Approximation and invariance properties of one-dimensional probabilities

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Applied Mathematics

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Abstract

This thesis is based on four papers. The first two papers fall into the field of approximation of one-dimensional probabilities, the third into uniform distribution theory, and the last paper relates to ergodic theory.

The first paper (joint with Arno Berger), studies best finitely supported approximations of one-dimensional probability measures with respect to (w.r.t.) the L^r-Kantorovich (or transport) distance, where either the locations or the weights of the approximations' atoms are prescribed. Special attention is given to the case of best uniform approximations (i.e., all atoms having equal weight).

In the second paper (joint with Arno Berger), for arbitrary one-dimensional Borel probability measures with compact support, characterizations are established of the best finitely supported approximations, relative to three familiar probability metrics (Lévy, Kantorovich, and Kolmogorov), given any number of atoms, and allowing for additional constraints regarding weights or positions of atoms. As an application, best (constrained or unconstrained) approximations are identified for Benford's Law (logarithmic distribution of significands).

The third paper studies the distributional asymptotics of the slowly changing sequence of logarithms $(\log_b n)$ with integer base $b \geq 2$. An upper estimate $\left(N^{-1} (\log N)^{1/2}\right)$ is obtained for the rate of convergence w.r.t. the Kantorovich metric on the circle. Moreover, a sharp rate of convergence $(N^{-1} \log N)$ w.r.t. the Kantorovich and the discrepancy (or Kolmogorov) metrics on the real line is derived.

The last paper proves a threshold result on the existence of a circularly invariant and uniform probability measure (CIUPM) for non-constant linear transformations on the real line, which shows that there is a constant c depending only on the slope of the linear transformation such that there exists a CIUPM if and only if the diameter of the support is not smaller than c. Moreover, the CIUPM is unique up to translation when the diameter of the support is equal to c.

To my parents Baihe Xu and Jing Zhang

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Chapter 1

Introduction

This thesis provides an in-depth study of approximation and invariance properties of one-dimensional probability measures, a topic that touches on three important areas of mathematics: quantization of probability distributions, uniform distribution theory, and ergodic theory.

This introductory chapter gives an informal overview of some main results of the thesis, motivated by classical facts and recent developments in these areas.

Let us first have some basic understanding of quantization of probability measures by directly quoting from [40]: *"The term 'quantization' ... originates in the theory of signal processing. It was used by electrical engineers starting in the late 40's. In this context quantization means a process of discretising signals and should not be mistaken for the same term in quantum physics. As a mathematical topic quantization for probability distributions concerns the best approximation of a d-dimensional probability distribution P by a discrete probability with a given number n of supporting points or in other words, the best approximation of a d-dimensional random vector X with distribution P by a random vector Y with at most n values in its image. It turns out that for the error measures used in this book there is always a best approximation of the form $f(X)$, a 'quantized version of X'. The quantization problem can be rephrased as a partition problem of the underlying space which explains the term quantization."* In fact, the same mathematical problem arises in various other contexts, for instance, in cluster analysis, machine learning, numerical integration, stochastic processes, mathematical finance, convex geometry, optimal transport, and kinetic theory [15,18–20,40,45,74,78,90,97].

To be more concrete, let us describe the quantization problem on the m-dimensional Euclidean space $\mathbb{R}^m$ with the L^r-metric as a numerical measure of the quantization error. This problem has been extensively studied in the literature; for a systematic review, the reader is referred to the excellent monograph [40].

To fix notation, let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ be the set of all reals, integers and positive integers, respectively. Let $m \in \mathbb{N}$, $\|\cdot\|$ an arbitrary norm on $\mathbb{R}^m$, λ^m Lebesgue measure on $\mathbb{R}^m$ (with $\mathbb{R} := \mathbb{R}^1$, $\lambda := \lambda^1$, for simplicity), and $\mathcal{P}(\mathbb{R}^m)$ the space of all Borel probabilities on

$\mathbb{R}^m$, endowed with a complete and separable metric as needed. For any $\mu \in \mathcal{P}(\mathbb{R}^m)$, let $\operatorname{supp}\mu$ denote the *support* of μ, that is, the smallest closed set with full μ-measure. Let δ_a denote the Dirac (probability) measure concentrated at $a \in \mathbb{R}^m$. For any non-empty Borel set $A \subset \mathbb{R}^m$, $\#A$ stands for its *cardinality*, $\operatorname{diam} A := \sup_{x,y\in A} \|x - y\|$ its *diameter*, and $\mathbb{1}_A$ its *indicator function*; also, $\#\varnothing := \operatorname{diam}\varnothing := 0$. Write $\mathbb{I}$ for the compact unit interval $[0, 1]$. For $r \geq 1$, let $\mathcal{P}_r(\mathbb{R}^m) \subset \mathcal{P}(\mathbb{R}^m)$ be the set of all probabilities with finite r-th moment (that is, $\int \|x\|^r \mu(\mathrm{d}x) < +\infty$).

With these notations, let X be an $\mathbb{R}^m$-valued random vector with law $\mu \in \mathcal{P}_r(\mathbb{R}^m)$. Note that $\mathbb{E}\|X\|^r < +\infty$. For every $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the set of all Borel measurable maps $f : \mathbb{R}^m \to \mathbb{R}^m$ with $\#f(\mathbb{R}^m) \leq n$. The elements of $\mathcal{F}_n$ are called *n-quantizers*. The *n-th quantization error for μ of order r* is defined as

$$V_{n,r}(\mu) = \inf\nolimits_{f\in\mathcal{F}_n} \mathbb{E}\|X - f(X)\|^r.$$

A quantizer $f \in \mathcal{F}_n$ is *n-optimal for μ of order r* if

$$V_{n,r}(\mu) = \mathbb{E}\|X - f(X)\|^r.$$

The following property shows that indeed the quantization problem can be interpreted as an approximation of probability measures problem; recall the L^r-*Kantorovich* metric on $\mathcal{P}_r(\mathbb{R}^m)$,

$$d_r(\mu, \nu) = \inf\nolimits_\gamma \left(\int_{\mathbb{R}^m\times\mathbb{R}^m} \|x - y\|^r \gamma(\mathrm{d}x, \mathrm{d}y) \right)^{1/r},$$

with the infimum taken over all $\gamma \in \mathcal{P}_r(\mathbb{R}^m \times \mathbb{R}^m)$ with marginals μ and ν.

Proposition 1.1. *Assume $\mu \in \mathcal{P}_r(\mathbb{R}^m)$ for some $r \geq 1$. Then, for every $n \in \mathbb{N}$,*

$$V_{n,r}(\mu) = \inf\nolimits_{f\in\mathcal{F}_n} d_r\left(\mu, \mu \circ f^{-1}\right)^r = \inf\nolimits_{\nu\in\mathcal{P}_r(\mathbb{R}^m),\ \#\operatorname{supp}\nu\leq n} d_r(\mu, \nu)^r.$$

Thus f is an n-optimal quantizer for μ (of order r) if and only if $\mu\circ f^{-1}$ is a best approximation of μ (w.r.t. d_r), and to study the optimal quantization problem (w.r.t. the L^r-metric) means to study the best approximation (w.r.t. the d_r-metric) of a probability by finitely supported probabilities. The main focus then is on the characterization of best approximations (or equivalently, optimal quantizers) and the rate of convergence of best approximations as the number of atoms goes to infinity, i.e., the study of the asymptotic quantization error [40, 43].

Let us get some basic sense of the two topics by a simple example, namely the *standard exponential distribution* μ with *distribution function* $F_\mu(x) = \max\{0, 1 - e^{-x}\}$ for all $x \in \mathbb{R}$, with the underlying complete and separable space $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the absolute value on

$\mathbb{R}$. As it turns out, for every $n \in \mathbb{N}$, there exists a unique best d_r-approximation of μ for each $r \geq 1$. In fact, the best d_1-approximation[1] $\delta_{\bullet}^{\bullet,n}$ is given explicitly by

$$\delta_{\bullet}^{\bullet,n} = \sum_{i=1}^{n} \frac{2(n+1-i)}{n(n+1)} \delta_{x_{,i}} \text{ with } x_{,i} = -2\log \frac{n+1-i}{\sqrt{n(n+1)}}, \quad \forall\, i = 1, \cdots, n,$$

and $d_1(\mu, \delta_{\bullet}^{\bullet,n}) = \log(1 + n^{-1})$. While, by contrast, there is no explicit expression for the unique best approximation for $r > 1$, nevertheless, it can be shown that

$$\lim_{n\to\infty} n d_r(\mu, \delta_{\bullet}^{\bullet,n}) = (1 + r)/2, \quad \forall\, r \geq 1.$$

Thus the rate of convergence for best d_r-approximations of μ is (n^{-1}). Indeed, convergence at rate (n^{-1}) turns out to be *universal* for all probabilities under a mild moment condition (see Proposition 2.50, also stated below in Section 1.1). This is one of the celebrated results regarding the quantization of probability measures. It is worth noting, though, that there are probability metrics where the rate of convergence for the exponential distribution differs from (n^{-1}) [43, Ex.5.1(d)], which shows that the best approximations depend, presumably in a nontrivial way, on the underlying probability metric.

As a stochastic version of the approximation problem, consider a sequence of independent, identically distributed (iid.) random variables $(X_n)_{n\geq 1}$ with common law μ. The *random empirical measure* $\mu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{X_j}$, and in particular the (almost sure or mean) asymptotics of $(d_r(\mu, \mu_n))$, have attracted much interest recently [11]. Typically, $(d_r(\mu, \mu_n))$ decays much more slowly than (n^{-1}).

The above results for the exponential distribution motivate several aspects of the investigation carried out in this thesis. First, the dependence of all approximation results on the underlying probability metric will be emphasized, for instance by highlighting the r-dependence for d_r-related results (in Chapters 2 and 3) and by considering qualitatively different metrics such as, e.g., the Lévy and Kolmogorov metrics (notably in Chapters 4 and 5). Second, the deterministic analogues of random empirical measures, that is, uniform approximations $\frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$ will be considered in detail, as an important special case of a more comprehensive theory of best *constrained* approximations, developed in Chapters 2-5. Third, although best (or best uniform) approximations exist practically always, often even uniquely, they very rarely can be computed explicitly. This motivates the construction of easy-to-determine approximations that are best (or best uniform) approximations *asymptotically*, i.e., as $n \to \infty$,

[1] From now on, the symbol $\delta_{\bullet}^{\bullet,n}$ is used to denote a (or *the*, if unique) best approximation of a probability measure (w.r.t. a metric clear from the context).

a problem considered repeatedly in Chapters 2-5.

Beyond these basic motivations, we mention four general aspects of the quantization (or approximation) problem that allow us to put the results of this thesis into perspective. The first aspect pertains to the **method** used to study the quantization problem. The classical method, known as the *Voronoi partition approach* [40] will be reviewed briefly in Section 1.1. Roughly speaking, it is based on the geometry of the underlying metric space (say, the Euclidean space $\mathbb{R}^m$) as well as the given probability measure. Another method, the Γ-*convergence* of functional analysis [14, 15, 72], is used to establish the rates of convergence of best approximations in a different setting (optimal location problem). A recent dynamical system approach, the *gradient flow approach*, is proposed to study quantization of measures from the viewpoint of kinetic theory [18, 19, 57, 58]. In this thesis, in Chapter 2, a new elementary approach, based on *approximation of monotone functions by step functions*, is proposed to study best constrained approximations of one-dimensional probabilities.

A second important perspective is the **underlying space**. The classical quantization problem is formulated in finite-dimensional Euclidean spaces. Recently, the quantization problem has been generalized to *Riemannian manifolds* with a complete Riemannian metric [57, 61]. Asymptotics of the quantization error of absolutely continuous probabilities on a compact Riemannian manifold has been investigated in [61], and a generalization (in the spirit of Proposition 2.50) to all Borel probabilities on a (not necessarily compact) Riemannian manifold has been proved under a mild integrability assumption [57].

A third important aspect is the **metric** for the quantization problem, or correspondingly the underlying probability metric for the approximation problem. Though classical results are relative to the L^r-Kantorovich metric for $1 \le r < \infty$ [40], quantization of probability measures on $\mathbb{R}^m$ w.r.t. the L^∞- and *Ky Fan* metric, as well as the *Orlicz* norm, has also been studied [26, 40, 43]. As mentioned earlier, beyond an extensive analysis of d_r-approximations, this thesis also studies, in Chapters 4 and 5, best (constrained or unconstrained) approximations of one-dimensional probabilities w.r.t. the *Lévy* metric

$$d_{\mathsf{L}}(\mu,\nu) = \inf\left\{y \ge 0 : F_\mu(\cdot - y) - y \le F_\nu \le F_\mu(\cdot + y) + y\right\} ,$$

and the *Kolmogorov* metric

$$d_{\mathsf{K}}(\mu,\nu) = \sup_{x\in\mathbb{R}} |F_\mu(x) - F_\nu(x)| ,$$

among other metrics; here F_μ, F_ν denote the distribution functions of $\mu, \nu \in \mathcal{P}(\mathbb{R})$.

A fourth and final aspect is **randomness**. There now exists a sizable literature on the random quantization problem, see, e.g., [40, Sec.9] and [42], as well as on the approximation by empirical measures [11]. Rates of convergence for empirical measures, in particular, have become a hot topic recently, due to their many applications in such diverse fields as propagation of chaos, partial differential equations, and interacting particle systems [11, 13, 33, 34, 96, 97]. A main focus of research is on the rate of convergence for the *empirical quantization error*, or equivalently, the speed of convergence of *empirical measures*, in terms of either non-asymptotic estimates or concentration inequalities [11, 25, 33, 42]. An in-depth study of one-dimensional empirical measures w.r.t. the L^r-Kantorovich metrics is provided by [11], where the explicit formula

$$d_r(\mu,\nu)^r = \int_0^1 |F_\mu^{-1}(t) - F_\nu^{-1}(t)|^r \mathrm{d}t, \quad \forall\, \mu,\nu \in \mathcal{P}_r(\mathbb{R}),$$

plays a crucial role; here $F_\mu^{-1},\ F_\mu^{-1}$ are the *(upper) quantile functions* of $\mu,\ \nu$; see Chapter 2 below for precise definitions and details. For instance, it is known [11] that for the standard exponential distribution μ considered earlier, $(\mathbb{E}d_r(\mu,\mu_n))$ with $\mu_n = \frac{1}{n}\sum_{j=1}^n \delta_{X_j}$ decays like $\left(n^{-1/2}\right)$ for $1 \le r < 2$, but decays only like $\left(n^{-1/2}(\log n)^{1/2}\right)$ for $r = 2$, and like $\left(n^{-1/r}\right)$ for $r > 2$. By contrast, the theory of best uniform approximations developed in Chapters 2-3 yields that $(d_r(\mu, \delta_\bullet^{u_n}))$, with

$$d_r\left(\mu, \delta_\bullet^{u_n}\right) = \inf\nolimits_{x_1\le\ldots\le x_n\in\mathbb{R}} d_r\left(\mu, \frac{1}{n}\sum_{j=1}^n \delta_{x_j}\right),$$

decays like $(n^{-1}\log n)$ for $r = 1$, and like $\left(n^{-1/r}\right)$ for $r > 1$. Thus randomness leads to a poorer rate of convergence precisely if $1 \le r \le 2$.

Finally, quantization of probability measures also relates naturally to fractal geometry, notably via the quantization of singular probabilities supported on fractals generated by conformal IFS in $\mathbb{R}^m$ [67]. A classical moment condition guarantees that the quantization error decays faster than (n^{-1}). Theorem 2.56 below complements this by showing that without a moment condition, the quantization error may in fact decay arbitrarily slowly. Two important numbers describing the asymptotics of the quantization error are the *quantization dimension* and the *quantization coefficient*; see Section 2.4 for precise definitions. These numbers are of special interest for singular probabilities, with their relations to various concepts of geometric measure theory, notably dimension, continuing to be the focus of much research [22, 23, 40, 69]. A popular example of a continuous singular probability is the *Cantor distribution*, i.e., the uniform distribution on the classical Cantor middle-thirds set, where d_r-quantization coefficients do not exist [64, Rem.5.4]. In Chapters 2-3, a detailed analysis is presented of best constrained approximations of the Cantor distribution, as well as of its *inverse*. The latter, a discrete distribution with support $\mathbb{I}$ has many interesting properties but has rarely appeared in the literature so far.

As mentioned in the beginning, another important area of mathematics related to the investigations herein is uniform distribution theory (UDT). Historically, UDT originated from the study of the distribution of fractional parts of real sequences, the key classical concept being that of a sequence *uniformly distributed modulo one* (u.d. mod 1; see Definition 1.17 below). Let us first quote from [65]: *"The development of this theory started with Hermann Weyl's celebrated paper of 1916 titled:'Über die Gleichverteilung von Zahlen mod. Eins.' Weyl's work was primarily intended as a refinement of Kronecker's approximation theorem, and, therefore, in its initial stage, the theory was deeply rooted in diophantine approximations .... Today, the subject presents itself as a meeting ground for topics as diverse as number theory, probability theory, functional analysis, topological algebra ... "*. Let us also quote from the preface by H. Niederreiter to a new edition of [65], speaking to the prominence of UDT in various applications :*"These dynamic research activities on uniform distribution in concrete settings, and in particular on discrepancy theory, are driven by demands from applications ... the study of the discrepancy of sequences in unit cubes plays an important role in quasi-Monte Carlo methods for scientific computing. Other areas where discrepancy theory has made an impact are pseudorandom number generation, computer graphics, cryptology, and various part of number theory."* For an authoritative monograph on *discrepancy*, a key concept in quantifying distributional convergence of u.d. mod 1 sequences, the reader is referred to [28].

Although much emphasis has been put historically on u.d. mod 1 sequences, the distributional behaviour of sequences that are *not* u.d. mod 1 continues to attract attention also [65,93]. Traditionally, the behaviour of such sequences has often been studied using *asymptotic distribution functions*. However, many asymptotic distribution functions may correspond to the same element in $\mathcal{P}(\mathbb{T})$, the space of all Borel probabilities on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ endowed with the topology of weak convergence. A more transparent and conceptually satisfying approach, advocated in Chapter 6, is to work directly in $\mathcal{P}(\mathbb{T})$, making use of basic dynamical notions, specifically *Omega limit sets* [98]; see Section 1.2 for details.

One particularly interesting class of sequences that in general are not u.d. mod 1 consists of *logarithmically uniformly distributed* (l.u.d.) sequences [95] which have recently found applications in machine learning [9]. The very simplest l.u.d. sequence is $(\log_b n)$, with some integer $b \geq 2$. The distributional asymptotics of these sequences is well known: The Omega limit set is a closed loop in $\mathcal{P}(\mathbb{T})$ consisting entirely of rotated versions of one exponential distribution whose parameter depends on b [59, 99]. Though [99] studied a wider class of sequences that include $(\log_b n)$, the precise rate of convergence to the Omega limit set does not seem to ever have been analyzed. Chapter 6 provides this analysis. We point out that shortly after [100], on which Chapter 6 is based, [77] addressed a similar problem in terms of asymptotic distribution functions, deriving $(N^{-1} \log N)$ as an upper bound on the rate of convergence w.r.t. the discrepancy metric. As will be seen in Chapter 6, this bound can be strengthened to $\left(N^{-1} (\log N)^{1/2}\right)$ for the L^1-Kantorovich metric $d_{\mathbb{T}}$ on $\mathcal{P}(\mathbb{T})$, but is in fact sharp for both the discrepancy metric and the L^1-Kantorovich metric on $\mathcal{P}_1(\mathbb{R})$.

Given any map $T : \mathbb{R} \to \mathbb{R}$, a classical question in UDT is whether T preserves uniform distribution in the sense that if (x_n) is u.d. mod 1 then so is $(T(x_n))$; see, e.g., [65]. This question allows for a natural variation:

Question 1.2. *Given a (continuous) transformation $T : \mathbb{R} \to \mathbb{R}$, does there exist a u.d. mod 1 sequence (x_n) such that $(T(x_n))$ is also u.d. mod 1?*

It is easy to see that for many convex maps T the answer to Question 1.2 is YES. For example, if $T(x) = e^x$ then (αn) and $(T(\alpha n))$ are both u.d. mod 1 for all but countably many irrational numbers α [65, Cor.1.4.1], notwithstanding the fact that it remains an open problem whether $(T(\alpha n))$ is u.d. mod 1 for specific irrational α such as $\log \pi$ or, in fact, $\log 3/2$.

In the context of Question 1.2, notice that if the (non-random) empirical averages $\mu_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{x_n}$ were to converge in $\mathcal{P}(\mathbb{R})$, to μ say, then both μ and $\mu \circ T^{-1}$ would project onto Lebesgue measure on $\mathbb{T}$. As explained in Section 1.3 below, Chapter 7 analyzes in detail a class of (convex) maps T for which this situation does indeed occur. In particular, if μ and $\mu \circ T^{-1}$ are supported on an interval of length 1 then $\mu \circ T^{-1} = \mu$, i.e., μ is an (absolutely continuous) *invariant probability measure* for T. Recall that invariant (probability) measures arguably are the most fundamental objects in ergodic theory and continue to be studied extensively. Thus let us conclude this general introduction by quoting from [21]: *"Ergodic theory is one of the few branches of mathematics which has changed radically during the last two decades. Before this period, with a small number of exceptions, ergodic theory dealt primarily with averaging problems and general qualitative questions, while now it is a powerful amalgam of methods used for the analysis of statistical properties of dynamical systems. For this reason, the problems of ergodic theory now interest not only the mathematician, but also the research worker in physics, biology, chemistry, etc."*

In the three sections that follow, we describe in detail the respective contents of Chapters 2-7 in a relatively broad context.

1.1 Best finite constrained approximations of one-dimensional probabilities

The classical method for the optimal quantization problem is the Voronoi partition approach [29], based on the following equivalent representation.

Proposition 1.3. [40, Lem.3.1]. *Assume $\mu \in \mathcal{P}_r(\mathbb{R}^m)$ for some $r \geq 1$. For every $n \in \mathbb{N}$,*

$$V_{n,r}(\mu) = \inf_{A \subset \mathbb{R}^m, \# A \leq n} \mathbb{E} \min_{a \in A} \|X - a\|^r.$$

Let us briefly recall the notion of *Voronoi partition*. Assume a non-empty set A of $\mathbb{R}^m$ is *locally finite*, i.e., the number of elements of A within any bounded subset of $\mathbb{R}^m$ is finite. The *Voronoi region* generated by $a \in A$ is

$$W(a|A) = \{x \in \mathbb{R}^m : \|x - a\| = \min_{a' \in A} \|x - a'\|\},$$

and $\{W(a|A) : a \in A\}$ is called the *Voronoi diagram* of A. A Borel measurable partition $\{U_a : a \in A\}$ of $\mathbb{R}^m$ is called *Voronoi partition* of $\mathbb{R}^m$ w.r.t. A and $\mu \in \mathcal{P}(\mathbb{R}^m)$ if

$$\mu\left(W(a|A) \setminus U_a\right) = 0, \quad \forall\, a \in A.$$

We comment that the Voronoi partition depends on the geometry (in terms of the norm $\|\cdot\|$) of $\mathbb{R}^m$. This shows that optimal quantizers are metric-dependent also; or, from the viewpoint of approximation of probability measures, that best approximations may vary from one probability metric to another [40, 43]; see also Chapter 4.

Let $\mu \in \mathcal{P}_r(\mathbb{R}^m)$. If $A \subset \mathbb{R}^m$ is an n-optimal set of centers and $\{W_a : a \in A\}$ is a Voronoi partition of $\mathbb{R}^m$ w.r.t. A and μ, then $f = \sum_{a \in A} a \mathbb{1}_{W_a}$ is an n-optimal quantizer. In fact, the classical optimality conditions are often presented in terms of Voronoi partitions.

Proposition 1.4. [40, Thm.4.1]. *Let $\mu \in \mathcal{P}_r(\mathbb{R}^m)$, $A \subset \mathbb{R}^m$ an n-optimal set of centers and $\{W_a : a \in A\}$ a Voronoi partition of $\mathbb{R}^m$ w.r.t. A and μ. Assume that $\#\operatorname{supp}\mu \geq n$. Then*

$$\# A = n, \;\; \mu(W_a) > 0, \quad \forall\, a \in A,$$

and B is an m-optimal set of centers for $\mu\big|_{\cup_{a \in B} W_a}$, for every $B \subset A$ and $\#B = m$. In particular, $\mu(W_a) > 0$, and a is a 1-optimal set of center for $\mu\big|_{W_a}$ of order r, for every $a \in A$.

Note that there are two quantities that together determine a finitely supported probability μ: the *location vector* storing the information of the support of μ, and the *weight/probability* *vector* recording the relative mass of each atom in the support. Based on this consideration, we introduce, in the arguably simplest, one-dimensional setting, a notation for *finitely supported probabilities*: For $n \in \mathbb{N}$, let

$$\delta_x^p := \sum_{j=1}^{n} p_{,j}\delta_{x_{,j}}, \quad x = (x_{,1}, \cdots, x_{,n}) \in \Xi_n,\ p = (p_{,1}, \cdots, p_{,n}) \in \Pi_n,$$

where $\Xi_n := \{y \in \mathbb{R}^n : y_{,1} \le y_{,2} \le \cdots \le y_{,n}\}$ is the cone of all (non-decreasingly) ordered n-dimensional (location) vectors, and $\Pi_n := \{q \in \mathbb{R}^n : \sum_{j=1}^{n} q_{,j} = 1,\ q_{,j} \ge 0,\ \forall\ j = 1, \cdots, n\}$ the simplex of all n-dimensional weight/probability vectors.

Question 1.5. *Given $\mu \in \mathcal{P}(\mathbb{R})$, what is a best (constrained) finite approximation of μ with prescribed locations (respectively, weights)?*

Naturally, Question 1.5 may be divided into several specific questions. As mentioned earlier, one of the fundamental research interests in quantization is the existence and characterization of the optimal quantizer (or equivalently, the best approximation). Existence and necessary conditions for optimality are derived in the literature in terms of Voronoi partitions for the L^r-Kantorovich metric [40, Thm.4.1&Sec.5.2] and for the Prokhorov metric [43, Thms.2.1–2]. Analogously, we ask:

Question 1.6. *Given $\mu \in \mathcal{P}(\mathbb{R})$, do best approximations of μ with given locations (or weights) exist, and if so, can they be characterized in a simple way?*

This question is addressed in Sections 2.4, 4.3 and 4.5, where existence, as well as necessary and sufficient conditions for best approximations of probabilities on $\mathbb{R}$, are derived, w.r.t. three types of probability metrics, the L^r-Kantorovich metrics ($1 \le r < +\infty$), the Lévy metric, and the Kolmogorov metric. These results are given, respectively, in Theorems 2.25 and 2.29 for the Kantorovich metrics; in Theorems 4.4 and 4.5 for the Lévy metric, and in Theorem 4.20 and 4.23 for the Kolmogorov metric. Note that the method employed for these necessary and sufficient conditions w.r.t. the L^r-norm is based on an approach different from the classical Voronoi partitions, namely best approximation of monotone functions by step functions (see Section 2.2). For Lévy and Kolmogorov metrics the analysis is also based on specific characteristics of these metrics. Moreover, as a byproduct, necessary conditions for optimality w.r.t. Kantorovich metrics are derived (see Theorem 2.47), and necessary and *sufficient* conditions for optimality w.r.t. Lévy and Kolmogorov metrics are presented (see Theorems 4.9 and 4.25, respectively).

As n, the number of atoms of an approximation, goes to infinity, convergence to μ of the n-atom best approximations w.r.t. the L^r-Kantorovich metric is guaranteed [40, Lem.6.1]; see also [43, (4.1)] for the Prokhorov metric, and [11, Thm.2.14] for the mean convergence of empirical measures. A natural analogous question therefore is:

Question 1.7. *Do best approximations of $\mu \in \mathcal{P}(\mathbb{R})$ with a sequence of prescribed locations (respectively, weights) converge to μ?*

We answer this question by establishing necessary and sufficient conditions for the convergence of these best constrained approximations; see Theorems 2.27 and 2.33. We mention that our approach to establish these necessary and sufficient conditions is rather different from the classical one for unconstrained best approximations. The simplest and arguably most interesting special case of our analysis concerns the best approximation with *identical weights*, or more formally, the best approximation within $\{\delta_x^{u_n} : x \in \Xi_n\}$, where $u_n = (n^{-1}, \cdots, n^{-1}) \in \Pi_n$. Such *best uniform approximations* may be considered deterministic analogues of approximations by empirical measures, with the iid. random variables X_j replaced by the deterministic points $x_{,j}$. A substantial portion of Chapters 2 and 3 is devoted to best uniform approximations.

Once convergence is guaranteed, rates of convergence become the next concern. A celebrated result on asymptotics of the quantization error (or equivalently, the universal rate of convergence of best unconstrained approximations) already alluded to earlier is as follows; see also [40, Thm.6.2], [43, Thm.4.2].

Proposition (2.50).[2] *Assume $\mu \in \mathcal{P}_r(\mathbb{R}^m)$ for some $r \geq 1$. Then*

$$\liminf_{n\to\infty} n^{r/d} V_{n,r}(\mu) \geq Q_r([0,1]^m) \left\| \frac{\mathrm{d}\mu_a}{\mathrm{d}\lambda^m} \right\|_{d/(d+r)}.$$

If $\mu \in \mathcal{P}_s(\mathbb{R}^m)$ with $s > r$, then

$$\lim_{n\to\infty} n^{r/d} V_{n,r}(\mu) = Q_r([0,1]^m) \left\| \frac{\mathrm{d}\mu_a}{\mathrm{d}\lambda^m} \right\|_{d/(d+r)},$$

where μ_a is the absolutely continuous part *(w.r.t. λ^m) of μ, and $Q_r([0,1]^m)$, the quantization coefficient of the uniform distribution on the unit cube $[0,1]^m$, is a positive constant.*

The proof of this classical result is quite involved and based on the variational representation of the quantization error (Proposition 1.3). Analogous, weaker results w.r.t. the Prokhorov metric are obtained in [43, Thm.4.3&Prop.4.1]. As mentioned earlier, Proposition

[2] From now on, if a Proposition (or Theorem, Lemma, etc.) is shown with its label $(x.y)$ in parentheses then this indicates that the same Proposition (or Theorem, Lemma, etc.) is restated verbatim at the appropriate place in Chapter x, typically with further detail or proof.

2.50 has been generalized to compact Riemannian manifolds for absolutely continuous measures [57, Thm.1.4], and to the non-compact case for all probability measures [61, Thm.1.2]. Again, in the random setting, rates of convergence of empirical measures are intensively investigated, motivated by various applications [11, 33, 97].

In light of these, a natural analogous question for best uniform approximations is

Question 1.8. *How fast do best uniform approximations of $\mu \in \mathcal{P}(\mathbb{R})$ converge to μ?*

This question is fully discussed in Subsection 2.4.2. It turns out that unlike for best (unconstrained) approximations, the rates of convergence of best uniform approximations may vary drastically, a phenomenon that has been observed for empirical measures also [11]. As one of the main results of Chapter 2, we show that nevertheless, a universal rate of convergence (n^{-1}) can be guaranteed under mild conditions. To formulate this result, we need the following definitions.

Definition 1.9. [11, 102]. Given $\mu \in \mathcal{P}(\mathbb{R})$, the positive Borel measure μ^{-1} on $]0, 1[$ with

$$\mu^{-1}\left(]t, u]\right) = F_\mu^{-1}(u) - F_\mu^{-1}(t), \; 0 < t < u < 1,$$

is called the *inverse measure* of μ.

Definition 1.10. The sequence $\left(\delta_{x_n}^{u_n}\right)$ with $x_n \in \Xi_n$ for all $n \in \mathbb{N}$ is a sequence of *asymptotically best uniform r-approximations* of $\mu \in \mathcal{P}_r(\mathbb{R}) \setminus \{\delta_x^{u_i} : \; i \in \mathbb{N}, \; x \in \Xi_i\}$ if

$$\lim_{n\to\infty} \frac{d_r\left(\mu, \delta_{x_n}^{u_n}\right)}{d_r\left(\mu, \delta_\bullet^{u_n}\right)} = 1.$$

Note that finding asymptotically best uniform r-approximations is as important as finding asymptotically best r-approximations, simply because most best uniform r-approximations do not have closed-form expressions.

Theorem (2.39). *Assume that $\mu \in \mathcal{P}_r(\mathbb{R})$ for some $r \geq 1$. If μ^{-1} is absolutely continuous (w.r.t. λ) then*

$$\lim_{n\to\infty} n d_r\left(\mu, \delta_\bullet^{u_n}\right) = \frac{1}{2(r+1)^{1/r}} \left(\int_{\mathbb{I}} \left(\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda} \right)^r \right)^{1/r}.$$

Moreover, if $\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda} \in L^r(\mathbb{I})$ then $\left(\delta_{x_n}^{u_n}\right)$, with $x_{n,i} = F_\mu^{-1}\left(\frac{2i-1}{2n}\right)$ for $1 \leq i \leq n$, is a sequence of asymptotically best uniform r-approximations of μ, unless μ is degenerate, i.e., unless $\mu = \delta_a$ for some $a \in \mathbb{R}$.

We point out that in contrast to Proposition 2.50, the proof of Theorem 2.39 is neither straightforward nor directly amenable to the classical Voronoi partition approach, because the latter fails when, as in Theorem 2.39, the weights rather than the locations are prescribed. We also mention that the two assumptions, absolute continuity and integrability, are crucial in Theorem 2.39; see Examples 2.40 and 2.41 for details. Finally, it is worth noting that rates of convergence of best countable (possibly infinite) approximations (with constraints on the weights) of a compactly supported absolutely continuous probability on $\mathbb{R}^m$ have been investigated in [15], using the Γ-convergence approach.

As indicated earlier, one current line of research focuses on the quantization of singular probability distributions. Two important notions in quantifying the rate of convergence are *quantization dimension* and *quantization coefficient* [40, Ch.3]. The quantization dimension of a singular probability distribution, in particular, is studied for its relations to other notions of dimension, such as *Hausdorff, box*, and *packing dimensions* [40, 67]. The quantization coefficient measures the homogeneity (or the lack thereof) of convergence. To state a classical result on the quantization of the Cantor measure μ, let us recall these two notions more formally; see [40, Def.11.1]. For every $\mu \in \mathcal{P}_r(\mathbb{R})$, let $\delta_{\bullet}^{\bullet,n}$ be a best d_r-approximation of μ.

Definition 1.11. $\underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log d_r\left(\mu,\delta_{\bullet}^{\bullet,n}\right)}$ is the *lower quantization dimension* of μ of order r; and $\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log d_r\left(\mu,\delta_{\bullet}^{\bullet,n}\right)}$ is the *upper quantization dimension* of μ of order r. If $\underline{D}_r(\mu) = \overline{D}_r(\mu)$, then the common value, denoted $D_r(\mu)$, is the *quantization dimension* of μ of order r. If $\lim_{n\to\infty} n^{r/D_r(\mu)} d_r\left(\mu, \delta_{\bullet}^{\bullet,n}\right)^r$ exists, and is both positive and finite, then this limit is called the r-th *quantization coefficient* of μ.

The quantization dimension and *non-existence* of quantization coefficient for the Cantor distributions are known in the literature.

Theorem 1.12. [64, Prop.5.3&Rem.5.4]. *Let μ be the Cantor distribution and $r \geq 1$. Then $D_r(\mu) = \frac{\log 2}{\log 3}$, but the quantization coefficient of μ of order r does not exist.*

The inverse Cantor measure is also singular, in fact discrete, and has rarely appeared in the literature before. We ask

Question 1.13. *With μ denoting the Cantor distribution, what is the quantization dimension of μ^{-1}? Does the quantization coefficient exist for μ^{-1} ?*

Both parts of this question are addressed in Example 2.54, where it is shown that $D_r(\mu^{-1}) = \left(\left(1 - \frac{1}{r}\right) + \frac{1}{r}\frac{\log 3}{\log 2}\right)^{-1}$ for all $r \geq 1$. For reasons explained there, it remains an open problem whether the quantization coefficient exists for μ^{-1}.

Uniform quantization dimensions and *coefficients* can be defined analogously. For every $\mu \in \mathcal{P}_r(\mathbb{R})$, let $\delta_\bullet^{u_n}$ be a best uniform d_r-approximation of μ.

Definition 1.14. $\underline{D}_r^u(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log d_r(\mu, \delta_\bullet^{u_n})}$ is the *lower uniform quantization dimension* of μ of order r; and $\overline{D}_r^u(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log d_r(\mu, \delta_\bullet^{u_n})}$ is the *upper uniform quantization dimension* of μ of order r. If $\underline{D}_r^u(\mu) = \overline{D^u}_r(\mu)$, then the common value, denoted $D_r^u(\mu)$, is the *uniform quantization dimension* of μ of order r. If $\lim_{n\to\infty} n^{r/D_r^u(\mu)} d_r\left(\mu, \delta_\bullet^{u_n}\right)^r$ exists, and is both positive and finite, then this limit is called the r-th *uniform quantization coefficient* of μ.

We complement Question 1.13 by asking

Question 1.15. *What are the uniform quantization dimensions of μ and μ^{-1}? Do uniform quantization coefficients exist for μ and μ^{-1}?*

As will be seen in Section 2.4,

$$\underline{D}_r(\mu) = \frac{\log 2}{\log 3}, \quad \overline{D}_r(\mu) = r; \quad D_r\left(\mu^{-1}\right) = \left(\frac{1}{r} + \left(1 - \frac{1}{r}\right)\frac{\log 2}{\log 3}\right)^{-1}, \quad \forall\, r \geq 1,$$

and the first and second uniform quantization coefficients of μ^{-1} do not exist; see Examples 2.40 and the comments following Remark 2.42.

Next, let us further discuss the rate of convergence for best (unconstrained) approximations. Recall from Proposition 2.50 that $(d_r(\mu, \delta_\bullet^{\bullet,n}))$ decays not slower than (n^{-1}), provided that $\mu \in \mathcal{P}_s(\mathbb{R})$ and $\mu_a \neq 0$.

Question 1.16. *What if the moment condition $\mu \in \mathcal{P}_s(\mathbb{R})$ is not satisfied?*

As shown in Chapter 2, the quantization error can decay arbitrarily slowly without the moment condition. More precisely, we prove

Theorem (2.56). *Given $r \geq 1$ and any sequence (a_n) of non-negative real numbers with $\lim_{n\to\infty} a_n = 0$, there exists $\mu \in \mathcal{P}_r(\mathbb{R})$ such that $d_r(\mu, \delta_\bullet^{\bullet,n}) \geq a_n$ for every $n \in \mathbb{N}$.*

Naturally, one may also ask how all the results mentioned in this section so far are affected by a change of the underlying probability metric. This question motivates much of the analysis in Chapters 4 and 5. For instance, it will be shown that universality of the rate (n^{-1}) is not specific to the Kantorovich d_r-metrics, but rather prevails under more general circumstances; see Proposition 4.30. To illustrate many of our results in these chapters, we use one particular probability distribution, *Benford's Law*, as a recurring example, a choice that will be explained at the beginning of Chapter 4.

1.2 Distributional asymptotics of slowly changing sequences

Given a sequence (x_n) of real numbers, associate with it a sequence $(\nu_N(x_n))_{N\geq 1}$ of finitely supported probability measures on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$,

$$\nu_N(x_n) := \frac{1}{N}\sum_{n=1}^{N} \delta_{\langle x_n\rangle},$$

where $\langle a\rangle = a + \mathbb{Z}$ for every $a \in \mathbb{R}$. Let $\lambda_{\mathbb{T}}$ (respectively, $\lambda_{\mathbb{I}}$) denote the uniform probability measure on $\mathbb{T}$ (respectively, $\mathbb{I}$). Also, denote by $\langle\langle a\rangle\rangle \in [0,1[$ the fractional part of a. With this, $\iota : \langle a\rangle \mapsto \langle\langle a\rangle\rangle$ establishes a bijection from $\mathbb{T}$ onto $[0,1[$.

Definition 1.17. [65, Def.1.1.1]. A sequence (x_n) of real numbers is *uniformly distributed modulo one* (u.d. mod 1) if, for every pair α, β of real numbers with $0 \leq \alpha < \beta \leq 1$,

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \alpha \leq \langle\langle x_n\rangle\rangle < \beta\}}{N} = \beta - \alpha.$$

First, let us recall some equivalent definitions of a u.d. mod 1 sequence.

Proposition 1.18. [65, Ch.1]. *The followings are equivalent for every sequence (x_n) of real numbers:*

(i) *(x_n) is u.d. mod 1.*

(ii) *The sequence $(\nu_N(x_n))_{N\geq 1}$ converges weakly in $\mathcal{P}(\mathbb{T})$ to $\lambda_{\mathbb{T}}$.*

(iii) $\lim_{N\to\infty} d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}}) = 0$.

(iv) *Every real-valued continuous function f on $[0,1]$ satisfies*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\langle\langle x_n\rangle\rangle) = \int_0^1 f(s)\mathrm{d}x. \tag{1.1}$$

(v) *Every complex-valued continuous 1-periodic function f on $\mathbb{R}$ satisfies (1.1).*

(vi) *Every real-valued Riemann-integrable function f on $[0,1]$ satisfies (1.1).*

(vii) Weyl's Criterion: *(x_n) satisfies $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0$, $\forall\ h \in \mathbb{Z}\setminus\{0\}$.*

In the literature, any finer, quantitative analysis of u.d. mod 1 sequences is based on the fundamental concept of discrepancy. Specifically, given any sequence (x_n), for every $N \in \mathbb{N}$, the *discrepancy* $D_N(x_n)$ is defined as

$$D_N(x_n) = \sup_{0\le\alpha<\beta<1} \left| \frac{\#\{1 \le n \le N : \alpha \le \langle\langle x_n \rangle\rangle < \beta\}}{N} - (\beta - \alpha) \right| .$$

With this, it is easily checked that (x_n) is u.d. mod 1 if and only if $\lim_{N\to\infty} D_N(x_n) = 0$. To tie this classical concept in with the tools employed elsewhere in this thesis, note that

$$d_{\mathsf{K}}\left(\nu_N(x_n) \circ \iota^{-1}, \lambda_{\mathbb{I}}\right) \le D_N(x_n) \le 2 d_{\mathsf{K}}\left(\nu_N(x_n) \circ \iota^{-1}, \lambda_{\mathbb{I}}\right).$$

Thus the two sequences $(D_N)_{N\ge1}$ and $(d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}}))_{N\ge1}$ have the same asymptotics, and we will state our results in terms of the latter, for consistency and convenience. (Often, $d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}})$ is denoted D_N^* in the literature [65].) There is a vast literature on the estimation of discrepancy for u.d. mod 1 sequences, i.e., on the rate of convergence of $(D_N)_{N\ge1}$; see, e.g., the two authoritative monographs [28, 65], as well as the more than 2500(!) references therein. In general, two important methods, attributed, respectively, to K.F. Roth and W. Schmidt, provide lower bounds for D_N, whereas LeVeque's inequality and the Erdös-Turán Theorem yield upper bounds. A restriction to special classes of sequences naturally leads to more accurate estimates. A prominent example in this regard are *almost arithmetic progressions* (αn), where α is irrational (and may satisfy further, number-theoretical assumptions). Precise studies of $(D_N(\alpha n))$ play an important role in Diophantine approximation. As one basic result, let us mention [65, Thm.2.3.4] which asserts that

$$d_{\mathsf{K}}\left(\nu_N(\alpha n) \circ \iota^{-1}, \lambda_{\mathbb{I}}\right) = \mathcal{O}\left(N^{-1} \log N\right), \tag{1.2}$$

provided that α has *bounded partial quotients*; estimate (1.2) also holds for the *Van der Corput sequence* [65, Thm.2.3.5]. For a comprehensive study of discrepancy, the reader is referred to the two monographs mentioned earlier.

Notice that (x_n) is u.d. mod 1 precisely if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[0,t]}(\langle\langle x_n \rangle\rangle) = t, \quad \forall\, t \in \mathbb{I}. \tag{1.3}$$

A simple example not satisfying (1.3) is $(x_n) = (\log n)$. However, it is well known [95], and in fact easy to check, that the latter sequence satisfies

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbb{1}_{[0,t]}(\langle\langle x_n \rangle\rangle) = t, \quad \forall\, t \in \mathbb{I}. \tag{1.4}$$

Note that (1.3) implies (1.4). Any sequence with (1.4) is referred to as *logarithmically uni-*

formly distributed (l.u.d.) or as *uniformly distributed w.r.t. the logarithmic mean*. In analogy to D_N, when dealing with l.u.d. sequences one may consider the *logarithmic discrepancy*

$$\widetilde{D}_N(x_n) = d_{\mathsf{K}}\left(\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\delta_{\langle\langle x_n\rangle\rangle}, \lambda_{\mathbb{I}}\right) = \sup_{0\le t<1}\left|\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbb{1}_{[0,t]}(\langle\langle x_n\rangle\rangle) - t\right|.$$

For example, for the l.u.d. sequence $(\alpha n + \beta \log n)$ with real α and $\beta \neq 0$, which are relevant, e.g., for binomial and hypergeometric series on the periphery of their disc of convergence, it has been shown in [49] that

$$\widetilde{D}_N((\alpha n + \beta \log n)) = \mathcal{O}((\log N)^{-1/2}),$$

and this has later been improved to $\mathcal{O}((\log N)^{-1})$; see [75, 94].

Our analysis of the distributional asymptotics of sequences is motivated by a treatise by J.H.B. Kemperman [59], who, more generally and in fact prior to [49], studied the class of sequences (x_n) that are *slowly changing* in the sense that

$$\lim_{n\to\infty} n(x_{n+1} - x_n) = \xi \in \mathbb{R}.$$

For example, $(\log n)$ is slowly changing, with $\xi = 1$, and so are $(\log p_n)$ and (p_n/n), with p_n denoting the n-th prime number. As shown in [59], no slowly growing sequence is u.d. mod 1. Rather, the *Omega limit set*

$$\Omega[x_n] := \left\{\nu \in \mathcal{P}(\mathbb{T}) : \nu_{N_k}(x_n) \xrightarrow{k\to\infty} \nu \text{ weakly for some subsequence } (N_k) \text{ of } \mathbb{N}\right\}$$

consists of a 1-parameter loop in $\mathcal{P}(\mathbb{T})$ of rotated exponential (if $\xi \neq 0$) or Dirac distributions (if $\xi = 0$). This may be seen as a complement to the even earlier result [65, Thm.1.2.6] that $\limsup_{n\to\infty} n|x_{n+1} - x_n| = +\infty$ for every u.d. mod 1 sequence (x_n).

As the analysis in [59] is purely qualitative, our goal is to supplement it with precise quantitative estimates. In complete generality, this is a challenging task, and for much of our analysis we focus on one particular family of sequences, namely $(\log_b n)$ with $b \in \mathbb{N} \setminus \{1\}$. Some aspect of these sequences have been studied in [77, 94, 100] already, but sharp estimates and rates of convergence have been elusive so far. Note that $(\log_b n)$ is both l.u.d. and slowly changing, with $\xi = (\log b)^{-1}$.

To formulate one of our main results in this regard, for every $a > 0$ denote by $-\mathrm{Exp}\,(a)$ the negative exponential distribution on $\mathbb{R}$ with parameter a, with distribution function

$$F_{-\mathrm{Exp}(a)}(x) = \min\{1, e^{ax}\}, \quad \forall\, x \in \mathbb{R},$$

and let $E_a = -\mathrm{Exp}(a) \circ \pi^{-1}$, where $\pi : \mathbb{R} \to \mathbb{T}$ is the natural projection. Also, denote the rotation $\langle x \rangle \mapsto \langle x + a \rangle$ of $\mathbb{T}$ by R_a. In Chapter 6, we will establish the following estimate regarding the distributional asymptotics of $(\log_b n)$ w.r.t. the Kantorovich metric $d_{\mathbb{T}}$ on $\mathcal{P}(\mathbb{T})$.

Theorem (6.3). *Assume $b \in \mathbb{N} \setminus \{1\}$. Then, with $x_n = (\log_b n)$,*

$$\limsup_{N\to\infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}} \left(\nu_N, E_{\log b} \circ R^{-1}_{-x_N}\right) < +\infty.$$

As seen earlier, the metrics d_1 and d_{K} may also be utilized to describe this asymptotics. In fact, we prove even stronger quantitative estimates w.r.t. these metrics.

Theorem (6.5&6.6). *Assume $b \in \mathbb{N} \setminus \{1\}$. Then, with $x_n = (\log_b n)$,*

$$\lim_{N\to\infty} \frac{N}{\log N} d_1 \left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-x_N} \circ \iota^{-1}\right) = \frac{1}{2 \log b},$$

and the sequence $\left(\frac{N}{\log N} d_{\mathsf{K}} \left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-x_N} \circ \iota^{-1}\right)\right)_{N\geq 2}$ is bounded above and below by positive constants (which may depend on b).

1.3 Circularly invariant and uniform probability measures

Recall from the informal discussion accompanying Question 1.2 that, given a u.d. mod 1 sequence (x_n) and a convex map $T : \mathbb{R} \to \mathbb{R}$, the sequence $(T(x_n))$ may be u.d. mod 1 as well. For example, (πn) is u.d. mod 1, and so are $(T(\pi n))$ with $T(x) = 2x$ or $T(x) = x^2$. In general, consider the sequence $(\mu_N(x_n))_{N\geq 1}$ in $\mathcal{P}(\mathbb{R})$ with

$$\mu_N(x_n) := \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}.$$

Note that $\mu_N(x_n) \circ \pi^{-1} = \nu_N(x_n) \to \lambda_{\mathbb{T}}$ as $N \to \infty$, but also $\mu_N(x_n) \circ T^{-1} \circ \pi^{-1} = \nu_N(T(x_n)) \to \lambda_{\mathbb{T}}$, provided that (x_n) and $(T(x_n))$ both are u.d. mod 1. (Recall that $\pi : \mathbb{R} \to \mathbb{T}$ denotes the natural projection.) Now, suppose that $(\mu_N)_{N\geq 1}$ converges in $\mathcal{P}(\mathbb{R})$, to μ, say. Then $\mu \circ \pi^{-1} = \mu \circ T^{-1} \circ \pi^{-1} = \lambda_{\mathbb{T}}$, that is, the probability measures μ and $\mu \circ T^{-1}$ both are uniform when pushed forward onto $\mathbb{T}$ via π. As this seems to be a fairly strong invariance property that $\mu \in \mathcal{P}(\mathbb{R})$ may have, we ask

Question 1.19. *Given a convex map $T : \mathbb{R} \to \mathbb{R}$, does there exist $\mu \in \mathcal{P}(\mathbb{R})$ with the property*

$$\mu \circ \pi^{-1} = \mu \circ T^{-1} \circ \pi^{-1} = \lambda_{\mathbb{T}}? \tag{1.5}$$

For convenience, call any $\mu \in \mathcal{P}(\mathbb{R})$ satisfying (1.5) a *circularly invariant and uniform probability measure* (CIUPM) for T, and write $\mathcal{C}_T$ for the family of all CIUPM for T, i.e., $\mathcal{C}_T = \{\mu \in \mathcal{P}(\mathbb{R}) : \text{(1.5) holds for } \mu\}$. Note that every element of $\mathcal{C}_T$ is absolutely continuous (w.r.t. λ).

Simple examples show that the answer to Question 1.19 often is NO, that is, $\mathcal{C}_T = \varnothing$, especially if T is strictly convex. In general, it appears to be a very challenging problem to decide for which maps T exactly $\mathcal{C}_T \neq \varnothing$. As outlined below, in Chapter 7 we provide a partial, positive answer for the simplest class of convex maps, namely *linear* maps. We expect our results to be helpful for tackling Question 1.19 in greater generality in the future.

For every $\alpha, \beta \in \mathbb{R}$, consider the linear map $T_{\beta,\alpha}(x) = \beta x + \alpha$, and let $\mathcal{C}_{\beta,\alpha} = \mathcal{C}_{T_{\beta,\alpha}}$ for convenience. Clearly, $\mathcal{C}_{0,\alpha} = \varnothing$, so henceforth assume $\beta \neq 0$. It is not hard to see that in fact $\mathcal{C}_{\beta,\alpha} = \mathcal{C}_{\beta,0}$; see Proposition 7.4. Moreover, since

$$\mu \in \mathcal{C}_{\beta,\alpha} \Leftrightarrow \mu \circ T_{\beta,\alpha}^{-1} \in \mathcal{C}_{\beta^{-1},-\alpha\beta^{-1}} \Leftrightarrow \mu \circ T_{-1,0}^{-1} \in \mathcal{C}_{-\beta,-\alpha},$$

it suffices to consider the case of $\beta \geq 1$. Thus, when specialized to the family of linear maps $T_{\beta,\alpha}$, Question 1.19 really reads

Question 1.20. *Is $\mathcal{C}_{\beta,0} \neq \varnothing$ for every $\beta \geq 1$?*

The following result completely answers Question 1.20. In a way, it also provides a lower bound on the "size" of every CIUPM for $T_{\beta,0}$, and hence also for $T_{\beta,\alpha}$.

Theorem (7.9). *Assume $\beta \geq 1$. Then $\mathcal{C}_{\beta,0} \neq \varnothing$, and there exists $c_\beta \geq 1$ such that* $\operatorname{diam}\operatorname{supp}\mu \geq c_\beta$ *for every $\mu \in \mathcal{C}_{\beta,0}$. Moreover, there exists $\mu_\beta \in \mathcal{C}_{\beta,0}$ with* $\operatorname{diam}\operatorname{supp}\mu = c_\beta$, *and μ_β is uniquely determined up to translation, i.e., if $\mu \in \mathcal{C}_{\beta,0}$ satisfies* $\operatorname{diam}\operatorname{supp}\mu = c_\beta$ *then $\mu = \mu_\beta \circ T_{1,\alpha}^{-1}$ for some $\alpha \in \mathbb{R}$.*

In Chapter 7, the value of the threshold length c_β will be determined explicitly. In particular, it will be seen that $1 < c_\beta < 2$ unless $\beta \in \mathbb{N}$, in which case $c_\beta = 1$. Note also that μ_β automatically is absolutely continuous (w.r.t. λ).

Let us finally relate the topic of this subsection to a classical theme in ergodic theory. For this, notice that every linear map $T_{\beta,\alpha}$ induces a measurable map $\langle T_{\beta,\alpha}\rangle : \mathbb{T} \to \mathbb{T}$, via $\langle T_{\beta,\alpha}\rangle = \pi \circ T_{\beta,\alpha} \circ \iota$. (Recall that $\iota : \mathbb{T} \to \mathbb{R}$ denotes the natural inclusion given by $\iota(\langle x\rangle) = \langle\langle x\rangle\rangle$.) The map $\langle T_{\beta,\alpha}\rangle$ is continuous precisely if β is an integer. In this case, $\langle T_{\beta,\alpha}\rangle \circ \pi = \pi \circ T_{\beta,\alpha}$, and $\lambda_{\mathbb{T}}$ is $\langle T_{\beta,\alpha}\rangle$-invariant. This shows that simply $\mathcal{C}_{n,0} = \left\{\mu \in \mathcal{P}(\mathbb{R}) : \mu \circ \pi^{-1} = \lambda_{\mathbb{T}}\right\}$ for any $n \in \mathbb{N}$. In general, for $\beta > 1$ the dynamics of $\langle T_{\beta,\alpha}\rangle$ is very complicated, and unlike anything the simple dynamics of $T_{\beta,\alpha}$ may suggest. In fact, the detailed study of $\langle T_{\beta,\alpha}\rangle$, and in particular of the maps $\langle T_{\beta,0}\rangle$ often referred to as β-*transformations*, has been an important topic in the development of measurable dynamics and ergodic theory. It is now well known that, for every $\alpha, \beta \in \mathbb{R}$ with $|\beta| > 1$, the map $\langle T_{\beta,\alpha}\rangle$ preserves a unique absolutely continuous probability measure on $\mathbb{T}$. The latter is the unique measure of maximal entropy $\log|\beta|$, is supported on a finite union of arcs, and allows for an explicit representation via Parry's formula; for these and many other interesting dynamical properties of $\langle T_{\beta,\alpha}\rangle$, the reader is referred, e.g., to [16, 46, 50–52, 56, 80] and the references therein. Note that if β is an integer with $|\beta| \geq 2$ then the unique absolutely continuous $\langle T_{\beta,\alpha}\rangle$-invariant probability measure simply is $\lambda_{\mathbb{T}}$. This is consistent with the particular structure of $\mathcal{C}_{\beta,\alpha}$ noted earlier for integer β.

Organization of this thesis

The thesis is based on four papers. Chapters 2 and 3, as well as Chapters 4 and 5 are based on two joint papers with Arno Berger that have been submitted for publication. All results in both papers were obtained jointly, which makes it impossible to separate individual contributions item by item. My co-author has given permission to include these papers in the thesis. Chapter 6 (respectively, Chapter 7) is based on a submitted (respectively, published) paper by Chuang Xu. The bibliographical details for these four papers are as follows:

1. C. Xu and A. Berger, Best finite constrained approximations of one-dimensional probabilities, preprint (2017), arXiv:1704.07871.
2. A. Berger and C. Xu, Best finite approximations of Benford's Law, to appear in J. Theor. Probab., 2018.
3. C. Xu, The distributional asymptotics mod 1 of $(\log_b n)$, to appear in Unif. Distrib. Theory., 2018.
4. C. Xu, Circularly invariant uniformizable probability measures for linear transformations, J. Math. Anal. Appl., 455 (2017) 778–791.

Since the thesis is based on individual papers, some notational inconsistencies and redundancies are unavoidable, however, efforts have been made to keep them to a minimum.

Acknowledgements

Finally, it is with great pleasure and gratitude that I acknowledge the following persons who helped me in various ways. First and foremost, I like to thank my advisor, Arno Berger, for introducing me to the research topics covered in this thesis. Thanks to his persistence and patience in guiding me to let me know what a PhD student in mathematics should do, I benefited more than even I desired at the beginning of my PhD study, not only in the way of thinking mathematics, in looking for interesting mathematical problems, but particularly in mathematical writing. Beyond mathematics, I also have learned a lot imperceptibly from his manners – politeness, seriousness, persistence, patience, and wisdom. Honestly speaking, this thesis could not have been written as it is now without his patient guidance.

Besides, I have enjoyed and benefited a lot from the conversations with many people who passed to me their experience as well as provided me many valuable suggestions unselfishly. Here I like to express my sincere gratitude to them: Professor Michael Y. Li, Professor Yingfei Yi, Professor Hao Wang, Professor Brendan Pass, Professor Hassan Safouhi, Dr. Eric Woolgar, Professor Bin Han, Professor Feng Dai, Professor Michael A. Kouritzin, and Professor Zhongwei Shen, at the University of Alberta; Professor Hong Qian at the University of Washington-Seattle, Professor Zhisheng Shuai at the University of Central Florida, and Professor Yao Li at the University of Massachusetts-Amherst.

Special thanks to Professor Pawel Góra at Concordia University for his careful work as my external examiner.

Special thanks also are due my master thesis advisor, Professor Junjie Wei at Harbin Institute of Technology, who introduced me to the field of dynamical systems and mathematical biology, and provided me with constant support in my PhD study.

Special thanks also to Dr. Weifeng Shen who provided me with many invaluable suggestions and encouragement during the most difficult time of my PhD program.

I also like to thank all the faculty in the Department of Mathematical and Statistical Sciences who taught me in courses throughout my PhD program, and the staff in the general office who organized many activities for us graduate students, especially for the international students. Just because of these activities, my PhD life was not so boring. Special thanks also to Professor Jochen Kuttler and Ms. Tara Schuetz-Zawaduk for all their help, especially in the last year of the PhD program.

I want to express my thanks to all the members of the discussion group on *dynamical systems and stochastic analysis* in the last year of my PhD, from whom I learned a lot in the related areas of mathematics.

I like to say thank you to all my friends, without your appearance, my life would not have been as joyful and colorful. Special thanks to the following friends who provided me constant help and encouragement during the difficult times I experienced throughout my PhD: Dr. Xiaoyu Li, Dr. Kuize Zhang, Mr. Hui Wang and Mrs. Jingsi Chen, and Mr. Xi Zhang and Mrs. Jinyan Guo.

I owe thanks to many more people who are not mentioned here. It was my luck to know you all and I believe this is called “destiny”.

Last but not least, I would like to express my deepest gratitude and love to my dear parents, Baihe Xu and Jing Zhang. Without you, I would not have enjoyed my life. Without your constant support and encouragement, I would not be able to complete my PhD.

All the research in the thesis was supported in part by University of Alberta Doctoral Recruitment Scholarship, Pundit RD Sharma Memorial Graduate Award, Eoin L. Whitney Scholarship, Josephine Mitchell Graduate Scholarship, Pacific Institute for the Mathematical Sciences (PIMS) Graduate Scholarship, as well as the travel funding provided by Graduate Students’ Association (GSA) and Faculty of Graduate Studies and Research (FGSR).

Chapter 2

Best finite constrained approximations of one-dimensional probabilities

Finding best finitely supported approximations of a given (Borel) probability measure μ on $\mathbb{R}$ is an important basic problem that has been studied extensively and from several perspectives. Assuming for instance that $\int_{\mathbb{R}} |x|^r \mathrm{d}\mu(x) < +\infty$ for some $r \geq 1$, a classical question asks to minimize the L^r-Kantorovich (or transport) distance $d_r(\nu, \mu)$ over all discrete probabilities ν supported on at most n atoms, where n is a given positive integer. A rich theory of *quantization of probability measures* addresses this question, as well as applications thereof in such diverse fields as information theory, numerical integration, and optimal transport, among others; see, e.g., [15,40,78] and the many references therein. As is well known, a minimal value of $d_r(\nu, \mu)$ always is attained for some discrete probability $\nu = \delta_{\bullet}^{\bullet,n}$ which may or may not be determined uniquely by this minimality property. Moreover, $d_r(\delta_{\bullet}^{\bullet,n}, \mu) \to 0$ as $n \to \infty$, and the precise rate of convergence has attracted particular interest. A celebrated theorem (see Proposition 2.50 below) asserts that, under a mild moment condition, $\left(n d_r(\delta_{\bullet}^{\bullet,n}, \mu)\right)$ converges to a finite positive limit whenever μ is non-singular (w.r.t. Lebesgue measure). Results in a similar spirit have been established for important classes of singular measures, notably self-similar and -conformal probabilities [41, 60, 84]. While these classical results crucially employ Voronoi partitions (as developed in some detail, e.g., in [40]), alternative tools and extensions to other metrics have recently been studied as well [15, 18, 26].

A second important perspective on the approximation problem is that of *random empirical quantization* [12,25]. To illustrate it, let $(X_j)_{j\geq 1}$ be an iid. sequence of random variables with common law μ, and consider the (random) empirical measure $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{X_j}$; here and throughout, δ_a is a Dirac unit mass at $a \in \mathbb{R}$. Then $d_r(\mu_n, \mu) \to 0$ with probability one as $n \to \infty$, and $\lim_{n\to\infty} \mathbb{E} d_r(\mu_n, \mu) = 0$. A comprehensive analysis of the rate of convergence of $\left(\mathbb{E} d_r(\mu_n, \mu)\right)$ is provided by the recent monograph [11] which, in particular, identifies necessary and sufficient conditions for decay to occur at the "standard rate" $\left(n^{-1/2}\right)$, that is, for $\left(n^{1/2}\mathbb{E} d_r(\mu_n, \mu)\right)$ to be bounded above and below by finite positive constants. Beyond these one-dimensional results, rates of convergence for random empirical quantization have lately been studied in higher dimensions and other settings also; see, e.g., [12, 25, 33].

The purpose of the present chapter is to develop a third perspective on the approximation problem that in a sense lies between the two established perspectives briefly recalled above. Specifically, we present an in-depth study of finitely supported approximations that are *non-random* yet *constrained* in that either the locations or the weights of the approximations' atoms are prescribed. To the best of our knowledge, such approximations have not been studied systematically in the literature, though the recent papers [2] and [15] do consider (uniform) "$\mathcal{U}$-quantization" and discrete approximations of absolutely continuous probabilities μ, respectively. The necessary and sufficient conditions for best constrained approximations presented in this article make no assumptions on μ beyond $\int_{\mathbb{R}} |x|^r \mathrm{d}\mu(x) < +\infty$. They follow rather directly from elementary properties of monotone functions and exploit a certain duality between locations and weights of atoms. (In contrast, Voronoi partitions appear to be far less useful if weights, rather than positions, are prescribed.) Arguably the simplest special case where our results apply is that of best *uniform* approximations: Given μ and a positive integer n, for which $\nu = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$ is $d_r(\nu, \mu)$ minimal, where $x_1 \le x_2 \le \ldots \le x_n$? Theorem 2.29 below characterizes the (often unique) minimizer $\delta_{\bullet}^{u_n}$. This special case is of considerable interest in itself, as practical considerations often demand that all atoms have equal weights, or at least be integer multiples of one fixed unit weight [6]. Just as for the best unconstrained and the random empirical approximations mentioned earlier, $d_r(\delta_{\bullet}^{u_n}, \mu) \to 0$ as $n \to \infty$, which again makes the rate of convergence a natural object of study. Presented in Subsection 2.4.2, our results in this regard are quite similar to those of [11], despite their obviously different context. As a simple illustrative example, consider the standard exponential distribution, i.e., let $\mu(]-\infty, x]) = 1 - e^{-x}$ for all $x \ge 0$. From Proposition 2.50 below, it follows that in the case of best (unconstrained) approximation, for all $r \ge 1$,

$$d_r\left(\delta_{\bullet}^{\bullet, n}, \mu\right) = \mathcal{O}\Big(n^{-1}\Big) \quad \text{as } n \to \infty\,,$$

whereas in the case of random empirical approximations, [11, Sec.6.4] shows that

$$\mathbb{E} d_r(\mu_n, \mu) = \begin{cases} \mathcal{O}\Big(n^{-1/2}\Big) & \text{if } 1 \le r < 2\,, \\ \mathcal{O}\Big(n^{-1/2}(\log n)^{1/2}\Big) & \text{if } r = 2\,, \\ \mathcal{O}\Big(n^{-1/r}\Big) & \text{if } r > 2\,. \end{cases}$$

In contrast, the reader will learn in Section 2.4 that, in the case of best uniform approximations,

$$d_r\left(\delta_\bullet^{u_n}, \mu\right) = \begin{cases} \mathcal{O}(n^{-1}\log n) & \text{if } r = 1\,, \\ \mathcal{O}\Big(n^{-1/r}\Big) & \text{if } r > 1\,. \end{cases}$$

Moreover, all rates displayed above are sharp. Not too surprisingly, therefore, the rate of convergence of $\big(d_r(\delta_\bullet^{u_n}, \mu)\big)$ is slower than that of $\big(d_r(\delta_\bullet^{\bullet, n}, \mu)\big)$, but faster than that of $\big(\mathbb{E}d_r(\mu_n, \mu)\big)$, at least for $1 \le r \le 2$; based on our results, it is tempting to speculate why for $r > 2$ optimal (non-random) and random empirical approximations on average exhibit the same rate of convergence. Due to the nature of the underlying approximation problem for monotone functions, our approach is not restricted to d_r, and results in a similar spirit can be established for other important metrics and for discrete approximations with countable support. One-dimensionality, on the other hand, is crucial, and multidimensional analogues for our results may prove more challenging than for best (unconstrained) or random empirical approximations (with some caveats; see [11, p.8] and [33, p.709]).

This chapter is organized as follows. Section 2.1 introduces the notations used throughout, and recalls definition and basic properties of the metric d_r for the reader's convenience. Section 2.2 reviews several elementary facts about monotone functions and their quantile and growth sets, as well as the notion of a balanced function, to be used subsequently in Section 2.3 to characterize best approximations of (monotone) L^r-functions by step functions. While they may also be of independent interest, these results crucially serve as tools in Section 2.4, the main part of this chapter. In that section, necessary and sufficient conditions for best finite approximations with prescribed locations (Subsection 2.4.1) or weights (Subsection 2.4.2) are established. Much attention is devoted to the special case of best uniform approximations $\delta_\bullet^{u_n}$, and in particular to the rate of convergence of $\big(d_r(\delta_\bullet^{u_n}, \mu)\big)$. Convergence theorems and finite (upper and lower) bounds for such sequences are provided. All results are illustrated for simple examples of μ which include absolutely continuous (exponential, Beta) as well as singular (Cantor, inverse Cantor) probability measures.

For the reader's convenience, the proofs of all propositions in this chapter are assembled in Section 3.1.

2.1 Notations

The following, mostly standard notations are used throughout. The natural and real numbers are denoted $\mathbb{N}$ and $\mathbb{R}$, respectively. The extended real numbers are $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For any $a \in \overline{\mathbb{R}}$, sgn $a = 1$ if $a > 0$, sgn $0 = 0$, and sgn $a = -1$ if $a < 0$. The indicator function of any set $A \subset \overline{\mathbb{R}}$ is denoted $\mathbb{1}_A$, and log symbolizes the natural logarithm. For $x \in \mathbb{R}$, let $|x|$ be the absolute value, $\lfloor x \rfloor$ the floor (i.e., the largest integer $\leq x$), and $\langle x \rangle = x - \lfloor x \rfloor$ the fractional part of x, respectively. Lebesgue measure on $\overline{\mathbb{R}}$ is symbolized by λ, and δ_a stands for the Dirac measure concentrated at a, i.e., $\delta_a(A) = \mathbb{1}_A(a)$ for all A.

The usual notations for intervals, e.g., $[a, b[= \left\{x \in \overline{\mathbb{R}} : a \leq x < b\right\}$ are used. When endowed with the topology $\left\{[-\infty, a[\cup U \cup]b, +\infty] : a, b \in \overline{\mathbb{R}},\ U \subset \mathbb{R} \text{ open}\right\}$, the space $\overline{\mathbb{R}}$ is compact and homeomorphic to the unit interval $\mathbb{I} = [0, 1]$. Throughout, $I \subset \overline{\mathbb{R}}$ always denotes a closed (and hence compact) interval that is non-degenerate, i.e., $\lambda(I) > 0$. For $A \subset \overline{\mathbb{R}}$, denote by $\#\ A$, $\overset{\circ}{A}$ and $\overline{A}$ the cardinality (number of elements), interior and closure of A, respectively. Every non-empty A has an infimum $\inf A$ and a supremum $\sup A$; if A is closed, then $\inf A = \min A$ and $\sup A = \max A$. If $A \subset \overline{\mathbb{R}}$ is an interval and $f : A \to \overline{\mathbb{R}}$ is monotone, then $f(a-) = \lim_{\varepsilon \downarrow 0} f(a - \varepsilon)$ and $f(a+) = \lim_{\varepsilon \downarrow 0} f(a + \varepsilon)$ both exist for every $a \in \overset{\circ}{A}$.[1] For any set $A \subset \overline{\mathbb{R}}$ and any function $f : A \to \overline{\mathbb{R}}$, the image of A under f is $f(A) = \{f(a) : a \in A\}$, while the pre-image of $B \subset \overline{\mathbb{R}}$ is $f^{-1}(B) = \{a \in A : f(a) \in B\}$. Also, for every $b \in \overline{\mathbb{R}}$, let $\{f \leq b\} = f^{-1}([-\infty, b])$; the sets $\{f \geq b\}$, $\{f < b\}$, $\{f > b\}$ and $\{f = b\}$ are defined analogously. Denote by $\text{essinf}_A f$ and $\text{esssup}_A f$ the essential infimum and supremum of f on A, respectively. For $1 \leq r < +\infty$ and any (closed) interval $I \subset \overline{\mathbb{R}}$, let $L^r(I)$ be the space of all measurable functions $f : I \to \overline{\mathbb{R}}$ that are (absolutely) r-integrable with respect to λ, and $L^\infty(I)$ the space of all functions bounded λ-almost everywhere (a.e.). For $f \in L^r(I)$, let $f^+ = \max\{f, 0\}$ and $f^- = (-f)^+$, hence $f = f^+ - f^-$.

Let $\mathcal{P}$ be the family of all Borel probability measures on $\overline{\mathbb{R}}$ with $\mu(\mathbb{R}) = 1$. For every $\mu \in \mathcal{P}$, $F_\mu : \overline{\mathbb{R}} \to \mathbb{I}$ with $F_\mu(x) = \mu([-\infty, x])$ is the associated *distribution function*, F_μ^{-1} the associated *(upper) quantile function*, i.e.,

$$F_\mu^{-1}(t) = \sup\{F_\mu \leq t\},\ \forall\ t \in]0, 1[, \tag{2.1}$$

and $\operatorname{supp} \mu$ the *support* of μ, that is, the smallest closed set of μ-measure 1. Both F_μ and F_μ^{-1} are non-decreasing and right-continuous. As a consequence, F_μ^{-1} generates a positive Borel measure μ^{-1} on $]0, 1[$ via

$$\mu^{-1}(]t, u]) = F_\mu^{-1}(u) - F_\mu^{-1}(t),\ 0 < t < u < 1.$$

Note that μ^{-1}, referred to as the *inverse measure* of μ, is finite if and only if $\operatorname{supp} \mu$ is bounded, since in fact $\mu^{-1}(]0, 1[) = \max \operatorname{supp} \mu - \min \operatorname{supp} \mu$; see, e.g., [11, App.A] for further

[1] Note that without causing any confusion, we also write $f_\pm(a)$ for $f(a\pm)$, notably in Chapters 4-5.

basic properties of inverse measures.

For every $r \geq 1$, the set of probability measures with finite r-th moment is denoted $\mathcal{P}_r$, i.e.,

$$\mathcal{P}_r = \left\{ \mu \in \mathcal{P} : \int_{\mathbb{R}} |x|^r \mathrm{d}\mu(x) < +\infty \right\}.$$

Thus $\mu \in \mathcal{P}_r$ if and only if $F_\mu^{-1} \in L^r(\mathbb{I})$. On $\mathcal{P}_r$, the *L^r-Kantorovich distance* d_r is

$$d_r(\mu, \nu) = \left(\int_{\mathbb{I}} \left| F_\mu^{-1}(t) - F_\nu^{-1}(t) \right|^r \mathrm{d}t \right)^{1/r} = \left\| F_\mu^{-1} - F_\nu^{-1} \right\|_r, \quad \forall\, \mu, \nu \in \mathcal{P}_r. \tag{2.2}$$

For $r = 1$, by Fubini's theorem,

$$d_1(\mu, \nu) = \int_{\mathbb{R}} |F_\mu(x) - F_\nu(x)| \, \mathrm{d}x, \quad \forall\, \mu, \nu \in \mathcal{P}_1.$$

When endowed with the metric d_r, the space $\mathcal{P}_r$ is separable and complete, and $d_r(\mu_n, \mu) \to 0$ implies that $\mu_n \to \mu$ weakly. Note that $\mathcal{P}_r \supset \mathcal{P}_s$ and $d_r \leq d_s$ whenever $r < s$. On $\mathcal{P}_s$, the metrics d_r and d_s are not equivalent, as the example of $\mu_n = (1 - n^{-s})\delta_0 + n^{-s}\delta_n$ shows, for which $d_s(\mu_n, \delta_0) \equiv 1$, and yet $\lim_{n\to\infty} d_r(\mu_n, \delta_0) = 0$ for all $r < s$, and hence $\mu_n \to \delta_0$ weakly. The reader is referred to [30,88] for details on the mathematical background of the Kantorovich distance, and to [40, 88] for a discussion of its appropriateness for mass transportation and quantization problems.

2.2 Monotone and balanced functions and their inverses

Quantization, as informally alluded to in the Introduction, may be understood as the approximation of a given probability measure by finite weighted sums of point masses. Every quantile function is non-decreasing; in particular, the quantile function associated with a finitely supported probability measure is a monotone step function. Therefore, it is natural—not least in view of (2.2)—to formulate the ensuing approximation problem more generally as a problem about the best approximation of monotone L^r-functions by step functions. Towards this goal, we first present some relevant properties of monotone functions. For ease of exposition, the focus is on non-*decreasing* functions, but all subsequent arguments hold analogously for non-increasing functions as well.

Given an interval $I \subset \overline{\mathbb{R}}$ and a non-decreasing function $f : I \to \overline{\mathbb{R}}$, define the *$t$-quantile set* Q_t^f of f as

$$Q_t^f = [\inf\{f \geq t\}, \sup\{f \leq t\}], \ \forall\, t \in \overline{\mathbb{R}};$$

here and throughout, $\inf \varnothing := \max I$ and $\sup \varnothing := \min I$. Also remember that I is closed and non-degenerate, by convention. As a generalization of (2.1), the *(upper) inverse* function $f^{-1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ associated with f is

$$f^{-1}(t) := \sup\{f \le t\} = \max Q_t^f, \ \forall\, t \in \overline{\mathbb{R}}.$$

Some basic properties of inverse functions are summarized below.

Proposition 2.1. *Let $f : I \to \overline{\mathbb{R}}$ be non-decreasing. Then f^{-1} is non-decreasing, right-continuous and, on $f(I)$, coincides with the ordinary inverse of f whenever f is one-to-one. Moreover, $(f_\pm)^{-1} = f^{-1}$ and $(f^{-1})^{-1}(x) = f(x+)$ for all $x \in \mathring{I}$; in particular, therefore, $(f^{-1})^{-1}$ equals f a.e. on $\mathring{I}$, and in fact everywhere if f is right-continuous.*

A few other elementary properties of quantile sets are as follows.

Proposition 2.2. [6, Lem.2.7]. *Let $f : I \to \overline{\mathbb{R}}$ be non-decreasing. Then, for every $t \in \overline{\mathbb{R}}$, the set Q_t^f is a non-empty, compact (possibly one-point) subinterval of I, and $f(x) = t$ whenever $\min Q_t^f < x < \max Q_t^f$. Moreover, the following hold:*
(i) *If $t < u$ then $x \le y$ for every $x \in Q_t^f$ and every $y \in Q_u^f$, and the set $Q_t^f \cap Q_u^f$ contains at most one point.*
(ii) *For every $x \in I$ and $t \in \overline{\mathbb{R}}$, $x \in Q_t^f$ if and only if $t \in Q_x^{f^{-1}}$.*

For any non-decreasing function $f : I \to \overline{\mathbb{R}}$, call $x \in I$ a *growth point* of f if $f(y) < f(x)$ for all $y \in I$ with $y < x$, or $f(y) > f(x)$ for all $y > x$; see also [11, p.97]. Define the *growth set* of f as

$$G^f = \{x \in I : x \text{ is a growth point of } f\}.$$

Thus for example, $G^{F_\mu} = \operatorname{supp}\mu$ for every $\mu \in \mathcal{P}$, and $\{0,1\} \subset G^{F_\mu^{-1}} \subset \mathbb{I}$. An elementary relation between growth and quantile sets follows directly from the definitions.

