Z}^2_{\geq 0}$. Indeed,

$$|z|^{-n+1} D_x^\alpha \left[f\left(\frac{x}{|x|}, \frac{z}{|z|} \right) \right]$$

will consist of terms which are multiplied by derivatives of $\tilde{s}_\varepsilon$ of degree not greater than $|\alpha|$, and a term of the form

$$h_\alpha(z) := \frac{\langle D^\alpha \tilde{v}, z \rangle}{|z|^n} \int_0^{\tilde{\rho}_{K_\varepsilon}(z)} \frac{r^{n-1}}{\left(1 - r\langle v, z/|z| \rangle \tilde{s}_\varepsilon\right)^{n+1}}\, dr.$$

We can then bound

$$\int_{S^{n-1}} |\langle x, \xi \rangle|^{2k} \operatorname{sgn}(\langle x, \xi \rangle) \xi_j \Delta^{k+1} h_\alpha(\xi)\, d\xi$$

in the same way (4.13) was bounded. □

Lemma 4.10. *Let K_ε be the convex body defined in Lemma 4.5. Define*

$$g_\varepsilon(\xi) := \rho_{CI(K_\varepsilon)}(\xi) - \frac{\omega_{n-1}}{n-1}, \quad \xi \in S^{n-1}.$$

For small enough $\varepsilon > 0$, $g_\varepsilon \in C^\infty(S^{n-1})$. Furthermore, for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = k$,

$$|D^\alpha \tilde{g}_\varepsilon(\xi)| \leq C(k,n)\varepsilon \qquad \forall\ \xi \in S^{n-1}.$$

The constants $C(k,n) \geq 0$ depend only on k and the dimension n.

Proof. It is clear that g_ε is rotationally symmetric about the x_1 - axis, because $\rho_{CI(K_\varepsilon)}$ is so. We will show that the restriction of $\tilde{g}_\varepsilon$ to the x_1, x_2 - plane is infinitely smooth with bounded partial derivatives; the general result will then follow from the rotational symmetry.

From now on, suppose $x \in \mathbb{R}^n \backslash \{0\}$ lies in the x_1, x_2 - plane. Using the representation of $\rho_{CI(K_\varepsilon)}$ on the x_1, x_2 - plane given by equation (4.8), we have

$$\tilde{g}_\varepsilon(x) = \tilde{\rho}_{CI(K_\varepsilon)}(x) - \frac{\omega_{n-1}}{n-1} = \int_{S^{n-1}\cap(x/|x|)^\perp} f\left(\frac{x}{|x|}, \xi\right) d\xi$$

where

$$f(\phi, \xi) = \int_0^{\rho_{K_\varepsilon}(\xi)} \frac{\left[\rho^{n-1}_{K_\varepsilon}(\xi) - (1 - r\langle v, \xi\rangle s_\varepsilon)^n\right] r^{n-2}}{\rho^{n-1}_{K_\varepsilon}(\xi)(1 - r\langle v, \xi\rangle s_\varepsilon)^n}\, dr.$$

This function $f : S^1 \times S^{n-1} \to \mathbb{C}$ is infinitely smooth (for small enough $\varepsilon > 0$) because ρ_{K_ε} and s_ε are so. Observe that for all $\phi \in S^1$ and $\xi \in S^{n-1}$, we have

$$\begin{aligned}
\left|f(\phi, \xi)\right| &\leq \int_0^{\rho_{K_\varepsilon}(\xi)} \left| \frac{\left[\rho^{n-1}_{K_\varepsilon}(\xi) - (1 - r\langle v, \xi\rangle s_\varepsilon)^n\right] r^{n-2}}{\rho^{n-1}_{K_\varepsilon}(\xi)(1 - r\langle v, \xi\rangle s_\varepsilon)^n} \right| dr \\
&\leq \frac{(1+N\varepsilon)^{n-2}}{(1-N\varepsilon)^{n-1}(1 - C\varepsilon(1+\varepsilon N))^n} \int_0^{1+N\varepsilon} \left|\rho^{n-1}_{K_\varepsilon}(\xi) - (1 - r\langle v, \xi\rangle s_\varepsilon)^n\right| dr \\
&\leq h(\varepsilon),
\end{aligned}$$

where $C = C(0, n)$ is the constant from Lemma 4.9, N is the maximum value of $|P|$ on the interval $[-1,1]$, and

$$h(\varepsilon) := \frac{(1+N\varepsilon)^{n-1}}{(1-N\varepsilon)^{n-1}(1-C\varepsilon(1+\varepsilon N))^n}\Big[(1+N\varepsilon)^{n-1} - (1-C\varepsilon(1+N\varepsilon))^n\Big] + \frac{(1+N\varepsilon)^{n-1}}{(1-N\varepsilon)^{n-1}(1-C\varepsilon(1+\varepsilon N))^n}\Big[(1+C\varepsilon(1+N\varepsilon))^n - (1-N\varepsilon)^{n-1}\Big].$$

The definition of h is independent of ϕ and ξ, and it is smooth in a neighbourhood of $0 \in \mathbb{R}$ with $h(0) = 0$. Therefore, there is another constant $C > 0$ such that $|f(\phi,\xi)| \leq C\varepsilon$ for all $(\phi,\xi) \in S^1 \times S^{n-1}$, and which is independent of small enough $\varepsilon > 0$.

For any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$, it is easily seen that every term of

$$D_x^\alpha D_z^\beta \left[f\left(\frac{x}{|x|}, \frac{z}{|z|}\right)\right]$$

is multiplied by $\tilde{s}_\varepsilon$ or one of its derivatives, or by a derivative of $\tilde{\rho}_{K_\varepsilon}$. Using Lemma 4.9, such partial derivatives of f can be absolutely bounded by $\varepsilon > 0$ multiplied by a constant depending only on the dimension and the order of the derivative.

Using Corollary 4.7 to calculate the partial derivative of $\tilde{g}_\varepsilon$ with respect to x_j $(j = 1, 2)$, we obtain the same expression as (4.10). Given our previous remarks, we can appropriately bound this derivative of $\tilde{g}_\varepsilon$, as well as all partial derivatives of higher orders. $\square$

We can conclude from the previous lemma that

$$\lim_{m\to\infty} \left\| \rho_{CI(K_{1/m})} - \frac{\omega_{n-1}}{n-1} \right\|_{C^k(S^{n-1})} = 0 \quad \forall\, k \in \mathbb{N}.$$

Theorem 1.3 now follows immediately from Theorem 4.2.

4.4 Concluding Remarks

The authors of [45] proved that cross-section bodies do not uniquely determine the Euclidean ball in any dimension. It is natural to ask the corresponding question for convex intersection bodies.

Our construction for Theorem 1.3 does not show that the origin-symmetric convex body L can be taken to be a Euclidean ball. However, this is possible with a simpler construction in two dimensions. Consider $K \in \mathbb{R}^2$. For any $\phi \in S^1$, $K^{*g}\big|\phi^\perp$ is a line segment with length

$$h_{K^{*g}}(v(\phi)) + h_{K^{*g}}(-v(\phi)),$$

where $v \in SO(2)$ again denotes rotation about the origin by $\pi/2$ radians. With the simple observation that the Santaló point of a compact line segment is its center, we have

$$\begin{aligned}\rho_{CI(K)}(\phi) &= \mathrm{vol}_1\left[\left(K^{*g}\big|\phi^\perp\right)^{*s(\phi)}\right] \\ &= \frac{4}{h_{K^{*g}}((v(\phi)) + h_{K^{*g}}(-v(\phi))}, \quad \phi \in S^1.\end{aligned}$$

Let $L \subset \mathbb{R}^2$ be a convex body of constant width which is not centrally-symmetric, and put $K = L^{*s}$. It is a property of the Santaló point that $g(K) = s(L)$ (see [49], page 420), so $CI(K)$ is necessarily a Euclidean disk. Finally, we have $CI(K) = IB = CI(B)$ for some disk $B \subset \mathbb{R}^2$ with appropriate diameter.

This counter-example should be compared with the two dimensional case for cross-section bodies. Indeed, the cross-section body of a convex body in $\mathbb{R}^2$ of constant width is always a disk (see, for example, Theorem 8.3.5 in [14]).

Given the preceding comments, we ask the following:

Question. *Let $n \in \mathbb{Z}$, $n \geq 3$. Is there a convex body $K \subset \mathbb{R}^n$ which is not centrally-symmetric, and whose convex intersection body $CI(K)$ is a Euclidean ball?*

Chapter 5

Maximal Perimeters of Polytope Sections & Origin-Symmetry

The content of this chapter comes from my preprint [51]. The main goal is to prove that a convex polytope P, containing the origin in its interior, must be origin-symmetric if every hyperplane section of P through the origin has maximal $(n-2)$-dimensional surface area amongst all parallel sections. This was stated as Theorem 1.4 in the introduction.

I introduce some notation and simple lemmas in Section 5.1 which are specific to this chapter. The proof of Theorem 1.4 is presented in Section 5.2. Finally, in Section 4, I explain how to characterize the origin-symmetry of C^1 convex bodies using the dual quermassintegrals of sections; this is a dual version of the conjecture of Martini et al. [33].

5.1 Some Notation and Auxiliary Lemmas

We let $S^{n-1}(\xi, \varepsilon) := S^{n-1} \cap B_2^n(\xi, \varepsilon)$ for small $\varepsilon > 0$. The geodesic connecting linearly independent $\xi_1, \xi_2 \in S^{n-1}$ is given by

$$[\xi_1, \xi_2] := S^{n-1} \cap \{\alpha\xi_1 + \beta\xi_2 : \alpha, \beta \geq 0\}.$$

For any $(n-2)$-dimensional polytope $G \subset \mathbb{R}^n$ which does not contain the origin, define $\eta_G \in S^{n-1}$ to be the unique unit vector for which

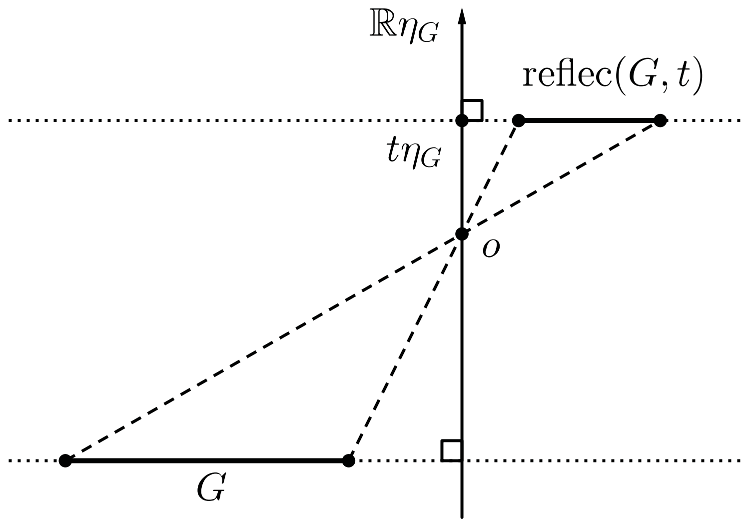

Figure 5.1: The geometric meaning of reflec(G, t).

- the line $\mathbb{R}\eta_G$ and aff(G) intersect orthogonally;
- $G \subset \eta_G^- := \{x \in \mathbb{R}^n : \langle x, \eta_G \rangle \leq 0\}$.

For each $t > 0$,

$$\text{reflec}(G, t) := \big\{x \in \mathbb{R}^n : \langle x, \eta_G \rangle = t \text{ and the line } \mathbb{R}x \text{ interesects } G\big\}$$

is an $(n-2)$-dimensional polytope in $\mathbb{R}^n$; see Figure 5.1. In words, reflec(G, t) is the homothetic copy of $-G$ lying in $\{t\eta_G + \eta_G^\perp\}$, so that every line connecting a vertex of reflec(G, t) to the corresponding vertex of G passes through the origin.

Lemma 5.1. *Let $Q \subset \mathbb{R}^n$ be a polytope for which the origin is not a vertex. Let $S^{n-1}(\theta_0, \varepsilon)$ be a spherical cap of radius $\varepsilon > 0$ centred at $\theta_0 \in S^{n-1}$. There exists $\theta \in S^{n-1}(\theta_0, \varepsilon)$ such that $\theta^\perp$ does not contain any vertices of Q.*

Proof. If $u_1, \dots, u_d$ are the vertices of Q, choose any θ from the non-empty set $S^{n-1}(\theta_0, \varepsilon) \setminus \big(u_1^\perp, \dots, u_d^\perp\big)$. $\square$

The proof of the following lemma is trivial.

Lemma 5.2. *Let $I \subset \mathbb{R}$ be an open interval. Let $\{f_j\}_{j=1}^n$ be a collection of differentiable $\mathbb{R}^n$-valued functions on I. Define $F(t) := \det\big(f_1(t), \dots, f_n(t)\big)$. Then F is differentiable on I with*

$$F'(t) = \sum_{j=1}^{n} \det\Big(f_1(t), \dots, f_{j-1}(t), f_j'(t), f_{j+1}(t), \dots, f_n(t)\Big).$$

5.2 Proof of Theorem 1.4

Let P be a convex polytope containing the origin in its interior and satisfying (1.4) for all $\xi \in S^{n-1}$. Our proof has two distinct parts.

We first need to prove that

$$\operatorname{reflec}(G, t) \text{ is an } (n-2)\text{-dimensional face of } P \text{ for some } t > 0 \tag{5.1}$$

whenever G is an $(n-2)$-dimensional face of P. To the contrary, we suppose G_0 is an $(n-2)$-dimensional face of P for which (5.1) is false. We find a special spherical cap $S^{n-1}(\xi_0, \varepsilon)$. For every $\xi \in S^{n-1}(\xi_0, \varepsilon)$, $\xi^\perp$ misses all the vertices of P, while intersecting G_0 and no other $(n-2)$-dimensional faces which are parallel to G_0. We derive a "nice" equation from (1.4) which is valid for all $\xi \in S^{n-1}(\xi_0, \varepsilon)$. Forgetting the geometric meaning, we analytically extend this nice equation to all $\xi \in S^{n-1}$, excluding a finite number of great subspheres. Studying the behaviour near one of these subspheres, we arrive at our contradiction.

We conclude that for every vertex v of P, the line $\mathbb{R}v$ contains another vertex $\widetilde{v}$ of P. In the second part of our proof, we prove that $\widetilde{v} = -v$. Hence, P is origin-symmetric.

5.2.1 First Part

Assume there is an $(n-2)$-dimensional face G_0 of P such that $\operatorname{reflec}(G_0, t)$ is not an $(n-2)$-dimensional face of P for any $t > 0$. By the convexity of P, the intersection of $\operatorname{aff}(o, G_0)$ with P contains at most one other $(n-2)$-dimensional face of P (besides G_0) which is parallel to G_0. If such a face exists, it must lie in $\{t\eta_G + \eta_G^\perp\}$ for some $t > 0$ because P contains the origin in its interior. We still allow that $\mathrm{reflec}(G_0, t)$ may have a non-empty intersection with another $(n-2)$-dimensional face of P for some $t > 0$. However, we can additionally assume without loss of generality that $\mathrm{reflec}(G_0, t)$ is not contained within an $(n-2)$-dimensional face of P for any $t > 0$.

Lemma 5.3. *There is a $\xi_0 \in S^{n-1}$ such that*

(i) the hyperplane $\xi_0^\perp$ does not contain any vertices of P;

(ii) $\xi_0^\perp$ intersects G_0 but no other $(n-2)$-dimensional faces of P parallel to G_0;

(iii) there is exactly one vertex v of G_0 contained in $\xi_0^+ := \{x \in \mathbb{R}^n : \langle x, \xi_0 \rangle \geq 0\}$.

Proof. Choose $\theta \in S^{n-1}$ so that $\theta^\perp = \mathrm{aff}(o, G_0)$. Let $\eta := \eta_{G_0}$ be the unit vector defined as before.

There are two possibilities: either $\{x \in \mathbb{R}^n : \langle x, \eta \rangle = t\} \cap \theta^\perp$ does not contain an $(n-2)$-dimensional face of P for any $t > 0$, or it does for exactly one $t_0 > 0$. If the first case is true, we can of course choose an affine $(n-3)$-dimensional subspace $\widetilde{L}$ lying within $\mathrm{aff}(G_0)$ which does not pass through any vertices of G_0, and separates exactly one vertex $v \in G_0$ from the others.

Suppose the second case is true, i.e. $H := \{x \in \mathbb{R}^n : \langle x, \eta \rangle = t_0\} \cap \theta^\perp$ contains an $(n-2)$-dimensional face G of P. By definition and assumption, $\mathrm{reflec}(G_0, t_0)$ lies in H and is not contained in G. We can choose $\widetilde{L} \subset H$ to be an $(n-3)$-dimensional affine subspace which, within H, strictly separates exactly one vertex $\widetilde{v} \in \mathrm{reflec}(G_0, t_0)$ from both G and the remaining vertices of $\mathrm{reflec}(G_0, t_0)$. Let v be the vertex of G_0 lying on the line $\mathbb{R}\widetilde{v}$.

Regardless of which case was true, set $L := \mathrm{aff}(o, \widetilde{L}) \subset \theta^\perp$. The $(n-2)$-dimensional subspace L intersects G_0 but no other $(n-2)$-dimensional faces of P parallel to G_0, and separates v from the remaining vertices of G_0. Perturbing L if necessary (see Lemma 5.1), L also does not intersect any vertices of P. Choose $\phi \in S^{n-1} \cap \theta^\perp \cap L^\perp$.

Define the slab $\theta_\alpha^\perp := \{x \in \mathbb{R}^n : |\langle x, \theta \rangle| \leq \alpha\}$, with $\alpha > 0$ small enough so that $\theta_\alpha^\perp$ only contains vertices of P lying in $\theta^\perp$. Necessarily, $\theta_\alpha^\perp$ also only contains the $(n-2)$-dimensional faces of P parallel to G_0 which lie entirely in $\theta^\perp$. Choose $\beta > 0$ large enough so that the slab $\phi_\beta^\perp$ contains P. Let $\xi_0 \in S^{n-1}$ be such that $\xi_0^\perp = \operatorname{aff}\big(o, \alpha\theta + \beta\phi + L\big)$ and $\langle v, \xi_0 \rangle \geq 0$. We then have $\xi_0^\perp \cap P \subset \theta_\alpha^\perp$ and $\xi_0^\perp \cap \theta^\perp = L$. It follows from the construction of $\theta_\alpha^\perp$ and L that ξ_0 has the desired properties. □

Let $\{E_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ respectively be the edges and facets (i.e. $(n-1)$-dimensional faces) of P intersecting $\xi_0^\perp$. Consider a spherical cap $S^{n-1}(\xi_0, \varepsilon)$ of radius $\varepsilon > 0$ centred at ξ_0. For $\varepsilon > 0$ small enough, the set

$$\big\{x \in \mathbb{R}^n : |\langle x, \xi \rangle| \leq \varepsilon \text{ for some } \xi \in S(\xi_0, \varepsilon)\big\}$$

does not contain any vertices of P. Consequently, the map

$$t \mapsto \operatorname{vol}_{n-2}\Big(\operatorname{relbd}\big(P \cap \{t\xi + \xi^\perp\}\big)\Big) = \sum_{j \in J} \operatorname{vol}_{n-2}\big(F_j \cap \{t\xi + \xi^\perp\}\big)$$

is differentiable in a neighbourhood of $t = 0$ for each $\xi \in S^{n-1}(\xi_0, \varepsilon)$. Therefore, (1.4) implies

$$\sum_{j \in J} \frac{d}{dt} \operatorname{vol}_{n-2}\big(F_j \cap \{t\xi + \xi^\perp\}\big)\bigg|_{t=0} = 0 \tag{5.2}$$

for every $\xi \in S^{n-1}(\xi_0, \varepsilon)$. We need to find an expression for this derivative.

