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Analysis of Shells of Revolution

by

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ABSTRACT

Two approaches to solving problems involving thin shells of revolution, based on standard methods of structural analysis are presented. With the stiffness method, the governing shell equations are transformed into eight first order differential equations corresponding to the eight displacement degrees of freedom formed at each meridian boundary and the corresponding stress resultants. A forward numerical integration technique is used to form the stiffness matrix and the particular solutions. With the flexibility method, the governing shell equations are simplified by limiting the analysis to axisymmetric shells of constant thickness so that closed form solutions may be obtained for the flexibility coefficients. Hence, only certain common shell geometries and loading cases are considered. Particular solutions are approximated by the appropriate membrane solution.

Computer programs were developed for each method and the user's manuals are included. Numerical examples to illustrate the program input and capabilities are presented.

PREFACE

This report is the result of a number of research studies conducted by the Department of Civil Engineering, The University of Alberta, under the direction of S.H. Simmonds. The objective of the original study, begun in 1969 as a doctorate thesis by S.H. Iyer, was to obtain closed form expressions for the influence coefficients of the stiffness matrix for higher harmonics of loading, to enable the consideration of lateral loading on selected shell shapes. This study led to the general formulation of the stiffness solution of Flügge's equations. This formulation was refined and coded to produce the first version of SASHELL in a Master's thesis by A.M. Shazly. As part of an analytical study of the behaviour of secondary containment structures by Murray, Rohardt, and Simmonds, the flexibility analysis for axisymmetric loading was developed and coded into program FLEXSHELL. This program was extensively revised to include other shell types and loadings in a Master's thesis by N. Hernandez. This report summarizes the theory developed and contains the user's manuals and the listings of the current versions of these shell programs.

NOMENCLATURE

a, b, c, d = parameters used in the Runge-Kutta numerical integration method, defined in Appendix C.

a = radius of curvature of a sphere

$[A_s]$ = matrix coefficients which are a function of the geometric and material properties of the shell

$[A]$ = Connectivity Matrix

$\{B_s\}$ = load vector coefficients

$\{C\}$ = constants of integration vector

CA = constant parameter which is a function of the rigidities of the shell and the principal shell curvature

D = flexural rigidity

$\{D\}$ = segment deformation vector

e = length of the interval between points of the numerical integration.

E = modulus of elasticity

F_o = fixed end forces

$\{F_s\}, \{F_e\}$ = primary and secondary force vectors respectively

$[F]$ = segment flexibility matrix

$[\bar{F}]$ = structure flexibility matrix

h = shell thickness

H = horizontal force, positive in the direction towards the axis of revolution

H_V = fictitional horizontal force due to a vertical edge load at the top of a cone or sphere

$[H_s]$ = transfer matrix arising from integrating $[A_s]$

k = subgrade modulus for the base segment

K = extensional rigidity

$[K]$ = segment stiffness matrix

l = length of a segment between points i and j.

$L(\ )$ = linear differential operator defined in Appendix A

λL = critical meridian length coefficient

M_s, M_ϕ, M_θ = in-plane bending moments

$M_{\phi\theta}, M_{\theta\phi}, M_{s\theta}, M_{\theta s}$ = twisting moments

n = harmonic number

N_s, N_ϕ, N_θ = normal in-plane forces

$N_{\phi\theta}, N_{\theta\phi}, N_{s\theta}, N_{\theta s}$ = in-plane shear forces

$p_s, p_\phi, p_\theta, p_z$ = intensity of the load components in the directions s, ϕ, θ, z respectively

$\{q_o\}$ = structure particular solution used in the flexibility analysis

$\{Q_s\}$ = vector arising from the integration of $\{B_s\}$

Q_d, Qf = coefficients of $\{Q_s\}$

Q_s, Q_ϕ, Q_θ = transverse shear forces

r = radius of the parallel circle for the cylindrical segment

r_o = radius of a parallel circle

r_1 = radius of curvature of a meridian

r_2 = length of the normal between any point on the midsurface and the axis of revolution

R_o = curvature of a parallel circle

R_1 = first principal curvature$=1/r_1$

R_2 = second principal curvature$=1/r_2$

R = total vertical load acting on a segment due to the applied loads

s = coordinate which measures the distance along the shell meridian

S_s = effective transverse shear force

T_s = effective tangential shear force

[TT] = matrix relating the constants of integration to the redundant vector for the segment; a function of the shell geometry

[TA] = matrix relating the constants of integration to the particular solution displacement vector; a function of the geometric and material properties of the shell

u = displacement component in the circumferential direction

U = change of variable in terms of Q_ϕ and r_2 used to form the homogeneous solution in the flexibility analysis

v = displacement component in the meridional direction

V = change of variable in terms of r_1, v, w, used to form the homogeneous solution in the flexibility analysis

w = displacement component in the radial direction

$\{y_s\}$ = vector of the eight dependent variables in the stiffness analysis consisting of four displacements and their associated stress resultants.

z = coordinate which measures the distance in the direction normal to the midsurface toward the axis of revolution

α = angle between the outer edge of the sphere and the axis of revolution, or the semi-vertex angle for a cone

α_0 = angle between the inner edge of the sphere and the axis of revolution

α_T = coefficient of thermal expansion

β = parameter which is a function of ν, r, and h in the flexibility analysis, or, the meridional rotation in the stiffness analysis

γ = specific weight of the shell or the liquid weight density

$\gamma_{\phi\theta}$ = shear strain

δ, Δ = particular and homogeneous deformations respectively

Δ_H = horizontal displacement of a shell

Δ_ϕ = meridional rotation of a shell

η = change of variable used to form the homogeneous solution for the cone, defined in Eqn. A.19

θ = coordinate which measures the angle in the circumferential direction

λ = dimensionless parameter which is a function of a/h and ν for the sphere, or the parameter in terms of h and the semi-vertex angle for the cone

ν = Poisson's ratio

ξ = dimensionless input parameter for the evaluation of the Kelvin functions for the spherical and conical segments.

$\sigma_\phi, \sigma_\theta$ = meridional and circumferential stresses

$\tau_{\theta\phi}$ = shear stress

ϕ = coordinate which measures the angle between any point on the midsurface and the axis of revolution

$(\ )^{\cdot} = \dfrac{\partial}{\partial \phi}$

$(\ )' = \dfrac{\partial}{\partial \theta}$

$(\ )^{\circ} = \dfrac{\partial}{\partial s}$

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1. INTRODUCTION

1.1 Introductory Remarks

A shell of revolution is a surface generated by rotating a plane curve about an axis lying in the same plane. Shells of revolution are commonly used in industry to form part of such structures as pressure vessels, storage tanks, silos, nuclear containment structures, and cooling towers. Apart from their attractive appearance, the widespread use of such shells as structural elements is attributed to their efficiency in resisting load. This leads to thinner sections and reduced material costs.

The general theory of shells of revolution, originally developed by Flügge, applies to any type of meridian geometry with either constant or variable thickness, and subjected to any type of loading. Two methods of obtaining a solution to these equations are discussed. In the stiffness method, the governing shell equations are transformed to a set of eight first order differential equations involving eight unknowns and their derivatives. A forward numerical integration technique is used to form the stiffness matrix and the fixed end forces vector. By expanding the terms using a Fourier series, loads varying in the circumferential direction can be considered. Consequently, this method is applicable to any type of shell of revolution of varying thickness and subjected to arbitrary loadings. However, for many practical applications the shell segments are of constant thickness and the loads are axisymmetric. Thus, the analysis of such shells can be simplified by separating the solution into two parts: firstly, the particular solution approximated by the membrane stresses due to the applied loads; and secondly, the bending stresses due to the edge effects. For shells of known geometry, closed form expressions can be developed for these bending effects. Since this solution is analogous to the method of consistent deformation in elastic frame analysis, it is referred to as the flexibility method.

1.2 The Scope of the Report

This report reviews the general theory developed by Flügge for shells of revolution, formulates methods of solution corresponding to the stiffness and flexibility methods of structural analysis, and presents computer programs based on these solutions.

1.3 Structure of Report

The thirteen basic differential equations of shells of revolution are formulated in detail in Chapter 2. Chapter 3 presents the two solution techniques to solve these governing shell equations based on standard methods of structural analysis. Computer programs SASHELL and FLEXSHELL, based on these two methods of analysis are described in Chapters 4 and 5, respectively. Example results obtained from the two shell programs are presented in

Chapter 6. Detailed derivations of equations, and the user's manuals and the listings for programs SASHELL and FLEXSHELL are found in the Appendices.

2. THEORY OF SHELLS OF REVOLUTION

2.1 Shell Geometry

A shell is geometrically defined by its midsurface which is the surface which bisects the shell thickness, h. A surface of revolution is generated by the rotation of a plane curve about an axis in its plane. This generating curve is called a meridian. Another term frequently used is the parallel circle, which is the intersection of the surface with a plane perpendicular to the axis of revolution. To specify an arbitrary point on the midsurface, two coordinates need be specified: θ, the angular distance of the point from the datum meridian, and ϕ, the angle between a normal to the shell and its axis of revolution, see Fig. 2.1. To measure the distance along a normal to the midsurface, a third coordinate z, may be specified. The radii of curvature of a shell of revolution are:

r_0 = radius of the parallel circle;

r_1 = radius of curvature of a meridian;

r_2 = length of the normal between any point on the midsurface and the axis of revolution.

The following relations can be derived from Fig. 2.1.

$$r_0 = r_2 \sin\phi \tag{2.1(a)}$$

$$ds = r_1 d\phi \tag{2.1(b)}$$

$$\therefore \frac{\partial}{\partial s} = \frac{1}{r_1}\frac{\partial}{\partial \phi} \tag{2.1(c)}$$

$$dr = ds \cos\phi \tag{2.1(d)}$$

$$dz = ds \sin\phi \tag{2.1(e)}$$

$$\frac{dr_2}{ds} = \frac{r_1-r_2\cot\phi}{r_1} \qquad 2.1(f)$$

The internal stress resultants in Fig. 2.2, is determined by integrating the internal stresses through the shell thickness as follows

$$N_\phi = \int_{-h/2}^{h/2} \sigma_\phi(1+z/r_2)dz \qquad 2.2(a)$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta(1+z/r_1)dz \qquad 2.2(b)$$

$$N_{\phi\theta} = \int_{-h/2}^{h/2} \tau_{\phi\theta}(1+z/r_2)dz \qquad 2.2(c)$$

$$N_{\theta\phi} = \int_{-h/2}^{h/2} \tau_{\theta\phi}(1+z/r_1)dz \qquad 2.2(d)$$

$$Q_\phi = \int_{-h/2}^{h/2} \tau_{\phi z}(1+z/r_2)dz \qquad 2.2(e)$$

$$Q_\theta = \int_{-h/2}^{h/2} \tau_{\theta z}(1+z/r_1)dz \qquad 2.2(f)$$

$$M_\phi = \int_{-h/2}^{h/2} z\sigma_\phi(1+z/r_2)dz \qquad 2.2(g)$$

$$M_\theta = \int_{-h/2}^{h/2} z\sigma_\theta(1+z/r_1)dz \qquad 2.2(h)$$

$$M_{\phi\theta} = \int_{-h/2}^{h/2} z\tau_{\phi\theta}(1+z/r_2)dz \qquad 2.2(i)$$

$$M_{\theta\phi} = \int_{-h/2}^{h/2} z\tau_{\theta\phi}(1+z/r_1)dz \qquad 2.2(j)$$

2.2 The Fundamental Assumptions

The fundamental equations of the general theory of shells of revolution first presented by Flügge (1) are based on the following set of assumptions:

1. Thin shell - the shell thickness is small in comparison to the other dimensions of the shell. Thus, the stresses on the z-face, and the twisting moments about the z-axis may be neglected.
2. Small deflection theory applies. The displacements of the shell due to the applied loads are sufficiently small that the equilibrium equations developed from the initial shell geometry do not change.
3. Material is linearly elastic, i.e., Hooke's law applies.
4. Plane sections remain plane after bending. i.e., the normals to the middle surface before bending remain normal after bending.
5. Deformations of the shell due to radial shears can be neglected.

Now, based on these set of assumptions and the shell geometry, the general theory of shells of revolution may be formulated by:

1. Determining the equilibrium of forces acting on the differential element shown in Fig. 2.2; (six equations with ten unknowns)
2. Establishing the strain-displacement relationships; (six equations with six unknowns)
3. Establishing the stress-strain relationships from

Hooke's Law; (three equations with six unknowns)

4. Transforming the stress-resultant equations into the force-displacement equations; (six equations with three unknowns)

5. Obtaining a complete formulation by combining the force-displacement equations with the equilibrium equations. (thirteen equations with thirteen unknowns)

2.3 Equations of Equilibrium

Consider the differential element shown in Fig. 2.2. From the summation of forces in each of the coordinate directions and moments about each of the coordinate axes, ϕ, θ, and z, the six equations of equilibrium are:

$$(r_0 N_\phi)^{\cdot} + r_1 (N_{\theta\phi})' - r_1 N_\theta \cos\phi - r_0 Q_\phi + r_0 r_1 p_\phi = 0 \qquad 2.3(a)$$

$$(r_0 N_{\phi\theta})^{\cdot} + r_1 (N_\theta)' + r_1 N_{\theta\phi} \cos\phi - r_1 Q_\theta \sin\phi + r_0 r_1 p_\theta = 0 \qquad 2.3(b)$$

$$r_1 N_\theta \sin\phi + r_0 N_\phi + r_1 (Q_\theta)' + (r_0 Q_\phi)^{\cdot} - r_0 r_1 p_z = 0 \qquad 2.3(c)$$

$$(r_0 M_\phi)^{\cdot} + r_1 (M_{\theta\phi})' - r_1 M_\theta \cos\phi - r_0 r_1 Q_\phi = 0 \qquad 2.3(d)$$

$$(r_0 M_{\phi\theta})^{\cdot} + r_1 (M_\theta)' + r_1 M_{\theta\phi} \cos\phi - r_0 r_1 Q_\theta = 0 \qquad 2.3(e)$$

$$\frac{M_{\phi\theta} - M_{\theta\phi}}{r_1} = \frac{N_{\phi\theta} - N_{\theta\phi}}{r_2} \qquad 2.3(f)$$

where

$$\frac{\partial(\ )}{\partial\phi} = (\ )^{\cdot}$$

$$\frac{\partial(\ )}{\partial\theta} = (\ )'$$

N_ϕ, N_θ = meridional and circumferential forces respectively;

$N_{\phi\theta}$, $N_{\theta\phi}$ = meridional and circumferential shear forces;

Q_ϕ, Q_θ = transverse shear forces;

M_ϕ, M_θ = meridional and circumferential moments, respectively;

$M_{\theta\phi}$, $M_{\phi\theta}$ = meridional and circumferential twisting moments, respectively.

Note that all forces and moments are expressed in units of force per unit length. The sign convention used is as shown in Fig. 2.2, where N_ϕ and N_θ are positive for tension along the meridian and circumference, respectively. M_ϕ and M_θ are positive when the outer shell surface is in compression.

2.4 Force-Displacement Equations

The deformation of a shell element consists of the change in length of the shell edges, $r_1 d\phi$ and $r_0 d\theta$, and of the change of the angle between these edges. In reference to Fig. 2.3, the midsurface strain-displacement relationships for a shell element are:

Meridional strain, $$\epsilon_\phi = \frac{1}{r_1}(v^{\cdot} - w) \qquad 2.4(a)$$

Hoop strain, $$\epsilon_\theta = \frac{1}{r_0}(u' + v\cos\phi - w\sin\phi) \qquad 2.4(b)$$

Shear strain, $$\gamma_{\phi\theta} = \frac{v'}{r_0} + \frac{u^{\cdot}}{r_1} - \frac{u}{r_0}\cos\phi \qquad 2.4(c)$$

where

u = midsurface displacement component in the circumferential direction, positive in the direction of increasing θ.

v = midsurface displacement component in the meridional direction, positive in the direction of increasing ϕ.

w = midsurface displacement component in the radial direction, positive in the direction away from the centre of curvature.

Consider a point i at a distance z to the midsurface, i.e., $(r_1)_i = r_1 + z$, and $(r_2)_i = r_2 + z$. From Eqn. 2.1(a), the strains at point i are:

$$(\epsilon_\phi)_i = \frac{(v_i' - w_i)}{(r_1 + z)} \tag{2.5(a)}$$

$$(\epsilon_\theta)_i = \frac{(u'_i + v_i\cos\phi - w_i\sin\phi)}{(r_2 + z)\sin\phi} \tag{2.5(b)}$$

$$(\gamma_{\phi\theta})_i = \frac{v_i'}{(r_2 + z)\sin\phi} + \frac{u_i'}{r_1 + z} - \frac{u_i\cos\phi}{(r_2 + z)\sin\phi} \tag{2.5(c)}$$

where

$$w_i = w \tag{2.6(a)}$$

$$v_i = \frac{v}{r_1}(r_1 + z) - \frac{z\,w'}{r_1} \tag{2.6(b)}$$

$$u_i = \frac{u}{r_2}(r_2 + z) - \frac{z\,w'}{r_2} \tag{2.6(c)}$$

Hooke's law forms the basis for the formulation of the stress-strain equations.

$$\sigma_\phi = \frac{E}{(1-\nu^2)}(\epsilon_\phi + \nu\epsilon_\theta) \tag{2.7(a)}$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}(\epsilon_\theta + \nu\epsilon_\phi) \tag{2.7(b)}$$

$$\tau_{\theta\phi} = \frac{E}{2(1+\nu)}\gamma_{\theta\phi} \tag{2.7(c)}$$

where E is the modulus of elasticity and ν is Poisson's ratio. Combining the strain-displacement relationships (Eqns. 2.5 and 2.6) and substituting these into the stress-strain equations (Eqns. 2.7), and finally, substituting these into the stress-resultant equations

(Eqns. 2.2) and integrating through the shell thickness, the force-displacement relationships are as follows:

$$N_\phi = K\left[\frac{\dot{v}+w}{r_1} + \frac{\nu(u'+v\cos\phi+w\sin\phi)}{r_0}\right]$$

$$+ \frac{D}{r_1^2}\,\frac{r_2-r_1}{r_2}\left[\frac{v-\dot{w}}{r_1}\,\frac{\dot{r}_1}{r_1} + \frac{\ddot{w}+w}{r_1}\right] \qquad 2.8(a)$$

$$N_\theta = K\left[\frac{u'+v\cos\phi+w\sin\phi}{r_0} + \frac{\nu(\dot{v}+w)}{r_1}\right]$$

$$- \frac{D}{r_0 r_1}\,\frac{r_2-r_1}{r_2}\left[-\frac{v}{r_1}\,\frac{r_2-r_1}{r_2}\cos\phi + \frac{w\sin\phi}{r_2} + \frac{w''}{r_0} + \frac{\dot{w}\cos\phi}{r_1}\right] \qquad 2.8(b)$$

$$N_{\phi\theta} = \frac{K(1-\nu)}{2}\left[\frac{\dot{u}}{r_1} + \frac{v'-u\cos\phi}{r_0}\right] + \frac{D}{r_1^2}\,\frac{1-\nu}{2}\,\frac{r_2-r_1}{r_2}\left[\frac{\dot{u}}{r_1}\,\frac{r_2-r_1}{r_2}\right.$$

$$\left. + \frac{u}{r_2}\,\frac{r_1-r_2}{r_2}\cot\phi + \frac{\dot{w}'}{r_0} - \frac{w'}{r_0}\,\frac{r_1\cos\phi}{r_0}\right] \qquad 2.8(c)$$

$$N_{\theta\phi} = \frac{K(1-\nu)}{2}\left[\frac{\dot{u}}{r_1} + \frac{v'-u\cos\phi}{r_0}\right]$$

$$+ \frac{D}{r_0 r_1}\,\frac{1-\nu}{2}\,\frac{r_2-r_1}{r_2}\left[\frac{v'}{r_1}\,\frac{r_2-r_1}{r_2} - \frac{\dot{w}'}{r_1} + \frac{w'\cos\phi}{r_0}\right] \qquad 2.8(d)$$

$$M_\phi = D\left[\frac{1}{r_1^2}\left(\ddot{w} - w\,\frac{\dot{r}_1}{r_1} - w\left(\frac{r_1-r_2}{r_2}\right)\right) - \frac{\dot{v}}{r_1 r_2} + \frac{v}{r_1^2}\,\frac{\dot{r}_1}{r_1}\right.$$

$$\left. + \frac{\nu w''}{r_0^2} + \frac{\nu\dot{w}\cos\phi}{r_0 r_1} - \frac{\nu u'}{r_0 r_1} - \frac{\nu v\cos\phi}{r_0 r_1}\right] \qquad 2.8(e)$$

$$M_\theta = D\left[\frac{w''}{r_0^2} + \frac{\dot{w}\cos\phi}{r_0 r_1} - \frac{w}{r_2^2}\,\frac{r_2-r_1}{r_1} - \frac{u'}{r_0 r_1} - \frac{v\cos\phi}{r_0 r_1}\,\frac{2r_2-r_1}{r_2}\right.$$

$$\left. + \frac{\nu}{r_1^2}\left(\ddot{w} - \dot{w}\,\frac{\dot{r}_1}{r_1}\right) - \frac{\nu\dot{v}}{r_1^2} + \frac{\nu v\dot{r}_1}{r_1^3}\right] \qquad 2.8(f)$$

$$M_{\phi\theta} = \frac{D(1-\nu)}{2}\left[\frac{2\dot{w}'}{r_0 r_1} - \frac{2w'\cos\phi}{r_2} - \frac{\dot{u}}{r_1 r_2}\,\frac{2r_1 - r_2}{r_1}\right.$$

$$\left. + \frac{u}{r_2^2}\,\frac{(2r_1 - r_2)\cot\phi}{r_1} - \frac{\dot{v}'}{r_0 r_1}\right] \qquad 2.8(g)$$

$$M_{\theta\phi} = \frac{D(1-\nu)}{2}\left[\frac{2\dot{w}'}{r_0 r_1} - \frac{2w'\cos\phi}{r_0^2} - \frac{\dot{u}}{r_1 r_2}\right.$$

$$\left. + \frac{u}{r_2^2}\cot\phi - \frac{\dot{v}'}{r_0 r_1}\,\frac{(2r_2 - r_1)}{r_2}\right] \qquad 2.8(h)$$

Where the extensional rigidity K and the flexural rigidity D, are defined respectively as

$$K = \frac{Eh}{(1-\nu^2)}$$

$$D = \frac{Eh^3}{(1-\nu^2)}$$

There are now fourteen equations (Eqns. 2.3 and 2.8) with thirteen unknowns, N_ϕ, N_θ, $N_{\theta\phi}$, $N_{\phi\theta}$, M_ϕ, M_θ, $M_{\theta\phi}$, $M_{\phi\theta}$, Q_ϕ, Q_θ, u, v, w. Note that there is one equation too many. Since both sides of Eqn. 2.3(f) are small differences between small quantities which are almost equal, this equation may be discarded. Thus, there is now a balance of unknowns and equations. The classical method of solution would be to reduce these differential equations into a single eighth order equation in terms of one variable. This procedure tends to be too complicated and cumbersome to solve. Therefore, alternative solutions to these equations based on the standard methods of structural analysis will be presented in the following chapter.

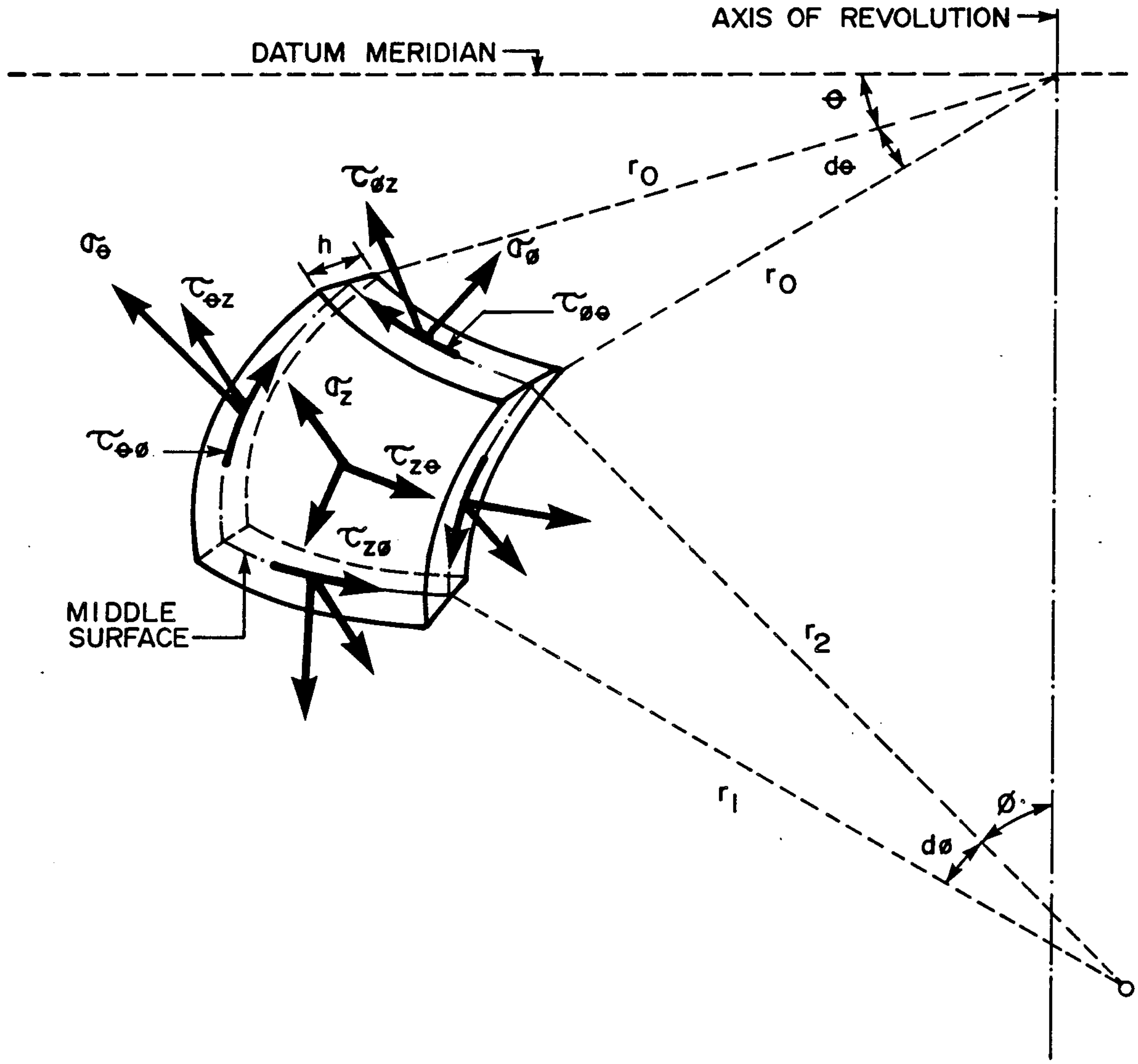


Figure 2.1 GEOMETRY OF SHELL

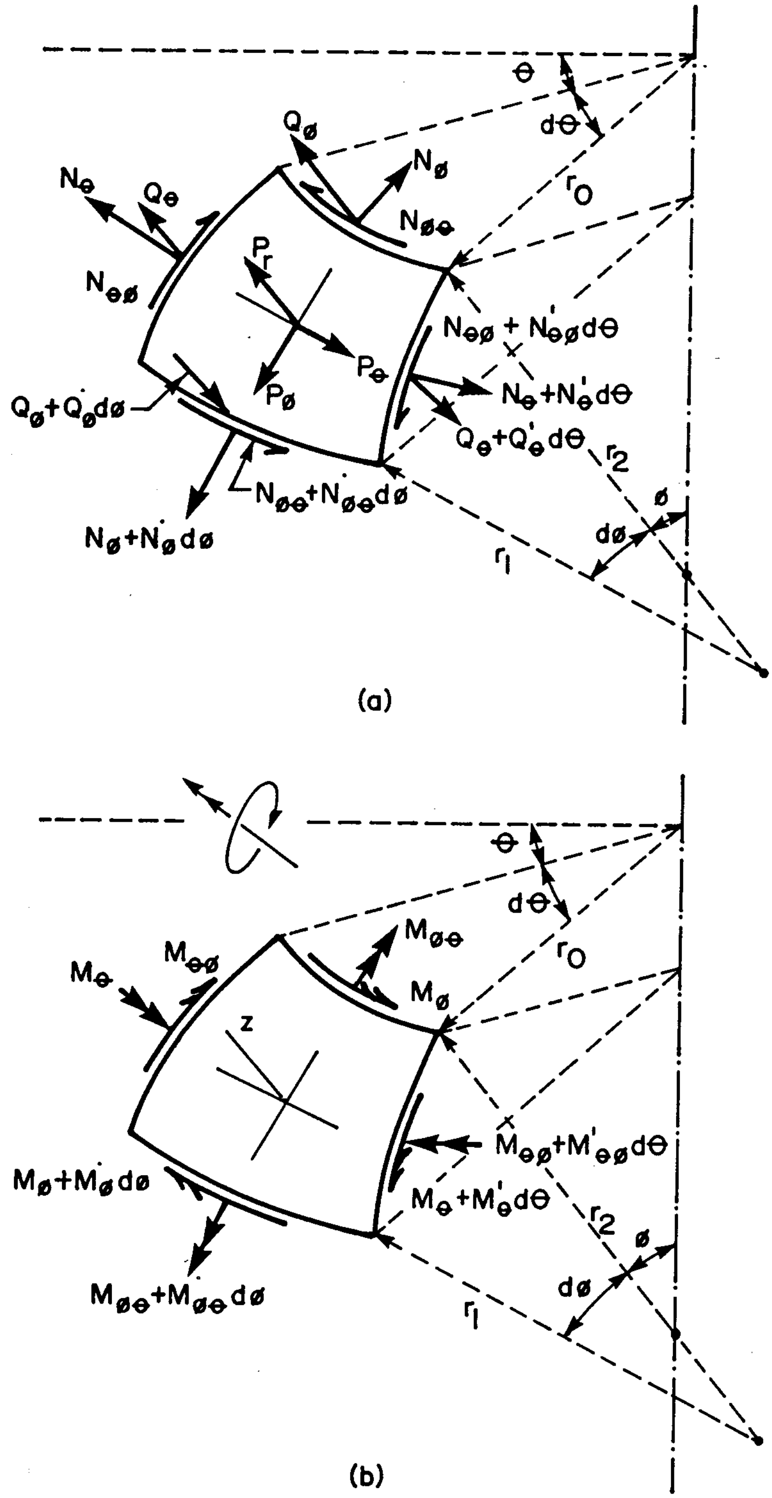


Figure 2.2 FORCES ACTING ON SHELL MIDSURFACE

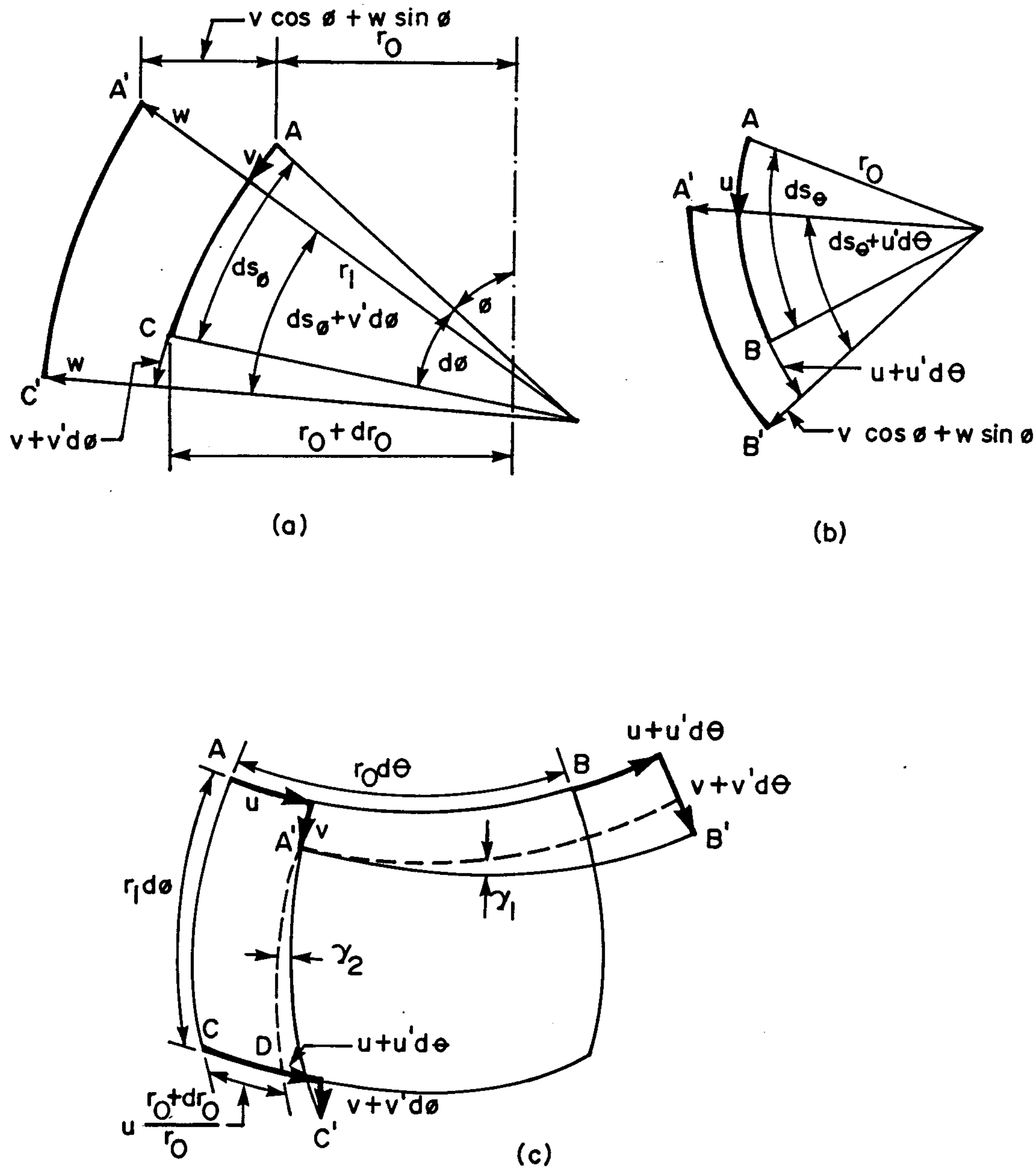


Figure 2.3 SHELL SEGMENTS BEFORE & AFTER DEFORMATION
(a) meridian
(b) parallel circle
(c) angle change

3. METHODS OF ANALYSIS

Structures that geometrically consist of several segments of shells of revolution can be analyzed by either of the two standard methods of structural analysis, namely: the stiffness method and the flexibility method. With the stiffness method, the stiffness matrix relating the forces and deformations at the edge of each shell segment are computed using the procedures outlined in Section 3.1. These segment stiffness matrices are then superposed to form the global stiffness matrix from which the segment edge deformations are computed. With the flexibility approach (Section 3.2), the flexibility influence coefficients for each shell segment are obtained. Equations of geometric compatibility at the segment boundaries are written to obtain the forces at segment junctions.