Proposition 2.3. *Let $f : I \to \overline{\mathbb{R}}$ be non-decreasing. Then G^f is a closed subset of I, non-empty unless f is constant, and $G^f \cup \{\min I, \max I\} = \bigcup_{t \in \overline{\mathbb{R}}} \left\{\min Q_t^f, \max Q_t^f\right\}$.*

Next, we recall a useful terminology from [17]: Given a bounded interval $I \subset \mathbb{R}$, call a measurable function $f : I \to \overline{\mathbb{R}}$ *balanced* if

$$\Big|\lambda(\{f > 0\}) - \lambda(\{f < 0\})\Big| \le \lambda(\{f = 0\}),$$

and denote by $B^f := \{t \in \mathbb{R} : f - t \text{ is balanced}\}$ the *set of all balanced values* of f. To establish a few basic properties of B^f (in Lemma 2.7 below), consider the auxiliary function $\ell_f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ given by

$$\ell_f(t) = \frac{1}{2}\left(\min I + \max I + \lambda(\{f < t\}) - \lambda(\{f > t\})\right).$$

The following properties of ℓ_f are straightforward to verify.

Proposition 2.4. *Let I be a bounded interval and $f : I \to \overline{\mathbb{R}}$ a measurable function. Assume that f is finite a.e.. Then the following hold:*
(i) *ℓ_f is non-decreasing;*
(ii) *For every $t \in \mathbb{R}$, $\ell_f(t\pm) = \ell_f(t) \pm \frac{1}{2}\lambda(\{f = t\})$, and hence ℓ_f is continuous at t if and only if $\lambda(\{f = t\}) = 0$. Moreover, $\lambda\left(\left\{\ell_f^{-1} < t\right\} \cap I\right) = \lambda(\{f < t\})$ and $\lambda\left(\left\{\ell_f^{-1} > t\right\} \cap I\right) = \lambda(\{f > t\})$;*
(iii) $\lim_{t\to-\infty} \ell_f(t) = \ell_f(-\infty) = \min I$ *and* $\lim_{t\to+\infty} \ell_f(t) = \ell_f(+\infty) = \max I$;
(iv) *If f is non-decreasing then*

$$\ell_f(t) = \frac{1}{2}\left(f^{-1}(t) + f^{-1}(t-)\right), \ \forall\, t \in \mathbb{R},$$

and also

$$\ell_f^{-1}(x) = \left(f^{-1}\right)^{-1}(x) = f(x+), \ \ell_f^{-1}(x-) = f(x-), \ \forall\, x \in \mathring{I};$$

(v) *If $f \in L^r(I)$ for some $1 \le r < +\infty$, then $\left\|\ell_f^{-1} - t\right\|_r = \|f - t\|_r$ for every $t \in \mathbb{R}$.*

Example 2.5. Let $I = \mathbb{I}$ and $f(x) = \begin{cases} 1 - 3x^2 & \text{if } 0 \le x \le 1/3, \\ 1/2 & \text{if } 1/3 < x < 2/3, \\ (3x-2)^2/3 & \text{if } 2/3 \le x \le 1. \end{cases}$ Here the functions $\ell_f,\ \ell_f^{-1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are given by

$$\ell_f(t) = \begin{cases} \sqrt{t^+/3} & \text{if } t < 1/3, \\ 1/3 & \text{if } 1/3 \le t < 1/2, \\ 1/2 & \text{if } t = 1/2, \\ 2/3 & \text{if } 1/2 < t < 2/3, \\ 1 - \sqrt{(1-t)^+/3} & \text{if } t \ge 2/3, \end{cases}$$

and

$$\ell_f^{-1}(x) = \begin{cases} -\infty & \text{if } x < 0, \\ 3x^2 & \text{if } 0 \le x < 1/3, \\ 1/2 & \text{if } 1/3 \le x < 2/3, \\ 1 - 3(1-x)^2 & \text{if } 2/3 \le x < 1, \\ +\infty & \text{if } x \ge 1, \end{cases}$$

respectively. From Figure 2.1, it is clear that indeed $\left\|\ell_f^{-1} - t\right\|_r = \|f - t\|_r$ for all t, as asserted by Proposition 2.4(v).

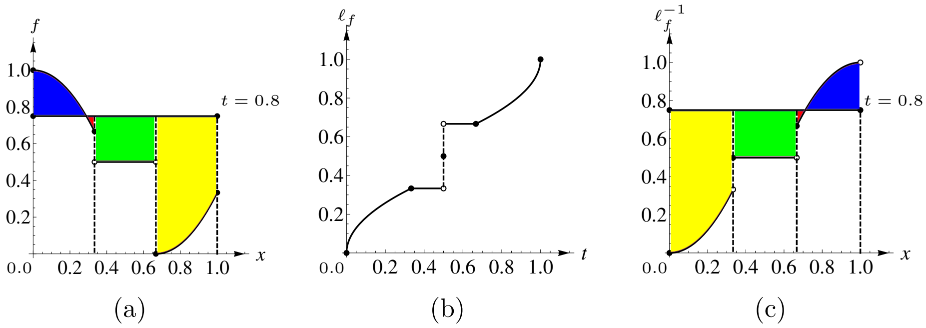

Figure 2.1: Graphing f, ℓ_f and ℓ_f^{-1}; see Example 2.5.

Remark 2.6. By Proposition 2.4(v), minimizing $t \mapsto \|f - t\|_r$ for $f \in L^r(I)$ is equivalent to minimizing $t \mapsto \left\|\ell_f^{-1} - t\right\|_r$ for the *monotone* function ℓ_f^{-1}. Note also that if $f \in L^r(I)$ is non-decreasing then f and ℓ_f^{-1} coincide a.e., by Proposition 2.4(iv).

Utilizing Propositions 2.2 and 2.4, we now establish a few basic properties of the sets B^f that will be used in the next section.

Lemma 2.7. *Let I be a bounded interval and $f : I \to \overline{\mathbb{R}}$ a measurable function. Assume that f is finite a.e.. Then $B^f = Q^{\ell_f}_{\frac{1}{2}(\min I + \max I)}$. Moreover, the following hold:*
(i) *For every $t \in \mathbb{R}$, $\lambda(\{f > t\}) > \lambda(\{f \leq t\})$ if $t < \min B^f$, and $\lambda(\{f < t\}) > \lambda(\{f \geq t\})$ if $t > \max B^f$;*
(ii) $\lambda\left(f^{-1}\left(\overset{\circ}{B^f}\right)\right) = 0$;
(iii) $\lambda\left(\left\{f \leq \min B^f\right\}\right) = \lambda\left(\left\{f \geq \max B^f\right\}\right)$.

Proof. For convenience, let $\xi = \frac{1}{2}(\min I + \max I)$, and note that, by definition,

$$B^f = \left\{t : |\ell_f(t) - \xi| \leq \frac{1}{2}\lambda(\{f = t\})\right\}.$$

Define $a = \inf\{\ell_f \geq \xi\}$, $b = \sup\{\ell_f \leq \xi\}$, and hence $[a, b] = Q^{\ell_f}_\xi$. It is easy to see that a and b are finite, with $a \leq b$, and

$$\ell_f(a-) \leq \xi \leq \ell_f(a+), \quad \ell_f(b-) \leq \xi \leq \ell_f(b+),$$

which implies that $a,\ b \in B^f$, by Proposition 2.4(ii). For every $t \in]a, b[$, $\ell_f(t) = \xi$, thus $t \in B^f$, and hence $[a, b] \subset B^f$. For every $t > b$, $\ell_f(t-) > \xi$, so again by Proposition 2.4(ii), $\ell_f(t) - \xi > \frac{1}{2}\lambda(\{f = t\})$, which implies that $t \notin B^f$ and $\lambda(\{f < t\}) > \lambda(\{f \geq t\})$. Similarly,

$t \notin B^f$ and $\lambda(\{f > t\}) > \lambda(\{f \le t\})$ for every $t < a$. This proves that $B^f = [a,b] = Q_\xi^{\ell_f}$, and also establishes (i).

To prove (ii) and (iii), assume that $\mathring{B}^f \neq \varnothing$, i.e., $a < b$. For every $t \in \mathring{B}^f$, $\ell_f(t) = \xi$, i.e., $\lambda(\{f > t\}) = \lambda(\{f < t\})$. Hence for all $a < t_1 < t_2 < b$,

$$\lambda(\{f < t_1\}) \le \lambda(\{f < t_2\}) = \lambda(\{f > t_2\}) \le \lambda(\{f > t_1\}) = \lambda(\{f < t_1\}).$$

Thus $\lambda(\{t_1 \le f < t_2\}) = \lambda(\{t_1 < f \le t_2\}) = 0$. Letting $t_1 \downarrow a$ and $t_2 \uparrow b$, properties (ii) and (iii) immediately follow from the continuity of λ. □

Remark 2.8. If, under the assumptions of Lemma 2.7, the function f is non-decreasing, then $B^f = Q^{f^{-1}}_{\frac{1}{2}(\min I + \max I)}$.

2.3 Approximating L^r-functions by step functions

This section characterizes the best approximations of a given function by step functions. Two main results (Lemma 2.9 and Theorem 2.11) will be used in Section 2.4 to identify best finitely supported approximations of a given probability measure $\mu \in \mathcal{P}$; they may also be of independent interest. Throughout this section, we assume that the closed interval $I \subset \mathbb{R}$ is *bounded*. (For unbounded I, most statements become either trivial or meaningless.)

First, we give a result on the best approximation of a monotone function by a (monotone) step function with a prescribed range and a single jump at a variable location.

Lemma 2.9. *Assume that $f : I \to \overline{\mathbb{R}}$ is non-decreasing, and $f \in L^r(I)$ for some $r \ge 1$. Let $a,\ b \in \mathbb{R}$ with $a < b$. Then the value of*

$$\left\| f - \left(a\mathbb{1}_{[\min I, \xi[} + b\mathbb{1}_{[\xi, \max I]} \right) \right\|_r, \ \forall\, \xi \in I,$$

is minimal if and only if $\xi \in Q^f_{\frac{1}{2}(a+b)}$.

Proof. Given $f \in L^r(I)$ and $a < b$, define $\psi(\xi) = \left\| f - \left(a\mathbb{1}_{[\min I, \xi[} + b\mathbb{1}_{[\xi, \max I]} \right) \right\|_r$ for all $\xi \in I$, and let $c = \frac{1}{2}(a+b)$. Clearly, the function ψ is non-negative and continuous, and so attains a minimal value. If $\xi > f^{-1}(c)$ then there exists $0 < \varepsilon < \xi - f^{-1}(c)$ such that $f(x) > c$ for all

$x \in [\xi - \varepsilon, \xi]$. Hence

$$\begin{aligned}\psi(\xi)^r - \psi\left(f^{-1}(c)\right)^r &= \int_{f^{-1}(c)}^{\xi} \left(|f(x)-a|^r - |f(x)-b|^r\right) \mathrm{d}x \\ &= \int_{f^{-1}(c)}^{\xi} \left((f(x)-a)^r - (f(x)-b)^r\right) \mathbb{1}_{\{f\geq b\}} \mathrm{d}x \\ &\quad + \int_{f^{-1}(c)}^{\xi} \left((f(x)-a)^r - (b-f(x))^r\right) \mathbb{1}_{\{f< b\}} \mathrm{d}x \\ &\geq \int_{\xi-\varepsilon}^{\xi} (b-a)^r \mathbb{1}_{\{f\geq b\}} \mathrm{d}x + \int_{\xi-\varepsilon}^{\xi} \left(2f(x)-a-b\right)^r \mathbb{1}_{\{f< b\}} \mathrm{d}x \\ &\geq \varepsilon \min\left\{b-a, 2(f(\xi-\varepsilon)-c)\right\}^r > 0,\end{aligned}$$

i.e., $\psi(\xi) > \psi\left(f^{-1}(c)\right)$. Similarly, $\psi(\xi) > \psi\left(f^{-1}(c)\right)$ whenever $\xi < \inf\{f \geq c\}$. Therefore ψ attains its minimal value on the interval $[\inf\{f \geq c\}, f^{-1}(c)] = Q_c^f$, and the proof will be complete once it is shown that in fact ψ is constant on Q_c^f. If Q_c^f is a singleton, then, trivially, this is the case. On the other hand, if $\xi,\ \eta \in \overset{\circ}{Q_c^f}$ with $\xi < \eta$, then $f([\xi, \eta]) = \{c\}$, by Proposition 2.2, and

$$\psi(\eta)^r - \psi(\xi)^r = \int_{\xi}^{\eta} \left(|f(x)-b|^r - |f(x)-a|^r\right) \mathrm{d}x = \int_{\xi}^{\eta} \left(|c-b|^r - |c-a|^r\right) \mathrm{d}x = 0.$$

Thus ψ is constant on Q_c^f, as claimed. □

Remark 2.10. (See Section 3.2 for details.) The monotonicity of f is essential in Lemma 2.9. To see this, take for instance $I = [0, 5]$ and the (non-monotone) function $f = 16 \cdot \mathbb{1}_{[0,1[} + 8 \cdot \mathbb{1}_{[1,2[} + 18 \cdot \mathbb{1}_{[2,3[} + 9 \cdot \mathbb{1}_{[3,5]}$. For $a = 0,\ b = 24$, it is straightforward to verify that $\left\|f - 24 \cdot \mathbb{1}_{[\xi,5]}\right\|_r$ is minimal precisely for $\xi \in \{0, 2, 5\}$ if $r = 1$ or $r = 2$, for $\xi = 5$ if $1 < r < 2$, and for $\xi \in \{0, 2\}$ if $r > 2$. In general, therefore, the set of minimizers is not an interval and may depend on r.

The remainder of this section deals with a problem dual to the one addressed by Lemma 2.9, namely the best approximation of an L^r-function f by a step function with prescribed locations but variable jumps. By considering intervals of constancy individually, clearly it is enough to consider the approximation of f by a *constant* function. Remember that the closed, non-degenerate interval $I \subset \mathbb{R}$ is assumed to be bounded throughout.

Theorem 2.11. *Assume that $f \in L^{r_0}(I)$ for some $r_0 \geq 1$. Then for every $1 \leq r \leq r_0$, there exists $\tau_r^f \in \mathbb{R}$ such that*

$$\left\|f - \tau_r^f\right\|_r \leq \|f - t\|_r\,, \quad \forall\ t \in \mathbb{R}.$$

Moreover, the following hold:

(i) $\tau_r^f \in [\operatorname{essinf}_I f, \operatorname{esssup}_I f]$;

(ii) $\|f-t\|_1 = \left\|f-\tau_1^f\right\|_1$ *if and only if* $t \in B^f$;

(iii) *For* $1 < r \leq r_0$, *the number* τ_r^f *is unique, and* $r \mapsto \tau_r^f$ *is continuous.*

Proof. Given $f \in L^{r_0}(I)$, recall that $f \in L^r(I)$ for every $1 \leq r \leq r_0$, since I is bounded. Hence the auxiliary function ϕ_r given by

$$\phi_r(t) = \lambda(I)^{-1/r} \|f-t\|_r \,, \ \forall\, t \in \mathbb{R}, \tag{2.3}$$

is well defined and real-valued. Note that $\lim_{|t|\to+\infty} \phi_r(t) = +\infty$. Since ϕ_r is convex, there exists $\tau_r^f \in \mathbb{R}$ such that $\phi_r\left(\tau_r^f\right) \leq \phi_r(t)$ for all $t \in \mathbb{R}$.

It remains to prove assertions (i)-(iii). To establish (i), let $b = \operatorname{esssup}_I f$ for convenience, and observe that, for all $t > b$,

$$\lambda(I)\left(\phi_r(t)^r - \phi_r(b)^r\right) = \int_I \left((t-f(x))^r - (b-f(x))^r\right) \mathrm{d}x \geq \int_I (t-b)^r \mathrm{d}x = \lambda(I)(t-b)^r > 0,$$

hence $\phi_r(t) > \phi_r(b)$. Similarly, $\phi_r(t) > \phi_r\left(\operatorname{essinf}_I f\right)$ whenever $t < \operatorname{essinf}_I f$. This shows that $\tau_r^f \in \left[\operatorname{essinf}_I f, \operatorname{esssup}_I f\right]$.

To prove (ii), given $t > \max B^f$, pick any u with $\max B^f < u < t$. Then,

$$\begin{aligned}
\lambda(I)\left(\phi_1(t) - \phi_1(u)\right) &= \int_I \left(|f(x)-t| - |f(x)-u|\right) \mathrm{d}x \\
&\geq \int_{\{f<u\}} \left(t - f(x) - (u - f(x))\right) \mathrm{d}x + \int_{\{f\geq u\}} \left(f(x) - t - (f(x)-u)\right) \mathrm{d}x \\
&= (t-u)\left(\lambda\left(\{f<u\}\right) - \lambda\left(\{f \geq u\}\right)\right) > 0,
\end{aligned}$$

by Lemma 2.7(i), and so $\tau_1^f \leq \max B^f$. Similarly, $\tau_1^f \geq \min B^f$. On the other hand, if $t,\ u \in B^f$, then

$$\begin{aligned}
\lambda(I)\left(\phi_1(u) - \phi_1(t)\right) &= \int_{\left\{f\leq \min B^f\right\}} \left(u - f(x) - (t - f(x))\right) \mathrm{d}x + \int_{f^{-1}\left(\overset{\circ}{B^f}\right)} \left(|f(x)-u|\right. \\
&\quad \left. - |f(x)-t|\right) \mathrm{d}x + \int_{\left\{f \geq \max B^f\right\}} \left(f(x) - u - (f(x)-t)\right) \mathrm{d}x \\
&= (u-t)\left(\lambda\left(\left\{f \leq \min B^f\right\}\right) - \lambda\left(\left\{f \geq \max B^f\right\}\right)\right) = 0,
\end{aligned}$$

by Lemma 2.7(ii) and (iii). Thus $\phi_1(t)$ is minimal if and only if $t \in B^f$.

Regarding (iii), we claim that the number τ_r^f is unique for $1 < r \leq r_0$. Trivially, this is true if f is essentially constant. In any other case, note that ϕ_r^r is differentiable w.r.t. t, and

$$\begin{aligned}\frac{\lambda(I)}{r}\frac{\mathrm{d}\phi_r^r(t)}{\mathrm{d}t} &= \int_I |f(x)-t|^{r-1}\,\mathrm{sgn}\,(t-f(x))\,\mathrm{d}x \\ &= \int_{\{f<t\}} (t-f(x))^{r-1}\,\mathrm{d}x - \int_{\{f>t\}} (f(x)-t)^{r-1}\,\mathrm{d}x \\ &= \int_{\left\{f\le \min B^f\right\}} (t-f(x))^{r-1}\,\mathrm{d}x - \int_{\left\{f\ge \max B^f\right\}} (f(x)-t)^{r-1}\,\mathrm{d}x\end{aligned} \tag{2.4}$$

is increasing in t. Thus ϕ_r^r is strictly convex, and τ_r^f is unique.

To show that $r\mapsto \tau_r^f$ is continuous on $]1,r_0]$, pick any $1<r\le r_0$ and any sequence (r_n) in $]1,r_0]$ with $\lim_{n\to\infty} r_n = r$. Given $\varepsilon>0$, by the strict convexity of ϕ_r, there exists $\delta>0$ such that $\phi_r\left(\tau_r^f\pm\varepsilon\right) > \phi_r\left(\tau_r^f\right)+3\delta$. On the other hand, $\lim_{n\to\infty}\phi_{r_n}(t)=\phi_r(t)$ for every $t\in\mathbb{R}$, by the Dominated Convergence Theorem. Hence for all sufficiently large n,

$$\phi_{r_n}\left(\tau_r^f\pm\varepsilon\right) > \phi_r\left(\tau_r^f\right)+2\delta \quad\text{and}\quad \phi_{r_n}\left(\tau_r^f\right) < \phi_r\left(\tau_r^f\right)+\delta,$$

from which it is clear that $\left|\tau_{r_n}^f-\tau_r^f\right|<\varepsilon$. Since $\varepsilon>0$ was arbitrary, $r\mapsto\tau_r^f$ is continuous. □

For monotone functions, Theorem 2.11 takes a particularly simple form.

Corollary 2.12. *Assume that $f: I\to\overline{\mathbb{R}}$ is non-decreasing, and $f\in L^{r_0}(I)$ for some $r_0\ge 1$. Then for every $1\le r\le r_0$, there exists $\tau_r^f\in\mathbb{R}$ such that*

$$\left\|f-\tau_r^f\right\|_r \le \|f-t\|_r\,, \quad \forall\, t\in\mathbb{R}.$$

Moreover, the following hold:
(i) $\tau_r^f\in[f(\min I+), f(\max I-)]$.
(ii) $\|f-t\|_1=\left\|f-\tau_1^f\right\|_1$ *if and only if* $t\in Q^{f^{-1}}_{\frac12(\min I+\max I)}$.
(iii) *For $1<r\le r_0$, the number τ_r^f is unique and $r\mapsto\tau_r^f$ is continuous.*

Remark 2.13. (i) If $f\in L^2(I)$ then simply $\tau_2^f=\frac{1}{\lambda(I)}\int_I f(x)\mathrm{d}x$.
(ii) For $r=1$, Corollary 2.12 immediately yields Lemma 2.9. Indeed, under the assumptions of the latter, $f^{-1}\big|_{[a,b]}\in L^1([a,b])$, and $\left\|f-\left(a\mathbb{1}_{[\min I,\xi[}+b\mathbb{1}_{[\xi,\max I]}\right)\right\|_1$ is minimal if and only if $\left\|f^{-1}\big|_{[a,b]}-\xi\right\|_1$ is minimal. By Corollary 2.12, this is the case precisely if $\xi\in Q^{g^{-1}}_{\frac12(a+b)}$ with $g=f^{-1}\big|_{[a,b]}$, which by Proposition 2.2 is equivalent to $\xi\in Q^f_{\frac12(a+b)}$.

Given $r>1$, the number τ_r^f depends on f in a monotone and continuous way, as the following two simple observations show.

Proposition 2.14. *Assume that $f,\ g\in L^r(I)$ for some $r>1$, and $f\le g$. Then $\tau_r^f\le\tau_r^g$, and $\tau_r^f=\tau_r^g$ if and only if $f=g$ a.e..*

Lemma 2.15. *Assume that $f,\ f_n \in L^{r_0}(I)$ for some $r_0 > 1$ and all $n \in \mathbb{N}$. If $\lim_{n\to\infty} f_n = f$ in $L^{r_0}(I)$, then $\lim_{n\to\infty} \tau_r^{f_n} = \tau_r^f$ locally uniformly on $]1, r_0]$.*

Proof. Since $f_n \to f$ in $L^{r_0}(I)$ and I is bounded, $\sup_{n\in\mathbb{N}} \|f_n\|_{r_0} < +\infty$ and, for all $r \in]1, r_0]$ and $n \in \mathbb{N}$,

$$\begin{aligned}\left|\tau_r^{f_n}\right| &= \lambda(I)^{-1/r} \left\|\tau_r^{f_n}\right\|_r \le \lambda(I)^{-1/r} \left(\left\|f_n - \tau_r^{f_n}\right\|_r + \|f_n\|_r\right) \\ &\le 2\lambda(I)^{-1/r} \|f_n\|_r \le 2\lambda(I)^{-1/r_0} \|f_n\|_{r_0},\end{aligned}$$

by Hölder's inequality. This shows that $\left(\tau_r^{f_n}\right)$ is uniformly bounded on $]1, r_0]$.

Fix any $1 < s < r_0$. To prove that $\lim_{n\uparrow\infty} \tau_r^{f_n} = \tau_r^f$ uniformly on $[s, r_0]$, suppose by way of contradiction that there exists $\varepsilon_0 > 0$, a sequence (r_j) in $[s, r_0]$ and an increasing sequence (n_j) in $\mathbb{N}$ such that

$$\left|\tau_{r_j}^f - \tau_{r_j}^{f_{n_j}}\right| \ge \varepsilon_0,\ \forall\ j \in \mathbb{N}.$$

Assume w.o.l.g. that $r_j \to r^*$ and, by the uniform boundedness of $\left(\tau_r^{f_n}\right)$, $\tau_{r_j}^{f_{n_j}} \to \tau^* \in \mathbb{R}$. Since $r \mapsto \tau_r^f$ is continuous at r^*, it follows that

$$\left|\tau_{r^*}^f - \tau^*\right| \ge \varepsilon_0. \tag{2.5}$$

On the other hand,

$$\begin{aligned}\left\|f - \tau_{r_j}^{f_{n_j}}\right\|_{r_j} &\le \left\|f - f_{n_j}\right\|_{r_j} + \left\|f_{n_j} - \tau_{r_j}^{f_{n_j}}\right\|_{r_j} \le \left\|f - f_{n_j}\right\|_{r_j} + \left\|f_{n_j} - \tau_{r_j}^f\right\|_{r_j} \\ &\le 2\left\|f - f_{n_j}\right\|_{r_j} + \left\|f - \tau_{r_j}^f\right\|_{r_j},\end{aligned}$$

and letting $j \to \infty$ yields, $\|f - \tau^*\|_{r^*} \le \left\|f - \tau_{r^*}^f\right\|_{r^*}$ since $(r, t) \mapsto \|f - t\|_r$ is continuous. By Theorem 2.11(iii), $\tau^* = \tau_{r^*}^f$, which clearly contradicts (2.5). □

Remark 2.16. In Lemma 2.9, the convergence $\tau_r^{f_n} \to \tau_r^f$ in general is not uniform on $]1, r_0]$. To see this, take for example $I = [0, 2]$ and $f_n = 2 \cdot \mathbb{1}_{[1+2^{-n},2]}$ for all $n \in \mathbb{N}$. With $f = 2 \cdot \mathbb{1}_{[1,2]}$, clearly, $f,\ f_n \in L^\infty(I)$ and $\lim_{n\to\infty} f_n = f$ in $L^r(I)$ for every $r \ge 1$. Still, $\lim_{r\downarrow 1} \tau_r^{f_n} = 0$ for every n, whereas $\tau_r^f = 1$ for all $r > 1$.

Note that if $f : I \to \mathbb{R}$ is affine, i.e., $f(x) = ax + b$ for all $x \in I$ and the appropriate $a,\ b \in \mathbb{R}$, then $\tau_r^f = f\left(\frac{1}{2}(\min I + \max I)\right)$ for all $r > 1$. In this context, Lemma 2.9 can be given a slightly stronger, quantitative form.

Proposition 2.17. *Assume that $f : I \to \mathbb{R}$ is measurable, and let $\xi = \frac{1}{2}(\min I + \max I)$. If,* *for some* $a,\ b,\ c \in \mathbb{R}$,

$$|f(x) - (ax+b)| \le c\,|x-\xi|\,,\ \forall\ x \in I,$$

then $f \in L^\infty(I)$, *and* $\left|\tau_r^f - f(\xi)\right| \le \frac{1}{2}c\lambda(I)$ *for every* $r > 1$.

The remainder of this section studies how, given f, the number τ_r^f depends on r. First, this dependence is illustrated by an example, where for simplicity $f \in L^\infty(I)$ is a non-decreasing step function.

Example 2.18. Let $I = [0,8]$.
(i) Consider the function $f = (-4)\cdot \mathbb{1}_{[0,1[} + 4\cdot \mathbb{1}_{[5,8]}$, for which $B^f = \{0\}$. By (2.4), for every $r > 1$,

$$\left(\tau_r^f + 4\right)^{r-1} + 4\left(\tau_r^f\right)^{r-1} = 3\left(4 - \tau_r^f\right)^{r-1}, \tag{2.6}$$

and using (2.6), it is readily deduced that $\tau_{1+}^f := \lim_{r\downarrow 1}\tau_r^f = 0$, but also $\tau_\infty^f := \lim_{r\to+\infty}\tau_r^f = 0$. On the other hand, $\tau_2^f = 1$, and hence $r \mapsto \tau_r^f$ is not monotone; see Figure 2.2. Note that in order for $r \mapsto \tau_r^f$ to be non-monotone, a step function f has to attain at least three different values.
(ii) Consider the function $f = (-a)\mathbb{1}_{[0,1[} + (-1)\mathbb{1}_{[1,4[} + \mathbb{1}_{[4,5[} + b\mathbb{1}_{[5,8]}$ with real parameters $a,\ b > 1$. In this case, $B^f = [-1,1]$, and (2.4) yields, for every $r > 1$,

$$\left(\tau_r^f + a\right)^{r-1} + 3\left(\tau_r^f + 1\right)^{r-1} = \left(1 - \tau_r^f\right)^{r-1} + 3\left(b - \tau_r^f\right)^{r-1},$$

from which it is straightforward to deduce that τ_{1+}^f exists and equals the unique real root of

$$g_{a,b}(\tau) := (3b+a+4)\,\tau^3 - 3\left(b^2+b-a-1\right)\tau^2 + \left(b^3+3b^2+3a+1\right)\tau - b^3 + a = 0. \tag{2.7}$$

Given any $\tau \in]-1,1[$, note that $\lim_{a\to+\infty} g_{a,b}(\tau) = +\infty$ for every $b > 1$, and $\lim_{b\to+\infty} g_{a,b}(\tau) = -\infty$ for every $a > 1$. By the Intermediate Value Theorem, there exists $\overline{a} = \overline{a}(\tau),\ \overline{b} = \overline{b}(\tau)$ such that $g_{\overline{a},\overline{b}}(\tau) = 0$. Since the real root of (2.7) is unique, $\tau_{1+}^f = \tau$. This shows that with $a,\ b > 1$ chosen appropriately, τ_{1+}^f can have any value in $]-1,1[$. Note that, similarly to (i), $\tau_\infty^f = \frac{1}{2}(b-a)$.

As seen in Example 2.18, the number τ_r^f may depend on r in a non-monotone way. In both cases considered, however, the limits $\tau_{1+}^f = \lim_{r\downarrow 1}\tau_r^f$ and $\tau_\infty^f = \lim_{r\to+\infty}\tau_r^f$ exist. Also, by modifying Example 2.18(ii) appropriately, it is clear that, given any compact interval $J \subset \mathbb{R}$ and any $\tau \in J$, one can find $f \in L^\infty(I)$ with $B^f = J$ and $\tau_{1+}^f = \tau$. In fact, one can choose f to be a non-decreasing step function.

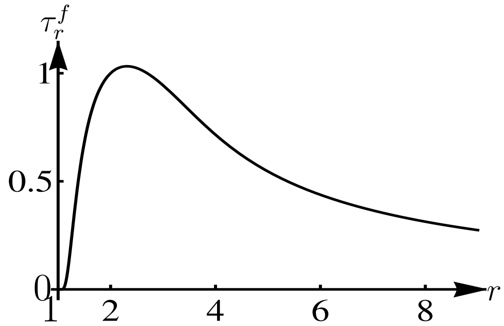

Figure 2.2: Profile of τ_r^f for $f = -4 \cdot \mathbb{1}_{[0,1[} + 4 \cdot \mathbb{1}_{[5,8]}$; see Example 2.18(i).

This section concludes with a demonstration that, just as in Example 2.18, τ_{1+}^f exists always (Theorem 2.19), whereas, unlike in Example 2.18, τ_∞^f may not exist (Example 2.23).

Theorem 2.19. *Assume that $f \in L^{r_0}(I)$ for some $r_0 > 1$. Then τ_{1+}^f exists, and $\tau_{1+}^f \in B^f$.*

Proof. We first show that

$$\left[\liminf_{r\downarrow 1} \tau_r^f, \limsup_{r\downarrow 1} \tau_r^f\right] \subset B^f, \tag{2.8}$$

and then that $\lim_{r\downarrow 1} \tau_r^f$ exists. For any $1 < r \le r_0$, let ϕ_r be defined as in (2.3). Recall that ϕ_r is convex, and $r \mapsto \phi_r(t)$ is continuous and non-decreasing for any $t \in \mathbb{R}$. Assume that $r_n \downarrow 1$ with $\tau_{r_n}^f \to \tau$. Then $\phi_1\left(\tau_{r_n}^f\right) \le \phi_{r_n}\left(\tau_{r_n}^f\right) \le \phi_{r_n}(t)$, and hence $\phi_1(\tau) = \lim_{n\to\infty} \phi_1\left(\tau_{r_n}^f\right) \le \lim_{n\to\infty} \phi_{r_n}(t) = \phi_1(t)$. Since $t \in \mathbb{R}$ was arbitrary, Theorem 2.11(ii) yields $\tau \in B^f$, which in turn establishes (2.8).

It remains to show that $\lim_{r\downarrow 1} \tau_r^f$ exists, which is non-trivial only if B^f is non-degenerate. In this case, define $\Psi : \mathring{B}^f \to \mathbb{R}$ as

$$\Psi(t) = \int_{\left\{f \le \min B^f\right\}} \log\left(t - f(x)\right) \mathrm{d}x - \int_{\left\{f \ge \max B^f\right\}} \log\left(f(x) - t\right) \mathrm{d}x, \ \forall\, t \in \mathring{B}^f.$$

Note that Ψ is well-defined and continuous. Moreover, if $t,\ u \in B^f$ with $t < u$ then, as $B^f \neq I$,

$$\Psi(t) - \Psi(u) = \int_{\left\{f \le \min B^f\right\}} \log\frac{t - f(x)}{u - f(x)} \mathrm{d}x + \int_{\left\{f \ge \max B^f\right\}} \log\frac{f(x) - u}{f(x) - t} \mathrm{d}x < 0,$$

showing that Ψ is increasing. By (2.4), $t \mapsto \frac{\lambda(I)}{r}\frac{\mathrm{d}\phi_r^r(t)}{\mathrm{d}t}$ is a real-valued increasing function. To compare the latter to Ψ, notice the following elementary inequality:

$$\left|y^{r-1} - 1 - (r-1)\log y\right| \le (r-1)^2 e^{|\log y|}, \ \forall\, y > 0, 1 \le r \le 2. \tag{2.9}$$

With Lemma 2.7 and (2.9), for any fixed $0 < \varepsilon < \min\left\{1, \frac{1}{2}\lambda\left(B^f\right)\right\}$, there exists $C_\varepsilon > 0$ such that

$$\left|\frac{\lambda(I)}{r}\frac{\mathrm{d}\phi_r^r(t)}{\mathrm{d}t} - (r-1)\Psi(t)\right| \le C_\varepsilon (r-1)^2,\ \forall\, 1 < r \le 2,\ t \in \left[\min B^f + \varepsilon, \max B^f - \varepsilon\right]. \quad (2.10)$$

Since Ψ is increasing, three cases have to be distinguished:
(i) $\Psi(\tau) = 0$ for a unique $\tau \in \mathring{B}^f$. Pick $\varepsilon > 0$ so that $\min B^f + \varepsilon < \tau < \max B^f - \varepsilon$. Then for every $\delta > 0$, (2.10) implies $\frac{\mathrm{d}\phi_r}{\mathrm{d}t}(\tau + \delta) > 0$ and $\frac{\mathrm{d}\phi_r}{\mathrm{d}t}(\tau - \delta) < 0$ for all $r > 1$ sufficiently small. It follows that $\tau_r^f \in [\tau - \delta, \tau + \delta]$ for all $r > 1$ sufficiently small, and since $\delta > 0$ was arbitrary, $\lim_{r\downarrow 1} \tau_r^f = \tau$.
(ii) $\Psi(\tau) > 0$ for all $\tau \in \mathring{B}^f$. Similarly to case (i), for every $\delta > 0$, (2.10) yields $\frac{\mathrm{d}\phi_r}{\mathrm{d}t}\left(\min B^f + \delta\right) > 0$ for all $r > 1$ sufficiently small. This implies that $\tau_r^f < \min B^f + \delta$ for all $r > 1$ sufficiently small and hence $\limsup_{r\downarrow 1} \tau_r^f \le \min B^f$. By (2.8), $\lim_{r\downarrow 1} \tau_r^f = \min B^f$.
(iii) $\Psi(\tau) < 0$ for all $\tau \in \mathring{B}^f$. This case is completely analogous to (ii), with $\lim_{r\downarrow 1} \tau_r^f = \max B^f$. □

Corollary 2.20. *Assume that $f : I \to \overline{\mathbb{R}}$ is non-decreasing, and $f \in L^{r_0}(I)$ for some $r_0 > 1$. Then τ_{1+}^f exists, and $\tau_{1+}^f \in Q^{f^{-1}}_{\frac{1}{2}(\min I + \max I)}$.*

Recall that in Example 2.18 the limit τ_∞^f also exists. This is a consequence of the fact that f is bounded, together with the following simple observation.

Theorem 2.21. *Assume that $f \in \bigcap_{r\ge 1} L^r(I)$. If $f^- \in L^\infty(I)$ or $f^+ \in L^\infty(I)$, then $\lim_{r\to+\infty} \tau_r^f = \frac{1}{2}\left(\operatorname{essinf}_I f + \operatorname{esssup}_I f\right)$.*

Proof. Let f be non-constant (otherwise, $r \mapsto \tau_r^f$ is constant, too), and assume that $f^- \in L^\infty(I)$, that is, $\operatorname{essinf}_I f > -\infty$. (The case $f^+ \in L^\infty(I)$ is completely analogous.) Let $u = \frac{1}{2}(\operatorname{essinf}_I f + \operatorname{esssup}_I f)$ for convenience, fix any $\operatorname{essinf}_I f < t < u$, and let $\delta = t - \operatorname{essinf}_I f$. For $\tau < t$, note that $\tau - \operatorname{essinf}_I f < \delta$ and $\lambda(\{f \ge \tau + \delta\}) > 0$, and hence, with (2.4),

$$\begin{aligned}
\frac{\lambda(I)}{r\delta^{r-1}}\frac{\mathrm{d}\phi_r^r(\tau)}{\mathrm{d}\tau} &= \int_{\{f<\tau\}} \left(\frac{\tau - f(x)}{\delta}\right)^{r-1} \mathrm{d}x - \int_{\{f>\tau\}} \left(\frac{f(x) - \tau}{\delta}\right)^{r-1} \mathrm{d}x \\
&\le \lambda(I)\left(\frac{\tau - \operatorname{essinf}_I f}{\delta}\right)^{r-1} - \int_{\{f\ge\tau+\delta\}} \left(\frac{f(x)-\tau}{\delta}\right)^{r-1} \mathrm{d}x \\
&\quad - \int_{\{\tau\le f<\tau+\delta\}} \left(\frac{f(x)-\tau}{\delta}\right)^{r-1} \mathrm{d}x \\
&\le \lambda(I)\left(\frac{\tau - \operatorname{essinf}_I f}{\delta}\right)^{r-1} - \lambda\left(\{f \ge \tau+\delta\}\right) < 0,
\end{aligned}$$

for all sufficiently large r. Thus $\liminf_{r\to+\infty} \tau_r^f \geq \tau$, and since t and $\tau < t$ were arbitrary, $\liminf_{r\to+\infty} \tau_r^f \geq u$. A similar argument shows $\limsup_{r\to+\infty} \tau_r^f \leq u$. □

Corollary 2.22. *If* $f \in \bigcap_{r\geq 1} L^r(I)$ *is non-decreasing and* $f(\min I+) > -\infty$ *or* $f(\max I-) < +\infty$, *then* $\lim_{r\to+\infty} \tau_r^f = \frac{1}{2}\left(f(\min I+) + f(\max I-)\right)$.

The final example shows that, unlike in Example 2.18, $\lim_{r\to+\infty} \tau_r^f$ may not exist if f is unbounded.

Example 2.23. Consider the function $f : I \to \mathbb{R}$ given by

$$f = \sum_{n=0}^{\infty} 2n(-1)^{n-1}\mathbb{1}_{I_n},$$

where $I := \overline{\bigcup_{n=0}^{\infty} I_n}$, and $I_0,\ I_1,\ \cdots$ are pairwise disjoint, contiguous half-open intervals, with I_1 to the right of I_0, and generally I_{2n+1} immediately to the right of I_{2n-1}, as well as I_{2n+2} immediately to the left of I_{2n}. (Clearly, f is non-decreasing on $\overset{\circ}{I}$.) The lengths $\lambda_n := \lambda(I_n) > 0$ will be determined by induction shortly, subject to the requirement that $\lambda_{n+1} \leq \frac{1}{2}\lambda_n$ for all $n \geq 0$. Thus I is a non-degenerate, closed interval of length $\sum_{n\geq 0}\lambda_n \leq 2\lambda_0$, and $f \in \bigcap_{r\geq 1} L^r(I)$ but clearly $f \notin L^\infty(I)$. For each $N \in \mathbb{N}$, let $f_N = \sum_{n=0}^{N} 2n(-1)^{n-1}\mathbb{1}_{I_n}$ and note that $\lim_{r\to+\infty} \tau_r^{f_N} = (-1)^{N-1}$, by Theorem 2.21. Moreover,

$$\|f_{N+1} - f_N\|_r = 2(N+1)\lambda_{N+1}^{1/r},\ \forall\, r > 1,\ N \in \mathbb{N}. \tag{2.11}$$

Let $\lambda_0 = 1,\ r_0 = 1$, and assume that $\lambda_1,\ \lambda_2, \cdots, \lambda_N$ with $\lambda_n \leq \frac{1}{2}\lambda_{n-1}$ as well as $r_0 < r_1 < \cdots < r_N$ with $r_n \geq \max\{r_{n-1}, n+1\}$ for $n = 1, \cdots, N$ have been chosen in such a way that, for every $1 \leq n \leq N$,

$$\left|\tau_{r_j}^{f_n} - (-1)^{j-1}\right| < 2^{1-j} - 2^{-n},\ \forall\, 1 \leq j \leq n. \tag{2.12}$$

For $N = 1$, clearly such a choice is possible. By Lemma 2.9 and (2.11), choosing $\lambda_{N+1} \leq \frac{1}{2}\lambda_N$ sufficiently small guarantees that

$$\left|\tau_r^{f_{N+1}} - \tau_r^{f_N}\right| < 2^{-(N+1)},\ \forall\, r \in [r_1, r_N],$$

and consequently

$$\begin{aligned}\left|\tau_{r_j}^{f_{N+1}} - (-1)^{j-1}\right| &\leq \left|\tau_{r_j}^{f_{N+1}} - \tau_{r_j}^{f_N}\right| + \left|\tau_{r_j}^{f_N} - (-1)^{j-1}\right| \\ &< 2^{-(N+1)} + 2^{1-j} - 2^{-N} = 2^{1-j} - 2^{-(N+1)},\ \forall\, j = 1, \cdots, N.\end{aligned}$$

Also, choose $r_{N+1} \geq \max\{r_N, N+2\}$ such that

$$\left|\tau_{r_{N+1}}^{f_{N+1}} - (-1)^N\right| < 2^{-(N+1)}.$$

Thus (2.12) holds for $n = N+1$, and in fact for all $n \in \mathbb{N}$, by induction. Furthermore, note that, given any $r > 1$,

$$\|f_N - f\|_r = \left(\sum_{n>N} (2n)^r \lambda_n\right)^{1/r} \leq 2\lambda_0^{1/r} \left(\sum_{n>N} n^r 2^{-n}\right)^{1/r} \to 0 \quad \text{as } N \to \infty,$$

and so in particular $\lim_{N\to\infty} \|f_N - f\|_{r_j} = 0$ for every $j \in \mathbb{N}$. By Lemma 2.9, $\left|\tau_{r_j}^{f_N} - \tau_{r_j}^{f}\right| < 2^{-j}$ for all sufficiently large N, which, together with (2.12), yields

$$\left|\tau_{r_j}^{f} - (-1)^{j-1}\right| < 3 \cdot 2^{-j}.$$

Since $j \in \mathbb{N}$ was arbitrary and $r_j \uparrow +\infty$, this shows that $\liminf_{r\to+\infty} \tau_r^f \leq -1$ and $\limsup_{r\to+\infty} \tau_r^f \geq 1$. On the other hand, using (2.4), it is readily confirmed that $t\frac{\mathrm{d}}{\mathrm{d}t}\|f - t\|_r^r > 0$ for $t = \pm 1$ and all $r > 1$, and consequently $\left|\tau_r^f\right| < 1$. Thus $\liminf_{r\to+\infty} \tau_r^f = -1$ and $\limsup_{r\to+\infty} \tau_r^f = 1$.

By modifying Example 2.23 appropriately, it is straightforward to establish

Proposition 2.24. *Given any (bounded) interval $I \subset \mathbb{R}$ and $-\infty \leq a \leq b \leq +\infty$, there exists a non-decreasing function $f \in \bigcap_{r\geq 1} L^r(I)$ such that $\liminf_{r\to+\infty} \tau_r^f = a$ and $\limsup_{r\to+\infty} \tau_r^f = b$.*

2.4 Best constrained approximations

In this section, we apply results established in previous sections, notably Lemma 2.9 and Theorem 2.11, to investigate best constrained approximations of $\mu \in \mathcal{P}_r$, i.e., approximations of μ by finitely supported probabilities for which either locations (Subsection 2.4.1) or weights (Subsection 2.4.1) are prescribed. We establish existence of best constrained approximations and study their behaviour as the number of atoms goes to infinity. Finally, in Subsection 2.4.3 we relate these results to the classical theory of best (unconstrained) approximations.

First, we fix a few notations specific to this section. Given $n \in \mathbb{N}$, let $\Xi_n = \{x \in \mathbb{R}^n : x_{,1} \leq \cdots \leq x_{,n}\}$ and $\Pi_n = \{p \in \mathbb{R}^n : p_{,i} \geq 0,\ \sum_{i=1}^n p_{,i} = 1\}$. For any $x \in \Xi_n$, the conventions $x_{,0} = -\infty$ and $x_{,n+1} = +\infty$ are adopted, and for any $p \in \Pi_n$, let $P_{,i} = \sum_{j=1}^i p_{,j}$, $i = 0, 1, \cdots, n$; note that $P_{,0} = 0$ and $P_{,n} = 1$. Given $x \in \Xi_n$ and $p \in \Pi_n$, let $\delta_x^p = \sum_{i=1}^n p_{,i}\delta_{x_{,i}}$.

Throughout, usage of the symbol δ_x^p tacitly assumes that $x \in \Xi_n$, $p \in \Pi_n$, with $n \in \mathbb{N}$ either specified explicitly or else clear from the context.

2.4.1 Best approximations with prescribed locations

Let $\mu \in \mathcal{P}_r$ for some $r \geq 1$, and $n \in \mathbb{N}$. Given $x \in \Xi_n$, call δ_x^p with $p \in \Pi_n$ a *best r-approximation of μ, given x* if

$$d_r\left(\delta_x^p, \mu\right) \leq d_r\left(\delta_x^q, \mu\right), \ \forall\, q \in \Pi_n.$$

Denote by $\delta_x^\bullet$ any (possibly not unique) best r-approximation of μ, given x. (Note that $\delta_x^\bullet$ also potentially depends on r. In the interest of readability, this dependence is made explicit by a subscript only when necessary to avoid ambiguities.)

The existence of best r-approximations with prescribed locations can be established using the results of Sections 2.2 and 2.3.

Theorem 2.25. *Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$, and $n \in \mathbb{N}$. For every $x \in \Xi_n$, there exists a best r-approximation of μ, given x. Moreover, $d_r\left(\delta_x^p, \mu\right) = d_r\left(\delta_x^\bullet, \mu\right)$ with $p \in \Pi_n$ if and only if, for every $i = 1, \cdots, n$,*

$$x_{,i} < x_{,i+1} \text{ implies } P_{,i} \in Q^{F_\mu^{-1}}_{\frac{1}{2}(x_{,i}+x_{,i+1})}. \tag{2.13}$$

Proof. For convenience, let $A_i = Q^{F_\mu^{-1}}_{\frac{1}{2}(x_{,i}+x_{,i+1})}$ for $0 \leq i \leq n$; note that $A_0 = [-\infty, 0]$, $A_n = [1, +\infty]$, and every A_i is a compact (possibly one-point) interval, by Proposition 2.2. Since the theorem trivially is correct for $n = 1$, henceforth assume $n \geq 2$. We first establish (2.13), as the asserted existence of best r-approximations will follow directly from it.

Labelling $x \in \Xi_n$ as

$$x_{,i_0+1} = \cdots = x_{,i_1} < x_{,i_1+1} = \cdots = x_{,i_2} < x_{,i_2+1} = \cdots < \cdots < x_{,i_{m-1}+1} = \cdots = x_{,i_m} \tag{2.14}$$

with integers $j \leq i_j \leq n$ for $1 \leq j \leq m \leq n$, and $i_0 = 0$, $i_m = n$, note first that $d_r\left(\delta_x^p, \mu\right) = d_r\left(\delta_{\overline{x}}^{\overline{p}}, \mu\right)$, where $\overline{x} \in \Xi_m$ and $\overline{p} \in \Pi_m$, with $\overline{x}_{,j} = x_{,i_j}$, and $\overline{P}_{,j} = P_{,i_j}$ for $1 \leq j \leq m$. Moreover, (2.13) reduces to $\overline{P}_{,j} \in Q^{F_\mu^{-1}}_{\frac{1}{2}(\overline{x}_{,j}+\overline{x}_{,j+1})}$ for all $1 \leq j \leq m-1$. To establish (2.13), therefore, it can be assumed w.o.l.g. that $x_{,i} < x_{,i+1}$ for all i.

To prove that (2.13) is necessary, let δ_x^p be a best r-approximation of μ, given x. Given any $1 \leq i \leq n-1$, let $\widetilde{p} \in \Pi_n$ satisfy $\widetilde{p}_{,j} = p_{,j}$ for all $j \neq i, i+1$, and $0 \leq \widetilde{p}_{,i} \leq p_{,i} + p_{,i+1}$. Note that $P_{,i-1} \leq \widetilde{P}_{,i} \leq P_{,i+1}$.

If $P_{,i-1} < P_{,i+1}$, then $d_r\left(\delta_x^p, \mu\right) \le d_r\left(\delta_x^{\widetilde{p}}, \mu\right)$ implies

$$\left\| f_i - \left(x_{,i} \mathbb{1}_{[P_{,i-1}, P_{,i}[} + x_{,i+1} \mathbb{1}_{[P_{,i}, P_{,i+1}]} \right) \right\|_r \le \left\| f_i - \left(x_{,i} \mathbb{1}_{[P_{,i-1}, \widetilde{P}_{,i}[} + x_{,i+1} \mathbb{1}_{[\widetilde{P}_{,i}, P_{,i+1}]} \right) \right\|_r ,$$

with $f_i = F_\mu^{-1}\Big|_{[P_{,i-1}, P_{,i+1}]}$. Since $\widetilde{P}_{,i} \in [P_{,i-1}, P_{,i+1}]$ was arbitrary, Lemma 2.9 and Proposition 2.2 yield $P_{,i} \in Q^{f_i}_{\frac{1}{2}(x_{,i}+x_{,i+1})} = A_i$.

If $P_{,i-1} = P_{,i+1}$, let i^- and i^+ be the minimum and maximum, respectively, of the (non-empty) set $\{0 \le j \le n : \; P_{,j} = P_{,i}\}$. Clearly, $0 \le i^- \le i-1$, $i+1 \le i^+ \le n$, and $i^+ - i^- \ge 2$. Assume first that $i^- = 0$, in which case $i^+ \le n-1$ and $P_{,i} = P_{,i^+} = 0$. Lemma 2.9, applied to f_{i^+} yields $0 \in A_{i^+}$. Recall that $A_i \subset \mathbb{I}$ and $\max A_i \le \min A_{i^+}$, by Proposition 2.2. Thus $0 \le \min A_i \le \min A_{i^+} \le 0$, and hence $0 = P_{,i} \in A_i$. By a completely analogous argument, the case of $i^+ = n$, where $i^- \ge 1$ and $P_{,i} = P_{,i^-} = 1$, leads to $1 = P_{,i} \in A_i$. Finally, assume that $1 \le i^- < i^+ \le n-1$. In this case, Lemma 2.9, applied to f_{i^-} and f_{i^+} yields $P_{,i^-} \in A_{i^-}$ and $P_{,i^+} \in A_{i^+}$, respectively. Thus $P_{,i} = P_{,i^-} = P_{,i^+} \in A_{i^-} \cap A_{i^+}$. Since $j \mapsto \frac{1}{2}\left(x_{,j} + x_{,j+1}\right)$ is increasing, Proposition 2.2 implies that $A_i = \{P_{,i}\}$, and hence trivially $P_{,i} \in A_i$. This completes the proof that (2.13) holds whenever $d_r\left(\delta_x^p, \mu\right)$ is minimal, i.e., (2.13) is necessary.

To see that (2.13) also is sufficient, let $p \in \Pi_n$ satisfy (2.13) and consider $\widetilde{p} \in \Pi_n$ with $\widetilde{P}_i = \max A_i$ for all i. It suffices to show that $d_r\left(\delta_x^p, \mu\right) = d_r\left(\delta_x^{\widetilde{p}}, \mu\right)$. To see the latter, note that by Proposition 2.2(i), $P_{,i} \le \widetilde{P}_{,i} \le P_{,i+1}$, for all $1 \le i \le n-1$, and $\left|x_{,i} - F_\mu^{-1}(t)\right| = \left|x_{,i+1} - F_\mu^{-1}(t)\right|$ for all $P_{,i} < t < \widetilde{P}_{,i}$. Consequently,

$$\begin{aligned}
d_r\left(\delta_x^p, \mu\right)^r &= \sum_{i=1}^n \int_{P_{,i-1}}^{P_{,i}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t \\
&= \sum_{i=1}^n \left(\int_{P_{,i-1}}^{\widetilde{P}_{,i-1}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t + \int_{\widetilde{P}_{,i-1}}^{P_{,i}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t \right) \\
&= \sum_{i=1}^n \left(\int_{P_{,i-1}}^{\widetilde{P}_{,i-1}} \left|x_{,i-1} - F_\mu^{-1}(t)\right|^r \mathrm{d}t + \int_{\widetilde{P}_{,i-1}}^{P_{,i}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t \right) \\
&= \sum_{i=1}^n \left(\int_{P_{,i}}^{\widetilde{P}_{,i}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t + \int_{\widetilde{P}_{,i-1}}^{P_{,i}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t \right) \\
&= \sum_{i=1}^n \int_{\widetilde{P}_{,i-1}}^{\widetilde{P}_{,i}} \left|x_{,i} - F_\mu^{-1}(t)\right|^r \mathrm{d}t = d_r\left(\delta_x^{\widetilde{p}}, \mu\right)^r .
\end{aligned}$$

As indicated earlier, the asserted existence of a best r-approximation of μ, given x, is a direct consequence of (2.13). Indeed, when $x \in \Xi_n$ is written as in (2.14), Proposition 2.2(i) guarantees that the m intervals $A_{i_1-1}, \; A_{i_2-1}, \cdots, \; A_{i_m-1} \subset \mathbb{I}$ are arranged in such a way that $t \le u$ for all $t \in A_{i_j-1}$ and $u \in A_{i_{j+1}-1}$. It is possible, therefore, to choose $p \in \Pi_n$ satisfying (2.13). □

Given $\mu \in \mathcal{P}_r$ and $x_n \in \Xi_n$ for all n, it is natural to ask whether $d_r\left(\delta^\bullet_{x_n}, \mu\right) \to 0$ as $n \to \infty$. The following example illustrates that this may or may not be the case.

Example 2.26. (See Subsection 3.3.1 for details.) Let μ be the standard *exponential distribution* with $F_\mu(x) = 1 - e^{-x}$ for all $x \ge 0$. Note that $\mu \in \bigcap_{r\ge 1} \mathcal{P}_r$. Given $x_n = (1, 2, \cdots, n)/\sqrt{n} \in \Xi_n$, Theorem 2.25 yields a unique best r-approximation of μ, namely, $\delta^{p_n}_{x_n}$ with $P_{n,i} = F_\mu\left(\frac{2i+1}{2\sqrt{n}}\right) = 1 - e^{-(2i+1)/(2\sqrt{n})}$ for $1 \le i \le n-1$. It is readily confirmed that $\lim_{n\to\infty} \sqrt{n} d_r\left(\delta^\bullet_{x_n}, \mu\right) = \frac{1}{2}(r+1)^{-1/r}$ for every $r \ge 1$; in particular, therefore, $\lim_{n\to\infty} d_r\left(\delta^{p_n}_{x_n}, \mu\right) = 0$. On the other hand, consider $y_n = (0, 2, \cdots, 2n-2) \in \Xi_n$, for which $\lim_{n\to\infty} d_r\left(\delta^\bullet_{y_n}, \mu\right) = d_r(\nu, \mu) > 0$ with $\nu = \left(1 - e^{-1}\right)\delta_0 + 2\sinh 1 \sum_{i=1}^\infty e^{-2i}\delta_{2i}$. Note that while every point in $\operatorname{supp}\mu = [0, +\infty]$ is the limit of an appropriate sequence (x_{n,i_n}), this clearly is not the case with (y_n).

As Example 2.26 suggests, a condition has to be imposed on (x_n), with $x_n \in \Xi_n$ for all n, in order to guarantee that $\lim_{n\to\infty} d_r\left(\delta^\bullet_{x_n}, \mu\right) = 0$.

Theorem 2.27. *Assume that $\mu \in \mathcal{P}_r$ for some $r \ge 1$, and $x_n \in \Xi_n$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d_r\left(\delta^\bullet_{x_n}, \mu\right) = 0$ if and only if*

$$\lim_{n\to\infty} \min\nolimits_{1\le i\le n} |x - x_{n,i}| = 0, \ \forall\, x \in \operatorname{supp}\mu. \tag{2.15}$$

In particular, (2.15) holds whenever

$$\lim_{n\to\infty} \left(F_\mu(x_{n,1}) + \max_{1\le i\le n-1}(x_{n,i+1} - x_{n,i}) + 1 - F_\mu(x_{n,n}) \right) = 0.$$

Proof. For convenience, let $P_{n,i} = F_\mu\left(\frac{1}{2}(x_{n,i} + x_{n,i+1})\right)$ for all $n \in \mathbb{N}$ and $0 \le i \le n$, as well as $A = \mathbb{I} \setminus \{P_{n,i} : n \in \mathbb{N}, 0 \le i \le n\}$ and $f_n = F_{\delta^{p_n}_{x_n}}$. Note that $\left|F_\mu^{-1}(t) - x_{n,i}\right| = \min_{1\le j\le n}\left|F_\mu^{-1}(t) - x_{n,j}\right|$ whenever $P_{n,i-1} < t < P_{n,i}$, and hence

$$\left|F_\mu^{-1}(t) - f_n^{-1}(t)\right| = \min\nolimits_{1\le j\le n}\left|F_\mu^{-1}(t) - x_{n,j}\right|, \ \forall\, t \in A. \tag{2.16}$$

We first show that (2.15) is necessary. To see this, assume that (2.15) fails. Then, with the appropriate $\varepsilon > 0$, $x \in \operatorname{supp}\mu$ and sequence (n_k),

$$\min\nolimits_{1\le i\le n_k} |x - x_{n_k,i}| \ge 2\varepsilon, \ \forall\, k \in \mathbb{N}.$$

Since f_n is constant on $[x-\varepsilon, x+\varepsilon]$ whereas F_μ is not,

$$d_1\left(\delta^\bullet_{x_{n_k}}, \mu\right) = d_1\left(\delta^{p_{n_k}}_{x_{n_k}}, \mu\right) \ge \min\nolimits_{c\in\mathbb{R}} \int_{x-\varepsilon}^{x+\varepsilon} |F_\mu(y) - c|\,\mathrm{d}y > 0, \ \forall\, k \in \mathbb{N},$$

and so $\limsup_{n\to\infty} d_r\left(\delta^{\bullet}_{x_n}, \mu\right) > 0$ as well.

To see that (2.15) also is sufficient, note first that if F_μ^{-1} is continuous at $t \in A$, then $F_\mu^{-1}(t) \in \operatorname{supp}\mu$, and hence $f_n^{-1}(t) \to F_\mu^{-1}(t)$, by (2.16). Since F_μ^{-1} is monotone, $f_n^{-1} \to F_\mu^{-1}$ a.e. on $\mathbb{I}$. If $\operatorname{supp}\mu$ is bounded then $f_n^{-1} \to F_\mu^{-1}$ in $L^r(\mathbb{I})$, by the Dominated Convergence Theorem, i.e., $\lim_{n\to\infty} d_r\left(\delta^{p_n}_{x_n}, \mu\right) = 0$, and thus $\lim_{n\to\infty} d_r\left(\delta^{\bullet}_{x_n}, \mu\right) = 0$. If, on the other hand, $\operatorname{supp}\mu$ is unbounded, then, given any $\varepsilon > 0$, choose $\nu \in \mathcal{P}$ with bounded support and $d_r(\mu, \nu) < \varepsilon$. Then $d_r\left(\delta^{\bullet}_{x_n}, \mu\right) \le d_r\left(\widetilde{\delta}^{\bullet}_{x_n}, \mu\right) \le d_r\left(\widetilde{\delta}^{\bullet}_{x_n}, \nu\right) + d_r(\nu, \mu)$, where $\widetilde{\delta}^{\bullet}_{x_n}$ denotes a best r-approximation of ν, given x_n. By the above, $\limsup_{n\to\infty} d_r\left(\delta^{\bullet}_{x_n}, \mu\right) \le \varepsilon$, and since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} d_r\left(\delta^{\bullet}_{x_n}, \mu\right) = 0$. □

Example 2.28. (See Subsection 3.4.1 for details.) Let μ be the Beta$(2, 1)$ distribution, i.e., $F_\mu(x) = x^2$ for all $x \in \mathbb{I}$, and consider $x_n = (1, \sqrt{2}, \cdots, \sqrt{n})/\sqrt{n} \in \Xi_n$. By Theorem 2.27, $\lim_{n\to\infty} d_r\left(\delta^{\bullet}_{x_n}, \mu\right) = 0$ for every $r \ge 1$. Unlike in Example 2.26, however, the rate of convergence depends on r: With $\gamma_r = \frac{1}{2} + \frac{1}{\max\{2,r\}}$ and the appropriate $0 < \widetilde{C}_r < +\infty$,

$$\lim_{n\to\infty} n^{\gamma_r} d_r\left(\delta^{\bullet}_{x_n}, \mu\right) = \widetilde{C}_r$$

whenever $r \ne 2$, whereas

$$\lim_{n\to\infty} \frac{n}{\sqrt{\log n}} d_2\left(\delta^{\bullet}_{x_n}, \mu\right) = \frac{1}{4\sqrt{3}}.$$

Thus $\left(d_r\left(\delta^{\bullet}_{x_n}, \mu\right)\right)$ decays like $(n^{-\gamma_r})$ and $\left(n^{-1}\sqrt{\log n}\right)$ if $r \ne 2$ and $r = 2$, respectively.

2.4.2 Best approximations with prescribed weights, notably equal weights

Let $\mu \in \mathcal{P}_r$ for some $r \ge 1$, and $n \in \mathbb{N}$. Given $p \in \Pi_n$, call δ^p_x with $x \in \Xi_n$ a *best r-approximation of μ, given p* if

$$d_r\left(\delta^p_x, \mu\right) \le d_r\left(\delta^p_y, \mu\right), \quad \forall\, y \in \Xi_n.$$

Denote by $\delta^p_\bullet$ any best r-approximation of μ, given p. (Again in the interest of readability, the r-dependence of $\delta^p_\bullet$ is made explicit by a subscript only when necessary to avoid ambiguity.) An important special case of $p \in \Pi_n$ is the uniform probability vector $u_n = (1, \cdots, 1)/n$. Best r-approximations of μ, given u_n, will be referred to as best *uniform* r-approximations, and denoted $\delta^{u_n}_\bullet$. As in the case of prescribed locations studied in Subsection 2.4.1, the existence of best r-approximations with prescribed weights follows from results in Sections 2.3-2.4. Due to the nature of (2.1), the proof of the following theorem even is simpler than that of its counterpart, Theorem 2.25.

Theorem 2.29. *Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$, and $n \in \mathbb{N}$. For every $p \in \Pi_n$, there exists a best r-approximation of μ, given p. Moreover, $d_1(\delta_x^p, \mu) = d_1(\delta_\bullet^p, \mu)$ if and only if, for every $i = 1, \cdots, n$,*

$$P_{,i-1} < P_{,i} \text{ implies } x_{,i} \in Q^{F_\mu}_{\frac{1}{2}(P_{,i-1}+P_{,i})}, \tag{2.17}$$

and for $r > 1$, $d_r(\delta_x^p, \mu) = d_r(\delta_\bullet^p, \mu)$ if and only if, for every $i = 1, \cdots, n$,

$$P_{,i-1} < P_{,i} \text{ implies } x_{,i} = \tau_r^{f_i}, \text{ where } f_i = F_\mu^{-1}\Big|_{[P_{,i-1}, P_{,i}]}. \tag{2.18}$$

Proof. As in the proof of Theorem 2.25, existence follows immediately, once (2.17) and (2.18) are established. Labelling P as

$$P_{,i_0} = \cdots = P_{,i_1-1} < P_{,i_1} = \cdots = P_{,i_2-1} < P_{,i_2} = \cdots < \cdots < P_{,i_{m-1}} = \cdots = P_{,i_m-1}$$

with integers $j \leq i_j \leq n+1$ for $1 \leq j \leq m \leq n$, and $i_0 = 0$, $i_m = n+1$, note that $d_r(\delta_x^p, \mu) = d_r\left(\delta_{\overline{x}}^{\overline{p}}, \mu\right)$, where $\overline{x} \in \Xi_m$ and $\overline{p} \in \Pi_m$, with $\overline{x}_{,j} = x_{,i_j}$, and $\overline{P}_{,j} = P_{,i_j}$ for $1 \leq j \leq m$. Moreover, (2.17) reduces to $\overline{x}_{,j} \in Q^{F_\mu}_{\frac{1}{2}(\overline{P}_{,j-1}+\overline{P}_{,j})}$ for all $1 \leq j \leq m$, whereas (2.18) reduces to $\overline{x}_{,j} = \tau_r^{f_j}$ with $f_j = F_\mu^{-1}\Big|_{[\overline{P}_{,j-1}, \overline{P}_{,j}]}$. Thus, to establish (2.17) and (2.18), it can be assumed w.o.l.g. that $P_{,i-1} < P_{,i}$ for all i.

Given $p \in \Pi_n$, it is clear from $d_r(\delta_x^p, \mu)^r = \sum_{i=1}^n \|x_{,i} - f_i\|_r^r$ that $d_r(\delta_x^p, \mu)$ is minimal if and only if $\|x_{,i} - f_i\|_r$ is minimal for every i. By Corollary 2.12, the latter is the case precisely if $x_{,i} \in Q^{f_i^{-1}}_{\frac{1}{2}(P_{,i-1}+P_{,i})} = Q^{F_\mu}_{\frac{1}{2}(P_{,i-1}+P_{,i})}$ for $r = 1$, and if $x_{,i} = \tau_r^{f_i}$ for $r > 1$. □

Remark 2.30. (i) For $r = 1$ and $p = u_n$, Theorem 2.29 reduces to [6, Thm.2.8]. In particular, $\frac{1}{n}\sum_{i=1}^n \delta_{F_\mu^{-1}\left(\frac{2i-1}{2n}\right)}$ is a best uniform 1-approximation of $\mu \in \mathcal{P}_1$. For $n = 1$, (2.17) yields the well-known fact that $d_1(\delta_a, \mu)$ is minimal if and only if $a \in \mathbb{R}$ is a *median* of μ.
(ii) For $r = 2$, if $\mu \in \mathcal{P}_2$ and $p \in \Pi_n$ with $p_{,i} > 0$ for all i, then by Remark 2.13(i), the unique best 2-approximation of μ, given p, is δ_x^p with $x_{,i} = p_{,i}^{-1}\int_{P_{,i-1}}^{P_{,i}} F_\mu^{-1}(t)\mathrm{d}t$. In particular, $d_2(\delta_a, \mu)$ is minimal precisely for $a = \int_0^1 F_\mu^{-1}(t)\mathrm{d}t$.

Example 2.31. Given $\mu \in \mathcal{P}_r$ and $p \in \Pi_n$, Theorem 2.29 can also be utilized to minimize $d_r\left(\sum_{i=1}^n p_{,i}\delta_{x_{,i}}, \mu\right)$ where $x \in \mathbb{R}^n$ but not necessarily $x \in \Xi_n$. For instance, with $\mu = \mathrm{Beta}(2,1)$ as in Example 2.28 and $p = (2/3, 1/3)$ as well as $q = (1/3, 2/3)$, for $r = 1$,

$$\delta_\bullet^p = \frac{2}{3}\delta_{1/\sqrt{3}} + \frac{1}{3}\delta_{\sqrt{5/6}}, \quad \delta_\bullet^q = \frac{1}{3}\delta_{1/\sqrt{6}} + \frac{2}{3}\delta_{2/\sqrt{6}}.$$

Since $d_1(\delta_\bullet^p, \mu) \approx 0.12154 > d_1(\delta_\bullet^q, \mu) \approx 0.10677$, it follows, that $\min_{x \in \mathbb{R}^2} d_1\left(\frac{2}{3}\delta_{x,1} + \frac{1}{3}\delta_{x,2}, \mu\right) = d_1(\delta_\bullet^q, \mu)$. In general, this minimizing problem can be solved by applying Theorem 2.29 to $\left(p_{,\sigma(1)}, \cdots, p_{,\sigma(n)}\right) \in \Pi_n$ for all permutations σ of $\{1, \cdots, n\}$. The permutations yielding the minimal value may depend on r. Often, not all $n!$ permutations σ have to be considered. For instance, if F_μ^{-1} is concave on $]0,1[$ as in the above example, then only the (unique) non-decreasing rearrangement of p is relevant.

Given $\mu \in \mathcal{P}_r$ and $p_n \in \Pi_n$ for all n, it again is natural to ask whether $d_r\left(\delta_\bullet^{p_n}, \mu\right) \to 0$ as $n \to \infty$. As in the dual situation of Subsection 2.4.1, this may or may not be the case, as illustrated by the following example.

Example 2.32. (See Subsection 3.3.2 for details.) Consider again the exponential distribution μ of Example 2.26. By (2.17), the unique best uniform 1-approximation of μ is $\delta_{x_n}^{u_n}$ with $x_{n,i} = F_\mu^{-1}\left(\frac{2i-1}{2n}\right) = \log \frac{2n}{2n-2i+1}$, for every $n \in \mathbb{N}$ and $1 \le i \le n$, and

$$n d_1\left(\delta_\bullet^{u_n}, \mu\right) = -2\sum_{i=1}^n i \log \frac{2i-1}{2i} + \log \frac{(2n)!}{2^{2n} n! n^n} = \frac{1}{4}\log n + \mathcal{O}(1) \ \text{ as } n \to \infty.$$

By Remark 2.30(ii), the unique best uniform 2-approximation of μ is $\delta_{y_n}^{u_n}$ with $y_{n,i} = n\int_{(i-1)/n}^{i/n} F_\mu^{-1}(t)\mathrm{d}t = \log \dfrac{en(n-i)^{n-i}}{(n-i+1)^{n-i+1}}$, and

$$\sqrt{n} d_2\left(\delta_\bullet^{u_n}, \mu\right) = \sqrt{n - \sum_{i=1}^{n-1} i(i+1)\left(\log \frac{i}{i+1}\right)^2} = C_2 + \mathcal{O}\left(n^{-1}\right) \ \text{ as } n \to \infty,$$

where $C_2^2 = 1 + \sum_{i=1}^\infty \left(1 - i(i+1)\left(\log \frac{i}{i+1}\right)^2\right) \approx 1.0803$. In fact, it can be shown that $\lim_{n\to\infty} n^{1/r} d_r\left(\delta_\bullet^{u_n}, \mu\right) = C_r$ whenever $r > 1$, with the appropriate $0 < C_r < +\infty$. Thus $d_r\left(\delta_\bullet^{u_n}, \mu\right) \to 0$ as $n \to \infty$, but the rate of convergence evidently depends on r, and is slower than $\left(n^{-1}\right)$. On the other hand, consider $p_n \in \Pi_n$ with $p_{n,i} = \frac{2^{i-1}}{2^n - 1}$ for $1 \le i \le n$. Then $\lim_{n\to\infty} d_r\left(\delta_\bullet^{p_n}, \mu\right) = d_r(\nu, \mu) > 0$ with $\nu = \sum_{i=1}^\infty 2^{-i}\delta_{a_i}$, and $a_i = F_\mu^{-1}\left(3 \cdot 2^{-i-1}\right)$ if $r = 1$ and $a_i = \tau_r^{F_\mu^{-1}\big|_{[2^{-i}, 2^{-i+1}]}}$ if $r > 1$.

Example 2.32 suggests a simple condition that may be imposed on (p_n), with $p_n \in \Pi_n$ for every n, in order to guarantee that $\lim_{n\to\infty} d_r\left(\delta_\bullet^{p_n}, \mu\right) = 0$. The following result is a counterpart of Theorem 2.27. Due to the nature of (2.1), the proof is similar but not identical; recall that $G^{F_\mu^{-1}} \subset \mathbb{I}$ for every $\mu \in \mathcal{P}$.

Theorem 2.33. *Assume that $\mu \in \mathcal{P}_r$ for some $r \ge 1$, and $p_n \in \Pi_n$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d_r\left(\delta_\bullet^{p_n}, \mu\right) = 0$ if and only if*

$$\lim_{n\to\infty} \min\nolimits_{1\le i\le n} |t - P_{n,i}| = 0, \ \forall\, t \in G^{F_\mu^{-1}}. \tag{2.19}$$

In particular, (2.19) *holds whenever* $\lim_{n\to\infty} \max_{1\le i\le n} p_{n,i} = 0$.

Proof. For every $n \in \mathbb{N}$, let $\delta_{x_n}^{p_n}$ be a best r-approximation of μ, given p_n, and also $f_n = F_{\delta_{x_n}^{p_n}}$, for convenience.

To see that (2.19) is necessary, suppose that

$$\min_{1\le i\le n_k} |t - P_{n_k,i}| \ge 2\varepsilon, \ \forall\, k \in \mathbb{N},$$

for some $0 < t < 1$, $0 < \varepsilon < \min\{t, 1-t\}$, and the appropriate sequence (n_k). (The other cases, $t = 0$ and $t = 1$, are analogous.) Since $f_{n_k}^{-1}$ is constant on $[t-\varepsilon, t+\varepsilon]$ whereas F_μ^{-1} is not,

$$d_r\left(\delta_\bullet^{p_{n_k}}, \mu\right)^r = d_r\left(\delta_{x_{n_k}}^{p_{n_k}}, \mu\right)^r \ge \min_{c\in\mathbb{R}} \int_{t-\varepsilon}^{t+\varepsilon} \left|F_\mu^{-1}(u) - c\right|^r \mathrm{d}u > 0, \ k \in \mathbb{N},$$

and hence $\limsup_{n\to\infty} d_r\left(\delta_\bullet^{p_n}, \mu\right) > 0$.

To show that (2.19) also is sufficient, assume that t is a continuity point of F_μ^{-1}. If $t \in G^{F_\mu^{-1}}$ then, given $\varepsilon > 0$, there exist $t_1,\ t_2 \in G^{F_\mu^{-1}}$ with $\left|F_\mu^{-1}(t_{1,2}) - F_\mu^{-1}(t)\right| < \varepsilon$ and either $t < t_1 < t_2$ or $t_1 < t_2 < t$. Assume w.o.l.g. that $t < t_1 < t_2$. (The other case is similar.) By (2.19), $t < P_{n,i_n} < P_{n,i_n+1} < t_2$ for all sufficiently large n and the appropriate $1 \le i_n \le n$. Since f_n^{-1} is constant on $[P_{n,i_n}, P_{n,i_n+1}]$ with a value between $F_\mu^{-1}(P_{n,i_n}) \ge F_\mu^{-1}(t)$ and $F_\mu^{-1}(P_{n,i_n+1}) \le F_\mu^{-1}(t) + \varepsilon$, clearly $f_n^{-1}(t) \to F_\mu^{-1}(t)$. If, on the other hand, $t \notin G^{F_\mu^{-1}}$, then let $]a, b[\subset \mathbb{I}$ be the largest interval that contains t but is disjoint from $G^{F_\mu^{-1}}$. Assume w.o.l.g. that $0 < a < b < 1$. (The cases $a = 0$ and $b = 1$ are analogous.) Then $a, b \in G^{F_\mu^{-1}}$. Given $\varepsilon > 0$, since $F_\mu^{-1} - F_\mu^{-1}(t) \in L^r(\mathbb{I})$, there exists $\delta > 0$ such that $\int_A \left|F_\mu^{-1}(u) - F_\mu^{-1}(t)\right|^r \mathrm{d}u < \varepsilon$ whenever $\lambda(A) < \delta$. Let $i_n = \min\{1 \le j \le n :\ P_{n,j} > t\}$. Note that $P_{n,i_n-1} \le t < P_{n,i_n}$. If $a \le P_{n,i_n-1} < P_{n,i_n} \le b$, then $f_n^{-1}(t) = F_\mu^{-1}(t)$. If $P_{n,i_n-1} < a$, then $|P_{n,i_n-1} - a| = \min_{1\le i\le n} |a - P_{n,i}|$, $\max\{b, P_{n,i_n}\} - b \le \min_{1\le i\le n} |b - P_{n,i}|$, and $(a - P_{n,i_n-1}) + \max\{b, P_{n,i_n}\} - b < \delta$ for all sufficiently large n, by (2.19). Hence

$$(t-a)\left|F_\mu^{-1}(t) - f_n^{-1}(t)\right|^r \le \int_{P_{n,i_n-1}}^{P_{n,i_n}} \left|F_\mu^{-1}(u) - f_n^{-1}(t)\right|^r \mathrm{d}u \le \int_{P_{n,i_n-1}}^{P_{n,i_n}} \left|F_\mu^{-1}(u) - F_\mu^{-1}(t)\right|^r \mathrm{d}u$$
$$= \int_{P_{n,i_n-1}}^{a} \left|F_\mu^{-1}(u) - F_\mu^{-1}(t)\right|^r \mathrm{d}u + \int_b^{\max\{b, P_{n,i_n}\}} \left|F_\mu^{-1}(u) - F_\mu^{-1}(t)\right|^r \mathrm{d}u < \varepsilon.$$

For $P_{n,i_n} > b$, an analogous argument applies. In summary, $f_n^{-1} \to F_\mu^{-1}$ a.e. on $\mathbb{I}$, and the remaining argument is identical to the one in the proof of Theorem 2.27. □

Since $\delta_\bullet^p$ is a best approximation of $\mu \in \mathcal{P}_r$ w.r.t. the metric d_r, given weights p, it is natural to ask whether $\delta_\bullet^p$ reflects any basic feature of μ. Most basically perhaps, how is $\operatorname{supp} \delta_\bullet^p$ related to $\operatorname{supp} \mu$? As the following example shows, it may not be possible to guarantee

$\operatorname{supp}\delta_\bullet^p \subset \operatorname{supp}\mu$.