For each $i \in I$, let $u_i + l_i s$ be the line in $\mathbb{R}^n$ containing E_i; u_i is a point on the line, l_i is a unit vector parallel to the line, and s is the parameter. Clearly, $\{t\xi + \xi^\perp\}$ intersects the same edges and facets as $\xi_0^\perp$ for every $\xi \in S^{n-1}(\xi_0, \varepsilon)$ and $|t| \leq \varepsilon$. The intersection point of $\{t\xi + \xi^\perp\}$ with the edge E_i is given by

$$p_i(\xi, t) := u_i + l_i \left(\frac{t - \langle u_i, \xi \rangle}{\langle l_i, \xi \rangle} \right).$$

Note that $\xi_0^\perp$ intersects exactly those edges of G_0 which are adjacent to the vertex v. Whenever E_i is an edge of G_0 adjacent to v, we put $u_i := v$ and choose l_i so that it gives the direction from another vertex of G_0 to v; this ensures $\langle l_i, \xi_0 \rangle > 0$.

For each $j \in J$, there is a pair of vertices from the facet F_j such that the line through them does not lie in a translate of $\mathrm{aff}(G_0)$, and with one of the vertices on either side of $\xi_0^\perp$. Translating this line if necessary, we obtain an auxiliary line $w_j + m_j s$ which

- lies within $\mathrm{aff}(F_j)$ and intersects the relative interior of F_j;
- is transversal to $\xi^\perp$ for every $\xi \in S^{n-1}(\xi_0, \varepsilon)$;
- does not lie within an $(n-2)$-dimensional affine subspace parallel to $\mathrm{aff}(G_0)$.

Again, w_j is a point on the line, m_j is a unit vector parallel to the line, and s is the parameter. The intersection point of $\{t\xi + \xi^\perp\}$ with $w_j + m_j s$ is given by

$$q_j(\xi, t) := w_j + m_j \left(\frac{t - \langle w_j, \xi \rangle}{\langle m_j, \xi \rangle} \right).$$

Note that we necessarily have $\xi_0 \not\perp m_j$ for all $j \in J$.

Consider a facet F_j, and an $(n-2)$-dimensional face G of P which intersects $\xi_0^\perp$ and is adjacent to F_j. Observe that $G \cap \{t\xi + \xi^\perp\}$ is an $(n-3)$-dimensional face of the $(n-2)$-dimensional polytope $F_j \cap \{t\xi + \xi^\perp\}$, for each $\xi \in S^{n-1}(\xi_0, \varepsilon)$ and $|t| \le \varepsilon$. Express $G \cap \{t\xi + \xi^\perp\}$ as a disjoint union of $(n-3)$-dimensional simplices whose vertices correspond to the vertices of $G \cap \{t\xi + \xi^\perp\}$; that is, each simplex has vertices $p_{i_1}(\xi, t), \ldots, p_{i_{n-2}}(\xi, t)$ for some $i_1, \ldots, i_{n-2} \in I$. Triangulating every such $(n-3)$-dimensional face $G \cap \{t\xi + \xi^\perp\}$ in this way, we get a triangulation of $F_j \cap \{t\xi + \xi^\perp\}$ by taking the convex hull of the simplices in its relative boundary with $q_j(\xi, t)$.

Remark 5.4. The description and orientation of a simplex Δ in the triangulation of $F_j \cap \{t\xi + \xi^\perp\}$ in terms of the ordered vertices

$$\{p_{i_1}(\xi, t), \ldots, p_{i_{n-2}}(\xi, t), q_j(\xi, t)\}$$

is independent of $\xi \in S(\xi_0, \varepsilon)$ and $|t| \le \varepsilon$.

Setting $n_j \in S^{n-1}$ to be the outer unit normal to F_j, $\mathrm{vol}_{n-2}\big(F_j \cap \{t\xi + \xi^\perp\}\big)$ is then a sum of terms of the form

$$\mathrm{vol}_{n-2}(\Delta) = \frac{\det\Big(p_{i_1}(\xi,t) - q_j(\xi,t), \ldots, p_{i_{n-2}}(\xi,t) - q_j(\xi,t), n_j, \xi\Big)}{(n-2)!\sqrt{1-\langle n_j, \xi\rangle^2}}; \qquad (5.3)$$

see page 14 in [14], for example, for the volume formula for a simplex. We assume the column vectors in the determinant are ordered so that the determinant is positive. Differentiating (5.3) at $t = 0$ with the help of Lemma 5.2 gives

$$\frac{1}{(n-2)!\sqrt{1-\langle n_j,\xi\rangle^2}} \sum_{\gamma=1}^{n-2} \det\Big(X_{i_1}(\xi), \ldots, \widetilde{X}_{i_\gamma}(\xi), \ldots, X_{i_{n-2}}(\xi), n_j, \xi\Big), \quad (5.4)$$

where

$$\begin{aligned} X_{i_\gamma}(\xi) &:= p_{i_\gamma}(\xi,0) - q_j(\xi,0) \\ &= u_{i_\gamma} - \left(\frac{\langle u_{i_\gamma}, \xi\rangle}{\langle l_{i_\gamma}, \xi\rangle}\right) l_{i_\gamma} - w_j + \left(\frac{\langle w_j,\xi\rangle}{\langle m_j,\xi\rangle}\right) m_j, \\ \widetilde{X}_{i_\gamma}(\xi) &:= \frac{d}{dt}\Big(p_{i_\gamma}(\xi,t) - q_j(\xi,t)\Big)\Big|_{t=0} = \frac{l_{i_\gamma}}{\langle l_{i_\gamma},\xi\rangle} - \frac{m_j}{\langle m_j,\xi\rangle}. \end{aligned}$$

The left hand side of equation (5.2) is a sum of expressions having the form (5.4). That is, (5.2) is equivalent to

$$\sum_{\Delta} \left(\frac{\sum_{\gamma=1}^{n-2} \det\Big(X_{i_1}(\xi), \ldots, \widetilde{X}_{i_\gamma}(\xi), \ldots, X_{i_{n-2}}(\xi), n_j, \xi\Big)}{(n-2)!\sqrt{1-\langle n_j,\xi\rangle^2}} \right) = 0, \qquad (5.5)$$

where the first summation is over all appropriately ordered indices

$$\{i_1, \ldots, i_{n-2}, j\}$$

corresponding to vertices of simplices Δ in our triangulation of $P \cap \xi_0^\perp$. Forget the geometric meaning of equation (5.5). Clearing denominators on the left side of equation (5.5) gives a function of ξ which we denote by $\Phi(\xi)$. Because

Φ is a sum of products of scalar products of ξ and terms $\sqrt{1-\langle n_j, \xi\rangle^2}$, we are able to consider Φ as a function on all of S^{n-1} such that $\Phi \equiv 0$ on $S^{n-1}(\xi_0, \varepsilon)$.

Lemma 5.5. $\Phi(\xi) = 0$ *for all* $\xi \in S^{n-1}$.

Proof. Suppose $\zeta \in S^{n-1}$ is such that $\Phi(\zeta) \neq 0$. We have $\Phi(n_j) = 0$ for all $j \in J$, so $\zeta \neq n_j$. There is a ζ_1 from the relative interior of $S^{n-1}(\xi_0, \varepsilon)$ which is not parallel to ζ, and is such that the geodesic $[\zeta_1, \zeta]$ connecting ζ_1 to ζ contains none of the n_j. Choose $\zeta_2 \in S^{n-1}$ which is perpendicular to ζ_1, lies in $\operatorname{aff}\big([\zeta_1, \zeta]\big)$, and is such that $\langle \zeta_2, \zeta\rangle > 0$. Let $\widetilde{\Phi}$ be the restriction of Φ to $\xi \in [\zeta_1, \zeta]$, and adopt polar coordinates $\xi = \zeta_1 \cos(\phi) + \zeta_2 \sin(\phi)$. As a function of $\phi \in \big[0, \arccos(\langle \zeta_1, \zeta\rangle)\big]$, $\widetilde{\Phi}$ is a sum of products of $\cos(\phi)$, $\sin(\phi)$, and $\sqrt{1 - \langle n_j, \zeta_1 \cos(\phi) + \zeta_2 \sin(\phi)\rangle^2}$. The radicals in the expression for $\widetilde{\Phi}$ are never zero because $[\zeta_1, \zeta]$ misses all of the n_j, so $\widetilde{\Phi}$ is analytic. Consequently, $\widetilde{\Phi}$ must be identically zero, as it vanishes in a neighbourhood of $\phi = 0$. This is a contradiction. $\square$

Lemma 5.5 implies that the equality in (5.5) holds for all $\xi \in S^{n-1}\backslash A$, where A is the union over $i \in I$ and $j \in J$ of the unit spheres in $l_i^\perp$ and $m_j^\perp$. Of course, we have $\pm n_j \in S^{n-1} \cap m_j^\perp$ for each $j \in J$, so $\{\pm n_j\}_{j\in J} \subset A$. We will consider the limit of the left side of equation (5.5) along a certain path in $S^{n-1}\backslash A$ which terminates at a point in A.

The $(n-2)$-dimensional face G_0 is the intersection of two facets of P belonging to $\{F_j\}_{j\in J}$, say F_1 and F_2. The normal space of G_0 is two dimensional and spanned by n_1 and n_2, the outer unit normals of F_1 and F_2. Consider the non-degenerate geodesic

$$[n_1, n_2] := \left\{ \widetilde{n}_s := \frac{(1-s)n_1 + sn_2}{\big|(1-s)n_1 + sn_2\big|_2} \;\middle|\; 0 \le s \le 1 \right\}$$

in $S^{n-1} \cap G_0^\perp \cong S^1$. This arc is not contained in the normal space of any other $(n-2)$-dimensional face of P intersected by $\xi_0^\perp$, because $\xi_0^\perp$ does not intersect any other $(n-2)$-dimensional faces parallel to G_0; nor is $[n_1, n_2]$ contained in $m_j^\perp$ for any $j \in J$, because m_j is not contained in a translate of $\operatorname{aff}(G_0)$. Therefore, we can fix $0 < s_0 < 1$ so that

- $\widetilde{n} := \widetilde{n}_{s_0}$ is not a unit normal for any $(n-2)$-dimensional face of P intersected by $\xi_0^\perp$, besides G_0;

- $\widetilde{n} \not\perp m_j$, hence $\widetilde{n} \neq \pm n_j$, for all $j \in J$.

We additionally select s_0 so that it is not among the finitely many roots of the function

$$(0,1) \ni s \mapsto \frac{-s}{\sqrt{1-\langle n_1, \widetilde{n}_s\rangle^2}} + \frac{1-s}{\sqrt{1-\langle n_2, \widetilde{n}_s\rangle^2}}. \tag{5.6}$$

Observe that $\langle v, \widetilde{n}\rangle > 0$ because $\langle v, n_1\rangle > 0$ and $\langle v, n_2\rangle > 0$.

For $\delta > 0$, define the unit vector

$$\xi_\delta := \frac{\widetilde{n} + \delta\xi_0}{|\widetilde{n} + \delta\xi_0|_2}.$$

Clearly,

$$\lim_{\delta\to 0^+} \xi_\delta = \widetilde{n} \in S^{n-1} \cap G_0^\perp \subset \bigcup_{i\in I} S^{n-1} \cap l_i^\perp \subset A.$$

We have $\langle \xi_\delta, l_i\rangle, \langle \xi_\delta, m_j\rangle \neq 0$ for all $i \in I, j \in J$ whenever

$$0 < \delta < \min\left\{ \frac{|\langle \widetilde{n}, l_i\rangle|}{|\langle \xi_0, l_i\rangle|}, \frac{|\langle \widetilde{n}, m_j\rangle|}{|\langle \xi_0, m_j\rangle|} \;\middle|\; i \in I \text{ such that } \langle \widetilde{n}, l_i\rangle \neq 0,\ j \in J \right\}.$$

The previous minimum is well-defined and positive, because $\xi_0 \not\perp l_i, m_j$ and $\widetilde{n} \not\perp m_j$ for all $i \in I$, $j \in J$. So $\xi_\delta \in S^{n-1} \backslash A$ for small enough $\delta > 0$.

Now, replace ξ with ξ_δ in (5.5), multiply both sides of the resulting equation by δ^{n-2}, and take the limit as δ goes to zero. Consider what happens to the expressions (5.4) multiplied by δ^{n-2} in this limit. We have

$$\begin{aligned}
\lim_{\delta\to 0^+} \delta X_{i_\gamma}(\xi_\delta) &= \delta u_{i_\gamma} - \delta\left(\frac{\langle u_{i_\gamma}, \widetilde{n} + \delta\xi_0\rangle}{\langle l_{i_\gamma}, \widetilde{n} + \delta\xi_0\rangle}\right) l_{i_\gamma} - \delta w_j + \delta\left(\frac{\langle w_j, \widetilde{n} + \delta\xi_0\rangle}{\langle m_j, \widetilde{n} + \delta\xi_0\rangle}\right) m_j \\
&= \begin{cases} o & \text{if} \quad \widetilde{n} \not\perp l_{i_\gamma}; \\ -\frac{\langle u_{i_\gamma}, \widetilde{n}\rangle}{\langle l_{i_\gamma}, \xi_0\rangle} l_{i_\gamma} & \text{if} \quad \widetilde{n} \perp l_{i_\gamma}, \end{cases}
\end{aligned}$$

and

$$\lim_{\delta\to 0^+} \delta \widetilde{X}_{i_\gamma}(\xi_\delta) = \delta|\widetilde{n}+\delta\xi_0|_2 \frac{l_{i_\gamma}}{\langle l_{i_\gamma}, \widetilde{n}+\delta\xi_0\rangle} - \delta|\widetilde{n}+\delta\xi_0|_2 \frac{m_j}{\langle m_j, \widetilde{n}+\delta\xi_0\rangle}$$
$$= \begin{cases} o & \text{if} \quad \widetilde{n} \not\perp l_{i_\gamma}; \\ \frac{l_{i_\gamma}}{\langle l_{i_\gamma}, \xi_0\rangle} & \text{if} \quad \widetilde{n} \perp l_{i_\gamma}, \end{cases}$$

because $\xi_0 \not\perp l_i$ for all $i \in I$ and $\widetilde{n} \not\perp m_j$ for all $j \in J$. Therefore, expression (5.4) vanishes in the limit if at least one index i_γ in (5.4) corresponds to an edge direction l_{i_γ} which is not perpendicular to $\widetilde{n}$. If $l_{i_1}, \ldots, l_{i_{n-2}}$ are all perpendicular to $\widetilde{n}$, then expression (5.4) becomes

$$\frac{(-1)^{n-3} \det\Big(l_{i_1}, \ldots, l_{i_{n-2}}, n_j, \widetilde{n}\Big)}{(n-2)!\sqrt{1-\langle n_j, \widetilde{n}\rangle^2}} \sum_{\omega=1}^{n-2} \left(\langle u_{i_\omega}, \widetilde{n}\rangle^{-1} \prod_{\gamma=1}^{n-2} \frac{\langle u_{i_\gamma}, \widetilde{n}\rangle}{\langle l_{i_\gamma}, \xi_0\rangle} \right). \tag{5.7}$$

If the determinant in (5.7) is non-zero, then $l_{i_1}, \ldots, l_{i_{n-2}}$ are linearly independent. Therefore, $l_{i_1}, \ldots, l_{i_{n-2}}$ span an $(n-2)$-dimensional plane which is parallel to the $(n-2)$-dimensional face G of P to which the edges $E_{i_1}, \ldots, E_{i_{n-2}}$ belong. Necessarily, $\widetilde{n}$ will be a unit normal for G, so $G = G_0$ by our choice of $\widetilde{n}$. We conclude that the limit of (5.4) only has a chance of being non-zero if (5.4) corresponds to an $(n-2)$-dimensional simplex Δ_j in our triangulation of $F_j \cap \xi_0^\perp$, $j = 1$ or $j = 2$, with the base of Δ_j being an $(n-3)$-dimensional simplex in the triangulation of $G_0 \cap \xi_0^\perp$.

If (5.4) comes from such a Δ_1 in the triangulation of $F_1 \cap \xi_0^\perp$, then its limit is given by (5.7), and simplifies further to the non-zero term

$$\frac{(-1)^{n-3}\langle v, \widetilde{n}\rangle^{n-3} s_0 \det\Big(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2\Big)}{(n-3)!\langle l_{i_1}, \xi_0\rangle \times \cdots \times \langle l_{i_{n-2}}, \xi_0\rangle \big|(1-s_0)n_1 + s_0 n_2\big|_2 \sqrt{1-\langle n_1, \widetilde{n}\rangle^2}} \neq 0. \tag{5.8}$$

The distinct indices $i_1, \ldots, i_{n-2}$ correspond to the vertices of a simplex in the triangulation of $G_0 \cap \xi_0^\perp$, ordered so that the expression in (5.3) for the facet F_1 is positive. The important fact that (5.8) is non-zero is clear once we observe that the determinant is non-zero. Indeed, the unit vectors $l_{i_1}, \ldots, l_{i_{n-2}}$ are necessarily linearly independent and perpendicular to both n_1 and n_2, because they give the directions for distinct edges of G_0 with the common vertex v. Similarly, when (5.4) comes from such a Δ_2 in the triangulation of $F_2 \cap \xi_0^\perp$, it has the non-zero limit

$$\frac{(-1)^{n-3}\langle v, \widetilde{n}\rangle^{n-3}(1-s_0)\det\Big(l_{k_1}, \dots, l_{k_{n-2}}, n_2, n_1\Big)}{(n-3)!\langle l_{k_1}, \xi_0\rangle \times \cdots \times \langle l_{k_{n-2}}, \xi_0\rangle\big|(1-s_0)n_1 + s_0 n_2\big|_2\sqrt{1-\langle n_2, \widetilde{n}\rangle^2}} \neq 0. \tag{5.9}$$

The distinct indices $k_1, \dots, k_{n-2}$ correspond to the vertices of a simplex in the triangulation of $G_0 \cap \xi_0^\perp$, ordered so that the expression in (5.3) for the facet F_2 is positive.

We will now consider the signs of the determinants in (5.8) and (5.9).

Lemma 5.6. *The determinants* $\det\big(l_{i_1}, \dots, l_{i_{n-2}}, n_1, n_2\big)$ *in (5.8) have the same sign for any collection of indices* $i_1, \dots, i_{n-2}$ *with the previously described properties. The determinants in (5.9) also all have the same sign. However, the signs of the determinants in (5.8) and (5.9) may differ.*

Proof. Let $y \in G_0 \cap \xi_0^\perp$. Consider the $(n-3)$ - dimensional subspace $L := \operatorname{span}(G_0 \cap \xi_0^\perp - y)$, which is orthogonal to $\operatorname{span}(n_1, n_2, \xi_0)$, and the $(n-2)$ - dimensional subspace $\widetilde{L} = \operatorname{span}(n_1, L)$. The projections $n_2|n_1^\perp$ and $\xi_0|n_1^\perp$ are non-zero and orthogonal to $\widetilde{L}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the special orthogonal matrix which leaves $\widetilde{L}$ fixed, and rotates $n_2|n_1^\perp$ through the two - dimensional plane $\operatorname{span}(n_2|n_1^\perp, \xi_0|n_1^\perp)$ to a vector parallel to, and with the same direction as, $\xi_0|n_1^\perp$. We have $\big(n_2|n_1^\perp\big) \perp (v-y)$, because $n_1, n_2 \perp (v-y)$. Since orthogonal transformations preserve inner products,

$$\begin{aligned}\big\langle \xi_0|n_1^\perp, T(v-y)\big\rangle &= \big|\xi_0|n_1^\perp\big|_2\big|T(n_2|n_1^\perp)\big|_2^{-1}\big\langle T(n_2|n_1^\perp), T(v-y)\big\rangle \\ &= \big|\xi_0|n_1^\perp\big|_2\big|T(n_2|n_1^\perp)\big|_2^{-1}\big\langle n_2|n_1^\perp, v-y\big\rangle \\ &= 0,\end{aligned}$$

and

$$\big\langle n_1, T(v-y)\big\rangle = \big\langle T(n_1), T(v-y)\big\rangle = \langle n_1, v-y\rangle = 0.$$

We have $(\xi_0|n_1^\perp)^\perp \cap n_1^\perp = \operatorname{span}\big(F_1 \cap \xi_0^\perp - y\big)$, because $n_1, \xi_0 \perp \operatorname{span}\big(F_1 \cap \xi_0^\perp - y\big)$. Therefore, T maps $v-y$ into $\operatorname{span}\big(F_1 \cap \xi_0^\perp - y\big)$, which also contains $q_1(\xi_0, 0) - y$.

