University of Alberta

Simple Components of Group Algebras

by

Allen William Herman ©

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Canada
T0J 0H0

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Dr. H. L. Abbott

Dr. M. Shirvani (Supervisor)

Dr. S. K. Sehgal

Dr. G. Cliff

Dr. L. K. Stewart

Prof. Dr. H. Opolka (TU Braunschweig)

Date: Aug. 18, 1995

ABSTRACT

The isomorphism types of simple components of group algebras of finite groups over fields of characteristic zero are investigated. The main objective of interest is to determine the isomorphism type of the division algebra part of the simple component, or at least the Schur index, given a knowledge of the group and the field up to isomorphism.

After a variety of background in Chapter One, a constructive approach to Clifford Theory is given in Chapter Two. Specific restrictions on the structure of groups that are terminal objects with respect to these reductions are given. It is shown that these terminal objects must be metabelian in the case where the original group is either supersolvable or nilpotent.

In Chapter Three, after restricting to algebraic number fields, it is shown that precise sections of the group determines the division algebra part of the simple component of a group algebra after Clifford Theory reductions, as long as the original group satisfies a certain character-theoretic condition. These sections are given precisely, one for each prime number that divides the Schur index. This is a constructive Brauer-Witt Theorem for groups that Clifford reduce to ones satisfying this condition. This result is then extended to show that Schur indices of division algebras occurring in simple components of the group algebras of nilpotent-by-abelian groups are determined by the same sections as those satisfying the condition mentioned above.

In Chapter Four, a constructive algorithm is given for computing the Schur index of a simple component of the group algebra of any group occurring as a result of the reductions of Chapter Three. Examples are given to acquaint the reader with the type of computations involved.

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LIST OF SYMBOLS

$\mathbb{C}$	the field of complex numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Q}_p$	the field of p-adic numbers, p some prime integer
$\mathbb{Z}$	the ring of integers
$\mathbb{N}$	the natural numbers (integers ≥ 1)
$\mathbb{Z}_p$	the ring of p-adic integers, p some prime integer
$\mathbb{F}_p$	the finite field of prime order p
$\mathbb{F}, \mathbb{K}$	notation for general fields
$Gal(\mathbb{F}/\mathbb{K})$	galois group of the galois extension $\mathbb{F}/\mathbb{K}$
G, H	notation for general finite groups
C_n	cyclic group of finite order n
$N \times H$	direct product of the groups N and H
$N \rtimes H$	split extension of the group N by the group H
$Z(G)$	the center of the group G
G'	the commutator subgroup (or derived subgroup) of G
$\Phi(G)$	the Frattini subgroup of G
$F(G)$	the Fitting subgroup of G
$C_G(x)$	the centralizer in G of the element x (x is either a group or a ring element)
$exp(G)$	the exponent of the group G
$Aut(G)$	the group of automorphisms of the group G
$Inn(G)$	the group of inner automorphisms of the group G
$Out(G)$	the group of outer automorphisms of the group G
ζ_n	primitive complex root of unity of order n
$\langle a \rangle$	cyclic group generated by a
$\mathbb{F}G$	ordinary group algebra of the group G over the field $\mathbb{F}$
V	notation for a general vector space
$End_{\mathbb{F}}(V)$	$\mathbb{F}$-linear endomorphisms on the vector space V
$End_A(V)$	endomorphisms of V that commute with the subalgebra A
$A \otimes_{\mathbb{F}} B$	tensor product of the $\mathbb{F}$-algebras A and B
$Z(A)$	center of the algebra A
D	general division algebra
$D^{n \times n}$	ring of $n \times n$ matrices over the division algebra D
$(\mathbb{F}/\mathbb{K}, \beta)$	crossed product algebra defined by the factor set β
$(\mathbb{F}/\mathbb{K}, \sigma, \epsilon)$	cyclic algebra defined by the galois group $\langle \sigma \rangle$ and norm element ϵ
$Br(\mathbb{F})$	the Brauer group of the field $\mathbb{F}$

$exp(A)$	the exponent of the central simple algebra A
$m(A)$	the Schur index of the algebra A
χ	notation for a general complex-valued character of a finite group
$Irr(G)$	set of characters afforded by irreducible representations of the finite group G
χ_H	restriction of the character χ to the subgroup H
χ^G	the induction of the character χ to G
$\mathbb{Z}[Irr(G)]$	the integral ring of generalized characters
(χ, ϕ)	the inner product on $\mathbb{Z}[Irr(G)]$

Introduction

Occasionally in the field of algebra one is able to find new ways to view past problems. Often this leads to new studies or even new areas of research by giving new proofs of old theorems through new perspectives.

The intention of this work is to look at a classical existence theorem, the Brauer-Witt theorem, in the study of representations of finite groups over the rational numbers, and try to find an explicit means of reaching the conclusion of that theorem. This amounts to presenting a constructive algorithm for reducing a simple component of the group algebra of a finite group over the rational numbers to its division algebra part. This thesis is an attempt to improve upon existing methods of doing this. The hope is that a new understanding of how group structure generates non-trivial division algebra parts in the simple components of rational group algebras will result, thus leading to an improved understanding of the structure of these algebras, allowing for progress on unsolved problems in the study of group algebras.

The study of abstract groups originated with Galois in 1824, whose ideas made possible one of the first proofs that the roots of general quintic equations cannot be found by algebraic means. The mathematicians of the late 1800's recognized the applicability of the theory of groups to other areas mathematics, a movement which attracted quite a bit of attention. At the turn of the century, Frobenius published the first study of representations of finite groups on vector spaces, the subject which we are still interested in almost 100 years later. Frobenius' study was of representations of finite groups over the field of complex numbers, a study which his student, Schur, extended to study representations of finite groups over subfields of the complex numbers, especially the rational numbers. One important observation of Schur was that certain irreducible representations over the complex numbers could not be realized over the rational numbers, but that a finite multiple of the irreducible representation could always be realized. The number associated with this multiple representation came to be called the *Schur index*. In the early 1900's, the appearance of the Artin-Wedderburn theorem showed that the questions of rational representations of a finite groups G could be thought of as questions of the algebraic structure of the rational group algebra $\mathbb{Q}G$. The simple two-sided ideals of $\mathbb{Q}G$, which we will call the *simple components* were known to be matrix rings over division algebras that were finite-dimensional over $\mathbb{Q}$, with the square root of this dimension being exactly the Schur index of the associated irreducible representation over the rationals. In that time, Schur indices could not be computed in general. However, Brauer's characterization of the character rings of finite groups led, in the early 1950's, to a reduction theorem, the Brauer-Witt theorem, that demonstrated the existence of a subgroup of the finite group G of a certain type that could be used to find the Schur index associated to a simple component of $\mathbb{Q}G$. An algorithm for computing Schur indices associated to groups of this type was achieved by Yamada in the late 1960's using the theory of crossed product algebras over local fields and several deep theorems of algebraic number theory. This has remained the state of methods for computing the Schur index associated to an irreducible rational representation of a finite group up to now, except for certain improvements to restrictions on the Schur index imposed by group structure that have continued to appear in the literature.

This work began with a study of my supervisor's attempts to find more explicit structural information on the structure of simple components of the group algebras of finite metabelian groups over arbitrary fields of characteristic zero [Sh]. The motivation for this was to determine how group structure is arranged in a simple component of a group algebra over an arbitrary field. This was achieved by finding certain characteristic subgroups of the finite metabelian group G that determine the center and the matrix degree to a large extent in the case where the group is faithfully represented in the simple component. This study led to the realization that an explicit version of the Brauer-Witt theorem was available for finite metabelian groups [H]. The main problem for this thesis is to find an explicit version of the Brauer-Witt theorem for a larger class of groups, thus giving improvements to methods for computing Schur indices associated to the simple components of rational group algebras, and to the study of the algebraic structure of these simple components.

In order to explain our main results, fix $\mathbb{Q}Ga_\chi$ to mean a simple component of the rational group algebra $\mathbb{Q}G$. Let D be a division algebra, finite-dimensional over Q, such that $\mathbb{Q}Ga_\chi$ is isomorphic to a matrix ring over D. The problem of finding an explicit means of computing the Schur index of D will be attacked by looking for a a precise subgroup H of the group G, depending on a prime p dividing the order of G, that can be used to directly compute largest p power divisor of the Schur index of D. This differs from the Brauer Witt theorem in that the choice of H will be precisely stated, and that H can be found quite easily using our knowledge of the multiplicative structure of G. The classical Brauer-Witt theorem only indicates the existence of subgroups with the properties of this H, but does little that leads to a procedure for finding such a subgroup starting with a general finite group G.

The first step in this (Chapter 2) is to reduce $\mathbb{Q}Ga_\chi$ as much as possible using Clifford theory. This means that we can assume that any subalgebra $\mathbb{Q}Na_\chi$ of $\mathbb{Q}a_\chi$ generated by a normal subgroup N of G is simple. Because this condition is (possibly) stronger than the condition that every normal abelian subgroup of G is cyclic, we use the known structure of groups that have every normal abelian subgroup cyclic as the starting point for our investigation. The structure of such a group G is determined to a large extent by the structure of its characteristic subgroups $F = C_{F(G)}(\Phi(F(G)))$, $U = Z(F)$, and $C = C_G(U)$, where $F(G)$ denotes the Fitting subgroup of G, the maximal normal nilpotent subgroup of G. In particular, U is a maximal normal abelian subgroup of G (hence cyclic), F/U is elementary abelian, G/C is abelian, and C/F is isomorphic to a subgroup of the automorphism group of F/U that preserves the commutator relations of F. (Thus C/F is the only section of G that is not under control.)

Under the assumption that $\mathbb{Q}Ga_\chi$ cannot be reduced further using Clifford theory (and hence we have the subgroups F, U, and C as indicated above), we attain our goal of finding a constructive version of the Brauer-Witt theorem in case $\mathbb{Q}Fa_\chi = \mathbb{Q}Ca_\chi$. (Theorem 3.4 (iii) and Corollary 3.4.1)

Main Result. *Suppose $\mathbb{Q}Ga_\chi$ is a simple component of $\mathbb{Q}G$ such that $Ga_\chi \simeq G$. Assume that every subalgebra of $\mathbb{Q}Ga_\chi$ of the form $\mathbb{Q}Na_\chi$ for $N \lhd G$ is simple. Let $F = C_{F(G)}(\Phi(F(G)))$, $U = Z(F)$, and $C = C_G(U)$, where $F(G)$ denotes the Fitting subgroup of G. Let $\Bbbk$ be a field isomorphic to the center of $\mathbb{Q}Ga_\chi$, and let p be a prime integer dividing the order of G*

Then whenever $\mathbb{Q}Fa_\chi = \mathbb{Q}Ca_\chi$, the p-part of the Schur index of $\mathbb{Q}Ga_\chi$ is equal to the Schur index of a simple component of $\Bbbk H$, where $H = UP$, for an arbitrary Sylow p-subgroup P of G.

As an application of the main result, we prove (pages 45 49) that the reduction given applies to finite nilpotent-by-abelian groups once certain units of finite order have been added to the group.

To complete the demonstration of how to compute the Schur index of the simple component $\mathbb{Q}Ga_\chi$, an algorithm is presented in Chapter 4 that demonstrates how to compute the index in the case of a group that is the extension of a cyclic group by a p-group, for a fixed prime integer p. (This is exactly the structure of the group H of the main theorem.) The algorithm is based on the computation of local indices at all primes of the center, and is valid only in the case where the center of the algebra is an algebraic number field. Such algorithms have been known to exist for several years–however, the algorithm that we present is accessible to a wide audience because it does not use homological algebra and a minimal background to algebraic number theory.

The steps in the algorithm are to first reduce the algebra $\mathbb{Q}Ga_\chi$ by localizing at a prime q. This means we will compute the Schur index of the algebra $\mathbb{Q}_q Ga_\chi$. We again carry out all Clifford theory reductions, which means we may assume the group G has a cyclic maximal abelian normal subgroup C, with G/C isomorphic to the Galois group of a cyclotomic extension of the center $\Bbbk_q$ of $\mathbb{Q}_q Ga_\chi$. In the case that q is a real prime, there is a fairly straightforward to determine whether the simple component is a matrix algebra over the real quaternions, an argument that is only necessary when $p = 2$. Under the assumption that q is an odd prime not equal to p, we can reduce to the case where G is the semi-direct product of a cyclic q-group with a cyclic p-group. If the order of the cyclic p-group is p^d, then we give a formula (Theorem 4.4) for the Schur index that depends only on the number of p-th power roots of unity of the center of $\mathbb{Q}_q Ga_\chi$.

Schur Index formula (tame case). *Suppose $G \cong C_{q^c} \rtimes C_{p^d}$. Then a simple component of $\mathbb{Q}_q Ga_\chi$ of $\mathbb{Q}G$ that satisfies $Ga_\chi \cong G$ has Schur index p^m, where $m = \max\{d - \ell, 0\}$, where ℓ is the highest p-th power of a root of unity in the center of $\mathbb{Q}Ga_\chi$.*

In the case $q = 2$ we may also assume $p = 2$. The Schur index in this case is at most two. The computation in this case is too complicated to describe at this stage, and so we refer the interested reader to Theorem 4.5.

Chapter 1: Simple Components of Group Algebras

This chapter is an overview of the background information that is needed for an understanding of the following chapters. It is assumed that the reader is already familiar with the basic theory of groups, rings, and modules, especially the notions of subgroup, normal subgroup, ideal, field, galois extensions of fields, vector spaces, and an algebra over a field. The reader is also assumed to be familiar with the terms homomorphism, isomorphism, and automorphism, in either a group-theoretic, ring-theoretic, or module-theoretic setting. Any of this basic terminology is available in any contemporary introductory text for abstract algebra. The main references used for this chapter will be Curtis and Reiner [CR], Reiner [R], and Isaacs [I].

§1. Representation Theory of Finite Groups

In practical applications, groups are usually identified with interactive sets of symmetries on an object. In physical applications, this object is usually geometric, like an icosahedron or a cube, whereas in mathematical applications the object is usually a vector space. Usually there are mathematical models available that can alter the interpretation of a physical application so that we can interpret the action of the group as being a mathematical group acting on a vector space. We will develop the theory from this point of view.

Let $\mathbb{F}$ be an arbitrary field, G be a finite group. We will think of the group operation in G as being a multiplication. By a *representation of G over $\mathbb{F}$* we will mean a homomorphism of multiplicative groups $\mathcal{X} : G \longrightarrow Aut_{\mathbb{F}}(V)$, the $\mathbb{F}$-automorphisms of V, where V is a finite-dimensional vector space over $\mathbb{F}$. In this case we say that *G acts on V*, or that *V is a G-module*. The dimension of V over $\mathbb{F}$ is referred to as the *degree* of the representation.

Let

$$\mathbb{F}G = \{\sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{F}\}$$

be the natural vector space over $\mathbb{F}$ that has G as a free basis. There is a natural multiplication on the space $\mathbb{F}G$ that extends the multiplication on the group and on the field, being given by

$$\begin{aligned}(\sum_{g\in G}\alpha_g g)(\sum_{h\in G}\beta_h h) &= \sum_{g\in G}\sum_{h\in G}\alpha_g\beta_h gh \\ &= \sum_{x\in G}(\sum_{g\in G}\alpha_g\beta_{g^{-1}x})x.\end{aligned}$$

This multiplication makes $\mathbb{F}G$ into a ring, and thus into an algebra over $\mathbb{F}$, called the group algebra of G over $\mathbb{F}$. The central copy of $\mathbb{F}$ in $\mathbb{F}G$ is $\mathbb{F}\cdot 1$, where 1 is the group identity in G.

Any representation $\mathcal{X}$ of G extends naturally to an algebra homomorphism

$$\mathcal{X} : \mathbb{F}G \longrightarrow End_{\mathbb{F}}(V).$$

As before, we will refer to V as an *$\mathbb{F}G$-module* in this case. (If we are to interpret the endomorphisms of V as being written on the left of elements of V, then we can avoid any confusion by calling V a left $\mathbb{F}G$-module. There is no significant difference in the theory for left modules and the theory for right modules, but once we have fixed the convention to be used, we must stick to it.) Existence of this extension homomorphism means that any representation of a group G over $\mathbb{F}$ gives rise to an algebra representation of the group algebra $\mathbb{F}G$. By restricting $\mathcal{X}$ to G, the reverse of this statement is easily seen to hold also. This means that we can investigate the representations of the group G over F by examining the representations of $\mathbb{F}G$.

In order to understand a representation of $\mathbb{F}G$, it is enough to understand its "smallest" parts. We call an $\mathbb{F}G$-module V *simple* if the only $\mathbb{F}G$-submodules of V are 0 and V. In this case, the representation of G associated to V is called *irreducible*. The endomorphism algebra $End_{\mathbb{F}}(V)$ of a simple $\mathbb{F}G$-module V has a subalgebra of an interesting type.

Schur's Lemma [CR, 3.17] *Every element of $End_{\mathbb{F}G}(V)$ is either 0 or is invertible.*

A ring that has every non-zero element invertible is called a *division ring*. When the division ring is also an algebra over a central subfield $\mathbb{F}$, it is called a *division algebra* over $\mathbb{F}$.

For many group algebras, every $\mathbb{F}G$-module is *semisimple*, that is, it can be written as a direct sum of simple $\mathbb{F}G$-modules.

Maschke's Theorem [CR, 3.14] *Every left $\mathbb{F}G$-module is semisimple, as long as the order of the finite group G is relatively prime to p whenever the field $\mathbb{F}$ has positive characteristic p.*

The study of representations of group algebras satisfying the conditions of Maschke's Theorem is called ordinary representation theory. The other situation, when the characteristic of $\mathbb{F}$ is positive and divides the order of G, is referred to as modular representation theory. We will be interested only in the case where the characteristic of the field $\mathbb{F}$ is zero, because that is the only case where non-trivial division algebras can occur in simple components of the group algebras of finite groups [I, Theorem 9.21(b)].

When the algebra $\mathbb{F}G$ is semisimple, then it can be decomposed into the direct sum of two-sided ideals by the Artin-Wedderburn Theorem.

Artin-Wedderburn Theorem. [R, 7.1 and 7.4] *Suppose that A is a finite-dimensional semisimple $\mathbb{F}$-algebra over a field $\mathbb{F}$. Then*

$$A = A_1 \oplus \cdots \oplus A_n,$$

where the A_i are the unique minimal two-sided ideals of A. Each A_i is equal to Ae_i, for a central idempotent e_i of A. Each A_i is isomorphic to a ring of matrices over a division algebra D over $\mathbb{F}$. This D is isomorphic to the opposite ring for $End_A(L)$, where L is a minimal left ideal of A that is contained in A_i.

The set $\{e_1, \ldots, e_n\}$ of central idempotents indicated in the above theorem is obtained by writing

$$1 = e_1 + \cdots + e_n,$$

for elements $e_i \in A_i$. These elements e_i will satisfy $e_i e_j = 0$ for $i \neq j$, and $e_i{}^2 = e_i$, for all i, because $e_i \sum_j e_j = e_i$ and $e_i e_j \in A_i \cap A_j$. A maximal set of central idempotents satisfying these conditions and summing to 1 is called a *full set of centrally primitive orthogonal idempotents of A.*

We call the minimal two-sided ideals of a semisimple algebra A the simple components of A. The goal of this work is to obtain results that relate the structure of G to the structure of a simple component $\mathbb{F}Ge$ of $\mathbb{F}G$, where e is a centrally primitive idempotent of $\mathbb{F}G$, for a given field $\mathbb{F}$ of characteristic zero. In order to proceed, we will first need to examine some of what is known about matrix rings over division algebras that are finite-dimensional algebras over their centers.

§2. Central Simple Algebras and the Schur index

For any field $\mathbb{F}$, a finite dimensional central simple algebra over $\mathbb{F}$ is a matrix ring over a division algebra with center precisely $\mathbb{F}$. By the Artin-Wedderburn Theorem, we know that in characteristic zero every simple component of $\mathbb{F}G$ is a central simple algebra over its center. The dimension of a division algebra over its center is an important structural invariant.

Theorem. [R, 7.15] *Suppose D is a finite dimensional division algebra over a field $\mathbb{F}$ with center $\mathbb{K}$. Then every maximal subfield $\mathbb{E}$ of D contains $\mathbb{K}$ and*

$$\mathbb{E} \otimes_{\mathbb{K}} D \cong \mathbb{E}^{m \times m},$$

where $m = |\mathbb{E} : \mathbb{K}|$. The dimension of $\mathbb{E}$ over $\mathbb{K}$ is the same integer m for every maximal subfield $\mathbb{E}$ of D, and $|D : \mathbb{K}| = m^2$.

This integer m is called the Schur index of the division algebra D. Although it does not determine the division algebra up to isomorphism, it "almost" does in the case of simple components of group algebras of finite groups over local fields. (We will outline this in the next section.)

A maximal subfield of the division algebra D is an example of a *splitting field*. A finite extension $\mathbb{E}$ of the center $\mathbb{K}$ of D is called a splitting field for D if the central simple $\mathbb{E}$-algebra $\mathbb{E} \otimes_{\mathbb{K}} D \cong \mathbb{E}^{m \times m}$. Thus the non-commutative division algebra part of D vanishes under tensoring with $\mathbb{E}$. (This is usually referred to as a *splitting* of D.) It is known that an extension of $\mathbb{K}$ is a splitting field exactly when it is isomorphic to a maximal subfield of some matrix ring over D, and that the Schur index always divides the dimension of a splitting field over $\mathbb{K}$ [R,28.5].

The *Brauer Group* is an algebraic object that classifies central simple algebras up to the isomorphism classes of their division algebra parts. The elements of the Brauer Group $Br(\mathbb{K})$ of a field $\mathbb{K}$ are equivalence classes in which all finite dimensional central simple algebras over $\mathbb{K}$ with isomorphic division algebra parts are identified [R, 28.2]. $Br(\mathbb{K})$ is an abelian group, with group operation

$$[A][B] = [A \otimes_{\mathbb{K}} B],$$

in which $[A]$ and $[B]$ are the equivalence classes determined by the central simple $\mathbb{K}$-algebras A and B, respectively. The identity element of $Br(\mathbb{K})$ is the class $[\mathbb{K}]$. Each element of $Br(\mathbb{K})$ has finite order which divides the Schur index of any central simple algebra in its class [R, 29.22]. The order of $[A] \in Br(\mathbb{K})$ is called the *exponent of* A and is denoted by $exp(A)$. $exp(A)$ is a divisor of the Schur index of A in general.

The subset of $Br(\mathbb{K})$ consisting of classes containing an algebra that occurs as a simple component of $\mathbb{K}G$ for some finite group G forms a subgroup of $Br(\mathbb{K})$ called the *Schur subgroup of* $\mathbb{K}$. For arbitrary fields of characteristic zero, the Schur subgroup has not received much attention in the literature. This is because representations of finite groups are always realizable in fields that possess enough roots of unity, and so the non-trivial division algebras that can occur will always be closely related to ones whose center is a finite extension of the rational numbers $\mathbb{Q}$. In the case where $\mathbb{K}$ is an *algebraic number field*, that is, a finite extension field of $\mathbb{Q}$, the Schur subgroup has been shown to be the subgroup of $Br(\mathbb{K})$ whose elements are classes containing cyclotomic crossed product algebras [Y, Corollary 3.10].

