University of Alberta

THE RICCI FLOW OF ASYMPTOTICALLY HYPERBOLIC MASS

by

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To my mother, Marilyn, and my grandmother, Minnie.

Abstract

In this thesis, we generalize the notion of asymptotically hyperbolic mass (first introduced by Wang in 2001) to manifolds with toroidal ends. Using this generalized definition, we show that under a normalized Ricci flow with asymptotically hyperbolic, conformally compact initial data with a well-defined mass, the mass will decay exponentially in time to zero, in contradistinction to the constant behaviour of asymptotically flat mass under Ricci flow. We then use this result for the evolution of asymptotically hyperbolic mass to prove that there does not exist a breather solution to the normalized Ricci flow with non-zero mass. Further, we provide a proof of the rigidity case of the Positive Mass Theorem in the asymptotically hyperbolic setting, using Ricci flow. We note that this result for the exponential behaviour of asymptotically hyperbolic mass provides support for a conjecture in general relativity stated by Horowitz and Myers.

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Chapter 1

Introduction

In the early 1980's, Richard Hamilton developed the Ricci flow as a means by which he hoped to obtain a proof of the three dimensional Poincaré Conjecture. Though Hamilton was unsuccessful in obtaining such a proof, his Ricci flow approach to such a proof was nonetheless an ingenious technique. In the early 2000's, Grigori Perelman released a series of preprints [28, 30, 29] based on Hamilton's work which provided conclusive Ricci flow arguments for the validity of the three dimensional Poincaré Conjecture.

Soon after Hamilton introduced his Ricci flow, several mathematicians began to ask natural questions such as "What topological properties, geometric properties, or invariants of the initial data are preserved under the Ricci flow?". One of the many qualities of the initial data that was investigated was the property of asymptotic flatness. From [27] and [13], it is known that solutions to the Ricci flow on asymptotically flat manifolds remained asymptotically flat. Additionally, an invariant of asymptotically flat manifolds that has been analyzed is the notion of mass. It was also shown in [27] and [13] that the Arnott-Deser-Misner (ADM) [5] mass of an asymptotically flat manifold is preserved under Ricci flow. One may then wonder, "Do similar results hold for Ricci flow on asymptotically hyperbolic manifolds?" (see definition 2.2 and

2.4). Very recently, it has been proven in [8] that the curvature-normalized Ricci flow given by

$$\frac{\partial g_{ij}}{\partial t} = -2E_{ij} := -2[Ric[g]_{ij} + (n-1)g_{ij}]\,, \tag{1.1}$$

$$g(0) = g_0\,, \tag{1.2}$$

exists for a short time, given asymptotically hyperbolic initial data (M, g_0); further it was also noted that its solutions $g(t)$ remained asymptotically hyperbolic. As well, analogously to asymptotically flat manifolds, there exists an invariant of asymptotically hyperbolic manifolds, called the mass (see definition 2.7). Thus, it is natural to address the question "Is the mass of an asymptotically hyperbolic manifold preserved under this curvature-normalized Ricci flow?". In this paper, we will show that in general the answer is "No.". Curiously, in contrast to the asymptotically flat case, we have found that the mass of an asymptotically hyperbolic manifold is not preserved under this Ricci flow in general, but rather evolves exponentially to zero. That is, we prove the following theorem:

Theorem 1.1: *Let (M, g_0) be an asymptotically hyperbolic manifold of dimension $n \geq 2$, with well-defined mass m_0. If $(M, g(t))$ for $t \in [0, T)$ is a solution of (1.1) arising from initial data (M, g_0), then the mass $m(t)$ of $(M, g(t))$ obeys*

$$m(t) = m_0 e^{-(n-2)t}\,.$$

From Theorem 1.1, we further deduce two immediate corollaries:

Corollary 1.2: *Let (M, g_0) be as in Theorem 1.1. Consider solutions to (1.1) arising from initial data (M, g_0). Then there exist times $t_1 < t_2$ such that $m(t_1) = m(t_2) \iff m(t_1) = m(t_2) = 0$.*

Corollary 1.3: (Non-existence of massive breathers). *Let (M, g_0) be as in*

Theorem 1.1. Let $g(t)$ for $t \in [0,T)$ be a solution of (1.1) arising from initial data (M, g_0). Suppose that $g(t_2) = \varphi^*(g(t_1))$, for $t_1 < t_2$, where φ is a diffeomorphism such that $\varphi - Id_M = o(\rho^{\frac{n}{2}})$, for ρ a defining function of M. Then $m(t_2) = m(t_1) = 0$.

Back in the asymptotically flat setting, recall we have the famous Positive Mass Theorem, proven by Schoen-Yau [36] and subsequently by Witten [40], as well as others (see also [34], [25] [12]), which states that given a complete, asymptotically flat manifold with non-negative scalar curvature, the ADM mass of the manifold is non-negative. Schoen and Yau obtained their proof for the Positive Mass Theorem by minimal surface arguments, while Witten adopted an approach from the perspective of spin structures. It is of interest to obtain an alternative, Ricci flow proof for the Positive Mass Theorem. As of the date of this thesis, no one has yet obtained such an argument. However, in 2010, Haselhofer [20] provided a Ricci flow argument for a corollary of the Positive Mass Theorem, referred to in the literature as the *Rigidity* (or *Scalar Curvature Rigidity*) statement (see [36] and [40]). The Rigidity statement asserts that for an asymptotically flat manifold M satisfying the conditions in the Positive Mass Theorem, if further the ADM mass of M is zero, then M is isometric to Euclidean space.