The subspace L splits $\operatorname{span}\big(F_1 \cap \xi_0^\perp - y\big)$ into two halves. If $T(v-y)$ and $q_1(\xi_0,0) - y$ lie in the same half, let $\widetilde{T} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity. If $T(v-y)$ and $q_1(\xi_0,0) - y$ lie in opposite halves, let $\widetilde{T}$ be the orthogonal transformation which leaves L and $\operatorname{span}\big(F_1 \cap \xi_0^\perp - y\big)^\perp$ fixed, and reflects $T(v-y)$ across L. In either case, set $u := \widetilde{T}T(v-y) + y \in \operatorname{aff}(F_1 \cap \xi_0^\perp)$. We have that u and $q_1(\xi_0, 0)$ lie on the same side of $\operatorname{aff}(G_0 \cap \xi_0)$ in $\operatorname{aff}(F_1 \cap \xi_0^\perp)$. Also, $\widetilde{T} n_1 = n_1$ and $\widetilde{T}(\xi_0|n_1^\perp) = \xi_0|n_1^\perp$.

For any indices $i_1, \ldots, i_{n-2}$ from (5.8), we find that

$$\begin{aligned}
&\det\Big(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2\Big)\\
&= \det\left(\frac{v - p_{i_1}(\xi_0,0)}{|v - p_{i_1}(\xi_0,0)|_2}, \ldots, \frac{v - p_{i_{n-2}}}{|v - p_{i_{n-2}}(\xi_0,0)|_2}, n_1, n_2 - \langle n_1, n_2\rangle n_1\right)\\
&= C \det\Big(\widetilde{T}T(v-y) - \widetilde{T}T(p_{i_1}(\xi_0,0) - y), \ldots\\
&\qquad\qquad \ldots, \widetilde{T}T(v-y) - \widetilde{T}T(p_{i_{n-2}}(\xi_0,0) - y), \widetilde{T}Tn_1, \widetilde{T}T(n_2|n_1^\perp)\Big)\\
&= \frac{C(-1)^{n-2}\big|\xi_0|n_1^\perp\big|_2}{\big|T(n_2|n_1^\perp)\big|_2} \det\Big(p_{i_1}(\xi_0,0) - u, \ldots, p_{i_{n-2}}(\xi_0,0) - u, n_1, \xi_0|n_1^\perp\Big)\\
&= \frac{C(-1)^{n-2}\big|\xi_0|n_1^\perp\big|_2}{\big|T(n_2|n_1^\perp)\big|_2} \det\Big(p_{i_1}(\xi_0,0) - u, \ldots, p_{i_{n-2}}(\xi_0,0) - u, n_1, \xi_0\Big), \qquad (5.10)
\end{aligned}$$

where

$$C = \pm\left(\prod_{\gamma=1}^{n-2} |v - p_{i_\gamma}(\xi_0,0)|_2\right)^{-1}.$$

The sign of C depends on the definition of $\widetilde{T}$. Importantly, the sign of

$$\frac{C(-1)^{n-2}\big|\xi_0|n_1^\perp\big|_2}{\big|T(n_2|n_1^\perp)\big|_2}$$

is independent of any particular choice of appropriate indices in (5.8). The function

$$t \mapsto \det\Big(p_{i_1}(\xi_0,0) - \big((1-t)u + tq_1(\xi_0,0)\big), \dots$$
$$\dots, p_{i_{n-2}}(\xi_0,0) - \big((1-t)u + tq_1(\xi_0,0)\big), n_1, \xi_0\Big)$$

is continuous for $t \in [0,1]$; it is also non-vanishing for such t because the line segment connecting u to $q_1(\xi_0,0)$ lies in $F_1 \cap \xi_0^\perp$ and does not intersect $G_0 \cap \xi_0^\perp$. By the Intermediate Value Theorem, the determinant in (5.10) must have the same sign as

$$\det\Big(p_{i_1}(\xi_0,0) - q_1(\xi_0,0), \dots, p_{i_{n-2}}(\xi_0,0) - q_1(\xi_0,0), n_1, \xi_0\Big).$$

Recalling formula (5.3), we recognize that the previous determinant is positive. We conclude that the sign of $\det\big(l_{i_1}, \dots, l_{i_{n-2}}, n_1, n_2\big)$ is independent of the choice of appropriate indices in (5.8).

A similar argument shows that the sign of the determinant in (5.9) is also independent of the choice of appropriate indices $k_1, \dots, k_{n-2}$. □

In view of Lemma 5.6 and the expressions (5.8) and (5.9), we see that

$$\begin{aligned}
&\lim_{\delta\to 0^+} \delta^{n-2} \sum_{\Delta} \left(\frac{\sum_{\gamma=1}^{n-2} \det\Big(X_{i_1}(\xi_\delta), \dots, \widetilde{X}_{i_\gamma}(\xi_\delta), \dots, X_{i_{n-2}}(\xi_\delta), n_j, \xi_\delta\Big)}{(n-2)!\sqrt{1-\langle n_j, \xi_\delta\rangle^2}} \right) \\
&= \sum \left(\frac{(-1)^{n-3}\langle v, \widetilde{n}\rangle^{n-3} s_0 \det\Big(l_{i_1}, \dots, l_{i_{n-2}}, n_1, n_2\Big)}{(n-3)!\langle l_{i_1}, \xi_0\rangle \times \cdots \times \langle l_{i_{n-2}}, \xi_0\rangle \big|(1-s_0)n_1 + s_0 n_2\big|_2 \sqrt{1-\langle n_1, \widetilde{n}\rangle^2}} \right. \\
&\qquad \left. + \frac{(-1)^{n-3}\langle v, \widetilde{n}\rangle^{n-3}(1-s_0) \det\Big(l_{k_1}, \dots, l_{k_{n-2}}, n_2, n_1\Big)}{(n-3)!\langle l_{k_1}, \xi_0\rangle \times \cdots \times \langle l_{k_{n-2}}, \xi_0\rangle \big|(1-s_0)n_1 + s_0 n_2\big|_2 \sqrt{1-\langle n_2, \widetilde{n}\rangle^2}} \right) \\
&= \frac{(-1)^{n-3}\langle v, \widetilde{n}\rangle^{n-3}}{(n-3)!\big|(1-s_0)n_1 + s_0 n_2\big|_2} \left(\frac{s_0}{\sqrt{1-\langle n_1, \widetilde{n}\rangle^2}} \pm \frac{1-s_0}{\sqrt{1-\langle n_2, \widetilde{n}\rangle^2}} \right) \\
&\qquad \cdot \sum \frac{\det\Big(l_{i_1}, \dots, l_{i_{n-2}}, n_1, n_2\Big)}{\langle l_{i_1}, \xi_0\rangle \times \cdots \times \langle l_{i_{n-2}}, \xi_0\rangle}.
\end{aligned} \tag{5.11}$$

The third and fourth summations are taken over indices $i_1, \dots, i_{n-2} \in I$ corresponding to the vertices $\{p_{i_1}(\xi_0, 0), \ldots, p_{i_{n-2}}(\xi_0, 0)\}$ of simplices in the triangulation of $G_0 \cap \xi_0^\perp$, ordered so that the expression in (5.3) is positive for F_1. For each set of indices $i_1, \ldots, i_{n-2}$, $k_1, \ldots, k_{n-2}$ is a suitable rearrangement so that (5.3) is positive for F_2. The $\pm$ in (5.11) depends on whether or not the determinants

$$\det\Big(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2\Big)$$

have the same sign as the determinants

$$\det\Big(l_{k_1}, \ldots, l_{k_{n-2}}, n_2, n_1\Big).$$

We see that (5.11) is non-zero because $\langle v, \widetilde{n}\rangle > 0$, s_0 is not a root of (5.6), $\langle l_i, \xi_0\rangle > 0$ for all edges E_i of G_0 intersected by $\xi_0^\perp$, and by Lemma 5.6. The limit being non-zero contradicts the equality in (5.5).

5.2.2 Second Part

Therefore, for every $(n-2)$-dimensional face G of P, reflec(G, t) is also an $(n-2)$-dimensional face of P for some $t > 0$. From this fact, we can immediately conclude the following:

- If v is a vertex of P, then the line $\mathbb{R}v$ contains exactly one other vertex of P. This second vertex, which we will denote by $\widetilde{v}$, necessarily lies on the opposite side of the origin as v.

- If u and v are vertices of P connected by an edge $E(u, v)$, then $\widetilde{u}$ and $\widetilde{v}$ are connected by an edge $E(\widetilde{u}, \widetilde{v})$ parallel to $E(u, v)$.

We prove $P = -P$ by showing $\widetilde{v} = -v$ for every vertex v.

To the contrary, suppose there is a vertex v for which $|v|_2 < |\widetilde{v}|_2$. Let $\{v_i\}_{i=0}^k$ be a sequence of vertices of P such that $v_0 = v$, $v_k = \widetilde{v}$, and the vertices v_i and v_{i+1} are connected by an edge $E(v_i, v_{i+1})$ for each $0 \le i \le k-1$. It follows from the previous itemized observations that the triangle $T(o, v, v_1)$ with vertices $\{o,\ v,\ v_1\}$ is similar to the triangle $T(o, \widetilde{v}, \widetilde{v}_1)$; see Figure 5.2. Given that $|v|_2 < |\widetilde{v}|_2$, we must also have $|v_1|_2 < |\widetilde{v}_1|_2$. Continuing this argument recursively,

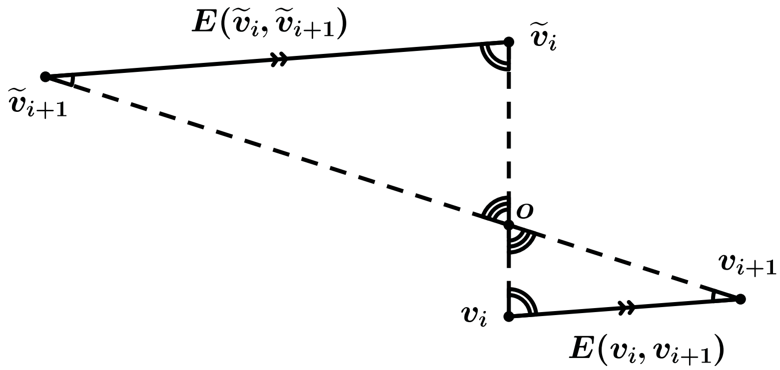

Figure 5.2: The triangle $T(o, v_i, v_{i+1})$ is similar to the triangle $T(o, \widetilde{v}_i, \widetilde{v}_{i+1})$ for $0 \leq i \leq k-1$.

the similarity of the triangle $T(o, v_i, v_{i+1})$ to the triangle $T(o, \widetilde{v}_i, \widetilde{v}_{i+1})$ implies $|v_{i+1}|_2 < |\widetilde{v}_{i+1}|_2$ for $1 \leq i \leq k-1$. But then $|\widetilde{v}|_2 = |v_k|_2 < |\widetilde{v}_k|_2 = |v|_2$, which is a contradiction.

5.3 Dual Quermassintegrals of Sections

Throughout this section, let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior. We consider the *radial sum*

$$K \,\widetilde{+}\, tB_2^n(o,1) := \{o\} \cup \big\{x \in \mathbb{R}^n \backslash \{o\} : |x|_2 \leq \rho_K(x/|x|_2) + t\big\}, \qquad t \geq 0.$$

The set $K \,\widetilde{+}\, tB_2^n(o,1)$ is the star body in $\mathbb{R}^n$ whose radial function is the sum of the radial function of K with t times the radial function of $B_2^n(o,1)$. The so-called *dual quermassintegrals* $\widetilde{W}_l(K)$ arise as coefficients in the expansion

$$\mathrm{vol}_n\Big(K \,\widetilde{+}\, tB_2^n(o,1)\Big) = \sum_{l=0}^{n} \binom{n}{l} \widetilde{W}_l(K)\, t^l, \qquad t \geq 0.$$

Dual quermassintegrals (and, more generally, dual mixed volumes) were introduced by Lutwak [28]. See [14, 49] for further details. There are many parallels between quermassintegrals and dual quermassintegrals, so it is natural to consider the conjecture of Makai et al. [33] in the dual setting. We pose and solve such a question.

For each integer $0 \le l \le n-2$ and $\xi \in S^{n-1}$, we define the function

$$\widetilde{W}_{l,\xi}(t) := \widetilde{W}_l\Big((K - t\xi) \cap \xi^{\perp}\Big), \qquad -\rho_K(-\xi) < t < \rho_K(\xi),$$

where $\widetilde{W}_l\big((K-t\xi)\cap\xi^{\perp}\big)$ is the dual quermassintegral of the $(n-1)$-dimensional convex body $(K - t\xi) \cap \xi^{\perp}$ in $\xi^{\perp}$. It follows from the dual Kubota formula (e.g. Theorem A.7.2 in [14]) and Brunn's Theorem that

$$\widetilde{W}_l(K \cap \xi^{\perp}) = \widetilde{W}_{l,\xi}(0) = \max_{-\rho_K(-\xi)<t<\rho_K(\xi)} \widetilde{W}_{l,\xi}(t) \quad \text{for all} \quad \xi \in S^{n-1} \tag{5.12}$$

whenever K is origin-symmetric. For $l = 0$, (5.12) is equivalent to (1.1). We prove the converse statement when K is a C^1 convex body; that is, the boundary of K is a C^1 manifold, or equivalently $\rho_K \in C^1(S^{n-1})$.

Theorem 5.7. *Suppose $K \subset \mathbb{R}^n$ is a C^1 convex body containing the origin in its interior. If K satisfies (5.12) for some $1 \le l \le n-2$, then necessarily $K = -K$.*

The proof of Theorem 5.7 follows from formulas derived in [58]. These formulas involve spherical harmonics, and the fractional derivatives of $\widetilde{W}_{l,\xi}$ at $t = 0$.

The definition of fractional derivatives that was used in [58] differs slightly from our definition in Chapter 2. Let h be an integrable function on $\mathbb{R}$ which is m times continuously differentiable in a neighbourhood of zero. Let $q \in \mathbb{C}\backslash\{0, 1, \ldots, m-1\}$ with real part $-1 < \mathrm{Re}(q) < m$. In [58], the fractional derivative of h of order q at zero is given by

$$\begin{aligned} h^{(q)}(0) =& \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left(h(t) - \sum_{k=0}^{m-1} \frac{d^k}{ds^k} h(s)\Big|_{s=0} \frac{t^k}{k!} \right) dt \\ &+ \frac{1}{\Gamma(-q)} \int_1^{\infty} t^{-1-q} h(t)\, dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{1}{k!(k-q)} \frac{d^k}{dt^k} h(t)\Big|_{t=0}; \end{aligned}$$

see, for example, [25]. Defining $h^{(k)}(0)$ by the limit for $k = 0, 1, \ldots, m-1$

gives analytic function $q \mapsto h^{(q)}(0)$ for $q \in \mathbb{C}$ with $-1 < \mathrm{Re}(q) < m$, and

$$h^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} h(t)\Big|_{t=0} \quad \text{for} \quad k = 0, 1, \ldots, m-1.$$

In the following, we use the definition from [58] for fractional derivatives, rather than our definition in Chapter 2.

Note that $\widetilde{W}_{l,\xi}$ is continuously differentiable in a neighbourhood of zero when K is C^1, so we can consider the fractional derivatives of $\widetilde{W}_{l,\xi}$ at zero of order q, $-1 < \mathrm{Re}(q) < 1$.

Proof of Theorem 5.7. It is proven in [58] (at the bottom of page 8, in their Theorem 2) that

$$\int_{S^{n-1}} H_m^n(\xi)\, \widetilde{W}_{l,\xi}^{(q)}(0)\, d\xi = \frac{(n-1-l)\lambda_m(q)}{(n-1-q-l)(n-1)} \int_{S^{n-1}} H_m^n(\xi)\, \rho_K^{n-1-q-l}(\xi)\, d\xi \tag{5.13}$$

for all $-1 < q < 0$ and spherical harmonics H_m^n of dimension n and odd degree m. The multipliers $\lambda_m(q)$ in (5.13) come from an application of the Funke-Hecke Theorem. Let P_m^n denote the Legendre polynomial of dimension n and odd degree m. It is shown in [58] (on page 7) that, explicitly, $\lambda_m(q)$ is the fractional derivative of $f(t) = P_m^n(t)(1-t^2)^{(n-2-l)/2}$ of order q at $t = 0$. Therefore, $q \mapsto \lambda_m(q)$ is analytic, and (5.13) can immediately be extended to $-1 < q < 1$, $q \neq n-1-l$.

Observe that for odd integers m and $l = n-2$,

$$\lim_{q \to 1} \int_{S^{n-1}} H_m^n(\xi)\, \rho_K^{n-1-q-l}(\xi)\, d\xi = \int_{S^{n-1}} H_m^n(\xi)\, d\xi = 0$$

because of the orthogonality of spherical harmonics with different degrees.

Taking the limit as q approaches 1 in (5.13), we get

$$\begin{aligned}
&-\int_{S^{n-1}} H^n_m(\xi)\widetilde{W}'_{l,\xi}(0)\,dt = \int_{S^{n-1}} H^n_m(\xi)\widetilde{W}^{(1)}_{l,\xi}(0)\,dt \\
&= \begin{cases} \frac{(n-1-l)\lambda_m(1)}{(n-2-l)(n-1)} \int_{S^{n-1}} H^n_m(\xi)\rho_K^{n-2-l}(\xi)\,d\xi & \text{if} \quad l \neq n-2; \\ \frac{\lambda_m(1)}{n-1} \int_{S^{n-1}} H^n_m(\xi)\log\big(\rho_K(\xi)\big)\,d\xi & \text{if} \quad l = n-2, \end{cases}
\end{aligned} \tag{5.14}$$

for all odd integers m. We use L'Hospital's rule to evaluate the limit for the case $l = n-2$.

Calculating

$$\lambda_m(1) = f^{(1)}(0) = -\frac{d}{dt}P^n_m(t)(1-t^2)^{(n-2-l)/2}\Big|_{t=0} = -\frac{d}{dt}P^n_m(t)\Big|_{t=0},$$

it then follows from Lemma 3.3.9 and Lemma 3.3.8 in [19] that $\lambda_m(1) \neq 0$ for odd m. As (5.12) implies $\widetilde{W}'_{l,\xi}(0) = 0$ for all $\xi \in S^{n-1}$, we conclude from (5.14) and $\lambda_m(1) \neq 0$ that

- if $l \neq n-2$, the spherical harmonic expansion of ρ_K^{n-2-l} does not have any harmonics of odd degree;
- if $l = n-2$, the spherical harmonic expansion of $\log(\rho_K)$ does not have any harmonics of odd degree.

Consequently, ρ_K must be an even function, so K is origin-symmetric. $\square$

Remark 5.8. It can be seen that Theorem 5.7 is actually true for C^1 star bodies, i.e. compact sets with positive and C^1 radial functions. However, it is not necessary for origin-symmetric star bodies to satisfy (5.12).

Chapter 6

Grünbaum's Inequality for Projections

The content of this chapter comes from my paper with N. Zhang [53]. In Section 6.1, we prove some auxiliary lemmas. In Section 6.2, we present our main results (Theorem 6.6 and Corollary 6.7) and their proofs.

