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WEAK CONVERGENCE OF EMPIRICAL PROCESSES FOR NON-STATIONARY STRONG MIXING SEQUENCES AND THEIR APPLICATIONS TO ASYMPTOTIC NORMALITY AND EFFICIENCY OF LINEAR RANK STATISTICS

by

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Dedicated to the memories of

my father MADIRAJU VENKATRAM RAO

and

my father-in-law JAMALAPURAM VENKAT RAO

ABSTRACT

In this thesis we are concerned with the weak convergence of generalized empirical processes for non-stationary strong mixing (s.m.) sequences of random variables (rv's) and their application to the asymptotic normality of simple linear rank statistics introduced by Hájek [*Ann. Math. Statist.* (1968) *39*, 325-346].

Let $\{Y_{iN}\}$ be a sequence of s.m. rv's and $\{C_{iN}\}$ an arbitrary (non-null) sequence of constants. Using the techniques developed by Billingsley [*Convergence of Probability Measures*, John Wiley (1968)] we prove, under certain regularity conditions, the weak convergence of $\{V_N(t)/q(t)$, $0 \leq t \leq 1$, $N \geq 1\}$, where $V_N(t) = N^{1/2}[H_N(t) - EH_N(t)]$, $H_N(t) = N^{-1} \sum C_{iN} I[F(Y_{iN}) \leq t]$, $F(x) = N^{-1} \sum P(Y_{iN} \leq x)$ and q is a real valued function defined on $[0,1]$. For stationary ϕ-mixing sequences this was proved by Fears and Mehra [*Ann. Statist.* (1974) *2*, 586-596] in the special case $C_{iN} \equiv 1$ and $q(t) = K[t(1-t)]^{\frac{1}{2}-\delta}$ and, has been extended by Mehra and Rao [*Abs. No.* 73t-69 *IMS Bull.* *2* (1973), and *Abs. No.* 144-3 *IMS Bull.* *3* (1974)] to stationary strong mixing sequences with more general q functions.

Utilising the weak convergence results for non-stationary strong mixing sequences proved in this thesis, the asymptotic normality of a simple linear rank statistic $T_N = \sum C_{iN} \psi_N(\frac{R_{iN}}{N+1})$ is established for general alternatives, where ψ_N denotes a scores-generating function and R_{iN} = rank (Y_{iN}), $1 \leq i \leq N$. The conditions imposed on ψ_N are sufficiently general in that they are satisfied by the Normal scores, Wilcoxon,

and Median test statistics. These results cover both t[illegible]nd c-sample problems and are based on a new approach to Chernoff-Savage theorems, as developed by Pyke and Shorack [*Ann. Math. Statist.* (196[illegible]) 39, 755–771]. The results of Pyke and Shorack were for the two sample problem in the case of independent rv's and were extended subsequently by [illegible] and Mehra. The one sample problem has been recently studied by Mehra [*Canad. Math. Bull.* (to appear)] and Sen and Ghosh [*Sankhyā* (1973) *Ser. A 35*, 153–172] for stationary ϕ-mixing sequences.

Finally, we consider the regression alternatives $Y_{iN} = d_{iN}\beta + X_{iN}$, where $\{X_{iN}\}$ is a stationary strong mixing (ϕ-mixing) sequence with $\max_{i\leq N} d_{iN} \to 0$ as $N \to \infty$. Under certain regularity conditions on $\{X_{iN}\}$, we derive the asymptotic relative efficiency of T_N relative to the classical t-test for testing $H_0 : \beta = 0$ against $H_1 : \beta > 0$. The regularity conditions assumed on $\{X_{iN}\}$ are shown to be satisfied for stationary strong mixing Gaussian sequences. For this case, the efficiency expressions are explicitly obtained.

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CHAPTER I

INTRODUCTION AND SUMMARY

I.1 INTRODUCTION. The object of this work is to study the asymptotic normality and efficiency of linear rank statistics using weak convergence methods, when the underlying random variables (rv's) form a non-stationary strong mixing sequence. The study of non-parametric methods under mixing and other types of dependence has gained considerable attention recently. The papers by Serfling (1968), Hoyland (1968), Gastwirth and Rubin (1969), Fears and Mehra (1974), Sen and Ghosh (1973), and Mehra and Rao (1973) should be mentioned in this connection.

As observed by Pyke (1970), much of the present research in rank theory is concerned with asymptotic results, since the exact non-null distributions of test statistics are usually intractable. In the case of independent rv's, the asymptotic normality of a general class of rank statistics for the two-sample problem was established by Chernoff and Savage (1958). Their results apply to some important test statistics, such as the Normal scores and the Wilcoxon, but do not cover statistics whose scores-generating functions either are discontinuous, or do not satisfy their stringent differentiability conditions. The Median test and the Ansari-Bradley test are examples of the latter type. The smoothness conditions imposed by Chernoff and Savage on the scores generating function are greatly relaxed by Govindarajulu et al. (1967), who made use of some deeper properties of empirical processes. Using similar properties of empirical processes. Pyke and Shorack (1968) gave a new approach to Chernoff-Savage theorems, and further relaxed the regularity conditions so as to include discontinuous scores. They employed a different representation of test statistics, and in conjunction with the Skorohod construction, proved sufficiently general Chernoff-Savage theorems for the two and c-sample problems. Hájek (1968) studied the problem of linear rank tests under complete non-stationarity of underlying rv's, and proved results which are applicable to various other situations besides the two and c-sample problems. His methods are different from those referred to above, and are based on certain variance inequalities and a projection technique. Hájek's results were further extended by Hájek and Dupac (1969).

The methods adopted in this thesis are based on the Pyke-Shorack approach. This approach has recently been used by Fears and Mehra (1974) to study the two-sample problem in the case of ϕ-mixing rv's. Referring to Pyke and Shorack's method for the case of independent but non-stationary rv's, Hájek (1970) remarked "... at present it does not extend to cases when each observation may have a different distribution, ...; there is no sign, however, that by further development the method could not be extended to such cases, too". We believe, our contribution is a step towards the solution of such cases, where the underlying rv's are not identically distributed, and also of the cases where they are dependent. The class of rank tests considered in this work include, among others, the two and c-sample tests, and tests for the regression parameter. The one-sample problem is not touched upon here; we refer to Mehra (1973) and Sen and Ghosh (1973), who have studied it in the stationary ϕ-mixing case.

In the Pyke-Shorack approach, rank statistics are represented by certain functionals of appropriately defined empirical processes. The asymptotic normality is established by approximating a given rank statistic by a functional on the limiting process of the corresponding empirical process. The weak convergence of such a process relative to a stronger metric d_q (see Section I.3), in conjunction with the Skorohod construction, is used to justify this approximating procedure. Consequently, the weak convergence of certain empirical processes in stronger metrics is crucial to this approach. For the one and two-sample problems, the appropriate process turns out to be the usual empirical process based on stationary rv's. The weak convergence of such processes, relative to stronger metrics d_q, was studied in the case of independent and identically distributed rv's, by Cibisov (1965), and Pyke and Shorack (1968). For stationary ϕ-mixing processes, similar results were proved by Fears and Mehra (1974), and Mehra and Rao (1973). In the case of stationary strong mixing rv's, the weak convergence of empirical processes relative to the usual metric d, has been independently proved by Deo (1973), Yokoyama (1973), and Mehra and Rao (1973). Recently, Mehra and Rao (1974) established, under stationarity and strong mixing, the weak convergence of more general empirical processes relative to stronger metrics d_q.

The regression and other problems involve non-stationary rv's and for such problems, an appropriate process is, what we call, a generalized empirical process (see Section I.3 below). The weak convergence, relative to d, of such processes, in the case of independent rv's, was considered by Koul (1970). With a view to adopting the Pyke-Shorack approach to the case of non-stationary strong mixing rv's, we have studied the weak convergence of generalized empirical processes, relative to stronger metrics d_q. These results are then applied to the study of asymptotic normality and efficiency of linear rank tests, whose scores-generating functions could be unbounded with at most a finite number of discontinuities.

It may be mentioned that the weak convergence results proved in this thesis can also be used for studying the asymptotic normality of linear combinations of order statistics under strong mixing. This can be seen from the corresponding work, in the case of independent rv's, by Bickel (1973), and Shorack [(1972), (1973)], and in the stationary ϕ-mixing case by Mehra and Rao (1973). However, we do not pursue this matter here.

1.2 A BRIEF SUMMARY OF THE RESULTS. Let $\{Y_{iN}\}$ be a sequence of non-stationary strong mixing rv's and $\{C_{iN}\}$, an arbitrary sequence of constants. Let R_{iN} denote the rank of Y_{iN} in the combined ranking (ascending order of magnitude) of $Y_{1N},\ldots,Y_{NN}$. Let α and ϕ denote mixing coefficients.

In chapter II, we study weak convergence relative to the d-metric. Mixing inequalities are discussed in Section 2.1 and in Section 2.2, a central limit theorem for $\{C_{iN}\ \xi_{iN}\}$, where $\{\xi_{iN}\}$ is a sequence of uniformly bounded, non-stationary strong mixing rv's, is proved. In Section 2.3, this result is utilised in the proof of the weak convergence, relative to d, of a generalized empirical process defined by

$$V_N(t) = N^{1/2}[H_N(t) - EH_N(t)] \quad , \quad 0 \le t \le 1 \; ,$$

where $H_N(t) = N^{-1} \sum C_{iN} I(F(Y_{iN}) \le t)$, and $F(x) = N^{-1} \sum P(Y_{iN} \le x)$.

The above results are derived under certain regularity conditions on C's , α and $\{Y_{iN}\}$.

Chapter III is concerned with the weak convergence of processes $\{V_N(t)/q(t) \; , \; 0 \le t \le 1\}$, relative to the metric d . The class of functions q , for which the results hold, include those functions having unbounded growth on $[0,1]$ (see Section 3.1).

In Chapter IV, we use the Pyke-Shorack approach to prove the asymptotic normality of a class of linear rank statistics represented by $T_N = N^{-1} \sum C_{iN} \; \psi_N \left(\frac{R_{iN}}{N+1}\right)$, where $\psi_N \left(\frac{i}{N+1}\right)$ for $1 \le i \le N$ denote certain scores.

Chapter V is devoted to the study of asymptotic relative efficiency. The first two sections are concerned with a Chernoff-Savage theorem and its application to regression alternatives. In the remainder of this chapter, the relative efficiencies (with respect to t-test) of the Normal scores, the Wilcoxon, and the Median tests are derived when the sequences Y_{iN} is Gaussian. Some general remarks are also included.

1.3 NOTATION AND TERMINOLOGY. Most of the notation is introduced in the text as and when necessary. We have tried to use standard notation whenever there is one. The following items should be noted, as they appear more frequently in the text.

(i) The general terminology on rank tests for such as the two-sample problem, the regression problem, the Normal scores, etc., is adopted from Hájek and Sidák (1967).

(ii) The function spaces $C[0,1]$ with the uniform metric ρ, and $D[0,1]$ with the Skorohod metric d, are as defined in Billingsley [(1968); 54, 109]. We define $\rho_q(x,y) = \rho(x/q, y/q)$ for elements x/q, y/q in $C[0,1]$; $d_q(x,y)$ is similarly defined. The standard notation $\Rightarrow_s$ is used to indicate the weak convergence relative to a metric s.

(iii) K_1, K_2 etc., denote absolute positive constants, not necessarily representing the same value in each appearance. Also, $K(\alpha)$, $K(\alpha,\delta)$ etc., represent generic constants, depending only on their arguments.

(iv) The inverse of a function is defined to be left continuous:

$$G^{-1}(t) = \inf \{x : G(x) \geq t\} .$$

(v) The sequence $\{C_{iN}\}$ represents a non-null sequence of regression constants [Hájek and Sidák (1967), 100], and is used only in this sense throughout. We also use

$$\Delta(N) = \max_{i \leq N} |C_{iN}|$$

$$\tau(x) = \sup_{N \geq 1} (N^{-1} \sum |C_{iN}|^{1/x})^x \quad \text{for } x \neq 0 .$$

(vi) The symbol $\uparrow$ is used as an abreviation to 'non-decreasing'.

I(A) denotes the indicator function of set A, and $\square$ signals the end of a proof. The following abreviations are also used:

$$FH \equiv F \circ H \qquad \text{(composition)}$$

$$\sum \equiv \sum_{1}^{N} \quad \text{unless, otherwise specified.}$$

CHAPTER II

WEAK CONVERGENCE RELATIVE TO THE SKOROHOD METRIC d

This chapter deals with the weak convergence of generalized empirical processes (see (2.3.2) below), relative to the Skorohod metric d , when the underlying rv's are strong mixing. The main results are proved in Section 2.3.

2.1 MIXING INEQUALITIES. In this section, a brief summary of mixing conditions and corresponding inequalities are given. These will be referred to in the subsequent chapters.

Let $\{Y_{iN}\}$ be a double sequence of random variables, defined on a probability space $(\Omega,\mathcal{A},P)$. Let $\mathcal{B}(Y)$ denote the sub σ-algebra generated by a random function Y, defined on Ω. We shall drop N from the double subscript throughout below for notational convenience. The sequence $\{Y_i\}$ is said to be ϕ-*mixing* if there exists a sequence of non-negative, monotone non-increasing functions $\{\phi_N\}$, defined on $\{0,1,2,\ldots\}$, for which

$$\phi_N(m) \to 0 \quad \text{as} \quad m \to \infty \quad \text{uniformly in } N ,$$

and

$$|P(A|B) - P(A)| \leq \phi_N(m) , \; m \geq 1 \tag{2.1.1}$$

for all $A \in \mathcal{B}(Y_1,\ldots,Y_k)$, $B \in \mathcal{B}(Y_{k+m},\ldots)$. We set $\phi_N(0) = 1$, and

$|P(A|B) - P(A)| = 0$ for $P(B) = 0$. Similarly, the sequence $\{Y_i\}$ is said to be *strong mixing* if there exists $\{\alpha_N\}$ for which

$$|P(A\cap B) - P(A)P(B)| \leq \alpha_N(m) \ , \ m \geq 1 \ , \tag{2.1.2}$$

and satisfying all other properties of ϕ_N as defined above. Clearly, ϕ-mixing implies strong mixing.

The concept of ϕ-mixing was introduced by Ibragimov [17], and that of strong mixing by Rosenblatt [30]. There are several other types of mixing conditions in the literature (see for example [25], section 2), but they are more restrictive than the two types described above.

By defining $\phi(k) = \sup_N \{\phi_N(k)\}$ and $\alpha(k) = \sup_N\{\alpha_N(k)\}$, it is easy to see that the functions ϕ_N and α_N of (2.1.1) and (2.1.2) can be replaced by ϕ and α respectively. Throughout below, the mixing coefficients ϕ and α are used in this sense without further mention. The following mixing inequalities will be needed for our work in later sections:

Suppose $\xi \in \mathcal{B}(Y_i, i\leq k)$, $\eta \in \mathcal{B}(Y_i, i\geq k+m)$, $E|\xi|^a < \infty$, and $E|\eta|^b < \infty$ where m is an integer (≥ 0). Then,

$$|\mathrm{cov}(\xi,\eta)| \leq 2(E|\xi|^a)^{1/a}(E|\eta|^b)^{1/b}[\phi(m)]^{1/a} \tag{2.1.3}$$

for all $a,b \geq 1$, $\frac{1}{a} + \frac{1}{b} = 1$;

$$|\mathrm{cov}(\xi,\eta)| \leq 12(E|\xi|^a)^{1/a}(E|\eta|^b)^{1/b}[\alpha(m)]^{1/c} \tag{2.1.4}$$

for all $a,b,c \geq 1$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. We refer to Ibragimov [17], and Davydov [8] for the proofs of (2.1.3) and (2.1.4) respectively. We state below an elementary result whose proof is omitted.

LEMMA 2.1.1. *If* f *is a non-negative and non-increasing function defined on* $\{0,1,2,...\}$ *, then,*

(i) $\sum_1^\infty f^{1/2}(m) < \infty \Rightarrow m^2 f(m) \to 0$ *as* $m \to \infty$, *and* $\sum_1^\infty m f(m) < \infty$.

(ii) $\sum_1^\infty m^2 [f(m)]^\delta < \infty \Rightarrow \sum_1^\infty [f(m)]^{\delta/2} < \infty$, *for* $0 < \delta \leq 1$.

2.2 A CENTRAL LIMIT THEOREM. We shall apply a theorem of Philipp [25] to prove a central limit theorem for non-stationary strong mixing rv's. First, we describe the theorem of Philipp.

Let $\{x_{iN}\}$ be a sequence of rv's with $E\, x_{iN} = 0$ and finite second moments. Let

$$\Sigma_N^2 = E(\sum x_{iN})^2$$

$$D(N) = \text{a.s.} \sup_{i \leq N} |x_{iN}| .$$

Throughout $\sum$ denotes summation 1 to N unless, otherwise specified. 'a.s.' is the abbreviation to 'almost surely'. Suppose

(2.2.1) $$\sum_1^\infty \alpha^{1/2}(j) < \infty$$

$$(2.2.2)\quad \begin{cases} \sup_N \Sigma_N^2 < \infty \\ D(N) \to 0 \quad \text{as} \quad N \to \infty \\ \Sigma_N / D(N) \to \infty \end{cases}$$

Let P_N , S_N be real sequences satisfying

$$(2.2.3)\quad \begin{cases} P_N \to 0 \ , \ \zeta_N = \dfrac{P_N S_N}{D^2(N)} \to \infty \\ \dfrac{\Sigma_N^2}{S_N} \to \infty \ , \ \alpha([\zeta_N]) \dfrac{\Sigma_N^2}{S_N} \to 0 \end{cases} \quad \text{as} \quad N \to \infty$$

A pair (P_N, S_N) is said to be *admissible* for the sequence $\{x_{iN}\}$ if (2.2.1), (2.2.2), and (2.2.3) are satisfied.

PROPOSITION (Philipp): *If conditions* (2.2.1) *and* (2.2.2) *hold, and* (P_N, S_N) *is any admissible pair for* $\{x_{iN}\}$ *then, we can represent*

$$\sum_1^N x_i = \sum_{j=1}^{n} y_j + \sum_{j=1}^{n+1} z_j \ ,$$

where (N is suppressed in the double subscript)

$$y_1 = x_1 + \cdots + x_{h_1} \qquad z_1 = x_{h_1+1} + \cdots + x_{h_1+\kappa}$$

$$y_2 = x_{\rho_2+1} + \cdots + x_{\rho_2+h_2} \qquad z_2 = x_{\rho_2+h_2+1} + \cdots + x_{\rho_2+h_2+\kappa}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_n = x_{\rho_n+1} + \cdots + x_{\rho_n+h_n} \qquad z_n = x_{\rho_n+h_n+1} + \cdots + x_{\rho_{n+1}}$$

$$z_{n+1} = x_{\rho_{n+1}+1} + \dots + x_N$$

$$\rho_j = \sum_{i<j} (h_i + \kappa) \ , \ j = 2,3,\dots,n+1 \ , \quad \text{and} \quad \rho_1 = 0$$

such that

$$(2.2.4) \quad \begin{cases} n = [\zeta_N] , \\ E\, y_j^2 = S_N\, (1 + o(1)) \\ \Sigma_N^2 = n\, S_N + o(n\, S_N) \ . \end{cases}$$

Moreover, if $\Sigma_N^2 \to 1$ *as* $N \to \infty$ *then,*

$$L(\sum x_i) \to N(0,1) \ ,$$

provided for every $\varepsilon > 0$

$$(2.2.5) \qquad \sum_{j \le n} \int_{|y| \ge \varepsilon} y^2\, d\, F_{jN} \to 0 \ \text{as} \ N \to \infty \ ,$$

where F_{jN} *is the distribution function of* y_j .

A proof of the above proposition is given in Philipp [25] (see his Lemma 4, and Theorem 3). Now, we shall apply this proposition to a special case. Let $\{\xi_{iN}\}$ be a sequence of strong mixing random variables with $E\, \xi_{iN} = 0$, $|\xi_{iN}| \le 1$ almost surely. Let $\{C_{iN}\}$ be a sequence of real numbers. We suppress the dependence on N as before. Set

$$s_N^2 = \text{Var}\left(\sum C_i \xi_i\right)^2$$

(2.2.6) $$Z_N = \sum x_i \quad \text{where} \quad x_i = \frac{C_i \xi_i}{s_N} .$$

Throughout below we assume

(2.2.7) $$\liminf_{N\to\infty} \frac{s_N^2}{\sum C_i^2} > 0 ,$$

which ensures the non-degeneracy of the limiting variance.

THEOREM 2.2.1. *If (in addition to (2.2.7))* $\sum_1^\infty j^2 \alpha(j) < \infty$ *, and*

(2.2.8) $$b_N^2 = \frac{\max_{i\le N} C_i^2}{\sum C_i^2} = o(1) \quad \text{as} \quad N \to \infty ,$$

then, Z_N *converges in distribution to* $N(0,1)$ *as* $N \to \infty$.

THEOREM 2.2.2. *If (in addition to (2.2.7))* $\sum_1^\infty \alpha^{1/2}(j) < \infty$ *and* $\alpha(m) \cdot m^{2/\delta} \to 0$ *as* $m \to \infty$ *for some* $0 < \delta \le 1$ *, and*

$$\frac{\max_{i\le N} C_i^2}{\sum C_i^2} = O(N^{-\delta_o}) \quad \text{for some} \quad \delta_o > \frac{2+\delta}{3+\delta} \quad \text{as} \quad N \to \infty ,$$

then, Z_N *converges in distribution to* $N(0,1)$.

In general, the hypothesis of Theorem 2.2.2 is stronger than that of Theorem 2.2.1. However, if the constants C_i's are uniformly bounded, then the second theorem is more general than the first. In this case, the asymptotic normality of Z_N holds under Theorem 2.2.2 if $\sum_1^\infty \alpha^{1/2}(j) < \infty$

(see also Theorem 18.5.4 of [18]), whereas, Theorem 2.2.1 still requires the stronger mixing $\sum_1^\infty j^2 \alpha(j) < \infty$. It may be mentioned that the condition (2.2.8) is the same as that usually imposed on C_i's in the case of independent rv's (see [14], Chapter V).

Before proving the theorems, we obtain some auxiliary results.

LEMMA 2.2.1. *If* $\sum_1^\infty \alpha(j) < \infty$ *and* $|\xi_1| < 1$ *a.s., then*

$$\mathrm{Var}\Big(\sum_{M+1}^{M+m} C_i \xi_i\Big) = O\Big(\sum_{M+1}^{M+m} C_i^2\Big)$$

where M *and* m *are arbitrary non-negative integers.*

PROOF: Note that

$$\mathrm{Var}\Big(\sum_{M+1}^{M+m} C_i \xi_i\Big) = \sum_{M+1}^{M+m} C_i^2\, E\xi_i^2 + 2 \sum_{i<j}^{M+m} C_i C_j \operatorname{cov}(\xi_i, \xi_j)$$

$$\leq \sum_{M+1}^{M+m} C_i^2 + 24 \sum_{i<j}^{M+m} |C_i C_j|\, \alpha(j-i) \text{ , using } |\xi_i| \leq 1 \text{ and } (2.1.4)$$

$$= \sum_{M+1}^{M+m} C_i^2 + 24 \sum_{j=1}^{m-1} \alpha(j) \sum_{i=M+1}^{M+m-j} |C_i C_{i+j}| \,.$$

By Schwarz's inequality, we have

$$\Big[\frac{1}{m-j} \sum_{M+1}^{M+m-j} |C_i C_{i+j}|\Big]^2 \leq \Big(\frac{1}{m-j} \sum_{M+1}^{M+m-j} C_i^2\Big) \Big(\frac{1}{m-j} \sum_{M+j+1}^{M+m} C_i^2\Big)$$

$$\leq \Big(\frac{1}{m-j} \sum_{M+1}^{M+m} C_i^2\Big)^2 \,.$$

$$\therefore \sum_{M+1}^{M+m-j} |c_i c_{i+j}| \leq \sum_{M+1}^{M+m} c_i^2 ,$$

and hence from above

$$\mathrm{Var}(\sum_{M+1}^{M+m} c_i \xi_i) \leq \sum_{M+1}^{M+m} c_i^2 \, [1 + 24 \sum \alpha(j)] ,$$

which proves the assertion of the lemma. □

LEMMA 2.2.2. *If* $\sum_1^\infty j^2 \alpha(j) < \infty$ *and* $M+m \leq N$ *, then*

$$E(\sum_{M+1}^{M+m} c_i \xi_i)^4 \leq K(\alpha) \, [(\sum_{M+1}^{M+m} c_i^2)^2 + (\sum_{M+1}^{M+m} c_i^2) \, \Delta^2(N)] ,$$

where $\Delta(N) = \max_{i \leq N} |c_i|$ *and* $K(\alpha) < \infty$ *is a constant depending only on* α *.*

PROOF: We have

$$(2.2.9) \qquad (\sum c_i \xi_i)^4 = \sum c_i^4 \xi_i^4 + 4 \sum_{i \neq j} c_i^3 c_j \, \xi_i^3 \xi_j$$
$$+ 6 \sum_{i<j} c_i^2 c_j^2 \, \xi_i^2 \xi_j^2 + 12 \sum_{i<j \neq k} c_k^2 c_i c_j \, \xi_i \xi_j \xi_k^2$$
$$+ 24 \sum_{i<j<k<\ell} c_i c_j c_k c_\ell \, \xi_i \xi_j \xi_k \xi_\ell .$$

Noting that (under the hypothesis)

$$(2.2.10) \qquad \sum_1^\infty j^p \alpha(j) < \infty \quad \text{for} \quad p = 0,1,2,$$

we estimate the expected values of terms on the right hand side of (2.2.9).

