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Ergodic theorems for certain Banach algebras associated to locally compact groups

by

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To my family

My grandparents: Marthe, René and Jules
My parents: Béatrice and Christian
My brother: Jérôme

Abstract

In this thesis, we establish some ergodic theorems related to $A_p(G)$, the Figà-Talamanca-Herz algebra of a locally compact group G. This thesis is divided in two main portions.

The first part is primarily concerned with the study of ergodic sequences in $A_p(G)$ and with a newly introduced notion of ergodic multipliers. After presenting a full description of the non-degenerate $*$-representations of $A_p(G)$ and of their extensions to the multiplier algebra $MA_p(G)$, it is shown that, for all locally compact groups, the weakly ergodic sequences in $MA_p(G)$ coincide with the strongly ergodic ones, and that they are, in a sense, approximating sequences for the topologically invariant means on some spaces of linear functionals on $A_p(G)$. Next, motivated by the study of ergodic sequences of iterates, we introduce a notion of ergodic multipliers, and we provide a solution to the dual version of the complete mixing problem for probability measures.

The second part is of a more abstract nature and deals with some ergodic and fixed point properties of φ-amenable Banach algebras. Among other things, we prove a mean ergodic theorem, establish the uniqueness of a two-sided φ-mean on the weakly almost periodic functionals, and provide a simpler proof of a fixed point theorem which is well known in the context of semigroups. We also study the norm spectrum of some linear functionals on $A_p(G)$ and present a new characterization of discrete groups.

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Contents

Chapter 1

Introduction.

In the first half of the twentieth century, many researchers in harmonic analysis were working with locally compact Abelian groups, especially with their group algebra $\mathrm{L}^1(G)$ and their measure algebra $\mathrm{M}(G)$. However, as many locally compact groups are not Abelian, it was desirable and natural to look for non-commutative versions of the classical results available in commutative harmonic analysis. Thus, a new direction of research was instigated by P. Eymard [28] in 1964, when he introduced and pioneered the study of the Fourier algebra $\mathrm{A}(G)$ and of the Fourier-Stieltjes algebra $\mathrm{B}(G)$ for all locally compact groups. Many analysts then became interested in these new algebras since, in the case when G is Abelian, $\mathrm{A}(G)$ can be identified with $\mathrm{L}^1(\widehat{G})$ via the Fourier transform and $\mathrm{B}(G)$ with $\mathrm{M}(\widehat{G})$ via the Fourier-Stieltjes transform, $\widehat{G}$ being the dual group of G.

In 1965, A. Figà-Talamanca [31] introduced an L^p-version of $\mathrm{A}(G)$ for locally compact Abelian groups, which he reasonably denoted $\mathrm{A_p}(G)$ as $\mathrm{A_2}(G) = \mathrm{A}(G)$. But Figà-Talamanca was mainly concerned with the study of multipliers of $\mathrm{L^p}(G)$ and so did not prove that $\mathrm{A_p}(G)$ is an algebra. Indeed, it was only a few years later that C. S. Herz [48] realized that $\mathrm{A_p}(G)$ is closed under pointwise multiplication via an application of the theory of Banach space tensor products. Since Herz's seminal paper, the algebra $\mathrm{A_p}(G)$ is known as the Figà-Talamanca-Herz algebra, and $\mathrm{A_p}(G)$, as well as its multiplier algebra $\mathrm{MA_p}(G)$, have enjoyed a great deal of attention and are still under current investigations. See [48, 24, 43, 45, 77, 17] and the references therein.

In this thesis, we establish several theorems related to $\mathrm{A_p}(G)$, all of them being of an ergodic nature. They are discussed independently in different chapters, which are organized as follows:

In Chapter 2, we collect the necessary definitions and preliminary results needed in the sequel. In Chapter 3, we describe all the non-degenerate $*$-representations of $\mathrm{A_p}(G)$, and we show in Section 3.2 that they are defined, up to unitary equivalence, by pointwise multiplication on some L^2-space. In Section 3.3, we prove the existence of a unique extension of any such representation to the multiplier algebra $\mathrm{MA_p}(G)$.

In Chapter 4, we apply the results of the previous chapter to study ergodic sequences in $\mathrm{MA_p}(G)$. In particular, we show in Section 4.1 that for all locally compact groups, the weakly ergodic sequences coincide with the strongly ergodic ones, and that these sequences approximate, in the weak* topology, the topologically invariant mean on some spaces of linear functionals on $\mathrm{A_p}(G)$. Some functorial properties are considered in Section 4.2, and examples of ergodic sequences are given in Section 4.3 using a notion of well distribution.

In Chapter 5, motivated by the study of ergodic sequences of iterates, we introduce a notion of ergodic and mixing multipliers of $\mathrm{A_p}(G)$. We show in Section 5.1 that the ergodic multipliers coincide with the mixing ones, and this provides a solution to a dual version of the complete mixing problem for probability measures. We also characterize the existence of such multipliers. In Section 5.2, we solve a similar problem in the more general context of a Tauberian regular semisimple commutative Banach algebra, in which the multipliers are ergodic (respectively mixing) with respect to a set of spectral synthesis.

In the second part, we deal with some ergodic and fixed point properties of φ-amenable Banach algebras, a class of Banach algebras which include the left amenable Lau algebras as well as the Figà-Talamanca-Herz algebras. As a tool for the next sections, we present in Section 6.1 a new characterization of left amenability for semigroups in terms of certain Hahn-Banach separation properties. In Section 6.2, we present new proofs of several known results (see [58, 96, 67]). In Section 6.3, we prove a mean ergodic theorem in the general setting of φ-amenable Banach algebras, and we also show the uniqueness of a two-sided φ-mean on the weakly almost periodic functionals. As an application, we obtain in Section 6.4 a direct sum decomposition for some subspaces of weakly almost periodic functionals. In Section 6.5, we consider various fixed point properties which characterize φ-amenability. Among other things, we prove an analogue of a fixed point theorem which is well known in the context of semigroups [79, Theorem 3.], and our proof greatly simplifies the original one.

In Chapter 7, by means of some of our previous results, we describe the norm spectrum of any weakly almost periodic functional on $\mathrm{A_p}(G)$ in terms of the topologically invariant mean. In particular, this leads to a new characterization of discrete groups.

A list of the main results of this thesis goes as follows:

Theorem 4.1.3. *Let (v_n) be a sequence in $\mathrm{S_M^p}$. The following assertions are equivalent:*

(i) (v_n) *is strongly ergodic.*

(ii) (v_n) *is weakly ergodic.*

(iii) *For every* $x \in G$, $x \neq e$, $v_n(x) \to 0$.

(iv) *For every* $T \in \mathrm{C}_{\delta,\mathrm{p}}(G)$, $\langle v_n, T\rangle \to \langle \Psi, T\rangle$.

(v) *For every* $T \in \mathrm{M_p}(G)$, $\langle v_n, T\rangle \to \langle \Psi, T\rangle$.

Theorem 5.2.1. *Let $T \in M(A)$ such that $||T|| = 1 = \widehat{T}(\varphi)$ for all $\varphi \in F$. If F is a set of spectral synthesis, then the following statements are equivalent:*

(i) $\widehat{T^n}(\gamma) \to 0$ *pointwise for all* $\gamma \in \triangle(A) \setminus F$.

(ii) $|\widehat{T}(\gamma)| < 1$ *for all* $\gamma \in \triangle(A) \setminus F$.

(iii) T *is completely* F*-mixing.*

(iv) T *is* F*-mixing.*

(v) T *is weakly* F*-mixing.*

(vi) T *is* F*-ergodic.*

(vii) $I_T = k(F)$.

(viii) $\{f \in A^* : T^*f = f\} = \overline{span(F)}^{w^*}$.

(ix) $F = F_T$, *where* $F_T := \{\gamma \in \triangle(A) : \widehat{T}(\gamma) = 1\}$.

Corollary 6.3.7. *Let C be a non-empty closed A-invariant subset of* $\mathrm{wap}(A)$ *such that $\mathbb{C}\varphi \subseteq C$. Suppose that A admits a two-sided φ-mean of norm one. Then there exists a unique non-expansive retraction P from C onto $Fix(S_\varphi)$ such that $PT_\mu = T_\mu P = P$ for each $\mu \in \mathscr{M}_\varphi$, and $Px \in \overline{\{u \cdot x : u \in S_\varphi\}}$ for each $x \in C$.*
Moreover, if (μ_α) is an ergodic net in $\mathscr{M}_\varphi$, then there exists a subnet (μ_β) of (μ_α) such that T_{μ_β} converges to P in the weak operator topology.

Theorem 6.5.5. *Let X be a topologically left invariant subspace of A^* with $\varphi \in X$, and such that condition (6.6) is satisfied. The following assertions are equivalent:*

(i) There exists a φ-mean Ψ on X with $||\Psi|| = 1$.

(ii) For each $x \in X$, there exists $\lambda \in \mathbb{C}$ such that $\lambda\varphi \in \overline{\{u \cdot x : u \in S_\varphi\}}^{w^*}$.

Theorem 7.1. *Let G be a discrete group, and $x \in G$ be arbitrary. The following assertions hold:*

(a) For any $T \in \mathrm{PM_p}(G)$, $T \neq 0$, $x \in \sigma_*(T)$ if and only if $\langle \Psi_x, T\rangle \neq 0$.

(b) For any $T \in \mathrm{WAP_p}(G)$, $T \neq 0$, $x \in \sigma(T)$ if and only if $\langle \Psi_x, T\rangle \neq 0$ if and only if $T = c\lambda_p(x) + T_0$ for some $c \in \mathbb{C}$, $c \neq 0$, and $T_0 \in \{S \in \mathrm{WAP_p}(G) : \Psi_x(S) = 0\}$.

(c) For any $T \in \mathrm{AP_p}(G)$, $T \neq 0$, $x \in \sigma(T)$ if and only if $\langle \Psi_x, T\rangle \neq 0$ if and only if $T = c\lambda_p(x) + T_0$ for some $c \in \mathbb{C}$, $c \neq 0$, and $T_0 \in \{S \in \mathrm{AP_p}(G) : \Psi_x(S) = 0\}$.

Chapter 2

Preliminaries.

2.1 Analysis on locally compact groups.

Given a Banach space X we denote its dual space by X^*. If Y is a subset of X^*, the $\sigma(X,Y)$-topology is the weakest topology on X such that every linear functional $\phi \in Y$ is continuous. Given a dual pair (X,Y), by $\phi(x)$ or $\langle \phi, x\rangle$ we will usually denote the value of ϕ at x.
For a locally compact Hausdorff space X, we denote by $\mathrm{CB}(X)$ (respectively $\mathrm{C}_0(X)$, $\mathrm{C}_{00}(X)$) the space of all complex-valued bounded continuous functions on X (resp. which vanish at infinity, with compact support), equipped with the sup-norm $||.||_{\mathrm{sup}}$.
A *locally compact group* is a pair (G,τ), where G is a group and τ is a locally compact Hausdorff topology on G, such that the map $G\times G \to G, (x,y)\mapsto xy^{-1}$ is continuous when $G\times G$ is equipped with the product topology. The identity element of the group will be denoted e.
Following Bourbaki's terminology [8, Chap. 3], we call an element $\mu \in \mathrm{C}_{00}(G)^*$ a (*complex*) *Radon measure* on G. Since the space $\mathrm{C}_{00}(G)$ can be viewed as the union of $\mathrm{C}_{00}(G,K)$, where K runs over all compact subsets of G and $\mathrm{C}_{00}(G,K) := \{f \in \mathrm{C}_{00}(G) : supp(f) \subseteq K\}$, we may define an inductive limit topology on $\mathrm{C}_{00}(G)$. Applying the criterion of continuity in inductive limits, we may as well describe a Radon measure as follows: a linear functional μ on $\mathrm{C}_{00}(G)$ is a *Radon measure* on G if for every compact subset K of G, there is a positive constant c_K such that $|\langle \mu, f\rangle| \leq c_K \sup_{x\in K} |f(x)|$ whenever $f \in \mathrm{C}_{00}(G,K)$. We say that μ is *bounded* if there is $C>0$ such that $|\langle \mu, f\rangle| \leq$

$C\,||f||_{\sup}$ for all $f \in \mathrm{C}_{00}(G)$, and we say that μ is *positive* if $\langle \mu, f\rangle \geq 0$ for every $f \in \mathrm{C}_{00}^{+}(G)$, where $\mathrm{C}_{00}^{+}(G)$ stands for the set of all real-valued functions in $\mathrm{C}_{00}(G)$ which are non-negative. We write $\mathrm{M}(G)$ for the space of all bounded Radon measures on G, and $\mathrm{M}^{+}(G)$ for the positive bounded Radon measures on G. Unless otherwise stated, by "measure" we always mean a bounded Radon measure, and for any $f \in \mathrm{C}_{00}(G)$ we may write $\langle \mu, f\rangle = \int f(x)\,d\mu(x)$. By the Riesz representation theorem, there is a one-to-one correspondence between positive Radon measures on G and positive regular Borel measures on G. For a complex-valued function f on G we will use the following notations:

$$\ell_x f(y) \,=\, f(xy), \quad r_x f(y) \,=\, f(yx), \qquad \text{for } x, y \in G,$$

$$\check{f}(x) \,=\, f(x^{-1}), \quad \widetilde{f}(x) \,=\, \overline{f(x^{-1})}, \qquad \text{for } x \in G.$$

A non-zero positive Radon measure m on G is called a *left Haar measure* on G if m is left invariant, *i.e.*, $\langle m, \ell_x f\rangle = \langle m, f\rangle$ for all $x \in G$, $f \in \mathrm{C}_{00}(G)$. The most remarkable feature of a locally compact group G is certainly the existence of a left Haar measure on G, which is unique up to multiplication by a positive constant. Thus, measure theory and functional analysis may be used as important tools for the study of locally compact groups.

Let G be a locally compact group, and let m be a fixed left Haar measure on G. For $f \in \mathrm{C}_{00}(G)$ we write $\langle m, f\rangle = \int f(x)\,dx$. For each $1 \leq p < \infty$, the space of all p-integrable functions on G is denoted by $\mathscr{L}^p(G)$, and $(\mathrm{L}^{\mathrm{p}}(G), ||.||_p)$ will denote the Banach space of all equivalence classes of p-integrable functions, where two functions $f, g \in \mathscr{L}^p(G)$ are said to be equivalent if $\int |f(x) - g(x)|^p\,dx = 0$. For $p = \infty$, $(\mathrm{L}^{\infty}(G), ||.||_{\infty})$ is the Banach space of all equivalence classes of (Haar) measurable functions which are uniformly bounded outside a locally negligible set, *i.e.*, a set A such that $m(A \cap K) = 0$ for every compact subset K of G). For $[f] \in \mathrm{L}^{\infty}(G)$,

$$||[f]||_{\infty} \,:=\, \inf\{C > 0 \,:\, |f(x)| \leq C \ \text{ for locally almost all } x \in G\}.$$

Let $\mathrm{WAP}(G)$ (resp. $\mathrm{AP}(G)$) denote the space of all *weakly almost periodic* (resp. *almost periodic*) functions on G, *i.e.*, all $f \in \mathrm{L}^{\infty}(G)$ for which the left orbit $\mathcal{L}_f := \{\ell_x f \,:\, x \in G\}$ is relatively weakly compact (resp. relatively norm compact). In general, the following inclusions can be verified:

$$\mathrm{AP}(G) \oplus \mathrm{C}_0(G) \subseteq \mathrm{WAP}(G) \subseteq \mathrm{CB}(G) \subseteq \mathrm{L}^\infty(G).$$

For a continuous function, we identify the function with its equivalence class. Furthermore, the spaces $\mathrm{C}_0(G)$, $\mathrm{CB}(G)$ and $\mathrm{L}^\infty(G)$ are commutative C^*-algebras with pointwise multiplication and involution given by conjugation. In order to define a structure of involutive Banach algebra on $\mathrm{L}^1(G)$ and $\mathrm{M}(G)$ one must consider the *convolution product*

$$\int f(x)\, d(\mu * \nu)(x) := \int\int f(xy)\, d\mu(x) d\nu(y), \text{ for } f \in \mathrm{C}_{00}(G), \mu, \nu \in \mathrm{M}(G);$$

$$f * g(x) := \int f(xy)\, g(y^{-1})\, dy, \text{ for } f, g \in \mathscr{L}^1(G), \text{ and almost all } x \in G.$$

Equipped with these products and the following involutions

$$\int f(x)\, d\mu^*(x) := \int \check{f}(x)\, d\overline{\mu}(x), \text{ for } f \in \mathrm{C}_{00}(G),\, \mu \in \mathrm{M}(G);$$

$$f^*(x) := \triangle_G(x^{-1})\, \widetilde{f}(x), \text{ for } f \in \mathscr{L}^1(G),\, x \in G,$$

where $\triangle_G$ is the modular function of G, $\mathrm{M}(G)$ and $\mathrm{L}^1(G)$ become involutive Banach algebras. For each $1 \le p \le \infty$, the *left regular representation* of G (resp. $\mathrm{M}(G)$) on $\mathrm{L}^\mathrm{p}(G)$ is given by

$$\lambda_p : G \to \mathscr{B}(\mathrm{L}^\mathrm{p}(G)),\, \lambda_p(x)([f]) = [\ell_{x^{-1}} f]$$

$$(\text{resp. } \lambda_p : \mathrm{M}(G) \to \mathscr{B}(\mathrm{L}^\mathrm{p}(G)),\, \lambda_p(\mu)([f]) = [\mu * f]),$$

where $\mathscr{B}(\mathrm{L}^\mathrm{p}(G))$ is the Banach algebra of all bounded linear operators on $\mathrm{L}^\mathrm{p}(G)$. A crucial property of the convolution product is the following: if $1 < p < \infty$ and $p' = \frac{p}{p-1}$, then for any $f \in \mathscr{L}^p(G)$, $g \in \mathscr{L}^{p'}(G)$, $f * \check{g}(x)$ exists everywhere and defines a function in $\mathrm{C}_0(G)$.
References: [8], [50], [33].

2.2 The Figà-Talamanca-Herz algebras and related spaces.

From now on and throughout this thesis, we let G be a locally compact group with a fixed left Haar measure $m = dx$, and $1 < p, p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\mathrm{A_p}(G)$ denote the space of all continuous functions $u : G \to \mathbb{C}$ which can be represented as

$$(2.1) \qquad u = \sum_{n=1}^{\infty} \ell_n * \check{k_n} \quad \text{for some } k_n \in \mathscr{L}^p(G),\, \ell_n \in \mathscr{L}^{p'}(G),$$

so that $\sum_{n=1}^{\infty} N_p(k_n)\, N_{p'}(\ell_n) < \infty$, where $N_q(f) := ||[f]||_q$. Then $\mathrm{A_p}(G)$ is a dense linear subspace of $\mathrm{C_0}(G)$. Under pointwise multiplication and with the norm given by

$$||u||_{\mathrm{A_p}} := \inf \sum_{n=1}^{\infty} N_p(k_n)\, N_{p'}(\ell_n),$$

where the infimum runs over all the possible representations of u as in (2.1), $\mathrm{A_p}(G)$ is a Tauberian semisimple regular commutative Banach algebra, nowadays known as the *the Figà-Talamanca-Herz algebra* of G, or sometimes called the *generalized Fourier algebra* of G.

Let $\mathrm{PM_p}(G)$ denote the w^*-closure of $\lambda_p(\mathrm{M}(G))$ in $\mathscr{B}(\mathrm{L^p}(G))$, where the w^*-topology is the $\sigma(\mathscr{B}(\mathrm{L^p}(G)), \mathrm{L^p}(G)\widehat{\otimes}\mathrm{L^{p'}}(G))$-topology, $\widehat{\otimes}$ being the completed projective tensor product. The elements in $\mathrm{PM_p}(G)$ are called *p-pseudo-measures*, and they form a Banach algebra when endowed with the operator norm. Moreover, $\mathrm{PM_p}(G)$ is isometrically isomorphic to the dual space $\mathrm{A_p}(G)^*$ and the w^*-topology $\sigma(\mathscr{B}(\mathrm{L^p}(G)), \mathrm{L^p}(G)\widehat{\otimes}\mathrm{L^{p'}}(G))$ is carried over to the w^*-topology $\sigma(\mathrm{A_p}(G)^*, \mathrm{A_p}(G))$. In particular, the measure algebra $\mathrm{M}(G)$ can be viewed as a subset of $\mathrm{A_p}(G)^*$ via

$$\langle \lambda_p(\mu), u\rangle = \int u(x)\, d\mu(x), \qquad \text{for } u \in \mathrm{A_p}(G),\, \mu \in \mathrm{M}(G).$$

Let $\mathrm{PF_p}(G)$ denote the norm-closure of $\lambda_p(\mathrm{L^1}(G))$ in $\mathscr{B}(\mathrm{L^p}(G))$. The elements in $\mathrm{PF_p}(G)$ are called *p-pseudofunctions*, and they form a closed subalgebra of $\mathrm{PM_p}(G)$. Under pointwise multiplication, the dual space $\mathrm{W_p}(G) := \mathrm{PF_p}(G)^*$

is a commutative Banach algebra of continuous functions on G.

Remark 2.2.1. In case $p = 2$, $\mathrm{A}_2(G)$ is the Fourier algebra $A(G)$, $\mathrm{PM}_2(G)$ is the group von Neumann algebra $\mathrm{VN}(G)$, $\mathrm{PF}_2(G)$ is the reduced group C^*-algebra $\mathrm{C}^*_\rho(G)$, and $\mathrm{W}_2(G)$ is the reduced Fourier-Stieltjes algebra $\mathrm{B}_\rho(G)$. See [28, 30].

A *multiplier function* of $\mathrm{A_p}(G)$ is a continuous function $v : G \to \mathbb{C}$ such that $vu \in \mathrm{A_p}(G)$ whenever $u \in \mathrm{A_p}(G)$. We write $\mathrm{MA_p}(G)$ for the commutative Banach algebra of all multiplier functions of $\mathrm{A_p}(G)$, equipped with the *multiplier norm*

$$||v||_M := \sup\{||vu||_{\mathrm{A_p}} : u \in \mathrm{A_p}(G), ||u||_{\mathrm{A_p}} \leq 1\}.$$

In general, the following inclusions hold:

$$\mathrm{A_p}(G) \subseteq \mathrm{W_p}(G) \subseteq \mathrm{MA_p}(G),$$

and

$$||u||_{\mathrm{sup}} \leq ||u||_M \leq ||u||_{\mathrm{W_p}} \leq ||u||_{\mathrm{A_p}} \quad \text{for } u \in \mathrm{A_p}(G).$$

If G is amenable (see definition below), then $\mathrm{W_p}(G) = \mathrm{MA_p}(G)$.
References: [29], [48], [13], [85], [23].