3.1 The Stiffness Approach

Program SASHELL analyzes a segmented shell structure based on the stiffness method. To establish the stiffness matrix and the corresponding fixed end forces vector, the basic shell equations must be transformed to a set of eight first order differential equations, corresponding to the eight natural boundary conditions of the shell segment. This is accomplished by the following steps:

1. Eliminate ϕ by introducing the coordinate s along the meridian, i.e., $\partial s = r_1 \partial \phi$
2. Expand the governing equations using a Fourier series to

eliminate the necessity of forming the equilibrium equations in the circumferential direction (Section 3.1.1).

3. Introduce auxilliary equations to transform the variables from shell theory to the eight selected degrees of freedom for the stiffness matrix and the corresponding stress resultants (Section 3.1.2).
4. Perform matrix operations to eliminate the forces in the circumferential direction (Section 3.1.3).
5. Numerically integrate the governing differential equations formed in step 4 (Section 3.1.4).
6. Form the stiffness matrix and the corresponding fixed end forces vector (Section 3.1.5).
7. Apply the consistent sign conventions to the matrices formed in step 6 (Section 3.1.6).

Introducing the s coordinate, which measures the distance along the shell meridian, the five independent equations of equilibrium (Eqns. 2.3) become

$$r_1(r_0N_s)^\circ + r_1N_{\theta s}' - r_1N_\theta\cos\phi - r_0Q_s + r_0r_1p_s = 0 \qquad 3.1(a)$$

$$r_1(r_0N_{s\theta})^\circ + r_1N_\theta' + r_1N_{\theta s}\cos\phi - r_1Q_\theta\sin\phi + r_0r_1p_\theta = 0 \qquad 3.1(b)$$

$$r_1N_\theta\sin\phi + r_0N_s + r_1Q_\theta' + r_1(r_0Q_s)^\circ - r_0r_1p_z = 0 \qquad 3.1(c)$$

$$r_1(r_0M_s)^\circ + r_1M_{\theta s}' - r_1M_\theta\cos\phi - r_0r_1Q_s = 0 \qquad 3.1(d)$$

$$r_1(r_0M_{s\theta})^\circ + r_1M_\theta' + r_1M_{\theta s}\cos\phi - r_0r_1Q_\theta = 0 \qquad 3.1(e)$$

where

$$\frac{1}{r_1}\frac{\partial(\ )}{\partial\phi} = \frac{\partial(\ )}{\partial s} = (\ )^\circ \qquad 3.1(f)$$

The force-displacement equations (Eqns. 2.8) in terms of the s-coordinate are

$$N_s = K\left[v^\circ+\frac{w}{r_1} + \frac{\nu(u'+v\ \cos\phi+w\sin\phi)}{r_o}\right]$$

$$+ \frac{D}{r_1^2}\ \frac{r_2-r_1}{r_2}\left[v-\frac{w^\circ}{r_1}\ r_1{}^\circ+\ w^\circ+\frac{w}{r_1}\right] \qquad 3.2(a)$$

$$N_\theta = K\left[\frac{u'+v\cos\phi+w\sin\phi}{r_o} + \nu v^\circ+\frac{w}{r_1}\right]$$

$$- \frac{D}{r_o r_1}\ \frac{r_2-r_1}{r_2}\left[-\frac{v}{r_1}\ \frac{r_2-r_1}{r_2}\cos\phi + \frac{w\sin\phi}{r_2}+ \frac{w''}{r_o} + w^\circ\cos\phi\right] \qquad 3.2(b)$$

$$N_{s\theta} = \frac{K(1-\nu)}{2}\left[u^\circ+ \frac{v'-u\cos\phi}{r_o}\right] + \frac{D}{r_1^2}\ \frac{1-\nu}{2}\ \frac{r_2-r_1}{r_2}\left[u^\circ\frac{r_2-r_1}{r_2}\right.$$

$$\left. + \frac{u}{r_2}\ \frac{r_1-r_2}{r_2}\cot\phi+\frac{r_1 w'^\circ}{r_o}- \frac{w'}{r_o}\ \frac{r_1}{r_o}\cos\phi\right] \qquad 3.2(c)$$

$$N_{\theta s} = \frac{K(1-\nu)}{2}\left[u^\circ+ \frac{v'-u\cos\phi}{r_o}\right]$$

$$+ \frac{D}{r_o r_1}\ \frac{1-\nu}{2}\ \frac{r_2-r_1}{r_2}\left[\frac{v'}{r_1}\ \frac{r_2-r_1}{r_2}\ -w'^\circ + \frac{w'\cos\phi}{r_o}\right] \qquad 3.2(d)$$

$$M_s = D\left[w^{\circ\circ} - w^\circ\frac{r_1{}^\circ}{r_1}- w\frac{(r_1-r_2)}{r_2 r_1^2} - \frac{v^\circ}{r_2}+ \frac{v}{r_1^2}\ r_1\right.$$

$$\left. + \frac{\nu w''}{r_o^2} + \frac{\nu w^\circ\cos\phi}{r_o} - \frac{\nu u'}{r_o r_1} - \frac{\nu v\cos\phi}{r_o r_1}\right] \qquad 3.2(e)$$

$$M_\theta = D\left[\frac{w''}{r_o^2} + \frac{w^\circ\cos\phi}{r_o} - \frac{w}{r_2^2}\ \frac{r_2-r_1}{r_1} - \frac{u'}{r_o r_1} - \frac{v\cos\phi}{r_o r_1}\ \frac{2r_2-r_1}{r_2}\right.$$

$$\left. + \nu w^{\circ\circ} - \nu w^\circ\frac{r_1{}^\circ}{r_1} - \frac{\nu v^\circ}{r_1} + \frac{\nu v r_1{}^\circ}{r_1^2}\right] \qquad 3.2(f)$$

$$M_{s\theta} = \frac{D(1-\nu)}{2}\left[\frac{2w'^\circ}{r_o} - \frac{2w'}{r_2}\cos\phi - \frac{u^\circ}{r_2}\ \frac{2r_1-r_2}{r_1}\right.$$

$$+ \frac{u}{r_2^2}\,\frac{(2r_1 - r_2)\cot\phi}{r_1} - \left.\frac{v'}{r_0 r_1}\right] \qquad 3.2(g)$$

$$M_{\theta s} = \frac{D(1-\nu)}{2}\left[\frac{2w'^{\circ}}{r_0} - \frac{2w'\cos\phi}{r_0^2} - \frac{u^{\circ}}{r_2}\right.$$

$$+ \left.\frac{u}{r_2^2}\cot\phi - \frac{v'}{r_0 r_1}\,\frac{(2r_2 - r_1)}{r_2}\right] \qquad 3.2(h)$$

3.1.1 Fourier Series

For any variable, say F(x,y) being arbitrary functions of x and y, may be represented in the form

$$F = \sum_{n=0}^{\infty} F_n(x)\cos(ny) + \sum_{n=1}^{\infty} F_n(x)\sin(ny) \qquad 3.3$$

where n is the harmonic number and variable F_n is a function of x only. Similarly, the load components p_s, p_θ, and p_z, and forces N_s, N_θ, $N_{s\theta}$, $N_{\theta s}$, M_s, M_θ, $M_{\theta s}$, $M_{s\theta}$, Q_s, and Q_θ, and displacement components u, v, and w, may be expressed as a Fourier series, where the variable components become a function of s only. The first and second series in each expression represent the portions of the variables which are respectively symmetric and anti-symmetric with respect to the meridian passing through the line $\theta = 0$.

$$p_s = \sum_{n=0}^{\infty} p_{sn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} p_{sn}(s)\sin(n\theta) \qquad 3.4(a)$$

$$p_\theta = \sum_{n=0}^{\infty} p_{\theta n}(s)\cos(n\theta) + \sum_{n=1}^{\infty} p_{\theta n}(s)\sin(n\theta) \qquad 3.4(b)$$

$$p_z = \sum_{n=0}^{\infty} p_{zn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} p_{zn}(s)\sin(n\theta) \qquad 3.4(c)$$

$$N_s = \sum_{n=0}^{\infty} N_{sn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} N_{sn}(s)\sin(n\theta) \qquad 3.4(e)$$

$$N_\theta = \sum_{n=0}^{\infty} N_{\theta n}(s)\cos(n\theta) + \sum_{n=1}^{\infty} N_{\theta n}(s)\sin(n\theta) \qquad 3.4(f)$$

$$N_{s\theta} = \sum_{n=0}^{\infty} N_{s\theta n}(s)\cos(n\theta) + \sum_{n=1}^{\infty} N_{s\theta n}(s)\sin(n\theta) \qquad 3.4(g)$$

$$N_{\theta s} = \sum_{n=0}^{\infty} N_{\theta sn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} N_{\theta sn}(s)\sin(n\theta) \qquad 3.4(h)$$

$$Q_s = \sum_{n=0}^{\infty} Q_{sn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} Q_{sn}(s)\sin(n\theta) \qquad 3.4(i)$$

$$Q_\theta = \sum_{n=0}^{\infty} Q_{\theta n}(s)\cos(n\theta) + \sum_{n=1}^{\infty} Q_{\theta n}(s)\sin(n\theta) \qquad 3.4(j)$$

$$M_s = \sum_{n=0}^{\infty} M_{sn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} M_{sn}(s)\sin(n\theta) \qquad 3.4(k)$$

$$M_\theta = \sum_{n=0}^{\infty} M_{\theta n}(s)\cos(n\theta) + \sum_{n=1}^{\infty} M_{\theta n}(s)\sin(n\theta) \qquad 3.4(l)$$

$$M_{s\theta} = \sum_{n=0}^{\infty} M_{s\theta n}(s)\cos(n\theta) + \sum_{n=1}^{\infty} M_{s\theta n}(s)\sin(n\theta) \qquad 3.4(m)$$

$$M_{\theta s} = \sum_{n=0}^{\infty} M_{\theta sn}(s)\cos(n\theta) + \sum_{n=1}^{\infty} M_{\theta sn}(s)\sin(n\theta) \qquad 3.4(n)$$

$$u = \sum_{n=0}^{\infty} u_n(s)\cos(n\theta) + \sum_{n=1}^{\infty} u_n(s)\sin(n\theta) \qquad 3.4(o)$$

$$v = \sum_{n=0}^{\infty} v_n(s)\cos(n\theta) + \sum_{n=1}^{\infty} v_n(s)\sin(n\theta) \qquad 3.4(p)$$

$$w = \sum_{n=0}^{\infty} w_n(s)\cos(n\theta) + \sum_{n=1}^{\infty} w_n(s)\sin(n\theta) \qquad 3.4(q)$$

For an arbitrary applied load expressed as a Fourier series of order N, there are 2N+1 terms that represent each component of the load; (n = 0,1,2,...,N) for the symmetric series and (n = 1,2,3,...,N) for the anti-symmetric series. For each value of n, the s-dependent variables with subscript n (Eqn. 3.4) can be substituted into the basic shell equations (Eqns. 3.1 and 3.2), because the sequences sin(nθ) and cos(nθ) are linearly independent. Differentiations with respect to θ can be performed and the terms grouped according to the common factors, cos(nθ) and sin(nθ). Since the coefficient of each of these factors must be zero, each factor produces a separate equation. For example, for any n, the cosine terms in Eqn. 3.1(a) become

$$r_o N^{\circ}_{sn}\cos(n\theta) + \cos\phi N_{sn}\cos(n\theta) + nN_{\theta sn}\cos(n\theta) - \cos\phi N_{\theta n}\cos(n\theta) - \frac{r_o}{r_1}Q_{sn}\cos(n\theta) + R_o p_{sn}\cos(n\theta) = 0 \qquad 3.5$$

which, upon factoring out the common term, yields

$$r_o N^{\circ}_{sn} + \cos\phi N_{sn} + nN_{\theta sn} - \cos\phi N_{\theta n} - \frac{r_o}{r_1}Q_{sn} + r_o p_{sn} = 0 \qquad 3.6$$

Similarly, for the sine terms, Eqn. 3.1(a) become

$$r_o N^{\circ}_{sn} + \cos\phi N_{sn} - nN_{\theta sn} - \cos\phi N_{\theta n} - \frac{r_o}{r_1}Q_{sn} + r_o p_{sn} = 0 \qquad 3.7$$

Let R_o, R_1, R_2 be defined as shell curvature, i.e.

$$R_o = \frac{1}{r_o}$$

$$R_1 = \frac{1}{r_1}$$

$$R_2 = \frac{1}{r_2}$$

Thus, for the nth set of equations, the five independent equilibrium equations derived from Eqns. 3.1 become

$$N_{sn}^{\circ} + R_0\cos\phi N_{sn} \pm nR_0N_{\theta sn} - R_0\cos\phi N_{\theta n} - R_1Q_{sn} + p_{sn} = 0 \quad 3.8(a)$$

$$N_{s\theta n}^{\circ} + R_0\cos\phi N_{s\theta n} \mp nR_0N_{\theta n} + R_0\cos\phi N_{\theta sn} - R_2Q_{\theta n} + p_{\theta n} = 0 \quad 3.8(b)$$

$$R_2N_{\theta n} + R_1N_{sn} \pm nR_0Q_{\theta n} + Q_{sn}^{\circ} + R_0\cos\phi Q_{sn} - p_{zn} = 0 \quad 3.8(c)$$

$$M_{sn}^{\circ} + R_0\cos\phi M_{sn} \pm nR_0M_{\theta sn} - R_0\cos\phi M_{\theta n} - Q_{sn} = 0 \quad 3.8(d)$$

$$M_{s\theta n}^{\circ} + R_0\cos\phi M_{s\theta n} \mp nR_0M_{\theta n} + R_0\cos\phi M_{\theta sn} - Q_{\theta n} = 0 \quad 3.8(e)$$

and the eight force-displacement equations obtained from Eqn. 3.2 become

$$N_{sn} = [DR_1(R_1-R_2)r_1^{\circ}]\beta_n - [D(R_1-R_2)]\beta_n^{\circ} + [K(R_1+\nu R_2) + DR_1^2(R_1-R_2)]w_n + [\nu KR_0\cos\phi - DR_1^2(R_1-R_2)r_1^{\circ}]v_n + [K + DR_1(R_1-R_2)]v_n^{\circ} \pm [\nu nKR_0]u_n \quad 3.9(a)$$

$$N_{\theta n} = [DR_0(R_1-R_2)\cos\phi]\beta_n + [K(R_2+\nu R_1) + D(R_1-R_2)(R_0^2n^2-R_2^2)]w_n + [KR_0\cos\phi - DR_0R_2\cos\phi(R_1-R_2)]v_n + [\nu K]v_n^{\circ} \pm [nKR_0]u_n \quad 3.9(b)$$

$$N_{s\theta n} = 0.5(1-\nu)\{\pm[nDR_0(R_1-R_2)]\beta_n \pm [nDR_0^2\cos\phi(R_1-R_2)]w_n \mp [nKR_0 + nDR_0R_1(R_1-R_2)]v_n - [KR_0\cos\phi - DR_0\cos\phi(R_1-R_2)^2]u_n + [K + D(R_1-R_2)^2]u_n^{\circ}\} \quad 3.9(c)$$

$$N_{\theta sn} = 0.5(1-\nu)\{\pm\{nDR_0(R_1-R_2)]\beta_n \mp [nDR_0^2\cos(\&v>(R_1-R_2)]w_n \mp [nKR_0 + nDR_0R_1(R_1-R_2)]v_n - [KR_0\cos\phi]u_n + [K]u_n^{\circ} \quad 3.9(d)$$

$$M_{sn} = [DR_1r_1^{\circ} - \nu DR_0\cos\phi]\beta_n - [D]\beta_n^{\circ} + [DR_1(R_1-R_2) - \nu DR_0^2n^2]w_n - [DR_1^2r_1^{\circ}]v_n + [D(R_1-R_2)]v_n^{\circ} \mp [\nu nDR_0R_2]u_n \quad 3.9(e)$$

$$M_{\theta n} = [\nu DR_1r_1^{\circ} - DR_0\cos\phi]\beta_n - [\nu D]\beta_n^{\circ} + [Dn^2R_0^2 + DR_2(R_1-R_2)]w_n - [\nu DR_1r_1^{\circ} + DR_0\cos\phi(R_1-R_2)]v_n \mp [nDR_0R_2]u_n \quad 3.9(f)$$

$$M_{s\theta n} = 0.5(1-\nu)\{\pm[2nDR_0]\beta_n \pm [2nDR_0^2\cos\phi]w_n \mp [nDR_0R_1]v_n -$$

$[DR_0\cos\phi(R_1-2R_2)]u_n + [D(R_1-2R_2)]u_n^\circ\}$ 3.9(g)

$M_{\theta s n} = 0.5(1-\nu)\{\pm[2nDR_0]\beta_n \pm [2nDR_0^2\cos\phi]w_n \mp [nDR_0R_1]v_n +$

$[DR_0R_2\cos\phi]u_n - [DR_2]u_n^\circ\}$ 3.9(h)

where β_n and β_n° is an auxilliary varible which will be defined in the following section. Note that there are two sets of equations, grouped according to the cosine and sine terms. For expressions with two signs, the upper and lower signs correspond to the cosine and sine coefficients respectively. The final solution is obtained by solving each set separately and superimposing the two solutions.

3.1.2 Auxilliary Equations

The quantities in the natural boundary conditions on the edges of a shell segment are the four displacement components, the rotation of the meridian (β), the radial displacement (w), the meridional displacement (v), and the circumferential displacement (u), and the four corresponding forces, the meridional moment (M_s), the effective transverse shear force (S_s), the normal in-plane meridional force (N_s), and the effective tangential shear force (T_s). Three of these variables β, T_s, and S_s do not appear in the basic shell equations. They may be introduced by setting up the so-called auxilliary equations which express these variables in terms of in-plane shear forces in the circumferential direction and the meridional twisting moment.

Consider the side view of the top edge of the shell segment shown on Fig. 3.1 with two adjacent elements of length ds = $r_0 d\theta$. (Note that ds = $r_2 d\theta$ for very small ds) The moments acting on the infinitesimal element ds can be replaced by a set of statically equivalent forces F_n and F_t (5), such that

$$F_n = M_{s\theta}$$

$$F_t = F_n d\theta$$

From Fig. 3.1, superimposing these forces with the transverse force Q_s, and the in-plane shear force, $N_{s\theta}$, respectively, yields and expression for the Kirchhoff shears, S_s and T_s

$$S_s = Q_s + R_0 M_{s\theta}^{\prime}$$

$$T_s = N_{s\theta} - R_2 M_{s\theta}$$

Expanding these into a Fourier series yield

$$S_{sn} = Q_{sn} \pm nR_0 M_{s\theta n} \qquad 3.10$$

$$T_{sn} = N_{s\theta n} - R_2 M_{s\theta n} \qquad 3.11$$

Using the geometrical relations in Eqns. 2.1 and 3.1(f), the derivatives of these forces with respect to the coordinate s may be written as

$$S_{sn}^{\circ} = Q_{sn}^{\circ} \pm nR_0 M_{s\theta n}^{\circ} \mp nR_0^{2}\cos\phi M_{s\theta n} \qquad 3.12$$

$$T_{sn}^{\circ} = N_{s\theta n}^{\circ} - R_2 M_{s\theta n}^{\circ} - R_2(R_1 - R_2)\cot\phi M_{s\theta n} \qquad 3.13$$

Also, by superimposing Figs. 3.2(a) and (b), the angle by which an element of the meridian rotates during deformation may be expressed in terms of the displacement components as follows,

$$\beta_n = -w^{\circ} + R_1 v \qquad 3.14$$

3.1.3 Reduction of the Shell Equations

Rewriting Eqn. 3.10 to form an expression for Q_{sn}, and substituting this into Eqns. 3.8(a) and (d) respectively, yields

$$N_{sn}^{\circ} = R_1 S_{sn} - R_0\cos\phi N_{sn} \mp nR_0R_1M_{s\theta n} + R_0\cos\phi N_{\theta n} \mp nR_0N_{\theta sn} - p_{sn} \qquad 3.15$$

$$M_{sn}^{\circ} = S_{sn} - R_0\cos\phi M_{sn} + R_0\cos\phi M_{\theta n} \mp nR_0(M_{\theta sn}+M_{s\theta n}) \qquad 3.16$$

Rewriting Eqns. 3.11, 3.13, and 3.8(e) to form expressions for $N_{s\theta n}$, $N_{s\theta n}^{\circ}$, and $Q_{\theta n}$, yields

$$N_{s\theta n} = T_{sn} + R_2M_{s\theta n}$$

$$N_{s\theta n}^{\circ} = T_{sn}^{\circ} + R_2M_{s\theta n}^{\circ} + R_2(R_1-R_2)\cot\phi M_{s\theta n}$$

$$Q_{\theta n} = M_{s\theta n} + R_0\cos\phi M_{s\theta n} \mp nR_0M_{\theta n} + R_0\cos\phi M_{\theta sn}$$

Substituting the above expressions into Eqn. 3.8(b), and using the relation, $R_0\sin\phi = R_2$, yields

$$T_{sn}^{\circ} = -R_0\cos\phi(R_1-R_2)M_{s\theta n} - R_0\cos\phi T_{sn} \pm nR_0N_{\theta n} - R_0\cos\phi N_{\theta sn} \mp nR_0R_2M_{\theta n} + R_0R_2\cos\phi M_{\theta sn} - p_{\theta n} \qquad 3.17$$

Finally, rewriting Eqn. 3.12 to form an expression for Q_{sn}°, and substituting this, in addition to the expressions for $Q_{\theta n}$ and Q_{sn} derived earlier, into Eqn. 3.8(a), gives

$$S_{sn}^{\circ} = -R_2N_{\theta n} - R_1N_{sn} + n^2R_0^2M_{\theta n} \mp nR_0^2\cos\phi(M_{\theta sn}+M_{s\theta n}) - R_0\cos\phi S_{sn} + p_{zn} \qquad 3.18$$

Eqns. 3.15 to 3.18 may be written in matrix form as,

$$\{F_s^{\circ}\} = [B_1 \;\; B_2]\begin{Bmatrix} F_s \\ F_\theta \end{Bmatrix} + \{B_3\} \qquad 3.19$$

where

$$\langle F_s\rangle = \langle M_{s\theta n} \;\; S_{sn} \;\; N_{sn} \;\; T_{sn}\rangle$$

$$\langle F_s^{\circ}\rangle = \langle M_{s\theta n}^{\circ} \;\; S_{sn}^{\circ} \;\; N_{sn}^{\circ} \;\; T_{sn}^{\circ}\rangle$$

$$\langle F_\theta\rangle = \langle M_{\theta n} \;\; M_{\theta sn} \;\; M_{s\theta n} \;\; N_{\theta n} \;\; N_{\theta sn}\rangle$$

and the coefficients of $[B_1\ B_2]$ is a function of the geometric and material properties of the shell; and $\{B_3\}$ is the load vector. These matrices are defined in Table 3.1. The plus and minus signs relate to the two sets of equations, grouped according to the cosine and sine terms in the Fourier series expansion.

To form expressions for the displacement variables, manipulate the force-displacement equations as follows

Let

$$CA_1 = K + DR_1(R_1-R_2) \qquad 3.20(a)$$

$$CA_2 = K + DR_2(R_1-R_2) \qquad 3.20(b)$$

Multiply Eqn. 3.9(a) by (R_1-R_2),

$$N_{sn}(R_1-R_2) = [DR_1(R_1-R_2)^2 r_o{}^2]\beta_n - [D(R_1-R_2)^2]\beta_n{}^\circ + [K(R_1+\nu R_2) + DR_1^2(R_1-R_2)](R_1-R_2)w_n + [\nu DR_o\cos\phi - DR_1^2(R_1-R_2)](R_1-R_2)v_n + [CA_1(R_1-R_2)]v_n{}^\circ \pm [\nu KnR_o(R_1-R_2)]u_n$$

and multiply Eqn. 3.9(e) by CA_1/D,

$$CA_1M_{sn}/D = [R_1r_1{}^\circ - \nu R_o\cos\phi]CA_1\beta_n - CA_1\beta_n{}^\circ + [R_1(R_1-R_2) - \nu n^2R_o^2]CA_1w_n - CA_1R_1^2r_1{}^\circ v_n + [CA_1(R_1-R_2)]v_n{}^\circ \mp [\nu nR_oR_2CA_1]u_n$$

Subtracting the first from the second expression, and simplifying by means of Eqns. 3.20 yields,

$$\beta_n{}^\circ = \{-CA_1M_{sn}/D + (R_1-R_2)N_{sn} + [R_1r_1{}^\circ CA_2 - \nu R_o\cos\phi CA_1]\beta_n - [CA_1\nu n^2R_o^2 + \nu KR_2(R_1-R_2)]w_n - [\nu KR_o\cos\phi(R_1-R_2) + R_1^2r_1{}^\circ CA_2]v_n \pm [\nu nR_oR_1CA_2]u_n\}/CA_2 \qquad 3.21$$

Similarly, subtracting Eqn 3.9(a) from the product of (R_1-R_2) and Eqn. 3.9(e), and simplify the expression using

Eqns. 3.20 yields

$$v_n^\circ = \{-(R_1-R_2)M_{sn} + N_{sn} - [\nu DR_0\cos\phi(R_1-R_2)]\beta_n - [\nu Dn^2R_0^2(R_1-R_2) + R_1CA_2 + \nu KR_2]w_n - [\nu KR_0\cos\phi]v_n \mp [\nu nR_0CA_2]u_n\}/CA_2 \qquad 3.22$$

Rewriting Eqn. 3.14 yields

$$w_n^\circ = v_nR_1 - \beta_n \qquad 3.23$$

Finally, substituting Eqn. 3.9(g) into 3.11, and rewriting the equation to form an expression for $N_{s\theta n}$, then substituting this into Eqn. 3.9(c), and simplifying,

$$u_n^\circ = \{2T_{sn}/(1-\nu) \mp [DnR_0(R_1-3R_2)]\beta_n \mp [DnR_0^2\cos\phi(R_1-3R_2)]w_n \pm [nCA_1R_0 - DnR_0R_1R_2]v_n + [R_0\cos\phi CA_3]u_n\}/CA_3 \qquad 3.24$$

where

$$CA_3 = K + D(R_1^2-3R_1R_2+3R_2^2) \qquad 3.25$$

Eqns. 3.21 to 3.25 may be written in matrix form as,

$$\{D^\circ\} = [A_1 \ \ A_2]\begin{Bmatrix} D \\ F_s \end{Bmatrix} \qquad 3.26$$

where {D} and {D°} consists of displacements β_n, w_n, u_n, and v_n, and their derivatives with respect to the coordinate s, respectively; $[A_1 \ A_2]$ is a function of the geometric and material properties of the shell, defined in Table 3.2. Again, the upper and lower signs relate to the set of equations, grouped according to the cosine and sine terms in the Fourier series.

In order to solve the eight first order differential equations by integrating only along the meridian, it is necessary to eliminate the stress resultant variables in the circumferential direction. These variables are represented by the vector $<F_\theta>$. This vector can be eliminated from Eqn.