For the convenience of notation, we write $\sum$ in place of $\sum_{M+1}^{M+m}$ from (2.2.11) through (2.2.16).

(2.2.11) $E(\sum c_i^4 \xi_i^4) \leq \Delta^2(N) \sum c_i^2$, since $|\xi_i| \leq 1$ a.s.

(2.2.12)
$$E(\sum_{i\neq j} c_i^3 c_j \, \xi_i^3 \xi_j) = \sum_{i\neq j} c_i^3 c_j \; \mathrm{cov}(\xi_i^3;\xi_j)$$
$$\leq 12\, \Delta^2(N) \sum_{i\neq j} |c_i c_j| \, \alpha(|j-i|) \quad \text{using (2.1.4)}$$
$$\leq 24\, \Delta^2(N) \sum_{j>i} |c_i c_j| \, \alpha(j-i)$$
$$\leq 24\, \Delta^2(N) \, (\sum c_i^2) \, (\sum_{1}^{m} \alpha(j))$$
$$\leq K_1(\alpha)\, \Delta^2(N) \, (\sum c_i^2), \qquad \text{using (2.2.10).}$$

(2.2.13)
$$E(\sum_{i<j} c_i^2 c_j^2 \, \xi_i^2 \xi_j^2) = \sum_{i<j} c_i^2 c_j^2 \, [E\, \xi_i^2 \, E\, \xi_j^2 + \mathrm{cov}(\xi_i^2;\xi_j^2)]$$
$$\leq (\sum c_i^2)^2 + 12\, \Delta^2(N) \sum_{i<j} |c_i c_j| \, \alpha(j-i)$$
$$\leq K_2(\alpha) \, [(\sum c_i^2)^2 + \Delta^2(N)(\sum c_i^2)] .$$

Since,

$$E\, \xi_i \xi_j \xi_k^2 = E\, \xi_k^2 \, E\, \xi_i \xi_j + \mathrm{cov}(\xi_i \xi_j;\xi_k^2)$$
$$= \mathrm{cov}(\xi_i;\xi_j \xi_k^2) ,$$

we have

$$\sum_{i<j<k} |c_k^2 c_i c_j| \, |E\, \xi_i \xi_j \xi_k^2| \le \sum c_k^2 \, E\, \xi_k^2 \sum_{i<j} |c_i c_j| \, |E\, \xi_i \xi_j|$$

$$+ \sum_{i<j<k} c_k^2 \, |c_i c_j| \, \min\{|\mathrm{cov}(\xi_i;\xi_j\xi_k^2)|, |\mathrm{cov}(\xi_i\xi_j;\xi_k^2)|\}$$

$$\le 12(\sum c_i^2)^2 \sum_1^m \alpha(j) + 12\Delta^2(N) \sum_{i<j<k} c_k^2 \min\{\alpha(j-i), \alpha(k-j)\} .$$

But, the second sum on the right is dominated by

$$\sum_k c_k^2 \sum_{i<j<k} \min\{\alpha(j-i), \alpha(k-j)\} \le \sum c_k^2 \sum_{j'+k' \le m} \min\{\alpha(j'), \alpha(k')\}$$

$$\le 2 \sum c_k^2 \sum_1^m j\, \alpha(j) .$$

Hence from above and (2.2.10)

$$(2.2.14) \qquad |E \sum_{i<j<k} c_k^2 c_i c_j \, \xi_i \xi_j \xi_k^2| \le K_3(\alpha) \, [(\sum c_i^2)^2 + \Delta^2(N) \sum c_i^2] .$$

Similarly, it can be shown that the expectations of

$$\sum_{k \le i<j} c_i c_j c_k^2 \, \xi_i \xi_j \xi_k^2 \quad \text{and} \quad \sum_{i \le k<j} c_i c_j c_k^2 \, \xi_i \xi_j \xi_k^2$$

are dominated by the same term on the right hand side of (2.2.14). Finally, we have

$$(2.2.15) \qquad \sum_{i<j<k<\ell} |c_i c_j c_k c_\ell| \, |E\, \xi_i \xi_j \xi_k \xi_\ell| \le$$

$$\sum_{i<j} |c_i c_j| \, |E\,\xi_i \xi_j| \sum_{k<\ell} |c_k c_\ell| \, |E\,\xi_k \xi_\ell|$$

$$+ \sum_{i<j<k<\ell} |c_i c_j c_k c_\ell| \min\{|\mathrm{cov}(\xi_i;\xi_j\xi_k\xi_\ell)|, |\mathrm{cov}(\xi_i\xi_j;\xi_k\xi_\ell)|, |\mathrm{cov}(\xi_i\xi_j\xi_k;\xi_\ell)|\}$$

$$\leq 144(\sum c_i^2)^2(\sum_1^m \alpha(j))^2 + 12 \sum_{i<j<k<\ell} |c_i c_j c_k c_\ell| \min\{\alpha(j-i),\alpha(k-j),\alpha(\ell-k)\} .$$

Now, the second term on the right is dominated by

$$12 \sum_{i+j+k+\ell \leq m} |c_{M+i} c_{M+i+j} c_{M+i+j+k} c_{M+i+j+k+\ell}| \min\{\alpha(j),\alpha(k),\alpha(\ell)\}$$

$$\leq 12\Delta^2(N) \sum_{j+k+\ell \leq m} [\min\{\alpha(j),\alpha(k),\alpha(\ell)\} \sum_{1 \leq i \leq m-j} |c_{M+i} c_{M+i+j}|] .$$

But, by Schwarz's inequality

$$\sum_{1 \leq i \leq m-j} |c_{M+i} c_{M+i+j}| \leq \sum_{M+1}^{M+m} c_i^2 ,$$

and

$$\sum_{j+k+\ell \leq m} \min\{\alpha(j),\alpha(k),\alpha(\ell)\} \leq \sum_{j \geq (k,\ell)} \alpha(j) + \sum_{k \geq (j,\ell)} \alpha(k) + \sum_{\ell \geq (j,k)} \alpha(\ell)$$

$$\leq 3 \sum_1^m j^2 \alpha(j) .$$

Thus, using these bounds in (2.2.15), we see

$$(2.2.16) \quad \sum_{i<j<k<\ell} |c_i c_j c_k c_\ell| \, |E\xi_i\xi_j\xi_k\xi_\ell| \leq K_4(\alpha)[(\sum c_i^2)^2 + \Delta^2(N) \sum c_i^2] .$$

Combining (2.2.11) to (2.2.16), and substituting them in the expectation of (2.2.9), we get the assertion of the lemma. □

REMARK 2.2.1. From the above proof, it is clear that the assertion of Lemma 2.2.2 can be stated as

$$(2.2.17)\qquad E\Big(\sum_{M+1}^{M+m} C_i\xi_i\Big)^4 \le K(\alpha)\Big(\sum_{M+1}^{M+m} C_i^2\Big)^2 + \Delta^2(N)\sum_{M+1}^{M+m} C_i^2\,[K(\alpha) + A_m]\ ,$$

where $K(\alpha) < \infty$ if $\sum_1^\infty j\,\alpha(j) < \infty$, and $A_m = A_m(\alpha) = K\sum_1^m j^2\,\alpha(j)$, K being an absolute constant.

LEMMA 2.2.3. *If* $\sum_1^\infty j\,\alpha(j) < \infty$, *then,*

$$E\Big(\sum_{M+1}^{M+m} C_i\xi_i\Big)^4 \le K(\alpha)\, m^2\Delta^4(N)\ .$$

PROOF: The only major change required in the proof of Lemma 2.2.2 is in the estimate (2.2.15). Notice that from (2.2.15)

$$(2.2.18)\qquad \sum_{i<j<k<\ell} |C_iC_jC_kC_\ell|\,|E\,\xi_i\xi_j\xi_k\xi_\ell|$$

$$\le \Delta^4(N)\Big(\sum_{i<j}|E\,\xi_i\xi_j|\Big)^2 + 12\Delta^4(N)\sum_{i<j<k<\ell}\min\{\alpha(j-i),\alpha(k-j),\alpha(\ell-k)\}$$

$$\le 144\Delta^4(N)\Big[\Big(\sum_{i<j}\alpha(j-i)\Big)^2 + m\sum_1^m j^2\,\alpha(j)\Big]$$

$$\le 144\Delta^4(N)\Big[m^2\Big(\sum_1^m \alpha(j)\Big)^2 + m^2\sum_1^m j\,\alpha(j)\Big]$$

$$\le K_5(\alpha)\,\Delta^4(N)\,m^2\ .$$

All the other terms in the expansion $E(\sum C_i\xi_i)^4$ are clearly dominated by a constant multiple of $m^2\Delta^2(N)$, as seen from (2.2.11) to (2.2.14). Thus, (2.2.18) completes the proof. □

PROOF OF THEOREM 2.2.1: We check the conditions of the proposition. We have from (2.2.6),

$$Z_N = \sum x_i = \frac{\sum c_i \xi_i}{s_N}$$

$$\therefore \quad \Sigma_N^2 = \mathrm{Var}(Z_N) = 1 \quad ,$$

and

$$D(N) = \text{a.s.} \sup_{i \leq N} |x_i| \leq \frac{\Delta(N)}{s_N} \, .$$

But, from (2.2.7) and Lemma 2.2.1, there exist constants K_1 and K_2 (>0) such that

$$K_1 \leq \frac{s_N^2}{\sum c_i^2} \leq K_2 \, , \quad \text{for sufficiently large } N \, . \tag{2.2.19}$$

Hence,

$$D(N) \leq \frac{\Delta(N)}{s_N} = \frac{(\sum c_i^2)^{1/2}}{s_N} \cdot b_N \leq K_1^{-1/2} b_N \to 0 \text{ as } N \to \infty \text{ from (2.2.8)} \, ,$$

and consequently, $\Sigma_N / D(N) = 1/D(N) \to \infty$ as $N \to \infty$. Thus, by Lemma 2.1.1 and above, the hypotheses of the theorem imply (2.2.1) and (2.2.2). Let $P_N = b_N^{1/2}$ and $S_N = b_N$. Clearly $P_N \to 0$ and $\Sigma_N / S_N \to \infty$ as $N \to \infty$. Moreover, since $D^2(N) \leq K_1^{-1} b_N^2$, we have

$$\zeta_N = \frac{P_N S_N}{D^2(N)} \geq \frac{K_1 b_N^{3/2}}{b_N^2} = K_1 b_N^{-1/2} \to \infty \text{ as } N \to \infty \, .$$

Set $\nu = [K_1 b_N^{-1/2}]$, where $[x]$ denotes the integral part of x . Then, due to monotonicity of $\alpha(\cdot)$,

$$\alpha([\zeta_N]) \frac{\sum_N^2}{s_N} \leq \alpha(\nu) \cdot \frac{1}{b_N}$$

$$= \alpha(\nu)(K_1 b_N^{-1/2})^2 \cdot K_1^{-2}$$

$$\leq \alpha(\nu)(\nu+1)^2 K_1^{-2} \to 0 \quad \text{as} \quad N \to \infty \quad ,$$

by the assumption $\sum_1^{\infty} j^2 \alpha(j) < \infty$. Thus, the conditions (2.2.3) are satisfied so that (P_N, S_N) is an admissible pair for $\{x_i\}$. Note that from the decomposition given by the proposition, $[\rho_j, \rho_j + h_j]$ for $1 \leq j \leq n$ are disjoint so that

(2.2.20)
$$\sum_{j=1}^{n} \sum_{\rho_j+1}^{\rho_j+h_j} C_i^2 \leq \sum_1^N C_i^2 .$$

Also, from (2.2.4) $\sum_N^2 = n S_N + o(n S_N)$. Since $\sum_N^2 = 1$ in our case, we have

(2.2.21)
$$n = S_N^{-1} (1 + o(1))$$

$$= b_N^{-1} (1 + o(1)) .$$

Now, we verify (2.2.5) of the proposition which completes the proof. Let $\varepsilon > 0$ be arbitrary. Then,

$$\sum_{j \leq n} \int_{|y| \geq \varepsilon} y^2 \, d F_{jN} = \sum_{j \leq n} E[y_j^2 \, I(y_j \geq \varepsilon)]$$

$$\leq \sum_{j\leq n} (E\, y_j^4)^{1/2} [P(y_j \geq \varepsilon)]^{1/2} \quad , \text{ by Schwarz's inequality,}$$

$$\leq \sum_{j\leq n} (E\, y_j^4)^{1/2} \frac{(E\, y_j^2)^{1/2}}{\varepsilon} \quad , \text{ by Markov's inequality.}$$

Now, from the proposition,

$$E\, y_j^2 = S_N\, (1 + o(1)) = b_N\, (1 + o(1)) \ ,$$

and from Lemma 2.2.2,

$$E\, y_j^4 = \frac{1}{s_N^4} E\Big(\sum_{\rho_j+1}^{\rho_j+h_j} C_i \xi_i \Big)^4$$

$$\leq \frac{K(\alpha)}{s_N^4} [h_j^{*2} + h_j^* \Delta^2(N)] \quad , \text{ where } h_j^* = \sum_{\rho_j+1}^{\rho_j+h_j} C_i^2$$

$$\leq \frac{K(\alpha)}{K_1^2(\sum C_i^2)^2} [h_j^{*2} + h_j^* \Delta^2(N)] \quad , \text{ from (2.2.19)} \ .$$

Hence, from above

$$\text{(2.2.22)} \qquad \sum_{j\leq n} \int_{|y|\geq\varepsilon} y^2\, d\, F_{jN} \leq \frac{1}{\varepsilon} \sum_{j\leq n} b_N^{1/2}(1+o(1)) \frac{K^{1/2}(\alpha)}{K_1(\sum C_i^2)} (h_j^* + \Delta^2(N))$$

$$\leq K_\varepsilon'(\alpha)\, b_N^{1/2}(1+o(1)) \Big[1 + \frac{n\, \Delta^2(N)}{\sum C_i^2}\Big] \quad , \text{ from (2.2.20)}$$

$$\leq K_\varepsilon'(\alpha)\, b_N^{1/2}(1+o(1)) \, [1 + b_N^{-1}(1+o(1)) b_N^2] \quad , \text{ from (2.2.21)}$$

$$\leq K_\varepsilon''(\alpha)\, b_N^{1/2}(1 + 2\, b_N) \to 0 \text{ as } N \to \infty \quad \text{from (2.2.8)} \ . \qquad \square$$

COROLLARY 2.2.2. The assertion of Theorem 2.2.1 holds under a weaker condition on the mixing coefficient: The condition $\sum_1^\infty j^2 \ (j) < \infty$ can be replaced by $\sum_1^\infty \alpha^{1/2}(j) < \infty$, and $A_N = K_o \sum j^2\alpha(j) = O(b_N^{-\theta})$, for some $0 \le \theta < \frac{5}{2}$.

PROOF: Since $\sum_1^\infty \alpha^{1/2}(j) < \infty$ implies $m^2\ \alpha(m) \to 0$ as $m \to \infty$ (see Lemma 2.1.1(i)), the pair (P_N, S_N), where $P_N = b_N^{1/2}$ and $S_N = b_N$, is admissible for $\{x_i\}$ as shown before. Using (2.2.17), we have as in (2.2.22)

$$\sum_{j\le n} \int_{|y|\ge\varepsilon} y^2 d\ F_{jN} \le \frac{1}{\varepsilon} \sum_{j\le n} b_N^{1/2}(1+o(1)) \frac{K^{1/2}(\alpha)}{K_1(\sum C_i^2)} [h_j^* + (1+A_j^*)\ \Delta^2(N)] \ ,$$

where

$$h_j^{*2} = \sum_{\rho_j+1}^{\rho_j+h_j} C_i^2 \ , \quad A_j^* = \sum_{h_{j-1}+1}^{h_j} i^2\ \alpha(i) \ , \quad \text{and} \quad K(\alpha) < \infty .$$

$K(\alpha)$ depends only on $\sum_1^\infty \alpha^{1/2}(i)$.

Proceeding as in (2.2.22), we see that the right hand side of the above inequality is dominated by

$$K_\varepsilon'(\alpha)\ [b_N^{1/2}(1 + 2\ b_N) + b_N^{5/2} \sum_1^n A_j^*]$$

$$\le K_\varepsilon'(\alpha)\ [o(1) + b_N^{5/2} \sum j^2\ \alpha(j)]$$

$$= K_\varepsilon'(\alpha)\ [o(1) + K_o b_N^{5/2} \cdot A_N]$$

$$= K_\varepsilon''(\alpha)\ [o(1) + b_N^{5/2-\theta}]$$

$= o(1)$ as $N \to \infty$, under the hypotheses. □

PROOF OF THEOREM 2.2.2: We proceed as in Theorem 2.2.1 by setting $P_N = N^{-\beta/p}$ and $S_N = N^{-\beta}$, where β and p are chosen to satisfy $2(1-\delta_o) < \beta < \frac{2\delta_o}{2+\delta}$, and $p > \max\{(\frac{\delta_o}{\beta} - 1)^{-1}, [\frac{2}{\delta}(\frac{\delta_o}{\beta} - 1)-1]^{-1}\}$ (such a choice is possible under the hypotheses of the theorem). Then the conditions (2.2.1) and (2.2.2) are satisfied so that (P_N, S_N) is an admissible pair. We verify (2.2.5). As in the proof of Theorem 2.2.1 (now using Lemma 2.2.3), we have for every $\varepsilon > 0$

$$\sum_{j\leq n} \int_{|y|\geq\varepsilon} y^2 d\, F_{jN} \leq K_\varepsilon(\alpha) \sum_{j\leq n} \frac{h_j\, \Delta^2(N)}{(\sum c_i^2)} N^{-\beta/2} (1 + o(1))$$

$$\leq K'_\varepsilon(\alpha)\, N^{1-\delta_o}\, N^{-\beta/2}, \quad \text{for large } N$$

$$= K'_\varepsilon(\alpha)\, N^{[2(1-\delta_o)-\beta]/2} \to 0 \text{ as } N \to \infty,$$

since $2(1-\delta_o) < \beta$ by our choice of β. □

REMARK 2.2.3. If the sequence $\{\xi_{iN}\}$ is ϕ-mixing, the mixing condition of Theorem 2.2.1 can be improved. Infact the conditions $\sum_1^\infty \phi^{1/2}(j) < \infty$ and (2.2.8) imply the conclusion of Theorem 2.2.1 in the ϕ-mixing case. The proof is much simpler and follows from Theorem 2.1 of Bergström [1].

2.3 GENERALIZED EMPIRICAL PROCESSES FOR NON-STATIONARY SEQUENCES UNDER STRONG MIXING.

Let $\{Y_{iN}\}$ be a sequence of rv's with corresponding distribution functions $\{F_i^{(N)}\}$ which are continuous. Let

$$(2.3.1)\quad \begin{cases} F^{(N)} = \frac{1}{N} \sum F_i^{(N)} \ , \ L_{iN} = F_i^{(N)} \ F^{-1} \\ \eta_{iN} = F^{(N)}(Y_{iN}) \ , \quad 1 \le i \le N \ . \end{cases}$$

We define *generalised empirical processes* by

$$(2.3.2)\qquad V_N(t) = N^{-1/2} \sum C_{iN} \ [I(\eta_{iN} \le t) - L_{iN}(t)] \ ,$$

where $\{C_{iN}\}$ is a sequence of arbitrary constants. The usual empirical process is given by

$$(2.3.3)\qquad U_N(t) = N^{-1/2} \sum [I(\eta_{iN} \le t) - L_{iN}(t)] \ .$$

Note that $V_N \in D[0,1]$ for all $N \ge 1$. We shall denote by V and U, the limiting processes of V_N and U_N respectively, whenever they exist. Throughout below, we shall drop N from the double subscript for notational convenience. Define

$$(2.3.4)\qquad \sigma_{V_N}(s,t) = N^{-1} \sum C_i^2 \ [L_i(s \wedge t) - L_i(t)L_i(s)] \ + \\ N^{-1} \sum_{i \ne j} C_i C_j \ [P(\eta_i \le t \ , \ \eta_j \le s) - L_i(t)L_j(s)]$$

$$(2.3.5)\qquad \sigma_V(s,t) = \lim_{N \to \infty} \sigma_{V_N}(s,t) \ ,$$

whenever the limit exists for $0 \le s,t \le 1$. σ_{U_N} and σ_U are similarly defined.

$$(2.3.6)\qquad \tau(x) = \sup_N \ (N^{-1} \sum |C_i|^{1/x})^x \ , \quad \text{for} \ \ x \ne 0 \ .$$

We state the main results:

THEOREM 2.3.1. *Suppose* $\sum_1^\infty j^2 \alpha^\delta(j) < \infty$, $\tau(\delta/4) < \infty$, *and*

(2.3.7) $$\frac{\Delta(N)}{(\sum c_i^2)^{1/2}} = O(N^{-(1+\delta_o)/4}) \ , \ \textit{for some} \ \delta_o > \frac{2\delta}{4-\delta} \ , \ 0 < \delta < 1 \ .$$

If also $\sigma_V(s,t)$, *defined by* (2.3.5) *exists for all* $0 \leq s,t \leq 1$, *then* $V_N \Rightarrow_d V$ *where* V *is a Gaussian random function (tied down at* 0 *and* 1*) defined by*

(2.3.8)
$$EV(t) = 0 ;$$
$$EV(t)\, V(s) = \sigma_V(s,t) \ .$$

Moreover, $P(V \in C[0,1]) = 1$.

THEOREM 2.3.2. *If* $\sum_1^\infty j^2 \phi(j) < \infty$, $\tau(\delta/4) < \infty$, *for some* $0 < \delta < 1$,

(2.3.9) $$\frac{\Delta(N)}{(\sum c_i^2)^{1/2}} = o(1) \ \textit{as} \ N \to \infty \ ,$$

and if the limit $\sigma_V(s,t)$ *exists for all* $0 \leq s,t \leq 1$, *then, the conclusion of Theorem* 2.3.1 *holds.*

In the case of stationary sequences, the conclusions of both the theorems are known when $c_i \equiv 1$. In this case, the conclusion of Theorem 2.3.2 was proved by Billingsley ([4], Theorem 22.1) under the mixing condition $\sum_1^\infty j^2 \phi^{1/2}(j) < \infty$. Sen [32] showed that the condition $\sum_1^\infty j\, \phi^{1/2}(j) < \infty$ is sufficient for Billingsley's theorem. For strong mixing stationary sequences, the conclusion of Theorem 2.3.1, when $C_i \equiv 1$, has been independently proved by Deo [9] under the mixing condition $\sum_1^\infty j^2 \alpha^{\delta/2}(j) < \infty$, $0 < \delta < 1$, by Yokoyama [36] under $\sum_1^\infty j\, \alpha^\beta(j) < \infty$, $0 < \beta < \frac{1}{2}$, and by Mehra and Rao [22] under $\sum_1^\infty j^2 \alpha^\delta(j) < \infty$, $0 < \delta < 1$. It may be noted that the last mixing condition, of Mehra and Rao, is weaker than the other two. Recently, Mehra and Rao [23] have shown that the above result is valid for generalized empirical processes also, under certain conditions on C_i's. Our theorems here, while weakening the above mixing conditions, cover the generalized empirical processes for non-stationary mixing rv's.

First, we prove some lemmas. Let

$$(2.3.10)\qquad \begin{cases} \xi_i(t) = I(\eta_i \le t) - L_i(t), \text{ where } L_i(t) = F_i^{(N)}F^{-1}(t), \\ u_i = \xi_i(t) - \xi_i(s). \end{cases}$$

LEMMA 2.3.1. *For* $0 < \delta_1 < 1$, $0 < \delta_2 < 1$, *and* $s < t$, *we have*

(i) $N^{-1} \sum L_i(t) = t$, *for all* $0 \le t \le 1$,

(ii) $\sum C_i\, E\,|u_i| \le 2N\, \tau(\delta_1)(t-s)^{1-\delta_1}$,

(ii)' $\sum C_i^2\, E\,|u_i|^2 \le N\, \tau^2(\delta_1/2)(t-s)^{1-\delta_1}$,

(iii) $\sum_{i<j} |C_iC_j\, E\, u_iu_j| \le 12N\, \tau^2(\frac{\delta_2}{2})(t-s)^{1-\delta_2} \sum \alpha^{\delta_2}(j)$,

where δ_1 *can be taken as* 0 *in* (ii) *and* (ii)' *if* $C_i \equiv 1$.