Analogous to the asymptotically flat case, we also have Positive Mass Theorems for asymptotically hyperbolic manifolds, proven in [2] and [11]. It is also known that Rigidity statements hold in the asymptotically hyperbolic setting, as shown in [2], [11], and [26]. All of these results were obtained from methods other than Ricci flow. However, we follow Haselhofer's idea and similarly obtain a Ricci flow argument for the Rigidity statement in the asymptotically hyperbolic case:

Proposition 1.4: (Rigidity). *Let (M, g_0) be a asymptotically hyperbolic manifold of dimension $3 \leq n \leq 6$. Further, let (M, g_0) be such that the boundary-at-inifinity of M (see definition 2.2) is isometric to S^{n-1}, the scalar curvature* *of M is bounded below by $-n(n-1)$, and (M, g_0) has a well-defined mass $m_0 := m[g_0]$. If M is not spin, further suppose that M has a mass aspect (see definition 2.7) of semi-definite sign. If $m_0 = 0$, then M is isometric to standard hyperbolic space.*

This proposition, as well as Corollary 1.3, then leads us to the following statement:

Corollary 1.5: *Let (M, g_0) be as in Proposition 1.4 but with non-negative mass, and let $g(t)$ be a solution to (1.1) arising from initial data (M, g_0). If $g(t_2) = \varphi^*(g(t_1))$, for some $t_1 < t_2$ and φ as in Corollary 1.3, then (M, g_0) is isometric to standard hyperbolic space.*

We mention here that our Theorem 1.1 provides support for a conjecture put forth by Horowitz and Myers in [22]. Namely, the conjecture states that in the class of all the asymptotically hyperbolic, n-dimensional manifolds which have scalar curvature greater than or equal to $-n(n-1)$ and asymptote to an Anti-deSitter (AdS) soliton (see example 2.5) outside a compact set, the AdS solitons minimize mass. This conjecture may be thought of as a "positive" mass theorem for asymptotically hyperbolic manifolds with boundary-at-inifinity isometric to an $(n-1)$-torus. We give the details of how Theorem 1.1 supports this conjecture in chapter 5 of this thesis.

This thesis is organized as follows. In section 2.A, we introduce models of hyperbolic space and define standard hyperbolic space. In section 2.B, we define conformally compact manifolds and asymptotically hyperbolic manifolds. In section 2.C, we motivate and define asymptotically hyperbolic mass. In section 3.A, we give a brief introduction to Ricci flow. In section 3.B, we introduce the normalized Ricci flow (1.1) on conformally compact manifolds, and calculate the asymptotic behaviour of this flow arising from asymptotically hyperbolic initial data. Then, we calculate the asymptotic behaviour of the E_{ij} term in (1.1), and prove that (1.1) preserves the asymptotic behaviour of the initial data. Finally, we prove Theorem 1.1. In chapter 4, we prove Corollaries 1.2, 1.3, and 1.5, as well as provide our Ricci flow argument for Proposition 1.4. In chapter 5, we provide a discussion of our results and mention applications of our results and areas of future work. We also provide appendices A-D on Riemannian geometry, the equivalence of the Chruściel-Herzlich and extended Wang masses, an alternate proof to Theorem 1.1 using Ricci-DeTurck flow, and the Riemann curvature tensor under a conformal change, respectively.

Conventions: In this thesis, we use *Einstein summation notation* throughout. Thus an appearance of an index in both an up and a down position implies a sum over that index (for example, $E^{ij}E_{ij} := \sum_i \sum_j E^{ij}E_{ij}$), unless explicitly stated otherwise. On a Riemannian manifold (M, g), norms of tensors T on M are given by the Euclidean norm $|T| := [tr(T \cdot T^*)]^{\frac{1}{2}}$, where T^* denotes the metric-dual tensor to T and tr denotes the trace. Further, we define the (rough) Laplacian of a tensor T as $\Delta T := g^{ij}\nabla_i(\nabla_j T)$, where ∇ denotes the Levi-Cività connection associated to g. Lastly, we mention that we choose the sign of the Riemann curvature tensor by defining $Rm[g](W, X, Y, Z) = Rm[g]_{ijkl}W^iX^jY^kZ^l =: g(R(W, X)Y, Z)$, where $R(X, Y)Z := \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$.

Chapter 2

Asymptotically Hyperbolic Manifolds

2.A Hyperbolic Space

As motivation for our definition of asymptotically hyperbolic manifolds, we first introduce the notion of hyperbolic space. We call any representative of the isometry class of simply-connected, complete, n-dimensional Riemannian manifolds which possess constant sectional curvature equal to -1 (up to rescaling) *hyperbolic n-space*. In this section, we shall present a "canonical" model for hyperbolic n-space, which we will define to be *standard hyperbolic space*. However, there are in fact several approaches one may take to model hyperbolic space; an excellent introduction to hyperbolic geometry and several of the models for hyperbolic space is given in [9]. For the purposes of this thesis, we will describe just two approaches to describing hyperbolic space, resulting in the *hyperboloid model* and the *Poincaré ball model*. We note that each of the models of hyperbolic space we present are isometric; thus one may choose any one of them to be the "canonical" reference of hyperbolic space.