A *crossed product algebra* is an algebra that can be constructed as follows: Let $\mathbb{E}$ be a finite galois extension of a field $\mathbb{K}$, with galois group $\mathcal{G}$. Construct a vector space over $\mathbb{E}$

$$(\mathbb{E}/\mathbb{K}, f) = \bigoplus_{\sigma\in\mathcal{G}} \mathbb{E}u_\sigma$$

that is indexed by representatives u_σ of the elements of $\mathcal{G}$. Define a multiplication on this space by setting

$$u_\sigma \alpha = \alpha^\sigma u_\sigma, \text{ for all } \alpha \in \mathbb{E},$$

and letting

$$u_\sigma u_\tau = f(\sigma, \tau) u_{\sigma\tau},$$

for all $\sigma, \tau \in \mathcal{G}$, for some $f(\sigma, \tau) \in \mathbb{E}^\times$. In order for this multiplication to be associative we need the map

$$f : \mathcal{G} \times \mathcal{G} \to \mathbb{E}^\times$$

to be a *cocycle* [R, 29.2]. When f is a cocycle, it is known that $(\mathbb{E}/\mathbb{K}, f)$ is a central simple $\mathbb{K}$-algebra [R, 29.6]. We call a crossed product algebra $(\mathbb{E}/\mathbb{K}, f)$ *cyclotomic* when $\mathbb{E}$ is contained in a cyclotomic extension of $\mathbb{K}$ and the values of the cocycle f are all roots of unity in $\mathbb{E}$.

A special type of crossed product algebra is one in which $Gal(\mathbb{E}/\mathbb{K})$ is a cyclic group, in which case we call $(\mathbb{E}/\mathbb{K}, f)$ a *cyclic algebra.* A cyclic algebra can be written as

$$(\mathbb{E}/\mathbb{K}, f) = \bigoplus_{i=0}^{n-1} \mathbb{E}u_{\sigma^i}$$

whenever $Gal(\mathbb{E}/\mathbb{K}) = \langle\sigma\rangle$ has order n. This cyclic algebra is always isomorphic to one in which the values of f are completely determined by $(u_\sigma)^n = \zeta$ in $\mathbb{K}$ [R, 30.3]. Thus it is customary to use the notation $(\mathbb{E}/\mathbb{K}, \sigma, \zeta)$ to denote a cyclic algebra of the above form. Of course, a cyclic algebra is a cyclotomic algebra exactly when it is possible for ζ to be a root of unity.

Cyclic algebras are important to us because it is possible to retrieve some information concerning their division algebra part directly from the element ζ.

Theorem [R, 30.4(iii) and 30.7]. *Suppose that $A = (\mathbb{E}/\mathbb{K}, \sigma, \zeta)$ is a cyclic algebra. Then $exp(A)$ is the least positive integer e such that ζ^e lies in $N_{\mathbb{E}/\mathbb{K}}(\mathbb{E}^\times)$.*

($N_{\mathbb{E}/\mathbb{K}}(\mathbb{E}^\times)$ is the set of norms from $\mathbb{E}$ to $\mathbb{K}$, i.e. the elements $\alpha \in \mathbb{K}$ that are of the form

$$\alpha = \prod_{i=0}^{n} \beta^{\sigma^i}, \text{ for some } \beta \in \mathbb{E}^\times.)$$

This theorem gives a lower bound for Schur indices of cyclic algebras in general. In the case where $\mathbb{K}$ is an algebraic number field, it is a fact that the values of the exponent and Schur index coincide. This fact is a consequence of the localization results presented in the next section. When $\mathbb{K}$ is an arbitrary field of characteristic zero, then it is known that the exponent and the Schur index can be vastly different [N], and so the above result is not always effective for calculating Schur indices.

§3 Localization.

In order to make sense of the equality of the exponent and the Schur index for algebraic number fields, we have to develop the theory of *local number fields.*

Let p be a prime integer. By writing each integer $z \in \mathbb{Z}$ as $z = p^k n$, for some $n \in \mathbb{Z}$ that is relatively prime to p, we can define a *p-adic absolute value* $|.|_p$ on $\mathbb{Z}$ by setting

$$|z|_p = p^{-k}$$

for $z \neq 0$ and $|0|_p = 0$. This absolute value makes $\mathbb{Z}$ into a compact topological space, with $|z|_p \leq 1$ for all $z \in \mathbb{Z}$. The completion of $\mathbb{Z}$ with respect to this topology is the ring of *p-adic integers* $\mathbb{Z}_p$. Non-zero elements of $\mathbb{Z}_p$ can be expressed as infinite sums

$$\sum_{i=0}^{\infty} \alpha_i p^i$$

with each $\alpha_i \in \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. The absolute value of such a sum is p^{-k} whenever k is the least integer for which $\alpha_k \neq 0$. $\mathbb{Z}_p$ is an integral domain, and its field of quotients is called the p-adic number field $\mathbb{Q}_p$. It is known that the completion of the algebraic closure of $\mathbb{Q}_p$ is a field that is algebraically isomorphic to the field of complex numbers $\mathbb{C}$. For this reason, it is sometimes customary to think of $\mathbb{Q}_p$ as being a subfield of $\mathbb{C}$, even though the topologies on the two are completely unrelated. Elements of $\mathbb{Q}_p$ can be written in the form

$$\sum_{i=n}^{\infty} \alpha_i p^i$$

for some $\alpha_i \in \mathbb{F}_p$ satisfying $\alpha_n \neq 0$, for some $n \in \mathbb{Z}$.

It is also possible to do this construction starting with an algebraic number field $\mathbb{F}$. The ring of integers $\mathcal{O}_\mathbb{F}$ in $\mathbb{F}$ is the set of all elements of $\mathbb{F}$ that are *integral* over $\mathbb{Z}$, that is, are roots of monic polynomials in $\mathbb{Z}[X]$. If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_\mathbb{F}$, then we can define an absolute value $|.|_\mathfrak{p}$ on $\mathcal{O}_\mathbb{F}$ by $|a|_\mathfrak{p}$ = the least power of the ideal $\mathfrak{p}$ that lies in the principal ideal $a\mathcal{O}_\mathbb{F}$ [R, page 52]. Completing with respect to this topology results in an integral domain, and so forming the field of fractions gives us a field, which we call $\mathbb{F}_\mathfrak{p}$. (Sometimes the prime ideal $\mathfrak{p}$ will be referred to as a *prime of* $\mathbb{F}$. This terminology also refers to inclusions of $\mathbb{F}$ into the field of complex numbers for algebraic number theory reasons.) We can view $\mathbb{F}$ in a natural way as a subfield of $\mathbb{F}_\mathfrak{p}$. Elements of $\mathbb{F}_\mathfrak{p}$ can all be written as infinite sums

$$\sum_{i=n}^{\infty} \alpha_i \pi^i,$$

with all $\alpha_i \in \mathcal{O}_\mathbb{F}/\mathfrak{p}$, $\alpha_n \neq 0$, for some $n \in \mathbb{Z}$, and π an element which generates the unique maximal ideal in the topological completion of $\mathcal{O}_\mathbb{F}$. We call such an element π a *uniformizer* of $\mathbb{F}_\mathfrak{p}$. If $\mathfrak{p}$ lies above the prime ideal $p\mathbb{Z}$, then $\mathcal{O}_\mathbb{F}/\mathfrak{p}$ is a finite extension of the finite field $\mathbb{F}_p$. We call $\mathcal{O}_\mathbb{F}/\mathfrak{p}$ the *residue class field* of $\mathbb{F}_\mathfrak{p}$. It is also always possible to choose the topology on $\mathbb{F}_\mathfrak{p}$ so that there is an integer e so that π^e generates the completion of $p\mathcal{O}_\mathbb{F}$, and so we can view the field $\mathbb{F}_\mathfrak{p}$ as being a finite extension of $\mathbb{Q}_p$. (It is also true that any finite extension of

$\mathbb{Q}_p$ is a field of the above type.) The degree of the extension $|\mathbb{F}_\mathfrak{p} : \mathbb{Q}_p|$ is the product of two parts, the ***residue degree*** $f = |\mathcal{O}_\mathbb{F}/\mathfrak{p} : \mathbb{F}_p|$ and the ***ramification index*** e, which is the least integer so that $|\pi^e|_\mathfrak{p} = |p|_\mathfrak{p}$. An extension of fields whose degree is precisely f is called ***unramified.*** An extension whose degree equals e is called ***totally ramified.*** A totally ramified extension of degree prime to p is called ***tamely ramified***, and if its degree is a power of p then it is called ***wildly ramified.***

By a *local field* we will mean either a finite extension of some p-adic number field $\mathbb{Q}_p$ (a *p-adic local field*) or the field of real numbers $\mathbb{R}$. (There are also some fields of prime characteristic that are commonly referred to as local fields, but these will not be needed here.) Most of the facts we need about p-adic local fields concern cyclotomic extensions of $\mathbb{Q}_p$, which we now list.

Theorem. *Let ζ_n be a fixed primitive complex n-th root of unity, for each integer $n \geq 1$.*

(i) $\mathbb{Q}_p(\zeta_{p^k})$ for $k \geq 1$ is totally ramified over $\mathbb{Q}_p$ of degree $p^{k-1}(p-1)$, with uniformizer $(1-\zeta_{p^k})$.

(ii) If n is relatively prime to p, then $\mathbb{Q}_p(\zeta_n)$ is unramified over $\mathbb{Q}_p$, and $\mathbb{Q}_p(\zeta_n) = \mathbb{Q}_p(\zeta_{p^f-1})$, where f is the order of n modulo p. (f is the residue class degree for this extension.)

If A is a simple algebra whose center is an algebraic number field $\mathbb{F}$, then we can also speak of *localizing A*. This will mean that we take a local field $\mathbb{F}_\mathfrak{p}$ obtained by localizing at a prime of $\mathbb{F}$ (possibly $\mathbb{F}_\mathfrak{p} = \mathbb{R}$ when $\mathbb{F}$ is a subfield of the reals), and form the tensor product

$$A_\mathfrak{p} = \mathbb{F}_\mathfrak{p} \otimes_\mathbb{F} A.$$

This new algebra can be viewed as a central simple algebra over $\mathbb{F}_\mathfrak{p}$. The Schur index $m(A_\mathfrak{p})$ is called the ***$\mathfrak{p}$-local index*** of A.

The advantage of localizing the algebra is that division algebras over $\mathbb{F}_\mathfrak{p}$ are much more restricted than those over $\mathbb{F}$. Each division algebra over a local field is determined by an interaction between two maximal subfields [We]. Each such division algebra contains maximal subfields that are unramified over the center $\mathbb{F}_\mathfrak{p}$. These maximal subfields will be isomorphic to $\mathbb{F}_\mathfrak{p}(\zeta_{p^f-1})$, for some root of unity ζ_{p^f-1} satisfying $|\mathbb{F}_\mathfrak{p}(\zeta_{p^f-1}) : \mathbb{F}_\mathfrak{p}| = m$, where m is the Schur index. There are also a maximal subfields of the division algebra that are totally ramified over $\mathbb{F}_\mathfrak{p}$. One of these totally ramified maximal subfields $\mathbb{F}_\mathfrak{p}(\pi)$ can be chosen so that its primitive element π is a uniformizer of $\mathbb{F}_{\mathfrak{p}}(\pi)$, and π normalizes the cyclic group $\langle \zeta_{p^f-1} \rangle$. In particular,

$$(\zeta_{p^f-1})^{\pi} = \zeta_{p^f-1}^{b},$$

for some integer b. The isomorphism class of the algebra is determined by the power of the Frobenius automorphism

$$\zeta_{p^f-1} \mapsto (\zeta_{p^f-1})^p$$

on the extension $\mathbb{F}_{\mathfrak{p}}(\zeta_{p^f-1})/\mathbb{F}_{\mathfrak{p}}$ that corresponds to conjugation by π. If this power is t, then the fraction $\frac{t}{m}$ is a lowest terms fraction lying in the interval $(0, 1]$. This fraction is called the *Hasse invariant* for the division algebra, and determines a division algebra over a local field up to isomorphism. (In fact, the isomorphism class of a division algebra over an algebraic number field is precisely determined by the collection of all the Hasse invariants over all possible primes of the center. This means that these are much more complicated algebraically than their localizations are.)

The information we get from the localized algebras has implications for the structure of the algebra A as given by the next result. This result is a consequence of the Hasse Norm Theorem, the Hasse-Brauer-Noether-Albert Theorem, and the Grunwald-Wang Theorem in algebraic number theory.

Theorem. [R, 32.17 and 32.19] *Suppose that A is a central simple algebra over an algebraic number field Then*

$$m(A) = exp(A) = L.C.M.\{m(A_{\mathfrak{p}})\},$$

where $\{A_{\mathfrak{p}}\}$ is the set of all possible localizations of the algebra A, including $\mathbb{R} \otimes_{\mathbb{F}} A$ if $\mathbb{F}$ is real.

In the case that A is a cyclotomic crossed product algebra over an algebraic number field, Benard [B] has shown that for any two primes $\mathfrak{p}_1$, $\mathfrak{p}_2$ of the central field $\mathbb{F}$ lying over the same rational prime p of $\mathbb{Q}$, the Schur indices of $A_{\mathfrak{p}_1}$ and $A_{\mathfrak{p}_2}$ will be the same. However, the Hasse invariants of the localized algebras can be different, and in fact when $\mathfrak{p}$ runs through all primes of the center lying over p, every fraction $\frac{t}{m}$ with $(t, m) = 1$ occurs equally often as a Hasse invariant of the algebra $A_{\mathfrak{p}}$ [BS, Corollary 1]. Thus the actual isomorphism class of a particular localization $\mathbb{F}_{\mathfrak{p}}Ge$ is determined only by the particular prime $\mathfrak{p}$ chosen to lie over p. This result indicates that if $\mathbb{F}$ is an algebraic number field and G

is a finite group, then determining the Schur index of a simple component $\mathbb{F}Ge$ also determines the set of isomorphism classes of the simple components of the localized algebras that occur over each rational prime. However, once the Schur index is known, the actual isomorphism class of a particular $\mathbb{F}Ge$ depends on a subtle interplay between the field arithmetic and the group operation. This is what we mean when we say that the Schur index "almost" determines the isomorphism class of the division algebra part of $\mathbb{F}Ge$.

The problem of classifying finite subgroups of division algebras has been completely solved [Am] (see also [SW]), and has been extended to a classification of finite subgroups of 2×2 matrices over a division algebra [Ban]. The groups that appear in this classification will be all those groups whose group algebra over some field has a simple component that is either a division algebra all by itself or a 2×2 matrix ring over a division algebra. These do not form a complete list of the division algebras that occur in simple components of group algebras over finite groups, however. For example, suppose p and q are odd primes such that $q-1$ is exactly divisible by p^n, for an integer $n \geq 2$. Define a group G by

$$G = C_q \rtimes C_{p^{n-1}(q-1)} = \langle a \rangle \rtimes \langle x \rangle$$

where x acts with order $q-1$ on $\langle a \rangle$. (See §5 for an explanation of this notation.) Then there is a simple component $\mathbb{Q}Ge$ of the group algebra $\mathbb{Q}G$ that is isomorphic to the cyclic algebra

$$(\mathbb{Q}(\zeta_{qp^{n-1}})/\mathbb{Q}(\zeta_{p^{n-1}}), x, x^{q-1} = \zeta_{p^{n-1}}).$$

(We will see in Chapter 4 how this is done.) The only primes of the center $\mathbb{Q}(\zeta_{p^{n-1}})$ for which the local index is non-trivial will be the ones lying over the prime q, and there norm calculations will yield an index of p^{n-1}. This means the matrix degree for this simple component is at least p. However, a theorem of Pendergrass [Pen, Corollary 2] indicates that the smallest matrix degree possible for a simple component of a group algebra of any finite group to have this particular division algebra part is $\frac{q-1}{p^{n-1}} \geq p$. Therefore, the simple component indicated by the above cyclic algebra is one for which its division algebra part occurs with minimal matrix degree. (This example is based on [Sh, Example 3.7]. At this point it is still unknown how to extract a presentation for this division algebra as a cyclic algebra from the above cyclic algebra presentation. Such presentations are known to be possible by [R, 32.20], but in this case the factor set will not consist of roots of unity.)

§4. Character Theory.

In this section we will outline the basic results in the character theory of finite groups that we will need in subsequent chapters. The advantage of using character theory for us will be that it is quite easy to keep track of the relationships between an irreducible representation of G with the restriction of that representation to its subgroups. The difficulties of doing character theory over non-algebraically closed fields will be avoided we will always work with complex characters.

Let G be a finite group, and suppose that V is an n-dimensional $\mathbb{C}G$-module, with associated representation

$$\mathcal{X} : \mathbb{C}G \longrightarrow End_{\mathbb{C}}(V) \cong \mathbb{C}^{n\times n}.$$

The *character* afforded by $\mathcal{X}$ (or V) is the map

$$\chi : G \longrightarrow \mathbb{C}$$

defined by

$$\chi(g) := tr(\mathcal{X}(g)), \text{ for all } g \in G.$$

It is easy to see that $\chi(1) = n$, and χ is constant on conjugacy classes of G. The most important property of characters is that whenever two complex G-modules are G-isomorphic, then their characters are exactly the same. Thus the set of irreducible complex G-modules is in bijective correspondence with the set of characters afforded by these modules. We call a character afforded by an irreducible complex G-module an *irreducible* character, and denote the set of irreducible characters of G by $Irr(G)$.

Because any complex representation of G can be decomposed into a sum of irreducible representations, any character of G can be decomposed into a sum of positive integer multiples of elements of $Irr(G)$. The most useful example of this is when we decompose $\mathbb{C}G$ as a G-module. We have

$$\mathbb{C}G = \mathbb{C}Ge_1 \oplus \cdots \oplus \mathbb{C}Ge_k$$

where $\{e_i : 1 \leq i \leq k\}$ is a full set of centrally primitive orthogonal idempotents of $\mathbb{C}G$, and each simple component $\mathbb{C}Ge_i$ is the opposite ring for the endomorphism ring of a simple $\mathbb{C}G$-module V_i. Thus the dimension of each simple component $\mathbb{C}Ge_i$ is the square of the dimension of V_i. As each simple component is the sum of $dim(V_i)$ G-isomorphic simple modules, the character afforded by $\mathbb{C}Ge_i$ is $\chi_i(1)\chi_i$, whenever V_i affords the character $\chi_i \in Irr(G)$. The character

$$\rho = \sum_{\chi \in Irr(G)} \chi(1)\chi$$

afforded by the G-module $\mathbb{C}G$ is called the *regular* character of G.

The irreducible characters of G not only determine the isomorphism classes of simple $\mathbb{C}G$-modules, they also determine the idempotents e_i as well. Given $\chi \in Irr(G)$, the centrally primitive idempotent for the simple component of $\mathbb{C}G$ determined by a simple G-module affording χ is [I, 2.12]

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g.$$

It is a fact that the values $\chi(g)$ are all algebraic over $\mathbb{Z}$ [I, 3.6]. Because G is finite this implies that all $\chi(g)$ are contained in a finite extension of $\mathbb{Q}$. We define the *field of character values* $\mathbb{Q}(\chi)$ to be $\mathbb{Q}(\{\chi(g) : g \in G\})$. This field will always be a galois extension of $\mathbb{Q}$ because it can be shown to lie in the cyclotomic extension $\mathbb{Q}(\zeta_e)$, with e being the exponent of the group G.

The idempotent e_χ always lies in the subalgebra $\mathbb{Q}(\chi)G$ of $\mathbb{C}G$. This means that $\mathbb{Q}(\chi)Ge_\chi$ must be a simple component of $\mathbb{Q}(\chi)G$, because otherwise e_χ would decompose as a sum of central idempotents in the smaller algebra. If $\mathbb{k}$ is a subfield of $\mathbb{Q}(\chi)$, then there is a unique centrally primitive idempotent for $\mathbb{k}G$ that is not orthogonal to e_χ. This idempotent is given by

$$a_\chi = \sum_{\sigma \in \mathcal{G}} {e_\chi}^\sigma,$$

with $\mathcal{G} = Gal(\mathbb{Q}(\chi)/\mathbb{k})$. (We extend the action of $\mathcal{G}$ on $\mathbb{Q}(\chi)$ to $\mathbb{Q}(\chi)G$ by acting trivially on elements of G.) That a_χ is a central idempotent of $\mathbb{k}G$ follows from all of the ${e_\chi}^\sigma$'s being central idempotents of $\mathbb{Q}(\chi)G$. It is primitive in $\mathbb{k}G$ because the sum of galois conjugates of e_χ is the minimal sum of centrally primitive idempotents of $\mathbb{Q}(\chi)$ that can lie in $\mathbb{k}G$. (Note that the notation a_χ depends on the particular field $\mathbb{k}$ chosen. Also, replacing $\mathbb{Q}$ by an arbitrary subfield $\mathbb{F}$ of $\mathbb{C}$ would not make much difference for this. The idempotents would be the e_χ's as long as $\mathbb{Q}(\chi) \subseteq \mathbb{F}$, and otherwise we can use a_χ's that are similarly defined sums of galois conjugates.)

So given a subfield $\mathbb{F}$ of $\mathbb{C}$ and a finite group G, each $\chi \in Irr(G)$ defines a unique simple component $\mathbb{F}Ga_\chi$ of $\mathbb{F}G$. (The same $\mathbb{F}Ga_\chi$ will be associated to more than one χ when $\mathbb{F} \neq \mathbb{F}(\chi)$.) The Schur index of the simple component $\mathbb{F}Ga_\chi$ we will denote by $m_\mathbb{F}(\chi)$. It is well known that the center of $\mathbb{F}Ga_\chi$ is always isomorphic to the field of character values $\mathbb{F}(\chi)$, and that there is an isomorphism of the simple components

$$\mathbb{F}Ga_\chi \cong \mathbb{F}(\chi)Ge_\chi$$

[Y, Propositions 1.4 and 1.5]. This means that we always have $m_\mathbb{F}(\chi) = m_{\mathbb{F}(\chi)}(\chi)$.