PROOF: (i) follows by the continuity of $F_i^{(N)}$ for $i = 1,2,\ldots,N$. Also, for $s < t$,

$$(2.3.11)\qquad \begin{cases} E|u_i| \le 2(L_i(t) - L_i(s)) \\ E\, u_i^2 \le (L_i(t) - L_i(s)) \end{cases}.$$

Hence

$$\sum C_i\, E\,|u_i| \le 2 \sum |C_i|\,(L_i(t) - L_i(s))$$

$$\le 2N\,\tau(\delta_1)\cdot N^{-1} \sum (L_i(t) - L_i(s))^{1-\delta_1}$$

$$\le 2N\,\tau(\delta_1)(t-s)^{1-\delta_1}\quad,\ \text{using (i)}.$$

The proof (ii)' is similar. Finally, writing $u_i^* = E(|u_i|^{2/1-\delta_2})$, we have from (2.1.4)

$$\sum_{i<j} |C_i C_j\, E\, u_i u_j| \le 12 \sum_{i<j} |C_i C_j|\,(u_i^*)^{\frac{1-\delta_2}{2}} \cdot (u_j^*)^{\frac{1-\delta_2}{2}}\ \alpha^{\delta_2}(j-i)$$

$$\le 12 \sum_i C_i^2\,(u_i^*)^{1-\delta_2} \sum_j \alpha^{\delta_2}(j)\ ,\ \text{as in the proof of Lemma (2.2.1)},$$

$$\le 12\,\tau^2\!\left(\frac{\delta_2}{2}\right)(N^{-1} \sum E\, u_i^2)^{1-\delta_2} \sum_j \alpha^{\delta_2}(j)$$

$$\le 12\,\tau^2\!\left(\frac{\delta_2}{2}\right)(t-s)^{1-\delta_2} \sum_j \alpha^{\delta_2}(j)\quad,\ \text{using (i)}.$$

If $C_i \equiv 1$, clearly we could take $\delta_1 = 0$ in view of (i) and (2.3.11). □

LEMMA 2.3.2.

(i) *If* $\sum_1^\infty j^2 \alpha^\delta(j) < \infty$, $\tau(\delta/4) < \infty$ *for some* $0 < \delta < 1$, *then for* $s < t$,

$$E(\sum C_i u_i)^4 \le K(\alpha,\delta)\ [N^2(t-s)^{2-\delta} + N\Delta^4(N)(t-s)^{1-\delta}] \ ;$$

(ii) *If* $\sum_1^\infty j^2 \phi(j) < \infty$, $\tau(\delta/4) < \infty$ *for some* $0 < \delta < 1$, *then for* $s < t$,

$$E(\sum C_i u_i)^4 \le K(\phi,\delta)\ [N^2(t-s)^{2-\delta} + N\Delta^4(N)(t-s)] \ ,$$

where $K(\alpha,\delta)$, $K(\phi,\delta)$ *denote generic finite constants, and* $\Delta(N) = \max_{i \le N} |C_i|$.

PROOF: We will make use of the expansion given in (2.2.9) to obtain the bounds. We have

$$(2.3.12) \qquad \sum C_i^4\ E\ u_i^4 \le \Delta^4(N) \sum E\ u_i^2 \ , \quad \text{since } |u_i| \le 1 \text{ a.s.,}$$

$$\le N\Delta^4(N)(t-s) \ , \quad \text{from (2.3.11) and (i).}$$

$$(2.3.13) \qquad \sum_{i<j} C_i^2 C_j^2\ E\ u_i^2 u_j^2 \le (\sum C_i^2\ E\ u_i^2)^2 + \Delta^4(N) \sum_{i<j} |\mathrm{cov}(u_i^2, u_j^2)|$$

$$\le N^2\ \tau^4(\tfrac{\delta}{2})(t-s)^{2-\delta} + 12\Delta^4(N)(t-s)^{1-\delta} N \sum \alpha^\delta(j) \ ,$$

using Lemma 2.3.1 with $\delta_1 = \frac{\delta}{2}$ and $\delta_2 = \delta$. Similarly, we get

$$(2.3.14) \qquad |\sum_{i \ne j} C_i^3 C_j\ E\ u_i^3 u_j| \le 24\Delta^4(N)\ N(t-s)^{1-\delta} \sum \alpha^\delta(j) \ .$$

$$(2.3.15)\quad \Big|\sum_{i<j<k} C_iC_jC_k^2\, E\, u_iu_ju_k^2\Big| \le \sum_{i<j} |C_iC_j|\,|Eu_iu_j| \sum_{k=1} C_k^2\, E\, u_k^2 +$$

$$\Delta^4(N) \sum_{i<j<k} \min\{|cov(u_iu_j;u_k^2)|,|cov(u_i;u_ju_k^2)|\} \quad .$$

Now, in veiw of Lemma 2.3.1 with $\delta_1 = \delta_2 = \frac{\delta}{2}$, the first term on the right is dominated by

$$12\tau^4(\tfrac{\delta}{4})\, N^2(t-s)^{2-\delta} \sum \alpha^{\delta/2}(j) \quad ,$$

and since $|u_i| \le 1$ a.s., the sum in the second term is dominated by

$$12 \sum_{i<j<k} \min\{\alpha^{\delta}(k-j),\alpha^{\delta}(j-i)\}(E|u_i|^{1/1-\delta})^{1-\delta}$$

$$\le 12 \sum_i (E|u_i|)^{1-\delta} \sum_{k>j>i} \min\{\alpha^{\delta}(k-j),\alpha^{\delta}(j-i)\}$$

$$\le 12N \Big(\sum_i \frac{E|u_i|}{N}\Big)^{1-\delta} \sum_{k+j\le N} \min\{\alpha^{\delta}(j),\alpha^{\delta}(k)\}$$

$$\le 48(t-s)^{1-\delta}\, N \sum j\, \alpha^{\delta}(j) \quad ,$$ using Lemma 2.3.1 with $C_i \equiv 1$

Similary, the sums $\sum_{k\le i<j}$ and $\sum_{i\le k<j}$ are bounded by the same terms abov[e] so that

$$(2.3.16)\quad \Big|\sum_{i<j\ne k} C_iC_jC_k^2 E\, u_iu_ju_k^2\Big| \le 36\tau^4(\tfrac{\delta}{4})\, N^2(t-s)^{2-\delta} \sum \alpha^{\delta/2}(j)$$

$$+ 144N\Delta^4(N)\, (t-s)^{1-\delta} \sum j\, \alpha^{\delta}(j) \quad .$$

Finally,

$$\left|\sum_{i<j<k<\ell} C_iC_jC_kC_\ell\, E\, u_iu_ju_ku_\ell\right| \le \left(\sum_{i<j} C_iC_j\, E\, u_iu_j\right)^2$$

$$+ \Delta^4(N) \sum_{i<j<k<\ell} \min\{|\mathrm{cov}(u_i;u_ju_ku_\ell)|, |\mathrm{cov}(u_iu_j;u_ku_\ell)|,$$

$$|\mathrm{cov}(u_iu_ju_k;u_\ell)|\}.$$

Now, the first term on the right is dominated by

$$144\tau^4(\tfrac{\delta}{4})\, N^2(t-s)^{2-\delta}\, (\sum \alpha^{\delta/2})^2 ,$$

using Lemma 2.3.1 with $\delta_2 = \frac{\delta}{2}$, and the sum in the second term by,

$$12 \sum_{i<j<k<\ell} \min\{\alpha^\delta(\ell-k), \alpha^\delta(k-j), \alpha^\delta(j-i)\}(E|u_i|^{1/1-\delta})^{1-\delta}$$

$$\le 12 \left(\frac{\sum E|u_i|}{N}\right)^{1-\delta} N \sum_{j+k+\ell\le N} \min\{\alpha^\delta(j), \alpha^\delta(k), \alpha^\delta(\ell)\}$$

$$\le 72\, N(t-s)^{1-\delta} \sum j^2\, \alpha^\delta(j) .$$

Thus,

$$(2.3.17)\quad \left|\sum_{i<j<k<\ell} C_iC_jC_kC_\ell\, E\, u_iu_ju_ku_\ell\right| \le 144\tau^4(\tfrac{\delta}{4})N^2(t-s)^{2-\delta}[\sum \alpha^{\delta/2}(j)]^2$$

$$+ 72\Delta^4(N)N(t-s)^{1-\delta} \sum j^2\, \alpha^\delta(j) .$$

Now, the assertion (i) follows from (2.3.12) to (2.3.17) above. The proof of (ii) is similar, using the inequality (2.1.3) and the fact

$$\left|\sum_{i<j} C_iC_j\, E\, u_iu_j\right| \le 2 \sum_{i<j} |C_iC_j|\, (E|u_i^2|)^{1/2}(E|u_j|^2)^{1/2}\, \phi^{1/2}(j-i)$$

$$\leq 2 \, (\sum c_i^2 \, E \, u_i^2) \sum \phi^{1/2}(j)$$

$$\leq 2 \, \tau^2(\tfrac{\delta}{2}) (t-s)^{1-\delta} \; N \; \sum \phi^{1/2}(j) \, . \qquad \square$$

LEMMA 2.3.3. *Under the conditions of Theorems 2.3.1, or Theorem 2.3.2, the finite dimensional distributions of* V_N *converge to those of* V, *which is Gaussian with the covariance function* σ *defined by* (2.3.5).

PROOF: Suppose the conditions of Theorem 2.3.1 are satisfied. Then, clearly the conditions of Theorem 2.2.1 hold for the sequence $V_N(t)$ defined by $V_N(t) = N^{-1/2} \sum c_i \xi_i(t)$, where t is fixed. Since the limit $\sigma_V(t)$ exists by hypothesis, we may assume $\sigma_V(t) > 0$ (for, otherwise the limiting distribution would be the degenerate normal $N(0,0)$). Hence by Theorem 2.2.1, $V_N(t)$ is asymptotically $N(0,\sigma_V(t))$. Let $0 < t_1 < t_2 < \ldots < t_m < 1$, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ (not all zero) be fixed. Consider

$$Z_N = \sum_{j=1}^{m} \lambda_j \, V_N(t_j) \, .$$

Since $\sum_1^m |\lambda_j| > 0$, we have

$$Z_N (\sum_1^m |\lambda_j|)^{-1} = N^{-1/2} \sum_{j=1}^{m} \lambda_j \sum_1^N c_i \xi_i(t_j) \cdot (\sum_1^m |\lambda_j|)^{-1}$$

$$= N^{-1/2} \sum c_i \xi_i'(t) \quad ,$$

where

$$\xi_i'(t) = \sum_1^m \lambda_j \xi_i(t_j) \; [\sum_1^m |\lambda_j|]^{-1} \, .$$

Note that $|\xi_i'| \leq 1$, and they are mixing with the same mixing coefficient as ξ_i's. Consequently, by Theorem 2.2.1, Z_N is asymptotically normal. The result now follows from Theorem 7.7 of Billingsley [4].

PROOF OF THEOREM 2.3.1: First we assume $C_i \geq 0$. Since the finite dimensional convergence has been established in Lemma 2.3.3, we need only prove the tightness of the sequence $\{V_N\}$ (see Theorem 15.1 of [4]). From (2.3.2), (2.3.10), and Lemma (2.3.2)(i), we have

$$(2.3.18) \quad E|V_N(t) - V_N(s)|^4 \leq K(\alpha,\delta)\ [|t-s|^{2-\delta} + |t-s|^{1-\delta}\frac{\Delta^4(N)}{N}]\ .$$

From (2.3.7), we have $\Delta^4(N)\ (\sum C_i^2)^{-2} \leq K\ N^{-(1+\delta_o)}$ for N large, where K is a constant, and $\delta_o > 2\delta/(4-\delta)$. This implies $\Delta^4(N) \leq K\tau^4(\frac{1}{2})N^{-\delta_o-1}$, and since $\tau(\frac{\delta}{4}) < \infty$, we have

$$(2.3.19) \quad \frac{\Delta^4(N)}{N} = O(N^{-\delta_o}) \quad \text{for} \quad \delta_o > \frac{2\delta}{4-\delta}\ .$$

Let $0 < \varepsilon < 1$. Since $\Delta^4(N)/N \to 0$ as $N \to \infty$, we can choose N large so that

$$(2.3.20) \quad (\frac{\varepsilon\ \Delta^4(N)}{N})^{1/\delta'} < |t-s|\ ,$$

where δ' satisfies the inequality,

$$(2.3.21) \quad \frac{2\delta}{4-\delta} < \frac{2\delta'}{4-\delta} < \delta_o\ .$$

Note that $\delta_o > \frac{2\delta}{4-\delta}$ (under the hypotheses of the theorem) so that δ' satisfying the above inequality, does exist. For $s < t$ and N, δ' satisfying (2.3.20), (2.3.21), from (2.3.18) we have

(2.3.22) $$E|V_N(t)-V_N(s)|^4 \le K(\alpha,\delta)\ [(t-s)^{2-\delta} + \frac{(t-s)^{1-\delta+\delta'}}{\varepsilon}]$$

$$\le \frac{2K(\alpha,\delta)}{\varepsilon}\ (t-s)^{1+\delta''}\ , \quad \text{where} \quad \delta'' = \delta'-\delta > 0\ .$$

Let p be a number satisfying

(2.3.23) $$\left(\frac{\varepsilon\ \Delta^4(N)}{N}\right)^{1/\delta'} < p < \left[\frac{\varepsilon}{4\tau\ (\frac{\delta}{4})}\right]^{4/(4-\delta)} N^{-2/(4-\delta)}\ .$$

Consider $V_N(s+ip) - V_N(s+(i-1)p)$ for $i = 1,2,\ldots,m$. From (2.3.22) and Theorem 12.2 of Billingsley [4], we get for $\lambda > 0$

(2.3.24) $$P[\max_{i\le m} |V_N(s+ip) - V_N(s)| \ge \lambda] \le \frac{2K(\alpha,\delta)}{\varepsilon\ \lambda^4}\ (mp)^{1+\delta''}\ .$$

Also, for $s \le t \le s+p$,

$$|V_N(t)-V_N(s)| = N^{-1/2}\ |\sum c_i\{[I(\eta_i\le t)-L_i(t)] - [I(\eta_i\le s)-L_i(s)]\}|$$

$$\le N^{-1/2}\ |\sum c_i[I(\eta_i\le t)-I(\eta_i\le s)]| + N^{-1/2}|\sum c_i(L_i(t)-L_i(s))|$$

$$\le N^{-1/2}\ |\sum c_i[I(\eta_i\le s+p)-I(\eta_i\le s)]| + N^{-1/2}|\sum c_i(L_i(t)-L_i(s))|$$

$$\le |V_N(s+p)-V_N(s)| + 2N^{-1/2}\ |\sum c_i(L_i(s+p)-L_i(s))|$$

$$= |V_N(s+p)-V_N(s)| + 2N^{1/2}\ \tau\ (\tfrac{\delta}{4})\ p^{1-(\delta/4)}\ .$$

Hence,

$$\sup_{s\le t\le s+mp} |V_N(t)-V_N(s)| \le 3 \max_{i\le m} |V_N(s+ip)-V_N(s)| + 2N^{1/2}\, \tau\, (\tfrac{\delta}{4})\, p^{\frac{4-\delta}{4}}$$

$$\le 3 \max_{i\le m} |V_N(s+ip)-V_N(s)| + \frac{\varepsilon}{2} ,$$

from (2.3.23). Consequently, from (2.3.24) with $\lambda = \frac{\varepsilon}{6}$, and above we have

$$P[\sup_{s\le t\le s+mp} |V_N(t)-V_N(s)| \le \varepsilon] \le \frac{K(\alpha,\delta)}{\varepsilon^5}\, (mp)^{1+\delta''} .$$

Now, the assertion of the theorem follows as in Theorem 22.1 of Billingsley [4], provided we show that the choice of p, satisfying (2.3.23), is possible under the hypotheses of our theorem. Clearly, the number p satisfying (2.3.23) exists if

$$(2.3.25) \qquad (\frac{\Delta^4(N)}{N})^{1/\delta'} \cdot N^{2/(4-\delta)} = o(1) \quad \text{as} \quad N \to \infty ,$$

since all other quantities involved in that inequality are finite, and do not depend on N. But, from (2.3.19) and (2.3.21), we have $[\Delta^4(N)/N] = O(N^{-\delta_o})$ and $\delta_o > \frac{2\delta'}{4-\delta}$, so that

$$(\frac{\Delta^4(N)}{N})^{1/\delta'} \cdot N^{2/(4-\delta)} = O(N^{-\frac{\delta_o}{\delta'} + \frac{2}{4-\delta}}) = o(1) \quad \text{as} \quad N \to \infty .$$

This complete the proof when $C_i \ge 0$. For general C_i's, consider $V_N = V_N^+ - V_N^-$, where $V_N^+(t) = N^{-1/2} \sum C_i^+ \xi_i(t)$, $V_N^-(t) = N^{-1/2} \sum C_i^- \xi_i(t)$, and C_i^+ and C_i^- are being the positive and negative parts of C_i. The tightness of V_N follows from that of V_N^+ and V_N^-. □

PROOF OF THEOREM 2.3.2: The proof runs parallel to that given above, making use of Lemma 2.3.2 (ii) in the place of Lemma 2.3.2 (i). The details are omitted. □

CHAPTER III

WEAK CONVERGENCE RELATIVE TO d_q METRIC

In this chapter we prove the weak convergence of $\{V_N/q\}$ relative to the Skorohod metric on $D[0,1]$, where V_N is as defined by (2.3.2) and q to be defined below. For stationary ϕ-mixing and a class of strong mixing sequences, such results were obtained by Fears and Mehra [11] and Mehra and Rao [22] in the case $C_{iN} \equiv 1$. Similar results were obtained recently, for general C_{iN}'s, by Mehra and Rao [23] in the case of stationary strong mixing sequences. The results of the present chapter extend those of [23] to non-stationary sequences, and can be applied to study the limiting distributions of test statistics, appropriate for regression and other problems, under mixing type of dependence (see Chapter IV). The main results are contained in Section 3.2. The assertion of Lemma 3.2.1 is basic for the proof of Theorem 3.2.1; however, it is not needed for Theorem 3.2.2.

3.1 THE q FUNCTIONS. For a fixed $r > 0$, let $[Q(r)\uparrow]$ denote the class of functions q defined on $[0,1]$ and satisfying:

(3.1.1)
(i) $q(t)$ is continuous with $q(0) \geq 0$ and $q(t) > 0$ for $t > 0$,
(ii) $q(t)\uparrow t$ and $t\, q^{-r}(t)\uparrow t$.

Let $[Q(r)]$ denote the class of functions q such that $q(t) = q(1-t) = \bar{q}(t)$, $0 \leq t \leq \frac{1}{2}$ for some $\bar{q} \in [Q(r)\uparrow]$. If the property (ii) above is replaced by

$$(ii)^* \quad q(t) \uparrow \text{ strictly, and } t\, q^{-r}(t) \uparrow \text{ strictly,}$$

then, the corresponding classes are denoted by $[Q^*(r)\uparrow]$ and $[Q^*(r)]$. Clearly, $[Q^*(r)] \subset [Q(r)]$ and $[Q(s)] \subset [Q(r)]$ for $1 \leq s \leq r$. An important example of $q \in [Q^*(r)]$ is given by

$$q(t) = K[t(1-t)]^{\frac{1}{r} - \delta} \quad , \; 0 < \delta < \frac{1}{r} ,$$

where K is a constant.

LEMMA 3.1.1. *For every* $q \in [Q(r)]$ *with* $r > 2$ *and* $0 < s < r$

$$\int_0^1 q^{-s}(t)dt < \infty \quad \text{and} \quad \int_0^1 [\frac{1}{t\, q^r(t)}]^{1/r} dt < \infty .$$

PROOF: By the symmetry of q it is enough to show that both the integrals, in the range $(0,\eta)$ with $\eta \leq \frac{1}{2}$, are finite. Now,

$$\int_0^\eta q^{-s}(t)dt = \int_0^\eta \frac{1}{t^{1-\delta}} \frac{t^{1-\delta}}{q^s(t)} dt = \int_0^\eta \frac{1}{t^{1-\delta}} (\frac{t}{q^r(t)})^{1-\delta} dt \quad \text{where } \delta = (1 - \frac{s}{r})$$

$$\leq \frac{\eta^{1-\delta}}{q^s(\eta)} \int_0^\eta t^{\delta-1} dt \qquad \text{since } q \in [Q(r)]$$

$$< \infty \quad , \quad \text{as } \delta > 0 .$$

Similarly,

$$\int_0^\eta \frac{dt}{t^{1/r} q(t)} = \int_0^\eta \frac{1}{t^{2/r}} [\frac{t}{q^r(t)}]^{1/r} dt$$

$$\leq \frac{\eta^{1/r}}{q(\eta)} \int_0^\eta t^{2/r} \qquad \text{since} \quad q \in [Q(r)]$$

$$< \infty \quad \text{as} \quad r > 2 \; . \qquad \square$$

The following lemma is needed for the next chapter.

LEMMA 3.1.2. *Let* q *be defined on* $[0,\infty)$ *and satisfy* (i) *and* (ii)* *of* (3.1.1) *with* $r \geq 2$. *For fixed* $c > 0$, *and* $b > 0$ *there exists a function* $\bar{q}$ *satisfying* (i) *and* (ii)* *with* $r' = \frac{r}{2} + 1$, *and for which*

$$q(x) = \bar{q}(cx + bx^{1/2}\, q(x)) \; . \tag{3.1.2}$$

PROOF: Define $Q(y) = c\,R(y) + b\,y[R(y)]^{1/2}$, where $R = q^{-1}$. Then $\bar{q} = Q^{-1}$ satisfies (3.1.2). Note that $r' = \frac{r}{2} + 1 \leq r$, and by hypothesis $x/q^r(x) \uparrow$ strictly. From the fact that $R(y)$ is strictly increasing in y it follows then that $R(y)/y^r$ and $R(y)/y^{r'}$ are strictly increasing in y . But,

$$\frac{Q(y)}{y^{r'}} = c[\frac{R(y)}{y^{r'}}] + b[R(y)]^{1/2} y \cdot y^{-r'}$$

$$= c[\frac{R(y)}{y^{r'}}] + b[\frac{R(y)}{y^{r}}]^{\prime} \; .$$

Thus, $Q(y)/y^{r'}$, and consequently $y/\bar{q}^{r'}(y)$ are strictly increasing in y . $\square$

3.2 WEAK CONVERGENCE OF $\{V_N/q\}$. We continue to use the notation introduced in Section 2.3. We state the main results first.

THEOREM 3.2.1. *Suppose* $\sum_1^\infty j^2 \alpha^\delta(j) < \infty$, $\tau(\frac{\delta}{4}) < \infty$, *for some* $0 < \delta < 1$, *and*

$$(3.2.3) \qquad \frac{\Delta(N)}{(\sum c_i^2)^{1/2}} = O(N^{-(1+\delta_o)/4}), \quad \text{where } \delta_o > \delta \text{ as } N \to \infty .$$

If also, the limit σ_V *defined by* (2.3.5) *exists, then, for every* $q \in [Q(\frac{3}{1-\delta})]$, $V_N/q \Rightarrow_d V/q$ *holds, where* V *is the Gaussian random function defined by* (2.3.8). *Also,* $P(V/q \in C[0,1]) = 1$.

THEOREM 3.2.2. *Suppose* $\sum_1^\infty j^2(\phi(j))^{(1+\delta)/2} < \infty$, $\tau(\frac{\delta}{4}) < \infty$, *for some* $0 < \delta < 1$, *and*

$$\frac{\Delta(N)}{(\sum c_i^2)^{1/2}} = o(1) \quad \text{as } N \to \infty .$$

If also, σ_V *defined by* (2.3.5) *exists, then, for every* $q \in [Q(\frac{2}{1-\delta})]$, $V_N/q \Rightarrow_d V/q$ *holds with* $P(V/q \in C[0,1]) = 1$.

REMARKS 3.2.1.

(i) The conditions of Theorem 3.2.1 imply those of Theorem 2.3.1. Similar too is the case with Theorem 3.2.2 and Theorem 2.3.2.

(ii) If $c_i \equiv 1$, Theorem 3.2.2 has an improved version; in this case, its conclusion holds for every $q \in [Q(2)]$ satisfying $\int_0^1 q^{-2} < \infty$ under the mixing condition $\sum_1^\infty j^2 \phi^{1/2}(j) < \infty$. This result is not a corollary to Theorem 3.2.2, but its proof is similar to that of the latter.