We first present the hyperboloid model. To begin, consider

$$\mathbb{R}^{n+1} = \{(t, z) = (t, z^1, z^2, \ldots, z^n) : t\,, z^i \in \mathbb{R}\}\,.$$

Let $\mathbb{H}^n \subset \mathbb{R}^{n+1}$ denote the "upper" sheet of the embedded hyperboloid $t^2 - [(z^1)^2 + \cdots + (z^n)^2] = 1$. That is,

$$\mathbb{H}^n := \{(t, z) = (t, z^1, \ldots, z^n) \,:\, t > 0,\ z^i \in \mathbb{R},\ t^2 - \big[(z^1)^2 + \cdots + (z^n)^2\big] = 1\}\,.$$

Now, $\mathbb{R}^{n+1}$ may be equipped with a *pseudo*-Riemannian metric given by

$$\eta = -dt^2 + (dz^1)^2 + \cdots + (dz^n)^2\,.$$

The pair $(\mathbb{R}^{n+1}, \eta)$ is often referred to as $n+1$-dimensional *Minkowski space.*

Even though η is not a Riemannian metric (in particular, $\eta(X, Y)$ may be less than zero, or we may have $\eta(X, Y) = 0$ for non-zero X, Y), we may still use it to induce a Riemannian metric on $\mathbb{H}^n$. To achieve this induced Riemannian metric on $\mathbb{H}^n$, consider the "spherical coordinates" imbedding

$$F : \mathbb{H}^n \to \mathbb{R}^{n+1}\,, \qquad F(w, \theta^1, \ldots, \theta^{n-1}) = (t, z)\,,$$

where

$$\begin{aligned}
t &= \cosh w\,,\\
z^1 &= \sinh w \cos\theta^1\,,\\
z^2 &= \sinh w \sin\theta^1 \cos\theta^2\,,\\
&\cdots\\
z^{n-1} &= \sinh w \sin\theta^1 \sin\theta^2 \cdots \sin\theta^{n-2} \cos\theta^{n-1}\,,\\
z^n &= \sinh w \sin\theta^1 \sin\theta^2 \cdots \sin\theta^{n-2} \sin\theta^{n-1}\,,
\end{aligned}$$

and where $w \in \mathbb{R}$, $\theta^i \in [0, \pi]$ for $i = 1, \ldots, n-2$ and $\theta^{n-1} \in [0, 2\pi]$. We will use F to pull-back η to a metric on $\mathbb{H}^n$.

We thus compute the differentials

$$
\begin{aligned}
dt &= \sinh w dw \,, \\
dz^1 &= \cosh w \cos\theta^1 dw - \sinh w \sin\theta^1 d\theta^1 \,, \\
dz^2 &= \cosh w \sin\theta^1 \cos\theta^2 dw + \sinh w \sin\theta^1 \cos\theta^2 d\theta^1 - \sinh w \sin\theta^1 \sin\theta^2 d\theta^2 \,, \\
&\quad \ldots \\
dz^{n-1} &= \cosh w \sin\theta^1 \cdots \sin\theta^{n-1} \cos\theta^{n-1} dw \\
&\quad + \sinh w \sin\theta^1 \cos\theta^2 \sin\theta^3 \cdots \cos\theta^{n-1} d\theta^1 + \cdots + \\
&\quad - \sinh w \sin\theta^1 \sin\theta^2 \cdots \sin\theta^{n-2} \sin\theta^{n-1} d\theta^{n-1} \,, \\
dz^n &= \cosh w \sin\theta^1 \cdots \sin\theta^{n-1} \sin\theta^{n-1} dw \\
&\quad + \sinh w \cos\theta^1 \sin\theta^2 \sin\theta^3 \cdots \sin\theta^{n-1} d\theta^1 + \cdots + \\
&\quad - \sinh w \sin\theta^1 \sin\theta^2 \cdots \sin\theta^{n-2} \cos\theta^{n-1} d\theta^{n-1} \,.
\end{aligned}
$$

As well, we find

$$
\begin{aligned}
dt^2 &= \sinh^2 w dw^2 \,, \\
(dz^1)^2 &= \cosh^2 w \cos^2\theta^1 dw^2 - \cosh w \cos\theta^1 \sinh w \sin\theta^1 (d\theta^1)^2 \\
&\quad - \cosh w \cos\theta^1 \sinh w \sin\theta^1 d\theta^1 dw + \sinh^2 w \sin^2\theta^1 dw d\theta^1 \,, \\
&\quad \ldots \\
(dz^n)^2 &= \cosh^2 w \sin^2\theta^1 \cdots \sin^2\theta^{n-1} \sin^2\theta^{n-1} dw^2 \\
&\quad + \cosh w \sinh w \sin\theta^1 \cos\theta^1 \sin^2\theta^2 \cdots \sin^2\theta^{n-1} dw d\theta^1 \\
&\quad + \cosh w \sinh w \sin\theta^1 \cos\theta^1 \sin^2\theta^2 \cdots \sin^2\theta^{n-1} d\theta^1 dw + \cdots + \\
&\quad - \sinh^2 w \sin^2\theta^1 \sin^2\theta^2 \cdots \sin^2\theta^{n-2} \cos^2\theta^{n-1} (d\theta^{n-1})^2 \,.
\end{aligned}
$$