Arising from the theory of Banach algebras, $\mathrm{A_p}(G)$ naturally acts on its dual space $\mathrm{PM_p}(G)$, and this action is even well-defined for the multiplier algebra $\mathrm{MA_p}(G)$. Thus, the Banach spaces $\mathrm{PM_p}(G)$ and $\mathrm{PM_p}(G)^*$ are both $\mathrm{MA_p}(G)$-modules under the actions

$$\langle v \cdot T, u\rangle := \langle T, vu\rangle, \quad \text{and} \quad \langle v \cdot \Phi, T\rangle := \langle \Phi, v \cdot T\rangle,$$

for $v \in \mathrm{MA_p}(G)$, $u \in \mathrm{A_p}(G)$, $T \in \mathrm{PM_p}(G)$ and $\Phi \in \mathrm{PM_p}(G)^*$.
Next, we list four closed subspaces of $\mathrm{PM_p}(G)$ that are of interest to us. We follow the notations of [24].

$$\mathrm{C}_{\delta,\mathrm{p}}(G) := \overline{\mathrm{span}\{\lambda_p(x) : x \in G\}}^{||\cdot||_{\mathrm{PM_p}}};$$

$$\mathrm{M_p}(G) := \overline{\{\lambda_p(\mu) : \mu \in \mathrm{M}(G)\}}^{||\cdot||_{\mathrm{PM_p}}};$$

$$\mathrm{AP_p}(G) := \left\{ T \in \mathrm{PM_p}(G) : \overset{\mathrm{A_p}(G) \to \mathrm{PM_p}(G)}{u \mapsto u \cdot T} \quad \text{is a compact operator} \right\}.$$

$$\mathrm{WAP_p}(G) := \left\{ T \in \mathrm{PM_p}(G) : \overset{\mathrm{A_p}(G)\to\mathrm{PM_p}(G)}{u \mapsto u\cdot T} \quad \text{is a weakly compact operator} \right\}.$$

In general, the following inclusions are satisfied:

$$\mathrm{C}_{\delta,\mathrm{p}}(G) \subseteq \mathrm{M_p}(G) \subseteq \mathrm{WAP_p}(G) \subseteq \mathrm{PM_p}(G) \quad \text{and} \quad \mathrm{C}_{\delta,\mathrm{p}}(G) \subseteq \mathrm{AP_p}(G).$$

Moreover, $\mathrm{C}_{\delta,\mathrm{p}}(G)$, $\mathrm{PF_p}(G)$, $\mathrm{M_p}(G)$, $\mathrm{AP_p}(G)$ and $\mathrm{WAP_p}(G)$ are norm-closed $\mathrm{MA_p}(G)$-submodules of $\mathrm{PM_p}(G)$ [43], and $\mathrm{C}_{\delta,\mathrm{p}}(G)$ contains the identity $\lambda_p(e)$ of $\mathrm{PM_p}(G)$.

Remark 2.2.2. If G is Abelian and $\widehat{G}$ denotes its dual group, then $\mathrm{C}_{\delta,2}(G) = \mathrm{AP}(\widehat{G}) = \mathrm{AP}_2(G)$, $\mathrm{PF}_2(G) = \mathrm{C}_0(\widehat{G})$, $\mathrm{M}_2(G)$ is the $||.||_\infty$-closure of the Fourier-Stieltjes algebra $\mathrm{B}(\widehat{G})$ in $\mathrm{L}^\infty(\widehat{G})$, and $\mathrm{WAP}_2(G) = \mathrm{WAP}(\widehat{G})$ [26]. If G is not necessarily Abelian, then $\mathrm{C}_{\delta,2}(G)$ is sometimes denoted $\mathrm{C}^*_\delta(G)$, as in [65] and [70] for instance.

2.3 Amenable groups and topologically invariant means.

Let G be a locally compact group, and let X be a translation-invariant subspace of $\mathrm{L}^\infty(G)$ containing 1_G, the constant one function. A linear functional $m \in X^*$ is called a *mean* if $||m|| = 1 = m(1_G)$. If X is a C^*-subalgebra of $\mathrm{L}^\infty(G)$, then m is a mean if and only if $m \geq 0$ and $||m|| = 1$. A mean $m \in X^*$ is said to be *left invariant* if $\langle m, \ell_a f\rangle = \langle m, f\rangle$ for all $f \in X$, $a \in G$. If X is topologically left invariant, *i.e.*, $\varphi * f \in X$ for all $\varphi \in \mathrm{L}^1(G)$, $f \in X$, then a mean $m \in X^*$ is said to be *topologically left invariant* if $\langle m, \varphi * f\rangle = \langle m, f\rangle$ for all $\varphi \in \mathrm{L}^1(G)$, $\varphi \geq 0$, $||\varphi||_1 = 1$, and all $f \in X$. In particular, on $\mathrm{L}^\infty(G)$, the existence of a left invariant mean is equivalent to the existence of a topologically left invariant mean, and in case such a mean exists, then G is called *amenable*. The class of amenable groups includes all solvable groups and all compact groups. However, the free group on two generators is not amenable.

Remark 2.3.1. It is known that a locally compact group G is amenable if and only if $\mathrm{A_p}(G)$ admits a bounded approximate identity [48]. Other characterizations of the amenability of G in terms of the algebras $\mathrm{A_p}(G)$ are given in [38] (see also [34]).

References: [47], [84], [85], [92].

Analogously, a *mean* on $\mathrm{PM_p}(G)$ is defined as a linear functional $\Phi \in \mathrm{PM_p}(G)^*$ such that $||\Phi|| = 1 = \Phi(\lambda_p(e))$. In addition, if $v \cdot \Phi = v(e)\,\Phi$ for all $v \in \mathrm{MA_p}(G)$, then we say that Φ is *topologically invariant.* E. E. Granirer [43] showed that the set of topologically invariant means on $\mathrm{PM_p}(G)$ is never empty, and also that such a topologically invariant mean on $\mathrm{WAP_p}(G)$ is unique. By restriction, there is a unique topologically invariant mean on $\mathrm{C_{\delta,p}}(G)$ and $\mathrm{M_p}(G)$ [44, Proposition 3.].

Chapter 3

Non-degenerate $*$-representations of $\mathrm{A_p}(G)$ and $\mathrm{MA_p}(G)$.

In this chapter, we shall describe all non-degenerate $*$-representations of $\mathrm{A_p}(G)$ along with their extensions to the multiplier algebra $\mathrm{MA_p}(G)$. The main results are Theorem 3.2.4 and Theorem 3.3.3. Our study of $*$-representations of $\mathrm{A_p}(G)$ was motivated by the lemmas presented in section 2. of [70], where Theorem 3.3.3 is proved for the special case $p = 2$, under the additional assumption that G is amenable. We repeat these lemmas in section 3.1 below, but since their extensions to the case $1 < p < \infty$ do not require any major transformation of the proofs given in [70], we omit their proofs.
Throughout this chapter, let $1 < p < \infty$ and G be a locally compact group.

3.1 Two preliminary lemmas.

Lemma 3.1.1. *For any positive bounded Radon measure $\mu \in \mathrm{M}^+(G)$ we define the linear operator $S_\mu : \mathrm{A_p}(G) \to \mathscr{B}(\mathrm{L}^2(G,\mu))$ by*

$$S_\mu(u)([h]) = [uh] \qquad \text{for } u \in \mathrm{A_p}(G) \text{ and } [h] \in \mathrm{L}^2(G,\mu).$$

Then the map $u \mapsto S_\mu(u)$ is a cyclic $*$-representation of $\mathrm{A_p}(G)$ as bounded linear operators on $\mathrm{L}^2(G,\mu)$, with cyclic vector $[1] \in \mathrm{L}^2(G,\mu)$.
Furthermore,

$$\langle u,\mu\rangle_{\mathrm{A_p}(G),\mathrm{M}(G)} = \langle S_\mu(u)([1]),[1]\rangle \qquad \text{for any } u \in \mathrm{A_p}(G).$$

For a locally compact Hausdorff space X, it is known that every cyclic $*$-representation of $\mathrm{C}_0(X)$ as bounded linear operators on a Hilbert space $\mathscr{H}$ is unitarily equivalent to a representation of the form $\{S_\mu, \mathrm{L}^2(X,\mu)\}$ for a positive bounded measure μ on X [50, Theorem (C.36)]. A similar result for the Fourier algebra $\mathrm{A}(G)$ appears in [70, Lemma 2.2.]. The analogous version for $\mathrm{A_p}(G)$ is:

Lemma 3.1.2. *Let $\{T, \mathscr{H}\}$ be a cyclic $*$-representation of $\mathrm{A_p}(G)$. Then there exists a positive measure $\mu \in \mathrm{M}^+(G)$ such that $\{T, \mathscr{H}\}$ is unitarily equivalent to $\{S_\mu, \mathrm{L}^2(G,\mu)\}$, i.e., there exists a surjective linear isometry W from $\mathscr{H}$ onto $\mathrm{L}^2(G,\mu)$ such that $WT(u) = S_\mu(u)W$ for all $u \in \mathrm{A_p}(G)$.*

3.2 Description of the $*$-representations of $\mathrm{A_p}(G)$.

By means of the lemmas presented in the first section, we completely characterize all the $*$-representations of $\mathrm{A_p}(G)$, up to unitary equivalence. Before stating the main results, the following preliminary lemmas are needed.

Lemma 3.2.1. *Let $\mu_\alpha, \mu_\beta \in \mathrm{M}^+(G)$ be such that $supp(\mu_\alpha) \cap supp(\mu_\beta) = \varnothing$. Then we have:*

$$\mathrm{L}^2(G,\mu_\alpha) \cap \mathrm{L}^2(G,\mu_\beta) \;=\; \{[0]\}.$$

Proof. Let $[f] \in \mathrm{L}^2(G,\mu_\alpha) \cap \mathrm{L}^2(G,\mu_\beta)$ and assume that $[f] \neq [0]$. In particular, $|f(x)|^2 \neq 0$ almost everywhere. Then it follows that $\int |f(x)|^2\, d\mu_\alpha(x) \neq 0$ and $\int |f(x)|^2\, d\mu_\beta(x) \neq 0$ [8, chap. IV, §2, n^o 3, Theorem 1]. Therefore, $|f|^2 \neq 0$ in $supp(\mu_\alpha)$ and $|f|^2 \neq 0$ in $supp(\mu_\beta)$, hence $|f|^2 \neq 0$ in $supp(\mu_\alpha) \cap supp(\mu_\beta)$ [8, chap. III, §2, n^o 3, Proposition 8]. So there exists $x \in supp(\mu_\alpha) \cap supp(\mu_\beta)$ such that $|f(x)|^2 \neq 0$. Contradiction.

■

Lemma 3.2.2. *Let $(\mu_\alpha)_{\alpha \in I}$ be a summable family of positive bounded measures on G and assume that the supports of the measures μ_α are pairwise disjoint. If μ is the sum of the measures μ_α, then $\mathrm{L}^2(G,\mu)$ is isometrically isomorphic to $\sum_{\alpha \in I}^{\oplus} \mathrm{L}^2(G,\mu_\alpha)$.*

Proof. Let $[h] \in \mathrm{L}^2(G,\mu)$. Then $|h|^2$ is μ_α-integrable for each $\alpha \in I$ [9, §2, n^o 2, Proposition 3]. By Lemma 3.2.1, $[h] \in \mathrm{L}^2(G,\mu_\alpha)$ for a unique $\alpha \in I$ unless

$[h] = [0]$. In every instance,

$$\mathrm{L}^2(G,\mu) \subseteq \sum\nolimits_{\alpha\in I}^{\oplus} \mathrm{L}^2(G,\mu_\alpha).$$

Now using the definition of the direct sum of Hilbert spaces, and by [9, §2, n^o 2, Proposition 3], the conclusion follows from an argument similar to that used in the proof of [50, Theorem (C.37)].

■

Proposition 3.2.3. *For any positive Radon measure μ on G, μ not necessarily bounded, we define the linear operator $S_\mu : \mathrm{A_p}(G) \to \mathscr{B}(\mathrm{L}^2(G,\mu))$ by*

$$S_\mu(u)([h]) = [uh] \quad \textit{for } u \in \mathrm{A_p}(G) \textit{ and } [h] \in \mathrm{L}^2(G,\mu).$$

Then the map $u \mapsto S_\mu(u)$ is a non-degenerate $*$-representation of $\mathrm{A_p}(G)$ in $\mathscr{B}(\mathrm{L}^2(G,\mu))$.

Proof. Let μ be a positive Radon measure on G, and let $\mathscr{K}_G$ denote the set of all compact subsets of G. Since $\mathscr{K}_G$ is a μ-dense set for G [8, chap. IV, §5, n^o 8], by applying [9, §2, n^o 3, Proposition 4.] we may write $\mu = \sum_{\alpha\in I}\mu_\alpha$ for a summable family $(\mu_\alpha)_{\alpha\in I}$ of positive bounded measures on G, such that the supports $\mathit{supp}(\mu_\alpha)$ are pairwise disjoint. It is then straightforward to verify that S_μ is a $*$-representation of $\mathrm{A_p}(G)$ in $\mathscr{B}(\mathrm{L}^2(G,\mu))$. To prove its non-degeneracy, we will show that S_μ is a direct sum of cyclic $*$-representations. By Lemma 3.1.1, the operators S_{μ_α} are cyclic $*$-representations, and for any $[h] = \sum_\alpha [h_\alpha] \in \sum_{\alpha\in I}^{\oplus} \mathrm{L}^2(G,\mu_\alpha)$ we have by Lemma 3.2.2 that

$$\sum\nolimits_{\alpha\in I} S_{\mu_\alpha}(u)([h_\alpha]) = \sum_{\alpha\in I}[u\,h_\alpha] = [u\,h] = S_\mu(u)([h]).$$

That is, $\{S_\mu, \mathrm{L}^2(G,\mu)\}$ is the direct sum of $\{S_{\mu_\alpha}, \mathrm{L}^2(G,\mu_\alpha)\}$, and it follows that $\{S_\mu, \mathrm{L}^2(G,\mu)\}$ is non-degenerate by [100, Proposition 9.17].

■

The main result of this section is the following:

Theorem 3.2.4. *For any non-degenerate $*$-representation $\{T,\mathscr{H}\}$ of $\mathrm{A_p}(G)$, there exists a positive Radon measure μ on G, possibly unbounded, such that $\{T,\mathscr{H}\}$ is unitarily equivalent to $\{S_\mu, \mathrm{L}^2(G,\mu)\}$.*

Proof. By [100, Proposition 9.17] we may write $\{T,\mathscr{H}\}=\sum_{\alpha\in I}^{\oplus}\{T_\alpha,\mathscr{H}_\alpha\}$ for some cyclic $*$-representations $\{T_\alpha,\mathscr{H}_\alpha\}$. By Lemma 3.1.2, each $\{T_\alpha,\mathscr{H}_\alpha\}$ is unitarily equivalent, for some $\mu_\alpha\in \mathrm{M}^+(G)$, to $\{S_{\mu_\alpha},\mathrm{L}^2(G,\mu_\alpha)\}$. We now claim that the family $(\mu_\alpha)_{\alpha\in I}$ is summable. Indeed, for $([h_\alpha])_{\alpha\in I}\in\sum_{\alpha\in I}^{\oplus}\mathrm{L}^2(G,\mu_\alpha)$ we have

$$\sum_{\alpha\in I}\int_G |h_\alpha(x)|^2\,d\mu_\alpha(x)\;<\;\infty.$$

In particular,

$$\sum_{\alpha\in I}\int_G |f(x)|^2\,d\mu_\alpha(x)\;<\;\infty,\qquad \text{for any }\; f\in \mathrm{C}_{00}^{+}(G). \tag{3.1}$$

Thus, applying the Cauchy-Schwarz inequality for any $f\in\mathrm{C}_{00}^{+}(G)$, we obtain

$$\begin{aligned}\sum_{\alpha\in I}\int_G |f(x)|\,d\mu_\alpha(x) \;&\le\; \sum_{\alpha\in I}\left(\int_G |f(x)|^2\,d\mu_\alpha(x)\right)^{\frac{1}{2}}\left(\int_G 1\,d\mu_\alpha(x)\right)^{\frac{1}{2}}\\ &\le\; \sup_{\alpha\in I}(\mu_\alpha(G))^{\frac{1}{2}}\sum_{\alpha\in I}\left(\int_G |f(x)|^2\,d\mu_\alpha(x)\right)^{\frac{1}{2}}\;<\;\infty,\end{aligned}$$

after making use of the boundedness of each μ_α and by (3.1). Hence $(\mu_\alpha)_{\alpha\in I}$ is summable [9, §2, n^o 1]. We set $\mu=\sum_{\alpha\in I}\mu_\alpha$, so that μ is a positive Radon measure, which is not necessarily bounded [9, §2, n^o 1]. Therefore, by Proposition 3.2.3 we get

$$\{T,\mathscr{H}\}\;=\;\sum\nolimits_{\alpha\in I}^{\oplus}\{T_\alpha,\mathscr{H}_\alpha\}\;\cong\;\sum\nolimits_{\alpha\in I}^{\oplus}\{S_{\mu_\alpha},\mathrm{L}^2(G,\mu_\alpha)\}\;=\;\{S_\mu,\mathrm{L}^2(G,\mu)\}.$$

■

3.3 Extension to $*$-representations of $\mathrm{MA_p}(G)$.

We start this section by developing a general process for extending any non-degenerate $*$-representation of $\mathrm{A_p}(G)$ to a unique non-degenerate $*$-representation of $\mathrm{MA_p}(G)$. This work was inspired by the comments after Lemma 2.2 in [70], where our next lemma and proposition were proved for the case $p=2$ and G amenable. We here give an improvement of these results since we are able to get rid of the assumption of amenability.

Lemma 3.3.1. *Let $\{T, \mathscr{H}\}$ be a non-degenerate $*$-representation of $\mathrm{A_p}(G)$. For any $v \in \mathrm{MA_p}(G)$ and any $u \in \mathrm{A_p}(G)$, the following inequality holds:*

$$||T(vu)\,\xi|| \;\leq\; ||v||_M \, ||T(u)\,\xi|| \qquad \textit{for every } \xi \in \mathscr{H}.$$

Proof. We first consider the representation $\{S_\mu, \mathrm{L}^2(G,\mu)\}$ as defined in Proposition 3.2.3. Let $v \in \mathrm{MA_p}(G)$, $u \in \mathrm{A_p}(G)$ and $[h] \in \mathrm{L}^2(G,\mu)$ be arbitrary. Then we have

$$\begin{aligned} ||S_\mu(vu)([h])||_2^2 &= \int_G |v(x)u(x)h(x)|^2 \, d\mu(x) \\ &\leq ||v||_{\sup}^2 \int_G |u(x)h(x)|^2 d\mu(x) \;\leq\; ||v||_M^2 \, ||S_\mu(u)([h])||_2^2. \end{aligned}$$

Now let $\{T, \mathscr{H}\}$ be an arbitrary non-degenerate $*$-representation of $\mathrm{A_p}(G)$. By Theorem 3.2.4, $\{T, \mathscr{H}\}$ is unitarily equivalent to $\{S_\mu, \mathrm{L}^2(G,\mu)\}$ for some positive Radon measure μ on G. Let W be the corresponding linear isometry from $\mathscr{H}$ onto $\mathrm{L}^2(G,\mu)$. Thus, for any $\xi \in \mathscr{H}$,

$$\begin{aligned} ||T(vu)\xi|| &= ||W^* S_\mu(vu) W\xi|| \\ &\leq ||W^*|| \, ||S_\mu(vu)W\xi|| \;=\; ||S_\mu(vu)W\xi|| \\ &\leq ||v||_M \, ||S_\mu(u)W\xi|| \;=\; ||v||_M \, ||WT(u)\xi|| \\ &\leq ||v||_M \, ||W|| \, ||T(u)\xi|| \;=\; ||v||_M \, ||T(u)\xi||. \end{aligned}$$

■

Proposition 3.3.2. *For any non-degenerate $*$-representation T of $\mathrm{A_p}(G)$ and any $v \in \mathrm{MA_p}(G)$, there exists a unique bounded linear operator on $\mathscr{H}$, denoted $S = \widetilde{T}(v)$, such that*

$$S \circ T(u) \;=\; T(vu) \qquad \textit{for every } u \in \mathrm{A_p}(G). \tag{3.2}$$

Furthermore, $\widetilde{T}(u) = T(u)$ for all $u \in \mathrm{A_p}(G)$.

Proof. First we consider T cyclic, with cyclic vector $\xi_0 \in \mathscr{H}$. In this case, it suffices to define S on $span(T(\mathrm{A_p}(G))\xi_0)$ since $\overline{span(T(\mathrm{A_p}(G))\xi_0)} = \mathscr{H}$. Then for any $v \in \mathrm{MA_p}(G)$ we may define

$$S(T(u)\xi_0) := T(vu)\xi_0 \quad \text{ for } u \in \mathrm{A_p}(G).$$

Fix $v \in \mathrm{MA_p}(G)$. We notice that S is linear since T is a homomorphism, and by Lemma 3.3.1, S is a bounded linear operator on $\langle T(\mathrm{A_p}(G))\xi_0\rangle$ with $||S|| \leq ||v||_M$. As $span(T(\mathrm{A_p}(G))\xi_0)$ is dense in $\mathscr{H}$, S may be extended uniquely to a bounded linear operator on $\mathscr{H}$, which we also denote by S. Moreover, by a similar density argument, it suffices to verify (3.2) for a vector of the form $T(\phi)\xi_0$, $\phi \in \mathrm{A_p}(G)$. In fact, for $u \in \mathrm{A_p}(G)$ we have

$$S \circ T(u)(T(\phi)\xi_0) = S(T(u\phi)\xi_0) = T(vu\phi)\xi_0 = T(vu)(T(\phi)\xi_0).$$

Next, let $\{T, \mathscr{H}\}$ be a non-degenerate $*$-representation of $\mathrm{A_p}(G)$. Again, we may write $\{T, \mathscr{H}\} = \sum_{\alpha\in I}^{\oplus}\{T_\alpha, \mathscr{H}_\alpha\}$ for some cyclic $*$-representations $\{T_\alpha, \mathscr{H}_\alpha\}$ [100, Proposition 9.17]. For any $v \in \mathrm{MA_p}(G)$ we define

$$S := \sum_{\alpha\in I}^{\oplus} S_\alpha,$$

where S_α is the bounded linear operator on $\mathscr{H}_\alpha$ associated to $\{T_\alpha, \mathscr{H}_\alpha\}$ as above. Thus, for any $u \in \mathrm{A_p}(G)$ we have

$$S \circ T(u) = \sum_{\alpha\in I}^{\oplus} S_\alpha \circ T_\alpha(u) = \sum_{\alpha\in I}^{\oplus} T_\alpha(vu) = T(vu).$$

Finally, we observe that S is uniquely determined by T using (3.2).

■

Remark 3.3.1. We observe that the procedure developed in Lemma 3.3.1 and Proposition 3.3.2 apply similarly to $\mathrm{W_p}(G)$ or to any $*$-algebra $\mathscr{A}$ which satisfy $\mathrm{A_p}(G) \subseteq \mathscr{A} \subseteq \mathrm{MA_p}(G)$.

A well known result states that every non-degenerate $*$-representation of $\mathrm{L}^1(G)$ admits a unique extension to $\mathrm{M}(G)$. Our next theorem, which is the main result of this section, presents an analogue of this result in the "dual framework" of the Figà-Talamanca-Herz algebras. Although the proof for $\mathrm{L}^1(G)$ makes use of an approximate identity, we do not require the group G to be amenable in our setting, thanks to Lemma 3.3.1 and Proposition 3.3.2. We recall that a locally compact group G is amenable if and only if $\mathrm{A_p}(G)$ admits a bounded approximate identity [48, Theorem 6.].