(iii) The conclusions of both the theorems are valid with any function g in place of q, provided, there is a $q \in [Q(\cdot)]$ for which $q(t) \le g(t)$, for all $t \in [0,1]$ and all other conditions remain unchanged. In particular, the results hold for uniformly bounded real functions on $[0,1]$.

We recall the notation of Section 2.3. $\xi_i(t) = I(\eta_i \le t) - L_i(t)$, $\tau(x) = \sup_N (N^{-1} \sum |C_i|^{1/x})^x$. Define

$$v_i = \frac{\xi_i(t)}{q(t)} - \frac{\xi_i(s)}{q(s)}, \quad 0 \le s,t \le 1, \tag{3.2.1}$$

where $\xi_i(x)/q(x)$ is defined to be zero if $x = 0$ or 1. As before, all summations run from 1 to N unless, otherwise specified.

LEMMA 3.2.1. *Suppose* $q \in [Q(r)\uparrow]$ *for a fixed* $r \ge 1$. *Then for* $0 < s \le t < 1$, $1 \le m \le r$

(i) $$E\,|v_i|^m \le K\left[\frac{L_i(t)-L_i(s)}{q^m(t)} + \frac{L_i(s)}{q^m(s)}\left(\frac{t-s}{t}\right)\right]$$

(ii) $$\sum E\,|v_i|^m \le \frac{K(t-s)}{q^m(t)}\, N,$$

where $K = K(m)$ *denotes a generic constant, depending only on* m.

PROOF: First note that (ii) can be obtained by summing both sides of (i) from 1 to N, and using $N^{-1} \sum L_i(x) = x$ $(0 \le x \le 1)$ and the fact $s/q^m(s) \le t/q^m(t)$. We prove (i).

Since $I(\eta_i \leq t)$ is a Bernoulli rv taking on values 1 or 0 with probabilities $L_i(t)$ or $1 - L_i(t)$ respectively, we have

$$(3.2.2) \qquad E|v_i|^m = \left|\frac{1 - L_i(t)}{q(t)} - \frac{1 - L_i(s)}{q(s)}\right|^m L_i(s) + \left|\frac{1 - L_i(t)}{q(t)} + \frac{L_i(s)}{q(s)}\right|^m [L_i(t) - L_i(s)] + \left|\frac{L_i(t)}{q(t)} - \frac{L_i(s)}{q(s)}\right|^m [1 - L_i(t)] \; .$$

Note that $q \in [Q(r)\uparrow]$, $r \geq 1$ implies

$$(3.2.3) \qquad q(s)/q(t) \geq s/t \; .$$

Hence

$$(3.2.4) \qquad \left|\frac{1}{q(t)} - \frac{1}{q(s)}\right| = \frac{1}{q(s)} \left(1 - \frac{q(s)}{q(t)}\right) \leq \frac{1}{q(s)} \left(1 - \frac{s}{t}\right) ,$$

using (3.2.3). Similarly,

$$\left|\frac{L_i(t)}{q(t)} - \frac{L_i(s)}{q(s)}\right| \leq \frac{L_i(t) - L_i(s)}{q(t)} + L_i(s) \left|\frac{1}{q(t)} - \frac{1}{q(s)}\right|$$
$$\leq \frac{L_i(t) - L_i(s)}{q(t)} + \frac{L_i(s)}{q(s)} \left(1 - \frac{s}{t}\right) ,$$

from (3.2.4). Now, using the fact $|a+b|^m \leq 2^{m-1} [|a|^m + |b|^m]$ for $m \geq 1$, we have

$$\left|\frac{1-L_i(t)}{q(t)}-\frac{1-L_i(s)}{q(s)}\right|^m L_i(s) \le 2^{m-1}\left[\left|\frac{1}{q(t)}-\frac{1}{q(s)}\right|^m L_i(s) + \left|\frac{L_i(t)}{q(t)}-\frac{L_i(s)}{q(s)}\right|^m L_i(s)\right]$$

$$\le K\left[\frac{L_i(s)}{q^m(s)}\frac{(t-s)}{t} + \frac{L_i(t)-L_i(s)}{q^m(t)}\right] ,$$

from (3.2.3), (3.2.4), and the fact $0 \le L_i(x) \le 1$ for all x. Similarly, using the above mentioned inequality, it is easy to see that the second and third terms on the right of (3.2.2) are also bounded by the same quantity as above. This proves (i). □

LEMMA 3.2.2. *Suppose* $q \in [Q(r)\uparrow]$ *and* $0 < s \le t < 1$. *Then*

(i) $\sum C_i^2 \, E\, v_i^2 \le 2\, N\, \tau^2(\delta_1/2) \dfrac{(t-s)^{1-\delta_1}}{q^2(t)}$, *for* $r \ge \dfrac{2}{1-\delta_1}$

(ii) $\sum_{i<j} |C_i C_j \, E\, v_i v_j| \le K\, \tau^2(\delta_1/2) \dfrac{N(t-s)^{1-\delta_1}}{q^2(t)} \sum \alpha^{\delta_1}(j)$, *for* $r \ge \dfrac{2}{1-\delta_1}$.

(iii) $\sum_{i<j} |C_i C_j \, E\, v_i v_j| \le 4N\, \tau^2(\delta_1/2) \dfrac{(t-s)^{1-\delta_1}}{q^2(t)} \sum \phi^{(1+\delta_1)/2}(j)$, *for* $r \ge \dfrac{2}{1-\delta_1}$.

(iii)' $\sum_{i<j} |E\, v_i v_j| \le \dfrac{2N(t-s)}{q^2(t)} \sum \phi^{1/2}(j)$, *for* $r \ge 2$,

where $0 < \delta_1 < 1$ *is arbitrary, and* $K = K(\delta_1) < \infty$.

PROOF: By letting $a = b = \frac{2}{1-\delta_1}$ and $c = 1/\delta_1$ in (2.1.4), we have

$$\sum_{i<j} |C_i C_j E v_i v_j| \le 12 \sum_{i<j} |C_i C_j| (E|v_i|^a)^{1/a} (E|v_j|^a)^{1/a} [\alpha(j-i)]^{\delta_1}$$

$$= 12 \sum_{i<j} a_i a_j \alpha^{\delta_1}(j-i) \text{ , where } a_i = |C_i| (E|v_i|^a)^{\frac{1}{a}}$$

$$\le 12 \sum_i a_i^2 \sum_j \alpha^{\delta_1}(j) .$$

But

$$\sum a_i^2 = \sum C_i^2 (E|v_i|^a)^{1-\delta_1}$$

$$\le N \tau^2(\tfrac{\delta_1}{2}) (\tfrac{1}{N} \sum_{i=1}^{N} E|v_i|^a)^{1-\delta_1} \text{ , by Hölder's inequality}$$

$$\le K N \tau^2(\tfrac{\delta_1}{2}) \frac{(t-s)^{1-\delta_1}}{q^{a(1-\delta_1)}(t)} \text{ , from (ii) Lemma 3.2.1, since } a > 1$$

$$= K N \tau^2(\tfrac{\delta_1}{2}) \frac{(t-s)^{1-\delta_1}}{q^2(t)} .$$

By substituting this in the above, we get (ii). The proofs of (iii) and (iii)' are similar to above using the ϕ-mixing inequality (2.1.3). Finally, to prove (i) note that from Lemma 3.2.1(i), we have

$$\sum C_i^2 E v_i^2 \le K \sum C_i^2 \{\frac{L_i(t)-L_i(s)}{q^2(t)} + \frac{L_i(s)}{q^2(s)} \frac{(t-s)}{t}\}$$

$$\le N K \tau^2(\tfrac{\delta_1}{2}) [\frac{1}{q^2(t)} \{\frac{1}{N} \sum (L_i(t)-L_i(s))^{1/(1-\delta_1)}\}^{1-\delta_1} +$$

$$+ \frac{1}{q^2(s)} \{\frac{1}{N} \sum L_i^{1/(1-\delta_1)}(s)\}^{1-\delta_1} \frac{(t-s)}{t}]$$

$$\leq N\, K\, \tau^2(\frac{\delta_1}{2}) \, [\frac{(t-s)^{1-\delta_1}}{q^2(t)} + \frac{s^{1-\delta_1}}{q^2(s)} \frac{(t-s)}{t}] \, .$$

Now,

$$s^{1-\delta_1} q^{-2}(s) = [s(q(s))^{-2/(1-\delta_1)}]^{(1-\delta_1)}$$

$$\leq [t(q(t))^{-2/(1-\delta_1)}]^{(1-\delta_1)} \quad , \quad \text{since } \frac{2}{1-\delta_1} \leq r \, .$$

Using this, and the fact $t^{1-\delta_1}(t-s) \leq (t-s)^{1-\delta_1}$, for $0 \leq s \leq t \leq 1$, we get (i) from above. □

LEMMA 3.2.3. *Suppose* $q \in [Q(r)\uparrow]$ *and* $0 < s \leq t < 1$.

(i) *If* $\sum_1^\infty j^2 \alpha^\delta(j) < \infty$, $\tau(\frac{\delta}{4}) < \infty$, *and* $r \geq \frac{3}{1-\delta}$, $0 < \delta < 1$, *then*

$$E(\sum C_i v_i)^4 \leq K(\alpha,\delta) \, [\frac{(t-s)^{2-\delta} N^2}{q^4(t)} + \frac{N(t-s)^{1-\delta} \Delta^4(N)}{q^3(t)\, q(s)}] \, .$$

(ii) *If* $\sum_1^\infty j^2 [\phi(j)]^{(1+\delta)/2} < \infty$, $\tau(\frac{\delta}{4}) < \infty$, *and* $r \geq \frac{2}{1-\delta}$, $0 < \delta < 1$, *then*

$$E(\sum C_i v_i)^4 \leq K(\phi,\delta) \, [\frac{(t-s)^{2-\delta} N^2}{q^4(t)} + \frac{N(t-s)\Delta^4(N)}{q^2(t) q^2(s)}] \, ,$$

where $K(\alpha,\delta)$ *and* $K(\phi,\delta)$ *are both finite; they depend on* τ *also, but this dependence is suppressed for convenience. If* $C_i \equiv 1$, δ *in* (ii) *can be taken as zero.*

PROOF; We use the expansion

$$(\sum c_i v_i)^4 = \sum c_i^4 v_i^4 + 4 \sum_{i \neq j} c_i^3 c_j v_i^3 v_j + 6 \sum_{i<j} c_i^2 c_j^2 v_i^2 v_j^2$$

$$+ 12 \sum_{i<j \neq k} c_k^2 c_i c_j v_i v_j v_k^2 + 24 \sum_{i<j<k<\ell} c_i c_j c_k c_\ell v_i v_j v_k v_\ell .$$

We estimate the expectations term by term. First note that

$$\text{(3.2.5)} \qquad E\, v_i = 0 \ , \ |v_i| \leq \frac{2}{q(s)} \ , \ 1 \leq i \leq N \ .$$

Now,

$$\text{(3.2.6)} \qquad \sum c_i^4 \, E\, v_i^4 \leq \Delta^4(N) \sum E\, v_i^4 \leq \frac{K_1 \Delta^4(N)(t-s)N}{q(s)q^3(t)} \ ,$$

using (3.2.5) and (ii) of Lemma 3.2.1. Again, using (2.1.4), (3.2.5) and (ii) of Lemma 3.2.1,

$$\text{(3.2.7)} \qquad \sum_{i \neq j} |c_i^3 c_j| \, |E\, v_i^3 v_j| \leq \frac{24\Delta^4(N)}{q(s)} \Big[\sum_{i<j} (E|v_i|^{\frac{3}{1-\delta}})^{1-\delta} \alpha^\delta (j-i)$$

$$+ \sum_{j<i} (E|v_i|^{\frac{3}{1-\delta}})^{1-\delta} \alpha^\delta (i-j) \Big]$$

$$\leq \frac{K_2 \, \Delta^4(N)}{q(s)q^3(t)} \, N(t-s)^{1-\delta} \sum \alpha^\delta (j) \ .$$

Further,

$$\sum_{i<j} c_i^2 c_j^2 E\, v_i^2 v_j^2 \leq (\sum c_i^2 \, E\, v_i^2)^2 + \sum_{i<j} c_i^2 c_j^2 |\operatorname{cov}(v_i^2, v_j^2)| \ .$$

Noting that $\frac{4}{2-\delta} \leq \frac{3}{1-\delta}$ $(0 < \delta < 1)$, from Lemma 3.2.2 (i) we have $(\sum C_i^2 E v_i^2)^2 \leq 4N^2\tau^4(\frac{\delta}{4})(t-s)^{2-\delta}q^{-4}(t)$. Also, from (2.1.4), (3.2.5), and Lemma 3.2.1 (ii), we have

$$\sum_{i<j} C_i^2 C_j^2 |\mathrm{cov}(v_i^2;v_j^2)| \leq \frac{24\Delta^4(N)}{q(s)} \sum_{i<j} (E|v_i|^{\frac{3}{1-\delta}})^{\frac{1-\delta}{2}} (E|v_j|^{\frac{3}{1-\delta}})^{\frac{1-\delta}{2}} \alpha^\delta(j-i)$$

$$\leq \frac{24\Delta^4(N)}{q(s)} \sum_i (E|v_i|^{\frac{3}{1-\delta}})^{1-\delta} \sum_j \alpha^\delta(j)$$

$$\leq \frac{24K_3\Delta^4(N)}{q(s)q^3(t)} N(t-s)^{1-\delta} \sum \alpha^\delta(j) .$$

Therefore,

$$\sum_{i<j} C_i^2 C_j^2 E v_i^2 v_j^2 \leq \frac{K_4\Delta^4(N)}{q(s)q^3(t)} N(t-s)^{1-\delta} \sum \alpha^\delta(j) + 4N^2 \tau^4(\tfrac{\delta}{4}) \frac{(t-s)^{2-\delta}}{q^4(t)} . \tag{3.2.8}$$

Now, consider

$$A = \sum_{i<j<k} \min\{|\mathrm{cov}(v_i v_j;v_k^2)| , |\mathrm{cov}(v_i;v_j v_k^2)|\} ,$$

and

$$B = \sum_{i<j<k<\ell} \min\{|\mathrm{cov}(v_i;v_j v_k v_\ell)|,|\mathrm{cov}(v_i v_j;v_k v_\ell)|,|\mathrm{cov}(v_i v_j v_k;v_\ell)|\} .$$

We have

$$|\mathrm{cov}(v_i v_j;v_k^2)| \leq \frac{24}{q(s)} (E|v_i|^{\frac{3}{1-\delta}})^{\frac{1-\delta}{3}} (E|v_k^2|^{\frac{3}{2(1-\delta)}})^{\frac{2(1-\delta)}{3}} \alpha^\delta(k-j)$$

$$|\mathrm{cov}(v_i;v_jv_k^2)| \le \frac{24}{q(s)} (E|v_i|^{\frac{3}{1-\delta}})^{\frac{1-\delta}{3}} (E|v_k^2|^{\frac{3}{2(1-\delta)}})^{\frac{2(1-\delta)}{3}} \alpha^\delta(j-i) .$$

By letting $a_i = E|v_i|^{\frac{3}{1-\delta}}$, we have

$$A \le \frac{24}{q(s)} \sum_{i<j<k} a_i^{\frac{1-\delta}{3}} a_k^{\frac{2(1-\delta)}{3}} \min\{\alpha^\delta(k-j),\alpha^\delta(j-i)\}$$

$$\le \frac{24}{q(s)} \sum_{i+j+k\le N} a_i^{\frac{1-\delta}{3}} a_{i+j+k}^{\frac{2(1-\delta)}{3}} \min\{\alpha^\delta(k),\alpha^\delta(j)\}$$

$$\le \frac{24}{q(s)} \sum_{j+k\le N} \min\{\alpha^\delta(k),\alpha^\delta(j)\} \sum_{i\le N-k-j} a_i^{\frac{1-\delta}{3}} a_{i+j+k}^{\frac{2(1-\delta)}{3}} .$$

Also,

$$\sum_{i\le N-k-j} a_i^{\frac{1-\delta}{3}} a_{i+j+k}^{\frac{2(1-\delta)}{3}} \le (\sum a_i^{\frac{2(1-\delta)}{3}})^{1/2} (\sum a_{i+j+k}^{\frac{4(1-\delta)}{3}})^{1/2}$$

$$\le (\sum a_i^{\frac{2(1-\delta)}{3}})^{1/2} (\sum a_i^{\frac{4(1-\delta)}{3}})^{1/2}$$

$$\le N (\frac{1}{N}\sum a_i)^{\frac{1-\delta}{3}} \cdot (\frac{1}{N}\sum a_i)^{\frac{2}{3}(1-\delta)}$$

$$\le K_5 N \frac{(t-s)^{1-\delta}}{q^3(t)} ,$$

from Lemma 3.2.1 (ii). Thus,

$$(3.2.9) \qquad \sum_{i<j<k} \min\{|\mathrm{cov}(v_iv_j;v_k^2)|,|\mathrm{cov}(v_i;v_jv_k^2)|\}$$

$$\le K_6 N \frac{(t-s)^{1-\delta}}{q(s)q^3(t)} \sum_{j=1}^{N} j\,\alpha^\delta(j) .$$

Now by the mixing inequality (2.1.4) and Hölder's inequality, we have

$$B \leq \frac{24}{q(s)} \sum_{i<j<k<\ell} (a_i a_k a_\ell)^{\frac{1-\delta}{3}} \min\{\alpha^\delta(\ell-k), \alpha^\delta(k-j), \alpha^\delta(j-i)\} ,$$

where $a_i = E|v_i|^{3/(1-\delta)}$,

$$\leq \frac{24}{q(s)} \sum_{j+k+\ell \leq N} \min\{\alpha^\delta(\ell), \alpha^\delta(k), \alpha^\delta(j)\} \sum_{i \leq N-\ell-k-j} (a_i a_{i+j+k} a_{i+j+k+\ell})^{\frac{1-\delta}{3}} .$$

Further, by the generalized Hölder's inequality

$$\sum_{i \leq N-\ell-k-j} (a_i a_{i+j+k} a_{i+j+k+\ell})^{\frac{1-\delta}{3}} \leq (\sum_i a_i^{1-\delta})^{\frac{1}{3}} (\sum_i a_{i+j+k}^{1-\delta})^{\frac{1}{3}} (\sum_i a_{i+j+k+\ell}^{1-\delta})^{\frac{1}{3}}$$

$$\leq \sum a_i^{1-\delta}$$

$$\leq N(\frac{1}{N} \sum a_i)^{1-\delta}$$

$$= N K_7 \frac{(t-s)^{1-\delta}}{q^3(t)}$$

from Lemma 3.2.1 (ii). Thus, from above

$$(3.2.10) \quad \sum_{i<j<k<\ell} \min\{|\mathrm{cov}(v_i;v_j v_k v_\ell)|, |\mathrm{cov}(v_i v_j;v_k v_\ell)|, |\mathrm{cov}(v_i v_j v_k;v_\ell)|\}$$

$$\leq 72\ K_7 N \frac{(t-s)^{1-\delta}}{q(s)q^3(t)} \sum j^2 \alpha^\delta(j) .$$

From (3.2.9), (3.2.10), Lemma 3.2.2, and using similar arguments as for the first term on the right of (2.3.15) and (2.3.17),

$$(3.2.11)\quad \sum_{i<j\neq k} |C_k C_j C_k^2 \; E \; v_i v_j v_k^2| \leq K_8 \left[\frac{\tau^4(\frac{\delta}{4}) N^2 (t-s)^{2-\delta}}{q^4(t)} \sum \alpha^{\delta/2}(j) + \frac{\Delta^4(N) N (t-s)^{1-\delta}}{q(s) q^3(t)} \sum j \, \alpha^{\delta}(j)\right]$$

and

$$(3.2.12)\quad \sum_{i<j<k<\ell} |C_i C_j C_k C_\ell \; E \; v_i v_j v_k v_\ell| \leq K_9 \left[\frac{\tau^4(\frac{\delta}{4}) N^2 (t-s)^{2-\delta}}{q^4(t)} \left(\sum \alpha^{\delta/2}(j)\right)^2 + \frac{\Delta^4(N) N (t-s)^{1-\delta}}{q(s) q^3(t)} \sum j^2 \alpha^{\delta}(j)\right] .$$

Thus, (i) follows by combining (3.2.6) to (3.2.12). The proof of (ii) is similar to above making use of (iii), (iii)' of Lemma 3.2.2 in place of (i). We omit the details. □

PROOF OF THEOREM 3.2.1: Without loss of generality, we assume $C_i \geq 0$ and $\delta_o \leq 1$. First, we will show that under the hypothesis of the theorem, there exist $\theta = \theta(\varepsilon,\alpha,\delta,q,\tau) > 0$, $\theta' = \theta'(\varepsilon,\alpha,\delta,q,\tau) < 1$, and $n_o = n_o(\varepsilon,\delta,q,\tau)$ such that

$$(3.2.13)\qquad P\left[\sup_{\{0<t<\theta\}\cup\{1-\theta'<t<1\}} \frac{|V_N(t)|}{q(t)} \leq \varepsilon\right] \leq 2\varepsilon ,$$

for $N \geq n_o$, where $\varepsilon > 0$ is arbitrary. Consider the supremum of $V_N(t)/q(t)$ over $0 < t < \theta$. In this case, there is no loss of generality in assuming $q \in [Q(\frac{3}{1-\delta})\uparrow]$. Then, for $s < \frac{1}{2}$, from the definition of V_N, Lemma 3.2.3 (i), and Lemma 2.3.2 (i), we have.