Then, the pull-back of η by the mapping F is the Riemannian metric

$$\begin{aligned} g_H &:= F^*(\eta) \\ &= F^*(-dt^2 + (dz^1)^2 + \cdots + (dz^n)^2) \\ &= \left(-\sinh^2 w + \cosh^2 w \cos^2 \theta^1 + \cdots + \sinh^2 w \sin^2 \theta^1 \cdots \sin \theta^{n-1}\right) dw^2 \\ &\quad + \cdots + \sinh^2 w \sin \theta^1 \cdots \sin \theta^{n-1} (d\theta^{n-1})^2 \\ &= dw^2 + \sinh^2 w \cdot g(S^{n-1}, can)\,, \end{aligned}$$

where of course $g(S^{n-1}, can)$ denotes the round metric on the sphere S^{n-1} with constant sectional curvature equal to $+1$. We call $\mathbb{H}^n$ endowed with the Riemannian metric g_H the *hyperboloid model* of hyperbolic space. Indeed, $(\mathbb{H}^n, g_H)$ is a Riemannian manifold. By construction, $(\mathbb{H}^n, g_H)$ carries the topology of $\mathbb{R}^n$, and hence $(\mathbb{H}^n, g_H)$ is simply connected. Further, the metric g_H is is equivalent to the flat Euclidean metric $\delta = dx_1^2 + \cdots + dx_n^2$, in the sense that there exists a constant $C > 0$ such that $C^{-1} \cdot \delta \leq g_H \leq C \cdot \delta$; hence since δ is complete, g_H is complete. We shall now show that $(\mathbb{H}^n, g_H)$ has constant sectional curvature equal to -1, and thus affirm that $(\mathbb{H}^n, g_H)$ is indeed a model of hyperbolic space.

For computational ease, express $g_H = dw^2 + g_w$, where $g_w = \sinh^2 w \cdot g(S^{n-1}, can)$, and re-label the coordinates as $x^1 = w$, $x^{i+1} = \theta^i$, $i = 1, \ldots, n-1$. We will require the identities

$$\mathcal{L}_{\frac{\partial}{\partial w}} g_w = 2\mathrm{Hess}\, w\,, \tag{2.1}$$

$$\nabla_{\frac{\partial}{\partial w}} \mathrm{Hess}\, w + \mathcal{L}_{\frac{\partial}{\partial w}} \mathrm{Hess}\, w = Rm[g_H]_{i11j}\,. \tag{2.2}$$

Here $\mathrm{Hess}\, w$ denotes the Hessian of the function w, ∇ denotes the Levi-Cività connection of g_H, and $\mathcal{L}$ denotes the Lie derivative. We then compute

$$
\begin{aligned}
2\mathrm{Hess}\, w &= \mathcal{L}_{\frac{\partial}{\partial w}} g_w \\
&= \frac{\partial}{\partial w}\left(\sinh^2 w\right) + \sinh^2 w \cdot \mathcal{L}_{\frac{\partial}{\partial w}} g(S^{n-1}, can) \\
&= 2\frac{\cosh w}{\sinh w} \cdot g_w \,,
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial w}} \mathrm{Hess}\, w &= \nabla_{\frac{\partial}{\partial w}} \left(\frac{\cosh w}{\sinh w} \cdot g_w\right) \\
&= \frac{\partial}{\partial w}\left(\frac{\cosh w}{\sinh w}\right) \cdot g_w + \frac{\cosh w}{\sinh w} \cdot \nabla_{\frac{\partial}{\partial w}} g_w \\
&= \left(1 - \frac{\cosh^2 w}{\sinh^2 w}\right) \cdot g_w + \frac{\cosh w}{\sinh w} \cdot 0 \\
&= g_w - \left(\frac{\cosh w}{\sinh w}\right)^2 \cdot g_w \,,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\frac{\partial}{\partial w}} \mathrm{Hess}\, w &= \mathcal{L}_{\frac{\partial}{\partial w}} \left(\frac{\cosh w}{\sinh w} g_w\right) \\
&= \frac{\partial}{\partial w}\left(\frac{\cosh w}{\sinh w}\right) \cdot g_w + \frac{\cosh w}{\sinh w} \cdot \mathcal{L}_{\frac{\partial}{\partial w}} g_w \\
&= \left(1 - \frac{\cosh^2 w}{\sinh^2 w}\right) \cdot g_w + \frac{\cosh w}{\sinh w} \cdot \left(2\frac{\cosh w}{\sinh w} \cdot g_w\right) \\
&= g_w + \left(\frac{\cosh w}{\sinh w}\right)^2 \cdot g_w \,,
\end{aligned}
$$