3.19 by the following procedure. Write the force-displacement equations in the following order, 3.9(f), (h), (g), (b), (d). In matrix form,

$$\{F_\theta\} = [B_4 \ B_5]\begin{Bmatrix} D^\circ \\ D \end{Bmatrix} \qquad 3.27$$

where $\{D\}$ and $\{D^\circ\}$ are defined as before. The cofficients of $[B_4 \ B_5]$ is a function of the geometric and material properties of the shell, defined in Tables 3.3.

Substituting Eqn. 3.27 into 3.19 yields

$$\{F_s{}^\circ\} = [B_1]\{F_s\} + [B_2]([B_4]\{D^\circ\} + [B_5]\{D\}) + \{B_3\}$$

Thus, after eliminating the terms involving θ, Eqn. 3.19 simplifies to

$$\{F_s{}^\circ\} = [A_3]\{D\} + [A_4]\{F_s\} + \{B_3\} \qquad 3.28$$

where

$$[A_3] = [B_2][B_4][A_1] + [B_2][B_5]$$

$$[A_4] = [B_1] + [B_2][B_4][A_2]$$

Combining Eqns. 3.26 and 3.28 to form a single matrix equation yields

$$\begin{Bmatrix} D^\circ \\ F_s{}^\circ \end{Bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{Bmatrix} D \\ F_s \end{Bmatrix} + \begin{Bmatrix} 0 \\ B_3 \end{Bmatrix} \qquad 3.29$$

Matrix equation 3.29 relates, at any point, the eight fundamental dependent variables, that appear in the natural boundary conditions of shells of revolution, and their derivatives with respect to the independent variable s only.

3.1.4 Solution of the Governing System of Equations

To establish the stiffness matrix, the eight first order differential equations represented by matrix Eqn. 3.29, must be solved numerically. In general, Eqn. 3.29 can be written as

$$\{y_s^{\circ}\} = [A_s]\{y_s\} + \{B_s\} \qquad 3.30$$

where $\{y_s\}$ and $\{y_s^{\circ}\}$ are vectors of eight dependent variables, four displacement components and four corresponding forces, and their derivatives, respectively. $[A_s]$ is the coefficient matrix relating the variables and their derivatives consisting only of functions of the material properties and geometry of the shell. $\{B_s\}$ is a function of the applied loads.

Consider a segment of length l in the region $i \le s \le j$. Now, divide the segment into, say twenty equal parts, each of length e. Let the first interval be bounded by points m_1 and n_1, with midpoint o_1, the second interval be bounded by points m_2 and n_2, with midpoint o_2, and so on. Note that point m_2 of the second interval coincides with n_1 of the first interval, m_3 coincides with n_2, and so on. Also, points m_1 of the first interval and n_{20} of the last interval coincide with segment edges i and j respectively.

The general solution of Eqn. 3.30 may be considered to consist of two parts: the homogeneous solution and the particular solution. The form of the homogeneous part is

$$\{y_s^{\circ}\}_h = [A_s]\{y_s\}_h \qquad 3.31$$

and the form of the particular part is

$$\{y_s^{\circ}\}_p = [A_s]\{y_s\}_p + \{B_s\} \qquad 3.32$$

The solution of Eqn. 3.30 is done in steps by integrating over the intervals selected. By assigning initial boundary values to $\{y_s\}_h$ at each starting point m, it is possible to relate these initial values with the corresponding values at n. For the first interval, the initial boundary value is the vector $\{y_i\}_h$ which is selected as a vector of unit values. For the second interval, the initial boundary value is the vector $\{y_{n1}\}_h$ which was computed from the preceding interval, and so on.

The following integration step is demonstrated for the first interval. The same procedure is repeated for subsequent intervals. Let the eight initial boundary values at edge i, which coincides with m_1, be the vector $\{y_i\}_h$, then Eqn. 3.31 becomes

$$\{y_i^{\circ}\}_h = [A_i]\{y_i\}_h \qquad 3.33$$

Integrating this numerically allows the values of $\{y_s\}_h$ at any point in the region $m_1 \le s \le n_1$ to be determined as

$$\{y_s\}_h = [H_s]\{y_i\}_h \qquad 3.34$$

where $[H_s]$ represents the matrix arising from the integration of $[A_s]$ along the meridian. Similarly, the integration of Eqn. 3.31 for the second interval in the region $m_2 \le s \le n_2$ can be written as

$$\{y_s\}_h = [H_s]\{y_{m2}\}_h \qquad 3.35$$

where the vector $\{y_{m2}\}_h$ coincides with $\{y_{n1}\}_h$ which is a function of $\{y_i\}_h$, derived from Eqn. 3.34. Similar expressions can be formed for subsequent intervals until the point n_{20} which coincides with edge j is reached. At this stage the homogeneous solution for Eqn. 3.30 can be written as

$$\{y_j\}_h = [H_j]\{y_i\}_h \qquad 3.36$$

A similar procedure can be used to obtain the particular solution. Integrating over the interval of length e in the region $m \leq s \leq n$ yields

$$\{y_s\}_p = [H_s]\{y_i\}_p + \{Q_s\} \qquad 3.37$$

where $\{Q_s\}$ is a vector arising from the integration of $\{B_s\}$. Since the particular solution is any solution which satisfies the inhomogeneous equations, it is adequate to select $\{y_i\}_p = 0$.

Hence, for the region $m \leq s \leq n$, Eqn. 3.37 reduce to

$$\{y_s\}_p = \{Q_s\} \qquad 3.38$$

By integrating successively along the meridian until the edge j is reached, Eqn. 3.38 becomes

$$\{y_j\}_p = \{Q_j\} \qquad 3.39$$

Superimposing the two solutions, the form of the general solution to Eqn. 3.30 in the region $m \leq s \leq n$ is

$$\{y_s\} = [H_s]\{y_i\} + \{Q_s\} \qquad 3.40$$

Upon integrating over all the intervals, an expression relating the values of $\{y_s\}$ at j to the values at i is formed as follows

$$\{y_j\} = [H_j]\{y_i\} + \{Q_j\} \qquad 3.41$$

where each column vector of the transfer matrix $[H_j]$ represents the variables at 'j' corresponding to each unit variable applied at 'i' in the absence of any external loads. $\{Q_j\}$ represents the variables at 'j' corresponding to zero displacements, D, and forces, F at 'i' in the presence of the external loads.

3.1.5 Segment Stiffness Matrix

Eqn. 3.41 can be expanded into

$$\begin{Bmatrix} D_j \\ F_j \end{Bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{Bmatrix} D_i \\ F_i \end{Bmatrix} + \begin{Bmatrix} Q_d \\ Qf \end{Bmatrix} \tag{3.42}$$

where Q_d and Qf are the displacements and forces from the particular solution respectively. Expanding Eqn. 3.42 into two equations

$$\begin{Bmatrix} D_i \\ D_j \end{Bmatrix} = \begin{bmatrix} I & 0 \\ H_1 & H_2 \end{bmatrix} \begin{Bmatrix} D_i \\ F_i \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q_d \end{Bmatrix} = [Y_1] \begin{Bmatrix} D_i \\ F_i \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q_d \end{Bmatrix} \tag{3.43}$$

and

$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \begin{bmatrix} 0 & I \\ H_3 & H_4 \end{bmatrix} \begin{Bmatrix} D_i \\ F_i \end{Bmatrix} + \begin{Bmatrix} 0 \\ Qf \end{Bmatrix} = [Y_2] \begin{Bmatrix} D_i \\ F_i \end{Bmatrix} + \begin{Bmatrix} 0 \\ Qf \end{Bmatrix} \tag{3.44}$$

Solving for $\langle D_i \ F_i \rangle$ in Eqn. 3.43, and substituting into 3.44 yields

$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = [Y_2][Y_1]^{-1} \begin{Bmatrix} D_i \\ D_j - Q_d \end{Bmatrix} + \begin{Bmatrix} 0 \\ Qf \end{Bmatrix} \tag{3.45}$$

and

$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = [K] \begin{Bmatrix} \bar{D}_i \\ \bar{D}_j \end{Bmatrix} + \begin{Bmatrix} F_{oi} \\ F_{oj} \end{Bmatrix} \tag{3.46}$$

where $\bar{D}_i = D_i$ and $\bar{D}_j = D_j - Q_d$. The coefficients of [K] represent the forces at each shell edge due to a unit displacement at each end while all other displacements are restrained. This matrix is known as the stiffness matrix. $\{F_0\}$ represents the fixed end forces.

3.1.6 Stiffness Matrix Sign Convention

In the derivation of the segment stiffness matrix and the fixed end stresses, the sign convention used corresponds to that generally used in shell theory as given in Fig. 2.2. As a result, the stiffness matrix will have some negative elements on the main diagonal. This can be corrected by adapting the so-called 'stiffness matrix sign convention', described in Chapter 5, and shown in Fig. 3.3.

3.2 The Flexibility Approach

The solution to the basic shell equations may be split into two parts, namely: the particular solution, which can be simplified to the membrane solution with negligible loss of accuracy, and the homogeneous solution which considers the bending stresses. This procedure is analogous to the flexibility method of analysis for a statically indeterminate structure. Program FLEXSHELL was developed based on this approach. To simplify the shell equations and limit the particular solutions, the following assumptions will be made:

1. Loads are axisymmetric, i.e., $\partial/\partial\theta = 0$, $p_\theta = 0$, thus, $\partial\phi = d\phi$;
2. The shell segment has uniform thickness; and,

3. z (from Eqns. 2.5 and 2.6 is small compared with the radii of curvature, i.e., $r_1+z \simeq r_1$ and $r_2+z \simeq r_2$.

Thus, the equations of equilibrium become

$$\frac{d(r_0N_\phi)}{d\phi} - r_1N_\theta\cos\phi - Q_\phi r_0 + r_0r_1p_\phi = 0 \qquad 3.47(a)$$

$$\frac{d(r_0Q_\phi)}{d\phi} + N_\theta r_1\sin\phi + r_0N_\phi - r_0r_1p_z = 0 \qquad 3.47(b)$$

$$-\frac{d(r_0M_\phi)}{d\phi} + r_1M_\theta\cos\phi + Q_\phi r_0r_1 = 0 \qquad 3.47(c)$$

and the force-displacement relations become

$$N_\phi = K\left[\frac{1}{r_1}\left(\frac{dv}{d\phi}-w\right) + \frac{\nu}{r_0}(v\cos\phi - w\sin\phi)\right] \qquad 3.48(a)$$

$$N_\theta = K\left[\frac{\nu}{r_1}\left(\frac{dv}{d\phi}-w\right) + \frac{1}{r_0}(v\cos\phi - w\sin\phi)\right] \qquad 3.48(b)$$

$$M_\phi = -D\left[\frac{1}{r_1}\frac{d}{d\phi}\left(\frac{1}{r_1}\frac{dw}{d\phi}\right) + \frac{\nu\cos\phi}{r_0r_1}\frac{dw}{d\phi}\right] \qquad 3.48(c)$$

$$M_\theta = -D\left[\frac{\nu}{r_1}\frac{d}{d\phi}\left(\frac{1}{r_1}\frac{dw}{d\phi}\right) + \frac{\cos\phi\, dw}{r_0r_1d\phi}\right] \qquad 3.48(d)$$

The method of analysis is outlined as follows:

1. Determine the particular solution forces and the deformations at the edges of the shell due to the applied loads;
2. Establish the flexibility matrix;
3. Solve for the edge forces and moments necessary to restore the incompatibilities of the deformations between adjoining segments;
4. Determine the final stresses by superimposing the particular solution stresses and the stresses due to the

incompatibilities.

3.2.1 The Particular Solution

As mentioned earlier, the particular solution is approximated by the membrane solution. The membrane theory for axisymmetrically loaded shells can be obtained from Eqns. 3.47 and 3.48 by neglecting the bending components, based on the assumption that the displacements due to the membrane stresses do not induce any appreciable bending. Thus, Eqn. 3.47 and 3.48 reduce to two equations with two unknowns as shown:

$$(r_0 N_\phi)^{\cdot} - r_1 N_\theta \cos\phi + r_0 r_1 p_\phi = 0 \qquad 3.49(a)$$

$$r_1 N_\theta \sin\phi + r_0 N_\phi + r_0 r_1 p_z = 0 \qquad 3.49(b)$$

The in-plane forces N_ϕ and N_θ are obtained more simply from the vertical and normal equilibrium of the statically determinate shell segment under the applied loads. Since the radii of curvature r_1 and r_2 vary in form depending on the type of shell of revolution, so does the form of the membrane solution.

1. Cylinder

$$N_s = -\int_s p_s ds \qquad 3.50(a)$$

$$N_\theta = -p_z r \qquad 3.50(b)$$

2. Sphere

$$N_\phi = \frac{-R}{2\pi r_0 \sin\phi} \qquad 3.51(a)$$

$$N_\theta = \frac{\pm R}{2\pi r_1 \sin^2\phi} \mp p_z r_2 \qquad 3.51(b)$$

3. Cone

$$N_s = \frac{-R}{2\pi s \cos\alpha} \tag{3.52(a)}$$

$$N_\theta = \mp p_z r_2 \tag{3.52(b)}$$

where R is the total vertical load, positive when directed toward the supports; p_z is the component of the external load per unit area normal to the shell surface in the direction towards the axis of revolution. The upper and lower signs relate to Figs. (a) and (b) respectively, of Tables 3.4 and 3.6. The expression for the membrane in-plane forces for the spherical, cylindrical, and conical segments, subjected to various loading conditions shown in Tables 3.4 to 3.6 were derived from Eqns. 3.50 to 3.52. The solution due to the thermal effects were obtained from Billington(3).

3.2.2 The Homogeneous Solution

Consider the vertical equilibrium of a shell segment, then

$$2\pi r_0 N_\phi \sin\phi + 2\pi r_0 Q_\phi \cos\phi + R = 0$$

from which

$$N_\phi = -Q_\phi \cot\phi - \frac{R}{2\pi r_0 \sin\phi} \tag{3.53}$$

where R is defined as before. Note that the second term is the membrane force which can be evaluated separately as shown earlier. Therefore, the homogeneous solution is obtained by solving the simplified shell equations (Eqns. 3.47 to 3.48) ignoring all load terms. Thus, the homogeneous solution for the meridional force is

$$N_\phi = -Q_\phi \cot\phi \tag{3.54}$$

Substituting this into Eqn. 3.47(b), ignoring the load term p_z, and using the relation,

$$r_0 = r_2 \sin\phi \qquad 3.55$$

then

$$N_\theta = -\frac{r_2}{r_1}\frac{dQ_\phi}{d\phi} \qquad 3.56$$

Let

$$U = r_2 Q_\phi \qquad 3.57$$

$$V = \frac{1}{r_1}\left[v + \frac{dw}{d\phi}\right] \qquad 3.58$$

Eqns. 3.54 and 3.56 become

$$N_\phi = -\frac{1}{r_2} U \cot\phi \qquad 3.59$$

$$N_\theta = -\frac{1}{r_1}\frac{dU}{d\phi} \qquad 3.60$$

Rearranging Eqns. 3.48(a) and 3.48(b), and substituting Eqn. 3.55 yields,

$$\frac{dv}{d\phi} - w = \frac{r_1}{Eh}(N_\phi - \nu N_\theta) \qquad 3.61$$

$$v\cot\phi - w = \frac{r_2}{Eh}(N_\theta - \nu N_\phi) = \Delta_H \qquad 3.62$$

where Δ_H is the horizontal displacement. Combining Eqns. 3.61 and 3.62, w may be eliminated to yield,

$$\frac{dv}{d\phi} - v\cot\phi = \frac{1}{Eh}[(r_1+\nu r_2)N_\phi - (r_2+\nu r_1)N_\theta] \qquad 3.63$$

Differentiating Eqn. 3.62, and combining with Eqn. 3.63 gives,

$$v + \frac{dw}{d\phi} = r_1 V = \frac{\cot\phi}{Eh}[(r_1+\nu r_2)N_\phi - (r_2+\nu r_1)N_\theta]$$

$$-\frac{d}{d\phi}\left[\frac{r_2}{Eh}(N_\theta - \nu N_\phi)\right] = \Delta_\phi \qquad 3.64$$

where Δ_ϕ is the rotation of the edge of the shell. Substituting Eqns. 3.59 and 3.60 into Eqn. 3.64 yields one equation with U and V terms only.

$$\frac{r_2}{r_1^2}\frac{d^2U}{d\phi^2} + \frac{1}{r_1}\left[\frac{d}{d\phi}\left(\frac{r_2}{r_1}\right) + \frac{r_2\cot\phi}{r_1} - \frac{r_2}{r_1 h}\frac{dh}{d\phi}\right]\frac{dU}{d\phi}$$

$$- \frac{1}{r_1}\left[\frac{r_1\cot\phi}{r_2} - \nu - \frac{\nu}{h}\frac{dh}{d\phi}\cot\phi\right]U = EhV \qquad 3.65$$

Substituting Eqns. 3.57 and 3.58 into Eqns. 3.48(c) and 3.48(d),

$$M_\theta = -D\left[\frac{V}{r_2}\cot\phi + \frac{\nu}{r_1}\frac{dV}{d\phi}\right] \qquad 3.66$$

$$M_\phi = -D\left[\frac{\nu V\cot\phi}{r_2} + \frac{1}{r_1}\frac{dV}{d\phi}\right] \qquad 3.67$$

Substituting these two equations and Eqn. 3.57 into Eqn. 3.47(c) yields,

$$\frac{r_2}{r_1^2}\frac{d^2V}{d\phi^2} + \frac{1}{r_1}\left[\frac{d}{d\phi}\left(\frac{r_2}{r_1}\right) + \frac{r_2\cot\phi}{r_1} + \frac{3r_2}{r_1 h}\frac{dh}{d\phi}\right]\frac{dV}{d\phi}$$

$$-\frac{1}{r_1}\left[\nu - \frac{3\nu\cot\phi}{h}\frac{dh}{d\phi} + \frac{r_1}{r_2}\cot^2\phi\right]V = -\frac{U}{D} \qquad 3.68$$

Eqns. 3.65 and 3.68 permit a closed form solution of the equations of shells of revolution. The solution of these equations may be further simplified by applying the geometrical properties of each shell.

For the cylindrical segment with $r_1 = \infty$ and $r_0 = r_2 = r$, these equations reduce to

$$r_2\frac{d^2U}{ds^2} = EtV \qquad 3.69$$

$$r_2\frac{d^2V}{ds^2} = -\frac{U}{D} \qquad 3.70$$

Substituting Eqns. 3.57 and 3.58 into the above equations, and using the expression for the horizontal displacement (Eqn. 3.62), Eqns. 3.69 and 3.70 can be combined to form

$$\frac{d^4\Delta_H}{ds^4} + 4\beta^4\Delta_H = 0 \qquad 3.71$$

where

$$\beta^4 = \frac{3(1-\nu^2)}{r^2h^2} \qquad 3.72$$

and s is as defined in Section 3.1. The final solution expressed in closed form is

$$\Delta_H = e^{\beta s}(C_1\cos\beta s + C_2\sin\beta s) + e^{-\beta s}(C_3\cos\beta s + C_4\sin\beta s) \qquad 3.73$$

where C_1, C_2, C_3, and C_4 are arbitrary constants of integration.

For a spherical segment with $r_1 = r_2 = a$ and $r_0 = a \sin\phi$, and using Eqn. 3.57, Eqns. 3.65 and 3.68 become

$$\frac{d^2Q_\phi}{d\phi^2} + \cot\phi\frac{dQ_\phi}{d\phi} - (\cot^2\phi - \nu)Q_\phi = EhV \qquad 3.74$$

$$\frac{d^2V}{d\phi^2} + \cot\phi\frac{dV}{d\phi} - (\cot^2\phi - \nu)V = -\frac{a^2Q_\phi}{D} \qquad 3.75$$

As shown in detail in Appendix A, the two unknowns Q_ϕ and V may be separated to form

$$\ddot{Q}_\phi + \dot{Q}_\phi\cot\phi - Q_\phi\cot^2\phi \pm 2i\lambda^2Q_\phi = 0$$

Using the Langer technique (16), the solution of the above equation in terms of the derivatives of the Kelvin functions of order zero is

$$Q_\phi = \frac{K}{a}\sqrt{\frac{\phi}{\sin\phi}}(C_1\mathrm{ber}'\xi + C_2\mathrm{bei}'\xi + C_3\mathrm{ker}'\xi + C_4\mathrm{kei}'\xi) \qquad 3.76$$

where $\xi = \lambda\phi\sqrt{2}$

$$\lambda^4 = 3(1-\nu^2)\frac{a^2}{h^2} - \frac{\nu^2}{4} \qquad 3.77$$

where the prime in Eqn. 3.76 represents derivatives with respect to ξ.

For the conical segment with $r_0 = s \sin\alpha$, $r_1 = \infty$, $r_2 = s \tan\alpha$, and $\phi = \pi/2 - \alpha$, using Eqn. 3.57, Eqns. 3.65 and 3.68 become

$$s \frac{d^2(sQ_s)}{ds^2} + \frac{d(sQ_s)}{ds} - \frac{(sQ_s)}{s} = EhV\cot^2\alpha \qquad 3.78$$

$$s \frac{d^2V}{ds^2} + \frac{dV}{ds} - \frac{V}{s} = -\frac{(sQ_s)}{D} \qquad 3.79$$

As with the spherical segment, the unknowns may be separated to form

$$s(sQ_s)^{\circ\circ} + (sQ_s)^{\circ} - Q_s \pm i\lambda^2 sQ_s = 0$$

for which the closed form solution in terms of the Kelvin functions of order two, as shown in detail in Appendix A is

$$Q_s = \frac{1}{s}(C_1 \mathrm{ber}_2\xi + C_2 \mathrm{bei}_2\xi + C_3 \mathrm{ker}_2\xi + C_4 \mathrm{kei}_2\xi) \qquad 3.80$$

where

$$\lambda^4 = \frac{12(1-\nu^2)}{h^2\tan^2\alpha} \qquad 3.81$$

$$\xi = 2\lambda\sqrt{s} \qquad 3.82$$

3.2.3 Segment Flexibility Matrix

The construction of the flexibility matrix for the cylindrical, conical, and spherical segments will be discussed in this section. By definition, the flexibility matrix coefficient, say F(i,j), is the deformation of the segment at i due to a unit value of the load applied to the segment at j.

Only those deformations which violate continuity and the corresponding forces which produce these deformations need be identified in the formulation of the flexibility matrix. For axisymmetric loading, these are the horizontal displacement Δ_H and the meridional rotation Δ_ϕ and the corresponding forces H and M_ϕ at each discontinuous edge of the shell. From Eqn. 3.67, the expression for the meridional moment is

$$M_\phi = -D\left[\frac{\nu V \cot\phi}{r_2} + \frac{1}{r_1}\frac{dV}{d\phi}\right] \tag{3.67}$$

and depending on the geometry of the shell, the horizontal force, H, can be expressed as a function of the transverse shear force Q_ϕ.

Consistent with the sign conventions shown in Fig. 3.4, expressions for the moment and horizontal force at each shell edge may be expressed in terms of the homogeneous solution, as shown in matrix form below for the cylindrical segment

Let

$$\beta^4 = \frac{3(1-\nu^2)}{r^2h^2}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

and

$$\phi_1 = e^{\beta s}\cos\beta s \qquad \theta_1 = e^{\beta s}(\cos\beta s + \sin\beta s)$$

$$\phi_2 = e^{\beta s}\sin\beta s \qquad \theta_2 = e^{\beta s}(\cos\beta s - \sin\beta s)$$

$\phi_3 = e^{-\beta s}\cos\beta s \qquad \theta_3 = e^{-\beta s}(\cos\beta s + \sin\beta s)$

$\phi_4 = e^{-\beta s}\sin\beta s \qquad \theta_4 = e^{-\beta s}(\cos\beta s - \sin\beta s)$

then

$$\begin{Bmatrix} H^i \\ M_\phi^i \\ H^j \\ M_\phi^j \end{Bmatrix} = \begin{bmatrix} 2D\beta^3 & 0 & 0 & 0 \\ 0 & 2D\beta^3 & 0 & 0 \\ 0 & 0 & 2D\beta^3 & 0 \\ 0 & 0 & 0 & 2D\beta^3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ \theta_1^j & -\theta_2^j & -\theta_4^j & -\theta_3^j \\ -\phi_2^j & \phi_1^j & \phi_4^j & \phi_3^j \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

or simply,

$$\{V\} = [T_1][T_2]\{C\}$$

Multiplying matrices $[T_1]$and $[T_2]$ simplifies to

$$\{V\} = [TT]\{C\} \qquad 3.83$$

Similarly, expressions for the deformations at each shell edge may be expressed in terms of the homogeneous solution as follows:

$$\begin{Bmatrix} \Delta_H^i \\ \Delta_\phi^i \\ \Delta_H^j \\ \Delta_\phi^j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ \phi_1^j & \phi_2^j & \phi_3^j & \phi_4^j \\ \theta_2^j & \theta_1^j & -\theta_3^j & \theta_4^j \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

or simply,

$$\{\Delta\} = [T_3][T_4]\{C\}$$

Multiplying matrices $[T_3]$ and $[T_4]$ simplifies to

$$\{\Delta\} = [TA]\{C\} \qquad 3.84$$

Combining Eqn. 3.83 and 3.84 yields

$$\{\Delta\} = [TA][TT]^{-1}\{V\} \qquad 3.85$$

$$\{\Delta\} = [F]\{V\} \qquad 3.86$$

where [F] is the segment flexibility matrix, such that

$$[F] = [TA][TT]^{-1} \tag{3.87}$$

Similarly, the flexibility matrices for the spherical and conical segments are constructed using the homogeneous solutions, Eqns. 3.73 and 3.76 respectively, as shown in detail in Appendix B.

The base segment is considered to be a circular plate supported on a Winkler type foundation, whose stiffness is expressed as the subgrade modulus, k (4). The segment flexibility matrix was developed in the same manner as the spherical, cylindrical, and conical segments, based on the asymptotic solution to the fourth order plate equation.

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{q - kw}{D}$$

where w is the deformation component, r is the radius of the circular plate, q is the load term, and D is the flexural rigidity.

TABLE 3.1 Coefficients of Matrices B_1, B_2 and Load Vector B_3 in Eqn. 3.19

$R_o \cos\phi$	$\mp R_o n$	$\mp R_o n$		
$R_o^2 n^2$	$\mp R_o^2 n \cos\phi$	$\mp R_o^2 n \cos\phi$	$-R_2$	
		$\mp R_o R_1 n$	$R_o \cos\phi$	$\mp R_o n$
$\mp R_o R_2 n$	$R_o R_2 \cos\phi$	$-R_o \cos\phi$	$\pm R_o n$	$-R_o \cos\phi$

Matrix B_2

$-R_o \cos\phi$			
	$-R_o \cos\phi$	$-R_1$	
	R_1	$-R_o \cos\phi$	
			$-R_o \cos\phi$

Matrix B_1

P_r
$-P_s$
$-P_\theta$

Vector B_3

TABLE 3.2 Coefficients of Matrix A_1 and A_2 in Eqn. 3.26

$R_1 r_2^o - \nu R_o \cos\phi \dfrac{CA_1}{CA_2}$	$- \dfrac{\nu K R_2 (R_1 - R_2)}{CA_2}$ $- \nu R_o^2 n^2 \dfrac{CA_1}{CA_2}$	$-R_1^2 r_1^o$ $- \dfrac{\nu K R_o \cos\phi (R_1 - R_2)}{CA_2}$	$\mp \nu R_0 R_1 n$
-1		R_1	
$-\nu D \dfrac{R_o \cos\phi (R_1 - R_2)}{CA_2}$	$-R_1 - \dfrac{\nu K R_2}{CA_2}$ $- \dfrac{\nu D R_o^2 n^2 (R_1 - R_2)}{CA_2}$	$- \dfrac{\nu K R_o}{CA_2} \cos\phi$	$\mp \nu R_o n$
$\mp \dfrac{D R_o n (R_1 - 3R_2)}{CA_3}$	$\mp \dfrac{D R_o^2 n}{CA_3} \cos\phi (R_1 - 3R_2)$	$\pm \dfrac{R_o n CA_1}{CA_3}$ $- \dfrac{D n R_o R_1 R_2}{CA_3}$	$R_o \cos\phi$

Matrix A_1

$- \dfrac{CA_1}{DCA_2}$		$\dfrac{R_1 - R_2}{CA_2}$	
$- \dfrac{(R_1 - R_2)}{CA_2}$		$\dfrac{1}{CA_2}$	
			$\left(\dfrac{2}{1 - \nu}\right) \dfrac{1}{CA_3}$

Matrix A_2

TABLE 3.3 (a) Coeffiecients of Matrix B_4 in Eq. 3.27

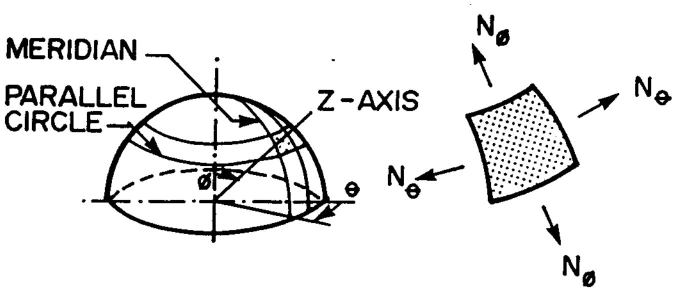
$-\nu D$			
			$-(\frac{1-\nu}{2}) D\ R_2$
			$(\frac{1-\nu}{2})\{D(R_1-R_2\}$
		νK	
			$(\frac{1-\nu}{2})\ K$

Matrix B_4

$\nu DR_1 r_1^o - DR_o \cos\phi$	$-DR_o^2 n^2 - DR_2(R_1 - R_2)$	$-\gamma DR_1^2 r_1^o$ $-DR_o \cos\phi\ (R_1 - R_2)$	$\mp DR_o R_1 n$
$\pm(1-\nu)DR_o n$	$\mp(1-\nu)DR_o^2 n \cos\phi$	$\mp\left(\frac{1-\nu}{2}\right) R_o R_2 n$	$\left(\frac{1-\nu}{2}\right) DR_o R_2 \cos\phi$
$\pm(1-\nu)DR_o n$	$(1-\nu)DR_o^2 n \cos\phi$	$\mp\left(\frac{1-\nu}{2}\right) DR_o R_1 n$	$-\left(\frac{1-\nu}{2}\right) DR_o \cos\phi\ (R_1 - 2R_2)$
$DR_o \cos\phi\ (R_1 - R_2)$	$K(R_2 + \nu R_1)$ $+D(R_1 - R_2)\ (R_o^2 n^2 - R^2)$	$R_o \cos\phi\{K - DR_2\ (R_1 - R_2)\}$	$\pm KR_o n$
$\mp\left(\frac{1-\nu}{2}\right) DR_o n\ (R_1 - R_2)$	$\mp\left(\frac{1-\nu}{2}\right) DR_o^2 n \cos\phi (R_1 - R_2)$	$\mp\left(\frac{1-\nu}{2}\right) R_o n[K - D(R_1 - R_2)^2]$	$-\left(\frac{1-\nu}{2}\right) KR_o \cos\phi$

