Modified unrolled small quantum groups

by

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Abstract

We consider the category of modules over certain subalgebras of the unrolled restricted quantum group associated to any reductive Lie algebra and show some progress towards the proof of an equivalence of categories of this with the category of local representations of a simple current extension.

Aknowledgements

Firstly, I would like to thank my supervisor professor Thomas Creutzig for all the time he spent answering my questions and guiding me throughout this work; this would not have been possible without his advice and patience. I also would like to thank my family and friends for all their support during the adverse times. To all of the above, all of my gratitude.

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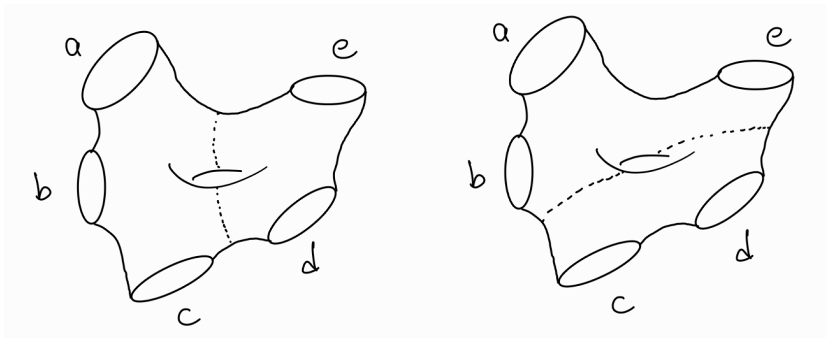
1 Introduction

Motivation

Conformal Field Theories in two dimensions are rare examples of interacting and exactly solvable quantum field theories [1], this is due to its rich structure of symmetries, which is usually infinite dimensional, in contrast to the finite dimensional space of symmetries of other conformal field theories. However, higher dimensional CFTs have lately attracted much attention due to the Maldacena duality (or AdS/CFT correspondence), which relates quantum gravity formulated in the language of string theory and CFTs that include theories similar to Yang-Mills theories, which describe elementary particles.

In 1989, Moore and Seiberg axiomized Rational Conformal Field Theory, which is a special type of two-dimensional CFT (it is worth noting that 2dCFTs provide a way of constructing string theories in less than twenty-six dimensions [2]). Consider a block, that is, a punctured Riemann surface where every puncture has a label corresponding to a representation space of some chiral algebra (a Virasoro algebra or any of its extensions, namely W-algebras, WZW-theories (or current algebras), etc [3]) and a vector space assigned to it; a Rational CFT arises when the underlying vector space is a strongly rational vertex operator algebra, that is, a $\mathbb{Z}$-graded, simple, CFT-type, self-contragredient rational and C_2-cofinite VOA. This Riemann surface can be formed by glueing together three holed spheres (also known as pairs of pants); however, different ways of glueing the same surface (see image below) give rise to different direct sums over the intermediate states passing through the glued holes; the assumption of duality states that every vector space spanned by these blocks is independent of the way the surface was obtained [4]. Some quantum groups will be introduced later, but these are basically modifications of the universal enveloping algebra associated to a Lie algebra. For now, it is worth noting that solutions of Rational CFTs are given by representations of quantum groups at roots of unity [5], and although the category of quantum group modules at a root of unity frecuently contains infinitely many simple objects and not every short exact sequence of morphisms splits (so it is non-semisimple), it decomposes into blocks with finitely many simple objects. Additionally, the category of finite-dimensional modules over any quantum group is abelian, and so the derived category can be constructed by considering the homotopy category $\mathcal{H}(\mathcal{C})$ whose objects are chain complexes of objects in $\mathcal{C}$ and the morphisms are chain maps modulo homotopy. Then one deems equivalent any two objects related by a morphism that induces an isomorphism on their cohomology. It can be shown that $\mathcal{H}(\mathcal{C})$ becomes a differential graded category, which is necessary to connect with topologically twisted QFTs. This is desirable since only differential graded categories (which are tipically non-abelian) make sense physically and behave well under dualities such as 3d mirror symmetry [6].



The same surface obtained by glueing in two different ways

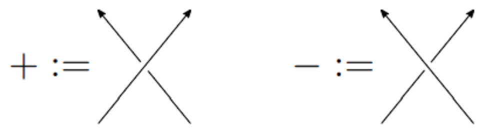
The same year, in [7], Witten showed that the expectation of a collection of Wilson loops in the Chern-Simons theory (which is a 3-dimensional Topological Quantum Field Theory) is related to the Jones polynomial of the corresponding link and also gives link invariants [8].

Later on, in 1991, Turaev and Reshetikhin described quantum invariants of 3-manifolds inspired by Jones polynomials. Schematically, consider the subalgebra $\overline{U}$ of $U_q(\mathfrak{sl}(2))$ defined by setting $E^r = F^r = 0$ and $K^r = 1$. Consider the quotient of the category of finite dimensional $\overline{U}$-modules by those with zero quantum dimension; this is a modular category $\mathcal{D}$, which means it is a semisimple ribbon category and is characterized by having a finite number of isomorphism classes of simple objects satisfying a series of axioms [9]. It is very well known that one can reduce the topology of any 3-manifold to the theory of links in S^3 since there is a one-to-one correspondence assigning a closed, oriented, connected 3-manifold M_L to a link L by surgering S^3 along L and each closed, oriented, connected 3-manifold M is homeomorphic to some M_L by a degree 1 homeomorphism [10]. Let M_L be a manifold and assume that the i-th component of L is coloured by some simple module V_i of $\mathcal{D}$ then define the weighted link invariant

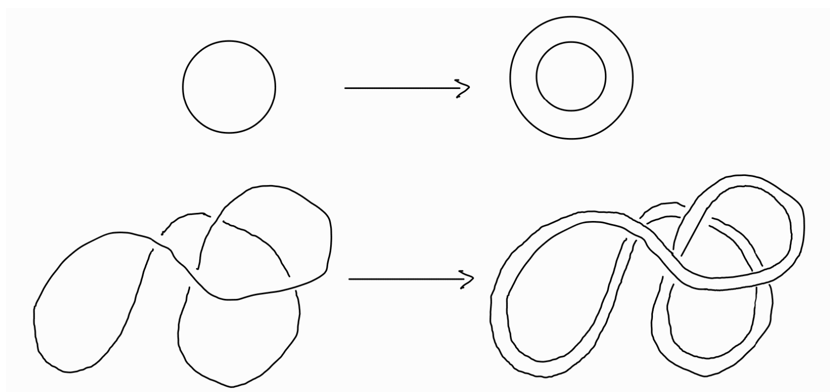
$$\left(\prod_i qdim_{\mathcal{D}}(V_i)\right) F(L)$$

where F is the Reshetikin-Turaev invariant associated to $\mathcal{D}$. The invariant of M_L is then the finite sum of these weighted link invariants over all possible colourings of L [9] [10].

Now, let us see a link invariant computed by Jones which involves $U_q(\mathfrak{sl}(2))$, where its Hopf algebra structure plays a fundamental role. Consider a link L colored by an object in $U_q(\mathfrak{sl}(2))$-mod, that is, an embeding of circles into $\mathbb{R}^3$ where each circle has the same element V of $U_q(\mathfrak{sl}(2))$-mod assigned to it. Define the positive and negative crossings as



Then define the writhe of L, denoted by $\omega(L)$, as the total number of possitive crossings minus the total number of negative crossings. Now, let L^b be the blackboard framed link associated to L



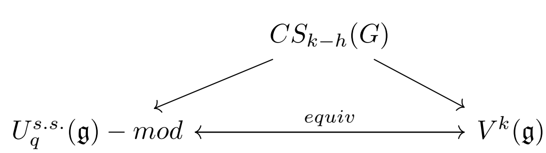
Examples of Blackboard Framing

then, the link invariant of L is

$$\theta_V^{\omega(L)} F^V(L^b)$$

where $F^V(L)$ is the invariant of ribbon tangles described in [11]. It is worth noting that the Jones Polynomial can be recovered from this construction.

Surprisingly, there is an interplay among three perspectives on quantum invariants related by braided tensor categories: topological quantum field theory, quantum groups and vertex operator algebras that has played an important role in both mathematical physics and mathematics. A fundamental object in these perspectives, which contains all the necessary data to construct invariants of 3-manifolds and links is the category $\mathcal{C}$ of line operators in 3d QFT, which corresponds to categories of modules from the quantum group and VOA perspectives [6]. This equivalence can be exemplified with the following diagram, where each vertex corresponds to the semisimplified version of $\mathcal{C}$



Where CS_{k-h} is the Chern-Simons theory with compact gauge group G at level $k-h$, $V^k(\mathfrak{g})$ is a simple quotient of $\hat{\mathfrak{g}}_k$, and $U_q^{s.s.}(\mathfrak{g})$ is the semisimplification of modules for $U_q(\mathfrak{g})$ at a $2k$ root of unity.

Most of the research on quantum invariants has been based on semisimple categories and most of the non-semisimple work has been done primarily on quantum variants of $\mathfrak{sl}(2)$ [9]. The current goal is to extend the diagram above to the non-semisimple case and get a diagram of the form

$$\begin{array}{ccc} & \text{non-semisimple TFT} & \\ \swarrow & & \searrow \\ QG-mod & \xleftrightarrow{equiv} & VOA-mod \end{array}$$

Examples of this type of equivalences are conjectures 1.2 and 1.4 in [12], relating the categories of modules over a modification of the restricted quantum group and the triplet vertex operator algebra, namely

$$\overline{U}_q^{(\phi)}(\mathfrak{sl}(2)) - mod \cong W(p) - mod$$

and

$$Rep_{wt}\overline{U}_q^{H}(\mathfrak{sl}(2)) \cong Rep_{\langle s\rangle}\mathcal{M}(p)$$

where these are equivalences of ribbon categories [13] [14].

This thesis studies primarily the category $\mathcal{C}$ of weight modules over the unrolled restricted quantum group associated to any reductive finite dimensional Lie algebra, $\overline{U}_q^{H}(\mathfrak{g})$ (as well as some subcategories of it), which has generators $X_{\pm i}, H_i, K_\gamma$ and non-trivial relations

$$K_0 = 1, \quad X_{\pm i}^{r_i} = 0, \quad K_{\gamma_1}K_{\gamma_2} = K_{\gamma_1+\gamma_2}, \quad K_\gamma X_{\pm j} K_{-\gamma} = q^{\pm\langle\gamma,\alpha_j\rangle}X_{\sigma j},$$

$$[H_i, X_{\pm j}] = \pm a_{ij}X_{\pm j}, \quad [H_i, K_\gamma] = 0, \quad [H_i, H_j] = 0, \quad [X_i, X_{-j}] = \delta_{i,j}\frac{K_{\alpha_j}-K_{\alpha_j}^{-1}}{q_j-q_j^{-1}}$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_i}X_{\pm i}^k X_{\pm j} X_{\pm i}^{1-a_{ij}-k} = 0 \text{ if } i \neq j$$

Let G be a simply connected complex Lie group, let the central elements of $U_q(\mathfrak{g})$ act by fixed constants on any indecomposable module and let there be no morphism between modules with different values of the center, then $\mathcal{C} = U_q(\mathfrak{g})$-mod decomposes as

$$\mathcal{C} = \bigoplus_{g\in G} U_q(\mathfrak{g})_g - mod$$

where $U_q(\mathfrak{g})_g$ is the quantum group with the Frobenius center set equal to $g \in G$ [6]; for example, see conjecture 4, or more generally, conjecture 6.

Results

In general terms, we have made significant progress towards the proofs of the commutativity of the following diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}} & & \\ \downarrow{\scriptstyle Forg} & \searrow^{\mathcal{F}} & \\ \hat{\mathcal{C}} & \xrightarrow{\hat{\mathcal{F}}} & Rep^{loc}(A) \end{array}$$

as well as the, presumably, equivalence of categories $\hat{\mathcal{F}}$.

To understand what this all means, we need to first introduce some terminology.

Consider a set of indices $\lambda \in I$ in the abelian group I and a set of simple objects $\{\mathbb{C}_\lambda | \lambda \in I, \mathbb{C}_\lambda \text{ invertible }, \mathbb{C}_\lambda \otimes \mathbb{C}_\mu = \mathbb{C}_{\lambda+\mu}\}$ such that this is closed under tensor products and duals. Define the simple current extension A, which can be given the structure of a commutative associative algebra in $\mathcal{C}$:

$$A = \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$$

The monodromy $M_{V,W} : V \otimes W \to V \otimes W$ of two elements in a braided tensor category, V and W is equal to the doble brading $c_{V,W} \circ c_{W,V}$, where $c_{X,Y}$ corresponds to the usual braiding map $c_{X,Y} : X \otimes Y \to Y \otimes X$. This monodromy corresponds to the picture

Now, $Rep^{loc}(A)$ is the braided tensor category of representations of A that have monodromy with A equal to 1.

In the diagram, $\tilde{\mathcal{C}}$ is the subcategory of $\mathcal{C}$ of Z/L-modules, with Z being the algebra generated by $X_{\pm i}, H_i, K_\gamma$ and $L = \langle q^{2t_\lambda} - Id_Z | \lambda \in I\rangle$, and $\hat{\mathcal{C}}$ is the subcategory of Z_1/L-modules, where Z_1 is the Z-subalgebra generated by $X_{\pm i}, Ann_{\mathfrak{h}}(I), K_\gamma$. Additionally, $\mathcal{F}$ and $Forg$ are the induction and forgetful functors, such that $\mathcal{F}(X) = X \otimes A$ and $Forg$ forgets the action of the complement of $Ann_{\mathfrak{h}}(I)$.

Creutzig and Rupert made some advances in the treatment of the induction functor in [15]. Now, most of our important results are about the forgetful functor; these are:

1 Let $P \in \tilde{\mathcal{C}}$ be projective, then $Forg(P)$ is projective (Corollary 4).

2 Let $P \in \hat{\mathcal{C}}$ be projective, then there is a projective $\tilde{P} \in \tilde{\mathcal{C}}$ such that $Forg(\tilde{P}) \cong P$ (Corollary 5).

These two corollaries give a correspondence between projective modules. Furthermore, we have a correspondence between projective covers of simple modules given by the next two lemmas

3 Let $\tilde{L}_\lambda \in \tilde{\mathcal{C}}$ be a lift of the simple module $L_\lambda \in \hat{\mathcal{C}}$, then $Forg(\tilde{P}_\lambda)$ is isomorphic to the projective cover of L_λ, where $\tilde{P}_\lambda$ is the projective cover of $\tilde{L}_\lambda$ (Lemma 19).

4 Let $\pi : P_M \to M$ be the projective cover of $M \in \tilde{\mathcal{C}}$, then $Forg(P_M)$ is the projective cover of $Forg(M)$ (Lemma 20).

The next lemma guarantees that any simple module can be lifted under the forgetful functor

5 Let $S \in \hat{\mathcal{C}}$ be simple, then there is a simple $\tilde{S} \in \tilde{\mathcal{C}}$ such that $Forg(\tilde{S}) \cong S$ (Lemma 21).

Remark 1. *It is worth noting that the respective statements for 1-5 hold for the induction functor $\mathcal{F}$ [15].*

Additionally, we have the following results:

6 Let $X, Y \in \tilde{\mathcal{C}}$, then $Forg(X) \cong Forg(Y)$ if and only if $\mathcal{F}(X) \cong \mathcal{F}(Y)$ (Corollary 2).

7 We have a bijection $Hom_{\hat{\mathcal{C}}}(Forg(X), Forg(Y)) \cong Hom_{Rep^{loc}(A)}(\mathcal{F}(X), \mathcal{F}(Y))$ (Corollary 6).

These two results are a significant step forward towards the proof of the commutativity of the diagram above and the equivalence $\hat{\mathcal{F}}$ since 6 ensures that two induced modules are isomorphic if and only if the forgetful of their preimages are in $\hat{\mathcal{C}}$ and are isomorphic, which is a necessary condition for the diagram to commute and for $\hat{\mathcal{F}}$ to exist. Additionally, 7 supposes some progress in light of proposition 1.

To prove the equivalence $\hat{F}$, one needs correspondences between simple modules, projective modules and extensions of modules. The first two were adressed in this thesis and the last one was not possible to be proved due to time constraints.

Finally, we have the following conjecture:

Conjecture 1. *Let $L = \langle L_1, ...L_m \rangle$ and $\mu_1, ...\mu_m \in \mathbb{C}$, then if*

$$\hat{Z}_{\mu_1,\mu_2,...\mu_m} = Z_1 / \langle L_1 - \mu_1, ...L_n - \mu_m \rangle$$

and $\mathcal{C}(\mu_1, ...\mu_m)$ is the category with objects

$$Obj(\mathcal{C}(\mu_1, ...\mu_m)) = \{(V, \mu_V) \in Rep(A) : M_{V,S_1} = \mu_1 Id_{V\otimes S_1}, ...M_{V,S_m} = \mu_m Id_{V\otimes S_m}\}$$

For $S_i = \mathbb{C}_{\lambda_i}$ and I being generated by the λ_i.

Then, we have an equivalence of module categories

$$\hat{Z}_{\mu_1,\mu_2,...\mu_m} - mod \cong \mathcal{C}(\mu_1, ...\mu_m)$$

If it can be proven that $\hat{\mathcal{C}} \cong Rep^{loc}(A)$, the proof of the conjecture would be analogous.

2 Basics

Here we give some basic notions on category theory and algebra which are fundamental to understand this work.

2.1 Category theory

Categories are natural generalizations of algebraic structures, for example, a monoidal category is a category with a product where associativity holds, which is the analogue of a monoid. Loosely speaking, a category is the generalization of a set with arrows between objects; these have become ubiquitous in modern mathematics and have shown importance as an abstract structure. For example, if we consider the categories of Lie groups and Lie algebras, there is a functor (intuitively, a morphism between categories) from the subcategory of simply connected Lie groups to the category of Lie algebras that assigns the tangent space at 1 to the underlying Lie group; furthermore, this functor is an equivalence of categories, which gives us a dictionary to translate properties from simply connected Lie groups to Lie algebras and viceversa.

***Definition* 1.** *A category $\mathcal{C}$ consists of the following data:*

- *A collection of objects, denoted by $Ob(\mathcal{C})$.*
- *For every pair of objects $X, Y \in Ob(\mathcal{C})$, a collection of arrows $Hom(X, Y)$.*
- *A composition rule*

$$(f, g) \to f \circ g : Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$$

- *An identity morphism $1_X \in Hom(X, X)$ for every object X.*

such that composition of morphisms is associative and the identity morphism 1_X is a two-sided identity for compsition of morphisms.

Example 1. *Some examples of categories are the following:*

* *The category* **Set** *of sets, where the morphisms functions.*

* *The category* **Top** *of topological spaces, where morphisms are continuous maps.*

* *The category* **Group** *of groups, where morphisms are homomorphisms of groups.*

* *The category* $\mathbf{Vec}_{\mathbb{K}}$ *of vector spaces over* $\mathbb{K}$*, with morphisms being linear maps.*

* *The category* $\overline{\mathbf{U_q^H}}(\mathfrak{g}) - \mathbf{mod}$ *of* $\overline{U_q^H}(\mathfrak{g})$*-modules, where morphisms are morphisms of modules.*

* *The category* $\mathbf{Rep(A)}$ *of representations of an associative algebra* A*, where morphisms are homomorphisms of representations.*

Now, like homomorphisms of groups, rings, etc, it is desirable to define morphisms between categories; these are called functors and are defined as follows:

***Definition* 2.** *Let* $\mathcal{C}, \mathcal{D}$ *be two categories. A functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *consists of the following data:*

- *A map* $\mathcal{F} : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
- *For every* $X, Y \in \mathcal{C}$*, a map* $\mathcal{F} : Hom(X, Y) \to Hom(\mathcal{F}(X), \mathcal{F}(Y))$

subject to the following axioms:

- $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ *for every* $X \in \mathcal{C}$
- *For all composable morphisms* $f, g \in \mathcal{C}$ *, one has* $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Example 2. *Some well known functors are:*

* *The power set functor* $P : \mathbf{Set} \to \mathbf{Set}$ *that sends* X *to* $P(X)$.
* *The forgetful functor* $Forg : \mathbf{Group} \to \mathbf{Set}$.
* *The fundamental group* $\pi_1 : \mathbf{Top} \to \mathbf{Group}$ *and more generally, the n-th homotopy group* π_n.
* *The n-th homology* $H_n : \mathbf{Top} \to \mathbf{Group}$ *and cohomology* $H^n : \mathbf{Top} \to \mathbf{Group}$.
* *The Lie functor* $Lie : \mathbf{LieG} \to \mathbf{LieA}$.
* *For any module* N*, the* $Tor^n(\_, N)$ *functors and the* $Ext^n(\_, N)$ *functors.*

***Definition* 3.** *Suppose that* $\mathcal{C}$ *and* $\mathcal{D}$ *are categories and let* $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ *be functors. We define a natural transformation as a rule* $\nu : \mathcal{F} \to \mathcal{G}$ *which assigns a morphism* $\nu_X : \mathcal{F}(X) \to \mathcal{G}(X)$ *to every* $X \in \mathcal{C}$ *. This rule must satisfy that, for every morphism* $f \in Hom(X, Y)$*, the diagram*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_Y} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

commutes.

If every ν_X *is an isomorphism,* ν *is called a natural isomorphism and we denote* $\mathcal{F} \cong_\nu \mathcal{G}$.

One could have several notions of two categories being "equal", but following the generalization of algebraic structures logic, we define two categories to be equivalent if the following is satisfied:

***Definition* 4.** *Let $\mathcal{C}, \mathcal{D}$ be two categories. An equivalence of categories is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ such that there is another functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{G} \cong Id$ and $\mathcal{G} \circ \mathcal{F} \cong Id$.*

Example 3. *Some equivalences of categories are the following:*

* *Let $\mathcal{C}$ be a category and $S \subset ob\mathcal{C}$ be a subcollection of objects such that every object of $\mathcal{C}$ is isomorphic to some object in S. Let S denote the full subcategory of $\mathcal{C}$ spanned by S. Then the inclusion $I : S \to \mathcal{C}$ is an equivalence.*
* $\mathbf{LieG_{simply}} \cong \mathbf{LieA}$
* $\mathbf{Vec^{fin}_{\mathbb{R}}} \cong \mathbf{Mat}(\mathbb{R})$

***Definition* 5.** *A functor $\mathcal{F} : C \to D$ is called full resp. faithful if for all $X, Y \in C$, the map*

$$\mathcal{F} : Hom_C(X, Y) \to Hom_D(\mathcal{F}(X), \mathcal{F}(Y))$$

is surjective resp. injective. A fully faithful functor is a functor which is both full and faithful.

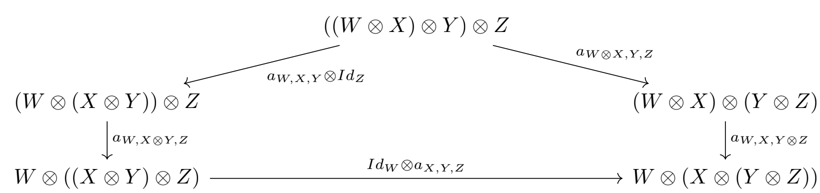
The following proposition characterizes the equivalences of categories:

Proposition 1. *(Theorem 1 of subsection IV.4 in [16]) Let $\mathcal{F} : C \to D$ be a functor. The following are equivalent:*

- *$\mathcal{F}$ is an equivalence of categories.*
- *$\mathcal{F}$ is fully faithful and essentially surjective, i.e. for every object $Y \in D$ there exists an object $X \in C$ such that $\mathcal{F}(X) \cong Y$.*

***Definition* 6.** *A tensor category or monoidal category is a quintuple $(\mathcal{C}, \otimes, a, 1, i)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ is a natural isomorphism called the associativity constraint and $i : 1 \otimes 1 \to 1$ is an isomorphism, subject to the following axioms:*

- *The pentagon axiom. The diagram*



is commutative for all objects $W, X, Y, Z \in \mathcal{C}$.