We therefore obtain by equation (2.2) that the Riemann curvature of $(\mathbb{H}^n, g)$ has components

$$
Rm[g_H]_{i11j} = (g_w)_{ij} \,,
$$

which gives

$$Rm[g_H]_{i11l} = (g_w)_{il} \neq 0 \quad \text{if } i, l \neq 1, \tag{2.3}$$

$$Rm[g_H]_{i11l} = 0 \quad \text{if either of } i, l \text{ equals } 1\,. \tag{2.4}$$

To compute the other components of $Rm[g_H]$ we employ the Gauss and Codazzi equations, which are respectively

$$Rm[g_w]_{ijkl} = Rm[g]_{ijkl} + K_{il}K_{jk} - K_{ik}K_{jl}\,,$$

$$Rm[g_H]_{ijk1} = \nabla_j K_{ik} - \nabla_i K_{jk}\,,$$

where $K_{ij} := (\mathrm{Hess}\, w)_{ij}$ is the extrinsic curvature of the $w = \textit{constant}$ hypersurfaces of $\mathbb{H}^n$, and $i, j, k, l \neq 1$.

Since K_{ij} only depends on w, the Codazzi equation tells us $Rm[g_H]_{ijk1} = 0$; the symmetry properties of the Riemann curvature tensor of g_H then imply $Rm[g_H]_{ijkl} = 0$ if any one of the indices i, j, k, l is 1. Also, since the $(n-1)$-sphere with the canonical round metric $g(S^{n-1}, can)$ has constant sectional curvature $+1$, and $g_w = \sinh^2 w \cdot g(S^{n-1}, can)$, we have $Rm[g_w]_{ijkl} = \frac{1}{\sinh^2 w}[(g_w)_{ik}(g_w)_{jl} - (g_w)_{il}(g_w)_{jk}]$. Thus from Gauss' equation, for $i, j, k, l \neq 1$ we have

$$Rm[g_H]_{ijkl} = Rm[g_w]_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \tag{2.5}$$

$$= \frac{1}{\sinh^2 w}\left[(g_w)_{ik}(g_w)_{jl} - (g_w)_{il}(g_w)_{jk}\right] \tag{2.6}$$

$$+ \frac{\cosh w}{\sinh w}\left[(g_w)_{ik}(g_w)_{jl} - (g_w)_{il}(g_w)_{jk}\right]$$

$$= \frac{1 - \cosh^2 w}{\sinh^2 w}\left[(g_w)_{ik}(g_w)_{jl} - (g_w)_{il}(g_w)_{jk}\right] \tag{2.7}$$

$$= -\left[(g_w)_{ik}(g_w)_{jl} - (g_w)_{il}(g_w)_{jk}\right]\,. \tag{2.8}$$

Thus, since $g_H = dw^2 + g_w$, from (2.3), (2.4), and (2.8) we find

$$Rm[g_H]_{ijkl} = -\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) .$$

This implies that $(\mathbb{H}^n, g_H)$ has constant sectional curvature equal to -1. As well, we note that the Ricci curvature obeys $Ric[g] = -(n-1)g$ and the scalar curvature obeys $R[g] = -n(n-1)$.

We now derive a re-expression of g_H which will be a useful reference in later chapters. First, define the diffeomorphism

$$G : \mathbb{H}^n \to \mathbb{H}^n , \qquad G(x, \theta) = \left(\log\left(\frac{e^x + 1}{e^x - 1}\right), \theta\right) =: (w, \theta) ,$$

where $\theta = (\theta^1, \ldots, \theta^n)$. Then, pulling g_H back by G, we get another Riemannian metric on $\mathbb{H}^n$, isometric to g_H, given by

$$\begin{aligned} h &= G^*(g_H) \\ &= \frac{1}{\sinh^2 x}\left(dx^2 + g(S^{n-1}, can)\right) . \end{aligned}$$

This now leads us to our first definition.

Definition 2.1: Throughout this thesis, we define the model of hyperbolic space given by $(\mathbb{H}^n, h)$ as *standard hyperbolic space.*

Alternatively, we may construct another model of hyperbolic space, called the *Poincaré ball model.* To construct the Poincaré ball model, consider again $\mathbb{H}^n = \{(t, z) = (t, z^1, \ldots, z^n) \,:\, t > 0,\ z^i \in \mathbb{R}\,, t^2 - [(z^1)^2 + \cdots + (z^n)^2] = 1\}$, but now equipped with the pseudo-Riemannian metric η restricted to $\mathbb{H}^n$. As well, consider the n-ball $\mathbb{B}^n := \{y = (y^1, \ldots, y^n) : y^i \in \mathbb{R},\ (y^1)^2 + \cdots + (y^n)^2 < 1\}$. Again, we will derive a hyperbolic, Riemannian metric on $\mathbb{B}^n$ by pulling back η by a diffeomorphism. The diffeomorphism we shall employ is the stereographic projection

$$S : \mathbb{B}^n \to \mathbb{H}^n \,, \qquad S(y^1, \dots, y^n) = \left(\frac{1 + ||y||^2}{1 - ||y||^2}, \frac{2y}{1 - ||y||^2} \right) =: (t, z) \,,$$

where $||y||^2 := (y^1)^2 + \cdots + (y^n)^2$.