The rest of what we need from character theory is considered quite standard. If H is a subgroup of the group G and χ is a character of G, then we can define the *restriction* χ_H of χ to H by $\chi_H(h) = \chi(h)$, for all $h \in H$. Of course, χ_H is the character of H afforded by the restriction of a G-representation affording χ to H. Conversely, given an H-module V over $\mathbb{C}$, we can define the induced G-module from V by

$$V^G = \oplus_{t \in T}(V \otimes t)$$

where T is a right transversal of H in G. (A transversal is a set of coset representatives that contains the element 1.) If $g \in G$, then by writing $g = h_g t_g$, for $h \in H$, $t \in T$, we get an action of G on V^G by letting $g.(v \otimes t) = h_g h(t_g, t).v \otimes t(t_g, t)$, where $t_g t = h(t_g, t)t(t_g, t)$ for $h(t_g, t) \in H$ and $t(t_g, t) \in T$. The character afforded by V^G is denoted by ψ^G whenever ψ is the character of H afforded by V. We call ψ^G the *induced character*. The formula for calculating the values of ψ^G from those of ψ is

$$\psi^G(g) = \sum_{t \in T} \psi^\circ(t^{-1}gt),$$

where

$$\psi^\circ(g) = \begin{cases} \psi(g), & \text{if } g \in H \\ 0, & \text{otherwise.} \end{cases}$$

Induction and restriction are related by Frobenius reciprocity, which we will now explain. If we consider the abstract character ring

$$\mathbb{Z}[Irr(G)] = \{ \sum_{\chi \in Irr(G)} z_\chi \chi : \text{ all } z_\chi \in \mathbb{Z}\}$$

then we can define an inner product $(\cdot, \cdot)$ on this ring by extending the following orthogonality relation on the irreducible characters bilinearly to all of $\mathbb{Z}[Irr(G)]$. The orthogonality relation is

$$(\chi, \varphi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} \varphi(g),$$

which is known to have the properties $(\chi, \chi) = 1$ and $(\chi, \varphi) = 0$ whenever $\varphi \in Irr(G)$ is distinct from χ [I, 2.13]. If φ is any character of G, and $\chi \in Irr(G)$, then (φ, χ) is the multiplicity to which the irreducible representation affording χ occurs in a representation affording φ. When (φ, χ) is non-zero, then we call χ an *irreducible constituent* of φ.

A natural relation between induced and restricted characters is given in terms of the inner products on $\mathbb{Z}[Irr(H)]$ and $\mathbb{Z}[Irr(G)]$.

Frobenius Reciprocity [I, 5.2] *Let H be a subgroup of G, χ be a character of G, and ψ be a character of H. Then*

$$(\chi, \psi^G) = (\chi_H, \psi).$$

Given an irreducible character χ of G, and a subgroup H of G, the restricted character χ_H always decomposes as

$$\chi_H = \sum_{\psi \in Irr(H)} (\chi_H, \psi)\psi,$$

so that only the irreducible constituents of ψ appear in the sum.

When H is a normal subgroup of G, then there is an action of G on $Irr(H)$ that is given by $\psi^g(h) = \psi(ghg^{-1})$, for all $h \in H$, $g \in G$. (Of course, $\psi^h = \psi$ for all $h \in H$ because characters are constant on the conjugacy classes of H.) In this case, we call ψ^g a *G-conjugate* of ψ. We can use this to make χ_H even simpler when H is a normal subgroup of G.

Clifford's Theorem [I, 6.2] *Let H be a normal subgroup of G and let $\chi \in Irr(G)$. Let $\psi^{g_1}, \dots, \psi^{g_k}$ be all of the G-conjugates of ψ. Then $\psi^{g_1}, \dots, \psi^{g_k}$ are all of the irreducible constituents of χ_H, and*

$$\chi_H = e\sum_{i=1}^{k} \psi^{g_i}.$$

(Thus the multiplicity of ψ^g in χ_H is the same integer e for every G-conjugates of ψ.)

The situations where we obtain the most information occur when either an irreducible character on a normal subgroup H induces to an irreducible character of G, or when an irreducible character of G restricts to an irreducible character of H.

Proposition. *Suppose that H is a normal subgroup of the finite group G.*
(i) If $\psi \in Irr(H)$ and $\psi^G = \chi \in Irr(G)$, then

$$\chi_H = \sum_{t \in T} \psi^t$$

for a transversal T of H in G.

(ii) (Gallagher, [I, 6.17]*) If $\chi \in Irr(G)$ and $\chi_H = \psi \in Irr(H)$, then the characters $\chi\alpha$ for $\alpha \in Irr(G/H)$ are distinct irreducible characters of G, and are all of the irreducible constituents of ψ^G.*

(By $\chi\alpha$ we mean the pointwise product of the characters χ and α. This will also be a character of G. We consider each α as an element of $Irr(G)$ by using the fact that a representation of G/H affording α has H in its kernel, and thus can be interpreted as being a representation of G that maps elements of H to the identity.)

In the situation where H is a normal subgroup of G and $\chi \in Irr(G)$ is induced from H, it follows from the formula for induced characters that $\chi(g) = 0$, for all $g \in G \setminus H$. A character of G that has this property with respect to some subgroup N is said to *vanish off* N.

In the situations of the above proposition, something can also be said about the relation of the Schur index $m_{\mathbb{F}}(\chi)$ to $m_{\mathbb{F}}(\psi)$.

Proposition. [I, Problem 10.1] *Suppose that H is a normal subgroup of the finite group G. Let $\mathbb{F}$ be a subfield of $\mathbb{C}$.*
(i) If $\chi \in Irr(G)$ is such that $\chi_H = \psi \in Irr(H)$, then

$$m_{\mathbb{F}}(\chi) \leq m_{\mathbb{F}}(\psi) \leq [G:H]m_{\mathbb{F}}(\chi).$$

(ii) If $\psi \in Irr(H)$ is such that $\psi^G = \chi \in Irr(G)$, then

$$m_{\mathbb{F}}(\psi) \leq m_{\mathbb{F}}(\chi) \leq [G:H]m_{\mathbb{F}}(\psi).$$

§5. Group Theory

Quite a bit of familiarity with standard group theory will be needed for an appreciation of this work. Here we will give all of the relevant definitions and background results.

A *cyclic* group is a group in which every element is a power of some fixed element. We denote a cyclic group of order n by C_n, and the cyclic group generated by the element a by $\langle a \rangle$. An *abelian* group is a group that has a commutative group operation. Otherwise, the group is called *non-abelian.* The Fundamental Theorem of Abelian Groups states that every finite abelian group can be written as the direct product of cyclic groups. A group is called *elementary abelian* if it is the direct product of cyclic groups all having prime order.

A group G that has order a power of the prime number p is called a *p-group.* If the order of a finite group is divisible by a prime number p, then G always has subgroups that have order the maximal power of p dividing the order of the group, called *Sylow p-subgroups.*

In chapter 2, certain types of non-abelian 2-groups will come to our attention. A *dihedral* 2-group D_{2^n} $(n \geq 3)$ is a group that has an abstract presentation of the form

$$\langle X, Y : X^{2^{n-1}} = 1, Y^2 = 1, XY = YX^{-1} \rangle.$$

(This means that the group D_{2^n} consists of all possible combinations of the abstract generators X and Y, subject to the relations indicated. In particular, the first two relations indicate that the elements represented by X and Y have finite order. The last is a rule that shows how to shuffle X's and Y's past one another, because it indicates that $XY = YX^{-1}$. This implies that the subgroup $\langle X \rangle$ is normal in D_{2^n}, and every element of D_{2^n} can be expressed as $Y^i X^j$, with $i = 0, 1$ and $j = 0, \ldots, 2^{n-1} - 1$.)

A *generalized quaternion* group Q_{2^n} $(n \geq 3)$ has presentation of the form

$$\langle X, Y : X^{2^{n-1}} = 1, Y^2 = X^{2^{n-2}}, XY = YX^{-1} \rangle.$$

A *semi-dihedral* group SD_{2^n} $(n \geq 4)$ has presentation of the form

$$\langle X, Y : X^{2^{n-1}} = 1, Y^2 = X^{2^{n-2}}, XY = YX^{2^{n-2}-1} \rangle.$$

A group G is called *nilpotent* if it is the direct product of its Sylow subgroups. The *Fitting Subgroup* $F(G)$ of a group G is the unique maximal nilpotent normal subgroup of G. The *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all of the maximal proper subgroups of G. For finite groups G, $\Phi(G)$ is always a nilpotent normal subgroup of G, and $F(G)/\Phi(G)$ is always elementary abelian.

The center $Z(G)$ of a finite group G is the subgroup of G consisting of all elements that commute with every element of G. $Z(G)$ is always an abelian normal (and characteristic) subgroup of G. Nilpotent groups always have non-trivial centers. If we consider the series of normal subgroups Z_i of G defined by $Z_1 = Z(G)$, $Z_i/Z_{i-1} = Z(G/Z_{i-1})$ for $i > 1$, we see that for nilpotent groups G, $Z_n = G$ for some n. The smallest n for which this holds is called the *nilpotency class* of the nilpotent group G. Groups of nilpotency class 1 are of course abelian. Certain characters of a group of nilpotency class 2 have special properties.

Proposition. [I, 2.31] *Suppose G is a group of nilpotency class 2. Suppose $\chi \in Irr(G)$ is a faithful irreducible character of G.* (Faithful means that the kernel of a representation affording χ is 1.)

Then χ vanishes off $Z(G)$ and $\chi(1)^2 = [G : Z(G)]$.

The *commutator* subgroup (or *derived* subgroup) G' of a group G is the subgroup of G generated by all *commutators* $[x, y] = x^{-1}y^{-1}xy$, $x, y \in G$. The factor group G/G' is always abelian, and in fact if G/H is abelian for some normal subgroup H of G, then $G' \subseteq H$. A group is called *solvable* when the series of derived subgroups $G^{(1)} = G'$, $G^{(i)} = (G^{(i-1)})'$ for $i > 1$ terminates with $G^{(n)} = 1$ for some n.

A *section* of a group G is a factor group of the form H/K, where H, K are subgroups of G and K is a normal subgroup of the group H. We call a section H/K of G a *chief factor* of G when both H and K are normal subgroups of G and there is no normal subgroup of G that lies properly between H and K. Finite solvable groups are precisely the finite groups whose chief factors are all abelian. When a group is not solvable, it has a chief factor that contains a non-abelian simple group. A *simple* group is a group that has no normal subgroups other than 1 and itself.

When G is solvable, the least integer n with $G^{(n)} = 1$ is called the *derived length* of G. Groups of derived length 1 are again abelian groups. Groups of derived length 2 are called *metabelian*. Metabelian groups also have a nice character-theoretic property in that any faithful irreducible character has to be induced from a maximal abelian subgroup.

In chapter 3, we will come to be interested in *nilpotent-by-abelian* groups. These are finite groups G that satisfy $G' \subseteq F(G)$, or alternatively, finite groups G that have a nilpotent normal subgroup N with G/N abelian. In general, to call a group (something)-by-(something) means that there is a normal subgroup N of G such that the group N satisfies the first property and G/N satisfies the second. Whenever G has a normal subgroup N, we say that G *extends* N. When G extends N and G has subgroup H such that $G = NH$ and $N \cap H = 1$, then we call G the *split extension* of N by H, and write $G = N \rtimes H$. The split extension of a cyclic group by another cyclic group is called a *metacyclic* group.

Finally, we will need to be familiar with the structure and the character theory of the extraspecial p-groups that we will encounter in Chapters 2 and 3. A p-group E is *extraspecial* if

$$E' = Z(E) = \Phi(E) \cong C_p.$$

In this case, we have that $E/Z(E)$ is the direct product of cyclic groups of order p because $E/\Phi(E)$ is elementary abelian. This means that the exponent of E has to be either p or p^2. Extraspecial p-groups always have order greater than or equal to p^3, because groups of order p^2 are always abelian. When $p = 2$, the extraspecial 2-groups of order 8 are just D_8 and Q_8, both of exponent 4. For odd primes p, the extraspecial p-groups of order p^3 have either of the presentations [G, Theorem 5.1]

$$\langle X, Y, Z : X^p = Y^p = Z^p = 1, [X, Z] = [Y, Z] = 1, [X, Y] = Z \rangle$$

or

$$\langle X, Y : X^{p^2} = Y^p = 1, X^Y = X^{p+1} \rangle.$$

The first of these is a presentation of a group having exponent p, the other will give a group of exponent p^2.

A *central product* of two groups G_1, G_2 having isomorphic centers is a homomorphic image of the direct product of the two groups with the homomorphism having kernel $\{z_1^{-1}\phi(z) : z \in Z(G_1)\}$ with $\phi : Z(G_1) \to Z(G_2)$ a fixed isomorphism. It is known that any extraspecial p-group E of order p^{2r+1} is the central product of r extraspecial p-groups of order p^3 [G, Theorem 5.2]. If $p = 2$, then E will either be the central product of r dihedral groups or the central product of $r-1$ copies of D_8 with one copy of Q_8. If p is odd and E has exponent p, then E is the direct product of r extraspecial groups of order p^3 and exponent p. If E has odd exponent p^2, then E is the central product of $r-1$ groups of exponent p with one of exponent p^2 [Hup, §III, 13.7 and 13.8].

The extraspecial p-groups that we will encounter will all have exponent 4 or odd exponent p. For these groups, the simple components of their rational group algebras are quite well known.

Proposition. [G, Theorems 5.4 and 5.5] *Let E be an extraspecial p-group of order p^{2r+1}. Then there are $p-1$ faithful irreducible characters $\chi \in Irr(E)$, and for each such χ,*

(i) $\chi(1) = p^r$;
(ii) $\chi_{Z(E)} = p^r\lambda$, *for a faithful linear character λ of $Z(E)$;*
(iii) $\mathbb{Q}(\chi) = \mathbb{Q}(\zeta_p)$;
(iv) $m_{\mathbb{Q}}(\chi) = 1$ *unless $p = 2$ and E is the central product of $r-1$ dihedral groups of order 8 with a quaternion group of order 8, in which case $m_{\mathbb{Q}}(\chi) = 2$.*

The property (ii) above indicates that each of these faithful characters is *fully ramified* with respect to the subgroup $Z(E)$ [I, Problem 6.3]. In this situation, $\lambda^E = p^r\chi$ also holds.

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$, H be a normal subgroup of G, and $\chi \in Irr(G)$. We say that $\mathbb{F}Ga_\chi$ is a *crossed product* over its subalgebra $\mathbb{F}Ha_\chi$ by G/H whenever

$$\mathbb{F}Ga_\chi = \oplus_{t\in T}\mathbb{F}Ha_\chi t,$$

where T is a transversal of H in G. When χ is fully ramified with respect to H, this is an example of this situation. For a proof of this and other instances of $\mathbb{F}Ga_\chi$ being a crossed product algebra, we refer the reader to Theorem 2.3.

Chapter 2: Clifford Theory and its Consequences

Clifford Theory is a description of how induction and restriction of modules with respect to a finite group G behave with respect to normal subgroups of G. In this chapter we give a version of Clifford's Theorem in terms of idempotents and simple components of group algebras. We will use this to establish an isomorphism from a simple component of the group algebra $\mathbb{Q}G$ to a matrix ring over a simple component of the group algebra of a subgroup of G. The division algebra parts of these two simple components will be the same, and so we can interpret this as a reduction of the division algebra problem for a character of G to the division algebra problem for a character of a subgroup of G. We conclude the chapter with a partial characterization of the types of groups that occur as the end results of non-trivial reductions of this type, based on [MW,Section 1].

Let G be a finite group and let $\chi \in Irr(G)$ be a fixed irreducible complex character of G. We might as well assume that χ is a faithful character of G. Let $\mathbb{k} = \mathbb{Q}(\chi)$ be the field of character values of χ. Let

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$$

denote the centrally primitive idempotent of $\mathbb{k}G$ determined by χ. The simple component $\mathbb{k}Ge_\chi$ is a G-module affording the character $\chi(1)\chi$ because the multiplicity of χ as a constituent of the regular character of G is $\chi(1)$. Since $\mathbb{k}$ is the centre of $\mathbb{k}Ge_\chi$, this is another way to say that the dimension of $\mathbb{k}Ge_\chi$ over $\mathbb{k}$ is $\chi(1)^2$.

Suppose that N is a normal subgroup of G. Consider the subalgebra $\mathbb{k}Ne_\chi$ of $\mathbb{k}Ge_\chi$ that is generated by N. $\mathbb{k}Ne_\chi$ is semisimple because it is a homomorphic image of the semisimple algebra $\Bbbk N$. Thus we can decompose $\Bbbk Ne_\chi$ as a direct sum of simple algebras

$$\Bbbk Ne_\chi = \oplus_{i=1}^m \Bbbk Ne_i,$$

where the e_i are a full set of centrally primitive orthogonal idempotents of $\Bbbk Ne_\chi$. These e_i can in fact be determined from the characters of N. Let $\{a_\lambda\}$ be a full set of centrally primitive orthogonal idempotents of $\Bbbk N$. Since

$$\Bbbk Ne_\chi = (\sum_{\lambda \in Irr(N)} \Bbbk Na_\lambda)e_\chi,$$

and e_χ is the multiplicative identity in $\Bbbk Ge_\chi$, the only simple components $\Bbbk Na_\lambda$ of $\Bbbk N$ that will be non-zero after multiplication by e_χ are those for which

$$a_\lambda e_\chi \neq 0.$$

When this happens, the algebra $\Bbbk Na_\lambda e_\chi$ has to be isomorphic to the simple algebra $\Bbbk Na_\lambda$. Thus $e_i = a_{\lambda_i} e_\chi$ for some $\lambda_i \in Irr(N)$, for all $i = 1, \ldots, m$. The next result shows that $\{\lambda_i : i = 1, \ldots, m\}$ is exactly the set of irreducible characters of N for which $(\chi_N, \lambda) \neq 0$.

Proposition 2.1. *If $\lambda \in Irr(N)$ then $a_\lambda e_\chi \neq 0$ if and only if $(\chi_N, \lambda) \neq 0$.*

Proof: If $a_\lambda e_\chi \neq 0$, then $\Bbbk Ga_\lambda e_\chi$ is a non-zero left ideal of $\Bbbk Ge_\chi$. Thus $\Bbbk Ga_\lambda e_\chi$ is a sum of minimal left ideals of $\Bbbk Ge_\chi$, all of which afford the character χ as a G-module, so $\Bbbk Ga_\lambda e_\chi$ affords a multiple $n\chi$ of χ. Since $\Bbbk Na_\lambda$ affords the character $|\Bbbk(\lambda) : \Bbbk|\lambda(1)\lambda$ as an N-module, we know that $\Bbbk Ga_\lambda e_\chi$ will afford the induced character $|\Bbbk(\lambda) : \Bbbk|\lambda(1)\lambda^G$ as a G-module. A comparison of these characters shows that $(\lambda^G, \chi) \neq 0$, and so Frobenius reciprocity implies that $(\lambda, \chi_N) \neq 0$, as required.

On the other hand, if $(\chi_N, \lambda) \neq 0$, then $(\lambda^G, \chi) \neq 0$. Since $\Bbbk Ga_\lambda$ affords the induced character $|\Bbbk(\lambda) : \Bbbk|\lambda(1)\lambda^G$ as a G-module, $\Bbbk Ga_\lambda$ must contain a minimal left ideal affording the character χ. Since multiplication by e_χ will not annihilate this ideal, we see that $\Bbbk Ge_\chi a_\lambda \neq 0$, and hence $e_\chi a_\lambda \neq 0$. ■

Fix a constituent λ of χ_N, and put $e_1 := a_\lambda e_\chi$. Identify G with the set $Ge_\chi = \{ge_\chi : g \in G\}$. Because G is a set of units in $\Bbbk Ge_\chi$ and N is a normal subgroup of G, each $a_\lambda{}^g e_\lambda$ is another centrally primitive idempotent of $\Bbbk Ne_\chi$. Let $C_G(e_1)$ be the centralizer of this idempotent, and let T be a transversal of $C_G(e_1)$ in G. The element

$$\sum_{t \in T} a_\lambda{}^t e_\chi$$

will be a central idempotent of the algebra $\mathbb{k}Ge_\chi$, and so must be equal to e_χ. Since $\{e_i : i = 1 \dots m\}$ is a full set of centrally primitive orthogonal idempotents of $\mathbb{k}Ne_\chi$, we have

$$\sum_{i=1}^{m} e_i = e_\chi,$$

and no proper subset of these e_i's can sum to the identity e_χ. This forces G to act transitively on the set $\{e_i : i = 1, \dots, m\}$, and hence on the simple components $\mathbb{k}Na_\lambda e_\chi$ of the subalgebra $\mathbb{k}Ne_\chi$. In particular, each simple component $\mathbb{k}Na_\lambda e_\chi$ has the same dimension $|\mathbb{k}(\lambda) : \mathbb{k}|\lambda(1)^2$ over $\mathbb{k}$ and affords a character that is G-conjugate to $|\mathbb{k}(\lambda) : \mathbb{k}|\lambda(1)\lambda$. When e_1 is not central in G, the next result applies to give a reduction of the problem of computing the Schur index of $\mathbb{k}Ge_\chi$ to that of computing the Schur index of a simple component of $\mathbb{k}C_G(e_1)$.

Proposition 2.2 *Let $T = \{t_i : i = 1, \dots, m\}$ be a transversal of $C_G(e_1)$ in G.*
(i) The set of elements

$$\{t_i^{-1} e_1 t_j : 1 \leq i, j \leq m\}$$

is a set of matrix units in the algebra $\mathbb{k}Ge_\chi$.
(ii) e_1 is a centrally primitive idempotent of $\mathbb{k}C_G(e_1)$.
(iii) $\mathbb{k}Ge_\chi \cong (\mathbb{k}C_G(e_1)e_1)^{m\times m}$, with $m = |G : C_G(e_1)|$.

Proof: Because the action of G on the set of conjugates of $a_\lambda e_\chi$ is well-defined, we see that the the number of G-conjugates of e_1 is the index of $C_G(e_1) = C_G(a_\lambda)$ in G. To prove (i), we need to show that for any $i, j, k, \ell \in \{1, \dots, m\}$,

$$t_i^{-1} e_1 t_j t_k^{-1} e_1 t_\ell = \begin{cases} t_i^{-1} e_1 t_\ell, & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases},$$

and

$$\sum_{i=1}^{m} t_i^{-1} e_1 t_i = e_\chi.$$

Both of these properties are consequences of the set $\{a_\lambda{}^t e_\chi : t \in T\}$ being a full set of centrally primitive orthogonal idempotents of $\mathbb{k}Ne_\chi$. We have

$$t_i^{-1} e_1 t_j t_k^{-1} e_1 t_\ell = t_i^{-1} t_j (e_1)^{t_j} (e_1)^{t_k} t_k^{-1} t_\ell,$$

which is 0 unless $j = k$, in which case this becomes $t_i^{-1} e_1 t_\ell$. Also, we know that $\sum_{t\in T} e_1{}^t = \sum_{t \in T} a_\lambda{}^t e_\chi = e_\chi$, the multiplicative identity of $\mathbb{k}Ne_\chi$.