- *The unit axiom. The functors*

$$l_1 : X \to 1 \otimes X$$

$$r_1 : X \to X \otimes 1$$

of left and right multiplication by 1 are autoequivalences of $\mathcal{C}$.

Let $\mathcal{C}$ be a tensor category. A *simple current* is a simple object $J \in \mathcal{C}$ which is invertible with respect to the tensor product, that is, there is another object $J^{-1} \in \mathcal{C}$ such that $J \otimes J^{-1} = \mathbf{1}$. An element which is its own inverse is called *self dual.*

Let $\mathcal{C}$ be a monoidal category. Let $V, W \in \mathcal{C}$, then their braiding is an isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ satisfying the hexagon diagrams below.

$$\begin{array}{ccccccc} & & U \otimes (V \otimes W) & \xrightarrow{c_{U,V \otimes W}} & (V \otimes W) \otimes U & & \\ & \nearrow^{a_{U,V,W}} & & & & \searrow^{c_{V,W} \otimes Id_U} & \\ (U \otimes V) \otimes W & & & & & & (W \otimes V) \otimes U \\ & \searrow_{c_{U \otimes V,W}} & & & & \nearrow_{a^{-1}_{W,V,U}} & \\ & & W \otimes (U \otimes V) & \xrightarrow{Id_W \otimes c_{U,V}} & W \otimes (V \otimes U) & & \end{array}$$

$$\begin{array}{ccccccc} & & (U \otimes V) \otimes W & \xrightarrow{c_{U \otimes V,W}} & W \otimes (U \otimes V) & & \\ & \nearrow^{a^{-1}_{U,V,W}} & & & & \searrow^{a^{-1}_{W,U,V}} & \\ U \otimes (V \otimes W) & & & & & & (W \otimes U) \otimes V \\ & \searrow_{Id_U \otimes c_{V,W}} & & & & \nearrow_{c_{U,W} \otimes Id_V} & \\ & & U \otimes (W \otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U \otimes W) \otimes V & & \end{array}$$

where $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is the associativity isomorphism.

This braiding can be represented as

this kind of representations are read from bottom to top, where each strand is coloured by an element in $\mathcal{C}$.

Definition 7. *A left module category over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with an action (or module product) bifunctor: $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and a natural associativity isomorphism such that the usual pentagon diagram commutes.*

Suppose $\mathcal{C}$ is a locally finite k-linear abelian rigid monodial category, if the bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is k-bilinear on morphisms, then we call $\mathcal{C}$ a multitensor category.

Definition 8. *Let $\mathcal{C}$ be a multitensor category. Let $\mathcal{M}$ be a $\mathcal{C}$-module category and fix objects $M_1, M_2 \in \mathcal{M}$. Consider the functor*

$$X \to Hom_{\mathcal{M}}(X \otimes M_1, M_2)$$

This functor is representable, i.e., there exists an object $\underline{Hom}(M_1, M_2) \in \mathcal{C}$ and a natural isomorphism

$$Hom_{\mathcal{M}}(X \otimes M_1, M_2) \cong Hom_{\mathcal{C}}(X, \underline{Hom}(M_1, M_2))$$

2.2 Algebra

Now, we introduce some algebraic notions that are used throughout this work.

Let (A, μ, ι) be an commutative associative unital algebra in a braided monoidal category $\mathcal{C}$, with product given by μ and unit $\iota : \mathbf{1} \to A$.

We denote by $Rep(A)$, the category whose objects are given by pairs (V, μ_V) where $V \in Obj(\mathcal{C})$ and $\mu_V \in Hom(A \otimes V, V)$ satisfying the following assumptions:

- $\mu_V \circ (Id_A \otimes \mu_V) = \mu_V \circ (\mu \otimes Id_V) \circ a^{-1}_{A,A,V}$.
- $\mu_V \circ (\iota \otimes Id_V) \circ l_V^{-1} = Id_V$.

Where $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is the associativity isomorphism and $l_V : \mathbf{1} \otimes V \to V$ is the left unit isomorphism.

Define the *induction functor* $\mathcal{F} : \mathcal{C} \to Rep(A)$ by $\mathcal{F}(V) = (A \otimes V, \mu_{\mathcal{F}(V)})$, where $\mu_{\mathcal{F}(V)} = (\mu \otimes Id_V) \circ a^{-1}_{A,A,V}$, and $\mathcal{F}(f) = Id_A \otimes f$.

Define $Rep^0(A)$ or $Rep^{loc}(A)$ to be the full subcategory of $Rep(A)$ whose objects (V, μ_V) satisfy

$$\mu_V \circ M_{A,V} = \mu_V$$

with $M_{A,V} = c_{V,A} \circ c_{A,V}$ and $c_{X,Y} : X \otimes Y \to Y \otimes X$ is the braiding in $\mathcal{C}$.

Definition* 9.** ***(Essential epimorphism) *An epimorphism $f : M \to N$ is called* essential *if no proper submodule of M is mapped onto N.*

Definition* 10.** ***(Simple current) *A simple current is a simple object which is invertible with respect to the tensor product. Objects which are their own inverse are called self-dual.*

Definition* 11.** ***(Simple currents extension) *Consider a set of indices $\lambda \in I$ in the abelian group I and a set of simple objects $\{\mathbb{C}_\lambda | \lambda \in I, \mathbb{C}_\lambda$ invertible $, \mathbb{C}_\lambda \otimes \mathbb{C}_\mu = \mathbb{C}_{\lambda+\mu}\}$ such that this is closed under tensor products and duals. Define the simple current extension, associative algebra A as:*

$$A = \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$$

Proposition 2. ***(Frobenius Reciprocity)*** *Let $\mathcal{G} : Rep(A) \to \mathcal{C}$ be the forgetful functor sending (V, μ_V) to V. Then the induction functor $\mathcal{F}$ and $\mathcal{G}$ are adjoint, that is*

$$Hom_{Rep(A)}(\mathcal{F}(V), W) \cong Hom_{\mathcal{C}}(V, \mathcal{G}(W))$$

Proposition 3. *Let A be a simple algebra, then $\mathcal{F}(X)$, the induction of X, is simple if X is simple.*

Proof. Let X, Y be simple modules, then by Frobenius reciprocity, we have

$$Hom_{Rep(A)}(\mathcal{F}(X), \mathcal{F}(Y)) \cong Hom_{\mathcal{C}}(X, A \otimes Y) = \begin{cases} \mathbb{C} & X \cong \mathbb{C}_\lambda \otimes Y \\ 0 & otherwise \end{cases}$$

if the first case holds, then

$$\mathcal{F}(Y) = \mathcal{F}(\mathbb{C}_\lambda \otimes X) = \mathcal{F}(\mathbb{C}_\lambda) \otimes \mathcal{F}(X) \cong \mathcal{F}(X)$$

where the last congruence holds since

$$Hom_{Rep(A)}(\mathcal{F}(\mathbb{C}_\lambda), A) \cong Hom_{\mathcal{C}}(\mathbb{C}_\lambda, \mathcal{G}(A)) = \mathbb{C}$$

by Frobenius reciprocity and as A is simple, $\mathcal{F}(\mathbb{C}_\lambda)$ surjects onto A. As

$$Hom_{Rep(A)}(\mathcal{F}(\mathbb{C}_\lambda), A) = Hom_{Rep(A)}(\mathcal{F}(\mathbb{C}_\mu \otimes \mathbb{C}_\lambda), A) \cong Hom_{\mathcal{C}}(\mathbb{C}_{\mu+\lambda}, A) \neq 0$$

Hence, $\mathcal{F}(\mathbb{C}_\lambda)$ is a summand of A; therefore $\mathcal{F}(\mathbb{C}_\lambda) \cong A$. Now, to see that this implies that $\mathcal{F}(X)$ is simple, consider $Z \subset \mathcal{F}(X)$, then there exists λ such that $\mathbb{C}_\lambda \otimes X$ is a summand of Z and

$$Hom_{Rep(A)}(\mathcal{F}(X), Z) = Hom_{Rep(A)}(\mathcal{F}(\mathbb{C}_\lambda \otimes X), Z) \cong Hom_{\mathcal{C}}(\mathbb{C}_\lambda \otimes X, Z) \neq 0$$

Then $\mathcal{F}(X)$ is a direct summand of Z, which implies $\mathcal{F}(X) = Z$ and therefore, $\mathcal{F}(X)$ is simple.

□

Theorem 1. ***(Schur lemma)*** *If M and N are two simple modules over a ring R, then any homomorphism $f : M \to N$ of R-modules is either invertible or zero.*

Loewy diagrams

***Definition* 12.** *The socle of a module M over a ring R is defined to be the direct sum of the minimal nonzero submodules of M, that is,*

$$soc(M) = \bigoplus_{N \subset M \ simple} N$$

Equivalently, $soc(M)$ is the unique maximal semisimple submodule.

Let a module M have a filtration

$$0 = M_0 \subset M_1 \subset ...M_{l-1} \subset M_l = M$$

such that each subquotient M_i/M_{i-1} is the socle of M/M_{i-1}, The Loewy diagram *of M is then constructed by piling up with the i-th pile consisting of the direct summands of the quotient M_i/M_{i-1}.*

BGG Reciprocity

Definition* 13.** ***(Verma module) *Recall that for a Lie algebra $\mathfrak{g}$ we have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Any $\lambda \in \mathfrak{h}^*$ defines a 1-dimensional $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$- module with trivial $\mathfrak{n}^+$ action, denoted by $\mathbb{C}_\lambda$. If we let $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, it has a natural structure of $U(\mathfrak{g})$-module; this is called a Verma module.*

For each Verma module M_λ there is a unique simple quotient L_λ such that every simple module is isomorphic to a L_λ (section 1.3 of [17]).

Definition* 14.** ***(Projective cover) *Let $M \in \mathcal{C}$. The projective cover of M is a pair (P_M, π_M), with P_M a projective object in $\mathcal{C}$ and $\pi_M : P_M \to M$ an essential epimorphism, that is, no proper submodule of P_M is mapped onto M.*

The projective cover is unique up to isomorphism having this property (Proposition 6.20 in [18]); furthermore, every indecomposable projective module is isomorphic to some projective cover P_λ (Section 3.9 of [17]). For any $\lambda \in \mathfrak{h}^*$ denote by $\pi_\lambda : P_\lambda \to L_\lambda$ a fixed projective cover and by the projective nature of P_λ, we have the epimorphisms $P_\lambda \to M_\lambda \to L_\lambda$, which in particular implies that P_λ is also the projective cover of M_λ.

Definition* 15.** ***(Standard filtration) *Let $M \in \mathcal{O}$, the BGG category defined in [17]. A standard filtration of M is a sequence of submodules $0 = M_0 \subset M_1 \subset M_2 \subset ...M_n = M$ for which each $M^i = M_i/M_{i-1}$ is isomorphic to a Verma module.*

By Section 3.7 in [17], the multiplicity with which each Verma module occurs as a subquotient is well defined and is denoted by $(M : M_\lambda)$.

Theorem 2. ***(BGG Reciprocity, Section 3.11 of [17])***

For any $\lambda, \mu \in \mathfrak{h}^*$, then

$$(P_\lambda : M_\mu) = [M_\mu : L_\lambda]$$

where $[M_\mu : L_\lambda]$ is the multiplicity of L_λ in a Jordan-Hölder series of M_μ.

3 Quantum Groups

Quantum groups where originally developed to provide solutions to the Yang-Baxter equation [19] that first appear in statistical mechanics and then in confrmal and topological quantum field theory, knots and links and braid groups. It was well known that semisimple Lie algebras are rigid objects and cannot be deformed. Quantum groups appeared as equivalent but larger objects where a quantization (a modification) can be performed.

There are different quantizations of $\mathfrak{sl}(2)$ depending on two main features, namely if the quantum parameter q is a root of unity or not, and the part of the center of $U(\mathfrak{sl}(2))$ that is being killed. In this section, we will consider an intermediate quotient between the small quantum group and the non-restricted quantum group called the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{sl}(2))$ as well as its generalization to any reductive finite dimensional Lie algebra. This has shown to be useful in the construction of links and 3-manifold invariants as introduced in the motivation section.

We will see that $\overline{U}_q^H(\mathfrak{sl}(2))$ has an additional generator H with respect to the Concini-Kac quantum group that acts as a kind of logarithm with respect to K and is used to define a braiding on $\overline{U}_q^H(\mathfrak{sl}(2)) - mod$; and unlike the small quantum group does not restrict K^p, which allows modules in $\overline{U}_q^H(\mathfrak{sl}(2)) - mod$ to have non-integral weights; additionally, the restrictions $E^p = 0 = F^p$ force modules to be highest weight modules.

3.1 The unrolled restricted quantum group of $\mathfrak{g}$

This section is primarily based on [20].

3.1.1 Definition of $\overline{U}_q^H(\mathfrak{g})$

Let $\mathfrak{g}$ be a reductive finite dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}$.

We know that the restriction of the killing form $k(-,-)$ of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerate (Corollary 8.2 in [21]), which allows us to identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by sending $\phi \in \mathfrak{h}^*$ to $t_\phi \in \mathfrak{h}$ such that $\phi(h) = k(t_\phi, h)$ for all $h \in \mathfrak{h}$.
Additionally, we know that the killing form in $\mathfrak{h}$ may be used to define a symmetric bilinear form in $\mathfrak{h}^*$ by letting $\langle \lambda, \gamma \rangle = k(t_\lambda, t_\gamma)$.

We follow [15] and [22] to define the unrolled restricted quantum group associated to $\mathfrak{g}$, $\overline{U}_q^H(\mathfrak{g}) := Z$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$, $C = (a_{ij})_{i,j=1}^n$ its Cartan matrix and $\Delta := \{\alpha_1, ...\alpha_n\} \subset \mathfrak{h}^*$ be the set of simple roots of $\mathfrak{g}$. Let $\{H_1, ...H_n\}$ be the basis of $\mathfrak{h}$ such that $\alpha_i(H_j) = a_{ij}$ and $d_i = \langle \alpha_i, \alpha_i \rangle /2$. Define the root and weight lattices $Q = \bigoplus_{i=1}^n \alpha_i \mathbb{Z}$ and $P = \bigoplus_{i=1}^n \omega_i \mathbb{Z}$, respectively, where $\{\omega_1, ...\omega_n\} \subset \mathfrak{h}^*$ is the dual basis of $\{d_1 H_1, ...d_n H_n\}$.

Now, let $l \geq 3$ such that $r = 2l/(3 + (-1)^l) > max\{g_1, ...g_n\}$, where $g_k = gcd(d_k, r)$. Let $q \in \mathbb{C}$ be a primitive l-th root of unity and let $q_i = q^{d_i}$. Recall the definitions

$$\{x\} = q^x - q^{-x} \qquad \{n\}! = \{n\}\{n-1\}...\{1\} \qquad [x] = \{x\}/\{1\} \qquad \binom{n}{m} = \frac{\{n\}!}{\{m\}!\{n-m\}!}$$

The unrolled restricted quantum group associated to $\mathfrak{g}$ at root of unity q, $\overline{U}_q^H(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators $X_{\pm i}, H_i, K_\gamma$ with $i = 1, 2, ...n$ and $\gamma \in R$ with $Q \subset R \subset P$, and relations

$$K_0 = 1, \qquad X_{\pm i}^{r_i} = 0, \qquad K_{\gamma_1} K_{\gamma_2} = K_{\gamma_1 + \gamma_2}, \qquad K_\gamma X_{\pm j} K_{-\gamma} = q^{\pm \langle \gamma, \alpha_j \rangle} X_{\sigma j},$$

$$[H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, \qquad [H_i, K_\gamma] = 0, \qquad [H_i, H_j] = 0, \qquad [X_i, X_{-j}] = \delta_{i,j} \frac{K_{\alpha_j} - K_{\alpha_j}^{-1}}{q_j - q_j^{-1}}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} X_{\pm i}^k X_{\pm j} X_{\pm i}^{1-a_{ij}-k} = 0 \text{ if } i \neq j$$

There is a Hopf-algebra structure on $\overline{U}_q^H(\mathfrak{g})$ with coproduct Δ, counit ϵ and antipode S defined by

$$\begin{array}{lll}
\Delta(K_\gamma) = K_\gamma \otimes K_\gamma & \epsilon(K_\gamma) = 1 & S(K_\gamma) = K_{-\gamma} \\
\Delta(X_i) = 1 \otimes X_i + X_i \otimes 1 & \epsilon(X_i) = 0 & S(X_i) = -X_i K_{-\alpha_i} \\
\Delta(X_{-i}) = K_{-\alpha_i} \otimes X_{-i} + X_{-i} \otimes 1 & \epsilon(X_{-i}) = 0 & S(X_{-i}) = -K_{\alpha_i} X_{-i} \\
\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1 & \epsilon(H_i) = 0 & S(H_i) = -H_i
\end{array}$$

Denote the category of $\overline{U}_q^H(\mathfrak{g})$-modules, $\overline{U}_q^H(\mathfrak{g})$-mod, by $\mathcal{C}$.

3.1.2 Representation theory of $\overline{U}_q^H(\mathfrak{g})$

The Verma modules have the following form:

If I_λ is the ideal of $\overline{U}_q^H(\mathfrak{g})$ generated by the relations

$$H_i 1 = \lambda(H_i) \qquad K_\gamma = \prod_{i=1}^n q^{d_i k_i \lambda(H_i)} \qquad X_i 1 = 0$$

for each $\gamma \in R$ and $i \in \{1, 2, ...n\}$.

Then $M_\lambda = \overline{U}_q^H(\mathfrak{g})/I_\lambda$, which means that M_λ is generated as a module by the coset $v_\lambda = 1 + I_\lambda$ with the relations

$$H_i v_\lambda = \lambda(H_i) v_\lambda \qquad K_\gamma v_\lambda = \prod_{i=1}^{n} q^{d_i k_i \lambda(H_i)} v_\lambda \qquad X_i v_\lambda = 0$$

for each $\gamma \in R$ and $i \in \{1, 2, ...n\}$ and that the set $\{X_{-i} : i \in \{1, 2, ...n\}\}$ acts freely on M_λ.

Let $V \in \overline{U}_q^H(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$. The weight space of weight λ is the space $V(\lambda) = \{v \in V : H_i v = \lambda(H_i) v\}$ and any element of it is called a weight vector of weight λ.

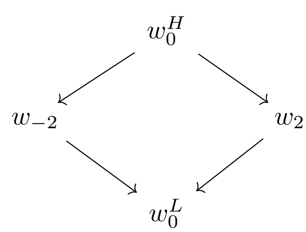
As noted in [20], we can unambiguously write N_λ and L_λ for the unique maximal proper submodule and unique irreducible quotient of weight λ, respectively.

Definition 16. ***(Weight module)*** *We say that a $\overline{U}_q^H(\mathfrak{g})$-module V is a weight module if $K_\gamma = \prod_{i=1}^{n} q^{d_i c_i H_i}$ as operators on V for all $\gamma = \oplus_{i=1}^{n} c_i \alpha_i \in R$ and V decomposes as a direct sum of eigenspaces*

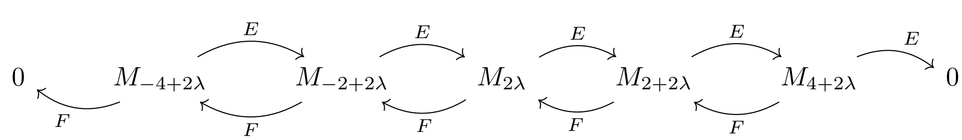
$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where $V_\lambda = \{v \in V | H_i v = \lambda(H_i) v\}$. We denote the category of finite dimensional weight modules by $\mathcal{C}$.

Example 4. *Recall that the projectives in $\overline{U}_{e^{\frac{i\pi}{2}}}^H(\mathfrak{sl}(2))$ are of the form $P_0 \otimes \mathbb{C}_{2k}^H$, where P_0 has the form*



Consider the module $M = P_0 \otimes P_0 \otimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$, then the weight decomposition of M as Z_1-module is of the form



where both $M_{-4+2\lambda}$ and $M_{4+2\lambda}$ have dimension 1 and are generated by $w_{-2} \otimes w_{-2} \otimes v$ and $w_2 \otimes w_2 \otimes v$, respectively; $M_{-2+2\lambda}$ and $M_{2+2\lambda}$ have *dimension 4 and are generated by* $\{w_{-2}\otimes w_0^X\otimes v, w_0^X\otimes w_{-2}\otimes v\}$ *and* $\{w_2\otimes w_0^X\otimes v, w_0^X\otimes w_2\otimes v\}$ *(with* $X=H,L$*), respectively;* $M_{2\lambda}$ *has dimension 6 and is generated by* $w_{-2}\otimes w_2\otimes v$*,* $w_2\otimes w_{-2}\otimes v$ *and the remaining vectors of the base of* M*.*

Simple $\overline{U}_q^H(\mathfrak{g})$-modules

From [20] we have the following characterization of simple $\overline{U}_q^H(\mathfrak{g})$-modules

Proposition 4. $V\in\overline{U}_q^H(\mathfrak{g})$*-mod is irreducible iff* $V\cong L_\lambda$ *for some* $\lambda\in\mathfrak{h}^*$*.*

Now, according to Remark 4.7 of [20], the simple currents in $\mathcal{C}$ are the 1-dimensional modules given in the set

$$\{L_\lambda|\lambda\in\mathcal{L}\}$$

where $\mathcal{L}=\{\lambda\in\mathfrak{h}^*|\lambda(H_i)\in\frac{l}{2d_i}\mathbb{Z}\}$.

We adopt the notation $\mathbb{C}_\lambda=L_\lambda$ when $\lambda\in\mathcal{L}$.

3.1.3 Monodromy

Recall that the monodromy is defined as the doble braiding, that is, $M_{X,Y}=c_{Y,X}\circ c_{X,Y}$, where $c_{X,Y}:X\otimes Y\to Y\otimes X$ is the usual braiding.

According to [15], for any vector $w_\gamma\in X\in\mathcal{C}$ of weight $\gamma\in\mathfrak{h}^*$ we have that the monodromy $M_{\mathbb{C}_\lambda,X}$ is given by

$$M_{\mathbb{C}_\lambda,X}(v_\lambda\otimes w_\gamma)=q^{2\langle\lambda,\gamma\rangle}(v_\lambda\otimes w_\gamma)$$

3.2 The unrolled restricted quantum group of $\mathfrak{sl}(2)$

This section is primarily based on [23].