Theorem 6.6 says the following: for a convex body $K \subset \mathbb{R}^n$, $p > 0$, and a not identically zero concave function $\psi : K \to [0, \infty)$ with

$$g(\psi^p, K) := \frac{\int_K x\, \psi^p\, dx}{\int_K \psi^p\, dx} \in \operatorname{int}(K) = o,$$

we have

$$\operatorname{vol}_n\big(K \cap \xi^+\big) \geq \left(\frac{n}{n+1+p}\right)^n \operatorname{vol}_n(K) \quad \forall\, \xi \in S^{n-1}.$$

As a particular case, we get Grünbaum's inequality for projections (Corollary 6.7): for integers $1 \leq k \leq n$ and convex body $K \subset \mathbb{R}^n$ with $g(K) = o$,

$$\operatorname{vol}_k\Big((K|E) \cap \xi^+\Big) \geq \left(\frac{k}{n+1}\right)^k \operatorname{vol}_k\big(K|E\big) \quad \forall\, E \in G(n,k),\ \xi \in S^{n-1} \cap E.$$

6.1 Auxiliary Lemmas

We associate with a convex body $K \subset \mathbb{R}^n$, $z \in \operatorname{int}(K)$, and $\xi \in S^{n-1}$ the unique cone

$$G = G(K, z, \xi) = \operatorname{conv}\{a\xi + B, b\xi\}$$

in $\mathbb{R}^n$ for which

- $B \subset \xi^{\perp}$ is an $(n-1)$-dimensional Euclidean ball centred at the origin;
- $a, b \in \mathbb{R}$ and $a < b$;
- $\operatorname{vol}_{n-1}\Big((K - z) \cap \xi^{\perp}\Big) = \operatorname{vol}_{n-1}\Big((G - z) \cap \xi^{\perp}\Big)$;
- $\operatorname{vol}_{n}\Big((K - z) \cap \xi^{+}\Big) = \operatorname{vol}_{n}\Big((G - z) \cap \xi^{+}\Big)$;
- $\operatorname{vol}_n(K) = \operatorname{vol}_n(G)$.

We summarize some simple properties of G in the following lemma.

Lemma 6.1. *Let K be a convex body in $\mathbb{R}^n$, $z \in int(K)$, and $\xi \in S^{n-1}$. Let $G = G(K, z, \xi)$ be the previously defined cone. Then*

$$h_G(-\xi) \leq h_K(-\xi) \quad and \quad h_K(\xi) \leq h_G(\xi).$$

Furthermore,

$$vol_n\Big(\{x \in K : \langle x, \xi\rangle \geq t\}\Big) \leq vol_n\Big(\{x \in G : \langle x, \xi\rangle \geq t\}\Big) \quad \forall\ t \in \mathbb{R}; \tag{6.1}$$

if there is equality for all $t \in \mathbb{R}$, then $K = conv\{y_1 + L, y_2\}$ where

$$\begin{cases} L \subset \xi^{\perp} \text{ is an } (n-1)\text{-dimensional convex body;} \\ \langle y_1, -\xi\rangle = h_G(-\xi) \quad and \quad \langle y_2, \xi\rangle = h_G(\xi). \end{cases}$$

Proof. Assume without loss of generality that z is the origin. Let $\widetilde{K}$ be the Schwarz symmetral of K with respect to the direction ξ (see e.g. [14]). That is, $\widetilde{K}$ is the convex body in $\mathbb{R}^n$ for which $(\widetilde{K}-t\xi)\cap\xi^\perp$ is an $(n-1)$-dimensional Euclidean ball centred at the origin in $\xi^\perp$ with

$$\mathrm{vol}_{n-1}\Big((\widetilde{K}-t\xi)\cap\xi^\perp\Big)=\mathrm{vol}_{n-1}\Big((K-t\xi)\cap\xi^\perp\Big)\quad\forall\, t\in[-h_K(-\xi),h_K(\xi)].$$

It is easy to see that

$$h_{\widetilde{K}}(\pm\xi)=h_K(\pm\xi),\qquad G=G(K,0,\xi)=G(\widetilde{K},0,\xi),$$

and

$$\mathrm{vol}_n\Big(\{x\in\widetilde{K}:\,\langle x,\xi\rangle\geq t\}\Big)=\mathrm{vol}_n\Big(\{x\in K:\,\langle x,\xi\rangle\geq t\}\Big)\quad\forall\, t\in\mathbb{R}.$$

Suppose $h_{\widetilde{K}}(\xi)>h_G(\xi)$. We then have

$$G\cap\xi^+=\mathrm{conv}\{G\cap\xi^\perp,h_G(\xi)\xi\}\subsetneq\mathrm{conv}\{G\cap\xi^\perp,h_{\widetilde{K}}(\xi)\xi\}\subset\widetilde{K}\cap\xi^+,$$

which implies $\mathrm{vol}_n\big(G\cap\xi^+\big)<\mathrm{vol}_n\big(\widetilde{K}\cap\xi^+\big)$. This is a contradiction, so $h_{\widetilde{K}}(\xi)\leq h_G(\xi)$. Now, there is a $t_0\in(0,h_K(\xi)]$ for which

$$\{x\in G:\,0\leq\langle x,\xi\rangle\leq t_0\}\subset\{x\in\widetilde{K}:\,0\leq\langle x,\xi\rangle\leq t_0\}\tag{6.2}$$

and

$$\{x\in\widetilde{K}:\,t_0<\langle x,\xi\rangle\leq h_{\widetilde{K}}(\xi)\}\subset\{x\in G:\,t_0<\langle x,\xi\rangle\leq h_G(\xi)\};\tag{6.3}$$

otherwise, we will get a contradiction of the convexity of $\widetilde{K}$, or find that $\mathrm{vol}_n\big(\widetilde{K}\cap\xi^+\big)<\mathrm{vol}_n\big(G\cap\xi^+\big)$. The convexity of $\widetilde{K}$, the containment (6.2), and $\widetilde{K}\cap\xi^\perp=G\cap\xi^\perp$ together imply

$$\widetilde{K}\cap\big\{t\xi+\xi^\perp\big\}\subset G\cap\big\{t\xi+\xi^\perp\big\}\quad\forall\, t\in[-h_G(\xi),0].\tag{6.4}$$

Suppose $h_G(-\xi)>h_{\widetilde{K}}(-\xi)$. With (6.4), we then get

$$\{x\in\widetilde{K}:\,-h_{\widetilde{K}}(-\xi)\leq\langle x,\xi\rangle\leq0\}\subsetneq\{x\in G:\,-h_G(-\xi)\leq\langle x,\xi\rangle\leq0\}$$

and

$$\mathrm{vol}_n\big(\widetilde{K}\big) - \mathrm{vol}_n\big(\widetilde{K} \cap \xi^+\big) < \mathrm{vol}_n\big(G\big) - \mathrm{vol}_n\big(G \cap \xi^+\big),$$

which is again a contradiction. So $h_G(-\xi) \le h_{\widetilde{K}}(-\xi)$. Finally, we see that inequality (6.1) follows from the facts $\mathrm{vol}_n(\widetilde{K}) = \mathrm{vol}_n(G)$ and $\mathrm{vol}_n\big(\widetilde{K} \cap \xi^+\big) = \mathrm{vol}_n\big(G \cap \xi^+\big)$ combined with (6.2), (6.3), and (6.4).

If there is equality in inequality (6.1) for all $t \in \mathbb{R}$, then there will be equality in (6.2), (6.3), and (6.4). This shows $\widetilde{K} = G$. Because its Schwarz symmetral is a cone, K itself must be the cone given in the lemma statement. $\square$

Note. The concave functions in this chapter are always assumed to be continuous on their supports. Of course, the concavity of a function guarantees its continuity on the interior of its support in general.

Lemma 6.2. *Let K be a convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$, and $p > 0$. Let $\psi : K \to \mathbb{R}^+$ be a concave function, not identically zero. Put $G = G(K, g(\psi^p, K), \xi)$. There is a unique function $\Psi : G \to \mathbb{R}^+$ for which*

$$\begin{cases} \Psi \equiv f(\langle \cdot\,, \xi\rangle) \text{ for some non-decreasing } f : [-h_G(-\xi), h_G(\xi)] \to \mathbb{R}^+; \\ vol_n\Big(\{x \in K :\ \psi(x) \ge \tau\}\Big) = vol_n\Big(\{x \in G :\ \Psi(x) \ge \tau\}\Big) \quad \forall\, \tau \in \mathbb{R}. \end{cases}$$

This Ψ is concave. Furthermore,

$$\langle g(\psi^p, K), \xi\rangle \le \langle g(\Psi^p, G), \xi\rangle;$$

if there is equality, then

$$\begin{cases} K \text{ is the cone from the equality case of Lemma 6.1}; \\ \psi(x) = f(\langle x, \xi\rangle) \quad \forall\, x \in K. \end{cases}$$

Proof. Put

$$m := \min_{x \in K} \psi(x), \qquad M := \max_{x \in K} \psi(x).$$

Define functions $w : [m, M] \to [-h_G(-\xi), h_G(\xi)]$ and $W : [m, M] \to \mathbb{R}^+$ by

$$W(\tau) := \mathrm{vol}_n\Big(\{x \in K : \psi(x) \geq \tau\}\Big) = \mathrm{vol}_n\Big(\{x \in G : \langle x, \xi\rangle \geq w(\tau)\}\Big) \quad (6.5)$$

for all $\tau \in [m, M]$. Note that $|K| = |G|$ ensures w is well-defined.

The function $W^{\frac{1}{n}}$ is concave and strictly decreasing. As ψ is concave, we have

$$\begin{aligned}&\lambda\{x \in K : \psi(x) \geq \tau_1\} + (1-\lambda)\{x \in K : \psi(x) \geq \tau_2\}\\ &\subset \{x \in K : \psi(x) \geq \lambda\tau_1 + (1-\lambda)\tau_2\}\end{aligned}$$

for all $\lambda \in [0, 1]$ and $\tau_1, \tau_2 \in [m, M]$. Applying the Brunn-Minkowski inequality to these level sets shows $W^{\frac{1}{n}}$ is concave. The connectedness of K and the continuity of ψ guarantee $W^{\frac{1}{n}}$ is strictly decreasing.

The function w is convex and strictly increasing. Let $H > 0$ denote the height of the cone G, and let $V > 0$ denote the $(n-1)$-dimensional volume of its base. The set

$$\{x \in G : \langle x, \xi\rangle \geq w(\tau)\}$$

is a cone homothetic to G, with height $h_G(\xi) - w(\tau)$ and a base of some $(n-1)$-dimensional volume $v > 0$. It is necessary that

$$\frac{v}{V} = \left(\frac{h_G(\xi) - w(\tau)}{H}\right)^{n-1} \quad \text{and} \quad \frac{v(h_G(\xi) - w(\tau))}{n} = W(\tau),$$

so

$$w(\tau) = h_G(\xi) - \left(\frac{nH^{n-1}}{V}\right)^{\frac{1}{n}} W^{\frac{1}{n}}(\tau).$$

As $W^{\frac{1}{n}}$ is concave and strictly decreasing, w is convex and strictly increasing. It is then necessary that w has an inverse $w^{-1} : [-h_G(-\xi), \delta] \to [m, M]$ which is concave and strictly increasing, where $\delta := \max w \leq h_G(\xi)$.

Define $f : [-h_G(-\xi), h_G(\xi)] \to \mathbb{R}^+$ by

$$f(t) := w^{-1}(t) \quad \forall\, t \in [-h_G(-\xi), \delta], \quad \text{and} \quad f(t) := M \quad \forall\, t \in [\delta, h_G(\xi)].$$

By construction, f is non-decreasing with

$$\mathrm{vol}_n\Big(\{x \in K :\, \psi(x) \geq \tau\}\Big) = \mathrm{vol}_n\Big(\{x \in G :\, f(\langle x, \xi\rangle) \geq \tau\}\Big) \quad \forall\, \tau \in \mathbb{R}.$$

The uniqueness of f is easy to verify. As w^{-1} is concave and increasing, f is concave.

Although the upper level sets for $\Psi := f(\langle\,\cdot\,, \xi\rangle)$ have the same volume as the corresponding sets for ψ, they are "pushed" further in the direction ξ. More precisely, by equation (6.5) and Lemma 6.1,

$$\begin{aligned}
&\mathrm{vol}_n\Big(\{x \in K :\, \psi(x) \geq \tau\} \cap \{x \in K :\, \langle x, \xi\rangle \geq t\}\Big) \\
&\leq \min\Big\{\mathrm{vol}_n\Big(\{x \in K :\, \psi(x) \geq \tau\}\Big), \mathrm{vol}_n\Big(\{x \in K :\, \langle x, \xi\rangle \geq t\}\Big)\Big\} \\
&\leq \min\Big\{\mathrm{vol}_n\Big(\{x \in G :\, \Psi(x) \geq \tau\}\Big), \mathrm{vol}_n\Big(\{x \in G :\, \langle x, \xi\rangle \geq t\}\Big)\Big\} \\
&= \mathrm{vol}_n\Big(\{x \in G :\, \Psi(x) \geq \tau\} \cap \{x \in G :\, \langle x, \xi\rangle \geq t\}\Big) \qquad (6.6)
\end{aligned}$$

for all $\tau, t \in \mathbb{R}$. We have

$$\int_K \psi^p\, dx = p \int_0^\infty \tau^{p-1} W(\tau)\, d\tau = \int_G \Psi^p\, dx$$

using the "layer cake representation" for the L_p-norm of a function (e.g. Theorem 1.13 of [27]). The obvious generalization of Theorem 1.13 to products of functions, and inequality (6.6), give

$$\begin{aligned}
&\int_K \langle x,\xi\rangle \psi^p \, dx \\
&= p\int_0^\infty \int_0^\infty \tau^{p-1}\mathrm{vol}_n\Big(\{x\in K:\, \psi(x)\ge\tau\}\cap\{x\in K:\, \langle x,\xi\rangle\ge t\}\Big)\, dt\, d\tau \\
&\le p\int_0^\infty \int_0^\infty \tau^{p-1}\mathrm{vol}_n\Big(\{x\in G:\, \Psi(x)\ge\tau\}\cap\{x\in G:\, \langle x,\xi\rangle\ge t\}\Big)\, dt\, d\tau \\
&= \int_G \langle x,\xi\rangle \Psi^p \, dx, \qquad (6.7)
\end{aligned}$$

where we now assume without loss of generality that $h_K(-\xi)=0$.

Observe that equality in (6.7) implies equality in (6.6) for all $\tau, t\in\mathbb{R}$. Choosing $\tau=m$ gives

$$\mathrm{vol}_n\Big(\{x\in K:\, \langle x,\xi\rangle\ge t\}\Big) = \mathrm{vol}_n\Big(\{x\in G:\, \langle x,\xi\rangle\ge t\}\Big) \quad \forall\, t\in\mathbb{R}, \qquad (6.8)$$

so K is the cone from the equality case of Lemma 6.1. We need to show that

$$\psi(x) = f(\langle x,\xi\rangle) \quad \forall\, x\in K;$$

this is obvious when $m=M$, so assume $m<M$. Now, choosing $t=w(\tau)$ for $\tau\in[m,M]$ gives

$$\begin{aligned}
&\mathrm{vol}_n\Big(\{x\in K:\, \psi(x)\ge\tau\}\cap\{x\in K:\, \langle x,\xi\rangle\ge w(\tau)\}\Big) \\
&= \mathrm{vol}_n\Big(\{x\in G:\, \langle x,\xi\rangle\ge w(\tau)\}\Big), \qquad (6.9)
\end{aligned}$$

because

$$\{x\in G:\, \Psi(x)\ge\tau\} = \{x\in G:\, \langle x,\xi\rangle\ge w(\tau)\}.$$

Equalities (6.5), (6.8), and (6.9) show, respectively, that the sets

$$A_\tau := \{x\in K:\, \psi(x)\ge\tau\}, \qquad B_\tau := \{x\in K:\, \langle x,\xi\rangle\ge w(\tau)\} = \{x\in K:\, f(\langle x,\xi\rangle)\ge\tau\},$$

and $A_\tau \cap B_\tau$ each have the same volume as

$$C_\tau := \{x \in G : \langle x, \xi \rangle \geq w(\tau)\}$$

for $\tau \in [m, M]$. Therefore, A_τ and B_τ must coincide up to a set of measure zero. We also have

$$A_\tau = \overline{\{x \in K : \psi(x) > \tau\}} = \overline{\operatorname{int}(A_\tau)} \qquad \text{and} \qquad B_\tau = \overline{\operatorname{int}(B_\tau)} \tag{6.10}$$

for all $\tau \in [m, M)$, because ψ is continuous and concave, and B_τ is always an n-dimensional cone for $\tau < M$. If $A_\tau \neq B_\tau$ for a given $\tau \in [m, M)$, then (6.10) contradicts the fact that A_τ and B_τ only differ by a set of measure zero. It then follows that

$$A_M = \bigcap_{m \leq \tau < M} A_\tau = \bigcap_{m \leq \tau < M} B_\tau = B_M.$$

Because the upper level sets for ψ coincide exactly with those for $f(\langle \cdot, \xi \rangle)$, we must have $\psi \equiv f(\langle \cdot, \xi \rangle)$. $\square$

Remark 6.3. An inspection of Lemma 6.2 and its proof shows there is also a unique function $\widetilde{\Psi} : K \to \mathbb{R}^+$ whose upper level sets have the same volume as those for ψ, and which has the form $\widetilde{\Psi} \equiv \widetilde{f}(\langle \cdot, \xi \rangle)$ for some non-decreasing $\widetilde{f} : [-h_K(-\xi), h_K(\xi)] \to \mathbb{R}^+$. However, it is interesting to note that this $\widetilde{\Psi}$ is not concave in general. For a specific example, take $K = \operatorname{conv}\{(0,0),\ (1,0),\ (1,1)\} \subset \mathbb{R}^2$, $\xi = (1,0) \in S^1$, and

$$\psi(x) := 1 - \langle x, \xi \rangle \qquad \forall\, x \in K.$$

One will find that

$$\widetilde{\Psi}(x) = 1 - \sqrt{1 - \langle x, \xi \rangle^2} \qquad \forall\, x \in K,$$

which is in fact convex.