TABLE 3.3 (b) Coefficients of Matrix B_5 in Eqn. 3.27

Table 3.4 MEMBRANE SOLUTION FOR A SPHERICAL SEGMENT



— UPPER AND LOWER SIGNS RELATE TO Figs. (a) AND (b) RESPECTIVELY

— $\psi_a = \alpha_o$
$\psi_b = \alpha$

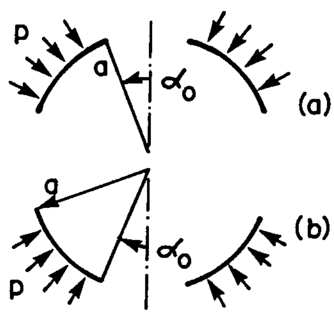
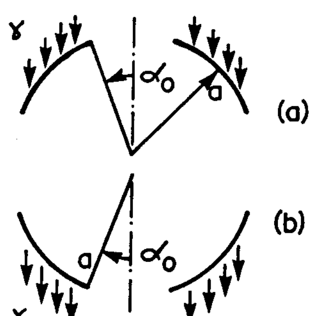
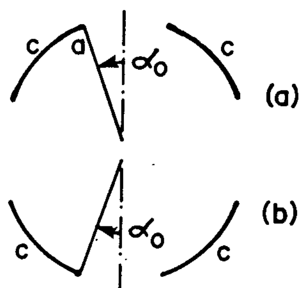
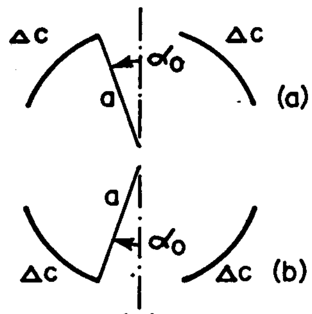
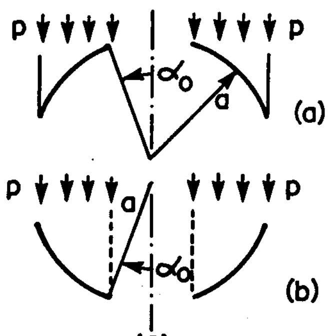
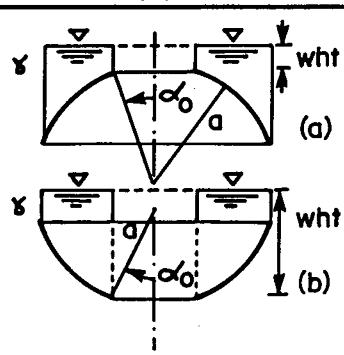
LOAD CASE	IN-PLANE FORCES & DEFORMATIONS
	$N_\phi = -\frac{pa}{2}\left(1 - \frac{\sin^2\psi}{\sin^2\phi}\right)$ $N_\theta = -\frac{pa}{2}\left(1 + \frac{\sin^2\psi}{\sin^2\phi}\right)$
	$N_\phi = \mp \gamma ha \frac{(\cos\psi - \cos\phi)}{\sin^2\phi}$ $N_\theta = \gamma ha \left[\frac{(\cos\psi - \cos\phi)}{\sin^2\phi} \mp \cos\phi\right]$
	$\Delta_H = -C\,\alpha_T\,a \sin\phi$

Table 3.4 (cont'd)

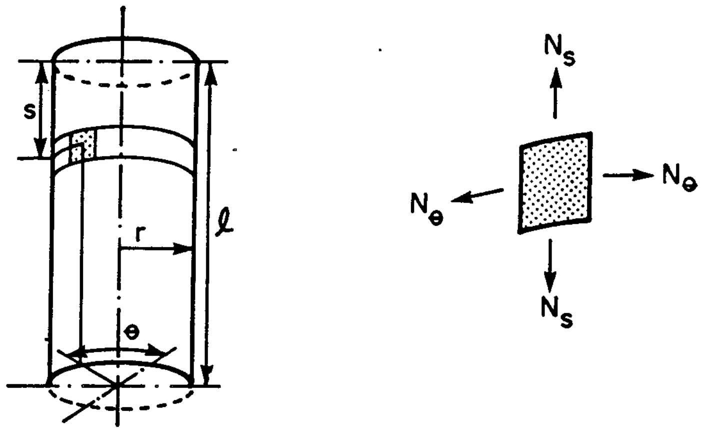
LOAD CASE	IN-PLANE FORCES & DEFORMATIONS
(5)	$M_\phi = \dfrac{\Delta c\,\alpha_T E h^2}{12(1-\nu)}$ $M_\theta = M_\phi$
(6)	$N_\phi = \mp \dfrac{pa}{2}\left(1 - \dfrac{\sin^2\psi}{\sin^2\phi}\right)$ $N_\theta = \pm \dfrac{pa}{2}\left[\left(1 - \dfrac{\sin^2\psi}{\sin^2\phi}\right) \mp 2\cos^2\phi\right]$
(7)	$N_\phi = \mp \gamma a\left[\dfrac{(wht \pm a\cos\psi)}{2}\left(1 - \dfrac{\sin^2\psi}{\sin^2\phi}\right) \pm \dfrac{a}{3}\,\dfrac{(\cos^3\phi - \cos^3\psi)}{\sin^2\phi}\right]$ $N_\theta = \mp \gamma a\left[\dfrac{(wht \pm a\cos\psi)}{2}\left(1 + \dfrac{\sin^2\psi}{\sin^2\phi}\right) \mp \dfrac{a}{3}\,\dfrac{(\cos^3\phi - \cos^3\psi)}{\sin^2\phi} \mp a\cos\phi\right]$

NOTE:

$$\Delta_H = \frac{a\sin\phi}{Eh}\,(N_\theta - \nu N_\phi)$$

$$\Delta_\phi = \frac{\cot\phi}{Eh}\,(1+\nu)(N_\theta - N_\phi) - \frac{d}{d\phi}\,[N_\theta - \nu N_\phi]\,\frac{1}{Eh}$$

Table 3.5 MEMBRANE SOLUTION FOR A CYLINDRICAL SEGMENT



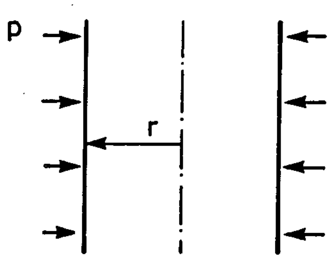
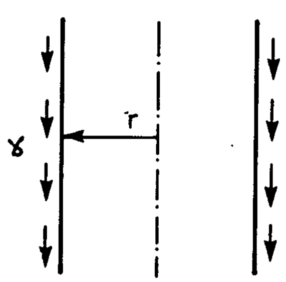
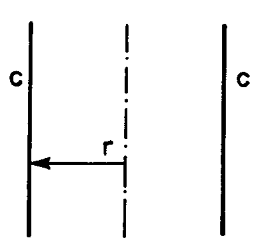
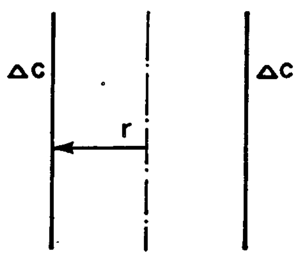
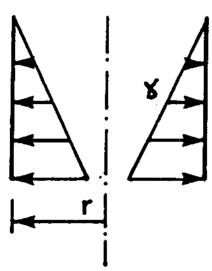
LOAD CASE	IN-PLANE FORCES & DEFORMATIONS
 (1) & (3)	$N_s = 0$ $N_\theta = -pr$
 (2)	$N_s = -\gamma hs$ $N_\theta = 0$
 (4)	$\Delta_H = -c\alpha_T r$

Table 3.5 (cont'd)

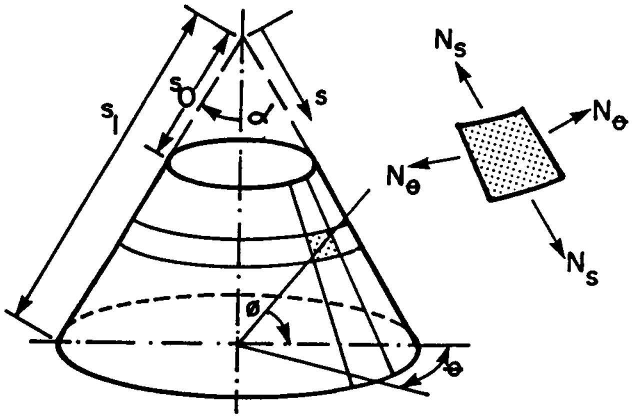
LOAD CASE	IN-PLANE FORCES & DEFORMATION
 (5)	$M_s = \dfrac{\Delta c \alpha_T E h^2}{12(1-\nu)}$ $M_\theta = M_s$
 (7)	$N_s = 0$ $N_\theta = \gamma r s$

NOTE:

$$\Delta_H = \frac{r}{Eh}(N_\theta - \nu N_s)$$

$$\Delta_\phi = \frac{-d}{ds}(N_\theta - \nu N_s)$$

Table 3.6 MEMBRANE SOLUTION FOR A CONICAL SEGMENT



- UPPER AND LOWER SIGNS RELATE TO Figs (a) AND (b) RESPECTIVELY
- $y_{(a)} = s_0$

  $y_{(b)} = s_1$

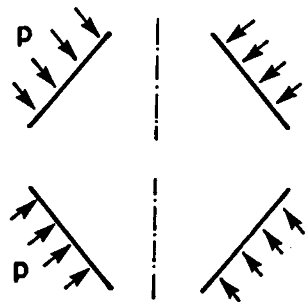
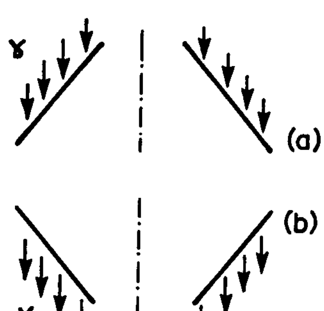
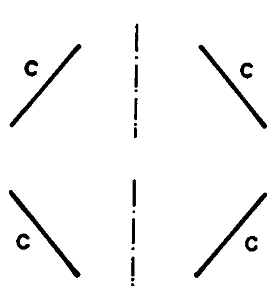
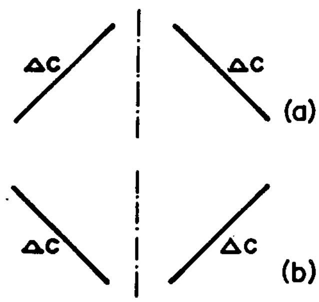
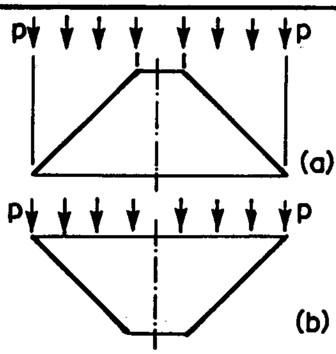
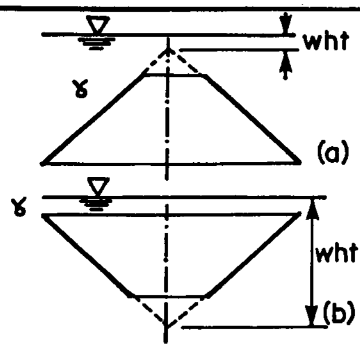
LOAD CASE	IN-PLANE FORCES & DEFORMATIOS
 (1) & (3)	$N_s = -p \tan\alpha \dfrac{(s^2-y^2)}{2S}$ $N_\theta = -ps \tan\alpha$
 (2)	$N_s = \mp \dfrac{\gamma h (s^2-y^2)}{2s \cos\alpha}$ $N_\theta = \mp \gamma hs \tan\alpha \sin\alpha$
 (4)	$\Delta_H = -c\alpha_T s \sin\alpha$

Table 3.6 (cont'd)

LOAD CASE	IN - PLANE FORCES & DEFORMATIONS
 (5)	$M_s = \dfrac{\Delta c \alpha_T E h^2}{12(1-\nu)}$ $M_\theta = M_s$
 (6)	$N_s = \mp p \tan\alpha \dfrac{(s^2 - y^2)}{2s}$ $N_\theta = \mp ps \sin^2\alpha \tan\alpha$
 (7)	$N_s = \dfrac{\mp \gamma s \tan\alpha}{6}\left[3\,wht\left(1 - \dfrac{y^2}{s^2}\right) \pm 2s\cos\alpha\left(1 - \dfrac{y^3}{s^3}\right)\right]$ $N_\theta = \mp \gamma s \tan\alpha\,[wht \pm s\cos\alpha]$

NOTE:

$$\Delta H = \frac{s \sin\alpha}{Eh}(N_\theta - N_s)$$

$$\Delta_\phi = \frac{\tan\alpha}{Eh}\left[(1+\nu)(N_s - N_\theta) - \frac{1}{s}\frac{d}{ds}(N_\theta - \nu N_s)\right]$$

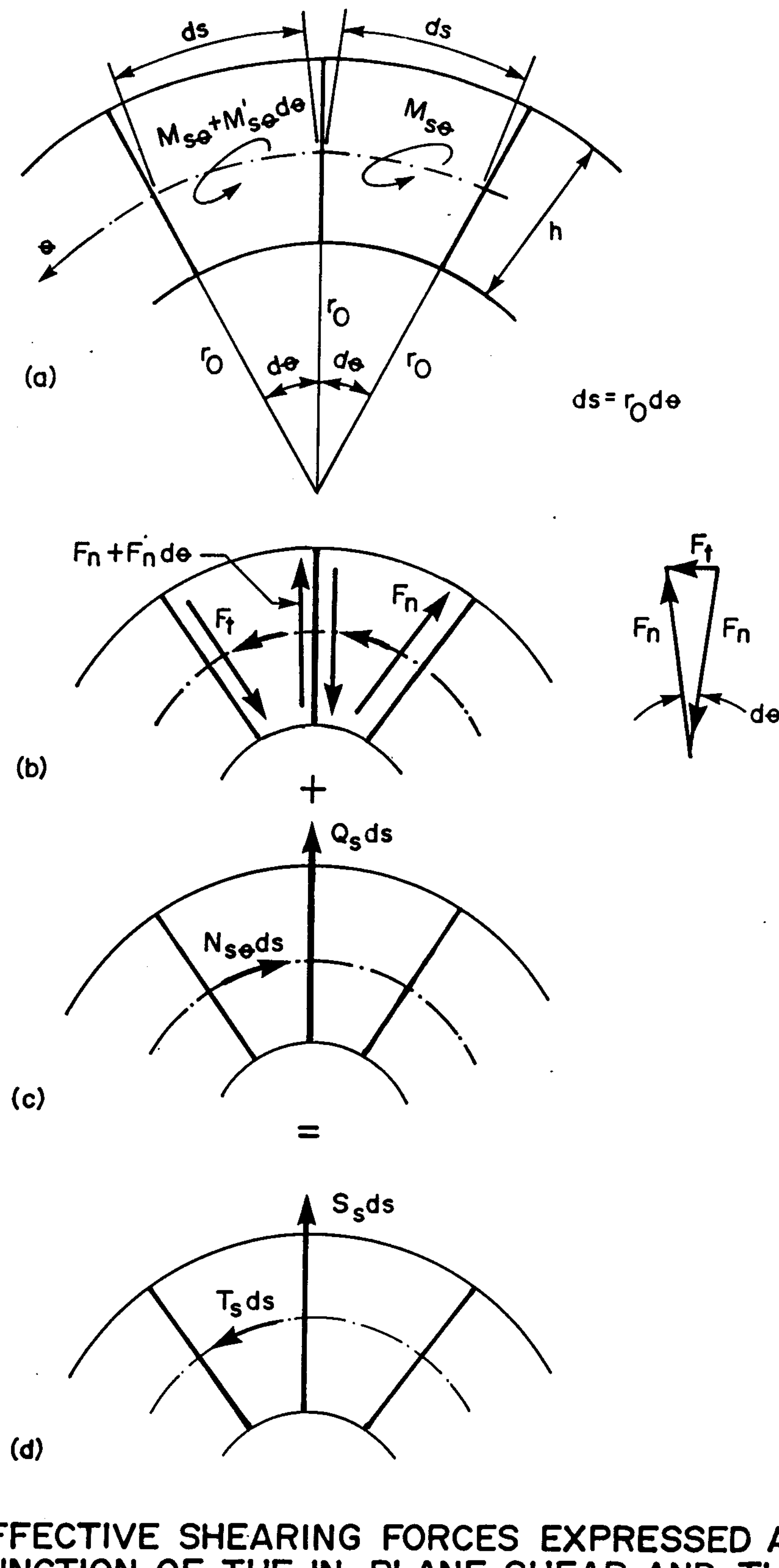


Figure 3.1 EFFECTIVE SHEARING FORCES EXPRESSED AS A FUNCTION OF THE IN-PLANE SHEAR AND THE TWISTING MOMENT

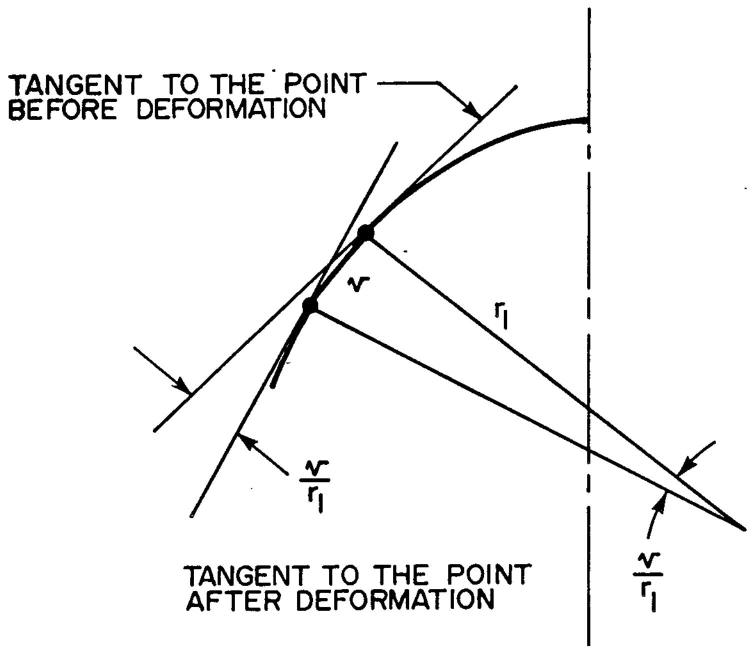


Figure 3.2(a) MERIDIONAL ROTATION β DUE TO DISPLACEMENT v

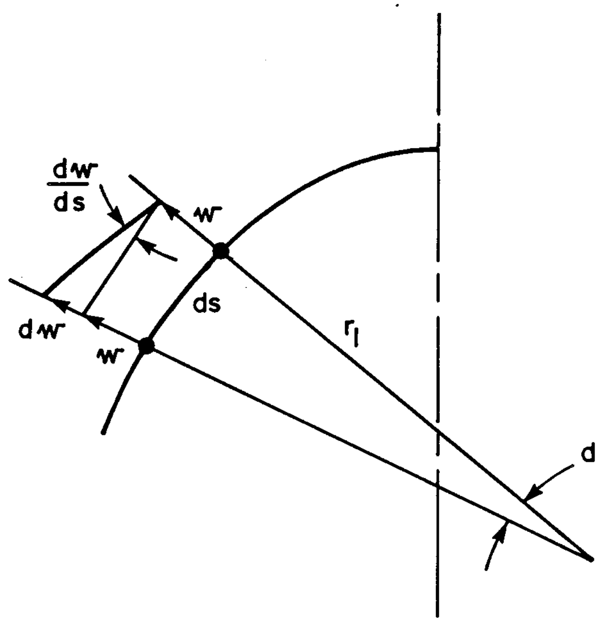


Figure 3.2(b) MERIDIONAL ROTATION β DUE TO DISPLACEMENT w

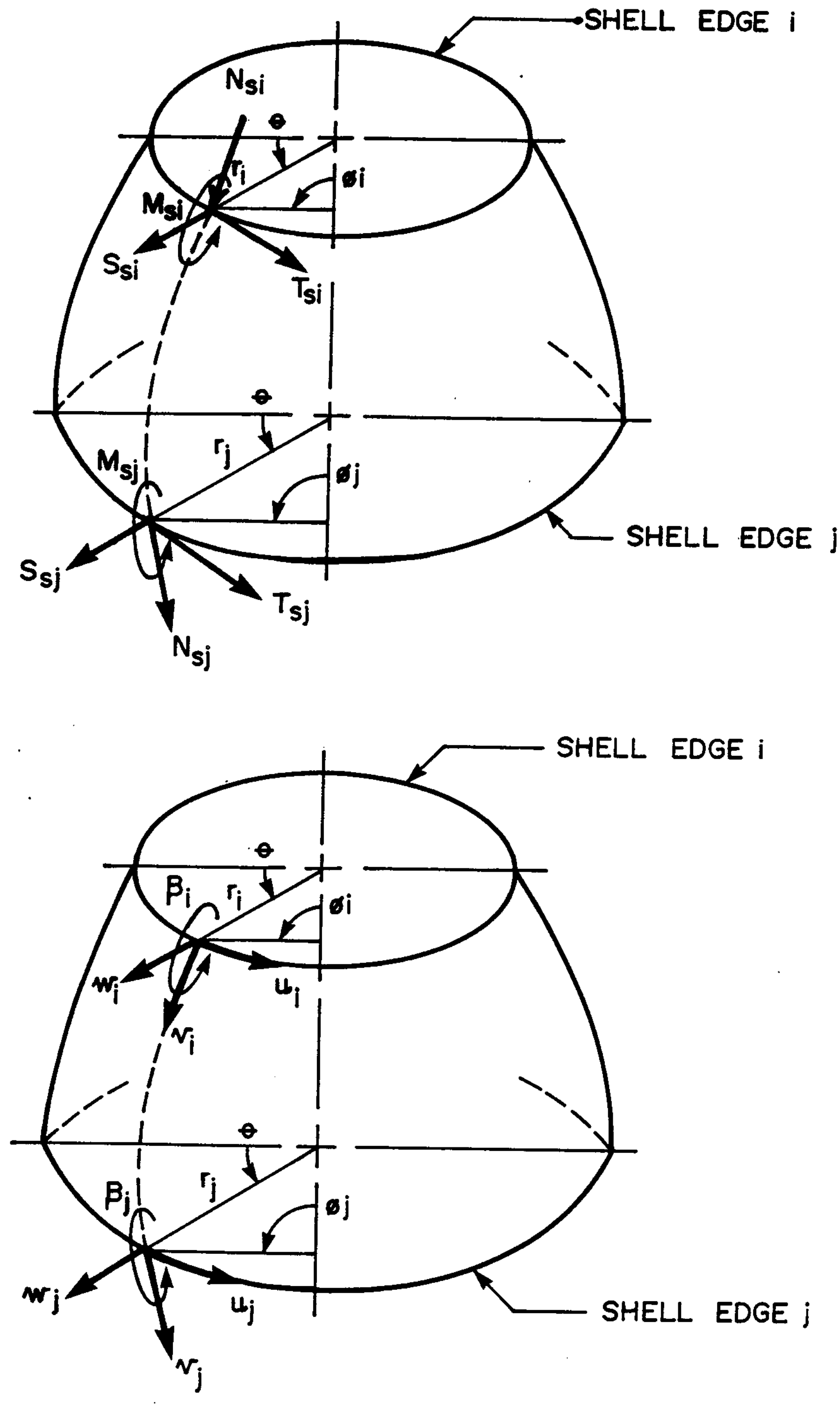


Figure 3.3 SASHELL STIFFNESS MATRIX SIGN CONVENTION

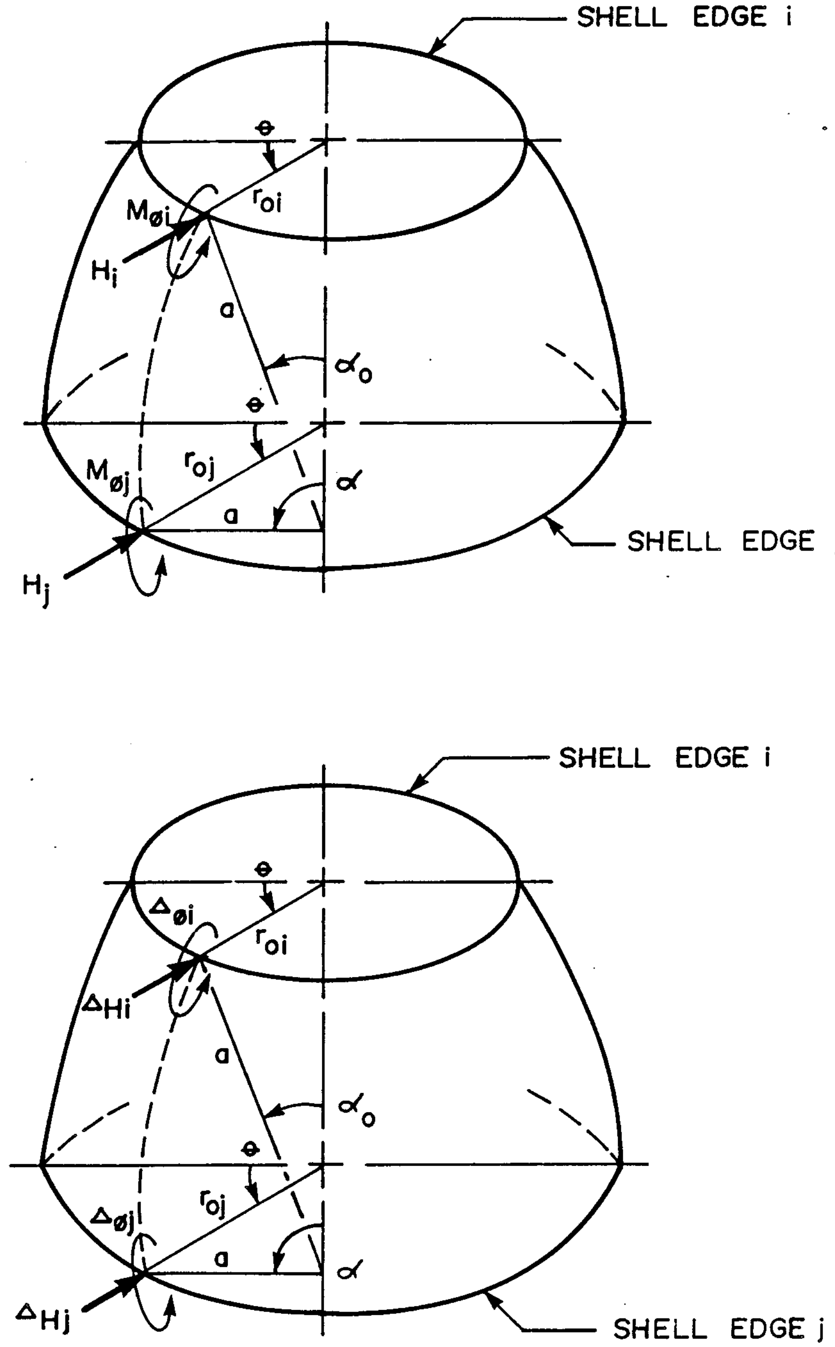


Figure 3.4 FLEXSHELL FLEXIBILITY MATRIX SIGN CONVENTION

4. STIFFNESS ANALYSIS SHELL PROGRAM

Based on the stiffness approach (Section 3.1), program SASHELL analyzes any structure that consists of a number of arbitrary shell of revolution segments assembled along a single axis. The solution conforms to the general theory of shells of revolution (Chapter 2), subject to small inaccuracies that may be introduced in the arbitrary load definition and numerical integration technique used. The original program was developed by Shazly (5) and a listing of the revised version is given in Appendix F. This chapter contains a description of the strategy used to program the solution given in Section 3.1.

The various steps of the program are summarized as follows:

1. Define nodal coordinates and segment properties.
2. Define loading. If the loading is not axisymmetric, the variation along the circumference is defined as a Fourier series in terms of n harmonics.
3. For each segment, repeat steps 3 to 8 for each harmonic of the loading.
4. Calculate the applied loads at integration points.
5. Integrate the governing set of equations using the fourth-order Runge-Kutta technique to obtain the segment transfer matrix.
6. Establish the segment stiffness matrix and fixed end forces vector from the transfer matrix as described in Section 3.1.5.

7. Modify the segment stiffness matrix and fixed end forces vector for the following: sign convention, joint eccentricity, and coordinate transformation.
8. Assemble the segment stiffness matrix and fixed end forces vector into the global matrices, subtracting the fixed end forces from the applied loads including self weight.
9. Impose the known boundary conditions.
10. Solve the equilibrium equations.
11. Calculate the final displacements and stress resultants.

The sign convention consistent throughout the program is shown in Fig. 3.3.

1. Moments and rotations are positive as shown in the figure.
2. In-plane forces and corresponding displacements tangent to the meridian are positive downwards from the top of the structure.
3. Shear forces and corresponding displacements tangent to the parallel circle are positive in the counterclockwise direction.
4. Shear forces and corresponding displacements perpendicular to a point on the parallel circle are positive in the direction away from the axis of revolution.

The program was written to be unit independent, that is any consistent set of units can be used. The user must select a unit of length and force, and all input quantities must be in these two units. Output will be in the same units. For example, if the unit of length selected is feet and the unit of force in kips, then all input quantities must be consistent with these units, such as thickness in feet, modulus of elasticity in kips per square feet, etc. Output will be in terms of kips per foot and foot-kips per foot, etc.

4.1 Geometric Definition

The structure is divided into segments each of which is a single shell of revolution connected along nodal parallel circles or 'nodes'. Each 'node' is defined by two global coordinates, x, which measures distances along the axis of revolution directed downwards from the top of the structure, and r_o, which is the radius of the parallel circle passing through the 'node'. Segment location and connectivity is established by identifying the 'nodes' at each end of a segment.

The theory on which the program is based is applicable to any arbitrary shell of revolution, that is the shape of the meridian and the thickness the meridian can vary in any arbitrary manner. However, the input required to define such an arbitrary segment is considered to be excessive and for many practical applications, the shell segments consist of well-defined geometric shapes for which a simple input is possible. Thus it was decided to pre-define a series of standard geometric shapes in which the thickness could vary linearly from one edge to the other. The meridian geometries that are pre-defined are cylinders, cones, circular plates, spheres, ogives, and hyperboloids of revolution. Arbitrary shaped segments may be approximated by a combination of these pre-defined segments.

4.2 Load Definition

A general load on a segment in SASHELL is defined by expressing the variation of the loading in the circumferential direction as a Fourier series at each edge and assuming a linear variation along a meridian. Thus the Fourier series at each end of a segment must contain the same number of harmonics but the magnitudes of the coefficients may vary. A complete definition may involve up to five terms corresponding to each loading component namely: loads in the direction tangent to the meridian (s), tangent to the parallel circle (θ), and perpendicular to the tangent to the meridian (z), and temperature at the outer and inner surface of the shell surface.

The user may input the loading directly by entering the Fourier coefficients for each harmonic number for each non-zero loading component. Alternatively, the user may enter the magnitude of the loads at a prescribed number of equally spaced discrete points along the circumference for each non-zero loading component. Program SASHELL calls up subroutine FORIT to generate the required series to any user-specified number of harmonics.