Example 2.34. Let μ be the Cantor probability measure, i.e., the $\frac{\log 2}{\log 3}$-dimensional Hausdorff measure on the classical Cantor middle third set. Using the fact that $Q_t^{F_\mu}$ is a non-degenerate interval for every dyadic rational $0 < t < 1$, it is readily seen that $\delta_\bullet^{u_n}$ is not unique for any $n \in \mathbb{N}$ whenever $r = 1$. For instance, $\frac{1}{2}\left(\delta_{1/5} + \delta_{4/5}\right)$ and $\frac{1}{2}\left(\delta_{1/9} + \delta_{8/9}\right)$ both are best uniform 1-approximations of μ, and $\{1/5, 4/5\} \cap \operatorname{supp}\mu = \varnothing$ whereas $\{1/9, 8/9\} \subset \operatorname{supp}\mu$. For $r > 1$, on the other hand, $\delta_\bullet^{u_n}$ always is unique. In fact, $\delta_\bullet^{u_{2^k}}$ even is independent of $r > 1$, due to symmetry, and $\operatorname{supp}\delta_\bullet^{u_{2^k}} \cap \operatorname{supp}\mu = \varnothing$. For example, $\delta_\bullet^{u_2} = \frac{1}{2}\left(\delta_{1/6} + \delta_{5/6}\right)$ for all $r > 1$, and $\{1/6, 5/6\} \cap \operatorname{supp}\mu = \varnothing$.

To formalize the observations in Example 2.34, note that if δ_x^p is a best 1-approximation of μ, given $p \in \Pi_n$, then, by Theorem 2.29, $x_{,i} \in Q^{F_\mu}_{\frac{1}{2}(P_{,i-1}+P_{,i})}$ whenever $P_{,i-1} < P_{,i}$. Since the endpoints of all quantile sets $Q_t^{F_\mu}$ belong to $\operatorname{supp}\mu$, by Proposition 2.3, it is possible to choose $y \in \Xi_n$ with $d_1\left(\delta_y^p, \mu\right) = d_1\left(\delta_x^p, \mu\right)$ and $\operatorname{supp}\delta_y^p \subset \operatorname{supp}\mu$. Similarly, if $r > 1$, then $x_{,i} = \tau_r^{f_i}$ with $f_i = F_\mu^{-1}\big|_{[P_{,i-1}, P_{,i}]}$, and consequently $x_{,i} \in \left[F_\mu^{-1}\left(P_{,i-1}\right), F_\mu^{-1}\left(P_{,i}-\right)\right]$. By Corollary 2.12(i), it follows that

$$\min\ \operatorname{supp}\mu = F_\mu^{-1}\left(P_{,0}+\right) \le x_{,i} \le F_\mu^{-1}\left(P_{,n}-\right) = \max\ \operatorname{supp}\mu,\ \forall\ i = 1, \cdots, n.$$

This establishes

Proposition 2.35. *Assume that $\mu \in \mathcal{P}_r$ for some $r \ge 1$, and $n \in \mathbb{N}$. If $r = 1$ or $\operatorname{supp}\mu$ is connected (and hence an interval), then there exists a best r-approximation $\delta_\bullet^p$ of μ, given $p \in \Pi_n$, with $\operatorname{supp}\delta_\bullet^p \subset \operatorname{supp}\mu$.*

Among the best approximations of μ, given $p \in \Pi_n$, the case of *uniform* approximations, i.e., $p = u_n$, arguably is the most important. In this case, Theorem 2.33 has the following corollary.

Corollary 2.36. *Assume that $\mu \in \mathcal{P}_r$ for some $r \ge 1$, and $1 \le s \le r$. For every $n \in \mathbb{N}$, let $\delta_{\bullet,s}^{u_n}$ be a best uniform s-approximation of μ. Then $\lim_{n\to\infty} d_r\left(\delta_{\bullet,s}^{u_n}, \mu\right) = 0$. In particular, $\lim_{n\to\infty} d_r\left(\delta_{x_n}^{u_n}, \mu\right) = 0$ for $x_{n,i} = F_\mu^{-1}\left(\frac{2i-1}{2n}\right)$.*

Remark 2.37. For $r = s = 2$, Corollary 2.36 yields [2, Thm.3.6]. In [2], a *convex order* on $\mathcal{P}$ is considered, shown to be preserved by best uniform 2-approximations (termed $\mathcal{U}$-*quantization*), and applied to the numerical construction of martingales. We conjecture that best uniform r-approximations preserve this order for all $r > 1$. By contrast, best (unconstrained) 2-approximations, considered in Subsection 2.4.3 below, do not in general preserve the convex order; see [2, Thm.2.1].

The remainder of this subsection is devoted to a study of $d_r\left(\delta_\bullet^{u_n}, \mu\right)$ as $n \to \infty$. Since best uniform r-approximations may be hard to identify explicitly, we will also consider *asymptotically* best uniform r-approximations. Formally, $\left(\delta_{x_n}^{u_n}\right)$ with $x_n \in \Xi_n$ for all $n \in \mathbb{N}$ is a sequence of *asymptotically best uniform r-approximations* of $\mu \in \mathcal{P}_r \setminus \{\delta_x^{u_i} : \ i \in \mathbb{N}, \ x \in \Xi_i\}$ if

$$\lim_{n\to\infty} \frac{d_r\left(\delta_{x_n}^{u_n}, \mu\right)}{d_r\left(\delta_\bullet^{u_n}, \mu\right)} = 1.$$

To illustrate a possible behaviour of $\left(d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$, as well as the practical relevance of asymptotically best uniform approximations, we first consider a simple example.

Example 2.38. (See Subsection 3.4.2 for details.) Let $\mu = \text{Beta}(2,1)$ as in Example 2.28. Theorem 2.29 yields a unique best uniform r-approximation of μ for every $r \geq 1$. For $r = 1$, a short calculation shows that

$$nd_1\left(\delta_\bullet^{u_n}, \mu\right) = \frac{1}{4} + \mathcal{O}\left(n^{-1/2}\right) \quad \text{as } n \to \infty,$$

whereas for $r = 2$,

$$nd_2\left(\delta_\bullet^{u_n}, \mu\right) = \frac{1}{4\sqrt{3}}\sqrt{\log n} + \mathcal{O}(1) \quad \text{as } n \to \infty.$$

For $1 < r < 2$, however, $\delta_\bullet^{u_n}$ is not easy to calculate explicitly. This not only makes the rate of convergence of $\left(d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ hard to determine, but it also emphasizes the need for simple *asymptotically* best uniform approximations. In fact, Theorem 2.39 below shows that, for every $1 \leq r < 2$, $\lim_{n\to\infty} nd_r\left(\delta_\bullet^{u_n}, \mu\right) = \left(\frac{2^{1-2r}}{(r+1)(2-r)}\right)^{1/r}$, and $\left(\delta_{x_n}^{u_n}\right)$, with $x_{n,i} = \sqrt{\frac{2i-1}{2n}}$ for $1 \leq i \leq n$, is a sequence of asymptotically best uniform r-approximations. By contrast, it turns out that $\lim_{n\to\infty} n^{1/2+1/r} d_r\left(\delta_\bullet^{u_n}, \mu\right)$ is finite and positive whenever $r > 2$.

The observations in Example 2.38 are a special case of a general principle: If the quantile function of $\mu \in \mathcal{P}_r$ is absolutely continuous (and not constant), then $\left(nd_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ converges to a positive limit. This fact may be seen as an analogue, in the context of best uniform approximations, of a classical result regarding best approximations; cf. Proposition 2.50 below.

Theorem 2.39. *Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$. If μ^{-1} is absolutely continuous (w.r.t. λ) then*

$$\lim_{n\to\infty} nd_r\left(\delta_\bullet^{u_n}, \mu\right) = \frac{1}{2(r+1)^{1/r}} \left(\int_{\mathbb{I}} \left(\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda}\right)^r\right)^{1/r}. \tag{2.20}$$

Moreover, if $\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda} \in L^r(\mathbb{I})$ then $\left(\delta_{x_n}^{u_n}\right)$, with $x_{n,i} = F_\mu^{-1}\left(\frac{2i-1}{2n}\right)$ for $1 \leq i \leq n$, is a sequence of asymptotically best uniform r-approximations of μ, unless μ is degenerate, i.e., unless $\mu = \delta_a$

for some $a \in \mathbb{R}$.

Proof. For convenience, let $f = F_\mu^{-1}|_{]0,1[}$, as well as $J_{n,i} = \left[\frac{i-1}{n}, \frac{i}{n}\right]$ and $x_{n,i} = f\left(\frac{2i-1}{2n}\right)$ for $n \in \mathbb{N}$ and $1 \le i \le n$. Note that the non-decreasing function f is absolutely continuous, by assumption. For the reader's convenience, the following proof is divided into four steps: First, (2.20) will be established assuming that f has a C^1-extension to $\mathbb{I}$; then (2.20) will be shown to hold in general, regardless of whether both sides are finite (Step 2) or infinite (Step 3); finally, the assertion regarding asymptotically best uniform approximations will be proved (Step 4).

Step 1. Assume f can be extended to a C^1-function on $\mathbb{I}$. Then

$$\begin{aligned} n^r d_r\left(\delta_{x_n}^{u_n}, \mu\right)^r &= n^r \sum_{i=1}^n \int_{J_{n,i}} |f(t) - x_{n,i}|^r \,\mathrm{d}t \le n^r \sum_{i=1}^n \left(\max\nolimits_{J_{n,i}} f'\right)^r \int_{J_{n,i}} \left|t - \frac{2i-1}{2n}\right|^r \mathrm{d}t \\ &= \frac{1}{2^r(r+1)} \cdot \frac{1}{n} \sum_{i=1}^n \left(\max\nolimits_{J_{n,i}} f'\right)^r . \end{aligned}$$

Since $(f')^r$ is Riemann integrable, $\lambda(J_{n,i}) = 1/n$, and similarly

$$n^r d_r\left(\delta_{x_n}^{u_n}, \mu\right)^r \ge \frac{1}{2^r(r+1)} \cdot \frac{1}{n} \sum_{i=1}^n \left(\min\nolimits_{J_{n,i}} f'\right)^r ,$$

it follows that $\lim_{n\to\infty} n d_r\left(\delta_\bullet^{u_n}, \mu\right) = \frac{1}{2(r+1)^{1/r}}\left(\int_{\mathbb{I}} f'(t)^r \mathrm{d}t\right)^{1/r} < +\infty$. Moreover, f' is uniformly continuous, hence given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f'(t) - f'(u)| \le \varepsilon, \ \forall\, t, u \in \mathbb{I}, \ |t-u| < \frac{1}{N}.$$

Whenever $n \ge N$, therefore, the Mean Value Theorem yields

$$\left| f(t) - x_{n,i} - f'\left(\frac{2i-1}{2n}\right)\left(t - \frac{2i-1}{2n}\right)\right| \le \varepsilon \left| t - \frac{2i-1}{2n}\right|, \ \forall\, t \in J_{n,i},$$

and consequently, with $y_{n,i} = \tau_r^{f|_{J_{n,i}}}$,

$$|y_{n,i} - x_{n,i}| \le \frac{\varepsilon}{n}, \ \forall\, 1 \le i \le n,$$

by Proposition 2.17. It follows that

$$n^r d_r\left(\delta_\bullet^{u_n}, \delta_{x_n}^{u_n}\right)^r = n^r d_r\left(\delta_{y_n}^{u_n}, \delta_{x_n}^{u_n}\right)^r = n^r \sum_{i=1}^n \int_{J_{n,i}} |y_{n,i} - x_{n,i}|^r \le \varepsilon^r,$$

and since $\varepsilon > 0$ was arbitrary,

$$\limsup_{n\to\infty} \left| n^r d_r \left(\delta_\bullet^{u_n}, \mu\right)^r - n^r d_r \left(\delta_{x_n}^{u_n}, \mu\right)^r \right| \le \lim_{n\to\infty} n^r d_r \left(\delta_\bullet^{u_n}, \delta_{x_n}^{u_n}\right)^r = 0,$$

which establishes (2.20), with the same finite value on either side.

Step 2. Let the non-decreasing and absolutely continuous function f be arbitrary, but assume that $f' \in L^r(\mathbb{I})$. Similarly, let $\widetilde{\mu} \in \mathcal{P}_r$ be such that $\widetilde{\mu}^{-1}$ is absolutely continuous, with $\widetilde{f} := F_{\widetilde{\mu}}^{-1}$ and $\widetilde{f}' \in L^r(\mathbb{I})$. For $n \in \mathbb{N}$ and $1 \le i \le n$, pick any $t_{n,i} \in J_{n,i}$ and define $z_{n,i} = f(t_{n,i})$, $\widetilde{z}_{n,i} = \widetilde{f}(t_{n,i})$. Below, it will be shown that, for any $r \ge 1$,

$$\left| n^r d_r \left(\delta_{z_n}^{u_n}, \mu\right)^r - n^r d_r \left(\delta_{\widetilde{z}_n}^{u_n}, \widetilde{\mu}\right)^r \right| \le 2 \left\| f' - \widetilde{f}' \right\|_r \left\| f' + \widetilde{f}' \right\|_r^{r-1}, \ \forall\, n \in \mathbb{N}. \tag{2.21}$$

To see that (2.20) follows easily from (2.21), at least under the current assumption that $f' \in L^r(\mathbb{I})$, fix $r \ge 1$ and w.o.l.g. $0 < \varepsilon < \|f'\|_r$. There exists $\widetilde{\mu} \in \mathcal{P}_r$ such that $\widetilde{f}$ has a C^1-extension to $\mathbb{I}$, and $\|f' - \widetilde{f}'\|_r < \varepsilon$. With the appropriate t_n, let $\delta_{z_n}^{u_n}$ be a best uniform r-approximation of μ, and $\delta_{x_n}^{u_n}$ a best uniform r-approximation of $\widetilde{\mu}$. For all sufficiently large n, Step 1 and (2.21) yield

$$\begin{aligned} n^r d_r \left(\delta_{z_n}^{u_n}, \mu\right)^r &\le n^r d_r \left(\delta_{x_n}^{u_n}, \mu\right)^r \le n^r d_r \left(\delta_{\widetilde{x}_n}, \mu\right)^r + 2\varepsilon \left(\varepsilon + 2\|f'\|_r\right)^{r-1} \\ &\le \frac{1}{2^r(1+r)} \left(\|f'\|_r + \varepsilon\right)^r + \varepsilon + 2\varepsilon \left(\varepsilon + 2\|f'\|_r\right)^{r-1}, \end{aligned}$$

but also

$$\begin{aligned} n^r d_r \left(\delta_{z_n}^{u_n}, \mu\right)^r &\ge n^r d_r \left(\delta_{\widetilde{z}_n}^{u_n}, \widetilde{\mu}\right)^r - 2\varepsilon \left(\varepsilon + 2\left\|f'\right\|_r\right)^{r-1} \\ &\ge \frac{1}{2^r(1+r)} \left(\left\|f'\right\|_r - \varepsilon\right)^r - \varepsilon - 2\varepsilon \left(\varepsilon + 2\left\|f'\right\|_r\right)^{r-1}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this establishes (2.20).

It remains to verify (2.21), which only requires the elementary inequality, valid for all $r \ge 1$,

$$|a^r - b^r| \le r\,|a - b| \left(a^{r-1} + b^{r-1}\right), \ \forall\, a,\ b \ge 0, \tag{2.22}$$

together with a repeated application of Hölder's inequality, as follows: Note first that

$$d_r \left(\delta_{z_n}^{u_n}, \mu\right)^r = \sum_{i=1}^{n} \left(\int_{(i-1)/n}^{t_{n,i}} \left(\int_t^{t_{n,i}} f'(u)\mathrm{d}u \right)^r \mathrm{d}t + \int_{t_{n,i}}^{i/n} \left(\int_{t_{n,i}}^{t} f'(u)\mathrm{d}u \right)^r \mathrm{d}t \right), \tag{2.23}$$

and consequently

$$\left|d_r\left(\delta_{z_n}^{u_n},\mu\right)^r - d_r\left(\delta_{\widetilde{z}_n}^{u_n},\widetilde{\mu}\right)^r\right| \le \sum_{i=1}^n \left\{\int_{(i-1)/n}^{t_{n,i}} \left|\left(\int_t^{t_{n,i}} f'(u)\mathrm{d}u\right)^r - \left(\int_t^{t_{n,i}} \widetilde{f}'(u)\mathrm{d}u\right)^r\right|\mathrm{d}t \right.$$
$$\left. + \int_{t_{n,i}}^{i/n} \left|\left(\int_{t_{n,i}}^{t} f'(u)\mathrm{d}u\right)^r - \left(\int_{t_{n,i}}^{t} \widetilde{f}'(u)\mathrm{d}u\right)^r\right|\mathrm{d}t\right\}.$$

With (2.22), therefore,

$$\begin{aligned}
&\int_{(i-1)/n}^{t_{n,i}} \left|\left(\int_t^{t_{n,i}} f'(u)\mathrm{d}u\right)^r - \left(\int_t^{t_{n,i}} \widetilde{f}'(u)\mathrm{d}u\right)^r\right|\mathrm{d}t \\
\le& r\int_{(i-1)/n}^{t_{n,i}} \left|\int_t^{t_{n,i}} \left(f'(u)-\widetilde{f}'(u)\right)\mathrm{d}u\right| \left(\left(\int_t^{t_{n,i}} f'(u)\mathrm{d}u\right)^{r-1} + \left(\int_t^{t_{n,i}} \widetilde{f}'(u)\mathrm{d}u\right)^{r-1}\right)\mathrm{d}t \\
\le& 2r\int_{(i-1)/n}^{t_{n,i}} \left|\int_t^{t_{n,i}} \left(f'(u)-\widetilde{f}'(u)\right)\mathrm{d}u\right| \left(\int_t^{t_{n,i}} \left(f'(u)+\widetilde{f}'(u)\right)\mathrm{d}u\right)^{r-1}\mathrm{d}t \\
\le& 2r(a_i^-)^{1/r}(b_i^-)^{(r-1)/r},
\end{aligned}$$

where, using Hölder's inequality again

$$a_i^- = \int_{(i-1)/n}^{t_{n,i}} \left|\int_t^{t_{n,i}} \left(f'(u)-\widetilde{f}'(u)\right)\mathrm{d}u\right|^r \mathrm{d}t \le \frac{1}{rn^r}\int_{(i-1)/n}^{t_{n,i}} \left|f'(t)-\widetilde{f}'(t)\right|^r\mathrm{d}t,$$

$$b_i^- = \int_{(i-1)/n}^{t_{n,i}} \left(\int_t^{t_{n,i}} \left(f'(u)+\widetilde{f}'(u)\right)\mathrm{d}u\right)^r \mathrm{d}t \le \frac{1}{rn^r}\int_{(i-1)/n}^{t_{n,i}} \left(f'(t)+\widetilde{f}'(t)\right)^r\mathrm{d}t.$$

By a completely analogous argument,

$$\int_{t_{n,i}}^{i/n} \left|\left(\int_{t_{n,i}}^{t} f'(u)\mathrm{d}u\right)^r - \left(\int_{t_{n,i}}^{t} \widetilde{f}'(u)\mathrm{d}u\right)^r\right|\mathrm{d}t \le 2r\left(a_i^+\right)^{1/r}\left(b_i^+\right)^{(r-1)/r}, \ \forall\ 1\le i\le n,$$

where

$$a_i^+ = \int_{t_{n,i}}^{i/n} \left|\int_{t_{n,i}}^{t} \left(f'(u)-\widetilde{f}'(u)\right)\mathrm{d}u\right|^r \mathrm{d}t \le \frac{1}{rn^r}\int_{t_{n,i}}^{i/n} \left|f'(t)-\widetilde{f}'(t)\right|^r\mathrm{d}t,$$

$$b_i^+ = \int_{t_{n,i}}^{i/n} \left(\int_{t_{n,i}}^{t} \left(f'(u)+\widetilde{f}'(u)\right)\mathrm{d}u\right)^r \mathrm{d}t \le \frac{1}{rn^r}\int_{t_{n,i}}^{i/n} \left(f'(t)+\widetilde{f}'(t)\right)^r\mathrm{d}t.$$

In summary, therefore,

$$\begin{aligned}
&n^r\left|d_r\left(\delta_{z_n}^{u_n},\mu\right)^r - d_r\left(\delta_{\widetilde{z}_n}^{u_n},\widetilde{\mu}\right)^r\right| \le 2rn^r\sum_{i=1}^n\left(\left(a_i^-\right)^{1/r}\left(b_i^-\right)^{(r-1)/r} + \left(a_i^+\right)^{1/r}\left(b_i^+\right)^{(r-1)/r}\right) \\
\le& 2rn^r\left(\sum_{i=1}^n\left(a_i^- + a_i^+\right)\right)^{1/r}\left(\sum_{i=1}^n\left(b_i^- + b_i^+\right)\right)^{(r-1)/r} \\
\le& 2rn^r\left(\frac{1}{rn^r}\int_{\mathbb{I}}\left|f'(t)-\widetilde{f}'(t)\right|^r\mathrm{d}t\right)^{1/r}\left(\frac{1}{rn^r}\int_{\mathbb{I}}\left(f'(t)+\widetilde{f}'(t)\right)^r\mathrm{d}t\right)^{(r-1)/r} \\
=& 2\left\|f'-\widetilde{f}'\right\|_r\left\|f'+\widetilde{f}'\right\|_r^{r-1},
\end{aligned}$$

which is just (2.21).

Step 3. To establish (2.20) in case the value on the right is $+\infty$, assume that $f' \notin L^r(\mathbb{I})$. For $N \in \mathbb{N}$, let $g_N = \min\{f', N\}$ and, given $C > 0$, choose N so large that $\|g_N\|_r^r \geq 2^r(1+r)C$. Let μ_N be a probability measure with $\left(F_{\mu_N}^{-1}\right)' = g_N$. By (2.23),

$$d_r\left(\delta_{z_n}^{u_n}, \mu\right)^r \geq \sum_{i=1}^n \left(\int_{(i-1)/n}^{t_i} \left(\int_t^{t_i} g_N(u)\mathrm{d}u\right)^r \mathrm{d}t + \int_{t_i}^{i/n} \left(\int_{t_i}^t g_N(u)\mathrm{d}u\right)^r \mathrm{d}t\right) \geq d_r\left(\delta_\bullet^{u_n}, \mu_N\right)^r,$$

and since Step 2 applies to μ_N,

$$\liminf_{n\to\infty} n^r d_r\left(\delta_\bullet^{u_n}, \mu\right)^r \geq \lim_{n\to\infty} n^r d_r\left(\delta_\bullet^{u_n}, \mu_N\right)^r = \frac{1}{2^r(r+1)}\|g_N\|_r^r \geq C.$$

As $C > 0$ was arbitrary, $n^r d_r\left(\delta_\bullet^{u_n}, \mu\right)^r \to +\infty$ whenever $f' \notin L^r(\mathbb{I})$, i.e., (2.20) is valid in this case also.

Step 4. Finally, to prove the assertion regarding asymptotically best uniform approximations, assume that $f' \in L^r(\mathbb{I})$. Note that $\|f'\|_r > 0$ whenever $\mu \neq \delta_a$ for all $a \in \mathbb{R}$. In this case, given $\varepsilon > 0$, pick $\widetilde{\mu} \in \mathcal{P}_r$ such that $\widetilde{f} = F_{\widetilde{\mu}}^{-1}$ has a C^1-extension to $\mathbb{I}$ and $\left\|f' - \widetilde{f}'\right\|_r < \varepsilon$. By Step 1, $\lim_{n\to\infty} n^r d_r\left(\delta_{\widetilde{x}_n}^{u_n}, \widetilde{\mu}\right)^r = \frac{\|\widetilde{f}'\|_r^r}{2^r(r+1)}$, whereas Step 2 guarantees that $\lim_{n\to\infty} n^r d_r\left(\delta_\bullet^{u_n}, \mu\right)^r = \frac{\|f'\|_r^r}{2^r(r+1)}$, and (2.21) yields

$$\left|n^r d_r\left(\delta_{\widetilde{x}_n}^{u_n}, \widetilde{\mu}\right)^r - n^r d_r\left(\delta_{\widetilde{x}_n}^{u_n}, \mu\right)^r\right| \leq 2\varepsilon\left(1 + 2\|f'\|_r\right)^{r-1}.$$

Combining these three facts leads to

$$\begin{aligned}\limsup_{n\to\infty} \frac{d_r\left(\delta_{\widetilde{x}_n}^{u_n}, \mu\right)^r}{d_r\left(\delta_\bullet^{u_n}, \mu\right)^r} &\leq \limsup_{n\to\infty} \frac{n^r d_r\left(\delta_\bullet^{u_n}, \mu\right)^r + 2\varepsilon\left(1 + 2\left\|f'\right\|_r\right)^{r-1}}{2^{-r}\left\|f'\right\|_r^r (r+1)^{-1}} \\ &\leq \left(1 + \frac{\varepsilon}{\|f'\|_r}\right)^r + 2^{r+1}(r+1)\varepsilon\frac{\left(1 + 2\left\|f'\right\|_r\right)^{r-1}}{\|f'\|_r^r},\end{aligned}$$

as well as to an analogous lower bound for $\liminf_{n\to\infty} \frac{d_r(\delta_{\widetilde{x}_n}^{u_n},\mu)^r}{d_r(\delta_\bullet^{u_n},\mu)^r}$. Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} \frac{d_r(\delta_{\widetilde{x}_n}^{u_n},\mu)}{d_r(\delta_\bullet^{u_n},\mu)} = 1$, i.e., $\left(\delta_{\widetilde{x}_n}^{u_n}\right)$ is a sequence of asymptotically best uniform r-approximations of μ, as claimed. □

The following examples highlight the importance of the absolute continuity and integrability assumptions, respectively, in Theorem 2.39.

Example 2.40. (See Subsection 3.6.1 for details.) Let μ be the inverse of the Cantor probability measure in Example 2.34. Explicitly, μ is purely atomic, with $\mu\left(\{j2^{-m}\}\right) = 3^{-m}$ for every $m \in \mathbb{N}$ and every odd $1 \leq j \leq 2^m$. Note that $F_\mu^{-1}\big|_{]0,1[}$ simply equals the classical Cantor function, hence is continuous, in fact, $\frac{\log 2}{\log 3}$-Hölder, and $\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda} = 0$ a.e.. While Theorem 2.39, if it did apply, would seem to suggest that $\lim_{n\to\infty} n d_r\left(\delta_\bullet^{u_n}, \mu\right) = 0$, a detailed but elementary analysis shows that this is not the case. In fact, $\left(n^\alpha d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ may not converge to a finite positive limit for any $r \ge 1$ and $\alpha > 0$. More specifically, let $\alpha_r = \frac{1}{r} + \left(1 - \frac{1}{r}\right)\frac{\log 2}{\log 3}$ for $r \ge 1$. With this, $3^{\alpha_r} d_r\left(\delta_\bullet^{u_{3n}}, \mu\right) = d_r\left(\delta_\bullet^{u_n}, \mu\right)$ for all n, and hence

$$d_r\left(\delta_{1/2}, \mu\right) \cdot \frac{1}{4}\left(\frac{2}{9}\right)^{1/r} \le \liminf_{n\to\infty} n^{\alpha_r} d_r\left(\delta_\bullet^{u_n}, \mu\right) = \inf_{n\in\mathbb{N}} n^{\alpha_r} d_r\left(\delta_\bullet^{u_n}, \mu\right),$$

as well as

$$\limsup_{n\to\infty} n^{\alpha_r} d_r\left(\delta_\bullet^{u_n}, \mu\right) = \sup_{n\in\mathbb{N}} n^{\alpha_r} d_r(\delta_\bullet^{u_n}, \mu) \le 2^{1/r}.$$

For $r = 1$, for instance, $\alpha_1 = 1$, and

$$d_1\left(\delta_\bullet^{u_1}, \mu\right) = 1/6, \ d_1\left(\delta_\bullet^{u_2}, \mu\right) = 2/15,$$

whereas for $r = 2$, $\alpha_2 = \frac{1}{2}\left(1 + \frac{\log 2}{\log 3}\right)$, and

$$d_2\left(\delta_\bullet^{u_1}, \mu\right) = \frac{1}{2\sqrt{5}}, \ d_2\left(\delta_\bullet^{u_2}, \mu\right) = \frac{1}{3\sqrt{5}},$$

which shows that $\left(n^{\alpha_r} d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ is not constant when $r = 1, 2$, and hence divergent. (It is conjectured that $\left(n^{\alpha_r} d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ is divergent for every $r \ge 1$.) This illustrates that the conclusion of Theorem 2.39 may fail if μ^{-1} is not absolutely continuous.

Example 2.41. (See Subsection 3.3.3 for details.) The integrability assumption also is crucial (for the second assertion) in Theorem 2.39. To see this, consider μ as in Examples 2.26 and 2.32, where $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \notin L^1(\mathbb{I})$, and (2.20) yields $\lim_{n\to\infty} n d_r\left(\delta_{\bullet,r}^{u_n}, \mu\right) = +\infty$ for all $r \ge 1$, in perfect agreement with earlier observations. Deduce from a short calculation that

$$\sqrt{n} d_2\left(\delta_{\bullet,1}^{u_n}, \mu\right) = D_2 + \mathcal{O}\left(n^{-1}\right) \text{ as } n \to \infty,$$

where $D_2^2 = 1 + 2\sum_{i=1}^\infty \left(1 - i\left(1 + \log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right) \approx 1.1749$. Recall from Example 2.32 that $\sqrt{n} d_2\left(\delta_{\bullet,2}^{u_n}, \mu\right) = C_2 + \mathcal{O}(n^{-1})$ with $C_2^2 \approx 1.0803$. Thus while $\left(\delta_{\bullet,1}^{u_n}\right)$ identifies the correct rate of decay for $\left(d_2\left(\delta_{\bullet,2}^{u_n}, \mu\right)\right)$, namely $\left(n^{-1/2}\right)$, it is *not* a sequence of asymptotically best uniform 2-approximations of μ, since

$$\lim_{n\to\infty} \frac{d_2\left(\delta_{\bullet,1}^{u_n}, \mu\right)}{d_2\left(\delta_{\bullet,2}^{u_n}, \mu\right)} = \frac{D_2}{C_2} > 1.$$

Similarly, for any $r > 1$ it can be shown that $\lim_{n\to\infty} n^{1/r} d_r\left(\delta_{\bullet,1}^{u_n}, \mu\right) = D_r$ with the appropriate constant $D_r > C_r$, and C_r as in Example 2.32.

Remark 2.42. For any (non-degenerate) $\mu \in \mathcal{P}$, [11, Prop.A.17] asserts that μ^{-1} is absolutely continuous if and only if $\operatorname{supp}\mu$ is connected and $\frac{\mathrm{d}\mu_a}{\mathrm{d}\lambda} > 0$ a.e. on $\operatorname{supp}\mu$, where μ_a is the absolutely continuous part (w.r.t. λ) of μ; in this case, moreover, $\int_{\mathbb{I}} \left(\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda}\right)^r = \int_{\operatorname{supp}\mu} \left(\frac{\mathrm{d}\mu_a}{\mathrm{d}\lambda}\right)^{1-r}$.

If F_μ^{-1} not even is continuous, then the decay of $(d_r(\delta_\bullet^{u_n}, \mu))$ may be less homogeneous than in Example 2.40. For instance, for the Cantor measure of Example 2.34, for any $r \geq 1$, both numbers

$$\liminf_{n\to\infty} n^{\frac{\log 3}{\log 2}} d_r(\delta_\bullet^{u_n}, \mu) \quad \text{and} \quad \limsup_{n\to\infty} n^{1/r} d_r(\delta_\bullet^{u_n}, \mu)$$

are finite and positive; for verification of this statement, see Section 3.5.1. Thus, in general it cannot be expected that for some $\alpha_r > 0$, the sequence $(n^{\alpha_r} d_r(\delta_\bullet^{u_n}, \mu))$ is bounded below and above by positive constants, let alone convergent. Still, it is possible to identify a universal lower bound for $(d_r(\delta_\bullet^{u_n}, \mu))$ with $\mu \in \mathcal{P}_r$: But for trivial exceptions, this sequence never decays faster than (n^{-1}).

Theorem 2.43. *Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$. Then*

$$\limsup_{n\to\infty} n d_r(\delta_\bullet^{u_n}, \mu) > 0, \tag{2.24}$$

unless $\mu = \delta_a$ for some $a \in \mathbb{R}$.

Proof. Denote F_μ^{-1} by f for convenience, and for every $n \in \mathbb{N}$, let $a_i = f\left(\frac{2i-1}{4n}\right)$ and $b_i = f\left(\frac{2i-1}{4n+2}\right)$ for $1 \leq i \leq 2n$. Then $b_1 \leq a_1 \leq b_2 \leq a_2 \cdots \leq b_{2n} \leq a_{2n} \leq b_{2n+1}$, and

$$\begin{aligned}
&2n d_1(\delta_\bullet^{u_{2n}}, \delta_\bullet^{u_{2n+1}}) \\
&= 2n \sum_{i=1}^{2n} \left((a_i - b_i)\left(\frac{i}{2n+1} - \frac{i-1}{2n}\right) + (b_{i+1} - a_i)\left(\frac{i}{2n} - \frac{i}{2n+1}\right)\right) \\
&= \frac{1}{2n+1} \sum_{i=1}^{2n} \left((2n+1-i)(a_i - b_i) + i(b_{i+1} - a_i)\right) \\
&\geq \frac{1}{2n+1} \left\{ \sum_{i=1}^{n} i(b_{i+1} - b_i) + \sum_{i=n+1}^{2n} \left((2n+1-2i)(a_i - b_i) + i(b_{i+1} - b_i)\right) \right\} \\
&= \frac{1}{2n+1} \left\{ \sum_{i=1}^{n} i(b_{i+1} - b_i) + \sum_{i=n+1}^{2n} \left((2n+1-i)(b_{i+1} - b_i) + (2i-2n-1)(b_{i+1} - a_i)\right) \right\} \\
&\geq \frac{1}{2n+1} \left\{ \sum_{i=1}^{n} i(b_{i+1} - b_i) + \sum_{i=n+1}^{2n} (2n+1-i)(b_{i+1} - b_i) \right\} \\
&= \sum_{i=n+2}^{2n+1} \frac{b_i}{2n+1} - \sum_{i=1}^{n} \frac{b_i}{2n+1}.
\end{aligned}$$

Since f is locally Riemann integrable on $]0,1[$, it follows that

$$\limsup_{n\to\infty} 2nd_1\left(\delta_\bullet^{u_{2n}}, \delta_\bullet^{u_{2n+1}}\right) \geq \int_0^{1/2} \left(f\left(t+\frac{1}{2}\right) - f(t)\right) \mathrm{d}t,$$

and consequently

$$\begin{aligned}\limsup_{n\to\infty} nd_r\left(\delta_\bullet^{u_n}, \mu\right) \geq \limsup_{n\to\infty} nd_1\left(\delta_\bullet^{u_n}, \mu\right) \geq \frac{1}{2}\limsup_{n\to\infty} 2nd_1\left(\delta_\bullet^{u_{2n}}, \delta_\bullet^{u_{2n+1}}\right)\\ \geq \frac{1}{2}\int_0^{1/2} \left(f\left(t+\frac{1}{2}\right) - f(t)\right) \mathrm{d}t > 0\end{aligned}$$

unless f is constant, i.e., unless $\mu = \delta_a$ for some $a \in \mathbb{R}$. $\square$

It is natural to ask whether Theorem 2.43 has a counterpart in that there also exists a universal *upper* bound on $\left(d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$. In general, this is not the case: As an immediate consequence of Theorem 2.56 below, given $r \geq 1$ and any sequence (a_n) of positive real numbers with $\lim_{n\to\infty} a_n = 0$, there exists $\mu \in \mathcal{P}_r$ such that $d_r\left(\delta_\bullet^{u_n}, \mu\right) \geq a_n$, for all $n \in \mathbb{N}$. Under additional assumptions, however, an upper bound on $\left(d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ can be established.

Theorem 2.44. *Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$.*
(i) *If $\mu \in \mathcal{P}_s$ with $s > r$ then $\lim_{n\to\infty} n^{1/r-1/s} d_r\left(\delta_\bullet^{u_n}, \mu\right) = 0$.*
(ii) *If $\operatorname{supp}\mu$ is bounded then $\limsup_{n\to\infty} n^{1/r} d_r\left(\delta_\bullet^{u_n}, \mu\right) < +\infty$.*

Proof. Again, for convenience, let $f = F_\mu^{-1}$, and $x_{n,i} = f\left(\frac{2i-1}{2n}\right)$ for all $n \in \mathbb{N}$ and $1 \leq i \leq n$. With $t_0 = F_\mu(0)$, assume w.o.l.g. that $0 < t_0 < 1$. (The cases $t_0 = 0$ and $t_0 = 1$ are completely analogous.) Recall that f is non-decreasing and right-continuous, $(t - t_0) f(t) \geq 0$ for all $t \in \mathbb{I}$, and $0 \leq f(t_0)$, $-f(t_0-) < +\infty$. For all sufficiently large n, therefore,

$$\begin{aligned}
&d_r\left(\delta_\bullet^{u_n}, \mu\right)^r \leq d_r\left(\delta_{x_n}^{u_n}, \mu\right)^r = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{2i-1}{2n}} \left((x_{n,i} - f(t))^r + \left(f\left(t + \frac{1}{2n}\right) - x_{n,i}\right)^r \right) \mathrm{d}t\\
&\leq \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{2i-1}{2n}} \left(f\left(t+\frac{1}{2n}\right) - f(t)\right)^r \mathrm{d}t \leq \int_{\frac{1}{4n}}^{1-\frac{1}{4n}} \left(f\left(t+\frac{1}{4n}\right) - f\left(t - \frac{1}{4n}\right)\right)^r \mathrm{d}t\\
&= \int_{\frac{1}{4n}}^{t_0 - \frac{1}{4n}} \left(\left|f\left(t - \frac{1}{4n}\right)\right| - \left|f\left(t+\frac{1}{4n}\right)\right|\right)^r \mathrm{d}t\\
&\quad + \int_{t_0-\frac{1}{4n}}^{t_0+\frac{1}{4n}} \left(f\left(t+\frac{1}{4n}\right) + \left|f\left(t-\frac{1}{4n}\right)\right|\right)^r \mathrm{d}t + \int_{t_0+\frac{1}{4n}}^{1-\frac{1}{4n}} \left(f\left(t+\frac{1}{4n}\right) - f\left(t-\frac{1}{4n}\right)\right)^r \mathrm{d}t\\
&\leq \int_0^{t_0-\frac{1}{2n}} |f(t)|^r \mathrm{d}t - \int_{\frac{1}{2n}}^{t_0} |f(t)|^r \mathrm{d}t + 2^{r-1}\int_{t_0-\frac{1}{2n}}^{t_0+\frac{1}{2n}} |f(t)|^r \mathrm{d}t + \int_{t_0+\frac{1}{2n}}^1 |f(t)|^r \mathrm{d}t - \int_{t_0}^{1-\frac{1}{2n}} |f(t)|^r \mathrm{d}t\\
&= a_n + \left(2^{r-1} - 1\right) b_n,
\end{aligned}$$

where the numbers $a_n,\ b_n$ are given by

$$a_n = \int_0^{\frac{1}{2n}} |f(t)|^r \,\mathrm{d}t + \int_{1-\frac{1}{2n}}^1 |f(t)|^r \,\mathrm{d}t \qquad \text{and} \qquad b_n = \int_{t_0-\frac{1}{2n}}^{t_0+\frac{1}{2n}} |f(t)|^r \,\mathrm{d}t,$$

respectively. Note that

$$0 \le nb_n \le \max\left\{f\left(t_0 + \frac{1}{2n}\right), -f\left(t_0 - \frac{1}{2n}\right)\right\}^r \to \max\left\{f(t_0), -f(t_0-)\right\}^r \text{ as } n \to \infty,$$

and hence (nb_n) is bounded.

(i) If $\mu \in \mathcal{P}_s$ for some $s > r$ then, by virtue of Hölder's inequality,

$$0 \le a_n \le \left(\left(\int_0^{\frac{1}{2n}} |f(t)|^s \,\mathrm{d}t\right)^{r/s} + \left(\int_{1-\frac{1}{2n}}^1 |f(t)|^s \,\mathrm{d}t\right)^{r/s}\right) 2^{r/s} n^{r/s-1},$$

which shows that $\lim_{n\to\infty} n^{1-r/s} a_n = 0$. It follows that

$$0 \le n^{1-r/s} d_r\left(\delta_\bullet^{u_n}, \mu\right)^r \le n^{1-r/s} a_n + \left(2^{r-1} - 1\right) n^{1-r/s} b_n \to 0 \text{ as } n \to \infty,$$

and hence $\lim_{n\to\infty} n^{1/r-1/s} d_r\left(\delta_\bullet^{u_n}, \mu\right) = 0$, as claimed.

(ii) If $\operatorname{supp}\mu$ is bounded then $\operatorname{esssup}_{\mathbb{I}}|f|$ is finite. In this case, (na_n) is bounded, and so is $\left(n^{1/r} d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$. □

Remark 2.45. (i) Boundedness of $\operatorname{supp}\mu$ is essential in Theorem 2.44(ii), as evidenced, e.g., by Example 2.41 for $r = 1$. Notice, however, that the conclusion of Theorem 2.44(ii) remains valid in this example whenever $r > 1$.

(ii) If $\operatorname{supp}\mu$ is disconnected, and hence F_μ^{-1} is discontinuous at some $0 < t < 1$, then there exists (n_k) such that $\langle n_k t\rangle \in [1/3, 2/3]$ for all k. For all sufficiently large k, therefore,

$$\begin{aligned} n_k d_r\left(\delta_\bullet^{u_{n_k}}, \mu\right)^r &\ge n_k \min_{c\in\mathbb{R}} \int_{\lfloor n_k t\rfloor/n_k}^{(\lfloor n_k t\rfloor+1)/n_k} \left|F_\mu^{-1}(s) - c\right|^r \mathrm{d}s \\ &\ge \min_{c\in\left[F_\mu^{-1}(t-), F_\mu^{-1}(t)\right]} \frac{1}{3}\left(\left(F_\mu^{-1}(t) - c\right)^r + \left(c - F_\mu^{-1}(t-)\right)^r\right) \\ &\ge \frac{2^{1-r}}{3}\left(F_\mu^{-1}(t) - F_\mu^{-1}(t-)\right)^r. \end{aligned}$$

Hence (2.24) can be strengthened to $\limsup_{n\to\infty} n^{1/r} d_r\left(\delta_\bullet^{u_n}, \mu\right) > 0$ whenever $\operatorname{supp}\mu$ is disconnected. In fact, by Theorem 2.44(ii), $\left(n^{-1/r}\right)$ is the sharp upper rate of $\left(d_r\left(\delta_\bullet^{u_n}, \mu\right)\right)$ in case $\operatorname{supp}\mu$ is bounded and disconnected, a situation observed for instance for the Cantor measure of Example 2.34.

2.4.3 Best approximations

This final subsection relates the results presented earlier to the classical theory of best (unconstrained) approximations. Let $\mu \in \mathcal{P}_r$ for some $r \geq 1$. Given $n \in \mathbb{N}$, call the probability measure δ_x^p with $x \in \Xi_n$ and $p \in \Pi_n$ a *best r-approximation* of μ if

$$d_r\left(\delta_x^p, \mu\right) \leq d_r\left(\delta_y^q, \mu\right), \ \forall\, y \in \Xi_n, \ q \in \Pi_n.$$

Denote by $\delta_\bullet^{\bullet,n}$ any best r-approximation of μ. (As before, the dependence of $\delta_\bullet^{\bullet,n}$ on r is made explicit by a subscript only where necessary to avoid ambiguities.) It is well known that best r-approximations exist always.

Proposition 2.46. [40, Sec.4.1].*Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$. For every $n \in \mathbb{N}$, there exists a best r-approximation $\delta_\bullet^{\bullet,n}$ of μ. If $\#\mathrm{supp}\,\mu \geq n$ then $\#\mathrm{supp}\,\delta_\bullet^{\bullet,n} = n$.*

By combining Proposition 2.46 with Theorems 2.25 and 2.29, a description of all best r-approximations is easily established.

Theorem 2.47. *Assume that $\mu \in \mathcal{P}_r$ for some $r \geq 1$, and $n \in \mathbb{N}$. Let δ_x^p with $x \in \Xi_n, \ p \in \Pi_n$ be a best r-approximation of μ. Then, for every $i = 1, \cdots, n$,*
(i) *$x_{,i} < x_{,i+1}$ implies $P_{,i} \in Q^{F_\mu^{-1}}_{\frac{1}{2}(x_{,i}+x_{,i+1})}$; and*
(ii) *$P_{,i-1} < P_{,i}$ implies $x_{,i} \in Q^{F_\mu}_{\frac{1}{2}(P_{,i-1}+P_{,i})}$ if $r = 1$, or $x_{,i} = \tau_r^{f_i}$ with $f_i = F_\mu^{-1}\big|_{[P_{,i-1},P_{,i}]}$ if $r > 1$.*
Moreover, if $\#\mathrm{supp}\,\mu \leq n$ then $\delta_x^p = \mu$, whereas if $\#\mathrm{supp}\,\mu > n$ then $x_{,i} < x_{,i+1}$ and $P_{,i-1} < P_{,i}$ for all $i = 1, \cdots, n$.

Proof. Note that δ_x^p is both a best r-approximation of μ, given p, and a best r-approximation of μ, given x, and thus conclusions (i) and (ii) follow directly from Theorems 2.25 and 2.29, respectively. For the non-trivial case where $\#\mathrm{supp}\,\mu > n$, Proposition 2.46 implies that $\#\mathrm{supp}\,\delta_x^p = n$, or equivalently, $x_{,i} < x_{,i+1}$ and $P_{,i-1} < P_{,i}$ for all $i = 1, \cdots, n$. ☐

As an important special case of Theorem 2.47, assume that $\mu \in \mathcal{P}_r$ is continuous. Then $Q_a^{F_\mu^{-1}}$ is a singleton for every $a \in \mathbb{R}$, and Theorem 2.47 asserts that every best 1-approximation δ_x^p of μ satisfies

$$F_\mu\left(\frac{x_{,i} + x_{,i+1}}{2}\right) = P_{,i}, \text{ and } F_\mu(x_{,i}) = \frac{P_{,i-1} + P_{,i}}{2}, \ \forall\, i = 1, \cdots, n,$$

and hence in particular

$$2F_\mu(x_{,i}) = F_\mu\left(\frac{x_{,i-1} + x_{,i}}{2}\right) + F_\mu\left(\frac{x_{,i} + x_{,i+1}}{2}\right), \ \forall\, i = 1, \cdots, n. \tag{2.25}$$

Similarly, every best 2-approximation of μ satisfies

$$F_\mu\left(\frac{x_{,i}+x_{,i+1}}{2}\right) = P_{,i}, \text{ and } \left(P_{,i}-P_{,i-1}\right)x_{,i} = \int_{P_{,i-1}}^{P_{,i}} F_\mu^{-1}(t)\mathrm{d}t, \ \forall\ i = 1, \cdots, n,$$

and consequently

$$x_{,i}F_\mu\left(\frac{x_{,i}+x_{,i+1}}{2}\right) - x_{,i}F_\mu\left(\frac{x_{,i-1}+x_{,i}}{2}\right) = \int_{\frac{1}{2}(x_{,i-1}+x_{,i})}^{\frac{1}{2}(x_{,i}+x_{,i+1})} x\mathrm{d}F_\mu(x), \ \forall\ i = 1, \cdots, n. \tag{2.26}$$

Note that (2.25) and (2.26) each yield n equations for $x_{,1}, \cdots, x_{,n}$. These equations are exactly the classical optimality conditions, derived, e.g., in [40, Sec.5.2] by means of Voronoi partitions.

Example 2.48. (See Subsection 3.7 for details.) Let $\mu = \frac{1}{2}\lambda\big|_{[0,1]} + \frac{1}{2}\delta_1$. While μ is not continuous, and hence not directly amenable to the classical conditions (2.25) and (2.26), Theorem 2.47 applies and yields, for instance, $\delta_{\bullet,r}^{\bullet,2} = \xi(r)\delta_{\xi(r)} + \left(1-\xi(r)\right)\delta_{3\xi(r)}$ for all $r \geq 1$, where $r \mapsto \xi(r)$ is smooth, decreasing, with

$$\xi(1) = \frac{1}{3}, \ \xi(2) = \frac{3-\sqrt{3}}{4}, \text{ and } \lim_{r\to+\infty}\xi(r) = \frac{1}{4}.$$

If (i) and (ii) in Theorem 2.47 identify only a single probability measure δ_x^p then the latter clearly is a best r-approximation. In general, however, and unlike in Theorems 2.25 and 2.29, the conditions of Theorem 2.47 are not sufficient, as the following example shows. Moreover, best r-approximations in general are not unique, not even when $r > 1$.

Example 2.49. (See Subsection 3.8 for details.) Consider $\mu = \frac{1}{3}\lambda_{[-1,1]} + \frac{1}{3}\delta_0$ and let $n = 2$. For $r = 1$, Theorem 2.47 identifies exactly three potential best 1-approximations $\delta_{x_j}^{p_j}$, $j = 1, 2, 3$, namely

$$x_1 = \left(-\frac{2}{3}, 0\right), \ p_1 = \left(\frac{2}{9}, \frac{7}{9}\right),$$
$$x_2 = \left(-\frac{1}{4}, \frac{1}{4}\right), \ p_2 = \left(\frac{1}{2}, \frac{1}{2}\right),$$
$$x_3 = \left(0, \frac{2}{3}\right), \ p_3 = \left(\frac{7}{9}, \frac{2}{9}\right).$$

It is clear from

$$d_1\left(\delta_{x_1}^{p_1}, \mu\right) = d_1\left(\delta_{x_3}^{p_3}, \mu\right) = \frac{2}{9} < \frac{7}{24} = d_1\left(\delta_{x_2}^{p_2}, \mu\right)$$

that the two (non-symmetric) probability measures $\delta_{x_1}^{p_1}$, $\delta_{x_3}^{p_3}$ are best 1-approximations of μ, whereas the (symmetric) $\delta_{x_2}^{p_2}$ is not. Similarly, for $r = 2$, Theorem 2.47 yields three candidates $\delta_{y_j}^{q_j}$,

$$y_1 = \left(\frac{1-\sqrt{33}}{8}, \frac{19-3\sqrt{33}}{8}\right), \ q_1 = \left(\frac{9-\sqrt{33}}{12}, \frac{3+\sqrt{33}}{12}\right),$$

$$y_2 = \left(-\frac{1}{3}, \frac{1}{3}\right), \; q_2 = \left(\frac{1}{2}, \frac{1}{2}\right),$$
$$y_3 = \left(\frac{3\sqrt{33}-19}{8}, \frac{\sqrt{33}-1}{8}\right), \; q_3 = \left(\frac{3+\sqrt{33}}{12}, \frac{9-\sqrt{33}}{12}\right),$$

where again only $\delta_{y_1}^{q_1}$ and $\delta_{y_3}^{q_3}$ turn out to be best 2-approximations of μ.

Since $d_r(\delta_\bullet^{\bullet,n}, \mu) \le d_r(\delta_\bullet^{u_n}, \mu)$ for every $\mu \in \mathcal{P}_r$ and $n \in \mathbb{N}$, it is clear that $\lim_{n\to\infty} d_r(\delta_\bullet^{\bullet,n}, \mu) = 0$. The rate of convergence of $(d_r(\delta_\bullet^{\bullet,n}, \mu))$ has been, and continues to be studied extensively; see, e.g., [40, 44, 60, 61, 64, 79] and the references therein. Arguably the simplest situation occurs if $\mu \in \mathcal{P}_r$ has a non-trivial absolutely continuous part and satisfies a mild moment condition. In this case, $(d_r(\delta_\bullet^{\bullet,n}, \mu))$ decays like (n^{-1}) for every r.

Proposition 2.50. [40, Thm.6.2].*Assume that $\mu \in \mathcal{P}_r$ for some $r \ge 1$. If $\mu \in \mathcal{P}_s$ with $s > r$ then*

$$\lim_{n\to\infty} n d_r(\delta_\bullet^{\bullet,n}, \mu) = \frac{1}{2(r+1)^{1/r}} \left(\int_{\mathbb{R}} \left(\frac{d\mu_a}{d\lambda}\right)^{\frac{1}{r+1}} \right)^{\frac{r+1}{r}},$$

where μ_a is the absolutely continuous part (w.r.t. λ) of μ.

It is instructive to compare Proposition 2.50 to Theorem 2.39. To do so, assume that $\mu \in \mathcal{P}_s$ for some $s > r$ and that μ^{-1} is absolutely continuous. Then $\lim_{n\to\infty} n d_r(\delta_\bullet^{\bullet,n}, \mu)$ and $\lim_{n\to\infty} n d_r(\delta_\bullet^{u_n}, \mu)$ both are finite and positive, provided that μ is non-singular and $\frac{d\mu^{-1}}{d\lambda} \in L^r(\mathbb{I})$. Thus $(d_r(\delta_\bullet^{\bullet,n}, \mu))$ and $(d_r(\delta_\bullet^{u_n}, \mu))$ exhibit the same rate of decay, namely (n^{-1}). Note that while the latter rate is a universal *upper* bound on $(d_r(\delta_\bullet^{\bullet,n}, \mu))$, at least under the mild assumption that $\mu \in \mathcal{P}_s$ for some $s > r$, it is a universal *lower* bound on $(d_r(\delta_\bullet^{u_n}, \mu))$, by Theorem 2.43. Even if both sequences decay at the same rate, however, $\lim_{n\to\infty} n d_r(\delta_\bullet^{\bullet,n}, \mu) \le \lim_{n\to\infty} n d_r(\delta_\bullet^{u_n}, \mu)$, and equality holds only if either $\mu = \frac{1}{\lambda(I)} \lambda|_I$ for some bounded, non-degenerate interval $I \subset \mathbb{R}$ or else $\mu = \delta_a$ for some $a \in \mathbb{R}$. Thus only in the trivial case of a (possibly degenerate) uniform distribution μ does $(\delta_\bullet^{u_n})$ provide a sequence of asymptotically best r-approximations of μ (as defined below).

Example 2.51. (See Subsection 3.3.4 for details.) Let μ be the exponential distribution of Example 2.26. For $r = 1$ and every $n \in \mathbb{N}$, (2.25) identifies a unique best 1-approximation $\delta_{x_n}^{p_n}$, with

$$x_{n,i} = -2\log\frac{n+1-i}{\sqrt{n(n+1)}}, \; P_{n,i} = \frac{i(2n+1-i)}{n(n+1)}, \; \forall\, i = 1, \cdots, n.$$

Here $\delta_\bullet^{\bullet,n}$ is unique, and

$$n d_1(\delta_\bullet^{\bullet,n}, \mu) = n \log\left(1 + \frac{1}{n}\right) = 1 + \mathcal{O}\left(n^{-1}\right) \text{ as } n \to \infty,$$

in agreement with Proposition 2.50. For comparison, recall from Example 2.32 that $\lim_{n\to\infty} \frac{n}{\log n}\, d_1\left(\delta_{\bullet}^{u_n}, \mu\right) = \frac{1}{4}$. For $r > 1$, no explicit expression seems to be known for $\delta_{\bullet}^{\bullet,n}$, not even for $r = 2$. However, in a sense made precise below, $\left(\delta_{y_n}^{\widetilde{p}_n}\right)$ with

$$y_{n,i} = (r+1)\log\frac{n+1}{n-i+1}, \quad \widetilde{P}_{n,i} = 1 - \left(\frac{(n+1-i)(n-i)}{(n+1)^2}\right)^{\frac{r+1}{2}}, \quad \forall\, i = 1, \cdots, n$$

yields a sequence of asymptotically best r-approximations of μ for any $r > 1$.

Example 2.51 illustrates that even in very simple situations it may be difficult to compute $\delta_{\bullet}^{\bullet,n}$ explicitly. Not least from a computational point of view, therefore, it is natural to seek r-approximations that at least are optimal *asymptotically*. Specifically, call $\left(\delta_{x_n}^{p_n}\right)$ with $x_n \in \Xi_n$, $p_n \in \Pi_n$ for all $n \in \mathbb{N}$ a sequence of *asymptotically best r-approximations* of $\mu \in \mathcal{P}_r$ with $\#\operatorname{supp}\mu = \infty$, if

$$\lim_{n\to\infty} \frac{d_r\left(\delta_{x_n}^{p_n}, \mu\right)}{d_r\left(\delta_{\bullet}^{\bullet,n}, \mu\right)} = 1.$$

There exists a large literature on asymptotically best approximations. Specifically, mild conditions (such as $\mu \in \mathcal{P}_r$ being absolutely continuous with $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ Hölder continuous and positive on $\mathring{\operatorname{supp}}\,\mu$, among others) have been established which guarantee that $\left(\delta_{x_n}^{\bullet}\right)$ is a sequence of asymptotically best approximations of μ, where

$$x_{n,i} = F_{\mu_r}^{-1}\left(\frac{i}{n+1}\right), \quad \forall\, i = 1, \cdots, n \tag{2.27}$$

with $\frac{\mathrm{d}\mu_r}{\mathrm{d}\lambda} = \frac{\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}^{\frac{1}{r+1}}}{\int_{\mathbb{R}} \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}^{\frac{1}{r+1}}}$; see, e.g., [66,85] and the references therein.

Example 2.52. (See Subsection 3.4.3 for details.) Let $\mu = \operatorname{Beta}(2,1)$ as in Examples 2.28 and 2.38. While for arbitrary $n \in \mathbb{N}$ the author does not know of an explicit expression for $\delta_{\bullet}^{\bullet,n}$ for any $r \geq 1$, (2.27) yields a sequence $\left(\delta_{x_n}^{p_n}\right)$ of asymptotically best r-approximations of μ, with

$$x_{n,i} = \left(\frac{i}{n+1}\right)^{\frac{r+1}{r+2}}, \quad P_{n,i} = \frac{1}{4(n+1)^{\frac{2(r+1)}{r+2}}}\left(i^{\frac{r+1}{r+2}} + (i+1)^{\frac{r+1}{r+2}}\right)^2, \quad \forall\, i = 1, \cdots, n-1,$$

and $x_{n,n} = \left(\frac{n}{n+1}\right)^{\frac{r+1}{r+2}}$. For instance, for $r = 2$ this specializes to

$$x_{n,i} = \left(\frac{i}{n+1}\right)^{3/4}, \quad P_{n,i} = \frac{1}{4(n+1)^{3/2}}\left(i^{3/4} + (i+1)^{3/4}\right)^2, \quad \forall\, i = 1, \cdots, n-1,$$

and $x_{n,n} = \left(\frac{n}{n+1}\right)^{3/4}$, and a short calculation yields

$$nd_2\left(\delta_{x_n}^{p_n}, \mu\right) = \frac{3}{8\sqrt{2}} + \mathcal{O}\left(n^{-1}\right) \quad \text{as } n \to \infty,$$

which is consistent with Proposition 2.50.

If $\mu \in \mathcal{P}_s$ with $s > r$ is singular then Proposition 2.50 only yields $\lim_{n\to\infty} nd_r\left(\delta_\bullet^{\bullet,n}, \mu\right) = 0$. The detailed analysis of $\left(d_r\left(\delta_\bullet^{\bullet,n}, \mu\right)\right)$ in this case is an active research area, for which already a substantial literature exists, notably for important classes of singular probabilities such as self-similar and -conformal measures; see, e.g., [40, 41, 60, 83, 84, 88]. A key notion in this context is the so-called *quantization dimension* of $\mu \in \mathcal{P}_r$ of order r, defined as

$$D_r(\mu) = \lim_{n\to\infty} \frac{\log n}{-\log d_r\left(\delta_\bullet^{\bullet,n}, \mu\right)},$$

provided that this limit exists. For instance, Proposition 2.50 implies that $D_r(\mu) = 1$ whenever $\mu_a \neq 0$. The relations of $D_r(\mu)$ to various other concepts of dimension have attracted considerable attention [40, 60, 83, 88].

Example 2.53. For the Cantor measure μ of Example 2.34, [64, Cor.4.7,Rem.6.1] show that, for every $r > 1$,

$$0 < \liminf_{n\to\infty} n^{\log 3/\log 2} d_r(\delta_\bullet^{\bullet,n}, \mu) < \limsup_{n\to\infty} n^{\log 3/\log 2} d_r\left(\delta_\bullet^{\bullet,n}, \mu\right) < +\infty.$$

From this, it is clear that $D_r(\mu) = \frac{\log 2}{\log 3}$, which is independent of r and coincides with the Hausdorff dimension of $\operatorname{supp}\mu$.

Example 2.54. (See Subsection 3.6.2 for details.) Let μ be the inverse Cantor measure of Example 2.40. Note that μ is not a self-similar, and hence the classical results for self-similar probabilities do not apply. Still, μ is the unique fixed point of a contraction on $\mathcal{P}_1$, namely $\nu \mapsto \frac{1}{3}\left(\nu \circ T_1^{-1} + \delta_{1/2} + \nu \circ T_2^{-1}\right)$, with the similarities $T_1(x) = \frac{1}{2}x$ and $T_2(x) = \frac{1}{2}(1+x)$. This property enables a fairly complete analysis of $\left(d_r\left(\delta_\bullet^{\bullet,n}, \mu\right)\right)$ which will be presented elsewhere. Specifically, with $\beta_r = \left(1 - \frac{1}{r}\right) + \frac{1}{r}\frac{\log 3}{\log 2}$, it can be shown that, for every $r \geq 1$, the numbers $\liminf_{n\to\infty} n^{\beta_r} d_r\left(\delta_\bullet^{\bullet,n}, \mu\right)$ and $\limsup_{n\to\infty} n^{\beta_r} d_r\left(\delta_\bullet^{\bullet,n}, \mu\right)$ both are finite and positive. In particular, $D_r(\mu) = \beta_r^{-1}$. Note that, unlike in the previous example, $D_r(\mu)$ depends on r, and $\frac{\log 2}{\log 3} \leq D_r(\mu) < 1$. Thus $D_r(\mu)$ is larger than 0, the Hausdorff dimension of μ, but smaller than 1, the Hausdorff dimension of $\operatorname{supp}\mu = \mathbb{I}$.

Proposition 2.50 guarantees that under a mild moment condition, $\left(d_r(\delta_\bullet^{\bullet,n}, \mu)\right)$ decays at least like $\left(n^{-1}\right)$, and in fact may decay faster, as Examples 2.53 and 2.54 illustrate. Even for purely atomic μ, however, the decay of $(d_r(\delta_{\bullet}^{\bullet,n},\mu))$ can be arbitrarily slow. This final observation, a refinement of [40, Ex.6.4], uses the following simple calculus fact; cf. also [11, Thm.3.3].

Proposition 2.55. *Given any sequence (a_n) of real numbers with $\lim_{n\to\infty} a_n = 0$, there exists a decreasing sequence (b_n) with $\lim_{n\to\infty} b_n = 0$ such that $(b_n - b_{n+1})$ is decreasing also, and $b_n \geq a_n$ for all n.*

Theorem 2.56. *Given $r \geq 1$ and any sequence (a_n) of non-negative real numbers with $\lim_{n\to\infty} a_n = 0$, there exists $\mu \in \mathcal{P}_r$ such that $d_r(\delta_{\bullet}^{\bullet,n},\mu) \geq a_n$ for every $n \in \mathbb{N}$.*

Proof. In view of Proposition 2.55, assume w.o.l.g. that (a_n) and $\left(a_n^r - a_{n+1}^r\right)$ both are decreasing. Pick $a_0 > a_1$ such that $a_0^r - a_1^r > a_1^r - a_2^r$, and let $c_r = \sum_{k=1}^{\infty} 2^{-(k-1)r}\left(a_{k-1}^r - a_k^r\right)$. Note that c_r is finite and positive. Consider $\mu = \sum_{k=1}^{\infty} p_k \delta_{x_k}$, where $p_k = c_r^{-1} 2^{-(k-1)r}\left(a_{k-1}^r - a_k^r\right)$ and $x_k = 3 \cdot 2^{k-1} c_r^{1/r}$ for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} p_k x_k^r = 3^r a_0^r < +\infty$, clearly $\mu \in \mathcal{P}_r$. For every $n \in \mathbb{N}$, define $K_n \subset \mathbb{N}$ as

$$K_n = \left\{k \in \mathbb{N} : \operatorname{supp} \delta_{\bullet}^{\bullet,n} \ \cap\ \left[2^k c_r^{1/r}, 2^{k+1} c_r^{1/r}\right[= \varnothing\right\}.$$

Since $\#\operatorname{supp} \delta_{\bullet}^{\bullet,n} \leq n$ and the intervals $\left\{\left[2^k c_r^{1/r}, 2^{k+1} c_r^{1/r}\right[\right\}$, $k \in \mathbb{N}$, are disjoint, $\#\left(\mathbb{N} \backslash K_n\right) \leq n$. Moreover,

$$\min_{y \in \operatorname{supp} \delta_{\bullet}^{\bullet,n}} |x_k - y|^r \geq 2^{(k-1)r} c_r \text{ for every } k \in K_n.$$

Recall from [18, (ii),p.1847] that $d_r(\delta_{\bullet}^{\bullet,n},\mu)^r = \int_{\mathbb{R}} \min_{y \in \operatorname{supp} \delta_{\bullet}^{\bullet,n}} |x - y|^r \mathrm{d}\mu(x)$; see also [40, Lem.3.1]. It follows that, for every $n \in \mathbb{N}$,

$$d_r(\delta_{\bullet}^{\bullet,n},\mu)^r = \sum_{k=1}^{\infty} p_k \min_{y \in \operatorname{supp} \delta_{\bullet}^{\bullet,n}} |x_k - y|^r \geq \sum_{k \in K_n} p_k 2^{(k-1)r} c_r = \sum_{k \in K_n} \left(a_{k-1}^r - a_k^r\right).$$

Moreover, recall that $\left(a_{n-1}^r - a_n^r\right)$ is decreasing, and $\#\left(\mathbb{N} \setminus K_n\right) \leq n$. Thus

$$d_r(\delta_{\bullet}^{\bullet,n},\mu)^r \geq \sum_{k=n+1}^{\infty} \left(a_{k-1}^r - a_k^r\right) = a_n^r,$$

and hence $d_r(\delta_{\bullet}^{\bullet,n},\mu) \geq a_n$ for every $n \in \mathbb{N}$. $\square$

Chapter 3

Supplements to Chapter 2

In this chapter, we mainly provide supplementary details omitted in Chapter 2. Section 3.1 provides elementary proofs of propositions as well as justifications of the statements in the remarks of Chapter 2. For the reader's convenience, we will restate the propositions before proceeding to their proofs. Sections 3.2-3.6 are devoted to computational details regarding remarks and examples in Chapter 2.

3.1 Proofs of propositions

In this section, we give proofs of propositions in Chapter 2. First, we establish some basic properties of the quantile functions. Recall that $I \subset \overline{\mathbb{R}}$ denotes a non-degenerate closed interval throughout.

Proposition (2.1). *Let $f : I \to \overline{\mathbb{R}}$ be non-decreasing. Then f^{-1} is non-decreasing, right-continuous and, on $f(I)$, coincides with the ordinary inverse of f whenever f is one-to-one. Moreover, $(f_{\pm})^{-1} = f^{-1}$ on $\overline{\mathbb{R}}$ and $\left(f^{-1}\right)^{-1}(x) = f(x+)$ for all $x \in \mathring{I}$; in particular, therefore, $\left(f^{-1}\right)^{-1}$ equals f a.e. on $\mathring{I}$, and in fact everywhere if f is right-continuous.*

Proof. We first show that f^{-1} is non-decreasing and right-continuous. Since

$$\{f \le t\} \subset \{f \le s\}, \quad \forall\, t < s,$$

f^{-1} is non-decreasing. Suppose by way of contradiction that $f^{-1}(t_*) \neq f_+^{-1}(t_*)$ for some $t_* \in \overline{\mathbb{R}}$, i.e., $f^{-1}(t_*) < f_+^{-1}(t_*)$. Then for all sufficiently small $\varepsilon_i > 0$ for $i = 1, 2$,

$$f^{-1}(t_*) + \varepsilon_1 < f^{-1}(t_* + \varepsilon_2),$$

which implies that $f\left(f^{-1}(t_*) + \varepsilon_1\right) \le t_* + \varepsilon_2$. Letting $\varepsilon_2 \downarrow 0$ yields $f\left(f^{-1}(t_*) + \varepsilon_1\right) \le t_*$, which implies by the definition of f^{-1} that $f^{-1}(t_*) + \varepsilon_1 \le f^{-1}(t_*)$, a contradiction. This shows the right-continuity of f^{-1}. Moreover, since f is one-to-one, $\{x \in I : f(x) = t\}$ is a singleton. Hence $\sup\{x \in I : f(x) = t\} = \sup\{x \in I : f(x) \le t\} = f^{-1}(t)$, i.e., f^{-1} coincides with the ordinary inverse of f on $f(I)$.

Next we verify $(f_\pm)^{-1} = f^{-1}$: On the one hand, note that for every $t \in \overline{\mathbb{R}}$, since $f \le f_+$,

$$(f_+)^{-1}(t) = \sup\{x \in \mathbb{R} : f_+(x) \le t\} \le f^{-1}(t).$$

On the other hand, given any $\varepsilon > 0$,

$$f\left((f_+)^{-1}(t) + 2\varepsilon\right) \ge f_+\left((f_+)^{-1}(t) + \varepsilon\right) > t,$$

which implies that $(f_+)^{-1}(t) + 2\varepsilon > f^{-1}(t)$. Since $\varepsilon > 0$ was arbitrary, $(f_+)^{-1}(t) \ge f^{-1}(t)$. Thus $(f_+)^{-1}(t) = f^{-1}(t)$. Analogously, one can show that $(f_-)^{-1} = f^{-1}$.

Finally, we prove $\left(f^{-1}\right)^{-1} = f_+$: Fix $x \in \mathbb{R}$, then

$$\begin{aligned}\left(f^{-1}\right)^{-1}(x) &= \left(f_+^{-1}\right)^{-1}(x) = \sup\left\{t \in \mathbb{R} : f_+^{-1}(t) \le x\right\} \\ &= \sup\left\{t \in \mathbb{R} : \sup\{y \in \mathbb{R} : f_+(y) \le t\} \le x\right\}.\end{aligned} \tag{3.1}$$

Note that given any $\varepsilon > 0$, it follows from (3.1) that $f_+(x) - \varepsilon \le \left(f^{-1}\right)^{-1}(x)$. Since f_+ is non-decreasing and right-continuous,

$$x < \sup\{y \in \mathbb{R} : f_+(y) \le f_+(x) + \varepsilon\},$$

which implies that $\left(f^{-1}\right)^{-1}(x) \le f_+(x) + \varepsilon$. Hence $\left(f^{-1}\right)^{-1}(x) = f_+(x)$, as $\varepsilon > 0$ was arbitrary. □

Next, we present the proof for the asserted properties of the quantile sets.

Proposition (2.2). *Let $f : I \to \overline{\mathbb{R}}$ be non-decreasing. Then, for every $t \in \overline{\mathbb{R}}$, the set Q_t^f is a non-empty, compact (possibly one-point) subinterval of I, and $f(x) = t$ whenever $\min Q_t^f < x < \max Q_t^f$. Moreover, the following hold:*

(i) *If $t < u$, then $x \le y$ for every $x \in Q_t^f$ and $y \in Q_u^f$, and the set $Q_t^f \bigcap Q_u^f$ contains at most one point.*

(ii) *For every $x \in I$ and $t \in \overline{\mathbb{R}}$, $x \in Q_t^f$ if and only if $t \in Q_x^{f^{-1}}$.*

Proof. Fix $t \in \overline{\mathbb{R}}$, let $a = \inf\{x \in I : f(x) \ge t\}$ and $b = \sup\{x \in I : f(x) \le t\}$. For all $x > b$, $f(x) > t$ and thus $x \ge a$. Since $x > b$ was arbitrary, $b \ge a$. This shows Q_t^f is non-empty. By definition, Q_t^f is closed and thus compact in I. For all $x > \min Q_t^f$, $f(x) \ge t$. Analogously, for all $x < \max Q_t^f$, $f(x) \le t$. Thus $f(x) = t$ if $\min Q_t^f < x < \max Q_t^f$.

(i) For all $x \in Q_t^f$, $y \in Q_u^f$,

$$f(x - \varepsilon) \le t < u \le f(y + \varepsilon), \ \forall\, \varepsilon > 0. \tag{3.2}$$

Thus $x \le y$. (Otherwise, if $x > y$, then $f\left(x - \frac{x-y}{2}\right) = f\left(y + \frac{x-y}{2}\right)$, a contradiction by choosing $\varepsilon = \frac{x-y}{2}$ in (3.2)). Hence $\max Q_t^f \le \min Q_u^f$, i.e., $Q_t^f \cap Q_u^f$ contains at most one point.
(ii) Note that $x \in Q_t^f$ if and only if

$$f(x-\varepsilon) \le t \le f(x+\varepsilon), \ \forall\, \varepsilon > 0. \tag{3.3}$$

If $x \in Q_t^f$, by the definition of f^{-1}, $x - \varepsilon \le f^{-1}(t)$. Since $\varepsilon > 0$ was arbitrary, $f^{-1}(t) \ge x$, i.e., $t \ge \min Q_x^{f^{-1}}$. On the other hand, we claim that

$$f^{-1}(t-\sigma) \le x, \ \forall\, \sigma > 0.$$

To see this, note that otherwise there exists $\sigma_0 > 0$ such that $f^{-1}(t-\sigma_0) > x$, and thus there exists $\varepsilon_0 > 0$ such that $x + \varepsilon_0 < f^{-1}(t-\sigma_0)$, yielding $f(x+\varepsilon_0) \le t - \sigma_0 < t$. This contradicts (3.3) with $\varepsilon = \varepsilon_0$. Conversely, if $t \in Q_x^{f^{-1}}$,

$$f^{-1}(t-\varepsilon) \le x \le f^{-1}(t+\varepsilon), \ \forall\, \varepsilon > 0.$$

By the definition of inverse function,

$$f(x+\sigma) > t - \varepsilon, \ f(x-\sigma) \le t + \varepsilon, \ \forall\, \sigma > 0.$$

Since ε were arbitrary,

$$f(x-\sigma) \le t \le f(x+\sigma), \ \forall\, \sigma > 0,$$

and so, by (3.3), $x \in Q_t^f$. □

Now we prove some elementary properties of the auxiliary function ℓ_f. Recall that

$$\ell_f(t) = \frac{1}{2}\left(\min I + \max I + \lambda(\{f < t\}) - \lambda(\{f > t\})\right). \tag{3.4}$$

Proposition (2.4). *Let I be a bounded interval and $f : I \to \overline{\mathbb{R}}$ a measurable function. Assume that f is finite a.e.. Then the following hold:*
(i) *ℓ_f is non-decreasing;*
(ii) *For every $t \in \mathbb{R}$, $\ell_f(t\pm) = \ell_f(t) \pm \frac{1}{2}\lambda(\{f = t\})$, and hence ℓ_f is continuous at t if and only if $\lambda(\{f = t\}) = 0$. Moreover, $\lambda\left(\left\{\ell_f^{-1} < t\right\} \cap I\right) = \lambda(\{f < t\})$ and $\lambda\left(\left\{\ell_f^{-1} > t\right\} \cap I\right) = \lambda(\{f > t\})$;*
(iii) $\lim_{t\to-\infty} \ell_f(t) = \ell_f(-\infty) = \min I$ *and* $\lim_{t\to+\infty} \ell_f(t) = \ell_f(+\infty) = \max I$;

(iv) *If f is non-decreasing then*

$$\ell_f(t) = \frac{1}{2}\left(f^{-1}(t) + f^{-1}(t-)\right),\ \forall\ t \in \mathbb{R},$$

and also

$$\ell_f^{-1}(x) = \left(f^{-1}\right)^{-1}(x) = f(x+),\ \ell_f^{-1}(x-) = f(x-),\ \forall\ x \in \mathring{I};$$

(v) *If $f \in L^r(I)$ for some $1 \le r < +\infty$, then $\left\|\ell_f^{-1} - t\right\|_r = \|f - t\|_r$ for every $t \in \mathbb{R}$.*

Proof. (i) The result follows directly from the monotonicity of $\lambda(\{f > t\})$ and $\lambda(\{f < t\})$.
(ii) This follows directly from

$$\ell_f(t+) = \frac{1}{2}\left(\inf I + \sup I + \lambda(\{f \le t\}) - \lambda(\{f > t\})\right)$$

and

$$\ell_f(t-) = \frac{1}{2}\left(\inf I + \sup I + \lambda(\{f < t\}) - \lambda(\{f \ge t\})\right).$$

Next, we show $\lambda\left(\left\{\ell_f^{-1} < t\right\} \cap I\right) = \lambda\left(\{f < t\}\right)$. It suffices to show

$$\lambda\left(\left\{\ell_f^{-1} < t\right\} \cap I\right) = \ell_f(t-) - \min I,$$

i.e.,

$$]\inf I, \ell_f(t-)[\subset \left\{\ell_f^{-1} < t\right\} \cap I \subset [\inf I, \ell_f(t-)].$$

On the one hand, $\forall\ x \in\ ]\inf I, \ell_f(t-)[$, by the definition of ℓ_f^{-1}, $t > \ell_f^{-1}(x)$. On the other hand, $\forall\ x > \ell_f(t-)$, again by the definition of ℓ_f^{-1}, $t \le \ell^{-1}(x)$. Note that $\lambda\left(\left\{\ell_f^{-1} > t\right\} \cap I\right) = \lambda\left(\{f > t\}\right)$ can be proved analogously.
(iii) By continuity of λ,

$$\lambda\left(\{f < -\infty\}\right) = 0 \quad \text{and} \quad \lambda\left(\{f > -\infty\}\right) = \lambda(I) = \max I - \min I,$$

yielding $\lim_{t\to-\infty} \ell_f(t) = \ell_f(-\infty) = \min I$. The other statement can be proved analogously.
(iv) Since f is non-decreasing, by the definition of f^{-1},

$$f^{-1}(t) = \lambda\left(\{f \le t\}\right) = \max I - \min I - \lambda\left(\{f > t\}\right), \quad f^{-1}(t-) = \lambda\left(\{f < t\}\right).$$

Thus

$$\ell_f(t) = \frac{1}{2}\left(f^{-1}(t) + f^{-1}(t-)\right),\ \forall\ t \in \mathbb{R}$$

follows from (3.4). To establish the remaining identities, it suffices to prove

$$\ell_f^{-1}(x) = f^{-1}\left(f^{-1}(x)\right), \; \forall\, x \in \mathring{I}, \tag{3.5}$$

and the other part follows directly from a limiting argument. To see (3.5), note that $\ell_f(t+) = f^{-1}(t)$. By the definition of inverse function, it suffices to show

$$\sup\left\{t \in \overline{\mathbb{R}} : \; \ell_f(t) \leq x\right\} = \sup\left\{t \in \overline{\mathbb{R}} : \; \ell_f(t+) \leq x\right\}.$$

By (i),

$$\ell_f(t+) \geq \ell_f(t), \; \forall\, t \in \overline{\mathbb{R}},$$

and thus

$$\sup\left\{t \in \overline{\mathbb{R}} : \; \ell_f(t) \leq x\right\} \geq \sup\left\{t \in \overline{\mathbb{R}} : \; \ell_f(t+) \leq x\right\}.$$

To show the reverse inequality, note that $\ell_f(t) \leq x$ implies that $\ell_f((t-\varepsilon)+) \leq x$, $\forall\, \varepsilon > 0$. Thus $\sup\left\{t \in \overline{\mathbb{R}} : \; \ell_f(t) \leq x\right\} \leq \sup\left\{t \in \overline{\mathbb{R}} : \; \ell_f(t+) \leq x\right\} + \varepsilon$. Since ε was arbitrary, the reverse inequality holds.