3.2.1 Definition of $\overline{U}_q^H(\mathfrak{sl}(2))$

Let $p\in\mathbb{Z}$ and $q=e^{\pi i/p}$, that is, q is a $2p$-th root of unity. Define the unrolled quantum group $U_q^H(\mathfrak{sl}(2))$ as the $\mathbb{C}$-algebra generated by E,F,H,K,K^{-1}, subject to the relations

$$KK^{-1}=1=K^{-1}K \qquad KEK^{-1}=q^2E \qquad KFK^{-1}=q^{-2}F \qquad HK^{\pm1}=K^{\pm1}H$$

$$[H,E]=2E \qquad [H,F]=-2F \qquad [E,F]=\frac{K-K^{-1}}{q-q^{-1}}$$

We have that $U_q^H(\mathfrak{sl}(2))$ has a Hopf algebra structure, where the coproduct, counit and antipode are given by

$$\begin{array}{lll}
\Delta(E) = 1\otimes E + E\otimes 1 & \epsilon(E)=0 & S(E) = -EK^{-1}\\
\Delta(H) = 1\otimes H + H\otimes 1 & \epsilon(H)=0 & S(H) = -H\\
\Delta(F) = K^{-1}\otimes F + F\otimes 1 & \epsilon(F)=0 & S(F) = -KF\\
\Delta(K) = K\otimes K & \epsilon(K)=1 & S(K) = K^{-1}\\
\Delta(K^{-1}) = K^{-1}\otimes K^{-1} & \epsilon(K^{-1})=1 & S(K^{-1}) = K
\end{array}$$

The unrolled restricted quantum group of $\mathfrak{sl}(2)$, $\overline{U}_q^H(\mathfrak{sl}(2))$, is then obtained by imposing the additional relations

$$E^p = 0 = F^p$$

Let V be a finite dimensional $\overline{U}_q^H(\mathfrak{sl}(2))$-module. A vector $v\in V$ is called a *weight vector* if $Hv=\lambda v$ for some $\lambda\in\mathbb{C}$, where λ is called the weight of v; a weight vector v is a *heighest weighty vector* if $Ev=0$. V is called a *weight module* if it is the direct sum of its H-eigenspaces and $K=q^H$ as an operator on V, that is, if v is a weight vector, then $Kv=q^\lambda v$.

3.2.2 Representation theory of $\overline{U}_q^H(\mathfrak{sl}(2))$

Simple and projective $\overline{U}_q^H(\mathfrak{sl}(2))$-modules

A classification of simple and projective $\overline{U}_q^H(\mathfrak{sl}(2))$-modules is given in [23].

For each $n\in\{0,1,...p-1\}$, let S_n be the usual $(n+1)$-dimensional simple highest weight $\overline{U}_q^H(\mathfrak{sl}(2))$-module, that is, the module with basis $\{s_0,s_1,...s_n\}$ and actions

$$Fs_i = s_{i+1} \qquad Es_i = [i][n-(i-1)]s_{i-1} \qquad Hs_i = (n-2i)s_i \qquad Es_0 = 0 = Fs_n$$

where $[m] = \frac{q^m-q^{-m}}{q-q^{-1}}$.

Now, for $k\in\mathbb{Z}$ define $\mathbb{C}^H_{kp}$ to be the one dimensional module where E,F act as 0 and H acts as multiplication by kp.

Finally, for $\alpha\in\mathbb{C}$ define the p-dimensional heighest weight module V_α of heighest weight vector v_0 with weight $\alpha+p-1$; where the action is given by

$$Fv_i = v_{i+1} \qquad Ev_i = [i][i-\alpha]v_{i-1} \qquad Hv_i = (\alpha+p-1-2i)v_i \qquad Ev_0 = 0 = Fv_{p-1}$$

Theorem 3. *(Lemma 5.3 in [23]) Given a simple $\overline{U}_q^H(\mathfrak{sl}(2))$-module, it is isomorphic to one of the following:*

1. $S_n \otimes \mathbb{C}^H_{kp}$, *with* $n \in \{0, 1, ...p-2\}$ *and* $k \in \mathbb{Z}$, *which has heighest weight* $n + kp$.

2. V_α *for some* $\alpha \in (\mathbb{C} - \mathbb{Z}) \cup p\mathbb{Z}$, *which has highest weight* $\alpha + p - 1$.

A weight vector v is called *dominant* if $(FE)^2 v = 0$. In [23], the authors prove the following theorem.

Theorem 4. *(Proposition 6.1 in [23]) Let* $v \in V$ *be a dominant vector of weight* $i \in \{0, 1, ...p-2\}$. *Consider the following* $2p$ *vectors of* V, *defined by*

$$w^H_i = v \qquad w^R_{i+2} = Ew^H_i \qquad w^S_i = Fw^H_{i+2} \qquad w^L_{-i-2} = F^{i+1}w^H_i$$

$$w^H_{i-2k} = F^k w^H_i \qquad \textit{and} \qquad w^S_{i-2k} = F^k w^S_i \qquad \textit{for } k \in \{0, 1, ...i\}$$

$$w^R_{i+2+2k} = E^k w^R_{i+2} \qquad \textit{and} \qquad w^L_{-i-2-2k} = F^k w^L_{-i-2} \qquad \textit{for } k \in \{0, 1, ...i\}$$

Then the vector space they generate is a submodule of V *and the following relations holds in* V *(whenever the involved vectors are defined):*

$$Hw^X_k = kw^X_k \qquad \textit{and} \qquad Kw^X_k = q^k w^X_k \quad \textit{for} \quad X \in \{L, R, H, S\}$$

$$Ew^R_k = w^R_{k+2} \qquad Fw^X_k = w^X_{k-2} \quad \textit{for} \quad X \in \{L, H, S\}$$

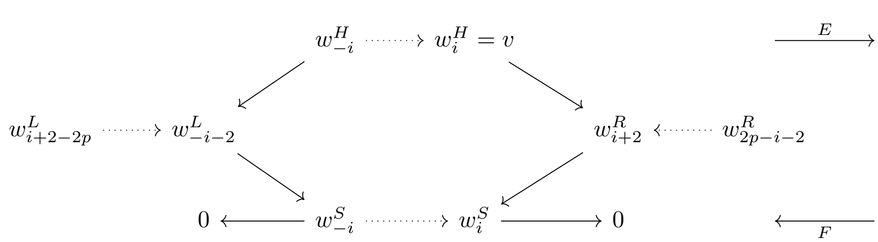
$$Fw^H_{-i} = w^L_{-i-2} \quad Ew^L_{-i-2} = w^S_{-i} \quad Ew^R_{2p-i-2} = Ew^S_i = Fw^S_{-i} = Fw^L_{i+2-2p} = 0$$

$$Ew^H_{i-2k} = \gamma_{i,k} w^H_{i-2k+2} + w^S_{i-2k+2} \qquad Ew^S_{i-2k} = \gamma_{i,k} w^S_{i-2k+2}$$

$$Fw^R_{2k+i+2} = -\gamma_{i,k} w^R_{2k+i} \qquad Ew^L_{-2k-i-2} = -\gamma_{i,k} w^L_{-2k-i}$$

where $\gamma_{i,k} = [k][n-k+1]$ *([*m*] as previously defined).*

Graphicaly, one would get something like



Where the marked arrows indicate when the action is given by the previous formulas and is different from the usual actions $Ew_j^X = w_{j+2}^X$ and $Fw_j^X = w_{j-2}^X$.

It is straightforward to check that there are $2p$ vectors of the form w_j^X in the previous construction.

Now, let $i \in \{0, 1, ...p-2\}$. Denote the vectors of the canonical base of $\mathbb{C}^{2p}$ by $(w^L_{i+2-2p}, w^L_{i-2p}, ...w^L_{-i-2}, w^H_{-i}, w^H_{-i+2}, ...w^H_i, w^S_{-i}, w^S_{-i+2}, ...w^S_i, w^R_{i+2}, w^R_{i+4}, ...w^R_{2p-i-2})$. Then the formulas of the previous theorem endows $\mathbb{C}^{2p}$ with a structure of weight module, which is denoted by P_i. We let $P_{p-1} = S_{p-1} = V_0$.

We have the following theorem, which characterizes the indecomposable projective weight $\overline{U}_q^H(\mathfrak{sl}(2))$-modules.

Theorem 5. *(Proposition 6.2 in [23]) Any projective indecomposable weight module P with heighest weight $(k+1)p - i - 2$ is isomorphic to $P_i \otimes \mathbb{C}^H_{kp}$.*

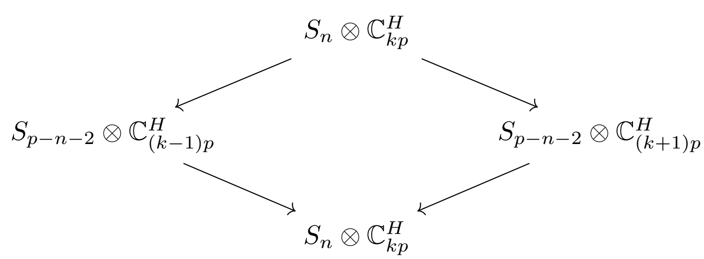
Lemma 1. *([24]) For any $k \in \{1, ...p-1\}$ there are short exact sequences of modules*

$$0 \to S_{p-1-k} \otimes \mathbb{C}^H_{lp} \to V_{k+lp} \to S_{k-1} \otimes \mathbb{C}^H_{(l+1)p} \to 0$$

$$0 \to V_{p-1-i+lp} \to P_i \otimes \mathbb{C}^H_{lp} \to V_{1+i-p+lp} \to 0$$

Example 5. *The Loewy diagram of $P_n \otimes \mathbb{C}^H_{kp}$ (the projective cover of $S_n \otimes \mathbb{C}^H_{kp}$) is given in [24] by*

$P_n \otimes \mathbb{C}^H_{kp}$

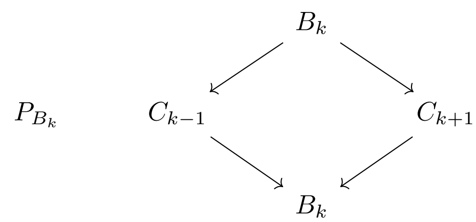


Let $S = S_n \otimes \mathbb{C}^H_{lp}$ be simple. Let us find a projective resolution for S. First, let

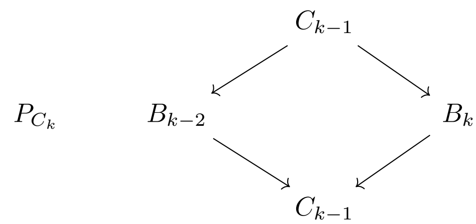
$$B_k = S_n \otimes \mathbb{C}^H_{kp}$$

$$C_k = S_{p-n-2} \otimes \mathbb{C}^H_{kp}$$

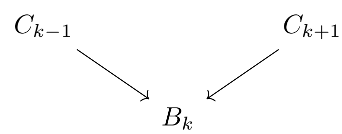
and P_{B_k}, P_{C_k} be their respective projective covers. These projective covers have the form



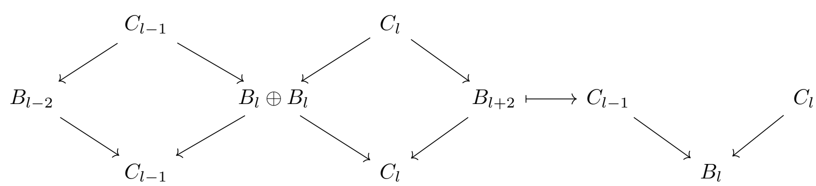
and



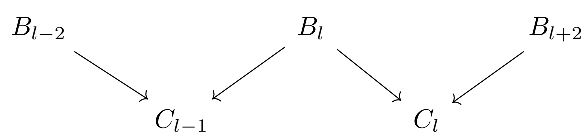
The kernel of the map $P_{B_l} \to B_l$ *is given by the diagram*



which has projective cover $P_{C_{k-1}} \oplus P_{C_{k+1}}$. *Now, The kernel of the map*



has the form



this has projective cover $P_{B_{l-2}} \oplus P_{B_l} \oplus P_{B_{l+2}}$.

Inductively, we have that

$$... \to P_{C_{l-3}} \oplus P_{C_{l-1}} \oplus P_{C_{l+1}} \oplus P_{C_{l+3}} \to P_{B_{l-2}} \oplus P_{B_l} \oplus P_{B_{l+2}} \to P_{C_{l-1}} \oplus P_{C_{l+1}} \to P_{B_l} \to B_l$$

is a projective resolution of $S = B_l$.

3.2.3 Twist θ

Let V be a $\overline{U}_q^H(\mathfrak{sl}(2))$-module. We want to define the twist $\theta_V : V \to V$; to do so, let us first define the operator $\tilde{\theta}$ as

$$\tilde{\theta} = K^{p-1} \sum_{n=0}^{p-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} S(F^n) q^{-H^2/2} E^n$$

where $\{j\} = q^j - q^{-j}$, $\{n\}! = \{n\}\{n-1\}...\{1\}$ and S is the antipode of $\overline{U}_q^H(\mathfrak{sl}(2))$.

Define the twist θ_V as $\theta_V(v) = \tilde{\theta}^{-1} v$.

Now, let us explicitly calculate the twist on simple modules.

Let $S \in \overline{U}_q^H(\mathfrak{sl}(2))$-Mod be a simple module, then by Schur's lemma the twist must act as scalar multiplication, that is, $\theta_S S = \theta^S S$, for some $\theta^S \in \mathbb{C}$. If v_λ is a highest weight vector (Ev_λ=0), then

$$\tilde{\theta} v_\lambda = K^{p-1} \sum_{n=0}^{p-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} S(F^n) q^{-H^2/2} E^n v_\lambda = K^{p-1} q^{-H^2/2} v_\lambda$$

As $K = q^H$ as operator and $Hv_\lambda = \lambda v_\lambda$, then

$$\tilde{\theta} v_\lambda = q^{\lambda(p-1)-\lambda^2/2} v_\lambda$$

Hence,

$$\theta v_\lambda = q^{\lambda^2/2-\lambda(p-1)} v_\lambda$$

The twist θ acts as multiplication by $q^{\lambda^2/2-\lambda(p-1)}$ on simple modules with weight λ.

3.2.4 Tensor products

Let us consider the tensor products $S \otimes \mathbb{C}^H_{kp}$ for all simple and projective modules S.

If $S \cong S_i \otimes \mathbb{C}_{k'p}$, then

$$S \otimes \mathbb{C}^H_{kp} \cong S_i \otimes \mathbb{C}_{(k+k')p}$$

If $S \cong P_i \otimes \mathbb{C}_{k'p}$, then

$$S \otimes \mathbb{C}^H_{kp} \cong P_i \otimes \mathbb{C}_{(k+k')p}$$

If $S \cong V_\alpha$

Let $w_\sigma \in \mathbb{C}^H_{kp}$ and $v_\alpha \in V_\alpha$ be highest weight vectors with weights kp and $\alpha + p - 1$, respectively. We have that

$$E(v_\alpha \otimes w_\sigma) = \Delta(E)(v_\alpha \otimes w_\sigma) = (1 \otimes E + E \otimes 1)(v_\alpha \otimes w_\sigma) =$$
$$v_\alpha \otimes Ew_\sigma + Ev_\alpha \otimes w_\sigma = 0$$

$$H(v_\alpha \otimes w_\sigma) = \Delta(H)(v_\alpha \otimes w_\sigma) = (1 \otimes H + H \otimes 1)(v_\alpha \otimes w_\sigma) =$$
$$v_\alpha \otimes Hw_\sigma + Hv_\alpha \otimes w_\sigma = v_\alpha \otimes kpw_\sigma + (\alpha + p - 1)v_\alpha \otimes w_\sigma =$$
$$(\alpha + p + kp - 1)w_\sigma \otimes v_\alpha$$

Hence, $v_\alpha \otimes w_\sigma$ is a highest weight vector of weight $\alpha + p + kp - 1$.

Thus, if $\alpha \in \mathbb{C} - \mathbb{Z}$, then $V_\alpha \otimes \mathbb{C}^H_{kp} = V_{\alpha+kp}$ since $V_\alpha \otimes \mathbb{C}^H_{kp}$ is also p-dimensional.

4 Vertex Operator Algebras

Vertex operator algebras were first introduced by Borcherds in an attempt to formalize conformal field theories. These provide a mathematical aproach to string theory and 2d-CFT by a formulation for chiral algebras. VOAs have shown importance in connecting seemingly unrelated areas of mathematics and has also applications in Lie theory, algebraic geometry and topology [24].

As mentioned earlier, there are interactions between VOAs and quantum groups; in this section we will present the definition of a VOA, some fundamental examples and a sample of such interactions that are used to consider the motivating examples for this work.

Let V be a complex vector space, and $End(V)$ the collection of linear operators $f : V \to V$. The formal power series

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$$

with coefficients $A_n \in End(V)$ is called a field if for all $v \in V$, $A(z)v$ is a Laurent series, that is,

$$A(z)v = \sum_{n\in\mathbb{Z}} A_n v z^{-n-1} \in V((z))$$

***Definition* 17.** *Two fields $A(z), B(w)$ are said to be local to each other if there exists an $N \in \mathbb{Z}_+$ such that*

$$(z - w)^N A(z)B(w) = (z - w)^N B(w)A(z)$$

Following [25], a *Vertex Algebra* (VA) is a $\mathbb{Z}_+$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n$ where $dim(V_n) < \infty$ equipped with the following data:

1. A *vacuum* vector $|0\rangle \in V$.
2. A *translation linear operator* $T : V \to V$.
3. A linear operator (*vertex operators*)

   $$Y(.,z) : V \to End(V)[[z^{\pm}]]$$

   given by

   $$Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$$

   acting on V. Where for each $B \in V$ there is n_0 such that $A_{(n)}B = 0$ for any $n \geq n_0$ (this is equivalent to the locality axiom below). The element $\sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$ is called *field*.

These data are subject to the following axioms:

- (*Vacuum axiom*) $Y(|0\rangle, z) = Id$. Furthermore, for any $A \in V$, we have

$$Y(A, z)|0\rangle \in V[[z]]$$

- (*Translation axiom*) For any $A \in V$

$$[T, Y(A, z)] = \partial_z Y(A, z)$$

  and $T|0\rangle = 0$.

- (*Locality axiom*) For any $u, v \in V$, there exists a positive integer N such that

$$(z-x)^N Y(u, z)Y(v, x) = (z-x)^N Y(v, z)Y(u, x)$$

  that is, $Y(u, z)$ and $Y(v, x)$ are local to each other.

A *vertex operator algebra (VOA)* is V endowed with a vertex algebra structure $(V, |0\rangle, T, Y)$ together with a vector w (conformal vector) such that

$$Y(w, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \left(= \sum_{n\in\mathbb{Z}} w_n z^{-n-1} \right)$$

and satisfy the following axioms:

VOA1 The Virasoro modes L_n form the Virasoro Lie algebra

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{(m+n)0}c_V Id_V$$

Where c_V is called *the central charge*, or *rank* of V.

VOA2 $L_0 v = wt(v)v$

VOA3 $T = L_{-1}$

Proposition 5. *(Theorem 5.5 in [26]) Suppose U and V are locally finite abelian categories, one of U and V is semisimple, and the other is closed under submodules and quotients. Then $\mathcal{C}$ is braided tensor equivalent to the Deligne product category $U \boxtimes V$.*

4.1 The Heisenberg VOA

The Heisenberg Lie algebra $\mathfrak{h}$ has vector space basis $\mathbf{c}, b_n$ for $n \in \mathbb{Z}$ and relations

$$[b_n, b_m] = \delta_{n+m,0} n\mathbf{c} \qquad\qquad [\mathbf{c}, b_n] = 0 = [\mathbf{c}, \mathbf{c}]$$

where $\delta_{i,j}$ is the Kronecker delta function.

Consider the infinite-dimensional representation of $\mathfrak{h}$ called the Fock space $\mathscr{F}$. As a vector space, $\mathscr{F}$ is the polynomial ring in the variables $b_{-1}, b_{-2}, ...$ which we denote by

$$\mathscr{F} = \mathbb{C}[b_{-1}, b_{-2}, ...]$$

The action of the Heisenberg algebra on $\mathscr{F}$ is given by

$$\mathbf{c} = Id_{\mathscr{F}} \qquad b_0 = 0$$

If $n < 0$, we let b_n act as multiplication as a monomial on $\mathscr{F}$ and if $n > 0$ we let b_n act as the operator

$$n\frac{\partial}{\partial b_{-n}}$$

Proposition 6. *The representation $\mathscr{F}$ is an irreducible $\mathfrak{h}$-module.*

Now, consider the following variant of the above representation of $\mathfrak{h}$. Take $\lambda \in \mathbb{C}^*$ and let the representation $\mathscr{F}_\lambda$ be the representation with the same underlying vector space as $\mathscr{F}$ and actions given by: $\mathbf{c} = aId_{\mathscr{F}}$, b_0 acts as multiplication by λ, b_n as multiplication as a monomial for $n < 0$ and b_n as

$$\lambda n\frac{\partial}{\partial b_{-n}}$$

for $n > 0$.

***Definition* 18.** *A graded $\mathfrak{h}$-module M is by definition equipped with a grading*

$$M = \bigoplus_{d\in\mathbb{Z}} M(d)$$

such that $M(d) = 0$ for d sufficiently small, and such that b_n is an operator of degree $-n(b_n : M(d) \to M(d-n))$.

Proposition 7. *Each $\mathscr{F}_\lambda$ is simple. Furthermore, any irreducible graded module of $\mathfrak{h}$ whose grading is bounded below is isomorphic to $\mathscr{F}_\lambda$ for some $\lambda \in \mathbb{C}^*$.*

***Definition* 19.** *The* Heisenberg Vertex Operator Algebra*, H= $(\mathscr{F}, 1, T, Y, w)$, has underlying vector space the Fock space $\mathscr{F}$ of the Heisenberg Lie algebra and has the following data:*

- *A $\mathbb{Z}$-graduation $deg(b_{-j_1}...b_{-j_k}) = \sum_{i=1}^{k} j_i$*
- $|0\rangle = 1$
- *$T(1) = 0$ and $[T, b_{-k}] = kb_{-k-1}$, where $T : \mathscr{F} \to \mathscr{F}$ is a derivation.*

- *$Y(-,z)$ defined by*

$$Y(b_{-j_1}...b_{-j_k}|0\rangle, z) = \frac{:\partial_z^{j_1-1}b(z)...\partial_z^{j_k-1}b(z):}{(j_1-1)!...(j_k-1)!}$$

  where $b(z) = Y(b_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$ and $:X(z)Y(z):$ denotes the normally ordered product.

- *a conformal vector $w = b_{-1}^2$ of central charge 1.*

Consider the expression

$$L_{-1} = \frac{1}{2}\sum_{m\in\mathbb{Z}} b_m b_{-m-1}$$

Lemma 2. *As operators on $\mathscr{F}$ and $\mathscr{F}_\lambda$ we have $T = L_{-1}$, that is,*

$$T = \sum_{n>0} n b_{-n-1} \frac{\partial}{\partial b_{-n}}$$

And the remaining Virasoro modes are

$$L_k = \frac{1}{2}\sum_{m\in\mathbb{Z}} b_m b_{k-m}$$

Lemma 3. $[L_m, b_n] = -n b_{n+m}$

Proof. A proof can be found in [27].

□

Now, let $\mathcal{H}^\oplus$ denote the category of $\mathbb{C}$-graded vector spaces of finite or countable dimension and $V = \bigoplus_{v\in\mathbb{C}} V_v \in \mathcal{H}^\oplus$. Define the degree map $H_V : V \to V$ by $H_V|_{V_v} = vId_{V_v}$.

$\mathcal{H}^\oplus$ can be endowed with a (non-unique) ribbon structure by defining the braiding c and the twist θ by

$$c_{U,V} = \tau_{U,V} \circ e^{\pi i H_U \otimes H_V}$$

$$\theta_V = e^{\pi i H_V^2} Id_V$$

where $\tau_{U,V}$ is the usual flip map. The full subcategory $\mathcal{H}^\oplus_{\mathrm{i}\mathbb{R}}$ of purely imaginary indexes is isomorphic to the category H-Mod and whose simple objects are the one dimensional spaces $\mathbb{C}_{ix}$, with $x \in \mathbb{R}$.