Lemma 6.4. *Let K be a convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$, and $p > 0$. Consider* *functions* $\phi, \Phi : K \to \mathbb{R}^+$ *defined by*

$$\phi(x) := h(\langle x, \xi\rangle) \qquad \text{and} \qquad \Phi(x) := \langle x, \xi\rangle + h_K(-\xi),$$

for some concave function $h : [-h_K(-\xi), h_K(\xi)] \to \mathbb{R}^+$*, not identically zero. Then*

$$\langle g(\phi^p, K), \xi\rangle \le \langle g(\Phi^p, K), \xi\rangle;$$

if there is equality, $\phi \equiv \tau \cdot \Phi$ *for some* $\tau > 0$*.*

Proof. As Φ is not identically zero, there is a unique $\tau > 0$ so that

$$\int_K \phi^p\, dx = \tau^p \int_K \Phi^p\, dx = \int_K (\tau \cdot \Phi)^p\, dx.$$

Assuming without loss of generality that $h_K(-\xi) = 0$ and $b := h_K(\xi) > 0$,

$$\begin{aligned} &\int_0^b h^p(t)\, \mathrm{vol}_{n-1}\Big(\{x \in K : \langle x, \xi\rangle = t\}\Big)\, dt \qquad (6.11)\\ &= \int_K \phi^p\, dx = \int_K (\tau\cdot\Phi)^p\, dx = \int_0^b (\tau \cdot t)^p\, \mathrm{vol}_{n-1}\Big(\{x \in K : \langle x, \xi\rangle = t\}\Big)\, dt. \end{aligned}$$

There exists $t_0 \in (0, b)$ such that $h(t_0) = \tau \cdot t_0$; otherwise, equation (6.11) is contradicted. Because h is concave with $h(0) \ge 0$,

$$h(t) \ge \tau \cdot t \quad \forall\, t \in [0, t_0], \quad h(t) \le \tau \cdot t \quad \forall\, t \in [t_0, b].$$

We then have

$$\int_0^b (t - t_0)\big(h^p(t) - (\tau \cdot t)^p\big)\, \mathrm{vol}_{n-1}\Big(\{x \in K : \langle x, \xi\rangle = t\}\Big)\, dt \le 0,$$

with $h(t) \equiv \tau \cdot t$ when there is equality. That is,

$$\begin{aligned}
&\int_0^b t\, h^p(t)\, \mathrm{vol}_{n-1}\Big(\{x \in K :\, \langle x, \xi\rangle = t\}\Big)\, dt \\
&\leq \int_0^b t\, (\tau \cdot t)^p\, \mathrm{vol}_{n-1}\Big(\{x \in K :\, \langle x, \xi\rangle = t\}\Big)\, dt,
\end{aligned}$$

or rather

$$\frac{\int_K \langle x, \xi\rangle \phi^p\, dx}{\int_K \phi^p\, dx} \leq \frac{\int_K \langle x, \xi\rangle (\tau \cdot \Phi)^p\, dx}{\int_K (\tau \cdot \Phi)^p\, dx} = \frac{\int_K \langle x, \xi\rangle \Phi^p\, dx}{\int_K \Phi^p\, dx},$$

with $\phi \equiv \tau \cdot \Phi$ when there is equality. $\square$

Remark 6.5. If we alter the statement of Lemma 6.4 so that $\phi : K \to \mathbb{R}^+$ is a concave function without necessarily having the particular form $\phi \equiv h(\langle\,\cdot\,, \xi\rangle)$, then it is possible that

$$\langle g(\phi^p, K), \xi\rangle > \langle g(\Phi^p, K), \xi\rangle.$$

For example, consider the closed curves

$$\begin{aligned}
\mathcal{C}_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_1 \in [-1, 1],\ x_2 = 1 - \sqrt{1 - x_1^2},\ x_3 = 0\}, \\
\mathcal{C}_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_1 \in [-1, 1],\ x_2 = 1,\ x_3 = \sqrt{1 - x_1^2}\},
\end{aligned}$$

which are arcs on a sphere in $\mathbb{R}^3$ of radius one and centred at $(0, 1, 0)$. Let $E, H \in G(3, 2)$ denote the x_1, x_2 - plane and the x_2, x_3 - plane, respectively. Then $K := \mathrm{conv}\{\mathcal{C}_1\}$ is half of a Euclidean disk in E, $L := \mathrm{conv}\{\mathcal{C}_1, \mathcal{C}_2\}$ is a convex body in $\mathbb{R}^3$, and $K = L|E$. For $(x_1, x_2, x_3) \in K$, define

$$\phi(x_1, x_2, x_3) := \mathrm{vol}_1\big(L \cap \{(x_1, x_2, x_3) + E^{\perp}\big) \quad \text{and} \quad \Phi(x_1, x_2, x_3) := x_2.$$

By the Brunn-Minkowski inequality, ϕ is concave. It can be shown for each $t \in [-1, 1]$ that

$$L \cap \{(t, 0, 0) + H\} = \mathrm{conv}\{(t, 1 - \sqrt{1 - t^2}, 0),\, (t, 1, 0),\, (t, 1, \sqrt{1 - t^2})\},$$

which is a right-angled triangle. With this more explicit representation for L, we can calculate

$$\frac{\int_K x\,\phi\,dx}{\int_K \phi\,dx} \approx (0,\ 0.705,\ 0) \quad \text{and} \quad \frac{\int_K x\,\Phi\,dx}{\int_K \Phi\,dx} \approx (0,\ 0.697,\ 0).$$

6.2 Main Results

Theorem 6.6. *Let K be a convex body in $\mathbb{R}^n$, and $p > 0$. Let $\psi : K \to \mathbb{R}^+$ be a concave function, not identically zero, with $g(\psi^p, K)$ at the origin. Then*

$$\frac{vol_n\big(K \cap \xi^+\big)}{vol_n(K)} \geq \left(\frac{n}{n+1+p}\right)^n \qquad \forall\ \xi \in S^{n-1};$$

there is equality for some ξ if and only if

$$\begin{cases} K = conv\{y_1 + L, y_2\}; \\ L \subset \xi^\perp \text{ is an } (n-1)\text{-dimensional convex body}; \\ y_1, y_2 \in \mathbb{R}^n \quad \text{with} \quad \langle y_1, \xi\rangle < 0 < \langle y_2, \xi\rangle; \\ \psi(x) = \tau[\langle x, \xi\rangle + h_K(-\xi)] \quad \forall\ x \in K, \quad \text{for some } \tau > 0; \\ g(\psi^p, K) = 0. \end{cases}$$

Proof. Put $G = G(K, 0, \xi)$. Define $\Phi : G \to \mathbb{R}^+$ by

$$\Phi(x) := \langle x, \xi\rangle + h_G(-\xi) \qquad \forall\ x \in G.$$

By Lemma 6.2 and Lemma 6.4,

$$0 = \frac{\int_K \langle x, \xi\rangle \psi^p\,dx}{\int_K \psi^p\,dx} \leq \frac{\int_G \langle x, \xi\rangle \Phi^p\,dx}{\int_G \Phi^p\,dx} =: c; \tag{6.12}$$

equality implies K and ψ satisfy the equality conditions given in the theorem statement. Given the definition of G and Lemma 6.1,

$$\frac{\mathrm{vol}_n\big(K\cap\xi^+\big)}{\mathrm{vol}_n(K)}=\frac{\mathrm{vol}_n\big(G\cap\xi^+\big)}{\mathrm{vol}_n(G)}\geq\frac{\mathrm{vol}_n\Big(\{x\in G:\,\langle x,\xi\rangle\geq c\}\Big)}{\mathrm{vol}_n(G)};$$

equality implies equality in (6.12).

Now suppose K and ψ satisfy these equality conditions, but without the requirement that the centroid of ψ^p is at the origin. Assume without loss of generality that $h_K(-\xi)=0$ and $b:=h_K(\xi)>0$. For some $\tau>0$, we have

$$\begin{aligned}&\int_K\langle x,\xi\rangle\psi^p\,dx=\int_0^b t(\tau\cdot t)^p\left(\mathrm{vol}_{n-1}(L)\left(1-\frac{t}{b}\right)^{n-1}\right)dt\\&=b^{2+p}\tau^p\mathrm{vol}_{n-1}(L)\frac{\Gamma(2+p)\Gamma(n)}{\Gamma(n+2+p)}\end{aligned}$$

and

$$\begin{aligned}\int_K\psi^p\,dx&=\int_0^b(\tau\cdot t)^p\left(\mathrm{vol}_{n-1}(L)\left(1-\frac{t}{b}\right)^{n-1}\right)dt\\&=b^{1+p}\tau^p\mathrm{vol}_{n-1}(L)\frac{\Gamma(1+p)\Gamma(n)}{\Gamma(n+1+p)},\end{aligned}$$

where Γ is the gamma function. So

$$d:=\frac{\int_K\langle x,\xi\rangle\psi^p\,dx}{\int_K\psi^p\,dx}=\left(\frac{1+p}{n+1+p}\right)b.$$

We can then calculate

$$\frac{\mathrm{vol}_n\Big(\{x\in K:\,\langle x,\xi\rangle\geq d\}\Big)}{\mathrm{vol}_n(K)}=\left(\frac{n}{n+1+p}\right)^n.$$

□

Corollary 6.7. *Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin,*

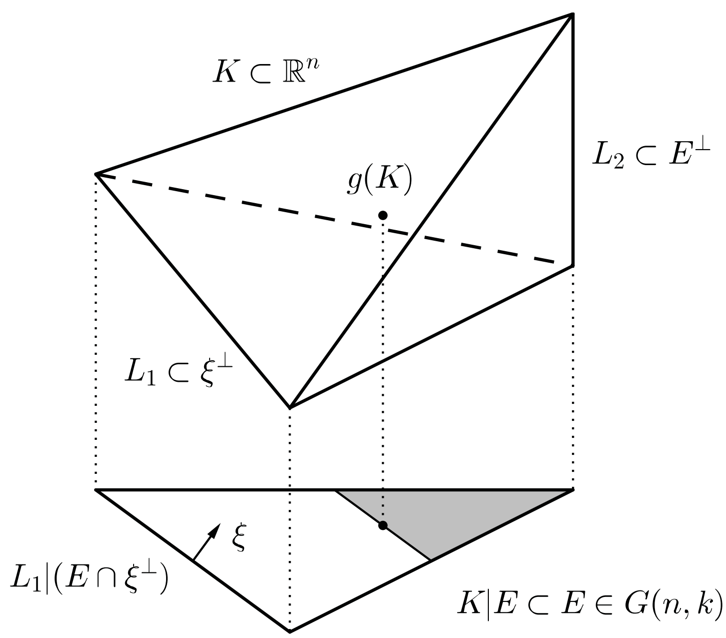

Figure 6.1: The equality conditions for Corollary 6.7.

and let $k \in \mathbb{Z}$ be such that $1 \le k \le n$. Then

$$\frac{vol_k\big((K|E) \cap \xi^+\big)}{vol_k\big(K|E\big)} \ge \left(\frac{k}{n+1}\right)^k \quad \forall\ E \in G(n,k), \quad \forall\ \xi \in S^{n-1} \cap E;$$

there is equality for some E and ξ if and only if $K = conv\{y_1 + L_1, y_2 + L_2\}$ where

$$\begin{cases} L_1 \subset \xi^\perp \text{ and } L_1|(E \cap \xi^\perp) \text{ are } (k-1)\text{-dimensional convex bodies;} \\ L_2 \subset E^\perp \text{ is an } (n-k)\text{-dimensional convex body;} \\ y_1, y_2 \in \mathbb{R}^n \quad \text{with} \quad \langle y_1, \xi \rangle < 0 < \langle y_2, \xi \rangle; \\ g(K) = 0. \end{cases}$$

See Figure 6.1 for an example of the equality case.

Proof. Suppose $1 \le k < n$. For $E \in G(n,k)$, define $\psi : K|E \to \mathbb{R}^+$ by

$$\psi(x) := \left[\mathrm{vol}_{n-k}\Big(K \cap \{x + E^\perp\}\Big)\right]^{\frac{1}{n-k}}.$$

By the Brunn-Minkowski inequality, ψ is concave. For all $\xi \in S^{n-1} \cap E$

$$\frac{\int_{K|E} \langle x, \xi \rangle \psi^{n-k}\, dx}{\int_{K|E} \psi^{n-k}\, dx} = \frac{\int_K \langle x, \xi \rangle\, dx}{\mathrm{vol}_n(K)} = 0,$$

so the centroid of ψ^{n-k} is at the origin. Therefore, by Theorem 6.6,

$$\frac{\mathrm{vol}_k\Big((K|E) \cap \xi^+\Big)}{\mathrm{vol}_k\big(K|E\big)} \geq \left(\frac{k}{k+1+(n-k)}\right)^k = \left(\frac{k}{n+1}\right)^k \quad \forall\, \xi \in S^{n-1} \cap E;$$

there is equality for some ξ if and only if

$$\begin{cases} K|E = \mathrm{conv}\{y_1 + L, y_2\}; \\ L \subset E \cap \xi^\perp \in G(n, k-1) \text{ is a } (k-1)\text{-dimensional convex body;} \\ y_1, y_2 \in E \quad \text{with} \quad \langle y_1, \xi \rangle < 0 < \langle y_2, \xi \rangle; \\ \psi(x) = \tau[\langle x, \xi \rangle + h_K(-\xi)] \quad \forall\, x \in K|E, \quad \text{for some } \tau > 0; \\ g(\psi^{n-k}, K|E) = 0. \end{cases} \tag{6.13}$$

The conditions (6.13) are equivalent to the equality conditions in the corollary statement. $\square$

Chapter 7

Grünbaum's Inequality for Sections

The content of this chapter comes from my preprint with S. Myroshnychenko and N. Zhang [43]. Our main result, Theorem 7.1, is formerly presented and proven in Section 7.1. We then state and prove Corollary 7.7 in Section 7.2, and Corollary 7.8 in Section 7.3.

Theorem 7.1 says the following:

$$\int_0^\infty f(s\theta)\,ds \geq \left(\frac{\gamma+1}{\gamma n+\gamma+1}\right)^{\frac{\gamma+1}{\gamma}} \int_{-\infty}^\infty f(s\theta)\,ds \quad \forall\ \theta \in S^{n-1}$$

for every γ-concave function $f : \mathbb{R}^n \to [0,\infty)$, $\gamma > 0$, with $0 < \int_{\mathbb{R}^n} f(x)\,dx < \infty$ and $\int_{\mathbb{R}^n} x f(x)\,dx = o$. From Theorem 7.1, we get Corollary 7.7:

$$\int_{E\cap\theta^+} f(x)dx \geq \left(\frac{k\gamma+1}{(n+1)\gamma+1}\right)^{\frac{k\gamma+1}{\gamma}} \int_E f(x)dx \ \ \forall\ E \in G(n,k),\ \theta \in S^{n-1}\cap E$$

for every γ-concave $f : \mathbb{R}^n \to [0,\infty)$ with $\gamma > 0$, $0 < \int_{\mathbb{R}^n} f(x)\,dx < \infty$, and $\int_{\mathbb{R}^n} x f(x)\,dx = o$. Corollary 7.8 is Grünbaum's inequality for sections: for integers $1 \leq k \leq n$ and convex body $K \subset \mathbb{R}^n$ with $g(K) = o$,

$$\mathrm{vol}_k\big(K\cap E\cap\theta^+\big) \geq \left(\frac{k}{n+1}\right)^k \mathrm{vol}_k\big(K\cap E\big) \quad \forall\ E \in G(n,k),\ \theta \in S^{n-1}\cap E.$$

7.1 One Dimensional Sections of γ-Concave Functions

A function $f : \mathbb{R}^n \to [0,\infty)$ is γ-*concave* for $\gamma \in (-\infty,0) \cup (0,\infty)$ if

$$f\big(\lambda x + (1-\lambda)y\big) \geq \big[\lambda f^\gamma(x) + (1-\lambda) f^\gamma(y)\big]^{\frac{1}{\gamma}} \tag{7.1}$$

for all $0 \leq \lambda \leq 1$ and all $x, y \in \mathbb{R}^n$ such that $f(x) \cdot f(y) \neq 0$. We say f is γ-*affine* if inequality (7.1) is always an equality. These definitions are extended to $\gamma = 0, \pm\infty$ by continuity, and log-concavity corresponds to the case $\gamma = 0$. The support of a function f will be denoted by $K_f := \operatorname{supp}(f)$. If f is γ-concave, then K_f is a convex set. If f is γ-concave for some $\gamma \in (0,\infty)$ with a positive and finite integral, then K_f is a convex body in $\mathbb{R}^n$ (see Remark 2.2.7 (i) in [4]); in this case, we define the *centroid* of f by

$$g(f) := \int_{\mathbb{R}^n} x f(x)\, dx \Big/ \int_{\mathbb{R}^n} f(x)\, dx \in \operatorname{int}(K_f).$$

Note. We will always implicitly assume that a γ-concave function is continuous on its support. This does not lead to a real loss of generality in our results. Indeed, a γ-concave f must be continuous on the (relative) interior of K_f; assuming f is continuous on K_f at most requires a redefinition of f on a set of measure zero.

Our main result is the following theorem:

Theorem 7.1. *Fix $\theta \in S^{n-1}$ and $\gamma \in (0,\infty)$. Let $f : \mathbb{R}^n \to [0,\infty)$ be a γ-concave function with $0 < \int_{\mathbb{R}^n} f(x)\, dx < \infty$ and centroid at the origin. Then*

$$\frac{\int_0^\infty f(s\theta)\, ds}{\int_{-\infty}^\infty f(s\theta)\, ds} \geq \left(\frac{\gamma+1}{\gamma n + \gamma + 1}\right)^{\frac{\gamma+1}{\gamma}}.$$

There is equality if and only if

- $f(x) = m\mathcal{X}_{K_f}(x)\Big(-\langle x, \xi\rangle + r\langle \theta, \xi\rangle\Big)^{\frac{1}{\gamma}}$ *for some constants $m, r > 0$ and a unit vector $\xi \in S^{n-1}$ such that $\langle \theta, \xi\rangle > 0$;*

- $K_f = conv\left(-\left(\frac{n\gamma}{\gamma+1}\right) r\theta,\ r\theta + D\right)$ *for some* $(n-1)$*-dimensional convex body* $D \subset \xi^{\perp}$ *whose centroid (taken in* $\xi^{\perp}$*) is at the origin.*

For the remainder of Section 2 we fix $\theta \in S^{n-1}$, $\gamma \in (0,\infty)$, and a γ-concave $f : \mathbb{R}^n \to [0,\infty)$ satisfying the hypotheses of Theorem 7.1. We prove Theorem 7.1 in subsections 2.1 to 2.3 by transforming f into a function having the form from the equality case, while showing that the ratio $\int_{\langle g(f),\theta\rangle}^{\infty} f(s\theta)\,ds / \int_{-\infty}^{\infty} f(s\theta)\,ds$ can only decrease.

7.1.1 Replacing γ-Concave Slices with γ-Affine Slices

For each $x' \in K_f|\theta^{\perp}$, define $f_{x'} : \mathbb{R} \to [0,\infty)$ to be the one dimensional restriction $f_{x'}(s) := f(x' + s\theta)$. We will transform each slice $f_{x'}$ into a γ-affine function of the form

$$F_{x'}(s) := \mathcal{X}_{\left[\Psi(x'),\frac{H(x')}{\beta}\right]}(s)(-\beta s + H(x'))^{\frac{1}{\gamma}}, \tag{7.2}$$

where $\Psi, H : K_f|\theta^{\perp} \to \mathbb{R}$ are functions and $\beta > 0$ is a constant. As the first step in constructing $F_{x'}$, choose

$$\beta := \frac{\gamma f_o^{\gamma+1}(0)}{(\gamma+1)\int_0^{\infty} f_o(s)\,ds} > 0$$

so that

$$\int_0^{\infty} f_o(s)\,ds = \left(\frac{\gamma f_o^{\gamma+1}(0)}{\gamma+1}\right)\frac{1}{\beta} = \int_0^{\frac{f_o^{\gamma}(0)}{\beta}} (-\beta s + f_o^{\gamma}(0))^{\frac{1}{\gamma}}\,ds. \tag{7.3}$$

Before describing H, we introduce the auxiliary function $\widetilde{H} : K_f|\theta^{\perp} \to \mathbb{R}$ defined by

$$\widetilde{H}(x') := \max_{a \in \operatorname{supp}(f_{x'})} \widetilde{H}(x'; a) \qquad \text{for} \qquad x' \in K_f|\theta^{\perp},$$

where

$$\widetilde{H}(x';a) := \left[\frac{\beta(\gamma+1)}{\gamma}\int_a^\infty f_{x'}(s)\,ds\right]^{\frac{\gamma}{\gamma+1}} + \beta a.$$

The function $\widetilde{H}$ is well-defined with $\widetilde{H}(x') \in \mathbb{R}$ for every $x' \in K_f|\theta^\perp$, because $\operatorname{supp}(f_{x'})$ is a compact interval and $\widetilde{H}(x';\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous. For the moment, fix $a \in \operatorname{supp}(f_{x'})$ and $h \geq \widetilde{H}(x')$. It follows from the definition of $\widetilde{H}$ that $h \geq \beta a$. Furthermore,

$$\int_a^\infty f_{x'}(s)\,ds < \frac{\gamma}{\beta(\gamma+1)}(-\beta a + h)^{\frac{\gamma+1}{\gamma}} = \int_a^{\frac{h}{\beta}} (-\beta s + h)^{\frac{1}{\gamma}}\,ds \tag{7.4}$$

if and only if $h > \widetilde{H}(x')$ or $h = \widetilde{H}(x') > \widetilde{H}(x';a)$, and

$$\int_a^\infty f_{x'}(s)\,ds = \frac{\gamma}{\beta(\gamma+1)}(-\beta a + h)^{\frac{\gamma+1}{\gamma}} = \int_a^{\frac{h}{\beta}} (-\beta s + h)^{\frac{1}{\gamma}}\,ds \tag{7.5}$$

if and only if $h = \widetilde{H}(x') = \widetilde{H}(x';a)$. More generally,

$$\int_a^\infty f_{x'}(s)\,ds \leq \int_a^\infty \chi_{\left(-\infty,\frac{h}{\beta}\right]}(s)(-\beta s + h)^{\frac{1}{\gamma}}\,ds \tag{7.6}$$

for all $a \in \mathbb{R}$ and $h \geq \widetilde{H}(x')$.