(v). By (ii) and

$$\|g\|_r^r = r\int_0^\infty s^{r-1}\lambda(\{|g| > s\})\mathrm{d}s, \; g \in L^r(I),$$

letting $g = f - t$ yields

$$\begin{aligned}\|f-t\|_r^r =& r\int_0^\infty s^{r-1}\left(\lambda(\{f-t > s\}) + \lambda(\{f-t < -s\})\right)\mathrm{d}s \\ =& r\int_0^\infty s^{r-1}\left(\lambda\left(\left\{\ell_f^{-1} - t > s\right\}\right) + \lambda\left(\left\{\ell_f^{-1} - t < -s\right\}\right)\right)\mathrm{d}s = \left\|\ell_f^{-1} - t\right\|_r^r.\end{aligned}$$

□

The following two propositions address some properties of τ_r^f. First, the monotonicity of τ_r^f w.r.t. f is addressed.

Proposition (2.14). *Assume that* $f,\; g \in L^r(I)$ *for some* $r > 1$, *and* $f \leq g$. *Then* $\tau_r^f \leq \tau_r^g$, *and* $\tau_r^f = \tau_r^g$ *if and only if* $f = g$ *a.e..*

Proof. Recall that $\phi_r(t) = \|f - t\|_r$. To stress the dependence of ϕ_r on f, denote ϕ_r by $\phi_{f,r}$. By (2.4),

$$\int_{\left\{f > \tau_r^f\right\}} \left(f(x) - \tau_r^f\right)^{r-1}\mathrm{d}x - \int_{\left\{f < \tau_r^f\right\}} \left(\tau_r^f - f(x)\right)^{r-1}\mathrm{d}x = 0.$$

Then

$$\begin{aligned}
&\left.\frac{\mathrm{d}\phi^r_{g,r}(t)}{\mathrm{d}t}\right|_{t=\tau^f_r} = \int_{\left\{g<\tau^f_r\right\}} \left(\tau^f_r - g(x)\right)^{r-1} \mathrm{d}x - \int_{\left\{g>\tau^f_r\right\}} \left(g(x) - \tau^f_r\right)^{r-1} \mathrm{d}x \\
&\le \int_{\left\{g<\tau^f_r\right\}} \left(\tau^f_r - f(x)\right)^{r-1} \mathrm{d}x - \int_{\left\{g<\tau^f_r\right\}} (g(x) - f(x))^{r-1} \mathrm{d}x \\
&\quad - \int_{\left\{g>\tau^f_r\ge f\right\}} \left(g(x) - \tau^f_r\right)^{r-1} \mathrm{d}x - \int_{\left\{\tau^f_r<f\right\}} \left(g(x) - \tau^f_r\right)^{r-1} \mathrm{d}x \\
&\le \int_{\left\{g<\tau^f_r\right\}} \left(\tau^f_r - f(x)\right)^{r-1} \mathrm{d}x - \int_{\left\{g<\tau^f_r\right\}} (g(x) - f(x))^{r-1} \mathrm{d}x - \int_{\left\{g>\tau^f_r\ge f\right\}} \left(g(x) - \tau^f_r\right)^{r-1} \mathrm{d}x \\
&\quad - \int_{\left\{\tau^f_r<f\right\}} \left(f(x) - \tau^f_r\right)^{r-1} \mathrm{d}x - \int_{\left\{\tau^f_r<f\right\}} (g(x) - f(x))^{r-1} \mathrm{d}x \\
&= \int_{\left\{f<\tau^f_r\right\}} \left(\tau^f_r - f(x)\right)^{r-1} \mathrm{d}x - \int_{\left\{f<\tau^f_r\le g\right\}} \left(\tau^f_r - f(x)\right)^{r-1} \mathrm{d}x - \int_{\left\{\tau^f_r<f\right\}} \left(f(x) - \tau^f_r\right)^{r-1} \mathrm{d}x \\
&\quad - \int_{\left\{g<\tau^f_r\right\}\cup\left\{\tau^f_r<f\right\}} (g(x) - f(x))^{r-1} \mathrm{d}x - \int_{\left\{g>\tau^f_r\ge f\right\}} \left(g(x) - \tau^f_r\right)^{r-1} \mathrm{d}x \\
&= - \int_{\left\{g<\tau^f_r\right\}\cup\left\{\tau^f_r<f\right\}} (g(x) - f(x))^{r-1} \mathrm{d}x - \int_{\left\{f\le\tau^f_r\le g\right\}} \left(\tau^f_r - f(x)\right)^{r-1} \left(g - \tau^f_r\right) \mathrm{d}x \le 0,
\end{aligned}$$

and the equality holds if and only if $g = f$ on I a.e.. □

Next, the zeroth order (constant) approximation of τ^f_r is considered.

Proposition (2.17). *Assume that $f : I \to \mathbb{R}$ is measurable, and let $\xi = \frac{1}{2}(\min I + \max I)$. If, for some $a,\ b,\ c \in \mathbb{R}$,*

$$|f(x) - (ax + b)| \le c\,|x - \xi|\,,\ \forall\ x \in I, \tag{3.6}$$

then $f \in L^\infty(I)$, and $\left|\tau^f_r - f(\xi)\right| \le \frac{1}{2}c\lambda(I)$ for every $r > 1$.

Proof. Notice that $f(\xi) = a\xi + b$. Assume w.o.l.g. that f is non-decreasing. Otherwise consider ℓ^{-1}_f instead. It follows from (3.6) that $f \in L^\infty(I)$ and $|(f(x) - f(\xi)) - a(t - \xi)| \le c|x - \xi|$. Let $\widetilde{\xi}$ be such that $f(\xi-) \le \tau^f_r \le f(\widetilde{\xi})$. Suppose by way of contradiction that $\tau^f_r < f(\xi) - \frac{1}{2}c\lambda(I)$. Note that $\widetilde{\xi} < \xi$ by (3.6). Also note that $\frac{\mathrm{d}\phi^r_r(t)}{\mathrm{d}t}$ is increasing in t for every $r > 1$. Then, by (3.6), for every $r > 1$,

$$\begin{aligned}
0 &= \int_{\inf I}^{\widetilde{\xi}} \left(\tau^f_r - f(x)\right)^{r-1} \mathrm{d}x - \int_{\widetilde{\xi}}^{\sup I} \left(f(x) - \tau^f_r\right)^{r-1} \mathrm{d}x \\
&< \int_{\inf I}^{\widetilde{\xi}} \left(f(\xi) - f(x) - \frac{1}{2}c\lambda(I)\right)^{r-1} \mathrm{d}x - \int_{\xi}^{\sup I} \left(f(x) + \frac{1}{2}c\lambda(I) - f(\xi)\right)^{r-1} \mathrm{d}x \\
&\le \int_{\inf I}^{\widetilde{\xi}} \max\left\{(a + c)(\xi - x) - \frac{1}{2}c\lambda(I), 0\right\}^{r-1} \mathrm{d}x - \int_{\xi}^{\sup I} \max\left\{(a - c)(x - \xi) + \frac{1}{2}c\lambda(I), 0\right\}^{r-1} \mathrm{d}x
\end{aligned}$$

$$\leq \int_{\inf I}^{\xi} \max\left\{(a+c)(\xi - x) - \frac{1}{2}c\lambda(I), 0\right\}^{r-1} \mathrm{d}x - \int_{\xi}^{\sup I} \max\left\{(a+c)(x-\xi) - \frac{1}{2}c\lambda(I), 0\right\}^{r-1} \mathrm{d}x = 0,$$

a contradiction. Thus $\tau_r^f \geq f(\xi) - \frac{1}{2}c\lambda(I)$. Similarly, $\tau_r^f \leq f(\xi) + \frac{1}{2}c\lambda(I)$ and thus

$$\left|\tau_r^f - f(\xi)\right| \leq \frac{1}{2}c\lambda(I), \quad \forall\, r > 1.$$

□

Finally, a preparatory proposition on the construction of a decreasing sequence used in the last proof of Chapter 2 is verified.

Proposition (2.55). *Given any sequence (a_n) of real numbers with $\lim_{n\to\infty} a_n = 0$, there exists a decreasing sequence (b_n) with $\lim_{n\to\infty} b_n = 0$ such that $(b_n - b_{n+1})$ is decreasing also, and $b_n \geq a_n$ for all n.*

Proof. Assume w.o.l.g. that (a_n) is decreasing, and also that integers $1 \leq N_1 < N_2 \ldots < N_k$ as well as real numbers $b_1 > b_2 > \cdots > b_{N_k}$ have already been constructed such that

$$b_n \geq a_n,\ \forall\, n = 1, \ldots, N_k;\ b_{N_j} = a_{N_j},\ \forall\, j = 1, \ldots, k;\ b_1 - b_2 > b_2 - b_3 > \ldots > b_{N_k - 1} - b_{N_k}, \tag{3.7}$$

with $N_1 = 1$, $N_2 = 2$, and $b_1 = a_1$, $b_2 = a_2$, clearly (3.7) holds for $k = 2$. Letting

$$N_{k+1} = \min\left\{n \geq N_k : b_{N_k} - \frac{1}{2}\min\{b_{N_k} - a_{N_k+1}, b_{N_k - 1} - b_{N_k}\}\sqrt{n - N_k} < a_n\right\}$$

as well as

$$b_n = \begin{cases} b_{N_k} - \frac{1}{2}\min\left\{b_{N_k} - a_{N_k+1}, b_{N_k-1} - b_{N_k}\right\}\sqrt{n - N_k} & \text{if } n = N_k + 1, \ldots, N_{k+1} - 1, \\ a_{N_{k+1}} & \text{if } n = N_{k+1}, \end{cases}$$

it is readily confirmed that $N_{k+1} \geq N_k + 2$, and (3.7) holds with $k+1$ instead of k. By induction, therefore, the sequence (b_n) thus constructed has all desired properties. □

3.2 Details of Remark 2.10

Let $\phi_r(\xi) = \left\|f - 24 \cdot \mathbb{1}_{[\xi,5]}\right\|_r^r$, and $\operatorname{Argmin} \phi_r := \{\xi \in \mathbb{R} : \phi_r(\xi) \leq \phi_r(\eta),\ \forall \eta \in \mathbb{R}\}$. In this section, we show that

$$\operatorname{Argmin} \phi_r = \begin{cases} \{0, 2, 5\} & \text{if } r = 1, 2, \\ \{5\} & \text{if } 1 < r < 2, \\ \{0\} & \text{if } r > 2. \end{cases} \tag{3.8}$$

It is straightforward to verify that $\xi = 0, 2, 5$ are the only three local minimizers of ϕ_r. Thus it suffices to compare the three values: $\phi_r(0),\ \phi_r(2),\ \phi_r(5)$. Note that

$$\phi_r(0) = \phi_r(2) = 8^r + 16^r + 6^r + 2\cdot 15^r, \quad \phi_r(5) = 16^r + 8^r + 18^r + 2\cdot 9^r,$$

and hence

$$f(r) := (2\cdot 9^r)^{-1}\left(\phi_r(0) - \phi_r(5)\right) = \left(\frac{5}{3}\right)^r - 1 + \frac{1}{2}\left\{\left(\frac{2}{3}\right)^r - 2^r\right\}.$$

It is easy to deduce that $f(1) = f(2) = 0$. Hence, to verify (3.8), it suffices to prove

$$f(r) > 0, \quad \forall\ 1 < r < 2; \qquad f(r) < 0, \quad \forall\ r > 2. \tag{3.9}$$

Direct calculations yield

$$f'(r) = \frac{1}{2}\left(\frac{2}{3}\right)^r \log\frac{2}{3} + \left(\frac{5}{3}\right)^r \log\frac{5}{3} - \frac{1}{2}2^r\log 2.$$

Consider the auxiliary function $g(r) := 2^{1-r}f'(r) = \left(\frac{1}{3}\right)^r \log\frac{2}{3} + 2\left(\frac{5}{6}\right)^r\log\frac{5}{3} - \log 2$. Obviously, g is decreasing. If

$$g(1) > 0 > g(2), \tag{3.10}$$

then there exists $1 < r_* < 2$ such that f is increasing on $[1, r_*]$ while decreasing on $[r_*, +\infty[$, which yields (3.9). A straightforward calculation reveals that (3.10) indeed holds:

$$g(1) = \frac{1}{3}\log\frac{3125}{2916} > 0, \quad g(2) = \frac{1}{18}\log\frac{1220703125}{1224440064} < 0.$$

3.3 The exponential distribution

In this section, we provide computational details for statements on approximations of the standard exponential distribution μ. Denote $f = F_\mu^{-1}$ for convenience within this section.

3.3.1 Details of Example 2.26

Recall that $x_n = (1, 2, \cdots, n)/\sqrt{n} \in \Xi_n$ and $y_n = (0, 2, \cdots, 2n-2) \in \Xi_n$.

First, it follows from Theorem 2.25 that $\delta_{y_n}^{q_n}$ with $Q_{n,i} = 1 - e^{\frac{y_{n,i}+y_{n,i+1}}{2}} = 1 - e^{-(2i-1)}$ for all $1 \le i \le n-1$, or equivalently, $q_{n,i} = e^{-(2i-3)} - e^{-(2i-1)} = 2e^{-2(i-1)}\sinh 1$ for all $\le i \le n$, is the unique best r-approximation of μ given y_n, which simply yields

$$\delta^{\bullet}_{y_n} \to \nu := \left(1 - e^{-1}\right)\delta_0 + 2\sinh 1 \sum_{i=1}^{\infty} e^{-2i}\delta_{2i}.$$

It remains to study the asymptotics for $\delta^{p_n}_{x_n}$, the unique best r-approximation of μ given x_n (due to Theorem 2.25) with

$$P_{n,i} = F_\mu\left(\frac{2i+1}{2\sqrt{n}}\right) = 1 - e^{-(2i+1)/(2\sqrt{n})}, \quad \forall\, 1 \le i \le n-1.$$

By virtue of (2.2), i.e., the formula for d_r, straightforward calculations yield the following recursive relation between $d_r\left(\delta^{p_n}_{x_n}, \mu\right)^r$ and $d_{r+1}\left(\delta^{p_n}_{x_n}, \mu\right)^{r+1}$:

$$\begin{aligned}
d_r\left(\delta^{p_n}_{x_n}, \mu\right)^r &= \int_0^1 \left|f(t) - F^{-1}_{\delta^{p_n}_{x_n}}(t)\right|^r \mathrm{d}t = \sum_{i=1}^{n} \int_{1-e^{-\frac{2i-1}{2\sqrt{n}}}}^{1-e^{-\frac{2i+1}{2\sqrt{n}}}} \left|-\log(1-t) - \frac{i}{\sqrt{n}}\right|^r \mathrm{d}t \\
=& \sum_{i=1}^{n}\left\{\int_{1-e^{-\frac{2i-1}{2\sqrt{n}}}}^{1-e^{-\frac{i}{\sqrt{n}}}} \left(\frac{i}{\sqrt{n}} + \log(1-t)\right)^r \mathrm{d}t + \int_{1-e^{-\frac{i}{\sqrt{n}}}}^{1-e^{-\frac{2i+1}{2\sqrt{n}}}} \left(\frac{i}{\sqrt{n}} - \log(1-t)\right)^r \mathrm{d}t\right\} \\
=& \sum_{i=1}^{n}\left\{\int_{-e^{-\frac{i}{\sqrt{n}}}}^{e^{-\frac{2i-1}{2\sqrt{n}}}} \log\left(te^{\frac{i}{\sqrt{n}}}\right)^r \mathrm{d}t + \int_{1-e^{-\frac{2i+1}{2\sqrt{n}}}}^{1-e^{-\frac{i}{\sqrt{n}}}} \left(-\log\left(te^{\frac{i}{\sqrt{n}}}\right)\right)^r \mathrm{d}t\right\} \\
=& \sum_{i=1}^{n} e^{-\frac{i}{\sqrt{n}}}\left\{\int_1^{e^{\frac{1}{2\sqrt{n}}}} (\log t)^r \,\mathrm{d}t - \int_{e^{-\frac{1}{2\sqrt{n}}}}^{1} (-\log t)^r \,\mathrm{d}t\right\} \qquad (3.11) \\
=& \sum_{i=1}^{n} e^{-\frac{i}{\sqrt{n}}} \frac{1}{r+1}\left\{t(\log t)^{r+1}\Big|_1^{e^{\frac{1}{2\sqrt{n}}}} - \int_1^{e^{\frac{1}{2\sqrt{n}}}} (\log t)^{r+1}\,\mathrm{d}t - t(-\log t)^{r+1}\Big|_{-e^{\frac{1}{2\sqrt{n}}}}^{1}\right. \\
&\left. + \int_{e^{-\frac{1}{2\sqrt{n}}}}^{1} (-\log t)^{r+1}\,\mathrm{d}t\right\} \\
=& \frac{e^{-\frac{1}{\sqrt{n}}}\left(1 - e^{-\frac{n}{\sqrt{n}}}\right)}{1 - e^{-\frac{1}{\sqrt{n}}}}\left\{\frac{1}{r+1}\left(\left(e^{-\frac{1}{2\sqrt{n}}} + e^{\frac{1}{2\sqrt{n}}}\right)\left(\frac{1}{2\sqrt{n}}\right)^{r+1}\right)\right\} - \frac{1}{r+1} d_{r+1}\left(\delta^{p_n}_{x_n}, \mu\right)^{r+1} \\
=& \frac{e^{-\frac{1}{\sqrt{n}}}\left(1 - e^{-\sqrt{n}}\right)\left(e^{-\frac{1}{2\sqrt{n}}} + e^{\frac{1}{2\sqrt{n}}}\right)}{2^{r+1}(r+1)} \frac{\frac{1}{\sqrt{n}}}{1 - e^{-\frac{1}{\sqrt{n}}}}\left(\frac{1}{\sqrt{n}}\right)^r - \frac{1}{r+1} d_{r+1}\left(\delta^{p_n}_{x_n}, \mu\right)^{r+1} \\
=& \frac{1}{2^r(r+1)}\left(\frac{1}{n}\right)^{r/2} \frac{\left(1 - e^{-\sqrt{n}}\right)\left(e^{-\frac{1}{2\sqrt{n}}} + e^{-\frac{3}{2\sqrt{n}}}\right)}{2\sqrt{n}\left(1 - e^{-\frac{1}{\sqrt{n}}}\right)} - \frac{1}{r+1} d_{r+1}\left(\delta^{p_n}_{x_n}, \mu\right)^{r+1}.
\end{aligned}$$

With

$$\int_1^{e^{\frac{1}{2\sqrt{n}}}} (\log t)^{r+1}\,\mathrm{d}t - \int_{e^{-\frac{1}{2\sqrt{n}}}}^1 (-\log t)^{r+1}\,\mathrm{d}t \leq \frac{1}{2\sqrt{n}} \left(\int_1^{e^{\frac{1}{2\sqrt{n}}}} (\log t)^{r}\,\mathrm{d}t - \int_{e^{-\frac{1}{2\sqrt{n}}}}^1 (-\log t)^{r}\,\mathrm{d}t \right),$$

it easily follows from (3.11) that

$$d_{r+1}\left(\delta_{x_n}^{p_n}, \mu\right)^{r+1} \leq \frac{1}{2\sqrt{n}} d_r\left(\delta_{x_n}^{p_n}, \mu\right)^{r},$$

which further yields

$$d_r\left(\delta_{x_n}^{p_n}, \mu\right)^{r} = \frac{1}{2^r(r+1)} \left(\frac{1}{n}\right)^{r/2} \frac{\left(1-e^{-\sqrt{n}}\right)\left(e^{-\frac{1}{2\sqrt{n}}} + e^{-\frac{3}{2\sqrt{n}}}\right)}{2\sqrt{n}\left(1-e^{-\frac{1}{\sqrt{n}}}\right)} + o\left(d_r\left(\delta_{x_n}^{p_n}, \mu\right)^{r}\right).$$

Note that

$$\begin{aligned} \frac{\left(1-e^{-\sqrt{n}}\right)\left(e^{-\frac{1}{2\sqrt{n}}} + e^{-\frac{3}{2\sqrt{n}}}\right)}{2\sqrt{n}\left(1-e^{-\frac{1}{\sqrt{n}}}\right)} &= \frac{1}{2}\left(2 - \frac{2}{\sqrt{n}} + \mathcal{O}\left(n^{-1}\right)\right)\left(1 - \frac{1}{2\sqrt{n}} + \mathcal{O}\left(n^{-1}\right)\right) + \mathcal{O}\left(e^{-\sqrt{n}}\right) \\ &= 1 - \frac{3}{2\sqrt{n}} + \mathcal{O}\left(n^{-1}\right), \end{aligned}$$

from which it is clear that

$$\lim_{n\to\infty} \sqrt{n} d_r\left(\delta_{x_n}^{p_n}, \mu\right) = \frac{1}{2(r+1)^{1/r}}.$$

3.3.2 Details of Example 2.32

In this subsection, we present the asymptotics of $(d_r(\delta_\bullet^{u_n}, \mu))$ and $(d_r(\delta_\bullet^{p_n}, \mu))$, with

$$p_{n,i} = \frac{2^{i-1}}{2^n - 1} \quad \forall\ 1 \leq i \leq n,$$

for all $r \geq 1$. First, by Theorem 2.29, $\delta_{x_n}^{p_n} = \delta_\bullet^{p_n}$ is the unique best r-approximation of μ given p_n with

$$x_{n,n-i} = \begin{cases} F_\mu^{-1}\left(\frac{3\cdot 2^{-i-1} - 2^{-n}}{1-2^{-n}}\right) & \text{if } r = 1, \\ x_{n,n-i} = \tau_r^{F_\mu\big|_{[P_{n,n-i-1}, P_{n,n-i}]}} = \tau_r^{F_\mu\big|_{\left[\frac{2^{n-i-1}-1}{2^n-1}, \frac{2^{n-i}-1}{2^n-1}\right]}} = \tau_r^{F_\mu\big|_{\left[\frac{2^{-i-1}-2^{-n}}{1-2^{-n}}, \frac{2^{-i}-2^{-n}}{1-2^{-n}}\right]}} & \text{if } r > 1. \end{cases}$$

This yields

$$p_{n,n-i} = \frac{2^{n-i}-1}{2^n-1} = \frac{2^{-i}-2^{-n}}{1-2^{-n}} \to 2^{-i}$$

and

$$x_{n,n-i} \to \begin{cases} F_\mu^{-1}\left(3\cdot 2^{-i-1}\right) & \text{if } r=1, \\ \tau_r^{F_\mu\big|_{[2^{-i-1},2^{-i}]}} & \text{if } r>1. \end{cases}$$

Hence $\delta_\bullet^{p_n} \to \nu = \sum_{i=1}^\infty 2^{-i}\delta_{a_i}$ with $a_i = F_\mu^{-1}\left(3\cdot 2^{-i-1}\right)$ if $r=1$ and $a_i = \tau_r^{F_\mu\big|_{[2^{-i},2^{-i+1}]}}$ if $r>1$.

The remainder of this subsection is devoted to the asymptotics of $(d_r(\delta_\bullet^{u_n},\mu))$. Note that for all $r\geq 1$,

$$\begin{aligned} &\int_{(i-1)/n}^{i/n} \left|\log t - \tau_r^{\widetilde{f}|_{[(i-1)/n,i/n]}}\right|^r \mathrm{d}t = \min_{c\in\mathbb{R}} \int_{(i-1)/n}^{i/n} |\log t - c|^r\, \mathrm{d}t \\ =&\frac{1}{n}\min_{c\in\mathbb{R}}\int_{i-1}^{i} |\log t - \log n - c|^r\,\mathrm{d}t = \frac{1}{n}\int_{i-1}^{i}\left|\log t - \tau_r^{f|_{[i-1,i]}}\right|^r \mathrm{d}t. \end{aligned}$$

This implies for all $r\geq 1$,

$$\begin{aligned} d_r\left(\delta_\bullet^{u_n},\mu\right)^r =& \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left|-\log(1-t)+\tau_r^{f|_{[(i-1)/n,i/n]}}\right|^r \mathrm{d}t \\ =&\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left|\log t - \tau_r^{f|_{[(i-1)/n,i/n]}}\right|^r \mathrm{d}t = \frac{1}{n}\sum_{i=1}^{n}\int_{i-1}^{i}\left|\log t-\tau_r^{f|_{[i-1,i]}}\right|^r\mathrm{d}t, \end{aligned} \tag{3.12}$$

from which it follows that $\lim_{n\to\infty} n d_r\left(\delta_\bullet^{u_n},\mu\right)^r = \sum_{i=1}^{\infty}\int_{i-1}^{i}\left|\log t - \tau_r^{f|_{[i-1,i]}}\right|^r \mathrm{d}t =: C_r^r$. To obtain the finiteness of the positive constant C_r, it suffices to show the convergence of

$$D_r^r := \sum_{i=1}^{\infty}\int_{i-1}^{i}\left|\log t - \log\left(i-\frac{1}{2}\right)\right|^r \mathrm{d}t, \quad \forall\, r>1. \tag{3.13}$$

Note that then $C_r < D_r < +\infty$.

Indeed,

$$\begin{aligned} &\int_0^1 \left|\log t - \log\left(\frac{1}{2}\right)\right|^r \mathrm{d}t = \frac{1}{2}\int_0^2 |\log t|^r \mathrm{d}t \\ \overset{t\mapsto e^{-t}}{=\!=\!=}&\frac{1}{2}\int_{-\log 2}^{+\infty} e^{-t}|t|^r \mathrm{d}t = \frac{1}{2}\Gamma(r+1) + \frac{1}{2}\int_0^{\log 2} e^t t^r \mathrm{d}t < +\infty, \quad \forall\, r>1, \end{aligned}$$

and

$$\begin{aligned} &\int_{i-1}^{i}\left|\log t - \log\left(i-\frac{1}{2}\right)\right|^r \mathrm{d}t \leq \max\left\{\log\left(1+\frac{1}{2i-1}\right), -\log\left(1-\frac{1}{2i-1}\right)\right\}^r \\ =&\left\{\log\left(1+\frac{1}{2(i-1)}\right)\right\}^r \sim \frac{1}{2^r(i-1)^r}, \quad \forall\, i\geq 2, \end{aligned}$$

which implies that $D_r < +\infty$, as claimed. Having established the convergence of $(nd_r(\delta_\bullet^{u_n}, \mu))$ for all $r > 1$, we now consider the special cases $r = 1, 2$. First observe that

$$\int_{i-1}^{i} \left|\log t - \log\left(i - \frac{1}{2}\right)\right| \mathrm{d}t = \left(i - \frac{1}{2}\right) \int_{\frac{2i-2}{2i-1}}^{\frac{2i}{2i-1}} |\log t| \mathrm{d}t$$
$$= \left(i - \frac{1}{2}\right) \left\{ t(\log t - 1)\Big|_{1}^{\frac{2i}{2i-1}} - (t(\log t - 1)\Big|_{\frac{2i-2}{2i-1}}^{1} \right\} = i \log \frac{2i}{2i-1} + (i-1)\log\frac{2i-2}{2i-1},$$

which yields

$$\begin{aligned}
nd_1(\delta_\bullet^{u_n}, \mu) &= \sum_{i=1}^{n} \int_{i-1}^{i} \left|\log t - \log\left(i - \frac{1}{2}\right)\right| \mathrm{d}t \\
&= \sum_{i=1}^{n} \left\{ i \log\frac{2i}{2i-1} + (i-1)\log\frac{2i-2}{2i-1} \right\} \\
&= -2\sum_{i=1}^{n} i \log\frac{2i-1}{2i} - \sum_{i=1}^{n} \left\{ i \log\frac{2i}{2i-1} - (i-1)\log\frac{2i-2}{2i-1} \right\} \\
&= -2\sum_{i=1}^{n} i \log\frac{2i-1}{2i} - \left\{ n\log n - \sum_{i=1}^{n} \log\left(i - \frac{1}{2}\right) \right\} \\
&= -2\sum_{i=1}^{n} i \log\frac{2i-1}{2i} + \log\frac{(2n)!}{2^{2n} n! n^n}.
\end{aligned}$$

Applying *Stirling's formula*,

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}},$$

as well as the asymptotic expansion

$$i \log\frac{2i-1}{2i} \sim -\frac{1}{2} - \frac{1}{8i} + \mathcal{O}(i^{-2}),$$

it follows that

$$-2\sum_{i=1}^{n} i \log\frac{2i-1}{2i} + \log\frac{(2n)!}{2^{2n} n! n^n} = \frac{1}{4}\log n + \mathcal{O}(1)\,, \text{ as } n \to \infty.$$

Finally, we deal with the case $r = 2$. By Remark 2.30(ii) and (3.12),

$$nd_2\left(\delta_{x_n}^{u_n}, \mu\right)^2 = \sum_{i=1}^{n} \int_{i-1}^{i} \left(\log t - \int_{i-1}^{i} \log t \mathrm{d}t\right)^2 \mathrm{d}t = \sum_{i=1}^{n} \left\{ \int_{i-1}^{i} (\log t)^2 \mathrm{d}t - \left(\int_{i-1}^{i} \log t \mathrm{d}t\right)^2 \right\}$$
$$= \sum_{i=1}^{n} \left\{ t\left((\log t - 1)^2 + 1\right) \Big|_{i-1}^{i} - \left(t(\log t - 1)\Big|_{i-1}^{i}\right)^2 \right\}$$
$$= n + \sum_{i=1}^{n} \left\{ \left(i(\log i - 1)^2 - (i-1)\left(\log(i-1) - 1\right)^2\right) - \left(i\log i - (i-1)\log(i-1) - 1\right)^2 \right\}$$

$$
\begin{aligned}
&= n + n\left(\log n - 1\right)^2 - \sum_{i=1}^{n}\left\{\left(i\log i - (i-1)\log(i-1)\right)^2 - 2\left(i\log i - (i-1)\log(i-1)\right) + 1\right\}\\
&= n(\log n - 1)^2 + 2n\log n - \sum_{i=1}^{n-1}\left((i+1)\log(i+1) - i\log i\right)^2\\
&= n + \sum_{i=1}^{n-1}\left\{(i+1)\left(\log(i+1)\right)^2 - i\left(\log i\right)^2 - \left((i+1)\log(i+1) - i\log i\right)^2\right\}\\
&= n + \sum_{i=1}^{n-1}\left\{(i+1)(\log(i+1))^2 - i(\log i)^2 - (i+1)^2(\log(i+1))^2\right.\\
&\quad \left. - i^2(\log i)^2 + 2i(i+1)\log i\log(i+1)\right\}\\
&= n + \sum_{i=1}^{n-1} i(i+1)\left\{2\log i\log(i+1) - (\log i)^2 - (\log(i+1))^2\right\}\\
&= n - \sum_{i=1}^{n-1} i(i+1)\left(\log\frac{i}{i+1}\right)^2\\
&= n - \sum_{i=1}^{n-1} i(i+1)\left(\log\frac{i}{i+1}\right)^2 = 1 + \sum_{i=1}^{n-1}\left(1 - i(i+1)\left(\log\frac{i}{i+1}\right)^2\right).
\end{aligned}
$$

A straightforward but tedious calculation yields the elementary asymptotic expansion:

$$1 - i(i+1)\left(\log\frac{i}{i+1}\right)^2 = \mathcal{O}(i^{-2}),$$

which implies that

$$\sum_{i=n}^{\infty}\left\{1 - i(i+1)\left(\log\frac{i}{i+1}\right)^2\right\} = \mathcal{O}\left(\sum_{i=n}^{\infty}\frac{1}{i^2}\right) = \mathcal{O}\left(n^{-1}\right),$$

which again shows $C_2^2 = 1 + \sum_{i=1}^{\infty}\left\{1 - i(i+1)\left(\log\frac{i}{i+1}\right)^2\right\} < +\infty$. Moreover,

$$\sqrt{n}d_2\left(\delta_{\bullet}^{u_n}, \mu\right) = C_2 + \mathcal{O}(n^{-1}).$$

3.3.3 Details of Example 2.41

As in the previous section, it is readily verified, again by virtue of (2.2), that

$$\lim_{n\to\infty} n^{1/r} d_r\left(\delta_{\bullet,1}^{u_n}, \mu\right) = D_r,$$

with D_r being defined in (3.13).

Since it has already been established in the previous subsection that

$$\lim_{n\to\infty} nd_r\left(\delta_{\bullet,1}^{u_n}, \mu\right) = D_r < C_r = \lim_{n\to\infty} nd_r\left(\delta_{\bullet}^{u_n}, \mu\right),$$

this subsection only contributes to the proof of

$$\sqrt{n} d_2\left(\delta_{\bullet,1}^{u_n},\mu\right) = D_2 + \mathcal{O}(n^{-1})$$

with $D_2^2 = 1 + 2\sum_{i=1}^{\infty}\left(1 - i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right)$.

Analogous to the calculation of $d_2\left(\delta_{\bullet}^{u_n},\mu\right)$,

$$\begin{aligned}
d_2\left(\delta_{\bullet,1}^{u_n},\mu\right)^2 =& \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(-\log(1-t) + \log\left(1-\frac{2i-1}{2n}\right)\right)^2 \mathrm{d}t = \sum_{i=1}^n \int_{1-\frac{i}{n}}^{1-\frac{i-1}{n}} \left(\log\frac{t}{1-\frac{2i-1}{2n}}\right)^2 \mathrm{d}t \\
=& \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\log\frac{t}{\frac{2i-1}{2n}}\right)^2 \mathrm{d}t = \frac{1}{2n}\sum_{i=1}^n \int_{\frac{2i-2}{2i-1}}^{\frac{2i}{2i-1}} (2i-1)\left(\log t\right)^2 \mathrm{d}t \\
=& \frac{1}{2n}\sum_{i=1}^n (2i-1)t\left((\log t - 1)^2 + 1\right)\Big|_{\frac{2i-2}{2i-1}}^{\frac{2i}{2i-1}} \\
=& \frac{1}{2n}\sum_{i=1}^n \left\{2i\left(\left(\log\frac{2i}{2i-1}-1\right)^2+1\right) - (2i-2)\left(\left(\log\frac{2i-2}{2i-1}-1\right)^2+1\right)\right\},
\end{aligned}$$

and hence

$$\begin{aligned}
nd_2\left(\delta_{\bullet,1}^{u_n},\mu\right)^2 =& \sum_{i=1}^n \left\{i\left(\left(\log\frac{2i}{2i-1}-1\right)^2+1\right) - (i-1)\left(\left(\log\frac{2i-2}{2i-1}-1\right)^2+1\right)\right\} \\
=& \sum_{i=1}^n i\left\{\left(\log\frac{2i}{2i-1}-1\right)^2+1\right\} - \sum_{i=1}^{n-1} i\left\{\left(\log\frac{2i}{2i+1}-1\right)^2+1\right\} \\
=& n\left\{\left(\log\frac{2n}{2n+1}-1\right)^2+1\right\} + \sum_{i=1}^n i\left\{\left(\log\frac{2i}{2i-1}-1\right)^2 - \left(\log\frac{2i}{2i+1}-1\right)^2\right\} \\
=& n\left\{\left(\log\frac{2n}{2n+1}\right)^2 - 2\log\frac{2n}{2n+1}\right\} + \sum_{i=1}^n \left\{2 + i\log\frac{2i+1}{2i-1}\left(\log\frac{4i^2}{4i^2-1}-2\right)\right\} \\
=& 1 + n\left\{\log\left(1+\frac{1}{2n}\right)\right\}^2 + 2n\left\{\log\left(1+\frac{1}{2n}\right) - \frac{1}{2n}\right\} \\
&+ 2\sum_{i=1}^n \left\{1 - i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right\} \\
=& 1 + 2\sum_{i=1}^\infty \left\{1 - i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right\} + n\left\{\log\left(1+\frac{1}{2n}\right)\right\}^2 \\
&+ 2n\left\{\log\left(1+\frac{1}{2n}\right) - \frac{1}{2n}\right\} - 2\sum_{i=n+1}^\infty \left\{1 - i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right\}.
\end{aligned}$$

It remains to determine the asymptotics of this sum. First, a direct calculation shows

$$n\left\{\log\left(1+\frac{1}{2n}\right)\right\}^2 + 2n\left\{\log\left(1+\frac{1}{2n}\right) - \frac{1}{2n}\right\} = \mathcal{O}\left(n^{-2}\right).$$

To see the asymptotics of the remaining terms, it suffices to obtain the asymptotics of

$$1-i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}.$$

In fact,

$$\begin{aligned}
&1-i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\\
=&1-i\left\{\log\left(1+\frac{1}{2i}\right)-\log\left(1-\frac{1}{2i}\right)\right\}\left\{1+\frac{1}{2}\log\left(1-\frac{1}{4i^2}\right)\right\}\\
=&1-i\left\{\log\left(1+\frac{1}{2i}\right)-\log\left(1-\frac{1}{2i}\right)\right\}-\frac{1}{2}i\left\{\log\left(1+\frac{1}{2i}\right)-\log\left(1-\frac{1}{2i}\right)\right\}\log\left(1-\frac{1}{4i^2}\right)\\
=&-2i\frac{1}{3}\left(\frac{1}{2i}\right)^3+\mathcal{O}\left(i^{-4}\right)-\frac{1}{2}\left(-\frac{1}{4i^2}\right)+\mathcal{O}\left(i^{-4}\right)=\frac{1}{24}\frac{1}{i^2}+\mathcal{O}\left(i^{-4}\right),
\end{aligned}$$

which implies that

$$-2\sum_{i=n+1}^{\infty}\left\{1-i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right\}=-\frac{1}{12}\sum_{i=n+1}^{\infty}\frac{1}{i^2}+\mathcal{O}\left(n^{-3}\right)=-\frac{1}{12}\frac{1}{n}+\mathcal{O}\left(n^{-2}\right),$$

thus yielding the convergence of $\sum_{i=1}^{\infty}\left\{1-i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right\}$ as well as

$$nd_2\left(\delta^{u_n}_{\bullet,1},\mu\right)^2=1+2\sum_{i=1}^{\infty}\left\{1-i\left(1+\log\frac{\sqrt{4i^2-1}}{2i}\right)\log\frac{2i+1}{2i-1}\right\}+\mathcal{O}\left(n^{-1}\right).$$

3.3.4 Details of Example 2.51

Finally, we investigate the best r-approximation of μ.

Let $\delta^{p_n}_{x_n}$ be a best 1-approximation of μ. By Theorem 2.47,

$$\begin{cases}
e^{-\frac{x_{n,i-1}}{2}}+e^{-\frac{x_{n,i+1}}{2}}=2e^{-\frac{x_{n,i}}{2}}, & \forall\ 2\le i\le n-1\\
\frac{1}{2}\left(1-e^{-\frac{x_{n,1}+x_{n,2}}{2}}\right)=1-e^{-x_{n,1}}, & \\
\frac{1}{2}\left(1+1-e^{-\frac{x_{n,n-1}+x_{n,n}}{2}}\right)=1-e^{-x_{n,n}}, &
\end{cases}$$

and $P_{n,i}=1-e^{-\frac{x_{n,i}+x_{n,i+1}}{2}}$, $i=1,\cdots,n-1$. It is straightforward to deduce that

$$x_{n,i}=-2\log\frac{n+1-i}{\sqrt{n(n+1)}},\quad P_{n,i}=\frac{i(2n+1-i)}{n(n+1)},\quad \forall\ i=1,\cdots,n.$$

By the variational representation of the quantization error again [40, Lem.3.1],

$$
\begin{aligned}
nd_1(\delta^{\bullet,n}_{\bullet}, \mu) = &\int_0^{\frac{x_{n,1}+x_{n,2}}{2}} |y - x_{n,1}| e^{-y} \mathrm{d}y + \sum_{i=2}^{n} \int_{\frac{x_{n,i-1}+x_{n,i}}{2}}^{\frac{x_{n,i}+x_{n,i+1}}{2}} |y - x_{n,i}| e^{-y} \mathrm{d}y \\
= &\int_0^{x_{n,1}} y e^{y - x_{n,1}} \mathrm{d}y + \int_0^{\frac{x_{n,2}-x_{n,1}}{2}} y e^{-y - x_{n,1}} \mathrm{d}y \\
&+ \sum_{i=2}^{n} \left(\int_0^{\frac{x_{n,i}-x_{n,i-1}}{2}} y e^{y - x_{n,i}} \mathrm{d}y + \int_0^{\frac{x_{n,i+1}-x_{n,i}}{2}} y e^{-y - x_{n,i}} \mathrm{d}y \right) \\
= &e^{-x_{n,1}} \left\{ (y-1)e^{y} \Big|_0^{x_{n,1}} - (1+y)e^{-y} \Big|_0^{\frac{x_{n,2}-x_{n,1}}{2}} \right\} \\
&+ \sum_{i=2}^{n} e^{-x_{n,i}} \left\{ (y-1)e^{y} \Big|_0^{\frac{x_{n,i}-x_{n,i-1}}{2}} - (1+y)e^{-y} \Big|_0^{\frac{x_{n,i+1}-x_{n,i}}{2}} \right\} \\
= &-1 + x_{n,1} - \frac{x_{n,2}-x_{n,1}}{2} e^{-\frac{x_{n,1}+x_{n,2}}{2}} - e^{-\frac{x_{n,1}+x_{n,2}}{2}} + 2e^{-x_{n,1}} \\
&+ \sum_{i=2}^{n} e^{-x_{n,i}} \left\{ 2 + \left(\frac{x_{n,i}-x_{n,i-1}}{2} - 1 \right) e^{\frac{x_{n,i}-x_{n,i-1}}{2}} - \left(\frac{x_{n,i+1}-x_{n,i}}{2} + 1 \right) e^{\frac{x_{n,i}-x_{n,i+1}}{2}} \right\} \\
= &-\sum_{i=2}^{n} \left(e^{-\frac{x_{n,i-1}+x_{n,i}}{2}} + e^{-\frac{x_{n,i}+x_{n,i+1}}{2}} \right) + 2\sum_{i=1}^{n} e^{-x_{n,i}} - \left(1 + \frac{x_{n,2}-x_{n,1}}{2} \right) e^{-\frac{x_{n,1}+x_{n,2}}{2}} \\
&+ x_{n,1} - 1 + \sum_{i=2}^{n} \left(\frac{x_{n,i}-x_{n,i-1}}{2} e^{-\frac{x_{n,i}+x_{n,i-1}}{2}} - \frac{x_{n,i+1}-x_{n,i}}{2} e^{-\frac{x_{n,i+1}+x_{n,i}}{2}} \right) \\
= &2\sum_{i=1}^{n} e^{-x_{n,i}} - 2\sum_{i=2}^{n} e^{-\frac{x_{n,i-1}+x_{n,i}}{2}} + x_{n,1} - 1 \\
= &2\sum_{i=1}^{n} \frac{(n+1-i)^2}{n(n+1)} - 2\sum_{i=1}^{n} \frac{(n+1-i)(n-i)}{n(n+1)} - \log \frac{(n+1-1)^2}{n(n+1)} - 1 \\
= &2\sum_{i=1}^{n} \frac{n+1-i}{n(n+1)} - 1 - \log \frac{(n+1-1)^2}{n(n+1)} = \log\left(1 + \frac{1}{n}\right).
\end{aligned}
$$

Next we construct a sequence of asymptotically best r-approximations of μ, for all $r \geq 1$. Define an auxiliary probability measure μ_r on $\overline{\mathbb{R}}$ by its density:

$$
F'_{\mu_r}(x) := \frac{F'_\mu(x)^{\frac{1}{r+1}}}{\int_0^\infty F'_\mu(y)^{\frac{1}{r+1}} \mathrm{d}y} = \frac{1}{r+1} e^{-\frac{x}{r+1}}, \quad \forall\, x \geq 0.
$$

Note that $F_{\mu_r}(x) = 1 - e^{-\frac{x}{r+1}}$. This implies $F^{-1}_{\mu_r}(t) = -(r+1)\log(1-t)$. By [85, Prop.5], $\left(\delta^{\widetilde{p}_n}_{y_n}\right)$ is a sequence of asymptotically best r-approximations of μ for any $r > 1$, with

$$
y_{n,i} = F^{-1}_{\mu_r}\left(\frac{i}{n+1}\right) = (1+r)\log\frac{n+1}{n-i+1},\ \widetilde{P}_{n,i} = F_\mu\left(\frac{y_{n,i}+y_{n,i+1}}{2}\right) = 1 - \left(\frac{(n+1-i)(n-i)}{(n+1)^2}\right)^{\frac{1+r}{2}},
$$

for every $i = 1, \cdots, n$.

3.4 The Beta(2, 1) distribution

In this section, we present computational details for $\mu = \mathrm{Beta}(2,1)$, and denote $f = F_\mu^{-1}$.

3.4.1 Details of Example 2.28

In this subsection, we investigate the asymptotics of $d_r\left(\delta_{x_n}^{\bullet}, \mu\right)$ with $x_n = (1, \sqrt{2}, \cdots, \sqrt{n})/\sqrt{n}$. Note that $\delta_{x_n}^{p_n}$ is the unique best r-approximation of μ given x_n, with

$$P_{n,i} = \left(\frac{\sqrt{i}+\sqrt{i+1}}{2\sqrt{n}}\right)^2,\ 1 \le i \le n-1.$$

Analogous to Subsection 3.3.1,

$$d_r\left(\delta_{x_n}^{p_n}, \mu\right)^r = \sum_{i=2}^{n-1} \int_{\left(\frac{\sqrt{i-1}+\sqrt{i}}{2\sqrt{n}}\right)^2}^{\left(\frac{\sqrt{i}+\sqrt{i+1}}{2\sqrt{n}}\right)^2} \left|\sqrt{t}-\sqrt{\frac{i}{n}}\right|^r \mathrm{d}t + \int_0^{\left(\frac{1+\sqrt{2}}{2\sqrt{n}}\right)^2} \left|\sqrt{t}-\sqrt{\frac{1}{n}}\right|^r \mathrm{d}t + \int_{\left(\frac{\sqrt{n-1}+\sqrt{n}}{2\sqrt{n}}\right)^2}^1 (1-t)^r \mathrm{d}t.$$

First, observe that

$$\begin{aligned}
&\int_{\left(\frac{\sqrt{i-1}+\sqrt{i}}{2\sqrt{n}}\right)^2}^{\left(\frac{\sqrt{i}+\sqrt{i+1}}{2\sqrt{n}}\right)^2} \left|\sqrt{t}-\sqrt{\frac{i}{n}}\right|^r \mathrm{d}t = 2\int_{\frac{\sqrt{i-1}+\sqrt{i}}{2\sqrt{n}}}^{\frac{\sqrt{i}+\sqrt{i+1}}{2\sqrt{n}}} t\left|t-\sqrt{\frac{i}{n}}\right|^r \mathrm{d}t\\
=&2\left\{\int_{\sqrt{\frac{i}{n}}}^{\frac{\sqrt{i}+\sqrt{i+1}}{2\sqrt{n}}} t\left(t-\sqrt{\frac{i}{n}}\right)^r \mathrm{d}t + \int_{\frac{\sqrt{i-1}+\sqrt{i}}{2\sqrt{n}}}^{\sqrt{\frac{i}{n}}} t\left(\sqrt{\frac{i}{n}}-t\right)^r \mathrm{d}t\right\}\\
=&2\left\{\int_{\sqrt{\frac{i}{n}}}^{\frac{\sqrt{i}+\sqrt{i+1}}{2\sqrt{n}}} \left\{\sqrt{\frac{i}{n}}\left(t-\sqrt{\frac{i}{n}}\right)^r + \left(t-\sqrt{\frac{i}{n}}\right)^{r+1}\right\}\mathrm{d}t\right.\\
&\left.+\int_{\frac{\sqrt{i-1}+\sqrt{i}}{2\sqrt{n}}}^{\sqrt{\frac{i}{n}}} \left\{\sqrt{\frac{i}{n}}\left(\sqrt{\frac{i}{n}}-t\right)^r - \left(\sqrt{\frac{i}{n}}-t\right)^{r+1}\right\}\mathrm{d}t\right\}\\
=&\frac{2}{r+1}\sqrt{\frac{i}{n}}\left\{\left(\frac{\sqrt{i+1}-\sqrt{i}}{2\sqrt{n}}\right)^{r+1} + \left(\frac{\sqrt{i}-\sqrt{i-1}}{2\sqrt{n}}\right)^{r+1}\right\}\\
&-\frac{2}{r+2}\left\{\left(\frac{\sqrt{i+1}-\sqrt{i}}{2\sqrt{n}}\right)^{r+2} + \left(\frac{\sqrt{i}-\sqrt{i-1}}{2\sqrt{n}}\right)^{r+2}\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^{\left(\frac{1+\sqrt{2}}{2\sqrt{n}}\right)^2} \left|\sqrt{t}-\frac{1}{\sqrt{n}}\right|^r \mathrm{d}t = 2\int_0^{\frac{1+\sqrt{2}}{2\sqrt{n}}} t\left|t-\frac{1}{\sqrt{n}}\right|^r \mathrm{d}t\\
=&\frac{2}{r+2}n^{-\frac{r+2}{2}}2^{-r-2}\left(\sqrt{2}+1\right)^{r+2} + \frac{2}{r+1}n^{-\frac{r+2}{2}}2^{-r-1}\left(\sqrt{2}+1\right)^{r+1};
\end{aligned}$$

$$\int_{\left(\frac{\sqrt{n-1}+\sqrt{n}}{2\sqrt{n}}\right)^2}^{1}\left(1-\sqrt{t}\right)^r \mathrm{d}t = 2\int_{\frac{\sqrt{n-1}+\sqrt{n}}{2\sqrt{n}}}^{1} t(1-t)^r \mathrm{d}t$$
$$=\frac{2}{r+1}n^{-\frac{r+1}{2}}2^{-r-1}\left(\sqrt{n}-\sqrt{n-1}\right)^{r+1} - \frac{2}{r+2}n^{-\frac{r+2}{2}}2^{-r-2}\left(\sqrt{n}-\sqrt{n-1}\right)^{r+2}.$$

Hence

$$\begin{aligned}
d_r\left(\delta_{x_n}^{p_n},\mu\right)^r =&\frac{2}{r+1}n^{-\frac{r+2}{2}}2^{-r-1}\sum_{i=2}^{n-1}\sqrt{i}\left(\sqrt{i+1}-\sqrt{i}\right)^{r+1}\\
&+\frac{2}{r+1}n^{-\frac{r+2}{2}}2^{-r-1}\sum_{i=2}^{n-1}\sqrt{i+1}\left(\sqrt{i+1}-\sqrt{i}\right)^{r+1}\\
&+\frac{2}{r+2}\left\{\left(\frac{\sqrt{n}-\sqrt{n-1}}{2\sqrt{n}}\right)^{r+2}-\left(\frac{\sqrt{2}-1}{2\sqrt{n}}\right)^{r+2}\right\}\\
&+\frac{2}{r+1}n^{-\frac{r+2}{2}}2^{-r-1}\left(\sqrt{2}+1\right)^{r+1}+\frac{2}{r+1}n^{-\frac{r+2}{2}}2^{-r-1}\sqrt{n}\left(\sqrt{n}-\sqrt{n-1}\right)^{r+1}\\
&+\frac{2}{r+2}n^{-\frac{r+2}{2}}2^{-r-2}\left(\sqrt{2}+1\right)^{r+2}-\frac{2}{r+2}\left(\frac{\sqrt{n}-\sqrt{n-1}}{2\sqrt{n}}\right)^{r+2}\\
=&\frac{2^{-r-1}}{r+2}n^{-\frac{r+2}{2}}\left\{\left(1+\sqrt{2}\right)^{r+2}-\left(\sqrt{2}-1\right)^{r+2}\right\}\\
&+\frac{2^{-r}}{r+1}n^{-\frac{r+2}{2}}\sum_{i=2}^{n-1}\left(\sqrt{i+1}-\sqrt{i}\right)^{r}+\frac{2^{-r}}{r+1}n^{-\frac{r+2}{2}}\left\{\left(\sqrt{2}-1\right)^{r+2}+\left(\sqrt{2}-1\right)^{r+1}\right\},
\end{aligned}$$

i.e.,

$$\begin{aligned}
2^r n^{\frac{r+2}{2}}d_r\left(\delta_{x_n}^{p_n},\mu\right)^r &= \frac{1}{2(r+2)}\left\{\left(\sqrt{2}-1\right)^{r+2}-\left(\sqrt{2}-1\right)^{r+2}\right\}\\
+\frac{1}{r+1}\left\{\left(\sqrt{2}-1\right)^{r+1}+\left(\sqrt{2}-1\right)^{r+1}\right\}&+\frac{1}{r+1}\sum_{i=2}^{n-1}\frac{1}{\left(\sqrt{i}+\sqrt{i+1}\right)^r}.
\end{aligned} \tag{3.14}$$

By the Euler-Maclaurin formula,

$$\sum_{i=2}^{n-1}\frac{1}{\left(\sqrt{i}+\sqrt{i+1}\right)^r} = \begin{cases} \frac{2^{1-r}}{2-r}n^{\frac{r-2}{2}}+\mathcal{O}(1) & \text{if } 1\le r<2,\\ \frac{1}{4}\log n+\mathcal{O}(1) & \text{if } r=2,\\ c_r+o(1) & \text{if } r>2,\end{cases}$$

where c_r is some positive (finite) constant. It then follows from (3.14) that, with $\gamma_r = \frac{1}{2}+\frac{1}{\max\{2,r\}}$ and the appropriate $0<\widetilde{C}_r<+\infty$,

$$\lim_{n\to\infty} n^{\gamma_r} d_r\left(\delta_{x_n}^{\bullet},\mu\right) = \widetilde{C}_r$$

whenever $r \neq 2$, whereas

$$\lim_{n\to\infty} \frac{n}{\sqrt{\log n}} d_2\left(\delta^{\bullet}_{x_n}, \mu\right) = \frac{1}{4\sqrt{3}}.$$

3.4.2 Details of Example 2.38

In this subsection, we present the asymptotics of $(d_r(\delta^{u_n}_{\bullet}, \mu))$ for $r = 1$, $r = 2$, and $r > 2$, respectively. For every $n \in \mathbb{N}$, let

$$J_i = \left[\frac{i-1}{n}, \frac{i}{n}\right], \ f_i = f\big|_{J_i} \quad \forall\, 1 \leq i \leq n.$$

Note that

$$d_r(\delta^{u_n}_{\bullet}, \mu)^r = \sum_{i=1}^{n} \int_{J_i} \left|\sqrt{t} - \tau_r^{f_i}\right|^r \mathrm{d}t.$$

By the minimality and homogeneity, for every $i = 1, \cdots, n$,

$$\begin{aligned}\int_{J_i} \left|\sqrt{t} - \tau_r^{f_i}\right|^r \mathrm{d}t &= \min\nolimits_{c\in\mathbb{R}} \int_{J_i} \left|\sqrt{t} - c\right|^r \mathrm{d}t = \frac{1}{n^{\frac{r}{2}+1}} \min\nolimits_{c\in\mathbb{R}} \int_{i-1}^{i} \left|\sqrt{t} - \sqrt{n}c\right|^r \mathrm{d}t \\ &= \frac{1}{n^{\frac{r}{2}+1}} \int_{i-1}^{i} \left|\sqrt{t} - \tau_r^{f|_{[i-1,i]}}\right|^r \mathrm{d}t,\end{aligned}$$

which yields

$$d_r(\delta^{u_n}_{\bullet}, \mu)^r = \frac{1}{n^{\frac{r}{2}+1}} \sum_{i=1}^{n} \int_{i-1}^{i} \left|\sqrt{t} - \tau_r^{f|_{[i-1,i]}}\right|^r \mathrm{d}t, \quad \forall\, r \geq 1.$$

First, consider the case $r > 2$. To show the limit $\lim_{n\to\infty} n^{1/2+1/r} d_r(\delta^{u_n}_{\bullet}, \mu)^r$ exists and equals a finite positive constant, it suffices to obtain the convergence of $\sum_{i=1}^{\infty} \int_{i-1}^{i} \left|\sqrt{t} - \tau_r^{f|_{[i-1,i]}}\right|^r \mathrm{d}t$, for $r > 2$. Analogous to Subsection 3.3.3, note that

$$\sum_{i=1}^{\infty} \int_{i-1}^{i} \left|\sqrt{t} - \sqrt{i - \frac{1}{2}}\right|^r \mathrm{d}t < +\infty \quad \forall\, r > 2,$$

which follows from the fact below:

$$\begin{aligned}\int_{i-1}^{i} \left|\sqrt{t} - \sqrt{i-1/2}\right|^r \mathrm{d}t &= \int_{i-1}^{i} \left|\frac{t-(i-1/2)}{\sqrt{t} + \sqrt{i-1/2}}\right|^r \mathrm{d}t \leq 2^{-r} \int_{i-1}^{i} \left|\sqrt{t} + \sqrt{i-1/2}\right|^{-r} \mathrm{d}t \\ &\leq 2^{-r} \left(\sqrt{i-1} + \sqrt{i-1/2}\right)^{-r}, \quad \forall\, i \in \mathbb{N}.\end{aligned}$$

Next, we show that

$$nd_2(\delta^{u_n}_{\bullet}, \mu) = \frac{1}{4\sqrt{3}} \sqrt{\log n} + \mathcal{O}(1) \ \text{ as } n \to \infty.$$

By Remark 2.30(ii), $\delta_{x_n}^{u_n}$ is the unique best uniform 2-approximation of μ when

$$x_{n,i} = n\int_{(i-1)/n}^{i/n} \sqrt{t}\mathrm{d}t = \frac{2}{3}\frac{i^{3/2}-(i-1)^{3/2}}{n^{1/2}}, \quad \forall\ 1 \le i \le n,$$

which yields

$$\begin{aligned}
d_2\left(\delta_\bullet^{u_n},\mu\right)^2 =& \sum_{i=1}^n \int_{(i-1)/n}^{i/n} \left(\sqrt{t} - \frac{2}{3}\frac{i^{3/2}-(i-1)^{3/2}}{n^{1/2}}\right)^2 \mathrm{d}t \\
=& \sum_{i=1}^n \left\{ \frac{2i-1}{2n^2} - \frac{4}{3}\frac{i^{3/2}-(i-1)^{3/2}}{n^{1/2}}\frac{2}{3}\frac{i^{3/2}-(i-1)^{3/2}}{n^{3/2}} + \frac{4}{9}\frac{\left(i^{3/2}-(i-1)^{3/2}\right)^2}{n^2} \right\} \\
=& \frac{1}{n^2}\sum_{i=1}^n \left\{ i - \frac{1}{2} - \frac{4}{9}\left(i^{3/2}-(i-1)^{3/2}\right)^2 \right\} \\
=& \frac{1}{n^2}\left\{ \frac{(1/2+n-1/2)\,n}{2} - \frac{4}{9}\sum_{i=1}^n \left(i^{3/2}-(i-1)^{3/2}\right)^2 \right\} \\
=& \frac{1}{2} - \frac{4}{9n^2}\sum_{i=1}^n \frac{\left(3i^2-3i+1\right)^2}{i^3\left(1+(1-1/i)^{3/2}\right)^2} = \frac{1}{2} - \frac{4}{9n^2}\sum_{i=1}^n \frac{i\left(3-3/i+1/i^2\right)^2}{\left(1+(1-1/i)^{3/2}\right)^2}.
\end{aligned}$$

From the elementary asymptotic expansion as $x \to 0$,

$$\frac{1}{1+(1-x)^{3/2}} = \frac{1}{2} + \frac{3}{8}x + \frac{3}{16}x^2 + \mathcal{O}\left(x^3\right),$$ [1]

it follows that

$$\frac{3-3x+x^2}{1+(1-x)^{3/2}} = \left(3-3x+x^2\right)\left\{\frac{1}{2} + \frac{3}{8}x + \frac{3}{16}x^2 + \mathcal{O}\left(x^3\right)\right\} = \frac{3}{2} - \frac{3}{8}x - \frac{1}{16}x^2 + \mathcal{O}\left(x^3\right),$$

and

$$\frac{(3-3x+x^2)^2}{\left(1+(1-x)^{3/2}\right)^2} = \left(\frac{3}{2} - \frac{3}{8}x - \frac{1}{16}x^2 + \mathcal{O}\left(x^3\right)\right)^2 = \frac{9}{4} - \frac{9}{8}x - \frac{3}{64}x^2 + \mathcal{O}\left(x^3\right).$$

This expansion further implies that

$$d_2\left(\delta_\bullet^{u_n},\mu\right)^2 = \frac{1}{2} - \frac{4}{9n^2}\sum_{i=1}^n \left(\frac{9}{4}i - \frac{9}{8} - \frac{3}{64}\frac{1}{i} + \mathcal{O}\left(i^{-2}\right)\right) = \frac{1}{2} - \frac{4}{9n^2}\frac{9}{8}n^2 + \frac{1}{48}\log n + \mathcal{O}(1),$$

i.e.,

$$\lim_{n\to\infty} \frac{n}{\sqrt{\log n}} d_2(\delta_\bullet^{u_n},\mu) = \frac{1}{4\sqrt{3}}.$$

[1] Here (and analogously throughout) this asymptotic expansion should be interpreted as $\left|\frac{1}{1+(1-x)^{3/2}} - \left(\frac{1}{2}+\frac{3}{8}x+\frac{3}{16}x^2\right)\right| \le C|x^3|$ for some positive constant C and all $|x|<1$.

Finally, we show that

$$nd_1\left(\delta_\bullet^{u_n}, \mu\right) = \frac{1}{4} + \mathcal{O}\Big(n^{-1/2}\Big) \text{ as } n \to \infty.$$

To this end, note that

$$\begin{aligned}
d_1\left(\delta_\bullet^{u_n}, \mu\right) &= \sum_{i=1}^n \int_{(i-1)/n}^{i/n} \left| \sqrt{t} - \sqrt{\frac{2i-1}{2n}} \right| \mathrm{d}t \\
=& 2\sum_{i=1}^n \left\{ \int_{\sqrt{\frac{i-1}{n}}}^{\sqrt{\frac{2i-1}{2n}}} t \left(\sqrt{\frac{2i-1}{2n}} - t \right) + \int_{\sqrt{\frac{2i-1}{2n}}}^{\sqrt{\frac{i}{n}}} t \left(t - \sqrt{\frac{2i-1}{2n}} \right) \right\} \\
=& \frac{2}{3}\sum_{i=1}^n \left\{ \left(\frac{i}{n}\right)^{3/2} + \left(\frac{i-1}{n}\right)^{3/2} - 2\left(\frac{2i-1}{2n}\right)^{3/2} \right\} \\
=& \frac{2}{3}\sum_{i=1}^n \left\{ \frac{i}{n}\left(\sqrt{\frac{i}{n}} - \sqrt{\frac{2i-1}{2n}} \right) + \frac{i-1}{n}\left(\sqrt{\frac{i-1}{n}} - \sqrt{\frac{2i-1}{2n}} \right) \right\} \\
=& \frac{1}{3n}\sum_{i=1}^n \left\{ \frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}} - \frac{\frac{i-1}{n}}{\sqrt{\frac{i-1}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}} \right\} \\
=& \frac{1}{3n}\sum_{i=1}^n \left\{ \left(\frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}} - \frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} + \frac{1}{2n}}} \right) + \frac{\frac{n}{n}}{\sqrt{\frac{n}{n}} + \sqrt{\frac{n}{n} + \frac{1}{2n}}} \right\} \\
=& \frac{1}{3n}\sum_{i=1}^n \left\{ \left(\frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}} - \frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n}}} \right) + \left(\frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n}}} - \frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} + \frac{1}{2n}}} \right) \right\} \\
&+ \frac{1}{3n}\frac{1}{1 + \sqrt{1 + \frac{1}{2n}}}.
\end{aligned}$$

For all $1 \leq i \leq n$,

$$\begin{aligned}
\frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}} - \frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n}}} =& \frac{1}{4n}\frac{\frac{i}{n}}{\left(\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}\right)^2} \\
=& \frac{1}{4n}\frac{\frac{i}{n}}{\left(\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n}}\right)^2} + \frac{1}{4n}\left\{ \frac{\frac{i}{n}}{\left(\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}\right)^2} - \frac{\frac{i}{n}}{\left(\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n}}\right)^2} \right\} \\
=& \frac{1}{16n}\sqrt{\frac{i}{n}} + \frac{1}{32n^2}\frac{3\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}}{\sqrt{\frac{i}{n}}\left(\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}\right)^3};
\end{aligned} \tag{3.15}$$

analogously,

$$\frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} - \frac{1}{2n}}} - \frac{\frac{i}{n}}{\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n}}} = \frac{1}{16n}\sqrt{\frac{i}{n}} - \frac{1}{32n^2}\frac{3\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} + \frac{1}{2n}}}{\sqrt{\frac{i}{n}}\left(\sqrt{\frac{i}{n}} + \sqrt{\frac{i}{n} + \frac{1}{2n}}\right)^3}. \tag{3.16}$$

Eqs. (3.15) and (3.16) together with

$$\frac{1}{1+\sqrt{1+\frac{1}{2n}}} = \frac{1}{2} + \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty,$$

yield

$$nd_1\left(\delta_{\bullet}^{u_n}, \mu\right) = \frac{1}{6} + \frac{1}{24n^{1/2}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \mathcal{O}(n^{-1}).$$

With the elementary estimate $\left|\sum_{i=1}^{n} \frac{1}{\sqrt{i}} - 2\sqrt{n}\right| \leq 2$ for all $n \in \mathbb{N}$, it follows that

$$nd_1\left(\delta_{\bullet}^{u_n}, \mu\right) = \frac{1}{4} + \mathcal{O}(n^{-1/2}).$$

3.4.3 Details of Example 2.52

This subsection is devoted to the proof of

$$nd_2\left(\delta_{x_n}^{p_n}, \mu\right) = \frac{3}{8\sqrt{2}} + \mathcal{O}\left(n^{-1}\right) \quad \text{as } n \to \infty, \tag{3.17}$$

where

$$x_{n,i} = \left(\frac{i}{n+1}\right)^{3/4}, \; P_{n,i} = \frac{1}{4(n+1)^{3/2}}\left(i^{3/4} + (i+1)^{3/4}\right)^2, \; \forall\, i = 1, \cdots, n-1,$$

and $x_{n,n} = \left(\frac{n}{n+1}\right)^{3/4}$. By virtue of (2.2),

$$\begin{aligned} &\sum_{i=1}^{n} \int_{P_{n,i-1}}^{P_{n,i}} \left(F_\mu^{-1}(t) - x_{n,i}\right)^2 \mathrm{d}t \\ =&\sum_{i=2}^{n-1} \int_{\left(\frac{1}{2}(x_{n,i-1}+x_{n,i})\right)^2}^{\left(\frac{1}{2}(x_{n,i}+x_{n,i+1})\right)^2} \left(\sqrt{t} - x_{n,i}\right)^2 \mathrm{d}t + \int_0^{\left(\frac{1}{2}(x_{n,1}+x_{n,2})\right)^2} \left(\sqrt{t} - x_{n,1}\right)^2 \mathrm{d}t \\ &+ \int_{\left(\frac{1}{2}(x_{n,n-1}+x_{n,n})\right)^2}^{1} \left(\sqrt{t} - x_{n,n}\right)^2 \mathrm{d}t. \end{aligned} \tag{3.18}$$

In the following, we provide the asymptotics of the summation in (3.18) term by term. First, let us estimate the last two terms:

$$\int_0^{\left(\frac{1}{2}(x_{n,1}+x_{n,2})\right)^2} \left(\sqrt{t} - x_{n,1}\right)^2 \mathrm{d}t \leq x_{n,2}^2 \cdot \frac{1}{4}\left(x_{n,2} + x_{n,1}\right)^2 \leq x_{n,2}^4 = \mathcal{O}\left(n^{-3}\right);$$

$$\int_{\left(\frac{1}{2}(x_{n,n-1}+x_{n,n})\right)^2}^{1}\left(\sqrt{t}-x_{n,n}\right)^2\mathrm{d}t\le\left(1-x_{n,n-1}\right)^2\left\{1-\left(\frac{1}{2}\left(x_{n,n-1}+x_{n,n}\right)\right)^2\right\}$$
$$=\left(1-x_{n,n-1}\right)^2\left\{1-\left(\frac{1}{2}\left(x_{n,n-1}+x_{n,n}\right)\right)^2\right\}\le 2\left(1-x_{n,n-1}\right)^2\left\{1-\left(\frac{1}{2}\left(x_{n,n-1}+x_{n,n}\right)\right)\right\}$$
$$\le 2\left(1-x_{n,n-1}\right)^2\left(1-x_{n,n-1}\right)=2\left(1-x_{n,n-1}\right)^3=2\left(1-\left(1-\frac{2}{n+1}\right)^{3/4}\right)^3=\mathcal{O}\left(n^{-3}\right).$$

Next, we present the asymptotics of the first summation term in (3.18). To this end, notice that

$$\sum_{i=2}^{n-1}\int_{\left(\frac{1}{2}(x_{n,i-1}+x_{n,i})\right)^2}^{\left(\frac{1}{2}(x_{n,i}+x_{n,i+1})\right)^2}\left(\sqrt{t}-x_{n,i}\right)^2\mathrm{d}t=2\sum_{i=2}^{n-1}\int_{\frac{1}{2}(x_{n,i-1}-x_{n,i})}^{\frac{1}{2}(x_{n,i+1}-x_{n,i})}t^2(t+x_{n,i})\mathrm{d}t$$
$$=\sum_{i=2}^{n-1}\left\{\frac{1}{2}\left\{\frac{1}{2^4}\left(x_{n,i+1}-x_{n,i}\right)^4-\frac{1}{2^4}\left(x_{n,i}-x_{n,i-1}\right)^4\right\}\right.$$
$$\left.+\frac{2}{3}x_{n,i}\left\{\frac{1}{2^3}\left(x_{n,i+1}-x_{n,i}\right)^3+\frac{1}{2^3}\left(x_{n,i}-x_{n,i-1}\right)^3\right\}\right\}$$
$$=\frac{1}{2^5}\left\{\left(x_{n,n}-x_{n,n-1}\right)^4-\left(x_{n,2}-x_{n,1}\right)^4\right\}+\frac{1}{12}\sum_{i=2}^{n-1}\left\{x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3+x_{n,i}\left(x_{n,i}-x_{n,i-1}\right)^3\right\}$$
$$=\frac{1}{2^5}\left\{\left(x_{n,n}-x_{n,n-1}\right)^4-\left(x_{n,2}-x_{n,1}\right)^4\right\}+\frac{1}{12}\sum_{i=2}^{n-1}\left\{x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3+x_{n,i-1}\left(x_{n,i}-x_{n,i-1}\right)^3\right.$$
$$\left.+\left(x_{n,i}-x_{n,i-1}\right)^4\right\}$$
$$=\frac{1}{2^5}\left\{\left(x_{n,n}-x_{n,n-1}\right)^4-\left(x_{n,2}-x_{n,1}\right)^4\right\}+\frac{1}{12}\sum_{i=2}^{n-1}x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3$$
$$=\frac{1}{2^5}\left\{\left(x_{n,n}-x_{n,n-1}\right)^4-\left(x_{n,2}-x_{n,1}\right)^4\right\}+\frac{1}{6}\sum_{i=2}^{n-1}x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3$$
$$+\frac{1}{12}\sum_{i=2}^{n-1}\left\{\left(x_{n,i}-x_{n,i-1}\right)^4-x_{n,n-1}\left(x_{n,n}-x_{n,n-1}\right)^3+x_{n,1}\left(x_{n,2}-x_{n,1}\right)^3\right\}.$$

Observe that

$$\left(x_{n,n}-x_{n,n-1}\right)^4-\left(x_{n,2}-x_{n,1}\right)^4$$
$$=\left\{\left(\frac{n}{n+1}\right)^{3/4}-\left(\frac{n-1}{n+1}\right)^{3/4}\right\}^4-\left\{\left(\frac{2}{n+1}\right)^{3/4}-\left(\frac{1}{n+1}\right)^{3/4}\right\}^4$$
$$=\frac{1}{(n+1)^3}\frac{1}{n}\left\{\frac{3-\frac{3}{n}+\frac{1}{n^2}}{\left(1+\left(1-\frac{1}{n}\right)^{3/4}\right)\left(1+\left(1-\frac{1}{n}\right)^{3/2}\right)}\right\}^4-\frac{\left(2^{3/4}-1\right)^4}{(n+1)^3}=\mathcal{O}\left(n^{-3}\right).$$

Similarly, one can show

$$x_{n,n-1}\left(x_{n,n}-x_{n,n-1}\right)^3=\mathcal{O}\left(n^{-3}\right),\quad x_{n,1}\left(x_{n,2}-x_{n,1}\right)^3=\mathcal{O}\left(n^{-3}\right).$$

It remains to estimate $\sum_{i=2}^{n-1} x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3$ and $\sum_{i=2}^{n-1}(x_{n,i}-x_{n,i-1})^4$, respectively. Substituting

$$x_{n,i}=\left(\frac{i}{n+1}\right)^{3/4}, \quad \forall\ 1\le i\le n,$$

we simplify these two summations as follows:

$$\begin{aligned}\sum_{i=2}^{n-1} x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3=&\sum_{i=2}^{n-1}\left(\frac{i}{n+1}\right)^{3/4}\left\{\left(\frac{i+1}{n+1}\right)^{3/4}-\left(\frac{i}{n+1}\right)^{3/4}\right\}^3\\ =&(n+1)^{-3}\sum_{i=2}^{n-1} i^{3/4}\left((i+1)^{3/4}-i^{3/4}\right)^3\\ =&(n+1)^{-3}\sum_{i=2}^{n-1} i^{3/4}\left\{\frac{3i^2+3i+1}{((i+1)^{3/4}+i^{3/4})\,((i+1)^{3/2}+i^{3/2})}\right\}^3\\ =&\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\left\{\frac{3+3/i+1/i^2}{\left(1+(1+1/i)^{3/4}\right)\left(1+(1+1/i)^{3/2}\right)}\right\}^3;\end{aligned}$$

$$\begin{aligned}\sum_{i=2}^{n-1}(x_{n,i}-x_{n,i-1})^4=&\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\left(i^{3/4}-(i-1)^{3/4}\right)^4=\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\left(\frac{i^{3/2}-(i-1)^{3/2}}{i^{3/4}+(i-1)^{3/4}}\right)^4\\ =&\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\left\{\frac{i^3-(i-1)^3}{(i^{3/4}+(i-1)^{3/4})\,(i^{3/2}+(i-1)^{3/2})}\right\}^4\\ =&\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\frac{1}{i}\left\{\frac{3-3/i+1/i^2}{\left(1+(1-1/i)^{3/4}\right)\left(1+(1-1/i)^{3/2}\right)}\right\}^4.\end{aligned}$$

Utilizing the asymptotic expansion for $\alpha>0$ as $x\to 0$

$$\frac{1}{1+(1+x)^\alpha}=\frac{1}{2}-\frac{\alpha}{4}x+\mathcal{O}\left(x^2\right),$$

one concludes that for small x,

$$\frac{3-3x+x^2}{(1+(1-x)^{3/4})\,(1+(1-x)^{3/2})}=\frac{3}{4}+\frac{3}{32}x+\mathcal{O}\left(x^2\right).$$

This further implies by tedious calculation that

$$2\left(\frac{3+3x+x^2}{(1+(1+x)^{3/4})\,(1+(1+x)^{3/2})}\right)^3+\left(\frac{3-3x+x^2}{(1+(1-x)^{3/4})\,(1+(1-x)^{3/2})}\right)^4=2\cdot\left(\frac{3}{4}\right)^3+\mathcal{O}\left(x^2\right).$$

With this, finally, the asymptotics of $\left(d_2\left(\delta_{x_n}^{p_n},\mu\right)\right)$ can be established:

$$
\begin{aligned}
d_2\left(\delta_{x_n}^{p_n},\mu\right)^2 =& \frac{1}{12}\sum_{i=2}^{n-1}\left\{2x_{n,i}\left(x_{n,i+1}-x_{n,i}\right)^3+\left(x_{n,i}-x_{n,i-1}\right)^4\right\}+\mathcal{O}\left(n^{-3}\right)\\
=&\frac{1}{12}\left\{\frac{2}{(n+1)^3}\sum_{i=2}^{n-1}\left(\frac{3+3/i+1/i^2}{\left(1+(1+1/i)^{3/4}\right)\left(1+(1+1/i)^{3/2}\right)}\right)^3\right.\\
&\left.+\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\frac{1}{i}\left(\frac{3-3/i+1/i^2}{\left(1+(1-1/i)^{3/4}\right)\left(1+(1-1/i)^{3/2}\right)}\right)^4\right\}\\
=&\frac{1}{12}\frac{1}{(n+1)^3}\sum_{i=2}^{n-1}\left\{2\cdot\left(\frac{3}{4}\right)^3+\mathcal{O}\left(i^{-2}\right)\right\}=\frac{9}{128}\frac{1}{n^2}+\mathcal{O}\left(n^{-3}\right),
\end{aligned}
$$

which yields (3.17).

3.5 The Cantor measure

In this section, we investigate best uniform r-approximations of the classical Cantor measure μ. Let $f=F_\mu^{-1}$ within this section.