In $\mathcal{H}^\oplus_{\mathrm{i}\mathbb{R}}$, we have the properties

$$\mathbb{C}_{ix}^* = \mathbb{C}_{-ix}$$

$$\mathbb{C}_{ix} \otimes \mathbb{C}_{iy} = \mathbb{C}_{i(x+y)}$$

4.2 The Virasoro VOA

The Virasoro algebra has generators $\{L_n : n \in \mathbb{Z}\} \cup \{C\}$ with non-trivial relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C$$

It has a grading given by

$$\mathcal{L}_0 = \mathbb{C}L_0 \oplus \mathbb{C}C \qquad \mathcal{L}_n = L_{-n}$$

such that

$$\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$$

The Verma module of highest weight h and central charge c is

$$V(h,c) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{\leq 0})} \mathbb{C}|h,c\rangle$$

where $\mathbb{C}|h,c\rangle$ is the module with actions

$$C|h,c\rangle = c|h,c\rangle \qquad L_0|h,c\rangle = h|h,c\rangle \qquad L_n|h,c\rangle = 0 \text{ for } n \geq 1$$

The VOA structure for the Virasoro algebra is given by

$$T = L_{-1} \qquad w(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$
$$\omega = L_{-2}|h,c\rangle \qquad |0\rangle = |h,c\rangle$$

4.3 The singlet VOA $\mathcal{M}(p)$

The following is primarily based on [28].

$\mathcal{M}(p)$

Let $\mathcal{V}ir$ be the Virasoro algebra and consider $\mathcal{V}ir$ modules of central charge

$$c_{p,1} = 13 - 6p - 6p^{-1}$$

with $p \in \mathbb{Z}_{\geq 1}$. The Verma module $\mathcal{V}_h$ for $h \in \mathbb{C}$ is the module

$$\mathcal{V}_h = U(\mathcal{V}ir) \otimes_{U(\mathcal{V}ir_{\geq 0})} \mathbb{C}v_h$$

where $\mathbb{C}v_h$ is the one-dimensional $\mathcal{V}ir_{\geq 0}$-module on which $\mathbf{c}$ acts by the central charge $c_{p,1}$, L_0 acts by h and L_n by 0 for $n \geq 0$. The module $\mathcal{V}_h$ is reducible if and only if $h = h_{r,s}$, where

$$h_{r,s} = \frac{(pr-s)^2 - (p-1)^2}{4p}$$

for some $r, s \in \mathbb{Z}_+$. We let $\mathcal{V}_{r,s} := \mathcal{V}_{h_{r,s}}$.

Proposition 8. *The unique irreducible quotient of $\mathcal{V}_{r,s}$ is*

$$\mathcal{L}_{r,s} = \begin{cases} \mathcal{V}_{r,s}/\mathcal{V}_{r+1,p-s} & \text{if } 1 \leq s \leq p-1 \\ \mathcal{V}_{r,s}/\mathcal{V}_{r+2,p} & \text{if } s = p \end{cases}$$

As a $\mathcal{V}ir$-module, we have

$$\mathcal{M}(p) \cong \bigoplus_{n=0}^{\infty} \mathcal{L}_{2n+1,1}$$

and as a vertex algebra, $\mathcal{M}(p)$ is generated by

$$\omega = \frac{1}{2} h(-1)^2 \mathbf{1} + \frac{p-1}{\sqrt{2p}} h(-2)\mathbf{1}$$

together with a Virasoro singular vector H of conformal weight $h_{3,1} = 2p-1$. That is, H generates the Vir- submodule $\mathcal{L}_{3,1} \subset \mathcal{M}(p)$.

Alternatively, $\mathcal{M}(p)$ can be defined as the kernel of the screening operator [29] [30]

$$ker\left(\oint Q_-(z)dz : \mathcal{F}_0 \to \mathcal{F}_{\alpha_+}\right)$$

where an explicit formula for Q_- can be found on [31], $\alpha_+ = \sqrt{2p}$ and the $\mathcal{F}$ are Fock modules.

We have a one-to-one correspondence between irreducible $\mathcal{M}(p) - modules$ and Heisenberg Fock modules, so that the following are irreducible $\mathcal{M}(p) - modules$:

- $\mathcal{F}_\lambda$ for $\lambda \in \mathbb{C} - L^\circ$, where L° is the lattice $\mathbb{Z}\frac{\alpha_-}{2}$ and $\alpha_- = -\sqrt{\frac{2}{p}}$.
- $\mathcal{M}_{r,s} := Soc(\mathcal{F}_{\alpha_{r,s}})$ for $r, s \in \mathbb{Z}$ and $1 \leq s \leq p$, where $\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-$.

and these are such that

- $\mathcal{M}_{r,p} = \mathcal{F}_{\alpha_{r,p}}$
- For $1 \leq s \leq p-1$, there is a non-split short exact sequence

$$0 \to \mathcal{M}_{r,s} \to \mathcal{F}_{\alpha_{r,s}} \to \mathcal{M}_{r+1,p-s} \to 0$$

$\mathcal{C}_{\mathcal{M}(p)}$ **and** $\mathcal{O}_{\mathcal{M}(p)}$

***Definition* 20.** *(Generalized V-module)*

Let V be a vertex operator algebra. A generalized V-module $W = \bigoplus_{h\in\mathbb{C}} W_{[h]}$ *is a graded vector space such that each* $W_{[h]}$ *is the generalized* L_0*-eigenspace with generalized eigenvalue* h*.*

A generalized V-module is grading resticted *if each* $W_{[h]}$ *is finite dimensional and for any* $h \in \mathbb{C}$*,*$W_{[h-n]} = 0$ *for all* $n >> 0$*,* $n \in \mathbb{Z}$*.*

Consider the quotient space

$$\hat{V} = V \otimes \mathbb{C}[t, t^{-1}]/D(V \otimes \mathbb{C}[t, t^{-1}])$$

with D being the operator $L_{-1} \otimes 1 + 1 \otimes \frac{d}{dt}$*. We have that* $\hat{V}$ *is* $\mathbb{Z}$*-graded by defining the degree of* $v \otimes t^m$ *to be* $wt(v) - m - 1$ *for homogeneous* v*; and denote the homogeneous space of degree* m *to be* $\hat{V}(m)$*. We have a Lie algebra structure on* $\hat{V}$ *with bracket*

$$[u(m), v(n)] = \sum_{i\in\mathbb{Z}_{\geq 0}} \binom{m}{i} u_i v(m+n-i)$$

From here we have that $\hat{V}(0)$ *is a Lie subalgebra. Consider a* $\hat{V}(0)$*-module* U*; it can be viewed as a* $\bigoplus_{i\geq 0} \hat{V}(i)$*-module* W *by letting* $\hat{V}(i)$ *act trivially. The module*

$$F(U) = (\mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(W)} U)/J(U)$$

where $J(U)$ *is the intersection of all* $ker(\alpha)$ *for alpha running over all* $\hat{V}$*-homomorphisms from* $\mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(W)} U$ *to* V*-modules.*

Then $F(U)$ *is a* V*-module called* generalized Verma module.

Denote the category of finite-length generalized $\mathcal{M}(p)$-modules by $\mathcal{C}_{\mathcal{M}(p)}$ and the category of C_1-cofinite grading-restricted generalized $\mathcal{M}(p)$-modules by $\mathcal{O}_{\mathcal{M}(p)}$. Let us see that these categories are equal and admit the vertex algebraic braided tensor category structure of Huang-Lepowsky-Zhang.

Conjecture 3.3.1 in [32] states that for a VOA V such that all irreducible ordinary V-modules are C_1-cofinite, then W is a lower bounded C_1-cofinite modules if and only if W is a finite length generalized module.

Now, we state theorems 3.3.5 and 3.3.4 in [32]

Theorem 6. *Let V be a vertex operator algebra such that all the irreducible ordinary (*$W_{[n]}$ *is an ordinary eigenspace for all n) V-modules are* C_1*-cofinite. If all the grading-restricted generalized Verma modules for V are of finite* *length, then Conjecture 3.3.1 holds.*

Theorem 7. *Suppose conjecture 3.3.1 in [32] holds, then $\mathcal{C}_{\mathcal{M}(p)}$ has a braided tensor category structure of the type HLZ.*

From theorem 13 in [33], we have that all irreducible $\mathcal{M}(p)$-modules are C_1-cofinite. Now, by theorem 3.6 in [28], for $r \in \mathbb{Z}$ and $1 \leq s \leq p$, the generalized Verma $\mathcal{M}(p)$-module $\mathcal{G}_{r,s}$ is a finite-length $\mathcal{M}(p)$-module in $\mathcal{C}_{\mathcal{M}(p)}$. We then conclude from theorem 6 that conjecture 3.3.1 holds and then $\mathcal{C}_{\mathcal{M}(p)}$ has a braided tensor category structure of the type HLZ by theorem 7.
We also conclude that $\mathcal{C}_{\mathcal{M}(p)}$ and $\mathcal{O}_{\mathcal{M}(p)}$ are equal.

Let $K = \{\alpha_{2n+1,1} | n \in \mathbb{Z}\}$ and define

$$T = \mathbb{C}/2K^\circ$$

for $t = \beta + 2K^\circ \in T$ define $\mathcal{O}^t_{\mathcal{M}(p)}$ to be the full subcategory of $\mathcal{O}_{\mathcal{M}(p)}$ consisting of $\mathcal{M}(p)$-modules M such that the monodromy satisfies

$$\mu^2_{\mathcal{M}_{2,1},M} = e^{-2\pi i \alpha_{2,1}\beta}$$

and define $\mathcal{O}^T_{\mathcal{M}(p)}$ to be the direct sum

$$\bigoplus_{t\in T} \mathcal{O}^t_{\mathcal{M}(p)}$$

By theorem 3.13 in [28] we have that $\mathcal{O}^T_{\mathcal{M}(p)}$ is a full subcategory of $\mathcal{O}_{\mathcal{M}(p)}$, it is abelian and a tensor subcategory of $\mathcal{O}_{\mathcal{M}(p)}$.

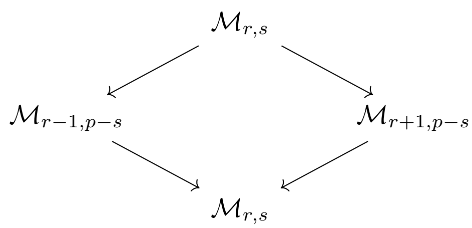
Projective $\mathcal{M}(p)$-modules

Theorems 3.19 and 3.18 in [28] tell us what the projective covers of indecomposable modules are:

Theorem 8. *For $\lambda \in \mathbb{C} - L^\circ \cup \{\alpha_{r,p} | r \in \mathbb{Z}\}$, the irreducible $\mathcal{M}(p)$-module $\mathcal{F}_\lambda$ is its own projective cover. In particular, it is projective.*

Now, define the $\mathcal{M}(p)$-module $\mathcal{P}_{r,s}$ by:

- $\mathcal{P}_{r,p} := \mathcal{M}_{r,p} (= \mathcal{F}_{\alpha_{r,p}})$ for all $r \in \mathbb{Z}$.
- For $1 \leq s \leq p-1$, $\mathcal{P}_{r,s}$ has length 4, is indecomposable and has Loewy diagram:



By theorem 5.1.3 in [30], the $\mathcal{P}_{r,s}$ are projective; furthermore, they are a projective cover for $\mathcal{M}_{r,s}$ (by theorem 1.2 in [28]).

By theorem 1.2 in [28], these are all the projective indecomposable modules in $\mathcal{O}^T_{\mathcal{M}(p)}$.

Theorem 9. *For $1 \leq s \leq p-1$, the indecomposable $\mathcal{M}(p)$-module $\mathcal{P}_{r,s}$ is a projective cover of $\mathcal{M}_{r,s}$ in $\mathcal{O}^T_{\mathcal{M}(p)}$.*

Fusion rules

By theorem 5.2.1 in [30] and section 4 of [28], we have the fusion rules for the previous indecomposable and/or projective modules given by:

- For $r, r' \in \mathbb{Z}$ and $1 \leq s, s' \leq p$

$$\mathcal{M}_{r,s} \boxtimes \mathcal{M}_{r',s'} = \left(\bigoplus_{\substack{l=|s-s'|+1 \\ l+s+s' \equiv 1 \bmod 2}}^{min\{s+s'-1, 2p-1-s-s'\}} \mathcal{M}_{r+r'-1,l} \right) \oplus \left(\bigoplus_{\substack{l=2p+1-s-s' \\ l+s+s' \equiv 1 \bmod 2}}^{p} \mathcal{P}_{r+r'-1,l} \right)$$

- For $r, r' \in \mathbb{Z}, 1 \leq s \leq p-1$ and $1 \leq s' \leq p$

$$\mathcal{P}_{r,s} \boxtimes \mathcal{M}_{r',s'} = \left(\bigoplus_{\substack{l=|s-s'|+1 \\ l+s+s' \equiv 1 \bmod 2}}^{min\{s+s'-1, p\}} \mathcal{P}_{r+r'-1,l} \right) \oplus \left(\bigoplus_{\substack{l=2p+1-s-s' \\ l+s+s' \equiv 1 \bmod 2}}^{p} \mathcal{P}_{r+r'-1,l} \right)$$
$$\oplus \bigoplus_{\substack{l=p+s-s'+1 \\ l+p+s+s' \equiv 1 \bmod 2}}^{p} \left(\mathcal{P}_{r+r',l} \oplus \mathcal{P}_{r+r'-2,l} \right)$$

- For $r, r' \in \mathbb{Z}$ and $1 \leq s, s' \leq p-1$

$$\mathcal{P}_{r,s}\boxtimes\mathcal{P}_{r',s'} = \left(2.\bigoplus_{\substack{l=|s-s'|+1\\ l+s+s'\equiv 1 \bmod 2}}^{min\{s+s'-1,p\}} \mathcal{P}_{r+r'-1,l}\right)\oplus\left(2.\bigoplus_{\substack{l=2p+1-s-s'\\ l+s+s'\equiv 1 \bmod 2}}^{p} \mathcal{P}_{r+r'-1,l}\right)$$

$$\oplus\, 2.\bigoplus_{\substack{l=p+s-s'+1\\ l+p+s+s'\equiv 1 \bmod 2}}^{p} (\mathcal{P}_{r+r',l}\oplus\mathcal{P}_{r+r'-2,l})$$

$$\oplus\bigoplus_{\substack{l=|s+s'-p|+1\\ l+p+s+s'\equiv 1 \bmod 2}}^{min\{s-s'+p-1,p\}} (\mathcal{P}_{r+r',l}\oplus\mathcal{P}_{r+r'-2,l})$$

$$\oplus\bigoplus_{\substack{l=p-s+s'+1\\ l+p+s+s'\equiv 1 \bmod 2}}^{p} (\mathcal{P}_{r+r',l}\oplus\mathcal{P}_{r+r'-2,l})$$

$$\oplus\bigoplus_{\substack{l=s+s'+1\\ l+s+s'\equiv 1 \bmod 2}}^{p} (\mathcal{P}_{r+r'+1,l}\oplus 2.\mathcal{P}_{r+r'-1,l}\oplus\mathcal{P}_{r+r'-3,l})$$

- For $r\in\mathbb{Z}, 1\leq s\leq p$ and $\lambda\in\mathbb{C}-L^\circ$

$$\mathcal{M}_{r,s}\boxtimes\mathcal{F}_\lambda\cong\bigoplus_{l=0}^{s-1}\mathcal{F}_{\lambda+\alpha_{r,s}+l\alpha_-}$$

- For $r\in\mathbb{Z}, 1\leq s\leq p-1$ and $\lambda\in\mathbb{C}-L^\circ$

$$\begin{aligned}\mathcal{P}_{r,s}\boxtimes\mathcal{F}_\lambda &\cong (\mathcal{M}_{r+1,p-s}\boxtimes\mathcal{F}_\lambda)\oplus 2.(\mathcal{M}_{r,s}\boxtimes\mathcal{F}_\lambda)\oplus(\mathcal{M}_{r-1,p-s}\boxtimes\mathcal{F}_\lambda)\\ &\cong\bigoplus_{l=0}^{p-1}(\mathcal{F}_{\lambda+\alpha_{r,s}+l\alpha_-}\oplus\mathcal{F}_{\lambda+\alpha_{r-1,p-s}+l\alpha_-})\end{aligned}$$

For the case $\mathcal{F}_\lambda\boxtimes\mathcal{F}_\mu$ for $\lambda,\mu\in\mathbb{C}-L^\circ$ we must consider two cases:$\lambda+\mu\in L^\circ$ and $\lambda+\nu\in\mathbb{C}-L^\circ$.

- For $\lambda,\mu\in\mathbb{C}-L^\circ$ such that $\lambda+\mu=\alpha_0+\alpha_{r,s}$ for some $1\leq s\leq p$

$$\mathcal{F}_\lambda\boxtimes\mathcal{F}_\mu\cong\bigoplus_{\substack{s'=s\\ s'\equiv s \bmod 2}}^{p}\mathcal{P}_{r,s'}\oplus\bigoplus_{\substack{s'=p+2-s\\ s'\equiv p-s \bmod 2}}\mathcal{P}_{r-1,s'}$$

- For $\lambda, \mu \in \mathbb{C} - L^\circ$ such that $\lambda + \mu \in \mathbb{C} - L^\circ$

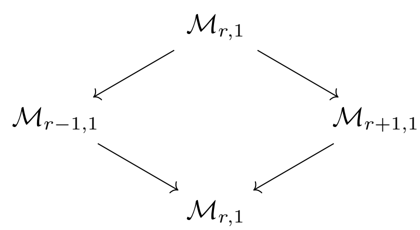
$$\mathcal{F}_\lambda \boxtimes \mathcal{F}_\mu \cong \bigoplus_{l=0}^{p-1} \mathcal{F}_{\lambda+\mu+l\alpha_-}$$

Computations for the case p=2

If $p = 2$, then $\alpha_+ = 2$ and $\alpha_- = -1$, which implies that $L^\circ = \mathbb{Z}(-\frac{1}{2})$ and $\alpha_{r,s} = \frac{1+s}{2} - r$.

Then, we have the following indecomposable $\mathcal{M}(2)$-modules

- $\mathcal{F}_\lambda$, where $\lambda \in \mathbb{C} - \mathbb{Z}(\frac{1}{2})$ which is its own projective cover.
- $\mathcal{M}_{r,1} = soc(\mathcal{F}_{1-r})$ for $r \in \mathbb{Z}$ with projective cover $\mathcal{P}_{r,1}$, which has Loewy diagram



- $\mathcal{M}_{r,2} := \mathcal{F}_{\frac{3}{2}-r}$ for $r \in \mathbb{Z}$, which is also its projective cover.
- $\mathcal{P}_{r,1}$
- $\mathcal{P}_{r,2}$

The tensor products among these are then given by

- $\mathcal{M}_{r,1} \boxtimes \mathcal{M}_{r',1} = \mathcal{M}_{r+r'-1,1}$
- $\mathcal{M}_{r,1} \boxtimes \mathcal{M}_{r',2} = \mathcal{M}_{r+r'-1,2}$
- $\mathcal{M}_{r,2} \boxtimes \mathcal{M}_{r',2} = \mathcal{P}_{r+r'-1,1}$
- $\mathcal{P}_{r,1} \boxtimes \mathcal{M}_{r',1} = \mathcal{P}_{r+r'-1,1}$
- $\mathcal{M}_{r,1} \boxtimes \mathcal{P}_{r',2} = \mathcal{P}_{r+r'-1,2}$
- $\mathcal{P}_{r,1} \boxtimes \mathcal{P}_{r',1} = \mathcal{P}_{r+r'-1,1} \oplus \mathcal{P}_{r+r'-1,1} \oplus \mathcal{P}_{r+r',1} \oplus \mathcal{P}_{r+r'-2,1}$
- $\mathcal{P}_{r,2} \boxtimes \mathcal{P}_{r',2} = \mathcal{P}_{r+r'-1,1}$
- $\mathcal{M}_{r,1} \boxtimes \mathcal{F}_\lambda = \mathcal{F}_{\lambda+1-r}$

- $\mathcal{M}_{r,2} \boxtimes \mathcal{F}_\lambda = \mathcal{F}_{\lambda+\frac{3}{2}-r} \oplus \mathcal{F}_{\lambda+\frac{1}{2}-r}$
- $\mathcal{P}_{r,1} \boxtimes \mathcal{F}_\lambda = \mathcal{F}_{\lambda+1-r} \oplus \mathcal{F}_{\lambda+2-r} \oplus \mathcal{F}_{\lambda-r} \oplus \mathcal{F}_{\lambda+1-r}$
- $\mathcal{P}_{r,2} \boxtimes \mathcal{F}_\lambda = \mathcal{F}_{\lambda+2-r} \oplus \mathcal{F}_{\lambda+1-r}$

Rigidity and Ribbon structure

Lemma 4. *The modules $\mathcal{M}_{2n+1,1}$ for all $n \in \mathbb{Z}$ and $\mathcal{M}_{2,1}$ are simple currents, i.e., they have inverses under the tensor product.*

***Definition* 21.** *We say that a module V with a right dual V^* is* rigid *if*

$$Id_V \otimes e_V \circ i_V \otimes Id_V = Id_V$$

and

$$e_V \otimes Id_{V^*} \circ Id_{V^*} \otimes i_V = Id_{V^*}$$

where e_V and i_V are the evaluation and unit morphisms.

***Definition* 22.** *A* ribbon *category is a rigid braided tensor category with functorial isomorphisms*

$$\delta_V : V \to V^{**}$$

such that

$$\delta_{V\otimes W} = \delta_V \otimes \delta_W$$
$$\delta_1 = Id$$
$$\delta_{V^*} = (\delta_V^*)^{-1}$$

for all objects V, W in the category.

In subsection 4.2 of [30], the authors show the rigidity of $\mathcal{M}_{1,2}$ for all $p \geq 2$. Additionally, by proposition 4.3.2, they show that $\mathcal{C}_{\mathcal{M}(p)}$ is closed under contragradients.

We have the following standard lemmas

Lemma 5. *If V_1 and V_2 are rigid objects in a tensor category, then $V_1 \boxtimes V_2$ is also rigid with dual $V_2^* \boxtimes V_1^*$.*

Lemma 6. *If $\mathcal{C}$ is a braided tensor category of modules for a self-contragredient vertex operator algebra V that is closed under contragredients, then any direct summand of a rigid module in $\mathcal{C}$ is rigid.*

Theorem 10. *Every simple module $\mathcal{M}_{r,s}$ in $\mathcal{C}_{\mathcal{M}(p)}$ is rigid.*

The proof uses the fact that $\mathcal{M}_{2n+1,1}$ for all $n \in \mathbb{Z}$ and $\mathcal{M}_{2,1}$ are simple currents (and therefore rigid), which implies that $\mathcal{M}_{2n,1} \cong \mathcal{M}_{2,1} \boxtimes \mathcal{M}_{2n-1,1}$ is rigid by lemma 5. Additionally, rigidity of $\mathcal{M}_{1,2}$ implies rigidity of $\mathcal{M}_{1,s}$ and the fusion rule $\mathcal{M}_{r,s} = \mathcal{M}_{r,1} \boxtimes \mathcal{M}_{1,s}$ helps conclude the proof.

By theorem 4.4.1 in [30], below, $\mathcal{C}_{\mathcal{M}(p)}$ is rigid.