We now prove that $\widetilde{H}(o) = f_o^\gamma(0)$. The function $f_o^\gamma : \mathbb{R} \to [0,\infty)$ is concave on its support,

$$l(s) := \mathcal{X}_{\left(-\infty,\frac{f_o^\gamma(0)}{\beta}\right]}(s)(-\beta s + f_o^\gamma(0))$$

is affine on its support, and $f_o^\gamma(0) = l(0)$; these facts and equality (7.3) imply there is some $0 < s' < f_o^\gamma(0)/\beta$ for which

$$\begin{aligned}
& f_o(s) < (-\beta s + f_o^\gamma(0))^{\frac{1}{\gamma}} \quad \text{whenever} \quad s < 0,\\
& f_o(s) > (-\beta s + f_o^\gamma(0))^{\frac{1}{\gamma}} \quad \text{whenever} \quad 0 < s < s',\\
\text{and} \quad & f_o(s) < (-\beta s + f_o^\gamma(0))^{\frac{1}{\gamma}} \quad \text{whenever} \quad s' < s < \frac{f_o^\gamma(0)}{\beta}.
\end{aligned}$$

It follows that $\operatorname{supp}(f_o) \subset \big(-\infty, f_o^\gamma(0)/\beta\big]$ and

$$\int_a^\infty \Big(l^{\frac{1}{\gamma}}(s) - f_o(s)\Big)\,ds \geq 0 \quad \text{for all} \quad a \in \mathbb{R}.$$

Therefore, if $\widetilde{H}(o) > f_o^\gamma(0)$, then

$$\int_a^{\frac{\widetilde{H}(o)}{\beta}} (-\beta s + \widetilde{H}(o))^{\frac{1}{\gamma}}\,ds > \int_a^{\frac{f_o^\gamma(0)}{\beta}} (-\beta s + f_o^\gamma(0))^{\frac{1}{\gamma}}\,ds \geq \int_a^\infty f_o(s)\,ds$$

for every $a \in \operatorname{supp}(f_o)$. Choosing $a \in \operatorname{supp}(f_o)$ so that $\widetilde{H}(o) = \widetilde{H}(o, a)$, this last inequality contradicts (7.5). On the other hand, if $\widetilde{H}(o) < f_o^\gamma(0)$, then equation (7.3) contradicts (7.4).

We claim $\widetilde{H}$ is concave on $K_f|\theta^\perp$. Indeed, let $0 \leq \lambda \leq 1$ and $x_1', x_2' \in K_f|\theta^\perp$. For $j = 1, 2$, choose $a_j \in \operatorname{supp}(f_{x_j'})$ so that $\widetilde{H}(x_j') = \widetilde{H}(x_j'; a_j)$. The Borell-Brascamp-Lieb inequality (see, for example, Theorem 10.1 in [13]), equality (7.5), and inequality (7.6) then imply

$$\begin{aligned}
&\left[\frac{\gamma}{\beta(\gamma+1)}\right]^{\frac{\gamma}{\gamma+1}} \Big(-\beta\big(\lambda a_1 + (1-\lambda)a_2\big) + \widetilde{H}\big(\lambda x_1' + (1-\lambda)x_2'\big)\Big) \\
=& \left[\int_{\lambda a_1 + (1-\lambda)a_2}^{\frac{\widetilde{H}\left(\lambda x_1' + (1-\lambda)x_2'\right)}{\beta}} (-\beta s + \widetilde{H}\big(\lambda x_1' + (1-\lambda)x_2'\big))^{\frac{1}{\gamma}}\,ds\right]^{\frac{\gamma}{\gamma+1}} \\
\geq& \left[\int_{\lambda a_1 + (1-\lambda)a_2}^{\infty} f(\lambda x_1' + (1-\lambda)x_2' + s\theta)\,ds\right]^{\frac{\gamma}{\gamma+1}} \\
\geq& \lambda \left[\int_{a_1}^\infty f(x_1' + s\theta)\,ds\right]^{\frac{\gamma}{\gamma+1}} + (1-\lambda)\left[\int_{a_2}^\infty f(x_2' + s\theta)\,ds\right]^{\frac{\gamma}{\gamma+1}} \\
=& \lambda \left[\int_{a_1}^{\frac{\widetilde{H}(x_1')}{\beta}} (-\beta s + \widetilde{H}(x_1'))^{\frac{1}{\gamma}}\,ds\right]^{\frac{\gamma}{\gamma+1}} + (1-\lambda)\left[\int_{a_2}^{\frac{\widetilde{H}(x_2')}{\beta}} (-\beta s + \widetilde{H}(x_2'))^{\frac{1}{\gamma}}\,ds\right]^{\frac{\gamma}{\gamma+1}} \\
=& \lambda \left[\frac{\gamma}{\beta(\gamma+1)}\right]^{\frac{\gamma}{\gamma+1}} (-\beta a_1 + \widetilde{H}(x_1')) + (1-\lambda)\left[\frac{\gamma}{\beta(\gamma+1)}\right]^{\frac{\gamma}{\gamma+1}} (-\beta a_2 + \widetilde{H}(x_2')) \\
=& \left[\frac{\gamma}{\beta(\gamma+1)}\right]^{\frac{\gamma}{\gamma+1}} \Big(-\beta\big(\lambda a_1 + (1-\lambda)a_2\big) + \lambda\widetilde{H}(x_1') + (1-\lambda)\widetilde{H}(x_2')\Big).
\end{aligned}$$

Therefore, we must have

$$\widetilde{H}\big(\lambda x_1' + (1-\lambda)x_2'\big) \geq \lambda \widetilde{H}(x_1') + (1-\lambda)\widetilde{H}(x_2').$$

As $\widetilde{H} : K_f|\theta^\perp \to \mathbb{R}$ is concave, there is a linear function $L : \theta^\perp \to \mathbb{R}$ such that $\widetilde{H}(x') \leq \widetilde{H}(o) + L(x')$ for every $x' \in K_f|\theta^\perp$. Recalling that $\widetilde{H}(o) = f^\gamma(o)$, we now put

$$H(x') := f^\gamma(o) + L(x') \quad \text{for all} \quad x' \in K_f|\theta^\perp,$$

so that H is an affine function on $K_f|\theta^\perp$.

Having defined $\beta > 0$ and $H : K_f|\theta^\perp \to \mathbb{R}$, we finally choose $\Psi(x') \in \mathbb{R}$ so that $\Psi(x') \leq H(x')/\beta$ and

$$\int_{-\infty}^{\infty} F_{x'}(s)\, ds = \int_{-\infty}^{\infty} f_{x'}(s)\, ds, \tag{7.7}$$

where $F_{x'}(s)$ is defined as in (7.2). Then,

$$\frac{\gamma}{\beta(\gamma+1)}(-\beta\Psi(x') + H(x'))^{\frac{\gamma+1}{\gamma}} = \int_{\Psi(x')}^{\frac{H(x')}{\beta}} (-\beta s + H(x'))^{\frac{1}{\gamma}}\, ds = \int_{-\infty}^{\infty} f(x' + s\theta)\, ds,$$

which gives

$$\Psi(x') = \frac{1}{\beta}\left[f^\gamma(o) + L(x') - \left(\frac{\beta(\gamma+1)}{\gamma}\int_{-\infty}^{\infty} f(x'+s\theta)\, ds\right)^{\frac{\gamma}{\gamma+1}}\right].$$

Since $L(x')$ is linear and $x' \mapsto \int_{\mathbb{R}} f(x'+s\theta)\, ds$ is $\frac{\gamma}{\gamma+1}$-concave (again by the Borell-Brascamp-Lieb inequality), we have that $\Psi : K_f|\theta^\perp \to \mathbb{R}$ is convex.

Now, define the function $F : \mathbb{R}^n \to [0,\infty)$ by

$$F(x) := \mathcal{X}_{K_F}(x)\Big(-\beta\langle x,\theta\rangle + H\big(x - \langle x,\theta\rangle\theta\big)\Big)^{\frac{1}{\gamma}} \quad \text{for} \quad x \in \mathbb{R}^n. \tag{7.8}$$

Here,

$$K_F = \left\{ x \in \mathbb{R}^n : x' = \big(x - \langle x, \theta\rangle\theta\big) \in K_f|\theta^\perp \text{ and } \Psi(x') \le \langle x, \theta\rangle \le \frac{H(x')}{\beta} \right\},$$

and β, H, Ψ are as previously constructed. The set K_F is a convex body in $\mathbb{R}^n$ with $K_F|\theta^\perp = K_f|\theta^\perp$, because $\Psi, H : K_f|\theta^\perp \to \mathbb{R}$ are, respectively, convex and concave with $\Psi < H/\beta$ on the relative interior of $K_f|\theta^\perp$, and $\Psi \le H/\beta$ on the relative boundary. Therefore, it is clear that F is γ-affine with $\mathrm{supp}(F) = K_F$.

Also note that $F(x' + s\theta) \equiv F_{x'}(s)$ for each $x' \in K_F|\theta^\perp$, where $F_{x'}$ is the γ-affine slice defined in (7.2). Equality (7.3) remains true if the right-hand side is replaced with $\int_0^\infty F_o(s)\,ds$ because of $H(o) = f^\gamma(o)$ and the choice of $\Psi(o)$ in (7.7). Similarly, (7.6) is still valid when the right-hand side is replaced with $\int_a^\infty F_{x'}(s)\,ds$. When we reference (7.3) and (7.6) in the proof of the next lemma, we will be referring to their altered forms.

Lemma 7.2. *The centroid $g(F)$ lies on the θ-axis and $\langle g(F), \theta\rangle \ge 0$. Furthermore,*

$$\frac{\int_{\langle g(F),\theta\rangle}^\infty F(s\theta)\,ds}{\int_{-\infty}^\infty F(s\theta)\,ds} \le \frac{\int_0^\infty F(s\theta)\,ds}{\int_{-\infty}^\infty F(s\theta)\,ds} = \frac{\int_0^\infty f(s\theta)\,ds}{\int_{-\infty}^\infty f(s\theta)\,ds}, \tag{7.9}$$

with equality if and only if $K_F = K_f$ and $F \equiv f$.

Proof. Because the mass of F along lines parallel to $\mathbb{R}\theta$ is the same as for f (see equation (7.7)), $g(F)$ will lie on the θ-axis. Integration by parts, the fact that $H(x') \ge \widetilde{H}(x')$, and inequality (7.6) together imply

$$\int_{-\infty}^\infty s\big(F_{x'}(s) - f_{x'}(s)\big)\,ds = \int_{-\infty}^\infty \int_t^\infty \big(F_{x'}(s) - f_{x'}(s)\big)\,ds\,dt \ge 0 \tag{7.10}$$

for all $x' \in K_f|\theta^\perp$. Inequality (7.10), equation (7.7), and $K_F|\theta^\perp = K_f|\theta^\perp$ now show

$$\langle g(F),\theta\rangle = \frac{\int_{\mathbb{R}^n}\langle x,\theta\rangle F(x)\,dx}{\int_{\mathbb{R}^n} F(x)\,dx} = \frac{\int_{K_F|\theta^\perp}\int_{-\infty}^{\infty} sF_{x'}(s)\,ds\,dx'}{\int_{K_F|\theta^\perp}\int_{-\infty}^{\infty} F_{x'}(s)\,ds\,dx'}$$
$$\geq \frac{\int_{K_f|\theta^\perp}\int_{-\infty}^{\infty} sf_{x'}(s)\,ds\,dx'}{\int_{K_f|\theta^\perp}\int_{-\infty}^{\infty} f_{x'}(s)\,ds\,dx'} = \langle g(f),\theta\rangle = 0. \tag{7.11}$$

Inequality (7.11), equation (7.3), and equation (7.7) immediately give (7.9).

Suppose there is equality in (7.9). Equality in (7.9) is only possible if $\langle g(F),\theta\rangle = 0$, which implies equality in (7.11). It then follows from inequality (7.10) and the equality in (7.11) that

$$\int_{K_f|\theta^\perp}\left|\int_{-\infty}^{\infty} s\big(F_{x'}(s)-f_{x'}(s)\big)\,ds\right|\,dx'$$
$$= \int_{K_F|\theta^\perp}\int_{-\infty}^{\infty} sF_{x'}(s)\,ds\,dx' - \int_{K_f|\theta^\perp}\int_{-\infty}^{\infty} sf_{x'}(s)\,ds\,dx' = 0.$$

With continuity, we necessarily have

$$\int_{-\infty}^{\infty} s\big(F_{x'}(s)-f_{x'}(s)\big)\,ds = 0$$

for every $x'\in K_f|\theta^\perp$, so there is equality in (7.10). Inequality (7.6) and the equality in (7.10) imply

$$\int_{-\infty}^{\infty}\left|\int_t^{\infty}\big(F_{x'}(s)-f_{x'}(s)\big)\,ds\right|\,dt = \int_{-\infty}^{\infty}\int_t^{\infty}\big(F_{x'}(s)-f_{x'}(s)\big)\,ds\,dt = 0.$$

Again invoking continuity, we get that

$$\int_t^{\infty} f_{x'}(s)\,ds = \int_t^{\infty} F_{x'}(s)\,ds$$

for all $x'\in K_f|\theta^\perp$ and $t\in\mathbb{R}$, so the supports of F and f must coincide. We conclude $F\equiv f$, after differentiating both sides of the last equation with respect to t. □

7.1.2 Replacing the Domain with a Cone

Let $q : \mathbb{R}^n \to [0, \infty)$ be a γ-affine function with centroid at the origin, and having the form

$$q(x) = \mathcal{X}_{K_q}(x)\Big(- \alpha\langle x, \theta\rangle + G\big(x - \langle x, \theta\rangle\theta\big)\Big)^{\frac{1}{\gamma}};$$

$\alpha > 0$ is any positive constant, and $K_q = \operatorname{supp}(q)$ is any convex body such that

$$K_q = \left\{ x \in \mathbb{R}^n : x' = \big(x - \langle x, \theta\rangle\theta\big) \in K_q|\theta^{\perp} \text{ and } \Phi(x') \le \langle x, \theta\rangle \le \frac{G(x')}{\alpha} \right\}$$

for some respectively convex and affine functions Φ, $G : K_q|\theta^{\perp} \to \mathbb{R}$. Distinct level sets of q lie within distinct but parallel hyperplanes, because q is γ-affine. Also, the set $\{q(x) = 0\} \cap K_q$ lies entirely within the boundary of K_q and intersects the positive θ-axis, because of the particular form of q. Let $\eta \in S^{n-1}$ be the outward facing unit normal to $\{q(x) = 0\} \cap K_q$ (see Figure 7.1). We then have

$$\{q(x) = 0\} \cap K_q = K_q \cap \{h_{K_q}(\eta)\, \eta + \eta^{\perp}\}$$

and $\langle \theta, \eta\rangle > 0$.

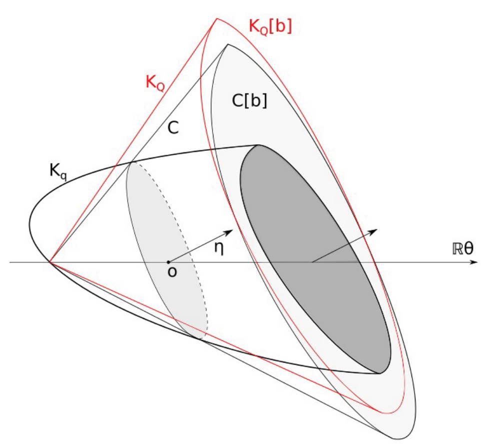

Figure 7.1: The construction of cones C and K_Q.

Let C be the n-dimensional cone with vertex $-\rho_{K_q}(-\theta)\,\theta \in K_q$, base lying in the hyperplane $\{h_{K_q}(\eta)\,\eta + \eta^\perp\}$, and for which $C \cap \eta^\perp = K_q \cap \eta^\perp$. Note that

$$-h_{K_q}(-\eta) \leq -h_C(-\eta) = -\rho_{K_q}(-\theta)\langle\theta,\eta\rangle < 0 < h_C(\eta) = h_{K_q}(\eta).$$

The "section volume" functions

$$A_{C,\eta}(t) = \mathrm{vol}_{n-1}\big(C \cap \{t\eta + \eta^\perp\}\big), \quad A_{K_q,\eta}(t) = \mathrm{vol}_{n-1}\big(K_q \cap \{t\eta + \eta^\perp\}\big), \quad t \in \mathbb{R},$$

are $1/(n-1)$-concave by the Brunn-Minkowski inequality. In fact, an explicit calculation shows $A_{C,\eta}$ is $1/(n-1)$-affine. As we also have

$$A_{C,\eta}\Big(-h_C(-\eta)\Big) = 0 \leq A_{K_q,\eta}\Big(-h_C(-\eta)\Big) \quad \text{and} \quad A_{C,\eta}(0) = A_{K_q,\eta}(0) > 0,$$

it is necessary that

$$\begin{aligned} A_{C,\eta}(t) &\leq A_{K_q,\eta}(t) \quad \text{for all} \quad t \leq 0, \\ A_{C,\eta}(t) &\geq A_{K_q,\eta}(t) \quad \text{for all} \quad t \geq 0. \end{aligned} \tag{7.12}$$

For convenience put $a := -h_C(-\eta)$, $b := h_C(\eta)$, and define

$$C[t] := C \cap \{t\eta + \eta^\perp\}, \quad t \in \mathbb{R}.$$

For each $t \in (a,b]$, $C[t]$ is an $(n-1)$-dimensional convex body whose centroid within the hyperplane $\{t\eta + \eta^\perp\}$ is given by

$$g\big(C[t]\big) = \left(\frac{1}{\mathrm{vol}_{n-1}\big(C[t]\big)} \int_{C[t]} x\,dx\right) \in \{t\eta + \eta^\perp\} \subset \mathbb{R}^n,$$

in terms of the ambient coordinates of $\mathbb{R}^n$.