3.5.1 Inhomogeneity of decay of the best uniform r-approximations

In this subsection, we verify the comments following Remark 2.42 by showing that for all $r\geq 1$,

$$\liminf_{n\to\infty} n^{\frac{\log 3}{\log 2}} d_r\left(\delta_\bullet^{u_n},\mu\right) \text{ and } \limsup_{n\to\infty} n^{1/r} d_r\left(\delta_\bullet^{u_n},\mu\right) \text{ are both positive and finite.}$$

We first establish bounds for lim inf. To begin with, to state some useful facts of f, let us introduce some handy notations: For every $n\in\mathbb{N}$, let $k=\lfloor\log_2 n\rfloor+2$,

$$I_j=\left[(j-1)2^{-k},j2^{-k}\right[,\text{ and } f_j=f|_{I_j},\quad \forall\, 1\leq j\leq 2^k.$$

Note that (I_j) is an equi-partition of $[0,1[$ with $\lambda(I_j)=2^{-k}$ for all $1\leq j\leq 2^k$. By the self-similar property of f,

$$f_j(\cdot)=f\left(\cdot-(j-1)2^{-k}\right)+f\left((j-1)2^{-k}\right),\ \forall\, 1<j\leq 2^k;\ f_1(\cdot)=3^{-k}f\left(2^k\cdot\right). \tag{3.19}$$

Let us identify all the pieces of subintervals where $F_{\delta_\bullet^{u_n}}^{-1}$ remains constant, and denote the index set by $J=\left\{1\leq j\leq 2^k : F_{\delta_\bullet^{u_n}}^{-1}\Big|_{I_j} \text{ is constant}\right\}$. Since $F_{\delta_\bullet^{u_n}}^{-1}$ has at most $n-1$ jumps,

$\#J \geq 2^k - n + 1$. By (3.19),

$$\begin{aligned} d_r\left(\delta_\bullet^{u_n}, \mu\right)^r &= \int_0^1 \left|f(t) - F^{-1}_{\delta_\bullet^{u_n}}(t)\right|^r \mathrm{d}t \geq \sum_{j\in J} \cdot \min\nolimits_{c_j\in\mathbb{R}} \int_{I_j} |f_j(t) - c_j|^r \,\mathrm{d}t \\ &= \#J \min\nolimits_{c_1\in\mathbb{R}} \int_{I_1} |f_1(t) - c_1|^r \,\mathrm{d}t \geq (2^k - n + 1)2^{-k}3^{-kr} \int_0^1 \left|f(t) - \tau_r^f\right|^r \mathrm{d}t \\ &\geq \frac{1}{2\cdot 9^r} n^{-r\frac{\log 3}{\log 2}} d_r\left(\delta_{1/2}, \mu\right)^r, \end{aligned}$$

where we employed $d_r\left(\delta_{1/2}, \mu\right)^r = \int_0^1 \left|f(t) - \tau_r^f\right|^r \mathrm{d}t$ in the second-to-last step. This immediately yields

$$\liminf_{n\to\infty} n^{\frac{\log 3}{\log 2}} d_r\left(\delta_\bullet^{u_n}, \mu\right) \geq \frac{1}{9\cdot 2^{1/r}} d_r\left(\delta_{1/2}, \mu\right).$$

On the other hand, again by self-similarity of μ,

$$3^m d_r\left(\delta_\bullet^{u_{2^m}}, \mu\right) = d_r\left(\delta_{1/2}, \mu\right), \quad \forall\, m \in \mathbb{N},$$

which in turn establishes the upper bound:

$$\liminf_{n\to\infty} n^{\frac{\log 3}{\log 2}} d_r\left(\delta_\bullet^{u_n}, \mu\right) \leq d_r\left(\delta_{1/2}, \mu\right).$$

To construct the bounds for the upper limit, we use a new equi-partition of $]0, 1[$, namely

$$\widetilde{I}_i = \left]\frac{i-1}{n}, \frac{i}{n}\right[, \quad \forall\, 1 \leq i \leq n.$$

For every $n \in \mathbb{N}$, let $\widetilde{k} = \lfloor \log_2 n \rfloor$, and

$$A_{\widetilde{k}} = \left\{m2^{-\widetilde{k}} : 1 \leq m \leq 2^{\widetilde{k}} - 1\right\} = \left\{j2^{-l} : 1 \leq l \leq \widetilde{k},\ 1 \leq j \leq 2^l \text{ odd}\right\}.$$

By self-similarity once again,

$$f\left(j2^{-l}\right) - f\left(j2^{-l}-\right) = 3^{-l}, \quad \forall\, 1 \leq j \leq 2^l \text{ odd}, \tag{3.20}$$

$$f\left(m2^{-\widetilde{k}}-\right) - f\left((m-1)2^{-\widetilde{k}}\right) = 3^{-\widetilde{k}}, \quad \forall\, 1 \leq m \leq 2^{\widetilde{k}}. \tag{3.21}$$

Since $\frac{1}{n} \leq 2^{-\widetilde{k}}$,

$$\#\left(\widetilde{I}_i \cap A_{\widetilde{k}}\right) \leq 1, \quad \forall\, 1 \leq i \leq n.$$

In the following, we distinguish two cases: (i) $\widetilde{I}_i \cap A_{\widetilde{k}} = \emptyset$; (ii) $\widetilde{I}_i \cap A_{\widetilde{k}}$ is a *singleton*.

Denote $\widetilde{J} = \left\{1 \le i \le n : \widetilde{I}_i \cap A_{\widetilde{k}} = \emptyset\right\}$ for convenience. If $i \in \widetilde{J}$, then either

$$j2^{-l} \le \frac{i-1}{n} < \frac{i}{n} \le j2^{-l} + 2^{-\widetilde{k}},$$

or

$$j2^{-l} - 2^{-\widetilde{k}} \le \frac{i-1}{n} < \frac{i}{n} \le j2^{-l}$$

holds for some $1 \le j \le 2^l$ odd and $1 \le l \le \widetilde{k}$; by (3.21), in either case, the following inequality holds:

$$f\left(\frac{i}{n}-\right) - f\left(\frac{i-1}{n}\right) \le 3^{-\widetilde{k}}.$$

If $i \notin \widetilde{J}$, then $\widetilde{I}_i \cap A_{\widetilde{k}} = \left\{j2^{-l}\right\}$ for some $1 \le j \le 2^l$ odd and $1 \le l \le \widetilde{k}$, and therefore

$$j \cdot 2^{-l} - 2^{-\widetilde{k}} < \frac{i-1}{n} < j \cdot 2^{-l} < \frac{i}{n} < j \cdot 2^{-l} + 2^{-\widetilde{k}};$$

by (3.20) and (3.21),

$$\begin{aligned}
&f\left(\frac{i}{n}-\right) - f\left(\frac{i-1}{n}\right) \\
\le& f\left(\left(j \cdot 2^{-l} + 2^{-\widetilde{k}}\right)-\right) - f\left(j2^{-l}\right) + f\left(j \cdot 2^{-l}\right) - f\left(j \cdot 2^{-l}-\right) + f\left(j \cdot 2^{-l}-\right) - f\left(j \cdot 2^{-l} - 2^{-\widetilde{k}}\right) \\
=& 3^{-l} + 2 \cdot 3^{-\widetilde{k}}.
\end{aligned}$$

This shows that

$$\begin{aligned}
d_r\left(\delta_{\bullet}^{u_n}, \mu\right)^r =& \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|f(t) - F^{-1}_{\delta_{\bullet}^{u_n}}(t)\right|^r \mathrm{d}t \le \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|f(t) - \frac{1}{2}\left(f\left(\frac{i}{n}-\right) + f\left(\frac{i-1}{n}\right)\right)\right|^r \mathrm{d}t \\
\le& \sum_{i=1}^{n} \frac{1}{n} \left\{\frac{1}{2}\left(f\left(\frac{i}{n}-\right) - f\left(\frac{i-1}{n}\right)\right)\right\}^r \\
=& \frac{1}{2^r n} \sum_{i \in J} \left(f\left(\frac{i}{n}-\right) - f\left(\frac{i-1}{n}\right)\right)^r + \frac{1}{2^r n} \sum_{i \notin J} \left(f\left(\frac{i}{n}-\right) - f\left(\frac{i-1}{n}\right)\right)^r \\
\le& \frac{1}{2^r n} \left(\sum_{i \in J} 3^{-r\widetilde{k}} + \sum_{l=1}^{\widetilde{k}} \sum_{1 \le j \le 2^l \text{ odd}} \left(3^{-l} + 2 \cdot 3^{-\widetilde{k}}\right)^{-r}\right) \\
\le& \frac{1}{2^r n} 3^{-r\widetilde{k}} n + \frac{1}{2^r n} \sum_{l=1}^{\widetilde{k}} \sum_{1 \le j \le 2^l \text{ odd}} 3^{r(-l+1)} = \frac{1}{2^r} 3^{-r\widetilde{k}} + \frac{1}{2^r n} \frac{1 - \left(2 \cdot 3^{-r}\right)^{\widetilde{k}}}{1 - 2 \cdot 3^{-r}},
\end{aligned}$$

which implies that

$$\limsup_{n \to \infty} n^{1/r} d_r\left(\delta_{\bullet}^{u_n}, \mu\right) \le \frac{1}{2}\left(1 - 2 \cdot 3^{-r}\right)^{-1/r}.$$

The lower bound

$$\limsup_{n\to\infty} n^{1/r} d_r\left(\delta_\bullet^{u_n}, \mu\right) \geq \frac{1}{6}$$

is readily obtained by Remark 2.45. Indeed, consider the subsequence (2^m+1) of (n). Note that $\frac{1}{2}$ is the midpoint of $I_0 := \left[\frac{2^{m-1}}{2^m+1}, \frac{2^{m-1}+1}{2^m+1}\right]$; by symmetry, $F^{-1}_{\delta_\bullet^{2^m m+1}}\Big|_{I_0} = \tau_r^{f|_{I_0}} = \frac{1}{2}\left(f\left(\frac{1}{2}-\right)+f\left(\frac{1}{2}\right)\right)$. Hence

$$d_r\left(\delta_\bullet^{u_n}, \mu\right)^r \geq \int_{I_0} \left|f(t)-\tau_r^{f|_{I_0}}\right|^r \mathrm{d}t \geq \frac{1}{n}\left\{\frac{1}{2}\left(f\left(\frac{1}{2}\right)-f\left(\frac{1}{2}-\right)\right)\right\}^r = \frac{1}{6^r n}.$$

3.6 The inverse Cantor measure

In this section, we study best r-approximations and best uniform r-approximations of the inverse Cantor measure μ. Let $f = F_\mu^{-1}$, i.e., f equals the distribution function of the Cantor measure, within this section, and recall that $\alpha_r = \frac{1}{r} + \left(1-\frac{1}{r}\right)\frac{\log 2}{\log 3}$, $\beta_r = \left(1-\frac{1}{r}\right) + \frac{1}{r}\frac{\log 3}{\log 2}$.

3.6.1 Details of Example 2.40

We first consider the rate of decay of the best uniform r-approximations, i.e., we establish estimates for $d_r\left(\delta_\bullet^{u_n}, \mu\right)$.

To bound $\liminf_{n\to\infty} n^{\alpha_r} d_r(\delta_\bullet^{u_n}, \mu)$ from below, let $k = \lfloor \log_3 n \rfloor + 1$ for every $n \in \mathbb{N}$, and

$$I_j = \left]f^{-1}((j-1)2^{-k}), f^{-1}(j2^{-k}-)\right[, \quad \forall\, 1 \leq j \leq 2^k.$$

Note that $\lambda(I_j) = 3^{-k}$, for all $1 \leq j \leq 2^k$. Since $\frac{1}{n} > 3^{-k}$,

$$\#\left(I_j \cap \frac{1}{n}\mathbb{Z}\right) \leq 1, \quad \forall\, 1 \leq j \leq 2^k,$$

which implies that $F^{-1}_{\delta_\bullet^{u_n}}$ is constant either on $I_{j,L}$ or on $I_{j,R}$, where for every $j = 1, \cdots, 2^k$,

$$I_{j,L} = \left]f^{-1}\left((j-1)2^{-k}\right), f^{-1}\left((j-1)2^{-k}\right)+3^{-k-1}\right[, \;\; I_{j,R} = \left]f^{-1}\left(j2^{-k}-\right)-3^{-k-1}, f^{-1}\left(j2^{-k}-\right)\right[.$$

From self-similarity of μ^{-1} and symmetry, it follows that

$$\min_{c\in\mathbb{R}} \int_{I_{j,L}} |f(t)-c|^r\, \mathrm{d}t = \min_{c\in\mathbb{R}} \int_{I_{j,R}} |f(t)-c|^r\, \mathrm{d}t = 3^{-k-1}2^{-(k+1)r} d_r\left(\delta_{1/2}, \mu\right)^r, \quad (3.22)$$

which further implies that

$$\int_{I_j} \left|f(t) - F^{-1}_{\delta_\bullet^{u_n}}(t)\right|^r \mathrm{d}t \geq 3^{-k-1}2^{-(k+1)r} d_r\left(\delta_{1/2}, \mu\right)^r.$$

Therefore

$$d_r(\delta_\bullet^{u_n},\mu)^r \geq \sum_{j=1}^{2^k}\int_{I_j}\left|f(t)-F^{-1}_{\delta_\bullet^{u_n}}(t)\right|^r \mathrm{d}t$$
$$\geq \sum_{j=1}^{2^k}\left(\min_{c\in\mathbb{R}}\int_{I_{j,L}}|f(t)-c|^r\,\mathrm{d}t+\min_{c\in\mathbb{R}}\int_{I_{j,R}}|f(t)-c|^r\,\mathrm{d}t\right)=\sum_{j=1}^{2^k}(3\cdot 2^r)^{-k-1}\,d_r\left(\delta_{1/2},\mu\right)^r,$$

where (3.22) was employed. This establishes the lower bound,

$$\liminf_{n\to\infty} n^{\alpha_r} d_r\left(\delta_\bullet^{u_n},\mu\right)\geq \frac{1}{4}\left(\frac{2}{9}\right)^{1/r} d_r\left(\delta_{1/2},\mu\right).$$

To bound $\limsup_{n\to\infty} n^{\alpha_r} d_r(\delta_\bullet^{u_n},\mu)$ from above, for every $n\in\mathbb{N}$, let $\widetilde{k}=\lfloor \log_3 n\rfloor$, and

$$A_{\widetilde{k}}=\left\{f^{-1}\left(m2^{-\widetilde{k}}-\right), f^{-1}\left(m2^{-\widetilde{k}}\right): 1\leq m\leq 2^{\widetilde{k}}-1\right\}$$
$$=\left\{f^{-1}\left(j\cdot 2^{-l}\right), f^{-1}\left(j\cdot 2^{-l}-\right): 1\leq j\leq 2^l \text{ odd},\ 1\leq l\leq \widetilde{k}\right\}.$$

Again by self-similarity of μ^{-1},

$$\min_{x,y\in A_{\widetilde{k}},\ x\neq y}|x-y|=3^{-\widetilde{k}};\quad f^{-1}\left(m\cdot 2^{-\widetilde{k}}-\right)-f^{-1}\left((m-1)2^{-\widetilde{k}}\right)=3^{-\widetilde{k}}.$$

Let $J=\left\{1\leq i\leq n: f^{-1}\left(m\cdot 2^{-\widetilde{k}}-\right)\leq \frac{i-1}{n}<\frac{i}{n}\leq f^{-1}\left(m\cdot 2^{-\widetilde{k}}\right) \text{ for some } 1\leq m\leq 2^{\widetilde{k}}\right\}$. If $i\in J$, then

$$f\left(\frac{i}{n}-\right)-f\left(\frac{i-1}{n}\right)=0.$$

If $i\notin J$, then

$$f^{-1}\left((m-1)\cdot 2^{-\widetilde{k}}\right)\leq \frac{i-1}{n}<f^{-1}\left(m\cdot 2^{-\widetilde{k}}-\right) \text{ or } f^{-1}\left((m-1)\cdot 2^{-\widetilde{k}}\right)<\frac{i}{n}\leq f^{-1}\left(m\cdot 2^{-\widetilde{k}}-\right),$$

for some $1\leq m\leq 2^{\widetilde{k}}$, which further yields

$$0\leq f\left(\frac{i}{n}-\right)-f\left(\frac{i-1}{n}\right)\leq 2^{-\widetilde{k}}.$$

Since $3^{-\widetilde{k}}<\frac{3}{n}$,

$$\#\left\{i:\left]\frac{i-1}{n},\frac{i}{n}\right[\subset\left[f^{-1}\left((m-1)\cdot 2^{-\widetilde{k}}\right), f^{-1}\left(m\cdot 2^{-\widetilde{k}}-\right)\right]\right\}\leq 4,$$

for all $1\leq m\leq 2^{\widetilde{k}}$, which implies that $n-\#J\leq 2^{\widetilde{k}}\cdot 4$. Hence

$$d_r\left(\delta_\bullet^{u_n},\mu\right)^r=\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left|f(t)-F^{-1}_{\delta_\bullet^{u_n}}(t)\right|^r\mathrm{d}t=\sum_{i\notin J}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left|f(t)-F^{-1}_{\delta_\bullet^{u_n}}(t)\right|^r\mathrm{d}t$$

$$\leq \sum_{i \notin J} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left| f(t) - \frac{1}{2}\left(f\left(\frac{i}{n}-\right) + f\left(\frac{i-1}{n}\right)\right)\right|^r \mathrm{d}t \leq \frac{1}{2^r n} \sum_{i \notin J} \left\{ f\left(\frac{i}{n}-\right) - f\left(\frac{i-1}{n}\right)\right\}^r$$

$$\leq \frac{1}{2^r n} \sum_{i \notin J} 2^{-r\widetilde{k}} = \frac{1}{2^{\left(\widetilde{k}+1\right) r} n} (n - \#J) \leq \frac{2^{\widetilde{k}+2}}{2^{\left(\widetilde{k}+1\right) r} n} \leq \frac{2}{n} 2^{\left(\widetilde{k}+1\right)(1-r)} < 2 \cdot n^{-r\alpha_r},$$

which implies that

$$\limsup_{n \to \infty} n^{\alpha_r} d_r \left(\delta_\bullet^{u_n}, \mu\right) \leq 2^{1/r}.$$

Putting things together, for every $r \geq 1$,

$$d_r\left(\delta_{1/2}, \mu\right) \cdot \frac{1}{4}\left(\frac{2}{9}\right)^{1/r} \leq \liminf_{n \to \infty} n^{\alpha_r} d_r \left(\delta_\bullet^{u_n}, \mu\right) \leq \limsup_{n \to \infty} n^{\alpha_r} d_r \left(\delta_\bullet^{u_n}, \mu\right) \leq 2^{1/r}.$$

Finally, we show that the sequence $\left(n^{\alpha_r} d_r \left(\delta_\bullet^{u_n}, \mu\right)\right)$ is not constant for $r = 1,\ 2$. Obviously, $\delta_{\bullet,r}^{u_1} = \delta_{1/2}$ for all $r \geq 1$. Observe that

$$1/4 = 2 \sum_{i=1}^{\infty} 3^{-2i}, \quad f\left(\frac{1}{4}\right) = \sum_{i=2}^{\infty} 2^{-2i} = \frac{1}{3},$$

which implies by symmetry that $\delta_{\bullet,1}^{u_2} = \frac{1}{2}\left(\delta_{1/3} + \delta_{2/3}\right)$.

By self-similarity of μ^{-1} and symmetry,

$$\begin{aligned}
d_1\left(\delta_{\bullet,1}^{u_n}, \mu\right) &= 2 \int_0^{1/3} \left(\frac{1}{2} - f\right) \\
=& 2\left\{ \int_0^{1/9} \left(\frac{1}{4} + \frac{1}{4} - f\right) + \int_{1/9}^{2/9} \left(\frac{1}{2} - \frac{1}{4}\right) + \int_{2/9}^{1/3} \left(\frac{1}{2} - f\right)\right\} \\
=& 2\left\{ \int_0^{1/9} \frac{1}{4} + \int_{1/9}^{2/9} \left(\frac{1}{2} - \frac{1}{4}\right) + \int_0^{1/9} \left(\frac{1}{4} - f\right) + \int_{2/9}^{1/3} \left(f - \frac{1}{4}\right)\right\} \\
=& 2\left\{ \frac{2}{9} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{2} d_1\left(\delta_{1/2}, \mu\right)\right\},
\end{aligned}$$

yielding $d_1\left(\delta_{1/2}, \mu\right) = 1/6$. Analogously,

$$\begin{aligned}
d_1\left(\delta_{\bullet,1}^{u_2}, \mu\right) =& 2 \int_0^{1/2} \left| f - \frac{1}{3}\right| \\
=& 2\left\{ \int_0^{1/6} \left(\frac{1}{3} - f\right) + \int_{1/6}^{1/3} \left| f - \frac{1}{3}\right| + \int_{1/3}^{1/2} \left(\frac{1}{2} - \frac{1}{3}\right)\right\} \\
=& 2\left\{ \int_0^{1/6} \left(\frac{1}{3} - \frac{1}{4}\right) + \int_0^{1/6} \left(\frac{1}{4} - f\right) + \int_{1/6}^{1/3} \left| f - \frac{1}{3}\right| + \frac{1}{36}\right\} \\
=& 2\left\{ \frac{1}{72} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} d_r\left(\delta_{1/2}, \mu\right) + \frac{1}{3} \cdot \frac{1}{2} \int_{1/6}^{1/3} \left| f - \frac{2}{3}\right| + \frac{1}{36}\right\} \\
=& 2\left\{ \frac{1}{72} + \frac{1}{12} \cdot \frac{1}{6} + \frac{1}{6} d_1\left(\delta_\bullet^{u_2}, \mu\right) + \frac{1}{36}\right\} = \frac{1}{9} + \frac{1}{6} d_1\left(\delta_\bullet^{u_2}, \mu\right),
\end{aligned}$$

yielding $2^{\alpha_1} d_1\left(\delta_{\bullet,1}^{u_2}, \mu\right) = 4/15$. Thus $(n^{\alpha_1} d_1(\delta_\bullet^{u_n}, \mu))$ is not constant.

Next we deal with the case $r = 2$. Since

$$2\int_0^{1/2} f = 2\left(\int_0^{1/3} f + \int_{1/3}^{1/2} \frac{1}{2}\right) = 2\left(\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{2}\right) = \frac{1}{3}, \quad 2\int_{1/2}^{1} f = 2\left(\int_0^{1/2} f + \frac{1}{3}\cdot\frac{1}{2}\right) = \frac{2}{3},$$

it follows from Remark 2.30(ii) that $\delta_\bullet^{u_2} = \frac{1}{2}(\delta_{1/3} + \delta_{2/3})$. Like $r = 1$, straightforward calculations yield:

$$\begin{aligned} d_2\left(\delta_{\bullet,2}^{u_1}, \mu\right)^2 =& 2\int_0^{1/3}\left(\frac{1}{2} - f\right)^2 = 2\left\{\int_0^{1/3}\left(\frac{1}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4} - f\right) + \left(\frac{1}{4} - f\right)^2\right\} \\ =& 2\left\{\frac{1}{48} + \frac{1}{2}\int_0^{1/3}\left(\frac{1}{4} - f\right) + \int_0^{1/3}\left(\frac{1}{4} - f\right)^2\right\} \\ =& 2\left\{\frac{1}{48} + 0 + \frac{1}{3}\left(\frac{1}{2}\right)^2\int_0^1\left(f - \frac{1}{2}\right)^2\right\} = \frac{1}{24} + \frac{1}{6} d_2\left(\delta_{1/2}, \mu\right)^2, \end{aligned}$$

i.e., $d_2\left(\delta_{1/2}, \mu\right) = \frac{1}{2\sqrt{5}}$. Hence

$$\begin{aligned} d_2\left(\delta_{\bullet,2}^{u_2}, \mu\right)^2 = 2\int_0^{\frac{1}{2}}\left(f - \frac{1}{3}\right)^2 &= 2\left\{\int_0^{1/2}\left(\frac{1}{2} - f\right)^2 - \frac{1}{3}\int_0^{1/2}\left(\frac{1}{2} - f\right) + \frac{1}{2}\cdot\frac{1}{36}\right\} \\ &= d_2\left(\delta_{1/2}, \mu\right)^2 - \frac{1}{3} d_1\left(\delta_{1/2}, \mu\right) + \frac{1}{36} = \frac{1}{45}, \end{aligned}$$

i.e., $2^{\alpha_2} d_2(\delta_\bullet^{u_2}, \mu) = \frac{2^{\log 6/\log 9}}{3\sqrt{5}}$. Thus $(n^{\alpha_2} d_2(\delta_\bullet^{u_n}, \mu))$ is not constant either.

For $r = 1,\ 2$, therefore, $(n^{\alpha_r} d_r(\delta_\bullet^{u_n}, \mu))$ is not constant, and consequently

$$\liminf_{n\to\infty} n^{\alpha_r} d_r(\delta_\bullet^{u_n}, \mu) < \limsup_{n\to\infty} n^{\alpha_r} d_r(\delta_\bullet^{u_n}, \mu),$$

i.e., the d_r-uniform quantization coefficient of μ does not exist.

3.6.2 Details of Example 2.54

In this subsection, we investigate best r-approximations of μ, and show that $\left(n^{\beta_r} d_r(\delta_\bullet^{\bullet,n}, \mu)\right)$ is bounded above and below by positive constants.

To estimate a lower bound, for every $n \in \mathbb{N}$, let $k = \lfloor \log_2 n \rfloor + 2$, and

$$I_j = \left[f^{-1}\left((j-1)2^{-k}\right), f^{-1}\left(j2^{-k}-\right)\right[\text{ and } f_j = f|_{I_j}, \quad \forall\, 1 \le j \le 2^k.$$

Note that $\lambda(I_j) = 3^{-k}$. By self-similarity of μ^{-1},

$$f_j(\cdot) = f\left(\cdot - f^{-1}\left((j-1)2^{-k}\right)\right) + (j-1)2^{-k}, \quad \forall\, 1 \le j \le 2^k.$$

Let $J = \left\{ 1 \leq j \leq 2^k : F^{-1}_{\delta_\bullet^{\bullet,n}}\Big|_{I_j} \text{ is constant} \right\}$. Since $F^{-1}_{\delta_\bullet^{\bullet,n}}$ has at most $n-1$ jumps on $[0,1[$, $\#J \geq 2^k - n + 1 > 2^{k-1}$. Again by self-similarity of μ^{-1} and symmetry,

$$
\begin{aligned}
d_r\left(\delta_\bullet^{\bullet,n}, \mu\right)^r \geq & \sum_{j\in J} \int_{I_j} \left| f_j(t) - F^{-1}_{\delta_\bullet^{\bullet,n}}(t) \right|^r \mathrm{d}t \geq \sum_{j\in J} \min\nolimits_{c_j\in\mathbb{R}} \int_{I_j} |f_j(t) - c_j|^r \,\mathrm{d}t \\
=& \#J \cdot \min\nolimits_{c_1\in\mathbb{R}} \int_{I_1} |f(t) - c_1|^r \,\mathrm{d}t = \#J \cdot (2^r \cdot 3)^{-k} d_r(\delta_{1/2}, \mu)^r \\
\geq & \left(2^k - n + 1\right) (2^r \cdot 3)^{-k} d_r(\delta_{1/2}, \mu)^r \geq n^{-r\beta_r} 2^{-1-2r\beta_r} d_r(\delta_{1/2}, \mu)^r,
\end{aligned}
$$

which yields

$$
\liminf_{n\to\infty} n^{\beta_r} d_r\left(\delta_\bullet^{\bullet,n}, \mu\right) \geq 2^{-\frac{1}{r}-2\beta_r} d_r\left(\delta_{1/2}, \mu\right).
$$

To establish an upper bound, first pick a subsequence $(2^m - 1)$ of (n): For every $m \in \mathbb{N}$, let $\nu_m = \sum_{l=1}^{m-1} 3^{-l} \sum_{j=1}^{2^{l-1}} \delta_{(2j-1)2^{-l}} + 3^{-m+1} \sum_{j=1}^{2^{m-1}} \delta_{(2j-1)\cdot 2^{-m}}$. By similarity of μ^{-1} and symmetry, $d_r\left(\nu_{m+1}, \mu\right)^r = 2 \cdot 2^{-r} 3^{-1} d_r\left(\nu_m, \mu\right)^r$. Thus $d_r\left(\nu_m, \mu\right)^r = 2^{r\beta_r(1-m)} d_r\left(\delta_{1/2}, \mu\right)^r$, for all $m \in \mathbb{N}$. Now for every $n \in \mathbb{N}$, let $\widetilde{k} = \lfloor \log_2(n+1) \rfloor$, and thus

$$
n \geq 2^{\widetilde{k}} - 1, \quad 2^{\widetilde{k}+1} \geq n + 2.
$$

By monotonicity,

$$
d_r\left(\delta_\bullet^{\bullet,n}, \mu\right) \leq d_r\left(\delta_\bullet^{\bullet,2^{\widetilde{k}}-1}, \mu\right) \leq d_r\left(\nu_{\widetilde{k}}, \mu\right) = 2^{\beta_r(1-\widetilde{k})} d_r\left(\delta_{1/2}, \mu\right) \leq 4^{\beta_r} n^{-\beta_r} d_r\left(\delta_{1/2}, \mu\right),
$$

which yields

$$
\limsup_{n\to\infty} n^{\beta_r} d_r\left(\delta_\bullet^{\bullet,n}, \mu\right) \leq 4^{\beta_r} d_r\left(\delta_{1/2}, \mu\right).
$$

3.7 Details of Example 2.48

Recall that $\mu = \frac{1}{2}\lambda\Big|_{[0,1]} + \frac{1}{2}\delta_1$. Let $n = 2$, and δ_x^p a best r-approximation of μ. By Theorem 2.47 as well as (2.2), $0 \leq x_{,1} < x_{,2} \leq 1$. By virtue of the variational expression for the n-th quantization error [40],

$$
\begin{aligned}
d_r(\delta_x^p, \mu)^r = & \int_0^1 \min_{i=1,2} |y - x_{,i}|^r \,\mu(\mathrm{d}y) = \frac{1}{2} \int_0^1 \min_{i=1,2} |y - x_{,i}|^r \,\mathrm{d}y + \frac{1}{2} \min_{i=1,2} |1 - x_{,i}|^r \\
=& \frac{1}{2} \left\{ \int_0^{x_{,1}} (x_{,1} - y)^r \mathrm{d}y + \int_{x_{,1}}^{\frac{1}{2}(x_{,1}+x_{,2})} (y - x_{,1})^r \mathrm{d}y + \int_{\frac{1}{2}(x_{,1}+x_{,2})}^{x_2} (x_2 - y)^r \mathrm{d}y \right. \\
& \left. + \int_{x_{,2}}^1 (y - x_{,2})^r \mathrm{d}y + (1 - x_{,2})^r \right\}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\left\{\frac{1}{1+r}\left\{x_{,1}^{1+r}+\left(\frac{1}{2}(x_{,2}-x_{,1})\right)^{1+r}+\left(\frac{1}{2}(x_{,2}-x_{,1})\right)^{1+r}+(1-x_{,2})^{1+r}\right\}+(1-x_{,2})^{r}\right\}\\
=&\frac{1}{2(1+r)}x_{,1}^{1+r}+\frac{1}{2^{1+r}(1+r)}(x_{,2}-x_{,1})^{1+r}+\frac{1}{2(1+r)}(1-x_{,2})^{1+r}+\frac{1}{2}(1-x_{,2})^{r}.
\end{aligned}
$$

Since δ_x^p is a best r-approximation of μ,

$$\nabla_x d_r\left(\delta_x^p,\mu\right)^r=0,$$

i.e.,

$$
\begin{cases}
\frac{1}{2}x_{,1}^{r}-\frac{1}{2^{1+r}}(x_{,2}-x_{,1})^{r}=0,\\
\frac{1}{2^{1+r}}(x_{,2}-x_{,1})^{r}-\frac{1}{2}(1-x_{,2})^{r}-\frac{r}{2}(1-x_{,2})^{r-1}=0.
\end{cases}
\tag{3.23}
$$

Let $x_{,1}=\xi(r)$. By (3.23), $x_{,2}=3\xi$, and

$$\xi^r=(1-3\xi)^r+r(1-3\xi)^{r-1}. \tag{3.24}$$

Note that $\xi\in]0,1/3[$ for all $r\geq 1$. By Theorem 2.47(i),

$$\delta_{\bullet,r}^{\bullet;2}=\xi(r)\delta_{\xi(r)}+\left(1-\xi(r)\right)\delta_{3\xi(r)}.$$

To simplify (3.24), let $\eta=\xi^{-1}-3$. Then (3.24) is equivalent to

$$1=(1+r)\eta^r+3r\eta^{r-1}. \tag{3.25}$$

Obviously, $\eta\in]0,1[$ and $\eta(2)=\frac{2}{\sqrt{3}}-1$. Observe that

$$\frac{\mathrm{d}\eta}{\mathrm{d}r}=-\frac{\eta^r+3\eta^{r-1}}{\log\eta},$$

which shows that $\eta=\eta(r)$ is increasing on $]1,+\infty[$, and thus $\eta(1+):=\eta_1$ and $\eta(+\infty):=\eta_\infty$ both exist. Suppose $\eta_1>0$. Letting $r\downarrow 1$, it follows from (3.25) that $\eta=-1$,, which contradicts $\eta_1\geq 0$. Hence $\eta_1=0$. Analogously, one can prove $\eta_\infty=1$. In sum, converting η back to ξ, it is clear that $r\mapsto\xi(r)$ is smooth, decreasing, with

$$\xi(1)=\frac{1}{3},\ \xi(2)=\frac{3-\sqrt{3}}{4},\ \text{and}\ \lim_{r\to+\infty}\xi(r)=\frac{1}{4}.$$

3.8 Details of Example 2.49

Recall that $\mu = \frac{1}{3}\lambda\Big|_{[-1,1]} + \frac{1}{3}\delta_0$. In the following ,we characterize all δ_x^p satisfying Theorem 2.47 for $n = 2$. Analogous to the previous section, $-1 \leq x_{,1} < x_{,2} \leq 1$.

First, consider the case $r = 1$. By symmetry, it suffices to consider only the four situations below.

(i) Assume that $x_{,1} < 0 < x_{,2}$ and $x_{,1} + x_{,2} < 0$. Then Theorem 2.47 yields

$$p_{,1} = \frac{\frac{1}{2}(x_{,1} + x_{,2}) + 1}{3}, \qquad \frac{x_{,1} + 1}{3} = \frac{1}{2}p_{,1}, \qquad \frac{x_{,2} + 2}{3} = \frac{1}{2}\left(1 + p_{,1}\right). \tag{3.26}$$

The unique solution to (3.26) is

$$x_{,1} = -3/4,\ x_{,2} = -1/4,\ p_{,1} = 1/6.$$

This yields a contradiction with the assumption that $x_{,2} > 0$.

Analogous arguments also verify that contradiction occurs as long as $x_{,1} < 0 < x_{,2}$ and $x_{,1} + x_{,2} > 0$ are assumed simultaneously.

(ii) Assume that $x_{,1} < 0 < x_{,2}$ and $x_{,1} + x_{,2} = 0$. Again by Theorem 2.47,

$$p_{,1} \in [1/3, 2/3], \qquad \frac{x_{,1} + 1}{3} = \frac{1}{2}p_{,1}, \qquad \frac{x_{,2} + 2}{3} = \frac{1}{2}(1 + p_{,1}),$$

whose unique solution is

$$x_{,1} = -1/4,\ x_{,2} = 1/4,\ p_{,1} = 1/2.$$

This corresponds to the second candidate $\delta_{x_2}^{p_2} := \frac{1}{2}\delta_{-1/4} + \frac{1}{2}\delta_{1/4}$ in Example 2.49.

(iii) Assume that $x_{,1} < x_{,2} < 0$. Similarly, one concludes that

$$p_{,1} = \frac{\frac{1}{2}\left(x_{,1} + x_{,2}\right) + 1}{3}, \qquad \frac{x_{,1} + 1}{3} = \frac{1}{2}p_{,1}, \qquad \frac{x_{,2} + 1}{3} = \frac{1}{2}\left(1 + p_{,1}\right),$$

and hence

$$x_{,1} = -1/4,\ x_{,2} = 5/4,\ p_{,1} = 1/2.$$

This contradicts the assumption that $x_{,2} < 0$. A similar contradiction will appear if $0 < x_{,1} < x_{,2}$ is assumed.

(iv) Assume $x_{,1} < 0 = x_{,2}$. One obtains

$$p_{,1} = \frac{\frac{1}{2}x_{,1} + 1}{3}, \qquad \frac{x_{,1} + 1}{3} = \frac{1}{2}p_{,1}, \qquad \frac{1}{2}(1 + p_{,1}) \in \left[\frac{1}{3}, \frac{2}{3}\right],$$

and thus

$$x_{,1} = -2/3,\ p_{,1} = 2/9.$$

corresponds to the first candidate $\delta_{x_1}^{p_1} := \frac{2}{9}\delta_{-2/3} + \frac{7}{9}\delta_0$ in Example 2.49.

By symmetry, $\delta_{x_3}^{p_3} := \frac{7}{9}\delta_0 + \frac{2}{9}\delta_{2/3}$ is the third candidate.

Now we compare $d_1(\delta_{x_i}^{p_i}, \mu)$ for $i = 1, 2, 3$. Straightforward calculations yield

$$d_1\left(\delta_{x_1}^{p_1}, \mu\right) = d_1\left(\delta_{x_3}^{p_3}, \mu\right) = \frac{2}{9} < \frac{7}{24} = d_1\left(\delta_{x_2}^{p_2}, \mu\right),$$

and thus only the two (non-symmetric) probability measures $\delta_{x_1}^{p_1}$, $\delta_{x_3}^{p_3}$ are best 1-approximations of μ.

Analogously for $r = 2$, consider the following three cases.

(i) Assume $x_{,1} + x_{,2} < 0$. Then Theorem 2.47 yields

$$p_{,1} = \frac{\frac{1}{2}(x_{,1} + x_{,2}) + 1}{3} < 1/3,\ x_{,1} = \frac{1}{p_{,1}} \int_0^{p_{,1}} (3t-1)\mathrm{d}t = \frac{3p_{,1}}{2} - 1,$$

$$x_{,2} = \frac{1}{1-p_{,1}} \left(\int_{p_{,1}}^{1/3} (3t-1)\mathrm{d}t + \int_{2/3}^{1} (3t-2)\mathrm{d}t \right) = \frac{p_{,1} - \frac{3}{2}p_{,1}^2}{1-p_{,1}},$$

and thus

$$x_{,1} = -\frac{1-\sqrt{33}}{8},\ x_{,2} = \frac{19-3\sqrt{33}}{8},\ p_{,1} = 1/6.$$

This corresponds to $\delta_{y_1}^{q_1} := \frac{9-\sqrt{33}}{12}\delta_{\frac{1-\sqrt{33}}{8}} + \frac{3+\sqrt{33}}{12}\delta_{\frac{19-3\sqrt{33}}{8}}$.

By symmetry, $\delta_{y_3}^{q_3} := \frac{3+\sqrt{33}}{12}\delta_{\frac{3\sqrt{33}-19}{8}} + \frac{9-\sqrt{33}}{12}\delta_{\frac{\sqrt{33}-1}{8}}$ is also a candidate.

(ii) Assume $x_{,1} + x_{,2} = 0$. It follows from Theorem 2.47 that

$$p_{,1} \in [1/3, 2/3],\ x_{,1} = \frac{1}{p_{,1}} \int_0^{1/3} (3t-1)\mathrm{d}t = -\frac{1}{6p_{,1}},$$

$$x_{,2} = \frac{1}{1-p_{,1}} \int_{2/3}^{1} (3t-2)\mathrm{d}t = \frac{1}{6(1-p_{,1})},$$

and thus

$$x_{,1} = -1/3,\ x_{,2} = 1/3,\ p_{,1} = 1/2.$$

This corresponds to $\delta_{y_2}^{q_2} := \frac{1}{2}\delta_{\frac{3\sqrt{33}-19}{8}} + \frac{1}{2}\delta_{\frac{\sqrt{33}-1}{8}}$. Tedious but elementary computations yield

$$\begin{aligned} d_2(\delta_{y_1}^{q_1}, \mu)^2 =& d_2(\delta_{y_3}^{q_3}, \mu)^2 = \frac{(3\sqrt{33}-11)^3 + 3(9-\sqrt{33})^3}{9 \cdot 8^3} + \frac{(19-3\sqrt{33})^2}{3 \cdot 8^2} \\ \approx& 0.0912788 < d_2\left(\delta_{y_2}^{q_2}, \mu\right)^2 = \frac{1}{9}. \end{aligned}$$

Thus again only $\delta_{y_1}^{q_1}$ and $\delta_{y_3}^{q_3}$ are best 2-approximations of μ.

Chapter 4

Best finite approximations of Benford's Law

Given real numbers $b > 1$ and $x \neq 0$, denote by $S_b(x)$ the unique number in $[1, b[$ such that $|x| = S_b(x)b^k$ for some (necessarily unique) integer k; for convenience, let $S_b(0) = 0$. The number $S_b(x)$ often is referred to as the *base-b significand* of x, a terminology particularly well-established in the case of b being an integer. (Unlike in much of the literature [3,5,47,89], the case of integer b does not carry special significance in this chapter.) A Borel probability measure μ on $\mathbb{R}$ is *Benford base b*, or *b-Benford* for short, if

$$\mu\Big(\{x \in \mathbb{R} : S_b(x) \leq s\}\Big) = \frac{\log s}{\log b}, \quad \forall\, s \in [1, b[\,; \tag{4.1}$$

here and throughout, log denotes the natural logarithm. Benford probabilities (or random variables) exhibit many interesting properties and have been studied extensively [1,32,48,70, 82]. They provide one major pathway into the study of *Benford's Law*, an intriguing, multi-faceted phenomenon that attracts interest from a wide range of disciplines; see, e.g., [5] for an introduction, and [70] for a panorama of recent developments. Specifically, denoting by β_b the Borel probability measure with

$$\beta_b([1, s]) = \frac{\log s}{\log b}, \quad \forall\, s \in [1, b[\,,$$

note that μ is b-Benford if and only if $\mu \circ S_b^{-1} = \beta_b$.

Historically, the case of *decimal* (i.e., base-10) significands has been the most prominent, with early empirical studies on the distribution of decimal significands (or significant digits) going back to Newcomb [73] and Benford [3]. If μ is 10-Benford, note that in particular

$$\mu\Big(\{x \in \mathbb{R} : \text{leading decimal digit of } x = D\}\Big) = \frac{\log(1 + D^{-1})}{\log 10}, \quad \forall\, D = 1, \ldots, 9\,. \tag{4.2}$$

For theoretical as well as practical reasons, mathematical objects such as random variables or sequences, but also concrete, finite numerical data sets that conform, at least approximately, to (4.1) or (4.2) have attracted much interest [27, 62, 89, 91]. Time and again, Benford's Law has emerged as a perplexingly prevalent phenomenon. One popular approach to understand this prevalence seeks to establish (mild) conditions on a probability measure that make (4.1) or (4.2) hold with good accuracy, perhaps even exactly [8, 31, 32, 35, 82]. It is the goal of this chapter to provide precise quantitative information for this approach.

Concretely, notice that while a finitely supported probability measure, such as, e.g., the empirical measure associated with a finite data set [6], may conform to the *first-digit law* (4.2), it cannot possibly satisfy (4.1) exactly. For such measures, therefore, it is natural to quantify, as accurately as possible, the failure of equality in (4.1), that is, the discrepancy between $\mu \circ S_b^{-1}$ and β_b. Utilizing three different familiar metrics d_* on probabilities (Lévy, Kantorovich, and Kolmogorov metrics; see Section 4.1 for details), this chapter does this in a systematic way: For every $n \in \mathbb{N}$, the value of $\min_\nu d_*(\beta_b, \nu)$ is identified, where ν is assumed to be supported on no more than n points (and may be subject to further restrictions such as, e.g., having only atoms of equal weight, as in the case of empirical measures); the minimizers of $d_*(\beta_b, \nu)$ are also characterized explicitly.

The scope of the results presented herein, however, extends far beyond Benford probabilities. In fact, a general theory of best (constrained or unconstrained) d_*-approximations is developed for probabilities with compact support. As far as the author can tell, no such theories exist for the Lévy and Kolmogorov metrics — unlike in the case of the Kantorovich metric where it (mostly) suffices to rephrase pertinent known facts [40, 102]. Once the general results are established, the desired quantitative insights for Benford probabilities are but straightforward corollaries. (Even in the context of Kantorovich distance, the study of β_b yields a rare new, explicit example of an *optimal quantizer* [40].) In particular, it turns out that, under all the various constraints considered here, the limit $Q_* = \lim_{n\to\infty} n \min_\nu d_*(\beta_b, \nu)$ always exists, is finite and positive, and can be computed more or less explicitly. This greatly extends earlier results, notably of [6], and also suggests that $n^{-1}Q_*$ may be an appropriate quantity against which to evaluate the many heuristic claims of closeness to Benford's Law for empirical data sets found in the literature [4, 70, 71].

This chapter is organized as follows: Section 4.1 reviews relevant basic properties of one-dimensional probabilities and the three main probability metrics used throughout. Each of the Sections 4.2 to 4.4 then is devoted specifically to one single metric. As indicated earlier, in each case the problem of best (constrained or unconstrained) approximation by finitely supported probability measures is first addressed in complete generality, and then the results are specialized to β_b. Section 4.5 summarizes and discusses the quantitative results obtained, and also mentions a few natural questions for subsequent studies. For the reader's convenience, proofs of propositions and informal claims in this chapter are assembled in Chapter 5.

4.1 Probability metrics

Throughout Chapters 4-5, let $\mathbb{J} \subset \mathbb{R}$ be a compact interval with $\lambda(\mathbb{J}) > 0$, and $\mathcal{P}$[1] the set of all Borel probability measures on $\mathbb{J}$. Associate with every $\mu \in \mathcal{P}$ its *distribution function* $F_\mu : \mathbb{R} \to \mathbb{R}$, given by

$$F_\mu(x) = \mu(\{y \in \mathbb{J} : y \le x\}), \quad \forall\, x \in \mathbb{R}\,,$$

as well as its (upper) *quantile function* $F_\mu^{-1} : [0,1[\to \mathbb{R}$, given by

$$F_\mu^{-1}(x) = \begin{cases} \min \mathbb{J} & \text{if } 0 \le x < \mu(\{\min \mathbb{J}\})\,, \\ \sup\{y \in \mathbb{J} : F_\mu(y) \le x\} & \text{if } \mu(\{\min \mathbb{J}\}) \le x < 1\,. \end{cases} \tag{4.3}$$

Note that F_μ and F_μ^{-1} both are non-decreasing, right-continuous, and bounded. The *support* of μ, denoted $\operatorname{supp}\mu$, is the smallest closed subset of $\mathbb{J}$ with μ-measure 1. Endowed with the weak topology, the space $\mathcal{P}$ is compact and metrizable.

Three important different metrics on $\mathcal{P}$ are discussed in detail in this chapter; for a panorama of other metrics the reader is referred, e.g., to [36, 87] and the references therein. Given probabilities $\mu, \nu \in \mathcal{P}$, their *Lévy distance* is

$$d_{\mathsf{L}}(\mu,\nu) = \omega \inf\{y \ge 0 : F_\mu(\cdot - y) - y \le F_\nu \le F_\mu(\cdot + y) + y\}\,, \tag{4.4}$$

with $\omega = \max\{1, \lambda(\mathbb{J})\}/\lambda(\mathbb{J})$; their L^r-*Kantorovich* (or *transport*) *distance*, with $r \ge 1$, is

$$d_r(\mu,\nu) = \lambda(\mathbb{J})^{-1} \left(\int_0^1 \left| F_\mu^{-1}(y) - F_\nu^{-1}(y) \right|^r \mathrm{d}y \right)^{1/r} = \lambda(\mathbb{J})^{-1} \|F_\mu^{-1} - F_\nu^{-1}\|_r; \tag{4.5}$$

and their *Kolmogorov* (or *uniform*) *distance* is

$$d_{\mathsf{K}}(\mu,\nu) = \sup\nolimits_{x \in \mathbb{R}} |F_\mu(x) - F_\nu(x)| = \|F_\mu - F_\nu\|_\infty\,.$$

Henceforth, the symbol d_* summarily refers to any of d_{L}, d_r, and d_{K}. The (unusual) normalizing factors in (4.4) and (4.5) guarantee that all three metrics are comparable numerically in that

[1] Note that this usage of $\mathcal{P}$ is different from the one in Chapters 2 and 3, where $\mathcal{P}$ denotes the set of all Borel probability measures μ on $\overline{\mathbb{R}}$ with $\mu(\mathbb{R}) = 1$.

$\sup_{\mu,\nu\in\mathcal{P}} d_*(\mu,\nu) = 1$ in either case. Note that

$$d_1(\mu,\nu) = \lambda(\mathbb{J})^{-1}\int_{\mathbb{J}} |F_\mu(x) - F_\nu(x)|\, \mathrm{d}x, \quad \forall\, \mu,\nu \in \mathcal{P}\,,$$

by virtue of Fubini's Theorem. The metrics d_{L} and d_r are equivalent: They both metrize the weak topology on $\mathcal{P}$, and hence are separable and complete. By contrast, the complete metric d_{K} induces a finer topology and is non-separable. However, when restricted to $\mathcal{P}_{\mathsf{cts}} := \{\mu \in \mathcal{P} : \mu(\{x\}) = 0 \ \ \forall\, x \in \mathbb{J}\}$, a dense G_δ-set in $\mathcal{P}$, the metric d_{K} does metrize the weak topology on $\mathcal{P}_{\mathsf{cts}}$ and is separable. The values of d_{L}, d_r, and d_{K} are not completely unrelated since, as is easily checked,

$$d_1 \le \frac{1+\lambda(\mathbb{J})}{\omega\lambda(\mathbb{J})} d_{\mathsf{L}}\,, \quad d_1 \le d_r\,, \quad d_1 \le d_{\mathsf{K}}\,, \quad d_{\mathsf{L}} \le \omega d_{\mathsf{K}}\,, \tag{4.6}$$

and all bounds in (4.6) are best possible. Beyond (4.6), however, no relative bounds exist between d_{L}, d_r, and d_{K} in general: If $* \ne 1$, $* \ne \circ$, and $(*,\circ) \ne (\mathsf{L},\mathsf{K})$ then

$$\sup_{\mu,\nu\in\mathcal{P}:\mu\ne\nu} \frac{d_*(\mu,\nu)}{d_\circ(\mu,\nu)} = +\infty\,.$$

For a detailed verification of the relations among these probability metrics, the reader is referred to Proposition 5.1. Each metric d_*, therefore, captures a different aspect of $\mathcal{P}$ and deserves to be studied independently. To illustrate this further, let $\mathbb{J} = [0,1]$, $\mu = \delta_0 \in \mathcal{P}$, and $\mu_k = (1-k^{-1})\,\delta_0 + k^{-1}\delta_{k^{-2}}$ for $k \in \mathbb{N}$; here and throughout, recall that δ_a denotes the Dirac (probability) measure concentrated at $a \in \mathbb{R}$. Then $\lim_{k\to\infty} d_*(\mu,\mu_k) = 0$, but the rate of convergence differs between metrics:

$$d_{\mathsf{L}}(\mu,\mu_k) = k^{-2}\,, \quad d_r(\mu,\mu_k) = k^{-2-1/r}\,, \quad d_{\mathsf{K}}(\mu,\mu_k) = k^{-1}, \quad \forall\, k \in \mathbb{N}.$$

The goal of this chapter is first to identify, for each metric d_* introduced earlier, the best finitely supported d_*-approximation(s) of any given $\mu \in \mathcal{P}$. The general results are then applied to Benford's Law. Specifically, if $\mu = \beta_b$ for some $b > 1$ then it is automatically assumed that $\mathbb{J} = [1,b]$. The following unified notation and terminology is used throughout Chapters 4 and 5: For every $n \in \mathbb{N}$, let $\Xi_n = \{x \in \mathbb{J}^n : x_{,1} \le \ldots \le x_{,n}\}$, $\Pi_n = \{p \in \mathbb{R}^n : p_{,j} \ge 0,\ \sum_{j=1}^n p_{,j} = 1\}$, and for each $x \in \Xi_n$ and $p \in \Pi_n$ define $\delta_x^p = \sum_{j=1}^n p_{,j}\delta_{x_{,j}}$. For convenience, $x_{,0} := -\infty$ and $x_{,n+1} := +\infty$ for every $x \in \Xi_n$, as well as $P_{,i} = \sum_{j=1}^i p_{,j}$ for $i = 0,\ldots,n$ and $p \in \Pi_n$; note that $P_{,0} = 0$ and $P_{,n} = 1$. Henceforth, usage of the symbol δ_x^p tacitly assumes that $x \in \Xi_n$ and $p \in \Pi_n$, for some $n \in \mathbb{N}$ either specified explicitly or else clear from the context. Call δ_x^p a *best d_*-approximation of $\mu \in \mathcal{P}$, given $x \in \Xi_n$* if

$$d_*\left(\mu, \delta_x^p\right) \le d_*\left(\mu, \delta_x^q\right), \quad \forall\, q \in \Pi_n .$$

Similarly, call δ_x^p a *best d_*-approximation of μ, given $p \in \Pi_n$* if

$$d_*\left(\mu, \delta_x^p\right) \le d_*\left(\mu, \delta_y^p\right), \quad \forall\, y \in \Xi_n .$$

Denote by $\delta_x^\bullet$ and $\delta_\bullet^p$ any best d_*-approximation of μ, given x and p, respectively. Best d_*-approximations, given $p = u_n = (n^{-1}, \ldots, n^{-1})$ are referred to as best *uniform* d_*-approximations, and denoted $\delta_\bullet^{u_n}$. Finally, δ_x^p is a *best d_*-approximation of* $\mu \in \mathcal{P}$, denoted $\delta_\bullet^{\bullet,n}$, if

$$d_*\left(\mu, \delta_x^p\right) \le d_*\left(\mu, \delta_y^q\right), \quad \forall\, y \in \Xi_n, q \in \Pi_n .$$

Notice that usage of the symbols $\delta_x^\bullet$, $\delta_\bullet^p$, and $\delta_\bullet^{\bullet,n}$ always refers to a specific metric d_* and probability measure $\mu \in \mathcal{P}$, both usually clear from the context.

Information theory sometimes refers to $d_*\left(\mu, \delta_\bullet^{\bullet,n}\right)$ as the *n-th quantization error*, and to $\lim_{n\to\infty} n d_*\left(\mu, \delta_\bullet^{\bullet,n}\right)$, if it exists, as the *quantization coefficient* of μ; see, e.g., [40]. By analogy, $d_*(\mu, \delta_\bullet^{u_n})$ and $\lim_{n\to\infty} n d_*(\mu, \delta_\bullet^{u_n})$, if it exists, may be called the *n-th uniform quantization error* and the *uniform quantization coefficient*, respectively.

4.2 Lévy approximations

This section identifies best finitely supported d_{L}-approximations (constrained or unconstrained) of a given $\mu \in \mathcal{P}$. To do this in a transparent way, it is helpful to first consider more generally a few elementary properties of non-decreasing functions. These properties are subsequently specialized to either F_μ or F_μ^{-1}.

Throughout, let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing, and define $f(\pm\infty) = \lim_{x\to\pm\infty} f(x) \in \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real line with the usual order and topology. Associate with f two non-decreasing functions $f_\pm : \mathbb{R} \to \overline{\mathbb{R}}$, defined as $f_\pm(x) = \lim_{\varepsilon\downarrow 0} f(x \pm \varepsilon)$. Clearly, f_- is left-continuous whereas f_+ is right-continuous, with $f_\pm(-\infty) = f(-\infty)$, $f_\pm(+\infty) = f(+\infty)$, as well as $f_- \le f \le f_+$, and $f_+(x) \le f_-(y)$ whenever $x < y$; in particular, $f_-(x) = f_+(x)$ if and only if f is continuous at x. Recall that the (upper) *inverse function* $f^{-1} : \mathbb{R} \to \overline{\mathbb{R}}$ is given by

$$f^{-1}(t) = \sup\{x \in \mathbb{R} : f(x) \le t\}, \quad \forall\, t \in \mathbb{R} ;$$

by convention, $\sup\varnothing := -\infty$ (and $\inf\varnothing := +\infty$). Note that (4.3) is consistent with this notation. For what follows, it is useful to recall a few basic properties of inverse functions.

Proposition (2.1)**.** *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing. Then f^{-1} is non-decreasing and right-continuous. Also, $(f_\pm)^{-1} = f^{-1}$, and $(f^{-1})^{-1} = f_+$.*

Given two non-decreasing functions $f, g : \mathbb{R} \to \overline{\mathbb{R}}$, by a slight abuse of notation, and inspired by (4.4), let

$$d_{\mathsf{L}}(f,g) = \inf\{y \ge 0 : f(\,\cdot\, - y) - y \le g \le f(\,\cdot\, + y) + y\} \in [0, +\infty]\,.$$

For instance, $d_{\mathsf{L}}(\mu,\nu) = \omega d_{\mathsf{L}}(F_\mu, F_\nu)$ for all $\mu,\nu \in \mathcal{P}$. It is readily checked that d_{L} is symmetric, satisfies the triangle inequality, and $d_{\mathsf{L}}(f,g) > 0$ unless $f_- = g_-$, or equivalently, $f_+ = g_+$. Crucially, the quantity d_{L} is invariant under inversion.

Proposition 4.1. *Let $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing. Then $d_{\mathsf{L}}(f^{-1}, g^{-1}) = d_{\mathsf{L}}(f,g)$.*

Thus, for instance, $d_{\mathsf{L}}(\mu,\nu) = \omega d_{\mathsf{L}}(F_\mu^{-1}, F_\nu^{-1})$ for all $\mu,\nu \in \mathcal{P}$. In general, the value of $d_{\mathsf{L}}(f,g)$ may equal $+\infty$. (For instance, take $f(x) \equiv x$ and $g(x) \equiv 0$.) However, if the set $\{f \ne g\} := \{x \in \mathbb{R} : f(x) \ne g(x)\}$ is bounded then $d_{\mathsf{L}}(f,g) < +\infty$. Specifically, notice that $\{F_\mu \ne F_\nu\} \subset \mathbb{J}$ and $\{F_\mu^{-1} \ne F_\nu^{-1}\} \subset [0,1[$ both are bounded for all $\mu,\nu \in \mathcal{P}$.

Given a non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$, let $I \subset \overline{\mathbb{R}}$ be any interval with the property that

$$f_-(\sup I), -f_+(\inf I) < +\infty\,, \tag{4.7}$$

and define an auxiliary function $\ell_{f,I} : \mathbb{R} \to \mathbb{R}$ as

$$\ell_{f,I}(x) = \inf\{y \ge 0 : f_-(\sup I - y) - y \le x \le f_+(\inf I + y) + y\}\,.$$

Note that for each $x \in \mathbb{R}$, the set on the right equals $[a, +\infty[$ with the appropriate $a \ge 0$, and hence simply $\ell_{f,I}(x) = a$. Clearly, $\ell_{f,J} \le \ell_{f,I}$ whenever $J \subset I$. Also, for every $a \in \mathbb{R}$, the function $\ell_{f,\{a\}}$ is non-increasing on $]-\infty, f_-(a)]$, vanishes on $[f_-(a), f_+(a)]$, and is non-decreasing on $[f_+(a), +\infty[$; see Proposition 5.2. A few elementary properties of $\ell_{f,I}$ are straightforward to check; they are used below to establish the main results of this section.

Proposition 4.2. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing, and $I \subset \overline{\mathbb{R}}$ an interval satisfying (4.7). Then $\ell_{f,I}$ is Lipschitz continuous, and*

$$0 \le \ell_{f,I}(x) \le |x| + \max\{0, f_-(\sup I), -f_+(\inf I)\}, \quad \forall\, x \in \mathbb{R}\,.$$

Moreover, $\ell_{f,I}$ attains a minimal value

$$\ell^*_{f,I} := \min_{x\in\mathbb{R}} \ell_{f,I}(x) = \min\{y \geq 0 : f_-(\sup I - y) - y \leq f_+(\inf I + y) + y\} \ \geq \ 0$$

which is positive unless $f_-(\sup I) \leq f_+(\inf I)$.

For $\mu \in \mathcal{P}$, note that (4.7) automatically holds if $f = F_\mu$, or if $f = F_\mu^{-1}$ and $I \subset [0,1]$. In these cases, therefore, $\ell_{f,I}$ has the properties stated in Proposition 4.2, and $\ell^*_{f,I} \leq \frac{1}{2}$.

When formulating the main results, the following quantities are useful: Given $\mu \in \mathcal{P}$, $n \in \mathbb{N}$, and $x \in \Xi_n$, let

$$\mathsf{L}^\bullet(x) = \max\left\{\ell_{F_\mu,[-\infty,x_{,1}]}(0), \ell^*_{F_\mu,[x_{,1},x_{,2}]}, \ldots, \ell^*_{F_\mu,[x_{,n-1},x_{,n}]}, \ell_{F_\mu,[x_{,n},+\infty]}(1)\right\} ;$$

similarly, given $p \in \Pi_n$, let

$$\mathsf{L}_\bullet(p) = \max\nolimits_{j=1}^{n} \ell^*_{F_\mu^{-1},[P_{,j-1},P_{,j}]} \,.$$

To illustrate these quantities for a concrete example, consider $\mu = \beta_b$, where $\ell^*_{F_\mu,[x_{,j},x_{,j+1}]}$ is the unique solution of

$$b^{2\ell} = \frac{x_{,j+1} - \ell}{x_{,j} + \ell}, \quad j = 1, \ldots, n-1\,,$$

whereas $\ell_{F_\mu,[-\infty,x_{,1}]}(0)$ and $\ell_{F_\mu,[x_{,n},+\infty]}(1)$ solve $b^\ell = x_{,1} - \ell$ and $b^\ell = b/(x_{,n} + \ell)$, respectively. (Recall that $1 \leq x_{,1} \leq \ldots \leq x_{,n} \leq b$.) Similarly, $\ell^*_{F_\mu^{-1},[P_{,j-1},P_{,j}]}$ is the unique solution of

$$2\ell = b^{P_{,j}-\ell} - b^{P_{,j-1}+\ell}, \quad j = 1, \ldots, n\,;$$

in particular, $j \mapsto \ell^*_{F_\mu^{-1},[(j-1)/n,j/n]}$ is increasing, and hence $\mathsf{L}_\bullet(u_n)$ is the unique solution of

$$2L = b^{1-L} - b^{1+L-1/n}\,. \tag{4.8}$$

By using functions of the form $\ell_{f,I}$, the value of $d_{\mathsf{L}}(\mu,\nu)$ can easily be computed whenever ν has finite support.

Lemma 4.3. *Let $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$. For every $x \in \Xi_n$ and $p \in \Pi_n$,*

$$d_{\mathsf{L}}\left(\mu, \delta_x^p\right) = \omega \max\nolimits_{j=0}^{n} \ell_{F_\mu,[x_{,j},x_{,j+1}]}(P_{,j}) = \omega \max\nolimits_{j=1}^{n} \ell_{F_\mu^{-1},[P_{,j-1},P_{,j}]}(x_{,j})\,. \tag{4.9}$$

Proof. Label $x \in \Xi_n$ uniquely as

$$x_{,j_0+1} = \ldots = x_{,j_1} < x_{,j_1+1} = \ldots = x_{,j_2} < x_{,j_2+1} = \ldots$$
$$< \ldots = x_{,j_{m-1}} < x_{,j_{m-1}+1} = \ldots = x_{,j_m}\,,$$

with integers $i \le j_i \le n$ for $1 \le i \le m$, and $j_0 = 0$, $j_m = n$, and define $y \in \Xi_m$ and $q \in \Pi_m$ as $y_{,i} = x_{,j_i}$ and $q_{,i} = P_{,j_i} - P_{,j_{i-1}}$, respectively, for $i = 1, \ldots, m$. For convenience, let $I_j = [x_{,j}, x_{,j+1}]$ for $j = 0, \ldots, n$, and $J_i = [y_{,i}, y_{,i+1}] = I_{j_i}$ for $i = 0, \ldots, m$. With this, $\delta_y^q = \delta_x^p$, and

$$\begin{aligned}\omega^{-1} d_{\mathsf{L}}\left(\mu, \delta_x^p\right) &= d_{\mathsf{L}}(F_\mu, F_{\delta_y^q}) \\ &= \inf\{t \ge 0 : F_{\mu-}(y_{,i+1} - t) - t \le Q_{,i} \le F_\mu(y_{,i} + t) + t \quad \forall\, i = 0, \ldots, m\} \\ &= \max\nolimits_{i=0}^m \ell_{F_\mu, J_i}(Q_{,i}) \\ &\le \max\nolimits_{j=0}^n \ell_{F_\mu, I_j}(P_{,j}) \,.\end{aligned}$$

To prove the reverse inequality, pick any $j = 0, \ldots, n$. If $x_{,j} < x_{,j+1}$ then $I_j = J_i$ and $P_{,j} = Q_{,i}$, with the appropriate i, and hence $\ell_{F_\mu, I_j}(P_{,j}) = \ell_{F_\mu, J_i}(Q_{,i})$. If $x_{,j} = x_{,j+1}$ then $I_j = \{y_{,i}\}$ for some i. In this case either $P_{,j} < F_{\mu-}(y_{,i})$ and $Q_{,i-1} \le P_{,j}$, and hence

$$\ell_{F_\mu, I_j}(P_{,j}) = \ell_{F_\mu, \{y_{,i}\}}(P_{,j}) \le \ell_{F_\mu, \{y_{,i}\}}(Q_{,i-1}) \le \ell_{F_\mu, J_{i-1}}(Q_{,i-1})\,;$$

or $F_{\mu-}(y_{,i}) \le P_{,j} \le F_\mu(y_{,i})$, and hence $\ell_{F_\mu, I_j}(P_{,j}) = \ell_{F_\mu, \{y_{,i}\}}(P_{,j}) = 0$; or $P_{,j} > F_\mu(y_{,i})$ and $Q_{,i} \ge P_{,j}$, and hence

$$\ell_{F_\mu, I_j}(P_{,j}) = \ell_{F_\mu, \{y_{,i}\}}(P_{,j}) \le \ell_{F_\mu, \{y_{,i}\}}(Q_{,i}) \le \ell_{F_\mu, J_i}(Q_{,i})\,.$$

In all three cases, therefore, $\omega^{-1} d_{\mathsf{L}}\left(\mu, \delta_x^p\right) \ge \max_{j=0}^n \ell_{F_\mu, I_j}(P_{,j})$, which establishes the first equality in (4.9). The second equality, a consequence of Proposition 4.1, is proved analogously. □

Utilizing Lemma 4.3, it is straightforward to characterize the best finitely supported d_{L}-approximations of $\mu \in \mathcal{P}$ with prescribed locations.

Theorem 4.4. *Let $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$. For every $x \in \Xi_n$, there exists a best d_{L}-approximation of μ, given x. Moreover, $d_{\mathsf{L}}\left(\mu, \delta_x^p\right) = d_{\mathsf{L}}\left(\mu, \delta_x^{\bullet}\right)$ if and only if, for every $j = 0, \ldots, n$,*

$$x_{,j} < x_{,j+1} \;\; \textit{implies} \;\; \ell_{F_\mu, [x_{,j}, x_{,j+1}]}(P_{,j}) \le \mathsf{L}^{\bullet}(x)\,, \tag{4.10}$$

and in this case $d_{\mathsf{L}}\left(\mu, \delta_x^p\right) = \omega \mathsf{L}^{\bullet}(x)$.

Proof. Fix $\mu \in \mathcal{P}$, $n \in \mathbb{N}$, and $x \in \Xi_n$. As in the proof of Lemma 4.3, write $I_j = [x_{,j}, x_{,j+1}]$ for convenience. By (4.9), for every $p \in \Pi_n$,

$$d_{\mathsf{L}}\left(\mu, \delta_x^p\right) = \omega \max\nolimits_{j=0}^n \ell_{F_\mu, I_j}(P_{,j}) \ge \omega \max\{\ell_{F_\mu, I_0}(0), \ell^*_{F_\mu, I_1}, \ldots, \ell^*_{F_\mu, I_{n-1}}, \ell_{F_\mu, I_n}(1)\} = \omega \mathsf{L}^{\bullet}(x)\,.$$

As seen in the proof of Lemma 4.3, validity of (4.10) implies $\ell_{F_\mu,[x_{,j},x_{,j+1}]}(P_{,j}) \le \mathsf{L}^\bullet(x)$ for *all* $j = 0,\ldots,n$. Thus δ_x^p is a best d_L-approximation of μ, given x, whenever (4.10) holds, i.e., the latter is sufficient for optimality. On the other hand, consider $q \in \Pi_n$ with

$$Q_{,j} = \frac{1}{2}\Big(F_{\mu-}\big(x_{,j+1} - \mathsf{L}^\bullet(x)\big) + F_\mu\big(x_{,j} + \mathsf{L}^\bullet(x)\big)\Big), \quad \forall\, j = 1,\ldots,n-1\,.$$

Note that q is well-defined, since $j \mapsto Q_{,j}$ is non-decreasing, and $0 \le Q_{,j} \le 1$ for all $j = 1,\ldots,n-1$. Moreover, by the definition of $\mathsf{L}^\bullet(x)$,

$$\ell_{F_\mu,I_j}\left(Q_{,j}\right) \le \mathsf{L}^\bullet(x), \quad \forall\, j = 0,\ldots,n\,,$$

and hence $d_\mathsf{L}\left(\delta_x^q,\mu\right) = \omega \mathsf{L}^\bullet(x)$. This shows that best d_L-approximations of μ, given x, do exist, and (4.10) also is necessary for optimality. □

Best finitely supported d_L-approximations of any $\mu \in \mathcal{P}$ with prescribed weights can be characterized in a similar manner. By virtue of (4.9), the proof of the following is analogous to the proof of Theorem 4.4 above, but included for the reader's convenience.

Theorem 4.5. *Let $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$. For every $p \in \Pi_n$, there exists a best d_L-approximation of μ, given p. Moreover, $d_\mathsf{L}\left(\mu,\delta_x^p\right) = d_\mathsf{L}\left(\mu,\delta_\bullet^p\right)$ if and only if, for every $j = 1,\ldots,n$,*

$$P_{,j-1} < P_{,j} \ \textit{implies}\ \ \ell_{F_\mu^{-1},[P_{,j-1},P_{,j}]}(x_{,j}) \le \mathsf{L}_\bullet(p), \tag{4.11}$$

and in this case $d_\mathsf{L}\left(\mu,\delta_x^p\right) = \omega \mathsf{L}_\bullet(p)$.

Proof. Fix $\mu \in \mathcal{P}$, $n \in \mathbb{N}$, and $p \in \Pi_n$. As in the proof of Lemma 4.3, write $J_i = [P_{,i-1},P_{,i}]$ for convenience. By (4.9), for every $x \in \Xi_n$,

$$d_\mathsf{L}(\mu,\delta_x^p) = \omega \max\nolimits_{i=1}^n \ell_{F_\mu^{-1},J_i}(x_{,i}) \ge \omega \max\nolimits_{i=1}^n \ell^*_{F_\mu^{-1},J_i} = \omega \mathsf{L}_\bullet(p)\,.$$

Thus δ_x^p is a best d_L-approximation of μ, given p, whenever (4.11) holds, i.e., the latter is sufficient for optimality. On the other hand, consider $y \in \Xi_n$ with

$$y_{,i} = \frac{1}{2}\Big(F_{\mu-}^{-1}\left(P_{,i} - \mathsf{L}_\bullet(p)\right) + F_\mu^{-1}\left(P_{,i-1} + \mathsf{L}_\bullet(p)\right)\Big), \quad \forall\, i = 1,\ldots,n\,.$$

Note that y is well-defined, since $i \mapsto y_{,i}$ is non-decreasing. Moreover, by the definition of $\mathsf{L}_\bullet(p)$,

$$\ell_{F_\mu^{-1},J_i}\left(y_{,i}\right) \le \mathsf{L}_\bullet(p), \quad \forall\, i = 1,\ldots,n\,,$$

and hence $d_\mathsf{L}\left(\delta_y^p,\mu\right) = \omega \mathsf{L}_\bullet(p)$. This shows that best d_L-approximations of μ, given p, do exist, and thus (4.11) also is necessary for optimality. □

Remark 4.6. (i) With f, I as in Proposition 4.2, for every $a \in \mathbb{R}$ the set $\{\ell_{f,I} \le a\}$ is a (possibly empty or one-point) interval. Thus, conditions (4.10) and (4.11) are very similar in spirit to the requirements of Theorems 2.25 and 2.29, restated in Proposition 4.12 below, though the latter may be quite a bit easier to work with in concrete calculations.

(ii) Note that if $n = 1$ then (4.10) holds automatically, whereas (4.11) shows that $d_{\mathsf{L}}(\mu, \delta_a)$ is minimal precisely if the function $\ell_{F_\mu^{-1},[0,1]}$ attains its minimal value at a.

As a corollary (for a proof, see Section 5.2), Theorem 4.5 identifies all best *uniform* d_{L}-approximations of β_b with $b > 1$. Recall that $\mathbb{J} = [1, b]$, and hence $\omega = \dfrac{\max\{b, 2\} - 1}{b - 1} =: \omega_b$ in this case.

Corollary 4.7. *Let $b > 1$ and $n \in \mathbb{N}$. Then $\delta_x^{u_n}$ is a best uniform d_{L}-approximation of β_b if and only if*

$$b^{j/n-L} - L \le x_{,j} \le b^{(j-1)/n+L} + L, \quad \forall\, j = 1, \ldots, n\,,$$

where L is the unique solution of (4.8); in particular, $\#\text{supp}\, \delta_\bullet^{u_n} = n$. Moreover, $d_{\mathsf{L}}\left(\beta_b, \delta_\bullet^{u_n}\right) = \omega_b L$, and

$$\lim_{n\to\infty} n d_{\mathsf{L}}\left(\beta_b, \delta_\bullet^{u_n}\right) = \frac{\max\{b, 2\} - 1}{2b - 2} \cdot \frac{b \log b}{1 + b \log b}\,.$$

By combining Theorems 4.4 and 4.5, it is possible to characterize the best d_{L}-approximations of $\mu \in \mathcal{P}$ as well, that is, to identify the minimizers of $\nu \mapsto d_{\mathsf{L}}(\mu, \nu)$ subject only to the requirement that $\#\text{supp}\, \nu \le n$. To this end, associate with every non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$ and every number $a \ge 0$ a map $T_{f,a} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, according to

$$T_{f,a}(x) = f_+\left(f^{-1}(x + a) + 2a\right) + a, \quad \forall\, x \in \overline{\mathbb{R}}\,.$$

For every $n \in \mathbb{N}$, denote by $T_{f,a}^{[n]}$ the n-fold composition of $T_{f,a}$ with itself. The following properties of $T_{f,a}$ are readily verified.

Proposition 4.8. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing, $a \ge 0$, and $n \in \mathbb{N}$. Then $T_{f,a}^{[n]}$ is non-decreasing and right-continuous. Also, $a \mapsto T_{f,a}^{[n]}(x)$ is increasing and right-continuous for every $x \in \mathbb{R}$, and if $x \le a + f(+\infty)$ then the sequence $\left(T_{f,a}^{[k]}(x)\right)$ is non-decreasing.*

To utilize Proposition 4.8 for the d_{L}-approximation problem, let $f = F_\mu$ with $\mu \in \mathcal{P}$. Then $\left(T_{F_\mu,a}^{[k]}(0)\right)$ is non-decreasing; in fact, $\lim_{k\to\infty} T_{F_\mu,a}^{[k]}(0) = a + 1$. On the other hand, given $n \in \mathbb{N}$, clearly $T_{F_\mu,a}^{[n]}(0) \ge 1$ for all $a \ge 1$, and hence

$$\mathsf{L}_\bullet^{\bullet,n} := \min\left\{a \ge 0 : T_{F_\mu,a}^{[n]}(0) \ge 1\right\} < +\infty\,.$$

Note that $\mathsf{L}_{\bullet}^{\bullet,n}$ only depends on μ and n. Clearly, the sequence $(\mathsf{L}_{\bullet}^{\bullet,n})$ is non-increasing, and $n\mathsf{L}_{\bullet}^{\bullet,n} \le \frac{1}{2}$ for every n. Also, $\mathsf{L}_{\bullet}^{\bullet,n} = 0$ if and only if $\#\mathrm{supp}\,\mu \le n$.

For a concrete example, consider $\mu = \beta_b$ with $a < \frac{1}{2}(b-1)$, where

$$T_{F_\mu,a}(x) = \begin{cases} a & \text{if } x < -a\,, \\ a + \log_b(b^{x+a} + 2a) & \text{if } -a \le x < -a + \log_b(b-2a)\,, \\ a+1 & \text{if } x \ge -a + \log_b(b-2a)\,, \end{cases}$$

from which it is easily deduced that $\mathsf{L}_{\bullet}^{\bullet,n}$ is the unique solution of

$$b^{2nL} = \frac{2L + b(b^L - b^{-L})}{2L + b^L - b^{-L}}\,. \tag{4.12}$$

As the following result shows, the quantity $\mathsf{L}_{\bullet}^{\bullet,n}$ always plays a central role in identifying best (unconstrained) d_{L}-approximations of a given $\mu \in \mathcal{P}$.

Theorem 4.9. *Let $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$. There exists a best d_{L}-approximation of μ, and $d_{\mathsf{L}}(\mu, \delta_{\bullet}^{\bullet,n}) = \omega\mathsf{L}_{\bullet}^{\bullet,n}$. Moreover, for every $x \in \Xi_n$ and $p \in \Pi_n$, the following are equivalent:*

(i) *$d_{\mathsf{L}}(\mu, \delta_x^p) = d_{\mathsf{L}}(\mu, \delta_{\bullet}^{\bullet,n})$;*

(ii) *all implications in (4.10) are valid with $\mathsf{L}^{\bullet}(x)$ replaced by $\mathsf{L}_{\bullet}^{\bullet,n}$;*

(iii) *all implications in (4.11) are valid with $\mathsf{L}_{\bullet}(p)$ replaced by $\mathsf{L}_{\bullet}^{\bullet,n}$.*

Proof. To see that best d_{L}-approximations of μ do exist, simply note that the set $\{\nu \in \mathcal{P} : \#\mathrm{supp}\,\nu \le n\}$ is compact, and the function $\nu \mapsto d_{\mathsf{L}}(\mu,\nu)$ is continuous, hence attains a minimal value for some $\nu = \delta_x^p$ with $x \in \Xi_n$ and $p \in \Pi_n$. Clearly, any such δ_x^p also is a best approximation of μ, given p. By Theorem 4.5, therefore, $d_{\mathsf{L}}(\mu, \delta_x^p) = \omega\mathsf{L}_{\bullet}(p)$, as well as

$$F_{\mu-}^{-1}\big(P_{,j} - \mathsf{L}_{\bullet}(p)\big) - \mathsf{L}_{\bullet}(p) \le x_{,j} \le F_\mu^{-1}\big(P_{,j-1} + \mathsf{L}_{\bullet}(p)\big) + \mathsf{L}_{\bullet}(p)$$

whenever $P_{,j-1} < P_{,j}$, and indeed for every $j = 1, \ldots, n$. It follows that $P_{,j} \le T_{F_\mu,\mathsf{L}_{\bullet}(p)}(P_{,j-1})$ for all j, and hence $1 = P_{,n} \le T_{F_\mu,\mathsf{L}_{\bullet}(p)}^{[n]}(0)$, that is, $\mathsf{L}_{\bullet}^{\bullet,n} \le \mathsf{L}_{\bullet}(p)$. This shows that $d_{\mathsf{L}}(\mu, \delta_x^p) \ge \omega\mathsf{L}_{\bullet}^{\bullet,n}$. To establish the reverse inequality, let

$$m = \min\left\{i \ge 1 : T_{F_\mu,\mathsf{L}_{\bullet}^{\bullet,n}}^{[i]}(0) \ge 1\right\}\,.$$

Clearly, $1 \le m \le n$, and $\mathsf{L}_{\bullet}^{\bullet,m} = \mathsf{L}_{\bullet}^{\bullet,n}$. Define $q \in \Pi_m$ via

$$Q_{,i} = T_{F_\mu,\mathsf{L}_{\bullet}^{\bullet,n}}^{[i]}(0), \quad \forall\, i = 1, \ldots, m-1\,.$$

Note that $i \mapsto Q_{,i}$ is non-decreasing, and $0 \leq Q_{,i} \leq 1$, so q is well-defined. Also, consider $y \in \Xi_m$ with

$$y_{,i} = \frac{1}{2}\Big(F_{\mu-}^{-1}(Q_{,i} - \mathsf{L}_\bullet^{\bullet,m}) + F_\mu^{-1}(Q_{,i-1} + \mathsf{L}_\bullet^{\bullet,m})\Big), \quad \forall\, i = 1, \ldots, m\,.$$

By the definitions of $\mathsf{L}_\bullet^{\bullet,m}$, q, and y,

$$\ell_{F_\mu^{-1},[Q_{,i-1},Q_{,i}]}(y_{,i}) \leq \mathsf{L}_\bullet^{\bullet,m}, \quad \forall\, i = 1, \ldots, m\,,$$

and hence

$$d_{\mathsf{L}}\left(\mu, \delta_x^p\right) \leq d_{\mathsf{L}}\left(\mu, \delta_y^q\right) = \omega \max\nolimits_{i=1}^n \ell_{F_\mu^{-1},[Q_{,i-1},Q_{,i}]}(y_{,i}) \leq \omega \mathsf{L}_\bullet^{\bullet,m} = \omega \mathsf{L}_\bullet^{\bullet,n}\,.$$

This shows that indeed $d_{\mathsf{L}}\left(\mu, \delta_x^p\right) = \omega \mathsf{L}_\bullet^{\bullet,n}$ and also proves (i)⇒(iii). The implication (i)⇒(ii) follows by a similar argument. That, conversely, either of (ii) and (iii) implies (i) is evident from (4.9), together with the fact that, as seen in the proof of Lemma 4.3 above, validity of (4.10) and (4.11) implies $\max_{j=0}^n \ell_{F_\mu,[x_{,j},x_{,j+1}]}(P_{,j}) \leq \mathsf{L}^\bullet(x)$ and $\max_{j=1}^n \ell_{F_\mu^{-1},[P_{,j-1},P_{,j}]}(x_{,j}) \leq \mathsf{L}_\bullet(p)$, respectively. □

Remark 4.10. (i) The above proof of Theorem 4.9 shows that in fact

$$\mathsf{L}_\bullet^{\bullet,n} = \min\nolimits_{x \in \Xi_n} \mathsf{L}^\bullet(x) = \min\nolimits_{p \in \Pi_n} \mathsf{L}_\bullet(p)\,.$$

(ii) Theorem 4.9 is similar to classical one-dimensional quantization results as presented, e.g., in [40, Sec.5.2]. What makes the theorem (and its analogue, Theorem 4.25 in Section 4.4) particularly appealing is that its conditions (ii) and (iii) not only are necessary for optimality, but also sufficient. By contrast, it is well known that sufficient conditions for best d_*-approximations may be hard to come by in general; see, e.g., [40, Sec.4.1], and also Proposition 4.12(iii) below, regarding the case of $* = 1$.