Theorem 11. *Assume that V is a self-contragredient vertex operator algebra and $\mathcal{C}$ is a category of grading-restricted generalized V-modules such that:*

- *The category is closed under submodules, quotients and contragradients, and every module has finite length.*
- *The category has braided tensor category structure as introduced by Huang-Lepowski-Zhang [34].*
- *Every simple module is rigid.*

Then $\mathcal{C}$ is a rigid tensor category.

Theorem 12. *The tensor category $\mathcal{O}_{\mathcal{M}(p)}$ is rigid and ribbon, with duals given by contragredient modules and ribbon twist $\theta = e^{2\pi i L_0}$.*

Theorem 13. *The tensor category $\mathcal{O}^T_{\mathcal{M}(p)}$ is rigid and ribbon.*

Proposition 10 in [35] gives a bijection between representatives of equivalence classes of simple $\overline{U}^H_q(\mathfrak{sl}(2))$-modules and simple $\mathcal{M}(p)$-modules up to isomorphism, which is fundamental to the motivating examples.

Theorem 14. *For $\alpha \in (\mathbb{C} - \mathbb{Z}) \cup p\mathbb{Z}$, $i \in \{0, 1, ...p-2\}$ and $k \in \mathbb{Z}$, the map*

$$\varphi : S_i \otimes \mathbb{C}^H_{kp} \to M_{1-k,i+1}$$
$$\varphi : V_\alpha \to F_{\frac{\alpha+p-1}{\sqrt{2p}}}$$

between simple $\overline{U}^H_q(\mathfrak{sl}(2))$-modules and simple $\mathcal{M}(p)$-modules is a bijection of the sets of representatives of equivalence classes of simple modules up to isomorphism.

Based on this theorem, it is expected that $\mathcal{M}(p) \cong \overline{U}^H_q(\mathfrak{sl}(2))$ as monoidal, and perhaps braided, categories.

Example 6. *Let us see the correspondence in the case $p = 2$.*

If $p = 2$, then $q = e^{\frac{i\pi}{2}}$.

The simple $\overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$-modules are of one of the following forms:

- V_α, *with* $\alpha \in (\mathbb{C} - \mathbb{Z}) \cup 2\mathbb{Z}$ *and highest weight* $\alpha + 1$.
- $S_0 \otimes \mathbb{C}^H_{2k} \cong \mathbb{C}^H_{2k}$ *with* $k \in \mathbb{Z}$ *with highest weight* $2k$.

The simple $\mathcal{M}(2)$*-modules are of one of the following forms:*

- $F_{\alpha_{r,2}} = F_{3/2-r}$ *with* $r \in \mathbb{Z}$.
- $M_{r,1} = soc(F_{\alpha_{r,1}}) = soc(F_{1-r})$ *with* $r \in \mathbb{Z}$.

According to the previous theorem, we have the bijective correspondence:

$\overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2)) - Mod$	$\mathcal{M}(2) - Mod$
$\mathbb{C}^H_{2k}$	$M_{1-k,1} = soc(F_k)$
V_α	$F_{\frac{\alpha+1}{2}}$

5 The motivating examples

Before we proceed with the examples, we will take a look into the strategy we follow. This strategy can be split into the following steps:

1. Take a quantum group or quantum groups and consider the categories of modules over them. Now, consider the category $\mathcal{H}^{\oplus}_{i\mathbb{R}}$-Mod, which is the full subcategory of $\mathcal{H}^{\oplus}$ (the category of $\mathbb{C}$-graded complex vector spaces with finite or countable dimension) with purely imaginary indices.
2. Consider the Deligne product of the categories above. This product will be our base category $\mathcal{C}$. The tensor product on $\mathcal{C}$ is componentwise and the brading, twist and rigidity morphisms are given by the product of the corresponding morphisms.
3. Take an abelian group I of indices and a set of simple objects coloured by these indices such that the set $\{\mathbb{C}_\lambda \text{ invertible} | \lambda \in I, \mathbb{C}_\lambda \otimes \mathbb{C}_\mu = \mathbb{C}_{\lambda+\mu}\}$ is closed under tensor products and duals and consider the simple currents extension associated to it, call it A.
4. Require A to have the structure of commutative associative unital algebra in $\mathcal{C}$ so we can consider the category $Rep(A)$ and the induction functor $\mathcal{F} : \mathcal{C} \to Rep(A)$.
5. Find the full subcategory of $Rep(A)$, $Rep^0(A) = Rep^{loc}(A)$ of local representations, that is, those where the monodromy with A is the identity. This is a braided tensor category.
6. Take the quantum groups chosen above and take their tensor product along with $X = \mathbb{C}$ under the action $X\mathbb{C}_\sigma = \sigma\mathbb{C}_\sigma$ and call this product Z.
7. Define a subalgebra $\tilde{Z}$ wisely so that a Z-module is a $\tilde{Z}$-module iff it is local and that $\mathcal{F}(X) \cong \mathcal{F}(Y)$ implies that X, Y are shifted by a simple module, that is, $X = \mathbb{C}_\lambda \otimes Y$ for some $\lambda \in I$.

5.1 $\mathcal{H}^{\oplus}_{i\mathbb{R}} \boxtimes \overline{U}^H_q(\mathfrak{sl}(2))$-Mod

Steps 1-2

Let $\mathcal{C} = \mathcal{H}^{\oplus}_{i\mathbb{R}} \boxtimes \overline{U}^H_q(\mathfrak{sl}(2))$-Mod be the Deligne product. Notice that

$$\mathbb{C}_{ix} \boxtimes \mathbb{C}^H_p \otimes \mathbb{C}_{-ix} \boxtimes \mathbb{C}^H_{-p} = \mathbb{C}_0 \boxtimes \mathbb{C}^H_0 = \mathbf{1}_{\mathcal{C}}$$

Hence, $\mathbb{C}_{ix} \boxtimes \mathbb{C}^H_p$ is a simple current.

From now on, let λ_p be fixed such that $\lambda_p^2 = -\frac{p}{2}$. Additionally, let $\mathcal{C}_\oplus$ be as in [36], which is an extension of $\mathcal{C}$ that allows infinite direct sums.

Steps 3-4

Now, define the object $\mathcal{A}_p \in \mathcal{C}_\oplus$ as

$$\mathcal{A}_p = \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}_{i\lambda_p} \boxtimes \mathbb{C}_p^H)^{\otimes k} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}_{ik\lambda_p} \boxtimes \mathbb{C}_{kp}^H$$

The $\mathcal{B}_p - algebra$ is the H$\otimes\mathcal{M}(p)$-module defined by

$$\mathcal{B}_p \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{C}_{ikp} \boxtimes M_{1-k,1}$$

which is the image of $\mathcal{A}_p$ under the correspondance φ of theorem 14.

$$\mathcal{B}_p = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}_{ikp} \boxtimes \varphi(\mathbb{C}_{kp}^H)$$

The $\mathcal{B}_p$-algebras are interesting in physics because they are the chiral algebras of certain four-dimensional superconformal field theories called Argyres-Douglas theories of type (A_1, A_{2p-3})[28][37][38][39].

In [40], it is shown that $\mathcal{A}_p$ has a unique (up to isomorphism) structure of commutative algebra in $\mathcal{C}_\oplus$ with a non-degenerate invariant pairing. This fact is used along with the definition of the induction functor to characterize simple objects X in $\mathcal{C}$ such that $M_{\mathcal{A}_p,X} = Id_{\mathcal{A}_p\otimes X}$.

Step 5

We want to see when $M_{V,A_p} = Id_{V\otimes A_p}$ occurs. According to [35], this equation holds if and only if $M_{V,\mathbb{C}_{i\lambda_p}\otimes\mathbb{C}_p^H} = Id_{V\otimes\mathbb{C}_{i\lambda_p}\otimes\mathbb{C}_p^H}$ and we have the following:

If $V = \mathbb{C}_{i\gamma} \otimes V_\alpha$, then $M_{V,\mathbb{C}_{i\lambda_p}\otimes\mathbb{C}_p^H} = e^{\pi i(\alpha+p-1+2\lambda_p\gamma)}$.

If $V = S_j \otimes \mathbb{C}_{lp}^H$, then $M_{V,\mathbb{C}_{i\lambda_p}\otimes\mathbb{C}_p^H} = e^{\pi i(j+lp+2\lambda_p\gamma)}$.

Hence, the simple objects X in $\mathcal{C}$ such that $M_{\mathcal{A}_p,X} = Id_{\mathcal{A}_p\otimes X}$ (induce to $Rep^{loc}(A_p)$) are of one of the following forms:

1. $\mathbb{C}_{i\gamma} \boxtimes V_\alpha$ with $\alpha + p - 1 + 2\lambda_p\gamma$ an even integer.
2. $\mathbb{C}_{i\gamma} \boxtimes (S_j \otimes \mathbb{C}_{lp}^H)$ with $j + lp + 2\lambda_p\gamma$ an even integer.

Now, we want an operator $L = e^{\frac{\pi i}{p}(x_1H_1+x_2H_2)}$ such that $L(V) = M_{V,\mathbb{C}_{i\lambda_p}\otimes\mathbb{C}_p^H}(V \otimes \mathbb{C}_{i\lambda_p} \otimes \mathbb{C}_p^H)$ for all simple (hence all) modules.

Steps 6-7

If we let $x_1 = 2p\lambda_p$ and $x_2 = p$, we get such an operator.

Now, define the algebra $Z = X \boxtimes \overline{U}_q^H(\mathfrak{sl}(2))$, where $X = \mathbb{C}$ with action $X\mathbb{C}_\sigma = \sigma\mathbb{C}_\sigma$.

Define the quotient algebra $\widetilde{Z} = Z/\langle L - Id_Z\rangle$.

Now, we want to find a $\widetilde{Z}$-subalgebra in which all the Z-modules having isomorphic images under the induction functor are isomorphic; to do so, suppose that $\mathcal{F}(V) \cong \mathcal{F}(V')$.

If $V = \mathbb{C}_{i\gamma} \otimes V_\alpha$ and $V' = \mathbb{C}_{i\gamma'} \otimes V_{\alpha'}$, then one must have that $\alpha' = \alpha + mp$ and $\gamma' = \gamma + m\lambda_p$ for some $m \in \mathbb{Z}$, that is, $V = V' \otimes (\mathbb{C}_{m\lambda_p} \otimes \mathbb{C}_{mp})$.

Now, we then need some linear combination $x_1H_1 + x_2H_2$ that acts equally on V and V', that is:

$$(x_1H_1 + x_2H_2)V = (x_1H_1 + x_2H_2)V'$$
$$x_1\gamma + x_2(\alpha + p - 1) = x_1(\gamma + m\lambda_p) + x_2(\alpha + mp + p - 1)$$
$$0 = x_1\lambda_p + x_2p$$

Similarly, if $V = S_j \otimes \mathbb{C}^H_{lp}$ and $V' = S'_j \otimes \mathbb{C}^H_{l'p}$, then $j' = j$, $l' = l + m$ and $\gamma' = \gamma + m\lambda_p$ and as before, we get

$$(x_1H_1 + x_2H_2)V = (x_1H_1 + x_2H_2)V'$$
$$x_1\gamma + x_2(j + lp) = x_1(\gamma + m\lambda_p) + x_2(j + (l + m)p)$$
$$0 = x_1\lambda_p + x_2p$$

We conclude that if $x_1 = p$ and $x_2 = -\lambda_p$, the equation hold.

We know that $Z = \langle H_1, K_1, E_2, F_2, H_2, K_2\rangle$. Define the Z-subalgebra $Z_1 = \langle K_1, E_2, F_2, K_2, H\rangle$, where $H = pH_1 - \lambda_pH_2$.

Now, define the $\widetilde{Z}$-subalgebra $\hat{Z} = Z_1/\langle L - Id_Z\rangle$.

Conjecture 2. *There is an equivalence of categories*

$$\hat{Z} - mod \cong Rep^{loc}(A_p)$$

under the induction functor.

5.2 $\overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$-Mod $\boxtimes$ $\overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$-Mod $\boxtimes$ $\mathcal{H}^{\oplus}_{\mathbf{i}\mathbb{R}}$-Mod

Steps 1-2

Consider the Deligne product $\mathcal{C} = \overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$-Mod $\boxtimes$ $\overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$-Mod $\boxtimes$ $\mathcal{H}^{\oplus}_{\mathrm{i}\mathbb{R}}$-Mod; according to theorem 14, this corresponds to $\mathcal{M}(2)$-Mod $\boxtimes$ $\mathcal{M}(2)$-Mod $\boxtimes$ H-Mod.

Step 3

Denote the simple module $\mathbb{C}^H_{2a} \boxtimes \mathbb{C}^H_{2b} \boxtimes \mathbb{C}_{i\frac{a+b}{\sqrt{2}}}$ by $S_{a,b}$ and let $I = \{(a,b) \in \mathbb{Z}^2 | a+b \in 2\mathbb{Z}\}$.

We have that any $S_{a,b}$ is a simple current since

$$S_{a,b} \otimes S_{-a,-b} = \mathbb{C}^H_{2a} \boxtimes \mathbb{C}^H_{2b} \boxtimes \mathbb{C}_{i\frac{a+b}{\sqrt{2}}} \otimes \mathbb{C}^H_{-2a} \boxtimes \mathbb{C}^H_{-2b} \boxtimes \mathbb{C}_{i(-\frac{a+b}{\sqrt{2}})} = \mathbb{C}^H_0 \boxtimes \mathbb{C}^H_0 \boxtimes \mathbb{C}_0 = \mathbf{1}_{\mathcal{C}}$$

Step 4

Consider the algebra

$$A = \bigoplus_{a,b\in\mathbb{Z}, a+b\in 2\mathbb{Z}} \mathbb{C}^H_{2a} \boxtimes \mathbb{C}^H_{2b} \boxtimes \mathbb{C}_{i\frac{a+b}{\sqrt{2}}} = \bigoplus_{(a,b)\in I} S_{a,b}$$

According to [40], we have that the algebra A is commutative if and only if $\theta^2_{S_{a,b}} = Id_{S_{a,b}}$ for all $(a,b) \in I$.

Now, we have that

$$\theta_{S_{a,b}} = \theta_{\mathbb{C}^H_{2a}} \otimes \theta_{\mathbb{C}^H_{2b}} \otimes \theta_{\mathbb{C}_{i\frac{a+b}{\sqrt{2}}}} = Id_{S_{a,b}}$$

Since

- $\theta_{\mathbb{C}^H_{2a}}$ acts as multiplication by $e^{\frac{i\pi}{2}(\frac{(2a)^2}{2} - 2a)}$.
- $\theta_{\mathbb{C}^H_{2b}}$ acts as multiplication by $e^{\frac{i\pi}{2}(\frac{(2b)^2}{2} - 2b)}$.
- $\theta_{\mathbb{C}_{i\frac{a+b}{\sqrt{2}}}}$ acts as multiplication by $e^{-i\pi\frac{(a+b)^2}{2}}$.

As $a+b = 2n$ for some $n \in \mathbb{Z}$, then $a^2 + b^2 = 2(2n^2 - ab)$ and $\frac{(a+b)^2}{2} = 2n^2$. Thus,

$$\theta_{S_{a,b}} = e^{\frac{i\pi}{2}(\frac{(2a)^2+(2b)^2}{2} - 2(a+b))} e^{-i\pi\frac{(a+b)^2}{2}} Id_{S_{a,b}} = e^{\frac{i\pi}{2}4(2n^2-ab-n)} e^{-2\pi i n^2} Id_{S_{a,b}} = Id_{S_{a,b}}$$

Hence, A can be endowed with a structure of commutative associative unital algebra in $\mathcal{C}$ and we can consider the category $Rep(A)$ and the induction functor $\mathcal{F} : \mathcal{C} \to Rep(A)$ and the full subcategory $Rep^0(A) = Rep^{loc}(A)$.

Step 5

We have that a simple object $V \in \mathcal{C}$ induces to $Rep^{loc}(A)$ if and only if $M_{V,A} = Id_{V\otimes A}$, where $M_{V,A}$ is the monodromy.

Recall the balancing equation 2.2.8 in [41]

$$\theta_{X\otimes Y} = \theta_X \otimes \theta_Y \circ M_{X,Y}$$

Therefore

$$M_{X,Y} = (\theta_X \otimes \theta_Y)^{-1} \circ \theta_{X\otimes Y}$$

Recall that if $S \in \overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$-Mod is simple with weight w, then θ_S acts as multiplication by $e^{\frac{i\pi}{2}(\frac{w^2}{2}-w)}$ and if S is a simple H-module with weight w, then θ_S acts as multiplication by $e^{\pi i w^2}$.

Let V be a simple $\mathcal{C}$-module of the form $\mathbb{C}^H_{2k} \boxtimes \mathbb{C}^H_{2j} \boxtimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$. We have

$$V \otimes A = \bigoplus_{a,b\in I} \mathbb{C}^H_{2(a+k)} \boxtimes \mathbb{C}^H_{2(b+j)} \boxtimes \mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}$$

We have that $M_{V,A} = Id_{V\otimes A}$ if and only if $M_{V,S_{a,b}} = Id_{V\otimes S_{a,b}}$ for all $(a, b) \in I$. Let us calculate $M_{V,S_{a,b}}$.

- $\theta_{S_{a,b}} = Id_{S_{a,b}}$
- $\theta_V = e^{i\pi(k^2+j^2-k-j)} e^{-\pi i\frac{\lambda^2}{2}} Id_V$ since
  - $\theta_{\mathbb{C}^H_{2k}} = e^{\frac{i\pi}{2}(\frac{(2k)^2}{2}-2k)} Id_{\mathbb{C}^H_{2k}}$
  - $\theta_{\mathbb{C}^H_{2j}} = e^{\frac{i\pi}{2}(\frac{(2j)^2}{2}-2j)} Id_{\mathbb{C}^H_{2j}}$
  - $\theta_{\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}} = e^{-\pi i\frac{\lambda^2}{2}} Id_{\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}}$
- $\theta_{V\otimes S_{a,b}} = e^{i\pi((a+k)^2+(b+j)^2-a-k-b-j)} e^{-\pi i\frac{(a+b+\lambda)^2}{2}} Id_{V\otimes S_{a,b}}$ since
  - $\theta_{\mathbb{C}^H_{2(a+k)}} = e^{\frac{i\pi}{2}(2(a+k)^2-2(a+k))} Id_{\mathbb{C}^H_{2(a+k)}}$
  - $\theta_{\mathbb{C}^H_{2(b+j)}} = e^{\frac{i\pi}{2}(2(b+j)^2-2(b+j))} Id_{\mathbb{C}^H_{2(b+j)}}$
  - $\theta_{\mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}} = e^{-\pi i\frac{(a+b+\lambda)^2}{2}} Id_{\mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}}$

Hence,

$$M_{V,S_{a,b}} = (e^{i\pi(k^2+j^2-k-j)}e^{-\pi i\frac{\lambda^2}{2}}Id_{V\otimes S_{a,b}})^{-1}\circ$$
$$(e^{i\pi((a+k)^2+(b+j)^2-a-k-b-j)}e^{-\pi i\frac{(a+b+\lambda)^2}{2}}Id_{V\otimes S_{a,b}}) =$$

$$(e^{-i\pi(k^2+j^2-k-j)}e^{\pi i\frac{\lambda^2}{2}}Id_{V\otimes S_{a,b}})\circ$$
$$(e^{i\pi((a+k)^2+(b+j)^2-a-k-b-j)}e^{-\pi i\frac{(a+b+\lambda)^2}{2}}Id_{V\otimes S_{a,b}}) =$$

$$e^{-i\pi(k^2+j^2-k-j)+\pi i\frac{\lambda^2}{2}+i\pi((a+k)^2+(b+j)^2-a-k-b-j)-\pi i\frac{(a+b+\lambda)^2}{2}}Id_{V\otimes S_{a,b}} =$$

$$e^{-i\pi(\frac{(a-b)^2}{2}-(a+b)(1+\lambda))}Id_{V\otimes S_{a,b}}$$

as $a+b=2n$ for some $n\in\mathbb{Z}$; then it is sufficient and necessary to have $\lambda\in\mathbb{Z}$ so that $M_{V,S_{a,b}} = Id_{V\otimes S_{a,b}}$.

Hence, for any such module V, we have $M_{V,A} = Id_{V\otimes A}$.

Now, suppose that $V = V_\alpha \boxtimes V_\beta \boxtimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$, then

$$V\otimes A = \bigoplus_{a,b\in I} V_{\alpha+2a}\boxtimes V_{\beta+2b}\boxtimes \mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}$$

- $\theta_{S_{a,b}} = Id_{S_{a,b}}$
- $\theta_V = e^{\frac{i\pi}{2}(\frac{(\alpha+1)^2+(\beta+1)^2}{2}-(\alpha+1+\beta+1))}e^{-\pi i\frac{\lambda^2}{2}}Id_V$ since
  - $\theta_{V_\alpha} = e^{\frac{i\pi}{2}(\frac{(\alpha+1)^2}{2}-(\alpha+1))}Id_{V_\alpha}$
  - $\theta_{V_\beta} = e^{\frac{i\pi}{2}(\frac{(\beta+1)^2}{2}-(\beta+1))}Id_{V_\beta}$
  - $\theta_{\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}} = e^{-\pi i\frac{\lambda^2}{2}}Id_{\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}}$
- $\theta_{V\otimes S_{a,b}} = e^{\frac{i\pi}{2}(\frac{(\alpha+2a+1)^2+(\beta+2b+1)^2}{2}-(\alpha+\beta+2(a+b+1)))}e^{-\pi i\frac{(a+b+\lambda)^2}{2}}Id_{V\otimes S_{a,b}}$ since
  - $\theta_{V_{\alpha+2a}} = e^{\frac{i\pi}{2}(\frac{(\alpha+2a+1)^2}{2}-(\alpha+2a+1))}Id_{V_{\alpha+2a}}$
  - $\theta_{V_{\beta+2b}} = e^{\frac{i\pi}{2}(\frac{(\beta+2b+1)^2}{2}-(\beta+2b+1))}Id_{V_{\beta+2b}}$
  - $\theta_{\mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}} = e^{-\pi i\frac{(a+b+\lambda)^2}{2}}Id_{\mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}}$

Thus,

$$M_{V,S_{a,b}} = (e^{\frac{i\pi}{2}(\frac{(\alpha+1)^2+(\beta+1)^2}{2} - (\alpha+1+\beta+1))} e^{-\pi i \frac{\lambda^2}{2}} Id_{V\otimes S_{a,b}})^{-1} \circ$$

$$(e^{\frac{i\pi}{2}(\frac{(\alpha+2a+1)^2+(\beta+2b+1)^2}{2} - (\alpha+\beta+2(a+b+1)))} e^{-\pi i \frac{(a+b+\lambda)^2}{2}} Id_{V\otimes S_{a,b}}) =$$

$$e^{\frac{i\pi}{2}(-\frac{(\alpha+1)^2+(\beta+1)^2}{2} + (\alpha+1+\beta+1) + \frac{(\alpha+2a+1)^2+(\beta+2b+1)^2}{2} - (\alpha+\beta+2(a+b+1))) + \lambda^2 - (a+b+\lambda)^2)} Id_{V\otimes S_{a,b}}$$

After some cancellations and noticing that $a^2 + b^2 - 2ab \cong 0 \mod 4$, then the last expression becomes

$$e^{\frac{i\pi}{2}(2a(\alpha+1)+2b(\beta+1)-2(a+b)\lambda)} Id_{V\otimes S_{a,b}} \tag{-1}$$

Now, we must have that

$$2a(\alpha + 1) + 2b(\beta + 1) - 2(a + b)\lambda \in 4\mathbb{Z} \text{ for all } (a, b) \in I$$

which happens if and only if

$$a(\alpha + 1) + b(\beta + 1) - (a + b)\lambda \in 2\mathbb{Z} \text{ for all } (a, b) \in I$$

In particular, for all $(a, b) \in I$ such that $a + b = 0$, the last expression becomes

$$a(\alpha - \beta) \in 2\mathbb{Z} \text{ for all } a \in \mathbb{Z}$$

This holds if and only if $\alpha - \beta \in 2\mathbb{Z}$.