We compute the differentials dt, dz^i, $i = 1, \dots, n$ to be

$$\begin{aligned} dt &= \frac{\sum_{k=1}^n 2y^k dy^k}{1 - ||y||^2} + \frac{(1 + ||y||^2) \sum_{k=1}^n 2y^k dy^k}{(1 - ||y||^2)^2} \,, \\ dz^i &= \frac{2dy^i}{1 - ||y||^2} + \frac{2y^i \sum_{k=1}^n 2y^k dy^k}{(1 - ||y||^2)^2} \,. \end{aligned}$$

Hence,

$$\begin{aligned} dt^2 &= \frac{16(\sum_{k=1}^n 2y^k dy^k)^2}{(1 - ||y||^2)^4} \,, \\ (dz^i)^2 &= \frac{4(dy^i)^2}{(1 - ||y||^2)^2} + \frac{8(1 - ||y||^2) y^i dy^i \sum_{k=1}^n 2y^k dy^k}{(1 - ||y||^2)^4} + \frac{16(y^i)^2 (\sum_{k=1}^n 2y^k dy^k)^2}{(1 - ||y||^2)^4} \,. \end{aligned}$$

The metric pulled-back via the diffeomorphism S is then

$$\begin{aligned} g_P &:= S^*(\eta) \\ &= S^*(-dt^2 + (dz^1)^2 + \cdots + (dz^n)^2) \\ &= -\frac{16(\sum_{k=1}^n 2y^k dy^k)^2}{(1 - ||y||^2)^4} + \frac{4||dy||^2}{(1 - ||y||^2)^2} + \frac{8(\sum_{k=1}^n 2y^k dy^k)^2}{(1 - ||y||^2)^4} \\ &\quad - \frac{8||y||^2)(\sum_{k=1}^n 2y^k dy^k)^2}{(1 - ||y||^2)^4} + \frac{16||y||^2 (\sum_{k=1}^n 2y^k dy^k)^2}{(1 - ||y||^2)^4} \\ &= \frac{4||dy||^2}{(1 - ||y||^2)^2} \,. \end{aligned}$$

We call $(\mathbb{B}^n, g_P)$ the *Poincaré ball model* of hyperbolic space. Considering the diffeomorphism

$$G \circ F^{-1} \circ S : \mathbb{B}^n \to \mathbb{H}^n \,,$$

we have the pullback satisfies

$$\begin{aligned}(G \circ F^{-1} \circ S)^*(h) &= S^* \circ F^{-1^*} \circ G^*(h) \\ &= S^* \circ F^{-1^*}(g_H) \\ &= S^*(\eta) \\ &= g_P\,;\end{aligned}$$

hence $(\mathbb{B}^n, g_P)$ and $(\mathbb{H}^n, h)$ are isometric as Riemannian manifolds. Thus, $(\mathbb{B}^n, g_P)$ is also a model of hyperbolic space.

Switching perspective slightly, we highlight here that, given the Poincaré model $(\mathbb{B}^n, g_P)$, we may obtain a smooth metric $\tilde{g}_P := (dy^1)^2 + \cdots + (dy^n)^2$ extended on the closed ball $\bar{\mathbb{B}}^n := \mathbb{B}^n \cup \partial\mathbb{B}^n = \mathbb{B}^n \cup S^{n-1}$, upon multiplying the hyperbolic metric g_P by the square of the smooth function $\rho(y) := \frac{1}{2}(1 - ||y||^2)$. Alternatively, from the hyperboloid model point-of-veiw, by conformally transforming h by the smooth function $\rho(x) := \sinh x$, we achieve a smooth metric $\tilde{h}$ on $\mathbb{B}^n$, given by

$$\tilde{h} := \rho^2(x)h = dx^2 + g(S^{n-1}, can)\,,$$

which we may smoothly extend to the closed ball $\bar{\mathbb{B}}^n$. Thus, in some sense, we have just "compactified" these models of hyperbolic space given by either the open manifold $(\mathbb{B}^n, g_P)$ or the open manifold $(\mathbb{H}^n, h)$, to the respective closed manifold $(\bar{\mathbb{B}}^n, \tilde{g}_P)$ or $(\bar{\mathbb{B}}^n, \tilde{h})$, by a conformal transformation. We will elaborate on this observation in the next section.

2.B Asymptotically Hyperbolic Manifolds[1]

Let (M, g) be a smooth, n-dimensional Riemannian manifold.

There are two main approaches to defining an asymptotically hyperbolic man-

[1] A version of this section has been published. T Balehowsky, E Woolgar 2012. Journal of Mathematical Physics. 53: 072501.

ifold. One method is to first define what is called a conformal completion of the manifold (see [21], [11], and [2]), and then impose certain restrictions on the conformal completion. The alternative is a coordinate-dependent approach which specifies an asymptotic expansion for the Riemannian metric in a given neighbourhood of infinity. This latter method is presented in [21] and [11]. In [21], it is proven that these two definitions are equivalent. Thus, we may view asymptotically hyperbolic manifolds (M, g) as either Riemannian manifolds which possess a conformal completion with restrictions, or alternatively as Riemannian manifolds for which the metric g realizes a certain asymptotic expansion in some local coordinate system. We shall take the conformal completion approach.