When specialized to $\mu = \beta_b$, Theorem 4.9 yields the best finitely supported d_{L}-approximations of Benford's Law.

Corollary 4.11. *Let $b > 1$ and $n \in \mathbb{N}$. Then the best d_{L}-approximation of β_b is δ_x^p, with*

$$x_{,j} = b^{(2j-1)L} + 2L\frac{b^{2jL}-1}{b^{2L}-1} - L = b^{P_{,j}-L} - L\,,$$
$$P_{,j} = \frac{1}{\log b}\log\left(b^{(2j-1)L} + 2L\frac{b^{2jL}-1}{b^{2L}-1}\right) + L = \frac{\log(x_{,j}+L)}{\log b} + L\,,$$

for all $j = 1, \ldots, n$, where L is the unique solution of (4.12); in particular, $\#\mathrm{supp}\,\delta_\bullet^{\bullet,n} = n$.

Moreover, $d_{\mathsf{L}}\left(\beta_b, \delta_{\bullet}^{\bullet,n}\right) = \omega_b L$*, and*

$$\lim_{n\to\infty} n d_{\mathsf{L}}\left(\beta_b, \delta_{\bullet}^{\bullet,n}\right) = \frac{\max\{b,2\}-1}{2b-2} \cdot \frac{\log(1+b\log b) - \log(1+\log b)}{\log b}\,.$$

To compare this to Corollary 4.7, note that $P_{,j} \not\equiv j/n$ whenever $n \ge 2$, and then the n-th quantization error $d_{\mathsf{L}}\left(\beta_b, \delta_{\bullet}^{\bullet,n}\right)$ is smaller than the n-th *uniform* quantization error $d_{\mathsf{L}}\left(\beta_b, \delta_{\bullet}^{u_n}\right)$. The d_{L}-quantization coefficient of β_b also is smaller than its uniform counterpart, since

$$\frac{\log(1+b\log b) - \log(1+\log b)}{\log b} < \frac{b\log b}{1+b\log b}\,, \quad \forall\, b > 1\,.$$

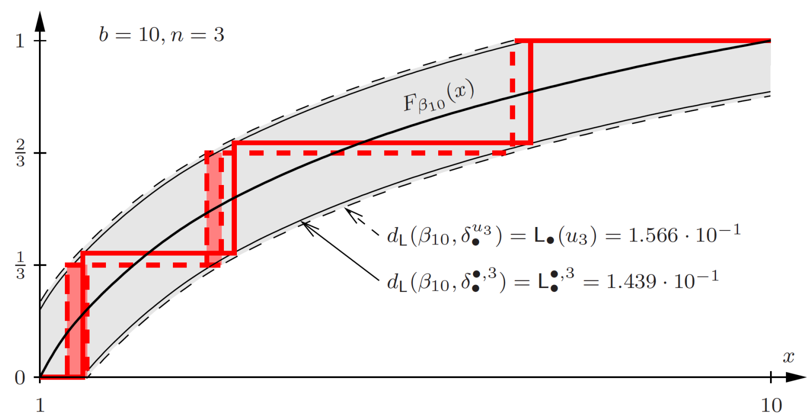

Figure 4.1: The best d_{L}-approximation (solid red line) of β_{10} is unique, whereas best uniform d_{L}-approximations (broken red lines) are not; see Corollaries 4.11 and 4.7, respectively.

4.3 Kantorovich approximations

This section studies best finitely supported d_r-approximations of Benford's Law. Mostly, the results are special cases of more general facts taken from the study on d_r-approximations in Chapters 2 and 3.

4.3.1 d_1-approximations

With d_{L} replaced by d_1, the main results of the previous section have the following analogues, stated here for the reader's convenience; see Chapters 2 and 3 for details.

Proposition 4.12. *Let* $\mu \in \mathcal{P}$ *and* $n \in \mathbb{N}$.

(i) *For every* $x \in \Xi_n$, *there exists a best* d_1*-approximation of* μ, *given* x. *Moreover,* $d_1(\mu, \delta_x^p) = d_1(\mu, \delta_x^\bullet)$ *if and only if, for every* $j = 0, \ldots, n$,

$$x_{,j} < x_{,j+1} \text{ implies } F_{\mu-}\left(\tfrac{1}{2}(x_{,j} + x_{,j+1})\right) \le P_{,j} \le F_\mu\left(\tfrac{1}{2}(x_{,j} + x_{,j+1})\right). \tag{4.13}$$

(ii) *For every* $p \in \Pi_n$, *there exists a best* d_1*-approximation of* μ, *given* p. *Moreover,* $d_1(\mu, \delta_x^p) = d_1(\mu, \delta_\bullet^p)$ *if and only if, for every* $j = 1, \ldots, n$,

$$P_{,j-1} < P_{,j} \text{ implies } F_{\mu-}^{-1}\left(\tfrac{1}{2}(P_{,j-1} + P_{,j})\right) \le x_{,j} \le F_\mu^{-1}\left(\tfrac{1}{2}(P_{,j-1} + P_{,j})\right). \tag{4.14}$$

(iii) *There exists a best* d_1*-approximation of* μ, *and if* $d_1(\mu, \delta_x^p) = d_1(\mu, \delta_\bullet^{\bullet,n})$ *then* (4.13) *and* (4.14) *are valid for every* $j = 1, \ldots, n$.

Remark 4.13. Though the phrasing of Proposition 4.12 emphasizes its analogy to Theorem 4.4 (and also to Theorem 4.20 below), there nevertheless is a subtle difference: While in (4.10) and (4.17) it can equivalently be stipulated that, respectively, $\ell_{F_\mu,[x_{,j},x_{,j+1}]}(P_{,j}) \le \mathsf{L}^\bullet(x)$ and $F_{\mu-}(x_{,j+1}) - \mathsf{K}^\bullet(x) \le P_{,j} \le F_\mu(x_{,j}) + \mathsf{K}^\bullet(x)$ for *all* $j = 0, \ldots, n$, simple examples show that the "only if" part of Proposition 4.12(i) may fail, should (4.13) be replaced by

$$F_{\mu-}\left(\tfrac{1}{2}(x_{,j} + x_{,j+1})\right) \le P_{,j} \le F_\mu\left(\tfrac{1}{2}(x_{,j} + x_{,j+1})\right), \quad \forall\, j = 0, \ldots, n.$$

Similar observations pertain to Proposition 4.12(ii) vis-à-vis Theorems 4.5 and 4.23.

Proposition 4.12 immediately yields the existence of unique best uniform d_1-approximations of β_b; see also [6, Cor.2.10].

Corollary 4.14. *Let* $b > 1$ *and* $n \in \mathbb{N}$. *Then the best uniform* d_1*-approximation of* β_b *is* $\delta_x^{u_n}$, *with* $x_{,j} = b^{(2j-1)/(2n)}$ *for all* $j = 1, \ldots, n$, *and* $\#\mathrm{supp}\, \delta_\bullet^{u_n} = n$. *Moreover,* $d_1(\beta_b, \delta_\bullet^{u_n}) = \dfrac{1}{\log b} \tanh\left(\dfrac{\log b}{4n}\right)$, *and* $\lim_{n\to\infty} n d_1(\beta_b, \delta_\bullet^{u_n}) = \frac{1}{4}$.

Proof. By Proposition 4.12(ii), $x_{,j} = b^{(2j-1)/(2n)}$ for all $j = 1, \ldots, n$, and

$$\begin{aligned} n d_1(\beta_b, \delta_\bullet^{u_n}) &= \frac{n}{b-1} \sum\nolimits_{j=1}^n \int_{(j-1)/n}^{j/n} \left| b^y - b^{(2j-1)/(2n)} \right| \mathrm{d}y \\ &= \frac{n\left(b^{1/(4n)} - b^{-1/(4n)}\right)^2}{(b-1)\log b} \sum\nolimits_{j=1}^n b^{(2j-1)/(2n)} = \frac{n}{\log b} \tanh\left(\frac{\log b}{4n}\right) \overset{n\to\infty}{\longrightarrow} \frac{1}{4}. \end{aligned}$$

□

Best (unconstrained) d_1-approximations of β_b exist and are unique, too, by virtue of Proposition 4.12 and a direct calculation.

Corollary 4.15. *Let $b > 1$ and $n \in \mathbb{N}$. Then the best d_1-approximation of β_b is δ_x^p, with*

$$x_{,j} = \left(1 + \frac{j-1}{n}\left(b^{1/2} - 1\right)\right)\left(1 + \frac{j}{n}\left(b^{1/2}-1\right)\right),$$
$$P_{,j} = \frac{2}{\log b}\log\left(1 + \frac{j}{n}\left(b^{1/2}-1\right)\right),$$

for all $j = 1, \ldots, n$; in particular, $\#\mathrm{supp}\,\delta_{\bullet}^{\bullet,n} = n$. Moreover, $d_1(\beta_b, \delta_{\bullet}^{\bullet,n}) = \dfrac{1}{n\log b}\tanh\left(\dfrac{\log b}{4}\right)$.

Proof. Let δ_x^p be a best d_1-approximation. Then, by Proposition 4.12(iii),

$$b^{P_{,j}} = \frac{x_{,j} + x_{,j+1}}{2}, \quad \forall\, j = 1, \ldots, n-1,$$

but also $x_{,j} = b^{(P_{,j-1}+P_{,j})/2}$ for all $j = 1, \ldots, n$, and hence $2b^{P_{,j}/2} = b^{P_{,j-1}/2} + b^{P_{j+1}/2}$. Since $P_0 = 0$, $P_n = 1$, it follows that $b^{P_{,j}/2} = 1 + j(b^{1/2}-1)n^{-1}$ for all $j = 0, \ldots, n$. This yields the asserted unique δ_x^p, and

$$d_1\left(\beta_b, \delta_{\bullet}^{\bullet,n}\right) = \frac{1}{b-1}\sum\nolimits_{j=1}^{n}\int_{P_{,j-1}}^{P_{,j}} |b^y - x_{,j}|\,\mathrm{d}y = \frac{b - x_{,n} - (x_{,1}-1)}{(b-1)\log b} = \frac{1}{n\log b}\tanh\left(\frac{\log b}{4}\right),$$

via a straightforward calculation. □

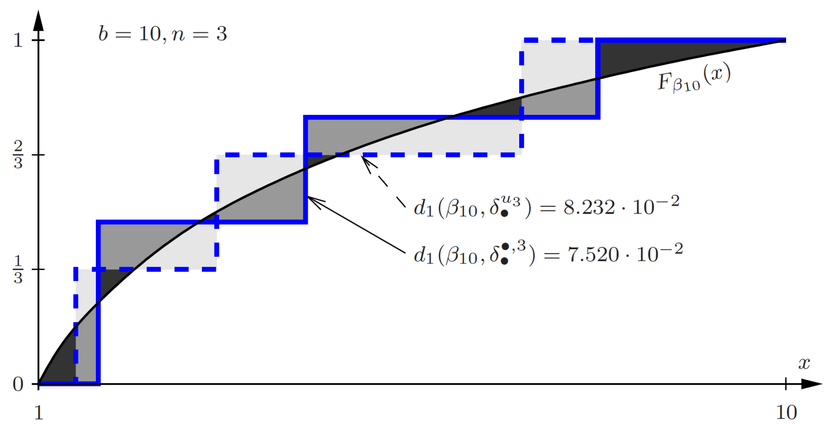

Figure 4.2: The best (solid blue line) and best uniform (broken blue line) d_1-approximations of β_{10} both are unique; see Corollaries 4.15 and 4.14, respectively. Coincidentally, best uniform d_1-approximations of β_{10} are best d_{K}-approximations as well; see Corollary 4.28.

Remark 4.16. (i) Due to the highly non-linear nature of the optimality conditions (4.13) and (4.14), best d_1-approximations are rarely given by explicit formulae such as those in Corollary 4.15. Aside from Benford's Law, the author knows of only two other families of continuous distributions that allow for similarly explicit formulae, namely uniform and (one- or two-sided) exponential distributions.

(ii) A popular family of metrics on $\mathcal{P}$ closely related to d_1 are the so-called *Fortet–Mourier r-distances* ($1 \le r < +\infty$), given by

$$d_{\mathsf{FM}_r}(\mu, \nu) = \int_{\mathbb{J}} \max\{1, |y|\}^{r-1} |F_\mu(y) - F_\nu(y)| \,\mathrm{d}y \,.$$

Like the Lévy and Kantorovich metrics, the Fortet–Mourier r-distance also induces the weak topology on $\mathcal{P}$. The reader is referred to [81, 86] for details on the mathematical background of d_{FM_r} and its use for stochastic optimization. Note that if $\mathbb{J} \subset [1, +\infty[$ then

$$d_{\mathsf{FM}_r}(\mu, \nu) = \frac{\lambda\big(T(\mathbb{J})\big)}{r} d_1\left(\mu \circ T^{-1}, \nu \circ T^{-1}\right) ,$$

with the homeomorphism $T : x \mapsto x^r$ of $[1, +\infty[$. For instance, $\beta_b \circ T^{-1} = \beta_{rb}$, and hence best (or best uniform) d_{FM_r}-approximations of β_b can easily be identified using Corollary 4.15 (or 4.14).

4.3.2 d_r-approximations ($1 < r < +\infty$)

Similarly to the case of $r = 1$, Theorem 2.29 guarantees that, given any $n \in \mathbb{N}$, there exists a (unique) best uniform d_r-approximation $\delta^{u_n}_{\bullet}$ of β_b. Except for $r = 2$, however, no explicit formula seems to be available for $\delta^{u_n}_{\bullet}$. It is desirable, therefore, to at least identify *asymptotically* best uniform d_r-approximations, that is, a sequence (x_n) with $x_n \in \Xi_n$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{d_r\left(\beta_b, \delta^{u_n}_{x_n}\right)}{d_r\left(\beta_b, \delta^{u_n}_{\bullet}\right)} = 1 \,.$$

Usage of Theorem 2.39 accomplishes this and also yields the uniform d_r-quantization coefficient of β_b. (Notice that, as $r \downarrow 1$, the latter is consistent with Corollary 4.14.)

Proposition 4.17. *Let $b, r > 1$. Then $\left(\delta^{u_n}_{x_n}\right)$, with $x_{n,j} = b^{(2j-1)/(2n)}$ for all $n \in \mathbb{N}$ and $j = 1, \ldots, n$, is a sequence of asymptotically best uniform d_r-approximations of β_b. Moreover,*

$$\lim_{n\to\infty} n d_r(\beta_b, \delta^{u_n}_{\bullet}) = \frac{(\log b)^{1-1/r}}{2(b-1)} \left(\frac{b^r - 1}{r(r+1)}\right)^{1/r} .$$

The remainder of this section studies best d_r-approximations of β_b. In general, the question of uniqueness of best d_r-approximations is a difficult one, for which only partial answers exist; see, e.g., [40, Sec.5]. Specifically, β_b does not seem to satisfy any known condition (such as, e.g., log-concavity) that would guarantee uniqueness. However, uniqueness can be established via a direct calculation.

Theorem 4.18. *Let $b, r > 1$ and $n \in \mathbb{N}$. There exists a unique best d_r-approximation $\delta_{\bullet}^{\bullet,n}$ of β_b, and $\#\text{supp}\,\delta_{\bullet}^{\bullet,n} = n$.*

Proof. Existence follows as in Theorem 4.9; alternatively, see [40, Sec.4.1] or Proposition 2.46. To avoid trivialities, henceforth assume $n \geq 2$. If $d_r(\beta_b, \delta_x^p) = d_r(\beta_b, \delta_{\bullet}^{\bullet,n})$, then by Theorem 2.47,

$$b^{P_{,j}} = \frac{x_{,j} + x_{,j+1}}{2}, \quad \forall\, j = 1, \ldots, n-1,$$

but also

$$\int_{P_{,j-1}}^{\log_b x_{,j}} (x_{,j} - b^y)^{r-1}\, \mathrm{d}y = \int_{\log_b x_{,j}}^{P_{,j}} (b^y - x_{,j})^{r-1}\, \mathrm{d}y, \quad \forall\, j = 1, \ldots, n. \tag{4.15}$$

Eliminating P and substituting $z = b^y / x_{,j}$ in (4.15) yields n equations for $x_{,1}, \ldots, x_{,n}$, namely

$$\begin{aligned} \int_1^{x_{,1}} (z-1)^{r-1} \frac{\mathrm{d}z}{z^r} &= 2^{1-r} g_0\left(\frac{x_{,2}}{x_{,1}}\right), \\ g_r\left(\frac{x_{,j}}{x_{,j-1}}\right) &= g_0\left(\frac{x_{,j+1}}{x_{,j}}\right), \quad \forall\, j = 2, \ldots, n-1, \\ g_r\left(\frac{x_{,n}}{x_{,n-1}}\right) &= g_0\left(\frac{2b - x_{,n}}{x_{,n}}\right), \end{aligned} \tag{4.16}$$

where the smooth, increasing function g_a, with $a \in \mathbb{R}$, is given by

$$g_a(x) = \int_1^x \frac{(z-1)^{r-1}}{z^a(z+1)}\, \mathrm{d}z, \quad x \geq 1.$$

Assume that $\widetilde{x} \in \Xi_n$ also solves (4.16). If $\widetilde{x}_{,1} > x_{,1}$ then $\widetilde{x}_{,j+1}/\widetilde{x}_{,j} > x_{,j+1}/x_{,j}$ and hence $\widetilde{x}_{,j+1} > x_{,j+1}$ for all $j = 0, \ldots, n-1$, but by the last equation in (4.16) also $2b/\widetilde{x}_{,n} > 2b/x_{,n}$, an obvious contradiction. Similarly, $\widetilde{x}_{,1} < x_{,1}$ leads to a contradiction. Thus, $\widetilde{x}_{,1} = x_{,1}$, and consequently $\widetilde{x} = x$. (If $n = 1$ then (4.16) reduces to

$$\int_1^{x_{,1}} (z-1)^{r-1} \frac{\mathrm{d}z}{z^r} = 2^{1-r} g_0\left(\frac{2b - x_{,1}}{x_{,1}}\right),$$

which also has a unique solution since, as $x_{,1}$ increases from 1 to b, the left side increases from 0 whereas the right side decreases to 0.) In summary, therefore, $x \in \Xi_n$ and $p \in \Pi_n$ are uniquely determined by $d_r(\beta_b, \delta_x^p) = d_r(\beta_b, \delta_{\bullet}^{\bullet,n})$. □

As in the case of best uniform d_r-approximations of β_b, no explicit formula is available for $\delta_{\bullet}^{\bullet,n}$, not even when $r = 2$. Still, it is possible to identify *asymptotically* best d_r-approximations, that is, a sequence $\left(\delta_{x_n}^{p_n}\right)$ with $x_n \in \Xi_n$ and $p_n \in \Pi_n$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{d_r\left(\beta_b, \delta_{x_n}^{p_n}\right)}{d_r\left(\beta_b, \delta_{\bullet}^{\bullet,n}\right)} = 1\,.$$

In addition, the d_r-quantization coefficient of β_b can be computed explicitly; for details see [102, Prop.5.26] and the references given there. Notice that, as $r \downarrow 1$, the result is consistent with Corollary 4.15.

Proposition 4.19. *Let $b, r > 1$. Then $\left(\delta_{x_n}^{p_n}\right)$, with*

$$x_{n,j} = \left(1 + \frac{j}{n+1}\left(b^{r/(r+1)} - 1\right)\right)^{1+1/r}, \quad P_{n,j} = \frac{1}{\log b}\log\frac{x_{n,j} + x_{n,j+1}}{2},$$

for all $n \in \mathbb{N}$ and $j = 1, \ldots, n-1$, and $x_{n,n} = \left(1 + (b^{r/(r+1)} - 1)\frac{n}{n+1}\right)^{1+1/r}$, is a sequence of asymptotically best d_r-approximations of β_b. Moreover,

$$\lim_{n\to\infty} n d_r(\beta_b, \delta_{\bullet}^{\bullet,n}) = \frac{r+1}{2(b-1)(\log b)^{1/r}}\left(\frac{b^{r/(r+1)} - 1}{r}\right)^{1+1/r}.$$

4.4 Kolmogorov approximations

This section discusses best finitely supported d_{K}-approximations. Though ultimately the results are true analogues of their counterparts in Sections 4.2 and 4.3, the underlying arguments are subtly different, which may be seen as a reflection of the fact that d_{K} metrizes a topology finer than the weak topology of $\mathcal{P}$. (Recall, however, that d_{K} does metrize the weak topology on $\mathcal{P}_{\mathsf{cts}}$.)

Given $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$, for every $x \in \Xi_n$, let

$$\mathsf{K}^{\bullet}(x) = \max\left\{F_{\mu-}(x_{,1}), \frac{1}{2}\max_{j=1}^{n-1}\left(F_{\mu-}(x_{,j+1}) - F_{\mu}(x_{,j})\right), 1 - F_{\mu}(x_{,n})\right\}.$$

Note that $\mathsf{K}^{\bullet}(x) = d_{\mathsf{K}}\left(\mu, \delta_x^{\pi(x)}\right)$ with $\Pi(x)_{,j} = \frac{1}{2}\left(F_{\mu}(x_{,j}) + F_{\mu-}(x_{,j+1})\right)$ for all $j = 1, \ldots, n-1$. Existence and characterization of best d_{K}-approximations with prescribed locations are analogous to Theorem 4.4.

Theorem 4.20. *Assume that $\mu \in \mathcal{P}$, and $n \in \mathbb{N}$. For every $x \in \Xi_n$, there exists a best d_{K}-approximation of μ, given x. Moreover, $d_{\mathsf{K}}(\mu, \delta_x^p) = d_{\mathsf{K}}(\mu, \delta_x^{\bullet})$ if and only if, for every $j = 0, \ldots, n$,*

$$x_{,j} < x_{,j+1} \text{ implies } F_{\mu-}(x_{,j+1}) - \mathsf{K}^{\bullet}(x) \le P_{,j} \le F_{\mu}(x_{,j}) + \mathsf{K}^{\bullet}(x), \tag{4.17}$$

and in this case $d_{\mathsf{K}}(\mu, \delta_x^\bullet) = \mathsf{K}^\bullet(x)$.

Proof. Given $x \in \Xi_n$ and $p \in \Pi_n$, let $y \in \Xi_m$ and $q \in \Pi_m$ as in the proof of Lemma 4.3. Then

$$\begin{aligned} d_{\mathsf{K}}(\mu, \delta_x^p) &= \max\nolimits_{i=0}^m \sup\nolimits_{t\in[y_{,i}, y_{,i+1}[} |F_\mu(t) - Q_{,i}| \\ &\geq \max\left\{F_{\mu-}(y_{,1}), \frac{1}{2}\max\nolimits_{i=1}^{m-1}\left(F_{\mu-}(y_{,i+1}) - F_\mu(y_{,i})\right), 1 - F_\mu(y_{,m})\right\} \\ &= \max\left\{F_{\mu-}(x_{,1}), \frac{1}{2}\max\nolimits_{j=1}^{n-1}\left(F_{\mu-}(x_{,j+1}) - F_\mu(x_{,j})\right), 1 - F_\mu(x_{,n})\right\} \\ &= \mathsf{K}^\bullet(x). \end{aligned}$$

This shows that $\delta_x^{\pi(x)}$ is a best d_{K}-approximation, given x, and $d_{\mathsf{K}}(\mu, \delta_x^\bullet) = \mathsf{K}^\bullet(x)$. Moreover, $d_{\mathsf{K}}(\mu, \delta_x^p) = \mathsf{K}^\bullet(x)$ if and only if

$$\max\left\{|F_{\mu-}(y_{,i+1}) - Q_{,i}|, |F_\mu(y_{,i}) - Q_{,i}|\right\} \leq \mathsf{K}^\bullet(x), \quad \forall\, i = 1, \ldots, m-1,$$

that is,

$$F_{\mu-}(y_{,i+1}) - \mathsf{K}^\bullet(x) \leq Q_{,i} \leq F_\mu(y_{,i}) + \mathsf{K}^\bullet(x), \quad \forall\, i = 0, \ldots, m,$$

which in turn is equivalent to the validity (4.17) for every j. □

To address the approximation problem with prescribed weights, an auxiliary function analogous to $\ell_{f,I}$ in Section 4.2 is useful. Specifically, given a non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$, let $I \subset \mathbb{R}$ be any bounded, non-empty interval, and define $\kappa_{f,I} : \mathbb{R} \to \overline{\mathbb{R}}$ as

$$\kappa_{f,I}(x) = \max\left\{\Big|f_-(x) - \inf I\Big|, \Big|f_+(x) - \sup I\Big|\right\}.$$

A few basic properties of $\kappa_{f,I}$ are easily established.

Proposition 4.21. *Let* $f : \mathbb{R} \to \overline{\mathbb{R}}$ *be non-decreasing, and* $\varnothing \neq I \subset \mathbb{R}$ *a bounded interval. Then, with* $s := f^{-1}\left(\frac{1}{2}(\inf I + \sup I)\right)$*, the function* $\kappa_{f,I}$ *is non-increasing on* $]-\infty, s[$*, and non-decreasing on* $]s, +\infty[$*. Moreover,* $\kappa_{f,I}$ *attains a minimal value whenever* $\inf I \leq \frac{1}{2}\Big(f_-(s) + f_+(s)\Big) \leq \sup I$*.*

It is worth noting that $\kappa_{f,I}$ may in general not attain its infimum, as the example of $f = 15F_\mu$, with $\mu = \frac{1}{15}\lambda\Big|_{[0,5]} + \frac{2}{3}\delta_5$, and $I = [6, 8]$ shows, for which $s = 5$, and $\kappa_{f,I}(5-) = 3$, $\kappa_{f,I}(5) = 7$, $\kappa_{f,I}(5+) = 9$; correspondingly, $\frac{1}{2}\Big(f_-(5) + f_+(5)\Big) \notin I$.

By using functions of the form $\kappa_{f,I}$, the value of $d_{\mathsf{K}}(\mu, \nu)$ can easily be bounded above whenever ν has finite support. For convenience, for every $n \in \mathbb{N}$ let $\Xi_n^+ = \{x \in \Xi_n :\ x_{,1} < \ldots < x_{,n}\}$. The proof of the following analogue of Lemma 4.3 is straightforward.

Proposition 4.22. *Let $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$. For every $x \in \Xi_n$ and $p \in \Pi_n$,*

$$d_{\mathsf{K}}\Big(\mu, \delta_x^p\Big) \le \max\nolimits_{j=1}^n \kappa_{F_\mu,[P_{,j-1},P_{,j}]}(x_{,j}), \tag{4.18}$$

and equality holds in (4.18) whenever $x \in \Xi_n^+$.

Consider for instance $\mu = \frac{1}{6}\lambda\Big|_{[0,2]} + \frac{2}{3}\delta_1$, and $x = (1,1)$. Then, for every $p \in \Pi_2$, clearly $d_{\mathsf{K}}(\mu, \delta_x^p) = \frac{1}{6}$, whereas $\max_{j=1}^2 \kappa_{F_\mu,[P_{,j-1},P_{,j}]}(x_{,j}) = \frac{1}{3} + \Big|p_{,1} - \frac{1}{2}\Big| \ge \frac{1}{3}$. Thus the inequality (4.18) may be strict if $x \notin \Xi_n^+$. This, together with the fact that a function $\kappa_{f,I}$ may not attain its infimum, suggests that d_{K}-approximations with prescribed weights are potentially somewhat fickle. Still, best approximations do exist and can be characterized in a spirit similar to Sections 4.2 and 4.3. To this end, given $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$, for every $p \in \Pi_n$, let

$$\mathsf{K}_\bullet(p) = d_{\mathsf{K}}\left(\mu, \delta_{\xi(p)}^p\right) \ \text{ with } \ \xi(p)_{,j} = F_\mu^{-1}\left(\frac{1}{2}\left(P_{,j-1} + P_{,j}\right)\right), \quad \forall\, j = 1, \ldots, n.$$

Note that $\mathsf{K}_\bullet(p) \le \frac{1}{2}\max_{j=1}^n p_{,j}$, and in fact $\mathsf{K}_\bullet(p) = \frac{1}{2}\max_{j=1}^n p_{,j}$ whenever $\mu \in \mathcal{P}_{\mathsf{cts}}$.

Theorem 4.23. *Assume that $\mu \in \mathcal{P}$, and $n \in \mathbb{N}$. For every $p \in \Pi_n$, there exists a best d_{K}-approximation of μ, given p. Moreover, $d_{\mathsf{K}}(\mu, \delta_x^p) = d_{\mathsf{K}}(\mu, \delta_\bullet^p)$ if and only if, for every $j = 1, \ldots, n$,*

$$P_{,j-1} < P_{,j} \ \text{ implies } \ F_{\mu-}^{-1}\Big(P_{,j} - \mathsf{K}_\bullet(p)\Big) \le x_{,j} \le F_\mu^{-1}\Big(P_{,j-1} + \mathsf{K}_\bullet(p)\Big), \tag{4.19}$$

and in this case $d_{\mathsf{K}}(\mu, \delta_\bullet^p) = \mathsf{K}_\bullet(p)$.

Proof. Note first that deleting all zero entries of p does not change the value of $\mathsf{K}_\bullet(p)$, and hence does not affect (4.19), nor of course the asserted existence of a best d_{K}-approximation, given p. Thus assume $\min_{j=1}^n p_{,j} > 0$ throughout. For convenience, write $\xi(p)$ simply as ξ, and for every $x \in \Xi_n$, write $F_{\delta_x^p}$ as G. To prove the existence of a best d_{K}-approximation of μ, given p, as well as $d_{\mathsf{K}}(\mu, \delta_\bullet^p) = \mathsf{K}_\bullet(p)$, clearly it suffices to show that

$$d_{\mathsf{K}}(\mu, \delta_x^p) \ge d_{\mathsf{K}}\left(\mu, \delta_\xi^p\right), \quad \forall\, x \in \Xi_n. \tag{4.20}$$

Similarly to the proof of Lemma 4.3, label ξ uniquely as

$$\xi_{,1} = \ldots = \xi_{,j_1} < \xi_{,j_1+1} = \ldots = \xi_{,j_2} < \xi_{,j_2+1} = \ldots < \ldots = \xi_{,j_{m-1}} < \xi_{,j_{m-1}+1} = \ldots = \xi_{,j_m},$$

with integers $i \le j_i \le m$ for $1 \le i \le m$, and $j_0 = 0$, $j_m = n$, and define $\eta \in \Xi_m$ and $q \in \Pi_m$ as $\eta_{,i} = \xi_{,j_i}$ and $q_{,i} = P_{,j_i} - P_{,j_{i-1}}$, respectively. With this, $\delta_\xi^p = \delta_\eta^q$, and by Proposition 4.22,

$$\mathsf{K}_\bullet(p) = d_{\mathsf{K}}\left(\mu, \delta_\eta^q\right) = \max\nolimits_{i=1}^m \kappa_{F_\mu,[Q_{,i-1},Q_{,i}]}\left(\eta_{,i}\right).$$

Pick i such that $\kappa_{F_\mu,[Q_{,i-1},Q_{,i}]}(\eta_{,i}) = \mathsf{K}_\bullet(p)$, that is,

$$\max\{|F_{\mu-}(\eta_{,i}) - Q_{,i-1}|, |F_\mu(\eta_{,i}) - Q_{,i}|\} = \mathsf{K}_\bullet(p).$$

Clearly, to establish (4.20) it is enough to show that

$$\max\{|F_{\mu-}(\eta_{,i}) - G_-(\eta_{,i})|, |F_\mu(\eta_{,i}) - G(\eta_{,i})|\} \geq \mathsf{K}_\bullet(p) \tag{4.21}$$

and this will now be done. To this end, notice that by the definition of η,

$$\frac{1}{2}\left(P_{,j_{i-1}-1} + P_{,j_{i-1}}\right) \leq F_{\mu-}(\eta_{,i}) \leq \frac{1}{2}\left(P_{,j_{i-1}} + P_{,j_{i-1}+1}\right), \tag{4.22}$$

but also

$$\frac{1}{2}(P_{,j_i-1} + P_{,j_i}) \leq F_\mu(\eta_{,i}) \leq \frac{1}{2}(P_{,j_i} + P_{,j_i+1}), \tag{4.23}$$

with the convention that $P_{,-1} = 0$ and $P_{,n+1} = 1$.

Assume first that $\mathsf{K}_\bullet(p) = |F_{\mu-}(\eta_{,i}) - Q_{,i-1}|$. If $\eta_{,i} \leq x_{,j_{i-1}}$ then $G_-(\eta_{,i}) \leq P_{,j_{i-1}-1}$, and hence $F_{\mu-}(\eta_{,i}) - G_-(\eta_{,i}) \geq F_{\mu-}(\eta_{,i}) - P_{,j_{i-1}}$, but also, by (4.22),

$$F_{\mu-}(\eta_{,i}) - G_-(\eta_{,i}) \geq F_{\mu-}(\eta_{,i}) - P_{,j_{i-1}} - \left(2F_{\mu-}(\eta_{,i}) - P_{,j_{i-1}-1} - P_{,j_{i-1}}\right) = P_{,j_{i-1}} - F_{\mu-}(\eta_{,i}),$$

and consequently

$$F_{\mu-}(\eta_{,i}) - G_-(\eta_{,i}) \geq \left|F_{\mu-}(\eta_{,i}) - P_{,j_{i-1}}\right| = |F_{\mu-}(\eta_{,i}) - Q_{,i-1}| = \mathsf{K}_\bullet(p).$$

If $x_{,j_{i-1}} < \eta_{,i} \leq x_{,j_{i-1}+1}$ then $G_-(\eta_{,i}) = P_{,j_{i-1}}$ and hence

$$|F_{\mu-}(\eta_{,i}) - G_-(\eta_{,i})| = \mathsf{K}_\bullet(p).$$

Finally, if $\eta_{,i} > x_{,j_{i-1}+1}$ then $G_-(\eta_{,i}) \geq P_{,j_{i-1}+1}$, and hence $G_-(\eta_{,i}) - F_{\mu-}(\eta_{,i}) \geq P_{,j_{i-1}} - F_{\mu-}(\eta_{,i})$, but also, again by (4.22),

$$G_-(\eta_{,i}) - F_{\mu-}(\eta_{,i}) \geq P_{,j_{i-1}+1} - F_{\mu-}(\eta_{,i}) - \left(P_{,j_{i-1}} + P_{,j_{i-1}+1} - 2F_{\mu-}(\eta_{,i})\right) = F_{\mu-}(\eta_{,i}) - P_{,j_{i-1}},$$

and therefore

$$G_-(\eta_{,i}) - F_{\mu-}(\eta_{,i}) \geq \left|F_{\mu-}(\eta_{,i}) - P_{,j_{i-1}}\right| = \mathsf{K}_\bullet(p).$$

Thus (4.21) holds whenever $\mathsf{K}_\bullet(p) = |F_{\mu-}(\eta_{,i}) - Q_{,i-1}|$.

Next assume that $\mathsf{K}_\bullet(p) = |F_\mu(\eta_{,i}) - Q_{,i}|$. Utilizing (4.23) instead of (4.22), completely analogous arguments show that $|F_\mu(\eta_{,i}) - G(\eta_{,i})| \geq \mathsf{K}_\bullet(p)$ in this case as well, which again implies (4.21). The latter therefore holds in either case. As seen earlier, this proves the existence of a best d_{K}-approximation of μ, given p, and also that $d_{\mathsf{K}}(\mu, \delta_{\bullet}^{p}) = \mathsf{K}_{\bullet}(p)$.

Finally, with $y \in \Xi_m^+$ and $p \in \Pi_m$ as in the proof of Lemma 4.3, observe that $d_{\mathsf{K}}(\mu, \delta_x^p) = \mathsf{K}_{\bullet}(p)$ if and only if $\max_{i=1}^m \kappa_{F_\mu,[Q_{,i-1},Q_{,i}]}(y_{,i}) = \mathsf{K}_{\bullet}(p)$, by Proposition 4.22. As seen in the proof of Theorem 4.20, this means that

$$F_{\mu-}(y_{,i+1}) - \mathsf{K}_{\bullet}(p) \le Q_{,i} \le F_\mu(y_{,i}) + \mathsf{K}_{\bullet}(p), \quad \forall\, i = 0, \ldots, m\,,$$

or equivalently,

$$F_{\mu-}^{-1}\left(Q_{,i} - \mathsf{K}_{\bullet}(p)\right) \le y_{,i} \le F_\mu^{-1}\left(Q_{,i-1} + \mathsf{K}_{\bullet}(p)\right), \quad \forall\, i = 1, \ldots, m\,,$$

which in turn is equivalent to the validity of (4.19) for every j. $\square$

Corollary 4.24. *Assume $\mu \in \mathcal{P}_{\mathsf{cts}}$, and $n \in \mathbb{N}$. Then $d_{\mathsf{K}}(\mu, \delta_x^{u_n}) \ge \frac{1}{2}n^{-1}$ for all $x \in \Xi_n$, with equality holding if and only if*

$$F_{\mu-}^{-1}\left(\frac{2j-1}{2n}\right) \le x_{,j} \le F_\mu^{-1}\left(\frac{2j-1}{2n}\right), \quad \forall\, j = 1, \ldots, n.$$

By combining Theorems 4.20 and 4.23, it is possible to characterize best d_{K}-approximations of $\mu \in \mathcal{P}$ as well. For this, associate with every non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$ and every number $a \ge 0$ a map $S_{f,a} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, given by

$$S_{f,a}(x) = f_+\left(f^{-1}(x+a)\right) + a, \quad \forall\, x \in \overline{\mathbb{R}}\,.$$

This map is a true analogue of $T_{f,a}$ in Section 4.2, and in fact, Proposition 4.8, with $T_{f,a}$ replaced by $S_{f,a}$, remains fully valid. Identical reasoning then shows that

$$\mathsf{K}_{\bullet}^{\bullet,n} := \min\left\{a \ge 0 : S_{F_\mu,a}^{[n]}(0) \ge 1\right\} < +\infty\,;$$

again, $(\mathsf{K}_{\bullet}^{\bullet,n})$ is non-increasing, $n\mathsf{K}_{\bullet}^{\bullet,n} \le \frac{1}{2}$ for every n, and $\mathsf{K}_{\bullet}^{\bullet,n} = 0$ if and only if $\#\mathrm{supp}\,\mu \le n$. Notice that if $\mu \in \mathcal{P}_{\mathsf{cts}}$ then

$$S_{F_\mu,a}(x) = \begin{cases} a & \text{if } x < -a\,, \\ 2a + x & \text{if } -a \le x < 1-a\,, \\ a+1 & \text{if } x \ge 1-a\,, \end{cases}$$

from which it is clear that $\mathsf{K}_{\bullet}^{\bullet,n} = \frac{1}{2}n^{-1}$.

Theorem 4.25. *Let $\mu \in \mathcal{P}$ and $n \in \mathbb{N}$. There exists a best d_{K}-approximation of μ, and $d_{\mathsf{K}}(\mu, \delta_{\bullet}^{\bullet,n}) = \mathsf{K}_{\bullet}^{\bullet,n}$. Moreover, for every $x \in \Xi_n$ and $p \in \Pi_n$, the following are equivalent:*

(i) $d_{\mathsf{K}}(\mu, \delta_x^p) = d_{\mathsf{K}}(\mu, \delta_\bullet^{\bullet,n})$;

(ii) *all implications in* (4.17) *are valid with* $\mathsf{K}^\bullet(x)$ *replaced by* $\mathsf{K}_\bullet^{\bullet,n}$;

(iii) *all implications in* (4.19) *are valid with* $\mathsf{K}_\bullet(p)$ *replaced by* $\mathsf{K}_\bullet^{\bullet,n}$.

Proof. Note that once the existence of a best d_{K}-approximation of μ is established, the proof is virtually identical to that of Theorem 4.9. Thus, only the existence is to be proved here. To this end, let $a = \inf_{x\in\Xi_n, p\in\Pi_n} d_{\mathsf{K}}(\mu, \delta_x^p)$, and pick sequences (x_k) and (p_k) in Ξ_n and Π_n, respectively, with the property that $\lim_{k\to\infty} d_{\mathsf{K}}\left(\mu, \delta_{x_k}^{p_k}\right) = a$. By the compactness of Ξ_n, assume w.o.l.g. that $\lim_{k\to\infty} x_k = \eta \in \Xi_n$. Since $a \le \mathsf{K}^\bullet(x_k) \le d_{\mathsf{K}}\left(\mu, \delta_{x_k}^{p_k}\right)$, it suffices to show that $\mathsf{K}^\bullet(\eta) \le a$. To see the latter, assume that $\eta_{,j} < \eta_{,j+1}$ for any $j = 1, \ldots, n-1$. Then $x_{k,j} < x_{k,j+1}$ for all sufficiently large k, and hence by Theorem 4.20, $F_{\mu-}(x_{k,j+1}) - F_\mu(x_{k,j}) \le 2\mathsf{K}^\bullet(x_k)$, which in turn implies

$$F_{\mu-}(\eta_{,j+1}) - F_\mu(\eta_{,j}) \le \liminf_{k\to\infty} \left(F_{\mu-}(x_{k,j+1}) - F_\mu(x_{k,j})\right) \le 2a\,.$$

Since, similarly, $F_{\mu-}(\eta_{,1}) \le a$ and $1 - F_\mu(\eta_{,n}) \le a$, it follows that $\mathsf{K}^\bullet(\eta) \le a$, as claimed. □

Corollary 4.26. *Assume* $\mu \in \mathcal{P}_{\mathsf{cts}}$, *and* $n \in \mathbb{N}$. *Then* $\mathsf{K}_\bullet^{\bullet,n} = \mathsf{K}_\bullet(u_n) = \frac{1}{2}n^{-1}$, *and* δ_x^p *with* $x \in \Xi_n$, $p \in \Pi_n$ *is a best* d_{K}*-approximation of* μ *if and only if it is a best uniform* d_{K}*-approximation of* μ.

Remark 4.27. (i) By Theorem 4.25, $\mathsf{K}_\bullet^{\bullet,n} = \min_{x\in\Xi_n} \mathsf{K}^\bullet(x) = \min_{p\in\Pi_n} \mathsf{K}_\bullet(p)$.

(ii) If μ has even a single atom, then $\mathsf{K}_\bullet^{\bullet,n}$ may be smaller than $\mathsf{K}_\bullet(u_n)$, and thus a best uniform d_{K}-approximation may not be a best d_{K}-approximation. A simple example illustrating this is $\mu = \frac{3}{4}\delta_0 + \frac{1}{4}\lambda\big|_{[0,1]}$, where $\mathsf{K}_\bullet^{\bullet,n} = \frac{1}{4}(2n-1)^{-1}$ whereas $\mathsf{K}_\bullet(u_n) = \frac{1}{2}\max\{n,2\}^{-1}$, and hence $\mathsf{K}_\bullet^{\bullet,n} < \mathsf{K}_\bullet(u_n)$ for every $n \ge 2$.

For Benford's Law, the best d_{K}-approximations are the same as the best uniform d_1-approximations; see also Figure 4.2.

Corollary 4.28. *Assume* $b > 1$, *and* $n \in \mathbb{N}$. *Then* $\delta_{x_n}^{u_n}$ *with* $x_{n,i} = b^{(2j-1)/(2n)}$ *for all* $j = 1, \ldots, n$ *is the unique best (uniform)* d_{K}*-approximation of* β_b. *Moreover,* $d_{\mathsf{K}}(\beta_b, \delta_\bullet^{\bullet,n}) = \frac{1}{2}n^{-1}$.

4.5 Conclusion

The general results in this chapter have been motivated mainly by a quantitative analysis of Benford's Law, and the precise statements regarding the latter are but simple corollaries of the former. In particular, Sections 4.2 to 4.4 show that the quantization coefficients $Q_* = \lim_{n\to\infty} n d_*(\beta_b, \delta_{\bullet}^{\bullet,n})$ and their uniform counterparts $Q_{*,u} = \lim_{n\to\infty} n d_*(\beta_b, \delta_{\bullet}^{u_n})$ all are finite and positive for each metric d_* considered. Clearly, $Q_* \le Q_{*,u}$ for all $b > 1$. Also, note that $\left(n d_*(\beta_b, \delta_{\bullet}^{\bullet,n})\right)$ is non-increasing, possibly constant, whereas $\left(n d_*(\beta_b, \delta_{\bullet}^{u_n})\right)$ is non-decreasing; for a proof of the latter, see Proposition 5.8. Figure 4.3 summarizes the results obtained earlier. The dependence of Q_* and $Q_{*,u}$ on b is illustrated in Figure 4.4. On the one hand,

$*$	Q_*	$Q_{*,u}$
L	$\dfrac{\max\{b,2\}-1}{2b-2}\cdot\dfrac{\log(1+b\log b)-\log(1+\log b)}{\log b}$	$\dfrac{\max\{b,2\}-1}{2b-2}\cdot\dfrac{b\log b}{1+b\log b}$
$r \ge 1$	$\dfrac{r+1}{2(b-1)(\log b)^{1/r}}\left(\dfrac{b^{r/(r+1)}-1}{r}\right)^{1+1/r}$	$\dfrac{(\log b)^{1-1/r}}{2(b-1)}\left(\dfrac{b^r-1}{r(r+1)}\right)^{1/r}$
K	$\dfrac{1}{2}$	$\dfrac{1}{2}$

Figure 4.3: The quantization (Q_*) and uniform quantization ($Q_{*,u}$) coefficients of β_b for d_*; see also Figure 4.4.

Q_{L} and $Q_{\mathsf{L},u}$ tend to $\frac{1}{2}$ as $b \downarrow 1$, but also as $b \to +\infty$, both attaining their respective minimal value for $b = 2$. On the other hand, Q_r and $Q_{r,u}$ both tend to $\frac{1}{2}(r+1)^{-1/r}$ as $b \downarrow 1$, whereas $\lim_{b\to+\infty}(\log b)^{1/r} Q_r = \frac{1}{2}(r+1)r^{-(r+1)/r}$ and $\lim_{b\to+\infty}(\log b)^{1/r-1} Q_{r,u} = \frac{1}{2}r^{-1/r}(r+1)^{-1/r}$. Finally, $Q_{\mathsf{K}} = Q_{\mathsf{K},u} = \frac{1}{2}$ for all b.

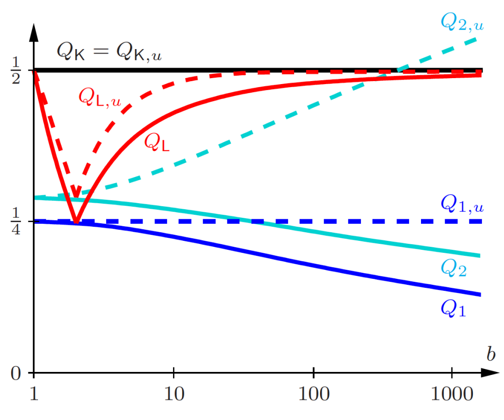

Figure 4.4: Comparing the quantization coefficients Q_* (solid curves) and uniform quantization coefficients $Q_{*,u}$ (broken curves) of β_b, for $* = \mathsf{L}$ (red), $* = 1, 2$ (blue), and $* = \mathsf{K}$ (black), respectively; see also Figure 4.3.

Remark 4.29. In the context of Benford's Law, $\mathbb{J} = [1, b]$, and since $S_b < b$ always, it may seem more natural to study the approximation problem not on all of $\mathcal{P}$, but rather on the (dense) subset $\widetilde{\mathcal{P}} := \left\{ \mu \in \mathcal{P} : \mu(\{b\}) = 0 \right\}$. Clearly, d_{L} and d_r both metrize the weak topology on $\widetilde{\mathcal{P}}$ but are not complete. (By contrast, d_{K} is complete but not separable, and induces a finer topology.) Since $\widetilde{\mathcal{P}}$ is a G_δ-set in $\mathcal{P}$, a classical theorem [30, Thm.2.5.4] yields, for instance,

$$\widetilde{d}(\mu, \nu) = \int_0^1 |G_\mu - G_\nu| + \sum\nolimits_{k=1}^\infty \frac{2^{-k} \left| \int_{1-k^{-1}}^1 (G_\mu - G_\nu) \right|}{\int_{1-k^{-1}}^1 G_\mu \int_{1-k^{-1}}^1 G_\nu + \left| \int_{1-k^{-1}}^1 (G_\mu - G_\nu) \right|},$$

with $G_\mu = b - F_\mu^{-1}$, $G_\nu = b - F_\nu^{-1}$, as an equivalent complete, separable metric on $\widetilde{\mathcal{P}}$. However, $\widetilde{d}$ appears to be quite unwieldy, and the author does not know of an equivalent *complete* metric on $\widetilde{\mathcal{P}}$ for which explicit results similar to those in Sections 4.2 and 4.3 could be established.

Also, it is readily confirmed that, given any $\mu \in \widetilde{\mathcal{P}}$, there exists a best (or best uniform) d_*-approximation $\delta_{\bullet}^{\bullet, n} \in \widetilde{\mathcal{P}}$ (or $\delta_{\bullet}^{u_n} \in \widetilde{\mathcal{P}}$), i.e., these approximation problems always have a solution in $\left(\widetilde{\mathcal{P}}, d_*\right)$, notwithstanding the fact that the latter space is not complete (if $* = \mathsf{L}, r$) or not separable (if $* = \mathsf{K}$).

In the case of Benford's Law, as seen above, all best (or best uniform) approximations considered converge at the same rate, namely (n^{-1}). This is not a coincidence. Rather, for many other probability metrics n^{-1} turns out to yield the correct order of magnitude of the n-th quantization error as well. Specifically, consider a metric d on $\mathcal{P}$ for which

$$a_1 \|F_\mu^{s_1} - F_\nu^{s_1}\|_1 \le d(\mu, \nu) \le a_2 \left(\epsilon \|F_\mu^{s_2} - F_\nu^{s_2}\|_\infty + (1 - \epsilon) \|F_\mu^{-1} - F_\nu^{-1}\|_\infty \right), \quad \forall \, \mu, \nu \in \mathcal{P}, \tag{4.24}$$

with positive constants a_1, a_2, s_1, s_2, and $\epsilon \in \{0, 1\}$; see, e.g., [10, 86, 87] for examples and properties of such metrics. Note that validity of (4.24) causes d to metrize a topology at least as fine as the weak topology, and clearly (4.24) holds for any $d = d_*$. The latter fact, together with [40, Thm.6.2] yields a simple observation regarding the prevalence of the rate (n^{-1}).

Proposition 4.30. *Let d be a metric on $\mathcal{P}$ satisfying (4.24). Then, for every $\mu \in \mathcal{P}$,*

$$\limsup\nolimits_{n \to \infty} n \inf\nolimits_{x \in \Xi_n, p \in \Pi_n} d\left(\mu, \delta_x^p\right) < +\infty \, ,$$

and if μ is non-singular (w.r.t. λ) then also

$$\liminf\nolimits_{n \to \infty} n \inf\nolimits_{x \in \Xi_n, p \in \Pi_n} d\left(\mu, \delta_x^p\right) > 0 \, .$$

Remark 4.31. (i) Apart from d_*, examples of familiar probability metrics that satisfy (4.24) include the discrepancy distance $\sup_{I \subset \mathbb{R}} |\mu(I) - \nu(I)|$ and the L^r-distance $\|F_\mu - F_\nu\|_r$ between distribution functions [86]. For the important Prokhorov distance, validity of the right-hand inequality in (4.24) appears to be unknown [36], but best approximations are suspected to converge at the rate $\left(n^{-1}\right)$ regardless [43, Sec.4]. Also, (n^{-1}) is established in [26] as the universal rate of convergence for best approximations under Orlicz norms, which contains d_r as a special case.

(ii) In [87, Sec.4.2], for any $a \geq 0$, the a-Lévy distance

$$d_{\mathsf{L}_a}(\mu,\nu) = \inf\{y \geq 0 : F_\mu(\cdot - ay) - y \leq F_\nu \leq F_\mu(\cdot + ay) + y\}$$

is considered. Every d_{L_a} satisfies (4.24), and $d_{\mathsf{L}_0} = d_{\mathsf{K}}$, $d_{\mathsf{L}_1} = \omega^{-1} d_{\mathsf{L}}$. Usage of a-Lévy distances may enable a unified treatment of the results in Sections 4.2 and 4.4.

(iii) Under additional assumptions on μ, the value of $n \inf_{x \in \Xi_n} d(\mu, \delta_x^{u_n})$ can similarly be bounded above and below by positive constants; cf. Theorem 2.39.

Finally, it is worth pointing out that, though motivated here by Benford's Law, compactness of the interval $\mathbb{J}$ was assumed largely for convenience, and can easily be dispensed with for many of the general results in this chapter. For instance, if $\mathbb{J}$ is (closed but) unbounded then (4.4), with $\omega = 1$, still yields d_{L} as a complete, separable metric inducing the weak topology on $\mathcal{P}$, though the latter no longer is compact. Clearly, Theorem 4.4 is valid in this situation, as (4.7) holds for $f = F_\mu$ and any interval $I \subset \overline{\mathbb{R}}$. Even though (4.7) may fail for $f = F_\mu^{-1}$ when $\operatorname{supp}\mu$ is unbounded, it is readily checked that nevertheless the conclusions of Proposition 4.2 remain intact for $\ell_{F_\mu^{-1},I}$, provided that $I \subset [0,1]$ but $I \neq \{0\}$ and $I \neq \{1\}$. With $\ell^*_{F_\mu^{-1},\{0\}} := \ell^*_{F_\mu^{-1},\{1\}} := 0$, then, Theorem 4.5 holds verbatim, and so does Theorem 4.9. Analogously, Theorems 4.20, 4.23, and 4.25 all can be seen to be correct, with the definition of $\mathsf{K}_\bullet(p)$ understood to assume that $p_{,1}p_{,n} > 0$. By contrast, the classical L^1-Kantorovich distance $d_1(\mu,\nu) = \|F_\mu^{-1} - F_\nu^{-1}\|_1$ is defined only on the (dense) subset $\mathcal{P}_1 = \left\{\mu \in \mathcal{P} : \int_{\mathbb{J}} |x| \, \mathrm{d}\mu(x) < +\infty\right\}$ where it metrizes a topology finer than the weak topology. Still, with $\mathcal{P}$ replaced by $\mathcal{P}_1$, Proposition 4.12 also remains intact; see Subsection 2.4.3. Note that the sequence $\left(nd_*(\mu, \delta_\bullet^{u_n})\right)$ is bounded when $* = \mathsf{L}, \mathsf{K}$ because $d_{\mathsf{L}} \leq d_{\mathsf{K}}$, whereas $\left(nd_1(\mu, \delta_\bullet^{\bullet,n})\right)$ may decay arbitrarily slowly, by Theorem 2.56. For a simple application of these results to a probability measure with unbounded support, let μ be the standard exponential distribution, i.e., $F_\mu(x) = \max\{0, 1 - e^{-x}\}$. Calculations quite similar to the ones shown earlier for Benford's Law yield (for details, see Propositions 5.10 and 5.9)

$$\lim_{n\to\infty} nd_{\mathsf{L}}\left(\mu, \delta_\bullet^{\bullet,n}\right) = \frac{\log 2}{2}, \quad \lim_{n\to\infty} nd_{\mathsf{L}}\left(\mu, \delta_\bullet^{u_n}\right) = \frac{1}{2},$$

whereas

$$\lim_{n\to\infty} nd_1\left(\mu, \delta_\bullet^{\bullet,n}\right) = 1 \quad \text{but} \quad \lim_{n\to\infty} \frac{n}{\log n} d_1(\mu, \delta_\bullet^{u_n}) = \frac{1}{4}\,,$$

and clearly $nd_{\mathsf{K}}(\mu, \delta_\bullet^{\bullet,n}) = nd_{\mathsf{K}}(\mu, \delta_\bullet^{u_n}) = \frac{1}{2}$ for all n. Even though μ has finite moments of all orders, there exist probability metrics for which $\left(nd(\mu, \delta_\bullet^{\bullet,n})\right)$ is unbounded; see [43, Ex.5.1(d)].

Chapter 5

Supplements to Chapter 4

In this chapter, we provide supplementary details to Chapter 4. More precisely, Section 5.1 is devoted to the proofs of propositions and informal claims in Chapter 4. Section 5.2 provides details regarding computation of approximations of Benford's Law, and Section 5.3 for the Lévy approximations of the standard exponential distribution. For the reader's convenience, we will restate each proposition and claim before proceeding to the proof.

5.1 Proofs of propositions

Recall that $\mathbb{J}$ denotes a compact interval with $\lambda(\mathbb{J}) > 0$, and $\mathcal{P}$ the set of all Borel probability measures on $\mathbb{J}$. Given probabilities $\mu, \nu \in \mathcal{P}$, their *Lévy distance* is

$$d_{\mathsf{L}}(\mu, \nu) = \omega \inf \{y \geq 0 : F_\mu(\cdot - y) - y \leq F_\nu \leq F_\mu(\cdot + y) + y\} ,$$

with $\omega = \max\{1, \lambda(\mathbb{J})\}/\lambda(\mathbb{J})$; their *$L^r$-Kantorovich distance*, with $r \geq 1$, is

$$d_r(\mu, \nu) = \lambda(\mathbb{J})^{-1} \left(\int_0^1 \left| F_\mu^{-1}(y) - F_\nu^{-1}(y) \right|^r \mathrm{d}y \right)^{1/r} = \lambda(\mathbb{J})^{-1} \|F_\mu^{-1} - F_\nu^{-1}\|_r;$$

and their *Kolmogorov distance* is

$$d_{\mathsf{K}}(\mu, \nu) = \sup\nolimits_{x \in \mathbb{R}} |F_\mu(x) - F_\nu(x)| = \|F_\mu - F_\nu\|_\infty .$$

We first address the inequalities among the three probability metrics: d_{L}, d_{K} and d_r for $r \geq 1$.

Proposition 5.1. *Let $\mathbb{J}$ be a non-degenerate compact interval. Then*

$$d_1(\mu, \nu) < \frac{1 + \lambda(\mathbb{J})}{\omega \lambda(\mathbb{J})} d_{\mathsf{L}}(\mu, \nu), \quad d_1(\mu, \nu) \leq d_r(\mu, \nu), \quad d_1(\mu, \nu) \leq d_{\mathsf{K}}(\mu, \nu), \quad \forall\ \mu \neq \nu, \mu, \nu \in \mathcal{P}. \tag{5.1}$$

Moreover, if $* \neq 1$, $* \neq \circ$, and $(*, \circ) \neq (\mathsf{L}, \mathsf{K})$ then

$$\sup\nolimits_{\mu, \nu \in \mathcal{P} : \mu \neq \nu} \frac{d_*(\mu, \nu)}{d_\circ(\mu, \nu)} = +\infty . \tag{5.2}$$

Proof. Note that the inequalities

$$d_1(\mu,\nu) \le d_r(\mu,\nu), \quad d_1(\mu,\nu) \le d_{\mathsf{K}}(\mu,\nu), \quad \forall\, \mu \ne \nu, \mu, \nu \in \mathcal{P}$$

are readily deduced from the definitions of the related probability metrics. To prove the first inequality in (5.1), we first show that

$$d_1(\mu,\nu) \le \frac{1+\lambda(\mathbb{J})}{\omega\lambda(\mathbb{J})} d_{\mathsf{L}}(\mu,\nu), \quad \forall\, \mu,\nu \in \mathcal{P}, \tag{5.3}$$

then we prove this inequality is strict.

Define the following three axillary (distribution) functions on $\mathbb{R}$:

$$\overline{F}(x) := \max\left\{F_\mu(x), F_\nu(x)\right\}, \ \underline{F}(x) := \min\left\{F_\mu(x), F_\nu(x)\right\},$$

$$G(x) =: \inf\left\{y: \ \overline{F}\left(x + \underline{F}(x) - y\right) - y \le 0\right\} = \inf\left\{\underline{F}(x) + y : \overline{F}(x-y) \le \underline{F}(x) + y\right\}.$$

It is straightforward to verify that $\overline{F}, \underline{F}$ are distribution functions on $\mathbb{J}$, that G is non-decreasing, and that

$$G(x) = 0, \quad \forall\, x < \min \mathbb{J} \quad \text{and} \quad G(x) = 1, \quad \forall\, x > \max \mathbb{J}.$$

Now we show that G is also right continuous by way of contradiction. Suppose G is not right continuous at some $x_* \in \mathbb{R}$. Then there exist $G(x_*) < t_1 < t_2 < G_+(x_*)$. Hence given any $\varepsilon > 0$,

$$\underline{F}\left(x_* + \overline{F}(x) - t_1\right) \le t_1, \ \underline{F}\left(x_* + \varepsilon + \overline{F}\left(x_* + \varepsilon\right) - t_2\right) > t_2,$$

which, letting $\varepsilon \downarrow 0$, yields

$$\underline{F}\left(x_* + \overline{F}(x) - t_2\right) \ge t_2 > t_1 \ge \underline{F}\left(x_* + \overline{F}(x_*) - t_1\right) \ge \underline{F}\left(x_* + \overline{F}(x_*) - t_2\right),$$

a contradiction. Moreover,

$$\underline{F}(x) \le G(x) \le \overline{F}(x), \quad \forall\, x \in \mathbb{R},$$

and thus

$$\overline{F}^{-1}(t) \le G^{-1}(t) \le \underline{F}^{-1}(t), \quad \forall\, t \in]0,1[.$$

Define

$$g(x) := \inf\left\{y : \overline{F}(x-y) \le \underline{F}(x) + y\right\}, \quad \forall\, x \in \mathbb{R}.$$

It is easy to verify that $G(x) = \underline{F}(x) + g(x)$, and g is right-continuous and has left limits on

$\mathbb{R}$; moreover,

$$\omega^{-1}d_{\mathsf{L}}(\mu,\nu) = \inf\{y : \overline{F}(\cdot - y) \le \underline{F}(\cdot) + y\} = \sup_{x\in\mathbb{R}} g(x).$$

In the following we show that

$$G^{-1}(t) - \overline{F}^{-1}(t) \le g_{-}\left(G^{-1}(t)\right) \le \omega^{-1}d_{\mathsf{L}}(\mu,\nu). \tag{5.4}$$

By the definition of G^{-1} and g, given any $\varepsilon_i > 0$ for $i = 1, 2$,

$$\begin{aligned} t \ge& G\left(G^{-1}(t) - \varepsilon_1\right) = \underline{F}\left(G^{-1}(t) - \varepsilon_1\right) + g\left(G^{-1}(t) - \varepsilon_1\right) + \varepsilon_2 - \varepsilon_2 \\ \ge& \overline{F}\left(G^{-1}(t) - \varepsilon_1 - \left(g\left(G^{-1}(t) - \varepsilon_1\right) + \varepsilon_2\right)\right) - \varepsilon_2, \end{aligned}$$

which implies by the definition of $\underline{F}^{-1}$ that

$$G^{-1}(t) - \varepsilon_1 - g\left(G^{-1}(t) - \varepsilon_1\right) \le \overline{F}^{-1}(t + \varepsilon_2) + \varepsilon_2,$$

and letting $\varepsilon_i \downarrow 0$ yields (5.4). By Fubini's theorem,

$$\begin{aligned} \lambda(\mathbb{J})d_1(\mu,\nu) &= \int_{\mathbb{J}} |F_\mu(x) - F_\nu(x)|\,\mathrm{d}x = \int_{\mathbb{J}} \left(\overline{F}(x) - \underline{F}(x)\right)\mathrm{d}x \\ &= \int_{\mathbb{J}} \left(\overline{F}(x) - G(x)\right)\mathrm{d}x + \int_{\mathbb{J}} (G(x) - \underline{F}(x))\,\mathrm{d}x \\ &= \int_0^1 \left(G^{-1}(t) - \overline{F}^{-1}(t)\right)\mathrm{d}t + \int_{\mathbb{J}} g(x)\mathrm{d}x \le \omega^{-1}(1 + \lambda(\mathbb{J}))\,d_{\mathsf{L}}(\mu,\nu), \end{aligned} \tag{5.5}$$

i.e., (5.3) holds.

Nest, we show by way of contradiction the inequality in (5.3) is strict whenever $\mu \ne \nu$. Suppose $d_1(\mu,\nu) = \frac{1+\lambda(\mathbb{J})}{\omega\lambda(\mathbb{J})}d_{\mathsf{L}}(\mu,\nu)$ for some $\mu \ne \nu$. Then by right-continuity, it follows from (5.5) that

$$G^{-1} - \overline{F}^{-1} = d_{\mathsf{L}}(\mu,\nu) \quad \text{on }]0,1[\tag{5.6}$$

and

$$g = d_{\mathsf{L}}(\mu,\nu), \quad \forall\, t \in [\min\mathbb{J}, \max\mathbb{J}[,$$

which implies that

$$G(\min\mathbb{J}) \ge g(\min\mathbb{J}) = d_{\mathsf{L}}(\mu,\nu). \tag{5.7}$$

Since

$$\overline{F}^{-1}(t) \ge \min\mathbb{J} \quad \text{on } \in]0,1[,$$

by the definition of G^{-1} and (5.7),

$$G^{-1}(t) \le \min\mathbb{J} \le \overline{F}^{-1}(t), \quad \forall\, 0 < t < d_{\mathsf{L}}(\mu,\nu).$$

This yields

$$G^{-1} - \overline{F}^{-1} \leq 0 \quad \text{on }]0, d_{\mathsf{L}}(\mu,\nu)[,$$

contradicting (5.6).

To establish (5.2), w.o.l.g. let $\mathbb{J} = [0,1]$, and consider $\mu = \delta_0$ and $\mu_k = (1-k^{-1})\,\delta_0 + k^{-1}\delta_{k^{-2}}$ for $k \in \mathbb{N}$. It is straightforward to verify that

$$d_{\mathsf{L}}(\mu_k,\mu) = k^{-2}, \quad d_r(\mu_k,\mu) = k^{-2-1/r}, \quad d_{\mathsf{K}}(\mu_k,\mu) = k^{-1},$$

which immediately yields

$$\sup_{\mu\neq\nu} \frac{d_{\mathsf{K}}(\mu,\nu)}{d_{\mathsf{L}}(\mu,\nu)} = \sup_{\mu\neq\nu} \frac{d_{\mathsf{L}}(\mu,\nu)}{d_r(\mu,\nu)} = \sup_{\mu\neq\nu} \frac{d_{\mathsf{K}}(\mu,\nu)}{d_r(\mu,\nu)} = +\infty, \quad \forall\, r \geq 1;$$

$$\sup_{\mu\neq\nu} \frac{d_r(\mu,\nu)}{d_1(\mu,\nu)} = +\infty, \quad \forall\, r > 1.$$

To establish (5.2) for the remaining possibilities of $(*,\circ)$, consider: $\widetilde{\mu}_k = (1-k^{-1})\,\delta_0 + k^{-1}\delta_1$ for $k \in \mathbb{N}$. It follows from direct calculation that

$$d_{\mathsf{L}}(\mu,\widetilde{\mu}_k) = d_{\mathsf{K}}(\mu,\widetilde{\mu}_k) = k^{-1}, \quad d_r(\mu,\widetilde{\mu}_k) = k^{-1/r},$$

and hence

$$\sup_{\mu\neq\nu} \frac{d_r(\mu,\nu)}{d_{\mathsf{L}}(\mu,\nu)} = \sup_{\mu\neq\nu} \frac{d_r(\mu,\nu)}{d_{\mathsf{K}}(\mu,\nu)} = +\infty.$$

□

Next we consider basic properties of d_{L}. Recall that given two non-decreasing functions $f, g : \mathbb{R} \to \overline{\mathbb{R}}$,

$$d_{\mathsf{L}}(f,g) = \inf\{y \geq 0 : f(\,\cdot\, - y) - y \leq g \leq f(\,\cdot\, + y) + y\} \in [0,+\infty]\,.$$

Proposition 5.2. *Let $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing. Then*
(i) (Symmetry) $d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}(g,f)$*;*
(ii) (Triangle inequality) $d_{\mathsf{L}}(f,g) + d_{\mathsf{L}}(g,h) \geq d_{\mathsf{L}}(f,h)$*;*
(iii) (Positive semi-definiteness) $d_{\mathsf{L}}(f,g) = 0$ *if and only if* $f_- = g_-$ *if and only if* $f_+ = g_+$*. In particular,* $d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}(f_-,g_-) = d_{\mathsf{L}}(f_+,g_+)$*.*

Proof. (i) For all $y \geq 0$,

$$f(\cdot - y) - y \leq g \leq f(\cdot + y) + y \Leftrightarrow g(\cdot - y) - y \leq f \leq g(\cdot + y) + y,$$

which yields that $d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}(g,f)$.
(ii) Given any $y_1,\ y_2 \geq 0$ such that

$$f(\cdot - y_1) - y_1 \leq g \leq f(\cdot + y_1) + y_1, \quad g(\cdot - y_2) - y_2 \leq h \leq g(\cdot + y_2) + y_2,$$

it follows that

$$f(\cdot - (y_1 + y_2)) - (y_1 + y_2) \leq h \leq f(\cdot + (y_1 + y_2)) + y_1 + y_2,$$

which, by the definition of d_{L}, implies that $d_{\mathsf{L}}(f,g) + d_{\mathsf{L}}(g,h) \geq d_{\mathsf{L}}(f,h)$.
(iii) Note that $d_{\mathsf{L}}(f,g) = 0$ if and only if

$$f(\cdot - y) - y \leq g \leq f(\cdot + y) + y, \quad \forall\ y > 0.$$

In particular, $d_{\mathsf{L}}(f,g) = 0$ implies

$$f(\cdot - 2y) - y \leq g(\cdot - y), \quad \forall\ y > 0,$$

and thus $f_- \leq g_-$ by letting $y \downarrow 0$. By (i), $g_- \leq f_-$, and hence $f_- = g_-$. Conversely, $f_- = g_-$ implies that

$$f(\cdot - y) - y \leq f_- = g_- \leq g, \quad \forall\ y > 0. \tag{5.8}$$

Again (i) yields

$$g \leq f(\cdot + y) + y. \tag{5.9}$$

With (5.8) and (5.9), it follows from the definition of d_{L} that $d_{\mathsf{L}}(f,g) = 0$. Analogously, $f_+ = g_+$ also is equivalent to $d_{\mathsf{L}}(f,g) = 0$. Finally, by (ii),

$$|d_{\mathsf{L}}(f,g) - d_{\mathsf{L}}(f_-, g_-)| \leq d_{\mathsf{L}}(f, f_-) + d_{\mathsf{L}}(g, g_-) = 0,$$

which implies that $d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}(f_-, g_-)$. Similarly, $d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}(f_+, g_+)$. □

Finally, we show the invariance of d_{L} under inversion.

Proposition (4.1). *Let $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing. Then $d_{\mathsf{L}}(f^{-1}, g^{-1}) = d_{\mathsf{L}}(f,g)$.*

Proof. We first show $d_{\mathsf{L}}(f,g) \geq d_{\mathsf{L}}(f^{-1}, g^{-1})$. By Proposition 5.2(iii), given any $y > d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}(f_-, g_-)$,

$$f_-(\cdot - y) - y \leq g_-. \tag{5.10}$$

If $t < g(-\infty)$ or $t \geq f(+\infty) - y$, then it holds trivially that

$$g^{-1}(t) \leq f^{-1}(t + y) + y. \tag{5.11}$$

If $t \geq g(-\infty)$ and $t < f(+\infty) - y$, then it follows from (5.10) that

$$f_- \left(g^{-1}(t) - y \right) - y \leq g_- \left(g^{-1}(t) \right) \leq t,$$

which also implies (5.11), by the definition of f_-^{-1} and Proposition 2.1. Analogously, one can show that

$$f^{-1}(t - y) - y \leq g^{-1}(t). \tag{5.12}$$

Combining (5.11) and (5.12) yields $d_{\mathsf{L}}\left(f^{-1}, g^{-1}\right) \leq y$. Since $y > d_{\mathsf{L}}(f,g)$ was arbitrary, $d_{\mathsf{L}}\left(f^{-1}, g^{-1}\right) \leq d_{\mathsf{L}}(f,g)$. From this, Proposition 2.1 and Proposition 5.2(iii),

$$d_{\mathsf{L}}\left(f^{-1}, g^{-1}\right) \geq d_{\mathsf{L}}\left(\left(f^{-1}\right)^{-1}, \left(g^{-1}\right)^{-1}\right) = d_{\mathsf{L}}(f_+, g_+) = d_{\mathsf{L}}(f,g).$$

This shows $d_{\mathsf{L}}(f,g) = d_{\mathsf{L}}\left(f^{-1}, g^{-1}\right)$. □

Next, we investigate some elementary properties of the auxiliary function $\ell_{f,I}$. Recall that for a non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$, and any interval $I \subset \overline{\mathbb{R}}$ with the property (4.7),

$$\ell_{f,I} : \begin{cases} \mathbb{R} \to \mathbb{R}, \\ y \mapsto \inf\{y \geq 0 : f_-(\sup I - y) - y \leq x \leq f_+(\inf I + y) + y\}. \end{cases}$$

Proposition 5.3. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing. For every $a \in \mathbb{R}$, the function $\ell_{f,\{a\}}$ is non-increasing on $]-\infty, f_-(a)]$, and is non-decreasing on $[f_+(a), +\infty[$.*

Proof. Let $x_1 < x_2 \leq f_-(a)$. By the definition of $\ell_{f,\{a\}}$,

$$f_-\left(a - \ell_{f,\{a\}}(x_1)\right) - \ell_{f,\{a\}}(x_1) \leq x_1 < x_2 \leq f_-(a) \leq f_+\left(a + \ell_{f,\{a\}}(x_1)\right) + \ell_{f,\{a\}}(x_1),$$

which implies that $\ell_{f,\{a\}}(x_2) \leq \ell_{f,\{a\}}(x_1)$. This shows $\ell_{f,\{a\}}$ is non-increasing on $]-\infty, f_-(a)]$. Analogously, $\ell_{f,\{a\}}$ is non-decreasing on $[f_+(a), +\infty[$. □

Proposition (4.2). *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing, and $I \subset \overline{\mathbb{R}}$ an interval satisfying (4.7). Then $\ell_{f,I}$ is Lipschitz continuous, and*

$$0 \leq \ell_{f,I}(x) \leq |x| + \max\{0, f_-(\sup I), -f_+(\inf I)\}, \quad \forall\, x \in \mathbb{R}.$$

Moreover, $\ell_{f,I}$ attains a minimal value

$$\ell^*_{f,I} := \min_{x \in \mathbb{R}} \ell_{f,I}(x) = \min\{y \geq 0 : f_-(\sup I - y) - y \leq f_+(\inf I + y) + y\}$$

which is positive unless $f_-(\sup I) \leq f_+(\inf I)$.

Proof. We first show that $\ell_{f,I}$ is Lipschitz continuous with Lipschitz constant 1.

Define the auxiliary functions $\underline{f}$ and $\overline{f}$ as

$$\underline{f}(y) = y - f_-(\sup I - y) \qquad \overline{f}(y) = f_+(\inf I + y) + y.$$

Note that $\underline{f}$ and $\overline{f}$ both are increasing and right-continuous; moreover,

$$\ell_{f,I}(x) = \max\Big\{\inf\left\{y \geq 0 : \ \underline{f}(y) \geq -x\right\}, \inf\left\{y \geq 0 : \ \overline{f}(y) \geq x\right\}\Big\} = \max\left\{\underline{f}_-^{-1}(-x), \overline{f}_-^{-1}(x)\right\}.$$

It is easy to verify that

$$\underline{f}_-^{-1}(x) - \varepsilon \leq \underline{f}_-^{-1}(x-\varepsilon) \leq \underline{f}_-^{-1}(x), \quad \overline{f}_-^{-1}(x) \leq \overline{f}_-^{-1}(x+\varepsilon) \leq \overline{f}_-^{-1}(x) + \varepsilon, \quad \forall\, \varepsilon > 0.$$

Thus, given any $\varepsilon > 0$,

$$\ell_{f,I}(x+\varepsilon) = \max\left\{\underline{f}_-^{-1}(-x-\varepsilon), \overline{f}_-^{-1}(x+\varepsilon)\right\} \leq \max\left\{\underline{f}_-^{-1}(-x)+\varepsilon, \overline{f}_-^{-1}(x)+\varepsilon\right\} = \ell_{f,I}(x) + \varepsilon.$$

Similarly, $\ell_{f,I}(x+\varepsilon) \geq \ell_{f,I}(x) - \varepsilon$. This shows that

$$|\ell_{f,I}(x_1) - \ell_{f,I}(x_2)| \leq |x_1 - x_2|,$$

i.e., $\ell_{f,I}$ is Lispschitz continuous with Lipschitz constant 1.