At this point, α and β could have a non-zero imaginary part, but as $\lambda \in \mathbb{R}$ and considering $(a, b) = (2n, 0)$, one gets that α must be a real number; analogously, β must also be a real number.

Now, for $(a, a) \in I$, one must have $a(\alpha + \beta - 2\lambda) \in \mathbb{Z}$, which must hold for all $a \in \mathbb{Z}$; this implies that $\alpha + \beta - 2\lambda$ must be even.

We conclude that the three conditions

- $\alpha - \beta \in 2\mathbb{Z}$
- $\alpha, \beta \in \mathbb{R}$
- $\alpha + \beta - 2\lambda \in 2\mathbb{Z}$

are necessary for V to be such that $M_{V,S_{a,b}} = Id_{V\otimes S_{a,b}}$; now, let us show that these are sufficient.

Suppose that V is such that the contidions above are met, that is, $\alpha-\beta = 2n$ and $\alpha + \beta - 2\lambda = 2m$. Hence

$$a(\alpha + 1) + b(\beta + 1) - (a + b)\lambda =$$

$$a\alpha + a + b\beta + b - \frac{a\alpha}{2} - \frac{a\beta}{2} + am - \frac{b\alpha}{2} - \frac{b\beta}{2} + bm =$$
$$\frac{a\alpha}{2} + a + \frac{b\beta}{2} + b - \frac{a\beta}{2} + am - \frac{b\alpha}{2} + bm =$$
$$\frac{a\alpha}{2} + a + \frac{b\beta}{2} + b - \frac{a(\alpha - 2n)}{2} + am - \frac{b(2n+\beta)}{2} + bm =$$
$$a + b + (a+b)m + (a+b)n$$

As $a + b \in 2\mathbb{Z}$,

$$a(\alpha+1) + b(\beta+1) - (a+b)\lambda \cong 0 \mod 2$$

and therefore $M_{V,S_{a,b}} = Id_{V\otimes S_{a,b}}$.

We conclude that $M_{V,A} = Id_{V\otimes A}$ with $V = V_\alpha \boxtimes V_\beta \boxtimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$ if and only if the three aforementioned conditions hold.

Finally, we need to consider the mixed cases, that is, if V is of one of the following forms

1. $\mathbb{C}^H_{2k} \boxtimes V_\beta \boxtimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$
2. $V_\alpha \boxtimes \mathbb{C}^H_{2k} \boxtimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$

with $\alpha, \beta \in (\mathbb{C} - \mathbb{Z}) \cup 2\mathbb{Z}$, $k \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$.

Let us only do case 1 since case 2 is completely analogous.

As

- $\theta_{S_{a,b}} = Id_{S_{a,b}}$
- $\theta_V = e^{\frac{i\pi}{2}(\frac{(2k)^2}{2} - 2k + \frac{(\beta+1)^2}{2} - (\beta+1) - \lambda^2)} Id_V$ since
  - $\theta_{\mathbb{C}^H_{2k}} = e^{\frac{i\pi}{2}(\frac{(2k)^2}{2} - 2k)} Id_{\mathbb{C}^H_{2k}}$
  - $\theta_{V_\beta} = e^{\frac{i\pi}{2}(\frac{(\beta+1)^2}{2} - (\beta+1))} Id_{V_\beta}$
  - $\theta_{\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}} = e^{-\pi i \frac{\lambda^2}{2}} Id_{\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}}$
- $\theta_{V\otimes S_{a,b}} = e^{\frac{i\pi}{2}(2(a+k)^2 - 2(a+k) + \frac{(\beta+2b+1)^2}{2} - (\beta+2b+1) - (a+b+\lambda)^2)} Id_{V\otimes S_{a,b}}$ since
  - $\theta_{\mathbb{C}^H_{2(a+k)}} = e^{\frac{i\pi}{2}(2(a+k)^2 - 2(a+k))} Id_{\mathbb{C}^H_{2(a+k)}}$
  - $\theta_{V_{\beta+2b}} = e^{\frac{i\pi}{2}(\frac{(\beta+2b+1)^2}{2} - (\beta+2b+1))} Id_{V_{\beta+2b}}$

* $\theta_{\mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}} = e^{-\pi i\frac{(a+b+\lambda)^2}{2}} Id_{\mathbb{C}_{i\frac{a+b+\lambda}{\sqrt{2}}}}$

Hence,

$$M_{V,S_{a,b}} = (e^{\frac{i\pi}{2}(\frac{(2k)^2}{2}-2k+\frac{(\beta+1)^2}{2}-(\beta+1)-\lambda^2)} Id_{V\otimes S_{a,b}})^{-1}\circ$$
$$(e^{\frac{i\pi}{2}(2(a+k)^2-2(a+k)+\frac{(\beta+2b+1)^2}{2}-(\beta+2b+1)-(a+b+\lambda)^2)} Id_{V\otimes S_{a,b}}) =$$
$$e^{\frac{i\pi}{2}(2b(\beta+1)-2(a+b)\lambda)} Id_{V\otimes S_{a,b}}$$

We must then have $b(\beta+1)-(a+b)\lambda \in 2\mathbb{Z}$ for all $(a,b)\in I$, in particular, for $(a,b)=(2n,0)$, which forces $\lambda\in\mathbb{Z}$ and therefore $b(\beta+1)\in 2\mathbb{Z}$, which implies that β must be odd.

We conclude that $M_{V,A} = Id_{V\otimes A}$ with $V = \mathbb{C}^H_{2k}\boxtimes V_\beta \boxtimes \mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$ if and only if β is odd and $\lambda\in\mathbb{Z}$.

As V_σ is not defined for σ odd, no such V exists.

Let $\mathcal{G} : Rep(A) \to \mathcal{C}$ be the forgetful functor sending (V,μ_V) to V. The Frobenius reciprocity tells us that $\mathcal{F}$ and $\mathcal{G}$ are adjoint, that is

$$Hom_{Rep(A)}(\mathcal{F}(V), W) \cong Hom_{\mathcal{C}}(V, \mathcal{G}(W))$$

Denote $V_\alpha\boxtimes V_\beta\boxtimes\mathbb{C}_{i\frac{\lambda}{\sqrt{2}}}$ by $V_{\alpha,\beta,\lambda}$ and let $M_{\alpha,\beta,\lambda} = A\otimes V_{\alpha,\beta,\lambda} = \mathcal{F}(V_{\alpha,\beta,\lambda})$. Therefore, we have

$$Hom_{Rep(A)}(M_{\alpha',\beta',\lambda'}, M_{\alpha,\beta,\lambda}) \cong Hom_{\mathcal{C}}(V_{\alpha',\beta',\lambda'}, \mathcal{G}(M_{\alpha,\beta,\lambda})) =$$
$$Hom_{\mathcal{C}}(V_{\alpha',\beta',\lambda'}, A\otimes V_{\alpha,\beta,\lambda}) = Hom_{\mathcal{C}}(V_{\alpha',\beta',\lambda'}, \bigoplus_{(a,b)\in I} S_{a,b}\otimes V_{\alpha,\beta,\lambda})$$

This last is trivial unless there is $(a,b)\in I$ such that $V_{\alpha',\beta',\lambda'}\cong S_{a,b}\otimes V_{\alpha,\beta,\lambda}$, in which case it is isomorphic to $\mathbb{C}$.

We then have $Hom_{\mathcal{C}}(V_{\alpha',\beta',\lambda'}, \bigoplus_{(a,b)\in I} S_{a,b}\otimes V_{\alpha,\beta,\lambda})\cong\mathbb{C}$ if and only if

$$\alpha-\alpha'\in 2\mathbb{Z}$$
$$\beta-\beta'\in 2\mathbb{Z}$$
$$\frac{\alpha-\alpha'}{2}+\frac{\beta-\beta'}{2} = \lambda-\lambda'$$

and $Hom_{\mathcal{C}}(V_{\alpha',\beta',\lambda'}, \bigoplus_{(a,b)\in I} S_{a,b}\otimes V_{\alpha,\beta,\lambda})$ is trivial otherwise.

By proposition 4.4 in [42], we have that the $M_{\alpha,\beta,\lambda}$ are simple in $Rep(A)$ since each $S_{a,b}$ is simple and $S_{a,b} \otimes V_{\alpha,\beta,\lambda} \neq S_{a',b'} \otimes V_{\alpha,\beta,\lambda}$ unless $a = a'$ and $b = b'$. Moreover, by proposition 3.4 in [15], each simple in $Rep(A)$ is the induction of a simple module in $\mathcal{C}$.

Now, let $U_j = \overline{U}^H_{e^{\frac{i\pi}{2}}}(\mathfrak{sl}(2))$ for $j \in \{1,2\}$ and $X = \mathbb{C}$, with action on $\mathbb{C}_\sigma$ given by $X\mathbb{C}_\sigma = \sigma\mathbb{C}_\sigma$.

Step 6

Define the algebra

$$Z = U_1 \boxtimes U_2 \boxtimes X$$

We want to see when the action $K_1^{x_1} \boxtimes K_2^{x_2} \boxtimes e^{x_3 X}$ is trivial. First, recall that $K_j V_\sigma = q^{H_j} Id_{V_\sigma} = e^{i\frac{\pi}{2}(\sigma+1)} Id_{V_\sigma}$.

We then have

$$K_1^{x_1} \boxtimes K_2^{x_2} \boxtimes e^{x_3 X} V_{\alpha,\beta,\lambda} = e^{i\frac{\pi}{2}(x_1(\alpha+1)+x_2(\beta+1))+x_3 i\frac{\lambda}{\sqrt{2}}} Id_{V_{\alpha,\beta,\lambda}} =$$

$$e^{i\frac{\pi}{2}(x_1(\alpha+1)+x_2(\beta+1)+x_3\frac{\lambda\sqrt{2}}{\pi})} Id_{V_{\alpha,\beta,\lambda}}$$

Let

$$A_{\alpha,\beta,\lambda}(a,b) = e^{i\frac{\pi}{2}(2a(\alpha+1)+2b(\beta+1)-2(a+b)\lambda)}$$

and

$$B_{\alpha,\beta,\lambda}(x_1,x_2,x_3) = e^{i\frac{\pi}{2}(x_1(\alpha+1)+x_2(\beta+1)+x_3\frac{\lambda\sqrt{2}}{\pi})}$$

We know from equation 5.2 that $V_{\alpha,\beta,\lambda}$ induces to a local module if and only if $A_{\alpha,\beta,\lambda}(a,b) = 1$ for all $(a,b) \in I$.

Now, if we let $x_1 = 2a$, $x_2 = 2b$ and $x_3 = -\frac{2\pi}{\sqrt{2}}(a+b)$, then we have that $V_{\alpha,\beta,\lambda}$ induces to a local module if and only if $B_{\alpha,\beta,\lambda}(x_1,x_2,x_3) = 1$ for all $(a,b) \in I$.

Step 7

Let L be the ideal of Z generated by $K_1^{2a} K_2^{2b} e^{-\frac{2\pi}{\sqrt{2}}(a+b)}$ with $(a,b) \in I$. Define $\tilde{Z} = Z/\langle L - Id_Z \rangle$.

We know that a Z-module V is a $\tilde{Z}$-module if and only if V is a trivial L-module.

Lemma 7. *A Z-module V restricts to a $\tilde{Z}$-module if and only if V is local.*

Proof. Suppose that V restricts to a $\tilde{Z}$-module, then it is a trivial L-module, that is, $K_1^{2a}K_2^{2b}e^{-\frac{2\pi}{\sqrt{2}}(a+b)} = Id_V$ for all $(a,b) \in I$. As

$$M_V, S_{a,b} = K_1^{2a}K_2^{2b}e^{-\frac{2\pi}{\sqrt{2}}(a+b)}Id_{V\otimes S_{a,b}} = Id_{V\otimes S_{a,b}}$$

then V is local.

Conversely, suppose that V is local, that is $M_V, S_{a,b} = Id_{V\otimes S_{a,b}}$ for all $(a,b) \in I$, but as $M_V, S_{a,b} = K_1^{2a}K_2^{2b}e^{-\frac{2\pi}{\sqrt{2}}(a+b)}Id_{V\otimes S_{a,b}}$, se have that $K_1^{2a}K_2^{2b}e^{-\frac{2\pi}{\sqrt{2}}(a+b)} = Id_V$ for all $(a,b) \in I$, which implies that V is a trivial L-module, and therefore, a $\tilde{Z}$-module.

□

Hence, the $\tilde{Z}$-modules are the local modules, that is, those of the form $V_{\alpha,\beta,\lambda}$ such that $\alpha - \beta$ and $\alpha + \beta - 2\lambda$ are even, and $\alpha, \beta \in \mathbb{R}$.

Lemma 8. *Let $M_{\alpha,\beta,\lambda} = \mathcal{F}(V_{\alpha,\beta,\lambda})$ be the induction of $V_{\alpha,\beta,\lambda}$. If $V_{\alpha,\beta,\lambda}$ and $V_{\alpha',\beta',\lambda'}$ are simple Z-modules, then $M_{\alpha,\beta,\lambda} \cong M_{\alpha',\beta',\lambda'}$ if and only if $V_{\alpha,\beta,\lambda} = V_{\alpha',\beta',\lambda'} \otimes S_{a,b}$ for some $(a,b) \in I$.*

Proof. If $M_{\alpha,\beta,\lambda} \cong M_{\alpha',\beta',\lambda'}$, then $Hom_{Rep(A)}(M_{\alpha,\beta,\lambda}, M_{\alpha',\beta',\lambda'})$ is not trivial and we already saw that this implies the existence of $(a,b) \in I$ such that $V_{\alpha,\beta,\lambda} = V_{\alpha',\beta',\lambda'} \otimes S_{a,b}$.

Conversely, suppose that $V_{\alpha,\beta,\lambda} = V_{\alpha',\beta',\lambda'} \otimes S_{a,b}$ for some $(a,b) \in I$. First, notice that $S_{e,f} \otimes S_{a,b} = S_{e+a,f+b}$. Now, we have

$$M_{\alpha,\beta,\lambda} = \bigoplus_{(c,d)\in I} V_{\alpha,\beta,\lambda} \otimes S_{c,d}$$

and

$$M_{\alpha',\beta',\lambda'} = \bigoplus_{(e,f)\in I} V_{\alpha',\beta',\lambda'} \otimes S_{e,f} = \bigoplus_{(e,f)\in I} V_{\alpha,\beta,\lambda} \otimes S_{e+a,f+b} = \bigoplus_{(c,d)\in I} V_{\alpha,\beta,\lambda} \otimes S_{c,d}$$

Where the last equation holds since $S_{a,b}$ is a simple current.

□

We know that the algebra Z is generated by $E_1, F_1, H_1, K_1, E_2, F_2, H_2, K_2, H_3, K_3$. Now, define Z_1 as the subalgebra of Z generated by $E_1, F_1, K_1, E_2, F_2, K_2, K_3, H$, where $H = H_1 + H_2 + i2\sqrt{2}H_3$; and let $\hat{Z} = Z_1/\langle L - Id_Z\rangle$. Denote the category of $\hat{Z}$-modules by $\hat{\mathcal{C}}$.

Lemma 9. *If $M = V_{\alpha,\beta,\lambda}$ and $N = V_{\alpha',\beta',\lambda'}$ are Z-modules such that $M = N \otimes S_{a,b}$ for some $(a,b) \in I$, then $M \cong N$ as Z_1-modules.*

Proof. Notice that $V_\sigma = \mathbb{C}_{\sigma+1} \bigoplus \mathbb{C}_{\sigma-1}$, hence
$V_{\alpha,\beta,\lambda} = \mathbb{C}_{\alpha+1,\beta+1,\lambda} \bigoplus \mathbb{C}_{\alpha-1,\beta+1,\lambda} \bigoplus \mathbb{C}_{\alpha+1,\beta-1,\lambda} \bigoplus \mathbb{C}_{\alpha-1,\beta-1,\lambda}$.

As we need $M \cong N$ as Z_1-modules, we must have

$$HM = HN$$

that is,

$$(\alpha + 2a \pm 1) + (\beta + 2b \pm 1) - 2(a + b + \lambda) = (\alpha \pm 1) + (\beta \pm 1) - 2\lambda$$

As the last equation holds, $M \cong N$ as Z_1-modules.

□

Example 7. *Let φ be the golden ratio and $W = V_{\varphi+3,\varphi-9,\varphi+2}$. Notice that $W \in \hat{\mathcal{C}}$ since*

$$(\varphi + 3) - (\varphi - 9) = 12 \in 2\mathbb{Z}$$

$$\varphi + 3, \varphi - 9 \in \mathbb{R}$$

$$(\varphi + 3) + (\varphi - 9) - 2(\varphi + 2) = -10 \in 2\mathbb{Z}$$

and for any $(a,b) \in I$ we have

$$K_1^{2a} K_2^{2b} e^{-\frac{2\pi}{\sqrt{2}}(a+b)} = e^{\pi i(a\varphi+4a+b\varphi-8b-a\varphi-2a-b\varphi-2b)} = e^{\pi i(2a-10b)} = 1$$

note that the second equation follows from $(a,b) \in I$. By lemma 7 we have that W is a local representation of A.

Now, if $w \in W$ is a highest weight vector, then we have

$$Hw = (\varphi + 3 + 1 + \varphi - 9 + 1 + i2\sqrt{2}\frac{i(\varphi + 2)}{\sqrt{2}})w = -8w$$

Now, consider $W \otimes S_{a,b} = V_{\varphi+3+2a,\varphi-9+2b,\varphi+2+a+b}$ for some $(a,b) \in I$, then we have the action of H given by

$$Hw = (\varphi + 3 + 2a + 1 + \varphi - 9 + 2b + 1 + i2\sqrt{2}\frac{i(\varphi + 2 + a + b)}{\sqrt{2}})w = -8w$$

That is, $W \cong W \otimes S_{a,b}$ as Z_1-modules for any $(a,b) \in I$ (which was known from lemma 9) and therefore $\mathcal{F}(W) \cong \mathcal{F}(W \otimes S_{a,b})$ by lemma 8.

Conjecture 3. *There is an equivalence of categories*

$$\hat{Z} - mod \cong Rep^{loc}(A)$$

Lemma 10. *I is an abelian group under component addition and is generated by $\langle (1,1),(1,-1)\rangle$ as $\mathbb{Z}$-module.*

Proof. Let $(a,b) \in I$, then $a+b = 2n$ for some $n \in \mathbb{Z}$, and therefore $(a,b) = (a-n)(1,-1) + n(1,1)$.

□

Lemma 11. *Let $L_1 = K_1^2K_2^{-2}$ and $L_2 = K_1^2K_2^2e^{-4\pi/\sqrt{2}}$, then $L = \langle L_1, L_2\rangle$.*

Proof. Let $K_1^{2a}K_2^{2b}e^{-\frac{2\pi}{\sqrt{2}}(a+b)}$ with $(a,b) \in I$, then $a+b = 2n$.

We have that

$$K_1^{2a}K_2^{2b}e^{-\frac{2\pi}{\sqrt{2}}(a+b)} = (K_1^2K_2^{-2})^{a-n}(K_1^2K_2^2e^{-4\pi/\sqrt{2}})^n$$

□

Notice that both L_1 and L_2 act trivially on any simple current $S_{a,b}$, which is expected since they must act equally on all the orbit.

Define the category $\mathcal{C}(\mu_1,\mu_2)$, where the objects are

$$Obj(\mathcal{C}(\mu_1,\mu_2)) = \{(V,\mu_V) \in Rep(A) : M_{V,S_{1,-1}} = e^{2\pi i\mu_1}Id_{V\otimes S_{1,-1}}, M_{V,S_{1,1}} = e^{2\pi i\mu_2}Id_{V\otimes S_{1,1}}\}$$

and the morphisms are morphisms of representations; and the algebras

$$\tilde{Z}_{\mu_1,\mu_2} = Z/\langle L_1 - e^{2\pi i\mu_1}Id_Z, L_2 - e^{2\pi i\mu_2}Id_Z\rangle$$

$$\hat{Z}_{\mu_1,\mu_2} = Z_1/\langle L_1 - e^{2\pi i\mu_1}Id_Z, L_2 - e^{2\pi i\mu_2}Id_Z\rangle$$

Where $Z_1 = \langle E_1, F_1, K_1, E_2, F_2, K_2, K_3, H\rangle$, where $H = H_1 + H_2 + i2\sqrt{2}H_3$, as before.

We know that V restricts to a $\tilde{Z}_{\mu_1,\mu_2}$-module if and only if $L_1 = e^{2\pi i\mu_1}Id_V$ and $L_2 = e^{2\pi i\mu_2}Id_V$.

Lemma 12. *A Z-module V restricts to a $\tilde{Z}_{\mu_1,\mu_2}$-module if and only if $M_{V,S_{1,-1}} = e^{2\pi i\mu_1}Id_{V\otimes S_{1,-1}}$ and $M_{V,S_{1,1}} = e^{2\pi i\mu_2}Id_{V\otimes S_{1,1}}$*

Proof. First, suppose that V restricts to a $\tilde{Z}_{\mu_1,\mu_2}$-module, then $L_1 = e^{2\pi i\mu_1}Id_V$ and $L_2 = e^{2\pi i\mu_2}Id_V$, but $M_{V,S_{1,-1}} = L_1 = e^{2\pi i\mu_1}Id_{V\otimes S_{1,-1}}$ and $M_{V,S_{1,1}} = L_2 = e^{2\pi i\mu_2}Id_{V\otimes S_{1,1}}$.

Conversely, if $M_{V,S_{1,-1}} = e^{2\pi i \mu_1} Id_{V \otimes S_{1,-1}}$ and $M_{V,S_{1,1}} = e^{2\pi i \mu_2} Id_{V \otimes S_{1,1}}$, as $M_{V,S_{1,-1}} = L_1$ and $M_{V,S_{1,1}} = L_2$, V is a $\tilde{Z}_{\mu_1,\mu_2}$-module.

□

Conjecture 4. *There is an equivalence of categories*

$$\hat{Z}_{\mu_1,\mu_2} - Mod \cong \mathcal{C}(\mu_1, \mu_2)$$

6 General Case

6.1 The setup

The desired result is to have a commutative diagram of the form

$$\begin{array}{ccc} \tilde{\mathcal{C}} & & \\ \Big\downarrow Forg & \searrow^{\mathcal{F}} & \\ \hat{\mathcal{C}} & \xrightarrow{\hat{\mathcal{F}}} & Rep^{loc}(A) \end{array}$$

where $\hat{\mathcal{F}}$ is an equivalence of categories. The purpose of this section is to present all the elements of this diagram and some results relating them.

To begin with, let $I \subset \mathcal{L} = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \frac{l}{2d_i}\mathbb{Z}\}$ so that $\{\mathbb{C}_\lambda | \lambda \in I\}$ is closed under the tensor product and duals, and such that for any $\mathbb{C}_\lambda, \mathbb{C}_\mu \in \{\mathbb{C}_\lambda | \lambda \in I\}$, $M_{\mathbb{C}_\lambda, \mathbb{C}_\mu} = Id_{\mathbb{C}_\lambda \otimes \mathbb{C}_\mu}$. Existence of such an I is guaranteed by theorem 3.1 in [15].