For loads that vary randomly in the circumferential direction, for example, a line load along the meridian, it may not be apparent initially as to the number of harmonics required to adequately represent the loading. Program SASHELL can be used to generate the Fourier coefficients from which the magnitude of the loads can be computed and compared with the input load values, without forming the solution, by using the appropriate output control parameter (IPRINT = 0).

While the above describes the entry of a complex loading, many loadings encountered in practice are relatively simple, independent of the segment geometry, or axisymmetric. The input for such loads is, of course, much simpler. For convenience, certain commonly occurring loadings are pre-defined (LDW) and can be entered merely by calling the appropriate loading number (LDE). Such loadings include self weight, snow load that is a uniform pressure over the horizontal projection of the shell surface, and hydrostatic loading. The NBC snow drift for arch roofs (20) as approximated for spherical domes is also included.

The only provision for load superposition is the case of all pre-defined loads including self-weight and the first of the user-defined loads. If more than one user-defined load is input, there is no provision for superposing the results of these two user-defined loads. If only a portion of the pre-defined loads are to be superposed with a user-defined load, a separate run is necessary.

4.3 Formulation of the Segment Stiffness Matrix and Segment Fixed End Forces Vector

In Section 3.1.3, the Flügge shell equations were transformed into eight first order differential equations involving only the four natural shell displacements, the associated stress resultants and their derivatives with respect to the single coordinate along the meridian, s. These equations are given in matrix form as Eqn. 3.30 (subroutine FLUGGE) which is rewritten as follows.

$$\{y_s^\circ\} = [A_s]\{y_s\} + \{B_s\} \qquad 3.30$$

where $\{y_s\}$ are the four natural shell displacements and their associated stress resultants, $\{y_s^\circ\}$ are their derivatives with respect to s, $[A_s]$ is the coefficient matrix involving only functions of the material and geometric properties of the shell, and $\{B_s\}$ is a function of the applied loads.

To form the segment stiffness matrix, it is necessary to relate the displacements at both edges of the segment to the corresponding stress resultants. While this cannot be done directly, it is possible to relate the displacements and stress resultants at one edge to the displacements and stress resultants at the other edge, i.e., to form the transfer matrix by assuming a set of initial values at the near edge and integrating Eqn. 3.30 over the segment length to obtain the corresponding values at the far edge. The numerical integration procedure outlined in Section 3.1.4 is coded in program SASHELL using subroutine RNGKT.

Again consider a segment of length l in the region $i \leq s \leq j$. Now, divide the segment into, say twenty equal parts, each of length e. Let the first interval be bounded by points m_1 and n_1, with midpoint o_1, the second interval be bounded by points m_2 and n_2, with midpoint o_2, and so on. Note that point m_2 of the second interval coincides with n_1 of the first interval, m_3 coincides with n_2, and so on. Also, points m_1 of the first interval and n_{20} of the last interval coincide with segment edges i and j respectively.

An integration of Eqn. 3.30 with respect to the coordinate s in the region $m_1 \leq s \leq n_1$ of length e with initial boundary value vector $\{y_i\}$, would yield a solution of the form

$$\{y_s\} = [H_s]\{y_i\} + \{Q_s\} \qquad 3.40$$

where $[H_s]$ and $\{Q_s\}$ represent the matrices arising from the integration of matrices $[A_s]$ and $\{B_s\}$, respectively. Substituting Eqn. 3.40 into 3.30

$$\{y_s^\circ\} = [A_s][H_s]\{y_i\} + ([A_s]\{Q_s\} + \{B_s\})$$

$$\{y_s^\circ\} = [A_s][H_s]\{y_i\} + \{P_s\} \qquad 4.1$$

This equation is of the form that can be integrated numerically using the Runge-Kutta fourth order method described in Appendix C. Eqns. C.2 to C.5 become

$$a = e([A_i][H_i]\{y_i\} + \{P_i\}) \qquad 4.2$$

$$b = e([A_{o1}][H_{o1}](\{y_i\} + a/2) + \{P_{o1}\}) \qquad 4.3$$

$$c = e([A_{o1}][H_{o1}](\{y_i\} + b/2) + \{P_{o1}\}) \qquad 4.4$$

$$d = e([A_{n1}][H_{n1}](\{y_i\} + c) + \{P_{n1}\}) \qquad 4.5$$

therefore,

$$\{y_{n1}\} = \{y_i\} + (a + 2b + 2c + d)/6 \qquad 4.6$$

But from Eqn. 3.40

$$\{y_{n1}\} = [H_{n1}]\{y_i\} + \{Q_{n1}\} \qquad 4.7$$

Substituting this into Eqn. 4.6 yields

$$[H_{n1}]\{y_i\} + \{Q_{n1}\} = \{y_i\} + (a + 2b + 2c + d)/6 \qquad 4.8$$

A similar procedure is repeated for subsequent intervals using initial boundary values computed from the preceding intervals until the last point j in the segment is reached. For example, for the second interval in the region $m_2 \le s \le n_2$, the vector $\{y_i\}$ in Eqns. 4.1 to 4.8 inclusive is replaced with $\{y_{n1}\}$ determined from Eqn. 4.7.

Upon integrating over all intervals in the segment, Eqn. 4.8 is divided into two individual expressions: one for the transfer matrix $[H_n]$ and the other for the load vector $\{Q_n\}$ for each interval of length e, in the region $m \le s \le n$. The segment transfer matrix $[H_j]$ and the segment load vector $\{Q_j\}$ is then formed from these expressions, from which the segment stiffness matrix [SM] and the fixed end forces vector {FE} respectively are evaluated (subroutine STIFIX) as described in Section 3.1.5.

In the derivation of the segment stiffness matrix and fixed end forces (Section 3.1), the sign convention used corresponded to that used by Flügge (Fig. 2.2). As a result, the stiffness matrix will have some negative elements in the main diagonal. To solve the equilibrium equations in an efficient manner, it is important that the matrix be symmetric and positive definite. This is easily corrected by changing the sign conventions to those described in Fig. 3.3. Consequently, the matrices [SM] and {FE} is premultiplied by the following matrix

$$\begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && -1 &&&&& \\ &&& -1 &&&& \\ &&&& -1 &&& \\ &&&&& -1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{bmatrix}$$

Before the [SM] and the {FE} are assembled into the global matrices, two additional factors must be considered as described in the following sections.

4.4 Shell Eccentricity

In many cases, the middle surface of two adjacent elements do not coincide. A transformation of the stiffness coefficients and fixed end forces at a common reference point is thus necessary before assembling the segment matrices into the global matrices. Eqns. 2.6 relate the displacement components of a point i at a distance z from the middle surface to that of a point on the middle surface lying in the same plane. Expanding Eqn. 2.6 by means of a

Fourier series yield

$$\begin{Bmatrix} \beta_i \\ w_i \\ v_i \\ u_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z & 0 & 1 & 0 \\ 0 & \frac{nz}{r} & 0 & \frac{r_2+z}{r_2} \end{bmatrix} \begin{Bmatrix} \beta \\ w \\ v \\ u \end{Bmatrix}$$

Solving the displacement components of a point on the middle surface yield

$$\begin{Bmatrix} \beta \\ w \\ v \\ u \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -z & 0 & 1 & 0 \\ 0 & \frac{-nzr_2}{r_o(r_2+z)} & 0 & \frac{r_2}{r_2+z} \end{bmatrix} \begin{Bmatrix} \beta_i \\ w_i \\ v_i \\ u_i \end{Bmatrix}$$

or

$$\{D\} = [EC]\{D_i\} \qquad 4.9$$

From the work equivalence requirements, the relation between the stress resultants at the two points can be written as

$$\{F_i\} = [EC]^T\{F\} \qquad 4.10$$

4.5 Coordinate Transformation Matrix

Displacements and stress resultants, at any point along the generator, are in the direction tangent to the meridian at this point and the direction perpendicular to it. Due to possible discontinuity of the meridian curve at the junction between two segments, it is necessary to transform the influence coefficients and the fixed end forces at this junction, from the local to the global coordinate system, defined by the direction of the structure's axis of revolution, x, and that perpendicular to it, r_o. The transformation equations consistent with the sign convention described in Section 4.1 are

$$\{D_L\} = [L]\{D_G\} \tag{4.11}$$

$$\{F_G\} = [L]^T\{F_L\} \tag{4.12}$$

where

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_i & -\cos\phi_i \\ 0 & \cos\phi_i & \sin\phi_i \\ 0 & 0 & 0 \end{bmatrix} \tag{4.13}$$

{D} and {F} represent the displacements and forces, respectively, and the subscripts L and G represent the local and global coordinates. ϕ_i is the angle measured from the global axis of rotation to the meridian at the point s = i.

4.6 Segment Matrix Assembly

The segment stiffness matrix and the fixed end forces vector are assembled into the global matrices to form a set of equilibrium equations for the structure (subroutine STORE). Boundary conditions are imposed (subroutine BOUNDC), and the final displacements at the boundaries of each segment are obtained from the solution of the equilibrium equations (subroutine SOLVER). By substituting the final known boundary displacements of each segment into the corresponding segment equilibrium equation (Eqn. 3.46), the final stress resultants at the boundaries can be obtained.

4.7 Stress Resultants and Displacements at Intermediate Points

In order to evaluate the displacement and stress resultant at any number of intermediate points within a segment, the final boundary conditions at one end of the segment, say at end s = a, are used as the initial conditions in integrating the governing set of differential equations. At each intermediate point, the secondary stress resultants (which were eliminated from the governing differential equations) can be evaluated, first by evaluating the derivative of the displacements using Eqn. 3.26,

$$\{D^{\circ}\} = [A_1 \;\; A_2]\begin{Bmatrix} D \\ F_s \end{Bmatrix} \qquad 3.26$$

and then substitute into Eqn. 3.27

$$\{F_{\theta}\} = [B_4 \;\; B_5]\begin{Bmatrix} D^{\circ} \\ D \end{Bmatrix} \qquad 3.27$$

A simple check on the results of the integration is that the displacements and primary stress resultants at the terminal end of the segment should agree with the known boundary conditions at this end.

4.8 Limitations

Three factors affect the solution technique presented in the preceding section.

1. Singularity of the Governing Equations at the Apex

If the shell has a pole ($r_0 = 0$), the coefficients in the governing differential equations become singular. When such is the case, subroutine APEX arbitrarily chooses a boundary s = 0 not at the pole, but a very short distance away, determined from the length of the interval between the points of integration, and impose the boundary conditions at s = 0 as follows:

For harmonic number n = 0,

$$\beta = v = u = S_s = 0$$

For harmonic number n = 1

$$w = M_s = N_s = T_s = 0$$

For harmonic number n > 1

$$w = u = v = M_s = 0$$

An alternative is to introduce at the apex, a small fictitious hole, say x = 0.01 feet, as a free boundary. This yields a more satisfactory solution throughout the shell meridian, except for the single point at the apex, which can easily be extrapolated from the other values.

2. Stability of the Numerical Integration Process

The integration of the governing differential equations outlined in Section 4.3 is carried out using arbitrary boundary conditions (along with the known boundary conditions) at one end as the starting values. The calculations are repeated with the revised values of the unknowns until the boundary conditions are satisfied. Serious difficulties will arise if the solution dies out very rapidly, since whatever boundary conditions that are assumed at one end, will have become ineffective at the far end.

The resulting loss of accuracy cannot be eliminated by reducing the mesh for the integration. As more steps are taken, the numerical solution deviates from the actual solution, resulting in partial instability.

The solution is to limit the length over which the integration is taken so that the transfer matrix will not suffer a loss of accuracy. This critical length is dependent on the shell geometry (19) as initial boundary disturbances will die out more rapidly with doubly curved surfaces with small radii.

$$\lambda^4 L = \left(\frac{3(1-\nu^2)}{r^2h^2}\right)L$$

If the segment length exceeds the critical length, the segment is merely divided into smaller segments each having a length less than the critical length (subroutine SEGEOM).

Therefore, the upper bound of the critical length shall not exceed 25, and the number of points of integration, NP required for convergence, is conservatively limited as follows (5):

$$NP \geq 21 \text{ for } \lambda L \leq 20$$

$$NP \geq 31 \text{ for } 20 < \lambda L < 25$$

As an alternative, it is possible to manually alter the number of subsegments. If this results in a significant change in the solution, then the subsegments must be further subdivided.

3. Convergence of the Fourier Expansions

Loads varying along the circumference are expanded into a Fourier series (Section 4.2). Theoretically, the number of harmonics required for an arbitrary periodic function to be represented 'exactly' is infinity. However, the load can always be described at a sufficient number of discrete points to approximate the actual load conditions. Generally, if N is the number of harmonics required for the analysis, the load coefficients along the circumference may be described at a set of points, 2N equally spaced over the interval $0 \leq \theta \leq 2\pi$ such that

$$\theta_i = \frac{2\pi i}{2N} \quad \text{for } i = 0,1,2,\ldots,2N-1.$$

A termination of the higher harmonics or exemption of a harmonic number in the series can be decided upon by comparing the load values computed from the input or generated Fourier coefficients with the actual load values.

5. FLEXIBILITY ANALYSIS SHELL PROGRAM

Program FLEXSHELL analyzes structures composed of segments of certain specified shells of revolution for axisymmetrical loading only. Based on the flexibility solution (Section 3.2) to the equations of the general theory of shells of revolution, the program listed in Appendix F is a substantial extension of an earlier version developed by Murray, et al(4) for the analysis of the Gentilly type containment structure.

The implementation of the flexibility method used in FLEXSHELL requires a closed form solution for the influence coefficients and the particular (membrane) solutions. For these reasons, only a limited number of particular shell configurations and loadings are included. But the range provided is sufficient to handle most of the commonly occurring situations in practice. Provisions were made to incorporate easily new segment types and additional load cases.

Presently, the program consists of three types of 'short' shell segments: cylinders, spheres, and cones, and a base segment. In the analysis of a 'short' segment, the interaction of the two meridian boundaries are taken into account, i.e. four redundant forces and their corresponding displacements, two at each boundary, are considered when forming the segment flexibility matrix. Spherical and conical segments can be right-side-up or 'inverted'. An 'inverted' segment is that which forms a cup-like shape as illustrated by the lower half of Figs. 5.1(a) and (b).

The program can handle seven load cases: uniform pressure, self-weight, uniform prestressing, constant temperature change, thermal gradient, snow load (a uniform pressure over the horizontal projection of the shell), and hydrostatic load.

The logic flow of program FLEXSHELL is as follows:

1. Define the segment connectivities;
2. Satisfy the rigid body motion requirement by evaluating the effect of the vertical load components on the segment;
3. Evaluate the joint eccentricity effects;
4. Establish the segment flexibility matrix;
5. Solve the compatibility equations;
6. Solve for the final stress resultant values by superimposing the particular solution stresses with the stresses due to bending.

Consider a shell segment cut by a vertical plane shown in Fig. 5.1, the sign conventions consistent throughout the program are as follows:

1. Moments and rotations are positive as shown in the figure.
2. Horizontal forces, displacements, and eccentricities are positive in the direction towards the line of symmetry, also known as the axis of revolution.
3. Vertical forces are positive downward.
4. For base segments, vertical displacements are positive

downward, whereas, vertical eccentricities are positive upwards.

The program was written to be unit independent, that is any consistent set of units can be used. The user must select a unit of length and force, and all input quantities must be in these two units. Output will be in the same units. For example, if the unit of length selected is feet and the unit of force in kips, then all input quantities must be consistent with these units, such as thickness in feet, modulus of elasticity in kips per square feet, etc. Output will be in terms of kips per foot and foot-kips per foot, etc.

5.1 Definitions and Notations

Segments are defined with reference to a coordinate starting at the apex on the axis of revolution of a structure, and traversing the midsurface of the shell segment downwards, until again reaching the axis of revolution at the base of the structure. The coordinate for branches which do not fall in this primary circuit may be defined in the same manner, starting at the free edge, increasing downwards. Therefore, with the exception of the last segment in the primary circuit, the 'bottom' of each segment is always supported by the 'top' of the adjacent segment. For reasons which will be explained later, the segments must be numbered sequentially in such manner that any segment always has a higher number than any of the segments which it supports.

5.2 Connectivity Matrix

The connectivity matrix is established by satisfying the geometric compatibility requirements between adjacent segments. This is accomplished by forming the algebraic summation of the horizontal displacements and meridional rotations at adjacent edges of the shell segments. To express these equations in matrix form, it is necessary to number the segments as described earlier, to ensure the consistency in the order of assembly of the end deformations. Furthermore, associated with each segment is a flag indicating the presence or absence of a connection at the 'top' and 'bottom' of the segment, input as IR and JR, respectively. The connection between segments is specified by IDCO(I,1) and IDCO(I,2), which is the number of the 'top' segment and the adjacent 'bottom' segment, respectively. The compatibility equations expressed in matrix form is

$$[A]\{\Delta\}_t = \{0\} \tag{5.1}$$

where [A] is the connectivity matrix expressing the compatibility requirements between adjacent segments; $\{\Delta\}_t$ is the total segment deformation vector. A detailed derivation of the connectivity matrix [A] is found in a report by Murray, et al(4).

5.3 Vertical Edge Load

This section demonstrates how the rigid body motion of the shell structure which have been ignored up to this point is taken into account. Loads from the 'top' segment may be transmitted to the segment below it as a vertical edge load P, as shown in Fig. 5.2. Unlike the cylindrical segment which can carry this load by membrane action alone, for the case of the spherical and conical shells, a horizontal force H_V must be added vectorially, so that a resultant force N_ϕ is formed (1,6,14). This horizontal force must be compensated later by subtracting this value from the real horizontal loads PSF(N,1) and PSF(N,3) acting on segment N.

5.4 Shell Eccentricity

Since segments at a joint may not always end at the same point, a horizontal segment eccentricity may be specified in the input data. This results in the eccentricity of the edge horizontal and vertical loads, which in turn produces a moment which must be added to the existing moments PSF(N,2) and PSF(N,4) at the edges of segment N. This moment is automatically calculated in the program.

5.5 The Particular Solution

The particular solution is approximated by the membrane solution. The computation of the membrane in-plane forces N_ϕ and N_θ, for the spherical and conical segments are incorporated into function subprograms FN1, FN2, FN3, and FN4, respectively. The equations used in these subprograms are found in Tables 3.4 and 3.6. The solution for the cylindrical segment, found in Table 3.5, is simple enough, that a separate subroutine is not necessary. The particular solution displacements PSD are obtained from evaluating the equations for Δ_H and Δ_ϕ found at the end of Tables 3.4 to 3.6. These computations are incorporated into subroutines PCYLIN, PDOME, and PCONE, respectively. The particular solutions for the base segment derived in (4) are incorporated into subroutine PBASE.

5.6 The Flexibility Matrix

As derived earlier, the flexibility matrix for a shell of revolution may be expressed as follows

$$[F] = [TA][TT]^{-1} \qquad 3.87$$

These matrix operation is performed by subroutines CYLIN, DOME, CONE, and BASE, for the cylindrical, spherical, conical, and base segments respectively.

The first step is to initialize the coefficients of [TA] and [TT]. For subroutine CONE, this necessitates the use of another subroutine MMKEL2, which computes the Kelvin functions of order 2 using published recurrence formulas (10). Subroutines MMKEL2 and DOME call up a system-dependent subroutine which evaluates the Kelvin functions of order zero and one and their derivatives. Secondly, a check is made to determine if the segment is 'inverted'. By definition, an 'inverted' spherical or conical segment is that which forms a cup-like shape as illustrated by the lower half of Figs. 5.1(a) and (b) and as shown in Figs. (b) of Tables 3.4 and 3.6. If the segment is 'inverted', subroutine ROWEX is called. This performs row interchanges in the [TA] and [TT] to conform to the 'inverted' configuration. Furthermore, a check is made to determine whether the segment is a closed spherical or conical dome. If so, the four by four segment flexibility matrix degenerates into a two by two matrix. The next step is to invert the [TT] matrix which is performed by subroutine TTINV which is capable of inverting a four by four or a degenerated two by two matrix. Finally, the matrix multiplication

$$[TA][TT]^{-1}$$

is performed, thus forming the segment flexibility matrix.

5.7 Matrix Formulation of the Solution Procedure

Let $[F]_i$ be the flexibility matrix of segment i, then from Eqn. 3.86,

$$\{\Delta\}_i = [F]_i\{V\}_i \tag{5.2}$$

Similarly, for the entire structure, the equations are

$$\{\Delta\} = [F]\{V\} \tag{5.3}$$

where the end displacements $\{\Delta\}_i$, end forces $\{V\}_i$, and flexibility matrix $[F]_i$ of element i are assembled into the global matrices $\{\Delta\}$, $\{V\}$, and $[F]$, respectively, in the order consistent with the sequence of segment numbering.

The particular solution displacements and the vertical edge load displacements $\{\delta\}$ in the corresponding order as $\{\Delta\}$. The total displacement vector is

$$\{\Delta\}_t = \{\Delta\} + \{\delta\} \tag{5.4}$$

Substituting Eqn. 5.5 into 5.1,

$$[A](\{\Delta\} + \{\delta\}) = \{0\} \tag{5.5}$$

and multiplying Eqn. 5.3 by [A],

$$[A][F]\{V\} = [A]\{\Delta\} \tag{5.6}$$

Let $\{q\}$ be a set of relative displacements in terms of the homogeneous solution $\{\Delta\}$ such that

$$\{q\} = [A]\{\Delta\} \tag{5.7}$$

From a general theorem in structural analysis (13), if a set of forces $\{V\}$ is associated with a set of displacements $\{v\}$, and if in another coordinate system, the same set of forces may be described as $\{U\}$, and their associated displacements as $\{u\}$, then the work done in the two systems must be identical when undergoing equivalent displacements, i.e.,

$$\langle u\rangle\{U\} = \langle v\rangle\{V\}, \tag{5.8}$$

similarly,

$$\langle q\rangle\{Q\} = \langle \Delta\rangle\{V\}, \tag{5.9}$$

where $\{V\}$ are the forces associated with displacements $\{\Delta\}$ and $\{Q\}$ are the redundant forces associated with the relative displacements $\{q\}$. Substituting the transpose of Eqn. 5.7 into 5.9,

$$\langle \Delta\rangle[A]^T\{Q\} = \langle \Delta\rangle\{V\} \tag{5.10(a)}$$

$$\langle \Delta\rangle([A]^T\{Q\} - \{V\}) = 0 \tag{5.10(b)}$$

since Eqn. 5.9 must be true for all $<\Delta>$, Eqn. 5.10(b) becomes

$$\{V\} = [A]^T\{Q\}, \tag{5.11}$$

Substituting Eqn. 5.11 into 5.6 yields

$$[A][F][A]^T\{Q\} = -[A]\{\delta\} \tag{5.12(a)}$$

$$[\bar{F}]\{Q\} = \{q_o\} \tag{5.12(b)}$$

where the structure flexibility matrix and structure particular solution displacements, respectively, are

$$[\bar{F}] = [A][F][A]^T \tag{5.13}$$

$$\{q_o\} = -[A]\{\delta\} \tag{5.14}$$

The set of simultaneous equations (Eqns. 5.12(b)), can then be solved for the redundants Q, by Gauss elimination. Once evaluated, the value of the redundants may be back-substituted into Eqn. 5.11, to find the edge forces V, which in turn can be substituted into Eqn. 5.3, to find the displacements Δ. The final solution can then be obtained by superimposing these values on the particular solution.

5.8 Limitations

In FLEXSHELL, it is necessary to limit the range of values of ξ for which the Kelvin functions and their derivatives are evaluated, such that(17)

$$0 < \xi \leq 119.0$$

For the spherical segment,

$$\xi = \lambda\phi\sqrt{2}$$

where $\lambda^4 \simeq 3(1-\nu^2)\frac{a^2}{h^2}$

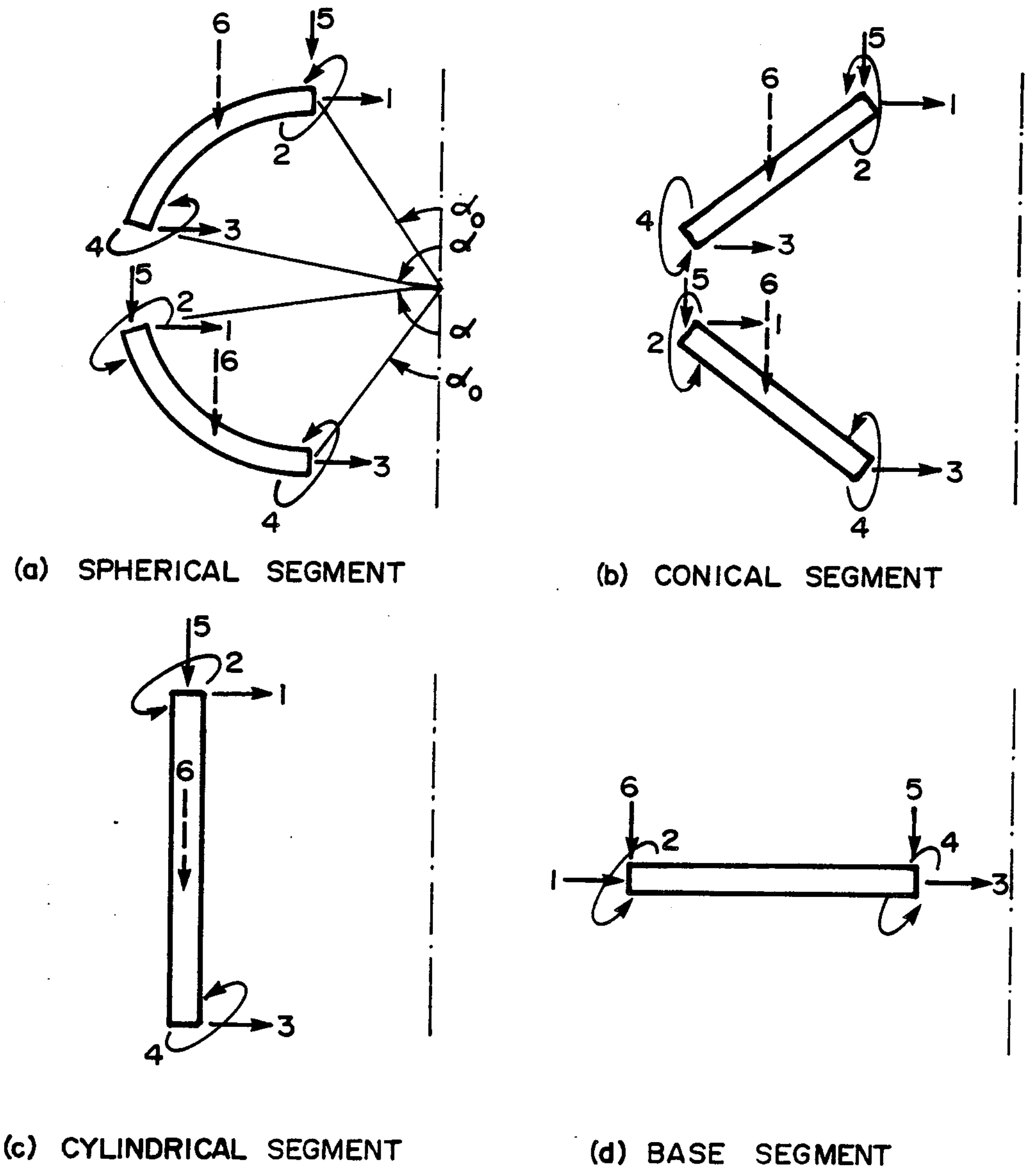
and for the conical segment,

$$\xi = 2\lambda\sqrt{s}$$

where $\lambda^4 = \dfrac{12(1-\nu^2)}{h^2\tan^2\alpha}$

These limits may be better understood as follows. As the angle ϕ for the sphere approaches zero, i.e., ξ approaches zero, the shell degenerates into a point. And as the semi-vertex angle α for the cone approaches zero, i.e., ξ approaches ∞, the shell degenerates into a line. For both limiting cases, no shell action is physically possible. Furthermore, as the semi-vertex angle approaches 90°, i.e., ξ approaches zero, the cone becomes a circular plate. Consequently, a lower limit on the range of values of α is imposed depending on the thinness of the conical shell. For instance, when s/h = 500, α must be greater than 26°, and when s/h = 100, α must be greater than 5.5°. Conical segments may be broken up accordingly into two or more segments to satisfy these limits. For example, a shell with s/h = 500 and α = 10°, may be divided into five segments each with s/h = 100 and α = 10°. Note that α can be very close to, but not equal to 90°, say 89.5°.

Similarly for the sphere, as the shell becomes increasingly thin, a limit must be imposed on the ϕ values. For instance, when a/h = 1000, ϕ must not exceed 116°, and when a/h = 1680, ϕ must not exceed 90°.



(a) SPHERICAL SEGMENT

(b) CONICAL SEGMENT

(c) CYLINDRICAL SEGMENT

(d) BASE SEGMENT

ARRAYS: P B F – PARTICULAR SOLUTION BASE FORCES
P S D – PARTICULAR SOLUTION DISPLACEMENTS
P S F – PARTICULAR SOLUTION SEGMENT FORCES
S F – SEGMENT FORCES

Figure 5.1 SUBSCRIPTING OF SEGMENT ARRAYS FOR FLEXSHELL

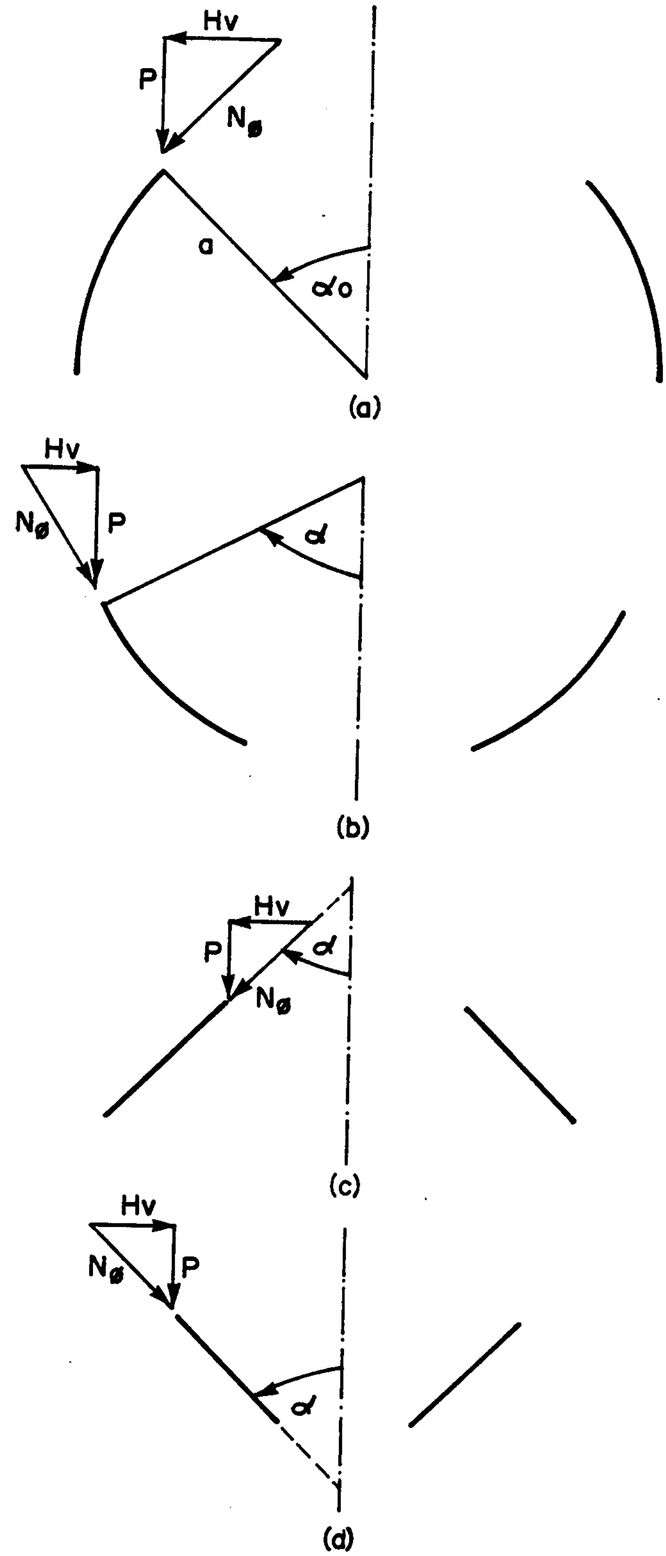


Figure 5.2 VERTICAL EDGE LOAD EFFECT

6. APPLICATION OF THE SHELL PROGRAMS

To demonstrate the capabilities of both computer programs, two example applications are presented. Both FLEXSHELL and SASHELL were run on identical axisymmetric problems using the Intze Tank. A non-axisymmetric problem using the hyperboloid tower presented by Shazly (5) was incorporated to illustrate the use of SASHELL for such types of problems. Input and output descriptions are included.