Now, define the cone

$$K_Q := \mathrm{conv}\left(-\rho_{K_q}(-\theta)\,\theta,\ C[b] - g\big(C[b]\big) + \frac{b}{\langle\theta,\eta\rangle}\theta\right).$$

The cones K_Q and C have the same vertex, they have the same width in the direction η, and their sections $K_Q[t]$, $C[t]$ are translates lying in the same hyperplane $\{t\eta + \eta^\perp\}$. Therefore, the inequalities in (7.12) are valid for $A_{K_Q,\eta}$ in place of $A_{C,\eta}$. We also have

$$\begin{aligned} g(K_Q[t]) &= \left(\frac{b-t}{b-a}\right)\big(-\rho_{K_q}(-\theta)\theta\big) + \left(1 - \frac{b-t}{b-a}\right) g(K_Q[b]) \\ &= \left(\frac{b-t}{b-a}\right)\big(-\rho_{K_q}(-\theta)\theta\big) + \left(\frac{t-a}{b-a}\right) g(K_Q[b]) \end{aligned} \tag{7.13}$$

for all $t \in [a, b]$, because $K_Q[t]$ is a dilated and translated copy of $K_Q[b]$ with

$$K_Q[t] = \left(\frac{b-t}{b-a}\right)\big(-\rho_{K_q}(-\theta)\theta\big) + \left(\frac{t-a}{b-a}\right) K_Q[b].$$

Define the γ-affine function $Q : \mathbb{R}^n \to [0, \infty)$ by

$$Q(x) = \mathcal{X}_{K_Q}(x)\, q\left(\frac{\langle x, \eta\rangle}{\langle \theta, \eta\rangle}\theta\right).$$

The support of Q is K_Q, Q is constant on the sections $K_Q[t]$, and

$$Q(t\theta) = q(t\theta) \qquad \text{for all} \qquad t \in \mathbb{R}. \tag{7.14}$$

Lemma 7.3. *There is a* $0 < \lambda_0 < 1$ *so that*

$$g(Q) = \lambda_0\big(-\rho_{K_q}(-\theta)\theta\big) + (1-\lambda_0) g(K_Q[b])$$

Proof. First, note that

$$\int_{K_Q} x\, Q(x)\, dx = \int_a^b \int_{K_Q[t]} y\, Q(y)\, dy\, dt = \int_a^b q\left(\frac{t\theta}{\langle \theta, \eta\rangle}\right) A_{K_Q,\eta}(t) g(K_Q[t])\, dt,$$

because Q is constant on the sections $K_Q[t]$. Integration by parts then gives

$$\begin{aligned}\int_{K_Q} x\, Q(x)\, dx = &\left(\int_a^b q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right) A_{K_Q,\eta}(t)\, dt\right) g(K_Q[b]) \\ &- \left(\int_a^b \int_a^t q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right) A_{K_Q,\eta}(s)\, ds\, dt\right) \left(\frac{g(K_Q[b]) + \rho_{K_q}(-\theta)\theta}{b-a}\right),\end{aligned}$$

where we use the representation of $g(K_Q[t])$ in (7.13) to its derivative. Dividing both sides of the last equation by

$$\int_{K_Q} Q(x)\, dx = \int_a^b \int_{K_Q[t]} Q(y)\, dy\, dt = \int_a^b q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right) A_{K_Q,\eta}(t)\, dt$$

and then rearranging the right-hand side shows

$$g(Q) = \frac{\int_{K_Q} x\, Q(x)\, dx}{\int_{K_Q} Q(x)\, dx} = \lambda_0\big(-\rho_{K_q}(-\theta)\theta\big) + (1-\lambda_0) g(K_Q[b]),$$

where

$$0 < \lambda_0 := \frac{\int_a^b \int_a^t q\big(s\theta/\langle\theta,\eta\rangle\big) A_{K_Q,\eta}(s)\, ds\, dt}{(b-a)\int_a^b q\big(t\theta/\langle\theta,\eta\rangle\big) A_{K_Q,\eta}(t)\, dt} < 1.$$

□

Remark 7.4. It can be seen from the proof of Lemma 7.3 that any function which is

- integrable with a positive integral;
- supported by a cone;
- constant on hyperplane sections of the cone parallel to the base;

will have its centroid on the line connecting the vertex of the cone to the centroid of the base.

Lemma 7.5. *The centroid $g(Q)$ lies on the θ-axis and $\langle g(Q), \theta\rangle \geq 0$. Fur-*

thermore,

$$\frac{\int_{\langle g(Q),\theta\rangle}^{\infty} Q(s\theta)\,ds}{\int_{-\infty}^{\infty} Q(s\theta)\,ds} \le \frac{\int_{0}^{\infty} Q(s\theta)\,ds}{\int_{-\infty}^{\infty} Q(s\theta)\,ds} = \frac{\int_{0}^{\infty} q(s\theta)\,ds}{\int_{-\infty}^{\infty} q(s\theta)\,ds}, \tag{7.15}$$

with equality if and only if $K_Q = K_q$ *and* $Q \equiv q$.

Proof. Both the vertex of K_Q and the centroid $g(K_Q[b])$ lie on the θ-axis, so $g(Q) = t_0\theta$ for some $t_0 \in \mathbb{R}$ by Lemma 7.3. We have

$$\begin{aligned} 0 = \int_{K_q} \langle x,\eta\rangle q(x)\,dx &= \int_{-\infty}^{\infty}\int_{K_q[t]} \langle y,\eta\rangle q(y)\,dy\,dt \qquad (7.16)\\ &\le \int_{a}^{\infty}\int_{K_q[t]} \langle y,\eta\rangle q(y)\,dy\,dt = \int_a^\infty t\,q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right) A_{K_q,\eta}(t)\,dt, \end{aligned}$$

because $g(q) = o$, $-h_{K_q}(-\eta) \le a := -h_C(-\eta) < 0$, and q has the constant value $q\big(t\theta/\langle\theta,\eta\rangle\big)$ on the section $K_q[t]$. Similarly,

$$\int_{K_Q} \langle x,\eta\rangle Q(x)\,dx = \int_a^\infty \int_{K_Q[t]} \langle y,\eta\rangle Q(y)\,dy\,dt = \int_a^\infty t\,q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right) A_{K_Q,\eta}(t)\,dt,$$

because Q has the constant value $q\big(t\theta/\langle\theta,\eta\rangle\big)$ on the section $K_Q[t]$. Therefore,

$$\begin{aligned} &\int_{K_Q} \langle x,\eta\rangle Q(x)\,dx\\ &\ge \int_a^\infty t\,q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right)\Big(A_{K_Q,\eta}(t) - A_{K_q,\eta}(t)\Big)\,dt\\ &= \int_a^0 t\,q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right)\Big(A_{K_Q,\eta}(t) - A_{K_q,\eta}(t)\Big)\,dt\\ &\qquad\qquad + \int_0^\infty t\,q\left(\frac{t\theta}{\langle\theta,\eta\rangle}\right)\Big(A_{K_Q,\eta}(t) - A_{K_q,\eta}(t)\Big)\,dt\\ &\ge 0, \end{aligned} \tag{7.17}$$

using inequality (7.12) and the fact that $A_{K_Q,\eta}(t) = A_{C,\eta}(t)$. This shows

$$0 \le \langle g(Q),\eta\rangle = t_0\langle\theta,\eta\rangle,$$

which then implies $t_0 \geq 0$ because $\langle \theta, \eta \rangle \geq 0$. That is, $\langle g(Q), \theta \rangle \geq 0$. We get (7.15) from $\langle g(Q), \theta \rangle \geq 0$ and (7.14).

Suppose there is equality in (7.15). Necessarily $\langle g(Q), \theta \rangle = 0$, so there must also be equality in (7.17) and (7.16). Therefore, $-a = h_{K_Q}(-\eta) = h_{K_q}(-\eta)$ and $A_{K_Q,\eta} \equiv A_{K_q,\eta}$. This means $A_{K_q,\eta}$ is γ-affine and increasing from zero on $[a, b]$, which is only possible if K_q is a cone with vertex $-\rho_{K_q}(-\theta)\theta$ and base $K_q[b] = K_q \cap \{h_{K_q}(\eta)\eta + \eta^\perp\}$. Recalling the construction of the cone C, we see that $C = K_q$. Because the centroid $g(q)$ and the vertex of K_q are on the θ-axis, Remark 7.4 implies $g(K_q[b]) = g(C[b])$ is also on the θ-axis. The choice of vertex and base for K_Q now implies $K_Q = C = K_q$. Since for each $c > 0$, $\{q(x) = c\}$ and $\{Q(x) = c\}$ lie in the same translate of $\eta^\perp$ and the supports of both functions coincide, we must have $Q \equiv q$. $\square$

Remark 7.6. By applying the argument in this subsection to the function $q(x) = F(x + g(F))$ (where F is defined in (7.8)), we can conclude

$$\frac{\int_0^\infty f(s\theta)ds}{\int_{-\infty}^\infty f(s\theta)ds} \geq \frac{\int_{\langle g(F),\theta\rangle}^\infty F(s\theta)ds}{\int_{-\infty}^\infty F(s\theta)ds} \geq \frac{\int_{\langle g(Q),\theta\rangle}^\infty Q(s\theta)ds}{\int_{-\infty}^\infty Q(s\theta)ds}.$$

7.1.3 Equality Case

We will evaluate the last of the integrals in the previous remark. Fix any unit vector $\xi \in S^{n-1}$ with $\langle \theta, \xi \rangle > 0$. Consider any n-dimensional cone

$$K_T = \operatorname{conv}\big(r_0\theta,\ r_1\theta + D\big),$$

where r_0, $r_1 \in \mathbb{R}$ with $r_0 < r_1$, and D is an $(n-1)$-dimensional convex body in $\xi^\perp$ with $g(D)$ at the origin. Let $T : \mathbb{R}^n \to [0, \infty)$ be any γ-affine function having the form

$$T(x) = m\mathcal{X}_{K_T}(x)\big(- \langle x, \xi \rangle + r_1\langle \theta, \xi \rangle\big)^{\frac{1}{\gamma}},$$

where $m > 0$ is a constant. We now determine the coordinates of $g(T)$. Compute

$$\begin{aligned}
&\int_{K_T} \langle x, \xi\rangle T(x)\,dx \\
&= \int_{r_0\langle\theta,\xi\rangle}^{r_1\langle\theta,\xi\rangle} s \cdot m \cdot (-s + r_1\langle\theta,\xi\rangle)^{\frac{1}{\gamma}} A_{K_T,\xi}(s)\,ds \\
&= \int_{r_0\langle\theta,\xi\rangle}^{r_1\langle\theta,\xi\rangle} s \cdot m \cdot (-s + r_1\langle\theta,\xi\rangle)^{\frac{1}{\gamma}} \left(\frac{s - r_0\langle\theta,\xi\rangle}{(r_1 - r_0)\langle\theta,\xi\rangle}\right)^{n-1} \mathrm{vol}_{n-1}(D)\,ds \\
&= m\Big((r_1 - r_0)\langle\theta,\xi\rangle\Big)^{\frac{1}{\gamma}+2} \mathrm{vol}_{n-1}(D) \int_0^1 t^n (1-t)^{\frac{1}{\gamma}}\,dt \\
&\qquad\qquad + m r_0\langle\theta,\xi\rangle\Big((r_1 - r_0)\langle\theta,\xi\rangle\Big)^{\frac{1}{\gamma}+1} \mathrm{vol}_{n-1}(D) \int_0^1 t^{n-1}(1-t)^{\frac{1}{\gamma}}\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{K_T} T(x)\,dx \\
&= \int_{r_0\langle\theta,\xi\rangle}^{r_1\langle\theta,\xi\rangle} m(-s + r_1\langle\theta,\xi\rangle)^{\frac{1}{\gamma}} \left(\frac{s - r_0\langle\theta,\xi\rangle}{(r_1 - r_0)\langle\theta,\xi\rangle}\right)^{n-1} \mathrm{vol}_{n-1}(D)\,ds \\
&= m\Big((r_1 - r_0)\langle\theta,\xi\rangle\Big)^{\frac{1}{\gamma}+1} \mathrm{vol}_{n-1}(D) \int_0^1 t^{n-1}(1-t)^{\frac{1}{\gamma}}\,dt
\end{aligned}$$

using the change of variables $t = \frac{s - r_0\langle\theta,\xi\rangle}{(r_1 - r_0)\langle\theta,\xi\rangle}$. Combining these calculations gives

$$\begin{aligned}
\langle g(T), \xi\rangle = \frac{\int_{K_T}\langle x,\xi\rangle T(x)\,dx}{\int_{K_T} T(x)\,dx} &= (r_1 - r_0)\langle\theta,\xi\rangle \left(\frac{\int_0^1 t^n(1-t)^{\frac{1}{\gamma}}\,dt}{\int_0^1 t^{n-1}(1-t)^{\frac{1}{\gamma}}\,dt}\right) + r_0\langle\theta,\xi\rangle \\
&= (r_1 - r_0)\langle\theta,\xi\rangle \left(\frac{n\gamma}{(n+1)\gamma + 1}\right) + r_0\langle\theta,\xi\rangle,
\end{aligned}$$

where we use the fact that for the Gamma function $\Gamma(z)$ one has

$$\int_0^1 t^{u-1}(1-t)^{v-1}dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \qquad \text{for all} \qquad u > 0,\ v > 0.$$

Both the vertex of K_T and the centroid of its base are on the θ-axis, so

$$g(T) = \left[(r_1 - r_0)\left(\frac{n\gamma}{(n+1)\gamma+1}\right) + r_0\right]\theta = \left[\frac{n\gamma r_1 + (\gamma+1)r_0}{(n+1)\gamma+1}\right]\theta \in \mathbb{R}\theta$$

by Remark 7.4. Note that the centroid $g(T)$ will be at the origin if and only if

$$r_0 = -\left(\frac{n\gamma}{\gamma+1}\right) r_1.$$

Finally, calculate

$$\begin{aligned}\int_{\langle g(T),\theta\rangle}^{\infty} T(s\theta)\,ds &= \int_{\langle g(T),\theta\rangle}^{r_1} \Big(-s\langle\theta,\xi\rangle + r_1\langle\theta,\xi\rangle\Big)^{\frac{1}{\gamma}} ds \\ &= \langle\theta,\xi\rangle^{\frac{1}{\gamma}}\left(\frac{\gamma}{\gamma+1}\right)(r_1 - r_0)^{\frac{\gamma+1}{\gamma}}\left(\frac{\gamma+1}{n\gamma+\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\end{aligned}$$

and

$$\int_{-\infty}^{\infty} T(s\theta)\,ds = \int_{r_0}^{r_1}\Big(-s\langle\theta,\xi\rangle + r_1\langle\theta,\xi\rangle\Big)^{\frac{1}{\gamma}} ds = \langle\theta,\xi\rangle^{\frac{1}{\gamma}}\left(\frac{\gamma}{\gamma+1}\right)(r_1 - r_0)^{\frac{\gamma+1}{\gamma}}$$

to see that

$$\frac{\int_{\langle g(T),\theta\rangle}^{\infty} T(s\theta)\,ds}{\int_{-\infty}^{\infty} T(s\theta)\,ds} = \left(\frac{\gamma+1}{n\gamma+\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}.$$

This concludes the proof of Theorem 7.1.

7.2 k-Dimensional Sections of γ-Concave Functions

Recall $\theta^+ := \{x \in \mathbb{R}^n : \langle x,\theta\rangle \geq 0\}$ for $\theta \in S^{n-1}$. We have the following generalization:

Corollary 7.7. *Fix a k-dimensional subspace E of $\mathbb{R}^n$, $\theta \in E \cap S^{n-1}$, and $\gamma \in (0,\infty)$. Let $f : \mathbb{R}^n \to [0,\infty)$ be a γ-concave function with $0 < \int_{\mathbb{R}^n} f(x)\,dx < \infty$*

and centroid at the origin. Then

$$\frac{\int_{E\cap\theta^+} f(x)dx}{\int_E f(x)dx} \geq \left(\frac{k\gamma+1}{(n+1)\gamma+1}\right)^{\frac{k\gamma+1}{\gamma}}.$$

There is equality when

- $f(x) = \mathcal{X}_{K_f}(x)\Big(-\langle x,\theta\rangle+1\Big)^{\frac{1}{\gamma}}$;
- $K_f = conv\left(-(n-k+1)\left(\frac{\gamma}{\gamma+1}\right)\theta + \delta B_2^{k-1},\ \theta + B_2^{n-k}\right)$ *where* B_2^{n-k} *is the centred Euclidean ball of unit radius in* $E^\perp$, B_2^{k-1} *is the centred Euclidean ball of unit radius in* $\widetilde{E}^\perp = E\cap\theta^\perp$, *and*

$$\delta = \left((n-k+1)\left(\frac{\gamma}{\gamma+1}\right)+1\right)\left[vol_{k-1}\big(B_2^{k-1}\big)\right]^{\frac{-1}{k-1}}.$$

Proof. Put $\widetilde{E} = \mathrm{span}\{E^\perp,\theta\}$, and define the function $F:\widetilde{E}\to[0,\infty)$ by

$$F(y) := \int_{\widetilde{E}^\perp} f(z+y)\,dz.$$

We claim F is a $\widetilde{\gamma} := \frac{\gamma}{(k-1)\gamma+1}$-concave function on the $d := (n-k+1)$-dimensional space $\widetilde{E}$. Fix any $y_1,\ y_2\in\widetilde{E}$ with $F(y_1)\cdot F(y_2)\neq 0$ and $0<\lambda<1$. The γ-concavity of f allows us to apply the Borell-Brascamp-Lieb inequality to the functions

$$z\mapsto f\big(z+\lambda y_1+(1-\lambda)y_2\big),\quad z\mapsto f(z+y_1),\quad z\mapsto f(z+y_2)$$

on $\widetilde{E}^\perp\in G(n,k-1)$ to get

$$F\big(\lambda y_1+(1-\lambda)y_2\big) = \int_{\widetilde{E}^\perp} f\big(z+\lambda y_1+(1-\lambda)y_2\big)\,dz$$

$$\geq \left(\lambda\left(\int_{\widetilde{E}^\perp} f(z+y_1)\,dz\right)^{\frac{\gamma}{(k-1)\gamma+1}} + (1-\lambda)\left(\int_{\widetilde{E}^\perp} f(z+y_2)\,dz\right)^{\frac{\gamma}{(k-1)\gamma+1}}\right)^{\frac{(k-1)\gamma+1}{\gamma}}$$

$$= \left(\lambda F^{\frac{\gamma}{(k-1)\gamma+1}}(y_1) + (1-\lambda)F^{\frac{\gamma}{(k-1)\gamma+1}}(y_2)\right)^{\frac{(k-1)\gamma+1}{\gamma}}.$$

Observe that $g(F) = o$. Indeed,

$$\begin{aligned}\int_{\widetilde{E}} y\, F(y)\, dy &= \int_{\widetilde{E}} y \int_{\widetilde{E}^{\perp}} f(z+y)\, dz\, dy \\ &= \int_{\mathbb{R}^n} (x|\widetilde{E}) f(x)\, dx = \left(\int_{\mathbb{R}^n} x\, f(x)\, dx\right) \Big| \widetilde{E} = o.\end{aligned}$$

By Theorem 7.1, we have

$$\frac{\int_{E\cap\theta^+} f(x)\, dx}{\int_E f(x)\, dx} = \frac{\int_0^\infty F(s\theta)\, ds}{\int_{-\infty}^\infty F(s\theta)\, ds} \geq \left(\frac{\widetilde{\gamma}+1}{(d+1)\widetilde{\gamma}+1}\right)^{(\widetilde{\gamma}+1)/\widetilde{\gamma}} = \left(\frac{k\gamma+1}{(n+1)\gamma+1}\right)^{\frac{k\gamma+1}{\gamma}}.$$

Assume $f(x) = \mathcal{X}_{K_f}(x)\Big(-\langle x,\theta\rangle + 1\Big)^{\frac{1}{\gamma}}$ and

$$K_f = \mathrm{conv}\left(-(n-k+1)\left(\frac{\gamma}{\gamma+1}\right)\theta + \delta B_2^{k-1},\ \theta + B_2^{n-k}\right).$$

Let y be any point lying in the $(n-k+1)$-dimensional cone

$$K_f|\widetilde{E} = \mathrm{conv}\left(-(n-k+1)\left(\frac{\gamma}{\gamma+1}\right)\theta,\ \theta + B_2^{n-k}\right)$$

in $\widetilde{E}$. There is a point v_1 in the base $\theta + B_2^{n-k}$ of $K_f|\widetilde{E}$ so that y lies on the line segment connecting v_1 to the vertex $v_0 := -(n-k+1)\left(\frac{\gamma}{\gamma+1}\right)\theta$ of $K_f|\widetilde{E}$. Then

$$K_f \cap \mathrm{aff}\left(v_0 + \widetilde{E}^{\perp},\ v_1\right) = \mathrm{conv}\left(v_0 + \delta B_2^{k-1},\ v_1\right),$$

and so

$$\begin{aligned}\mathrm{vol}_{k-1}\left(K_f \cap \left\{y + \widetilde{E}^{\perp}\right\}\right) &= \mathrm{vol}_{k-1}\left(\mathrm{conv}\left(v_0 + \delta B_2^{k-1},\ v_1\right) \cap \left\{y + \widetilde{E}^{\perp}\right\}\right) \\ &= \left(\frac{\langle v_1 - y, \theta\rangle}{\langle v_1 - v_0, \theta\rangle}\right)^{k-1} \mathrm{vol}_{k-1}\big(\delta B_2^{k-1}\big) \\ &= \Big(-\langle y,\theta\rangle + 1\Big)^{k-1}.\end{aligned}$$