Definition 2.2: We say that the manifold M is *(smoothly) conformally compactifiable* if there exists a smooth, compact, n-dimensional manifold with boundary $\widetilde{M} := M \cup \partial M$, and a smooth function $\rho : \widetilde{M} \to \mathbb{R}$ satisfying $\rho(p) = 0 \iff p \in \partial M$ and $d\rho(p) \neq 0 \; \forall p \in \partial M$, such that the metric $\tilde{g} := \rho^2 g$ extends to a smooth metric on $\widetilde{M}$. In this case, we call the triple $(\widetilde{M}, \tilde{g}, \rho)$ a *conformal completion* of the manifold M. The manifold ∂M is referred to as the *boundary-at-infinity* of M; we also refer to the function ρ as a *defining function* for M.

Example 2.3: The compact Riemannian manifold $(\bar{\mathbb{B}}^n, \tilde{g}_P)$, as given in section 2.A, along with the function $\rho(y) := \frac{1}{2}(1 - ||y||^2)$ defines a conformal completion $(\bar{\mathbb{B}}^n, \tilde{g}_P = \delta, \rho)$ of the Poincaré ball. Likewise, $(\bar{\mathbb{B}}^n, \tilde{h}, \rho)$ where ρ is the function $\rho(x) = \sinh x$ is a conformal completion of standard hyperbolic space $(\mathbb{H}^n, h)$.

Now, suppose (M, g) is a conformally compactifiable, n-dimensional manifold.

Definition 2.4: Given a conformal completion $(\widetilde{M}, \tilde{g}, \rho)$ of our manifold

(M, g) as above, if we have in addition $|d\rho|^2_{\tilde{g}} = 1 + O(\rho^l)$, [2] [3] for some $l \geq 1$ on ∂M, we define (M, g) to be *asymptotically hyperbolic (of order l)*.

Remark 2.5: If (M, g) is asymptotically hyperbolic with defining function ρ, and ρ satisfies $|d\rho|^2_{\tilde{g}} \equiv 1$ on a neighbourhood of ∂M, then ρ is called a *special defining function*. However, for the rest of this thesis, we will generally not assume that the defining function of an asymptotically hyperbolic manifold is special.

We now provide the motivation behind the above definition of "asymptotically hyperbolic". From a standard calculation (see appendix D), we have that in a nieghbourhood of some $p \in M$, if $\tilde{g} = \rho^2 g$, the Riemann curvature changes according to

$$\begin{aligned} Rm[g]_{ijkl} = {} & \frac{1}{\rho^2} Rm[\tilde{g}]_{ijkl} + \frac{1}{\rho^4}|d\rho|^2_{\tilde{g}}(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}) \\ & + \frac{1}{\rho^3}\left[\tilde{g}_{jk}\tilde{\nabla}_i\tilde{\nabla}_l\rho + \tilde{g}_{il}\tilde{\nabla}_j\tilde{\nabla}_k\rho - \tilde{g}_{ik}\tilde{\nabla}_j\tilde{\nabla}_l\rho - \tilde{g}_{lj}\tilde{\nabla}_i\tilde{\nabla}_k\rho\right] . \end{aligned}$$

Hence, the components of the sectional curvatures of any 2-plane at p may be expressed as

$$\begin{aligned} \frac{Rm[g]_{ijkl}}{g_{ik}g_{jl} - g_{il}g_{jk}} = {} & |d\rho|^2_{\tilde{g}} + \rho^2 \frac{Rm[\tilde{g}]_{ijkl}}{\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}} \\ & + \rho\, \frac{\tilde{g}_{jk}\tilde{\nabla}_i\tilde{\nabla}_l\rho + \tilde{g}_{il}\tilde{\nabla}_j\tilde{\nabla}_k\rho - \tilde{g}_{ik}\tilde{\nabla}_j\tilde{\nabla}_l\rho - \tilde{g}_{lj}\tilde{\nabla}_i\tilde{\nabla}_k\rho}{\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}} . \end{aligned} \tag{2.9}$$

Now, since $\tilde{g}$ extends smoothly to ∂M, if we take the limit as $\rho \to 0$ (or equivalently the limit as $p \to \partial M$), we obtain that the sectional curvature, denoted $sec(X, Y)$, of a 2-plane at the point p spanned by tangent vectors X, Y, obeys

$$sec(X, Y) = -1 + O(\rho^l) ,$$

[2] We define $|d\rho|^2_{\tilde{g}} := \tilde{g}^{ij}\partial_i\rho\partial_j\rho$.

[3] Here $u = O(\rho^s)$ means $|u| < C|\rho^s|$ as $\rho \to 0$ for some constant $C > 0$.

for some $l \geq 1$ if and only if $|d\rho|^2_{\tilde{g}} = 1 + O(\rho^l)$ for some $l \geq 1$. It is this behaviour of the sectional curvatures that leads to our notion of "asymptotically hyperbolic".

We further comment that if $|d\rho|^2_{\tilde{g}} = 1 + O(\rho^l)$ for some $l \geq 1$, the components of the Riemann curvature tensor satisfy

$$Rm[g]_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk}) + O(\rho^l)\,;$$

and so we have also

$$|E_{ij}| := |Ric[g]_{ij} + (n-1)g_{ij}| = O(\rho^l)\,.$$

In other words, the Ricci curvature of asymptotically hyperbolic manifolds satisfy $Ric[g]_{ij} = -(n-1)g_{ij} + O(\rho^l)$ and scalar curvature satisfies $R = -n(n-1) + O(\rho^l)$, where $l \geq 1$.