Example input and output files are listed in Appendices D and E.

6.1 The Intze Tank

Fig. 6.1 illustrates an Intze tank which are typically used for water storage. The structure is 114.35 feet high, consisting of cylindrical, spherical, and conical segments, and ring beams. The base of the structure is considered to be fully fixed. The material properties used are:

$$E = 0.5804 \times 10^{9} \text{ psf}$$

$$\nu = 0.167$$

$$\gamma = 150.0 \text{ pcf}$$

For input to FLEXSHELL, the separate shell segments were numbered in an increasing order from top to bottom, as shown in Fig. 6.1. The ring beams were modelled as cylindrical segments. To simulate the fully fixed condition at the base, the base segment was given a high modulus of elasticity, i.e.,

$$E(\text{base}) = 1.0 \times 10^{20} \text{ psf}$$

For input to SASHELL, the separate shell segments were numbered in the same order, strictly for convenience when comparing the two solutions. Due to the limitations in SASHELL for closed spheres (Section 4.8), a small fictitious hole was introduced at the apex with x = 0.01 feet and r_o = 1.37 feet. And because the bottom cone segment is quite 'long', it was manually divided into two segments. For the fully fixed condition, the following boundary conditions were imposed on the bottom end of the last segment at the base:

$$\beta = v = w = u = 0$$

Hydrostatic pressure for a liquid weight density, γ_W = 62.4 pcf, was input by specifying LDW = 1.

6.2 The Hyperboloid Tower

Fig. 6.2 illustrates a typical hyperboloid natural draft cooling tower. The 355-foot high structure is supported by columns evenly on a 290-foot diameter base. The throat of the tower, located 60 feet below the top of the structure is 165 feet in diameter. The thickness vary from 30 inches at the bottom of the structure to 6 inches at 25 feet from the bottom. The thickness at the top 10 feet of the structure also vary from 24 inches at the top to 6 inches. For the regions other than those mentioned, the thickness is constant at 6 inches.

The geometric equation for a hyperbola is

$$\frac{r^2}{a^2} - \frac{x^2}{b^2} = 1$$

where r is the horizontal radius, x is the vertical coordinate measured from the origin at the throat of the shell, a is the throat radius at x = 0, and b is a constant such that the ratio b/a is equal to the slope of the asymptotes to the hyperbola. The constant b required in the calculation of the principal radii of curvature r_1 and r_2 is determined by substituting the values for the structure shown in Fig. 6.2, where r = 145 feet, a = 82.5 feet, and x = 295 feet, which yields b = 204.1 feet.

The following concrete properties were used for the example problem.

$$E = 4.0 \times 10^6 \text{ psi}$$

$$\nu = 0.15$$

$$\gamma = 150.0 \text{ pcf}$$

The structure is subjected to wind load based on the ACI-ASCE Committee 334 recommendations using a normalized wind pressure distribution in the circumferential direction as shown in Fig. 6.3. And assuming the structure will be located in an open area with rough terrain, the wind pressure profile used is as shown in Fig. 6.4.

The tower was input to SASHELL as five hyperboloids of revolution. The base of the structure was considered to be hinged, i.e., the following boundary conditions were imposed on the last segment at the base:

$$M_s = w = v = u = 0$$

The geometric and material properties are input as shown in Appendix D.

As illustrated on Fig. 6.3, the load is non-axisymmetric in the circumferential direction, and symmetric with respect to the meridian $\theta = 0$. Thus, the final results need only be printed out for points along half the circumference of the shell, say NPC = 8. Also, because of the symmetry of the load with respect to the meridian, the analysis is required for the cosine coefficients of the Fourier series only, i.e., NTL = 2.

Generally, with increasing variation of the loading in the circumferential direction, the number of points at which the loading must be described increases. It was found that for the wind pressure distribution shown in Fig. 6.3, 24 circumferential points, i.e., NHPL = 24, adequately represents the actual loading conditions. Initially, the maximum number of harmonics, NHL = 12, the initial harmonic, NHS = 0, and the harmonics increment, NHIN = 1. However, after comparing the load values for 12 harmonics and those for 8 harmonics, it was found that only 8 harmonics are required for a reasonable approximation of the wind loading (Table 6.1). Furthermore, as illustrated by the wind pressure profile in Fig. 6.4, the load also varies along the height of the tower. This variation is approximated by a linearly varying load, acting along the meridian of each of the five hyperboloids of revolution. Thus, the magnitude of the load along the circumference of the shell is specified for the top and bottom ends of each segment, i.e., IDL(3) = 2.

Table 6.1 Comparison of the Input and Calculated Load Values for the Hyperboloid Cooling Tower

Circumfr	Input Load Values	Calculated 8 Harmonics	Load Values 12 Harmonics
1	-0.05300	-0.05353	-0.05388
2	-0.04240	-0.04171	-0.04152
3	-0.01060	-0.01162	-0.01148
4	0.02650	0.02770	0.02738
5	0.06360	0.06247	0.06272
6	0.06890	0.06992	0.06978
7	0.04770	0.04654	0.04682
8	0.02120	0.02275	0.02208
9	0.02120	0.01940	0.02032
10	0.02120	0.02265	0.02208
11	0.02120	0.02073	0.02032
12	0.02120	0.02059	0.02208
13	0.02120	0.02228	0.02032
14	0.02120	0.02059	0.02208
15	0.02120	0.02073	0.02032
16	0.02120	0.02265	0.02208
17	0.02120	0.01940	0.02032
18	0.02120	0.02265	0.02208
19	0.04770	0.04654	0.04682
20	0.06890	0.06992	0.06978
21	0.06360	0.06247	0.06272
22	0.02650	0.02770	0.02738
23	-0.01060	-0.01162	-0.01148
24	-0.04240	-0.04171	-0.04152

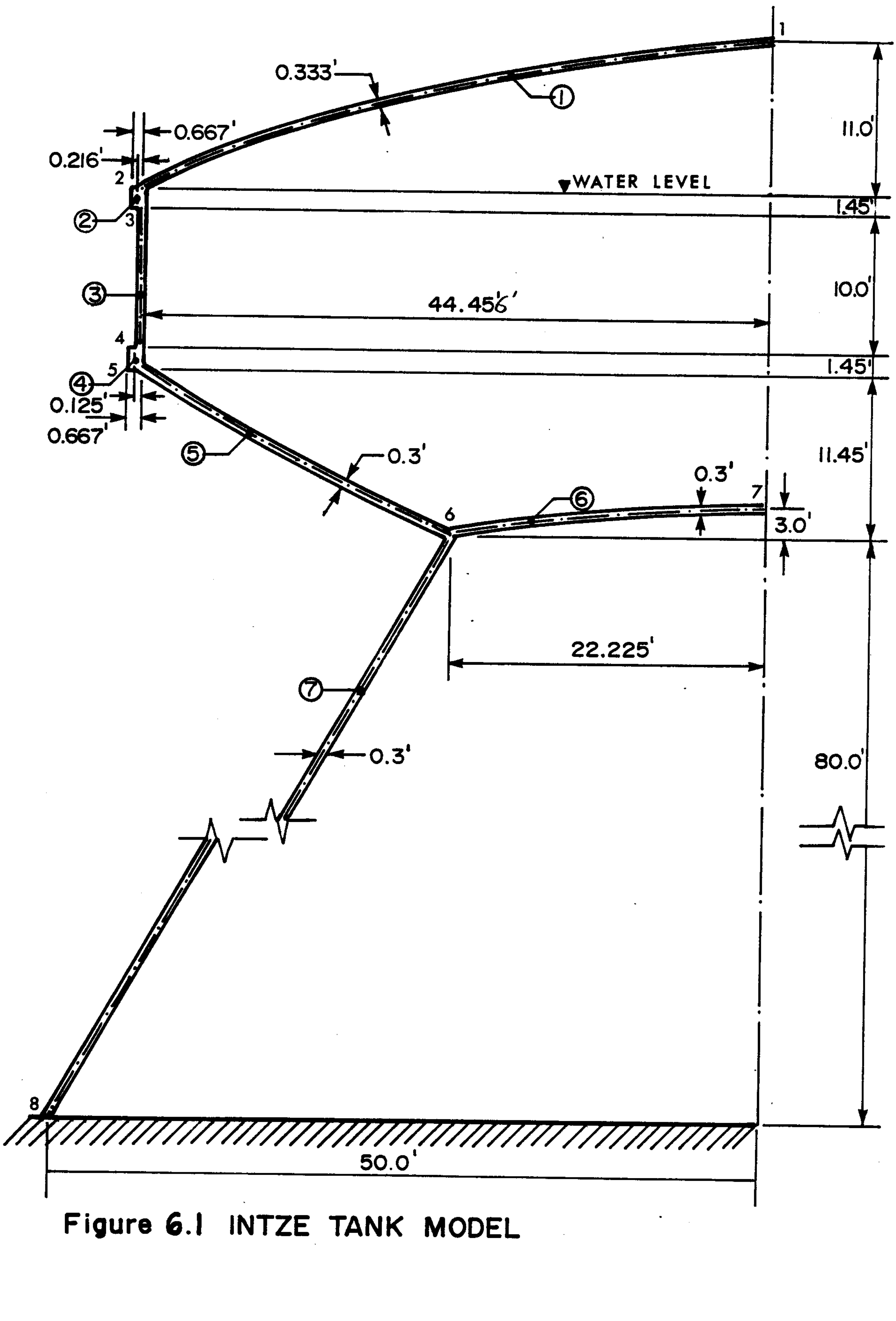


Figure 6.1 INTZE TANK MODEL

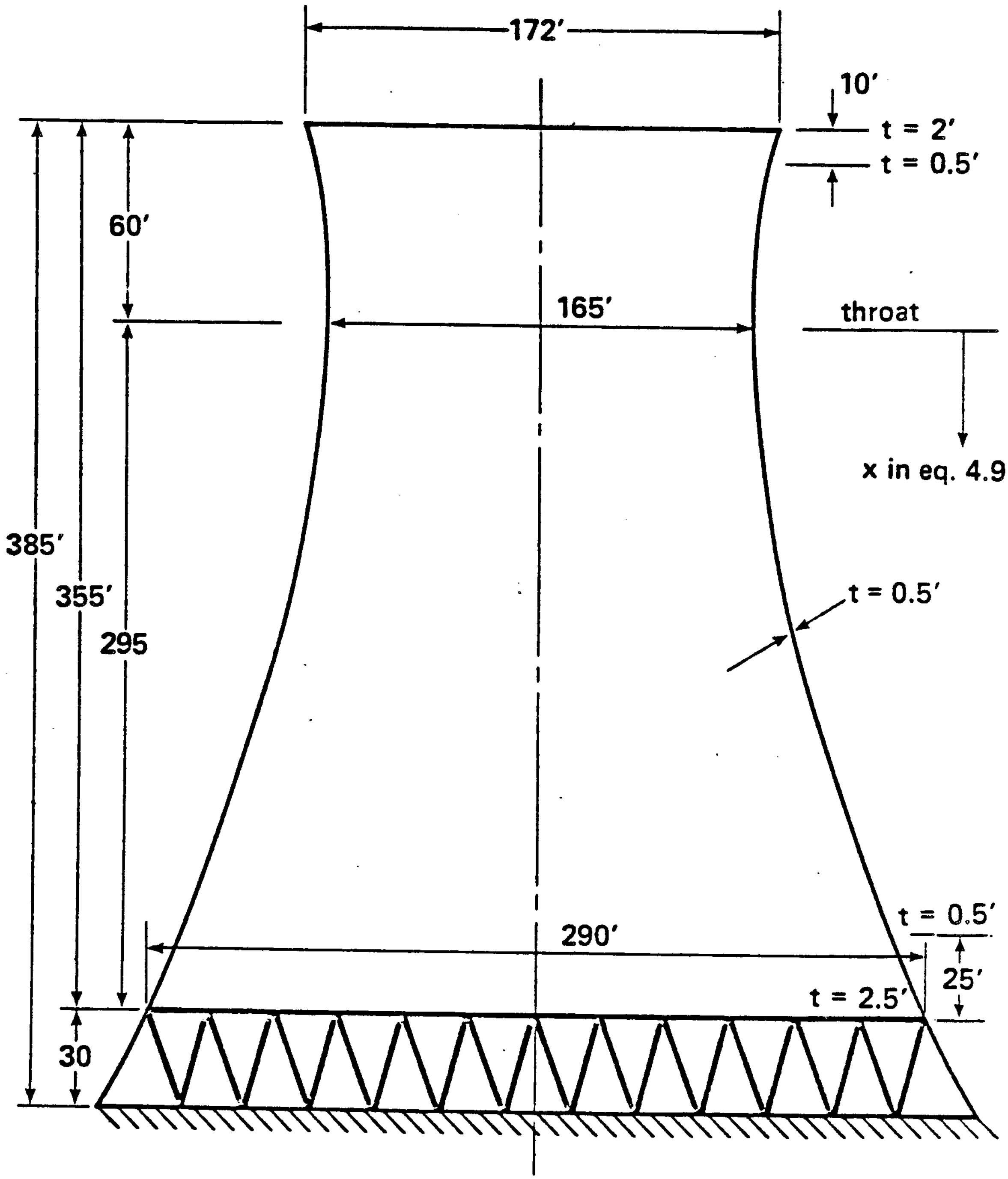


Fig. 6.2 Typical Hyperboloid Natrual Draft Cooling Tower

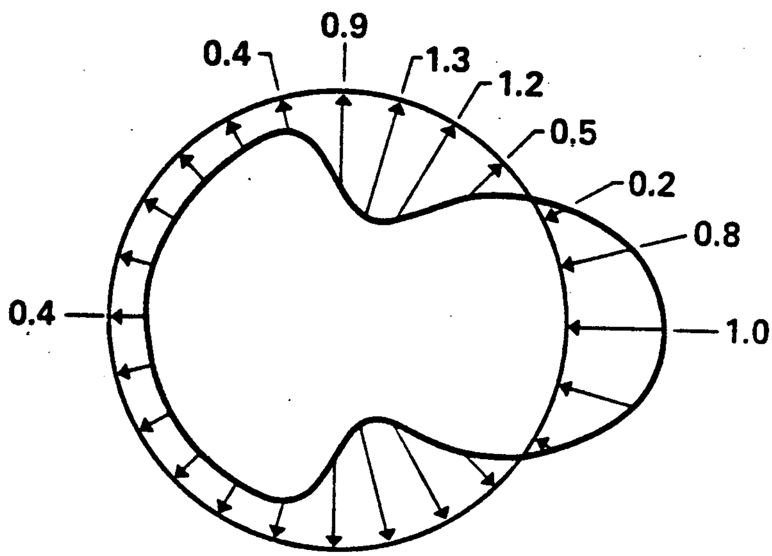


Fig. 6.3 Wind Pressure Coefficients

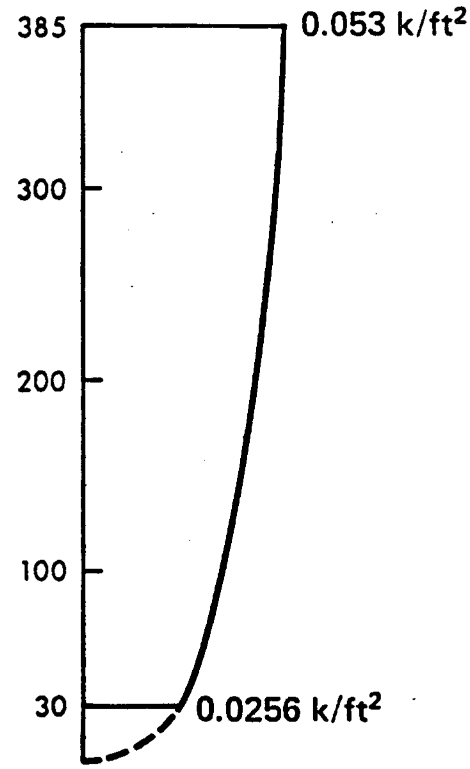


Fig. 6.4 Wind Pressure Profile

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APPENDIX A–HOMOGENEOUS SOLUTION FOR THE SPHERE AND CONE

For a spherical segment with $r_1 = r_2 = a$ and $r_0 = a \sin\phi$, and using Eqn. 3.57, Eqns. 3.65 and 3.68 become

$$\frac{d^2Q_\phi}{d\phi^2} + \cot\phi\frac{dQ_\phi}{d\phi} - (\cot^2\phi - \nu)Q_\phi = EhV \qquad 3.74$$

$$\frac{d^2V}{d\phi^2} + \cot\phi\frac{dV}{d\phi} - (\cot^2\phi - \nu)V = -\frac{a^2Q_\phi}{D} \qquad 3.75$$

With some simplifying assumptions these equations can be solved to form a fourth order equation in terms of a single variable (2,3,6,9,15). However, an alternative approach (16) is possible by introducing a linear differential operator as follows:

$$L(\ ) = \frac{d^2(\ )}{d\phi^2} + \cot\phi\frac{d(\ )}{d\phi} - \cot^2\phi(\ )$$

thus, Eqns. 3.74 and 3.75 become

$$L(Q_\phi) + \nu Q_\phi = EhV \qquad A.1$$

$$L(V) + \nu V = -\frac{a^2Q_\phi}{D} \qquad A.2$$

Substituting Eqn. A.1 into A.2 yields

$$LL(Q_\phi) + 4\lambda^4Q_\phi = 0 \qquad A.3$$

where $\lambda^4 = 3(1-\nu^2)\frac{a^2}{h^2} - \frac{\nu^2}{4}$

Eqn. A.3 may be written in either of the following forms:

$$L[L(Q_\phi) + 2i\lambda^2(Q_\phi)] - 2i\lambda^2[L(Q_\phi) + 2i\lambda^2(Q_\phi)] = 0$$

$$L[L(Q_\phi) - 2i\lambda^2(Q_\phi)] + 2i\lambda^2[L(Q_\phi) + 2i\lambda^2(Q_\phi)] = 0$$

which show that the solutions of the two second order equations are

$$L(Q_\phi) \pm 2i\lambda^2Q_\phi = 0$$

Writing this equation in full

$$Q_\phi^{\cdot\cdot} + Q_\phi^{\cdot}\cot\phi - Q_\phi\cot^2\phi \pm 2i\lambda^2 Q_\phi = 0 \qquad \text{A.4(a,b)}$$

Using the Langer technique, the solution of Eqn. A.4(a) is

$$Q_\phi = \frac{K}{a}\sqrt{\frac{\phi}{\sin\phi}}\,\{C_1(\mathrm{ber}'\xi + i\,\mathrm{bei}'\xi) + C_2(\mathrm{ker}'\xi + i\,\mathrm{kei}'\xi)\}$$

where $\xi = \lambda\phi\sqrt{2}$ and the prime indicates derivatives with respect to ξ. Similarly, the solution for Eqn. A.4(b) is

$$Q_\phi = \frac{K}{a}\sqrt{\frac{\phi}{\sin\phi}}\,\{C_3(\mathrm{ber}'\xi - i\,\mathrm{bei}'\xi) + C_4(\mathrm{ker}'\xi - i\,\mathrm{kei}'\xi)\}$$

These solutions are valid for the range $\pi > \phi > 0$. With $C_1 = C_1 + C_2$, $C_2 = (C_1 - C_2)i$, $C_3 = C_3 + C_4$, and $C_4 = (C_3 - C_4)i$, the expression for Q_ϕ become

$$Q_\phi = \frac{K}{a}\sqrt{\frac{\phi}{\sin\phi}}(C_1\mathrm{ber}'\xi + C_2\mathrm{bei}'\xi + C_3\mathrm{ker}'\xi + C_4\mathrm{kei}'\xi) \qquad \text{A.5}$$

The geometrical properties of a conical segment is

$$\begin{array}{lll} r_0 = s\sin\alpha & \phi = \pi/2 - \alpha & \dfrac{d}{d\phi} = r_1\dfrac{d}{ds} \\ r_1 = \infty & N_\phi = N_s & \\ r_2 = s\tan\alpha & p\phi = p_s & \end{array}$$

Substituting these relations into Eqns. 3.65 and 3.68 yield

$$r_2\frac{d^2U}{ds^2} + \frac{dU}{ds}\tan\alpha - \frac{U}{r_2}\tan^2\alpha = EhV \qquad \text{A.6}$$

$$r_2\frac{d^2V}{ds^2} + \frac{dV}{ds}\tan\alpha - \frac{V}{r_2}\tan^2\alpha = -\frac{U}{D} \qquad \text{A.7}$$

and Eqns. 3.57 and 3.58 become

$$U = s\,Q_s\tan\alpha \qquad \text{A.8}$$

Substituting this into A.6 and A.7,

$$s\,\frac{d^2(sQ_s)}{ds^2} + \frac{d(sQ_s)}{ds} - \frac{(sQ_s)}{s} = EhV\cot^2\alpha \qquad \text{A.9}$$

$$s\,\frac{d^2V}{ds^2} + \frac{dV}{ds} - \frac{V}{s} = -\frac{(sQ_s)}{D} \qquad \text{A.10}$$

These equations can be solved to form a fourth order equation in terms of a single variable (7). However, an alternative approach (8) is possible by introducing a linear differential operator (1) as follows:

$$L(\ ) = s\,\frac{d^2(\ )}{ds^2} + \frac{d(\ )}{ds} - \frac{(\ )}{s} \qquad \text{A.11}$$

thus, eqns. A.9 and A.10 become,

$$L(sQ_s) = EhV\cot^2\alpha \qquad \text{A.12}$$

$$L(V) = -\frac{(sQ_s)}{D} \qquad \text{A.13}$$

Operating on Eqn. A.12, and substituting back into Eqn. A.13 yields,

$$LL(sQ_s) + \lambda^4(sQ_s) = 0 \qquad \text{A.14}$$

where

$$\lambda^4 = \frac{12(1-\nu^2)}{h^2\tan^2\alpha}$$

This may be written in either of the following forms,

$$L[L(sQ_s) + i\lambda^2(sQ_s)] - i\lambda^2[L(sQ_s) + i\lambda^2(sQ_s)] = 0 \qquad \text{A.15}$$

$$L[L(sQ_s) - i\lambda^2(sQ_s)] + i\lambda^2[L(sQ_s) + i\lambda^2(sQ_s)] = 0 \qquad \text{A.16}$$

which show that the solutions of the two second-order equations are

$$L(sQ_s) \pm i\lambda^2(sQ_s) = 0 \qquad \text{A.17}$$

Expanding this equation yields,

$$s\,\frac{d^2(sQ_s)}{ds^2} + \frac{d(sQ_s)}{ds} - \frac{(sQ_s)}{s} \pm i\lambda^2(sQ_s) = 0 \qquad \text{A.18(a,b)}$$

The solution to Eqns. A.18 is complex, and it will be enough to solve one of the equations, and then use the real and imaginary parts of this solution separately as the solution of a fourth-order equation mentioned earlier. Introducing a new variable,

$$\eta = 2\lambda\sqrt{is}, \qquad \text{A.19}$$

Eqn. A.18(a) become,

$$\frac{d^2(sQ_s)}{d\eta^2} + \frac{1}{\eta}\frac{d(sQ_s)}{d\eta} + \left(1 - \frac{4}{\eta^2}\right)(sQ_s) = 0 \qquad \text{A.20}$$

The solution of this equation consists of Bessel functions of the second kind.

$$J_2(\eta) = \frac{2}{\eta} J_1(\eta) - J_0(\eta) \qquad \text{A.21(a)}$$

$$H_2^{(1)}(\eta) = \frac{2}{\eta} H_1^{(1)}(\eta) - H_0^{(1)}(\eta) \qquad \text{A.22(b)}$$

Let $\xi = 2\lambda\sqrt{s}$, then rewriting Eqns. A.21 in terms of the Kelvin functions of order zero yield

$$J_2(\eta) = \frac{2}{\xi}\text{bei}'\xi - \text{ber}\,\xi + i\left(\frac{2}{\xi}\text{ber}'\xi + \text{bei}\,\xi\right) \qquad \text{A.23(a)}$$

$$H_2^{(1)}(\eta) = \frac{2}{\eta}\left(\frac{2}{\xi}\text{ker}'\xi + \text{kei}\,\xi\right) - i\,\frac{2}{\eta}\left(\frac{2}{\xi}\text{kei}'\xi - \text{ker}\,\xi\right) \qquad \text{A.23(b)}$$

These two functions are independent solutions of Eqn. A.18, and their real and imaginary parts separately will satisfy the fourth-order equation formed by combining Eqns. A.9 and A.10. The general solution for a conical shell is

$$Q_s = \frac{1}{s}\left[A_1\left(\text{ber}\,\xi - \frac{2}{\xi}\text{bei}'\xi\right) + A_2\left(\text{bei}\,\xi + \frac{2}{\xi}\text{ber}'\xi\right)\right.$$
$$\left. + B_1\left(\text{ker}\,\xi - \frac{2}{\xi}\text{kei}'\xi\right) + B_2\left(\text{kei}\,\xi + \frac{2}{\xi}\text{ker}'\xi\right)\right] \qquad \text{A.24}$$

Using the recurrence formulas for the Kelvin functions (10) Eqn. A.24 can be rewritten as follows:

$$Q_s = \frac{1}{s}(C_1\text{ber}_2\xi + C_2\text{bei}_2\xi + C_3\text{ker}_2\xi + C_4\text{kei}_2\xi) \qquad \text{A.25}$$

APPENDIX B-CONSTRUCTION OF THE SEGMENT FLEXIBILITY MATRIX

In general, the flexibility matrix of a shell segment is of the form

$$[F] = [TA][TT]^{-1} \qquad B.1$$

The [TA] and [TT] matrices are a function of the geometrical and material properties of the shell segment.

Based on the geometrical properties, the [TA] and [TT] matrices for the spherical and conical segments, derived in a similar manner as for the cylindrical segment in Chapter 3, are as follows:

Assume $\lambda^4 \simeq 3(1-\nu^2)\frac{a^2}{h^2}$

Let

$$\xi = \lambda\phi\sqrt{2}$$

$$m = (1-\nu^2)$$

$$(\ )' = \frac{d(\ )}{d\xi}$$

$$\phi_1 = \frac{\phi}{\sin\phi} \text{ ber}'\xi$$

$$\phi_2 = \frac{\phi}{\sin\phi} \text{ bei}'\xi$$

$$\phi_3 = \frac{\phi}{\sin\phi} \text{ ker}'\xi$$

$$\phi_4 = \frac{\phi}{\sin\phi} \text{ kei}'\xi$$

$$\theta_1 = \frac{d\phi_2}{d\xi} + \nu\phi_2\cot\phi - \frac{\nu}{2\lambda^2}\frac{d\phi_1}{d\xi} - \nu\phi_1\cot\phi$$

$$\theta_2 = \frac{-d\phi_1}{d\xi} - \nu\phi_1\cot\phi - \frac{\nu}{2\lambda^2}\frac{d\phi_2}{d\xi} - \nu\phi_2\cot\phi$$

$$\theta_3 = \frac{d\phi_4}{d\xi} + \nu\phi_4\cot\phi - \frac{\nu}{2\lambda^2}\frac{d\phi_3}{d\xi} - \nu\phi_3\cot\phi$$

$$\theta_4 = -\frac{d\phi_3}{d\xi} - \nu\phi_3\cot\phi - \frac{\nu}{2\lambda^2}\frac{d\phi_4}{d\xi} - \nu\phi_4\cot\phi$$

then

$$\begin{Bmatrix} H^i \\ M_\phi^i \\ H^j \\ M_\phi^j \end{Bmatrix} = \begin{bmatrix} \frac{-K}{a\sin\alpha_o} & 0 & 0 & 0 \\ 0 & \frac{Eha}{2\lambda^2 m} & 0 & 0 \\ 0 & 0 & \frac{K}{a\sin\alpha} & 0 \\ 0 & 0 & 0 & \frac{-Eha}{2\lambda^2 m} \end{bmatrix} \begin{bmatrix} \phi_1^i & \phi_2^i & \phi_3^i & \phi_4^i \\ \theta_1^i & \theta_2^i & \theta_3^i & \phi_4^i \\ \phi_1^j & \phi_2^j & \phi_3^j & \phi_4^j \\ \theta_1^i & \theta_2^i & \theta_3^i & \phi_4^i \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_2 \end{Bmatrix}$$

or simply

$$\{V\} = [T_5][T_6]\{C\}$$

$$\{V\} = [TT]\{C\} \qquad \text{B.2}$$

and let

$$\Phi_1 = \nu\phi_1\cot\phi - \frac{d\phi_1}{d\xi}$$

$$\Phi_2 = \nu\phi_2\cot\phi - \frac{d\phi_2}{d\xi}$$

$$\Phi_3 = \nu\phi_3\cot\phi - \frac{d\phi_3}{d\xi}$$

$$\Phi_4 = \nu\phi_4\cot\phi - \frac{d\phi_4}{d\xi}$$

$$\Theta_1 = \phi_2 - \frac{\nu\phi_1}{2\lambda^2}$$

$$\Theta_2 = -\phi_1 - \frac{\nu\phi_2}{2\lambda^2}$$

$$\Theta_3 = \phi_4 - \frac{\nu\phi_3}{2\lambda^2}$$

$$\Theta_4 = -\phi_3 - \frac{\nu\phi_4}{2\lambda^2}$$

$$\begin{Bmatrix} \Delta_H^i \\ \Delta_\phi^i \\ \Delta_H^j \\ \Delta_\phi^j \end{Bmatrix} = \frac{1}{m} \begin{bmatrix} -\sin\alpha_0 & 0 & 0 & 0 \\ 0 & -2\lambda^2/a & 0 & 0 \\ 0 & 0 & -\sin\alpha & 0 \\ 0 & 0 & 0 & -2\lambda^2/a \end{bmatrix} \begin{bmatrix} \Phi_1^i & \Phi_2^i & \Phi_3^i & \Phi_4^i \\ \Theta_1^i & \Theta_2^i & \Theta_3^i & \Theta_4^i \\ \Phi_1^j & \Phi_2^j & \Phi_3^j & \Phi_4^j \\ \Theta_1^j & \Theta_2^j & \Theta_3^j & \Theta_4^j \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

or

$$\{\Delta\} = [T_7][T_8]\{C\}$$

$$\{\Delta\} = [TA]\{C\} \qquad \text{B.3}$$

The [TA] and [TT] matrices for the conical segment is as follows (8).