Theorem 3.3.3. *For any non-degenerate $*$-representation $\{T, \mathscr{H}\}$ of $\mathrm{A_p}(G)$, the map $\widetilde{T} : \mathrm{MA_p}(G) \to \mathscr{B}(\mathscr{H})$ is a non-degenerate $*$-representation of $\mathrm{MA_p}(G)$ in $\mathscr{B}(\mathscr{H})$.*

Proof. We start by verifying that $\widetilde{T}$ is a homomorphism. Let $v, w \in \mathrm{MA_p}(G)$. We have, for any $u \in \mathrm{A_p}(G)$:

$$\widetilde{T}(v)\widetilde{T}(w)T(u) \;=\; \widetilde{T}(v)T(wu) \;=\; T(vwu) \;=\; \widetilde{T}(vw)T(u). \tag{3.3}$$

By non-degeneracy of T, *i.e.*, $\mathrm{span}\{T(u)\xi \;:\; u \in \mathrm{A_p}(G), \xi \in \mathscr{H}\}$ is dense in $\mathscr{H}$, (3.3) shows that $\widetilde{T}$ is a homomorphism. Next, for any $v \in \mathrm{MA_p}(G)$, $u \in \mathrm{A_p}(G)$, and $\xi, \eta \in \mathscr{H}$, we establish the following identity:

$$\langle \widetilde{T}(\bar{v})T(u)\xi, \eta\rangle \;=\; \langle T(u)\widetilde{T}(v)^*\xi, \eta\rangle. \tag{3.4}$$

Indeed,

$$\begin{aligned}
\langle \widetilde{T}(\bar{v})T(u)\xi, \eta\rangle &= \langle T(\bar{v}u)\xi, \eta\rangle \;=\; \langle T(\overline{v\bar{u}})\xi, \eta\rangle \\
&= \langle T(v\bar{u})^*\xi, \eta\rangle \;=\; \langle \xi, T(v\bar{u})\eta\rangle \\
&= \langle \xi, \widetilde{T}(v)T(\bar{u})\eta\rangle \;=\; \langle \widetilde{T}(v)^*\xi, T(u)^*\eta\rangle \\
&= \langle T(u)\widetilde{T}(v)^*\xi, \eta\rangle,
\end{aligned}$$

where we used the fact that T is a $*$-representation of $\mathrm{A_p}(G)$. Since $\mathrm{A_p}(G)$ and $\mathrm{MA_p}(G)$ are commutative Banach algebras, and since T and $\widetilde{T}$ are homomorphisms, it follows that

$$T(u)\widetilde{T}(v) \;=\; \widetilde{T}(v)T(u) \qquad \text{for every } \; u \in \mathrm{A_p}(G), v \in \mathrm{MA_p}(G),$$

and via (3.4)

$$\langle \widetilde{T}(\bar{v})T(u)\xi, \eta\rangle \;=\; \langle T(u)\widetilde{T}(\bar{v})\xi, \eta\rangle \;=\; \langle \widetilde{T}(v)^*T(u)\xi, \eta\rangle.$$

This shows that $\widetilde{T}$ preserves involution, hence is a $*$-representation of $\mathrm{MA_p}(G)$ in $\mathscr{B}(\mathscr{H})$. The non-degeneracy of $\widetilde{T}$ is an immediate consequence of the non-degeneracy of T.

■

Corollary 3.3.4. *Let $\{T, \mathscr{H}\}$ be a $*$-representation of $\mathrm{MA_p}(G)$. Assume that the restriction, denoted T_A, of T to $\mathrm{A_p}(G)$ is non-degenerate. Then $\widetilde{T_A} = T$, and T is non-degenerate.*

Proof. For any $v \in \mathrm{MA_p}(G)$, $u \in \mathrm{A_p}(G)$, we have:

$$\widetilde{T_A}(v)T_A(u) \;=\; T_A(vu) \;=\; T(vu) \;=\; T(v)T(u) \;=\; T(v)T_A(u).$$

By non-degeneracy of $\{T_A, \mathscr{H}\}$, this shows that $\widetilde{T_A}(v) = T(v)$ for all $v \in \mathrm{MA_p}(G)$. Hence $\widetilde{T_A} = T$. By Theorem 3.3.3, $\{T, \mathscr{H}\}$ is non-degenerate.

■

Corollary 3.3.5. *Let $\{T, \mathscr{H}\}$ be a $*$-representation of $\mathrm{MA_p}(G)$ such that the restriction of T to $\mathrm{A_p}(G)$ is non-degenerate. Then $\{T, \mathscr{H}\}$ is unitarily equivalent, for a positive Radon measure μ on G, to the non-degenerate $*$-representation $\{\widetilde{S_\mu}, \mathrm{L}^2(G, \mu)\}$, defined by*

$$\widetilde{S_\mu}(v)([h]) \;=\; [v\,h], \qquad \textit{for } v \in \mathrm{MA_p}(G), [h] \in \mathrm{L}^2(G, \mu).$$

Proof. Let μ be a positive Radon measure on G and $\{S_\mu, \mathrm{L}^2(G, \mu)\}$ be the non-degenerate $*$-representation of $\mathrm{A_p}(G)$ defined as in Proposition 3.2.3. Then the extension $\widetilde{S_\mu}$, defined on $\mathrm{MA_p}(G)$, is given by $\widetilde{S_\mu}(v)([h]) = [v\,h]$ for all $v \in \mathrm{MA_p}(G)$, $[h] \in \mathrm{L}^2(G, \mu)$. Indeed, for any $u \in \mathrm{A_p}(G)$ we have

$$\widetilde{S_\mu}(v)\, S_\mu(u)([h]) \;=\; S_\mu(vu)([h]) \;=\; [vuh] \;=\; [v] \cdot S_\mu(u)([h]).$$

Now the corollary follows from Theorem 3.2.4 and Corollary 3.3.4.

■

Remark 3.3.2. (a) Through all this section, we may replace $\mathrm{MA_p}(G)$ by $\mathrm{W_p}(G)$, or by any $*$-algebra $\mathscr{A}$ which satisfy $\mathrm{A_p}(G) \subseteq \mathscr{A} \subseteq \mathrm{MA_p}(G)$, without any transformation in the proofs. For the case $p = 2$, $\mathrm{MA_2}(G)$ may be replaced by the Fourier-Stieltjes algebra $\mathrm{B}(G)$ or by the reduced Fourier-Stieltjes algebra $\mathrm{B}_\rho(G)$.

(b) In the previous two corollaries, we note that the condition "the restriction of T to $\mathrm{A_p}(G)$ is non-degenerate" cannot generally be dropped as the following example vindicates: Let G be a non-discrete locally compact group, identify $\mathrm{L}^1(G)$ with the subalgebra of $\mathrm{M}(G)$ consisting of the measures that are absolutely continuous with respect to the left Haar measure, and let π be a continuous unitary representation of G. Then the representation T of $\mathrm{M}(G)$ given by $T(\nu) = \sum_{x \in G} \nu(\{x\})\pi(x)$ is non-degenerate but will be identically zero when restricted to $\mathrm{L}^1(G)$. In this case, T is not unitarily equivalent to a representation of the form $\widetilde{S_\mu}$.

The next result is analogous to Theorem (22.11) in [50], where this result is proved for the measure algebra.

Corollary 3.3.6. *If λ denotes the left Haar measure on G, then the non-degenerate $*$-representation $\{\widetilde{S_\lambda}, \mathrm{L}^2(G,\lambda)\}$ of $\mathrm{MA_p}(G)$ is faithful.*

Proof. Let $u \in \mathrm{MA_p}(G)$ such that $u \neq 0$ and let $x \in G$ such that $u(x) \neq 0$. Let $h \in \mathrm{C}_{00}(G)$ such that $h(x) \neq 0$. Then $uh(x) \neq 0$ and uh is continuous with compact support so that $|uh|^2 \in \mathrm{C}_{00}^+(G)$ and $|uh|^2 \neq 0$. Therefore, by the property of the Haar integral [50, Theorem (15.5)(i)] we obtain:

$$\int_G |uh(x)|^2 \, dx \; = \; ||[uh]||_2^2 \; > \; 0,$$

hence $[uh] \neq 0$.

■

This leads us to the following interesting observation concerning the (multiplier algebra of the) Figà-Talamanca-Herz algebras:

Corollary 3.3.7. *For any $1 < p < \infty$ and any locally compact group G, $\mathrm{MA_p}(G)$ is embedded in a self-adjoint subalgebra of $\mathscr{B}(\mathrm{L}^2(G))$.*

Remark 3.3.3. Along with the results of N. J. Young [108], Proposition 3.3.6 implies that $\mathrm{WAP_p}(G)$, the weakly almost periodic functionals on $\mathrm{A_p}(G)$, form a separating set for $\mathrm{A_p}(G)$. Similar statements can be made for the weakly almost periodic functionals on $\mathrm{MA_p}(G)$. This result is new even in the case $p = 2$ and G non-Abelian, or when G is non-discrete, in which case $\mathrm{A_p}(G)$ is not Arens regular [36, Theorem 3.2] (see also [37]). For related material, we refer the reader to [16] and [24].

Chapter 4

Ergodic sequences in $\mathrm{A_p}(G)$ and $\mathrm{MA_p}(G)$.

Let G be a locally compact group and let $\mathrm{P}_1(G)$ be the set of continuous positive definite functions ϕ on G such that $\phi(e) = 1$. Improving results of J. Blum and B. Eisenberg [7] and of P. Milnes and A. L. T. Paterson [78], A. T.-M. Lau and V. Losert [70] proved that if G is amenable, the following are equivalent for a sequence (u_n) in $\mathrm{P}_1(G)$:

(i) (u_n) is strongly ergodic.

(ii) (u_n) is weakly ergodic.

(iii) For each $x \in G$, $x \neq e$, $u_n(x) \to 0$.

(iv) For each $T \in \mathrm{C}^*_\delta(G)$, $\langle u_n, T\rangle \to \langle \Psi, T\rangle$, where Ψ is the unique topologically invariant mean on $\mathrm{C}^*_\delta(G)$.

In this section, by means of the results proved in Chapter 3, we shall extend Lau and Losert's theorem to the setting of $\mathrm{A_p}(G)$ and $\mathrm{MA_p}(G)$: we shall give an improvement to condition (iv) above by replacing $\mathrm{C}_{\delta,\mathrm{p}}(G)$ by $\mathrm{M_p}(G)$, and we also remove the assumption of amenability (Theorem 4.1.3).
Throughout this chapter, we let $1 < p < \infty$, G be a locally compact group, and Ψ be the unique topologically invariant mean on $\mathrm{WAP_p}(G)$.

4.1 The main result.

In order to allow the discussion of ergodic sequences in $\mathrm{A_p}(G)$ and $\mathrm{MA_p}(G)$, we introduce the following notation:

$$\mathrm{S_A^p} \;=\; \{u \in \mathrm{A_p}(G) \,:\, ||u||_{\mathrm{A_p}} \;=\; 1 \;=\; u(e)\},$$

$$\mathrm{S_M^p} \;=\; \{v \in \mathrm{MA_p}(G) \,:\, ||v||_{M} \;=\; 1 \;=\; v(e)\}.$$

$\mathrm{S_A^p}$ and $\mathrm{S_M^p}$ are commutative semigroups with pointwise multiplication and are convex sets [43, Proposition 1.].

A typical example of an element in $\mathrm{S_A^p}$ is the function φ_U, U being any compact open subset of G, defined by

$$\varphi_U(x) \;=\; \frac{1}{m(U)}\, 1_U * \widetilde{1_U}(x) \;=\; \frac{m(xU \cap U)}{m(U)}.$$

Indeed,

$$\begin{aligned} 0 \le \varphi_U(x) \le 1 \;&=\; \varphi_U(e) \;\le\; ||\varphi_U||_\infty \\ &\le\; ||\varphi_U||_{\mathrm{A_p}} \;=\; \inf\left\{\sum ||k_n||_p\, ||\ell_n||_{p'} \;:\; \varphi_U = \sum \ell_n * \check{k}_n\right\} \\ &\le\; \frac{1}{m(U)}\, ||1_U||_p\, ||1_U||_{p'} \\ &=\; \frac{1}{m(U)}\, (m(U))^{\frac{1}{p}}\, (m(U))^{\frac{1}{p'}} \;=\; 1. \end{aligned}$$

Lemma 4.1.1. *For every $x \in G$, $x \neq e$, we have*

$$\langle \Psi, \lambda_p(x)\rangle \;=\; 0.$$

Proof. Fix $x \in G$, $x \neq e$. By semisimplicity of $\mathrm{A_p}(G)$ and since the spectrum of $\mathrm{A_p}(G)$ is identified with G [48, Theorem 3.], the Gelfand representation theorem [62, Theorem 3.1.1] insures the existence of some $u \in \mathrm{A_p}(G)$ such that $u(x) \neq u(e)$. In fact, there exists $u \in \mathrm{A_p}(G)$ such that $u(e) = 0$ and $u(x) \neq 0$ (see the proof of Lemma 6.2.3). Then $\langle u \cdot \Psi, \lambda_p(x)\rangle = u(e)\langle \Psi, \lambda_p(x)\rangle = 0$. But

$$\langle u \cdot \Psi, \lambda_p(x)\rangle = \langle \Psi, u \cdot \lambda_p(x)\rangle = \langle \Psi, u(x)\lambda_p(x)\rangle = u(x)\langle \Psi, \lambda_p(x)\rangle,$$

so that $u(x)\langle\Psi, \lambda_p(x)\rangle = 0$. Therefore, $\langle\Psi, \lambda_p(x)\rangle = 0$ since $u(x) \neq 0$. ■

Lemma 4.1.2. *Let (v_n) be a sequence in $\mathrm{MA_p}(G)$. The following are equivalent:*

(a) $v_n \to 1_{\{e\}}$ pointwise, where $1_{\{e\}}$ is the characteristic function of $\{e\}$.

(b) For every $\varphi \in \mathrm{C}_{\delta,\mathrm{p}}(G)$, $\langle v_n, \varphi\rangle \to \langle\Psi, \varphi\rangle$.

Proof. (a) $\Rightarrow$ (b): Let $x \in G$. If $x \neq e$, then $\langle\Psi, \lambda_p(x)\rangle = 0$ by Lemma 4.1.1. Since $v_n(x) = \langle v_n, \lambda_p(x)\rangle \to 0$, we obtain that $\langle v_n, \lambda_p(x)\rangle \to \langle\Psi, \lambda_p(x)\rangle$. If $x = e$, then $v_n(e) = \langle v_n, \lambda_p(e)\rangle \to 1 = \langle\Psi, \lambda_p(e)\rangle$.
(b) $\Rightarrow$ (a): Let $x \in G$. If $x = e$, then $\langle\Psi, \lambda_p(x)\rangle = 1$, and $v_n(e) = \langle v_n, \lambda_p(e)\rangle \to \langle\Psi, \lambda_p(e)\rangle = 1$. If $x \neq e$, then $\langle\Psi, \lambda_p(x)\rangle = 0$ by Lemma 4.1.1, and $v_n(x) = \langle v_n, \lambda_p(x)\rangle \to \langle\Psi, \lambda_p(x)\rangle = 0$.

■

With the notation of Proposition 3.3.2 and Theorem 3.3.3 we introduce the following:

Definition 1. Let (v_n) be a sequence in $\mathrm{S}^{\mathrm{p}}_{\mathrm{M}}$. We say that (v_n) is *strongly* (resp. *weakly*) *ergodic* if for any non-degenerate $*$-representation $\{\pi, \mathscr{H}\}$ of $\mathrm{A_p}(G)$ and any $\xi \in \mathscr{H}$, the sequence $(\widetilde{\pi}(v_n)\xi)$ converges strongly (resp. weakly) to some $\theta \in \mathscr{H}_f$, where

$$\mathscr{H}_f := \{\vartheta \in \mathscr{H} : \pi(u)\vartheta = \vartheta \text{ for all } u \in \mathrm{S}^{\mathrm{p}}_{\mathrm{A}}\}.$$

Example. In view of Theorem 4.1.3 and Proposition 4.1.5 below, if G is first countable and $\{U_n\}$ is a collection of compact symmetric open neighborhoods of e such that $U_{n+1}^2 \subseteq U_n$, then the sequence (φ_{U_n}) is ergodic in $\mathrm{S}^{\mathrm{p}}_{\mathrm{A}}$, where $\varphi_{U_n} = \frac{1}{m(U_n)} 1_{U_n} * 1_{\check{U}_n}$.

The next theorem extends [70, Theorem 3.3] to all locally compact groups, and is viewed as the main result of this section. We also consider an additional condition (condition (v)), which turns out to characterize ergodicity as well, and which seems to be new even for the case $p = 2$ and G Abelian.

Theorem 4.1.3. *Let (v_n) be a sequence in $\mathrm{S}^{\mathrm{p}}_{\mathrm{M}}$. The following assertions are equivalent:*

(i) (v_n) *is strongly ergodic.*

(ii) (v_n) *is weakly ergodic.*

(iii) *For every* $x \in G$, $x \neq e$, $v_n(x) \to 0$.

(iv) *For every* $T \in \mathrm{C}_{\delta,\mathrm{p}}(G)$, $\langle v_n, T\rangle \to \langle \Psi, T\rangle$.

(v) *For every* $T \in \mathrm{M}_{\mathrm{p}}(G)$, $\langle v_n, T\rangle \to \langle \Psi, T\rangle$.

Proof. The implications (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (iv) are obvious, whereas the equivalence (iii) $\Leftrightarrow$ (iv) is proved in Lemma 4.1.2.
(ii) $\Rightarrow$ (iii): Fix $x \in G$, $x \neq e$, and let $\{\pi, \mathbb{C}\}$ be the $*$-representation of $\mathrm{A}_{\mathrm{p}}(G)$ given by $\pi(u)(z) = u(x)z$, for $u \in \mathrm{A}_{\mathrm{p}}(G)$, $z \in \mathbb{C}$. By (3.2) of Proposition 3.3.2, there exists a unique extension $\widetilde{\pi}$ with $\widetilde{\pi}(v)(z) = v(x)z$ for all $v \in \mathrm{MA}_{\mathrm{p}}(G)$, $z \in \mathbb{C}$. The fixed point set of π is

$$\begin{aligned}\{\lambda \in \mathbb{C} \,:\, \pi(u)(\lambda) = \lambda \text{ for all } u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}\} = \{\lambda \in \mathbb{C} \,:\, u(x)\lambda = \lambda \text{ for all } u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}\} \\ = \{0\},\end{aligned}$$

where the last equality follows since $\mathrm{A}_{\mathrm{p}}(G)$ is semisimple with spectrum G. By weak ergodicity of (v_n), it follows that $\langle \pi(v_n)(z_1), z_2\rangle = v_n(x)z_1\overline{z_2} \to 0$ for all $z_1, z_2 \in \mathbb{C}$, hence $v_n(x) \to 0$.
(iii) $\Rightarrow$ (i): Let $\{\pi, \mathscr{H}\}$ be a non-degenerate $*$-representation of $\mathrm{A}_{\mathrm{p}}(G)$. We first consider the representation $\{S_\mu, \mathrm{L}^2(G,\mu)\}$ of Proposition 3.2.3. By Proposition 3.3.2, S_μ extends uniquely to $\mathrm{MA}_{\mathrm{p}}(G)$, and $\widetilde{S_\mu}(v)([h]) = [vh]$ for all $v \in \mathrm{MA}_{\mathrm{p}}(G)$, $[h] \in \mathrm{L}^2(G,\mu)$. For any $[h] \in \mathrm{L}^2(G,\mu)$ we have:

$$\begin{aligned}||\widetilde{S_\mu}(v_n)([h]) - \widetilde{S_\mu}(v_m)([h])||_2^2 &= \int_G |(v_n\, h \,-\, v_m\, h)(x)|^2\, d\mu(x) \\ &= \int_G |v_n(x) - v_m(x)|^2\, |h(x)|^2\, d\mu(x).\end{aligned}$$

As $n, m \to \infty$, $|v_n(x) - v_m(x)|^2\, |h(x)|^2 \to 0$ by (iii). Moreover,

$$\begin{aligned}|v_n(x) - v_m(x)|^2\, |h(x)|^2 &\leq (|v_n(x)| + |v_m(x)|)^2\, |h(x)|^2 \\ &\leq (||v_n||_{\sup} + ||v_m||_{\sup})^2\, |h(x)|^2 \\ &\leq (||v_n||_M + ||v_m||_M)^2\, |h(x)|^2 \\ &\leq (1+1)^2\, |h(x)|^2 \,=\, 4\, |h(x)|^2.\end{aligned}$$

Since the function $4|h|^2$ belongs to $\mathscr{L}^1(G,\mu)$, we can apply the Lebesgue dominated convergence theorem, and we obtain that

$$\lim_{n,m\to\infty} ||\widetilde{S_\mu}(v_n)([h])-\widetilde{S_\mu}(v_m)([h])||_2^2 = \int_G \lim_{n,m\to\infty} |v_n(x)-v_m(x)|^2 \, |h(x)|^2 \, d\mu(x) = 0.$$

That is, $(\widetilde{S_\mu}(v_n)([h]))$ is a Cauchy sequence in $\mathrm{L}^2(G,\mu)$, and consequently converges to some $[f] \in \mathrm{L}^2(G,\mu)$. Another application of the dominated convergence theorem yields that, for any $v \in \mathrm{S}^{\mathrm{p}}_{\mathrm{M}}$, $[h] \in \mathrm{L}^2(G,\mu)$,

(4.1)

$$||\widetilde{S_\mu}(v)(\widetilde{S_\mu}(v_n)([h]))-\widetilde{S_\mu}(v_n)([h])||_2^2 = \int_G |v(x)v_n(x)-v_n(x)|^2|h(x)|^2 d\mu(x) \to 0.$$

Hence the limit $[f]$ is a fixed point of $\{S_\mu, \mathrm{L}^2(G,\mu)\}$.

Now take an arbitrary non-degenerate $*$-representation $\{\pi, \mathscr{H}\}$. For all $u \in \mathrm{A_p}(G)$, it follows by Theorem 3.2.4 that $\pi(u) = W^* S_\mu(u) W$ for a linear isometry W from $\mathscr{H}$ onto $\mathrm{L}^2(G,\mu)$. Then, for any $u \in \mathrm{S}^{\mathrm{p}}_{\mathrm{A}}$, $v \in \mathrm{S}^{\mathrm{p}}_{\mathrm{M}}$, $\xi \in \mathscr{H}$, we have

$$\begin{aligned}
||\widetilde{\pi}(v)(\widetilde{\pi}(v_n)(\pi(u)\xi)) - \widetilde{\pi}(v_n)(\pi(u)\xi)|| &= ||\pi(vv_nu)\xi - \pi(v_nu)\xi|| \\
&= ||W^*S_\mu(vv_nu)W\xi - W^*S_\mu(v_nu)W\xi|| \\
&= ||W^*(S_\mu(vv_nu)W\xi - S_\mu(v_nu)W\xi)|| \\
&\leq ||W^*|| \, ||S_\mu(vv_nu)W\xi - S_\mu(v_nu)W\xi||_2 \\
&= ||S_\mu(vv_nu)W\xi - S_\mu(v_nu)W\xi||_2 \\
&= ||\widetilde{S_\mu}(v)(\widetilde{S_\mu}(v_n)(S_\mu(u)W\xi)) - \widetilde{S_\mu}(v_n)(S_\mu(u)W\xi)||_2 \\
&\to 0 \quad \text{by (4.1)}.
\end{aligned}$$

Therefore, by non-degeneracy of π it follows that, for all $\xi \in \mathscr{H}$, $(\widetilde{\pi}(v_n)\xi)$ converges to a fixed point of π. Indeed, $(\widetilde{\pi}(v_n)\xi)$ converges to an element in $\{\xi \in \mathscr{H} : \widetilde{\pi}(v)\xi = \xi \text{ for all } v \in \mathrm{S}^{\mathrm{p}}_{\mathrm{M}}\}$, which is contained in $\mathscr{H}_f$. Thus, (v_n) is strongly ergodic.