(3.2.14) $$E\left|\frac{V_N(t)}{q(t)} - \frac{V_N(s)}{q(s)}\right|^4 \leq K(\alpha,\delta)\left[\frac{(t-s)^{2-\delta}}{q^4(t)} + \frac{(t-s)^{1-\delta}}{q^3(t)q(s)}\frac{\Delta^4(N)}{N}\right],$$

and similarly,

(3.2.15) $$E\left|\frac{V_N(t)}{q(t)}\right|^4 \leq \frac{K(\alpha,\delta)}{q^4(t)}\left[t^{2-\delta} + \frac{\Delta^4(N)}{N}t^{1-\delta}\right]$$

$$\leq K(\alpha,\delta)\left[\frac{(t-s)^{2-\delta}}{q^4(t)} + \frac{(t-s)^{1-\delta}}{q^3(t)q(s)}\frac{\Delta^4(N)}{N}\right],$$

since for $s < t$, $q(s) < q(t)$. Since $\delta_o > \max\{\delta, \frac{1-\delta}{3}\}$ under the hypothesis, there exists δ' satisfying

$$\max\{0, \frac{1-4\delta}{3}\} \leq \delta' < \delta_o - \delta .$$

Then, $4(1+\delta')^{-1} < 3(1-\delta)^{-1}$, so that

$$(t-s)^{1+\delta'} q^{-4}(t) = [(t-s)(q(t))^{-4/1+\delta'}]^{1+\delta'}$$

$$\leq \left(\int_s^t q_{(t)}^{-4/1+\delta'} dt\right)^{1+\delta'}$$

$$= [S(s,t)]^{1+\delta'},$$

where the integral $S(s,t)$ is finite from Lemma 3.1.1. Also, since $q \in [Q(\frac{3}{1-\delta})\uparrow]$, for $s < t$

$$q(t)[q(s)]^{-1} \leq (t \cdot s^{-1})^{(1-\delta)/3}$$

Hence from above and (3.2.14), we get

(3.2.16)
$$E\left|\frac{V_N(t)}{q(t)}-\frac{V_N'(s)}{q(s)}\right|^4$$

$$\leq K[1+\frac{\Delta^4(N)}{N}(t-s)^{-(\delta+\delta')}(\frac{t}{s})^{(1-\delta)/3}][S(s,t)]^{1+\delta'}$$

where K denotes $K(\alpha,\delta)$ for short. Note that $S(0,1)<\infty$, so that $S(0,\theta)\to 0$ as $\theta\to 0$. We choose $0<\theta<\frac{1}{2}$ to satisfy

(3.2.17)
$$\frac{16^4K}{\varepsilon^4}[1+2(\frac{4}{\varepsilon})^{\delta+\delta L}][S(0,\theta)]^{1+\delta'}<\frac{\varepsilon}{2}\ ,$$

where $K<\infty$, is a constant (see (3.2.23), (3.2.26)) depending only on α, δ, δ_o, and τ. Throughout below, we use θ as a fixed value satisfying (3.2.17). Since $t/q^{3/1-\delta}(t)\uparrow t$, we have

(3.2.18)
$$q(t)\geq\frac{t^{(1-\delta)/3}}{K(\delta,q)}\ ,$$

for $t\leq\frac{1}{2}$, where $K(\delta,q)$ is a finite constant, depending only on δ and q. Let $n_o=n_o(\varepsilon,\delta,q,\tau)$ be such that

(3.2.19)
$$8\,\tau^2(\frac{\delta}{2})\,K(\delta,q)\,N^{-(1-\delta)/6}<\varepsilon^{(2+\delta)/6}\ ,$$

for $N\geq n_o$, where $K(\delta,q)$ is as given in (3.2.18). For a fixed $N\geq n_o$ of (3.2.19), and θ given by (3.2.17), choose an integer M satisfying

(3.2.20)
$$\frac{\varepsilon}{4}<\frac{N\theta}{M}<\frac{\varepsilon}{2}\ .$$

Clearly $M=M(N,\theta,\varepsilon)$, and it is of the same order as N (as $N\to\infty$), since ε and θ are fixed positive numbers. For later use, we note that if θ, n_o and M are as defined above, then, for fixed N $(\geq n_o)$

$$(3.2.21)\quad 2N^{1/2}\tau^2(\tfrac{\delta}{2})(\tfrac{\theta}{M})^{\frac{2-\delta}{2}}[q(\tfrac{\theta}{M})]^{-1} = 2(\tfrac{N\theta}{M})^{1/2}\tau^2(\tfrac{\delta}{2})(\tfrac{\theta}{M})^{\frac{1-\delta}{2}}[q(\theta)]^{-1}$$

$$\leq 2(\tfrac{N\theta}{M})^{1/2}\tau^2(\tfrac{\delta}{2})K(\delta,q)(\tfrac{\theta}{M})^{\frac{1-\delta}{6}},$$

from (3.2.18),

$$\leq 2(\tfrac{\varepsilon}{2})^{1/2}(\tfrac{\varepsilon}{2})^{\frac{1-\delta}{6}}\tau^2(\tfrac{\delta}{2})K(\delta,q)N^{-\frac{1-\delta}{6}},$$

from (3.2.20),

$$< \frac{\varepsilon}{4},$$

from (3.2.19). Now, let $0 < s_1 < s_2 < \dots < s_M = \theta$, $s_\ell = \frac{\ell\theta}{M}$, where θ and M are as defined in (3.2.17) and (3.2.20) respectively. For $i \leq j < k \leq M$, we have

$$(s_k-s_j)^{-(\delta+\delta')}(\tfrac{s_k}{s_j})^{\frac{1-\delta}{3}} = (k-j)^{-(\delta+\delta')}(\tfrac{\theta}{M})^{-(\delta+\delta')}(\tfrac{k}{j})^{\frac{1-\delta}{3}}$$

$$= (\tfrac{k}{j(k-j)})^{\frac{1-\delta}{3}}(\tfrac{M}{\theta})^{\delta+\delta'}(k-j)^{\frac{(1-\delta)}{3}-(\delta+\delta')}.$$

But, from our choice of δ', $\frac{1-\delta}{3} < \delta + \delta'$, and $k[j(k-j)]^{-1} \leq 2$ for $1 \leq j < k \leq M$, so that the right hand side of the equation above is dominated by

$$2^{\frac{1-\delta}{3}}(\tfrac{M}{\theta})^{\delta+\delta'}.$$

Further, from the hypotheses (3.2.3) and $\tau(\frac{\delta}{4}) < \infty$, it can be seen that

(3.2.22) $$\frac{\Delta^4(N)}{N} = O(N^{-\delta_o}) \quad , \quad \text{where} \quad \delta_o > \max\{\delta \, , \, \frac{1-\delta}{3}\} \, .$$

Hence, from (3.2.16), (3.2.22), and the fact $\delta + \delta' < \delta_o$, we have

$$E \left|\frac{V_N(s_k)}{q(s_k)} - \frac{V_N(s_j)}{q(s_j)}\right|^4 \leq K[1 + 2(\frac{M}{N\theta})^{\delta+\delta'}] \, [S(s_j,s_k)]^{1+\delta'} \, .$$

Similarly, from (3.2.15) we get

$$E \left|\frac{V_N(s_k)}{q(s_k)}\right|^4 \leq K[1 + 2(\frac{M}{N\theta})^{\delta+\delta'}] \, [S(0,s_k)]^{1+\delta'} \, .$$

Now, an application of Theorem 12.2 of Billingsley [4] yields, for $\lambda > 0$

(3.2.23) $$P[\max_{i \leq M} \left|\frac{V_N(s_i)}{q(s_i)}\right| \geq \lambda] \leq \frac{K}{\lambda^4} \, [1 + 2(\frac{M}{N\theta})^{\delta+\delta'}] \, [S(0,\theta)]^{1+\delta'} \, ,$$

where $K = K(\alpha,\delta,\delta') = K(\alpha,\delta,\delta_o) < \infty$. Further, for $s_i \leq t \leq s_{i+1}$, we have

$$|\sum C_i[I(\eta_i \leq t) - L_i(t)]|$$

$$= |\sum C_i[I(\eta_i \leq t) - I(\eta_i \leq s_i)] + \sum C_i[I(\eta_i \leq s_i) - L_i(s)] - \sum C_i[L_i(t) - L_i(s)]|$$

$$\leq |\sum C_i[I(\eta_i \leq s_{i+1}) - I(\eta_i \leq s_i)]| + N^{1/2}|V_N(s_i)| + \sum C_i[L_i(s_{i+1}) - L_i(s_i)]$$

$$\leq N^{1/2} \, |V_N(s_{i+1})| + 2N^{1/2}|V_N(s_i)| + 2\tau^2(\frac{\delta}{2})(s_{i+1} - s_i)^{(2-\delta)/2} \, N \, .$$

Also, $s_{i+1} - s_i = \theta/M$ and

$$\frac{q(s_{i+1})}{q(s_i)} \le \frac{s_{i+1}}{s_i} = \frac{i+1}{i} \le 2 \text{ , for all } i \ge 1 .$$

Hence from above, for $s_i \le t \le s_{i+1}$, we get

$$\left|\frac{V_N(t)}{q(t)}\right| = \frac{N^{-1/2}}{q(t)} \left|\sum C_i[I(\eta_i \le t) - L_i(t)]\right|$$

$$\le \frac{2|V_N(s_{i+1})|}{q(s_{i+1})} + \frac{2|V_N(s_i)|}{q(s_i)} + \frac{2N^{1/2}\tau^2(\frac{\delta}{2})}{q(\frac{\theta}{M})} (\frac{\theta}{M})^{(2-\delta)/2} ,$$

consequently,

$$(3.2.24) \qquad \sup_{\frac{\theta}{M} \le t \le \theta} \left|\frac{V_N(t)}{q(t)}\right| \le 4 \max_{i \le M} \left|\frac{V_N(s_i)}{q(s_i)}\right| + \frac{2N^{1/2}\tau^2(\frac{\delta}{2})}{q(\frac{\theta}{M})} (\frac{\theta}{M})^{(2-\delta)/2}$$

$$\le 4 \max_{i \le M} \left|\frac{V_N(s_i)}{q(s_i)}\right| + \frac{\varepsilon}{4} ,$$

from (3.2.21). Finally, observe that $I(\eta_i \le t) = 0$ for $i = 1,2,\ldots,N \Rightarrow \sum C_i I(\eta_i \le t) = 0$, so that,

$$(3.2.25) \qquad P[\sup_{0 < t < \frac{\theta}{M}} \left|\frac{V_N(t)}{q(t)}\right| < \frac{\varepsilon}{2}]$$

$$\ge P[\{\sup_{0 < t < \frac{\theta}{M}} N^{-1/2} \frac{\sum C_i L_i(t)}{q(t)} < \frac{\varepsilon}{2}\} \cap \{\sum C_i I(\eta_i \le \frac{\theta}{M}) = 0\}]$$

$$\ge P[\{\sup_{0 < t < \frac{\theta}{M}} \frac{N^{1/2}\tau^2(\frac{\theta}{2}) t^{(2-\delta)/2}}{q(t)} < \frac{\varepsilon}{2}\} \cap \{\bigcap_{i=1}^{N} \{I(\eta_i \le \frac{\theta}{M}) = 0\}\}]$$

$$\ge P[\{\frac{N^{1/2}\tau^2(\frac{\delta}{2})(\frac{\theta}{M})^{(2-\delta)/2}}{q(\frac{\theta}{M})} < \frac{\varepsilon}{2}\} \cap \bigcap_{i=I}^{N} \{I(\eta_i \le \frac{\theta}{M}) = 0\}] .$$

$$= P[\bigcap_{i=1}^{N} \{I(\eta_i \leq \frac{\theta}{M}) = 0\}] \quad , \text{ from (3.2.21)},$$

$$\geq 1 - \sum P[\eta_i \leq \frac{\theta}{M}]$$

$$= 1 - \sum L_i(\frac{\theta}{M})$$

$$= 1 - \frac{N\theta}{M} \quad , \text{ from Lemma 2.3.1 (i)} ,$$

$$> 1 - \frac{\varepsilon}{2} \quad , \text{ from (3.2.20)}.$$

Now, from (3.2.24) and (3.2.25)

(3.2.26)
$$P[\sup_{0<t<\theta} |\frac{V_N(t)}{q(t)}| > \varepsilon]$$

$$\leq P[4 \max_{i \leq M} |\frac{V_N(s_i)}{q(s_i)}| > (\frac{\varepsilon}{2} - \frac{\varepsilon}{4})] + \frac{\varepsilon}{2}$$

$$\leq \frac{16^4 K}{\varepsilon^4} [1 + 2(\frac{M}{N\theta})^{\delta+\delta'}] [S(0,\theta)]^{1+\delta'} + \frac{\varepsilon}{2} \quad ,$$

from (3.2.23),

$$\leq \frac{16^4 K}{\varepsilon^4} [1 + 2(\frac{4}{\varepsilon})^{\delta+\delta'}] [S(0,\theta)]^{1+\delta'} + \frac{\varepsilon}{2} \quad ,$$

from (3.2.20),

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad ,$$

from (3.2.17). By similar arguments as above, it can shown (by considering $\overline{V}_N(t) = V_N(1-t)$ process) that there exists a $\theta' < 1$, satisfying

$$P[\sup_{1-\theta'<t<1} |\frac{V_N(t)}{q(t)}| > \varepsilon] \leq \varepsilon \quad , \text{ for } N \geq n_o .$$

It is also clear from (3.2.17) to (3.2.19) that the choice of θ (and similarly of θ') is independent of N. This completes the proof of (3.2.13). Since, the rest of the proof is similar to that of Theorem 2.1 of [11], we omit it. □

PROOF OF THEOREM 3.2.2: The proof is similar to that of Theorem 3.2.1 and makes use of Lemma 3.2.2 (iii) and Lemmas 3.2.3 (ii). It may also be noted that Lemma 3.2.2 (iii)' and the special case of Lemma 3.2.3 (ii) for $C_i \equiv 1$, are needed for the proof of Remarks 3.2.1 (ii). The details are omitted. □

3.3 JOINT WEAK CONVERGENCE OF U_N AND V_N. Let $D^k[0,1]$ denote the k-fold cartesian product space of $(D[0,1],d)$ with the usual product topology. Let F be a family of probability measures on $D^k[0,1]$ and $F^{(i)}$, the family of corresponding marginal measures on $D[0,1]$. The following proposition is well known (see [4] problem 6, p. 41), but for completeness we include a proof.

PROPOSITION. F *is tight if and only if* $F^{(i)}$ *is tight for* $i = 1,2,..,k$.

PROOF: Suppose $F^{(i)}$ is tight for each i. For $\varepsilon > 0$, there is a $K_\varepsilon^{(i)}$ compact such that $P^{(i)}(K_\varepsilon^{(i)}) > 1 - \frac{\varepsilon}{k}$ for all $P^{(i)} \in F^{(i)}$. Setting $K_\varepsilon = K_\varepsilon^{(1)} \times \ldots \times K_\varepsilon^{(k)}$, we see that K_ε is compact in D^k and

$$P(K_\varepsilon) = 1 - P(K_\varepsilon^c)$$

$$\geq 1 - P[\bigcup_{i=1}^{k} (D \times \dots \times K_\varepsilon^{(i)^c} \times \dots D)$$

$$\geq 1 - \sum_1^k P^{(i)} (K_{\varepsilon_i}^{(i)^c})$$

$$> 1 - \varepsilon ,$$

for all $P \in F$, where c denotes complementation. If F is tight then for $\varepsilon > 0$ there exists K_ε such that $P(K_\varepsilon) > 1-\varepsilon$ for all $P \in F$. Since the projection is continuous in the product topology the projection of K_ε on D say K_ε^* is compact in D. Moreover

$$P^{(i)}(K_\varepsilon^*) = P(D \times \dots \times K_\varepsilon^* \times \dots \times D)$$

$$\geq P(K_\varepsilon) > 1-\varepsilon \quad \text{for all } i . \qquad \Box$$

THEOREM 3.3.1. *Under the conditions of Theorem* 3.2.1 *or Theorem* 3.2.2, $(U_N,V_N) \underset{d_q \times d_q}{\Rightarrow} (U,V)$, *where* U's *and* V's *are as defined in section* 3.2. *Moreover* $P\{(U/q,V/q) \in \tilde{C}^2[0,1]\} = 1$.

PROOF: Assume $q \equiv 1$ for simplicity since the proof is exactly the same for general q. First note that the tightness of the sequence $\{(U_N,V_N)\}$ follows from the tightness of the marginal sequences $\{U_N\}$ and $\{V_N\}$ as noted in the proposition above. Now, as in Theorem 15.1 of [4], it is sufficient to show that the finite dimensional convergence holds. In fact from (2.3.10)

$$\lambda_1 U_N(t) + \lambda_2 V_N(t) = \lambda_1 N^{-1/2} \sum \xi_i(t) + \lambda_2 N^{-1/2} \sum C_i \xi_i(t)$$

$$= N^{-1/2} \sum C_i' \xi_i(t)$$

where $c_i' = \lambda_1 + \lambda_2 C_i$. Since λ_1 and λ_2 are fixed constants, C_i''s satisfy the same conditions as C_i's . Consequently, by the same arguments as in Lemma 2.3.3, $\lambda_1 U_N(t) + \lambda_2 V_N(t) \Rightarrow \lambda_1 U(t) + \lambda_2 V(t)$ for each fixed t . By an extension of this argument as in Lemma 2.3.3, we see that the finite dimensional distributions of (U_N, V_N) converge to those of (U,V) . This completes the first part of the Theorem. The second assertion follows from the fact $P(\frac{U}{q} \in C[0,1]) = P(\frac{V}{q} \in C[0,1]) = 1$.

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CHAPTER IV

ASYMPTOTIC NORMALITY OF SIMPLE LINEAR RANK STATISTICS

This chapter is devoted to the asymptotic normality of a simple linear rank statistic $\sum C_i \psi_N (\frac{R_i}{N+1})$ in the case of non-stationary strong mixing rv's. For the independent not identically distributed rv's, the asymptotic normality of such statistics was established by Hájek [15].

This chapter is based on the ideas developed by Pyke and Shorack [26] (see also [29]), and in particular, the basic idea underlying Lemma 4.4.2 is due to them. All the results that are obtained here are valid for both the strong mixing and ϕ-mixing rv's - the only difference being in terms of the conditions, under which the weak convergence results of Chapter III are valid. It should be noted (see Section 4.5 below) that the two and c-sample problems fall as special cases of our results. However, these are not strictly generalizations of Pyke and Shorack's results, since their class of q functions is larger than ours. Also, we have not attempted to formulate different versions of the main theorem as in [26] and [27], but it is clear that this can be done with the tools we have developed in Chapters II and III.

The main results of this chapter are contained in Section 4.4.

4.1 NOTATION AND PRELIMINARIES. We recall from Section 2.3 that $\{Y_{iN}\}$ is a sequence of rv's having continuous distributions $\{F_i^{(N)}\}$. Throughout below, we assume that the finite dimensional joint-distributions of $\{Y_{iN}\}$ are absolutely continuous with respect to Lebesgue measure. This assumption is needed in order to make the ranks of Y_{iN} unique (upto measure zero). We recall also from Section 2.3 that $F = N^{-1} \sum F_i^{(N)}$, $L_i = F_i^{(N)} F^{-1}$, $\eta_i = F(Y_i)$, $V_N(t) = N^{-1/2} \sum C_i [I(\eta_i \le t) - L_i(t)]$, and $U_N(t) = N^{-1/2} \sum [I(\eta_i \le t) - L_i(t)]$, where N is suppressed in the double subscript as before. We need some additional notation:

$$(4.1.1) \quad \begin{cases} F_N(x) = \frac{1}{N} \sum I(Y_i \le x) \\ H_N(x) = \frac{1}{N} \sum C_i \, I(Y_i \le x) \end{cases}$$

$$(4.1.2) \quad \begin{cases} F^{(N)}(x) = E\, F_N(x) = \frac{1}{N} \sum F_i^{(N)} \\ H^{(N)}(x) = E\, H_N(x) = \frac{1}{N} \sum C_i \, F_i^{(N)} \end{cases}$$

We write F in place of $F^{(N)}$, and H in place of $H^{(N)}$ for the convenience of notation, whenever this does not lead to misunderstanding. Define,

$$(4.1.3) \qquad T_N = \frac{1}{N} \sum C_i \, \psi_N\left(\frac{R_i}{N+1}\right) ,$$

where R_i is the rank of Y_i in the combined ranking of Y_i, $i = 1,2,\ldots,N$, and $\psi_N(\frac{i}{N+1})$ denote certain scores. Let

$$(4.1.4) \quad \begin{cases} r_i = \psi_N\left(\frac{i}{N+1}\right) \\ r_i^* = r_i - r_{i+1} , \quad 1 \le i \le N-1 \\ r_N^* = r_N \end{cases}$$

Let ν_N be the signed measure giving mass r_i^* to i/N ; for $1 \leq i \leq N$ and zero elsewhere. Then from (4.1.4), we have

(4.1.5)
$$\sum_{j=i}^{N} r_j^* = \sum_{j=i}^{N-1} (r_j - r_{j+1}) + r_N$$
$$= r_i \quad , \quad 1 \leq i \leq N \ .$$

If $\{R^{-1}(i)\}$ denote the antiranks (see [14], 63) of $\{R_i\}$, then

$$T_N = \frac{1}{N} \sum r_i \, C^{(i)} , \text{ where } C^{(i)} = C_{R^{-1}(i)}$$

Set $R_o^* = 0$, and

$$R_i^* = N \, H_N \, F_N^{-1}(\tfrac{i}{N})$$
$$= \sum C_j \, I[Y_j \leq F_N^{-1}(\tfrac{i}{N})]$$
$$= \sum_{\{j: Y_j \leq Y^{(i)}\}} C_j \quad ,$$

where $Y^{(i)}$ = ith order statistic of Y_i's . Then, we see that for $N \geq i \geq 1$

$$R_i^* - R_{i-1}^* = C^{(i)}$$

and hence,

(4.1.6)
$$T_N = \frac{1}{N} \sum r_i (R_i^* - R_{i-1}^*)$$
$$= \frac{1}{N} \sum \left(\sum_{j=i}^{N} r_j^*\right)(R_i^* - R_{i-1}^*) \quad \text{by (4.1.5)}$$

$$= \frac{1}{N} \sum r_j^* \sum_{i=1}^{j} (R_i^* - R_{i-1}^*)$$

$$= \frac{1}{N} \sum r_j^* R_j^*$$

$$= \frac{1}{N} \sum r_j^* N H_N F_N^{-1}(\tfrac{j}{N})$$

$$= \int_0^1 H_N F_N^{-1}(t)\, d\nu_N(t) \quad .$$

Define

(4.1.7)
$$\begin{cases} \mu_N = \int_0^1 H F^{-1}(t)\, d\nu_N(t) \\ T_N^* = N^{1/2}(T_N - \mu_N) \\ \Pi_N(t) = N^{1/2}[H_N F_N^{-1}(t) - H F^{-1}(t)] \quad . \end{cases}$$

Then, from above we have

(4.1.8)
$$T_N^* = \int_0^1 \Pi_N(t)\, d\nu_N(t) \quad .$$

Our main concern in the following sections will be studying the asymptotic distribution of T_N^*, via the weak convergence of Π_N, using the techniques developed in Chapter III.

For an f defined on $[0,1]$ we set

(4.1.9)
$$f^o(t) = f(t) \quad , \quad \frac{1}{N} \le t \le 1 - \frac{1}{N}$$
$$= 0 \quad , \quad \text{otherwise.}$$

Since $L_i(t) = F_i^{(N)} F^{-1}(t)$ is absolutely continuous (note that $|L_i(t) - L_i(s)| \leq N|t-s|$) , $\ell_i(t) = d\, L_i(t)/dt$ exists a.e. We set

$$a_N(t) = \frac{1}{N} \sum C_i \, \ell_i(t) \; , \tag{4.1.10}$$

$$\lambda_o = \lim_{N\to\infty} a_N^o(t) \; , \tag{4.1.11}$$

whenever the limit exists,

$$\Pi_{oN}(t) = V(t) - a_N(t)U(t) \; , \tag{4.1.12}$$

and

$$\Pi_o(t) = V(t) - \lambda_o U(t) \; , \tag{4.1.13}$$

where V and U are as defined in Section 2.3.

4.2 A GLIVENKO-CANTELLI LEMMA. Let $\{Y_i\}$ be a sequence of strong mixing rv's with corresponding continuous distribution functions $\{F_i^{(N)}\}$. Let $H_N(x) = N^{-1} \sum C_i \, I(Y_i \leq x)$, $H^{(N)}(x) = EH_N(x)$ as defined in (4.1.1). We need the following

DEFINITION 4.2.1. *A sequence of non-decreasing right continuous functions* $\{G_N(x) \; , \; N \geq 1\}$ *, defined on* $(-\infty, \infty)$ *, is said to be uniformly equi-distributive (ued) if for every* $\varepsilon > 0$ *, there exists* N_ε *, and a positive integer* M *with points* $x_1 < x_2 < \ldots < x_M$ *not depending on* N *, such that*

$G_N(x_j-) - G_N(x_{j-1}) \leq \varepsilon$ *for* $N \geq N_\varepsilon$, $j = 1,2,\ldots,M+1$, *where* $x_o = -\infty$, $x_{M+1} = +\infty$, *and* $G_N(\pm\infty) = \lim_{x \to \pm\infty} G_N(x)$.

REMARK 4.2.1. Some examples of ued families:

(i) If $G_N \equiv G$, where G is a non-decreasing right continuous function with $G(-\infty) = 0$ and $G(\infty) \leq K < \infty$, then $\{G_N\}$ is ued .

(ii) If a sequence of non-decreasing continuous functions $\{G_N\}$, with $G_N(-\infty) = 0$ and $G_N(\infty) \leq K < \infty$ for all N , converges uniformly to a function G , then $\{G_N\}$ is ued.

Proof of (i) follows from the properties of non-decreasing functions and that of (ii) follows from (i), and the fact that the limit of a uniformly convergent sequence of continuous functions is continuous. The following is a version of Glivenko-Cantelli lemma for non-stationary rv's. Let $H^+(x) = N^{-1} \sum C_i^+ F_i^{(N)}$, $H^-(x) = N^{-1} \sum C_i^- F_i^{(N)}$, where C_i^+ and C_i^- are the positive and negative parts of C_i respectively. We have written H in place of $H^{(N)}$ for notational convenience.

LEMMA 4.2.1. *If* $\sum_1^\infty j\,\alpha(j) < \infty$ *and* $\tau(\frac{1}{2}) = \sup_{N \geq 1} (N^{-1} \sum C_i^2)^{1/2} < \infty$, *then*

$$\lim_{N \to \infty} |H_N(x) - H^{(N)}(x)| = 0$$

with probability 1 . *If also,* $\{H^+(x)\}$ *and* $\{H^-(x)\}$ *are ued families, then*

$$P[\lim_{N \to \infty} \sup_{-\infty < x < \infty} |H_N(x) - H^{(N)}(x)| = 0] = 1 .$$

PROOF: Without loss of generality, we assume $C_i \geq 0$. Setting $\xi_i = [I(Y_i \leq x) - F_i^{(N)}(x)]$, we see that $H_N(x) - H^{(N)}(x) = N^{-1} \sum C_i \xi_i$. Following the method of proof of Lemma 2.2.2, and using strong mixing inequality (2.1.4), we get under the hypotheses

$$E(\sum C_i \xi_i)^4 \leq K N^{5/2} ,$$

where K is a constant depending on α and τ only. (This bound is not as sharp as that of Lemma 2.2.2, but this is derived under a weaker mixing condition, and is enough for the present purpose.) The above inequality is equivalent to

$$(4.2.1) \qquad E\,|H_N(x) - H^{(N)}(x)|^4 \leq K N^{-3/2} .$$

Hence, by Markov's inequality, for $\eta > 0$ we have

$$P[|H_N(x) - H^{(N)}(x)| > \eta] \leq \frac{E|H_N(x) - H^{(N)}(x)|^4}{\eta^4}$$

$$\leq \frac{K N^{-3/2}}{\eta^4} ,$$

from (4.2.1). Since $\sum_1^\infty N^{-3/2} < \infty$, the Borel-Cantelli Lemma now implies $P[\limsup_{N\to\infty} |H_N(x) - H^{(N)}(x)| = 0] = 1$ for each x . This proves the first part of the lemma.