Let

$m^4 = 12(1-\nu^2)$

$\lambda^4 = \dfrac{12(1-\nu^2)}{h^2\tan^2\alpha}$

$\xi = 2\lambda\sqrt{s}$

$(\ )' = \dfrac{d(\ )}{d\xi}$

and

$\phi_1 = \mathrm{ber}_2\xi \qquad \theta_1 = \xi\mathrm{ber}_2'\xi + 2\nu\mathrm{ber}_2\xi$

$\phi_2 = \mathrm{bei}_2\xi \qquad \theta_2 = \xi\mathrm{bei}_2'\xi + 2\nu\mathrm{bei}_2\xi$

$\phi_3 = \mathrm{ker}_2\xi \qquad \theta_3 = \xi\mathrm{ker}_2'\xi + 2\nu\mathrm{ker}_2\xi$

$\phi_4 = \mathrm{kei}_2\xi \qquad \theta_4 = \xi\mathrm{kei}_2'\xi + 2\nu\mathrm{kei}_2\xi$

$\gamma_1 = \xi\mathrm{ber}_2'\xi - 2\nu\mathrm{ber}_2\xi$

$\gamma_2 = \xi\mathrm{bei}_2'\xi - 2\nu\mathrm{bei}_2\xi$

$\gamma_3 = \xi\mathrm{ker}_2'\xi - 2\nu\mathrm{ker}_2\xi$

$\gamma_4 = \xi\mathrm{kei}_2'\xi - 2\nu\mathrm{kei}_2\xi$

then

$$\begin{Bmatrix} H^i \\ M_\phi^i \\ H^j \\ M_\phi^j \end{Bmatrix} = \begin{bmatrix} \frac{-1}{s_i \sin\alpha} & 0 & 0 & 0 \\ 0 & \frac{h}{2m^2 s_i} & 0 & 0 \\ 0 & 0 & \frac{1}{s_j \sin\alpha} & 0 \\ 0 & 0 & 0 & \frac{-h}{2m^2 s_j} \end{bmatrix} \begin{bmatrix} \phi_1^i & \phi_2^i & \phi_3^i & \phi_4^i \\ \theta_2^i & -\theta_1^i & \theta_4^i & -\phi_3^i \\ \phi_1^j & \phi_2^j & \phi_3^j & \phi_4^j \\ \theta_2^j & -\theta_1^j & \theta_4^j & -\phi_3^j \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_2 \end{Bmatrix}$$

or

$$\{V\} = [T_9][T_{10}]\{C\}$$

$$\{V\} = [TT]\{C\} \tag{B.4}$$

and

$$\begin{Bmatrix} \Delta_H^i \\ \Delta_\phi^i \\ \Delta_H^j \\ \Delta_\phi^j \end{Bmatrix} = \frac{1}{2Eh} \begin{bmatrix} \sin\alpha & 0 & 0 & 0 \\ 0 & -2m^2/h & 0 & 0 \\ 0 & 0 & \sin\alpha & 0 \\ 0 & 0 & 0 & -2m^2/h \end{bmatrix} \begin{bmatrix} \gamma_1^i & \gamma_2^i & \gamma_3^i & \gamma_4^i \\ \phi_2^i & -\phi_1^i & \phi_4^i & -\phi_3^i \\ \gamma_1^j & \gamma_2^j & \gamma_3^j & \gamma_4^j \\ \phi_2^j & -\phi_1^j & \phi_4^j & -\phi_3^j \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

or

$$\{\Delta\} = [T_{11}][T_{12}\{C\}$$

$$\{\Delta\} = [TA]\{C\} \tag{B.5}$$

Inverted shell

An inverted cone or sphere is shown on Fig. (b) of Tables 3.5 and 3.6 respectively. Note that in the above derivations, i and j, relates to the 'top' and 'bottom' of a shell segment. So, for an inverted shell, the top becomes the bottom and vice versa. Thus, to find the flexibility matrix, it is a simple matter of interchanging rows one with three, and rows two with four, of matrices [TA] and [TT].

APPENDIX C-THE RUNGE-KUTTA METHOD

Given $y' = f(x,y)$ where the prime represents the derivative with respect to x and where the initial boundary conditions $y(i)$ are known. Then integrating $f(x,y)$ with respect to x yields

$$y(n) = y(i) + \int_m^n f(x,y)dx$$

where n is the point located at a distance e, which is the length of the interval, from the starting point m. Expanding the above equation using Taylor's series yields

$$y(n) = y(i) + e\, y'(i) + \frac{e^2}{2!} y''(i) + \frac{e^3}{3!} y'''(i) + \ldots$$

By expressing the various derivatives, y', y'', y''',... in terms of $f(x,y)$, Runge and Kutta approximated the series up to and including the term involving e^4 using the formula

$$y(n) = y(i) + (a + 2b + 2c + d)/6 \qquad \text{C.1}$$

where

$$a = e\, f(m,y(i)) \qquad \text{C.2}$$

$$b = e\, f(o,y(i)+a/2) \qquad \text{C.3}$$

$$c = e\, f(o,y(i)+b/2) \qquad \text{C.4}$$

$$d = e\, f(n,y(i)+c) \qquad \text{C.5}$$

For which o is the midpoint between points m and n.

APPENDIX D - SASHELL USER'S MANUAL

Program SASHELL computes stress resultants and displacements for any multi-shell structure composed of shell of revolution segments due to self-weight, external loads, and differential temperature variation along the meridional or circumferential direction.

The program recognizes six types of shell of revolution segments, namely: cylinders, spheres, ogives, cones, circular plates, and hyperboloids of revolution. An ogival segment is formed by a circular arc whose center of curvature does not fall on the axis of revolution, i.e., $r_1 \neq r_2$. Each segment may vary linearly in thickness along the meridian. Arbitrarily shaped segments may be approximated from a combination of these segments. Loads may vary linearly along the meridian and along the circumferential direction.

The input to SASHELL consists of multiple input records which may be lines in a datafile or a set of punched cards. For convenience, input records will be referred to as cards for the remainder of this manual. There are fourteen card types and certain card types are repeated as required.

The program is unit independent, that is any consistent set of units can be used. The user must select a unit of length and force, and all input quantities must be in these two units. Output will be in the same units. For example, if the unit of length selected is feet and the unit of force in kips, then all input quantities must be consistent with these units, such as thickness in feet, modulus of elasticity in kips per square feet, etc. Output will be in terms of kips per foot and foot-kips per foot, etc.

Several output options are available to the user. Following a description of the input and output, two examples are presented. While the user's manual is intended to be sufficiently complete to enable use of the program for most problems, for more complex cases, Chapter 4 may be particularly helpful.

D.1 Input

For ease of reference in the following input description, each card type is numbered from 1 to 14. The description consists of a descriptive name in bold type indicating the nature of the data and the format required, if fixed format is being used, followed by a sentence explaining the purpose of the input. This is followed by a symbolic line of input, in bold type, indicating the order of entries, followed by a description of each entry, and the options available, if any. When format free input is used, individual entries may be separated by either a comma or a blank, but entry fields must not exceed the format field specified. All entries in an input line are required but all zero entries occurring at the end of a line may be omitted.

1. **TITLE** card — Format 10A8

   Any identifier string up to 80 characters.

2. **CONTROL** card — Format 7I4,F7.0

   Defines the parameters which control the output and computations.

IPRINT NPC LT LDC LDW LDS NPM BETA

IPRINT = print control parameter.

0 - the output contains an echo check of the input data. The load values are computed from the input or generated Fourier coefficients to compare with the actual loading conditions. This is used to verify input only, no solution is executed.

1 - the output contains an echo check of the input data and the final stress resultants only;

2 - the output contains an echo check of the input data and the final stress resultants and displacements;

3 - the stress resultants and displacements for each harmonic are included in the output, in addition to the final values of the stress resultants and displacements. This is to be used for checking purposes only.

NPC = number of the circumferential points at which the final results are to be printed, not exceeding 21. There is no default value.

If the loads are symmetric or anti-symmetric with respect to the meridian passing through $\theta = 0$, NPC is the number of points along half the circumference $(0,\pi)$;

If the loads vary randomly in the circumferential direction, NPC is the number of points along the full circumference of the segment $(0,2\pi)$;

If the loads are constant in the circumferential direction (axisymmetric loading), NPC = 1.

LT = number of user-defined loads, not exceeding 6.

LDC = self-weight control parameter.

1 - self-weight is to be calculated from the input unit weight and included in the analysis;
0 - self-weight is not to be included in the analysis.

LDW = number of pre-defined loads, other than self-weight

LDS = load superposition control parameter.

0 - all the pre-defined loads, including self-weight, are to be superimposed only on the first user-defined load;
1 - each user-defined load, self-weight, and each pre-defined load are to be considered as separately.

NOTE:

NPM and BETA are included in the user's manual for parameter study only. Since the accuracy of the results and the computation time depend primarily on these values, the user is advised to use the default values in the program, by specifying zero NPM and BETA values. See Section 4.8 for details.

NPM = minimum number of points along the segment meridian for which the Runge-Kutta integration process is used. The default value is 21.

$$31 \le \text{NPM} \le 51 \text{ for } 20 < \text{beta} < 25$$
$$\text{NPM} \ge 21 \text{ for } \text{beta} \le 20$$

BETA = critical length coefficient, not to exceed 25. The default value is 20.

3. **STRUCTURAL DATA** card Format 2I3,4F12.0

This card specifies the number of segments and nodes and the global material properties of the structure.

NE NJ EG PUG GAMG THERMG

NE = number of segments, not exceeding 20.

NJ = number of junctions (nodes) between segments, not exceeding 21.

EG = global modulus of elasticity

PUG = global Poisson's ratio

GAMG = global unit weight

THERMG = global coefficient of thermal expansion

4. **NODAL DATA** card Format I4,2F10.0,4I4

One card per node is required.

I **XCR(I) RCR(I) IDF(I,1) IDF(I,2) IDF(I,3) IDF(I,4)**

I = node number

XCR(I) = global x-coordinate of node I along the axis of revolution directed downward from the top of the structure.

RCR(I) = radius of the parallel circle passing through node I.

IDF(I,J) = identification of the jth degree of freedom at node I. J = 1,4 corresponding to the rotation of the meridian (β), the radial displacement(w), the meridional displacement (v), and the circumferential displacement (u), respectively.

0 - the corresponding degree of freedom is not restrained;
1 - the corresponding degree of freedom is restrained;
2 - the corresponding degree of freedom is specified and input in card type 14.

5. **SEGMENT DATA** card Format 7I3,4F10.0

One card per segment is required. Note if IC = 1, card types 5 and 6 occur sequentially followed by the next card type 5.

I IC **IT**(I) **NC**(I,1) **NC**(I,2) **NDIV**(I) **LDE**(I) **TH**(I,1) **TH**(I,2) **EC**(I,1) **EC**(I,2)

I = segment number

IC = parameter which indicates if this card is immediately followed by a SEGMENT PROPERTY card (IC=1), or not (IC=0). See card type 6.

IT(I) = segment type

1 - cylinder
2 - cone or circular plate
3 - sphere
4 - ogive
5 - hyperboloid of revolution

NC(I,1) = node number at the top of segment I.

NC(I,2) = node number at the bottom of segment I.

NDIV(I) = number of equally spaced points in the segment at which the final results are to be printed. The minimum value is 2.

LDE(I) = parameter which indicates the type of pre-defined load acting on the segment.

1 - uniform load on a horizontal projection of the segment surface (snow load);
2 - hydrostatic pressure on the segment;
3 - equivalent to 1+2;
4 - snow drift as specified by the National Building Code of Canada as approximated for spherical domes;

TH(I,1) = segment thickness at the top

TH(I,2) = segment thickness at the bottom

EC(I,1) = eccentricity of the top node of segment I from the middle surface of the segment above it.

EC(I,2) = eccentricity of the bottom node of segment I from the middle surface of the segment below it.

NOTE:
The eccentricity is defined such that the radius of the parallel circle passing through the middle surface of the segment is

$$EC(I,J) = r_0(\text{node}) - r_0(\text{midsurface})$$

Thus, EC(I,J) is positive when directed inward from the node to the middle surface of the segment.

6. **SEGMENT PROPERTY** card Format 7F10.0

This card is required if the segment is a hyperboloid or an ogive, and/or, if the segment material properties are different from the global properties defined in the STRUCTURAL DATA card.

HPCN(1,I) HPCN(2,I) HPCN(3,I) GAMA(I) PU(I) E(I) THERM(I)

HPCN(1,I) = radius of curvature of the meridian for an ogival segment (type 4), or throat radius of a hyperboloid segment (type 5); For other segment types, this value is zero.

HPCN(2,I) = angle in degrees measured from the axis of revolution to the top edge of an ogival segment (type 4), or hyperboloid constant in which the ratio

$$\frac{HPCN(2,I)}{HPCN(1,I)}$$

equals to the slope of the asymptotes of the hyperbola. For other segment types, this value is zero.

HPCN(3,I) = angle in degrees measured from the axis of revolution to the top edge of an ogival segment (type 4), or global x-coordinate of the throat of a hyperboloid segment. For other segment types, this value is zero.

GAMA(I) = segment unit weight, if different from the global unit weight, otherwise, this value may be set to zero.

PU(I) = segment Poisson's ratio, if different from the global Poisson's ratio, otherwise, this value may be set to zero.

E(I) = segment modulus of elasticity, if different from the global modulus of elasticity, otherwise, this value may be set to zero.

THERM(I) = segment coefficient of thermal expansion, if different from the global coefficient, otherwise, this value may be set to zero.

7. **PRE-DEFINED LOAD** card — Format I5,2F10.0

LDW cards are required. (if LDW = 0, no cards are required. See card type 2)

I A1 A2

I = specifies the type of pre-defined load

1 - snow load
2 - hydrostatic pressure
3 - snow drift as specified by the NBC of Canada in Commentary H Sections 52-53, as applied to spherical domes.

A1 = intensity of snow load or liquid weight density

A2 = global liquid surface elevation measured from the top of the structure. This parameter is ignored when A1 ≠ 2.

8. **USER-DEFINED LOAD DEFINITION** card — Format 5I5

One card is required if LT > 0.

NTL NHS NHL NHIN NHPL

NTL = load type parameter.

0 - load is axisymmetric.

1 - load varies randomly in the circumferential direction and input is provided at NHPL points along the full circumference of the shell.

2 - load is symmetric along the circumference with respect to the meridian passing through $\theta = 0$ and input is provided at NHPL points along the full circumference of the shell.

3 - the Fourier cosine and sine coefficients are input for NHL harmonics.

NHS = harmonic number at which the analysis is to start.

NHL = number of harmonics required for the analysis including the NHS harmonic, not exceeding 21.

NHIN = defines the increment in the harmonics. The default value is 1.

NHPL = number of points along the full circumference of the shell at which the load values are described by the user, not exceeding 40.

NOTE:
For axisymmetric loads (NTL = 0), or for the first program run with IPRINT = 0 for non-axisymmetric loads (NTL > 0), the values of NHS, NHL, and NHIN may be specified as zero, NHPL/2, and one, respectively. When IPRINT = 0, the load values along the circumference are computed from the input or generated Fourier coefficients as a check on the accuracy of the representation of the actual loading conditions. After which the user may decide on the values of NHS, NHL, and NHIN for the final run.

If LT > 0, the following cards are required for each user-defined load.

9. **STRUCTURE LOAD CONTROL** card Format 3I5

L NEL(L) NJL(L)

L = load number

NEL(L) = number of loaded segments in the structure.

NJL(L) = number of loaded nodes in the structure.

10. **SEGMENT LOAD CONTROL** card Format 6I5

Number of cards required is equal to NEL(L). (See card type 9) Loads are input in the segment local coordinates.

LE IDL(1) IDL(2) IDL(3) IDL(4) IDL(5)

LE = number of the loaded segment

IDL = load type parameter.

IDL(1) = load in the direction tangent to the meridian (s).
IDL(2) = load in the direction tangent to the parallel circle (θ).
IDL(3) = load in the direction perpendicular to the tangent to the meridian (z).
IDL(4) = for temperature at the shell exterior surface.
IDL(5) = for temperature at the shell interior surface.

The corresponding values of IDL are:

0 - no load IDL(*) is acting on the segment;

1 - the applied load IDL(*) is constant along the meridian, to be specified for one edge of the segment only;

2 - the applied load IDL(*) varies linearly along the meridian, and must be specified for the two segment edges.

11. **SEGMENT LOADING** card

There are three distinct SEGMENT LOADING card formats corresponding to NTL = 0, 1 or 2, and 3. (See card type 8) For each loaded segment, one set of cards is required for each non-zero IDL term, input in the order consistent with the IDL array.

a. Axisymmetrical Load Values NTL = 0 — Format F10.0

One card for the top load values (plus one card for the bottom, if required) are required.

AECL(K1)

AECL(K1) = magnitude of the load (The first card corresponds to the top load value, and followed by a card which corresponds to the bottom value if required.)

b. Non-Axisymmetrical Load Values   Format 8F10.0
NTL = 1 or 2

Because of the input format used (8F10.0) by the program, there must be eight or less values per input record. One set of cards for the top load values (plus one set for the bottom, if required) at NHPL circumferential points are required.

W(I) I=1,NHPL

W(I) = magnitude of the load at NHPL circumferential points. (The first NHPL values correspond to the top load values, and the second set of NHPL values corresponds to the bottom values, if required)

c. Load Fourier Coefficients   Format 8F10.0
NTL = 3

Because of the input format used (8F10.0) by the program, there must be eight or less values per input record. One set of cards for the top Fourier coefficients (plus one set for the bottom, if required) for NHL harmonics are required.

AECL(N) N=1,2NHL

AECL(N) = cosine coefficients for NHL harmonics (for N = 1,2,3,...,NHL-1,NHL), immediately followed by the sine coefficients for NHL harmonics (for N = NHL+1,NHL+2,...,2NHL-1,2NHL). (The first NHL values correspond to the cosine coefficients for the top edge of the segment, immediately followed by NHL values corresponding to the sine coefficients for the top edge of the segment. This is repeated for the bottom edge if required.)

12. **NODAL LOAD CONTROL** card   Format 5I5

Number of cards required equal to NJL(L). (See card type 9) Nodal loads are input in the global coordinates.

LE(I) IDL(1) IDL(2) IDL(3) IDL(4)

LE(I) = number of the loaded node in the structure.

IDL = load type parameter

IDL(1) = meridional bending moment (M_s), positive as shown in Fig. 3.3.

IDL(2) = force perpendicular to the meridian (S_s) at node I, positive in the direction away from the axis of revolution.

IDL(3) = force tangent to the meridian (N_s) at node I, positive downward from the top of the structure, along the axis of revolution.

IDL(4) = force in the direction tangent to the parallel circle (T_s), positive in the counterclockwise circumferential direction.

The corresponding values of IDL are:

0 - no load type IDL(*) is acting at the node;
1 - load type IDL(*) is applied at the node.

13. **NODAL LOAD** card

This type is classified into three types, for NTL = 0, 1 or 2, and 3. (See card type 8) One set of cards is required for each non-zero IDL term, input in the order consistent with the IDL array, for each loaded node.

a. Axisymmetrical Load Value — Format 4F10.0
NTL = 0

One card is required.

ANJL(II,1)

ANJL(II,1) = magnitude of the nodal load.

b. Non-Axisymmetrical Load Values — Format 8F10.0
NTL = 1 or 2

Because of the input format used (8F10.0) by the program, there must be eight or less values per input record. One set of cards for the load values at NHPL circumferential points are required.

W(I) I=1,NHPL

W(I) = magnitude of the nodal load (NHPL values are required)

c. Load Fourier Coefficients — Format 8F10.0
NTL = 3

Because of the input format used (8F10.0) by the program, there must be eight or less values per input record. One set of cards for the cosine coefficients for NHL harmonics and another set of cards for the sine coefficients for NHL harmonics are required.

ANJL(I,N) N = 1,NHL

ANJL(I,N) = cosine (or sine) coefficient of Fourier expansion for node I for NHL harmonics,(i.e., N = 1,2,3,...,NHL-1,NHL).

14. **DISPLACEMENT SPECIFICATION** card Format 8F10.0

One card is required for each node with IDF = 2. (See card type 4)

DSP(J,N) N = 1,NHL

DSP(J,N) = specified displacement at node J for NHL harmonics, (i.e., N = 1,2,3,...,NHL-1,NHL).

D.2 Output

SASHELL echo checks the input data. In addition, the output consists of the following:

1. When IPRINT = 0, the computed nodal or segment load values from the generated or input Fourier coefficients (for the top edge of the segment and bottom edge, included only if the loads vary along the meridian) for each load type, for each loaded node and/or segment are printed. No solution is executed.
2. When IPRINT > 0, the following output is generated. The geometric properties for each division of each segment, according to NDIV, which is a parameter indicating the number of equally spaced points at which the final results are to be printed as specified by the user.
3. The final results are printed in the following order:
   a. User-defined axisymmetric loads;
   b. Self-weight;
   c. Pre-defined axisymmetric loads;
   d. User-defined non-axisymmetric loads;
   e. Pre-defined non-axisymmetric loads.
4. If the load is axisymmetric, three displacement components (β, w, v) and five stress resultants (M_s, S_s, N_s, M_θ, N_θ), positive in the direction shown in Fig. 2.2 are printed at NDIV points along the meridian for each segment.
5. If the load is non-axisymmetric, four displacement components (β, w, v, u) and ten stress resultants (M_s,

S_s, N_s, T_s, M_θ, N_θ, $M_{\theta s}$, $M_{s\theta}$, $N_{\theta s}$, $N_{s\theta}$), positive in the direction shown in Fig. 2.2 are printed for NPC circumferential points (THETA), at NDIV points along the meridian, for each segment.

Example Input Files

```
INTZE TANK (HYDROSTATIC PRESSURE)
2,1,0,0,1,1,
8,9,.5804E09,.167,150.,.6E-05,
1,0.01,1.37655,
2,11.061,44.365,
3,12.511,44.456,
4,22.511,44.456,
5,23.961,44.581,
6,35.408,22.225,
7,32.408,1.,
8,75.408,36.1125,
9,115.408,50.,1,1,1,1,
1,0,3,1,2,6,0,.333,.333,
2,0,1,2,3,6,0,.667,.667,-.216,-.125,
3,0,1,3,4,6,2,.417,.417,
4,0,1,4,5,6,2,.667,.667,-.125,-.125,
5,0,2,5,6,6,2,.3,.3,
6,0,3,7,6,6,2,.3,.3,
7,0,2,6,8,2,0,.5,.5,
8,0,2,8,9,2,0,.5,.5,
2,62.4,11.45,
```

← *global properties* (line 3)

nodal data (lines 4–12)

segment properties (lines 13–20)

← *pre-defined Load data* (last line)

```
NATURAL DRAFT HYPERBOLOID COOLING TOWER (WIND LOAD)
2,8,1,0,0,1,0,25.,
5,6,4000.,.15,.15, <- global propeties
1,0.,85.99,
2,10.,84.94,
3,116.67,85.62,
4,223.33,105.7,        nodal data
5,330.,136.8,
6,355.,145.,0,1,1,1,
1,1,5,1,2,5,0,2.,.5,
82.5,204.1,60.,
2,1,5,2,3,5,0,.5,.5,
82.5,204.1,60.,
3,1,5,3,4,5,0,.5,.5,   segment properties
82.5,204.1,60.,
4,1,5,4,5,5,0,.5,.5,
82.5,204.1,60.,
5,1,5,5,6,5,0,.5,2.5,
82.5,204.1,60.,
2,0,8,1,24, <- user-defined load data
1,5,
1,0,0,2,
2,0,0,2,
3,0,0,2,   load control porameters
4,0,0,2,
5,0,0,2,
-.053,-.0424,-.0106,.0265,.0636,.0689,.0477,.0212,    top load volues
.0212,.0212,.0212,.0212,.0212,.0212,.0212,.0212,     for segment 1
.0212,.0212,.0477,.0689,.0636,.0265,-.0106,-.0424,
-.0526,-.0421,-.0105,.0263,.0632,.0684,.0474,.0210,   bott. load volues
.0210,.0210,.0210,.0210,.0210,.0210,.0210,.0210,     for segment 1
.0210,.0210,.0474,.0684,.0632,.0263,-.0105,-.0421,
-.0526,-.0421,-.0105,.0263,.0632,.0684,.0474,.0210,
.0210,.0210,.0210,.0210,.0210,.0210,.0210,.0210,     top ; segment 2
.0210,.0210,.0474,.0684,.0632,.0263,-.0105,-.0421,
-.0478,-.0383,-.0096,.0239,.0574,.0622,.0430,.0191,
.0191,.0191,.0191,.0191,.0191,.0191,.0191,.0191,     bott; segment 2
.0191,.0191,.0430,.0622,.0574,.0239,-.0096,-.0383,
-.0478,-.0383,-.0096,.0239,.0574,.0622,.0430,.0191,
.0191,.0191,.0191,.0191,.0191,.0191,.0191,.0191,
.0191,.0191,.0430,.0622,.0574,.0239,-.0096,-.0383,
-.0414,-.0331,-.0083,.0207,.0497,.0583,.0372,.0165,   etc.
.0165,.0165,.0165,.0165,.0165,.0165,.0165,.0165,
.0165,.0165,.0372,.0538,.0497,.0207,-.0883,-.0331,
-.0414,-.0331,-.0083,.0207,.0497,.0583,.0372,.0165,
.0165,.0165,.0165,.0165,.0165,.0165,.0165,.0165,
.0165,.0165,.0372,.0538,.0497,.0207,-.0883,-.0331,
-.0304,-.0243,-.0061,.0152,.0365,.0395,.0273,.0122,
.0122,.0122,.0122,.0122,.0122,.0122,.0122,.0122,
.0122,.0122,.0273,.0395,.0365,.0152,-.0061,-.0243,
-.0304,-.0243,-.0061,.0152,.0365,.0395,.0273,.0122,
.0122,.0122,.0122,.0122,.0122,.0122,.0122,.0122,
.0122,.0122,.0273,.0395,.0365,.0152,-.0061,-.0243,
-.0256,-.0205,-.0051,.0127,.0307,.0332,.0230,.0102,
```

.0102,.0102,.0102,.0102,.0102,.0102,.0102,.0102,
.0102,.0102,.0230,.0332,.0307,.0127,-.0051,-.0205,

Example Output Files

```
1INTZE TANK (HYDROSTATIC PRESSURE)

***   ANALYSIS DATA   ***

IPRINT                          =   2
NO. OF CIRCUMF. POINTS          =   1
NUMBER OF USER-DEFINED LOADS =   0
SELF WEIGHT 1=YES  O=NO         =   0
NUMBER OF PRE DEFINED LOADS  =   1
LOAD SUPERPOSITION 1=NO O=YES=   1
NO.OF INTEGRATION POINTS        =  21
CRITICAL LENGTH COEFFICIENT  = 20.000


***   STRUCTURAL DATA   ***

NUMBER OF SEGMENTS                          =  8
NUMBER OF NODES                             =  9
GLOBAL YOUNG S MODULUS                      =  0.5804E+09
GLOBAL POISSON S RATIO                      =  0.1670E+00
GLOBAL UNIT WEIGHT                          =  0.1500E+03
GLOBAL THERMAL COEFFICIENT                  =  0.6000E-05
```

*** NODAL DATA ***

NODE	XCOOR	RCOOR	R	W	V	U
1	0.0100	1.3765	0	0	0	0
2	11.0610	44.3650	0	0	0	0
3	12.5110	44.4560	0	0	0	0
4	22.5110	44.4560	0	0	0	0
5	23.9610	44.5810	0	0	0	0
6	35.4080	22.2250	0	0	0	0
7	32.4080	1.0000	0	0	0	0
8	75.4080	36.1125	0	0	0	0
9	115.4080	50.0000	1	1	1	1

*** SEGMENT DATA ***

SEG	TYP	TOP	BOT	NDIV	LDE	THICKTOP	THICKBOT	ECTOP	ECBOT	HP1	HP2	HP3	UNIT WT	PR	E	TCOEF
1	3	1	2	6	0	0.3330	0.3330	0.0	0.0	0.0	0.0	0.0	150.00	0.17	0.5804E+09	0.6000E-05
2	1	2	3	6	0	0.6670	0.6670	-0.2160	-0.1250	0.0	0.0	0.0	150.00	0.17	0.5804E+09	0.6000E-05

```
3  1  3  4  6  2  0.4170  0.4170   0.0      0.0     0.0  0.0  0.0  150.00  0.17  0.5804E+09  0.6000E-05
4  1  4  5  6  2  0.6670  0.6670  -0.1250  -0.1250  0.0  0.0  0.0  150.00  0.17  0.5804E+09  0.6000E-05
5  2  5  6  6  2  0.3000  0.3000   0.0      0.0     0.0  0.0  0.0  150.00  0.17  0.5804E+09  0.6000E-05
6  3  7  6  6  2  0.3000  0.3000   0.0      0.0     0.0  0.0  0.0  150.00  0.17  0.5804E+09  0.6000E-05
7  2  6  8  2  0  0.5000  0.5000   0.0      0.0     0.0  0.0  0.0  150.00  0.17  0.5804E+09  0.6000E-05
8  2  8  9  2  0  0.5000  0.5000   0.0      0.0     0.0  0.0  0.0  150.00  0.17  0.5804E+09  0.6000E-05

*** PRE DEFINED LOADS ***

SNOW LOAD INTENSITY                         =   0.0
LIQUID UN. WT. FOR HYDROSTATIC PRESSURE=  62.400
LIQUID SURFACE ELEVATION                    =  11.450
NBC DRIFT SNOW INTENSITY FOR SPHERES        =   0.0


***   LOADING DEFINITIONS   ***

LOADING DEFINITION CODE       =    0
NUMBER OF INITIAL HARMONIC    =    0
NUMBER OF HARMONICS TO FIT    =    1
HARMONIC INCREMENTAL          =    1
NUMBER OF CIRCUMFR POINTS     =    0


****   CONNECTIVITY SYSTEM   ***

   SEGMENT    TOP NODE    BOT NODE
         1           1           2
         2           2           3
         3           3           4
         4           4           5
         5           5           6
         6           7           6
         7           6           8
         8           8           9

***********************************
*   NUMBER OF EQUATIONS =   36    *
*   FULL BAND WIDTH     =   23    *
***********************************
```

```
          127        35.408   22.225    1.237    0.500   23.526    0.0     67.763    0.0
          147        75.408   36.112    1.237    0.500   38.227    0.0    110.105    0.0
 8
          148        75.408   36.113    1.237    0.500   38.227    0.0    110.105    0.0
          168       115.408   50.000    1.237    0.500   52.928    0.0    152.447    0.0

DISPLACEMENTS (R,W,V) IN RADIANS OR LENGTH UNITS
AND/OR STRESS RESULTANTS (MS,SS,NS,MT,NT) IN FORCE UNITS
PER UNIT LENGTH DUE TO THE AXISYMMETRIC LOADINGS
(I.E. HARMONIC NO.=0)--NUMBERED SEQUENTIALLY IN THE
FOLLOWING ORDER:  USER-DEFINED LOADINGS FOLLOWED BY THE
PRE-DEFINED LOADINGS.

LOAD NO.  =       1

SEGMENT  1 AT    6 MERIDIONAL POINTS  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

POINT  1

       R     0.2394E-06
       W    -0.6553E-01
       V     0.9545E-03
       MS   -0.7276E-11
       SS    0.9095E-11
       NS    0.2910E-10
       MT   -0.3105E+00
       NT   -0.1187E+02

POINT  2

       R     0.4421E-06
       W    -0.6514E-01
       V     0.7154E-02
       MS    0.4149E-01
       SS    0.4834E-01
       NS   -0.4197E+00
       MT   -0.8040E-01
       NT   -0.5847E+01

POINT  3

       R    -0.5477E-06
       W    -0.6417E-01
       V    ·0.1329E-01
```

```
1NATURAL DRAFT HYPERBOLOID COOLING TOWER (WIND;HINGED)

***   ANALYSIS DATA   ***

IPRINT                          =   0
NO. OF CIRCUMF. POINTS          =   7
NUMBER OF USER-DEFINED LOADS =   1
SELF WEIGHT 1=YES  O=NO         =   0
NUMBER OF PRE DEFINED LOADS  =   0
LOAD SUPERPOSITION 1=NO O=YES=   1
NO.OF INTEGRATION POINTS        =  21
CRITICAL LENGTH COEFFICIENT   = 20.000


***   STRUCTURAL DATA   ***

NUMBER OF SEGMENTS                    = 5
NUMBER OF NODES                       = 6
GLOBAL YOUNG S MODULUS                = 0.4000E+04
GLOBAL POISSON S RATIO                = 0.1500E+00
GLOBAL UNIT WEIGHT                    = 0.1500E+00
GLOBAL THERMAL COEFFICIENT            = 0.6000E-05
```