Obviously $\ell_{f,I}(x) \geq 0$, and thus we only need to prove

$$\ell_{f,I}(x) \leq |x| + \max\{0, f_-(\sup I), -f_+(\inf I)\}, \quad \forall\, x \in \mathbb{R}.$$

For convenience, let $c = \max\{0, f_-(\sup I), -f_+(\inf I)\}$. Note that from (4.7) it follows that $0 \leq c < +\infty$, and for every $x \in \mathbb{R}$,

$$x \leq |x| + c + f_+(\inf I) \leq f_+(\inf I + |x| + c) + |x| + c,$$

and

$$x \geq -|x| - c + f_-(\sup I) \geq -|x| - c + f_-(\sup I - |x| - c),$$

which implies $\ell_{f,I}(x) \leq |x| + c$.

Let $a := \min\{y \geq 0 : f_-(\sup I - y) - y \leq f_+(\inf I + y) + y\}$. It is easy to verify that

$$a = \left(\underline{f} + \overline{f}\right)_-^{-1}(0) \leq \ell_{f,I}(0) \leq c < +\infty;$$

$$\ell_{f,I}(x) \geq a = \ell_{f,I}(s), \quad \forall\, x \in \mathbb{R},\ s \in \left[-\underline{f}(a), \overline{f}(a)\right].$$

Hence $\ell^*_{f,I} = a$, and $a = 0$ if and only if $-\underline{f}(0) \leq \overline{f}(0)$, which is equivalent to $f_-(\sup I) \leq f_+(\inf I)$. ☐

Next, we prove some basic properties of $T_{f,a}$. Recall that for every non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$ and every number $a \geq 0$,

$$T_{f,a} : \begin{cases} \overline{\mathbb{R}} \to \overline{\mathbb{R}}, \\ x \mapsto f_+\left(f^{-1}(x+a) + 2a\right) + a. \end{cases}$$

Proposition (4.8). *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing, $\alpha \geq 0$, and $n \in \mathbb{N}$. Then $T^{[n]}_{f,a}$ is non-decreasing and right-continuous. Also, $a \mapsto T^{[n]}_{f,a}(x)$ is increasing and right-continuous for every $x \in \mathbb{R}$, and if $x \leq a + f(+\infty)$ then the sequence $\left(T^{[k]}_{f,a}(x)\right)$ is non-decreasing.*

Proof. The monotonicity and right-continuity of $T^{[n]}_{f,a}$ come from the corresponding properties of f_+, f^{-1} and the identity mapping $a \mapsto a$. If $x < f(+\infty) - a$, then

$$T_{f,a}(x) = f_+(f^{-1}(x+a) + 2a) + a > x + a + a \geq x.$$

If $f(+\infty) - a \leq x \leq f(+\infty) + a$, then

$$T_{f,a}(x) = f(+\infty) + a \geq x.$$

In sum, $T_{f,a}(x) \geq x$ whenever $x \leq a + f(+\infty)$. Since $T_{f,a}(x) \leq a + f(+\infty)$ for all $x \in \mathbb{R}$, by the monotonicity of $T_{f,a}$ and induction, the sequence $\left(T^{[n]}_{f,a}(x)\right)$ is non-decreasing. ☐

Next, we study some properties of the quantity

$$\mathsf{L}^{\bullet,n}_{\bullet} := \min\left\{a \geq 0 : T^{[n]}_{F_\mu,a}(0) \geq 1\right\}. \tag{5.13}$$

Proposition 5.4. *Let $\mu \in \mathcal{P}$, $n \in \mathbb{N}$, and $\mathsf{L}^{\bullet,n}_{\bullet}$ be defined in (5.13). Then*

$$n\mathsf{L}^{\bullet,n}_{\bullet} \leq \frac{1}{2},$$

and $\mathsf{L}^{\bullet,n}_{\bullet} = 0$ if and only if $\#\operatorname{supp}\mu \leq n$.

Proof. We use the properties of $T_{f,a}$ to provide an upper bound for $\mathsf{L}^{\bullet,n}_{\bullet}$. To see $n\mathsf{L}^{\bullet,n}_{\bullet} \leq \frac{1}{2}$, it suffices to show by induction that

$$T^{[i]}_{F_\mu,\frac{1}{2n}}(0) \geq \frac{i}{n}, \quad \forall\ 1 \leq i \leq n. \tag{5.14}$$

Note that

$$T_{F_\mu,\frac{1}{2n}}(0) = F_\mu\left(F_\mu^{-1}\left(\frac{1}{2n}\right) + \frac{1}{n}\right) + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

and assume $T^{[i]}_{F_\mu,\frac{1}{2n}}(0) \geq \frac{i}{n}$. If $T^{[i]}_{F_\mu,\frac{1}{2n}}(0) + \frac{1}{2n} < 1$, then

$$T^{[i+1]}_{F_\mu,\frac{1}{2n}}(0) = F_\mu\left(F_\mu^{-1}\left(T^{[i]}_{F_\mu,\frac{1}{2n}}(0) + \frac{1}{2n}\right) + \frac{1}{n}\right) + \frac{1}{2n} \geq T^{[i]}_{F_\mu,\frac{1}{2n}}(0) + \frac{1}{2n} + \frac{1}{2n} \geq \frac{i+1}{n};$$

if $T^{[i]}_{F_\mu,\frac{1}{2n}}(0) + \frac{1}{2n} \geq 1$, then

$$T^{[i+1]}_{F_\mu,\frac{1}{2n}}(0) = F_\mu\left(+\infty + \frac{1}{2n}\right) + \frac{1}{2n} = 1 + \frac{1}{2n} \geq \frac{i+1}{2n}.$$

By induction, (5.14) holds.

Next, we show the equivalence between $\mathsf{L}^{\bullet,n}_{\bullet} = 0$ and $\#\mathrm{supp}\,\mu \leq n$.

Assume that $\#\mathrm{supp}\,\mu \leq n$. Then there exist $x \in \Xi^+_m$ and $p \in \Pi^+_m$ for some $1 \leq m \leq n$ such that $\mu = \delta^p_x$. It is straightforward to deduce by induction that $T^{[i]}_{F_\mu,0}(0) = \min\left\{1, P_{,i}\right\}$, for $1 \leq i \leq n$. In particular, $\mathsf{L}^{\bullet,n}_{\bullet} = 0$.

Conversely, assume $\mathsf{L}^{\bullet,n}_{\bullet} = 0$. This simply yields $T^{[n]}_{F_\mu,0}(0) \geq 1$. Suppose by way of contradiction that $\#\mathrm{supp}\,\mu > n$. Then there exists $y \in \Xi^+_{n+1}$ such that

$$0 < F_\mu(y_{,1}) < \cdots < F_\mu(y_{,n}) < F_\mu(y_{,n+1}) \leq 1,$$

which implies that $F_\mu^{-1}(0) \leq y_{,1}$. Similarly, it is readily proved by induction that $T^{[i]}_{F_\mu,0}(0) \leq F_\mu(y_{,i})$ for all $i = 1, \ldots, n$; in particular, $T^{[n]}_{F_\mu,0}(0) \leq F_\mu(y_{,n}) < 1$. This contradicts $T^{[n]}_{F_\mu,0}(0) \geq 1$. □

As for $\ell_{f,I}$, we now establish a few basic properties of the function $\kappa_{f,I}$. Recall that, given a non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$ and any bounded, non-empty interval $I \subset \mathbb{R}$, $\kappa_{f,I} : \mathbb{R} \to \overline{\mathbb{R}}$ is defined as

$$\kappa_{f,I}(x) = \max\left\{\left|f_-(x) - \inf I\right|, \left|f_+(x) - \sup I\right|\right\}.$$

Proposition (4.21). *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be non-decreasing, and $\varnothing \neq I \subset \mathbb{R}$ a bounded interval. Then, with $s := f^{-1}\left(\frac{1}{2}(\inf I + \sup I)\right)$, the function $\kappa_{f,I}$ is non-increasing on $]-\infty, s[$, and non-decreasing on $]s, +\infty[$. Moreover, $\kappa_{f,I}$ attains a minimal value whenever $\inf I \leq \frac{1}{2}\Big(f_-(s) + f_+(s)\Big) \leq \sup I$.*

Proof. Let $a = \inf I$ and $c = \sup I$ for convenience. For any $x < s$,

$$f_+(x) \leq f_-(s) \leq \frac{1}{2}(a + c) < c,$$

and thus

$$|f_+(x) - c| = c - f_+(x) \geq f_-(x) - a,$$

which implies that

$$\kappa_{f,I}(x) = \max\{a - f_-(x), c - f_+(x)\},$$

is obviously non-increasing on $]-\infty, s[$. Analogously, for any $x > s$,

$$\kappa_{f,I}(x) = \max\{f_-(x) - a, f_+(x) - c\},$$

is non-decreasing on $]s, +\infty[$.

Note that the infimum $\inf_{x\in\mathbb{R}} \kappa_{f,I}(x)$ is attained whenever $\min\{\kappa_{f,I}(s-), \kappa_{f,I}(s+)\} \geq \kappa_{f,I}(s)$, that is, if $a \leq \frac{1}{2}(f_-(s) + f_+(s)) \leq c$ and $f_-(s) \leq \frac{a+c}{2} \leq f_+(s)$. Since $s = f^{-1}\left(\frac{a+c}{2}\right)$, the inequalities

$$f_-(s) \leq \frac{a+c}{2} \leq f_+(s)$$

are always valid. Thus the infimum is attained whenever $\inf I \leq \frac{1}{2}(f_-(s) + f_+(s)) \leq \sup I$. □

Next, we present a general result on the universal rate of convergence for best approximations w.r.t. a probability metric. Consider a metric d on $\mathcal{P}$ for which

$$a_1\|F_\mu^{s_1} - F_\nu^{s_1}\|_1 \leq d(\mu,\nu) \leq a_2\left(\epsilon\|F_\mu^{s_2} - F_\nu^{s_2}\|_\infty + (1-\epsilon)\|F_\mu^{-1} - F_\nu^{-1}\|_\infty\right), \quad \forall\, \mu,\nu \in \mathcal{P},$$

with positive constants a_1, a_2, s_1, s_2, and $\epsilon \in \{0, 1\}$.

Proposition (4.30). *Let d be a metric on $\mathcal{P}$ satisfying the above inequalities. Then, for every $\mu \in \mathcal{P}$,*

$$\limsup_{n\to\infty} n \inf_{x\in\Xi_n, p\in\Pi_n} d\left(\mu, \delta_x^p\right) < +\infty, \tag{5.15}$$

and if μ is non-singular (w.r.t. λ) then also

$$\liminf_{n\to\infty} n \inf_{x\in\Xi_n, p\in\Pi_n} d\left(\mu, \delta_x^p\right) > 0. \tag{5.16}$$

Proof. Note that for every $s > 0$, F_μ^s defines a probability measure μ_s via its distribution function,

$$F_{\mu_s}(x) = F_\mu^s(x) \quad \forall\, x \in \mathbb{R}.$$

Let $\delta_{\bullet,*}^{\bullet;n}(\mu)$ be a best d_*-approximation of μ for $* = \mathsf{K}$, 1. To see (5.15), first consider $\epsilon = 1$.

In this case, with Theorem 4.25,

$$d\left(\mu, \delta^{\bullet,n}_{\bullet,\mathsf{K}}\left(\mu_{s_2}\right)_{s_2^{-1}}\right) \le a_2 \left\| F^{s_2}_{\mu} - F^{s_2}_{\delta^{\bullet,n}_{\bullet,\mathsf{K}}\left(\mu_{s_2}\right)_{s_2^{-1}}} \right\|_{\infty} = a_2 d_{\mathsf{K}}\left(\mu_{s_2}, \delta^{\bullet,n}_{\bullet,\mathsf{K}}\left(\mu_{s_2}\right)\right) \le \frac{a_2}{2n},$$

which yields (5.15). Analogous arguments apply to the case $\epsilon = 0$. To prove (5.16), note that

$$d\left(\mu, \delta^{\bullet,n}_{\bullet,1}\left(\mu_{s_1}\right)_{s_1^{-1}}\right) \ge a_1 \left\| F^{s_1}_{\mu} - F^{s_1}_{\delta^{\bullet,n}_{\bullet,1}\left(\mu_{s_1}\right)_{s_1^{-1}}} \right\|_{1} = a_1 d_1\left(\mu_{s_1}, (\delta^{\bullet,n}_{\bullet,1}\left(\mu_{s_1}\right)\right). \tag{5.17}$$

Since μ is non-singular, (5.16) follows from (5.17) and Proposition 2.50. □

Finally, with $\mathbb{J} = [1, b]$ and $b > 1$, we show the existence of best and best uniform approximations restricted to the space $\widetilde{\mathcal{P}} := \left\{\mu \in \mathcal{P} : \mu(\{b\}) = 0\right\}$.

Proposition 5.5. *Assume that $\mu \in \widetilde{\mathcal{P}}$, and $n \in \mathbb{N}$. Let $* = \mathsf{L}, \mathsf{K}$, or $r \ge 1$. Then there exists a best uniform d_*-approximation of μ as well as a best d_*-approximation of μ in $\widetilde{\mathcal{P}}$.*

Proof. We only prove the existence of a best d_*-approximation in $\widetilde{\mathcal{P}}$. Entirely similar arguments apply to best uniform approximations. Let δ^p_x be a best d_*-approximation in $\mathcal{P}$ of μ. Assume w.o.l.g. that $\#\mathrm{supp}\,\mu > n$, i.e., $d_*\left(\mu, \delta^p_x\right) > 0$. It suffices to show that it is possible to choose $x_{,n} < b$, which we prove case by case.

First consider $* = \mathsf{L}$. By Theorem 4.5,

$$F^{-1}_{\mu-}\left(P_{,i}\right) - \mathsf{L}^{\bullet,n}_{\bullet} \le x_{,i} \le F^{-1}_{\mu}\left(P_{,n-1}\right) + \mathsf{L}^{\bullet,n}_{\bullet}, \quad \forall\, i = 1, \cdots, n.$$

Since $F^{-1}_{\mu-}(P_{,n}) \le b$ and $\mathsf{L}^{\bullet,n}_{\bullet} = \omega^{-1} d_{\mathsf{L}}\left(\mu, \delta^p_x\right) > 0$,

$$F^{-1}_{\mu-}\left(P_{,i}\right) - \mathsf{L}^{\bullet,n}_{\bullet} < b,$$

which shows that $x_{,n}$ can be chosen to be smaller than b.

Similarly, for the case $* = \mathsf{K}$, by Theorem 4.25,

$$F^{-1}_{\mu-}\left(P_{,i} - \mathsf{K}^{\bullet,n}_{\bullet}\right) \le x_{,i} \le F^{-1}_{\mu}\left(P_{,n-1} + \mathsf{K}^{\bullet,n}_{\bullet}\right), \quad \forall\, i = 1, \cdots, n.$$

Since $\mathsf{K}^{\bullet,n}_{\bullet} = d_{\mathsf{K}}\left(\mu, \delta^p_x\right) > 0$, $F^{-1}_{\mu-}\left(P_{,n} - \mathsf{K}^{\bullet,n}_{\bullet}\right) < b$, and $x_{,n}$ can again be chosen to be smaller than b.

Finally, we study the approximation w.r.t. the L^r-Kantorovich metric. Theorem 2.47 yields for every $i = 1, \cdots, n$,

$$F^{-1}_{\mu-}\left(\frac{P_{,i-1} + P_{,i}}{2}\right) \le x_{,i} \le F^{-1}_{\mu}\left(\frac{P_{,i-1} + P_{,i}}{2}\right) \quad \text{for } r = 1 \quad \text{and} \quad x_{,i} = \tau_r^{\left. F^{-1}_{\mu}\right|_{[P_{,i-1}, P_{,i}]}} \quad \text{for } r > 1,$$

and $\#\text{supp}\,\delta_x^p = n$ which implies that $\frac{P_{,n-1}+P_{,n}}{2} < 1$. For $* = 1$, $F_\mu^{-1}\left(\frac{P_{,n-1}+P_{,n}}{2}\right) < b$ and hence $x_{,n} < b$. For $* = r > 1$, let $f = F_\mu^{-1}\big|_{[P_{,n-1},P_{,n}]}$. Note that

$$\int_{\left\{f<\tau_r^f\right\}} \left(\tau_r^f - f(u)\right)^{r-1} \mathrm{d}u = \int_{\left\{f>\tau_r^f\right\}} \left(f(u) - \tau_r^f\right)^{r-1} \mathrm{d}u,$$

which implies that $x_{,n} = \tau_r^f < b$. Indeed, if $\tau_r^f = b$, then $\left\{f > \tau_r^f\right\} = \varnothing$, and thus $f \equiv b$ on $]P_{,n-1}, P_{,n}[$, which contradicts $F_\mu^{-1}\left(\frac{P_{,n-1}+P_{,n}}{2}\right) < b$. □

5.2 Approximations of Benford's Law

The following proposition summarizes some properties of best uniform d_{L}-approximations of Benford's Law, that is, for $\mu = \beta_b$ and $\mathbb{J} = [1, b]$ with $b > 1$; recall that $\omega = \dfrac{\max\{b,2\}-1}{b-1} =: \omega_b$. The following proposition covers Corollary 4.7.

Proposition 5.6. *Let $b > 1$ and $n \in \mathbb{N}$. Then $\delta_x^{u_n}$ is a best uniform d_{L}-approximation of β_b if and only if*

$$b^{j/n-L} - L \le x_{,j} \le b^{(j-1)/n+L} + L, \quad \forall\, j = 1, \ldots, n\,, \tag{5.18}$$

where L is the unique solution of

$$2\tau = b^{1-\tau} - b^{1+\tau-1/n}\,;$$

in particular, $1 < x_{,1} < \cdots < x_{,n} < b$ for every $x \in \mathbb{R}^n$ satisfying (5.18), and thus $\#\text{supp}\,\delta_\bullet^{u_n} = n$. Moreover, $j \mapsto \ell^*_{F_{\beta_b}^{-1},[(j-1)/n,j/n]}$ is increasing, and $d_{\mathsf{L}}\left(\beta_b, \delta_\bullet^{u_n}\right) = \omega_b L$. In addition, $(\mathsf{L}_\bullet(u_n))$ is decreasing, and

$$\lim_{n\to\infty} n d_{\mathsf{L}}\left(\beta_b, \delta_\bullet^{u_n}\right) = \frac{\max\{b,2\}-1}{2b-2} \cdot \frac{b\log b}{1+b\log b}\,. \tag{5.19}$$

Proof. We first show that for any $x \in \mathbb{R}^n$, (5.18) implies that

$$1 < x_{,1} < \cdots < x_{,n} < b.$$

To see this, let $f = F_{\beta_b}^{-1}$ for convenience. It suffices to show that

$$1 < b^{\frac{1}{n}-L} - L, \quad b^{1-\frac{1}{n}+L} + L < b, \tag{5.20}$$

$$b^{\frac{i-1}{n}+L} + L < b^{\frac{i+1}{n}-L} - L, \quad i = 2, \cdots, n-1. \tag{5.21}$$

Note that $b^{1-\frac{1}{n}+L} + L = b^{1-L} - L < b$. In the following, we first prove (5.21) and then the first inequality in 5.20.

Since f is convex in $]0,1[$, $\left(f\left(\frac{i+1}{n} - L\right) - f\left(\frac{i-1}{n} + L\right)\right)_{1\le i\le n-1}$ is increasing. Hence (5.20) follows from

$$f\left(\frac{2}{n} - L\right) - f(L) > 2L,$$

i.e.,

$$\frac{f\left(\frac{2}{n} - L\right) - f(L)}{2\left(\frac{1}{n} - L\right)} > \frac{1}{\frac{1}{nL} - 1}. \tag{5.22}$$

By the Mean Value Theorem, (5.22) is obtained from $\log b = f'(0) > \frac{1}{\frac{1}{nL}-1}$, equivalently,

$$L < \frac{\log b}{(1+\log b)n}. \tag{5.23}$$

Define the auxiliary function $g(\tau) = 2\tau - b^{1-\tau} + b^{1-\frac{1}{n}+\tau}$. Note that g is increasing and $g(L) = 0$. To prove (5.23), it suffices to show $g\left(\frac{\log b}{(1+\log b)n}\right) > 0$, i.e.,

$$\frac{2\log b}{b(1+\log b)n} > b^{-\frac{\log b}{(1+\log b)n}} - b^{\frac{\log b}{(1+\log b)n} - \frac{1}{n}} = e^{-\frac{(\log b)^2}{(1+\log b)n}} - e^{-\frac{\log b}{(1+\log b)n}}. \tag{5.24}$$

To show (5.24), we only need to verify that

$$h\left(\frac{\log b}{(1+\log b)n}\right) > 0$$

with $h(\tau) := \frac{2}{b}\tau + e^{-\tau} - e^{-\tau\log b}$. Observe that

$$h(\tau) \ge \frac{2}{b}\tau > 0, \quad \forall\, \tau > 0$$

holds as long as $b \ge e$. It remains to consider $b < e$ and show that h is increasing, with $h(0) = 0$. A direct calculation yields

$$h'(\tau) = \frac{2}{b} - e^{-\tau} + e^{-\tau\log b}\log b.$$

Notice that

$$h'(0) = \frac{2}{b} - 1 - \log\frac{1}{b} > \frac{1}{b},$$

due to the simple fact

$$\log\frac{1}{b} < \frac{1}{b} - 1, \quad \forall\, b > 1.$$

Thus h is increasing if $h''(\tau) > 0$ were true. Note that

$$h''(\tau) = e^{-\tau} - (\log b)^2 e^{-\tau \log b}.$$

For each fixed $\tau > 0$, let $K(s) = s^2 e^{-s\tau}$. It is easy to verify that K is increasing on $]0, 1[$. Since $0 < \log b < 1$, $K(\log b) < K(1) = e^{-\tau}$, which yields $h''(\tau) > 0$, and thus (5.23) is confirmed at last. From employing (5.23), the first inequality in (5.20) follows directly:

$$b^{\frac{1}{n} - L} - L > b^{\frac{1}{n} - \frac{\log b}{(1+\log b)n}} - \frac{\log b}{(1 + \log b)n} = e^{\frac{\log b}{(1+\log b)n}} - \frac{\log b}{(1 + \log b)n} > 1.$$

Next, we prove $j \mapsto \ell^*_{F^{-1}_{\beta_b}, [(j-1)/n, j/n]}$ is increasing. For this purpose, define the auxiliary function $\tau = \tau(x)$ implicitly by

$$2\tau = b^{x + \frac{1}{n} - \tau} - b^{x + \tau}, \quad \forall\, x \geq 0. \tag{5.25}$$

Obviously, $\tau(x) > 0$ for all $x \geq 0$. To prove the asserted monotonicity, it suffices to show that $\tau = \tau(x)$ is increasing. From (5.25) it follows that

$$2 = b^{x + \frac{1}{n} - \tau} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} - 1 \right) \log b - b^{x+\tau} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} + 1 \right) \log b,$$

which indeed yields

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{2 + (b^{x + \frac{1}{n} - \tau} + b^{x+\tau}) \log b}{\left(b^{x + \frac{1}{n} - \tau} - b^{x+\tau} \right) \log b} = \frac{2 + (b^{x + \frac{1}{n} - \tau} + b^{x+\tau}) \log b}{2\tau \log b} > 0.$$

Finally, to show $(\mathsf{L}_\bullet(u_n))$ is decreasing, it is sufficient to prove the monotonicity of the following auxiliary function $\widetilde{\tau} = \widetilde{\tau}(x)$ implicitly defined by

$$2b^{-1} \widetilde{\tau} = b^{-\widetilde{\tau}} - b^{\widetilde{\tau} - x}, \quad \forall\, x \geq 0.$$

Indeed, $\widetilde{\tau}(x) \geq 0$ for all $x \geq 0$, and

$$\frac{\mathrm{d}x}{\mathrm{d}\widetilde{\tau}} = \frac{2b^{-1} + b^{-\widetilde{\tau}} \log b}{b^{\widetilde{\tau} - x} \log b} + 1 > 0,$$

via straightforward calculations. Recall that $2b^{-1}L = b^{-L} - b^{L - \frac{1}{n}}$, and (5.19) follows from the asymptotic expansion:

$$2b^{-1}L = -L \log b - \left(L - \frac{1}{n} \right) \log b + \mathcal{O}\left(L^2, \left(L - \frac{1}{n} \right)^2 \right).$$

□

Corollary (4.11). *Let $b > 1$ and $n \in \mathbb{N}$. Then the best d_{L}-approximation of β_b is δ_x^p, with*

$$x_{,j} = b^{(2j-1)L} + 2L\frac{b^{2jL}-1}{b^{2L}-1} - L = b^{P_{,j}-L} - L\,, \tag{5.26}$$

$$P_{,j} = \frac{1}{\log b}\log\left(b^{(2j-1)L} + 2L\frac{b^{2jL}-1}{b^{2L}-1}\right) + L = \frac{\log(x_{,j}+L)}{\log b} + L\,, \tag{5.27}$$

for all $j = 1,\ldots,n$, where L is the unique solution of (4.12); in particular, $\#\mathrm{supp}\,\delta_{\bullet}^{\bullet,n} = n$. Moreover, $d_{\mathsf{L}}(\beta_b, \delta_{\bullet}^{\bullet,n}) = \omega_b L$, and

$$\lim_{n\to\infty} n d_{\mathsf{L}}(\beta_b, \delta_{\bullet}^{\bullet,n}) = \frac{\max\{b,2\}-1}{2b-2}\cdot\frac{\log(1+b\log b)-\log(1+\log b)}{\log b}\,. \tag{5.28}$$

Proof. We first verify the formula for the unique best d_{L}-approximation. Recall that given any $p \in \Pi_n$,

$$\mathsf{L}_{\bullet}(p) = \max_{j=1}^{n} \ell^*_{F_\mu^{-1},[P_{,j-1},P_{,j}]}$$

with $\ell^*_{F_\mu^{-1},[P_{,j-1},P_{,j}]}$ being the unique solution of

$$2\ell = b^{P_{,j}-\ell} - b^{P_{,j-1}+\ell}, \quad \forall\, j = 1,\ldots,n.$$

This implies that

$$\mathsf{L}_{\bullet}(p) \geq b^{P_{,j}-\mathsf{L}_{\bullet}(p)} - b^{P_{,j-1}+\mathsf{L}_{\bullet}(p)}, \quad \forall\, j = 1,\ldots,n.$$

Moreover, by Remark 4.27(i), $\mathsf{L}_{\bullet}^{\bullet,n} = \min_{p\in\Pi_n}\mathsf{L}_{\bullet}(p)$. Hence it is easy to see that $L := \mathsf{L}_{\bullet}^{\bullet,n}$ is the unique solution to the set of equations

$$2L = b^{P_{,j}-L} - b^{P_{,j-1}+L}, \quad j = 1,\ldots,n, \tag{5.29}$$

for some $p \in \Pi_n$. Rewrite (5.29):

$$b^{P_{,j}-2jL} = b^{P_{,j-1}-2(j-1)L} + 2Lb^{(1-2j)L}, \quad j = 1,\ldots,n,$$

from which it immediately follows that L must be the solution of (4.12), and thus p is uniquely determined by

$$P_{,j} = \frac{1}{\log b}\log\left(b^{(2j-1)L} + 2L\frac{b^{2jL}-1}{b^{2L}-1}\right) + L, \quad \forall\, j = 1,\ldots,n.$$

By Theorem 4.25, the unique best d_{L}-approximation is given by (5.26)-(5.27).

Note that $L \overset{n\to\infty}{\longrightarrow} 0$ follows from $b^{2nL} = \frac{2L+b(b^L-b^{-L})}{2L+b^L-b^{-L}}$, and thus by asymptotic expansion,

$$b^{2nL} = \frac{2L+2Lb\log b+\mathcal{O}(L^2)}{2L+2L\log b+\mathcal{O}(L^2)} = \frac{1+b\log b+\mathcal{O}(L^2)}{1+\log b+\mathcal{O}(L^2)},$$

namely,

$$L = \frac{1}{n}\log_b \frac{1+b\log b}{1+\log b} + o\left(n^{-1}\right).$$

This directly yields (5.28). □

Next, we compare the quantization coefficient to its uniform counterpart.

Proposition 5.7. *Let $b>1$. Then*

$$\frac{1}{2}\log_b\frac{1+b\log b}{1+\log b} < \frac{b\log b}{2(1+b\log b)}.$$

Proof.

$$\begin{aligned}
&\frac{1}{2}\log_b\frac{1+b\log b}{1+\log b} < \frac{b\log b}{2(1+b\log b)}\\
\iff&\log_b\frac{1+b\log b}{1+\log b} - 1 < -\frac{1}{1+b\log b}\\
\iff&\log\frac{b+b\log b}{1+b\log b} > \frac{\log b}{1+b\log b}\\
\overset{a=\log b}{\iff}&\frac{e^a(1+a)}{1+ae^a} - e^{\frac{a}{1+ae^a}} > 0\\
\iff&1+\frac{e^a-1}{1+ae^a} - 1 - \frac{a}{1+ae^a} - \sum_{k=2}^{\infty}\frac{a^k}{k!}\frac{1}{(1+ae^a)^k} > 0\\
\iff&e^a-1-a-\sum_{k=2}^{\infty}\frac{a^k}{k!}\frac{1}{(1+ae^a)^{k-1}} > 0 \iff \sum_{k=2}^{\infty}\frac{a^k}{k!}\left(1-\frac{1}{(1+ae^a)^{k-1}}\right) > 0.
\end{aligned}$$

□

For $r>1$, best d_r-approximations (of β_b) cannot be computed explicitly, and hence asymptotically best d_r-approximations are of interest.

Proposition (4.19). *Let $b,r>1$. Then $\left(\delta_{x_n}^{p_n}\right)$, with*

$$x_{n,j} = \left(1+\frac{j}{n+1}\left(b^{\frac{r}{r+1}}-1\right)\right)^{\frac{r+1}{r}}, \quad P_{n,j} = \frac{1}{\log b}\log\frac{x_{n,j}+x_{n,j+1}}{2},$$

for all $n\in\mathbb{N}$ and $j=1,\ldots,n-1$, and $x_{n,n} = \left(1+(b^{r/(r+1)}-1)\frac{n}{n+1}\right)^{1+1/r}$, is a sequence of *asymptotically best d_r-approximations of β_b. Moreover,*

$$\lim_{n\to\infty} n d_r(\beta_b, \delta_{\bullet}^{\bullet,n}) = \frac{r+1}{2(b-1)(\log b)^{1/r}} \left(\frac{b^{r/(r+1)}-1}{r}\right)^{1+1/r}.$$

Proof. Recall from [85] the probability measure $\beta_{b,r}$ with density

$$\frac{\mathrm{d}\beta_{b,r}}{\mathrm{d}\lambda} = \frac{\frac{\mathrm{d}\beta_b}{\mathrm{d}\lambda}^{\frac{1}{r+1}}}{\int_1^b \frac{\mathrm{d}\beta_{b}}{\mathrm{d}\lambda}^{\frac{1}{r+1}}} = \frac{r}{r+1}\frac{x^{-\frac{1}{r+1}}}{b^{\frac{r}{r+1}}-1}, \qquad 1 < x < b.$$

With this, asymptotically best d_r-approximations $\left(\delta_{x_n}^{p_n}\right)$ are given by

$$x_{n,j} = F_{\beta_{b,r}}^{-1}\left(\frac{j}{n+1}\right), \quad \forall\, j = 1, \ldots, n; \quad P_{n,j} = F_{\beta_b}\left(\frac{x_{n,j}+x_{n,j+1}}{2}\right), \quad \forall\, j = 1, \ldots, n-1.$$

The r-th quantization coefficient follows directly from Proposition 2.50:

$$\begin{aligned}
&\frac{1}{\omega_b}\frac{1}{2(r+1)^{1/r}}\left(\int_1^b \left(\frac{\mathrm{d}\beta_b}{\mathrm{d}\lambda}\right)^{1/(r+1)} \mathrm{d}\lambda\right)^{1+1/r} \\
=&\frac{1}{2(b-1)(r+1)^{1/r}}\left(\int_1^b \left(\frac{1}{x\log b}\right)^{1/(r+1)} \mathrm{d}x\right)^{1+1/r} \\
=&\frac{1}{2(b-1)(r+1)^{1/r}}\frac{1}{(\log b)^{1/r}}\left(\left((1+1/r)\left(b^{r/(r+1)}-1\right)\right)\right)^{1+1/r} \\
=&\frac{r+1}{2(b-1)(\log b)^{1/r}}\left(\frac{b^{r/(r+1)}-1}{r}\right)^{1+1/r}
\end{aligned}$$

□

Finally, we show the monotonicity of the sequences $(nd_*(\beta_b, \delta_{\bullet}^{u_n}))$ and $(nd_*(\beta_b, \delta_{\bullet}^{\bullet,n}))$ mentioned in Section 4.5.

Proposition 5.8. *Let $b > 1$. Then*
(i) $(nd_*(\beta_b, \delta_{\bullet}^{u_n}))$ *is increasing for $* = \mathsf{L}, 1, 2$, and is constant for $* = \mathsf{K}$;*
(ii) $(nd_*(\beta_b, \delta_{\bullet}^{\bullet,n}))$ *is decreasing for $* = \mathsf{L}$, and is constant for $* = 1, \mathsf{K}$.*

Proof. We only show the strict monotonicity; the cases of constant sequences are obvious from the formulae for the respective probability metrics given in Chapter 4.
(i) First, consider $* = \mathsf{L}$. Recall that $\mathsf{L}_{\bullet}(u_n) = \omega^{-1} d_{\mathsf{L}}(\beta_b, \delta_{\bullet}^{u_n})$ is the unique solution of

$$2L = b^{1-L} - b^{1-\frac{1}{n}+L}.$$

To see that $(nd_{\mathsf{L}}(\beta_b, \delta_\bullet^{u_n}))$ is increasing, it suffices to show that $x \mapsto \tau(x)/x$ is increasing, where

$$\tau = \tau(x) := 1 + x - \log_b(b^{1-x} - 2x).$$

Note that $2x = b^{1-x} - b^{1-\tau+x}$. Obviously τ is increasing, and in fact,

$$\begin{aligned}\tau/x =& 1 + \frac{1}{x}\left(1 - \log_b(b^{1-x} - 2x\right) = 1 + \frac{1}{x}\left(1 - \left(1 - x + \log_b\left(1 - 2xb^{x-1}\right)\right)\right)\\ =& 2 - \frac{1}{x}\log_b\left(1 - 2xb^{x-1}\right) = 2 + \frac{1}{x\log b}\sum_{k=1}^{\infty}\frac{\left(2xb^{x-1}\right)^k}{k} = 2 + \frac{2b^{x-1}}{\log b}\sum_{k=0}^{\infty}\frac{\left(2xb^{x-1}\right)^k}{k+1},\end{aligned}$$

which clearly is increasing. Hence $(n\mathsf{L}_\bullet(u_n))$ is increasing.

Next, we consider the case $* = 1$. Recall that

$$nd_1(\beta_b, \delta_\bullet^{u_n}) = \frac{n}{\log b}\tanh\left(\frac{\log b}{4n}\right).$$

Thus, it suffices to show $g(x) := \frac{1}{x}\tanh\frac{x}{2} = \frac{1}{x}\frac{e^x - 1}{e^x + 1}$ is decreasing on $]0, +\infty[$. Observe that

$$g'(x) = \frac{2xe^x + 1 - e^{2x}}{x^2(1 + e^x)^2},$$

and via a straightforward calculation, $g'(x) < 0$, for all $x > 0$, which shows that g is decreasing on $]0, \infty[$, as claimed.

For the case $* = 2$, note that

$$n^2 d_2(\beta_b, \delta_\bullet^{u_n})^2 = \frac{b-1}{b+1}\frac{n^2}{2\log b}\left(1 - \frac{2n}{\log b}\tanh\left(\frac{\log b}{2n}\right)\right).$$

To show $(nd_2(\beta_b, \delta_\bullet^{u_n}))$ is increasing, it suffices to prove $h(x) := \frac{1}{x^2}\left(1 - \frac{1}{x}\tanh x\right)$ is decreasing on $]0, +\infty[$. To see the latter, notice that

$$h'(x) = \frac{x\left(1 - (\operatorname{sech} x)^2\right) - 3(x - \tanh x)}{x^4} = \frac{(3 - 2x)e^{4x} - 8xe^{2x} - 2x - 3}{x^4(e^{2x} + 1)^2}.$$

By Taylor expansion,

$$(3 - 2x)e^{4x} - 8xe^{2x} - 2x - 3 = \sum_{k=5}^{\infty}\frac{(2x)^k}{k!}\left(2^{k-1}(6 - k) - 4k\right) < 0, \quad \forall\, x > 0.$$

It then follows from

$$\left(2^{k-1}(6 - k) - 4k\right)\Big|_{k=5} = -4 < 0, \quad 2^{k-1}(6 - k) - 4k \le -4k, \quad \forall\, k \ge 6$$

that $h'(x) < 0$ for all $x > 0$, as asserted.
(ii) To show $(n\mathsf{L}^{\bullet,n}_{\bullet})$ is decreasing, recall that $\mathsf{L}^{\bullet,n}_{\bullet} = \omega^{-1} d_{\mathsf{L}}\left(\beta_b, \delta^{\bullet,n}_{\bullet}\right)$ is the unique solution of Note that

$$b^{2nL} = \frac{2L + b\left(b^{L} - b^{-L}\right)}{2L + b^{L} - b^{-L}}.$$

It suffices to show that the function $\tau = \tau(x)$, determined by

$$b^{2\tau} = \frac{2x + b\left(b^{x} - b^{-x}\right)}{2x + b^{x} - b^{-x}},$$

is increasing on $]0, +\infty[$. This, however, is obvious from

$$b^{2\tau} = b - \frac{b-1}{1 + \frac{b^{x} - b^{-x}}{2x}} = b - \frac{b-1}{1 + \sum_{k=0}^{\infty} \frac{x^{2k} (\log b)^{2k+1}}{(2k+1)!}}.$$

□

5.3 Lévy approximations of the exponential distribution

In this section, we calculate the best and best uniform d_{L}-approximations of the standard exponential distribution μ, with $F_\mu(x) = 1 - e^{-x}$ for all $x \geq 0$. First, let us address the best uniform d_{L}-approximation.

Proposition 5.9. *For every $n \in \mathbb{N}$, $\delta^{u_n}_{x}$ is a best uniform d_{L}-approximation of μ if and only if*

$$-\log\left(1 - \frac{i}{n} + L\right) - L \leq x_{,i} \leq -\log\left(1 - \frac{i-1}{n} - L\right) + L \quad \forall\, i = 1, \cdots, n,$$

where L is the unique solution of

$$\frac{1}{n} = L\left(e^{2L} + 1\right); \tag{5.30}$$

in particular, $\#\mathrm{supp}\, \delta^{u_n}_{\bullet} = n$. Moreover,

$$\lim_{n\to\infty} n d_{\mathsf{L}}\left(\mu, \delta^{u_n}_{\bullet}\right) = \frac{1}{2}. \tag{5.31}$$

Proof. By (the appropriately generalized version of) Theorem 4.5, with $\omega = 1$, $\delta^{u_n}_{x}$ is a best uniform d_{L}-approximation of μ if and only if

$$-\log\left(1 - \frac{j}{n} + L_n\right) - L_n \leq y_{,j} \leq -\log\left(1 - \frac{j-1}{n} - L_n\right) + L_n, \quad \forall\, j = 1, \cdots, n,$$

where $L_n := d_{\mathsf{L}}(\mu, \delta_{\bullet}^{u_n}) = \max_{j=1}^{n} \tau_j$, and τ_j is the unique positive solution of

$$1 - \frac{j-1}{n} = \tau + e^{2\tau}\left(1 - \frac{j}{n} + \tau\right), \quad \forall\, j = 1, \cdots, n.$$

Note that, $L_n = \tau_n$, and hence L_n is the unique solution of (5.30) as claimed—provided that

$$\tau_1 < \cdots < \tau_n.$$

To see the latter, let $z = z(x)$ be defined by

$$1 - x + \frac{1}{n} = z + e^{2z}(1 - x + z).$$

With this, in order to show that $i \mapsto \tau_i$ is increasing, it suffices to prove $\frac{\mathrm{d}x}{\mathrm{d}z} > 0$, for $0 < x \le 1$, which, however, is evident from $\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1+e^{2z}+2e^{2z}(1+z-x)}{e^{2z}-1} > 0$.

Finally, a straightforward asymptotic expansion yields

$$L_n = \frac{1}{2n} - \frac{1}{4n^2} + \mathcal{O}\left(n^{-3}\right),$$

and hence establishes (5.31). □

Finally, we address the best d_{L}-approximation of μ.

Proposition 5.10. *For every $n \in \mathbb{N}$, the best d_{L}-approximation of μ is δ_x^p, with*

$$P_{,j} = \frac{1 - e^{-2Lj}}{1 - e^{-2Ln}}, \quad x_{,j} = -\log(1 - P_{,j} + L) - L,$$

for all $j = 1, \ldots, n$, where L is the unique positive solution of

$$1 + \frac{\tanh L}{L} = e^{2nL}; \tag{5.32}$$

in particular, $\#\mathrm{supp}\, \delta_{\bullet}^{\bullet, n} = n$. Moreover,

$$\lim_{n\to\infty} n d_{\mathsf{L}}(\mu, \delta_{\bullet}^{\bullet, n}) = \frac{\log 2}{2}. \tag{5.33}$$

Proof. Since μ is continuous, $T^{[n]}_{F_\mu, \mathsf{L}_{\bullet}^{\bullet, n}}(0) = 1$, and hence, by the generalized version of Theorem 4.25 alluded to in Section 4.5, δ_x^p is a best d_{L}-approximation of μ if and only if, for every $j = 1, \cdots, n$,

$$P_{,j-1} = P_{,j}e^{2L} + 1 - L - (1 + L)e^{2L}, \quad x_{,j} = -\log(1 - P_{,j} + L) - L. \tag{5.34}$$

Solving (5.34) with $P_{,0} = 0$, $P_{,n} = 1$ shows that L is the unique solution of

$$0 = e^{2nL} + \frac{e^{2nL} - 1}{e^{2L} - 1} \left(1 - L - (1 + L)e^{2L}\right),$$

which is equivalent to (5.32), and for every $j = 1, \cdots, n$,

$$P_{,j} = \left((1 + L)e^{2L} + L - 1\right) \frac{e^{2Lj} - 1}{e^{2L} - 1} e^{-2Lj} = \left(1 + \frac{L}{\tanh L}\right) \left(1 - e^{-2Lj}\right) = \frac{1 - e^{-2Lj}}{1 - e^{-2Ln}}$$

as well as

$$x_{,j} = -\log(1 - P_{,j} + L) - L\,.$$

Finally, from (5.32) it is clear that $L \to 0$ as $n \to \infty$, and

$$nL = \frac{1}{2} \log\left(1 + \frac{\tanh L}{L}\right) \to \frac{\log 2}{2},$$

estabishing (5.33). □

Chapter 6

The distributional asymptotics mod 1 of $(\log_b n)$

Given a sequence of real numbers (x_n), associate with it a sequence $(\nu_N(x_n))_{N\geq 1}$ of finitely supported probability measures

$$\nu_N(x_n) := \frac{1}{N}\sum_{n=1}^{N} \delta_{\langle x_n\rangle},$$

where $\delta_{\langle x_n\rangle}$ stands for the Dirac measure concentrated at $\langle x_n\rangle$, the natural projection of x_n onto the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Here and throughout, we write ν_N for $\nu_N(x_n)$ when (x_n) is clear from the context. Note that (ν_N) is a sequence in the space $\mathcal{P}(\mathbb{T})$ of all Borel probability measures on $\mathbb{T}$. As a set, $\mathcal{P}(\mathbb{T})$ can be identified with the subspace $\{\mu \in \mathcal{P}(\mathbb{I}) : \mu(\{1\}) = 0\}$ of $\mathcal{P}(\mathbb{I})$.[1] Lowercase Greek letters μ, ν are used henceforth to denote elements of both $\mathcal{P}(\mathbb{T})$ and $\mathcal{P}(\mathbb{I})$, but it will always be clear from the context which space of measures is meant. Recall that a sequence (x_n) in $\mathbb{R}$ is *uniformly distributed modulo one* (u.d. mod 1) [65, Ch.1] if ν_N converges weakly in $\mathcal{P}(\mathbb{T})$ to the uniform distribution $\lambda_{\mathbb{T}}$ on $\mathbb{T}$. (Recall our convention that $\lambda_{\mathbb{I}}$ denotes the uniform distribution on $\mathbb{I}$.) Let d_{K} denote the *discrepancy* (or *Kolmogorov*) metric on $\mathcal{P}(\mathbb{I})$, i.e.

$$d_{\mathsf{K}}(\mu,\nu) = \sup_{x\in\mathbb{I}} |\mu([0,x]) - \nu([0,x])|, \quad \forall\ \mu,\nu \in \mathcal{P}(\mathbb{I}).$$

Recall from [65, Cor.2.1.1] that (x_n) is u.d. mod 1 if and only if $\lim_{N\to\infty} d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}}) = 0$, where $\iota : \mathbb{T} \to \mathbb{I}$ is the natural inclusion; see Section 6.1 for details. It is well known [65, Cor.2.1.2&Thm.2.2.2] that $d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}}) \geq \frac{1}{2N}$ for every positive integer N; in fact, given any (x_n) there exists a constant $c > 0$ such that $d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}}) > c\log N/N$ for infinitely many N.

There is a vast literature on the estimation of discrepancy, especially for u.d. mod 1 sequences. For instance, for the sequence (an), where $a \in \mathbb{R}$ is irrational with bounded partial

[1]Recall that $\mathbb{I}$ always denotes the compact unit interval $[0,1]$.

quotients, [65, Thm.2.3.4] asserts that

$$d_{\mathsf{K}}(\nu_N \circ \iota^{-1}, \lambda_{\mathbb{I}}) = \mathcal{O}(\log N/N), \tag{6.1}$$

and (6.1) also holds for Van der Corput sequence [65, Thm.2.3.5]. However, much less research seems to have been undertaken on sequences that are not u.d. mod 1, for example, on *slowly changing sequences* [59].

Given a sequence (x_n) in $\mathbb{R}$, an improved notion with regard to the distributional asymptotics of x_n is the *Omega limit set* $\Omega[x_n]$, defined as

$$\Omega[x_n] = \Big\{\mu \in \mathcal{P}(\mathbb{T}) : \nu_{N_k}(x_n) \xrightarrow{k\to\infty} \mu \text{ weakly for some sequence } (N_k) \text{ in } \mathbb{N}\Big\}.$$

It is not hard to see that $\Omega[x_n]$ is non-empty, closed and connected [98]. For sequences (x_n) that are *slowly changing* in the sense that

$$\lim_{n\to\infty} n(x_{n+1} - x_n) = \xi \in \mathbb{R},$$

it has been shown in [59] that (x_n) is not u.d. mod 1; moreover, the elements of $\Omega[x_n]$, have been described in terms of asymptotic distribution functions. Similar results for slowly changing sequences in the literature include logarithms of natural numbers or prime numbers, iterated logarithms, and monotone functions of prime numbers [37, 59, 65, 68, 76, 77, 92, 98, 99]. As far as the author knows, however, there were virtually no results, in the case of slowly changing sequences, on the rate(s) of convergence for subsequences of (ν_N) to $\Omega[x_n]$, not even for very basic sequences such as $(\log_b n)$ with $b \in \mathbb{N} \setminus \{1\}$, prior to [100]. Only recently did the author learn that [77] establishes an upper bound $(\log N/N)$ for the latter, as well as their asymptotic distribution functions. Even there, however, the specific nature of limit points as well as the sharpness of the bound $(\log N/N)$ remains obscure. This chapter aims at resolving these obscurities. Specifically, for sequences $(\log_b n)$, every limit point in $\Omega[x_n]$ is clearly identified, and $(\log N/N)$ is shown to be the sharp rate of convergence w.r.t. d_{K}.

While the discrepancy metric (on $\mathcal{P}(\mathbb{T})$, as induced by d_{K}) has been used in UDT for decades, its usage for sequences that are *not* u.d. mod 1 appears debatable. In fact, when analyzing such sequences, it may be more natural to study $\Omega[x_n]$ with a metric metrizing the weak topology of $\mathcal{P}(\mathbb{T})$ such as, for instance, the *Kantorovich* (or *transport*) metric $d_{\mathbb{T}}$. In a recent note [101], the author obtained several results in this regard, including an upper bound $(\log N/N)$ for $d_{\mathbb{T}}$-convergence. As is shown in this chapter, however, this bound is not sharp, and better bounds are provided to replace it for $(x_n) = (\log_b n)$. From the arguments presented, it will also become evident that finding a good *lower* bound remains a formidable challenge, even for sequences as simple as $(\log_b n)$.

6.1 Preliminaries and notations

Let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ be the set of real numbers, integers, and positive integers, respectively. Recall that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be thought of geometrically as the unit circle $\{e^{2\pi i x} : x \in \mathbb{R}\}$ in the complex plane, with its usual topology. For $-\infty < a < b \le \infty$, let $[a, b[:= \{y \in \mathbb{R} : a \le y < b\}$; intervals $[a, b]$, $]a, b]$, $]a, b[$ are defined analogously. Let $\lfloor x \rfloor$, $\lceil x \rceil$, and $\langle\langle x \rangle\rangle = x - \lfloor x \rfloor$ be the floor (i.e., the largest integer $\le x$), the ceiling (i.e., the smallest integer $\ge x$), and the fractional part of $x \in \mathbb{R}$, respectively. In what follows, it will prove useful to denote by $\pi : \mathbb{R} \to \mathbb{T}$, with $\pi(x) = \langle x \rangle = x + \mathbb{Z}$, and by $\iota : \mathbb{T} \to \mathbb{I}$, with $\iota(\langle x \rangle) = \langle\langle x \rangle\rangle$, the natural projection and inclusion, respectively. Arguably the most fundamental maps on $\mathbb{T}$ are rotations: Given any $\theta \in \mathbb{R}$, let R_θ be the (counter-clockwise) rotation of $\mathbb{T}$ by $2\pi\theta$, that is, $R_\theta(\langle x \rangle) = \langle x + \theta \rangle$ for all $\langle x \rangle \in \mathbb{T}$. With this, clearly $R_\theta^k = R_{k\theta} = R_{\langle\langle k\theta \rangle\rangle}$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Let (X, ρ_X) be a compact metric space, and $\mathcal{P}(X)$ the space of all Borel probability measures on X, endowed with the weak topology. Recall that $\mathcal{P}(X)$ is compact and metrizable. The *Kantorovich distance* on X is

$$d_X(\mu, \nu) = \inf_\gamma \int_{X \times X} \rho_X(x, y) \mathrm{d}\gamma(x, y), \quad \forall\, \mu, \nu \in \mathcal{P}(X),$$

where the infimum is taken over all Borel probability measures γ on $X \times X$ with marginals μ and ν. Note that d_X metrizes the weak topology of $\mathcal{P}_X$. For $X = \mathbb{I}$ and $X = \mathbb{T}$, let $\rho_{\mathbb{I}} = |x - y|$, and $\rho_{\mathbb{T}}(x, y) = \min\{|\iota(x) - \iota(y)|, 1 - |\iota(x) - \iota(y)|\}$, $\forall\, x, y \in X$, respectively. Note also that $\mu \mapsto \mu \circ \pi^{-1}$ maps $\mathcal{P}(\mathbb{I})$ continuously onto $\mathcal{P}(\mathbb{T})$; when restricted to $\widetilde{\mathcal{P}}(\mathbb{I}) := \{\mu \in \mathcal{P}(\mathbb{I}) : \mu(\{1\}) = 0\}$, a dense G_δ-set in $\mathcal{P}(\mathbb{I})$, this even yields a continuous bijection, but not a homeomorphism, as $\widetilde{\mathcal{P}}(\mathbb{I})$ is not compact. In the opposite direction, $\mu \mapsto \mu \circ \iota^{-1}$ establishes a measurable bijection from $\mathcal{P}(\mathbb{T})$ onto $\widetilde{\mathcal{P}}(\mathbb{I})$. Note also that $\mu \mapsto \mu \circ R_\theta^{-1}$ defines a homeomorphism of $\mathcal{P}(\mathbb{T})$.

Recall from Chapter 4 that $d_{\mathbb{I}}$ can be expressed explicitly as

$$d_{\mathbb{I}}(\mu, \nu) = \int_0^1 |F_\mu(x) - F_\nu(x)|\, \mathrm{d}x, \quad \forall\, \mu, \nu \in \mathcal{P}(\mathbb{I}). \tag{6.2}$$

A method of computing $d_{\mathbb{T}}$ has been developed in [17]; only the following simple upper bound will be used here.

Proposition 6.1. [17, Cor.3.8]. *Assume that μ, $\nu \in \mathcal{P}(\mathbb{T})$. Then*

$$d_{\mathbb{T}}(\mu, \nu) \le \inf_{y \in \mathbb{I}} \int_0^1 \Big|(F_{\mu\circ\iota^{-1}}(x) - F_{\nu\circ\iota^{-1}}(x)) - (F_{\mu\circ\iota^{-1}}(y) - F_{\nu\circ\iota^{-1}}(y))\Big| \mathrm{d}x \le d_{\mathsf{K}}\left(\mu \circ \iota^{-1}, \nu \circ \iota^{-1}\right).$$

For every $a > 0$, consider the negative exponential distribution $-\mathrm{Exp}(a)$ on $\mathbb{R}$ with parameter a, that is,

$$F_{\mathrm{Exp}(a)}(x) = e^{ax}, \quad \forall\, x \le 0,$$

and let $E_a = -\mathrm{Exp}(a) \circ \pi^{-1} \in \mathcal{P}(\mathbb{T})$. Thus

$$F_{E_a \circ \iota^{-1}}(x) = \frac{e^{ax} - 1}{e^a - 1}, \quad \forall\, x \in \mathbb{I}.$$

Rotated versions of E_a, that is, probabilities $E_a \circ R_\theta^{-1}$ with $\theta \in \mathbb{R}$, play an important role in this chapter. For such probabilities, observe that

$$F_{E_a \circ R_\theta^{-1}} \circ \iota^{-1} = \begin{cases} \frac{e^{\langle\langle \theta \rangle\rangle}(e^{ax}-1)}{e^a - 1} & \text{if } x \in [0, 1 - \langle\langle \theta \rangle\rangle], \\ 1 + \frac{e^{\langle\langle \theta \rangle\rangle}\left(e^{a(x-1)}-1\right)}{e^a - 1} & \text{if } x \in [1 - \langle\langle \theta \rangle\rangle, 1[. \end{cases}$$

Henceforth, our analysis focuses on the sequences $(x_n) = (\log_b n)$ with $b \in \mathbb{N} \setminus \{1\}$, and the associated discrete measures $\nu_N = \nu_N(\log_b n) \in \mathcal{P}(\mathbb{T})$. A simple calculation yields an explicit formula for the distribution function of $\nu_N \circ \iota^{-1}$.

Proposition 6.2. *Assume that $b \in \mathbb{N} \setminus \{1\}$ and $N \in \mathbb{N}$. Then, with $L = \lfloor \log_b N \rfloor$,*

$$F_{\nu_N \circ \iota^{-1}}(x) = \begin{cases} \dfrac{L + 1 + \sum_{j=0}^{L} \left(\lfloor i b^{-j} \rfloor - b^{L-j} \right)}{N} & \textit{if } x \in \left[\log_b \dfrac{i}{b^L}, \log_b \dfrac{i+1}{b^L} \right[, \\ & i = b^L, \ldots, N-1, \\ 1 + \dfrac{L + 1 + \sum_{j=0}^{L} \left(\lfloor \lfloor N b^{-1} \rfloor b^{-j} \rfloor - b^{L-j} \right)}{N} & \textit{if } x \in \left[\log_b \dfrac{N}{b^L}, \log_b \dfrac{b \lfloor N b^{-1} \rfloor + b}{b^L} \right[, \\ 1 + \dfrac{L + 1 + \sum_{j=0}^{L} \left(\lfloor i b^{-j} \rfloor - b^{L-j} \right)}{N} & \textit{if } x \in \left[\log_b \dfrac{bi}{b^L}, \log_b \dfrac{b(i+1)}{b^L} \right[, \\ & i = \lfloor N b^{-1} \rfloor + 1, \ldots, b^L - 1. \end{cases} \tag{6.3}$$

6.2 Rates of convergence

In this section, we study the rate of convergence for subsequences of $(\nu_N)_{N \ge 1}$ w.r.t. $d_\mathbb{T}$, $d_\mathbb{I}$, and d_K. Throughout, for ease of exposition, all proofs are given for $b = 10$, but all arguments can easily be adjusted to any other base $b \in \mathbb{N} \setminus \{1\}$.

6.2.1 Upper bound for the rate of convergence w.r.t. $d_{\mathbb{T}}$

We first present our main result regarding an upper bound for the rate of convergence w.r.t. $d_{\mathbb{T}}$.

Theorem 6.3. *Assume $b \in \mathbb{N} \setminus \{1\}$. Then*

$$\limsup_{N\to\infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}}\left(\nu_N, E_{\log b} \circ R^{-1}_{-\log_b N}\right) < +\infty. \tag{6.4}$$

Proof. Recall that $b = 10$ is assumed throughout; for every $N \in \mathbb{N}$, let $n = \lfloor \log_{10} N \rfloor + 1$ for convenience, and thus $10^{n-1} \leq N \leq 10^n - 1$, as well as $\eta_N = E_{\log_b N} \in \mathcal{P}(\mathbb{T}) \circ R^{-1}_{-\log_b N} \in \mathcal{P}(\mathbb{T})$. By Proposition 6.1, it suffices to estimate

$$\int_0^1 \left| \left(F_{\nu_N \circ \iota^{-1}}(x) - F_{\eta_N \circ \iota^{-1}}(x)\right) - \left(F_{\nu_N \circ \iota^{-1}}(y) - F_{\eta_N \circ \iota^{-1}}(y)\right) \right| \mathrm{d}x,$$

for an appropriate $0 \leq y < 1$. Utilizing Proposition 6.2 we first simplify the latter expression as follows: For every $y \in [0, \log_{10} N - n + 1[$, let $i_0 = \lfloor 10^{y+n-1} \rfloor$. Then $10^{n-1} \leq i_0 \leq N - 1$ and $y \in [\log_{10} i_0 - n + 1, \log_{10}(i_0 + 1) - n + 1[$. Similarly, for $10^{n-1} \leq i \leq 10^n - 1$ and $x \in \mathbb{I}$,

$$x \in [\log_{10} i - n + 1, \log_{10}(i + 1) - n + 1[\quad \Leftrightarrow \quad i = \lfloor 10^{x+n-1} \rfloor. \tag{6.5}$$

With this, it follows from Proposition 6.2 that

$$\begin{aligned}
&\int_0^1 \left| \left(F_{\nu_N \circ \iota^{-1}}(x) - F_{\eta_N \circ \iota^{-1}}(x)\right) - \left(F_{\nu_N \circ \iota^{-1}}(y) - F_{\eta_N \circ \iota^{-1}}(y)\right) \right| \mathrm{d}x \\
&= \sum_{i=10^{n-1}}^{N-1} \int_{\log_{10} i - n+1}^{\log_{10}(i+1)-n+1} \left| \frac{\sum_{j=0}^{n-1} \left(\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor\right)}{N} - \frac{10^n \left(10^x - 10^y\right)}{9N} \right| \mathrm{d}x \\
&+ \int_{\log_{10}(N/10)-n+2}^{\log_{10}(\lfloor N/10 \rfloor+1)-n+2} \left| \frac{\sum_{j=0}^{n-1} \left(\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor\right)}{N} - \frac{10^n (10^{x-1} - 10^y)}{9N} \right| \mathrm{d}x \\
&+ \sum_{i=\lfloor N/10 \rfloor+1}^{10^{n-1}-1} \int_{\log_{10} i-n+2}^{\log_{10}(i+1)-n+2} \left| \frac{\sum_{j=0}^{n-1} \left(\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor\right)}{N} - \frac{10^n (10^{x-1} - 10^y)}{9N} \right| \mathrm{d}x \\
&= \sum_{i=\lfloor N/10 \rfloor+1}^{N-1} \int_{\log_{10} i-n+1}^{\log_{10}(i+1)-n+1} \left| \frac{\sum_{j=0}^{n-1} \left(\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor\right)}{N} - \frac{10^n (10^x - 10^y)}{9N} \right| \mathrm{d}x \\
&\quad + \int_{\log_{10}(N/10)-n+1}^{\log_{10}(\lfloor N/10 \rfloor+1)-n+1} \left| \frac{\sum_{j=0}^{n-1} \left(\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor\right)}{N} - \frac{10^n (10^x - 10^y)}{9N} \right| \mathrm{d}x.
\end{aligned}$$

Since $|F_{\nu_N \circ \iota^{-1}}(x) - F_{\eta_N \circ \iota^{-1}}(x)| \leq 1$ for all $x \in \mathbb{I}$, it easily follows that

$$\int_{\log_{10}(N/10)-n+1}^{\log_{10}(\lfloor N/10 \rfloor+1)-n+1} \left| \frac{\sum_{j=0}^{n-1}\left(\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor\right)}{N} - \frac{10^n (10^x - 10^y)}{9N} \right| \mathrm{d}x = \mathcal{O}\left(N^{-1}\right).$$

From (6.5) and $i_0 = \lfloor 10^{y+n-1} \rfloor$, it is readily verified that

$$\begin{aligned}
&\frac{\sum_{j=0}^{n-1}(\lfloor i10^{-j}\rfloor - \lfloor i_0 10^{-j}\rfloor)}{N} - \frac{10^n(10^x - 10^y)}{9N} \\
=&\frac{\sum_{j=0}^{n-1}\left((i_0 10^{-j} - \lfloor i_0 10^{-j}\rfloor) - (i10^{-j} - \lfloor i10^{-j}\rfloor)\right)}{N} \\
&+ \frac{10\left((\lfloor 10^{x+n-1}\rfloor - 10^{x+n-1}) - (\lfloor 10^{y+n-1}\rfloor - 10^{y+n-1})\right)}{9N} - \frac{10^{1-n}(\lfloor 10^{x+n-1}\rfloor - \lfloor 10^{y+n-1}\rfloor)}{9N}.
\end{aligned}$$

Since also

$$\left|\frac{10\left((\lfloor 10^{x+n-1}\rfloor - 10^{x+n-1}) - (\lfloor 10^{y+n-1}\rfloor - 10^{y+n-1})\right)}{9N} - \frac{10^{1-n}(\lfloor 10^{x+n-1}\rfloor - \lfloor 10^{y+n-1}\rfloor)}{9N}\right| = \mathcal{O}\left(N^{-1}\right),$$

for $y \in [0, \log_{10} N - n + 1[$, we obtain

$$\begin{aligned}
&\int_0^1 \left|\left(F_{\nu_N\circ\iota^{-1}}(x) - F_{\eta_N\circ\iota^{-1}}(x)\right) - \left(F_{\nu_N\circ\iota^{-1}}(y) - F_{\eta_N\circ\iota^{-1}}(y)\right)\right| \mathrm{d}x \\
=&\frac{1}{N}\sum_{i=\lfloor N/10\rfloor+1}^{N-1} \log_{10}\left(1 + \frac{1}{i}\right) \cdot \left|\sum_{j=0}^{n-1}\left((i_0 10^{-j} - \lfloor i_0 10^{-j}\rfloor) - (i10^{-j} - \lfloor i10^{-j}\rfloor)\right)\right| + \mathcal{O}\left(N^{-1}\right).
\end{aligned} \tag{6.6}$$

In the following, we further estimate the right hand side of (6.6). The elementary inequality

$$x - x^2/2 \le \log(x+1) \le x, \quad \forall\, x \ge 0$$

yields

$$\frac{1}{i\log 10} - \frac{1}{2i^2 \log 10} \le \log_{10}\left(1 + \frac{1}{i}\right) \le \frac{1}{i\log 10},$$

and we also have

$$\frac{1}{N}\sum_{i=\lfloor N/10\rfloor+1}^{N-1} \frac{1}{2i^2\log 10}\left|\sum_{j=0}^{n-1}\left((i_0 10^{-j} - \lfloor i_0 10^{-j}\rfloor) - (i10^{-j} - \lfloor i10^{-j}\rfloor)\right)\right| = \mathcal{O}\left(N^{-2}\log N\right).$$

Hence

$$\begin{aligned}
&\int_0^1 \left|\left(F_{\nu_N\circ\iota^{-1}}(x) - F_{\eta_N\circ\iota^{-1}}(x)\right) - \left(F_{\nu_N\circ\iota^{-1}}(y) - F_{\eta_N\circ\iota^{-1}}(y)\right)\right| \mathrm{d}x \\
=&\frac{1}{N\log 10}\sum_{i=\lfloor N/10\rfloor+1}^{N-1} \frac{1}{i}\left|\sum_{j=0}^{n-1}\left(i_0 10^{-j} - \lfloor i_0 10^{-j}\rfloor) - (i10^{-j} - \lfloor i10^{-j}\rfloor)\right)\right| + \mathcal{O}\left(N^{-1}\right).
\end{aligned}$$

From

$$\frac{1}{N\log 10}\frac{1}{i}\left|\sum_{j=0}^{n-1}\left((i_0 10^{-j} - \lfloor i_0 10^{-j}\rfloor) - (i10^{-j} - \lfloor i10^{-j}\rfloor)\right)\right| \le \frac{2n}{N\lfloor N/10\rfloor \log 10},$$

for $i = \lfloor N/10 \rfloor$ and $i = N$, it follows that

$$\begin{aligned}&\int_0^1 \left| \left(F_{\nu_N \circ \iota^{-1}}(x) - F_{\eta_N \circ \iota^{-1}}(x)\right) - \left(F_{\nu_N \circ \iota^{-1}}(y) - F_{\eta_N \circ \iota^{-1}}(y)\right) \right| \mathrm{d}x \\ =&\frac{1}{N \log 10} \sum_{i=\lfloor N/10 \rfloor}^{N} \frac{1}{i} \left| \sum_{j=0}^{n-1} \left((i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right) \right| + \mathcal{O}\left(N^{-1}\right).\end{aligned} \tag{6.7}$$

Completely analogous arguments show that (6.7) holds also for $y \in [\log_{10} N - n + 1, 1[$ with $i_0 = \lfloor 10^{y+n-2} \rfloor$. Thus it suffices to determine the constant order of amplitude of

$$\sum_{i=\lfloor N/10 \rfloor}^{N} \frac{1}{i} \left| \sum_{j=0}^{n-1} \left((i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right) \right|.$$

To get rid of the absolute value, one can use the Cauchy-Schwarz inequality:

$$\left\{ \sum_{i=\lfloor N/10 \rfloor}^{N} \frac{1}{i} \left| \sum_{j=0}^{n-1} \left((i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right) \right| \right\}^2 \tag{6.8}$$

$$\leq \sum_{i=\lfloor N/10 \rfloor}^{N} \frac{1}{i^2} \sum_{i=\lfloor N/10 \rfloor}^{N} \left\{ \sum_{j=0}^{n-1} \left((i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right) \right\}^2.$$

Note that

$$\sum_{i=\lfloor N/10 \rfloor}^{N} \frac{1}{i^2} = 9N^{-1} + \mathcal{O}(N^{-2}). \tag{6.9}$$

Hence it remains to estimate $\sum_{i=0}^{N} \left(\sum_{j=0}^{n-1} \left((i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right) \right)^2$, which can be rewritten as

$$\begin{aligned}&\sum_{i=0}^{N} \left\{ \sum_{j=0}^{n-1} \left((i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right) \right\}^2 = (N+1) \left(\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) \right)^2 \\ &+ \sum_{i=0}^{N} \left(\sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right)^2 - 2 \sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) \sum_{i=0}^{N} \sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor).\end{aligned} \tag{6.10}$$

In the following, we consider each term on the right-hand side of (6.10) individually.

First we consider $\sum_{i=0}^{N} \sum_{j=0}^{n-1} (i 10^{-j} - i \lfloor 10^{-j} \rfloor)$, by switching the order of the summations. For every $i = 0, \cdots, N$ and $j = 0, \cdots, n-1$, there exist nonnegative integers k, l with $l \leq 10^j - 1$ such that $i = k10^j + l$, and hence $i10^{-j} - \lfloor i10^{-j} \rfloor = l10^{-j}$. Therefore

$$\{i : 0 \leq i \leq N\} = \left\{ k10^j + l : 0 \leq k \leq \lfloor N10^{-j} \rfloor - 1, 0 \leq l \leq 10^j - 1 \right\}$$

$$\cup \left\{ \lfloor N10^{-j} \rfloor 10^j + l : 0 \le l \le N - \lfloor N10^{-j} \rfloor 10^j \right\}.$$

Let $N = \overline{a_{n-1} \cdots a_0} = \sum_{j=0}^{n-1} a_j 10^j$ with $0 \le a_j \le 9$ for all $0 \le j \le n-1$. Notice that

$$\lfloor N10^{-j} \rfloor = N10^{-j} - \sum_{r=0}^{j-1} a_r 10^{r-j}, \quad \forall\ j = 1, \cdots, n-1. \tag{6.11}$$

From the simple observations

$$\sum_{j=0}^{n-1} (1 - 10^{-j}) = n + \mathcal{O}(1) \quad \text{and} \quad \sum_{j=1}^{n-1} 10^{-j} \sum_{r=0}^{j-1} a_r 10^r = \mathcal{O}(n),$$

it is tedious but straightforward to deduce that

$$\sum_{i=0}^{N} \sum_{j=0}^{n-1} \left(i10^{-j} - \lfloor i10^{-j} \rfloor \right) = \frac{1}{2} \left(n - \frac{10}{9} \right) N - \frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=0}^{j-1} a_r 10^r + \frac{1}{2} \sum_{j=1}^{n-1} 10^{-j} \left(\sum_{r=0}^{j-1} a_r 10^r \right)^2 + \mathcal{O}(n). \tag{6.12}$$

Next, we deal with $\sum_{i=0}^{N} \left(\sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \right)^2$, which can be expanded as

$$\begin{aligned} &\sum_{i=0}^{N} \left(\sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \right)^2 \\ =&2 \sum_{i=0}^{N} \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor) + \sum_{i=0}^{N} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2. \end{aligned} \tag{6.13}$$

For every $1 \le j \le n-1$, let $K_j = N - \lfloor N10^{-j} \rfloor 10^j$ for notational convenience. Then similarly,

$$\begin{aligned} \{i : 0 \le i \le N\} = &\left\{ k10^j + p10^r + l : 0 \le k \le \lfloor N10^{-j} \rfloor - 1, 0 \le p \le 10^{j-r} - 1, 0 \le l \le 10^r - 1 \right\} \\ &\cup \left\{ \lfloor N10^{-j} \rfloor 10^j + p10^r + l : 0 \le p \le \lfloor K_j 10^{-r} \rfloor - 1, 0 \le l \le 10^r - 1 \right\} \\ &\cup \left\{ \lfloor N10^{-j} \rfloor 10^j + \lfloor K_j 10^{-r} \rfloor 10^r + l : 0 \le l \le K_j - \lfloor K_j 10^{-r} \rfloor 10^r \right\}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \sum_{i=0}^{N} \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor) = \sum_{j=1}^{n-1} \sum_{r=0}^{j-1} \Bigg(\sum_{k=0}^{\lfloor N10^{-j} \rfloor - 1} \sum_{p=0}^{10^{j-r}-1} \sum_{l=0}^{10^r - 1} (p10^r + l) \cdot 10^{-j} l \\ 10^{-r} + \sum_{p=0}^{\lfloor K_j 10^{-r} \rfloor - 1} \sum_{l=0}^{10^r - 1} (p10^r + l) 10^{-j} l 10^{-r} + \sum_{l=0}^{K_j - \lfloor K_j 10^{-r} \rfloor 10^r} \left(\lfloor K_j 10^{-r} \rfloor 10^r + l \right) 10^{-j} l 10^{-r} \Bigg). \end{aligned} \tag{6.14}$$

From (6.11) and (6.14), a lengthy but elementary calculation leads to

$$\sum_{i=0}^{N}\sum_{j=0}^{n-1}(i10^{-j}-\lfloor i10^{-j}\rfloor)\sum_{r=0}^{j-1}(i10^{-r}-\lfloor i10^{-r}\rfloor)$$
$$=\left(\frac{n^2}{8}-\frac{85n}{216}\right)N-\frac{1}{4}\sum_{j=1}^{n-1}j\sum_{l=0}^{j-1}a_l10^l+\frac{1}{4}\sum_{j=1}^{n-1}j10^{-j}\left(\sum_{l=0}^{j-1}a_l10^l\right)^2+\mathcal{O}(N).$$

Analogously, one obtains also $\sum_{i=0}^{N}\sum_{j=0}^{n-1}(i10^{-j}-\lfloor i10^{-j}\rfloor)^2=\frac{nN}{3}+\mathcal{O}(N)$. Note that (6.13) immediately leads to

$$\sum_{i=0}^{N}\left(\sum_{j=0}^{n-1}(i10^{-j}-\lfloor i10^{-j}\rfloor)\right)^2$$
$$=\left(\frac{n^2}{4}-\frac{49}{108}n\right)N-\frac{1}{2}\sum_{j=0}^{n-1}j\sum_{l=0}^{j-1}a_l10^l+\frac{1}{2}\sum_{j=0}^{n-1}j10^{-j}\left(\sum_{l=0}^{j-1}a_l10^l\right)^2+\mathcal{O}(N). \tag{6.15}$$

The rest of the proof consists of choosing an appropriate i_0 (or equivalently, $y=\langle\langle\log_{10} i_0\rangle\rangle$) to obtain a sufficiently precise bound for (6.10): Let $i_0=10^{n-1}-10^{\lfloor n/2\rfloor-1}+1$ if $10^{n-1}\leq N\leq 10^n-10^{\lfloor n/2\rfloor}$, and $i_0=10^n-10^{\lfloor n/2\rfloor}$ if $10^n-10^{\lfloor n/2\rfloor}<N\leq 10^n-1$. Note that $\lfloor N/10\rfloor+1\leq i_0\leq N-1$, and it is straightforward to verify that

$$\sum_{j=0}^{n-1}(i_010^{-j}-\lfloor i_010^{-j}\rfloor)=\frac{n}{2}+c+\mathcal{O}\left(N^{-1/2}\right). \tag{6.16}$$

for some finite constant c. Combining (6.12), (6.15) and (6.16) yields

$$\sum_{i=0}^{N}\left\{\sum_{j=0}^{n-1}\left((i10^{-j}-\lfloor i10^{-j}\rfloor)-(i_010^{-j}-\lfloor i_010^{-j}\rfloor)\right)\right\}^2$$
$$=\frac{11}{108}nN+\frac{1}{2}\sum_{j=0}^{n-1}(n-j)\left(\sum_{l=0}^{j-1}a_l10^l\right)\left(1-\sum_{l=0}^{j-1}a_l10^{l-j}\right)+\mathcal{O}(N).$$

Next, observe that

$$\frac{1}{2}\sum_{j=0}^{n-1}(n-j)\left(\sum_{l=0}^{j-1}a_l10^l\right)\left(1-\sum_{l=0}^{j-1}a_l10^{l-j}\right)=\mathcal{O}(N).$$

which implies that

$$\sum_{i=0}^{N}\left\{\sum_{j=0}^{n-1}\left((i10^{-j}-\lfloor i10^{-j}\rfloor)-(i_010^{-j}-\lfloor i_010^{-j}\rfloor)\right)\right\}^2=\frac{11}{108}nN+\mathcal{O}(N),$$

and hence

$$\sum_{i=\lfloor N/10\rfloor}^{N}\left\{\sum_{j=0}^{n-1}\Big((i10^{-j}-\lfloor i10^{-j}\rfloor)-(i_0 10^{-j}-\lfloor i_0 10^{-j}\rfloor)\Big)\right\}^2 \le \frac{11\log N}{108\log 10}N+\mathcal{O}(N). \quad (6.17)$$

Let $y=\langle\langle \log_{10} i_0\rangle\rangle$. Combining (6.7), (6.8), (6.9) and (6.17) yields

$$\begin{aligned}&\int_0^1\Big|\Big(F_{\nu_N\circ\iota^{-1}}(x)-F_{\eta_N\circ\iota^{-1}}(x)\Big)-\Big(F_{\nu_N\circ\iota^{-1}}(y)-F_{\eta_N\circ\iota^{-1}}(y)\Big)\Big|\mathrm{d}x\\ &\le \frac{1}{6\log 10}\sqrt{\frac{33}{\log 10}}\frac{\sqrt{\log N}}{N}+\mathcal{O}\left(N^{-1}\right);\end{aligned}$$

and hence with Proposition 6.1, it follows at long last that

$$\limsup_{N\to\infty}\frac{N}{\sqrt{\log N}}d_{\mathbb{T}}\left(\nu_N,\eta_N\right)\le\frac{1}{6\log 10}\sqrt{\frac{33}{\log 10}}.$$

□

Note that Theorem 6.3 describes the asymptotics of $(\nu_N(\log_b n))_{N\ge1}$, in that it not only gives the rate of convergence, but also identifies the exponential distribution with specific rotation that (ν_N) asymptotically approaches.

Remark 6.4. (i) It follows from a general result in [101] that

$$\limsup_{N\to\infty}\frac{N}{\log N}d_{\mathbb{T}}\left(\nu_N,E_{\log b}\circ R^{-1}_{-\log_b N}\right)<+\infty$$

for every $b\in\mathbb{N}\setminus\{1\}$. Obviously, this is weaker than (6.4).
(ii) From Zador's theorem on asymptotic quantization errors in $\mathcal{P}(\mathbb{T})$ [57, Thm.1.4], it follows that

$$\liminf_{N\to\infty} N d_{\mathbb{T}}\left(\nu_N,E_{\log b}\circ R^{-1}_{-\log_b N}\right)>0.$$

This shows that $\left(d_{\mathbb{T}}\left(\nu_N,E_{\log b}\circ R^{-1}_{-\log_b N}\right)\right)_{N\ge1}$ cannot decay faster than (N^{-1}), and [17, Cor.3.8] suggests that it may be challenging to improve this lower bound.
(iii) Even if the inequality (6.8) is replaced by the following Hölder inequality,

$$\begin{aligned}&\sum_{i=\lfloor N/10\rfloor}^{N}\frac{1}{i}\left|\sum_{j=0}^{n-1}\Big((i_0 10^{-j}-\lfloor i_0 10^{-j}\rfloor)-(i10^{-j}-\lfloor i10^{-j}\rfloor)\Big)\right|\\ &\le\left(\sum_{i=\lfloor N/10\rfloor}^{N}\frac{1}{i^{4/3}}\right)^{3/4}\left(\sum_{i=\lfloor N/10\rfloor}^{N}\left(\sum_{j=0}^{n-1}\Big((i_0 10^{-j}-\lfloor i_0 10^{-j}\rfloor)-(i10^{-j}-\lfloor i10^{-j}\rfloor)\Big)\right)^4\right)^{1/4},\end{aligned}$$

the upper bound for the rate of convergence does not improve. Indeed, a tedious computation similar to the one in the proof of Theorem 6.3 yields

$$\limsup_{N\to\infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}}\left(\nu_N, E_{\log b} \circ R^{-1}_{-\log_b N}\right) \le c, \tag{6.18}$$

where the constant c may be smaller than $\dfrac{1}{6\log 10}\sqrt{\dfrac{33}{\log 10}}$ but still is positive. From this, one may optimistically conjecture that for all $b > 1$ (not necessarily integers), the sequence

$$\left(\frac{N}{\sqrt{\log N}} d_{\mathbb{T}}\left(\nu_N, E_{\log b} \circ R^{-1}_{-\log_b N}\right)\right)_{N\ge 2}$$

is bounded above and below by positive constants. Especially for non-integer b, this is speculation only, since many of the explicit calculations and estimates leading to (6.4) do not apply directly.