(iii) $\Rightarrow$ (v): Assume that (v_n) converges pointwise to $1_{\{e\}}$. In particular, $|v_n(x)| \leq ||v_n||_{\sup} \leq ||v_n||_M = 1$, *i.e.*, $|v_n| \leq 1_G$. Let $\mu \in \mathrm{M}(G)$. Since μ is bounded, $1_G \in \mathscr{L}^1(G,\mu)$. Thus, the dominated convergence theorem is applicable, and we obtain

$$\begin{aligned}\lim_n \langle v_n, \mu\rangle &= \lim_n \int_G v_n(x)\, d\mu(x) \;=\; \int_G \lim_n v_n(x)\, d\mu(x)\\ &= \int_G 1_{\{e\}}(x)\, d\mu(x) \;=\; \mu(\{e\}) \;=\; \langle \Psi, \mu\rangle,\end{aligned}$$

where the last equality follows by [43, Proposition 10.].
Next, let $T \in \mathrm{M_p}(G)$ and let $(\mu_n) \subset \mathrm{M}(G)$ such that $||T - \mu_n||_{\mathrm{PM_p}} \to 0$. Then we have:

$$\begin{aligned}|\langle v_n, T\rangle \,-\, \langle \Psi, T\rangle| &\leq |\langle v_n, T\rangle - \langle v_n, \mu_n\rangle| + |\langle v_n, \mu_n\rangle - \langle \Psi, \mu_n\rangle|\\ &\quad + |\langle \Psi, \mu_n\rangle - \langle \Psi, T\rangle|\\ &\leq ||v_n||_M ||T - \mu_n||_{\mathrm{PM_p}} + |\langle v_n, \mu_n\rangle - \langle \Psi, \mu_n\rangle|\\ &\quad + ||\Psi||\, ||\mu_n - T||_{\mathrm{PM_p}}\\ &\to 0.\end{aligned}$$

Therefore, $\langle v_n, T\rangle \,\to\, \langle \Psi, T\rangle$.

■

The next corollary is now immediate and provides an analogue of [70, Theorem 3.1].

Corollary 4.1.4. *Let (u_n) be a sequence in $\mathrm{S_A^p}$. The following assertions are equivalent:*

(i) (u_n) is strongly ergodic.

(ii) (u_n) is weakly ergodic.

(iii) For all $x \in G$, $x \neq e$, $u_n(x) \to 0$.

(iv) For every $\varphi \in \mathrm{C_{\delta,p}}(G)$, $\langle u_n, \varphi\rangle \,\to\, \langle \Psi, \varphi\rangle$.

(v) For every $T \in \mathrm{M_p}(G)$, $\langle u_n, T\rangle \,\to\, \langle \Psi, T\rangle$.

In regards to the existence of ergodic sequences in $\mathrm{MA_p}(G)$, the proof of the next proposition may be patterned after that of [70, Corollary 3.2], and so we omit the details.

Proposition 4.1.5. *Let G be a locally compact group. Then $\mathrm{MA_p}(G)$ contains a strongly ergodic sequence if and only if G is first countable.*

4.2 Functorial properties.

Proposition 4.2.1. *Let G be a locally compact group and H be a closed subgroup of G. If (u_n) is an ergodic sequence in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G)$, then the sequence of restrictions $(Res_H\, u_n)$ is ergodic in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(H)$.*

Proof. Let $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G)$. By [48, Theorem 1.] we have

$$1 = Res_H\, u(e) \leq ||Res_H\, u||_{\mathrm{sup}} \leq ||Res_H\, u||_{\mathrm{A_p}(H)} \leq ||u||_{\mathrm{A_p}(G)} = 1,$$

hence $Res_H\, u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(H)$.
Now let (u_n) be an ergodic sequence in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G)$. By Corollary 4.1.4, $u_n(x) \to 0$ for each $x \in G$, $x \neq e$. In particular, for each $y \in H$, $y \neq e$, $(Res_H\, u_n)(y) = u_n(y) \to 0$, hence $(Res_H\, u_n)$ is ergodic in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(H)$.

■

Proposition 4.2.2. *Let G be a locally compact group and H be an open subgroup of G. If (u_n) is an ergodic sequence in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(H)$, then the sequence $(\mathring{u_n})$ is ergodic in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G)$, where for any $h \in \mathrm{A_p}(H)$ the function $\mathring{h} \in \mathrm{A_p}(G)$ is defined by $\mathring{h}(x) = h(x)$ if $x \in H$, and $\mathring{h}(x) = 0$ if $x \notin H$.*

Proof. Let $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(H)$. By [48, Proposition 5.] we have

$$1 = \mathring{u}(e) \leq ||\mathring{u}||_{\mathrm{sup}} \leq ||\mathring{u}||_{\mathrm{A_p}(G)} = ||u||_{\mathrm{A_p}(H)} = 1,$$

hence $\mathring{u} \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G)$. The conclusion now follows from Corollary 4.1.4.

■

4.3 Examples via well distributed sequences.

V. Losert and H. Rindler [75] introduced a notion of well distributed sequence in semitopological semigroups and showed the existence of a well distributed sequence generator. After recalling their definition, we will adapt it to get a notion of well distributed sequences in the Figà-Talamanca-Herz algebras, which lead to some examples of ergodic sequences.

Definition 2. A *semitopological semigroup* S is an (algebraic) semigroup equipped with a Hausdorff topology τ for which the multiplication in S is separately continuous, that is, for any fixed $s \in S$, the mappings $t \mapsto st$ and $t \mapsto ts$ are continuous (with respect to τ).

Examples. 1. Let A be a Banach algebra, and let S denote its (closed) unit ball. Then S, equipped with the (induced) norm topology, is a semitopological semigroup. In this case, the multiplication in S is even jointly continuous, that is, the map $S \times S \to S, (s,t) \mapsto st$, is continuous when $S \times S$ has the product topology.
As a consequence, if $A = \mathrm{A_p}(G)$ (resp. $A = \mathrm{MA_p}(G)$), then $\mathrm{S^p_A}$ (resp. $\mathrm{S^p_M}$) is a semitopological semigroup whose multiplication is jointly continuous.

2. Let G be a locally compact group, $A = \mathrm{PM_p}(G)$, and S denote the (closed) unit ball of A. Then S, equipped with the weak* topology, is a semitopological semigroup. However, if G is not compact, then the multiplication in S is not jointly continuous; the reverse is also true. See [69, Theorem 8.8].

Definition 3. Let S be a semitopological semigroup. A sequence (x_n) in S is said to be *well distributed* (respectively *uniformly distributed*) if the following condition is satisfied: for any $\varepsilon > 0$, whenever $\{T, \mathscr{H}\}$ is a continuous representation of S as contractions on a Hilbert space $\mathscr{H}$, and $\eta \in \mathscr{H}$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\left\| \frac{1}{N} \sum_{n=k+1}^{k+N} T(x_n)\eta \;-\; P_f\eta \right\| < \varepsilon \qquad \text{for all } \; k \in \mathbb{N}$$

$$\left(\text{respectively } \; \left\| \frac{1}{N} \sum_{n=1}^{N} T(x_n)\eta \;-\; P_f\eta \right\| < \varepsilon \right),$$

where P_f denotes the orthogonal projection from $\mathscr{H}$ onto the fixed point set $\mathscr{H}_f$ of T.

Theorem 4.3.1. [75, Theorem 1.] *There exists a sequence of integers (r_j) with the following universal property: if S is any semitopological semigroup and (a_n) any sequence in S generating a dense subsemigroup of S, then the sequence (x_n) is well distributed, where $x_1 = a_{r_1}$, $x_2 = a_{r_1} \cdot a_{r_2}, \ldots, x_n = a_{r_1} \cdots a_{r_n}, \ldots$*
Such a sequence (r_j) is called a well distributed sequence generator (w.d.s.g.).

Definition 4. A sequence (u_n) in $\mathrm{S}^{\mathrm{p}}_{\mathrm{A}}$ is said to be *well distributed* if the following condition is satisfied: for any $\varepsilon > 0$, whenever $\{T, \mathscr{H}\}$ is a non-degenerate $*$-representation of $\mathrm{A_p}(G)$, $\eta \in \mathscr{H}$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\left\| \frac{1}{N} \sum_{n=k+1}^{k+N} T(u_n)\eta \, - \, P_f \eta \right\| < \varepsilon \qquad \text{for all } \ k \in \mathbb{N},$$

where P_f denotes the orthogonal projection from $\mathscr{H}$ onto $\mathscr{H}_f$.

Remark 4.3.1. Let (u_n) be a sequence in $\mathrm{S}^{\mathrm{p}}_{\mathrm{A}}$. If (u_n) is well distributed, then the sequence $\left(\frac{1}{N} \sum_{n=k+1}^{k+N} u_n \right)_N$ is strongly ergodic for all $k \in \mathbb{N}$.

Lemma 4.3.2. *Let* $u \in \mathrm{A_p}(G)$ *with* $||u||_{\mathrm{A_p}} \leq 1$*, and let* $\{T, \mathscr{H}\}$ *be a cyclic* $*$*-representation of* $\mathrm{A_p}(G)$*. Then* $T(u)$ *is a contractive operator on* $\mathscr{H}$*.*

Proof. First consider the representation $\{S_\mu, \mathrm{L}^2(G, \mu)\}$ as in Lemma 3.1.1. For any $u \in \mathrm{A_p}(G)$ with $||u||_{\mathrm{A_p}} \leq 1$ and any $h \in \mathrm{L}^2(G, \mu)$ we have

$$||S_\mu(u)(h)||_2 \, = \, ||uh||_2 \, \leq \, ||u||_{\mathrm{A_p}} \, ||h||_2 \, \leq \, ||h||_2.$$

Hence $S_\mu(u)$ is a contraction on $\mathrm{L}^2(G, \mu)$.
If $\{T, \mathscr{H}\}$ is an arbitrary cyclic $*$-representation of $\mathrm{A_p}(G)$, then by Lemma 3.1.2, there exists an isometry W from $\mathscr{H}$ onto $\mathrm{L}^2(G, \mu)$, for some $\mu \in \mathrm{M}^+(G)$, such that $T(u) = W^* S_\mu(u) W$ for all $u \in \mathrm{A_p}(G)$. Thus, for any $u \in \mathrm{A_p}(G)$ with $||u||_{\mathrm{A_p}} \leq 1$ it ensues that

$$||T(u)|| \, = \, ||W^* S_\mu(u) W|| \, \leq \, ||W^*|| \, ||S_\mu(u)|| \, ||W|| \, = \, ||S_\mu(u)|| \, \leq \, ||u||_{\mathrm{A_p}} \, \leq \, 1,$$

hence $T(u)$ is a contraction on $\mathscr{H}$.

∎

Lemma 4.3.3. *Let* S *be a subsemigroup of* $\mathrm{S}^{\mathrm{p}}_{\mathrm{A}}$*,* (u_n) *be a sequence in* S *such that* (u_n) *generates a dense subsemigroup of* S*, and let* (r_j) *be a w.d.s.g.. If* (x_n) *denotes the sequence of Theorem 4.3.1, the following condition is satisfied:*

whenever $\{T, \mathscr{H}\}$ *is a cyclic* $*$*-representation of* $\mathrm{A_p}(G)$ *the equation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=k+1}^{k+N} T(x_n) \;=\; P_f$$

holds uniformly in k *with respect to the strong operator topology of* $\mathscr{B}(\mathscr{H})$*.*

Proof. It is a consequence of Theorem 4.3.1 and Lemma 4.3.2.

■

Theorem 4.3.4. *Let* S *be a subsemigroup of* $\mathrm{S_A^p}$*,* (u_n) *be a sequence in* S *such that* (u_n) *generates a dense subsemigroup of* S*, and let* (r_j) *be a w.d.s.g.. Then the sequence* (x_n) *is well distributed in the sense of Definition 4, where* $x_1 = u_{r_1}, x_2 = u_{r_1} \cdot u_{r_2}, \ldots, x_n = u_{r_1} \cdots u_{r_n}, \ldots$*.*

Proof. Let $\{T, \mathscr{H}\}$ be a non-degenerate $*$-representation of $\mathrm{A_p}(G)$, and let $\{T_\alpha, \mathscr{H}_\alpha\}$ be cyclic $*$-representations of $\mathrm{A_p}(G)$ such that $T = \sum_\alpha T_\alpha$ and $\mathscr{H} = \bigoplus_\alpha \mathscr{H}_\alpha$. For each α, let $P_{\alpha,f}$ be the orthogonal projection from $\mathscr{H}_\alpha$ onto the fixed point set $\mathscr{H}_{\alpha,f}$, and let P_f be the orthogonal projection from $\mathscr{H}$ onto $\mathscr{H}_f$. Then we have:

$$\begin{aligned}
\mathscr{H}_f &= \{\xi \in \mathscr{H} \,:\, T(u)\xi = \xi \text{ for all } u \in \mathrm{S_A^p}\} \\
&= \left\{(\xi_\alpha) \in \bigoplus_\alpha \mathscr{H}_\alpha \,:\, \left(\sum_\alpha T_\alpha(u)\right)(\xi_\alpha) = (\xi_\alpha) \text{ for all } u \in \mathrm{S_A^p}\right\} \\
&= \left\{(\xi_\alpha) \in \bigoplus_\alpha \mathscr{H}_\alpha \,:\, T_\alpha(u)\xi_\alpha = \xi_\alpha \text{ for all } u \in \mathrm{S_A^p} \text{ and for all } \alpha\right\} \\
&= \left\{(\xi_\alpha) \in \bigoplus_\alpha \mathscr{H}_\alpha \,:\, \xi_\alpha \in \mathscr{H}_{\alpha,f} \text{ for all } \alpha\right\} \;=\; \bigoplus_\alpha \mathscr{H}_{\alpha,f}.
\end{aligned}$$

Hence $P_f = \sum_\alpha P_{\alpha,f}$. Moreover, by Lemma 4.3.3,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=k+1}^{k+N} T_\alpha(x_n) \;=\; P_{\alpha,f} \;\text{ for all } \alpha \;\text{ and all } k \in \mathbb{N}.$$

So we conclude that

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{n=k+1}^{k+N} T(x_n) &= \lim_{N\to\infty} T\left(\frac{1}{N} \sum_{n=k+1}^{k+N} x_n\right) \\
&= \lim_{N\to\infty} \sum_{\alpha} T_\alpha \left(\frac{1}{N} \sum_{n=k+1}^{k+N} x_n\right) \\
&= \sum_{\alpha} \lim_{N\to\infty} T_\alpha \left(\frac{1}{N} \sum_{n=k+1}^{k+N} x_n\right) \\
&= \sum_{\alpha} P_{\alpha,f} = P_f,
\end{aligned}$$

that is, (x_n) is well distributed in the sense of Definition 4.

■

Chapter 5

Complete mixing problem for multipliers.

The motivation for this chapter is the following question: For a fixed $u \in \mathrm{S}^{\mathrm{p}}_{\mathrm{M}}$, when is the sequence (u^n) of iterates of u strongly ergodic? In considering this problem, we are led to introduce a natural notion of ergodicity for multipliers, as well as various notions of mixing, all reflecting the well known theory of ergodic and mixing probability measures on locally compact (Abelian) groups [10, 32] (see also [86, 90, 39, 53, 54, 2, 1, 74, 14, 4]). In particular, using the theory of random walks and boundaries on groups, W. Jaworski [53] shows that if G is a locally compact Abelian group, a probability measure on G is ergodic if and only if it is completely mixing. In this paper, we shall not only provide a different proof of Jaworski's result, which is very transparent from the point of view of harmonic analysis, but we shall also extend it to all multipliers of any Tauberian, semisimple, regular, commutative Banach algebra, hence to the algebra $\mathrm{MA}_{\mathrm{p}}(G)$ of any locally compact group (Theorem 5.1.1). Thus, we provide a solution to the dual version of the complete mixing problem, which states that ergodicity and mixing are equivalent notions, and in doing so, we unify the Foguel type theorem and the Choquet-Deny type theorem recently considered by E. Kaniuth, A. T.-M. Lau and A. Ülger in [56] (Theorem 5.2.1). In addition, we also get rid of the assumption that the algebra admits a bounded approximate identity in [56, Theorem 3.4].

Lately, the complete mixing problem for measures has been definitely solved in the case of [SIN] groups [54] and in the case of locally compact motion groups

[2]. However, the problem seems to be still open in the case of σ-compact locally compact groups. See [54, 2] for more information on this problem. Finally, we want to mention that it is Jaworski's papers [53, 54], Kaniuth et al. paper [56], and Lemma 3.2.6 in [11] that gave us the impetus to study these notions in our context.

Throughout this section, we let A be a Tauberian, semisimple, regular, commutative Banach algebra. We denote its Gelfand spectrum by $\triangle(A)$ and the Gelfand tranform by $a \mapsto \widehat{a}$, where $\widehat{a}(\varphi) = \varphi(a)$ for any $\varphi \in \triangle(A)$. For any closed subset F of $\triangle(A)$ we write

$$j(F) = \{a \in A : \mathit{supp}(\widehat{a}) \text{ is compact, } \mathit{supp}(\widehat{a}) \cap F = \varnothing\},$$

and

$$k(F) = \{a \in A : \widehat{a}(\varphi) = 0 \, \text{for all} \, \varphi \in F\}.$$

If F is a singleton, say $\{\phi\}$, we write $k(\phi)$ instead of $k(\{\phi\})$. We say that F is a *set of spectral synthesis* if $\overline{j(F)} = k(F)$. See [62, chapter 8.] or [55, chapter 5.] for more information. Our notation here agrees with the one used in [56].

Remark 5.1. We recall that $\mathrm{A_p}(G)$ is a Tauberian, semisimple, regular, commutative Banach algebra [48, Proposition 3.] and that the group G may be identified with the Gelfand spectrum of $\mathrm{A_p}(G)$ [48, Theorem 3.]. It is known that singletons in G are always sets of spectral synthesis [48, 29] and so are the closed normal subgroups of G [48, Proposition 2.] (see also [20, Corollary 4.] and [22]).

A linear operator $T : A \to A$ is called a *multiplier of* A if $T(ab) = aT(b)$ holds for all $a, b \in A$. We write $M(A)$ for the unital commutative Banach algebra of all multipliers of A. For each $T \in M(A)$ we let $\widehat{T}$ denote the unique function in $\mathrm{CB}(\triangle(A))$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\widehat{a}(\varphi)$ for all $a \in A$ and $\varphi \in \triangle(A)$ [61, Theorem 1.2.2]. The book of R. Larsen [61] is a standard reference for multipliers.

Remark 5.2. T. Miao [77, Proposition 3.1] showed that $\mathrm{MA_p}(G) = M(\mathrm{A_p}(G))$ for any locally compact group G.

We now introduce the notion of ergodicity and mixing for multipliers. Compare with [90] and [54].

Definition 5. Let F be a closed subset in $\triangle(A)$, and let $T \in M(A)$ such that $||T|| = 1 = \widehat{T}(\varphi)$ for all $\varphi \in F$. We say that T is *F-ergodic* if

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} T^k(a) \right\| = 0 \text{ for all } a \in k(F);$$

T is *weakly F-mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\langle T^k(a), f\rangle| = 0 \text{ for all } a \in k(F) \text{ and all } f \in A^*;$$

T is *F-mixing* if

$$\lim_{n\to\infty} |\langle T^n(a), f\rangle| = 0 \text{ for all } a \in k(F) \text{ and all } f \in A^*;$$

T is *completely F-mixing* if

$$\lim_{n\to\infty} ||T^n(a)|| = 0 \text{ for all } a \in k(F).$$

5.1 Ergodic multipliers of $\mathrm{A_p}(G)$.

We first consider the case when $A = \mathrm{A_p}(G)$ and $F = \{e\}$. Write $\mathrm{A_p}(G)_0 := \{v \in \mathrm{A_p}(G) : v(e) = 0\} = k(e)$, and for $u \in \mathrm{MA_p}(G)$, let I_u be the norm closure of $\{uf - f : f \in \mathrm{A_p}(G)\}$. According to [11], the elements of the annihilator $I_u^{\perp}$ are called *u-harmonic functionals.* In the sequel, we shall say that u is "ergodic" instead of "$\{e\}$-ergodic", and so on.
Then the complete mixing problem for multipliers of $\mathrm{A_p}(G)$ is solved in the next theorem.

Theorem 5.1.1. *Let G be a locally compact group and let $u \in \mathrm{S^p_M}$. The following statements are equivalent:*

(i) $(u^n)_{n\in\mathbb{N}}$ *is strongly ergodic.*

(ii) For any neighborhood V of e, $u^n(x) \to 0$ for all $x \in G \setminus V$.

(iii) $|u(x)| < 1$ *for all $x \in G$, $x \neq e$.*

(iv) u is completely mixing.

(v) *u is mixing.*

(vi) *u is weakly mixing.*

(vii) *u is ergodic.*

(viii) $I_u = \mathrm{A_p}(G)_0$.

(ix) $I_u^{\perp} = \mathbb{C}\,\lambda_p(e)$.

Proof. (i) $\Leftrightarrow$ (iii): Assume that (i) holds. We recall from Theorem 4.1.3 that (u^n) is ergodic if and only if $u^n(x) \to 0$ for all $x \in G$, $x \neq e$. Since $||u||_M = 1$, it follows that $|u(x)| \leq ||u||_{\sup} \leq ||u||_M = 1$ for all $x \in G$, hence $|u(x)| < 1$ whenever $x \neq e$. The reverse implication is trivial.
(ii) $\Leftrightarrow$ (iii) is a consequence of (i) $\Leftrightarrow$ (iii).
(iii) $\Leftrightarrow$ (iv): If (iii) holds, then $\lim_n ||u^n v||_{\mathrm{A_p}} = 0$ for all $v \in j(e)$ by [56, Theorem 2.1]. Let $a \in \mathrm{A_p}(G)_0$. Since $\{e\}$ is a set of spectral synthesis, for any $\varepsilon > 0$ there exists $b \in j(e)$ such that $||a - b||_{\mathrm{A_p}} < \varepsilon$. Then

$$\begin{aligned} ||u^n a||_{\mathrm{A_p}} &\leq ||u^n(a-b)||_{\mathrm{A_p}} + ||u^n b||_{\mathrm{A_p}} \\ &\leq ||u^n||_{\mathrm{M}} ||a-b||_{\mathrm{A_p}} + ||u^n b||_{\mathrm{A_p}} < \varepsilon + ||u^n b||_{\mathrm{A_p}}. \end{aligned}$$

Therefore, $\lim_n ||u^n a||_{\mathrm{A_p}} = 0$ for all $a \in \mathrm{A_p}(G)_0$. The reverse implication also follows from [56, Theorem 2.1].
The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) and (iv) $\Rightarrow$ (vii) may be proved with standard calculations.
(vi) $\Rightarrow$ (ix): Clearly, $\mathbb{C}\lambda_p(e) \subseteq I_u^{\perp}$. To prove the reverse inclusion, let $T \in I_u^{\perp}$ and $a \in \mathrm{A_p}(G)_0$. By assumption,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\langle u^k a, T\rangle| = 0. \tag{5.1}$$

Since $T \in I_u^{\perp}$, it may be seen that $\langle T, v\rangle = \langle T, uv\rangle$ and that $v \cdot T \in I_u^{\perp}$, for all $v \in \mathrm{A_p}(G)$. Hence, $\langle u^k a, T\rangle = \langle a, T\rangle$. By (5.1) we have $\langle a, T\rangle = 0$, and $a \cdot T = 0$ (using the fact that $\mathrm{A_p}(G)_0$ is a closed ideal). Thus, by the definition of the support of a p-pseudomeasure ([48, p. 119], [23, Chapter 6.]), it follows that

$$(5.2) \qquad a(x) = 0 \text{ for all } x \in supp(T), \quad \text{whenever } a \in \mathrm{A_p}(G)_0.$$

If $supp(T) \neq \{e\}$ and $x \in supp(T)$, $x \neq e$, by regularity of $\mathrm{A_p}(G)$ there exists $v \in \mathrm{A_p}(G)$ such that $v(x) = 1$ and $v(e) = 0$. But this contradicts (5.2). Therefore, $supp(T) = \{e\}$ and by the "Lemme de contraction des supports" [29, p. 63] we then conclude that $T \in \mathbb{C}\lambda_p(e)$.