Now, recall the identification $\phi \in \mathfrak{h}^*$ to $t_\phi \in \mathfrak{h}$ mentioned earlier and consider the operators q^{2t_λ} for $\lambda \in I$. If we apply this operator to a weight vector w_γ of $X \in \mathcal{C}$, then we get

$$q^{2t_\lambda}(w_\gamma) = q^{2\gamma(t_\lambda)}(w_\gamma) = q^{2k(t_\gamma, t_\lambda)}(w_\gamma) = q^{2\langle\gamma,\lambda\rangle}(w_\gamma) = q^{2\langle\lambda,\gamma\rangle}(w_\gamma)$$

Define the subalgebra $L = \langle q^{2t_\lambda} - Id_Z | \lambda \in I\rangle$ of Z and let $\tilde{Z} = Z/L$.

Now, we have that Z has generators $X_{\pm i}, H_i, K_\gamma$ with $i = 1, 2, ...n$ and $\gamma \in R$; consider Z_1, the Z-subalgebra generated by $X_{\pm i}, K_\gamma$, $Ann_{\mathfrak{h}}(I) = \{H \in \mathfrak{h} | H\mathbb{C}_\lambda = 0 \text{ for all } \lambda \in I\}$ with $i = 1, 2, ...n$ and $\gamma \in R$, and define $\hat{Z} = Z_1/L$.

Let $\hat{\mathcal{C}}$ denote the category of $\hat{Z}$-modules and $\tilde{\mathcal{C}}$ denote the tensor subcategory of $\mathcal{C}$ of $\tilde{Z}$-modules. Notice that have the forgetful functor $Forg : \tilde{\mathcal{C}} \to \hat{\mathcal{C}}$ that forgets the action of the complement (the choice of the complement does not matter) of $Ann_{\mathfrak{h}}(I)$ in $\mathfrak{h}$.

Finally, denote by A the algebra

$$A = \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$$

6.1.1 Induction functor

Recall that the induction functor $\mathcal{F} : \mathcal{C} \to Rep(A)$ is defined by $\mathcal{F}(V) = (A \otimes V, \mu_{\mathcal{F}(V)})$, where $\mu_{\mathcal{F}(V)} = (\mu \otimes Id_V) \circ a^{-1}_{A,A,V}$, and $\mathcal{F}(f) = Id_A \otimes f$.

Lemma 13. *Let $X \in \mathcal{C}$, then $\mathcal{F}(X) \in Rep^{loc}(A)$ if and only if $X \in \tilde{\mathcal{C}}$.*

Proof. First, suppose that $\mathcal{F}(X) \in Rep^{loc}(A)$. By lemma 4.1 in [15], for any $w_\gamma \in X$ of weight $\gamma \in \mathfrak{h}^*$, we have that

$$M_{\mathbb{C}_\lambda, X}(v_\lambda \otimes w_\gamma) = q^{2\langle \lambda, \gamma \rangle} Id_{\mathbb{C}_\lambda \otimes X}$$

Now, as $\mathcal{F}(X) \in Rep^{loc}(A)$, $M_{A\otimes X, A} = Id_{(A\otimes X)\otimes A}$, but by theorem 2.11 in [40] we must have that $q^{2\langle \lambda, \gamma \rangle} = 1$ for all $\lambda \in I$ and therefore X has a structure of Z/L-module, that is, $X \in \tilde{\mathcal{C}}$.

Converselly, suppose that $X \in \tilde{\mathcal{C}}$, then $q^{2\langle \lambda, \gamma \rangle} = 1$ for all $\lambda \in I$, then by lemma 4.1 in [15] $M_{\mathbb{C}_\lambda, X} = q^{2\langle \lambda, \gamma \rangle} Id_{\mathbb{C}_\lambda \otimes X} = Id_{\mathbb{C}_\lambda \otimes X}$ and by theorem 2.11 in [40], this is enough for $\mathcal{F}(X) \in Rep^{loc}(A)$.

□

By the previous lemma, we can consider the induction functor (abusing of the notation) $\mathcal{F} : \tilde{\mathcal{C}} \to Rep^{loc}(A)$, and we have the following propositions.

Lemma 14. *Let A be a simple commutative associative unital algebra in $\mathcal{C}$*

$$A = \bigoplus_{\lambda \in I} J_\lambda$$

where $J_\lambda \in \mathcal{C}$ is a simple current and I is an indexing set (closed under tensor product).

Let V, V' be simple Z-modules. We have that $\mathcal{F}(V) \cong \mathcal{F}(V')$ if and only if $V' = V \otimes J$ for some simple current $J \in I$, where $\mathcal{F}$ is the induction functor.

Proof. First, suppose that $\mathcal{F}(V) \cong \mathcal{F}(V')$, that is,

$$\bigoplus_{\lambda \in I} V \otimes J_\lambda \cong \bigoplus_{\lambda \in I} V' \otimes J_\lambda$$

Particularly, for any $\sigma \in I$ we must have that

$$\bigoplus_{\lambda \in I'} V \otimes J_\lambda \cong V' \otimes J_\sigma$$

for some $I' \subset I$. As J_σ is a simple current, we have

$$\bigoplus_{\lambda \in I^*} V \otimes J_\lambda = \bigoplus_{\lambda \in I'} V \otimes J_\lambda \otimes J_\sigma^{-1} \cong V' \otimes J_\sigma \otimes J_\sigma^{-1} \cong V'$$

for some $I^* \subset I'$. As V' is simple, then there must be a $\lambda \in I$ such that $V \otimes J_\lambda \cong V'$.

Conversely, suppose that $V' = V \otimes J_\lambda$, then we have the Frobenius reciprocity

$$Hom^{Rep(A)}(\mathcal{F}(V), \mathcal{F}(V')) \cong Hom^{\mathcal{C}}(V, V' \otimes A) = \mathbb{C}$$

since both V, V' are simple. Hence, $\mathcal{F}(V) \cong \mathcal{F}(V')$.

□

Corollary 1. *Let $V, V' \in \tilde{\mathcal{C}}$ be simple, then $\mathcal{F}(V) \cong \mathcal{F}(V')$ if and only if $V' = V \otimes \mathbb{C}_\lambda$ for some $\lambda \in I$.*

Proof. Notice that any $\tilde{Z}$-module is a Z-module. The result follows immediatly from lemma 14.

□

Theorem 15. *Let $X, Y \in \tilde{\mathcal{C}}$, then $X \cong Y$ as Z_1-modules if and only if $\mathcal{F}(X) \cong \mathcal{F}(Y)$.*

Proof. First, suppose that $X \cong Y$ as Z_1-modules, that is, there is an isomorphism $f : X \to Y$ of Z_1-modules.

Define $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ as $\hat{f}(\oplus_{i=1}^n x_i \otimes v_{\lambda_i}) = \oplus_{i=1}^n f(x_i) \otimes v_{\lambda_i}$ and extend it linearly.

Let us see that $\hat{f}$ is an homomorphism of Z_1-modules. Let $H_1, H_2 \in Ann_{\mathfrak{h}}(I)$, then we have (using the Hopf algebra structure)

$$\hat{f}(H_1(\oplus_{i=1}^n x_i \otimes v_{\lambda_i}) + H_2(\oplus_{j=1}^m x_j \otimes v_\gamma)) =$$

$$\hat{f}(\oplus_{i=1}^n H_1(x_i) \otimes v_{\lambda_i} + \oplus_{j=1}^m H_2(x_j) \otimes v_{\gamma_j}) =$$

$$\hat{f}(\oplus_{i=1}^n H_1(x_i) \otimes v_{\lambda_i}) + \hat{f}(\oplus_{j=1}^m H_2(x_j) \otimes v_{\gamma_j}) =$$

$$\oplus_{i=1}^n f(H_1(x_i)) \otimes v_{\lambda_i} + \oplus_{j=1}^m f(H_2(x_j)) \otimes v_{\gamma_j} =$$

$$\oplus_{i=1}^n H_1(f(x_i)) \otimes v_{\lambda_i} + \oplus_{j=1}^m H_2(f(x_j)) \otimes v_{\gamma_j} =$$

$$H_1(\oplus_{i=1}^n f(x_i) \otimes v_{\lambda_i}) + H_2(\oplus_{j=1}^m f(x_j) \otimes v_{\gamma_j}) =$$

$$H_1(\hat{f}(\oplus_{i=1}^n (x_i \otimes v_{\lambda_i})) + H_2(\hat{f}(\oplus_{j=1}^m x_j \otimes v_{\gamma_j}))$$

The Z_1 linearity is similarly shown for the other elements in Z_1. Hence, $\hat{f}$ is an homomorphism of Z_1-modules.

Let $\oplus_{j=1}^{m} y_j \otimes v_{\lambda_j} \in \mathcal{F}(Y)$, as f is an isomorphism, there are $x_j \in X$ such that $f(x_j) = y_j$, then $\hat{f}(\oplus_{j=1}^{m} x_j \otimes v_{\lambda_j}) = \oplus_{j=1}^{m} y_j \otimes v_{\lambda_j}$, thus, $\hat{f}$ is surjective.

As we are taking a restriction of an isomorphism so it has to be injective.

We conclude that $\hat{f}$ is a linear isomorphism and therefore $\mathcal{F}(X) \cong \mathcal{F}(Y)$.

Conversely, suppose that $\mathcal{F}(X) \cong \mathcal{F}(Y)$.

For any $X, Y \in \mathcal{C}$ we have the Frobenius reciprocity

$$Hom^{Rep(A)}(\mathcal{F}(X), Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda) \cong Hom^{\mathcal{C}}(X, \mathcal{G}(Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda))$$

which can be written as

$$Hom^{Rep(A)}(\mathcal{F}(X), \mathcal{F}(Y)) \cong Hom^{\mathcal{C}}(X, Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda)$$

Let $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ be the aforementioned isomorphism, then there is an isomorphism $f : X \to Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$ in $\mathcal{C}$.

Now, let $H \in Ann_{\mathfrak{h}}(I)$. As $X \cong Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$ in $\mathcal{C}$, then the action of H on X and on $Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$ must coincide; therefore we have that

$$H(X) = H(Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda) = H(Y) \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda + Y \otimes H(\bigoplus_{\lambda \in I} \mathbb{C}_\lambda) = H(Y) \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$$

Where the last equation follows from $H \in Ann_{\mathfrak{h}}(I)$.

As the action of H on $Y \otimes \bigoplus_{\lambda \in I} \mathbb{C}_\lambda$ only depends on how it acts on Y, we must then have that the action of H on X and on Y coincide, that is, $X \cong Y$ as Z_1-modules.

□

Here we restate proposition 3.4 and theorem 4.2 of [15] in our context.

Proposition 9. *Suppose every indecomposable object in $\mathcal{C}$ has a simple subobject. Then $N \in Rep^{loc}(A)$ is simple if and only if $N \cong \mathcal{F}(M)$ for a simple object $M \in \mathcal{C}$.*

Theorem 16. *We have:*

- *Let X and P_X in $\mathcal{C}$ be the projective cover of X. Then $\mathcal{F}(X) \in Rep^{loc}(A)$ if and only if $P_X \in Rep^{loc}(A)$.*

- $\mathcal{F}(P_\lambda)$ *is the projective cover of* $\mathcal{F}(L_\lambda)$.
- *The distinct irreducible objects in* $Rep^{loc}(A)$ *are given by the set* $\{\mathcal{F}(L_\lambda)|\lambda \in \frac{l}{2}I^*/(\frac{l}{2}I^* \cap I)\}$.

6.1.2 $Forg$ functor

The forgetful functor $Forg : \tilde{\mathcal{C}} \to \hat{\mathcal{C}}$ forgets the action of the complement of $Ann_{\mathfrak{h}}(I)$ and take restriction of morphisms.

Lemma 15. *Let* $X, Y \in \tilde{\mathcal{C}}$*, then* $Forg(X) \cong Forg(Y)$ *if and only if* $X \cong Y$ *as* Z_1*-modules.*

Proof. Suppose that $X \cong Y$ as Z_1-modules, then the actions of $Ann_{\mathfrak{h}}(I)$ on X and Y are the same, then $Forg(X) \cong Forg(Y)$. Now, suppose that $Forg(X) \cong Forg(Y)$, then the action of $Ann_{\mathfrak{h}}(I)$ on X and Y is the same, then $X \cong Y$ as Z_1-modules.

□

Corollary 2. *Let* $X, Y \in \tilde{\mathcal{C}}$*, then* $\mathcal{F}(X) \cong \mathcal{F}(Y)$ *if and only if* $Forg(X) \cong Forg(Y)$.

Proof. It is clear from lemma 15 and theorem 15.

□

Lemma 16. *Let* $W \in \hat{\mathcal{C}}$ *be a highest weight module. There exist an element* $\tilde{W} \in \tilde{\mathcal{C}}$ *such that* $W \cong Forg(\tilde{W})$.

Proof. We have well defined actions of $X_{\pm i}, K_\gamma, Ann_{\mathfrak{h}}(I)$ for $i \in \{1, 2, ...n\}$ and $\gamma \in R$ on W.

Let $w_\sigma \in W$ be a generating highest weight vector of weight σ. For $\alpha_i^* = 0 \oplus ...\alpha_i \oplus ...0 \in R$, we have

$$K_{\alpha_i^*} w_\sigma = q^{\lambda_i + lm_i} w_\sigma$$

for some $\lambda_i \in \mathbb{C}$ and any $m_i \in \mathbb{Z}$.

Let $\bar{W} = W$ as vector spaces and let $X_{\pm i}, K_\gamma$ for $i \in \{1, 2, ...n\}$ and $\gamma \in R$ act on $\bar{W}$ as they do on W. We can define the action of H_i on w_σ by

$$H_i w_\sigma = \frac{\lambda_i + lm_i}{d_i} w_\sigma$$

We have that $\bar{W} \in \mathcal{C}$ since it is clearly finite dimensional and for any $\gamma = \oplus_{i=1}^n s_i \alpha_i \in R$ we have that

$$K_\gamma w_\sigma = K_{\alpha_1^*}^{s_1}...K_{\alpha_n^*}^{s_n} w_\sigma = \prod_{i=1}^{n} q^{s_i(\lambda_i + lm_i)} w_\sigma = \prod_{i=1}^{n} q^{d_i s_i H_i} w_\sigma$$

Now, notice that we have the decomposition

$$\mathfrak{h} = Ann_{\mathfrak{h}}(I) \oplus Ann_{\mathfrak{h}}(I)^{\perp}$$

Let $a = dim(Ann_{\mathfrak{h}}(I))$, $b = dim(Ann_{\mathfrak{h}}(I)^{\perp})$ and $\langle A_i = a_{i,1}H_1 + ...a_{i,n}H_n : i \in \{1, 2, ...a\}\rangle$ a basis for $Ann_{\mathfrak{h}}(I)$.

Additionally, notice that if $\lambda \in I$, then $t_\lambda \in Ann_{\mathfrak{h}}(I)^{\perp}$ since $\lambda(t_\lambda) = \langle \lambda, \lambda\rangle \neq 0$ and therefore $dim(\langle t_\lambda : \lambda \in I\rangle) \leq b$.

We want to get a $\tilde{W} \in \hat{\mathcal{C}}$ by showing that there is a choice of $m_1, ...m_n$ such that $\tilde{W}$ with the actions defined above coincides with W. We have that $\tilde{W}$ also needs to suffice the relations $q^{2t_\lambda} = Id$, for $\lambda \in I$, that is $t_\lambda = lm_\lambda$ for some $m_\lambda \in \mathbb{Z}$.

Hence, we have the following system of equations on the variables $m_1, ...m_n, m_\lambda$ for $\lambda \in I$.

$$\begin{cases} A_i w_\sigma = a_{i,1}H_1 w_\sigma + ...a_{i,n}H_n w_\sigma \\ t_\lambda w_\sigma = lm_\lambda w_\sigma \end{cases}$$

for $i = 1, 2, ...a$ and $\lambda \in I$.

This system has a solution in $\mathbb{Q}$ since $dim(\langle t_\lambda : \lambda \in I\rangle) \leq b$ and therefore a solution in $\mathbb{Z}$.

Define $\tilde{W}$ as any of these solutions, which by construction has the same action of $Ann_{\mathfrak{h}}(I)$ as W and therefore $W \cong Forg(\tilde{W})$.

□

Example 8. *Let us revisit example 7. Recall that if we let φ be the golden ratio and $W = V_{\varphi+3,\varphi-9,\varphi+2}$, then $W \in \hat{\mathcal{C}}$.*

Let $w \in W$ be a highest weight vector, then

$$Hw = -8w$$
$$K_1 w = e^{\frac{\pi i}{2}(\varphi+4+4m_1)} w$$
$$K_2 w = e^{\frac{\pi i}{2}(\varphi-8+4m_2)} w$$
$$K_3 w = e^{\frac{\pi i}{2}(\frac{i(\varphi+2)}{\sqrt{2}}+4m_3)} w$$

Then, following the proof of lemma 16, we must define

$$H_1(w) = (\varphi + 4 + 4m_1)w$$
$$H_2(w) = (\varphi - 8 + 4m_2)w$$
$$H_3(w) = (\frac{i(\varphi+2)}{\sqrt{2}} + 4m_3)w$$

Now, if we want to lift W to some $\tilde{W} \in \tilde{C}$, we have to solve the system of equations in the variables m_1, m_2, m_3

$$\begin{cases} K_1^{2a} K_2^{2b} e^{-\frac{2\pi}{\sqrt{2}}(a+b)} = 1 & \textit{for all } (a,b) \in I \\ H = -8 \end{cases}$$

that is

$$\begin{cases} e^{\frac{\pi i}{2}(\varphi+4+4m_1)2a} e^{\frac{\pi i}{2}(\varphi-8+4m_2)2b} e^{-\frac{2\pi}{\sqrt{2}}(\frac{i(\varphi+2)}{\sqrt{2}}+4m_3)(a+b)} = 1 & \textit{for all } (a,b) \in I \\ H = \varphi + 4 + 4m_1 + \varphi - 8 + 4m_2 + i2\sqrt{2}(\frac{i(\varphi+2)}{\sqrt{2}} + 4m_3) = -8 \end{cases}$$

which turns into

$$\begin{cases} e^{\pi i(\varphi a+4a+4am_1+\varphi b-8b+4bm_2-\varphi a-2a-\varphi b-2b)} e^{-\frac{2\pi}{\sqrt{2}}(4m_3(a+b))} = 1 & \textit{for all } (a,b) \in I \\ H = 4(m_1+m_2) - 8 + i8\sqrt{2}m_3 = -8 \end{cases}$$

and finally becomes

$$\begin{cases} e^{\pi i(2a-10b+4(m_1+m_2))} e^{-\frac{2\pi}{\sqrt{2}}(4m_3(a+b))} = 1 & \textit{for all } (a,b) \in I \\ H = 4(m_1+m_2) - 8 + i8\sqrt{2}m_3 = -8 \end{cases}$$

It is clear that the solutions are $m_1 + m_2 = 0$ and $m_3 = 0$.

Then, for example, if we let $m_1 = -2$, then $\tilde{W} = V_{\varphi-5,\varphi-1,\varphi+2}$. Let us check that $W \cong Forg(\tilde{W})$, that is, the action of each element in Z_1 is the same in both modules. We have the action of H, K_1, K_2, K_3 on $\tilde{W}$ given by

$$Hw = \varphi - 5 + 1 + \varphi - 1 + 1 + i2\sqrt{2}(\frac{i(\varphi+2)}{\sqrt{2}})w = (2\varphi - 4 - 2\varphi - 4)w = -8w$$

$$K_1 w = e^{\frac{\pi i}{2}(\varphi+4-8)}w = e^{\frac{\pi i}{2}(\varphi+4)}w$$
$$K_2 w = e^{\frac{\pi i}{2}(\varphi-8+8)}w = e^{\frac{\pi i}{2}(\varphi-8)}w$$
$$K_3 w = e^{\frac{\pi i}{2}(\frac{i(\varphi+2)}{\sqrt{2}})}w$$

Which clearly coincide with the actions on W.

By lemma 15, $Forg(V_{\varphi+3,\varphi-9,\varphi+2}) \cong Forg(V_{\varphi-5,\varphi-1,\varphi+2})$ since $V_{\varphi-5,\varphi-1,\varphi+2} \cong V_{\varphi+3,\varphi-9,\varphi+2}$ as Z_1-modules, which was known by example 7 and corollary 2 and also by corrollaries 1 and 2, provided $V_{\varphi-5,\varphi-1,\varphi+2} = V_{\varphi+3,\varphi-9,\varphi+2} \otimes S_{-8,8}$.

Lemma 17. *Let $M \in \hat{\mathcal{C}}$ be a Verma module, then there is a Verma module $\tilde{M} \in \tilde{\mathcal{C}}$ such that $M \cong Forg(\tilde{M})$.*

Proof. As M is a Verma module, it is a highest weight module; let $v_\sigma \in M \in \hat{\mathcal{C}}$ be a maximal weight vector of weight σ, then following the same construction as in lemma 16, define $\tilde{M} \in \tilde{\mathcal{C}}$. As M is a Verma module, we have

$$K_\gamma v_\sigma = \prod_{i=1}^{n} q^{d_i s_i H_i} v_\sigma$$

for $\gamma \in R$.

$$A_i v_\sigma = \sigma(A_i) v_\sigma$$

$$X_i v_\sigma = 0$$

for $A_i \in Ann_{\mathfrak{h}}(I)$ and the action of the X_{-i} is free for $i = 1, 2...n$.

By construction, $\tilde{M}$ has the same properties and action of $Ann_{\mathfrak{h}}(I)$, and the additional relations

$$H_i v_\sigma = \sigma(H_i) v_\sigma$$

which implies that $\tilde{M}$ is a Verma module and $M \cong Forg(\tilde{M})$.

□

Lemma 18. *Let $M_\lambda \in \tilde{\mathcal{C}}$ be a Verma module. Then $Forg(M_\lambda) \in \hat{\mathcal{C}}$ is a Verma module.*

Proof. As $Forg$ only forgets the action of $Ann_{\mathfrak{h}}(I)^\perp$, the actions of $X_{\pm i}$, K_γ and $Ann_{\mathfrak{h}}(I)$ are the same (free on X_{-i}), which is exactly the definition of a Verma module. Hence, $Forg(M_\lambda)$ is a Verma module.

□

Notice that $Forg$ is exact and therefore $Forg(M/N) \cong Forg(M)/Forg(N)$. To see this, let us consider an exact sequence

$$0 \to N \to M \to Q \to 0$$

as $Forg$ only forgets the action of $Ann_{\mathfrak{h}}(I)^\perp$, kernels and images remain untouched, which implies that

$$0 \to Forg(N) \to Forg(M) \to Forg(Q) \to 0$$

is exact. Hence $Forg(M/N) \cong Forg(M)/Forg(N)$.

In particular, all this together with lemma 18, implies that $Forg$ sends standard filtrations to standard filtrations.

Additionally, proposition 4.12 in [20] guarantees that BGG reciprocity holds in $\mathcal{C}$, which is fundamental in the proof of the following lemma.