The function $F : \widetilde{E} \to [0, \infty)$, defined by

$$\begin{aligned} F(y) := \int_{\widetilde{E}^\perp} f(z+y)\,dz &= \int_{\widetilde{E}^\perp} \mathcal{X}_{K_f}(z+y)\Big(-\langle z+y, \theta\rangle + 1\Big)^{\frac{1}{\gamma}} dz \\ &= \Big(-\langle y,\theta\rangle + 1\Big)^{\frac{1}{\gamma}} \int_{\widetilde{E}^\perp} \mathcal{X}_{K_f}(z+y)\,dz \\ &= \Big(-\langle y,\theta\rangle + 1\Big)^{\frac{1}{\gamma}} \mathrm{vol}_{k-1}\left(K_f \cap \left\{y + \widetilde{E}^\perp\right\}\right) \\ &= \Big(-\langle y,\theta\rangle + 1\Big)^{\frac{1}{\gamma}} \mathcal{X}_{K_f|\widetilde{E}}(y)\Big(-\langle y,\theta\rangle + 1\Big)^{k-1} \\ &= \mathcal{X}_{K_f|\widetilde{E}}(y)\Big(-\langle y,\theta\rangle + 1\Big)^{\frac{(k-1)\gamma+1}{\gamma}}, \end{aligned}$$

is then $\widetilde{\gamma}$-affine with support

$$K_f|\widetilde{E} = \mathrm{conv}\left(-(n-k+1)\left(\frac{\gamma}{\gamma+1}\right)\theta,\ \theta + B_2^{n-k}\right).$$

The centroid of f must lie in $\widetilde{E}$, because f is symmetric with respect to $\widetilde{E}$. Also notice that F satisfies the equality conditions of Theorem 7.1 in dimension $n-k+1$ for $\theta = \xi$ and $r = 1$. Therefore, the centroids of F and f are at the origin, and

$$\frac{\int_{E\cap\theta^+} f(x)\,dx}{\int_E f(x)\,dx} = \frac{\int_0^\infty F(s\theta)\,ds}{\int_{-\infty}^\infty F(s\theta)\,ds} = \left(\frac{k\gamma+1}{(n+1)\gamma+1}\right)^{\frac{k\gamma+1}{\gamma}}.$$

$\square$

7.3 Sections of Convex Bodies

We have the following corollary to Theorem 7.1:

Corollary 7.8. *Fix a k-dimensional subspace E of $\mathbb{R}^n$, and $\theta \in E \cap S^{n-1}$. Let $\widetilde{E}$ be the $(n-k+1)$-dimensional subspace spanned by θ and $E^\perp$. Let K be a* *convex body in* $\mathbb{R}^n$ *with* $g(K) \in \widetilde{E}^{\perp} = E \cap \theta^{\perp}$*. Then*

$$\frac{vol_k(K \cap E \cap \theta^+)}{vol_k(K \cap E)} \geq \left(\frac{k}{n+1}\right)^k.$$

There is equality if and only if

$$K = conv\left(-\left(\frac{n-k+1}{k}\right) z + D_0,\ z + D_1\right),$$

where

- $z \in E$ *with* $\langle z, \theta \rangle > 0$*;*
- D_0 *is a* $(k-1)$*-dimensional convex body in* $\widetilde{E}^{\perp}$*;*
- D_1 *is an* $(n-k)$*-dimensional convex body in an* $(n-k)$*-dimensional subspace* $F \subset \mathbb{R}^n$ *for which* $\mathbb{R}^n = span(E, F)$*, and* $g(D_1)$ *is at the origin (see Figure 7.2).*

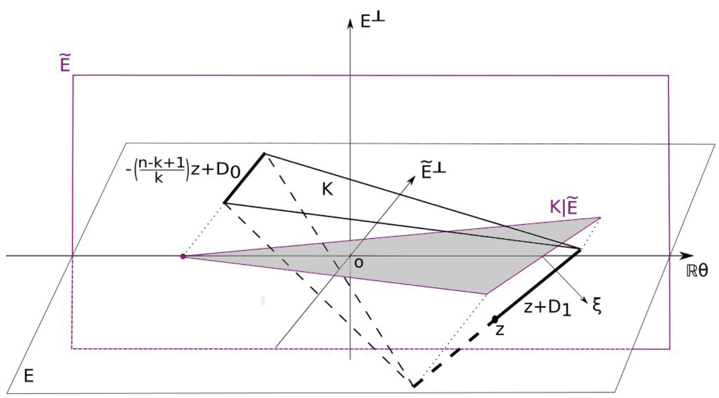

Figure 7.2: The equality case.

Proof. Define the section function

$$A_{K,\widetilde{E}}(y) := \mathrm{vol}_{k-1}\left(K \cap \{y + \widetilde{E}^{\perp}\}\right), \quad y \in \widetilde{E}.$$

It follows from the Brunn-Minkowski inequality that $A_{K,\widetilde{E}} : \widetilde{E} \to [0,\infty)$ is a $\gamma := 1/(k-1)$-concave function on the $d := (n-k+1)$-dimensional space $\widetilde{E}$. The centroid of $A_{K,\widetilde{E}}$ is at the origin; indeed,

$$g\big(A_{K,\widetilde{E}}\big) = \frac{\int_{K|\widetilde{E}} y\, A_{K,\widetilde{E}}(y)\, dy}{\int_{K|\widetilde{E}} A_{K,\widetilde{E}}\, dy} = \frac{\int_K (x|\widetilde{E})\, dx}{\mathrm{vol}_n(K)} = g(K)|\widetilde{E} = o.$$

From Theorem 7.1,

$$\frac{\mathrm{vol}_k(K\cap E\cap\theta^+)}{\mathrm{vol}_k(K\cap E)} = \frac{\int_0^\infty A_{K,\widetilde{E}}(s\theta)\, ds}{\int_{-\infty}^\infty A_{K,\widetilde{E}}(s\theta)\, ds} \geq \left(\frac{\gamma+1}{\gamma d+\gamma+1}\right)^{\frac{\gamma+1}{\gamma}} = \left(\frac{k}{n+1}\right)^k$$

with equality if and only if

- $A_{K,\widetilde{E}}(y) = m\mathcal{X}_{K|\widetilde{E}}(y)\Big(-\langle y,\xi\rangle + r\langle\theta,\xi\rangle\Big)^{\frac{1}{\gamma}}$ for some constants $m, r > 0$ and a unit vector $\xi \in \widetilde{E}\cap S^{n-1}$ such that $\langle\theta,\xi\rangle > 0$;
- $K|\widetilde{E} = \mathrm{conv}\left(-\left(\frac{d\gamma}{\gamma+1}\right)r\theta,\ r\theta + D\right)$ for some $(d-1)$-dimensional convex body $D \subset \widetilde{E}\cap\xi^\perp$ whose centroid is at the origin.

These equality conditions are equivalent to the ones given in the corollary statement, where m is the $(k-1)$-dimensional volume of D_0, $r = \langle z,\theta\rangle$, $\widetilde{E}\cap\xi^\perp = F|\widetilde{E}$, and $D = D_1|\widetilde{E}$. □

Remark 7.9. Observe that the inequality in Corollary 7.8 is the limiting case of the inequality in Corollary 7.7 as γ goes to infinity. This corresponds to the fact that ∞-concave functions, defined by taking the limit in (7.1), are the indicator functions of convex sets.

Remark 7.10. We are able to recover Grünbaum's inequality for projections from Grünbaum's inequality for sections. Consider any convex body $K \subset \mathbb{R}^n$ with its centroid at the origin. Let $\widetilde{K}$ be the Steiner symmetrization of K with respect to the k-dimensional subspace $E \subset \mathbb{R}^n$. Specifically,

$$\widetilde{K} = \bigcup_{y\in K|E}\left\{y + \left(\frac{\mathrm{vol}_{n-k}\left(K\cap\{y+E^\perp\}\right)}{\mathrm{vol}_{n-k}\left(B_2^{n-k}\right)}\right)^{\frac{1}{n-k}} B_2^{n-k}\right\},$$

where B_2^{n-k} is the centred Euclidean ball of unit radius in $E^\perp$. Now, $\widetilde{K}$ is a convex body with its centroid at the origin, and

$$\widetilde{K} \cap E \cap \theta^+ = \big(K|E\big) \cap \theta^+ \quad \text{for all} \quad \theta \in E \cap S^{n-1}.$$

Chapter 8

Conclusion

We answered several problems from convex geometry and geometric tomography. In Chapter 3, we proved two stability results. According to our first stability result, if the intersection and cross-section bodies of a convex body K are sufficiently close to each other with respect to the radial metric, then K is approximately origin-symmetric (cf. [33]). According to our second stability result, if K and L are smooth convex bodies so that the difference between $A^{(p)}_{K,\xi}(0)$ and $A^{(p)}_{L,\xi}(0)$ is small enough for all $\xi \in S^{n-1}$ (for some non-integer p), then K and L are close with respect to the radial metric (cf. [26]).

I showed in Chapter 4 that a convex body K is not uniquely determined up to congruency by its convex intersection body $CI(K)$, as defined by Meyer and Reisner [37].

In Chapter 5, I proved that a convex polytope $P \subset \mathbb{R}^n$ with the origin in its interior must be origin-symmetric if every hyperplane section through the origin has maximal $(n-2)$-dimensional surface area amongst all parallel sections. This gives partial confirmation to a conjecture made by Makai, Martini, and Ódor in [33]: a convex body K containing the origin in its interior must be origin-symmetric if the quermassintegral $W_l(K \cap \xi^\perp)$ of every hyperplane section through the origin is maximal amongst all parallel sections. My result provides some hope that the conjecture is true, and I will continue to work on the general problem. However, some questions with positive answers in the class of convex polytopes have negative answers for the class of convex bodies, cf. [42, 61].

We established Grünabaum's inequality for projections in Chapter 6. In Chapter 7, we proved Grünbaum's inequality for one-dimensional sections of γ-concave functions when $\gamma > 0$ (Theorem 7.1), which gave Grünbaum's inequality for sections as an immediate consequence. As future work, I will investigate whether Theorem 7.1 extends to $\gamma < 0$. I am also interested in developing a "Grünbaum's inequality for surface area". For a given function $F(K) = x_K$ mapping each convex body $K \subset \mathbb{R}^n$ to a point $x_K \in \mathbb{R}^n$, what is the largest constant $C = C(n, F) > 0$ so that

$$\mathrm{vol}_{n-1}\big(\partial(K - x_K) \cap \xi^+\big) \geq C\,\mathrm{vol}_{n-1}\big(\partial K\big) \quad \forall\, \xi \in S^{n-1}$$

for all convex bodies $K \subset \mathbb{R}^n$? Several choices of F provide reasonable conjectured extensions for Grünbaum's inequality, including $F(K) := g(K)$.

There is one final problem concerning the centroid which I wish to advertise. Informally, how far apart can the centroid of a convex body be from the centroid of the body's intersection with a subspace? Formally, what is the smallest constant $C = C(n, k) > 0$ such that

$$|g(K \cap E)|_2 \leq C\,\mathrm{vol}_1\Big(K \cap \mathbb{R}\,g(K \cap E)\Big)$$

for all convex bodies $K \subset \mathbb{R}^n$ with centroid at the origin, and all k-dimensional subspaces $E \subset \mathbb{R}^n$? Here, $g(K \cap E)$ is the centroid of the k-dimensional convex body $K \cap E$ taken within E, and $\mathbb{R}\,g(K \cap E)$ is the span of $g(K \cap E)$. This is a fundamental question concerning the centroid, but I believe it has gone unnoticed and is entirely open.

Bibliography

[1] S. Bobkov and A. Koldobsky, *On the central limit property of convex bodies*, Geometric aspects of functional analysis, 44–52, Lecture Notes in Math. **1807**, Springer, Berlin, 2003.

[2] T. Bonnesen and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow ID, 1987.

[3] K. J. Böröczky and R. Schneider, *Stable determination of convex bodies from sections*, Studia Sci. Math. Hungar. **46** (2009), 367–376.

[4] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.H. Vritsiou, *Geometry of Isotropic Convex Bodies*, American Mathematical Society, Providence RI, 2014.

[5] U. Brehm, *Convex bodies with non-convex cross-section bodies*, Mathematika **46** (1999), 127–129.

[6] U. Brehm and J. Voigt, *Asymptotics of cross-sections for convex bodies*, Beiträge Algebra Geom. **41** (2000), 437–454.

[7] H.T. Croft, *Two problems on convex bodies*, Proc. Cambridge Philos. Soc. **58** (1962), 1–7.

[8] K. J. Falconer, *Applications of a result on spherical integration to the theory of convex sets*, Amer. Math. Monthly **90** (1983), 690–693.

[9] K. J. Falconer, *X-ray problems for point sources*, Proc. London Math. Soc. (3) **46** (1983), 241–262.

[10] M. Fradelizi, *Sections of convex bodies through their centroid*, Arch. Math. **69** (1997), 515–522.

[11] M. Fradelizi, *Hyperplane sections of convex bodies in isotropic position*, Beiträge Algebra Geom. **40** (1999), 163-183.

[12] M. Fradelizi, M. Meyer, and V. Yaskin, *On the volume of sections of a convex body by cones*, Proc. Amer. Math. Soc. **145** (2017), 3153–3164.

[13] R. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), 355–405.

[14] R. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.

[15] R. J. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), 691–703.

[16] R. J. Gardner, D. Ryabogin, V. Yaskin, and A. Zvavitch, *A problem of Klee on inner section functions of convex bodies*, J. Differential Geom. **91** (2012), 261–279.

[17] I. M. Gelfand and G. E. Shilov, *Generalized Functions, Vol. 1: Properties and Operations*, Academic Press, New York, 1964.

[18] P. Goodey, V. Yaskin, and M. Yaskina, *Fourier transforms and the Funk-Hecke theorem in convex geometry*, J. London Math. Soc. (2) **80** (2009), 388–404.

[19] H. Groemer, *Geometric Applications of Fourier series and Spherical Harmonics*, Cambridge University Press, Cambridge, 1996.

[20] B. Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math. **10** (1960), 1257–1261.

[21] B. Grünbaum, *Measures of symmetry for convex sets*, In: Proc. Sympos. Pure Math., Vol. VII, pp. 233–270, Amer. Math. Soc., Providence RI, 1963.

[22] H. Huang, *John ellipsoid and the center of mass of a convex body*, available at arXiv:1605.06881 [math.MG].

[23] J. Kim and S. Reisner, *Local minimality of the volume-product at the simplex*, Mathematika **57** (2011), 121–134.

[24] V. Klee, *Research problems: Is a body spherical if its HA-measurements are constant?*, Amer. Math. Monthly **76** (1969), 539–542.

[25] A. Koldobsky, *Fourier Analysis in Convex Geometry*, American Mathematical Society, Providence RI, 2005.

[26] A. Koldobsky and C. Shane, *The determination of convex bodies from derivatives of section functions*, Arch. Math. **88** (2007), 279–288.

[27] E. Lieb and M. Loss, *Analysis*, second edition, Amer. Math. Soc., Providence, RI, 2001.

[28] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531–538.

[29] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

[30] E. Makai and H. Martini, *On bodies associated with a given convex body*, Canad. Math. Bull. **39** (1996), 448–459.

[31] E. Makai and H. Martini, *The cross-section body, plane sections of convex bodies and approximation of convex bodies, Part I*, Geom. Dedicata **63** (1996), 267–296.

[32] E. Makai and H. Martini, *Centrally symmetric convex bodies and sections having maximal quermassintegrals*, Studia Sci. Math. Hungar. **49** (2012), 189–199.

[33] E. Makai, H. Martini, T. Ódor, *Maximal sections and centrally symmetric bodies*, Mathematika **47** (2000), 19–30.

[34] H. Martini, *Cross-sectional measures, in: Intuitive geometry* (Szeged, 1991), 269-310, Colloq. Math. Soc. János Bolyai, 63, North-Holland, Amsterdam, 1994.

[35] M. Meyer, *Maximal hyperplane sections of convex bodies*, Mathematika **46** (1999), 131-136.

[36] M. Meyer, F. Nazarov, D. Ryabogin, and V. Yaskin, *Generalized Grünbaum inequality*, available at arXiv:1706.02373 [math.MG].

[37] M. Meyer and S. Reisner, *The convex intersection body of a convex body*, Glasgow Math. J. **53** (2011), 523–534.

[38] M. Meyer and E. Werner, *The Santaló-regions of a convex body*, Trans. Amer. Math. Soc. **350** (1998), 4569–4591.

[39] E. Milman, *Generalized intersection bodies*, J. Funct. Anal. **240** (2006), 530–567.

[40] E. Milman, *Generalized intersection bodies are not equivalent*, Adv. Math. **217** (2008), 2822–2840.

[41] B.S. Mityagin, *Two inequalities for volumes of convex bodies (Russian)*, Mat. Zametki **5** (1969), 99–106.

[42] S. Myroshnychenko and D. Ryabogin, *On polytopes with congruent projections or sections*, Adv. Math. **325** (2018), 482–504.

[43] S. Myroshnychenko, M. Stephen, and N. Zhang, *Grünbaum's inequality for sections*, J. Funct. Anal. (2018), DOI: 10.1016/j.jfa.2018.04.001.

[44] F. Nazarov, D. Ryabogin, and A. Zvavitch, *Non-uniqueness of convex bodies with prescribed volumes of sections and projections*, Mathematika **59** (2013), 213–221.

[45] F. Nazarov, D. Ryabogin, and A. Zvavitch, *An asymmetric convex body with maximal sections of constant volume*, Journal of Amer. Math. Soc. **27** (2014), 43–68.

[46] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[47] D. Ryabogin and V. Yaskin, *Detecting symmetry in star bodies*, J. Math. Anal. Appl. **395** (2012), 509–514.

[48] L. A. Santaló, *Un invariante afin para los cuerpos convexos del espacio de n dimensiones*, Portugal Math. **8** (1949), 155–161.

[49] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, second edition, Cambridge University Press, Cambridge, 2014.

[50] M. Stephen, *On convex intersection bodies and unique determination problems for convex bodies*, J. Math. Anal. Appl. **443** (2016), 295–312.

[51] M. Stephen, *Maximal perimeters of polytope sections and origin-symmetry*, available at arXiv:1803.01506 [math.MG].

[52] M. Stephen and V. Yaskin, *Stability results for sections of convex bodies*, Trans. Amer. Math. Soc. **369** (2017), 6239–6261.

[53] M. Stephen and N. Zhang, *Grünbaum's inequality for projections*, J. Funct. Anal. **272** (2017), 2628–2640.

[54] R. Vitale, *L_p metrics for compact, convex sets*, J. Approx. Theory **45** (1985), 280–287.

[55] V. Yaskin, *On strict inclusions in hierarchies of convex bodies*, Proc. Amer. Math. Soc. **136** (2008), 3281–3291.

[56] V. Yaskin, *On perimeters of sections of convex polytopes*, J. Math. Anal. Appl. **371** (2010), 447–453.

[57] V. Yaskin, *Unique determination of convex polytopes by non-central sections*, Math. Ann. **349** (2011), 647–655.

[58] V. Yaskin, *An extension of polynomial integrability to dual quermassintegrals*, available at arXiv:1803.00199 [math.MG].

[59] V. Yaskin and M. Yaskina, *Thick sections of convex bodies*, Adv. in Appl. Math. **71** (2015), 174–189.

[60] G. Zhang, *Sections of convex bodies*, Amer. J. Math. **118** (1996), 319–340.

[61] N. Zhang, *On bodies with congruent sections or projections*, available at arXiv:1711.10445 [math.FA].

[62] G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math. **262** (2014), 909-931.