*** NODAL DATA ***

NODE	XCOOR	RCOOR	R	W	V	U
1	0.0	85.9900	0	0	0	0
2	10.0000	84.9400	0	0	0	0
3	116.6700	85.6200	0	0	0	0
4	223.3300	105.7000	0	0	0	0
5	330.0000	136.8000	0	0	0	0
6	355.0000	145.0000	0	1	1	1

*** SEGMENT DATA ***

SEG	TYP	TOP	BOT	NDIV	LDE	THICKTOP	THICKBOT	ECTOP	ECBOT	HP1	HP2	HP3	UNIT WT	PR	E	TCOEF
1	5	1	2	5	0	2.0000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
2	5	2	3	5	0	0.5000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
3	5	3	4	5	0	0.5000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
4	5	4	5	5	0	0.5000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
5	5	5	6	5	0	0.5000	2.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05

```
***   LOADING DEFINITIONS   ***

LOADING DEFINITION CODE        =    2
NUMBER OF INITIAL HARMONIC     =    0
NUMBER OF HARMONICS TO FIT     =   12
HARMONIC INCREMENTAL           =    1
NUMBER OF CIRCUMFR POINTS      =   24


USER DEFINED LOAD NO.=    1     NO. OF LOADED SEGMENTS=    5


SEGMENT  LOAD TYPE     INPUT LOAD VALUES

   1        3
                      TOP       -0.5300E-01 -0.4240E-01 -0.1060E-01  0.2650E-01  0.6360E-01
                                 0.6890E-01  0.4770E-01  0.2120E-01  0.2120E-01  0.2120E-01
                                 0.2120E-01  0.2120E-01  0.2120E-01  0.2120E-01  0.2120E-01
                                 0.2120E-01  0.2120E-01  0.2120E-01  0.4770E-01  0.6890E-01
                                 0.6360E-01  0.2650E-01 -0.1060E-01 -0.4240E-01
                      BOT       -0.5260E-01 -0.4210E-01 -0.1050E-01  0.2630E-01  0.6320E-01
                                 0.6840E-01  0.4740E-01  0.2100E-01  0.2100E-01  0.2100E-01
                                 0.2100E-01  0.2100E-01  0.2100E-01  0.2100E-01  0.2100E-01
                                 0.2100E-01  0.2100E-01  0.2100E-01  0.4740E-01  0.6840E-01
                                 0.6320E-01  0.2630E-01 -0.1050E-01 -0.4210E-01

                      CALCULATED FOURIER COEFFICIENTS FOR   12 HARMONICS

                      TOP        0.2032E-01 -0.1480E-01 -0.3285E-01 -0.2699E-01 -0.4858E-02
                                 0.6251E-02  0.1767E-02 -0.2371E-02 -0.4416E-03  0.4915E-03
                                -0.7186E-03  0.3167E-03
                      BOT        0.2016E-01 -0.1465E-01 -0.3262E-01 -0.2681E-01 -0.4817E-02
                                 0.6214E-02  0.1750E-02 -0.2354E-02 -0.4333E-03  0.4805E-03
                                -0.7257E-03  0.3257E-03

                      CALCULATED LOAD VALUES AT   24 CIRCUMFERENTIAL POINTS

                      TOP       -0.5388E-01 -0.4152E-01 -0.1148E-01  0.2738E-01  0.6272E-01
                                 0.6978E-01  0.4682E-01  0.2208E-01  0.2032E-01  0.2208E-01
```

```
1NATURAL DRAFT HYPERBOLOID COOLING TOWER (WIND;HINGED)

***   ANALYSIS DATA   ***

IPRINT                          =   2
NO. OF CIRCUMF. POINTS          =   7
NUMBER OF USER-DEFINED LOADS =   1
SELF WEIGHT 1=YES  O=NO         =   O
NUMBER OF PRE DEFINED LOADS  =   O
LOAD SUPERPOSITION 1=NO O=YES=   1
NO.OF INTEGRATION POINTS        =  21
CRITICAL LENGTH COEFFICIENT  = 20.000


***   STRUCTURAL DATA   ***

NUMBER OF SEGMENTS                        =  5
NUMBER OF NODES                           =  6
GLOBAL YOUNG S MODULUS                    =  0.4000E+04
GLOBAL POISSON S RATIO                    =  0.1500E+00
GLOBAL UNIT WEIGHT                        =  0.1500E+00
GLOBAL THERMAL COEFFICIENT                =  0.6000E-05

***     NODAL DATA     ***
```

NODE	XCOOR	RCOOR	R	W	V	U
1	0.0	85.9900	0	0	0	0
2	10.0000	84.9400	0	0	0	0
3	116.6700	85.6200	0	0	0	0
4	223.3300	105.7000	0	0	0	0
5	330.0000	136.8000	0	0	0	0
6	355.0000	145.0000	0	1	1	1

```
***   SEGMENT DATA   ***
```

SEG	TYP	TOP	BOT	NDIV	LDE	THICKTOP	THICKBOT	ECTOP	ECBOT	HP1	HP2	HP3	UNIT WT	PR	E	TCOEF
1	5	1	2	5	0	2.0000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
2	5	2	3	5	0	0.5000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
3	5	3	4	5	0	0.5000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
4	5	4	5	5	0	0.5000	0.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05
5	5	5	6	5	0	0.5000	2.5000	0.0	0.0	82.5000	204.1000	60.0000	0.15	0.15	0.4000E+04	0.6000E-05

GEOMETRIC PARAMETERS

SEGMENT	POINT	XCOOR	RCOOR	SLOPE ANGLE	THICK	R2	R1	S.COR	DR1
1									
	1	0.0	85.991	1.684	2.000	86.548	-582.962	0.500	2.646
	6	2.500	85.711	1.680	1.625	86.225	-576.456	3.016	2.527
	11	5.000	85.443	1.676	1.250	85.914	-570.250	5.530	2.410
	16	7.500	85.186	1.671	0.875	85.616	-564.340	8.043	2.293
	21	10.000	84.940	1.667	0.500	85.331	-558.724	10.555	2.178
2									
	22	10.000	84.940	1.667	0.500	85.331	-558.724	12.667	2.178
	32	36.667	83.037	1.617	0.500	83.125	-516.490	39.405	0.993
	42	63.335	82.511	1.564	0.500	82.513	-505.166	66.080	-0.141
	52	90.002	83.387	1.512	0.500	83.531	-524.091	92.765	-1.283
	62	116.670	85.621	1.463	0.500	86.120	-574.363	119.529	-2.488
3									
	63	116.670	85.621	1.463	0.500	86.120	-574.363	119.337	-2.488
	73	143.335	89.112	1.419	0.500	90.146	-658.737	146.231	-3.808
	83	170.000	93.719	1.381	0.500	95.427	-781.414	173.293	-5.286
	93	196.665	99.287	1.350	0.500	101.767	-947.744	200.534	-6.958
	103	223.330	105.664	1.323	0.500	108.982	-1163.950	227.952	-8.850
4									
	104	223.330	105.664	1.323	0.500	108.982	-1163.950	228.664	-8.850
	109	249.997	112.714	1.302	0.500	116.911	-1436.923	256.248	-10.982
	114	276.665	120.318	1.285	0.500	125.417	-1773.959	283.978	-13.368
	119	303.332	128.377	1.270	0.500	134.392	-2182.696	311.837	-16.019
	124	330.000	136.811	1.259	0.500	143.748	-2670.996	339.807	-18.942
5									
	125	330.000	136.811	1.259	0.500	143.748	-2670.996	331.250	-18.942
	130	336.250	138.835	1.256	1.000	145.988	-2797.808	337.820	-19.667
	135	342.500	140.875	1.254	1.500	148.244	-2929.534	344.394	-20.407
	140	348.750	142.930	1.252	2.000	150.515	-3066.280	350.973	-21.163
	145	355.000	145.001	1.250	2.500	152.802	-3208.150	357.557	-21.935

DISPLACEMENTS IN RADIANS OR LENGTH UNITS DUE TO
THE NON-AXISYMMETRIC LOADS AT 7 CIRCUMFERENTIAL POINTS
DEFINED BY THE ANGLE THETA IN DEGREES

THETA = 0.0 30.00 60.00 90.00 120.00 150.00 180.00

SEGMENT 1 AT 5 MERIDIONAL POINTS *

POINT 1

R 0.2791E+00-0.1989E-01-0.2117E+00-0.2401E-01 0.1554E+00-0.7777E-02-0.1166E+00
W -0.1664E+02-0.5540E+01 0.9287E+01 0.9690E+01 0.1866E+01-0.5048E+01-0.3743E+01

```
STRESS RESULTANTS IN FORCE UNITS PER UNIT LENGTH
DUE TO THE NON-AXISYMMETRIC LOADS AT    7 CIRCUMFERENTIAL
POINTS DEFINED BY THE ANGLE THETA IN DEGREES

THETA=              0.0       30.00      60.00      90.00     120.00     150.00     180.00

SEGMENT   1 AT    5 MERIDIONAL POINTS  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *


POINT   1

       MS  -0.1308E-12 0.1138E-13 0.1693E-12-0.2875E-13-0.1836E-12 0.2143E-13 0.1592E-12
       SS   0.3777E-14 0.6415E-14-0.1262E-14-0.2626E-14-0.1422E-14-0.3933E-14 0.8769E-15
       NS   0.7673E-13-0.4440E-13-0.1024E-12 0.9682E-13 0.2751E-12-0.7659E-13-0.3596E-12
       TS   0.0         0.4096E-12 0.2708E-12-0.6551E-13-0.1049E-12 0.1611E-13 0.7547E-19
       MT   0.2938E+02 0.3018E+01-0.2087E+02-0.5841E+01 0.3588E+00 0.1442E+02-0.1167E+02
       NT  -0.3534E+02 0.8229E+01 0.1678E+02 0.5864E+01-0.1450E+02 0.4971E+01 0.7253E+01
       MTS 0.0         0.1810E+02-0.4657E+00-0.1415E+02 0.1601E+01 0.8731E+01 0.1353E-04
       MST 0.0         0.1832E+02-0.4362E+00-0.1430E+02 0.1570E+01 0.8815E+01 0.1369E-04
       NTS 0.0         0.4919E+00-0.5916E-02-0.3809E+00 0.3385E-01 0.2329E+00 0.3666E-06
       NST 0.0         0.2116E+00-0.5040E-02-0.1653E+00 0.1814E-01 0.1018E+00 0.1582E-06

POINT   2

       MS  -0.2678E+01 0.7976E+00 0.1162E+01 0.9736E-01-0.1307E+01 0.2816E+00 0.7462E+00
       SS   0.6354E+00-0.1758E+00-0.2716E+00-0.1485E-01 0.2836E+00-0.1373E-01-0.1815E+00
       NS   0.2314E+00-0.1146E+00-0.8640E-02-0.5782E-01 0.8302E-01-0.4600E-01-0.4454E-02
       TS   0.0        -0.1398E+01-0.3531E+00 0.1600E+01-0.6670E+00-0.2412E+00 0.8804E-07
       MT   0.1691E+02 0.1293E+01-0.1169E+02-0.3169E+01 0.7224E+00 0.7676E+01-0.6583E+01
       NT  -0.1451E+02 0.2139E+01 0.9146E+01 0.5269E+01-0.4235E+01 0.4446E+01 0.2943E+01
       MTS 0.0         0.9895E+01-0.2832E+00-0.7631E+01 0.7947E+00 0.4748E+01 0.7710E-05
       MST 0.0         0.1002E+02-0.2669E+00-0.7719E+01 0.7810E+00 0.4794E+01 0.7796E-05
       NTS 0.0        -0.1128E+01-0.3573E+00 0.1394E+01-0.6501E+00-0.1140E+00 0.2966E-06
       NST 0.0        -0.1282E+01-0.3562E+00 0.1510E+01-0.6579E+00-0.1856E+00 0.1785E-06

POINT   3

       MS  -0.2719E+01 0.8477E+00 0.1161E+01 0.9525E-01-0.1358E+01 0.3433E+00 0.7621E+00
       SS   0.7663E+00-0.2126E+00-0.3361E+00-0.3093E-01 0.3400E+00-0.3973E-01-0.2197E+00
       NS   0.6333E+00-0.3226E+00-0.3286E-02-0.1644E+00 0.2410E+00-0.1429E+00 0.1539E-01
       TS   0.0        -0.1886E+01-0.7138E+00 0.2232E+01-0.9220E+00-0.2063E+00 0.6853E-06
       MT   0.8203E+01 0.4454E+00-0.5535E+01-0.1456E+01 0.5494E+00 0.3484E+01-0.3144E+01
       NT  -0.2155E+01-0.1567E+01 0.4417E+01 0.4768E+01 0.2232E+01 0.3326E+01 0.4644E+00
       MTS 0.0         0.4661E+01-0.1459E+00-0.3567E+01 0.3663E+00 0.2214E+01 0.3748E-05
       MST 0.0         0.4718E+01-0.1380E+00-0.3609E+01 0.3612E+00 0.2235E+01 0.3786E-05
       NTS 0.0        -0.1759E+01-0.7161E+00 0.2135E+01-0.9141E+00-0.1468E+00 0.7865E-06
```

APPENDIX E - FLEXSHELL USER'S MANUAL

Using the flexibility method of analysis, program FLEXSHELL computes the in-plane forces, bending moments, and horizontal displacements for an axisymmetrically loaded shell structures due to various loads.

The program is capable of analyzing six types of shells of revolution of uniform thickness. These are cylinders, spheres, inverted spheres, cones, inverted cones, and base slabs on an elastic foundation. Seven axisymmetric loading cases are available. These are self-weight, uniform pressure, prestressing, snow load (a uniform vertical load over a horizontal projection), hydrostatic load, uniform temperature change, and temperature gradient through the shell thickness.

The input and output files for the example problem discussed in Chapter 6 are found in the latter part of this appendix.

The program is unit independent, that is any consistent set of units can be used. The user must select a unit of length and force, and all input quantities must be in these two units. Output will be in the same units. For example, if the unit of length selected is feet and the unit of force in kips, then all input quantities must be consistent with these units, such as thickness in feet, modulus of elasticity in kips per square feet, etc. Output will be in terms of kips per foot and foot-kips per foot, etc.

E.1 Input

The input to FLEXSHELL consists of multiple input records which may be lines in a datafile or a set of punched data cards. For convenience, input records will be referred to as card types in the context of this user's manual. There are six input card types and certain card types may be repeated as necessary.

A typical explanation of a card type consists of the card type number, a descriptive name indicating the nature of the data, the format used, and the number of cards of that type required. This is followed by a symbolic line of input, in bold type, indicating the order of the entries, followed by a description of each entry, and the options available ,if any, for each entry. Input is format free and individual entries may be separated by either a comma or blank. All entries in an input line are required but all zero entries occurring at the end of a line may be omitted.

1. **TITLE** card — Format 10A8

   Any identifier string up to 80 characters.

2. **CONTROL** card — Format 2I4

   NSEG IPRINT

   NSEG = number of shell segments in the structure, not exceeding 20.

   IPRINT = print control character

   0 - echos input data and prints final results only;
   1 - prints full output including the connectivity matrix, PSF and PBF arrays, and the element and structure flexibility matrices. (used for checking purposes only)

3. **SEGMENT DATA** card Format 5I4,2F10.4

One card per segment required. Note that the segments must be numbered sequentially in such manner that any segment always has a higher number than any of the segments which it supports.

I IT IR(1) IR(2) NDIV EC(I,1) EC(I,2)

I = segment number

IT = segment type

1 - cylinder
2 - sphere
3 - base on elastic foundation
4 - cone
5 - inverted sphere
6 - inverted cone

IR(1) = top connectivity flag for segment I

0 - top is not connected to another segment;
1 - top is connected to another segment.

IR(2) = bottom connectivity flag for segment I

0 - bottom is not connected to another segment;
1 - bottom is connected to another segment;
-1 - bottom is connected to another segment with a pure hinge.

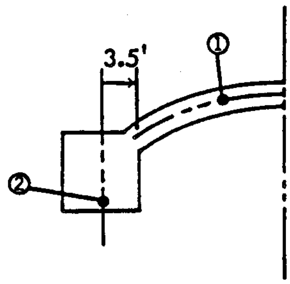
NDIV = number of divisions for segment I at which stress resultants are to be computed and printed. (max = 100)

EC(I,1) = eccentricity of joint connection at the top of the segment.

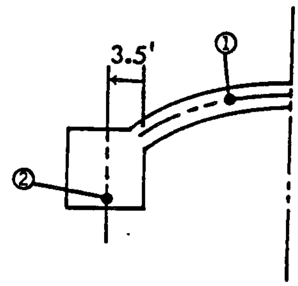
EC(I,2) = eccentricity of joint connection at the bottom of the segment.

NOTE:

When two shell segments are connected at a given elevation but have different midsurface radii, a horizontal eccentricity equal to the differences in horizontal radii to the midsurfaces will result. This eccentricity can be associated to either shell segment and is positive when directed inwards, that is, it is positive when associated with the segment having the larger radius. For the eccentricity between a spherical and cylindrical segment, EITHER of the following entries is permissible.



EC(1,2) = 0
EC(2,1) = 3.5



EC(1,2) = -3.5
EC(2,1) = 0

4. **CONNECTIVITY** specification card — Format 2I4

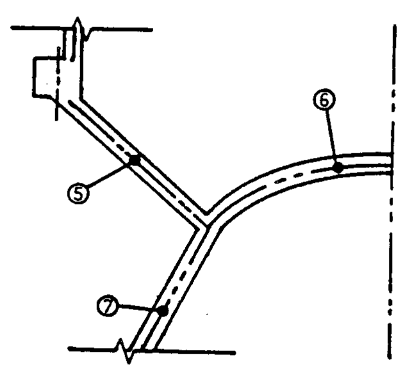
Specifies the connection between segments. Requires (NSEG - 1) cards.

IDCO(1) IDCO(2)

IDCO(1) = number of top segment.

IDCO(2) = number of segment to which the top segment is connected.

NOTE:
Where three shell segments intersect at the same elevation, two connectivity specification cards are required, one for each supported segment. For the Intze tank shown in Fig. 5.9, the entries are 5 7, on the first card, and 6 7, on the following card. Each segment number appears precisely once in IDCO(1) and these numbers must be arranged consecutively in increasing order, starting with segment 1 and ending with segment NSEG-1.



5. **SEGMENT PROPERTIES** card — Format I4,F6.0,F12.0, F8.0,5F10.0

I T R H HO E PR ALPHA UW

I = segment number

T = segment thickness

R = radius of the parallel circles for cylinders and spheres; or,
= subgrade coefficient for base segments; or,
= semi-vertex angle of cone in degrees.

H = length for cylinder; or,
= total angle in degrees from the axis of revolution to the outer edge for a sphere; or,
= outer radius of a circular base ring slab; or,
= distance from the apex of cone to the large end.

HO = 0.0 or blank for cylinder; or,
= angle in degrees from the axis of revolution to the inner edge for a sphere; or,
= inner radius of a circular base ring slab; or,
= distance from the apex of cone to inner edge; or,

E = Young's modulus for segment

PR = Poisson's ratio for segment

ALPHA = coefficient of thermal expansion

UW = unit weight of material for segment

6. **LOAD TYPE** card Format 2I4,7F10.0

One card per segment is required.

I IP PV WHT PSF(1) **PSF**(2) **PSF**(3) **PSF**(4) **PSF**(5) **PSF**(6)

I = segment number

IP = load type parameter

1 - uniform pressure
2 - self-weight
3 - prestress loading
4 - uniform temperature change across section
5 - temperature gradient across section
6 - uniformly distributed load over a horizontal projection, or snow load
7 - liquid pressure

NOTE:

A hydrostatic load applied to the base segment is simulated by using a uniform pressure equal to the product of the liquid weight density and the height of the water above the base.

PV = value of the applied load, depending on the type of load.

If IP=1, PV is the magnitude of uniform pressure. Positive for internally directed pressure and negative for externally directed pressure. For a base segment, this value is positive when pressure is directed downward and negative when directed upward.

If IP=2, value of PV is disregarded and a dead load analysis is carried out for the unit weights specified on the SEGMENT PROPERTIES cards.

If IP=3, PV is the magnitude of the uniformly distributed prestress pressure on the midsurface. Same sign convention as IP=1.

If IP=4, PV is the uniform temperature change in degree Celsius or Fahrenheit, depending on the units of ALPHA. (positive if the temperature rises above the reference temperature)

If IP=5, PV is the gradient of temperature across section in degrees per unit of thickness. (positive if the temperature rises above the reference temperature)

If IP=6, PV is the magnitude of uniform pressure distributed over a horizontal projection (snow load)

If IP=7, PV is the magnitude of the liquid weight density

WHT = height of liquid above the vertex of a cone or height of liquid above the inner edge of a sphere. Value is ignored for load types other than liquid pressure.

PSF(1) = magnitude of externally applied horizontal force at the top of the segment.

PSF(2) = magnitude of externally applied moment at the top of the segment.

PSF(3) = magnitude of externally applied horizontal force at the bottom of the segment.

PSF(4) = magnitude of externally applied moment at the bottom of the segment.

PSF(5) = magnitude of externally applied vertical force at the top of the segment.

PSF(6) = magnitude of externally applied vertical force at the bottom of the segment.

NOTE:

The PSF forces are forces and moments which, if necessary, are to be applied IN ADDITION TO the distributed loading effects identified by the PV values.

Prestressing effects are generally simulated as distributed loads but cable anchorages give rise to concentrated loads which are treated as PSF forces.

E.2 Output

FLEXSHELL echo checks the input data. If the output control parameter, IPRINT = 1, the particular basic forces (PBF), particular solution forces (PSF), and the incompatible displacements (PSD) vectors, the connectivity matrix, and the segment and global flexibility matrices are included in the output. If IPRINT = 0, only the final results are printed. The forces at the ends of each segment in the order SF(1), SF(2), SF(3), SF(4), SF(5), SF(6), positive in the direction shown on Fig. 5.1 are printed. The horizontal displacements, positive in the direction towards the axis of revolution, and the stress resultants in the order, N_ϕ, N_θ, M_ϕ, M_θ, positive in the direction shown on Fig. 3.4 are printed for each equally spaced point in each segment, as defined by the user from NDIV.

Example Input File

INTZE TANK MODEL (HYDROSTATIC PRESSURE)

```
8,
1,2,0,1,5,
2,1,1,1,5,.216,.125,
3,1,1,1,5,
4,1,1,1,5,.125,.125,
5,6,1,1,5,
6,2,0,1,5,
7,4,1,1,40,
8,3,1,0,1,
```

segment control data

```
1,2,
2,3,
3,4,
4,5,
5,7,
6,7,
7,8,
```

connectivity properties

```
1,.333,94.5,28.,0.,.5804E09,.167,.6E-5,150.,
2,.667,44.581,1.45,0.,.5804E9,.167,.6E-5,150.,
3,.417,44.456,10.,0.,.5804E9,.167,.6E-5,150.,
4,.667,44.581,1.45,0.,.5804E9,.167,.6E-5,150.,
5,.3,62.75,50.,25.,.5804E9,.167,.6E-5,150.,
6,.3,82.5,15.5,0.,.5804E9,.167,.6E-5,150.,
7,.5,19.146,152.447,67.763,.5804E9,.167,.6E-5,150.,
8,2.,450000.,50.,0.,1.E20,.167,.6E-5,150.,
```

segment properties

```
1,1,
2,1,
3,7,62.4,
4,1,-669.24,
5,7,62.4,34.35,
6,7,62.4,19.9,
7,1,
8,1,
```

load input data

Example Output File

1INTZE TANK (HYDROSTATIC PRESSURE)

1 **** OUTPUT FOR FLEXIBILITY ANALYSIS OF SEGMENTED SHELL ****

NUMBER OF SEGMENTS = 8

IPRINT = 0

SEG	TYPE	IR	JR	NDIV	EC1	EC2
1	2	0	1	5	0.0	0.0
2	1	1	1	5	0.2160	0.1250
3	1	1	1	5	0.0	0.0
4	1	1	1	5	0.1250	0.1250
5	6	1	1	5	0.0	0.0
6	2	0	1	5	0.0	0.0
7	4	1	1	2	0.0	0.0
8	3	1	0	1	0.0	0.0

CONNECTIVITY MATRIX

1	2
2	3
3	4
4	5
5	7
6	7
7	8

GEOMETRIC PARAMETERS

SEG	THICK	RADIUS	L OR ANG	ANGO	MODULUS	P RATIO	THERMCOEF	WEIGHT
1	0.333	94.500	28.000	0.0	0.5804E+09	0.167	0.6000E-05	150.000
2	0.667	44.581	1.450	0.0	0.5804E+09	0.167	0.6000E-05	150.000
3	0.417	44.456	10.000	0.0	0.5804E+09	0.167	0.6000E-05	150.000
4	0.667	44.581	1.450	0.0	0.5804E+09	0.167	0.6000E-05	150.000
5	0.300	62.750	50.000	25.000	0.5804E+09	0.167	0.6000E-05	150.000
6	0.300	82.500	15.500	0.0	0.5804E+09	0.167	0.6000E-05	150.000
7	0.500	19.146	152.447	67.763	0.5804E+09	0.167	0.6000E-05	150.000
8	2.000	450000.000	50.000	0.0	0.1000E+21	0.167	0.6000E-05	150.000

PARTICULAR SOLUTION INPUT INFORMATION

SEG	TYPE	VALUE	LIQ HT	TOP SHEAR	TOP MOM	BOT SHEAR	BOT MOM	TOP FORCE	VERT FORCE

1	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	7	62.400	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	1	-669.240	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	7	62.400	34.350	0.0	0.0	0.0	0.0	0.0	0.0
6	7	62.400	19.900	0.0	0.0	0.0	0.0	0.0	0.0
7	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FORCES ON ENDS OF SEGMENTS

SEG	J	IF	FORCE
1	3	1	0.59837E+02
1	4	2	-0.81341E+02
2	1	3	-0.59837E+02
2	2	4	0.81341E+02
2	3	5	0.71813E+02
2	4	6	-0.17960E+03
3	1	7	-0.71813E+02
3	2	8	0.17960E+03
3	3	9	-0.10400E+04
3	4	10	-0.26202E+04
4	1	11	0.10400E+04
4	2	12	0.26202E+04
4	3	13	-0.19213E+04
4	4	14	-0.65012E+03
5	1	15	0.19213E+04
5	2	16	0.65012E+03
5	3	17	0.18391E+05
5	4	18	-0.13140E+05
6	3	19	0.12479E+04
6	4	20	-0.97211E+03
7	1	21	-0.19639E+05
7	2	22	0.14112E+05
7	3	23	-0.58310E-06
7	4	24	0.20065E-05
8	1	25	0.58310E-06
8	2	26	-0.20065E-05
8	5	27	0.0

INDIVIDUAL END FORCES FOR SEGMENT 1

0.0	-0.0	0.59837E+02	-0.81341E+02
0.0	-0.0		

*** OUTPUT FOR DOME SEGMENT 1 ***

POINT	ANGLE	N1	N2	M1	M2
1	0.0	-0.0	-0.0	0.0	0.0
2	5.6000	0.35230E-01	-0.15776E+00	0.24813E-01	0.14989E-01
3	11.2000	-0.35215E+00	-0.10515E+01	-0.17631E+00	-0.36752E-01
4	16.8000	0.11789E+01	0.15270E+02	0.32328E-01	-0.11248E+00
5	22.4000	0.20833E+01	-0.64354E+02	0.94579E+01	0.24661E+01
6	28.0000	-0.52833E+02	-0.36714E+03	-0.81341E+02	-0.15467E+02

*** HORIZONTAL DISPLACEMENT ***

POINT	COORD	W
1	0.0	0.0
2	5.6000	0.78078E-08
3	11.2000	0.94278E-07
4	16.8000	-0.21302E-05
5	22.4000	0.12055E-04
6	28.0000	0.82249E-04

INDIVIDUAL END FORCES FOR SEGMENT 2

-0.59837E+02 0.81341E+02 0.71813E+02 -0.17960E+03
0.0 0.0

*** OUTPUT FOR CYLINDRICAL SEGMENT 2 ***

POINT	COORD	N1	N2	M1	M2
1	0.0	0.0	-0.71422E+03	-0.81341E+02	-0.13584E+02
2	0.2900	0.0	-0.58226E+03	-0.99327E+02	-0.16588E+02
3	0.5800	0.0	-0.44539E+03	-0.11841E+03	-0.19774E+02
4	0.8700	0.0	-0.30265E+03	-0.13833E+03	-0.23101E+02
5	1.1600	0.0	-0.15306E+03	-0.15882E+03	-0.26524E+02
6	1.4500	0.0	0.43762E+01	-0.17960E+03	-0.29994E+02

*** HORIZONTAL DISPLACEMENT ***

POINT	COORD	W
1	0.0	0.82249E-04
2	0.2900	0.67053E-04