A well-known result (see [P, Lemma 6.1.6]) concerning the existence of matrix units of the type described in (i) gives a natural isomorphism

$$\mathbb{k}Ge_\chi \cong (e_1\mathbb{k}Ge_\chi e_1)^{m\times m}.$$

The algebra $e_1\mathbb{k}Ge_\chi e_1$ is generated over $\mathbb{k} := \mathbb{k}e_1$ by the set of elements $\{e_1ge_1 : g \in G\}$. However, for any $g \in G$, writing $g = ct$ for some $c \in C_G(e_1)$ and $t \in T$ gives

$$e_1ge_1 = e_1cte_1 = ce_1te_1 = cte_1{}^te_1 = 0$$

unless $t = 1$ and $g \in C_G(e_1)$. Thus

$$e_1\mathbb{k}Ge_\chi e_1 = \mathbb{k}C_G(e_1)e_1,$$

proving (iii). Simplicity of $\mathbb{k}Ge_\chi$ and the above isomorphism implies that $\mathbb{k}C_G(e_1)e_1$ is also simple, and so e_1 must be a centrally primitive idempotent of $\mathbb{k}C_G(e_1)$. ■

In fact, more can be said concerning the idempotent e_1 than is done in Proposition 2.2. One of the consequences of Clifford's Theorem for characters is that χ is induced by a unique irreducible character lying over λ of the stabilizer $I_G(\lambda)$ of λ in G [I, Theorem 6.11]. Because $I_G(\lambda)$ is also the stabilizer of e_λ in the action of G by conjugation in $\mathbb{C}G$, $I_G(\lambda)$ naturally lies inside the stabilizer of a_λ. Thus χ is induced from $C_G(a_\lambda)$, and for this induction we can take a character $\psi \in Irr(C_G(a_\lambda))$ that lies over λ. Since $\chi = \psi^G$, from the formula for induced characters it follows that χ vanishes off $C_G(a_\lambda)$, and so $e_\chi \in \mathbb{k}C_G(a_\lambda)$. Thus e_χ is the sum of centrally primitive idempotents of $\mathbb{k}C_G(a_\lambda)$, one of which has to be a_ψ, the idempotent of $\mathbb{k}C_G(a_\lambda)$ determined by ψ. In fact, $\chi = \psi^G$ implies that

$$e_\chi = \sum_{t\in T} e_\psi{}^t,$$

because each non-identity $t \in T$ cannot stabilize e_ψ, and so this has to be the primitive central idempotent of $\mathbb{C}G$ determined by the idempotent e_ψ. Also, as a_λ is a central idempotent in $\mathbb{C}C_G(a_\lambda)$, we know that a_λ is a sum of centrally primitive idempotents of $\mathbb{C}C_G(a_\lambda)$. By Proposition 2.1 one of these idempotents will be e_ψ. Because multiplication by e_ψ annihilates all of the other idempotents in this sum, we have $a_\lambda e_\psi = e_\psi$. Furthermore,

$$\begin{aligned} a_\lambda e_\chi &= a_\lambda \sum_{t\in T} e_\psi{}^t \\ &= a_\lambda \sum_{t\in T} e_{\psi^t} \\ &= a_\lambda e_\psi + 0 \\ &= e_\psi, \end{aligned}$$

because the characters ψ^t do not lie over λ, for $t \neq 1$. As both of a_λ and e_χ belong to $\Bbbk G$, this proves that $e_\psi \in \Bbbk G$, and hence $\Bbbk(\psi) = \Bbbk$. (In particular, $a_\psi = e_\psi$ with respect to the field $\Bbbk$).

Thus part (iii) of the above proposition can be made more precise. There is an isomorphism

$$\Bbbk G e_\chi \cong (\Bbbk C_G(a_\lambda) e_\psi)^{m \times m},$$

where $m = |G : C_G(a_\lambda)|$, for any $\psi \in Irr(C_G(a_\lambda))$ for which $(\psi_N, \lambda) \neq 0$. This gives a non-trivial reduction of the Schur index problem for the faithful character χ of G to a Schur index problem for a faithful character ψ of a section $C_G(a_\lambda)/(\ker \psi)$ of G, whenever the subalgebra $\Bbbk N e_\chi$ of $\Bbbk G e_\chi$ is not simple. We will call a reduction of this type a *Clifford Theory reduction.*

Starting with G and its faithful character χ, we can iterate Clifford Theory reductions as follows: if we find a normal subgroup N of G for which $\Bbbk N e_\chi$ is not simple, then choose a constituent $\lambda \in Irr(N)$ of χ_N. We know that $\Bbbk G e_\chi$ is a matrix ring over $\Bbbk C_G(a_\lambda) e_\psi$, ψ as above, with the matrix degree being $|G : C_G(a_\lambda)| > 1$. Interpret ψ as being a faithful character of

$$\overline{N} = C_G(a_\lambda)/(\ker \psi).$$

Then the division algebra part of $\Bbbk G e_\chi$ is the same as that of $\Bbbk \overline{N} e_\psi$, where $e_\psi = a_\psi$ is now the centrally primitive idempotent of $\Bbbk \overline{N}$ determined by $\psi \in Irr(\overline{N})$. Now replace $\overline{N}$ by G, ψ by χ, and continue. This process of Clifford Theory reductions stops when we reach a group G and faithful character $\chi \in Irr(G)$ for which $\Bbbk N e_\chi$ is simple for every normal subgroup N of G.

Before proceeding to an algebra that will be the limit of Clifford theory reductions, we should indicate that it is sometimes evident from character-theoretic information that $\Bbbk G e_\chi$ is a crossed product over its subalgebra $\Bbbk N e_\chi$ by G/N, when N is a normal subgroup of G. Such simple criteria for crossed product results are sometimes useful.

Theorem 2.3 *Suppose that G is a finite group, $\chi \in Irr(G)$, N is a normal subgroup of G, and $\mathbb{F}$ is a subfield of the complex numbers.*

(i) If χ is fully ramified with respect to N, then $\mathbb{F} G a_\chi$ is a crossed product over $\mathbb{F} N a_\chi$ by G/N.

(ii) If $\chi = \psi^G$, for some $\psi \in Irr(N)$, then $\mathbb{F} G a_\chi$ is a crossed product over $\mathbb{F} N a_\chi$ by G/N.

(iii) If $\chi_N = \psi$*, for some* $\psi \in Irr(N)$*, then* $\mathbb{F}Ga_\chi = \mathbb{F}Na_\psi \otimes_{\mathbb{F}(\psi)} \mathbb{F}(\chi)$*. In particular,* $\mathbb{F}Ga_\chi$ *is a crossed product over* $\mathbb{F}Na_\chi$ *by* G/N *exactly when* $[G:N] = |\mathbb{F}(\chi) : \mathbb{F}(\psi)|$.

Proof. (i). When χ is fully ramified with respect to N, we have $\chi_N = e\psi$, for some $\psi \in Irr(N)$, and the integer e satisfies $e^2 = [G:N]$. Furthermore, the character χ vanishes off N, so $\mathbb{F}(\psi) = \mathbb{F}(\chi)$. This means that $a_\chi = a_\psi$ as elements of $\mathbb{F}G$.

We always have that

$$\mathbb{F}Ga_\chi = \sum_{t \in T} \mathbb{F}Na_\chi t,$$

for any transversal T of N in G. Thus

$$|\mathbb{F}Ga_\chi : \mathbb{F}Na_\chi| \le [G:N].$$

However, when χ is fully ramified over N,

$$\begin{aligned} |\mathbb{F}Ga_\chi : \mathbb{F}Na_\chi| &= \frac{|\mathbb{F}Ga_\chi : \mathbb{F}(\chi)|}{|\mathbb{F}Na_\chi : \mathbb{F}(\chi)|} \\ &= \frac{\chi(1)^2}{\psi(1)^2} \\ &= [G:N]. \end{aligned}$$

(ii). First, we will prove this in the case where $\mathbb{F} = \mathbb{C}$. When $\chi = \psi^G$, we know that $\chi(1) = [G:N]\psi(1)$. Thus the dimension of $\mathbb{C}Ge_\chi$ over $\mathbb{C}$ is equal to $[G:N]^2\psi(1)^2$. Since χ vanishes off N, we know that e_χ is an element of $\mathbb{C}N$. Since the G-conjugates ψ^t for $t \in T$ are all distinct, and are the only irreducible constituents of χ_N, by Proposition 2.1 we must have

$$e_\chi = \sum_{t \in T} e_{\psi^t} = \sum_{t \in T} (e_\psi)^t.$$

Therefore,

$$\mathbb{C}Ne_\chi = \oplus_{t \in T} (\mathbb{C}Ne_\psi)^t.$$

Now we have

$$\begin{aligned} \mathbb{C}Ge_\chi &= \sum_{u \in T} (\mathbb{C}Ne_\chi)u \\ &= \sum_{u \in T} (\oplus_{t \in T} (\mathbb{C}Ne_\psi)^t)u \end{aligned}$$

and so the dimension count implies that the first sum must also be direct.

If $\mathbb{F}$ is any subfield of $\mathbb{C}$ containing $\mathbb{Q}(\psi)$, the above proof will work with $\mathbb{F}$ replacing $\mathbb{C}$. Otherwise, we use the isomorphism

$$\mathbb{F}Ga_\chi \cong \mathbb{F}(\chi)Ge_\chi$$

to assume that $\mathbb{F} = \mathbb{F}(\chi)$. In this case, we know that

$$\mathbb{F}Ne_\chi = \oplus_{s\in S}(\mathbb{F}Na_\psi)^s$$

where S is a transversal of the stabilizer $C_G(a_\psi)$ of the idempotent a_ψ in G. Then

$$\mathbb{F}Ge_\chi = \sum_{t\in T}(\oplus_{s\in S}(\mathbb{C}Na_\psi)^s)t,$$

so

$$\chi(1)^2 \leq [G:N][G:C_G(a_\psi)]\psi(1)^2|\mathbb{F}(\psi):\mathbb{F}|.$$

The result then follows because $|\mathbb{F}(\psi):\mathbb{F}| = [C_G(a_\psi):N]$ when χ is induced from the normal subgroup N.

(iii). The assumption $\chi_N \in Irr(N)$ implies that restricting an irreducible representation affording χ to N gives an irreducible representation of N over $\mathbb{C}$. Thus $\mathbb{C}Ne_\chi$ is simple, and so $\mathbb{F}(\chi)Ne_\chi$ must also be simple. Since $\chi(1) = \psi(1)$, the algebras $\mathbb{F}(\chi)Ge_\chi$ and $\mathbb{F}(\psi)Ne_\psi$ have the same dimension over their centers. However, $\chi_N = \psi$ implies that $\mathbb{F}(\psi) \subseteq \mathbb{F}(\chi)$, and $e_\chi e_\psi = e_\chi$, so $\mathbb{F}(\chi)Ne_\psi \cong \mathbb{F}(\chi)Ne_\chi$. This forces

$$\mathbb{F}(\chi)Ge_\chi \cong \mathbb{F}(\chi)Ne_\psi$$

because both algebras have the same dimension over their central copy of $\mathbb{F}(\chi)$. Since $\mathbb{F}(\chi)Ne_\psi$ is a scalar extension of the simple algebra $\mathbb{F}(\psi)Ne_\psi$, we have an isomorphism

$$\mathbb{F}(\chi)Ge_\chi \cong \mathbb{F}(\psi)Ne_\psi \otimes_{\mathbb{F}(\psi)} \mathbb{F}(\chi).$$

Now the isomorphisms $\mathbb{F}(\psi)Ne_\psi \cong \mathbb{F}Na_\psi$ and $\mathbb{F}(\chi)Ga_\chi \cong \mathbb{F}Ga_\chi$ complete the proof of (iii). ■

We will now proceed by giving a partial classification of the groups that have no non-trivial Clifford Theory reductions on one of their faithful characters. This is achieved using a known classification of groups with the property that every normal abelian subgroup is cyclic [MW, Section 1]. If we have a group G and character χ that has been obtained as the terminal point of possible Clifford Theory reductions, then G can be seen to possess this property. (If N is an abelian normal subgroup of G, $\mathbb{k}Ne_\chi$ being simple implies that $\mathbb{k}Ne_\chi$

is commutative and simple, hence is a field. Thus $N = Ne_\chi$ is a finite multiplicative subgroup of this field, and so N has to be cyclic.) We should remark that, however, the property that every abelian normal subgroup of G is cyclic does not seem to characterize completely those groups that do not allow any non-trivial Clifford Theory reductions with respect to some faithful irreducible character--although the author has been unable to offer an example to illustrate this possible discrepancy. For our purposes in future chapters we must insist that this stronger condition be assumed.

Assume that G is a finite group, not necessarily solvable, having the property that all of its normal abelian subgroups are cyclic. Let $F(G)$ be the Fitting Subgroup of G, the maximal nilpotent normal subgroup of G. Our assumption about G implies that every characteristic abelian subgroup of $F(G)$ is cyclic. Since $F(G)$ is nilpotent, $F(G)$ is the direct product of Sylow subgroups $P_1, \ldots, P_k$, all of which will be characteristic in both G and $F(G)$. ($k > 0$ whenever $G \neq 1$.) These Sylow subgroups must also have the property that all of their characteristic abelian subgroups are cyclic. Our main theorem on the structure of G relies heavily on the classification of p-groups possessing this property.

Theorem 2.4 *Let p be a prime integer. Suppose that P is a non-trivial finite p-group possessing the property that all of the characteristic abelian subgroups of P are cyclic. Then there exist normal subgroups E and T of P such that*

(i) $P = ET$, $T = C_P(E)$, *and* $E \cap T = Z(E)$;

(ii) *E is an extraspecial p-group of exponent p; and*

(iii) *T is cyclic when p is odd, and T is either cyclic, dihedral, quaternion, or semi-dihedral when $p = 2$.*

Furthermore, when T is not cyclic, we can choose E, T so that $|T| > 8$. This implies that P has a characteristic cyclic subgroup U of index 2 in T whenever T is not cyclic.

Proof. The first three assertions are a theorem due to Philip Hall. A proof can be found in [Hup, III.13.10], or in [G, 5.4.9]. The final statement is part (iv) of [MW, Theorem 1.2]. (The characteristic cyclic subgroup U of P that has index at most 2 in T is identified there as $Z(C_P(\Phi(P)))$, where $\Phi(P)$ is the Frattini subgroup of P, i.e. the intersection of all maximal subgroups of P.) ∎

With our assumption on G, the structure of $F(G)$ is determined by the structure given by the above theorem for its Sylow subgroups. We have

$$F(G) = P_1 \times \cdots \times P_k,$$

with each P_i a central product of characteristic subgroups E_i and T_i, E_i an extraspecial p_i-group of prime exponent p_i, and T_i either a cyclic group or a non-abelian 2-group of the above types, for each $i = 1, \dots, k$. If any P_i is equal to T_i, just set E_i to be the central subgroup of T_i of order p_i. If we let

$$E = E_1 \times \cdots \times E_k, \text{ and}$$
$$T = T_1 \times \cdots \times T_k,$$

then E and T are normal subgroups of $F(G)$ such that $F(G) = ET$ and $E \cap T$ is cyclic of order $p_1 \dots p_k$. For each i with T_i cyclic, we have $T_i = Z(P_i)$, so in this case T_i is characteristic in both $F(G)$ and in G. If all of the T_i's are cyclic groups, then $F = F(G)$ is metabelian with $T = Z(F)$. If some T_i is not cyclic, say T_1, then P_1 has a characteristic subgroup $U_1 = Z(C_{P_1}(\Phi(P_1)))$ of index 2 in T_1. Therefore, when T_1 is not cyclic, T contains a characteristic subgroup

$$U = U_1 \times T_2 \times \cdots \times T_k,$$

of $F(G)$ such that $|T : U| = 2$. In this case, $F(G)$ has a characteristic subgroup $F = EU$ of index 2 in $F(G)$ such that $U = Z(F)$. This characteristic subgroup can be identified using the following: it follows from [Sc, 7.3.7] that

$$\Phi(F(G)) = \Phi(P_1) \times \dots \Phi(P_k).$$

Each $\Phi(P_i) \subseteq Z(P_i)$ when $T_i = Z(P_i)$ because P_i/T_i is elementary abelian. Thus $C_{P_i}(\Phi(P_i)) = P_i$ when T_i is cyclic. When T_1 is not cyclic, then $\Phi(P_1) \subseteq U_1$ because P_1/U_1 is elementary abelian. Because U_1 is a cyclic 2-group the index of $\Phi(P_1)$ in U_1 is 2. Since U_1 has order at least 8, $C_{P_1}(\Phi(P_1))$ cannot be all of P_1 because an element of T_1 that lies outside U_1 will not centralize an element of order 4. Since E_1U_1 centralizes U_1 and has index 2 in P_1, this forces $E_1U_1 = C_{P_1}(\Phi(P_1))$. Thus we can identify F as $C_{F(G)}(\Phi(F(G))$, when $F \neq F(G)$. (When all of the T_i's are cyclic, this definition will still work, but $F = F(G)$ is less complicated.) In either of these definitions of F, F is a characteristic metabelian subgroup of G and $F/Z(F) \cong E/Z(E)$ is elementary abelian. We have shown the following:

Proposition 2.5. *Suppose that G is a finite group with every normal abelian subgroup cyclic. Then $F(G)$ has a characteristic metabelian subgroup F of index at most 2. If $U = Z(F)$, then F/U is elementary abelian. Always, we have $F = C_{F(G)}(\Phi(F(G)))$.*

The next observations will be useful in case G is solvable.

Lemma 2.6. *Assume that a solvable group G has only cyclic normal abelian subgroups, and let F be as in Proposition 2.5. Then $Z(F) = C_G(F)$.*

Proof. Let $U = Z(F)$. If $F = F(G)$, then since $C_G(F(G)) \subseteq F(G)$ when G is solvable [Sc, 7.4.7], we must have

$$U = C_G(F) \cap F = C_G(F),$$

and we are done.

If F has index 2 in $F(G)$, then note that the Fitting subgroup of $C_G(F)$ must be $F \cap C_G(F) = U$, because the Fitting subgroup of $C_G(F)$ is a normal nilpotent subgroup of G and hence lies inside F.

Suppose $C_G(F)$ properly contains U. Let X/U be a chief factor of G with $X \subseteq C_G(F)$. Because G is solvable, X/U is a non-trivial p-group, for some prime p. Since U is the Fitting subgroup of $C_G(F)$, U is also the Fitting subgroup of X, and so X cannot centralize U. This contradicts the assumption that X centralizes F, and so we conclude that $U = C_G(F)$. ∎

Lemma 2.7. *Assume that every normal abelian subgroup of a finite group G is cyclic, and let F and U be as in Proposition 2.5. Assume that any extraspecial 2-subgroup of F is the central product of dihedral groups of order 8. Then every chief section of the group G that lies between U and F has square prime power order.*

Proof. Because F is a nilpotent normal subgroup of G, every chief factor of G that is a section of F must have order a power of some prime. Because F/U is elementary abelian, we can write

$$F/U = F_1/U \times \cdots \times F_n/U$$

as the direct product of chief sections $F_1/U, \ldots, F_n/U$ of G. If any F_i/U has odd prime order, then F_i is an abelian normal subgroup of G having exponent the same as that of U. But then F_i would be non-cyclic, a contradiction. If any F_i had order that is an odd power of an odd prime, then $Z(F_i)$ would have to properly contain U because F_i is a section of an extraspecial p-group of exponent p, for some odd prime p. Again, the existence of this group leads to a contradiction. If F_i/U had order an odd power of 2, then again F_i must have a central element that lies outside U. If the subgroup generated by U and this element is not cyclic, we get a contradiction because $Z(F_i)$ is a non-cyclic normal subgroup of G. If $Z(F_i)$ is cyclic, then the assumption that extraspecial 2-subgroups of F are always central products of dihedral groups implies that F

has an element of order 2 that inverts the 2-part of $Z(F_i)$. The smallest normal subgroup generated by U and this element will be abelian and non-cyclic, and so we get another contradiction.■

We will now let G, F, and U be as in Proposition 2.5, and examine the maps of G into $Aut(F)$ and $Aut(U)$ that are induced by conjugation.

First of all, let $\sigma : G \to Aut(U)$ be defined by

$$u^{\sigma(g)} = g^{-1}ug,$$

for all $u \in U$, for all $g \in G$. Clearly, σ is a well-defined group homomorphism, and $\ker\sigma = C_G(U)$. As U centralizes F, we have $F \subseteq C_G(U)$. Denote $C_G(U)$ by C. Because U is a cyclic group, $Aut(U)$ is abelian, and hence G/C is also abelian. (Further information on G/C, such as possible prime divisors, can be inferred from the order of U.)

Since F is a normal subgroup of C, the the action of C on F by conjugation defines a homomorphism

$$\phi : C \to Aut(F),$$

by $f^{\phi(c)} = c^{-1}fc$, for all $f \in F$, $c \in C$. Now,

$$Inn(F) \cong F/U,$$

so ϕ restricts to an isomorphism of F/U with $Inn(F)$. Of course, by the above we know that F/U is elementary abelian of square integer order, and every prime divisor of $|F : U|$ also divides $|U|$. If we let $\mathcal{K} = C_C(F)$ be the kernel of ϕ, then we must have $F\mathcal{K}/\mathcal{K} \cong F/U$. Also, ϕ factors to an injection of $C/F\mathcal{K}$ into $Out(F)$, or more precisely, into $C_{Out(F)}(U)$. Of course, when G is solvable, Lemma 2.6 implies that $\mathcal{K} = U$, so in this case C/F is isomorphic to a solvable subgroup of $C_{Out(F)}(U)$. In particular, if G is solvable, then $C = F$ whenever $F = U$. This observation leads us to the following results.

Proposition 2.8. [Fon, Lemme 3.3] *Suppose that G is a supersolvable group and let $\chi \in Irr(G)$. Suppose that $\mathbb{k}Ge_\chi$ has been fully reduced using Clifford theory. Then G is metabelian.*

Proof. Assume that $\mathbb{k}Ge_\chi$ has been fully reduced using Clifford theory, and let F, U, and C be as in Proposition 2.5. Note that if $F = U$, then our observation says that $F = C$, and so it would follow that G is metabelian because G/C is always abelian.

Suppose $F \neq U$. We know that every chief section of G contained in F/U has to have prime order. From the proof of Lemma 2.6, we see that this forces $F = Q_8U$, for a normal Q_8-subgroup of G. The section C/F of G has to be isomorphic to a subgroup of $C_{OutF}(U) \cong Out(Q_8) \cong S_3$. However, if the normal Q_8-subgroup of G is generated by elements x and y so that $\langle x\rangle/U$ and $\langle y\rangle/U$ are chief sections of G, then the outer automorphisms of Q_8 will not stabilize these sections. This implies that $C = F$ when G is supersolvable.

Let A be a maximal cyclic normal subgroup of G containing U. We have that $|F : A| = 2$ and $C_F(A) = A$. Since $U \subset A$, $C_G(A) \subset C_G(U) = C = F$. Thus A is self-centralizing in G. Since A is cyclic, G/A is abelian, and so we conclude that G is metabelian. ■

Proposition 2.9 *Suppose G is a finite nilpotent group and $\chi \in Irr(G)$. Then $\Bbbk Ge_\chi$ Clifford reduces to a simple component of the group algebra $\Bbbk H$, where H is either cyclic, or H is the direct product of an odd order cyclic group with either a quaternion group of order ≥ 8, or a dihedral or semi-dihedral group of order at least 16.*

Proof. Assume $\Bbbk Ge_\chi$ has been fully Clifford reduced, and let F, U, and C be as above. Because nilpotent groups are supersolvable, we must have $F = U = C$. Thus $G = F(G)$ has a cyclic subgroup of index at most 2. When G is not cyclic, then we are in the case where $|F(G) : F| = 2$, and the Sylow 2-subgroup of G is non-abelian. Thus the Sylow 2-subgroup of G is either Q_8, or a dihedral, quaternion, or semi-dihedral group of order at least 16 by Theorem 2.4. ■

(Roquette's Theorem states that simple components of the rational group algebra of a nilpotent group have Schur index at most 2 [I, 10.14]. Note that Roquette's Theorem follows directly from the above proposition.)