Let $\varepsilon > 0$ be arbitrary. Since $\{H^{(N)}(x)\}$ is eud, there exist M and $x_1 < x_2 < \ldots < x_M$, not depending on N , such that

$$(4.2.2) \qquad H^{(N)}(x_j) - H^{(N)}(x_{j-1}) \leq \frac{\varepsilon}{4} \text{ for } N \geq N_1(\varepsilon) \text{ and } j = 1,2,\ldots,M+1 ,$$

where $x_o = -\infty$, $x_{M+1} = +\infty$.

Also, from the first part of the lemma, we have

$$(4.2.3) \quad \max_{1 \le j \le M} |H_N(x_j) - H^{(N)}(x_j)| \le \frac{\varepsilon}{4} \quad \text{for } N \ge N_2(x_1,\ldots,x_M,\varepsilon) ,$$

except on a P-null set. Note that for each x, $x_{i-1} \le x \le x_i$,

$$|H_N(x) - H^{(N)}(x)| \le |H_N(x_i) - H^{(N)}(x_{i-1})| + |H_N(x_{i-1}) - H^{(N)}(x_i)|$$

$$\le |H_N(x_i) - H^{(N)}(x_i)| + |H_N(x_{i-1}) - H^{(N)}(x_{i-1})|$$

$$+ 2|H^{(N)}(x_i) - H^{(N)}(x_{i-1})| .$$

Hence,

$$\sup_{-\infty<x<\infty} |H_N(x) - H(x)| \le 2 \max_{1 \le i \le M} |H_N(x_i) - H^{(N)}(x_i)| + 2 \max_{1 \le i \le M+1} |H^{(N)}(x_i) - H_N(x_{i-1})|$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $N \ge \max(N_1, N_2)$, from (4.2.2) and (4.2.3). This proves the second part of the lemma. □

COROLLARY 4.2.1. *Under the conditions of the above lemma, we have*

$$\rho(HH_N^{-1}, HH^{-1}) \xrightarrow[\text{a.s.}]{} 0, \textit{ and}$$

$$\rho(FF_N^{-1}, FF^{-1}) \xrightarrow[\text{a.s.}]{} 0 , \textit{ as } N \to \infty ,$$

where ρ denotes the supremum metric.

PROOF: We have,

$$\rho(HH_N^{-1}, HH^{-1}) \leq \rho(HH_N^{-1}, H_N H_N^{-1}) + \rho(H_N H_N^{-1}, HH^{-1})$$

$$\leq \rho(H_N, H) + \frac{\Delta(N)}{N}, \quad \text{where } \Delta(N) = \max_{i \leq N} |C_i|$$

$$\xrightarrow[a.s.]{} 0,$$

as $N \to \infty$, because of Lemma 4.2.1, and the fact that $\tau(\frac{1}{2}) < \infty$ implies $\Delta(N)/N \to 0$ as $N \to \infty$. This proves the first part, and the second follows from this. □

NOTE: Since $H = N^{-1} \sum C_i F_i^{(N)}$ and $F = N^{-1} \sum F_i^{*(N)}$, the continuity of $F_i^{(N)}$ implies that $HH^{-1}(t) = FF^{-1}(t) = t$. In this case $\lim_{N\to\infty} \rho(HH_N^{-1}, t) = \lim_{N\to\infty} \rho(FF_N^{-1}, t) = 0$ a.s., under the conditions of the above lemma.

4.3 THE PROCESS $\{\Pi_N(t) : 0 \leq t \leq 1\}$. From (4.1.1), we note that

$$(4.3.1) \quad \begin{cases} U_N(t) = N^{1/2}[F_N F^{-1}(t) - t] \\ V_N(t) = N^{1/2}[H_N F^{-1}(t) - HF^{-1}(t)] \end{cases}$$

From (4.1.1) and (4.1.7) we then have

$$(4.3.2) \quad \Pi_N(t) = N^{1/2}[H_N F_N^{-1}(t) - HF^{-1}(t)]$$

$$\overset{a.s.}{=} N^{1/2}\{[H_N F^{-1}(FF_N^{-1}(t)) - HF^{-1}(FF_N^{-1}(t))]\}$$

$$+ [HF_N^{-1}(t) - HF^{-1}(t)]\}$$

$$\overset{a.s.}{=} V_N(FF_N^{-1}(t)) + \frac{HF^{-1}(FF_N^{-1}(t)) - HF^{-1}(t)}{FF_N^{-1}(t) - t} N^{1/2}[FF_N^{-1}(t)-t] \quad .$$

The ratio on the right side of the last equation is defined by left continuity, at those points (at most a finite number of them) where $FF_N^{-1}(t) = t$. Now, by letting

$$\delta_N^*(t) = N^{1/2}[F_N F_N^{-1}(t) - t]$$

we have,

$$(4.3.3) \quad N^{1/2}[FF_N^{-1}(t)-t] = N^{1/2}[FF_N^{-1}(t) - F_N F_N^{-1}(t)] + \delta_N^*(t)$$

$$\overset{a.s.}{=} -N^{1/2}[F_N F^{-1}(FF_N^{-1}(t)) - FF_N^{-1}(t)] + \delta_N^*(t)$$

$$= -U_N(FF_N^{-1}(t)) + \delta_N^*(t) \quad .$$

Substituting (4.3.3) in (4.3.2), we have

$$(4.3.4) \quad \Pi_N(t) \overset{a.s.}{=} V_N(FF_N^{-1}(t)) - A_N(t)\, U_N(FF_N^{-1}(t)) + \delta_N(t) \quad ,$$

where

$$(4.3.5) \quad \left.\begin{aligned} A_N(t) &= \frac{HF^{-1}(h_t) - HF^{-1}(t)}{h_t - t} \\ h_t &= FF_N^{-1}(t) \\ \delta_N(t) &= \delta_N^* A_N(t) \quad . \end{aligned}\right\}$$

For later use we note that

$$(4.3.6) \qquad |\delta_N^*(t)| \le N^{1/2} |F_N F_N^{-1}(t) - t| \le N^{1/2} \frac{1}{N} = N^{-1/2} .$$

We recall the class of functions $[Q^*(r)]$ defined in Section 3.1. Note that $[Q^*(r)] \subset [Q(r)]$.

LEMMA 4.3.1. *If* $q \in [Q^*(r)]$, $r \ge \frac{2}{1-\delta}$ *and* $\tau(\frac{\delta}{4}) < \infty$, *for some* δ $(0<\delta<1)$, *then with probability* 1

$$\sup_{1-\frac{1}{N} \le t \le 1} \frac{\Pi_N(t)}{q(t)} \to 0 \quad as \quad N \to \infty ,$$

PROOF: For $\frac{N-1}{N} \le t \le 1$ we have $Y_i \le F_N^{-1}(t)$ so that

$$H_N F_N^{-1}(t) = \frac{1}{N} \sum C_i \; I(Y_i \le F_N^{-1}(t)) = \frac{1}{N} \sum C_i .$$

Hence,

$$|\Pi_N(t)| = N^{1/2} |H_N F_N^{-1}(t) - HF^{-1}(t)|$$

$$\underset{a.s.}{=} N^{1/2} \left| \frac{1}{N} \sum C_i - \frac{1}{N} \sum C_i L_i(t) \right|$$

$$= N^{1/2} \left| \frac{1}{N} \sum C_i (1 - L_i(t)) \right|$$

$$\le N^{1/2} \tau^4 (\tfrac{\delta}{4}) (1-t)^{1 - \frac{\delta}{4}} ,$$

since $\sum L_i(t) = Nt$. Therefore,

$$\sup_{1-\frac{1}{N}\le t\le 1}\left|\frac{\Pi_N(t)}{q(t)}\right| \le \tau^4\left(\frac{\delta}{4}\right) N^{1/2} \sup_{1-\frac{1}{N}\le t\le 1} \frac{(1-t)^{1-\frac{\delta}{4}}}{q(t)}$$

$$\le \tau^4\left(\frac{\delta}{4}\right)\frac{N^{1/2}\cdot N^{\frac{\delta}{4}-1}}{q(1-\frac{1}{N})}\ , \quad \text{since } \frac{(1-t)^{\frac{2}{1-\delta}}}{q(t)} \downarrow \text{ in } [\tfrac{1}{2},1]$$

$$= \frac{\tau^4(\frac{\delta}{4})}{N^{\frac{2-\delta}{4}}\, q(\frac{1}{N})}$$

by the symmetry of q . Now,

$$\frac{1}{N\, q^{\frac{4}{2-\delta}}(\frac{1}{N})} \le \int_0^{1/N} q^{-\frac{4}{2-\delta}}(t)\, dt \to 0$$

as $N \to \infty$ by Lemma 3.1.1. Hence from above

$$\sup_{1-\frac{1}{N}\le t\le 1}\left|\frac{\Pi_N(t)}{q(t)}\right| \le \tau^4\left(\frac{\delta}{4}\right)\left[\frac{1}{N\, q^{\frac{4}{2-\delta}}(\frac{1}{N})}\right]^{\frac{2-\delta}{4}} \to 0 \text{ as } N\to\infty\ . \quad \square$$

4.4 ASYMPTOTIC NORMALITY OF T_N^*. In this section, we adopt the Pyke and Shorack [26] approach making use of equivalent processes $\tilde{U}_N$, $\tilde{V}_n$ in the sense of Item 3.1.1 of Skorohod [35]. As we have already shown (Theorem 3.3.1) that $x_N = (U_N/q, V_N/q)$ converges weakly to $x = (\frac{U}{q}, \frac{V}{q})$, and $D^2[0,1]$ is separable, there exist equivalent processes (see Corollary 2.1 of [11]) $\tilde{x}_N = (\tilde{U}_N/q, \tilde{V}_N/q)$ and $x = (\tilde{U}/q, \tilde{V}/q)$ for which $d_q(\tilde{x}_N, \tilde{x}) \to$ a.s. since $P(\tilde{x} \in C^2[0,1]) = 1$, $\rho_q(\tilde{x}_N, \tilde{x}) \to 0$ a.s. Consequently, we also have $\rho_q(\tilde{U}_N, \tilde{U}) \to 0$, $\rho_q(\tilde{V}_N, \tilde{V}) \to 0$ a.s. We define $\tilde{F}_N = N^{-1/2}\,\tilde{U}_N(F) + F$ and $\tilde{H}_N = N^{-1/2}\,\tilde{V}_N(F) + H$. It should be noted that $\tilde{F}_N$ and F_N have the same probabilistic properties, since $\tilde{U}_N$ and U_N

are equivalent. Similar remarks apply to $\tilde{H}_N$ and H_N. We also define $\tilde{\Pi}_N$ and $\tilde{\Pi}_{oN}$ of (4.3.4) and (4.3.7) similarly, by replacing U_N and V_N by $\tilde{U}_N$ and $\tilde{V}_N$ respectively. In what follows (in this section) we shall deal with these equivalent processes only - though '~' is omitted for notational simplicity.

We use the notation previously introduced.

LEMMA 4.4.1. *If* $\tau(\frac{\delta}{4}) < \infty$ *and* $q \in [Q^*(r)]$, $r \geq \frac{2}{1-\delta}$ *for some* $0 < \delta < 1$, *then*

$$\sup_{0 \leq t \leq 1} \frac{\delta_N^{*o}(t)}{q(t)} \xrightarrow[\text{a.s.}]{} 0 \quad as \quad N \to \infty .$$

where o - *notation is given by* (4.1.9).

PROOF: We recall from (4.3.5) and (4.3.6)

$$|\delta_N(t)| = |\delta_N^*(t)\, A_N(t)|$$

$$\leq N^{-1/2}\, |A_N(t)| .$$

Also from (4.3.5),

$$A_N(t) = \frac{H\, F^{-1}(h_t) - HF^{-1}(t)}{h_t - t}$$

$$= \frac{1}{N} \sum_i C_i \left[\frac{L_i(h_t) - L_i(t)}{h_t - t}\right]$$

$$\underset{\text{a.s.}}{\leq} \Delta(N) , \quad \text{where } \Delta(N) = \max_{i \leq N} C_i ,$$

since $\sum L_i(x) = Nx$, $0 \le x \le 1$. Moreover, $\tau(\frac{\delta}{4}) < \infty \Rightarrow \Delta(N) = O(N^{\delta/4})$ as $N \to \infty$. Hence from above,

$$\sup_{0\le t\le 1} \frac{|\delta_N^{*o}(t)|}{q(t)} = \sup_{\frac{1}{N}\le t\le 1-\frac{1}{N}} \frac{|\delta_N^{*}(t)|}{q(t)}$$

$$\le \sup_{\frac{1}{N}\le t\le 1-\frac{1}{N}} \frac{N^{-1/2}\,\Delta(N)}{q(t)}$$

$$\le K \cdot N^{-(\frac{1}{2}-\frac{\delta}{4})}\, q^{-1}(\tfrac{1}{N}) \to 0$$

as in Lemma 4.3.1. □

Now, we introduce the following assumptions on $F^{(N)} = N^{-1}\sum F_i^{(N)}$ and $a_N(t) = N^{-1}\sum c_i\, \ell_i(t)$ defined by (4.1.10).

ASSUMPTION 4.4.1.

(i) $a_N(t)$ exists at each point in $(0,1)$ with finite one sided limits at 0 and 1, and for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that

$$|a_N(t)-a_N(s)| < \varepsilon, \text{ for all } N \ge N_\varepsilon \text{ and } |t-s| < \delta_\varepsilon .$$

A sequence of functions satisfying (i) is called *uniformly equi-continuous* (or, uec, see condition (C4) of [27]).

(i)' $a_N(t)$ exists at each point in $(0,1)$ and is uniformly bounded with finite one sided limits at 0 and 1, and satisfies

$$a_N^o(t) \to \lambda_o \quad \text{uniformly in } t, \text{ as } N \to \infty,$$

where $\lambda_o < \infty$ is a constant.

(ii) The family $\{F^{(N)}(x)\}$ is ued (see Definition 4.2.1).

LEMMA 4.4.2. *Suppose the conditions of Theorem* 3.2.1 *and Assumption* 4.4.1 (ii) *are satisfied. Then, for every* $q \in [Q_\nu^*(r)]$, $r \geq \frac{2(2+\delta)}{1-\delta}$, $\rho_q((V_N FF_N^{-1})^o, V) \underset{P}{\to} 0$ *as* $N \to \infty$. *If the conditions of Theorem* 3.2.2 *hold, the above conclusion holds for every* $q \in [Q^*(r)]$, $r \geq \frac{2(1+\delta)}{1-\delta}$.

PROOF: Let $\bar{q}$ be the function as given by Lemma 3.1.2 with $r' = \frac{r}{2} + 1$, $b = 1$, and $c = 2$. As in the proof of Theorem 3.2.1 (see also the proof of Theorem 2.2. of [22]), it is sufficient to prove, for $\varepsilon > 0$ and N large,

$$(4.4.1) \qquad P[\sup_{\frac{1}{N} < t < \theta} |\frac{V_N(FF_N^{-1}(t))}{q(t)}| \leq \varepsilon] \geq 1 - 2\varepsilon$$

for some θ independent of N , where $0 < \theta < \frac{1}{2}$. Since $\bar{q} \in [Q^*(r')]$, $r' \geq \frac{3}{1-\delta}$, we have from (3.2.13) a θ^* (independent of N) > 0 and

$$S_N = \{|V_N(t)| \leq \varepsilon\bar{q}(t) \text{ and } |U_N(t)| \leq \varepsilon\bar{q}(t) , 0 \leq t \leq \theta^*\}$$

with $P(S_N) \geq 1-\varepsilon$ for N large. Without loss of generality (since we are considering lower tail probabilities), we assume $\bar{q} \in [Q^*(r')\uparrow]$. Now, by Corollary 4.2.1, $\sup_t |FF_N^{-1}(t) - t| \to 0$ a.s. as $N \to \infty$. Hence there exists $\theta(0 < \theta < \frac{1}{2})$ such that

$$P(S_N') \geq 1-\varepsilon , \quad \text{for } N \text{ large,}$$

where $S_N' = \{FF_N^{-1}(t) \leq \theta^* \text{ for } 0 \leq t \leq \theta\}$. Now, noting that $FF_N^{-1} = -N^{-1/2} U_N(FF_N^{-1}) + F_N F_N^{-1}$, we have on S_N

$$\bar{q}(FF_N^{-1}) = \bar{q}(F_N F_N^{-1} - N^{-1/2} U_N(FF_N^{-1}))$$

$$\leq \bar{q}(F_N F_N^{-1} + N^{-1/2} \varepsilon \bar{q}(FF_N^{-1})) \text{ , since } \bar{q} \in [Q^*(r')\uparrow] ,$$

$$= p(F_N F_N^{-1} : N^{-1/2} \varepsilon) ,$$

where $p(u:v) = \bar{q}(u+v\bar{q}(u+\ldots))$. The function p clearly satisfies the functional equation $p(u:v) = \bar{q}(u+vp(u,v))$, and it is increasing in both the arguments. Since, $t \geq \frac{1}{N}$ implies $F_N F_N^{-1}(t) \leq 2t$ and $N^{-1/2} \leq t^{1/2}$, we have

$$\bar{q}(FF_N^{-1}) \leq p(2t:t^{1/2}) = q(t) .$$

Hence,

$$P[|V_N(FF_N^{-1}(t))| \leq \varepsilon\, q(t) \text{ for } \tfrac{1}{N} \leq t \leq \theta]$$

$$\geq P[\{|V_N(FF_N^{-1}(t))| \leq \varepsilon\, q(t) , \tfrac{1}{N} \leq t \leq \theta\} \cap S_N \cap S_N']$$

$$\geq P[\{|V_N(FF_N^{-1}(t))| \leq \varepsilon\, \bar{q}(FF_N^{-1}(t)) , \tfrac{1}{N} \leq t \leq \theta\} \cap S_N \cap S_N']$$

$$= P[S_N \cap S_N']$$

$$\geq 1 - 2\varepsilon \text{ , for } N \text{ large.} \quad \square$$

COROLLARY 4.4.1. *Under the conditions of Lemma 4.4.2, we have*

$$\rho_q((U_N F F_N^{-1})^o, U) \underset{P}{\to} 0 \quad as \quad N \to \infty .$$

PROOF: It is enough to note that V_N is identical with U_N when [illegible] . □

Let Π'_N be defined by

$$\Pi'_N(t) = \Pi_N(t) \quad \text{for} \quad \frac{1}{N} \le t \le 1$$

$$= 0 \quad \text{for} \quad 0 \le t < \frac{1}{N} .$$

Since Π'_N is left continuous, ρ_q below is to be understood as the uniform metric on $D^-[0,1]$, the set of left continuous functions as defined in [26].

LEMMA 4.4.3. *Suppose the conditions of Theorem 3.2.1 (3.2.2) are satisfied. Then,*

Assumption 4.4.1 (i) and (ii) imply

$$\rho_q(\Pi'_N, \Pi_{oN}) \underset{P}{\to} 0 \quad as \quad N \to \infty , \quad and$$

Assumption 4.4.[illegible] *(i) and (*[illegible]*) imply*

$$\rho_q(\Pi'_N, \text{[illegible]}) \underset{P}{\to} 0 \quad as \quad N \to \infty \quad for$$

every $q \in [Q^*(r)]$, $r \ge \frac{2(2+\delta)}{1-\delta}$ $\left(r \ge \frac{2(1+\delta)}{1-\delta}\right)$, *where* Π'_N , Π_{oN} , *and* Π_o *are as defined by* (4.3.4), (4.1.12), *and* (4.1.13) *respectively.*

PROOF: We recall from (4.1.10) that $a_N(t) = N^{-1} \sum C_i \phi_i(t) = d\, HF_*^{-1}(t)/dt$. If Assumption 4.4.1 (ii) holds, then from Corollary 4.2.1, we have $h_t = FF_*^{-1}(t) \to t$ uniformly in t , except on a P-null set. Hence, by the mean value theorem, there exists t_N $(t \le t_N \le h_t$ or $h_t \le t_N \le t)$ for which (with probability 1)

(4.4.2) $$A_N(t) = \frac{HF_*^{-1}(h_t) - HF_*^{-1}(t)}{h_t - t} = a_N(t_N) ,$$

and

(4.4.3) $$|t_N - t| \le \delta_N , \text{ where } \delta_N \to 0 \text{ as } N \to \infty .$$

If Assumption 4.4.1 (i) also holds, then, clearly

(4.4.4) $$\rho(A_N, a_N) \underset{a.s.}{\le} \sup_{0\le t\le 1} |a_N(t_N) - a_N(t)| \to 0 \text{ as } N \to \infty ,$$

from (4.4.3). Also,

(4.4.5) $$\sup_{N\ge 1} \rho(A_N, 0) \underset{a.s.}{\le} \rho(a_N, 0) < \infty$$

Now

(4.4.6) $$\rho_q(\Pi_N, \Pi_{oN}) \le \rho_q(\Pi_N^o, \Pi_{oN}^o) + \sup_{0\le t<\frac{1}{N}} \frac{|\Pi_{oN}(t)|}{q(t)}$$
$$+ \sup_{1-\frac{1}{N}<t\le 1} \frac{|\Pi_{oN}(t)|}{q(t)} + \sup_{1-\frac{1}{N}<t\le 1} \frac{|\Pi_N(t)|}{q(t)}$$

and from (4.3.4) and (4.1.12),

(4.4.7) $$\rho_q(\Pi_N^o, \Pi_{oN}^o) \le \rho_q((V_N FF_N^{-1})^o, V) + \rho(A_N, a_N)\, \rho_q(U, o)$$
$$+ \rho(A_N, o)\, \rho_q((U_N FF_N^{-1})^o, U)$$

$$+ \rho_q(\delta_N^{*o}) \,.$$

The right side of (4.4.7) goes to zero in probability, as $N \to \infty$, by Lemma 4.4.2 (and its Corollary), (4.4.4), (4.4.5), and Lemma 4.4.1. Also, the second and third terms on the right side of (4.4.6) go to zero in probability because of Theorem 3.2.1 and (4.4.5). An application of Lemma 4.3.1 now shows that $\rho_q(\pi_N', \pi_{oN}) \to 0$ in probability as $N \to \infty$. This completes the proof of the first assertion.

Suppose now, Assumption 4.4.1 (i)' and (ii) hold. First note that (4.4.2) and (4.4.3) are valid in this case. Also, under the hypothesis (i)', it can be seen that

(4.4.8)
$$\sup_{\frac{1}{N} \le s \le 1-\frac{1}{N}} |a_N(t)-a_N(s)| \to 0 \,, \text{ as } N \to \infty \,.$$

Further,

(4.4.9)
$$\rho_q(\Pi_N', \Pi_o) \le \rho_q(\Pi_{oN}^o, \Pi_o) + \rho_q(\Pi_N^o, \Pi_{oN}^o) + \sup_{1-\frac{1}{N} < t \le 1} \frac{|\Pi_N(t)|}{q(t)}$$

Since $a_N(t)$ is uniformly bounded, the first and last terms on the right of (4.4.9) go to zero in probability as a consequence of Theorem 3.2.1 and Lemma 4.3.1. The proof will be completed by showing that the second term is also negligible under the hypothesis. Now, from (4.3.4), (4.1.12), and (4.4.2), we have

$$\Pi_N(t) - \Pi_{oN}(t) \underset{a.s.}{=} V_N(FF^{-1}(t)) - a_N(t_N)U_N(FF_N^{-1}(t)) - V(t) + a_N(t)U(t) \,.$$

Hence,

$$\rho_q(\Pi_N^o, \Pi_{oN}^o) \leq \rho_q((V_N F F_N^{-1})^o, V) + \rho(a_N, 0)\rho_q((U_N F F_N^{-1})^o, U)$$

$$+ \sup_{\frac{1}{N} \leq t \leq 1-\frac{1}{N}} [|a_n(t_N) - a_N(t)| \frac{|U(t)|}{q(t)}] .$$

But, from (4.4.3), the last term on the right is dominated by (with probability 1),

$$\sup_{\substack{\frac{1}{N} \leq t \leq 1-\frac{1}{N} \\ |t-s| \leq \delta_N}} [|a_N(t) - a_N(s)| \frac{|U(t)|}{q(t)}]$$

$$\leq \sup_{\frac{1}{N} \leq s,t \leq 1-\frac{1}{N}} |a_N(t) - a_N(s)| \rho_q(U,0) + \rho(a_N, 0) \sup_{0 \leq t \leq \frac{1}{N} + \delta_N} \frac{|U(t)|}{q(t)}$$

$$\xrightarrow{P} 0 ,$$

from (4.4.8), Theorem 3.2.1, and the hypothesis that a_N is uniformly bounded. Since the first and second terms also go to zero in probability by Lemma 4.4.2 and its corollary, it follows from above that $\rho_q(\Pi_N^o, \Pi_{oN}^o) \to 0$ in probability as required. □

Now we prove the main result of this chapter. We state the theorem for strong mixing rv's - the version of the theorem for ϕ-mixing rv's can be obtained by replacing the hypotheses that correspond to ϕ-mixing case (i.e. Theorem 3.2.2 and Lemma 4.4.3).