(vii) $\Rightarrow$ (ix) Let $T \in \mathrm{PM_p}(G)$ such that $u \cdot T = T$, and let $a \in \mathrm{A_p}(G)_0$. By assumption, $\frac{1}{n}\sum_{i=1}^{n} u^i a$ converges to 0 in norm, hence weakly; in particular,

$$\left|\left\langle \frac{1}{n}\sum_{i=1}^{n} u^i a, T \right\rangle\right| \to 0 \quad \text{as } n \to \infty.$$

However,

$$\begin{aligned} \left\langle \frac{1}{n}\sum_{i=1}^{n} u^i a, T \right\rangle &= \left\langle a, \frac{1}{n}\sum_{i=1}^{n} u^i \cdot T \right\rangle \\ &= \left\langle a, \frac{1}{n}\sum_{i=1}^{n} u^{i-1} \cdot T \right\rangle = \langle a, T\rangle. \end{aligned}$$

Thus, $\langle a, T\rangle = 0$, and (5.2) holds. The rest of the proof is now similar to (vi) $\Rightarrow$ (ix).

(viii) $\Leftrightarrow$ (ix) follows by [62, Theorem 7.1.2] since $\mathrm{A_p}(G)$ is regular.

(ix) $\Rightarrow$ (iii): We will show that $\{e\} = \{x \in G : u(x) = 1\}$ and the desired conclusion will be immediate. Since $u \in \mathrm{S_M^p}$, we only need to verify that, if $u(x) = 1$ for some $x \in G$, then $x = e$. To that matter, let $v \in \mathrm{A_p}(G)$ be arbitrary. Then we have $\langle v - uv, \lambda_p(x)\rangle = v(x) - u(x)v(x) = 0$, hence $\lambda_p(x) \in I_u^{\perp} = \mathbb{C}\lambda_p(e)$, and as $u(x) = 1$, it ensues that $\lambda_p(x) = \lambda_p(e)$. But by semisimplicity of $\mathrm{A_p}(G)$, the map $\lambda_p : G \to \mathrm{PM_p}(G)$ is injective, and therefore $x = e$.

■

Remark 5.1.1. We obtain Jaworski's result, mentioned in the introduction, by assuming G to be Abelian and by taking $p = 2$ in Theorem 5.1.1. Indeed, it is well known that the multipliers of $\mathrm{L}^1(G)$ are in one-to-one correspondence with the bounded Radon measures on G [61].

We now turn to the question of the existence of ergodic multipliers. The next lemma is certainly well known and is presented here without proof. The case in which $p = 2$ is the content of [40, Corollary 6.9].

Lemma 5.1.2. *Let G be a locally compact group. Then $\mathrm{A_p}(G)$ is separable if and only if G is second countable.*

Following the notation introduced in [105], we let $\mathscr{J} := \{I_u : u \in \mathrm{S^p_A}\}$. $\mathscr{J}$ is a set consisting of closed ideals in $\mathrm{A_p}(G)$, which we regard as being partially ordered by inclusion.

Theorem 5.1.3. *Let G be a locally compact group. The following statements are equivalent:*

(i) G is first countable.

(ii) There exists an ergodic $u \in \mathrm{S^p_A}$.

(iii) There exists an ergodic $v \in \mathrm{S^p_M}$.

(iv) $\mathscr{J}$ has a unique maximal ideal, namely $\mathrm{A_p}(G)_0$.

Proof. We start by noticing that (ii) $\Leftrightarrow$ (iii) holds once we know that, if $v \in \mathrm{S^p_M}$ is ergodic, then $vu \in \mathrm{S^p_A}$ is ergodic for all $u \in \mathrm{S^p_A}$.
(ii) $\Leftrightarrow$ (iv): If $u \in \mathrm{S^p_A}$ is ergodic, then $\mathrm{A_p}(G)_0 = I_u \in \mathscr{J}$, and $\mathrm{A_p}(G)_0$ is clearly the unique maximal ideal in $\mathscr{J}$. Conversely, if $\mathscr{J}$ has a unique maximal ideal, say I_u for some $u \in \mathrm{S^p_A}$, then $I_u = \mathrm{A_p}(G)_0$ and u is ergodic given Theorem 5.1.1. Indeed, using Lemma 6.2.4 in Chapter 6, one can show that $\mathrm{A_p}(G)_0 = \overline{span}\{I_\sigma : \sigma \in \mathrm{S^p_A}\}$ (see [11, Lemma 3.2.3] for the case $p = 2$), and it then follows by maximality of I_u that $\mathrm{A_p}(G)_0 \subseteq I_u$.
(iii) $\Rightarrow$ (i) is a consequence of Theorem 5.1.1 and Proposition 4.1.5.
(i) $\Rightarrow$ (ii): Let us first assume that G is second countable, and let $\varepsilon > 0$, $f_1, \ldots, f_n \in \mathrm{A_p}(G)_0$. By Lemma 5.1.2, $\mathrm{A_p}(G)_0$ is separable, and by Lemma 6.2.5 in Chapter 6, we have for each $i = 1, \ldots, n$,

$$0 \,=\, |f_i(e)| \,=\, \inf\{||uf_i||_{\mathrm{A_p}} : u \in \mathrm{S^p_A}\} \,<\, \varepsilon.$$

Let $u_i \in \mathrm{S^p_A}$ such that $||u_i f_i|| < \varepsilon$, and set $u = u_1 \cdots u_n \in \mathrm{S^p_A}$. So we have

$$d(f_i, I_u) \,=\, \inf\{||f_i - \psi|| : \psi \in I_u\} \,\leq\, ||f_i - (f_i - uf_i)|| \,<\, \varepsilon.$$

Therefore, applying [105, Lemma 1.1 and Remark (3), p. 210] we obtain the existence of some $u_0 \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$ such that $I_{u_0} = \mathrm{A}_{\mathrm{p}}(G)_0$, that is, u_0 is ergodic.
Now let $\{U_n\}$ be a countable basis of compact symmetric open neighborhoods of e such that $U_{n+1}^2 \subseteq U_n$, and assume that G is σ-compact. By the Kakutani-Kodaira Theorem [50, Theorem (8.7)], there exists a compact normal subgroup K of G such that G/K is second countable and $K \subseteq \bigcap_{n=1}^{\infty} U_n$. By above, there exists an ergodic $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G/K)$, *i.e.*, $u^n(xK) \to 0$ for all $x \in G$, $xK \neq eK$. Given [48, Proposition 6.], $u \circ q \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$ whenever $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(G/K)$, where $q : G \to G/K$ is the quotient map. And if V is an arbitrary neighborhood of e and $x \in G \setminus V$, then $xK \neq eK$ since $\bigcap_{n=1}^{\infty} U_n \subseteq V$ so that $x \notin K$. Therefore, $(u \circ q)^n(x) = u^n(xK) \to 0$ and $u \circ q$ is ergodic in $\mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$ by Theorem 5.1.1.
Finally, in the general case, pick an open σ-compact subgroup H of G [50, Theorem (7.5)]. Then there exists an ergodic $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}(H)$, and $\mathring{u} \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$ is ergodic by Proposition 4.2.2 and Theorem 5.1.1.

■

Remark 5.1.2. a) In the context of the group algebra $\mathrm{L}^1(G)$ of a second countable locally compact group, properties (iii) and (iv) were shown to be equivalent to amenability of G by J. Rosenblatt [90] and G. A. Willis [105] respectively.

b) For the case $p = 2$, Theorem 5.1.3 was proved in [83, Proposition 2.4]. See also [11, Proposition 3.2.7].

Corollary 5.1.4. *If G is a first countable locally compact group, then* $\mathrm{A}_{\mathrm{p}}(G)_0$ *is always of the form I_u for some $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$.*

The next corollary displays an interesting connection with fixed point properties of p-pseudomeasures.

Corollary 5.1.5. *Let $T \in \mathrm{PM}_{\mathrm{p}}(G)$ and let $u_0 \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$ be ergodic. Then T is a u_0-harmonic functional if and only if T is a u-harmonic functional for all $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$. In other words, T is a fixed point under the action of u_0 if and only if T is a common fixed point for all $u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}$.*

Proof.

$$I_{u_0}^{\perp} = \mathbb{C}\lambda_p(e) = \{T \in \mathrm{PM}_{\mathrm{p}}(G) : u \cdot T = T \text{ for all } u \in \mathrm{S}_{\mathrm{A}}^{\mathrm{p}}\}.$$

(See Lemma 6.2.4 in Chapter 6.).

■

5.2 F-ergodic multipliers.

In this section, we show that Theorem 5.1.1 is still true for any Tauberian, semisimple, regular, commutative Banach algebra A, and for any closed subset $F \subseteq \triangle(A)$. This leads to an improvement of [56, Theorem 3.4] where the algebra was assumed to have a bounded approximate identity, and it also shows that Theorem 2.1 and Theorem 3.4 in [56] are mutually equivalent if F is a set of spectral synthesis. Nevertheless, the properties (ii), (iii), (vii), (viii) and (ix), in Theorem 5.2.1 below were first considered by Kaniuth, Lau, and Ülger in [56].

We now fix a commutative Banach algebra A which is Tauberian, semisimple and regular, and we let F be a closed subset of $\triangle(A)$.

Theorem 5.2.1. *Let $T \in M(A)$ such that $||T|| = 1 = \widehat{T}(\varphi)$ for all $\varphi \in F$. If F is a set of spectral synthesis, then the following statements are equivalent:*

(i) $\widehat{T^n}(\gamma) \to 0$ *pointwise for all* $\gamma \in \triangle(A) \setminus F$.

(ii) $|\widehat{T}(\gamma)| < 1$ *for all* $\gamma \in \triangle(A) \setminus F$.

(iii) T *is completely F-mixing.*

(iv) T *is F-mixing.*

(v) T *is weakly F-mixing.*

(vi) T *is F-ergodic.*

(vii) $I_T = k(F)$.

(viii) $\{f \in A^* : T^* f = f\} = \overline{span(F)}^{w^*}$.

(ix) $F = F_T$, *where* $F_T := \{\gamma \in \triangle(A) : \widehat{T}(\gamma) = 1\}$.

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (vi) follows similarly as in the proof of Theorem 5.1.1. (ix) $\Rightarrow$ (ii) is obvious, and (vii) $\Leftrightarrow$ (viii) is a consequence of the regularity of A (see [62, Theorem 7.1.2]

and [56, Lemma 3.1.(v)]).
(viii) $\Rightarrow$ (ix): By our assumption on T, we only need to show that $F_T \subseteq F$. Let $\gamma \in F_T$ and assume that $\gamma \notin F$. Since A is regular, there exists $a \in A$ such that $\widehat{a}(\gamma) = 1$ and $\widehat{a}_{|F} \equiv 0$. Moreover, $\langle a - T(a), \gamma \rangle = \widehat{a}(\gamma) - \widehat{T}(\gamma)\widehat{a}(\gamma) = 0$, hence $\gamma \in I_T^{\perp} = \{f \in A^* : T^* f = f\} = \overline{\operatorname{span}(F)}^{w^*}$. So there exist $c_i \in \mathbb{C}$ and $\gamma_i \in F$ such that

$$\sum_{i=1}^{n_\alpha} c_i \gamma_i \xrightarrow{w^*} \gamma.$$

In particular, $\sum_{i=1}^{n_\alpha} c_i \langle \gamma_i, a \rangle \to \langle \gamma, a \rangle = 1$. But since $\gamma_i \in F$, $\langle \gamma_i, a \rangle = 0$, which is a contradiction. Therefore, $\gamma \in F$.
(vi) $\Rightarrow$ (viii): Let $f \in A^*$ such that $T^* f = f$. As in the proof of Theorem 5.1.1, we have for every $a \in k(F)$:

$$\left\langle \frac{1}{n} \sum_{k=1}^{n} T^k(a), f \right\rangle = \langle a, f \rangle. \tag{5.3}$$

By assumption, $\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} T^k(a) \right\| = 0$, hence we obtain that

$$\lim_{n\to\infty} \left| \left\langle \frac{1}{n} \sum_{k=1}^{n} T^k(a), f \right\rangle \right| = 0.$$

Therefore, $\langle a, f \rangle = 0$ by (5.3), so that $f \in k(F)^{\perp} = \overline{\operatorname{span}(F)}^{w^*}$ [56, Lemma 3.1.(v)].
The proof of (v) $\Rightarrow$ (viii) is similar to (vi) $\Rightarrow$ (viii).

■

Corollary 5.2.2. *Let $T \in M(A)$ with $||T|| = 1$. If F_T is a set of spectral synthesis, then T is F_T-ergodic.*

Remark 5.2.1. a) In Theorem 5.2.1, the assumption that F is a set of spectral synthesis cannot be removed, as Example 3.5 in [56] vindicates.

b) If G is an amenable locally compact group, then every $u \in \mathrm{B}(G)$ with $||u||_{\mathrm{B}(G)} = 1$ is F_u-ergodic. [56, Corollary 4.6].

c) If G is a non-discrete locally compact group, then there exists $u \in \mathrm{B}(G)$ such that $||u||_{\sup} = 1$ and F_u fails to be a set of spectral synthesis [56, Corollary 4.4]. Moreover, if G is also amenable, then u cannot be power bounded [56, Theorem 4.1]. In every instance, there is no closed subset $F \subseteq G$, which is of spectral synthesis, such that u is F-ergodic.

Proposition 5.2.3. *Let A be a Tauberian, semisimple and regular, commutative Banach algebra and $\phi \in \triangle(A)$. Assume that the following assertions hold:*

(i) $\{\phi\}$ is a set of spectral synthesis.

(ii) $S_\phi := \{u \in A : ||u|| = \widehat{u}(\phi) = 1\}$ is not empty and acts ergodically on A^*, i.e.,

$$\{\Phi \in A^* : \Phi \cdot u = \Phi \text{ for all } u \in S_\phi\} = \{\lambda\phi : \lambda \in \mathbb{C}\}.$$

(iii) A is separable.

Then there exists a ϕ-ergodic multiplier in S_ϕ.

For ease of notation we write $A_{0,\phi} := \{a \in A : \widehat{a}(\phi) = 0\}$, which is clearly a closed subspace of A, and for $T \in M(A)$ we write $I_T := \overline{(I-T)(A)}$.

Proof. Let $\varepsilon > 0$ and $f_1, \ldots, f_n \in A_{0,\phi}$. By Theorem 6.2.1 in Chapter 6. we have

$$0 = |\langle \phi, f_i \rangle| = \inf\{||uf_i|| : u \in S_\phi\} < \varepsilon, \qquad \text{for each } i = 1, \ldots, n.$$

Let $u_i \in S_\phi$ such that $||u_i f_i|| < \varepsilon$, and set $u = u_1 \cdots u_n \in S_\phi$. Then $||uf_i|| < \varepsilon$, and $f_i - uf_i \in I_u$ for each $i = 1, \ldots, n$. Therefore,

$$d(f_i, I_u) = \inf\{||f_i - \psi|| : \psi \in I_u\} \leq ||f_i - (f_i - uf_i)|| < \varepsilon.$$

Since A is separable, so is $A_{0,\phi}$. Thus, we can now apply [105, Lemma 1.1 & Remark (3), p. 210] to obtain the existence of some $u \in S_\phi$ such that $I_u = A_{0,\phi}$, that is, u is ϕ-ergodic given Theorem 5.2.1.

■

Corollary 5.2.4. *Let A, ϕ and S_ϕ as in Proposition 5.2.3. Let $T \in A^*$ and let $u_0 \in S_\phi$. Assume that u_0 is ϕ-ergodic. Then T is a fixed point under the action of u_0 if and only if T is a common fixed point for all $u \in S_\phi$.*

Proof. It is an immediate consequence of Theorem 5.2.1.(ix) and of the hypothesis on S_ϕ.

∎

We end this chapter with a list of Banach algebras for which the content of Theorem 5.2.1 is applicable:

1. $A = \mathrm{L}^1(G)$ for any locally compact Abelian group G.

2. $A = \mathrm{A}(G)$ for $G = \mathrm{SL_n}(\mathbb{R})$, or for any connected semisimple non-compact Lie group with finite center. In this case, Theorem 5.2.1 seems to be a new result since such groups do not admit a bounded approximate identity for $\mathrm{A}(G)$ [21, section 4.]. More generally, Theorem 5.2.1 applies to the Figà-Talamanca-Herz algebra of all non-amenable locally compact groups, and for any $1 < p < \infty$.

3. Let $1 < p < \infty$ and $1 \leq r \leq \infty$, and $A = \mathrm{A_p^r}(G) := \mathrm{A_p}(G) \cap \mathrm{L^r}(G)$ be the Figà-Talamanca-Herz-Lebesgue algebra of a locally compact group G, as studied by E. E. Granirer [45]. If there exists an approximate identity (u_α) in $\mathrm{A_p}(G)$ which is bounded in the multiplier norm $||.||_M$, and if (u_α) is in $\mathrm{A_p^r}(G)$, then the conclusion of Theorem 5.2.1 is also verified for any multiplier T of $\mathrm{A_p^r}(G)$ with $||T|| = 1 = T(e)$. Indeed, this is a consequence of [45, Theorem 1., Corollary 2. & Theorem 11.] and of Theorem 5.2.1.

4. $A = \mathrm{L^p}(G)$ for any compact Abelian group G and any $1 \leq p < \infty$, where the multiplication is given by the convolution product. In this case, every singleton in $\widehat{G}$, the dual group of G, is a set of spectral synthesis. In addition, if $p = 1$, then every closed subset of $\widehat{G}$ is a set of spectral synthesis [62, Corollary 8.3.2].

5. $A = \mathrm{C_0}(X)$ for a non-empty locally compact Hausdorff space X. In this case, every closed subset in X is a set of spectral synthesis [15, Theorem 4.2.1].

6. $A = \mathrm{S}^1(G)$, any Segal algebra of a locally compact Abelian group G [87, §6.2]. In this case, every singleton in $\widehat{G}$ is a set of spectral synthesis. Examples of such algebras include

(a) $A = \mathrm{L}^1(G) \cap \mathrm{L}^\mathrm{p}(G)$, $1 < p < \infty$, with the norm $||f|| = ||f||_1 + ||f||_p$.

(b) $A =$ the algebra of all functions $f \in \mathrm{L}^1(G)$ whose Fourier transform $\widehat{f}$ belongs to $\mathrm{L}^\mathrm{p}(\widehat{G})$, $1 \le p < \infty$, with the norm $||f|| = ||f||_1 + ||\widehat{f}||_p$.

7. $A = \mathrm{BVC}_0(\mathbb{R})$, the algebra of all complex-valued continuous functions on $\mathbb{R}$ which are of bounded variation and which vanish at infinity, endowed with pointwise multiplication and the norm $||f|| = ||f||_{\mathrm{sup}} + \mathrm{Var}_{\mathbb{R}}(f)$. In this case, every singleton in $\mathbb{R}$ is a set of spectral synthesis [80, Proposition 2.6].

8. Let (K, d) be a non-empty compact metric space, and let $0 < \alpha < 1$.

(a) $A = \mathrm{Lip}_\alpha K$, the algebra of Lipschitz functions of order α, endowed with pointwise multiplication and the norm $||f||_\alpha = ||f||_{\mathrm{sup}} + p_\alpha(f)$, where

$$p_\alpha(f) \;=\; \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \;:\; x, y \in K, x \neq y\right\}.$$

In this case, every clopen (*i.e.*, open and closed) subset of K is a set of spectral synthesis [15, Theorem 4.4.24, Theorem 4.4.31].

(b) $A = \mathrm{lip}_\alpha K$, the subalgebra of $\mathrm{Lip}_\alpha K$ consisting of all functions f such that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \quad \text{as} \quad d(x,y) \to 0.$$

In this case, every closed subset of K is a set of spectral synthesis [15, Theorem 4.4.30].

Chapter 6

Ergodic theorems for φ-amenable Banach algebras.

In this chapter, after proving a new characterization of left amenability for semigroups, we consider various ergodic and fixed point properties which characterize φ-amenability of Banach algebras. Among other things, we prove a mean ergodic theorem in that setting, and we establish the uniqueness of a φ-mean on the weakly almost periodic functionals.

6.1 Left amenability of semigroups.

Let S be a semitopological semigroup. We denote by $\mathrm{CB}(S)$ the space of all continuous bounded functions $f : S \to \mathbb{C}$. For $s \in S$ and $f \in \mathrm{CB}(S)$, the *left translate of f by s* is defined by $\ell_s f(t) = f(st)$ for $t \in S$. Then f is said to be *left uniformly continuous* if the map $s \mapsto \ell_s f$ is continuous with respect to the norm topology of $\mathrm{CB}(S)$. We write $\mathrm{LUC}(S)$ for the space of all left uniformly continuous functions on S. In particular, $\mathrm{LUC}(S)$ is a norm-closed, conjugate-closed, translation-invariant sub-C^*-algebra of $\mathrm{CB}(S)$ containing the constant functions.
Let $m \in \mathrm{LUC}(S)^*$. Then m is called a *mean* on $\mathrm{LUC}(S)$ if $m(1_S) = 1 = ||m||$, where 1_S denotes the constant one function. We say that m is *left invariant* if $m(\ell_s f) = m(f)$ for all $s \in S$, $f \in \mathrm{LUC}(S)$.

The notion of amenability for semigroups was first studied by M. M. Day [18, 19]. Since then, the subject has attracted many mathematicians (see [84, 85] and the references therein). The following definition is due to I. Namioka [81].

Definition 6. A semitopological semigroup S is called *left amenable* if $\mathrm{LUC}(S)$ admits a left invariant mean.

Remark 6.1.1. It is well known that every Abelian semitopological semigroup is left amenable.

A. T.-M. Lau [63, 68] studied various Hahn-Banach type properties which characterize left amenability of semitopological semigroups. See also [35].
In this section, we add two other Hahn-Banach separation properties to the list of properties characterizing left amenability. As an application, we will obtain in the next section some properties of $\mathrm{A_p}(G)$ which are "of an ergodic nature".
From now on and for the rest of this section, let S be a semitopological semigroup.

Definition 7. By a *continuous anti-representation* of S on a normed linear space X we mean an anti-homomorphism $T : S \to \mathscr{L}(X)$ such that T is continuous with respect to the strong operator topology, $\mathscr{L}(X)$ being the algebra of bounded linear operators on X.
For any $x \in X$ we define the *orbit of* x as the set $O_S(x) := \{T_s x \ : \ s \in S\}$, and we write $K(x) := \overline{\mathrm{co}}\, O_S(x)$.

Definition 8. S is said to have the *Hahn-Banach separation property I* (HBSP I) if the following is satisfied: for any continuous anti-representation T of S as linear contractions on a normed linear space X, and for any closed invariant subspace Y of X such that $X \setminus Y$ contains an invariant element x_0, there exists an invariant linear functional Φ on X such that $\Phi(x_0) = 1$, $\Phi(y) = 0$ for every $y \in Y$, and $||\Phi|| \leq \frac{1}{d(x_0,Y)}$.

Definition 9. S is said to have the *Hahn-Banach separation property II* (HBSP II) if the following is satisfied: for any continuous anti-representation T of S as linear contractions on a normed linear space X, any closed invariant subspace Y of X, and any $f \in X$ such that $d := d(K(f), Y) > 0$, there exists an invariant linear functional Ψ on X such that $\Psi = 0$ on Y, $\Psi(f) = 1$, and $||\Psi|| \leq \frac{1}{d}$.