Lemma 19. *Let $\tilde{L}_\lambda \in \tilde{\mathcal{C}}$ be a lift of the simple module $L_\lambda \in \hat{\mathcal{C}}$, then $Forg(\tilde{P}_\lambda)$ is isomorphic to the projective cover of L_λ, where $\tilde{P}_\lambda$ is the projective cover of $\tilde{L}_\lambda$.*

Proof. Let $M_\mu \in \hat{\mathcal{C}}$ and denote by $\tilde{M}_\mu$ one of its lifts, then by Corollaries 2 and 1 we have that any other lift of M_μ is of the form $\tilde{M}_\mu \otimes \mathbb{C}_\nu = \tilde{M}_{\mu+\nu}$. Let $S = \{\nu \in \mathfrak{h} \mid (\tilde{P}_\lambda : \tilde{M}_{\mu+\nu}) := s_\nu \neq 0 \text{ and } Forg(\tilde{M}_{\mu+\nu}) = M_\mu\}$ and $T = \{\nu \in \mathfrak{h} \mid [\tilde{M}_\mu : \tilde{L}_{\lambda-\nu}] := t_\nu \neq 0 \text{ and } Forg(\tilde{L}_{\lambda-\nu}) = L_\lambda\}$.

By Corollaries 2 and 1, and BGG reciprocity, we have the equation

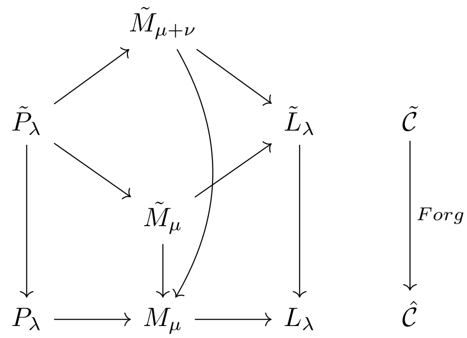
$$(\tilde{P}_\lambda : \tilde{M}_{\mu+\nu}) = [M_{\mu+\nu} : L_\lambda] = [M_\mu : L_{\lambda-\nu}]$$

that is, $s_\nu = t_\nu$ and therefore $S = T$.

We then have that

$$(P_\lambda : M_\mu) = \sum_{\nu \in S} (\tilde{P}_\lambda : \tilde{M}_{\mu+\nu}) = \sum_{\nu \in T} [\tilde{M}_\mu : \tilde{L}_{\lambda-\nu}] = [M_\mu : L_\lambda]$$

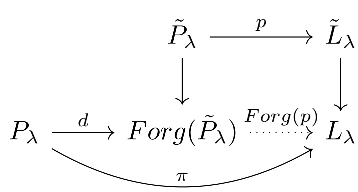
where the first equation holds since by lemma 17, there is a lift of any subquotient isomorphic to M_μ which is a Verma module and by corollaries 1 and 2 we have that it must be of the form $\tilde{M}_{\mu+\nu}$ and therefore $(P_\lambda : M_\mu) \leq \sum_{\nu \in S} (\tilde{P}_\lambda : \tilde{M}_{\mu+\nu})$; now, as every Z-module is a Z_1-module and $Forg(\tilde{M}_{\mu+\nu}) \cong M_\mu$, we must have that any filtration of $\tilde{P}_\lambda$ descends to a filtration of P_λ and clearly any appearance of $\tilde{M}_{\mu+\nu}$ as a subquotient of $\tilde{P}_\lambda$ translates to an appearance of M_μ as a subquotient of P_λ and therefore $\sum_{\nu \in S} (\tilde{P}_\lambda : \tilde{M}_{\mu+\nu}) \leq (P_\lambda : M_\mu)$.



The last equation holds by a similar argument, using lemma 16 and the fact that every simple module in $\mathcal{O}$ is a highest weight module (Theorem 1.3 in [17]) and the second equation follows from BGG reciprocity.

Now, let P_λ be the projective cover of L_λ. Let us see that $P_\lambda \cong Forg(\tilde{P}_\lambda)$. Notice that we have the surjective morphisms $Forg(p) : Forg(\tilde{P}_\lambda) \to L_\lambda$ and $\pi : P_\lambda \to L_\lambda$; as P_λ is projective, then there is $d : P_\lambda \to Forg(\tilde{P}_\lambda)$ such that $Forg(p) \circ d = \pi$. Notice that $Forg(p)$ is essential since otherwise there would be $N \subset Forg(M)$ surjecting onto L_λ, but N must be of the form $Forg(\tilde{N})$ and therefore p would not be essential, provided that $\tilde{N}$ would surject onto $\tilde{L}_\lambda$. Then we must have that d is surjective since otherwise $im(d)$ would be surjected onto L_λ, contradicting the fact of $Forg(p)$ being essential.

We have the diagram



We conclude that d is an isomorphism since P_λ and $Forg(\tilde{P}_\lambda)$ have the same composition factors.

□

Corollary 3. *Let $M \in \tilde{\mathcal{C}}$ and $\pi_M : P_M \to M$ be the projective cover of M, then $Forg(P_M)$ is projective in $\hat{\mathcal{C}}$.*

Proof. In $\mathcal{O}$, any projective module is the direct sum of copies of various $\tilde{P}_\lambda$, that is $P_M = \bigoplus \tilde{P}_\lambda$ (Theorem 3.9 of [17]). Using each L_λ in lemma 19 and clearly having $Forg(\bigoplus \tilde{P}_\lambda) = \bigoplus Forg(\tilde{P}_\lambda)$, then we have $Forg(P_M)$ is projective since each $Forg(\tilde{P}_\lambda)$ is projective.

□

Corollary 4. *Let $P \in \tilde{\mathcal{C}}$ be projective, then $Forg(P)$ is projective.*

Proof. As P is its own projective cover, it is projective by corollary 3.

□

Lemma 20. *Let $\pi : P_M \to M$ be the projective cover of $M \in \tilde{\mathcal{C}}$, then $Forg(P_M)$ is the projective cover of $Forg(M)$.*

Proof. By lemma 19, we have that the statement is true for any simple module.

Suppose that $Forg(P_M)$ is not the projective cover of $Forg(M)$; then let R_M be it.

Now, let M be any module and consider the short exact sequence $0 \to A \to M \to B \to 0$, with A, B simple, then we have the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Forg(P_A) & \longrightarrow & R_A & \longrightarrow & Forg(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Forg(P_M) & \longrightarrow & R_M & \longrightarrow & Forg(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Forg(P_B) & \longrightarrow & R_B & \longrightarrow & Forg(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $R_A = Forg(P_A) \cap R_M$ and $R_B = Forg(P_B) \cap R_M$. By the five lemma, we must have $R_M \cong Forg(P_M)$ since $R_A \cong Forg(P_A)$ and $R_B \cong Forg(P_B)$ (since A, B are simple).

Let us argue inductively on the length of M; suppose that the statement is true for any length up to n and let M have length $n + 1$, then if B is simple, A has length n and therefore $Forg(P_A)$ is the projective cover of $Forg(A)$ by induction hypothesis, and by the previous diagram and the five lemma, we have that $R_M \cong Forg(P_M)$ since $R_A \cong Forg(P_A)$ and $R_B \cong Forg(P_B)$.

□

Lemma 21. *Let $S \in \hat{\mathcal{C}}$ be simple, then there is a simple $\tilde{S} \in \tilde{\mathcal{C}}$ such that $Forg(\tilde{S}) \cong S$.*

Proof. As S is simple, it has a highest weight λ (since every simple module is a subquotient of a Verma module) and a morphism $M_\lambda \to S$. By lemma 17 there are a Verma module $\tilde{M}_\lambda \in \tilde{\mathcal{C}}$ and a morphism $\tilde{M}_\lambda \to \tilde{L}_\lambda$ (Proposition 4.1 in [17]); as both $\tilde{L}_\lambda$ and S have highest weight λ, the following diagram commutes

$$\begin{array}{ccc} \tilde{M}_\lambda & \dashrightarrow & \tilde{L}_\lambda \\ \downarrow & & \downarrow \\ M_\lambda & \longrightarrow & S \end{array}$$

we have that $\tilde{L}_\lambda$ is simple, then let $\tilde{S} = \tilde{L}_\lambda$.

□

Corollary 5. *Let $P \in \hat{\mathcal{C}}$ be projective, then there is a projective $\tilde{P} \in \tilde{\mathcal{C}}$ such that $Forg(\tilde{P}) \cong P$.*

Proof. We know that $P = \bigoplus P_\lambda$ (Theorem 3.9 of [17]) and that for each λ there are a simple L_λ and a morphism $P_\lambda \to L_\lambda$. By lemma 21 there is a lift $\tilde{L}_\lambda$ of L_λ and by lemma 19, we have $Forg(\tilde{P}_\lambda) = P_\lambda$ (where $\tilde{P}_\lambda$ is the projective cover of $\tilde{L}_\lambda$).

$$\begin{array}{ccc} \tilde{P}_\lambda & \dashrightarrow & \tilde{L}_\lambda \\ \downarrow & & \downarrow \\ P_\lambda & \longrightarrow & L_\lambda \end{array}$$

Let $\tilde{P} = \bigoplus \tilde{P}_\lambda$. Clearly $Forg(\tilde{P}) = Forg(\bigoplus \tilde{P}_\lambda) = \bigoplus Forg(\tilde{P}_\lambda) \cong P$.

□

Lemma 22. $\underline{Hom}(\mathbf{1}, \mathbf{1}) = A$

Proof. Let $X \in \tilde{\mathcal{C}}$. We have that $\hat{\mathcal{C}}$ is a $\tilde{\mathcal{C}}$-module category via $X \otimes M = Forg(X) \otimes M$, then

$$Hom_{\hat{\mathcal{C}}}(X, \mathbf{1}) \cong Hom_{\tilde{\mathcal{C}}}(X, \underline{Hom}(\mathbf{1}, \mathbf{1}))$$

and

$$Hom_{\hat{\mathcal{C}}}(X, \mathbf{1}) \cong \begin{cases} \mathbb{C} & \text{if } X = \mathbb{C}_\lambda \text{ for some } \lambda \in I \\ 0 & \textit{otherwise} \end{cases}$$

We then have that $A \subset \underline{Hom}(\mathbf{1}, \mathbf{1})$ since for any $\mathbb{C}_\lambda$, $Hom_{\tilde{\mathcal{C}}}(\mathbb{C}_\lambda, \underline{Hom}(\mathbf{1}, \mathbf{1})) = \mathbb{C}$.

Now, suppose that there is an $m \in \underline{Hom}(\mathbf{1}, \mathbf{1})$ not in A, and consider N, a direct summand of $\langle m \rangle$ not contained in A, then if P_N is its projective cover, we have

$$Hom_{\hat{\mathcal{C}}}(Forg(P_N), \mathbf{1}) \cong Hom_{\tilde{\mathcal{C}}}(P_N, \underline{Hom}(\mathbf{1}, \mathbf{1}))$$

but

$$Hom_{\hat{\mathcal{C}}}(Forg(P_N), \mathbf{1}) \cong \begin{cases} \mathbb{C} & \text{if } Forg(P_N) = \mathbb{C}_\lambda \text{ for some } \lambda \in I \\ 0 & otherwise \end{cases}$$

Hence, $\underline{Hom}(\mathbf{1}, \mathbf{1}) = A$.

□

Proposition 10. *We have a bijection* $Hom_{\tilde{\mathcal{C}}}(X, Y \otimes A) \cong Hom_{\hat{\mathcal{C}}}(Forg(X), Forg(Y))$

Proof. We have that $\hat{\mathcal{C}}$ is a $\tilde{\mathcal{C}}$-module category. Let $X, Y \in \tilde{\mathcal{C}}$, then

$$Hom_{\hat{\mathcal{C}}}(X \otimes \mathbf{1}, Y) \cong Hom_{\tilde{\mathcal{C}}}(X, \underline{Hom}(\mathbf{1}, Y))$$

By lemma 7.9.4 in [43], we have that

$$\underline{Hom}(\mathbf{1}, Y) \cong \underline{Hom}(\mathbf{1}, Y \otimes \mathbf{1}) \cong Y \otimes \underline{Hom}(\mathbf{1}, \mathbf{1})$$

By lemma 22

$$Hom_{\hat{\mathcal{C}}}(X, Y) \cong Hom_{\tilde{\mathcal{C}}}(X, Y \otimes A)$$

□

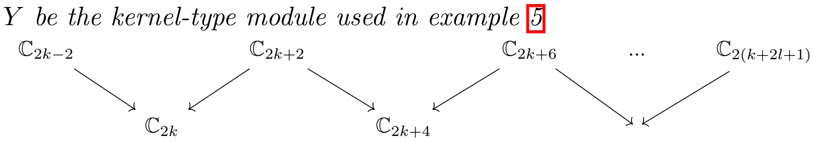
Example 9. *Let us revisit example 5 in the case* $p = 2$.

Let

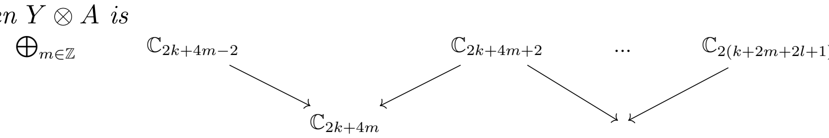
$$A = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}_{4m}$$

then $\theta_{\mathbb{C}_{4m}} = Id_{\mathbb{C}_{4m}}$ *and* A *is a commutative algebra.*

Let Y *be the kernel-type module used in example 5*



then $Y \otimes A$ *is*



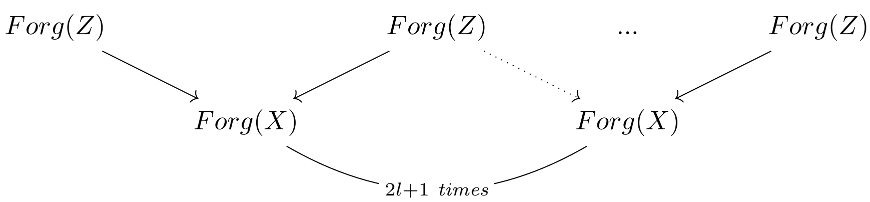
By corollary 2, we have that

$$Forg(Z) := Forg(\mathbb{C}_{2k-2}) = Forg(\mathbb{C}_{2k+2}) = Forg(\mathbb{C}_{2k+6}) = ... = Forg(\mathbb{C}_{2(k+2l+1)})$$

and

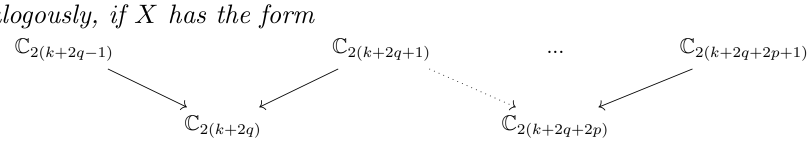
$$Forg(\mathbb{C}_{2k}) = Forg(\mathbb{C}_{2k+4}) = ... = Forg(\mathbb{C}_{2(k+2l)})$$

If $X = \mathbb{C}_{2(k+2q)}$*, then* $Hom(Forg(X), Forg(Y)) \cong \mathbb{C}^{2l+1}$ *since* $Forg(Y)$ *has the form*



and X *appears in exactly* $2l+1$ *direct summands of* $Y \otimes A$*, which implies that* $Hom(X, Y \otimes A) \cong \mathbb{C}^{2l+1}$.

Analogously, if X *has the form*



with $p \leq l$*. Then* $Hom(Forg(X), Forg(Y)) \cong \mathbb{C}^{2l-2p+1} \cong Hom(X, Y \otimes A)$.

Corollary 6. *We have a bijection* $Hom_{\hat{\mathcal{C}}}(Forg(X), Forg(Y)) \cong Hom_{Rep^{loc}(A)}(\mathcal{F}(X), \mathcal{F}(Y))$

Proof. Recall that we have Frobenius reciprocity, namely

$$Hom_{\hat{\mathcal{C}}}(X, Y \otimes A) \cong Hom_{Rep^{loc}(A)}(\mathcal{F}(X), \mathcal{F}(Y))$$

The result follows from proposition 10.

□

Conjecture 5. *There is an equivalence of categories*

$$\hat{\mathcal{C}} \cong Rep^{loc}(A)$$

Conjecture 6. *Let* $L = \langle L_1, ...L_m \rangle$ *and* $\mu_1, ...\mu_m \in \mathbb{C}$*, then if*

$$\hat{Z}_{\mu_1,\mu_2,...\mu_m} = Z_1 / \langle L_1 - \mu_1, ...L_m - \mu_m \rangle$$

and $\mathcal{C}(\mu_1, ...\mu_m)$ *is the category with objects*

$$Obj(\mathcal{C}(\mu_1, ...\mu_m)) = \{(V, \mu_V) \in Rep(A) : M_{V,S_1} = \mu_1 Id_{V\otimes S_1}, ...M_{V,S_m} = \mu_m Id_{V\otimes S_m}\}$$

For $S_i = \mathbb{C}_{\lambda_i}$ *and* I *being generated by the* λ_i.

Then, we have an equivalence of categories

$$\hat{Z}_{\mu_1,\mu_2,...\mu_m} - mod \cong \mathcal{C}(\mu_1, ...\mu_m)$$

References

[1] David Tong. Lecture notes on string theory, 2012.

[2] A. Sen. An Introduction to rational conformal field theories. *Comments Nucl. Part. Phys.*, 20(1+2):23–67, 1991.

[3] Roger E. Behrend, Paul A. Pearce, Valentina B. Petkova, and Jean-Bernard Zuber. Boundary conditions in rational conformal field theories. *Nuclear Physics B*, 570(3):525–589, mar 2000.

[4] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Communications in Mathematical Physics-Springer Verlag*, (123):177–254, 1989.

[5] Luis Alvarez-Gaumé, G. Sierra, and C. Gómez. Topics in conformal field theory. *Physics and Mathematics of Strings*, pages 16–184, 1989.

[6] Thomas Creutzig, Tudor Dimofte, Niklas Garner, and Nathan Geer. A qft for non-semisimple tqft. *arXiv:2112.01559*, 2021.

[7] Edward Witten. Quantum field theory and the Jones polynomial. *Communications in Mathematical Physics*, 121(3):351 – 399, 1989.

[8] J. Fröhlich and C. King. The Chern-Simons theory and knot polynomials. *Communications in Mathematical Physics*, 126(1):167 – 199, 1989.

[9] Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. *Journal of Topology*, 7:1005–1053, 2015.

[10] N. Reshetikhin and V. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones Mathematicae-Springer Verlag*, 103:547–598, 1991.

[11] Jenny August. Lecture notes of the working seminar at the hodge institute, topological field theories, chapter 3, 2016.

[12] Thomas Creutzig, Azat M. Gainutdinov, and Ingo Runkel. A quasi-hopf algebra for the triplet vertex operator algebra. *Communications in Contemporary Mathematics*, 22(03):1950024, apr 2019.

[13] Thomas Creutzig, Simon Lentner, and Matthew Rupert. Characterizing braided tensor categories associated to logarithmic vertex operator algebras. *arXiv:2104.13262*, 2021.

[14] Terry Gannon and Cris Negron. Quantum sl(2) and logarithmic vertex operator algebras at (p,1)-central charge. *arXiv:2104.12821*, 2021.

[15] Thomas Creutzig and Matthew Rupert. Uprolling unrolled quantum groups. *Communications in Contemporary Mathematics*, 24:04, 2150023, 2020.

[16] Saunders Mac Lane. *Categories for the working mathematician.* Springer-Verlag, 6 edition, 1971.

[17] J. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category O.* Graduate Studies in Mathematics. American Mathematical Society, 2008.

[18] Charles W. Curtis and Irving Reiner. *Methods of representation theory. With applications to finite groups and orders*, volume Vol.1 of *Pure and Applied Mathematics*. John Wiley and Sons, 1981.

[19] Vaughan F. R. Jones. In and around the origin of quantum groups. 437:101–126, 2007.

[20] Matthew Rupert. Categories of weight modules for unrolled restricted quantum groups at roots of unity. *arXiv:1910.05922*, 2020.

[21] J. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Graduate Texts in Mathematics. Springer New York, 1994.

[22] Nathan Geer and Bertrand Patureau-Mirand. The trace on projective representations of quantum groups. *Letters in Mathematical Physics*, 108(1):117–140, Oct 2017.

[23] Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Some remarks on the unrolled quantum group of sl(2). *Journal of Pure and Applied Algebra*, 219(8):3238–3262, Aug 2015.

[24] Matthew Rupert. Logarithmic hopf link invariants for the unrolled restricted quantum group of sl(2). *Master Thesis-Univeristy of Alberta*, 2016.

[25] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves.* Mathematical surveys and monographs 88. American Mathematical Society, 2001.

[26] Thomas Creutzig, Shashank Kanade, and Robert McRae. Gluing vertex algebras. *Advances in Mathematics*, 396:108174, feb 2022.

[27] Christoph A. Keller. Lecture notes. introduction to vertex operator algebras, 2017.

[28] Thomas Creutzig, Robert McRae, and Jinwei Yang. Ribbon tensor structure on the full representation categories of the singlet vertex algebras. *arXiv:2202.05496*, 2022.

[29] Thomas Creutzig, David Ridout, and Simon Wood. Coset constructions of logarithmic (1, p) models. *Letters in Mathematical Physics*, 104(5):553–583.

[30] Thomas Creutzig, Robert McRae, and Jinwei Yang. On ribbon categories for singlet vertex algebras. *Communications in Mathematical Physics*, 387(2):865–925, may 2021.

[31] Akihiro Tsuchiya and Simon Wood. On the Extended W-algebra of type 2 at positive rational level. *International Mathematics Research Notices*, 2015(14):5357–5435, jun 2014.

[32] Thomas Creutzig and Jinwei Yang. Tensor categories of affine lie algebras beyond admissible levels. *Mathematische Annalen*, 380:1991–2040, feb 2021.

[33] Thomas Creutzig, Antun Milas, and Matt Rupert. Logarithmic link invariants of $\overline{U}_q^H(\mathfrak{sl}_2)$ and asymptotic dimensions of singlet vertex algebras. *J. Pure Appl. Algebra*, 222(10):3224–3247, 2016.

[34] Yi-Zhi Huang, James Lepowsky, and Lin Zhang. *Logarithmic Tensor Category Theory for Generalized Modules for a Conformal Vertex Algebra, I: Introduction and Strongly Graded Algebras and Their Generalized Modules*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[35] Jean Auger, Thomas Creutzig, Shashank Kanade, and Matthew Rupert. Braided tensor categories related to B_p vertex algebras. *Communications in Mathematical Physics*, 378(1):219–260, Apr 2020.

[36] Jean Auger and Matt Rupert. On infinite order simple current extensions of vertex operator algebras. 711:143–168, [2018] ©2018.

[37] Thomas Creutzig. W-algebras for argyres-douglas theories. *European Journal of Mathematics*, 3:659–690, 2017.

[38] Drazen Adamovic, Thomas Creutzig, Naoki Genra, and Jinwei Yang. The vertex algebras $R^{(p)}$ and $V^{(p)}$. *Commun. Math. Phys.*, 383:1207–1241, 2020.

[39] Thomas Creutzig, Cuipo Jiang, Florencia Orosz Hunziker, David Ridout, and Jinwei Yang. Tensor categories arising from the virasoro algebra. *Advances in Mathematics*, 380:107601, 2021.

[40] Thomas Creutzig, Shashank Kanade, and Andrew R. Linshaw. Simple current extensions beyond semi-simplicity. *Communications in Contemporary Mathematics*, 22(01):1950001, Feb 2019.

[41] Bojko Bakalov and Alexander Kirillov. *Lectures on Tensor Categories and Modular Functors*. American Mathematical Society AMS, 2001.

[42] Thomas Creutzig, Shashank Kanade, and Robert McRae. Tensor categories for vertex operator superalgebra extensions. *arXiv:1705.05017*, 2021.

[43] Pavel Etingof et al. *Tensor categories*. American Mathematical Society, 2015.