THEOREM 4.4.1. *Suppose the conditions of Theorem 3.2.1 are satisfied and* [(0,r)], $r \geq \frac{2(2+\delta)}{1-\delta}$, $0 < \delta < 1$. *Also, the measures* ν_N *(defined* *just after (4.1.4) and* ν *satisfy*

(i) $\int_{1/N}^{1} \Pi_N \, d(\nu_N - \nu) \underset{P}{\to} 0$, *as* $N \to \infty$,

(ii) $\int_0^1 q \, d|\nu| < \infty$.

If also, Assumption 4.4.1 (i)' and (ii) are satisfied, then

$$|T_N^* - \int_0^1 \Pi_o \, d\nu| \underset{P}{\to} 0 \quad as \quad N \to \infty,$$

where T_N^* *and* Π_o *are given by (4.1.7) and (4.1.13), and*

$$\int_0^1 \Pi_o \, d\nu \quad is\ a \quad N(0,\sigma_o^2) \quad rv\ with$$

(4.4.10) $$\sigma_o^2 = \int_0^1 \int_0^1 E.\Pi_o(t)\Pi_o(s) \, d\nu(t) \, d\nu(s) < \infty .$$

If, Assumption 4.4.1 (i) and (ii) (instead of (i)' and (ii)) are satisfied, then the above conclusion holds with Π_{oN} *(defined by (4.1.12)) in the place of* Π_o.

PROOF: We have

$$|T_N^* - \int_0^1 \Pi_o \, d\nu| = |\int_0^1 \Pi_N' \, d(\nu_N - \nu) + \int_0^1 (\Pi_N' - \Pi_o) d\nu|$$

$$\leq |\int_0^1 \Pi_N' \, d(\nu_N - \nu)| + \rho_q(\Pi_N', \Pi_o) \int_0^1 q \, d|\nu|$$

$$\underset{P}{\to} 0$$

from (i) and (ii) above, and Lemma 4.4.3, as $N \to \infty$. Also, from the definition of Π_o, it is clear that $\int_0^1 \Pi_o \, d\nu$ is a Gaussian random variable, provided the double integral defined by σ_o^2 is finite. Now, since $\Pi_o(t) = V(t) - \lambda_o U(t)$ and $U(t)$ is a special case of $V(t)$ (when $C_i \equiv 1$), the finiteness of σ_o^2 follows from that of $\int_0^1 \int_0^1 E\, V(t)V(s)\, d\nu(t) d\nu(s)$. Note that from (2.3.4), (2.1.4), Hölder's inequality, and the fact $N^{-1} \sum L(x) = x$, $0 \leq x \leq 1$, we have for $s \leq t$,

$$|\sigma_{V_N}(x,t)| \leq \tau^2(\tfrac{\delta}{4}) \{[s(1-t)]^{\frac{2-\delta}{4}} + 2(st)^{\frac{2-\delta}{4}} \sum_1^\infty \alpha^{\delta/2}(j)\},$$

which from (2.3.5) and (2.3.8) implies

$$\begin{aligned} |EV(t)V(s)| &= |\sigma_V(s,t)| \\ &\leq \tau^2(\tfrac{\delta}{4}) \{[s(1-t)]^{\frac{2-\delta}{4}} + 2(st)^{\frac{2-\delta}{4}} \sum_1^\infty \alpha^{\delta/2}(j)\} \\ &\leq \tau^2(\tfrac{\delta}{4}) \{1 + 2 \sum_1^\infty \alpha^{\delta/2}(j)\} [s(1-s)t(1-t)]^{\frac{2-\delta}{8}} \\ &= K[s(1-s)t(1-t)]^{\frac{2-\delta}{8}}. \end{aligned}$$

Since $q \in [Q^*(r)]$, $r \geq \frac{2(2+\delta)}{1-\delta}$, and $\int_0^1 q \, d|\nu| < \infty$, it follows that

$$\begin{aligned} &\int_0^1 \int_0^1 |E\, V(t)V(s)| \, d|\nu(t)| \, d|\nu(s)| \\ &\leq K \int_0^1 \int_0^1 \frac{[s(1-s)t(1-t)]^{\frac{2-\delta}{8}}}{q(t)\, q(s)} \, q(t)q(s) \, d|\nu(t)| \, d|\nu(s)| \\ &< \infty \end{aligned}$$

$\square$

4.5 ON THE MAIN ASSUMPTIONS. In this section we discuss some conditions under which Assumption 4.4.1 is valid. We start with Assumption 4.4.1 (ii), which requires ued (Definition 4.2.1) property of the sequence $\{F^{(N)}(x)\}$, where $F^{(N)}(x) = N^{-1} \sum F_i^{(N)}$.

LEMMA 4.5.1. *The sequence* $\{F_i^{(N)}\}$ *is ued in the following two cases:*

(i) $F_i^{(N)} = F_o$ *for all* i *and* N,

(ii) $F_i^{(N)}(x) = F_o(x-d_{iN})$, *where* F_o *has a bounded continuous derivative, and* $\max_{i \leq N} d_{iN} \to 0$ *as* $N \to \infty$.

PROOF: (i) follows directly from Remark 4.2.1 (i). To prove (ii) note that by the Taylor expansion, we have $F(x-d_{iN}) = F_o(x) - d_{iN} F_o'(x_{iN})$, where x_{iN} depends on x and d_{iN}. Since $\max_{i \leq N} d_{iN} \to 0$ as $N \to \infty$, and F_o' is bounded, it follows that $|F_i^{(N)}(x) - F_o(x)| \to 0$ uniformly in x. The result (ii) follows from Remark 4.2.1 (ii). □

REMARK 4.5.1. The first part of Lemma 4.5.1 corresponds to the null hypothesis case in testing situations and the second corresponds to the regression alternatives (see [14], 210-219). It follows from Lemma 4.5.1 (ii) that for a general class of regression alternatives, the sequence $\{F^{(N)}\}$ is ued. It can also be seen that $\{F^{(N)}\}$ is ued in the one and two-sample cases: in the two sample case $F^{(N)} = \frac{m}{N} F_o + \frac{(N-m)}{N} G_o$, where F_o and G_o are distribution functions. Since $m/N \leq 1$, the result follows from Remark 4.2.1 (i). The C-sample case is similar.

ON ASSUMPTION 4.4.1 (i) AND (i)': Both the assumptions (i) and (i)' are difficult to verify in general, and require stringent regularity conditions on C_i's and $F_i^{(N)}$'s. However, the two and C-sample cases are simpler: In the two sample case, we have $C_i = 1$, $i = 1,2,\ldots,m$ and $C_i = 0$, $i = m+1,\ldots,N$. Also, $Y_1, Y_2, \ldots, Y_m$ have distribution F_o, $Y_{m+1}, Y_{m+2}, \ldots, Y_N$ have distribution G_o. Then

$$F = \frac{1}{N} \sum F_i^{(N)} = \frac{m}{N} F_o + \frac{(N-m)}{N} G_o$$

$$= \lambda_N F_o + (1-\lambda_N) G_o \text{, where } \lambda_N = \frac{m}{N}.$$

Further,

$$L_i(t) = F_i^{(N)} F^{-1}(t) = F_o F^{-1}(t) \quad , \quad i = 1,2,\ldots,m$$

$$= G_o F^{-1}(t) \text{, } i = m+1,\ldots,N$$

The functions $F_o F^{-1}$, $G_o F^{-1}$ are the same as [illegible] [26], and $\frac{d\, L_i(t)}{dt}$'s are the same as their a_λ [illegible]. Consequently, Assumption 4.4.1 (i) holds in this case from their Lemma 4.1 under quite general conditions on F_o and G_o (see Corollary 4.1 of [26]). The C-sample case is similar.

For general C_i's, we will show in Chapter V that Assumption 4.4.1 (i)' is satisfied if $F_i^{(N)}(x) = F_o\,(x-d_{iN})$, where F_o is Gaussian. It can also be easily verified when F_o is uniform. While in particular cases it is possible to verify the assumption, a general criterion seems to be difficult. However, there is one important general situation in which Assumption 4.4.1(i) is valid. This corresponds to the null hypothesis: $F_i^{(N)}(x) = F_o(x)$, where F_o is continuous. In this case, $F(x) = N^{-1} \sum F_i^{(N)}(x) = F_o(x)$, $L_i(t) = F_i^{(N)} F^{-1}(t) = t$, $0 \leq t \leq 1$. Consequently, $a_N(t) = N^{-1} \sum C_i \frac{d L_i}{dt} = N^{-1} \sum C_i$ satisfies Assumption 4.4.1 (i). It also satisfies Assumption 4.4.1 (i)' if $N^{-1} \sum C_i$ converges to a constant $\lambda_o < \infty$, as $N \to \infty$.

CHAPTER V

APPLICATIONS TO ASYMPTOTIC RELATIVE EFFICIENCY

In this chapter some applications of the theory developed in Chapters II to IV are given. In Section 1, we develop a Chernoff-Savage theorem for T_N, and in Section 2 this is used to obtain the asymptotic relative efficiency (ARE) of T_N relative to the classical t-test based on the sample regression coefficient. In Section 3, the important special case of Gaussian sequences is treated and explicit expressions for ARE's of some standard rank tests relative to t-test are obtained.

5.1 A CHERNOFF-SAVAGE THEOREM. Let $\psi_N \equiv J$ and define (see (4.1.4)),

$$J_N(t) = r_i = J\left(\frac{i}{N+1}\right) \quad \text{for} \quad \frac{i-1}{N} < t \leq \frac{i}{N}, \quad i = 1,2,\ldots,N$$

$$J_N(0) = J_N(0+) \quad ,$$

where J is a non-constant function of bounded variation inside $(0,1)$ inducing the Lebesgue-Stieltjes measure ν. Then from (4.1.6) it can be seen that

$$T_N = \int_{-\infty}^{\infty} J_N(F_N)\, d\, H_N \, .$$

Hence by letting

(5.1.1) $$\mu = \int_{-\infty}^{\infty} J(F)\, dH$$

we have,

(5.1.2) $$N^{1/2}(T_N-\mu) = T_N^* + N^{1/2}(\mu_N-\mu) \quad ,$$

where μ_N and T_N^* are as defined in (4.1.7). It can also be seen from (4.1.4) and the definition of J_N above that

$$-\nu_N\left(\left(\frac{i}{N}, \frac{i+1}{N}\right]\right) = J_N\left(\frac{i+1}{N}\right) - J_N\left(\frac{i}{N}\right) \quad ,$$

and

$$-\nu((a,b]) = J(b)-J(a) \quad , \quad 0 < a < b < 1 \quad .$$

Hence, from (4.1.7) and (5.1.1) we have

(5.1.3) $$N^{1/2}(\mu_N-\mu) = N^{1/2}\int_{-\infty}^{\infty} [J_N(F)-J(F)]\, dH \quad .$$

THEOREM 5.1.1. *Suppose the conditions of Theorem 3.2.1 on c_i's and α hold. Further, suppose*

(5.1.4) $$|J(t)| \le K[t(1-t)]^{-\frac{1}{r}+\theta} \quad \textit{for} \quad r \ge \frac{2}{1-\delta} \quad , \quad \theta > 0$$

(5.1.5) $$N^{1/2}\Delta(N)\int_{\frac{1}{N}}^{1-\frac{1}{N}} |J_N(t)-J(t^-)|\; d\, F_N F_N^{-1}(t) = o_p(1)$$

(5.1.6) $$N^{1/2}\Delta(N)\int_0^1 |J_N(t)-J(t)|\,dt = o(1) \quad .$$

Then,

(i) $N^{1/2}(\mu_N-\mu) = o(1)$

(ii) *there exists a* $q \in [Q(r)]$ *for which* $\int_0^1 q \, d|\nu| < \infty$

(iii) $\int_{1/N}^1 \Pi_N d(\nu_N - \nu) = o_p(1)$,

where all the order terms above are to be understood as $N \to \infty$ *, and* $\Delta(N) = \max_{i \leq N} |C_i|$.

PROOF: From (5.1.3) we have

$$|N^{1/2}(\mu_N - \mu)| \leq N^{1/2} \int_{-\infty}^{\infty} |J_N(F) - J(F)| \, d|H|$$

$$\leq N^{1/2} \Delta(N) \int_{-\infty}^{\infty} |J_N(F) - J(F)| \, dF$$

$$= o(1) \text{ by (5.1.6)} .$$

This proves (i). To prove (ii), taking $q = [t(1-t)]^{-\frac{1}{r} + \frac{\theta}{2}}$ we see that $q \in [Q(r)]$ and $\int_0^1 q \, d|\nu| = \int_0^1 q \, d|J| < \infty$ by (5.1.4) . Finally from (5.1.4) it is easy to see

$$J_N(0) = o(N^{\frac{1}{r} - \theta'}) , \quad J_N(1) = o(N^{\frac{1}{r} - \theta'}) ; \quad \theta' < \theta . \tag{5.1.7}$$

Now, by definition we have

$$P_N \equiv \int_{[\frac{1}{N}, 1]} \Pi_N d(\nu_N - \nu) = - \int_{[\frac{1}{N}, 1]} \Pi_N \, d(J_N - J) \tag{5.1.8}$$

$$\underset{a.s}{\leq} \int_{[\frac{1}{N}, 1 - \frac{1}{N}]} \Pi_N d(J_N - J) + \sup_{1 - \frac{1}{N} \leq t \leq 1} |\Pi_N(t)| \, |J_N(1)| + \sup_{1 - \frac{1}{N} \leq t} \frac{\Pi_N(t)}{q(t)} \int_0^1 q \, d|J$$

where q is as defined above. Now from Lemma 4.3.1 and (5.1.7) the last two terms on the right above are $o_p(1)$ as $N \to \infty$. Also the first term on the right is equal to

$$(5.1.9) \qquad \int_{[\frac{1}{N},1-\frac{1}{N}]} \Pi_N(t+)\,d(J_N-J) + \int_{[\frac{1}{N},1-\frac{1}{N}]} [\Pi_N \text{[illegible]}]\,d(J_N-J) \ .$$

But,

$$|\Pi_N(t+)-\Pi_N(t)| = |N^{1/2}H_N F_N^{-1}(t+)-H_N F_N^{-1}(t)| \leq N^{-1/2}\,\Delta(N) \ .$$

Also, using integration by parts and the bounds on J's given by (5.1.4) and (5.1.7), it can be seen as in (5.1.8) that

$$\text{[illegible]} = \int_{[\frac{1}{N},1-\frac{1}{N}]} [J_N(t)-J(t-)]\,d\,\Pi_N(t) + o_p(1) \ .$$

Hence, from (5.[illegible]) and (5.1.9), [illegible] have

$$|P_N| \underset{a.s}{\leq} N^{1/2} \int |J_N(t)-J(t-)|\; d|H_N F_N^{-1}|$$

$$+ N^{1/2} \int_{[\frac{1}{N},1-\frac{1}{N}]} |J_N(t)-J(t-)|\; d|HF^{-1}| + o_p(1)$$

$$\leq N^{1/2}\Delta(N)\Big[\int_{[\frac{1}{N},1-\frac{1}{N}]} |J_N(t)-J(t-)|\,d\,F_N F_N^{-1} + \int_0^1 |J_N(t)-J(t-)|\,dt\Big] + o_p(1)$$

$$= o_p(1) \quad \text{by (5.1.5) and (5.1.6).}$$

This completes the proof. □

REMARK 5.1.1. The condition (5.1.6) above is satisfied if the following holds: $J = J_c + J_d$, where J_d is a jump function with finite number of jumps at $a_1 < a_2 < \ldots < a_s$, and J_c has a continuous derivative J_c' on $(0,a_1),\ldots,(a_s,1)$ satisfying

(5.1.10) $$|J_c'(t)| \le K[t(1-t)]^{-\frac{r+1}{r}+\theta},$$

for $t \neq a_i$ and r,θ are as given in (5.1.4).

PROOF: The proof is similar to corollary 5.1 of [26] and hence is omitted.

COROLLARY 5.1.1. *Suppose Assumption 4.4.1 (i)', (ii) and the conditions of Theorem 3.2.1 are satisfied. If also, the conditions of Theorem 5.1.1 hold with some* $r \geq \frac{2(2+\delta)}{1-\delta}$, *then* $N^{1/2}(T_N-\mu)$ *converges in law to* $N(0,\sigma_o^2)$, *where* σ_o^2 *is given by* (4.4.10).

5.2 ASYMPTOTIC RELATIVE EFFICIENCY (ARE).

Let

(5.2.1) $$Y_{iN} = d_{iN}\beta + X_i ,$$

where $\{X_i\}$ is a stationary strong mixing sequence of rv's with $EX_i = 0$ and $Var(X_i) = 1$. In (5.2.1), the constant β is unknown regression coefficient and $\{d_{iN}\}$ is a sequence of known constants. Following Hájek (see [14], 267), we consider the following regression alternatives for studying ARE of T_N relative to t-test:

$$H_o : \beta = 0$$

(5.2.2)

$$H_{1N} : \beta > 0 \ , \ \max_{i \le N} d_{iN} = O(N^{-a}) \quad \text{for some} \quad 0 < a \le \frac{1}{2} \ .$$

We need the following assumptions as $N \to \infty$:

$$\bar{C}_N = N^{-1} \sum C_i \to \lambda_o \ ;$$

(5.2.3)

$$N^{-1} \sum C_i^2 \to \lambda_1 \quad \text{with} \quad \lambda_1 - \lambda_o^2 > 0 \ ;$$

$$\sum (d_i - \bar{d}_N)^2 \to \lambda_2 > 0 \ , \ \text{where} \quad \bar{d}_N = N^{-1} \sum d_{iN} \ ;$$

$$N^{-1} \sum_{i<j} (C_i - \bar{C}_N)(C_j - \bar{C}_N) \ \rho(j-i)$$

converges, where $\rho(i)$ is a sequence satisfying $\sum_1^{\infty} |\rho(i)| < \infty$. Similar conditions hold for d_{iN}'s also;

$$N^{-1/2} \sum (C_i - \bar{C}_N)(d_i - \bar{d}_N)$$

converges. Also, denoting by μ_o the value of μ under H_o , we assume

(5.2.4)

$$B = \lim_{N \to \infty} N^{1/2} (\mu - \mu_o)$$

$$= \lim_{N \to \infty} N^{1/2} \left[\int_{-\infty}^{\infty} J(F) dH - \lambda_o \int_0^1 J(t) dt \right]$$

exists and is finite under H_1 . Now, for the classical model (5.2.1), the t-test (for testing H_o) is based on the sample regression coefficient

$$\hat{\beta} = \sum (d_i - \bar{d}_N) Y_i \ / \ \sum (d_i - \bar{d}_N)^2 \ .$$

By letting $t_N = \hat{\beta} \sum (d_i - \bar{d}_N)^2$, we have under H_1,

(5.2.5) $$E\, t_N = \sum (d_i - \bar{d}_N) d_i \beta$$

$$= \sum (d_i - \bar{d}_N)^2 \beta$$

$$\to B^*, \text{ as } N \to \infty, \text{ where } B^* = \lambda_2 \beta .$$

Further,

(5.2.6) $$\mathrm{Var}(t_N) = \lambda_2 + A^* \text{ ; where}$$

$$A^* = \lim_{N\to\infty} \sum_{i \neq j} (d_i - \bar{d}_N)(d_j - \bar{d}_N)\, \rho(j-i) ,$$

$\rho(i)$ being the correlation between X_1 and X_{1+i}. We assume

(5.2.7) t_N is asymptotically normal with the parameters given by (5.2.5) and (5.2.6).

The following theorem is an immediate consequence of the assumptions made and hence its proof is omitted. We refer to [14] for the definition of ARE in the regression model.

THEOREM 5.2.1. *Suppose, for the model given by* (5.2.1), *the conditions of Corollary* 5.1.1 *are satisfied. If also, the limiting variance of* T_N *is the same as* σ_o^2 *given by* (4.4.10) *under* H_o, *and the assumptions* (5.2.3), (5.2.4), *and* (5.2.7) *hold, then the ARE of* T_N *relative to* t_N *is given by*

(5.2.8) $$e(T_N, t_N) = \left(\frac{B}{B^*}\right)^2 \cdot \frac{\lambda_2 + A^*}{\sigma_o^2},$$

where B, B^*, λ_2, *and* A^* *are given by* (5.2.4), (5.2.5), (5.2.3), *and* (5.2.6) *respectively.*

5.3 GAUSSIAN SEQUENCES AND SOME STANDARD RANK TESTS. Consider the model (5.2.1), where X_i's are stationary strong mixing Gaussian rv's with $E X_i = 0$ and $Var(X_i) = 1$. We use the following notation:

$$f(x) = (2\pi)^{-1/2} e^{-(1/2)x^2}$$

$$\Phi(x) = \int_{-\infty}^{x} f(t)\, dt$$

(5.3.1)

$$f_k(x,y) = [2\pi(1-\rho_k^2)]^{-1} \exp\left[\frac{-(1-\rho_k^2)^{-1}}{2} \{x^2 - 2xy\, \rho_k + y^2\}\right]$$

$$\Phi_k(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_k(x,y)\, dx\, dy .$$

First we prove some auxiliary results showing that the assumptions made in Theorem 5.2.1 are satisfied in the present case. The assertions of the following lemmas seem t[illegible] known, but for lack of a reference we include their proofs.

LEMMA 5.3.1. *For all* $t \in [0,1]$

$$|\Phi^{-1}(t)| \le K[t(1-t)]^{-\eta} ,$$

and

$$\left|\frac{d\Phi^{-1}(t)}{dt}\right| \le K[t(1-t)]^{-1} ,$$

where $\eta > 0$ *is arbitrary and* K *is a generic constant.*

PROOF: From the relation $\Phi^{-1}(t) = -\Phi^{-1}(1-t)$, clearly it suffices to prove the assertions for $t \ge \frac{1}{2}$. Suppose then $t \ge \frac{1}{2}$. We use the following two elementary inequalities (see [24]; 266, 291): For every $m > 0$, $x > 0$

$$-\log x \le \frac{1}{em\,x^m} ,$$

and

$$1 - \Phi(x) < e^{-\frac{x^2}{2}} .$$

Both of these can be easily proved by differentiation. Now, by letting $\Phi^{-1}(t) = x$, we have $t = \Phi(x)$ and from above

$$\Phi(x)[1 - \Phi(x)] < e^{-\frac{x^2}{2}}$$

$$\Rightarrow t(1-t) < e^{-[\Phi^{-1}(t)]^2/2}$$

$$\Rightarrow \Phi^{-1}(t) < \{-2 \log [t(1-t)]\}^{1/2}$$

$$< 2^{1/2}(2e\eta)^{-1/2}[t(1-t)]^{-\eta} ,$$

with $\frac{m}{2} = \eta$. This proves the first inequality of the lemma. To prove the second, observe that

$$\frac{d\Phi^{-1}(t)}{dt} = \frac{1}{f(\Phi^{-1}(t))} = (2\pi)^{1/2} \exp[+[\Phi^{-1}(t)]^2/2]$$

$$\leq (2\pi)^{1/2}[t(1-t)]^{-1} \text{ from above.} \quad \square$$

LEMMA 5.3.2. *Suppose* (5.2.3) *holds. Then, under* H_1 *of* (5.2.2), *Assumption* 4.4.1 (i)' *and* (ii) *are satisfied.*

PROOF: Lemma 4.5.1 (ii) shows that Assumption 4.4.1 (ii) is an obvious corollary in the present case. To prove the first part of the lemma, we recall from Chapter IV,

$$F = N^{-1} \sum F_i^{(N)} , \; L_i(t) = F_i^{(N)} F^{-1}(t) ,$$

and $a_N(t) = N^{-1} \sum C_i \ell_i(t)$, where $\ell_i(t) = d\, L_i(t)/dt$. Under (5.2.1),

$$\ell_i(t) = \frac{f(F^{-1}(t)-d_i)}{N^{-1} \sum f(F^{-1}(t)-d_j)} = \frac{e^{F^{-1}(t)d_i - d_i^2/2}}{N^{-1} \sum e^{F^{-1}(t)d_j - d_j^2/2}} ,$$

where we have set $\beta = 1$ for convenience. Hence, by writing $\hat{d}_N = \max_{i \leq N} |d_i| - \min_{i \leq N} |d_i|$, we see that

$$\ell_i(t) \in [\exp\{-\hat{d}_N - |F^{-1}(t)|\hat{d}_N\} , \exp\{\hat{d}_N + |F^{-1}(t)|\hat{d}_N\}] ,$$

for $1 \leq i \leq N$. Now, since $\Phi(x - \max|d_i|) \leq F(x) \leq \Phi(x - \min|d_i|)$, we have $|F^{-1}(t) - \Phi^{-1}(t)| \leq \max_{i \leq N} |d_i|$. Using this, the bound on Φ^{-1} given by Lemma 5.2.1, and the fact $\max_{i \leq N} d_i = O(N^{-a})$, $a > 0$, we see that $\ell_i^o(t) \to 1$ uniformly in i and t . Since by (5.2.3) $N^{-1} \sum C_i \to \lambda_o$ as $N \to \infty$, and $a_N(t) = N^{-1} \sum C_i \ell_i(t)$; it follows from above that $a_N^o(t) \to \lambda_o$ uniformly in t as $N \to \infty$. This proves that Assumption 4.4.1

(i)' is satisfied for the model under consideration. □

LEMMA 5.3.3. *Suppose, in addition to (5.2.3) the conditions of Theorem 2.3.1 hold. Then, under (5.2.1) and* H_1, *the limit in (4.4.10) exists and is equal to the limit under* H_o.