Remark 6.1.2. (i) If f is invariant, then $K(f) = \{f\}$, so that the HBSP II is clearly stronger than the HBSP I.
(ii) By [63, Theorem 1.(b)] and [12, Corollary 6.4, p.78], if S is left amenable, the following Hahn-Banach extension property is satisfied: for any right linear action of S on a topological vector space E, if p is a seminorm on X such that $p(s \cdot x) \le p(x)$ for all $s \in S$, $x \in E$, and if ϕ is an invariant linear functional on an invariant subspace F of E such that $|\phi| \le p$, then there exists an invariant extension $\widetilde{\phi}$ of ϕ to E such that $|\widetilde{\phi}| \le p$.

The next theorem generalizes Theorem A. in [91].

Theorem 6.1.1. *Assume that S has an identity. Then the following statements are equivalent:*
(i) *S is left amenable.*
(ii) *S has the* HBSP I.
(iii) *S has the* HBSP II.

Proof. (i) $\Rightarrow$ (iii): Let T, X, Y, and f as in the HBSP II. We may assume that S has an identity e such that T_e is the identity transformation on X. Let W be the linear span of Y and $K(f)$, *i.e.*, any $w \in W$ is of the form

$$w = y + \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{n} \lambda_{i,j} T_{s_{i,j}} f \qquad \text{with } y \in Y. \tag{6.1}$$

For such w, $T_s w = T_s y + \sum \alpha_j \sum \lambda_{i,j} T_{s_{i,j}s} f$, and $T_s w \in W$ since Y is invariant. Hence W is invariant. Now we define a linear functional ϕ on W by:

$$\phi(w) = \sum_{j=1}^{m} \alpha_j \, d \qquad \text{if } w = y + \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{n} \lambda_{i,j} T_{s_{i,j}} f,$$

where $d = d(K(f), Y) > 0$. Clearly, ϕ is invariant, $\phi(y) = 0$ for every $y \in Y$, and $\phi(f) = d$. Set $p(x) = ||x||$, for $x \in X$. Then $|\phi(y)| = 0 \le p(y)$ for every $y \in Y$, and for every w of the form (6.1) with $\sum_{j=1}^{m} \alpha_j \neq 0$,

$$|\phi(w)| = \left| \sum_{j=1}^{m} \alpha_j \right| d \le \left| \sum_{j=1}^{m} \alpha_j \right| \left\| \sum_{i=1}^{n} \lambda_{i,j} T_{s_{i,j}} f - \frac{-y}{\sum \alpha_j} \right\| = p(w).$$

Since S is left amenable, we may apply [63, Theorem 1.(b)] to obtain an invariant linear functional $\widetilde{\phi}$ on X with $\widetilde{\phi}(w) = \phi(w)$ for all $w \in W$, and $|\widetilde{\phi}(x)| \leq p(x)$ for all $x \in X$. Thus, the invariant linear functional $\Psi := \frac{\widetilde{\phi}}{d}$ meets the requirements for the HBSP II.
(ii) $\Rightarrow$ (i): Consider the continuous anti-representation of S given by left translation on $\mathrm{LUC}(S)$, *i.e.*, $T_s f = \ell_s f$ for $s \in S$, $f \in \mathrm{LUC}(S)$. For each $s \in S$, T_s is obviously a linear contraction. The constant function 1_S is (translation) invariant, so that $K(1_S) = \{1_S\}$, and $\{0\}$ is a closed invariant subspace of $\mathrm{LUC}(S)$ which does not contain 1_S. By assumption, there exists an invariant linear functional Ψ on $\mathrm{LUC}(S)$ such that $\Psi(0) = 0$, $\Psi(1_S) = 1$ and $||\Psi|| \leq 1$. So $||\Psi|| = 1$, and Ψ is a left invariant mean on $\mathrm{LUC}(S)$.

■

We recall that $\mathrm{S_A^p} = \{u \in \mathrm{A_p}(G) : u(e) = 1 = ||u||_{\mathrm{A_p}}\}$ is an Abelian semitopological semigroup, hence is left amenable. A first consequence of Theorem 6.1.1 is the following

Corollary 6.1.2. *Let $1 < p < \infty$ and let G be a locally compact group. For any closed $\mathrm{S_A^p}$-invariant subspace Y of $\mathrm{PM_p}(G)$ which does not contain $\lambda_p(e)$, there exists a topologically left invariant mean Ψ on $\mathrm{PM_p}(G)$ such that Ψ vanishes identically on Y.*

Remark 6.1.3. We may replace $\mathrm{PM_p}(G)$ by any topologically invariant closed subspace of $\mathrm{PM_p}(G)$ containing $\lambda_p(e)$.

6.2 Ergodic characterization.

As a consequence of Theorem 6.1.1 we find an ergodic property which is equivalent to a unital Banach algebra, or a semisimple commutative Banach algebra being φ-amenable with a φ-mean of norm one.
We start with the definition of φ-amenable Banach algebras [58, 57, 98, 52].

Definition 10. Let A be a Banach algebra, and let φ be a non-zero multiplicative linear functional from A into $\mathbb{C}$. A is called *φ-amenable* if A admits a *φ-mean*, *i.e.*, a continuous linear functional $m \in A^{**}$ such that $\langle m, \varphi \rangle = 1$ and $\langle m, f \cdot a \rangle = \varphi(a)\langle m, f \rangle$ for all $f \in A^*$ and $a \in A$, where $\langle f \cdot a, b \rangle = \langle f, ab \rangle$, $b \in A$.

The property we mentioned above is defined next.

Definition 11. Let A be a Banach algebra and let φ be a non-zero multiplicative linear functional from A into $\mathbb{C}$. Set $S_\varphi := \{u \in A : ||u|| = 1 = \varphi(u)\}$. We say that S_φ *acts ergodically* on A^* if

$$Fix(S_\varphi) := \{f \in A^* : u \cdot f = f \text{ for all } u \in S_\varphi\} = \{\lambda\varphi : \lambda \in \mathbb{C}\},$$

where $\langle u \cdot f, a\rangle = \langle f, au\rangle$, $u \in S_\varphi, a \in A, f \in A^*$. In other words, S_φ acts ergodically on A^* if the only common fixed points (for S_φ) are the constant multiples of φ.

Remark 6.2.1. Equipped with the relative norm topology of A, S_φ is a semi-topological semigroup and a convex set.

Let A be a Banach algebra and let φ be a non-zero multiplicative linear functional from A into $\mathbb{C}$ such that S_φ is not empty and left amenable as a semigroup. Consider the antilinear action of S_φ on A given by the usual *reverse* multiplication of A. Then this antilinear action of S_φ on A defines a continuous anti-representation of S_φ as linear contractions on A.

Theorem 6.2.1. *If A is unital, then the following assertions are equivalent:*

(i) A is φ-amenable with a φ-mean of norm one.

(ii) S_φ acts ergodically on A^*.

(iii) For every $f \in A$ we have:

$$|\langle\varphi, f\rangle| = \inf\{||fu|| : u \in S_\varphi\}.$$

Proof. The equivalence (i) $\Leftrightarrow$ (iii) is a consequence of [57, Theorem 2.4].
(ii) $\Rightarrow$ (iii): As S_φ is not empty, we first notice that $||\varphi|| = 1$. Let $f \in A$ with $d = d(0, K(f)) \geq 0$, where $K(f) = \overline{\{fu : u \in S_\varphi\}}$. By Theorem 6.1.1 there exists $\Phi \in A^*$ such that $\langle\Phi, f\rangle = d$, $||\Phi|| \leq 1$, and $u \cdot \Phi = \Phi$ for all $u \in S_\varphi$. By assumption, $\Phi = \lambda\varphi$ for some $\lambda \in \mathbb{C}$. Then we have:

$$|\lambda| = ||\lambda\varphi|| = ||\Phi|| \leq 1 \text{ and } d = |\langle\Phi, f\rangle| = |\langle\lambda\varphi, f\rangle| = |\lambda|\,|\langle\varphi, f\rangle| \leq |\langle\varphi, f\rangle|.$$

Since $d = \inf\{||fu|| \,:\, u \in S_\varphi\}$ and $|\langle\varphi, f\rangle| = |\varphi(u)\langle\varphi, f\rangle| \leq ||fu||$ for all $u \in S_\varphi$, we conclude that $|\langle\varphi, f\rangle| \leq d$.
(i) $\Rightarrow$ (ii): Let $\Phi \in Fix(S_\varphi)$. Then $\overline{\{u \cdot \Phi \,:\, u \in S_\varphi\}}^{w^*} = \{\Phi\}$. By [57, Corollary 2.3], $\Phi \in span(\varphi)$, hence S_φ acts ergodically on A^*.

■

In Theorem 6.2.1 we assumed A to be unital in order that S_φ has an identity and that Theorem 6.1.1 is applicable. It is natural to ask what happens in case A is not unital. As a partial answer, we present below an argument such that the conclusion of Theorem 6.2.1 is still valid when A is commutative and semisimple.
From now on, unless otherwise stated, let A be a semisimple commutative Banach algebra, $\mathrm{M}(A)$ be its multiplier algebra, and φ be a non-zero multiplicative linear functional from A into $\mathbb{C}$, abbreviated $\varphi \in \triangle(A)$. We write

$$S_A \,:=\, \{u \in A \,:\, ||u|| = 1 = \varphi(u)\}$$

and

$$S_M \,:=\, \{T \in \mathrm{M}(A) \,:\, ||T||_M = 1 = \varphi(T)\}.$$

We note that $\varphi(T)$, also denoted $\widehat{T}(\varphi)$, is well defined by [61, Theorem 1.2.2], and by [61, Theorem 1.1.4] we may identify any $u \in S_A$ with the multiplier $L_u \in S_M$ where $L_u v := uv$ for any $v \in A$. Also, since $\mathrm{M}(A)$ is a unital commutative Banach algebra, Theorem 6.2.1 holds for $\mathrm{M}(A)$ and $\varphi \in \triangle(\mathrm{M}(A))$.

Remark 6.2.2. By an easy application of [57, Theorem 2.4], the following assertions are equivalent for any semisimple commutative Banach algebra A and any $\varphi \in \triangle(A)$:

(i) There exists a φ-mean m on A such that $||m|| = 1$.

(ii) There exists a net (v_β) in $\mathrm{M}(A)$ such that $\varphi(v_\beta) = 1$ for all β, $||v_\beta|| \to 1$, and $||v_\beta a|| \to |\varphi(a)|$ for all $a \in A$.

By Remark 6.2.2 and by the proof of Theorem 6.2.1, if S_A acts ergodically on A^*, S_M also acts ergodically on A^*, and for any $f \in A$ we then have:

$$|\langle\varphi, f\rangle| \,=\, \inf\{||uf|| \,:\, u \in S_M\} \,=\, \inf\{||uf|| \,:\, u \in S_A\}.$$

Consequently, we obtain:

Theorem 6.2.2. *The following assertions are equivalent:*
(i) A *is* φ*-amenable with a* φ*-mean of norm one.*
(ii) S_A $(= S_\varphi)$ *acts ergodically on* A^**.*
(iii) S_M *acts ergodically on* A^**.*

By Theorem 6.2.2 we can now deduce that $\mathrm{A_p}(G)$ is φ-amenable with a φ-mean of norm one for any $\varphi \in \triangle(\mathrm{A_p}(G))$, an assertion that was first obtained by M. Sangani Monfared [96, Lemma 3.1].
The next lemma is certainly well known, but we repeat it here for the sake of completeness.

Lemma 6.2.3. *Let* G *be a locally compact group. For any* $x \in G$, $x \neq e$, *there exists* $u \in \mathrm{S^p_A}$ *such that* $u(x) = 0$.

Proof. Let $x \in G$, $x \neq e$. Let $U \subseteq G$ be an open neighborhood of e such that $x \notin U$, and let $V \subseteq G$ be a symmetric compact neighborhood of e such that $V^2 \subseteq U$. We set:

$$u(x) \;=\; \frac{1}{\lambda(V)}\, \chi_V * \widetilde{\chi_V}(x) \;=\; \frac{\lambda(xV \cap V)}{\lambda(V)}.$$

Then u belongs to $\mathrm{S^p_A}$ and $u(x) = 0$ since $xV \cap V = \varnothing$.

■

Remark 6.2.3. For the definition of the support of an element $T \in \mathrm{PM_p}(G)$ and its properties, we refer to [28, Propositions (4.4),(4.6),(4.8)], [48, Proposition 10.], [29, §6], [97, Lemmas 3.7, 3.8] and [23, Chapter 6.].

Lemma 6.2.4. *Let* G *be a locally compact group. Then* $\mathrm{S^p_A}$ *acts ergodically on* $\mathrm{PM_p}(G)$.

Proof. Let $T \in \mathrm{PM_p}(G)$. If $T = c\lambda_p(e)$ for some $c \in \mathbb{C}$, then it is clear that $T \in Fix(\mathrm{S^p_A})$.
On the other hand, if $T \neq 0$ and $u \cdot T = T$ for all $u \in \mathrm{S^p_A}$, then by [48, Proposition 10.],

$$supp(T) \;=\; supp(u \cdot T) \;\subseteq\; supp(u) \cap supp(T),$$

hence

$$supp(T) \subseteq supp(u) \qquad \text{for all } \; u \in \mathrm{S}^{\mathrm{p}}_{\mathrm{A}}.$$

So if $x \in supp(T)$ and $x \neq e$, then $x \in supp(u)$ for all $u \in \mathrm{S}^{\mathrm{p}}_{\mathrm{A}}$, but this is impossible by Lemma 6.2.3. Therefore $x = e$, $supp(T) = \{e\}$, and it follows by the "Lemme de contraction des supports" [29, §6.] that $T = c\lambda_p(e)$ for some $c \in \mathbb{C}$.

■

Corollary 6.2.5. *For any locally compact group G, and $\phi \in \mathrm{A_p}(G)$, we have:*

$$|\phi(e)| \; = \; \inf\{||\phi u||_{\mathrm{A_p}} \; : \; u \in \mathrm{S}^{\mathrm{p}}_{\mathrm{A}}\}. \tag{6.2}$$

Corollary 6.2.6. [96, Lemma 3.1] *For any locally compact group G and for every $\varphi \in \triangle(\mathrm{A_p}(G))$, $\mathrm{A_p}(G)$ is φ-amenable with a φ-mean of norm one.*

Remark 6.2.4. A. T.-M. Lau [67] introduced the notion of F-algebras, nowadays known as *Lau algebras*, as follows: an *F-algebra* is a pair (A, M), where A is a Banach algebra and M is a W^*-algebra, such that A is the predual of M, and the identity of M is a multiplicative linear functional on A. Examples of Lau algebras include the Fourier algebra $\mathrm{A}(G)$, the Fourier-Stieltjes algebra $\mathrm{B}(G)$, the group algebra $\mathrm{L}^1(G)$ of a locally compact group G, and the measure algebra $\mathrm{M}(X) = \mathrm{C}_0(X)^*$ of a locally compact semigroup X. Then, resulting from the study of these algebras, Lau [67, Corollary 4.9] showed that equation (6.2) holds for the Fourier algebra $\mathrm{A}(G)$ of any locally compact group. Earlier, H. Reiter showed that the validity of equation (6.2) for $\mathrm{L}^1(G)$ is equivalent to the amenability of the group [47, §3.7].

6.3 A mean ergodic theorem.

Providing a non-linear version of the celebrated von Neumann mean ergodic theorem, J.-B. Baillon [3] proved a mean ergodic theorem for a non-expansive mapping on a Hilbert space. Generalizing Baillon's theorem, W. Takahashi [99] and G. Rodé [89] established an analogous result for a semigroup of non-expansive mappings acting on a Hilbert space. Finally, A. T.-M. Lau, N. Shioji and W. Takahashi [72] extended these results to semigroups of non-expansive mappings acting on some Banach spaces. In this section, we shall obtain a similar mean ergodic theorem in the setting of φ-amenable Banach algebras, and in passing, we shall also prove the uniqueness of a two-sided φ-mean on the weakly almost periodic functionals.

Throughout the rest of this chapter, let A be a Banach algebra and φ be a non-zero multiplicative linear functional from A into $\mathbb{C}$. Set

$$S_\varphi := \{u \in A \,:\, ||u|| = 1 = \varphi(u)\} \text{ and } \mathscr{M}_\varphi := \{\mu \in A^{**} \,:\, ||\mu|| = 1 = \mu(\varphi)\}.$$

It follows by the Banach-Alaoglu Theorem that $\mathscr{M}_\varphi$ is a w^*-compact convex subset of A^{**}. In the sequel, we assume that S_φ is w^*-dense in $\mathscr{M}_\varphi$. In particular, this is satisfied if A is the predual of any W^*-algebra [41, Proposition 3.] or if $A = \mathrm{A_p}(G)$ [76, Lemma 1.1 & Remark 1.2].

Let $\mathrm{wap}(A)$ denote the space of all weakly almost periodic functionals on A. We recall that a continuous linear functional f on A belongs to $\mathrm{wap}(A)$ if and only if the map $L_f : A \to A^*$, $a \mapsto f \cdot a$, is weakly compact, or equivalently, if the map $R_f : A \to A^*$, $a \mapsto a \cdot f$, is weakly compact. For more information, see [16] and the references therein.

Let C be a non-empty A-invariant closed subset of $\mathrm{wap}(A)$ such that $\mathbb{C}\varphi$ is contained in C. Clearly, the action of S_φ on C is non-expansive, that is, $||u \cdot x - u \cdot y|| \leq ||x - y||$ for all $u \in S_\varphi$, $x, y \in C$.

For any $x \in \mathrm{wap}(A)$ and $\Phi \in A^{**}$ we define a mapping $\Pi_{x,\Phi} : A \to \mathbb{C}$ as follows: for $a \in A$,

$$\Pi_{x,\Phi}(a) := \langle a \cdot x, \Phi \rangle.$$

Then $\Pi_{x,\Phi}$ is a continuous linear functional on A, *i.e.*, $\Pi_{x,\Phi} \in A^*$.

Lemma 6.3.1. *For any $\mu \in \mathscr{M}_\varphi$ there exists a map $T_\mu : C \to C$ such that*

$$\langle T_\mu x, \Phi \rangle = \langle \mu, \Pi_{x,\Phi} \rangle \qquad \textit{for all } x \in C \textit{ and } \Phi \in A^{**}.$$

Proof. Let $\mu \in \mathscr{M}_\varphi$, $x \in C$, $\Phi \in A^{**}$ be arbitrary. Let (u_α) be a net in S_φ such that $\mu = w^* - \lim_\alpha u_\alpha$. Then

$$(6.3) \qquad \langle u_\alpha \cdot x, \Phi \rangle = \langle u_\alpha, \Pi_{x,\Phi} \rangle \to \langle \mu, \Pi_{x,\Phi} \rangle.$$

Let ν be a weak cluster point of $(u_\alpha \cdot x)$; such a point exists since the weak closure of $\{u \cdot x \,:\, u \in S_\varphi\}$ is weakly compact. So there is a subnet $(u_\beta \cdot x)$ of

$(u_\alpha \cdot x)$ such that $(u_\beta \cdot x)$ converges weakly to ν. Observe that $\nu \in \overline{\{u_\beta \cdot x\}}^w \subseteq \overline{\{u \cdot x \,:\, u \in S_\varphi\}}^w = \overline{\{u \cdot x \,:\, u \in S_\varphi\}}^{||\cdot||} \subseteq C$ by Mazur's theorem and since C is closed. Therefore,

$$\langle u_\beta \cdot x, \Phi\rangle \;=\; \langle u_\beta, \Pi_{x,\Phi}\rangle \;\to\; \langle \nu, \Phi\rangle,$$

and (6.3) implies $\langle \nu, \Phi\rangle = \langle \mu, \Pi_{x,\Phi}\rangle$. A fortiori, ν depends on μ and x, and

$$\langle u_\alpha, \Pi_{x,\Phi}\rangle \;\to\; \langle \nu, \Phi\rangle.$$

Thus, we may define T_μ by setting $T_\mu x := \nu$.

■

Remark 6.3.1. The map T_μ may be seen as an analogue of the Bochner integral defined in [89, II(f)].

We recall that the set of all common fixed points for the action of S_φ on A^* is denoted by

$$Fix(S_\varphi) \;:=\; \{f \in A^* \,:\, u \cdot f = f \;\text{ for all }\; u \in S_\varphi\}.$$

For the rest of this section, we assume that A is φ-amenable with a φ-mean of norm one, so that $Fix(S_\varphi) = \mathbb{C}\varphi$. See the proof of Theorem 6.2.1.

Lemma 6.3.2. *Let Ψ be a φ-mean on* wap(A) *with* $||\Psi|| = 1$.
(i) *For all $x \in$* wap(A), $T_\Psi x = \langle \Psi, x\rangle\, \varphi$.
(ii) T_Ψ *is a (linear) projection from* wap(A) *onto* $Fix(S_\varphi)$.
(iii) T_Ψ *is a non-expansive retraction from C onto* $Fix(S_\varphi)$.

Proof. (i) Let $x \in$ wap(A) and let (u_α) be a net in S_φ such that $w^* - \lim_\alpha u_\alpha = \Psi$. For any $\Phi \in A^{**}$ we have

$$\langle u_\alpha \cdot x, \Phi\rangle \;=\; \langle u_\alpha, \Pi_{x,\Phi}\rangle \;\to\; \langle \Psi, \Pi_{x,\Phi}\rangle \;=\; \langle T_\Psi x, \Phi\rangle,$$

that is, $u_\alpha \cdot x \to T_\Psi x$ in the weak topology. In particular, for any $\phi \in A$,

$$\langle u_\alpha \cdot x, \phi\rangle \;\to\; \langle T_\Psi x, \phi\rangle.$$

But

$$\langle u_\alpha \cdot x, \phi\rangle = \langle u_\alpha, x \cdot \phi\rangle \to \langle \Psi, x \cdot \phi\rangle,$$

and since Ψ is a φ-mean, $\langle \Psi, x \cdot \phi\rangle = \varphi(\phi)\langle \Psi, x\rangle$. Therefore, $T_\Psi x = \langle \Psi, x\rangle \varphi$.

(ii) Let $\lambda \in \mathbb{C}$ and $x, y \in \text{wap}(A)$. Then for any $\Phi \in A^{**}$, $\Pi_{\lambda x+y,\Phi} = \lambda\Pi_{x,\Phi} + \Pi_{y,\Phi}$. Thus, for each $\mu \in \mathscr{M}_\varphi$, $T_\mu(\lambda x + y) = \lambda T_\mu(x) + T_\mu(y)$, and $T_\mu : \text{wap}(A) \to \text{wap}(A)$ is linear.
It remains to show that T_Ψ is an idempotent. Using the fact that Ψ is a φ-mean, we have for any $x \in \text{wap}(A)$:

$$\begin{aligned} T_\Psi(T_\Psi x) &= \langle \Psi, T_\Psi x\rangle \varphi = \langle \Psi, \langle \Psi, x\rangle \varphi\rangle \varphi \\ &= \langle \Psi, x\rangle \langle \Psi, \varphi\rangle \varphi = \langle \Psi, x\rangle \varphi = T_\Psi x. \end{aligned}$$

(iii) We first notice that $T_\Psi : C \to C$ is not linear since C is not a subspace. Then, similarly as in (b), it follows that T_Ψ is a retraction from C onto $Fix(S_\varphi)$. Moreover, for any $x, y \in C$ we have

$$||T_\Psi x - T_\Psi y|| = |\langle \Psi, x - y\rangle| \, ||\varphi|| \le ||\Psi|| \, ||x - y|| = ||x - y||,$$

which shows that T_Ψ is non-expansive.