The factor C/F is the most complicated part of the group G. In [MW, Theorem 1.9], a complicated induction argument is used to show that E and T are normal subgroups of G. The ensuing homomorphism of C to $Aut(E)$ can be used to show that when G is solvable, C/F is a solvable subgroup of the direct product of symplectic groups over finite fields of prime power order [MW, Corollary 1.10(ix)]. This leads to an approach to several problems in the representation theory of solvable groups using the study of solvable subgroups of symplectic groups of this type. It should also be remarked that finite primitive solvable linear groups have the same structure as our group C [Sup, Chapter I]. It would be interesting to know if C is always primitive when G has no non-trivial Clifford Theory reductions.

Chapter 3:

Reducing to a p-quasi-elementary Subgroup for Certain Solvable Groups

This chapter is concerned with finding a new reduction for the Schur index problem once a group algebra has been reduced as much as possible using Clifford theory. Our main result gives a constructive reduction under the condition that a certain character on a subgroup extends as far as it can. We then show that for nilpotent-by-abelian groups, each prime part of the Schur index can be computed using a p-elementary group that can be constructively determined from our original group.

Let p be a fixed prime number. Let χ be a faithful irreducible character of a finite group G, and let $\mathbf{k}$ be an algebraic number field for which $\mathbf{k} = \mathbf{k}(\chi)$. Suppose the reductions of the previous chapter are all trivial on G, so that for every normal subgroup N of G, we have that $\mathbf{k}Ne_\chi$ is a simple $\mathbf{k}$-algebra. The following definition will make it easier to state our results.

Definition 3.1. Let p be a fixed prime number. Let χ be a faithful irreducible character of a finite group G, and let $\mathbf{k}$ be an algebraic number field for which $\mathbf{k} = \mathbf{k}(\chi)$. Let $e = exp(G)$. The *p'-splitting field of $\mathbf{k}Ge_\chi$ over $\mathbf{k}$* is the unique subextension K of $\mathbf{k}(\zeta_e)$ containing $\mathbf{k}$ such that

$$[\mathbf{k}(\zeta_e) : K] \text{ is a power of } p, \text{ and}$$
$$[K : \mathbf{k}] \text{ is relatively prime to } p.$$

Because cyclotomic extensions are always abelian Galois extensions when the base field has characteristic zero, it follows that the K defined above is the field fixed by the unique Sylow p-subgroup of $Gal(\mathbf{k}(\zeta_e)/\mathbf{k})$, and so K is unique. The rationale for calling K the p'-splitting field comes from Brauer's Splitting

Field Theorem [CR, 15.18], which says that $\mathbf{k}(\zeta_e)$ is always a splitting field for a group of exponent e. This indicates that the division algebra part of

$$KGe_\chi \cong K \otimes_{\mathbf{k}} \mathbf{k}Ge_\chi$$

is split by an extension of degree a power of p. It follows from [I, 10.2(g)] that KGe_χ has Schur index $m_{\mathbf{k}}(\chi)_p$ because the splitting field $\mathbf{k}(\zeta_e)$ has dimension over K a power of p, and the index of $\mathbf{k}$ in K is relatively prime to p. Thus K has split the p'-part of the division algebra part of $\mathbf{k}Ge_\chi$, but has left the p-part intact.

The best result to date on reducing Schur index problems for irreducible characters of general finite groups to Schur index problems for smaller groups is the Brauer-Witt Theorem [W]. (See also [Y, Theorem 3.8].) In order to state this theorem we need the following definition.

Definition 3.2 A finite group is called *p-quasi-elementary* if it is isomorphic to the split extension of a cyclic group of order relatively prime to p by a finite p-group.

Theorem (Brauer-Witt) *Let χ be an irreducible character of a finite group G. Let $\mathbf{k}$ be an algebraic number field satisfying $\mathbf{k}(\chi) = \mathbf{k}$. Let K be the p'-splitting field for $\mathbf{k}Ge_\chi$ over $\mathbf{k}$.*

Then there exists a p-quasi-elementary section H of G and a character $\xi \in Irr(H)$ such that

$$m_K(\xi) = (m_{\mathbf{k}}(\chi))_p.$$

Proofs of the Brauer-Witt Theorem are based on Brauer's Induction Theorem [CR, 15.9], which is used to establish the existence of the section H of the correct type and the desired character ξ [Y, Chapter 3]. This approach does not lead by itself to an algorithmic method of finding such a section H from a knowledge of the subgroup structure of G, and thus has limited practical applications.

The objective of this chapter is to find conditions on the group G for which the Brauer-Witt Theorem can be made constructive. Given G and $\chi \in Irr(G)$, we will present an algorithmic method of determining a p-quasi-elementary group that can be used to find the p-part of the Schur index of $\mathbf{k}Ge_\chi$, whenever G satisfies our conditions.

With this objective in mind, fix a prime integer p. Let χ be a faithful irreducible character of a finite group G, and let $\mathbf{k}$ be an algebraic number field for which $\Bbbk = \Bbbk(\chi)$. Let K be the p'-splitting field for $\Bbbk Ge_\chi$ over $\Bbbk$. Since the p-part of the Schur index of $\Bbbk Ge_\chi$ is exactly the Schur index of KGe_χ, we may replace $\Bbbk$ by K.

The first step is to re-do the Clifford Theory reductions of the previous chapter with respect to the new field K. Assume we have completed this process, so that for every normal subgroup N of G, KNe_χ is a simple algebra. In particular, every normal abelian subgroup of G is cyclic, so G has normal subgroups F, U, and C satisfying the following conditions of Chapter 2:

(1) F is a characteristic nilpotent subgroup of G having index at most 2 in $F(G)$;

(2) $U = Z(F)$ is cyclic, and F/U is elementary abelian;

(3) F is the central product of a group E with U, where E is a direct product of extraspecial q-groups having exponent q (or 4 when q is 2), for some prime integers q; and

(4) $C = C_G(U)$.

We now establish various character identities for irreducible characters of the above subgroups of G.

Proposition 3.3 *Let K be the p'-splitting field for the simple algebra $\Bbbk Ge_\chi$. Suppose that KGe_χ has been fully reduced using Clifford theory, and let F, U, and C be the subgroups of G defined above. Let S be a transversal of C in G. Then*

(i) $\chi_U = k\sum_{s\in S}\lambda^s$, *for some faithful irreducible character λ of U and some integer $k > 0$;*

(ii) $G/C \cong Gal(K(\lambda)/K)$;

(iii) $\chi = \psi^G$, *for some $\psi \in Irr(C)$;*

(iv) $\chi_F = d\sum_{s\in S}\varphi^s$, *for some faithful irreducible character φ of F, and some integer $d > 0$.*

(v) φ and ψ may be chosen so that $\varphi_U = f\lambda$, with $f^2 = [F : U]$, and $\psi_F = d\varphi$;

(vi) $K(\lambda) = K(\varphi) = K(\psi) = K(\zeta_u)$, *where $u = |U|$.*

Proof: Since $U \lhd G$, the algebra KUe_χ is simple under our assumptions. Thus

$$\chi_U = k\sum_{\sigma\in\mathcal{G}}\lambda^\sigma,$$

where λ is some irreducible character of U, $\mathcal{G} = Gal(K(\lambda)/K)$, and k is some positive integer. Because all of the characters λ^σ lying under χ are Galois conjugate, all of their kernels are the same. Thus

$$\ker \lambda = \bigcap_{\sigma \in \mathcal{G}} \ker \lambda^{\sigma} = \ker \chi_{U} = U \cap \ker \chi = U \cap 1 = 1.$$

so λ is a faithful irreducible character of U. Since λ is a faithful linear character, the stabilizer of λ in G must be the centralizer in G of the cyclic group U; namely, C. The character-theoretic version of Clifford's Theorem gives the decomposition

$$\chi_{U} = k \sum_{s \in S} \lambda^{s},$$

where S is a transversal of C in G. This proves (i). (iii) also follows because χ is always induced from the stabilizer in this situation [I, 6.11].

Mapping G onto permutations on the set $\{\lambda^{\sigma} : \sigma \in \mathcal{G}\}$ gives a natural isomorphism of G/C with $\mathcal{G}$, namely $gC \mapsto \sigma(g)$, where $\sigma(g)$ is defined by $\lambda^{g} = \lambda^{\sigma(g)}$. This proves (ii).

Similarly, $F \triangleleft G$ implies that

$$\chi_{F} = d \sum_{\tau \in \mathcal{H}} \varphi^{\tau},$$

where φ is some irreducible character of F, $\mathcal{H} = Gal(K(\varphi)/K)$, and d is some positive integer. As above, we can show that φ is faithful. Because F is nilpotent of class 2, it follows from [I, 2.31] that any faithful character of F vanishes off $Z(F) = U$. Thus we can choose $\varphi \in Irr(F)$ so that $\varphi_{U} = f\lambda$, with $f^2 = [F : U]$. Since φ must vanish off U, we have that $K(\varphi) = K(\lambda)$, and the stabilizer of φ is exactly C. In particular, the Galois groups $\mathcal{G}$ and $\mathcal{H}$ are isomorphic. (iv) now follows by applying the same argument as in the proof of (i).

G (χ)
|
C (ψ)
|
F (φ)
|
U (λ)
|
1

Figure 1: Table of subgroups and associated characters for Proposition 3.1.

Since $C = C_G(U)$ is the stabilizer of ψ when $\psi^G = \chi$, we can use the fact that $C \lhd G$ to use the above arguments again. This time we find that

$$\begin{aligned} \chi_C &= \sum_{s \in S} \psi^s, \text{ and} \\ &= \sum_{\beta \in \mathcal{B}} \psi^\beta; \end{aligned}$$

with $\mathcal{B} = Gal(K(\psi)/K)$. (The positive integer in front must be 1 because χ is induced by ψ.) Again, there is an isomorphism of G/C with $Gal(K(\psi)/K)$. Since C is also the stabilizer of φ, we can replace ψ by one of its Galois conjugates to get, $\psi_F = d\varphi$. This integer d is the same one as before because

$$\chi_F = d \sum_{s \in S} \varphi^s = (\chi_C)_F = \sum_{s \in S} (\psi_F)^s.$$

This finishes the proof of (v).

Since $\psi_F = d\varphi$ implies that $K(\varphi) \subseteq K(\psi)$, the equality of $|G : C|$ with both $|K(\psi) : K|$ and $|K(\varphi) : K|$ implies that these fields are equal. (vi) follows. ■

One final remark is needed before the proof of the main result of this section. From our assumption concerning K, we see that by part (ii) of the above proposition, G/C must be a p-group. Thus for any Sylow p-subgroup of G, we have $G = CP$. Since non-identity elements of a transversal of C in G must lie outside C, we can choose the transversal S such that $S \subseteq P$, for any previously chosen Sylow p-subgroup P.

Theorem 3.4. *Let K be the p'-splitting field for the simple algebra $\mathbf{k}Ge_\chi$. Suppose that KGe_χ has been fully reduced using Clifford theory, and let F, U, and C be the subgroups of G defined above. Let P be a Sylow p-subgroup of G for the fixed prime p. Suppose that*

$$H = UP = U_{p'} \rtimes P,$$

where $U_{p'}$ is the p-complement of the cyclic group U.

Then

(i) *H is a p-quasi-elementary subgroup of G;*

(ii) *If $\psi_F = d\varphi$, with $(d,p) = 1$, then there exists a $\xi \in Irr(H)$ such that $(\chi_H, \xi) \not\equiv 0 \mod p$.*

(iii) *If* $\psi_F = \varphi$*, then there exists a* $\xi \in Irr(H)$ *such that* $(\chi_H, \xi) \not\equiv 0 \mod p$ *and* $K(\xi) = K$.

Proof. (i) is clear, since U is a cyclic normal subgroup of G.

To prove (ii), we let Q be the normal subgroup $H \cap F$ of G, and we compare expressions for the characters $(\chi_F)_Q$ and $(\chi_H)_Q$. Suppose that we have chosen a transversal S of C in G so that $S \subseteq P$. Then we have that

$$\begin{aligned}(\chi_F)_Q &= d\sum_{s\in S}(\varphi^s)_Q \\ &= \frac{df}{\sqrt{[Q:U]}}\sum_{s\in S}\theta^s,\end{aligned}$$

for some $\theta \in Irr(Q)$ that is fully ramified with respect to U and lies over λ. On the other hand, suppose

$$\chi_H = \sum_{i=1}^{r} c_i\xi_i,$$

for some positive integers $c_1, \ldots, c_r$, and irreducible characters $\xi_1, \ldots, \xi_r$ of Q. Since the Galois conjugates of θ are the only irreducible characters of Q lying under χ, they are also the only irreducible characters of Q that lie under any of the characters $\xi_1, \ldots, \xi_r$. Since $\{\theta^s : s \in S\}$ is exactly the set of Galois conjugates of θ, and each $s \in H$, we conclude that for each $i = 1, \ldots, r$, we have

$$(\xi_i)_Q = p^{k_i}\sum_{s\in S}\theta^s,$$

for some integer $k_i \geq 0$. Therefore

$$\begin{aligned}(\chi_H)_Q &= \sum_{i=1}^{r} c_i(\xi_i)_Q, \\ &= \sum_{i=1}^{r} c_i(p^{k_i}\sum_{s\in S}\theta^s) \\ &= \sum_{i=1}^{r} c_i p^{k_i}(\sum_{s\in S}\theta^s).\end{aligned}$$

Comparing the two expressions for $\chi_Q = (\chi_F)_Q = (\chi_H)_Q$, we conclude that

$$\sum_{i=1}^{r} c_i p^{k_i} = \frac{df}{\sqrt{[Q:U]}}.$$

By our assumption that d is relatively prime to p, the right hand side has to be relatively prime to p. This implies that at least one of the numbers $c_i p^{k_i}$ in the sum on the left has to be relatively prime to p. Without loss of generality, assume $c_1 p^{k_1}$ is relatively prime to p. Then we must have $k_1 = 0$ and $c_1 = (\xi_1, \chi_H) \not\equiv 0 \mod p$. This $\xi = \xi_1$ is the irreducible character of H required for (ii). (Note that we have also shown that for this character ξ,

$$\xi_Q = \sum_{s \in S} \theta^s.)$$

To prove (iii), note that the assumption that $\psi_F = \varphi$ is equivalent to

$$KFa_\varphi \cong KCa_\psi,$$

since $K(\psi) \cong K(\varphi)$ and both algebras have dimension $\psi(1)^2 = \varphi(1)^2$ over their centers. Our assumption on G then gives $KFe_\chi = KCe_\chi$, and so

$$KCe_\chi \cap KHe_\chi = KFe_\chi \cap KHe_\chi.$$

The subalgebra KHe_χ need not be simple because H need not be normal in G. However, we know from $\chi_H = \sum_i c_i \xi_i$ that

$$KHe_\chi = \oplus_i KHa_{\xi_i} e_\chi,$$

because KHe_χ is a homomorphic image of KH as an algebra. (The simple components of KH that occur in this sum will be exactly those whose associated irreducible characters are constituents of χ_H.) Let $\theta \in Irr(Q)$ be as above. Since θ vanishes off U, it follows that $I_H(\theta) = H \cap C$. Let $R = H \cap C$, and note that FR/F is a Sylow p-subgro: p of C/F.

Now let $\xi \in Irr(H)$ be the character found in the proof of part (ii). From the remark that

$$\xi_H = \sum_{s \in S} \theta^s,$$

and $R = I_H(\theta)$ it follows that there is a character $\eta \in Irr(R)$ such that $\xi = \eta^H$ and $\eta_Q = \theta$.

Consider the subalgebra KRe_χ. We have that $KRe_\chi \subseteq KCe_\chi$, so there are algebra inclusions

$$KQe_\chi \subseteq KRe_\chi \subseteq KFe_\chi.$$

Since $\varphi_Q = f\theta$ with $f = \sqrt{|F : Q|}$, we know that

$$[KFe_\chi : KQe_\chi] = |F : Q|$$

is relatively prime to p. Thus the index of KQe_χ in the subalgebra KRe_χ of KFe_χ has to be relatively prime to p. But R/Q is a p-group, and K is the p'-splitting field, so $[KRe_\chi : KQe_\chi]$ must be a power of p. This forces $KRe_\chi = KQe_\chi$.

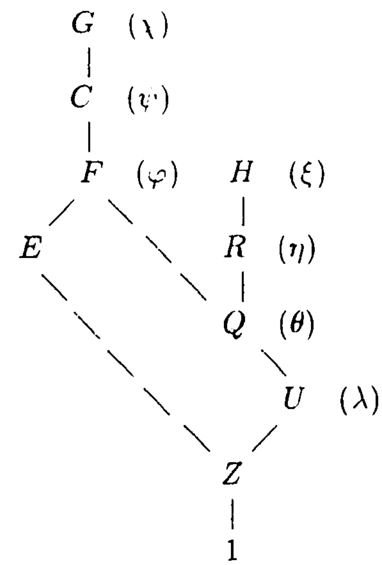

Figure 2: Table of subgroups and associated characters for Theorem 3.4.

Note that each of the characters ξ_i occurring in the above sum is faithful on Q by the formula $(\xi_i)_Q = p^{k_i} \sum_S \theta^s$. Since KQe_χ is a simple subalgebra of KHe_χ, this means that each of the maps $KQe_\chi \to KQa_{\xi_i}e_\chi$ has to be an isomorphism. (Here we can interpret the inclusion

$$KQe_\chi \to \oplus_i KHa_{\xi_i}e_\chi$$

as a diagonal embedding.) Since $KRe_\chi = KQe_\chi$, we have that $KRa_\xi e_\chi = KQa_\xi e_\chi$ is isomorphic to the simple component KRa_η of KR. The center of KRa_η is an isomorphic copy of $K(\eta)$, and so since $KQe_\chi \cong KQa_\xi e_\chi \cong KRa_\eta$, we conclude that $K(\theta) = K(\eta)$.

Finally, note that since ξ is induced from $I_H(\theta) = R$, ξ vanishes off R. Since H/R acts as galois automorphisms on the subfield $K(\theta) = KUa_\xi e_\chi$, H/R acts as galois automorphisms on $K(\eta) = Z(KRa_\eta e_\chi)$. Thus $K(\xi) \cong Z(KHa_\xi e_\chi)$ is the subfield of $K(\theta)$ fixed by H/R. Of course, this is exactly K, as required. ■

Corollary 3.4.1. *Under the conditions of Theorem 3.4, if we have* $\psi_F = \varphi$, *then* $(m_{\mathbb{k}}(\chi))_p = m_{\mathbb{k}}(\xi)$.

Proof: By [Y, Corollary 3.8], if ξ is a character of a subgroup H of G with $K(\xi) = K$ and (χ_H, ξ) relatively prime to a prime integer p, then the p-parts of the division algebra components of KGe_χ and KHe_ξ are isomorphic. Under the assumption $\psi_p = \varphi$, these conditions are satisfied for the subgroup H and its character ξ by Theorem 3.4. Because H is a p-elementary group, its irreducible characters all have degrees a power of p, and so $m_{\mathbb{k}}(\xi)$ is a power of p. Since $|K : \mathbb{k}|$ is relatively prime to p, $m_{\mathbb{k}}(\xi)$ is the same as $m_K(\xi)$. Since $m_K(\xi)$ is equal to $m_K(\chi)$ by the above, the assumption that K is the p'-splitting field gives $(m_{\mathbb{k}}(\chi))_p = m_K(\chi)$, which finishes the proof of the corollary. ■

We are now motivated to find conditions for the group G that ensure that the character φ of F extends to its stabilizer C once G has been reduced using Clifford theory. Exact conditions for this seem to be unknown. On the other hand, if C/F is cyclic, then it is well known that φ extends to C [I, 11.24].

We conclude this section by showing that if G is a nilpotent-by-abelian group, then the same p-quasi-elementary group H as in Theorem 3.4 can be used to compute the Schur index of χ. Janusz has shown in [J2] that in the case where N is a normal subgroup of G such that G/N is abelian and the particular irreducible character of N that we are interested in is invariant in G, then there is a method of decomposing $\mathbb{k}Ge_\chi$ into the tensor product of simple components of two *twisted* group algebras generated by N and G/N over $\mathbb{k}$. In the case where N is nilpotent, this would lead to a method of computing Schur indices of simple components of nilpotent-by-abelian groups. The methods demonstrated in this work are interesting in that this can be done without using any of the projective representation theory that is usually associated with representations that are invariant on a normal subgroup.

From the structure results in Chapter 2, we first note that the following holds for nilpotent-by-abelian groups.

Proposition 3.5. *Suppose that G is a finite nilpotent-by-abelian group and $\chi \in Irr(G)$. Then $\mathbb{k}Ge_\chi$ Clifford reduces to a simple component of the group algebra $\mathbb{k}\overline{G}$, where $\overline{G}$ is a nilpotent-by-abelian group of derived length at most 3.*

Proof. Assume that $\mathbb{k}Ge_\chi$ has been fully reduced using Clifford theory, and let the subgroups F, U, and C be as above. Since the class of nilpotent-by-abelian groups is closed under taking sections, we still have that G is nilpotent-by-abelian. We know that $F = F(G) \cap C$ and G/C is abelian, so our assumption that $G/F(G)$ is abelian implies that

$$G' \subseteq F(G) \cap C = F,$$

and so G/F is abelian. The proposition follows because F is a nilpotent group of class at most 2. ■

(We should note that if G is any solvable group, the Clifford theory reductions on $\mathbb{Q}Ga_\chi$ end in a simple component of the group algebra of a group having derived length at most the derived length of $G/F(G)$ plus 3.)

Suppose that G is a finite nilpotent-by-abelian group, with faithful irreducible character χ. Let K be the p'-splitting field for $\mathbb{Q}(\chi)Ge_\chi$, and suppose that KGe_χ has been fully reduced using Clifford Theory. Let F, U, and C be the characteristic subgroups of G determined in Chapter 2, and let $\varphi \in Irr(F)$, $\lambda \in Irr(U)$, and $\psi \in Irr(C)$ satisfy the conclusions of Proposition 3.3. By Proposition 3.5 we know that G/F is abelian.

Lemma 3.6 *Under the assumptions of the preceding paragraph, there exists a normal subgroup B and an irreducible character β of B such that*

(i) $F \subseteq B \subseteq C$;

(ii) $\chi = \beta^G$, $\psi = \beta^C$; *and*

(iii) $\beta_F = \varphi$.