6.2.2 Sharp rates of convergence w.r.t. $d_{\mathbb{I}}$ and d_{K}

In this subsection, we complement the results of Subsection 6.2.1 by characterizing the sharp rate of convergence of $(\nu_N(\log_b n))_{N\ge 1}$ w.r.t. both $d_{\mathbb{I}}$ and d_{K}.

Theorem 6.5. *Assume $b \in \mathbb{N}\setminus\{1\}$. Then*

$$\lim_{N\to\infty} \frac{N}{\log N} d_{\mathbb{I}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right) = \frac{1}{2\log b}.$$

Proof. Recall that $b = 10$. As in the proof of Theorem 6.3, by formula (6.2), it is easy to verify that for $10^{n-1} \le N \le 10^n - 1$,

$$d_{\mathbb{I}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right) = \frac{1}{N\log 10} \sum_{i=\lfloor N/10\rfloor}^{N} \frac{1}{i} \sum_{j=0}^{n-1} \left(i10^{-j} - \lfloor i10^{-j}\rfloor\right) + \mathcal{O}\left(N^{-1}\right). \tag{6.19}$$

Like the expression for $\sum_{i=0}^{N}\sum_{j=0}^{n-1}(i10^{-j} - \lfloor i10^{-j}\rfloor)$ as in the proof of Theorem 6.5, it is readily checked that

$$\sum_{i=\lfloor N/10\rfloor}^{N} \frac{1}{i} \sum_{j=0}^{n-1}(i10^{-j} - \lfloor i10^{-j}\rfloor) = \sum_{j=0}^{n-1}\left(\sum_{l=\lfloor N/10\rfloor - \lfloor\lfloor N/10\rfloor 10^{-j}\rfloor 10^j}^{10^j-1} \frac{l10^{-j}}{\lfloor\lfloor N/10\rfloor 10^{-j}\rfloor 10^j + l}\right.$$
$$\left. + \sum_{k=\lfloor\lfloor N/10\rfloor 10^{-j}\rfloor+1}^{\lfloor N10^{-j}\rfloor - 1} \sum_{l=0}^{10^j-1} \frac{l10^{-j}}{k10^j + l} + \sum_{l=0}^{N-\lfloor N10^{-j}\rfloor 10^j} \frac{l10^{-j}}{\lfloor N10^{-j}\rfloor 10^j + l}\right),$$

which implies that

$$\sum_{i=\lfloor N/10\rfloor}^{N}\frac{1}{i}\sum_{j=0}^{n-1}(i10^{-j}-\lfloor i10^{-j}\rfloor)\geq\sum_{j=0}^{n-1}\Bigg(\sum_{l=\lfloor N/10\rfloor-\lfloor\lfloor N/10\rfloor10^{-j}\rfloor10^{j}}^{10^{j}-1}\frac{l10^{-j}}{\lfloor\lfloor N/10\rfloor10^{-j}\rfloor10^{j}+10^{j}} \tag{6.20}$$
$$+\sum_{k=\lfloor\lfloor N/10\rfloor10^{-j}\rfloor+1}^{\lfloor N10^{-j}\rfloor-1}\sum_{l=0}^{10^{j}-1}\frac{l10^{-j}}{k10^{j}+10^{j}}+\sum_{l=0}^{N-\lfloor N10^{-j}\rfloor10^{j}}\frac{l10^{-j}}{N}\Bigg)$$
$$=\frac{1}{2}\sum_{j=0}^{n-1}\Bigg\{\frac{\left(1-10^{-j}+\lfloor N/10\rfloor10^{-j}-\lfloor\lfloor N/10\rfloor10^{-j}\rfloor\right)\left(1-\lfloor N/10\rfloor10^{-j}+\lfloor\lfloor N/10\rfloor10^{-j}\rfloor\right)}{\lfloor\lfloor N/10\rfloor10^{-j}\rfloor+1}$$
$$+\frac{\left(N10^{-j}-\lfloor N10^{-j}\rfloor\right)\left(N10^{-j}-\lfloor N10^{-j}\rfloor+1\right)10^{j}}{N}+(1-10^{-j})\sum_{k=\lfloor\lfloor N/10\rfloor10^{-j}\rfloor+1}^{\lfloor N10^{-j}\rfloor-1}\frac{1}{k+1}\Bigg\}.$$

Note that

$$\frac{1}{2}\sum_{j=0}^{n-1}\Bigg(\frac{(1-10^{-j}+\lfloor N/10\rfloor10^{-j}-\lfloor\lfloor N/10\rfloor10^{-j}\rfloor)(1-\lfloor N/10\rfloor10^{-j}+\lfloor\lfloor N/10\rfloor10^{-j}\rfloor)}{\lfloor\lfloor N/10\rfloor10^{-j}\rfloor+1}$$
$$+\frac{(N10^{-j}-\lfloor N10^{-j}\rfloor)(N10^{-j}-\lfloor N10^{-j}\rfloor+1)10^{j}}{N}\Bigg)=\mathcal{O}(1).$$

Moreover, it is tedious but straightforward to confirm that

$$\frac{1}{2}\sum_{j=0}^{n-1}(1-10^{-j})\sum_{k=\lfloor\lfloor N/10\rfloor10^{-j}\rfloor+1}^{\lfloor N10^{-j}\rfloor-1}\frac{1}{k+1}=\frac{1}{2}\log N+\mathcal{O}(1),$$

with which (6.20) takes the form

$$\sum_{i=\lfloor N/10\rfloor}^{N}\frac{1}{i}\sum_{j=0}^{n-1}(i10^{-j}-\lfloor i10^{-j}\rfloor)\geq\frac{1}{2}\log N+\mathcal{O}(1). \tag{6.21}$$

Analogously, one can also show that (6.21) holds with $\geq$ replaced by $\leq$, and hence

$$\sum_{i=\lfloor N/10\rfloor}^{N}\frac{1}{i}\sum_{j=0}^{n-1}(i10^{-j}-\lfloor i10^{-j}\rfloor)=\frac{1}{2}\log N+\mathcal{O}(1).$$

The conclusion now follows from (6.19). □

The following corollary is immediately obtained from Theorem 6.5, together with [77, Thm.5] and the fact that $d_{\mathbb{I}}\leq d_{\mathsf{K}}$.

Corollary 6.6. *Assume $b\in\mathbb{N}\setminus\{1\}$. Then*

$$0<\liminf_{N\to\infty}\frac{N}{\log N}d_{\mathsf{K}}\left(\nu_N\circ\iota^{-1},E_{\log b}\circ R^{-1}_{-\log_b N}\circ\iota^{-1}\right)$$

$$\leq \limsup_{N\to\infty} \frac{N}{\log N} d_{\mathsf{K}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right) < +\infty.$$

Comparing Theorems 6.3 and 6.5, as well as Corollary 6.6, notice how $\left(d_{\mathbb{T}}\left(\nu_N, E_{\log b} \circ R^{-1}_{-\log_b N}\right)\right)_{N\geq 1}$ decays somewhat faster than both $\left(d_{\mathbb{I}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right)\right)_{N\geq 1}$ and $\left(d_{\mathsf{K}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right)\right)_{N\geq 1}$. Moreover, the ratio

$$\frac{d_{\mathbb{I}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right)}{d_{\mathsf{K}}\left(\nu_N \circ \iota^{-1}, E_{\log b} \circ R^{-1}_{-\log_b N} \circ \iota^{-1}\right)}$$

is bounded above and below by positive constants. This is remarkable since

$$\inf_{\mu\neq\nu,\ \mu,\nu\in\mathcal{P}(\mathbb{T})} \frac{d_{\mathbb{I}}\left(\mu \circ \iota^{-1}, \nu \circ \iota^{-1}\right)}{d_{\mathsf{K}}\left(\mu \circ \iota^{-1}, \nu \circ \iota^{-1}\right)} = 0.$$

Chapter 7

Circularly invariant and uniform probability measures for linear maps

This chapter investigates a somewhat unusual form of invariant probability measures for linear maps on the line. Recall that λ and $\lambda_{\mathbb{T}}$ denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{T}$, respectively; $\lambda_{\mathbb{I}}$ denotes the Lebesgue measure on the compact unit interval $\mathbb{I}$. Note that $\operatorname{diam} A \geq \lambda(A)$ for every set A, and equality holds if and only if $\lambda(A) = +\infty$ or A is an interval up to a Lebesgue measure zero set (i.e., $\lambda([\inf A, \sup A] \setminus A) = 0$).

Given a u.d. mod 1 sequence (x_n) and a convex map $T : \mathbb{R} \to \mathbb{R}$, the sequence $(T(x_n))$ may be u.d. mod 1 as well. In this case, consider the sequence $(\mu_N(x_n))_{N\geq 1}$ in $\mathcal{P}(\mathbb{R})$ with

$$\mu_N(x_n) := \frac{1}{N}\sum_{n=1}^{N} \delta_{x_n},$$

and note that $\mu_N(x_n) \circ \pi^{-1} = \nu_N(x_n) \to \lambda_{\mathbb{T}}$. (Recall that $\pi : \mathbb{R} \to \mathbb{T}$ denotes the natural projection.) Now, suppose that $(\mu_N)_{N\geq 1}$ converges in $\mathcal{P}(\mathbb{R})$, to μ, say. Then $\mu \circ \pi^{-1} = \mu \circ T^{-1} \circ \pi^{-1} = \lambda_{\mathbb{T}}$. As this seems to be a fairly peculiar invariance property that $\mu \in \mathcal{P}(\mathbb{R})$ may have, we ask

Question (1.19). *Given a convex map $T : \mathbb{R} \to \mathbb{R}$, does there exist $\mu \in \mathcal{P}(\mathbb{R})$ with the property $\mu \circ \pi^{-1} = \mu \circ T^{-1} \circ \pi^{-1} = \lambda_{\mathbb{T}}$?*

Given T, call any $\mu \in \mathcal{P}(\mathbb{R})$ satisfying the property in Question (1.19) a *circularly invariant and uniform probability measure* (CIUPM) for T, and write $\mathcal{C}_T$ for the family of all such CIUPM, i.e., $\mathcal{C}_T = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu \circ \pi^{-1} = \mu \circ T^{-1} \circ \pi^{-1} = \lambda_{\mathbb{T}}\}$. Note that every element of $\mathcal{C}_T$ is absolutely continuous (w.r.t. λ).

While the answer to Question 1.19 may often be negative, in this chapter we provide a partial, positive answer for the simplest class of convex maps, namely *linear* maps. For every $\alpha, \beta \in \mathbb{R}$, consider the linear map $T_{\beta,\alpha}(x) = \beta x + \alpha$, and let $\mathcal{C}_{\beta,\alpha} = \mathcal{C}_{T_{\beta,\alpha}}$ for convenience. Clearly, $\mathcal{C}_{0,\alpha} = \varnothing$, so henceforth assume $\beta \neq 0$ throughout this chapter. It is not hard to see that in fact $\mathcal{C}_{\beta,\alpha} = \mathcal{C}_{\beta,0}$; see Proposition 7.4 below. Moreover, since

$$\mu \in \mathcal{C}_{\beta,\alpha} \Leftrightarrow \mu \circ T_{\beta,\alpha}^{-1} \in \mathcal{C}_{\beta^{-1},-\alpha\beta^{-1}} \Leftrightarrow \mu \circ T_{-1,0}^{-1} \in \mathcal{C}_{-\beta,-\alpha}$$

it suffices to consider the case of $\beta \geq 1$. Thus, when specialized to the family of linear maps $T_{\beta,\alpha}$, Question 1.19 takes the form of

Question (1.20). *Is $\mathcal{C}_{\beta,0} \neq \varnothing$ for every $\beta \geq 1$?*

Below, we answer Question 1.20 in the affirmative, and provide further information regarding the structure of $\mathcal{C}_{\beta,0}$. Let us mention in passing that for *nonlinear* convex maps like, e.g., the exponential map $T(x) = e^x$, answering Question 1.19 may be much more difficult. For such maps, the density of any CIUPM, if it exists at all, cannot be found explicitly, unlike in the linear case. In fact, even for piecewise linear maps such as, e.g., $T(x) = x\mathbb{1}_{]-\infty,0[} + \sqrt{2}x\mathbb{1}_{[0,+\infty[}$ the situation is considerably more involved than in the linear case. This will become apparent from solving the equations of a CIUPM in the proof of the main theorem.

Let us also mention some related work on invariant measures for "almost" linear transformations on $\mathbb{I}$. Kopf [63] gave a formula for the densities of invariant measures for piecewise linear transformations on $\mathbb{I}$. Góra [38, 39] found an explicit formula for the densities of invariant measures for arbitrary eventually expanding piecewise linear transformations whose slopes are not necessarily the same on $\mathbb{I}$.

7.1 Preliminaries

Let $A + x = \{y + x : \ y \in A\}$, for every $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, and denote by $\mathbb{Q}$ be the set of all rational numbers. Recall that two integers p, q are *coprime* if they have 1 as their *greatest common divisor* [55, p.5]. For every $\beta \in \mathbb{Q} \setminus \{0\}$, let (p_β, q_β) be the unique pair of coprime positive integers such that $|\beta| = p_\beta/q_\beta$ and let $s_\beta := q_\beta\langle\langle|\beta|\rangle\rangle$. Note that for $\beta \notin \mathbb{N}$, $1 \leq s_\beta \leq q_\beta - 1$ is an integer coprime with q_β.

For any complete metric space X, let $\mathcal{P}(X)$ be the family of all Borel probability measures on X. For convenience, write $\mathcal{P}(\mathbb{R})$ simply as $\mathcal{P}$ throughout this chapter. For every absolutely continuous $\mu \in \mathcal{P}$ (w.r.t. λ), denote by ρ_μ its density. Note that $\mu \circ \pi^{-1} \in \mathcal{P}(\mathbb{T})$, and $\mu \circ \pi^{-1} \circ \iota^{-1} \in \mathcal{P}(\mathbb{I})$. Let $\mathcal{S}_c = \{\mu \in \mathcal{P} : \operatorname{diam}\operatorname{supp}\mu = c\}$ for every $c \geq 0$.

As a quantitative refinement of Question (1.20), in this chapter we answer

Question 7.1. *What is* $\inf\{c : \ \mathcal{C}_{\beta,\alpha} \cap S_c \neq \varnothing\}$*?*

More specifically, we prove a threshold result on the existence of CIUPM for $T_{\beta,\alpha}$: For every $\beta \geq 1$, there exists $c_\beta > 0$ such that $\mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_c \neq \varnothing$ if and only if $c \geq c_\beta$. Moreover, $\mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{c_\beta} = \left\{\mu_\beta \circ T_{1,\eta}^{-1}\right\}_{\eta\in\mathbb{R}}$ for some $\mu_\beta \in \mathcal{P}$, and for every $c > c_\beta$, there exist $\mu_1, \mu_2 \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_c$ such that $\mu_1 \neq \mu_2 \circ T_{1,\eta}^{-1}$, for all $\eta \in \mathbb{R}$. In other words, by Proposition 7.4(i) below, CIUPM are unique up to translation in $\mathcal{S}_{c_\beta}$, but are *not* unique in $\mathcal{S}_c$ for $c > c_\beta$. To put this abundance of CIUPM into perspective, recall that $T_{\beta,\alpha}$ induces the measurable map $\langle T_{\beta,\alpha}\rangle = \pi \circ T_{\beta,\alpha} \circ \iota$ on $\mathbb{T}$ which for $|\beta| > 1$ allows for exactly one absolutely continuous invariant measure. In general, the fact that $\mu \in \mathcal{C}_{\beta,\alpha}$ does not imply that $\mu \circ \pi^{-1}$ is $\langle T_{\beta,\alpha}\rangle$-invariant. Such an implication is valid only for *integer* β, in which case $\mu \circ \pi^{-1} = \lambda_{\mathbb{T}}$.

Now we state several preliminary facts needed for the proof of our main result in the next section.

Proposition 7.2. *Assume $\mu \in \mathcal{P}$. If $\mu \circ \pi^{-1} = \lambda_{\mathbb{T}}$, then μ is absolutely continuous (w.r.t. λ), with $0 \leq \rho_\mu \leq 1$. In particular, every CIUPM is absolutely continuous with density bounded by 1.*

Proposition 7.3. *For every $\mu \in \mathcal{P}$,*

$$F_{\mu\circ\pi^{-1}\circ\iota^{-1}}(t) = \sum_{k\in\mathbb{Z}} \Big[F_\mu(t+k) - F_\mu(k-)\Big], \quad \forall\, t \in [0,1[.$$

If μ is absolutely continuous then so is $\mu \circ \pi^{-1} \circ \iota^{-1}$, and

$$\rho_{\mu\circ\pi^{-1}\circ\iota^{-1}}(t) = \sum_{k\in\mathbb{Z}} \rho_\mu(t+k), \qquad a.e.\ on\ [0,1[$$

The following properties of $\mathcal{C}_{\beta,\alpha}$ are easily checked.

Proposition 7.4. *Assume $\beta \neq 0$. Then:*
(i) *For every $\alpha \in \mathbb{R}$, $\mathcal{C}_{\beta,0} = \mathcal{C}_{\beta,\alpha}$ (translation invariance);*
(ii) *If $\mu_j \in \mathcal{C}_{\beta,0}$ for all $j = 1, \cdots, n$, then $\sum_{j=1}^n p_j\mu_j \in \mathcal{C}_{\beta,0}$ for all $p \in \Pi_n$ (convexity);*
(iii) *$\mu \in \mathcal{C}_{\beta,0}$ if and only if $\mu \circ T_{-1,0}^{-1} \in \mathcal{C}_{\beta,0}$ (symmetry);*
(iv) *$\mu \in \mathcal{C}_{\beta,0}$ if and only if $\mu \circ T_{\beta,0}^{-1} \in \mathcal{C}_{\beta^{-1},0}$.*

A straightforward consequence of Proposition 7.4 is

Corollary 7.5. *Assume $\beta \neq 0$ and $c \geq 0$. Suppose $\mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_c \neq \varnothing$. Then for every $\widetilde{c} > c$, $\mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\widetilde{c}} \neq \varnothing$; moreover,*

$$\left\{\mu \circ T_{1,\eta}^{-1}\right\}_{\eta\in\mathbb{R}} \subsetneq \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\widetilde{c}},\ \forall\ \mu \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}},$$

Proof. We prove it by construction. Let $\nu \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_c$. It follows from Proposition 7.4(i) that $\nu \circ T^{-1}_{1,\tilde{c}-c} \in \mathcal{C}_{\beta,\alpha}$. Let $\widetilde{\nu} = \frac{1}{2}\nu + \frac{1}{2}\nu \circ T^{-1}_{1,\tilde{c}-c}$. Then $\widetilde{\nu} \in \mathcal{C}_{\beta,\alpha}$ by Proposition 7.4(i) and (ii). Moreover, $\operatorname{diam}\operatorname{supp}\widetilde{\nu} = \operatorname{diam}\operatorname{supp}\nu + \tilde{c} - c = \tilde{c}$. This yields $\widetilde{\nu} \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}}$. Analogously, $\widetilde{\mu} = \frac{1}{3}\nu + \frac{1}{3}\nu \circ T^{-1}_{1,(\tilde{c}-c)/2} + \frac{1}{3}\nu \circ T^{-1}_{1,\tilde{c}-c} \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}}$. This yields $\left\{\mu \circ T^{-1}_{1,\eta}\right\}_{\eta\in\mathbb{R}} \neq \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}}$, for all $\mu \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}}$. On the other hand, it trivially follows from Proposition 7.4(i) that $\left\{\mu \circ T^{-1}_{1,\eta}\right\}_{\eta\in\mathbb{R}} \subset \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}}$, for all $\mu \in \mathcal{C}_{\beta,\alpha} \cap \mathcal{S}_{\tilde{c}}$. □

Recall that two real numbers x and y are *rationally independent* if one is a rational multiple of the other, i.e., the equation $r_1 x + r_2 y = 0$ only admits the trivial solution $r_1 = r_2 = 0$ in $\mathbb{Q}$. The following is a version of Kronecker's theorem.

Proposition 7.6. *Two numbers $x,\ y \in \mathbb{R} \setminus \{0\}$ are rationally independent, if and only if the set $\{mx + ny : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.*

7.2 An Answer to Question 7.1

In this section, we give an answer to Question 7.1 via a threshold result on the existence of CIUPM for linear maps $T_{\beta,\alpha}$. Before stating the result, let us look at two simple examples, which may give some intuitive picture of a "slimmest" CIUPM. Consider first a linear map with an irrational slope.

Example 7.7. Let $T_{\sqrt{2},0}(x) = \sqrt{2}x$. Using Proposition 7.3, it is easy to verify that μ with density

$$\rho_\mu(t) = \begin{cases} \sqrt{2}t & \text{if } t \in \left[0, 1/\sqrt{2}\right[, \\ 1 & \text{if } t \in \left[1/\sqrt{2}, 1\right[, \\ -\sqrt{2}t + 1 + \sqrt{2} & \text{if } t \in \left[1, 1 + 1/\sqrt{2}\right], \\ 0 & \text{otherwise,} \end{cases}$$

is a CIUPM for $T_{\sqrt{2},0}$. Notice that $\operatorname{supp}\mu$ is an interval and $\operatorname{diam}\operatorname{supp}\mu = \lambda(\operatorname{supp}\mu) = 1 + \frac{1}{\sqrt{2}}$; see also Figure 7.1(a).

Now we turn to a linear map with a rational slope.

Example 7.8. Let $T_{3/2,0}(x) = 3x/2$. Also Proposition 7.3 yields μ with density

$$\rho_\mu(t) = \begin{cases} \frac{1}{2} & \text{if } t \in [0, 1/3[\bigcup [1, 4/3[, \\ 1 & \text{if } t \in [1/3, 1[, \\ 0 & \text{otherwise,} \end{cases}$$

is a CIUPM for $T_{3/2,0}$. Note that $\operatorname{supp}\mu$ is an interval and $\operatorname{diam}\operatorname{supp}\mu = \lambda(\operatorname{supp}\mu) = 4/3 < 1 + \frac{1}{3/2}$.

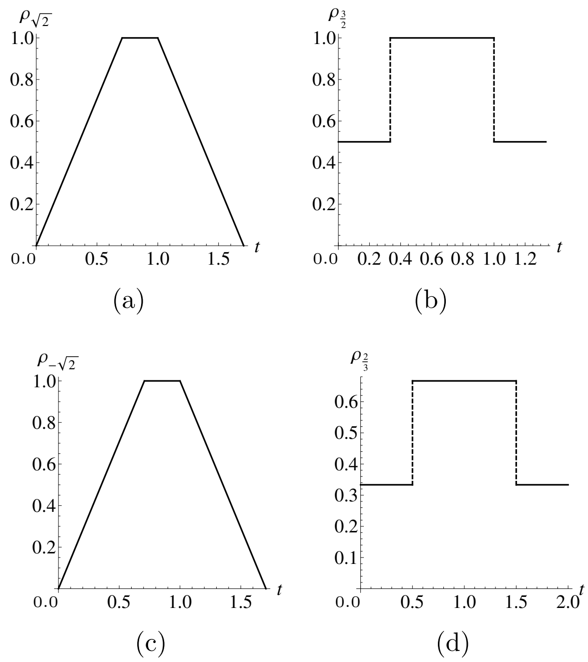

Figure 7.1: Profiles for ρ_β.

From the above two examples, one may expect that there always exists a CIUPM for a linear map with a non-zero slope. In fact, as illustrated by the following main result, the two CIUPM in Examples 7.7 and 7.8 are the "slimmest", in the sense that their support has minimal diameter.

For every $\beta \neq 0$, define

$$c_\beta = \begin{cases} 1 + \frac{1}{|\beta|} - \frac{1}{p_\beta} & \text{if } \beta \in \mathbb{Q}, \\ 1 + \frac{1}{|\beta|} & \text{if } \beta \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

and a probability measure μ_β by its density ρ_β :

(i)

$$\rho_\beta(t) = \begin{cases} \beta t & \text{if } t \in [0, 1/\beta[\,, \\ 1 & \text{if } t \in [1/\beta, 1[\,, \\ -\beta t + 1 + \beta & \text{if } t \in [1, 1 + 1/\beta]\,, \\ 0 & \text{otherwise,} \end{cases} \tag{7.1}$$

if $\beta \in [1, +\infty[\setminus\mathbb{Q}$;

(ii)

$$\rho_\beta(t) = \begin{cases} \frac{j}{q_\beta} & \text{if } t \in \left[\frac{j-1}{p_\beta}, \frac{j}{p_\beta}\right[\bigcup \left[1 + \frac{q_\beta - j - 1}{p_\beta}, 1 + \frac{q_\beta - j}{p_\beta}\right[, \ j = 1, \cdots, q_\beta - 1, \\ 1 & \text{if } t \in \left[\frac{q_\beta - 1}{p_\beta}, 1\right[, \\ 0 & \text{otherwise}, \end{cases} \tag{7.2}$$

if $\beta \in [1, +\infty[\cap \mathbb{Q}$;

(iii) $\rho_\beta(\cdot) = \beta\rho_{\beta^{-1}}(\beta\cdot)$ if $\beta \in]0, 1[$;

(iv) $\rho_\beta(\cdot) = \rho_{-\beta}(\cdot)$ if $\beta \in] - \infty, 0[$.

The following result completely answers Question 1.20. In a way, it also provides a lower bound on the "size" of every CIUPM for $T_{\beta,0}$, and hence also for $T_{\beta,\alpha}$.

Theorem 7.9. *Assume $\beta \geq 1$. Then $\mathcal{C}_{\beta,0} \neq \varnothing$, and there exists $c_\beta \geq 1$ such that* $\operatorname{diam}\operatorname{supp}\mu \geq c_\beta$ *for every $\mu \in \mathcal{C}_{\beta,0}$. Moreover, there exists $\mu_\beta \in \mathcal{C}_{\beta,0}$ with* $\operatorname{diam}\operatorname{supp}\mu = c_\beta$, *and μ_β is uniquely determined up to translation, i.e., if $\mu \in \mathcal{C}_{\beta,0}$ satisfies* $\operatorname{diam}\operatorname{supp}\mu = c_\beta$ *then $\mu = \mu_\beta \circ T_{1,\alpha}^{-1}$ for some $\alpha \in \mathbb{R}$.*

Proof. Throughout the proof, write $T_{\beta,0}$ and $\mathcal{C}_{\beta,0}$ simply as T_β and $\mathcal{C}_\beta$, respectively. Also, read all equations and inequalities for densities as holding a.e.. Note that all assertions clearly are correct for $\beta \in \mathbb{N}$, with $\beta_\beta = 1$, and hence it suffices to prove for every $\beta \in]1, +\infty[\setminus\mathbb{N}$ that:

(i) $\mathcal{C}_\beta \cap \mathcal{S}_c = \varnothing$ for $c < c_\beta$;

(ii) $\mathcal{C}_\beta \cap \mathcal{S}_{c_\beta} = \left\{\mu_\beta \circ T_{1,\eta}^{-1}\right\}_{\eta\in\mathbb{R}}$.

Note that $c \geq 1$ whenever $\mathcal{C}_\beta \cap \mathcal{S}_c \neq \varnothing$. In a first step, we establish the equations for density of a CIUPM for T_β which will be used throughout the proof. By Proposition 7.4(i), it suffices to consider μ with $\operatorname{supp}\mu \subset [0, c]$, and hence $\operatorname{diam}\operatorname{supp}\mu \leq c$. By Proposition 7.3 and the definition of a CIUPM, $\mu \in \mathcal{C}_\beta$ if and only if ρ_μ satisfies the following equations:

$$\sum_{k=0}^{\lfloor c \rfloor} \rho_\mu(t + k) = 1, \ t \in [0, \langle\!\langle c \rangle\!\rangle], \tag{7.3}$$

$$\sum_{k=0}^{\lfloor c \rfloor - 1} \rho_\mu(t + k) = 1, \ t \in [\langle\!\langle c \rangle\!\rangle, 1], \tag{7.4}$$

$$\frac{1}{\beta} \sum_{j=0}^{\lfloor \beta c \rfloor} \rho_\mu\left(\frac{t + j}{\beta}\right) = 1, \ t \in [0, \langle\!\langle \beta c \rangle\!\rangle], \tag{7.5}$$

$$\frac{1}{\beta} \sum_{j=0}^{\lfloor \beta c \rfloor - 1} \rho_\mu\left(\frac{t + j}{\beta}\right) = 1, \ t \in [\langle\!\langle \beta c \rangle\!\rangle, 1]. \tag{7.6}$$

By (7.3) and (7.4), $0 \le \rho_\mu(t) \le 1$ for $t \in [0, c]$. Note that for $c \le 1 + 1/\beta$,

$$\langle\langle \beta c \rangle\rangle \le \beta + 1 - \lfloor \beta c \rfloor,$$

and thus

$$(t + j)/\beta \in [c-1, 1],\ \forall\, t \in [0, \beta + 1 - \lfloor \beta c \rfloor],\ j = 1, \cdots, \lfloor \beta c \rfloor - 1,$$

from which it follows that (7.3)-(7.6) are equivalent to

$$\rho_\mu(t) + \rho_\mu(t+1) = 1,\ t \in [0, c-1], \tag{7.7}$$

$$\rho_\mu(t) = 1,\ t \in [c-1, 1], \tag{7.8}$$

$$\rho_\mu(t/\beta) + \rho_\mu((t + \lfloor \beta c)/\beta) = \beta - \lfloor \beta c \rfloor + 1,\ t \in [0, \langle\langle \beta c \rangle\rangle], \tag{7.9}$$

$$\rho_\mu(t/\beta) = \beta - \lfloor \beta c \rfloor + 1,\ t \in [\langle\langle \beta c \rangle\rangle, \min\{1, \beta + 1 - \lfloor \beta c \rfloor\}], \tag{7.10}$$

$$\rho_\mu(t/\beta) + \rho_\mu((t + \lfloor \beta c \rfloor - 1)/\beta) = \beta - \lfloor \beta c \rfloor + 2,\ t \in [\min\{1, \beta + 1 - \lfloor \beta c \rfloor\}, 1]. \tag{7.11}$$

By change of variables, (7.9)-(7.11) are equivalent to

$$\rho_\mu(t) + \rho_\mu(t + \lfloor \beta c \rfloor/\beta) = \beta - \lfloor \beta c \rfloor + 1,\ t \in [0, \langle\langle \beta c \rangle\rangle/\beta], \tag{7.12}$$

$$\rho_\mu(t) = \beta - \lfloor \beta c \rfloor + 1,\ t \in [(\langle\langle \beta c \rangle\rangle)/\beta, \min\{1/\beta, 1/\beta + 1 - \lfloor \beta c \rfloor/\beta\}], \tag{7.13}$$

$$\rho_\mu(t) + \rho_\mu(t + (\lfloor \beta c \rfloor - 1)/\beta) = \beta - \lfloor \beta c \rfloor + 2,\ t \in [\min\{1/\beta, 1/\beta + 1 - \lfloor \beta c \rfloor/\beta\}, 1/\beta]. \tag{7.14}$$

In the following, we first prove $\mathcal{C}_\beta \cap \mathcal{S}_c = \varnothing$ for $c < c_\beta$.

Suppose by way of contradiction that there exists $\mu \in \mathcal{C}_\beta \cap \mathcal{S}_c$. Then its associated density satisfies (7.7), (7.8), (7.12)-(7.14). Since $1 \le c < c_\beta = 1 + 1/\beta$ and $\beta \notin \mathbb{N}$,

$$\min\{1/\beta, 1/\beta + 1 - \lfloor \beta c \rfloor/\beta\} > \langle\langle \beta c \rangle\rangle/\beta.$$

By (7.13) and $\rho_\mu \le 1$,we have $\beta \le \lfloor \beta c \rfloor$, yielding that

$$\min\{1/\beta, 1/\beta + 1 - \lfloor \beta c \rfloor/\beta\} = 1/\beta + 1 - \langle\langle \beta c \rangle\rangle/\beta.$$

Hence (7.13) and (7.14) are equivalent to

$$\rho_\mu(t) = \beta - \lfloor \beta c \rfloor + 1,\ t \in [\langle\langle \beta c \rangle\rangle/\beta, 1/\beta + 1 - \lfloor \beta c \rfloor/\beta],$$

$$\rho_\mu(t) + \rho_\mu(t + \lfloor \beta c \rfloor - 1/\beta) = \beta - \lfloor \beta c \rfloor + 2,\ t \in [1/\beta + 1 - \lfloor \beta c \rfloor/\beta, 1/\beta].$$

Since $\beta \notin \mathbb{N}$, by $1 \le c < 1 + 1/\beta$ and $\beta \le \lfloor \beta c \rfloor$, we have $\lfloor \beta c \rfloor = \lfloor \beta \rfloor + 1$. This further implies that (7.7)-(7.11) are equivalent to

$$\rho_\mu(t) + \rho_\mu(t+1) = 1,\ t \in [0, c-1], \tag{7.15}$$

$$\rho_\mu(t) = 1,\ t \in [c-1, 1], \tag{7.16}$$

$$\rho_\mu(t) + \rho_\mu\left(t + 1 + \left(1 - \langle\langle\beta\rangle\rangle\right)/\beta\right) = \langle\langle\beta\rangle\rangle,\ t \in \left[0, c-1-1/\beta + \langle\langle\beta\rangle\rangle/\beta\right], \tag{7.17}$$

$$\rho_\mu(t) = \langle\langle\beta\rangle\rangle,\ t \in \left[c-1-1/\beta + \langle\langle\beta\rangle\rangle/\beta, \langle\langle\beta\rangle\rangle/\beta\right], \tag{7.18}$$

$$\rho_\mu(t) + \rho_\mu\left(t + 1 - \langle\langle\beta\rangle\rangle/\beta\right) = \langle\langle\beta\rangle\rangle + 1,\ t \in \left[\langle\langle\beta\rangle\rangle/\beta, 1/\beta\right]. \tag{7.19}$$

If $c - 1 < \langle\langle\beta\rangle\rangle/\beta$, (7.16) contradicts (7.18) simply because the corresponding intervals have a non-trivial intersection. For the rest of the argument, we assume $c - 1 \geq \langle\langle\beta\rangle\rangle/\beta$. Now we aim for a contradiction case by case.

Case I: $\beta \notin \mathbb{Q}$. Let $L_1 := [c-1-1/\beta + \langle\langle\beta\rangle\rangle/\beta, \langle\langle\beta\rangle\rangle/\beta]$. Note that $\lambda(L_1) = 1 + 1/\beta - c > 0$. By (7.15), (7.16) and (7.18), we have

$$\rho_\mu(t) = \langle\langle\beta\rangle\rangle,\ t \in L_1,$$

$$\rho_\mu(t) = 1 - \langle\langle\beta\rangle\rangle,\ t \in R_1 := L_1 + 1.$$

Since

$$\left(A_1 + \left(1 + \left(1 - \langle\langle\beta\rangle\rangle\right)/\beta\right)\right) \bigcup \left(A_2 + \left(1 - \langle\langle\beta\rangle\rangle/\beta\right)\right) = [1, c]$$

with $A_1 = [0, c-1-1/\beta + \langle\langle\beta\rangle\rangle/\beta]$ and $A_2 = \left[\langle\langle\beta\rangle\rangle/\beta, 1/\beta\right]$, by either (7.17) or (7.19), $0 \leq \rho_\mu(t) \leq 1$, as well as $\beta \notin \mathbb{Q}$, we deduce

$$\rho_\mu(t) = \langle\langle\rho_\mu(t)\rangle\rangle = \langle\langle 2\,\langle\langle\beta\rangle\rangle\rangle\rangle = \langle\langle 2\beta\rangle\rangle,\ t \in L_2 \subset [0, c-1],$$

where L_2 is a union of at most two subintervals of $[0, c-1]$ with $\lambda(L_2) = 1 + 1/\beta - c$. By induction, we can show that for every $k \in \mathbb{N}$, there exists L_k, a union of finite subintervals of $[0, c-1]$ with $\lambda(L_k) = 1 + 1/\beta - c$ such that

$$\rho_\mu(t) = \langle\langle k\beta\rangle\rangle,\ t \in L_k. \tag{7.20}$$

Since $\beta \notin \mathbb{Q}$,

$$\langle\langle i\beta\rangle\rangle \neq \langle\langle j\beta\rangle\rangle,\ \forall\ i \neq j,\ i, j \in \mathbb{N},$$

and thus by (7.20),

$$\lambda(L_i \cap L_j) = 0,\ \forall\ i \neq j,\ i, j \in \mathbb{N}.$$

Hence

$$\lambda\left(\cup_{j=1}^{k} L_j\right) = k\left(1 + 1/\beta - c\right), \ \forall\ k \in \mathbb{N}.$$

On the other hand, since $\cup_{j=1}^{k} L_j$ is a subset of $[0, c-1]$, we have $\lambda\left(\cup_{j=1}^{k} L_j\right) \leq c - 1$. Taking $k = \left\lfloor \frac{c-1}{1+1/\beta-c} \right\rfloor + 1$, we arrive at a contradiction.

Case II:$\beta \in \mathbb{Q}$. Similarly to case I, for $k = 1, \cdots, q_\beta - 1$, there exists $L_k \subset [0, c-1]$ with $\lambda(L_k) = 1 + 1/\beta - c$ such that

$$\rho_\mu(t) = \langle\langle k \langle\langle \beta \rangle\rangle \rangle\rangle = \langle\langle k s_\beta / q_\beta \rangle\rangle, \ t \in L_k.$$

Since s_β and q_β are coprime, by [55, Thm.1.5.1],

$$\langle\langle i s_\beta / q_\beta \rangle\rangle \neq \langle\langle j s_\beta / q_\beta \rangle\rangle, \ \forall\ i \neq j, \ 1 \leq i, j \leq q_\beta - 1,$$

which implies that

$$\lambda(L_i \cap L_j) = 0, \ \forall\ i \neq j, \ 1 \leq i, j \leq q_\beta - 1.$$

Hence $\lambda\left(\cup_{k=1}^{q_\beta - 1} L_k\right) = (q_\beta - 1)\left(1 + 1/\beta - c\right)$. On the other hand, since $\cup_{k=1}^{q_\beta-1} L_k \subset [0, c-1]$,

$$(q_\beta - 1)\left(1 + 1/\beta - c\right) \leq c - 1,$$

i.e., $c \geq 1 + 1/\beta - 1/p_\beta = c_\beta$, contradicting the assumption that $c < c_\beta$. This completes the proof of (i), i.e., $\mathcal{C}_\beta \cap \mathcal{S}_c = \varnothing$ whenever $c < c_\beta$.

It remains to prove (ii). For this, we first verify that $\mu_\beta \in \mathcal{C}_\beta \cap \mathcal{S}_{c_\beta}$, i.e., $\rho_{\mu_\beta \circ \pi^{-1} \circ \iota^{-1}} = \rho_{\mu_\beta \circ T_\beta^{-1} \circ \pi^{-1} \circ \iota^{-1}} \equiv 1$, treating the cases of irrational and rational β separately as before.

For $\beta \notin \mathbb{Q}$, by (7.1),

$$\rho_{\mu_\beta \circ \pi^{-1} \circ \iota^{-1}}(t) = \beta t + \left(-\beta(t+1) + 1 + \beta\right) = 1, \ t \in [0, 1/\beta[\,; \ \rho_{\mu_\beta \circ \pi^{-1} \circ \iota^{-1}}(t) = 1, \ t \in [1/\beta, 1[\,,$$

i.e., $\rho_{\mu_\beta \circ \pi^{-1} \circ \iota^{-1}} \equiv 1$. Note that

$$\rho_{\mu_\beta \circ T_\beta^{-1}}(t) = \begin{cases} \frac{1}{\beta}\left(\beta \frac{t}{\beta}\right) & \text{if } t \in [0, 1[\,, \\ \frac{1}{\beta} & \text{if } t \in [1, \beta[\,, \\ \frac{1}{\beta}\left(-\beta \frac{t}{\beta} + 1 + \beta\right) & \text{if } t \in [\beta, \beta+1[\,, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \frac{t}{\beta} & \text{if } t \in [0, 1[\,, \\ \frac{1}{\beta} & \text{if } t \in [1, \beta[\,, \\ -\frac{t}{\beta} + 1 + \frac{1}{\beta} & \text{if } t \in [\beta, \beta+1[\,, \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\rho_{\mu_\beta \circ T_\beta^{-1} \circ \pi^{-1} \circ \iota^{-1}}(t) = \frac{t}{\beta} + \lfloor \beta \rfloor \cdot \frac{1}{\beta} + 1 + \frac{1}{\beta} - \frac{t + \lfloor \beta \rfloor + 1}{\beta} = 1, \ t \in [0, \langle\langle \beta \rangle\rangle[\,;$$

$$\rho_{\mu_\beta\circ T_\beta^{-1}\circ\pi^{-1}\circ\iota^{-1}}(t)=\frac{t}{\beta}+(\lfloor\beta\rfloor-1)\cdot\frac{1}{\beta}+1+\frac{1}{\beta}-\frac{t+\lfloor\beta\rfloor}{\beta}=1,\ t\in[\langle\!\langle\beta\rangle\!\rangle,1[,$$

i.e., $\rho_{\mu_\beta\circ T_\beta^{-1}\circ\pi^{-1}\circ\iota^{-1}}\equiv 1$. Thus $\mu_\beta\in\mathcal{C}_\beta\cap\mathcal{S}_{c_\beta}$.

For $\beta\in\mathbb{Q}$, by (7.2) and induction, it is easy to confirm that

$$\rho_{\mu_\beta\circ\pi^{-1}\circ\iota^{-1}}(t)=\frac{j}{q_\beta}+\frac{q_\beta-j}{q_\beta}=1,\ t\in\left[\frac{j-1}{p_\beta},\frac{j}{p_\beta}\right[,\ j=1,\cdots,q_\beta-1;$$
$$\rho_{\mu_\beta\circ\pi^{-1}\circ\iota^{-1}}(t)=1,\ t\in\left[\frac{q_\beta-1}{p_\beta},1\right[,$$

i.e., $\rho_{\mu_\beta\circ\pi^{-1}\circ\iota^{-1}}\equiv 1$. Again by induction, one can show for $j=1,\cdots,s_\beta$,

$$\rho_{\mu_\beta\circ T_\beta^{-1}}(t)=\begin{cases}\frac{j}{p_\beta} & \text{if } t\in\left[\frac{j-1}{q_\beta},\frac{j}{q_\beta}\right[,\\ \frac{q_\beta}{p_\beta} & \text{if } t\in\left[k+\frac{j-1}{q_\beta},k+\frac{j}{q_\beta}\right[,\ k=1,\cdots,\lfloor\beta\rfloor,\\ \frac{s_\beta-j}{p_\beta} & \text{if } t\in\left[\lfloor\beta\rfloor+1+\frac{j-1}{q_\beta},\lfloor\beta\rfloor+1+\frac{j}{q_\beta}\right[,\end{cases}$$

and for $j=s_\beta+1,\cdots,q_\beta$,

$$\rho_{\mu_\beta\circ T_\beta^{-1}}(t)=\begin{cases}\frac{j}{p_\beta} & \text{if } t\in\left[\frac{j-1}{q_\beta},\frac{j}{q_\beta}\right[,\\ \frac{q_\beta}{p_\beta} & \text{if } t\in\left[k+\frac{j-1}{q_\beta},k+\frac{j}{q_\beta}\right[,\ k=1,\cdots,\lfloor\beta\rfloor-1,\\ \frac{q_\beta+s_\beta-j}{p_\beta} & \text{if } t\in\left[\lfloor\beta\rfloor+\frac{j-1}{q_\beta},\lfloor\beta\rfloor+\frac{j}{q_\beta}\right[,\end{cases}$$

yielding

$$\rho_{\mu_\beta\circ T_\beta^{-1}\circ\pi^{-1}\circ\iota^{-1}}(t)=\frac{j}{p_\beta}+\lfloor\beta\rfloor\cdot\frac{q_\beta}{p_\beta}+\frac{s_\beta-j}{p_\beta}=1,\ t\in\left[\frac{j-1}{q_\beta},\frac{j}{q_\beta}\right[,\ \text{for } j=1,\cdots,s_\beta,$$

$$\rho_{\mu_\beta\circ T_\beta^{-1}\circ\pi^{-1}\circ\iota^{-1}}(t)=\frac{j}{p_\beta}+(\lfloor\beta\rfloor-1)\cdot\frac{q_\beta}{p_\beta}+\frac{q_\beta+s_\beta-j}{p_\beta}=1,\ t\in\left[\frac{j-1}{q_\beta},\frac{j}{q_\beta}\right[,\ \text{for } j=s_\beta+1,\cdots,q_\beta,$$

i.e., $\rho_{\mu_\beta\circ T_\beta^{-1}\circ\pi^{-1}\circ\iota^{-1}}\equiv 1$, and again $\mu_\beta\in\mathcal{C}_\beta\cap\mathcal{S}_{c_\beta}$.

To complete the proof, therefore, it suffices to establish

Claim 7.10. *If $\mu\in\mathcal{C}_\beta\cap\mathcal{S}_{c_\beta}$ with* $\operatorname{supp}\mu\subset[0,c_\beta]$, *then $\mu=\mu_\beta$.*

In the following, we prove this claim, distinguishing cases as before.

Case I: $\beta\notin\mathbb{Q}$. In this case, it is not enough to only deal with equations and inequalities for the density (which only holds in the almost everywhere sense); we instead need to consider the distribution function. Recall that F_μ is continuous for all $\mu\in\mathcal{C}_\beta$, by Proposition 7.2.

It follows from (7.7), (7.8), (7.12)-(7.14) together with the continuity of F_μ that, for $c=$

$1+\frac{1}{\beta}$,

$$F_\mu(t)+F_\mu(t+1)=t+F_\mu(1),\ t\in[0,1/\beta], \tag{7.21}$$

$$F_\mu(t)=F_\mu(1/\beta)+t-1/\beta,\ t\in[1/\beta,1], \tag{7.22}$$

$$F_\mu(t)+F_\mu(t+1+(1-\langle\langle\beta\rangle\rangle)/\beta)=\langle\langle\beta\rangle\rangle t+F_\mu(1+(1-\langle\langle\beta\rangle\rangle)/\beta),\ t\in[0,\langle\langle\beta\rangle\rangle/\beta], \tag{7.23}$$

$$F_\mu(t)+F_\mu(t+1-\langle\langle\beta\rangle\rangle/\beta)=(\langle\langle\beta\rangle\rangle+1)(t-\langle\langle\beta\rangle\rangle/\beta)+F_\mu(\langle\langle\beta\rangle\rangle/\beta)+F_\mu(1),\ t\in[\langle\langle\beta\rangle\rangle/\beta,1/\beta]. \tag{7.24}$$

By (7.21) and (7.23),

$$\frac{F_\mu(t+(1-\langle\langle\beta\rangle\rangle)/\beta)-F_\mu(t)}{(1-\langle\langle\beta\rangle\rangle)/\beta}-\beta t=C_1,\ t\in[0,\langle\langle\beta\rangle\rangle/\beta], \tag{7.25}$$

with $C_1=\frac{F_\mu(1/\beta)-F_\mu((\langle\langle\beta\rangle\rangle)/\beta)}{(1-\langle\langle\beta\rangle\rangle)/\beta}-\langle\langle\beta\rangle\rangle$. Similarly, by (7.21) and (7.24),

$$\frac{F_\mu(t+(\langle\langle\beta\rangle\rangle)/\beta)-F_\mu(t)}{\langle\langle\beta\rangle\rangle/\beta}-\beta t=C_2,\ t\in[0,(1-\langle\langle\beta\rangle\rangle)/\beta], \tag{7.26}$$

with $C_2=\frac{F_\mu(\langle\langle\beta\rangle\rangle/\beta)}{\langle\langle\beta\rangle\rangle/\beta}$.

Furthermore, by (7.25) and (7.26), we can show by induction that for all $m,n\in\mathbb{Z}$ satisfying $m\langle\langle\beta\rangle\rangle/\beta+n(1-\langle\langle\beta\rangle\rangle)/\beta\in\left]0,\frac{1}{\beta}\right[$,

$$\begin{aligned}&F_\mu(m\langle\langle\beta\rangle\rangle/\beta+n(1-\langle\langle\beta\rangle\rangle)/\beta)\\=&\frac{\beta}{2}(m\langle\langle\beta\rangle\rangle/\beta+n(1-\langle\langle\beta\rangle\rangle)/\beta)^2+\left(C_1-\frac{1-\langle\langle\beta\rangle\rangle}{2}\right)n(1-\langle\langle\beta\rangle\rangle)/\beta+\left(C_2-\frac{\langle\langle\beta\rangle\rangle}{2}\right)m\langle\langle\beta\rangle\rangle/\beta.\end{aligned} \tag{7.27}$$

By Proposition 7.6, $\{m\langle\langle\beta\rangle\rangle/\beta+n(1-\langle\langle\beta\rangle\rangle)/\beta:m,n\in\mathbb{Z}\}\bigcap]0,1/\beta[$ is dense in $[0,1/\beta]$. Thus, for every $t\in[0,1/\beta]\setminus\{m\langle\langle\beta\rangle\rangle/\beta+n(1-\langle\langle\beta\rangle\rangle)/\beta:m,n\in\mathbb{Z}\}$, there exist two sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}m_k\langle\langle\beta\rangle\rangle/\beta+n_k(1-\langle\langle\beta\rangle\rangle)/\beta=t.$$

It is easy to see that $\lim_{k\to\infty}|m_k|=\lim_{k\to\infty}|n_k|=+\infty$. Otherwise, both $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ are bounded, and thus $t\in\{m\langle\langle\beta\rangle\rangle/\beta+n(1-\langle\langle\beta\rangle\rangle)/\beta:m,n\in\mathbb{Z}\}$. Substituting (m,n) in (7.27) by (m_k,n_k) and letting $k\to\infty$ on both sides of (7.27), by the continuity of F_μ,

$$C_1-\frac{1-\langle\langle\beta\rangle\rangle}{2}=C_2-\frac{\langle\langle\beta\rangle\rangle}{2}.$$

From (7.27) it follows that

$$F_\mu(t) = \beta t^2/2 + Ct, \quad \forall\, t \in [0, 1/\beta]\,, \tag{7.28}$$

where $C = C_1 - \frac{1-\langle\!\langle \beta \rangle\!\rangle}{2}$. Hence it follows from (7.28) that

$$F'_\mu(t) = \beta t + C, \quad \forall\, t \in\,]0, 1/\beta[\,.$$

Since F_μ is non-decreasing in $]0, 1/\beta[$, $\lim_{t\downarrow 0} F'_\mu(t) \geq 0$ implies that $C \geq 0$. By (7.21) and (7.28),

$$F_\mu(t) = -\beta(t-1)^2/2 + (1-C)(t-1) + F_\mu(1), \quad \forall\, t \in [1, 1+1/\beta]\,.$$

Similarly, $\lim_{t\uparrow(1+1/\beta)} F'_\mu(t) \geq 0$ yields $C \leq 0$. Thus $C = 0$.

By (7.22), F_μ is given by

$$F_\mu(t) = \begin{cases} \beta t^2/2 & \text{if } t \in [0, 1/\beta[\,, \\ t - 1/(2\beta) & \text{if } t \in [1/\beta, 1[\,, \\ -\beta(t-1)^2/2 + t - 1/(2\beta) & \text{if } t \in [1, 1+1/\beta[\,, \end{cases}$$

equivalently, $\rho_\mu = \rho_\beta$ and thus $\mu = \mu_\beta$.

Case II:$\beta \in \mathbb{Q}$. Recall the definitions of p_β, q_β and s_β for every $\beta \in \mathbb{Q} \setminus \{0\}$ from the previous section. We know $\lfloor \beta \rfloor = \left\lfloor \frac{p_\beta - 1}{q_\beta} \right\rfloor$ for $\beta \notin \mathbb{N}$. Hence

$$\rho_\mu(t) + \rho_\mu(t+1) = 1, t \in \left[0, \frac{q_\beta - 1}{p_\beta}\right], \tag{7.29}$$

$$\rho_\mu(t) = 1, \ t \in \left[\frac{q_\beta - 1}{p_\beta}, 1\right], \tag{7.30}$$

$$\rho_\mu(t) + \rho_\mu\left(t + 1 + \frac{q_\beta - s_\beta}{p_\beta}\right) = \frac{s_\beta}{q_\beta}, \ t \in \left[0, \frac{s_\beta - 1}{p_\beta}\right], \tag{7.31}$$

$$\rho_\mu(t) = \frac{s_\beta}{q_\beta}, \ t \in \left[\frac{s_\beta - 1}{p_\beta}, \frac{s_\beta}{p_\beta}\right], \tag{7.32}$$

$$\rho_\mu(t) + \rho_\mu\left(t + 1 - \frac{s_\beta}{p_\beta}\right) = 1 + \frac{s_\beta}{q_\beta}, \ t \in \left[\frac{s_\beta}{p_\beta}, \frac{q_\beta}{p_\beta}\right]. \tag{7.33}$$

It follows from (7.31) and (7.33) that,

$$\left(\left[0, \frac{s_\beta - 1}{p_\beta}\right[+ \left(1 + \frac{q_\beta - s_\beta}{p_\beta}\right)\right) \bigcup \left(\left[\frac{s_\beta}{p_\beta}, \frac{q_\beta}{p_\beta}\right[+ \left(1 - \frac{s_\beta}{p_\beta}\right)\right) = \left[1, 1 + \frac{q_\beta - 1}{p_\beta}\right[.$$

Using (7.29), (7.31) and (7.33),

$$\rho_\mu\left(t+\frac{s_\beta}{p_\beta}\right)-\rho_\mu(t)=\frac{s_\beta}{q_\beta},\ t\in\left[0,\frac{q_\beta-s_\beta}{q_\beta}\right],$$

$$\rho_\mu\left(t+\frac{q_\beta-s_\beta}{p_\beta}\right)-\rho_\mu(t)=1-\frac{s_\beta}{q_\beta},\ t\in\left[0,\frac{s_\beta-1}{q_\beta}\right].$$

Similarly to (7.27), we can show by induction that

$$\rho_\mu\left(t+m\frac{s_\beta}{p_\beta}+n\frac{q_\beta-s_\beta}{p_\beta}\right)=\rho_\mu(t)+m\frac{s_\beta}{q_\beta}+n\left(1-\frac{s_\beta}{q_\beta}\right), \tag{7.34}$$

for $m,n\in\mathbb{Z}$, $t\in\left[0,\frac{q_\beta-1}{p_\beta}\right]$ a.e. satisfying $t+m\frac{s_\beta}{p_\beta}+n\frac{q_\beta-s_\beta}{p_\beta}\in\left[0,\frac{q_\beta-1}{p_\beta}\right]$. Since s_β and q_β are coprime, by [55, Thm.1.4.4(i)] there exist $m_0,n_0\in\mathbb{Z}$ such that $m_0s_\beta+n_0(q_\beta-s_\beta)=1$. Then it follows from (7.34) that

$$\rho_\mu\left(t+\frac{j}{p_\beta}\right)=\rho_\mu(t)+\frac{j}{q_\beta}, \tag{7.35}$$

for $j\in\mathbb{Z}$, $t\in\left[0,\frac{q_\beta-1}{p_\beta}\right]$ a.e. satisfying $t+\frac{j}{p_\beta}\in\left[0,\frac{q_\beta-1}{p_\beta}\right]$. By (7.30), (7.29), (7.32) and (7.35), we can prove by induction that $\rho_\mu=\rho_\beta$ and thus $\mu=\mu_\beta$. □

Remark 7.11. (i) For $\beta\neq0$, $\alpha\in\mathbb{R}$, it follows from Theorem 7.9 that there exist CIUPM for $T_{\beta,\alpha}$ with arbitrarily long support. Moreover, from the proof of Theorem 7.9, one easily observes that if $\beta\in\mathbb{Q}\cap\ [1,+\infty[$, then $\widetilde{\mu}_\beta$ with

$$\rho_{\widetilde{\mu}_\beta}(t)=\begin{cases}\beta t & \text{if } t\in[0,1/\beta[\,,\\ 1, & \text{if } t\in[1/\beta,1[\,,\\ -\beta t+1+\beta & \text{if } t\in[1,1+1/\beta]\,,\\ 0 & \text{otherwise,}\end{cases}$$

is another CIUPM for $T_{\beta,\alpha}$, but with $\operatorname{diam}\operatorname{supp}\widetilde{\mu}_\beta>c_\beta$.

(ii) Notice that $\mathcal{C}_{\beta,\alpha}\cap\mathcal{S}_{c_\beta}$ may not contain every "slimmest" CIUPM for $T_{\beta,\alpha}$ in the sense that $\lambda(\operatorname{supp}\mu)=c_\beta$ may hold for some $\mu\in\mathcal{C}_{\beta,\alpha}\cap\mathcal{S}_c$ and $c>c_\beta$. For instance, $\mu:=\lambda|_{[0,1/2]}+\lambda|_{[3/2,2]}$ with $\lambda\left(\operatorname{supp}\mu\right)=1=c_k$ is a CIUPM for every linear map $T_{k,\alpha}$ with $\alpha\in\mathbb{R}$ and $k\in\mathbb{Z}\setminus\{0\}$.

Chapter 8

Concluding remarks

In this thesis, three different directions have been pursued. Chapters 2-5 investigate best finite approximations of probability measures on the real line, relative to three familiar probability metrics (Lévy, Kantorovich, and Kolmogorov), for any given number of atoms, and allowing for additional constraints regarding weights or positions of atoms. Chapter 6 provides a sharp upper bound for the rates of convergence of the slowly changing sequence $(\log_b n)$ w.r.t. the L^r-Kantorovich metrics on the circle and on the real line, respectively, as well as the discrepancy metric. Chapter 7 characterizes "slimmest" circularly invariant and uniform probability measures for non-constant linear maps on the real line.

All the results are concentrated around the topic of one-dimensional probabilities, their finite approximations and invariance properties. Most observations originate from, and are explored beyond explicit calculations of simple examples due to, the one-dimensionality. However, the questions considered in this thesis can be extended and pursued further in many different directions. Below, we list a few possible starting points for future investigations. This list, which is not meant to be exhaustive in any way, is loosely arranged in accordance with the general aspects of the thesis outlined in the Introduction.

Quantization/Approximation of probability measures

First, from the perspective of the **underlying space**, it is interesting to extend the results to the unique *compact* connected one-dimensional manifold — the circle.

Question 8.1. *Do best (constrained or unconstrained) approximations of $\mu \in \mathcal{P}(\mathbb{T})$ exist? Are they unique? How can they be characterized and computed?*

The problem of optimally matching circular distributions appears in a variety of applications, including image processing and computer vision, with image matching techniques for retrieval, classification, or stitching purposes as prominent examples [24]. These and other applications motivate a careful analysis of finitely supported approximations in $\mathcal{P}(\mathbb{T})$, in a spirit similar to this thesis. Such an analysis may have a *random* counterpart as well. Motivated by our study on rates of d_r-convergence of empirical measures for $\mu \in \mathcal{P}(\mathbb{R})$, we also ask

Question 8.2. *Is there a universal rate of d_r-convergence for empirical measures for $\mu \in \mathcal{P}(\mathbb{T})$?*

Another natural though challenging direction is to consider higher dimensional spaces, where distribution and quantile function techniques, essential tools in Chapters 2-5 of this thesis, are not available. Nevertheless, one may ask

Question 8.3. *Do best constrained approximations of $\mu \in \mathcal{P}(\mathbb{R}^m)$ with $m \geq 2$ exist? Are they unique? How can they be characterized and computed?*

In a similar spirit, given an $\mathbb{R}^m$-valued random variable X with law $\mu \in \mathcal{P}_r(\mathbb{R}^m)$, for any $n \in \mathbb{N}$, let

$$\mathcal{F}_n^u(\mu) := \left\{ f \in \mathcal{F}_n : \mu \circ f^{-1} = \delta_x^{u_n} \text{ for some } x \in \Xi_n \right\}.$$

We ask

Question 8.4. *Does there exist $f \in \mathcal{F}_n^u(\mu)$ such that*

$$\mathbb{E}\|X - f(X)\|^r \leq \mathbb{E}\|X - g(X)\|^r \quad \forall\, g \in \mathcal{F}_n^u(\mu)?$$

If so, how to characterize all functions f with this property?

In the one-dimensional setting, the following question connecting best constrained and best unconstrained approximations may be relevant for both theoretical and practical purposes: Fix $r \geq 1$, and given $\mu \in \mathcal{P}_r(\mathbb{R})$ and $n \in \mathbb{N}$, let $p_1 = u_n$, and define a sequence of best constrained approximations (ν_k) of μ as

$$\nu_{2k-1} := \delta_{x_k}^{p_k} = \delta_{\bullet}^{p_k}, \quad \nu_{2k} := \delta_{x_k}^{p_{k+1}} = \delta_{x_k}^{\bullet}, \quad k = 1, 2, \cdots .$$

With this, it is natural to ask

Question 8.5. *Under what condition(s) does $\lim_{k\to\infty} d_r\left(\nu_k, \delta_{\bullet}^{\bullet,n}\right) = 0$ hold? If it does, what is the rate of convergence?*

Second, future investigations may study in more detail the role of **probability metrics**. Let d be any metric on $\mathcal{P}(\mathbb{R})$ inducing a topology at least as fine as the weak topology. A set of rather open-ended questions then is

Question 8.6. *Do best (constrained or unconstrained) d-approximations of $\mu \in \mathcal{P}(\mathbb{R})$ exist? Are they unique? How can they be characterized and computed? What specific properties of d guarantee certain rates of convergence?*

Slowly changing sequences

In Chapter 6, an upper bound is established for the rate of $d_{\mathbb{T}}$-convergence for $(\log_b n)$ with integer base $b \geq 2$; however, it remains an open question whether this upper bound is sharp or not:

Question 8.7. *What is the sharp rate of convergence for $d_{\mathbb{T}}\left(\nu_N(\log_b n), E_{\log b} \circ R^{-1}_{-\log_b N}\right)$?*

Beyond $(\log_b n)$, it is natural to ask more generally

Question 8.8. *Is it possible to provide upper (and lower) bounds on the rate of convergence for a wider class of (or perhaps all) slowly changing sequences?*

Circular invariance for continuous maps

As explained in Chapter 7, the analysis there has been motivated mainly by Question 1.19. While Chapter 7 answered the latter in the affirmative for linear maps, a final general answer seems to be elusive. As a first potentially important step towards an answer beyond Chapter 7, we ask

Question 8.9. *Given a piecewise linear convex map $T : \mathbb{R} \to \mathbb{R}$, under what condition(s) is $\mathcal{C}_{\mathbb{T}} \neq \varnothing$, i.e., does there exist a CIUPM for T?*

Bibliography

[1] P.C. Allaart. An invariant-sum characterization of Benford's law. *J. Appl. Probab.*, 34 (1997), 288–291.

[2] D.M. Baker. Quantizations of probability measures and preservation of the convex order. *Stat. Probab. Lett.*, 107 (2015), 280–285.

[3] F. Benford. The law of anomalous numbers. *Proc. Amer. Philos. Soc.*, 78 (1938), 551–572.

[4] A. Berger and T.P. Hill. Benfords law strikes back: no simple explanation in sight for mathematical gem. *Math. Intelligencer*, 33 (2011), 85–91.

[5] A. Berger and T.P. Hill. *An Introduction to Benford's Law*, Princeton Univ. Press, Princeton, 2015.

[6] A. Berger, T.P. Hill, and K.E. Morrison. Scale-distortion inequalities for mantissas of finite data sets. *J. Theor. Probab.*, 21 (2008), 97–117.

[7] A. Berger, T.P. Hill, and E. Rogers. *Benford Online Bibliography*. http://www.benfordonline.net, 2009. (Last accessed Dec. 1st, 2017.)

[8] A. Berger and I. Twelves. On the significands of uniform random variables. Preprint (2017).

[9] J. Bergstra and Y. Bengio. Random search for hyper-parameter optimization. *J. Mach. Learn. Res.*, 13 (2012), 281–305.

[10] I. Bloch and J. Atif. Defining and computing Hausdorff distances between distributions on the real line and on the circle: link between optimal transport and morphological dilations. *Math. Morphol. Theory Appl.*, 1 (2016), 79–99.

[11] S.G. Bobkov and M. Ledoux. One-dimensional empirical measures, order statistics and Kantorovich transport distances. http://www-users.math.umn.edu/~bobko001/preprints/2016_BL_Order.statistics_Revised.version.pdf. To appear in: *Memoirs of the AMS.* Preprint (2016).

[12] E. Boissard and T. L. Gouic. On the mean speed of convergence of empirical and occupation measures in Wasserstein distance. *Ann. Inst. H. Poincaré Probab. Statist.*, 50 (2014), 539–563.

[13] F. Bolley, A Guillin, and C. Villani. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Relat. Fields*, 137 (2007), 541–593.

[14] G. Bouchitté, C. Jimenez, and R. Mahadevan. Asymptotique d'un problème de positionnement optimal. *C. R. Math. Acad. Sci. Paris*, 335 (2002), 853–858.

[15] G. Bouchitté, C. Jimenez, and R. Mahadevan. Asymptotic analysis of a class of optimal location problems. *J. Math. Pures Appl.*, 95 (2011), 382–419.

[16] A. Boyarsky and P. Góra. *Laws of Chaos, Invariant Measures and Dynamical Systems in One Dimension*, Probability and Its Applications, Birkhäuser, Boston, 1997.

[17] C.A. Cabrelli and U.M. Molter. The Kantorovich metric for probability measures on the circle. *J. Comput. Appl. Math.*, 57 (1995), 345–361.

[18] E. Caglioti, F. Golse, and M. Iacobelli. A gradient flow approach to quantization of measures. *Math. Models Methods Appl. Sci.*, 25 (2015), 1845–1885.

[19] E. Caglioti, F. Golse, and M. Iacobelli. Quantization of measures and gradient flows: a perturbative approach in the 2-dimensional case. https://arxiv.org/pdf/1607.01198.pdf. To appear in *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*. Preprint (2016).

[20] G.D. Cañas and L. Rosasco. Learning probability measures with respect to optimal transport metrics. In P. L. Bartlett, F. C. N. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 25: 26th Annual Conference on Neural Information Processing Systems 2012*. Lake Tahoe, Nevada, United States., 2501–2509, 2012.

[21] I.P. Cornfield, S.V. Fomin, and Ya.G. Sinai. *Ergodic Theory*, Grundlehren der mathematischen Wissenschaften 245, Springer, Berlin, New York, 1982.

[22] G. David and S. Semmes. *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Surveys and Monographs 38, AMS, Providence, RI, 1993.

[23] G. David and S. Semmes. *Fractured Fractals and Broken Dreams*, Clarendon Press, Oxford, 1997.

[24] J. Delon, J. Salomon, and A. Sobolevski. Fast transport optimization for Monge costs on the circle *SIAM J. Appl. Math.*, 70 (2010), 2239–2258.

[25] S. Dereich, M. Scheutzow, and R. Schottstedt. Constructive quantization: Approximation by empirical measures. *Ann. Inst. H. Poincaré Probab. Statist.*, 49 (2013), 1183–1203.

[26] S. Dereich and C. Vormoor. The high resolution vector quantization problem with Orlicz norm distortion. *J. Theor. Probab.*, 24 (2011), 517–544.

[27] P. Diaconis. The distribution of leading digits and uniform distribution mod 1. *Ann. Probab.*, 5 (1977), 72–81.

[28] M. Drmota and R. Tichy. *Sequences, Discrepancies and Applications*, Lecture Notes in Math. 1651, Springer, Berlin, Heidelberg, 1997.

[29] Q. Du, V. Faber, and M. Gunzburger. Centroidal Voronoi tessellations: Applications and algorithms. *SIAM Review*, 41 (1999), 637–676.

[30] R. Dudley. *Real Analysis and Probability*, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, 2004.

[31] L. Dümbgen and C. Leuenberger. Explicit bounds for the approximation error in Benford's law. *Elect. Comm. in Probab.*, 13 (2008), 99–112.

[32] W. Feller. *An Introduction to Probability Theory and Its Applications Vol. II*, John Wiley & Sons, New York, 1966.

[33] N. Fournier and A. Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Relat. Fields*, 162 (2015), 707–738.

[34] N. Fournier and S. Mischler. Rate of convergence of the Nanbu particle system for hard potentials and Maxwell molecules. *Ann. Probab.*, 44 (2016), 589–627.

[35] N. Gauvrit and J.-P. Delahaye. Scatter and Regularity Imply Benford's Law... and More. pp. 53–69 in: H. Zenil (ed.), *Randomness Through Complexity*, World Scientific, Singapore, 2011.

[36] A.L. Gibbs and F.E. Su. On choosing and bounding probability metrics. *Int. Stat. Rev.*, 70 (2002), 419–435.

[37] R. Giuliano Antonini and O. Strauch. On weighed distribution functions of sequences. *Unif. Distrib. Theory*, 3 (2008), 1–18.

[38] P. Góra. Invariant densities for generalized β-maps. *Ergod. Th. & Dynam. Sys.*, 27 (2007), 1583–1598.

[39] P. Góra. Invariant densities for piecewise linear maps of interval. *Ergod. Th. & Dynam. Sys.*, 29 (2009), 1549–1583.

[40] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*, Lecture Notes in Math. 1730, Springer, Berlin, 2000.

[41] S. Graf and H. Luschgy. The quantization dimension of self-similar probabilities. *Math. Nachr.*, 241 (2002), 103–109.

[42] S. Graf and H. Luschgy. Rates of convergence for the empirical quantization error. *Ann. Probab.*, 30 (2002), 874–897.

[43] S. Graf and H. Luschgy. Quantization for probability measures in the Prokhorov metric. *Theory Probab. Appl.*, 53 (2009), 216–241.

[44] S. Graf, H. Luschgy, and G. Pagès. The local quantization behavior of absolutely continuous probabilities. *Ann. Probab.*, 40 (2012), 1795–1828.

[45] P.M. Gruber. Optimum quantization and its applications. *Adv. Math.*, 186 (2004), 456–497.

[46] S. Halfin. Explicit construction of invariant measures for a class of continuous state Markov processes. *Ann. Prob.*, 3 (1975), 859–864.

[47] T.P. Hill. A statistical derivation of the significant-digit law. *Stat. Sci.*, 10 (1995), 354–363.

[48] T.P. Hill. Base-invariance implies Benford's law. *Proc. Amer. Math. Soc.*, 123 (1995), 887–895.

[49] E. Hlawka. Gleichverteilung und Konvergenzverhalten von Potenzreihen am Rande des Konvergenzkreises. *Manuscripta Math.*, 44 (1983), 231–263.

[50] F. Hofbauer. Maximal measures for piecewise monotonically increasing transformations on $[0,1]$. pp. 66–77 in: M. Denker and K. Jacobs (eds) *Ergodic Theory*, Lecture Notes in Math. 729, Springer, Berlin, Heidelberg, 1979.

[51] F. Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. *Isr. J. Math.*, 34 (1979), 213–237.

[52] F. Hofbauer. Maximal measures for simple piecewise monotonic transformations. *Z. Wahrsch. Verw. Gebiet*, 52 (1980), 289–300.

[53] F. Hofbauer. Generic properties of invariant measures for simple piecewise monotonic transformations. *Isr. J. Math.*, 59 (1987), 64–80.

[54] F. Hofbauer. The maximal measure for linear mod one transformation. *J. London Math. Soc.*, 23 (1981), 92–112.

[55] L.K. Hua. *Introduction to Number Theory*, Springer, New York, 1982.

[56] P.J. Huber. *Robust Statistics*, John Wiley & Sons, New York, 1981.

[57] M. Iacobelli. Asymptotic quantization for probability measures on Riemannian manifolds. *ESAIM: COCV*, 22 (2016), 770–785.

[58] M. Iacobelli. A gradient flow perspective on the quantization problem. *https://arxiv.org/pdf/1711.01652.pdf*. Preprint, 2017.

[59] J.H.B. Kemperman. Distributions modulo 1 of slowly changing sequences. *Nieuw Arch. Wisk.*, 21 (1973), 138–163.

[60] M. Kesseböhmer and S. Zhu. Some recent developments in quantization of fractal measures. pp. 105–120, In: C. Bandt, K. Falconer, and M. Zähle (eds), *Fractal Geometry and Stochastics V.*, Prog. Probab. 70, Birkhäuser, Cham, 2015.

[61] B. Kloeckner. Approximation by finitely supported measures. *ESAIM: COCV*, 18 (2012), 343–359.

[62] D.E. Knuth *The Art of Computer Programming*, Addison-Wesley, Reading, 1975.

[63] C. Kopf. Invariant measures for piecewise linear transformations of the interval. *Appl. Math. Comput.*, 39 (1990), 123–144.

[64] W. Kreitmeier. Optimal quantization for dyadic homogeneous Cantor distributions. *Math. Nachr.*, 281 (2008), 1307–1327.

[65] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. 2nd ed., John Wiley & Son Ltd, New York, 2006.

[66] T. Linder. On asymptotically companding quantization. *Probl. Control. Inform.*, 20 (1991), 475–484.

[67] L.J. Lindsay. Quantization Dimension for Probability Distributions. *PhD dissertation*, University of North Texas, Texas, 2001.

[68] J. Marklof and A. Strömbergsson. Gaps between logs. *Bull. London Math. Soc.*, 45 (2013), 1267–1280.

[69] P. Mattila. Geometry of Sets and Measures in Euclidean Spaces. Cambridge Univ. Press, Cambridge, 1995.

[70] S.J. Miller. *Benford's Law: Theory and Applications*, Princeton Univ. Press, Princeton, 2015.

[71] Y. Mori and K. Takashima. On the distribution of the leading digit of a^n: a study via χ^2 statistics. *Period. Math. Hung.*, 73 (2016), 224–239.

[72] S. Mosconi and P. Tilli. Γ-convergence for the irrigation problem. *J. Conv. Anal.*, 12 (2005), 145–158.

[73] S. Newcomb. Note on the frequency of use of the different digits in natural numbers. *Amer. J. Math.*, 4 (1881), 39–40.

[74] D. Nova and P.A. Estévez. A review of learning vector quantization classifiers. *Neural Computing and Applications*, 25 (2014), 511–524.

[75] Y. Ohkubo. The weighted discrepancies of some slowly increasing sequences. *Math. Nachr.*, 174 (1995), 239–251.

[76] Y. Ohkubo. On sequences involving primes. *Unif. Distrib. Theory*, 6 (2011), 221–238.

[77] Y. Ohkubo and O. Strauch. Distribution of leading digits of numbers. *Unif. Distrib. Theory*, 11 (2016), 23–45.

[78] G. Pagès. A space quantization method for numerical integration. *J. Comput. Appl. Math.*, 89 (1997), 1–38.

[79] G. Pagès and A. Sagna. Asymptotics of the maximal radius of an L^r-optimal sequence of quantizers. *Bernoulli*, 18 (2012), 360–389.

[80] W. Parry. Representations for real numbers. *Acta Math. Acad. Sci. Hungar.*, 15 (1964), 95–105.

[81] G.C. Pflug and A. Pichler. Approximations for probability distributions and stochastic optimization problems. pp. 343–387 in: M. Bertocchi, G. Consigli, and M. Dempster (eds.), *Stochastic Optimization Methods in Finance and Energy*, Internat. Ser. Oper. Res. Manage. Sci. 1633, Springer, New York, 2011.

[82] R.S. Pinkham. On the distribution of first significant digits. *Ann. Math. Stat.*, 32 (1961), 1223–1230.

[83] K. Pötzelberger. The quantization dimension of distributions. *Math. Proc. Camb. Phil. Soc.*, 131 (2001), 507–519.

[84] K. Pötzelberger. The quantization error of self-similar distributions. *Math. Proc. Camb. Phil. Soc.*, 137 (2004), 725–740.

[85] K. Pötzelberger and K. Felsenstein. An asymptotic result on principal points for univariate distributions. *Optimization.*, 28 (1994), 397–406.

[86] S.T. Rachev. *Probability Metrics and the Stability of Stochastic Models.* J. Wiley & Sons, New York, 1991.

[87] S.T. Rachev, L.B. Klebanov, S.V. Stoyanov, and F.J. Fabozzi. *The Methods of Distances in the Theory of Probability and Statistics.* Springer, New York, NY, 2013.

[88] S.T. Rachev and L. Rüschendorf. *Mass Transportation Problems. Vol. 1 and Vol. 2.* Springer, New York, 1998.

[89] R.A. Raimi. The first digit problem. *Amer. Math. Mon.*, 83 (1976), 521–538.

[90] J. Rosenblatt. Partitions for optimal approximations. *Int. J. Math. Anal.*, 7 (2013), 2861–2878.

[91] P. Schatte. On mantissa distributions in computing and Benford's law. *J. Inform. Process. Cybernet.*, 24 (1988), 443–455.

[92] O. Strauch and O. Blažeková. Distribution of the sequence p_n/n mod 1. *Unif. Distrib. Theory*, 1 (2006), 45–63.

[93] O. Strauch and Š. Porubský. *Distribution of Sequences: A Sampler.* Peter Lang, Bern, 2005.

[94] R.F. Tichy and G. Turnwald. Logorithmic uniform distribution of $(\alpha n + \beta \log n)$. *Tsukuba J. Math.*, 10 (1986), 351–366.

[95] M. Tsujii. On the uniform distribution of numbers mod. 1. *J. Math. Soc. Japan*, 4 (1952), 313–322.

[96] C. Villani. *Optimal Transport: Old And New.* Grundlehren der Mathematischen Wissenschaften 338, Springer, Berlin-Heidelberg, 2009.

[97] J. Weed and F. Bach. Sharp asymptotic and finite-sample rates of convergence of empirical measures in Wasserstein distance. https://arxiv.org/pdf/1707.00087.pdf. Preprint (2017).

[98] R. Winkler. On the distribution behaviour of sequences. *Math. Nachr.*, 186 (1997), 303–312.

[99] A. Wintner. On the cyclical distribution of the logarithms of the prime numbers. *Quart. J. Math. Oxford*, 6 (1935), 65–66.

[100] C. Xu. On the rate of convergence for $(\log_b n)$. https://arxiv.org/abs/1609.08207. Preprint (2016).

[101] C. Xu. Rates of convergence for a class of slowly changing sequences. In preparation.

[102] C. Xu and A. Berger. Best finite constrained approximations of one-dimensional probabilities. http://arxiv.org/abs/1704.07871. Preprint (2017).