■

Definition 12. Let $\Psi \in A^{**}$. Then Ψ is called a *two-sided φ-mean (on A^*) of norm one* if $\Psi \in \mathscr{M}_\varphi$ and $\langle \Psi, f \cdot a\rangle = \varphi(a)\langle \Psi, f\rangle = \langle \Psi, a \cdot f\rangle$ for all $a \in A$, $f \in A^*$. Obviously, if A is commutative, any φ-mean is automatically two-sided.

The next lemma is an analogue of [60, Lemma 2.].

Lemma 6.3.3. *Let Ψ be a two-sided φ-mean on* wap(A) *with* $||\Psi|| = 1$.
(i) *For any $u \in S_\varphi$, $x \in$* wap(A)*, we have:*

$$T_\Psi(x \cdot u) = (T_\Psi x) \cdot u = u \cdot T_\Psi x = T_\Psi(u \cdot x) = T_\Psi x.$$

(ii) *For any $\mu \in \mathscr{M}_\varphi$, $x \in$* wap(A)*, we have:*

$$T_\Psi T_\mu x = T_\mu T_\Psi x = T_\Psi x.$$

Proof. (i) Let $u \in S_\varphi$ and $x \in \mathrm{wap}(A)$. By Lemma 6.3.2, and since Ψ is a two-sided φ-mean, we have:

$$\begin{aligned} T_\Psi(x \cdot u) &= \langle \Psi, x \cdot u \rangle \, \varphi \,=\, \varphi(u) \, \langle \Psi, x \rangle \, \varphi \,=\, T_\Psi x \\ &= \langle \Psi, u \cdot x \rangle \, \varphi \,=\, T_\Psi(u \cdot x). \end{aligned}$$

Also, by multiplicativity of φ,

$$(T_\Psi x) \cdot u \,=\, \langle \Psi, x \rangle \, \varphi \cdot u \,=\, \varphi(u) \, \langle \Psi, x \rangle \, \varphi \,=\, T_\Psi x.$$

Similarly for $u \cdot (T_\Psi x)$.

(ii) Let $\mu \in \mathscr{M}_\varphi$ and $x \in \mathrm{wap}(A)$. For any $\Phi \in A^{**}$ and $a \in A$ we have:

$$\Pi_{T_\Psi x, \Phi}(a) \,=\, \langle a \cdot T_\Psi x, \Phi \rangle \,=\, \varphi(a) \, \langle T_\Psi x, \Phi \rangle; \tag{6.4}$$

and since Ψ is a two-sided φ-mean,

$$\Pi_{x, \Psi}(a) \,=\, \langle a \cdot x, \Psi \rangle \,=\, \varphi(a) \, \langle \Psi, x \rangle \,=\, \langle T_\Psi x, a \rangle. \tag{6.5}$$

By (6.4),

$$\langle T_\mu T_\Psi x, \Phi \rangle \,=\, \langle \mu, \Pi_{T_\Psi x, \Phi} \rangle \,=\, \langle T_\Psi x, \Phi \rangle \, \mu(\varphi) \,=\, \langle T_\Psi x, \Phi \rangle;$$

and by (6.5), Lemma 6.3.1, Lemma 6.3.2 and the Goldstine Theorem,

$$\begin{aligned} \langle T_\Psi T_\mu x, \Phi \rangle &= \langle \, \langle \Psi, T_\mu x \rangle \, \varphi, \Phi \rangle \,=\, \langle \, \langle \mu, \Pi_{x, \Psi} \rangle \, \varphi, \Phi \rangle \\ &= \langle \mu, \varphi \rangle \, \langle \Psi, x \rangle \, \langle \varphi, \Phi \rangle \,=\, \langle T_\Psi x, \Phi \rangle. \end{aligned}$$

■

Remark 6.3.2. We recall that a closed subspace X of A^* is *topologically left invariant* if $x \cdot a \in X$ for all $a \in A$, $x \in X$. Then we observe that in the above two lemmas, we may replace $\mathrm{wap}(A)$ by any topologically left invariant subspace X of $\mathrm{wap}(A)$ containing φ. We will show in Section 6.5 that Lemma 6.3.2 and Lemma 6.3.3.(i) actually characterize the existence of a φ-mean of norm one.

Now, as a consequence of Lemma 6.3.2.(i) and Lemma 6.3.3.(ii), we obtain the uniqueness of a two-sided φ-mean on $\text{wap}(A)$ of norm one.

Lemma 6.3.4. *Let Ψ_1, Ψ_2 be two two-sided φ-mean on* $\text{wap}(A)$ *with $||\Psi_1|| = ||\Psi_2|| = 1$. Then $T_{\Psi_1} = T_{\Psi_2}$.*

Proposition 6.3.5. *Let A be a Banach algebra, and suppose that A admits a two-sided φ-mean Ψ of norm one. Then the restriction of Ψ to* $\text{wap}(A)$ *is unique.*

Definition 13. Let (μ_α) be a net in $\mathscr{M}_\varphi$. We say that (μ_α) is *left ergodic* if $w^* - \lim_\alpha(a \cdot \mu_\alpha - \varphi(a)\mu_\alpha) = 0$ for all $a \in A$. We say that (μ_α) is *ergodic* if

$$w^* - \lim_{\alpha}(a \cdot \mu_\alpha - \varphi(a)\mu_\alpha) \;=\; w^* - \lim_{\alpha}(\mu_\alpha \cdot a - \varphi(a)\mu_\alpha) \;=\; 0, \;\text{ for all }\; a \in A.$$

Remark 6.3.3. It follows from [58, Theorem 1.4] that A is φ-amenable with a φ-mean of norm one if and only if there exists a left ergodic net in $\mathscr{M}_\varphi$. Similarly, A admits a two-sided φ-mean of norm one if and only if there exists an ergodic net in $\mathscr{M}_\varphi$.

Theorem 6.3.6. *Let X be a topologically left invariant subspace of* $\text{wap}(A)$ *containing φ, and let (μ_α) be a net in $\mathscr{M}_\varphi$. If (μ_α) is left ergodic, then there exists a subnet (μ_β) of (μ_α) such that T_{μ_β} converges in the weak operator topology to a linear projection from X onto $Fix(S_\varphi)$.*

Proof. Let Ψ be a weak* cluster point of (μ_α), and let (μ_β) be a subnet of (μ_α) such that $\mu_\beta \xrightarrow{w^*} \Psi$; such Ψ exists since $\mathscr{M}_\varphi$ is w^*-compact. By assumption, $\langle a \cdot \mu_\alpha - \varphi(a)\mu_\alpha, f\rangle \to 0$ for all $a \in A$, $f \in A^*$. So in particular,

$$\langle \mu_\beta, f \cdot a - \varphi(a) f\rangle \;\to\; 0 \;\text{ for all }\; a \in A,\; f \in A^*.$$

Then, $\langle \Psi, f \cdot a - \varphi(a) f\rangle = 0$ for all $a \in A$, $f \in A^*$, so Ψ is a φ-mean. Moreover, $\Psi \in \overline{\{\mu_\beta\}}^{w^*} \subseteq \overline{\mathscr{M}_\varphi}^{w^*} = \mathscr{M}_\varphi$ since $\mathscr{M}_\varphi$ is w^*-closed. Hence Ψ is a φ-mean of norm one. Finally, by Lemma 6.3.2.(ii) and Remark 6.3.2, T_Ψ is a projection onto $Fix(S_\varphi)$, and by Lemma 6.3.1 we have, for any $x \in X$ and any $\Phi \in A^{**}$,

$$\langle T_{\mu_\beta} x, \Phi\rangle \;=\; \langle \mu_\beta, \Pi_{x,\Phi}\rangle \;\to\; \langle \Psi, \Pi_{x,\Phi}\rangle \;=\; \langle T_\Psi x, \Phi\rangle,$$

that is, $T_{\mu_\beta} \to T_\Psi$ in the weak operator topology. ■

The next corollary follows from Lemma 6.3.1, Lemma 6.3.2.(iii), Proposition 6.3.5, and from the proof of Theorem 6.3.6.

Corollary 6.3.7. *Let C be a non-empty closed A-invariant subset of* wap(A) *such that $\mathbb{C}\varphi \subseteq C$. Suppose that A admits a two-sided φ-mean of norm one. Then there exists a unique non-expansive retraction P from C onto $Fix(S_\varphi)$ such that $PT_\mu = T_\mu P = P$ for each $\mu \in \mathscr{M}_\varphi$, and $Px \in \overline{\{u \cdot x \, : \, u \in S_\varphi\}}$ for each $x \in C$.*
Moreover, if (μ_α) is an ergodic net in $\mathscr{M}_\varphi$, then there exists a subnet (μ_β) of (μ_α) such that T_{μ_β} converges to P in the weak operator topology.

Remark 6.3.4. (a) Theorem 6.3.6 and Corollary 6.3.7 may be compared with [72, Theorem 1. and Theorem 2.], [60, Theorem 1.], [89, Theorem], [99, Theorem 1.], and [27, Theorem 4.1].

(b) In Corollary 6.3.7, if C is convex, then the retraction P is an affine map, and if C is a linear space, then P is a linear projection.

(c) Let C be a non-empty closed A-invariant subset of wap(A) such that $\mathbb{C}\varphi \subseteq C$. Then for every $x \in C$ we have $\overline{\{u \cdot x \, : \, u \in S_\varphi\}} \cap Fix(S_\varphi) \neq \varnothing$ by Lemma 6.3.1 and Lemma 6.3.2. Furthermore, a two-sided φ-mean Ψ on wap(A) with $||\Psi|| = 1$ is unique by Proposition 6.3.5. In this case,

$$\overline{\{u \cdot x \, : \, u \in S_\varphi\}} \cap Fix(S_\varphi) = \{\langle \Psi, x \rangle \, \varphi\}.$$

Indeed, if $\nu \in \overline{\{u \cdot x \, : \, u \in S_\varphi\}} \cap Fix(S_\varphi)$, then $\nu = \lambda\varphi$ for some $\lambda \in \mathbb{C}$ since S_φ acts ergodically on A^*, and there is a net (u_α) in S_φ such that $u_\alpha \cdot x \to \nu$ in norm, hence weakly. Now let $\mu \in \mathscr{M}_\varphi$ be a weak* cluster point of (u_α), and let (u_β) be a subnet of (u_α) such that $u_\beta \xrightarrow{w^*} \mu$. Thus, from the proof of Lemma 6.3.1 we deduce that $\nu = T_\mu x$, hence $T_\mu x = \lambda\varphi$. So we have:

$$\lambda \;=\; \langle \Psi, \lambda\varphi \rangle \;=\; \langle \Psi, T_\mu x \rangle \;=\; \langle \mu, \Pi_{x,\Psi} \rangle \;=\; \langle \mu, \langle \Psi, x \rangle \varphi \rangle \;=\; \langle \Psi, x \rangle$$

since $\langle \Pi_{x,\Psi}, a \rangle = \langle a \cdot x, \Psi \rangle = \varphi(a)\langle \Psi, x \rangle$ for all $a \in A$.
Compare with [27, Theorem 5.3] and [60, Lemma 4.].

Examples. 1. Let G be an amenable locally compact group, $A = \mathrm{L}^1(G)$ and $\varphi = \delta_e \in \mathrm{L}^\infty(G)$, the evaluation functional at e. In this case, the

φ-means of norm one are nothing but the topologically invariant means on $L^\infty(G) = A^*$. Therefore Proposition 6.3.5 provides a different proof, in the amenable case, of a famous result due to C. Ryll-Nardzewski [95, Theorem 4.(3)], which asserts the uniqueness of a topologically invariant mean on the space $\mathrm{WAP}(G)$ of all weakly almost periodic functions on G. We recall that $\mathrm{wap}(L^1(G)) = \mathrm{WAP}(G)$ by [102, Theorem 4.]. Note that Ryll-Nardzewski's result is true for any locally compact group.

2. Let G be a locally compact group, $A = A_p(G)$ and $\varphi = \lambda_p(e)$. Here again, the φ-means of norm one are nothing but the topologically invariant means on $PM_p(G) = A^*$. Then, Proposition 6.3.5 yields the uniqueness of a topologically invariant mean on $\mathrm{WAP}_p(G)$, a result that was first established in [43, Proposition 9.]. Also, we obtain the existence of a projection P from $\mathrm{WAP}_p(G)$ onto $Fix(S_A^p)$; this may have been deduced from [42, Theorem 6.] but we here obtain a precise definition of P (Lemma 6.3.2.(i)). Furthermore, with the terminology of [73] and by Remark 6.3.4.(c), if (u_α) is an ergodic net in S_A^p, the conclusion of Corollary 6.3.7 is equivalent to saying that the representation of S_A^p on $\mathrm{WAP}_p(G)$, given by the usual module action, is averageable. Thus, for $X = \mathrm{WAP}_p(G)$, Corollary 6.3.7 may be regarded as an analogue of a theorem due to C. Ryll-Nardzewski [94], who established the averageability of the representation given by left translation on the weakly almost periodic functions on a locally compact group. For $X = \mathrm{AP}_p(G)$, Corollary 6.3.7 may be seen as an analogue of a theorem of J. von Neumann [104].

3. Let A be an F-algebra as introduced in [67] (see Remark 6.2.4). If there is a topologically (two-sided) invariant mean on A^*, which is the case if A is commutative, then a topologically invariant mean on $\mathrm{wap}(A)$ is unique by Proposition 6.3.5. In particular, this applies when A is the measure algebra of a locally compact commutative semigroup, and when A is the measure algebra of an amenable locally compact group. In this case, these results are new.

6.4 Decomposition of subspaces of $\mathrm{wap}(A)$.

Let X be a topologically invariant subspace of $\mathrm{wap}(A)$ containing φ. Assume that A admits a two-sided φ-mean of norm one, which we denote by Ψ. Then Lemma 6.3.3 and the second statement of Corollary 6.3.7 assert that T_Ψ is the zero of the semigroup $\overline{\{T_u \,:\, u \in S_\varphi\}}^{wot}$, where $T_u x := u \cdot x$, $x \in X$. Moreover, for each $x \in X$, the closure of $\{u \cdot x \,:\, u \in S_\varphi\}$ contains a unique common fixed point for S_φ by Remark 6.3.4.(c). Thus, an application of [6, Theorem 3.6, Chapter 6.] yields the following decomposition of X:

Theorem 6.4.1. *The following statements hold:*

(i)

$$\begin{aligned} X_f &:= \{x \in X \,:\, u \cdot x = x \ \textit{for all} \ u \in S_\varphi\} \\ &= \{T_\Psi x \,:\, x \in X\} = \{\langle \Psi, x\rangle \varphi \,:\, x \in X\} = \mathbb{C}\,\varphi. \end{aligned}$$

(ii)

$$\begin{aligned} X_d &:= \overline{\mathrm{span}}\{u \cdot x - x \,:\, u \in S_\varphi, x \in X\} \\ &= \{x \in X \,:\, 0 \in \overline{\{u \cdot x \,:\, u \in S_\varphi\}}\} = \{x \in X \,:\, \langle \Psi, x\rangle = 0\}. \end{aligned}$$

(iii)

$$X = X_f \oplus X_d.$$

Remark 6.4.1. Let G be a locally compact group and let $x \in G$. The set of topologically invariant x-means on $\mathrm{PM_p}(G)$ is denoted by $\mathrm{TIM_p}(x)$. Recall that $\Psi \in \mathrm{TIM_p}(x)$ if and only if $\Psi \in \mathrm{PM_p}(G)^*$, $||\Psi|| = 1 = \langle \Psi, \lambda_p(x)\rangle$, and $\langle \Psi, T \cdot u\rangle = u(x)\langle \Psi, T\rangle$ for all $u \in \mathrm{A_p}(G)$, $T \in \mathrm{PM_p}(G)$. In particular, $\mathrm{TIM_p}(e) = \mathrm{TIM_p}(\widehat{G})$ is the set of all topologically invariant means on $\mathrm{PM_p}(G)$. Furthermore, for all $x \in G$, the following can easily be proved ([51, p. 216], [24, Proposition 5.2]):

$$\mathrm{TIM_p}(x) = \{\,{}_x m \,:\, m \in \mathrm{TIM_p}(\widehat{G})\},$$

where $\langle {}_x m, T\rangle = \langle m, {}_{x^{-1}}T\rangle$, $\langle {}_x T, u\rangle = \langle T, {}_x u\rangle$, and ${}_x u$ is the left translate of u by x, for $u \in \mathrm{A_p}(G)$, $T \in \mathrm{PM_p}(G)$. In particular, for any $x \in G$ there exists a unique topologically invariant x-mean on $\mathrm{WAP_p}(G)$ by Proposition 6.3.5.

Corollary 6.4.2. *Let G be a locally compact group, $x \in G$, X be a topologically invariant subspace of $\mathrm{WAP_p}(G)$ such that $\lambda_p(x) \in X$, and Ψ_x be the unique topologically invariant x-mean on $\mathrm{WAP_p}(G)$. Then*

$$X = \mathbb{C}\lambda_p(x) \oplus \{T \in X : \Psi_x(T) = 0\}.$$

In particular, $\langle \Psi_x, \lambda_p(z) \rangle = 0$ for every $z \in G$, $z \neq x$, and

$$\begin{aligned} \mathrm{WAP_p}(G) &= \mathbb{C}\lambda_p(x) \oplus \{T \in \mathrm{WAP_p}(G) : \Psi_x(T) = 0\} \\ &= \mathrm{AP_p}(G) + \{T \in \mathrm{WAP_p}(G) : \Psi_x(T) = 0\}. \end{aligned}$$

The third statement of Theorem 6.4.1 may be compared with [101, Theorem 2.11.(iii), p.23] whereas the second statement of Corollary 6.4.2 may be compared with [43, Proposition 10.], [51, Lemma 3.14.(a)], and Lemma 4.1.1 of the present thesis.

6.5 Fixed point characterizations.

Let X be a topologically left invariant subspace of A^* with $\varphi \in X$, and such that

(6.6) $$\text{for each } \ x \in X, \quad \overline{\{u \cdot x : u \in S_\varphi\}}^{w^*} \subset X.$$

Let $\mathscr{B}(X)$ denote the space of bounded linear operators from X into itself. For any $x \in X$, $a \in A$, if $\Pi_{x,a}(T) := \langle Tx, a \rangle$ for $T \in \mathscr{B}(X)$, then $\Pi_{x,a}$ defines a seminorm on $\mathscr{B}(X)$, and the locally convex topology generated by the family $\{\Pi_{x,a} : x \in X, a \in A\}$ is called the *weak* operator topology*, abbreviated w^*ot. In particular, if we regard S_φ as a semigroup of bounded linear operators on X - via the usual module action of A on A^* - then $\overline{S_\varphi}^{w^*ot}$ is w^*ot-compact since so is the closed unit ball of $\mathscr{B}(X)$. See [93].

Remark 6.5.1. We recall that a topologically left invariant subspace X of A^* is *topologically left introverted* if $m_L(x) \in X$ for any $x \in X$, where $\langle m_L(x), a \rangle = \langle m, x \cdot a \rangle$ for $a \in A$, $x \in X$, $m \in X^*$. Then, by [71, Lemma 1.2], any topologically left introverted subspace of A^* satisfies condition (6.6) above. In particular, this is the case if $X = \overline{A \cdot A^*}$, $X = \mathrm{wap}(A)$, or if X is any norm closed topologically left invariant subspace of $\mathrm{wap}(A)$.

Lemma 6.5.1. *For every $x \in X$, $\overline{S_\varphi x}^{w^*}$ is w^*-compact, and*

$$\overline{S_\varphi x}^{w^*} = \overline{S_\varphi}^{w^*ot} x.$$

Proof. Let $x \in X$ be fixed. Observe that the map $\phi_x : \mathscr{B}(X) \to X$, defined by $\phi_x(T) := Tx$, is w^*ot-w^* continuous. Then, since $\overline{S_\varphi}^{w^*ot}$ is w*ot compact, $\overline{S_\varphi}^{w^*ot} x$ is w^*-compact as the continuous image of a compact set. Now let $a \in \overline{S_\varphi x}^{w^*}$. Then there is a net (u_α) in S_φ such that $u_\alpha \cdot x \xrightarrow{w^*} a$. Let ν be a w*ot cluster point of (u_α) and let (u_β) be a subnet of (u_α) such that (u_β) converges to ν in the weak* operator topology, *i.e.*, $\Pi_{T,b}(u_\beta) \to \Pi_{T,b}(\nu)$ for all $T \in A^*$, $b \in A$. In particular, $\langle u_\beta \cdot x, b\rangle \to \langle \nu x, b\rangle$ for all $b \in A$. Hence $a = \nu x$, and $a \in \overline{S_\varphi}^{w^*ot} x$. It also follows that $\overline{S_\varphi x}^{w^*}$ is w^*-compact.
On the other hand, if $a \in \overline{S_\varphi}^{w^*ot} x$, there is $U \in \overline{S_\varphi}^{w^*ot}$ such that $a = Ux$, and there is a net (u_α) in S_φ such that $u_\alpha \xrightarrow{w^*ot} U$. In particular, $\langle u_\alpha \cdot x, b\rangle \to \langle Ux, b\rangle = \langle a, b\rangle$ for all $b \in A$, that is, $u_\alpha \cdot x \xrightarrow{w^*} a$. Hence $a \in \overline{S_\varphi x}^{w^*}$.

■

Imitating the proof of Lemma 6.3.1 and Lemma 6.3.2, the next lemma is easily verified.

Lemma 6.5.2. *For any $\mu \in \mathscr{M}_\varphi$ there exists a map $T_\mu \in \mathscr{B}(X)$ such that*

$$\langle T_\mu x, a\rangle = \langle \mu, \Pi_{x,a}\rangle = \langle \mu, a \cdot x\rangle \text{ for all } x \in X \text{ and } a \in A,$$

and $T_\mu x \in \overline{S_\varphi x}^{w^*}$. Moreover, $T_\Psi x = \langle \Psi, x\rangle \varphi$ for all $x \in X$, if Ψ is a φ-mean on A^* of norm one.

Theorem 6.5.3. *Let X be a topologically left invariant subspace of A^* with $\varphi \in X$, and such that condition (6.6) is satisfied. Assume that S_φ is left amenable as a semitopological semigroup and $\overline{\mathrm{span}}\, S_\varphi = A$. Then the following assertions are equivalent:*

(i) A is φ-amenable with a φ-mean of norm one.

(ii) There exists a φ-mean of norm one on X.

(iii) *For any topologically left invariant subspace* $Y \subseteq A^*$ *with* $\varphi \in Y$ *and such that* (6.6) *is satisfied, there exists* $P \in \overline{S_\varphi}^{w^*ot}$ *such that* $P(y \cdot a) = (Py) \cdot a = \varphi(a) Py$ *for all* $a \in A$, $y \in Y$, *where* w^*ot *designates the weak* operator topology on* $\mathscr{B}(Y)$.