Proof. Because G/F is abelian, G is a relative M-group with respect to F [I, 6.22], which means that for every character $\chi_0 \in Irr(G)$, there is a subgroup B_0 containing F and a $\beta_0 \in Irr(B_0)$ such that ${\beta_0}^G = \chi_0$ and $\beta_{0_F} \in Irr(F)$. Assuming B and β satisfy this for the character χ, we see that $\chi = \beta^G$. Since G/F is abelian, B has to be a normal subgroup of G. The assumption that KGe_χ is fully reduced using Clifford theory then implies that KBe_χ is simple, hence $KBe_\chi \cong KBa_\beta$. Since $\chi = \beta^G$, B is the stabilizer of β in G, and so

$$G/B \cong Gal(K(\beta)/K).$$

Since the G-conjugates of λ are the only irreducible characters of F lying under χ, we must have $\beta_F = \varphi^g$, for some $g \in G$. This φ^g is thus invariant in B. Since C is the stabilizer of every φ^g in G, this implies that $B \subseteq C$. As $\beta^{g^{-1}}$ also induces χ, replacing β by $\beta^{g^{-1}}$ gives us the condition $\beta_F = \varphi$. Since β^C is irreducible and lies over φ, we get that $\beta^C = \psi$ because $\psi \in Irr(C)$ is unique with this property. All three conclusions have now been shown. ■

Assume that B is the subgroup of G found in Lemma 3.6. Since B is a normal subgroup of G, its subgroups $F(B)$ and $Z(B)$ are also normal subgroups of G. Thus $F(B)$ is a nilpotent normal subgroup of B, and so $F(B) \subseteq F(G) \cap B \subseteq F(G) \cap C = F$. But F is a nilpotent normal subgroup of B, hence $F(B) = F$. Also, the inclusions $F \subseteq B \subseteq C$ imply that $U = Z(C) \subseteq Z(B) \subseteq Z(F) = U$, so $Z(B) = U$. Furthermore, the assumption that K is the p'-splitting field of $\mathbb{Q}(\chi)Ge_\chi$ implies that $G/B \cong Gal(K(\beta)/K)$ is a p-group.

In the case that the character field $K(\beta)$ is a cyclotomic field, we can "inflate" the group G in order to apply Theorem 3.4 directly. Let z be a root of unity in the center of KBe_χ so that

$$Z(KBe_\chi) = K(z).$$

Let $\hat{G}$ be the finite subgroup of $U(KBe_\chi)$ generated by z and G. Then it is easy to see that $KGe_\chi = K\hat{G}e_\chi$. Also, every normal abelian subgroup of $\hat{G}$ is cyclic, and the subgroups of $\hat{G}$ corresponding to the previous F, U, and C are $\langle z, F\rangle$, $\langle z\rangle$, and $\langle z, B\rangle$. The character on $\langle z, F\rangle$ extends to $\langle z, B\rangle$, so we can apply Theorem 3.4 to show that the division algebra part of KGe_χ is obtained as the division algebra part of a simple component of $K\langle z, H\rangle$.

In fact, we can show that this is the only method required for computing Schur indices, because we can always show that $K(\beta)$ will be a cyclotomic extension of K. The first step in doing this is to reduce to a group having a normal p-complement.

Lemma 3.7 *Suppose that G is a nilpotent-by-abelian group with faithful irreducible character χ. Assume KGe_χ has been fully reduced using Clifford theory, where K is the p'-splitting field of $\mathbb{Q}(\chi)Ge_\chi$. Let M be a subgroup of G so that M/F is the Sylow p-subgroup of G/F, F as above. Then $KMe_\chi = KGe_\chi$.*

Proof. Let B be the subgroup of G found in Lemma 3.6. Since $G/B \cong Gal(K(\beta)/K)$, G/B is a p-group, and so we can choose a transversal X of B in G consisting of elements of M. Since φ extends to $\beta \in Irr(B)$, we see that

$$KBe_\chi \cong KFe_\chi \otimes_{K(\varphi)} K(\beta),$$

because $\beta(1) = \varphi(1)$. Let D be the subgroup of B for which D/F is a Sylow p-subgroup of B/F. Note that KDe_χ must be simple because $D \lhd G$. Since φ extends to D, the same reasoning as above shows that

$$KDe_\chi \cong KFe_\chi \otimes_{K(\varphi)} K(\beta_D).$$

Now we see that $[KBe_\chi : KDe_\chi] = [K(\beta) : K(\beta_D)]$, which is a power of p by our assumption on K. However, $[KBe_\chi : KDe_\chi]$ divides $|B : D|$, so it is relatively prime to p. We conclude that $KDe_\chi = KBe_\chi$. Since $\beta^G = \chi$, we have

$$\begin{aligned} KGe_\chi &= \oplus_{x \in X} KBe_\chi x \\ &= \oplus_{x \in X} KDe_\chi x \\ &= KMe_\chi, \end{aligned}$$

since X is also a transversal of D in M. ■

Lemma 3.7 shows that we may assume G/F is a p-group by replacing G by M if necessary. If P is a Sylow p-subgroup of G, then we have that $G = FP$, and so since the p'-part of F is normal in G, G can be written as

$$G = N \rtimes P,$$

with the order of N being relatively prime to P.

We now use this to prove that the field $K(\beta)$ is a cyclotomic extension of K.

Theorem 3.8. *Suppose that B is a finite nilpotent-by-abelian group. Let β be a faithful irreducible character of B, and let K be any subfield of $\mathbb{C}$. Assume that KBa_β is fully reduced using Clifford theory, and let F, U, and C be the subgroups of B described above. Suppose further that B/F is a p-group, for a fixed prime p, and that β extends a faithful irreducible character φ of F. Then there exists a subgroup R of B such that $R \cap F = U$ and $RF = B$. Furthermore, $K(\beta) \cong K(\zeta_b)$, for some primitive root of unity $\zeta_b \in \mathbb{C}$.*

Proof. The existence of a subgroup R of B such that $R \cap F = U$ and $RF = B$ follows from [MW, Lemma 1.11], which only requires that B is solvable and $Z(B) = U$. (Our assumptions imply that $C = B$.)

Now, because F is a nilpotent group, the simple algebra KFa_β is isomorphic to $K(\varphi)^{f \times f}$, except possibly in the case where the extraspecial 2-subgroup of F is the central product of some dihedral groups of order 8 with a single copy of Q_8. In this exceptional case we will have

$$KFa_\beta \cong (\mathbb{H}(K(\varphi)))^{\frac{f}{2} \times \frac{f}{2}},$$

where $\mathbb{H}(K(\varphi))$ denotes the quaternion algebra over $K(\varphi)$, which is split if and only if $K(\varphi)$ is a splitting field for the rational quaternion algebra.

Whenever KFa_β is a split algebra, we know that

$$\begin{aligned} KBa_\beta &\cong KFa_\beta \otimes_{Z(KFa_\beta)} K(\beta) \\ &\cong KFa_\varphi \otimes_{K(\varphi)} K(\beta) \\ &\cong K(\varphi)^{f\times f} \otimes_{K(\varphi)} K(\beta) \\ &\cong K(\beta)^{f\times f}, \end{aligned}$$

where $f^2 = |F : U|$. If KFa_χ has Schur index 2, then the same reasoning shows that

$$KBa_\beta \cong (\mathbb{H}(K(\beta)))^{\frac{f}{2}\times\frac{f}{2}},$$

which could be a split algebra if $K(\beta)$ splits the quaternions.

On the other hand, if S is a transversal of F in B contained in R, we have

$$\begin{aligned} KBa_\beta &\cong \sum_{s\in S} KFa_\beta s \\ &\cong \sum_{s\in S} (K(\varphi)^{f\times f})s \\ &\cong \sum_{s\in S} ((KUa_\beta)^{f\times f})s \\ &\cong (KUa_\beta)^{f\times f} \otimes_{KUa_\beta} KRa_\beta \\ &\cong (KRa_\beta)^{f\times f}, \end{aligned}$$

using the fact that $Z(KFa_\beta) \cong KUa_\beta \cong K(\varphi)$, and that R is the subgroup generated in B by U and S. We conclude that KRa_β is simple, and $K(\beta) \cong KRa_\beta$.

Since U is central in R and R/U is an abelian p-group, we know that R is a nilpotent group of class at most 2. As faithful irreducible characters of nilpotent groups of class 2 vanish off of their centers, we have that the center of the simple algebra KRa_β generated by R is the field of character values of a linear character of $Z(\beta) = \{r \in R | |\beta(r)| = \beta(1)\}$. Thus $K(\beta)$ must be a cyclotomic extension of the field K, as required.

In the exceptional case,

$$\begin{aligned} KBa_\beta &\cong \sum_{s\in S} KFa_\beta s \\ &\cong \sum_{s\in S} (\mathbb{H}(K(\varphi)))^{\frac{\ell}{2}\times\frac{\ell}{2}} s \\ &\cong \sum_{s\in S} \mathbb{H}(K(\varphi)) \otimes_{K(\varphi)} K(\varphi)^{\frac{\ell}{2}\times\frac{\ell}{2}} s \\ &\cong K(UQ_8)a_\beta \otimes_{KUa_\beta} (KUa_\beta)^{\frac{\ell}{2}\times\frac{\ell}{2}} \otimes_{KUa_\beta} KRa_\beta \\ &\cong K(UQ_8)a_\beta \otimes_{KUa_\beta} (KHa_\beta)^{\frac{\ell}{2}\times\frac{\ell}{2}}. \end{aligned}$$

Again, the simplicity of KBa_β implies the simplicity of KRa_β, and the Schur index of KBa_β can be determined from the Schur index of KRa_β. (The Schur index of KBa_β will be twice that of KRa_β when the Schur index of the latter is odd, and half that of KRa_β when the latter has even index. Furthermore, the same reasoning as above shows that R is a nilpotent group of class 2, so the center of KRa_β, which is isomorphic to $K(\beta)$, is a cyclotomic extension of K, completing the proof of the theorem. ■

Thus we have found that the p-part of the simple algebra $\Bbbk Ge_\chi$ is generated by a simple component of the group algebra $K\hat{H}$, where K is the p'-splitting field of $\Bbbk Ge_\chi$ and $\hat{H}$ is any subgroup of the group of units of KGe_χ generated by U, a Sylow p-subgroup of G, and a central root of unity of maximal order in the subalgebra KBe_χ. Furthermore, this group $\hat{H}$ is p-quasi-elementary. This completes the reduction for a nilpotent-by-abelian group.

The class of nilpotent-by-abelian groups includes the classes of center- by-metabelian groups and all p-quasi-elementary groups, for any prime p. Supersolvable groups are also known to be nilpotent-by-abelian [Sc, 7.2.13].

Although the class of nilpotent-by-abelian groups is an improvement on the class of metabelian groups, for which the constructive reduction was explained previously in [H], it falls short of the class of all solvable groups. For example, let E be an extraspecial group of order 27 and exponent 3. The outer automorphisms of E that fix the center of E form a group isomorphic to $SL(2,3)$ [Win]. Let G be the finite subgroup of $\mathbb{C}^{3\times 3}$ generated by the matrices

$$a = \frac{1}{(2\zeta+1)}\begin{bmatrix} 1 & \zeta^2 & 1 \\ 1 & 1 & \zeta^2 \\ 1 & \zeta & \zeta \end{bmatrix} \quad \text{and} \quad b = \frac{\zeta}{(2\zeta+1)}\begin{bmatrix} 1 & 1 & 1 \\ \zeta^2 & 1 & \zeta \\ \zeta & 1 & \zeta^2 \end{bmatrix},$$

where ζ denotes a fixed cube root of unity in $\mathbb{C}$. G is a group of order 648 in which $F(G) \cong E$, $G/F(G) \cong SL(2,3)$, and $G/F(G)$ acts as the full group of outer automorphisms on $F(G)$. An examination of the character table of G yields that the irreducible characters of G which lie over a fixed faithful irreducible character φ of E are in one-to-one correspondence with characters of $SL(2,3)$, and all of these characters are all faithful on G. (All of this can be easily verified using the computer program **GAP** [Scho].) Simple components of $\mathbb{Q}(\zeta_3)G$ corresponding to these characters do not reduce using Clifford theory, in fact, $E = F$ and $G = C$ in the above notation. There are three characters of G that extend φ, corresponding to the linear characters of $SL(2,3)$. For these characters χ_i, $i = 1,2,3$, a reduction as in Corollary 3.1.1 is possible, and it shows that the 2- and 3-parts of $m_{\mathbb{Q}(\zeta_3)}(\chi_i)$ are Schur indices of simple components of rational group algebras over subgroups of G of the forms $C_3 \times Q_8$ and $E \rtimes C_3$, respectively. (These subgroups are exactly the groups H defined in Theorem 3.4, one for each of these two primes.) Since the quaternion component of $C_3 \times Q_8$ is not the one in question, and $E \rtimes C_3$ is a 3-group, we must have $m_{\mathbb{Q}}(\chi_i) = 1$ for $i = 1,2,3$. On the other hand, for the three characters of G lying over φ that satisfy $(\chi_4)_F = 2\varphi$, $(\chi_5)_F = 2\varphi$, and $(\chi_6)_F = 3\varphi$, the reduction fails because the hypothesis is not satisfied. Of course the group H can be computed, but we cannot guarantee the existence of a $\xi \in Irr(H)$ that will have the right Schur index.

For general finite groups, the gap between C and F can be large, and very arbitrary. We remark that Theorem 3.4 will hold regardless of whether or not the group G is solvable. The only condition is that the character φ of F extends to C in the case of the particular character χ of G chosen.

Chapter 4:

Computing Schur Indices of characters of p-quasi-elementary groups

The goal of this section is to give a precise algorithmic means of calculating the Schur index of a simple component of the rational group algebra of a p-quasi-elementary group, for a fixed prime number p. We will assume that H is a p-quasi-elementary group, with faithful irreducible character ξ, and that this H and ξ have been obtained from a larger group using the reductions of Chapters 2 and 3. In particular, H has a normal cyclic subgroup U, of index a power of p. If we let $R = C_H(U)$, then we have that

$$H/R \cong Gal(\Bbbk(\lambda)/\Bbbk),$$

where $\Bbbk = \mathbb{Q}(\xi)$, and λ is some faithful irreducible character of U.

The main idea of this chapter is that the simple algebra $\Bbbk He_\xi$ will Clifford reduce to a simple component of the rational group algebra of a metabelian group with a cyclic maximal abelian subgroup. Then we will give a constructive algorithm for determining the Schur index of a simple algebra generated by the latter type of group that only requires a knowledge of the multiplication in the group.

Proposition 4.1. *Let H be a p-elementary group with maximal cyclic normal subgroup U and faithful irreducible character $\xi \in Irr(H)$.*

Then $\mathbb{Q}Ha_\xi$ Clifford reduces to some $\mathbb{Q}Ga_\chi$, where G is a metabelian group with character χ.

Proof: By repeated application of Clifford Theory reductions, we may assume that $\Bbbk He_\xi$ is fully reduced using Clifford Theory. Thus H has a characteristic nilpotent subgroup F that is the central product of extraspecial groups with a cyclic group as in Chapter 2. Since the Sylow subgroups of H are cyclic for every prime except p, we know that $F/Z(F)$ is an elementary abelian p-group. Note that prior to Clifford reduction, H had a cyclic normal p-complement $U_{p'}$, and so H retains this property after Clifford reduction. If N is the cyclic normal p-complement of H, then it is easy to see that $F(H) = C_H(N)$. So even when F has index 2 in $F(H)$, we know that $N \subset Z(F)$. Thus $F/Z(F)$ is a p-section of the p-group $H/Z(F)$.

Assume that $F \neq Z(F)$. Because chief sections of p-groups all have order exactly p, there is a normal subgroup A of H such that $Z(F) \subset A \subset F$, and $|A : Z(F)| = p$. Since $Z(F)$ centralizes A, A is an abelian normal subgroup of H. When p is odd, the known facts about the structure of F force A to be non-cyclic, contradicting the assumption about Clifford reduction on $\mathbb{k}He_\xi$. This means that we must have had $F = U$. By our observations concerning Clifford theory reduced groups, we must have $U = C_H(U)$ also. Thus H/U is abelian, and U is cyclic, so H is metabelian, as required.

When $p = 2$, then the same will hold true if some choice of A is non-cyclic. However, the possibility exists that every possible choice of A results in a cyclic group. This can happen only when H has a normal Q_8-subgroup, the 2-part of U has order 2, and $|F : U|$ has order 4. In this case, we have $F = F(H)$ because the 2-part of U has order at least 8 when these are not equal. Fix a maximal cyclic normal subgroup A of H with $|A : U| = 2$. We know that $C_H(A)$ is a nilpotent normal subgroup of H because the p' part of $C_H(A)$ is centralized. Thus $C_H(A) \subseteq F(H) = F$. As F contains an element outside $C_H(A)$, and A has index 2 in F, we must have $C_H(A) = A$. Since $H/C_H(A)$ is abelian, this again forces H to be metabelian. ■

For the rest of this chapter, we will assume that G is a finite metabelian group with a cyclic maximal abelian normal subgroup C, and G/C is a p-group that acts faithfully as automorphisms of C, for a fixed prime number p. Let χ be an irreducible character of G that is induced from a faithful linear character λ of C, and put $u = |C|$. Our goal is to compute the Schur index of the simple algebra $\mathbb{Q}Ga_\chi$ in as constructive a manner as possible.

It is routine to use the isomorphism of $\mathbb{Q}Ga_\chi$ with $\mathbb{k}Ge_\chi$, where $\mathbb{k} = \mathbb{Q}(\chi)$, so for convenience we will use the latter algebra. Because of the isomorphism of G/C with $Gal(\mathbb{k}(\lambda)/\mathbb{k})$, there is a natural isomorphism of $\mathbb{k}Ge_\chi$ with the crossed product algebra

$$(\mathbb{k}(\zeta_u)/\mathbb{k}, \beta),$$

where $\beta : G/C \times G/C \longrightarrow \mathbb{k}(\zeta_u)$ is a factor set naturally defined by the multiplication in G and the character λ of C. Choose a transversal X of C in G, and note that

$$\mathbb{k}Ge_\chi = \oplus_{x \in X}(\mathbb{k}Ce_\chi)xe_\chi$$

because $\chi(1) = |G : C| = |\mathbb{k}(\zeta_u) : \mathbb{k}|$. Extension of the linear character $\lambda : C \to \mathbb{k}(\zeta_u)^\times$ to $\mathbb{k}C$ gives a homomorphism from $\mathbb{k}C$ to $\mathbb{k}(\zeta_u)$. Define $\sigma_x \in Gal(\mathbb{k}(\zeta_u)/\mathbb{k})$ by $(\zeta_u)^{\sigma_x} = \lambda(c^x)$, for all $x \in X$. Use this to define a natural isomorphism

$$\oplus_{x \in X}(\mathbb{k}Ce_\chi)xe_\chi \cong \oplus_{x \in X}\mathbb{k}(\zeta_u)u_x,$$

achieved by identifying the xe_χ's with the u_x's. Define conjugation by u_x in the algebra on the right so that $u_x\alpha = (\alpha)^{\sigma_x}u_x$, for all $x \in X$. Whenever x and y are elements of X and $xy = c_{x,y}z$ for some $c_{x,y} \in C$ and $z \in X$, then set

$$u_x u_y = \lambda(c_{x,y})u_z.$$

The set of elements $\{\lambda(c_{x,y}) : x, y \in X\}$ are the values of the factor set β. We use the notation $(\mathbb{k}(\zeta_u)/\mathbb{k}, \beta)$ for $\oplus_{x \in X}\mathbb{k}(\zeta_u)u_x$ in this case.

It is well known that the Schur index of a finite dimensional crossed product algebra whose center is an algebraic number field is the least common multiple of the Schur indices of the simple algebras obtained by localizing at all primes of the central field, including both finite and infinite primes [R, 32.19]. By "localizing" (or "completing"), we mean considering the tensor product algebra

$$\mathbb{k}_\mathfrak{q} \otimes_\mathbb{k} \mathbb{k}Ge_\chi$$

as a finite dimensional central simple $\mathbb{k}_\mathfrak{q}$-algebra, where $\mathbb{k}_\mathfrak{q}$ is the localization of the algebraic number field $\mathbb{k}$ over one of its primes $\mathfrak{q}$. Several authors, including Lorenz [L], Yamada [Y], Ford [F], and recently Schmidt [Sch] have developed methods for computing Schur indices of these algebras. Lorenz's work gives formulas for the Schur index of a cyclotomic crossed product algebra over a local field based on the order of certain elements of the factor set of the algebra [L, §2]. In our version, we will emphasize how the multiplicative structure of the group determines the structure of the division algebra part of its simple component. The formula for the Schur index of these simple components will be given in terms of the multiplication table of the group and the number of roots of unity contained in the central field. We believe that this makes available a formula for computing Schur indices for those with a limited understanding of homological algebra, or those that are not experts in algebraic number theory.

We can always restrict the set of the primes we need to consider to a finite set. For either infinite primes or primes lying over 2, the Schur indices of the localized algebras can only be 1 or 2. For infinite primes, this fact follows because $\mathbb{C}$ is a splitting field for G and $|\mathbb{C} : \mathbb{R}| = 2$. If $\mathfrak{q}$ lies over 2, then this follows from $|\mathbb{Q}_2(\zeta_4) : \mathbb{Q}_2| = 2$ and the Schur subgroup of the Brauer Group of $\mathbb{Q}_2(\zeta_4)$ being triv[illegible] [W, Satz 12] (see also [J, Proposition 3.2]). Thus when p is odd, we only have to localize at primes lying over odd rational primes. We can further restrict to the finitely many primes $\mathfrak{q}$ of $\mathbb{k}$ for which the extension $\mathbb{k}_{\mathfrak{q}}(\zeta_u)/\mathbb{k}_{\mathfrak{q}}$ is partly ramified. If the extension is unramified, then it is known to be a cyclic extension, and so the algebra above is naturally a cyclic extension with factor set consisting of roots of unity, all of which are norms for the unramified extension. By [R, 30.7 and 31.4], this implies that this algebra has trivial Schur index. Therefore, in the odd p case, we can restrict to primes $\mathfrak{q}$ of $\mathbb{k}$ that lie over an odd prime q dividing u. When $p = 2$, however, we must localize at a prime lying over 2, an infinite prime, and at primes lying over any odd prime that divides u. Of course, the Schur index of $\mathbf{k}Ge_\chi$ has to be a power of the fixed prime p, so there is a prime $\mathfrak{q}$ of $\mathbb{k}$ for which the local algebra $\mathbb{k}_{\mathfrak{q}}Ge_\chi$ has precisely the same Schur index.