PROOF: Under (5.2.1), we have

$$L_i(t) = \Phi(F^{-1}(t)-d_i\beta) ,$$

where

$$F(x) = N^{-1} \sum \Phi(x-d_i\beta) .$$

Setting $\beta = 1$ for convenience, we have for $s < t$,

$$L_i(s \wedge t) - L_i(t)L_i(s) = \Phi(F^{-1}(s)-d_i) - \Phi(F^{-1}(t)-d_i)\Phi(F^{-1}(s)-d_i) .$$

Noting that $|F^{-1}(t) - \Phi^{-1}(t)| \leq \max_{i \leq N} |d_i|$ and $\max_{i \leq N} |d_i| \to 0$, we see from uniform continuity of Φ,

$$(5.3.2) \qquad [L_i(s)-L_i(t)L_i(s)] \to s - st ,$$

uniformly in i as $N \to \infty$. Further,

$$(5.3.3) \qquad P[\eta_i \leq t, \eta_j \leq s] - L_i(t)\, L_i(s)$$

$$= \Phi_{j-i}(F^{-1}(t)-d_i\ ,\ F^{-1}(s)-d_i) - \Phi(F^{-1}(t)-d_i)\Phi(F^{-1}(s)-d_j)$$

$$\to \Phi_{j-i}(\Phi^{-1}(t),\Phi^{-1}(s)) - st ,$$

uniformly in i and j (for fixed s and t) as $N \to \infty$. Now, recall from (4.4.10) that

$$\sigma_\rho^2 = \int_0^1 \int_0^1 E\,\Pi_o(t)\Pi_o(s)\,dJ(t)\,dJ(s) .$$

Since, under the conditions of Theorem 2.3.1 and (5.2.3) $V_N - \overline{C}_N U_N \Rightarrow \Pi_o(t)$, we have from (2.3.4) and (2.3.8),

$$E\,\Pi_o(t)\Pi_o(s) = \lim_{N\to\infty} \{N^{-1} \sum (C_i-\overline{C}_N)^2 [L_i(s\wedge t) - L_i(t)\,L_i(s)]$$

$$+ N^{-1} \sum_{i\neq j} (C_i-\overline{C}_N)(C_j-\overline{C}_N)[P(\eta_i\leq t,\eta_j\leq s)-L_i(t)L_i(s)]\}$$

Under H_1, we have from (5.3.2), (5.3.3) and (5.2.3),

$$E\,\Pi_o(t)\Pi_o(s) = \lambda_1 - \lambda_o^2\,[s\wedge t - st] + A(s,t) ,$$

where

(5.3.4) $$A(s,t) = \lim_{N\to\infty} N^{-1} \sum_{i\neq j} (C_i-\overline{C}_N)(C_j-\overline{C}_N)[\Phi_{j-i}(\Phi^{-1}(t),\Phi^{-1}(s)) - ts] .$$

Since the last expression for $E\,\Pi_o(t)\Pi_o(s)$ under H_1 is the same as that under H_o, the assertion of the lemma follows. □

LEMMA 5.3.4. *Suppose* $J = \Phi^{-1}$. *If* C_i*'s satisfy the conditions of Theorem* 3.2.1, *then* (5.1.4), (5.1.5) *and* (5.1.6) *of Theorem* 5.1.1 *hold.*

PROOF: The conditions (5.1.4) and (5.1.6) follow from Lemma 5.3.1 and Remark 5.1.1. It remains to prove (5.1.5). We have

$$N^{1/2}\,\Delta(N)\int_{1/N}^{1-\frac{1}{N}} |J_N(t) - J(t-)|\, d\, F_N F_N^{-1}(t)$$

$$= N^{-1/2}\Delta(N) \sum_{1}^{N-1} |\Phi^{-1}(\tfrac{i}{N+1}) - \Phi^{-1}(\tfrac{i}{N})|$$

$$\leq N^{-1/2}\Delta(N) \sum_{1}^{N-1} \frac{i}{N^2} [\max_{\frac{i}{N+1} \leq u \leq \frac{i}{N}} |\frac{d\Phi^{-1}(u)}{du}|]$$

$$\leq K\, N^{-1/2}\Delta(N)\ [\sum_{1}^{N-1} \frac{i}{N^2} \{\frac{i}{N+1}(1-\frac{i}{N})\}^{-1}] \quad , \text{ from Lemma 5.3.1,}$$

$$\leq K\, N^{-1/2}\Delta(N)\ (\frac{N+1}{N})^2 \sum_{1}^{N-1} \frac{1}{N-i} \to 0 \quad , \text{ as } N \to \infty . \qquad \square$$

THEOREM 5.3.1. *Suppose* $\{X_i\}$ *of* (5.2.1) *is Gaussian and the conditions of Theorem* 3.2.1 *are satisfied. If* J *satisfies the conditions of Theorem* 5.1.1 *with* $r \geq 2(2+\delta)/(1-\delta)$, *then the ARE of* T_N *relative to* t_N *is given by*

(5.3.5) $$e(T_N, t_N) = (\frac{B}{B^*})^2 \cdot \frac{\lambda_2 + A^*}{(\lambda_1 - \lambda_0^2)D + A} ,$$

where

(5.3.6) $$A = \lim_{N\to\infty} N^{-1} \sum_{i\neq j} (C_i - \bar{C}_N)(C_j - \bar{C}_N)[\iint \Phi_{j-i}(x,y)\, dJ(\Phi(x))\, dJ(\Phi(y)) - (\int_0^1 J(u)du)^2]$$

(5.3.7) $$D = \int_0^1 J^2(u)du - (\int_0^1 J(u)du)^2 .$$

λ_1 , λ_2 , B , B^* , and A^* *are given by* (5.2.3) *to* (5.2.6).

PROOF: When $\{X_i\}$ is Gaussian, it follows from Lemmas (5.3.2) and (5.3.3) that the conditions of Corollary (5.1.1) are satisfied.

Since the conditions of Theorem 3.2.1 and Theorem 5.1.1, with $r \geq 2(r+\delta)/(1-\delta)$, are satisfied, we have from (5.2.8)

$$e(T_N, t_N) = \left(\frac{B}{B^*}\right)^2 \frac{\lambda_2 + A^*}{\sigma_o^2} .$$

But, from (4.4.10) and (5.3.4), we have

$$\sigma_o^2 = \int_0^1\int_0^1 E\, \Pi_o(t)\Pi_o(s)\; dJ(t)\; dJ(s)$$

$$= (\lambda_1 - \lambda_o^2) \int_0^1\int_0^1 (s \wedge t - st)\; dJ(t)\; dJ(s)$$

$$+ \int_0^1\int_0^1 A(s,t)\; dJ(t)\; dJ(s)$$

$$= (\lambda_1 - \lambda_o^2)D + A . \qquad \square$$

SOME STANDARD RANK TESTS:

I. NORMAL SCORES TEST. Here $J = \Phi^{-1}$. From Lemma 5.3.4, it follows that J satisfies the conditions of Theorem 5.3.1. From (5.2.4) it can be seen that

$$B = \lim_{N\to\infty} \{N^{1/2} \int_{-\infty}^{\infty} [J(F) - J(\Phi)]dH + N^{1/2} \int_{-\infty}^{\infty} J(\Phi)dH\} ,$$

since $\int_0^1 \Phi^{-1}(t)\; dt = 0$. Expanding the first integrand in Taylor series and noting that $F = N^{-1} \sum \Phi(x - d_i\beta)$, $H = N^{-1} \sum C_i\, \Phi(x - d_i\beta)$, it can be seen that the first term on the right is equal to $N^{1/2}\,\bar{C}_N\bar{d}_N\beta + o(1)$. Similarly, the second term is equal to $N^{-1/2}\sum C_i d_i\beta + o(1)$. Hence,

$$B = \lim_{N\to\infty} N^{-1/2}\sum (C_i-\bar{C}_N)(d_i-\bar{d}_N)\beta \ .$$

Further from (5.3.7) and (5.3.6)

$$D = \int_0^1 (\Phi^{-1}(u))^2 du - \left(\int_0^1 \Phi^{-1}(u)\,du\right)^2 = 1 \ ,$$

and

$$A = \lim_{N\to\infty} N^{-1}\sum_{i\neq j} (C_i-\bar{C}_N)(C_j-\bar{C}_N)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{j-i}(x,y)\,dx\,dy\right] ,$$

since $\int_0^1 \Phi^{-1}(u)\,du = 0$. But, it is well known (see [20] Lemma 2) that the double integral in A is equal to $\mathrm{corr}(X_i,X_j) = \rho(j-i)$. Hence

$$A = \lim_{N\to\infty} N^{-1}\sum_{i\neq j}(C_i-\bar{C}_N)(C_j-\bar{C}_N)\,\rho(j-i) \ .$$

Thus, by substituting these in (5.3.5), we get the relative efficiency of Normal scores test relative to t-test as

(5.3.8)
$$e(NS,t) = \left[\frac{\lim_{N\to\infty} N^{-1/2}\sum (C_i-\bar{C}_N)(d_i-\bar{d}_N)}{\lim_{N\to\infty}\sum (d_i-\bar{d}_N)^2}\right]^2 \times \frac{\lambda_2 + \lim_{N\to\infty}\sum_{i\neq j}(d_i-\bar{d}_N)(d_j-\bar{d}_N)\,\rho(j-i)}{\lambda_1-\lambda_0^2 + \lim_{N\to\infty} N^{-1}\sum_{i\neq j}(C_i-\bar{C}_N)(C_j-\bar{C}_N)\rho(j-i)} ,$$

where

$$\lambda_2 = \lim_{N\to\infty} \sum (d_i - \bar{d}_N)^2$$

$$\lambda_1 - \lambda_o^2 = \lim_{N\to\infty} N^{-1} \sum (C_i - \bar{C}_N)^2 .$$

If we take $d_i = N^{-1/2} C_i$, the conditions on d_i's are satisfied, and in this case $\lambda_1 - \lambda_o^2 = \lambda_2$, so that from (5.3.8) it is easy to see that $e(NS, t) = 1$ as should be expected. The same is true in the two sample case whether the variables are independent or not (in this case, $C_i = 1$, $i = 1,2,\ldots,m$, $C_i = 0$, $i = m+1,\ldots,N$, where $\frac{m}{N} \to \lambda_o$ and $d_i = N^{-1/2} C_i$).

II. WILCOXON TEST. Here $J(u) = u$. Since $J'(u) = 1$ all the conditions of Theorem 5.1.1 are satisfied. We have from (5.2.4)

$$B = \lim_{N\to\infty} N^{1/2} \left[\int_{-\infty}^{\infty} F \, dH - \bar{C}_N \int_0^1 t \, dt \right] ,$$

where $F = \frac{1}{N} \sum \Phi(x - d_i\beta)$, $H = \frac{1}{N} \sum C_i \Phi(x - d_i\beta)$. Using the Taylor series expansion to the first integrand, it can be seen

$$\int_{-\infty}^{\infty} F \, dH = \bar{C}_n \int_0^1 t \, dt + \frac{1}{N} \sum (C_i - \bar{C}_N)(d_i - \bar{d}_N) \int_{-\infty}^{\infty} f^2(x) dx + o(1) ,$$

where $f = d\Phi/dx$. Hence,

$$B = \left[\int f^2(x) dx \right] \lim_{N\to\infty} N^{-1/2} \sum (C_i - \bar{C}_N)(d_i - \bar{d}_N) \, \beta .$$

From (5.3.7) and (5.3.6),

$$D = \int_0^1 u^2 \, du - \left(\int_0^1 u \, du \right)^2 = \frac{1}{12} ,$$

and

$$A = \lim_{N\to\infty} N^{-1} \sum (C_i-\bar{C}_N)(C_j-\bar{C}_N) \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{j-i}(x,y)\,d\Phi(x)\,d\Phi(y) - \tfrac{1}{4}\right].$$

Now, following the method of proof of Theorem 5.2 in Bickel [2] and using the result of Cramer [7] page 290, it can be seen that the double integral above is equal to $\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho(j-i)$. Hence,

$$A = \lim_{N\to\infty} \frac{N^{-1}}{2\pi} \sum (C_i-\bar{C}_N)(C_j-\bar{C}_N) \sin^{-1}\rho(j-i).$$

By substituting these in (5.3.8) and using $\int f^2 = (2\sqrt{\pi})^{-1}$, we get

(5.3.9)
$$e(W,t) = \frac{1}{4\pi}\left[\frac{\lim_{N\to\infty} N^{-1/2} \sum (C_i-\bar{C}_N)(d_j-\bar{d}_N)}{\lim_{N\to\infty} \sum (d_i-\bar{d}_N)^2}\right]^2 \frac{\lambda_2 + \lim_{N\to\infty} \sum_{i\neq j} (d_i-\bar{d}_N)(d_j-\bar{d}_N)\,\rho(j-i)}{\frac{\lambda_1-\lambda_o^2}{12} + \lim_{N\to\infty} \frac{1}{2\pi N} \sum (C_i-\bar{C}_N)(C_j-\bar{C}_N)\sin^{-1}\rho(j-i)}$$

Independence Case: If the variables X_i are independent, then clearly

$$(5.3.10)\qquad e(W,t) = \frac{1}{4\pi}\left[\frac{\lim_{N\to\infty} N^{-1/2} \sum (C_i-\bar{C}_N)(d_j-\bar{d}_N)}{\lim_{N\to\infty} \sum (d_i-\bar{d}_N)^2}\right]^2 \frac{12\lambda_2}{\lambda_1-\lambda_o^2},$$

and taking $d_i = N^{-1/2} C_i$, we see that

$$e(W,t) = \frac{3}{\pi}.$$

This result is well known in the two sample case (see [14], 278).

Dependent Case: Taking $d_i = N^{-1/2} c_i$, we see from (5.3.9)

$$(5.3.11) \qquad e(W,t) = \frac{1}{4\pi}\left[\frac{\lambda_1-\lambda_o^2 + \lim_{N\to\infty} N^{-1} \sum_{i\neq j} (c_i-\bar{c}_N)(c_j-\bar{c}_N)\,\rho(j-i)}{\frac{\lambda_1-\lambda_o^2}{12} + \lim_{N\to\infty} \frac{1}{2\pi N} \sum_{i\neq j} (c_i-\bar{c}_N)(c_j-\bar{c}_N)\sin^{-1}\frac{\rho}{}}\right]$$

where $\lambda_1 - \lambda_o^2 = \lim_{N\to\infty} N^{-1} \sum(c_i-\bar{c}_N)^2$. For the *two sample problem* $i = 1,2,\ldots,m$ and 0 otherwise, where $\lim_{N\to\infty} \frac{m}{N} = \lambda_o > 0$. Hence $\lambda_1 - \lambda_o^2 = \lambda_o(1-\lambda_o)$, so that from above

$$(5.3.12) \qquad e(W,t) = \frac{1}{4\pi}\left[\frac{1 + 2\sum_1^\infty \rho(i)}{\frac{1}{12} + \frac{1}{\pi}\sum_1^\infty \sin^{-1}\frac{\rho(i)}{2}}\right]$$

$$= \frac{3}{\pi}\left[\frac{1 + 2\sum_1^\infty \rho(i)}{1 + \frac{12}{\pi}\sum_1^\infty \sin^{-1}\frac{\rho(i)}{2}}\right]$$

Hence if $\{X_i\}$ forms a *first order auto regressive (FOAR) process* $X_t = \rho X_{t-1} + \varepsilon_t$, where $0 < \rho < 1$, then ([12], 14)

$$\alpha(i) \leq K \rho^{2i} ,$$

so that our strong mixing conditions are satisfied. In this case $\mathrm{Corr}(X_1, X_{1+i}) = \rho^i$, consequently,

$$(5.3.13) \qquad e(W,t) = \frac{3(1+\rho)/(1-\rho)}{\pi(1 + \frac{12}{\pi}\sum_1^\infty \sin^{-1}\frac{\rho^i}{2})}$$

Now, $x \leq \sin^{-1} x \leq \frac{\pi}{3} x$, for $0 \leq x \leq \frac{1}{2}$ so that

$$\frac{12}{\pi}\sum_{1}^{\infty}\sin^{-1}\left(\frac{\rho^i}{2}\right) \leq 2\sum_{1}^{\infty}\rho^i = \frac{2\rho}{1-\rho}$$

and

$$\frac{12}{\pi}\sum_{1}^{\infty}\sin^{-1}\left(\frac{\rho^i}{2}\right) \geq \frac{6}{\pi}\sum_{1}^{\infty}\rho^i = \frac{6\rho}{\pi(1-\rho)} \quad .$$

Hence from (5.3.13), it follows that for the FOAR Gaussian process

$$\text{(5.3.14)} \qquad \frac{3}{\pi} \leq e(W,t) \leq \frac{3}{\pi} + \frac{(2-\frac{6}{\pi})\,\rho}{1+(\frac{6}{\pi}-1)\rho}\cdot\frac{3}{\pi} \quad .$$

Generally, from (5.3.12) it is clear that if $\rho(i) \geq 0$ for all i, then $e(W,t) \geq \frac{3}{\pi}$. The inequality is reversed if $\rho(i) \leq 0$. (This can be seen by using $x \leq \sin^{-1}x \leq \frac{\pi}{3}x$, $0 \leq x \leq \frac{1}{2}$.) The same is true with (5.3.11) if

$$\sum_{i=1}^{N-j}(C_i-\overline{C}_N)(C_{i+j}-\overline{C}_N) \geq 0 \quad \text{for all } j \; .$$

III. MEDIAN TEST. In this case

$$\begin{aligned} J(u) &= 0 \;, \quad \text{for } 0 \leq u \leq \tfrac{1}{2} \\ &= 1 \;, \quad \text{for } \tfrac{1}{2} < u \leq 1 \; . \end{aligned}$$

Here J satisfies the condition of Remark 5.1.1 and is bounded, so that the conditions of Theorem 5.1.1 are satisfied. We have from (5.2.4)

$$B = N^{1/2}\left[\int_{F(x)\geq\frac{1}{2}} dH - \frac{\overline{C}_N}{2}\right] ,$$

where $H' = \frac{1}{N} \sum C_i \; \Phi(x-d_i\beta)$, $F = \frac{1}{N} \sum \Phi(x-d_i\beta)$.

$$\therefore \quad B = N^{1/2} \left[\frac{1}{N} \sum \Phi(F^{-1}(\tfrac{1}{2})-d_i\beta) - \frac{\bar{C}_N}{2}\right] .$$

Now, using the Taylor series expansion of Φ and F , it can be seen that the right hand side is equal to

$$N^{1/2}\left[\frac{\bar{C}_N}{2} - \beta \, \bar{C}_N \bar{d}_N f(F^{-1}(\tfrac{1}{2})) + \frac{\sum C_i d_i}{N} \beta \, f(F^{-1}(\tfrac{1}{2}) + o(N^{-1/2}) - \frac{\bar{C}_N}{2}\right] .$$

Since, $f(F^{-1}(\frac{1}{2}) \to f(\Phi^{-1}(0)) = \frac{1}{\sqrt{2\pi}}$ we see that

$$B = N^{-1/2} \sum (C_i-\bar{C}_N)(d_i-\bar{d}_N)(2\pi)^{-1/2} \beta \quad .$$

Also, from (5.3.7) and (5.3.6),

$$D = \int_{1/2}^{1} du - \left(\int_{1/2}^{1} du\right)^2 = \frac{1}{4}$$

and

$$A = \lim_{N\to\infty} N^{-1} \sum_{i \neq j} (C_i-\bar{C}_N)(C_j-\bar{C}_N)\left[\int_0^\infty\int_0^\infty \Phi_{j-i}(x,y) \, dx \, dy - \frac{1}{4}\right]$$

$$= \lim_{N\to\infty} N^{-1} \sum_{i \neq j} (C_i-\bar{C}_N)(C_j-\bar{C}_N) \left[\frac{1}{2\pi} \sin^{-1} \rho(j-i)\right] ,$$

using an identity of Cramer ([7], 290). Hence, from (5.3.5) we have

(5.3.15) $e(M,t)$

$$= \frac{1}{2\pi}\left[\frac{\lim\limits_{N\to\infty} N^{-1/2}\sum(c_i-\bar{c}_N)(d_i-\bar{d}_N)}{\lim\limits_{N\to\infty}\sum(d_i-\bar{d}_N)^2}\right]^2 \frac{\lambda_2+\lim\limits_{N\to\infty}\sum\limits_{i\neq j}(d_i-\bar{d}_N)(d_j-\bar{d}_N)\,\rho(j-i)}{\frac{\lambda_1-\lambda_o^2}{4}+\lim\limits_{N\to\infty}\frac{1}{2\pi N}\sum(c_i-\bar{c}_N)(c_j-\bar{c}_N)\sin^{-1}\rho(j-i)}\,.$$

Independent Case: In this case, clearly

$$e(M,t) = \frac{1}{2\pi}\left[\frac{\lim N^{-1/2}(c_i-\bar{c}_N)(d_i-\bar{d}_N)}{\lim\sum(d_i-\bar{d}_N)^2}\right]^2 \frac{4\lambda_2}{\lambda_1-\lambda_o^2}\,.$$

If we take $d_i = N^{-1/2} c_i$,

$$e(M,t) = \frac{2}{\pi}\,,$$

which is well known.

Dependent Case: If we take $d_i = N^{-1/2} c_i$, we see from (5.3.15)

$$e(M,t) = \frac{1}{2\pi}\left[\frac{\lambda_1-\lambda_o^2+\lim N^{-1}\sum(c_i-\bar{c}_N)(c_j-\bar{c}_N)\,\rho(j-i)}{\frac{\lambda_1-\lambda_o^2}{4}+\lim\frac{1}{2\pi N}\sum(c_i-\bar{c}_N)(c_j-\bar{c}_N)\sin^{-1}\rho(j-i)}\right].$$

In particular, for the *two sample problem*

$$e(M,t) = \frac{1}{2\pi}\left[\frac{1+2\sum\limits_1^\infty\rho(i)}{\frac{1}{4}+\frac{1}{\pi}\sum\limits_1^\infty\sin^{-1}\rho(i)}\right]$$

$$\frac{2}{\pi} \frac{1 + 2 \sum_{1}^{\infty} \rho(i)}{1 + \frac{4}{\pi} \sum_{1}^{\infty} \sin^{-1} \rho(i)}$$

Since $x \leq \sin^{-1} x \leq \frac{\pi}{2} x$ for $0 \leq x \leq 1$, it can be seen from above $e(M,t) \geq \frac{2}{\pi}$ for $\rho(i) \geq 0$ and $e(M,t) \leq \frac{2}{\pi}$ for $\rho(i) \leq 0$. In particular, if $\{X_i\}$ forms a FOAR process with $0 < \rho < 1$ it can be seen (as in (5.3.14)),

$$\frac{2}{\pi} \leq e(M,t) \leq \frac{2}{\pi} + \frac{4\rho(1 - \frac{2}{\pi})}{\pi[1 + (\frac{4}{\pi} - 1)\rho]} \quad (5.3.16)$$

GENERAL REMARKS: The expressions for ARE derived in this section show that the usual rank tests for the regression problem (for Gaussian sequences) retain approximately the same relative efficiencies relative to t-test as in the independence case, whenever the correlations between successive observations are small. Furthermore, if the variables are positively correlated, their relative efficiencies are better than those in the case of independence. Indeed, for the first order auto-regressive Gaussian process with grade correlation coefficent $\rho (0 \leq \rho < 1)$, we have from (5.3.14) and (5.3.16)

$$e(W,t) \geq \frac{3}{\pi} \simeq 0.95$$

$$e(M,t) \geq \frac{2}{\pi} \simeq 0.64 ,$$

with equality in the independence case ($\rho = 0$). Moreover,

$$\lim_{\rho \uparrow 1} e(W,t) \uparrow = 1$$

$$\lim_{\rho \uparrow 1} e(M,t) \uparrow = 1 .$$

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