Proof. (i) $\Rightarrow$ (iii): Let $\Psi \in A^{**}$ be a φ-mean of norm one. Then $P := T_\Psi \in \overline{S_\varphi}^{w^*ot}$ by Lemma 6.5.2 and Lemma 6.5.1, and the remaining assertion follows from Lemma 6.3.3.
(iii) $\Rightarrow$ (ii): Let $P \in \overline{S_\varphi}^{w^*ot}$ such that $P(x \cdot a) = (Px) \cdot a = \varphi(a) Px$ for all $a \in A$, $x \in X$. By Lemma 6.5.1, $\overline{S_\varphi}^{w^*ot} x = \overline{S_\varphi x}^{w^*}$ for all $x \in X$. So for every $x \in X$, there exists a net (u_α) in S_φ such that $u_\alpha \cdot x \overset{w^*}{\to} Px$. Let $\mu \in \mathscr{M}_\varphi$ be a weak* cluster point of (u_α) and let (u_β) be a subnet of (u_α) such that $u_\beta \overset{w^*}{\to} \mu$. In particular, $\langle u_\beta, x\rangle \to \langle \mu, x\rangle$ for all $x \in X$. Then, for every $a \in A$ we have:

$$\begin{aligned} &\langle u_\beta \cdot x, a\rangle \;\to\; \langle Px, a\rangle \\ &= \langle u_\beta, x \cdot a\rangle \;\to\; \langle \mu, x \cdot a\rangle \;=\; \langle \mu \cdot x, a\rangle, \end{aligned}$$

hence $Px = \mu \cdot x$ for all $x \in X$. By assumption, $\mu \cdot (x \cdot a) = (\mu \cdot x) \cdot a = \varphi(a) \mu \cdot x$ for all $x \in X$, $a \in A$. Therefore, for every $u \in S_\varphi$, $u \cdot \mu$ is a φ-mean on X of norm one. Indeed,

$$\begin{aligned} \langle u \cdot \mu, x \cdot a\rangle &= \langle \mu, x \cdot a \cdot u\rangle \;=\; \langle \mu \cdot x, au\rangle \;=\; \langle \mu \cdot x \cdot a, u\rangle \\ &= \varphi(a)\, \langle \mu \cdot x, u\rangle \;=\; \varphi(a)\, \langle u \cdot \mu, x\rangle, \qquad \text{for all } x \in X,\, a \in A. \end{aligned}$$

(ii) $\Rightarrow$ (i): Let Ψ be a φ-mean on X with $||\Psi|| = 1$. By the Hahn-Banach Theorem, there exists an extension $\Psi_1 \in A^{**}$ of Ψ which preserves the norm. Furthermore, since $\varphi \in X$, $\Psi_1(\varphi) = \Psi(\varphi) = 1$, hence $\Psi_1 \in \mathscr{M}_\varphi$. Now let $M \in \ell^\infty(S_\varphi)^*$ be a translation invariant mean and define $\Psi_0 \in A^{**}$ by $\langle \Psi_0, f\rangle = M(\Pi_{f,\Psi_1})$, where $\Pi_{f,\Psi_1} \in \ell^\infty(S_\varphi)$ is given by $\Pi_{f,\Psi_1}(u) = \langle \Psi_1, f \cdot u\rangle$ for $u \in S_\varphi$. Then it is straightforward to see that $\Psi_0 \in \mathscr{M}_\varphi$, and for all $f \in A^*$, $u \in S_\varphi$, we have

$$\langle \Psi_0, f \cdot u\rangle \;=\; M(\Pi_{f \cdot u, \Psi_1}) \;=\; M(\ell_u \Pi_{f,\Psi_1}) \;=\; M(\Pi_{f,\Psi_1}) \;=\; \langle \Psi_0, f\rangle.$$

Therefore, since $\overline{\operatorname{span}}\, S_\varphi = A$, we conclude that Ψ_0 is a φ-mean on A^* which is of norm one. ■

Remark 6.5.2. 1. The equivalence (i) $\Leftrightarrow$ (iii) holds even without the assumption that S_φ is left amenable and that $\overline{\operatorname{span}}\, S_\varphi = A$.

2. For left amenable F-algebras, a fixed point characterization of a similar nature can be found in [82].

Corollary 6.5.4. *Let G be a locally compact group. For any topologically invariant topologically introverted subspace X of $\mathrm{PM_p}(G)$ containing $\lambda_p(e)$, there exists $P \in \overline{\mathrm{S_A^p}}^{w^*ot}$ such that $P(x \cdot u) = (Px) \cdot u = u(e)Px$ for all $u \in \mathrm{A_p}(G)$, $x \in X$.*

Next, by means of Lemma 6.5.2 and [64, Theorem 2.1], we are able to obtain an analogue of a famous fixed point theorem due to T. Mitchell [79, Theorem 3.]. It may also be compared with the main theorem in [46] and with [106, Theorem 5.4].

Theorem 6.5.5. *Let X be a topologically left invariant subspace of A^* with $\varphi \in X$, and such that condition (6.6) is satisfied. The following assertions are equivalent:*

(i) There exists a φ-mean Ψ on X with $||\Psi|| = 1$.

(ii) For each $x \in X$, there exists $\lambda \in \mathbb{C}$ such that $\lambda\varphi \in \overline{\{u \cdot x \,:\, u \in S_\varphi\}}^{w^*}$.

Proof. The implication (i) $\Rightarrow$ (ii) is a direct consequence of Lemma 6.5.2. For the reverse implication, let $X = A^*$ and set $F := \mathbb{C}\varphi$. By [64, Theorem 2.1], there exists an F-stationary operator P on A^* and $P \in \overline{S_\varphi}^{w^*ot}$. Let (u_α) in S_φ such that $u_\alpha \overset{w^*ot}{\to} P$, that is $\langle u_\alpha \cdot f, a\rangle \to \langle Pf, a\rangle$ for all $f \in A^*$, $a \in A$. Now, for a fixed $u_0 \in S_\varphi$, we define $\Psi(f) := \langle Pf, u_0\rangle$ for $f \in A^*$. It is then straightforward to verify that $||\Psi|| = 1$. For any $f \in A^*$, we denote by λ_f the complex number that satisfies $Pf = \lambda_f\varphi$; such λ_f exists by the definition of stationarity [64]. So for any $f \in A^*$, $a \in A$, we have:

$$\begin{aligned}\langle u_\alpha \cdot (f \cdot a), u_0\rangle = \langle u_\alpha \cdot f, au_0\rangle \;\to\; & \langle Pf, au_0\rangle = \langle \lambda_f\varphi, au_0\rangle \\ & = \varphi(a)\lambda_f = \varphi(a)\langle Pf, u_0\rangle = \varphi(a)\Psi(f)\end{aligned}$$

and

$$\begin{aligned}\langle u_\alpha \cdot (f \cdot a), u_0\rangle \;\to\; & \langle P(f \cdot a), u_0\rangle = \lambda_{f\cdot a} \\ & = \Psi(f \cdot a).\end{aligned}$$

Therefore, $\Psi(f \cdot a) = \lambda_{f \cdot a} = \varphi(a)\lambda_f = \varphi(a)\Psi(f)$, which shows that Ψ is a φ-mean on A^*. By restriction, Ψ is also a φ-mean on X, and this completes the proof.

■

We conclude this section with one last fixed point characterization of φ-amenability. First, analogously to [25, Lemma 7.1], we need the following lemma:

Lemma 6.5.6. *Let X be a topologically left invariant subspace of A^* with $\varphi \in X$. The following assertions are equivalent:*

(i) There exists $\Psi \in X^*$ such that $||\Psi|| = 1 = \Psi(\varphi)$ and $\langle \Psi, x \cdot u \rangle = \langle \Psi, x \rangle$ for all $x \in X$, $u \in S_\varphi$.

(ii) There exists a net (u_α) in S_φ such that, for each $u \in S_\varphi$, $uu_\alpha - u_\alpha \to 0$ in the weak topology $\sigma(A, X)$.

Proof. Assume that there exists $\Psi \in X^*$ as in (i). By the Hahn-Banach Theorem, Ψ has a norm-preserving extension to A^{**}, which we also denote by Ψ. Let (u_α) be a net in S_φ such that $u_\alpha \xrightarrow{w^*} \Psi$. Then, for any $x \in X$, $u \in S_\varphi$, we have:

$$\langle uu_\alpha - u_\alpha, x \rangle \;=\; \langle u_\alpha, x \cdot u - u \rangle \;\to\; \langle \Psi, x \cdot u - x \rangle \;=\; 0.$$

Conversely, assume that there exists a net (u_α) as in (ii), and let $\mu \in \mathscr{M}_\varphi$ be a weak* cluster point of (u_α). Denote the restriction of μ to X by Ψ. Then, for any $x \in X$, $u \in S_\varphi$, we have:

$$\langle \Psi, x \cdot u \rangle - \langle \Psi, x \rangle \;=\; \lim_\alpha \langle u_\alpha, x \cdot u \rangle - \lim_\alpha \langle u_\alpha, x \rangle \;=\; \lim_\alpha \langle uu_\alpha - u_\alpha, x \rangle \;=\; 0.$$

■

Theorem 6.5.7. *Let X be a topologically left introverted subspace of A^* with $\varphi \in X$. Assume that S_φ is left amenable as a semitopological semigroup and that $\overline{\operatorname{span}}\, S_\varphi = A$. Then the following assertions are equivalent:*

(i) A is φ-amenable with a φ-mean of norm one.

(ii) There exists a φ-mean on X of norm one.

(iii) Whenever S_φ acts on a compact convex subset K of a separated locally convex space E, and the action is linear and $\sigma(A,X)$-separately continuous, then K contains a common fixed point for S_φ.

Proof. The implications (i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii): Let E and K as in the hypotheses of (iii), let $\mathscr{Q}$ be a family of continuous seminorms on E which determines the topology of E, and let $s \in K$ be fixed. By Lemma 6.5.6, there exists (u_α) in S_φ such that $uu_\alpha - u_\alpha \overset{\sigma(A,X)}{\longrightarrow} 0$ for each $u \in S_\varphi$. Since K is S_φ-invariant, $u_\alpha \cdot s \in K$, and by compactness of K, we may assume, after passing to a subnet if necessary, that $u_\alpha \cdot s$ converges to some $s_0 \in K$, *i.e.*, $\rho(u_\alpha \cdot s - s_0) \to 0$ for all $\rho \in \mathscr{Q}$. Therefore, for any $u \in S_\varphi$, $\rho \in \mathscr{Q}$, we have:

$$\rho(u \cdot s_0 - s_0) \le \rho(u \cdot (s_0 - u_\alpha \cdot s)) + \rho((uu_\alpha - u_\alpha) \cdot s) + \rho(u_\alpha \cdot s - s_0) \to 0,$$

hence $u \cdot s_0 = s_0$ for all $u \in S_\varphi$.
(iii) $\Rightarrow$ (ii): Let $E = X^*$ endowed with the weak* topology. Then the action of A on E, given by the usual module action, is obviously linear. Moreover, if $a_\alpha \overset{\sigma(A,X)}{\longrightarrow} a$ in A, $m \in E$ and $x \in X$, then

$$\begin{aligned} |\langle a_\alpha \cdot m, x\rangle - \langle a \cdot m, x\rangle| &= |\langle m, x \cdot (a_\alpha - a)\rangle| \\ &= |\langle m_L(x), a_\alpha - a\rangle| \to 0, \end{aligned}$$

since X is topologically left introverted. Also, if $m_\beta \overset{w^*}{\longrightarrow} m$ in E, $a \in A$ and $x \in X$, then

$$\begin{aligned} |\langle a \cdot m_\beta, x\rangle - \langle a \cdot m, x\rangle| &= |\langle m_\beta, x \cdot a\rangle - \langle m, x \cdot a\rangle| \\ &= |\langle m_\beta - m, x \cdot a\rangle| \to 0. \end{aligned}$$

Therefore, the action of A on E is $\sigma(A,X)$-separately continuous.
Now set $K := \{\mu \in X^* : ||\mu|| = 1 = \mu(\varphi)\}$. Then K is a compact convex subset in E and, by assumption, contains a common fixed point for S_φ, say Ψ. Thus, $\langle \Psi, x \cdot u\rangle = \langle u \cdot \Psi, x\rangle = \langle \Psi, x\rangle$ for all $x \in X$, $u \in S_\varphi$, hence Ψ is a φ-mean on X, of norm one, as $\overline{\operatorname{span}}\, S_\varphi = A$.
The proof of (ii) $\Rightarrow$ (i) is similar to that of (ii) $\Rightarrow$ (i) in Theorem 6.5.3.

■

Remark 6.5.3. The equivalence (i) $\Leftrightarrow$ (iii) was obtained in [107] for $A = \mathrm{L}^1(G)$ and in [25] for F-algebras.

Corollary 6.5.8. *Let G be a locally compact group. For any topologically introverted subspace X of $\mathrm{PM_p}(G)$ containing $\lambda_p(e)$, if $\mathrm{S_A^p}$ acts on a compact convex subset K of a separated locally convex space E such that the action is linear and $\sigma(A, X)$-separately continuous, then K contains a common fixed point for $\mathrm{S_A^p}$.*

Chapter 7

Spectrums in $\mathrm{PM_p}(G)$ and discreteness of G.

Definition 14. Let A be a commutative Banach algebra. For $f \in A^*$ we define

$$\sigma(f) := \overline{\{a \cdot f \,:\, a \in A\}}^{\|\cdot\|} \cap \triangle(A), \qquad \text{the } \textit{norm spectrum} \text{ of } f;$$

$$\sigma_*(f) := \overline{\{a \cdot f \,:\, a \in A\}}^{w^*} \cap \triangle(A), \qquad \text{the } \textit{weak}^* \textit{ spectrum} \text{ of } f,$$

where $\triangle(A)$ denotes the Gelfand spectrum of A.

The norm spectrum has been studied in [51] and [103], and recently in [80]. The weak* spectrum is partly studied in [56] and is denominated as the *support* in that paper. In particular, it is proved in [56, Lemma 3.1.(i)] that if A is regular, Tauberian and semisimple, then $\sigma_*(f) \neq \varnothing$ for all $f \in A^*$, $f \neq 0$. See also [49, §40] and [59, chap. VI, §6] for the case when $A = \mathrm{L}^1(G)$ of a locally compact Abelian group. However, the norm spectrum $\sigma(f)$ may well be empty even if f is a non-zero function in $\mathrm{L}^\infty(\mathbb{R})$, as is the case if $f \in \mathrm{C}_0(\mathbb{R})$. But it is known that $\sigma(f) \neq \varnothing$ for every almost periodic function f on $\mathbb{R}$ [59, chap. VI, §5], and also for every almost periodic function on any locally compact Abelian group [5, Theorem 2.2.3, p. 110]. For the case when $A = \mathrm{A_p}(G)$, Z. Hu [51] showed that $\sigma(T) \neq \varnothing$ for all $T \in \mathrm{C}_{\delta,\mathrm{p}}(G)$, $T \neq 0$, whenever G is discrete or amenable as discrete.

In this chapter, we consider analogues of the aforementioned results, and using the decomposition of $\mathrm{WAP_p}(G)$ (Section 6.4) we obtain, in case G is a discrete group, a criterion for the non-emptiness of the spectrum of non-zero elements in $\mathrm{WAP_p}(G)$ in terms of topologically invariant means on $\mathrm{PM_p}(G)$. As a consequence, we show that the norm and weak* spectrum coincide for non-zero weakly almost periodic functionals, and this, in turn, characterize the discreteness of G.

Throughout this section, let G be a discrete group, and for any $x \in G$, let Ψ_x be the unique topologically invariant x-mean on $\mathrm{PM_p}(G)$, that is, $\Psi_x \in \{\Psi \in \mathrm{PM_p}(G)^* : ||\Psi|| = 1 = \langle \Psi, \lambda_p(x) \rangle\}$ and $\langle \Psi_x, u \cdot T \rangle = u(x)\langle \Psi_x, T\rangle$ for all $u \in \mathrm{A_p}(G)$, $T \in \mathrm{PM_p}(G)$. Since G is discrete, $\Psi_x(T) = \langle T, \delta_x \rangle$ for all $T \in \mathrm{PM_p}(G)$, where $\delta_x \in \mathrm{A_p}(G)$ is the point mass at x (see [88, Theorem 1.] for the case $p = 2$; the proof is similar for the general case).
The next theorem may be compared with [51, Lemma 3.14.(a)].

Theorem 7.1. *Let G be a discrete group, and $x \in G$ be arbitrary. The following assertions hold:*

(a) For any $T \in \mathrm{PM_p}(G)$, $T \neq 0$, $x \in \sigma_*(T)$ if and only if $\langle \Psi_x, T \rangle \neq 0$.

(b) For any $T \in \mathrm{WAP_p}(G)$, $T \neq 0$, $x \in \sigma(T)$ if and only if $\langle \Psi_x, T\rangle \neq 0$ if and only if $T = c\lambda_p(x) + T_0$ for some $c \in \mathbb{C}$, $c \neq 0$, and $T_0 \in \{S \in \mathrm{WAP_p}(G) : \Psi_x(S) = 0\}$.

(c) For any $T \in \mathrm{AP_p}(G)$, $T \neq 0$, $x \in \sigma(T)$ if and only if $\langle \Psi_x, T\rangle \neq 0$ if and only if $T = c\lambda_p(x) + T_0$ for some $c \in \mathbb{C}$, $c \neq 0$, and $T_0 \in \{S \in \mathrm{AP_p}(G) : \Psi_x(S) = 0\}$.

Proof. (a) Let $T \in \mathrm{PM_p}(G)$, $T \neq 0$. First assume that $x \in \sigma_*(T)$. Then, by definition of $\sigma_*(T)$, there is a net (f_α) in $\mathrm{A_p}(G)$ such that $f_\alpha \cdot T$ converges to $\lambda_p(x)$ in the weak* topology of $\mathrm{PM_p}(G)$. In particular,

$$\langle f_\alpha \cdot T, \delta_x \rangle \;=\; f_\alpha(x)\langle \Psi_x, T\rangle \;\to\; \langle \lambda_p(x), \delta_x\rangle \;=\; 1,$$

hence $\langle \Psi_x, T\rangle \neq 0$.
On the other hand, we assume that $\langle \Psi_x, T \rangle \neq 0$, and we let (V_α) be a basis for the neighborhood system of x, directed by inclusion. Proceeding as in [48, p. 100], we choose $K = \{x\}$ and $F_\alpha = G \setminus V_\alpha$, and we let U_α be a symmetric compact neighborhood of the identity e in G such that $KU_\alpha^2 \cap F_\alpha = \varnothing$. Then we define $u_\alpha = \frac{1}{m(U_\alpha)} 1_{xU_\alpha} * 1_{U_\alpha}^{\check{}}$, so that

$u_\alpha \in \{u \in \mathrm{A_p}(G) : ||u||_{\mathrm{A_p}} = 1 = u(x)\}$ and $u_\alpha(z) = 0$ for all $z \in F_\alpha$. Moreover, since $\bigcap_\alpha V_\alpha = \{x\}$, it follows that $supp(u_\alpha) \downarrow \{x\}$, hence $u_\alpha(z) \to 0$ pointwise for all $z \in G$, $z \neq x$. Now, with an argument similar to the proof of Lemma 4.1.2, we conclude that for any $\varphi \in \mathrm{C_{\delta,p}}(G)$, $\langle u_\alpha, \varphi\rangle \to \langle \Psi_x, \varphi\rangle$, where Ψ_x is the unique topologically invariant x-mean on $\mathrm{PM_p}(G)$. Since $\mathrm{C_{\delta,p}}(G) = \overline{\mathrm{A_p}(G) \cdot \mathrm{PM_p}(G)}$ [43, Proposition 15.(a)], we have for all $u \in \mathrm{A_p}(G)$,

$$\langle u_\alpha \cdot T, u\rangle = \langle u_\alpha, u \cdot T\rangle \to \langle \Psi_x, u \cdot T\rangle = \langle\langle \Psi_x, T\rangle \lambda_p(x), u\rangle,$$

that is, $u_\alpha \cdot T$ converges to $\langle \Psi_x, T\rangle\lambda_p(x)$ in the weak* topology of $\mathrm{PM_p}(G)$. Since $\langle \Psi_x, T\rangle \neq 0$, we may set $v_\alpha = \frac{1}{\langle \Psi_x, T\rangle} u_\alpha$. Then $v_\alpha \cdot T \xrightarrow{w^*} \lambda_p(x)$, hence $x \in \sigma_*(T)$.

(b) Let $T \in \mathrm{WAP_p}(G)$, $T \neq 0$. If $x \in \sigma(T)$, then $x \in \sigma_*(T)$, and $\langle \Psi_x, T\rangle \neq 0$ by (a). Conversely, if $\langle \Psi_x, T\rangle \neq 0$, then similarly as in (a), there exists a net (u_α) in $\{u \in \mathrm{A_p}(G) : ||u||_{\mathrm{A_p}} = 1 = u(x)\}$ such that $u_\alpha \cdot T \xrightarrow{w^*} \langle \Psi_x, T\rangle\lambda_p(x)$. Since T is a weakly almost periodic functional and the net (u_α) is bounded, we conclude that the convergence is in the weak topology, hence

$$\lambda_p(x) \in \overline{\{u \cdot T : u \in \mathrm{A_p}(G)\}}^{w} = \overline{\{u \cdot T : u \in \mathrm{A_p}(G)\}}^{||\cdot||},$$

where the last equality holds by Mazur's Theorem.

The last assertion is a consequence of Theorem 6.4.1.

(c) The proof is identical to (b).

■

The next corollary is an immediate consequence of Theorem 7.1 and [56, Lemma 3.1.(a)]. It generalizes [51, Corollary 3.15.(c)].

Corollary 7.2. *Let G be a discrete group. Then*

$$\sigma(T) = \sigma_*(T) \neq \varnothing \quad \textit{for all } T \in \mathrm{WAP_p}(G),\, T \neq 0.$$

In combination with results of A. Ülger [103] and B. Forrest [36], the next corollary is an application of Corollary 7.2 and of [51, Corollary 3.15.(b)]. It is actually an improvement of [51, Corollary 3.15.(b)] since we replace $C_{\delta,p}(G)$ by $WAP_p(G)$. Below, the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) were proved in [103, Theorem 3.6], whereas the equivalence (c) $\Leftrightarrow$ (d) was proved in [36, Lemma 3.3]. For the case $p = 2$, (c) $\Leftrightarrow$ (d) is due to A. T.-M. Lau [66, Theorem 3.7]. The properties (e) and (f) are new.

Corollary 7.3. *Let G be a locally compact group. The following assertions are equivalent:*

(a) For each $u \in A_p(G)$, the (left) multiplication operator τ_u, defined by $\tau_u(v) = uv$ for $v \in A_p(G)$, is compact.

(b) For each $u \in A_p(G)$, the multiplication operator τ_u is weakly compact.

(c) $A_p(G)$ is an ideal in $PM_p(G)^*$.

(d) G is discrete.

(e) For all $T \in AP_p(G)$, $T \neq 0$, $\sigma(T) = \sigma_*(T)$.

(f) For all $T \in WAP_p(G)$, $T \neq 0$, $\sigma(T) = \sigma_*(T)$.

Chapter 8

Open problems

1. Is it possible to find an $\mathrm{L^p}$-version of the results of Chapter 3? More precisely, could we find a description of the representations of $\mathrm{A_p}(G)$ as bounded operators on a $\mathrm{QSL_p}$-space? Where a $\mathrm{QSL_p}$-*space* is a Banach space that is isometrically isomorphic to a quotient of a subspace of an $\mathrm{L^p}$-space.

2. Given any φ-amenable Banach algebra, is it possible to introduce a notion of ergodic sequence? Could we find a characterization of such sequences similar to that of Theorem 4.1.3?

3. In order to solve the complete mixing problem for the measure algebra of a σ-compact locally compact group, could we adapt our approach used to solve the dual version of this problem (see Chapter 5)?

4. Do any of the results of Chapter 6 yield some interesting properties of (non-commutative) left amenable F-algebras? For instance, given a left amenable F-algebra (A, M) and a left Banach A-submodule X of $\mathrm{wap}(A)$ containing the identity $1 \in M$, is it possible to establish the uniqueness of a projection P from X onto $\mathbb{C}1$ such that for each $x \in X$, $Px \in \overline{\{u \cdot x : u \in \mathscr{S}_A\}}^w$, where $\mathscr{S}_A$ denotes the set of all normal states on M (see Corollary 6.3.7)? Furthermore, is it possible to find a direct sum decomposition for X similar to that of Theorem 6.4.1?

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