The main advantage of localizing the algebra is that the Galois group is usually smaller over the local field, and so we can obtain a further Clifford reduction of the algebra. It is well known that after localization, the new Galois group will be

$$Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_u)/\mathbb{k}_{\mathfrak{q}}) \cong Gal(\mathbb{k}(\zeta_u)/\mathbb{k}(\zeta_u) \cap \mathbb{k}_{\mathfrak{q}}).$$

Since the character λ induces χ, we know that χ vanishes off C, and so $e_\chi = a_\lambda$ in $\mathbf{k}Ge_\chi$. The idempotent a_λ has its support in the field $\mathbb{k}$, so extending the center to $\mathbf{k}_{\mathfrak{q}}$ has the effect of splitting the idempotent into a sum

$$a_\lambda = \sum_{\sigma \in \mathcal{G}} (\hat{a}_\lambda)^\sigma,$$

of centrally primitive idempotents of $\mathbf{k}_{\mathfrak{q}}Ge_\chi$, indexed by $\mathcal{G} = Gal(\mathbf{k}_{\mathfrak{q}} \cap \mathbf{k}(\zeta_u)/\mathbf{k})$. If we let $\hat{G}$ be the centralizer in G of the idempotent $\hat{a}_\lambda$, then we can apply the Clifford reduction to get an isomorphism

$$\mathbf{k}_{\mathfrak{q}}Ge_\chi \cong (\mathbf{k}_{\mathfrak{q}}\hat{G}\hat{a}_\lambda)^{n \times n},$$

where $n = |G : \hat{G}|$. As $\lambda^{\hat{G}} \in Irr(\hat{G})$, we also see that $\hat{a}_\lambda = e_{\hat{\chi}}$, for an irreducible character $\hat{\chi} = \lambda^{\hat{G}}$ of $\hat{G}$. Of course,

$$\hat{G}/C \cong Gal(\mathbf{k}_{\mathfrak{q}}(\zeta_u)/\mathbf{k}_{\mathfrak{q}}),$$

and we can view this isomorphism as being the restriction to $\hat{G}$ of the map $x \mapsto \sigma_x$ defined above.

Now $\hat{G}/C$ has been restricted to being the Galois group of the cyclotomic extension of a local number field. These Galois groups are not very complicated. If the prime $\mathfrak{q}$ of $\mathbb{k}$ is an infinite prime, then $\hat{G}/C$ is a subgroup of $Gal(\mathbb{C}/\mathbb{R})$, and so has order at most 2. If $\mathfrak{q}$ lies over a finite prime q, then

$$Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_u)/\mathbb{k}_{\mathfrak{q}}) \cong Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_r)/\mathbb{k}_{\mathfrak{q}}) \times Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_{q^s}/\mathbb{k}_{\mathfrak{q}}),$$

where $u = q^s r$, for an integer r that is relatively prime to q. Both of these extensions are now cyclic extensions when q is an odd prime, because $\mathbb{k}_{\mathfrak{q}}(\zeta_r)$ is the maximal unramified subextension of $\mathbb{k}_{\mathfrak{q}}(\zeta_u)/\mathbb{k}_{\mathfrak{q}}$, and the totally ramified extension $\mathbb{k}_{\mathfrak{q}}(\zeta_{q^s})/\mathbb{k}_{\mathfrak{q}}$ is cyclic. Thus when q is odd, we have that $\hat{G}/C$ has at most two generators, one fixing ζ_r and the other fixing ζ_{q^s}. If $\mathfrak{q}$ lies over the prime 2, then just as in the odd case we have

$$Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_u)/\mathbb{k}_{\mathfrak{q}}) \cong Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_r)/\mathbb{k}_{\mathfrak{q}}) \times Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_{2^s})/\mathbb{k}_{\mathfrak{q}}).$$

$Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_r)/\mathbb{k}_{\mathfrak{q}})$ is again cyclic, because $\mathbb{k}_{\mathfrak{q}}$ is the maximal unramified subextension. However, $Gal(\mathbb{k}_{\mathfrak{q}}(\zeta_{2^s}))$ is not cyclic when $s > 2$, being generated by the automorphisms $\zeta_4 \mapsto \zeta_4{}^{-1}$ and $\zeta_{2^s} \mapsto (\zeta_{2^s})^5$. So $\hat{G}/C$ can have at most three generators when $q = 2$.

In the case where $\mathfrak{q}$ is an infinite prime, we have $\mathbb{k}_{\mathfrak{q}} = \mathbb{R}$ if and only if $\mathbb{k}$ is real. If $\mathbb{k}$ is not real, then $\hat{G} = C$ and the Schur index has to be one. If $\mathbb{k}$ is real, then the Schur index can be determined directly from the structure of the group $\hat{G}$. If the Schur index is to be 2, then we must be able to construct an isomorphism from $\mathbb{R}\hat{G}e_{\hat{\chi}}$ to the real quaternion algebra, which has the cyclic algebra presentation

$$(\mathbb{C}/\mathbb{R}, \sigma, \sigma^2 = -1).$$

Thus we must be able to find an element $x \in \hat{G}$ that lies outside C, has order 4, and inverts all elements of C. In addition, $u = |C|$ has to be an even integer larger than 2, so that $\mathbb{R}(\lambda) = \mathbb{C}$.

Now suppose that $\mathfrak{q}$ lies over a finite prime q. The next theorem is useful in reducing the number of generators of the Galois group we need to consider. For any positive integer n and field $\mathbb{L}$, let $W(\mathbb{L}, n)$ be the group of roots of unity in $\mathbb{L}$ of order dividing some positive power of n.

Theorem 4.2. *Let $\mathbb{K}$ be a field of characteristic 0, $\mathbb{L}$ be a finite abelian extension of $\mathbb{K}$, with $W(\mathbb{L}, n)$ finite for some integer n. Suppose that $\mathbb{F}$ is a subfield of $\mathbb{L}$ such that $Gal(\mathbb{L}/\mathbb{F})$ is cyclic, and the norm map $N_{\mathbb{L}/\mathbb{F}}$ carries $W(\mathbb{L}, n)$ onto $W(\mathbb{F}, n)$.*

Then

(i) any crossed product algebra $(\mathbb{L}/\mathbb{K}, \alpha)$ whose factor set α consists of roots of unity in $W(\mathbb{L}, n)$, lies in the same class of $Br(\mathbb{K})$ as a crossed product algebra $(\mathbb{F}/\mathbb{K}, \gamma)$ with factor set γ consisting of roots of unity in $W(\mathbb{F}, n)$; and

(ii) If $Gal(\mathbb{F}/\mathbb{K})$ is also cyclic, and is a direct summand of $Gal(\mathbb{L}/\mathbb{K})$, then we can determine a precise cyclic algebra presentation for $(\mathbb{F}/\mathbb{K}, \gamma)$ from the proof of (i).

Proof. (i) is exactly [J, Theorem 1]. For the proof of (ii), we will repeat the steps of the proof of (i) with the assumption that $Gal(\mathbb{L}/\mathbb{K}) = Gal(\mathbb{F}/\mathbb{K}) \times Gal(\mathbb{L}/\mathbb{F})$ is the direct product of cyclic groups.

Suppose $\langle\sigma\rangle = Gal(\mathbb{L}/\mathbb{F})$ and $\langle\mu\rangle = Gal(\mathbb{F}/\mathbb{K})$. Let $\mathbb{V}$ be the subfield of $\mathbb{L}$ that is fixed by μ, so that we have $\mathbb{L} \cong \mathbb{F} \otimes_{\mathbb{K}} \mathbb{V}$, since these extensions are disjoint over $\mathbb{K}$.

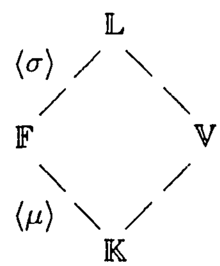

Figure 3: Field extensions and associated Galois groups for Theorem 4.2.

Suppose $W(\mathbb{L}, n)$ is generated by ζ_n, without loss of generality. Write the crossed product algebra as

$$(\mathbb{L}/\mathbb{K}, \alpha) = \oplus_{i=0}^{\ell-1} \oplus_{j=0}^{f-1} \mathbb{L} {u_\sigma}^i {u_\mu}^j,$$

where $(u_\sigma)^\ell = {\zeta_n}^{d_\sigma}$ and $(u_\mu)^f = {\zeta_n}^{d_\mu}$ are roots of unity in $W(\mathbb{L}, n)$. Because $N_{\mathbb{L}/\mathbb{F}}$ carries $W(\mathbb{L}, n)$ onto $W(\mathbb{F}, n)$, there is a $\zeta \in W(\mathbb{L}, n)$ such that

$$N_{\mathbb{L}/\mathbb{F}}(\zeta) = \zeta_n^{d_\sigma}.$$

Replacing u_σ by $v_\sigma = \zeta^{-1}u_\sigma$, we will not change the algebra, and now we have $(v_\sigma)^\ell = 1$. We now change u_μ so that it commutes with v_σ. We know that

$$u_\mu v_\sigma = \alpha_{\mu,\sigma} v_\sigma u_\mu,$$

for some $\alpha_{\mu,\sigma} \in W(\mathbb{L}, n)$. (Since ζ is in $W(\mathbb{L}, n)$, the factor set values are still in $W(\mathbb{L}, n)$.) Thus

$$u_\mu v_\sigma (u_\sigma)^{-1} = \alpha_{\mu,\sigma} v_\sigma,$$

and raising both sides to the ℓ-th power gives

$$1 = N_{\mathbb{L}/\mathbb{F}}(\alpha_{\mu,\sigma}).$$

Using Herbrand's Theory as in [J, page 527], we can conclude that there exists a root of unity $\delta \in W(\mathbb{L}, n)$ such that

$$\delta(\delta^{-1})^\sigma = \alpha_{\mu,\sigma}.$$

Now replacing u_μ by $v_\mu = \delta^{-1}u_\mu$, we have

$$\begin{aligned} v_\mu v_\sigma &= \delta^{-1} u_\mu v_\sigma \\ &= \delta^{-1}\alpha_{\mu,\sigma} v_\sigma u_\mu \\ &= (\delta^{-1})^\sigma v_\sigma u_\mu \\ &= v_\sigma(\delta^{-1})u_\mu \\ &= v_\sigma v_\mu. \end{aligned}$$

Now that v_μ and v_σ commute, we find that v_σ must also commute with ${\zeta_n}^{d_\mu}$. Since $(v_\sigma)^\ell = 1$, this means that the factor set generated by v_σ and v_μ is fixed by σ, and thus lies in $W(\mathbb{F}, n)$. Therefore,

$$\begin{aligned} (\mathbb{L}/\mathbb{K}, \alpha) &\cong \oplus_{i=0}^{\ell-1} \oplus_{j=0}^{f-1} \mathbb{L} {v_\sigma}^i {v_\mu}^j \\ &\cong \oplus_{i=0}^{\ell-1} \oplus_{j=0}^{f-1} (\mathbb{F} \otimes_{\mathbb{K}} \mathbb{V}) {v_\sigma}^i {v_\mu}^j \\ &\cong (\oplus_{i=0}^{\ell-1} \mathbb{V} {v_\sigma}^i) \otimes_{\mathbb{K}} (\oplus_{j=0}^{f-1} \mathbb{F} {v_\mu}^j) \end{aligned}$$

Since $(v_\mu)^\ell = 1$, $\oplus_{i=0}^{\ell-1}\mathbb{V}{v_\sigma}^i$ is the cyclic algebra $(\mathbb{V}/\mathbb{K}, \sigma, 1)$. Since 1 is always a norm, this algebra has Schur index 1, and thus is isomorphic to $(\mathbb{K})^{\ell\times\ell}$. The original crossed product algebra $(\mathbb{L}/\mathbb{K}, \alpha)$ thus lies in the same class of the Brauer group as $\oplus_{j=0}^{f-1}\mathbb{F}{v_\mu}^j$, which is the cyclic algebra $(\mathbb{F}/\mathbb{K}, \mu, (v_\mu)^f)$. We have

$$(v_\mu)^f = (\delta^{-1}u_\mu)^f = N_{\mathbb{F}/\mathbb{K}}(\delta)^{-1}\zeta_n^{d_\mu}.$$

This element can always be recovered from the original factor set α and from $W(\mathbb{L}, n)$. This proves (ii). ■

We now apply Theorem 4.2 in the ase where q is an odd prime. We know that $\Bbbk_q(\zeta_u)/\Bbbk_q(\zeta_{q^s})$ is an unramified extension, and the norm map will map $W(\Bbbk_q(\zeta_u), n)$ onto $W(\Bbbk_q(\zeta_{q^s}), n)$ as long as n is relatively prime to q. Thus Theorem 4.2 will apply as long as $p \neq q$, because this is the condition that will guarantee that the factor set generated by $\hat{G}/C$ consists of roots of unity of order prime to q. When $p = q$, however, $\Bbbk_q\hat{G}e_{\hat{\chi}}$ is a q-local algebra that has index a power of q. However, the following result of Janusz shows that such an algebra must necessarily have Schur index 1.

Lemma 4.3. [J, page 537] *Let $\mathbb{K}$ be a subfield of $\mathbb{L} = \mathbb{Q}_q(\zeta_{q^s}, \zeta_r)$ for some odd prime q, with $s \geq 0$ and $(r, q) = 1$. Then any crossed product algebra of the form*

$$(\mathbb{L}/\mathbb{K}, \alpha),$$

with factor set contained in $W(\mathbb{L}, q)$ must necessarily have Schur index one.

Proof. Suppose $(\mathbb{L}/\mathbb{K}, \alpha)$ has Schur index q^m, for some $m > 0$. By a theorem of Benard and Schacher [BS], we find that $\zeta_q \in \mathbb{K}$. This implies that $N_{\mathbb{L}/\mathbb{K}}$ maps $W(\mathbb{L}, q)$ onto $W(\mathbb{K}, q)$, because the constant term of the minimal polynomial of ζ_{q^s} over $\mathbb{K}$ will be a generator of $W(\mathbb{K}, q)$. We can now apply Theorem 4.2 with $\mathbb{F} = \mathbb{K}(\zeta_r)$. This implies that $(\mathbb{L}/\mathbb{K}, \alpha)$ has the same Schur index as a crossed product algebra of the form $(\mathbb{F}/\mathbb{K}, \gamma)$, with γ taking values in $W(\mathbb{F}, q)$. Since $\mathbb{F}/\mathbb{K}$ is unramified, and γ consists of roots of unity, this algebra is split, which proves the lemma. ■

It follows from Lemma 4.3 that if p is odd, it is only necessary to localize at odd primes q dividing u for which p divides $q - 1$ when computing the Schur index of $\Bbbk Ge_\chi$. (These are the only ones for which $Gal(\Bbbk_q(\zeta_{q^s})/\Bbbk_q)$ will have a non-trivial Sylow p-subgroup.)

As it is now safe to assume that $p \neq q$, we can apply Theorem 4.2 to the algebra $\Bbbk_q\hat{G}e_{\hat{\chi}}$. Choose elements x and y of a Sylow p-subgroup of $\hat{G}$ so that if $g \mapsto \sigma_g$ is the homomorphism of $\hat{G}$ to $Gal(\Bbbk_q(\zeta_u)/\Bbbk_q)$ satisfying

$$\lambda(c^g) = \lambda(c)^{\sigma_g}, \text{ for all } c \in C,$$

then

$$Gal(\Bbbk_q(\zeta_r)/\Bbbk_q) = \langle \sigma_x \rangle,$$

and

$$Gal(\mathbb{k}_{q}(\zeta_{q^s})/\mathbb{k}_{q}) = \langle\sigma_y\rangle.$$

Working through the proof of part (ii) of Theorem 4.2 with $n = p$, we find that by multiplying these by suitable elements of $W(\mathbb{k}_{q}(\zeta_u), p)$, we can create new units $\hat{x}$ and $\hat{y}$ that act as σ_x and σ_y, commute with each other, $(\hat{x})^{o(\sigma_x)} = 1$, and $(\hat{y})^{o(\sigma_y)} \in W(\mathbb{k}_{q}(\zeta_{q^s}, p)$. (Effectively, we are extending the group $\hat{G}$ to a larger group that has enough p-power roots of unity to guarantee the existence of a "nice" transversal of C in $\hat{G}$.) Now we apply Theorem 4.2 to get an isomorphism

$$\mathbb{k}_{q}\hat{G}e_{\hat{\chi}} \cong (\oplus_{i=0}^{o(\sigma_x)-1}\mathbb{k}_{q}(\zeta_r)(\hat{x})^i) \otimes_{\mathbb{k}_{q}} (\oplus_{j=0}^{o(\sigma_y)-1}\mathbb{k}_{q}(\zeta_{q^s})(\hat{y})^j).$$

The Schur index of $\mathbb{k}_{q}\hat{G}e_{\hat{\chi}}$ will be the same as the index of the second factor, which is the cyclic algebra

$$(\mathbb{k}_{q}(\zeta_{q^s})/\mathbb{k}_{q}, \hat{y}, (\hat{y})^{o(\sigma_y)}).$$

It is easy to see that this cyclic algebra is the simple component of the group algebra of a metacyclic group of the form $C_{q^s} \rtimes C_{p^{n+k}}$ over $\mathbb{k}_{q}$, where $p^n = |\mathbb{k}_{q}(\zeta_{q^s}) : \mathbb{k}_{q}|$, and p^k is the order of $(\hat{y})^{p^n}$. The faithful representation is chosen so that the generator of $C_{p^{n+k}}$ acts as σ_y on $\langle\zeta_{q^s}\rangle \cong C_{q^s}$. It is easy to compute the Schur index of such an algebra using the formula provided by the next result.

Theorem 4.4 *Suppose*

$$G \cong C_{q^c} \rtimes C_{p^{n+k}} = \langle a\rangle \rtimes \langle x\rangle,$$

where p and q are distinct primes with q odd, and x acts as an automorphism of $\langle a\rangle$ of order p^n. Let λ be a faithful irreducible complex character of the cyclic maximal abelian subgroup $U = \langle ax^{p^n}\rangle$ of G, and let χ be the faithful irreducible character λ^G of G. Let $\mathbb{k}_{q} = \mathbb{Q}_q(\chi)$, and suppose that ζ_{p^ℓ} generates $W(\mathbb{k}_{q}, p)$. Then the Schur index of the simple component $\mathbb{k}_{q}Ge_\chi$ of $\mathbb{Q}_qG$ is

$$p^m, \quad \text{where } m = \max\{n + k - \ell, 0\}.$$

Furthermore,

$$m_{\mathbb{Q}_q}(\chi) = 1 \iff \zeta_{p^{n+k}} \in \mathbb{Q}_q(\chi).$$

Proof. We know that the algebra $\mathbb{Q}_qGa_\chi$ is exactly the cyclic algebra

$$(\mathbb{k}_{q}(\lambda)/\mathbb{k}_{q}, \sigma_x, \zeta_{p^k}),$$

where σ_x is defined by

$$(\zeta_{q^c p^k})^{\sigma_x} = \lambda(v^x),$$

$v \in U$ chosen so that $\lambda(v) = \zeta_{q^c p^k}$. The Schur index of this algebra is well known to be the least power of ζ_{p^k} that is a norm for the given field extension. By [J, Lemma 3.1], since $p \neq q$, ζ_{p^k} is a norm if and only if it is a p^n-th power of a root of unity in $\Bbbk_{\mathfrak{q}}$, because the extension $\mathbb{Q}_q(\zeta_{q^c})/\Bbbk_{\mathfrak{q}}$ is totally ramified of degree p^n.

Thus if we set $p^m = m_{\mathbb{Q}_q}(\chi)$, then

$$\zeta_{p^k}^{p^m} = \zeta_{p^{\ell-d}}^{p^n},$$

for some integer d and primitive root of unity $\zeta_{p^{\ell-d}} = \zeta_{p^\ell}^{p^d}$. (Up to multiplication by a a power of ζ_{p^ℓ} that is relatively prime to p.) It follows that

$$k - m = \ell - d - n.$$

If the Schur index is 1, then $m = 0$, and we conclude that $d = \ell - n - k$, so $\zeta_{p^{n+k}} \in \langle\zeta_{p^\ell}\rangle$. If $m > 0$, then the least power of ζ_{p^k} that is a norm must come from a primitive p^ℓ-th root of unity, so d must be 0. Therefore, we have in this case that $m = n + k - \ell$. Since $m > 0$, this shows that $\zeta_{p^{n+k}} \notin \langle\zeta_{p^\ell}\rangle$ whenever the Schur index is non-trivial, which finishes the proof of the theorem.■

Corollary 4.4.1 *If χ is a faithful character of a metacyclic group of the form $C_{q^s} \rtimes C_{p^{n+k}}$ in which the order of the action is p^n, and $W(\mathbb{Q}_q(\chi), p) = \langle\zeta_{p^\ell}\rangle$, then*

$$m_{\mathbb{Q}_q}(\chi) \leq \min\{p^n, p^\ell\}.$$

Proof. Since $\zeta_{p^k} \in \mathbb{Q}_q(\chi)$, we must have $k \leq \ell$. Therefore, $n + k - \ell \leq n$. Also, $m_{\mathbb{Q}_q}(\chi)$ is the least power of ζ_{p^k} that is a norm from $\mathbb{Q}_q(\zeta_{q^s})$ to $\mathbb{Q}_q(\chi)$, so $k \leq \ell$ implies that $m_{\mathbb{Q}_q}(\chi) \leq p^\ell$. ■

Now suppose $\mathfrak{q}$ lies over the prime 2. We can assume $p = 2$ because as we have pointed out previously, the Schur index is at most 2 in this case. Let

$$\hat{G}/C = \langle\overline{x}\rangle \times \langle\overline{y}\rangle \times \langle\overline{z}\rangle,$$

for some $x, y, z \in \hat{G}$ such that

$$\langle\sigma_x\rangle = Gal(\Bbbk_{\mathfrak{q}}(\zeta_4)/\Bbbk_{\mathfrak{q}}),$$
$$\langle\sigma_y\rangle = Gal(\Bbbk_{\mathfrak{q}}(\zeta_{2^s})/\Bbbk_{\mathfrak{q}}(\zeta_4)), \text{ and}$$
$$\langle\sigma_z\rangle = Gal(\Bbbk_{\mathfrak{q}}(\zeta_r)/\Bbbk_{\mathfrak{q}}),$$

for some odd integer r with $u = 2^s r$, and $s \geq 2$. (Note that if 4 does not divide u, then the extension will be unramified, and thus the algebra $\mathbb{k}_q\hat{G}e_\chi$ will be split.) Since $\zeta_4 \in \mathbb{k}_q(\zeta_{4r})$ implies that $N_{\mathbb{k}_q(\zeta_u)/\mathbb{k}_q(\zeta_{4r})}$ maps ζ_{2^s} to the generator of $W(\mathbb{k}_q(\zeta_{4r}, 2)$, we can apply Theorem 4.2 to show that $\mathbb{k}_q\hat{G}e_\chi$ has the same Schur index as an algebra of the form

$$(\mathbb{k}_q(\zeta_{4r})/\mathbb{k}_q, \gamma)$$

in which values of the factor set γ lie in $W(\mathbb{k}_q(\zeta_{4r}), 2)$ and are a consequence of choosing x and z so that they commute with y. (No extension of the group is needed this time, as C generates enough roots of unity of order a power of 2.)

With this choice of x and y, we have

$$(\mathbb{k}_q(\zeta_{4r})/\mathbb{k}_q, \gamma) \cong \oplus_{i=0}^{1} \oplus_{j=0}^{2^k-1} \mathbb{k}_q(\zeta_{4r})(u_x)^i(u_z)^j,$$

where $2^k = |\mathbb{k}_q(\zeta_r)/\mathbb{k}_q|$. We now try to decompose this algebra into the product of two cyclic algebras.

First, determine the factor set values of γ in $W(\mathbb{k}_q(\zeta_{4r}), 2)$. We find that $(u_x)^2 = \zeta_x = \pm 1$, because $(u_x)^2$ has to be centralized by u_x, and ± 1 are the only roots of unity in $W(\mathbb{k}_q(\zeta_{4r}), 2)$ centralized by σ_x. Let $(u_z)^{2^k} = \zeta_z$, and $u_x u_z = \zeta_{x,z} u_z u_x$. If we can find an element $\omega \in \mathbb{k}_q(\zeta_{4r})$ such that

$$\omega(\omega^{-1})^{\sigma_x} = \zeta_{x,z},$$

then

$$\begin{aligned}(\omega^{-1}u_z)u_x &= \omega^{-1}{\zeta_{x,z}}^{-1}u_x u_z \\ &= u_x(\omega^{-1}{\zeta_{x,z}}^{-1})^{\sigma_x}u_z \\ &= u_x(\omega^{-1}u_z).\end{aligned}$$

So if we replaced u_z by $\omega^{-1}u_z$, then we could decompose the algebra as

$$(\mathbb{k}_q(\zeta_{4r})/\mathbb{k}_q, \gamma) \cong (\mathbb{k}_q(\zeta_4)/\mathbb{k}_q, \sigma_x, \zeta_x) \otimes_{\mathbb{k}_q} (\mathbb{k}_q(\zeta_r)/\mathbb{k}_q, \sigma_y, \epsilon),$$

with

$$\begin{aligned}\epsilon &= (\omega^{-1}u_z)^{2^k} \\ &= N_{\mathbb{k}_q(\zeta_r)/\mathbb{k}_q}(\omega)^{-1}\zeta_z.\end{aligned}$$

Different possibilities for $\zeta_{x,y}$ lead to different ω's and ϵ's, which determie the Schur index of each factor above. Of course, the Schur index of $(\Bbbk_q(\zeta_{4r})/\Bbbk_q, \gamma)$ will be 2 only when exactly one of the factors above has Schur index 2. (The algebra splits when both factors have index 2, because $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{H} \cong \mathbb{Q}^{4\times 4}$.)

If $\zeta_{x,z} = 1$, then u_x and u_z already commute, so the algebra decomposes with $\epsilon = \zeta_z$. Since $\Bbbk_q(\zeta_r)$ is unramified over $\Bbbk_q$ and ϵ is a root of unity, the second factor in this decomposition splits. Thus the Schur index will be 2 only when the first factor is not split. This will be the case if and only if $\zeta_x = -1$, and $|\Bbbk_q : \mathbb{Q}_2|$ is odd, because these are the conditions for the first factor to be a non-split quarternion algebra over $\Bbbk_q$ [FGS].

If $\zeta_{x,z} \neq 1$, then we have to examine the action of u_x on a generator ζ_{2^h} of $W(\Bbbk_q(\zeta_{4r}), 2)$. (The integer h here is at least 2, but can be greater than 2 depending on $\Bbbk_q$.) Since σ_x maps ζ_4 to its inverse, there are two possibilities whenever $h \geq 3$, these being

$$(\zeta_{2^h})^{\sigma_x} = (\zeta_{2^h})^{-1}, \text{ and}$$
$$(\zeta_{2^h})^{\sigma_x} = (\zeta_{2^h})^{(-1+2^{h-1})}.$$

We now need to establish the identity

$$(\zeta_{x,z})^{\sigma_x} = \zeta_{x,z}{}^{-1}.$$

Since $\zeta_x = \pm 1$, we have that $u_x{}^{-1} = \zeta_x u_x$. Thus $u_x u_z = \zeta_{x,z} u_z u_x$ implies that

$$\begin{aligned} u_z &= u_x{}^{-1} \zeta_{x,z} u_z u_x \\ &= \zeta_x (\zeta_{x,z})^{\sigma_x} u_x u_z u_x \\ &= (\zeta_{x,z})^{\sigma_x} \zeta_x \zeta_{x,z} u_z (u_x)^2 \\ &= (\zeta_{x,z})^{\sigma_x} \zeta_{x,z} u_z, \end{aligned}$$

establishing the identity.

If $h \geq 3$ and $(\zeta_{2^h})^{\sigma_x} = (\zeta_{2^h})^{(-1+2^{h-1})}$, then σ_x inverts only even powers of ζ_{2^h}. The above identity then implies that $\zeta_{x,z} = (\zeta_{2^h})^{2b}$ for some integer b. We can then choose $\omega = \zeta_{2^h}{}^{b(1+2^{h-2})}$, and achieve $\omega(\omega^{-1})^{\sigma_x} = \zeta_{x,z}$. Replacing u_z by $\omega^{-1} u_z$ allows for a decomposition of the algebra into a tensor product as above with ϵ being a root of unity. It follows that the second factor will be split, and so the Schur index will be 2 if and only if the Schur index of the first factor is 2.

Now suppose $\zeta_{x,z} \neq 1$ and σ_x inverts ζ_{2^h}. Set $\omega^{-1} = \zeta_4(1 - \zeta_{x,z}{}^{-1})$. Since $\zeta_{x,z} \neq 1$, we have that $\omega^{-1} \neq 0$, and $\omega^{-1}(\omega)^{\sigma_x} = \zeta_{x,z}^{-1}$. Thus if we replace u_z by $\omega^{-1}u_z$, then u_x and u_z will commute, and we get a decomposition of the algebra above. However, this time ϵ is not a root of unity, and so the second algebra may be non-split. Suppose that

$$\epsilon = \pi^t w,$$

with π a uniformizer for the field $\mathbb{k}_q$, and w an integer unit in $\mathbb{k}_q$. Since the extension $\mathbb{k}_q(\zeta_r)$ is unramified over $\mathbb{k}_q$, ϵ will be a norm if and only if π^t is a norm. The norms of elements in the maximal ideal of the integers of $\mathbb{k}_q(\zeta_r)$ are generated by π^{2^k}, because π is also a uniformizer of $\mathbb{k}_q(\zeta_r)$. Therefore, π^t is a norm if and only if 2^k divides t.

Suppose $\zeta_{x,z} = -1$. Then $\epsilon = 2^{2^k}w = (\pi^e)^{2^k}w$, for some integer unit w', with e being the ramification index for the extension $\mathbb{k}_q/\mathbb{Q}_2$. (Since $\mathbb{k}_q(\zeta_u)$ is the cyclotomic field $\mathbb{Q}_2(\zeta_u)$, we find that $\mathbb{k}_q(\zeta_{4r})$ is the field $\mathbb{Q}_2(\zeta_{2^h r})$. This implies that $e = 2^{h-2}$.) Thus $t = e2^k$, and the second factor splits when $\zeta_{x,z} = -1$.

If $\zeta_{x,z}$ has order 2^d, for some $d > 1$, then we use the fact that $N_{\mathbb{Q}_2(\zeta_{2^d})/\mathbb{Q}_2}(1 - \zeta_{2^d}) = 2$ [J, page 541]. This shows that

$$\begin{aligned} N_{\mathbb{k}_q(\zeta_{4r})/\mathbb{Q}_2}(1-\zeta_{2^d}) &= (N_{\mathbb{Q}_2(\zeta_{2^d})/\mathbb{Q}_2}(1-\zeta_{2^d}))^{|\mathbb{k}_q(\zeta_{4r}):\mathbb{Q}_2(\zeta_{2^d})|} \\ &= 2^{|\mathbb{k}_q(\zeta_{4r}):\mathbb{Q}_2(\zeta_{2^d})|}. \end{aligned}$$

On the other hand, up to multiplication by an integer unit in $\mathbb{k}_q$ we have

$$N_{\mathbb{k}_q(\zeta_{4r})/\mathbb{k}(\zeta_4)}(1 - \zeta_{2^d}) = \pi^t.$$

Thus

$$N_{\mathbb{k}_q(\zeta_{4r})/\mathbb{Q}_2}(1 - \zeta_{2^d}) = N_{\mathbb{k}_q/\mathbb{Q}_2}(\pi)^{2t} = 2^{2tf},$$

where f is the residue class degree of the extension $\mathbb{k}_q/\mathbb{Q}_2$. We conclude that $2tf = |\mathbb{k}_q(\zeta_{4r}) : \mathbb{Q}_2(\zeta_{2^d})|$. But

$$\begin{aligned} |\mathbb{k}_q(\zeta_{4r}) : \mathbb{Q}_2(\zeta_{2^d})| &= \frac{|\mathbb{k}_q(\zeta_{4r}) : \mathbb{k}_q|ef}{2^{d-1}} \\ &= \frac{2^{k+1}ef}{2^{d-1}}. \end{aligned}$$

Since $\mathbb{k}_q(\zeta_u)$ has to be the cyclotomic field $\mathbb{Q}_2(\zeta_u)$, we find that $\mathbb{k}_q(\zeta_{4r})$ is the field $\mathbb{Q}_2(\zeta_{2^h r})$. This implies that $e = 2^{h-2}$. Therefore, $t = 2^k 2^{h-d-1}$. Thus

2^k will divide t if $d < h$ and will not divide t only when $d = h$. Therefore, the conditions necessary and sufficient for the second factor to have index 2 are that $\zeta_{x,z}$ has order 2^h, where ζ_{2^h} generates $W(\Bbbk_{\mathfrak{q}}(\zeta_{4r}), 2)$, and σ_x inverts ζ_{2^h}.

We have now shown that the case for localizing at a prime lying over 2 is summarized by the following result.

Theorem 4.5. *Suppose* $\mathfrak{q}$ *is a prime lying over 2, and suppose* $p = 2$. *Write* $u = 2^s r$. *In order to determine the index of the algebra* $\Bbbk_{\mathfrak{q}}\hat{G}e_{\hat{\chi}}$, *we first use Theorem 4.2 to reduce to a crossed product algebra of the form*

$$\oplus_{i=0}^{1} \oplus_{j=0}^{2^k-1} \Bbbk_{\mathfrak{q}}(\zeta_{4r}) {u_x}^i {u_z}^j$$

with factor set contained in $W(\Bbbk_{\mathfrak{q}}(\zeta_{4r}), 2)$ *generated by* ${u_x}^2 = \zeta_x$, $(u_z)^{2^k} = \zeta_z$, *and* ${u_x}^{-1}{u_z}^{-1}u_x u_z = \zeta_{x,z}$. *Then we can find an* $\omega \in \Bbbk_{\mathfrak{q}}(\zeta_{4r})$ *such that replacing* u_z *by* $\omega^{-1}u_z$ *gives a decomposition of the above into cyclic algebras of the form*

$$(\Bbbk_{\mathfrak{q}}(\zeta_4), \sigma_x, \zeta_x) \otimes_{\Bbbk_{\mathfrak{q}}} (\Bbbk_{\mathfrak{q}}(\zeta_r), \sigma_z, \epsilon),$$

with $\epsilon = (N_{\Bbbk_{\mathfrak{q}}(\zeta_r)/\Bbbk_{\mathfrak{q}}}(\omega))^{-1}\zeta_z$. *In any case, the first factor of this decomposition has Schur index 2 if and only if* $|\Bbbk_{\mathfrak{q}} : \mathbb{Q}_2|$ *is odd and* $\zeta_x = -1$. *For the second algebra, the Schur index will be 2 if and only if* σ_x *inverts all roots of unity in* $W(\Bbbk_{\mathfrak{q}}(\zeta_{4r}), 2) = \langle \zeta_{2^h} \rangle$, *and* $\zeta_{x,z}$ *has maximal possible order* 2^h.

We now give examples to illustrate the process developed in Chapter 4.

Example 1: Let $G \cong C_{15} \rtimes Q_8$, where the action of Q_8 on $C_{15} = C_5 \times C_3$ is defined by the following: Let x and y be generators of Q_8, both of order 4, and suppose $a^x = a^{-1}$, $a^y = a$, $b^x = b$, $b^y = b^{-1}$, with $\langle a \rangle = C_5$, and $\langle b \rangle = C_3$. The maximal abelian subgroup of G is $C = C_{15} \times Z(Q_8)$, which has a faithful irreducible character that sends $a \mapsto \zeta_5$ and $b \mapsto \zeta_3$. λ induces to a faithful irreducible character χ of G. The center of the simple component $\mathbb{Q}Ga_\chi$ is the field of character values $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{5})$, which we will denote by $\Bbbk$. A natural isomorphism of G/C with $Gal(\Bbbk(\lambda)/\Bbbk)$ is implied by the above assumptions, given by $x \mapsto \sigma_x$, $y \mapsto \sigma_y$, where $\lambda(ab)^{\sigma_x} = \lambda(a^x b)$ and $\lambda(ab)^{\sigma_y} = \lambda(ab^y)$.

In order to compute the Schur index of the algebra $\Bbbk Ge_\chi$, we must compute the local indices at the primes 2, 3, 5, and an infinite prime, because G/C is a 2-group acting on λ as $Gal(\Bbbk(\lambda)/\Bbbk)$.

If we localize at an infinite prime of $\Bbbk$, we must expect a Clifford theory reduction because G/C has order larger than 2. The field fixed by $\sigma_x\sigma_y$ is real, being $\mathbb{Q}(\zeta_{15} + {\zeta_{15}}^{-1})$, and so a_λ breaks up into the sum of two idempotents in $\mathbb{R}G$. The ensuing Clifford theory reduction gives

$$\mathbb{R}Ge_\chi \cong (\mathbb{R}\hat{G}e_{\chi\hat{G}})^{2\times 2},$$

with $\hat{G} = C_{15} \rtimes \langle xy \rangle$. We have that xy inverts all elements of C_{15}, so since xy has order 4, we have a natural isomorphism of $\mathbb{R}\hat{G}e_{\chi\hat{G}}$ with the real quaternion algebra

$$(\mathbb{C}/\mathbb{R}, \sigma_x\sigma_y, -1).$$

So the Schur index of $\mathbb{k}Ge_\chi$ is at least 2, and is divisible by 2.

If $\mathfrak{q}$ is a prime of $\mathbb{k}$ lying over 2, then we find that $\mathbb{k}_\mathfrak{q}(\lambda) = \mathbb{k}_\mathfrak{q}(\zeta_{15})$. As this extension is unramified over $\mathbb{k}_\mathfrak{q}$, the algebra $\mathbb{k}_\mathfrak{q}Ge_\chi$ must be split, because it is naturally isomorphic to a crossed product algebra with factor set consisting of roots of unity. Thus the 2-local index is 1.

Suppose that $\mathfrak{q}$ lies over 3. There is a natural isomorphism

$$\mathbb{k}_\mathfrak{q}Ge_\chi \cong \oplus_{i,j=0}^{1}\mathbb{k}_\mathfrak{q}(\zeta_{15}){u_x}^i{u_y}^j,$$

where u_x and u_y are independent units satisfying

$$\begin{aligned}
(u_x)^2 &= -1 = (u_y)^2,\\
u_yu_x &= -u_xu_y,\\
(\zeta_5)^{u_x} &= {\zeta_5}^{-1},\\
(\zeta_3)^{u_x} &= \zeta_3,\\
(\zeta_3)^{u_y} &= {\zeta_3}^{-1}, \text{ and}\\
(\zeta_5)^{u_y} &= \zeta_5.
\end{aligned}$$

Now $\mathbb{k}_\mathfrak{q}(\zeta_{15})$ is unramified over $\mathbb{k}_\mathfrak{q}(\zeta_3)$, so by using Theorem 4.2 we can reduce the algebra.

First, we need to find a $\gamma \in W(\mathbb{k}_\mathfrak{q}(\zeta_{15}), 2)$ so that $(\gamma^{-1}u_x)^2 = 1$. This is equivalent to $\gamma(\gamma)^{\sigma_x} = -1$. Since $W(\mathbb{k}_\mathfrak{q}(\zeta_{15}), 2)$ is a subset of the maximal unramified subextension of $\mathbb{k}_\mathfrak{q}(\zeta_{15})/\mathbb{k}_\mathfrak{q}$, we know that $W(\mathbb{k}_\mathfrak{q}(\zeta_{15}), 2) = W(\mathbb{k}_\mathfrak{q}(\zeta_5), 2)$. Since $\mathbb{k} = \mathbb{Q}(\sqrt{5})$, the degree of the extension $\mathbb{k}_\mathfrak{q}(\zeta_5)/\mathbb{k}_\mathfrak{q}$ is 2, so $\mathbb{k}_\mathfrak{q}(\zeta_5) = \mathbb{k}_\mathfrak{q}(\zeta_{3^2-1})$. Thus ζ_8 is a generator for $W(\mathbb{k}_\mathfrak{q}(\zeta_5), 2)$. Since there is no ζ_4 in $\mathbb{k}_\mathfrak{q}$, we must have that

$${\zeta_8}^{\sigma_x} = {\zeta_8}^3.$$

(It has to be either a 3 or a 5, and it cannot fix ${\zeta_8}^2$.) Note that

$$\zeta_8 {\zeta_8}^3 = {\zeta_8}^4 = -1,$$

so ζ_8 is a suitable choice for γ. Put $v_x = {\zeta_8}^{-1} u_x$ as in the proof of Theorem 4.2 to get ${v_x}^2 = 1$.

Next, we need to replace u_y by a unit that will commute with v_x. We have that

$$\begin{aligned} u_y v_x &= u_y {\zeta_8}^{-1} u_x \\ &= {\zeta_8}^{-1} u_y u_x \\ &= -{\zeta_8}^{-1} u_x u_y \\ &= -v_x u_y, \end{aligned}$$

so by the proof of Theorem 4.2 we need to replace u_y by $\delta^{-1} u_y$, for some $\delta \in W(\mathbb{k}_{\mathfrak{q}}(\zeta_{15}), 2)$ satisfying

$$\delta (\delta^{-1})^{\sigma_x} = -1.$$

This time,

$$\zeta_8 ({\zeta_8}^{-1})^{\sigma_x} = \zeta_8 {\zeta_8}^{-3} = {\zeta_8}^{-2},$$

so ζ_4 is an appropriate choice for δ. $v_y = {\zeta_4}^{-1} u_y$ commutes with v_x, and so we have an isomorphism

$$\mathbb{k}_{\mathfrak{q}} G e_\chi \cong \oplus_{i=0}^{1} \mathbb{k}_{\mathfrak{q}}(\zeta_5) {v_x}^i \otimes_{\mathbb{k}_{\mathfrak{q}}} \oplus_{j=0}^{1} \mathbb{k}_{\mathfrak{q}}(\zeta_3) {v_y}^j.$$

Thus the 3-local index of $\mathbb{k} G e_\chi$ is the Schur index of the cyclic algebra

$$(\mathbb{k}_{\mathfrak{q}}(\zeta_3), \sigma_y, {v_y}^2).$$

We have that

$${v_y}^2 = ({\zeta_4}^{-1} u_y)^2 = {\zeta_4}^{-2} (u_y)^2 = 1,$$

so this is a norm, forcing the Schur index of the cyclic algebra to be 1.

Finally, suppose that $\mathfrak{q}$ is a prime lying over 5. Localizing $\mathbb{k} G e_\chi$ with respect to $\mathfrak{q}$ we find a similar isomorphism

$$\mathbb{k}_{\mathfrak{q}} G e_\chi \cong \oplus_{i,j=0}^{1} \mathbb{k}_{\mathfrak{q}}(\zeta_{15}) {u_x}^i {u_y}^j,$$

with u_x and u_y satisfying the same identities as before. Again $W(\mathbb{k}_q(\zeta_{15}),2)$ is contained in the maximal unramified subextension, which this time is $\mathbb{k}_q(\zeta_3) = \mathbb{k}_q(\zeta_{5^2-1})$. Hence ζ_8 again generates $W(\mathbb{k}_q(\zeta_{15}),2)$. Since ζ_4 is in $\mathbb{k}_q$ this time, we have that the galois action on ζ_8 is given by ${\zeta_8}^{u_y} = {\zeta_8}^5$. Replacing u_y by $v_y = {\zeta_4}^{-1}u_y$ gives

$${v_y}^2 = {\zeta_4}^{-1}({\zeta_4}^5)^{-1}{u_y}^2 = (-1)(-1) = 1.$$

Replacing u_x by $v_x = {\zeta_8}^{-1}u_x$ gives

$$\begin{aligned} v_y v_x &= {\zeta_4}^{-1}u_y{\zeta_8}^{-1}u_x \\ &= {\zeta_4}^{-1}{\zeta_8}^{-5}u_y u_x \\ &= -{\zeta_8}^{-5}u_x{\zeta_4}^{-1}u_y \\ &= {\zeta_8}^{-1}u_x v_y \\ &= v_x v_y. \end{aligned}$$

Therefore, we have an isomorphism

$$\mathbb{k}_q G e_\chi \cong \oplus_{i=0}^{1}\mathbb{k}_q(\zeta_5){v_x}^i \otimes_{\mathbb{k}_q} \oplus_{j=0}^{1}\mathbb{k}_q(\zeta_3){v_y}^j,$$

and the Schur index of this algebra is the same as that of the cyclic algebra

$$(\mathbb{k}_q(\zeta_5), \sigma_x, {v_x}^2).$$

We find that

$${v_x}^2 = ({\zeta_8}^{-1}u_x)^2 = {\zeta_8}^{-2}(u_x)^2 = -{\zeta_4}^{-1} = \zeta_4.$$

By [J, Lemma 3.1], ζ_4 is not a norm from $\mathbb{k}_q(\zeta_5)$ because it is not the square of a root of unity in $\mathbb{k}_q(\zeta_5)$. ($W(\mathbb{k}_q(\zeta_5),2)$ is generated by ζ_4.) Thus the 5-local index of $\mathbf{k}Ge_\chi$ is two. The Schur index of the original algebra is also 2, being achieved at primes of $\mathbb{k}$ lying above 5 or an infinite prime.

A final example shows that in the computation of the 2-local index, the second factor in the decomposition of Theorem 4.5 may indeed have Schur index 2. The example is a group having 3 generators that is of the type (QD, q) described in [Sch].

Example 2. Let X be a dihedral group of order 16, with $Z = \langle z \rangle$ its cyclic maximal subgroup of order 8, and x an element of order 2 such that $X = Z \rtimes \langle x \rangle$. Let $P = X \rtimes \langle y \rangle$ be a 2-group of order 32 in which the element y centralizes

Z and $[x, y] = z$. Let U be a cyclic group of order 3, and define G to be a group of the form $U \rtimes P$, with the action defined so that $C_P(U) = X$ and y inverts U. Then $C = U \times Z$ is a cyclic maximal abelian normal subgroup of G, and $G/C = \langle \overline{x}, \overline{y} \rangle$ is easily seen to act as Galois automorphisms on $\mathbb{Q}(\lambda)$, for any faithful irreducible character $\lambda \in Irr(C)$. Thus G has a faithful irreducible character $\chi = \lambda^G$. Working through the proof of Theorem 4.5, we see that ${u_x}^2 = 1$, ${u_y}^2 = 1$, and $u_x u_y = \zeta_8 u_y u_x$ when we choose λ so that $\lambda(z) = \zeta_8$. Using Theorem 4.5, we see that the Schur index of the first factor in the decomposition of $\mathbb{Q}_2(\sqrt{2})Ge_\chi$ will be 1 because ${u_x}^2 = 1$. However, the second factor will necessarily have Schur index 2 because ζ_8 is the maximal possible 2-th power root of unity in $\mathbb{Q}_2(\lambda)$.

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