Example 2.6: (Examples of Asymptotically Hyperbolic Manifolds) Let g be a Riemannian metric on a conformally compactifiable manifold M of dimension ≥ 3, such that in local radial coordinates, g may be expressed as[4]

$$g = \frac{dr^2}{r^2\left(1 - \frac{1}{r^n}\right)} + r^2\left[\left(1 - \frac{1}{r^n}\right)d\xi^2 + \sum_{i=3}^{n} d\theta_i^2\right]\,, \tag{2.10}$$

where $r \in [1, \infty)$, $\xi \in [0, \frac{4\pi}{n}]$, and $\theta_i \in [0, a_i]$ for some parameters $0 < a_1 \leq \cdots \leq a_n$. We call the family of Riemannian manifolds (M, g) *Anti-de Sitter (AdS) solitons*[5], since they are spacelike hypersurfaces of the Anti-de Sitter soliton spacetimes presented in [22] (see also [1] and [17]). AdS solitons are conformally compactifiable, with defining function $\rho = \frac{1}{r}$ and boundary-at-infinity given by an $(n-1)$-torus equipped with the flat metric $d\xi^2 + \sum_{i=3}^{n} d\theta_i^2$,

[4]The expression for g often contains a parameter r_0, which we have scaled away. As well, in the literature the expression (2.10) may also be written with a parameter ℓ, called the *radius of curvature at infinity*, which we have normalized to 1 in (2.10).

[5]Note that the noun "soliton" as used in this setting does not refer to *Ricci solitons*.

where the coordinates parametrizing the *cycles* of the torus take values in $\xi \in [0, \frac{4\pi}{n}]$ and $\theta_i \in [0, a_i]$, for $0 < a_1 \leq \cdots \leq a_n$. From the expression of the AdS soliton metric in (2.10), we see $|d\rho|^2_{\tilde{g}} = 1 + O(r^{-n})$. Making the change of coordinates $x = \frac{1}{r}$ for comparison purposes, we obtain $|d\rho|^2_{\tilde{g}} = 1 + O(x^n)$. Hence, AdS solitons are asymptotically hyperbolic.

Again, let (M, g) be a conformally compactifiable, n-dimensional manifold. In an *end*[6] of M given by $(0, \epsilon) \times \partial M$, we may define coordinates $(x, y^A)_{A \in \{1,\ldots,n-1\}}$ with respect to the boundary-at-infinity. That is, we choose x so that $\frac{\partial}{\partial x}$ defines a normal vector to ∂M, and $(y^A)_{A \in \{1,\ldots,n-1\}}$ define coordinates on the boundary ∂M. For notational convenience, we will occasionally write $x^1 = x$, $x^i = y^{i-1}$, $i \in \{2, \ldots, n\}$.

Let $\hat{g}$ be the metric induced by $\tilde{g}$ on the boundary-at-infinity ∂M. If (M, g) is asymptotically hyperbolic, we may express the metric g as the expansion

$$g = \rho^{-2} \left(dx^2 + \hat{g}_{AB} dy^A dy^B + \frac{x^l}{l} \kappa_{ij} dx^i dx^j + O(x^{l+1}) \right) \tag{2.11}$$

for some $l \geq 1$ and functions $\kappa_{ij} := \kappa(y)_{ij}$ of the coordinates $y := (y^A)$, $A = 1, \ldots, n-1$, such that $\kappa = \kappa_{ij} dx^i dx^j$ defines a symmetric (0,2)-tensor on the $x =$ *constant* hypersurfaces of M. In this setting, a tensor $v = v_{ij} dx^i dx^j \in O(x^p)$ means that the component functions v_{ij} are such that $|v_{ij}| \leq C|x|^p$ as $x \to 0$, for all i, j, where C is a positive constant.

For the majority of this thesis, we will be concerned with the class of asymptotically hyperbolic manifolds (M, g) for which ∂M may be equipped with either the round $(n-1)$-sphere metric (and thus $(\partial M, \hat{g})$ is isometric to S^{n-1}), or a metric of constant sectional curvature 0. In these cases, we have that the

[6] also called a *collar neighbourhood* of ∂M.

metric induced by g on ∂M is $\hat{g} = g_{(k)}$, for $k = 0$ or 1, where

$$g_{(1)} = \text{the canonical round metric on } S^{n-1}\text{, and} \tag{2.12}$$

$$g_{(0)} = \text{a metric of constant sectional curvature 0.} \tag{2.13}$$

As motivated by the calculations in the previous section, and as shown in [2] and [11], in these cases we may choose our defining functions to be respectively

$$\rho_{(1)}(x) = \sinh(x)\,, \text{ and} \tag{2.14}$$

$$\rho_{(0)}(x) = x\,. \tag{2.15}$$

Thus in these special cases the metric takes the form

$$g = \rho_{(k)}^{-2}\left(dx^2 + g_{(k)AB}dy^A dy^B + \frac{x^l}{l}\kappa_{ij}dx^i dx^j + O(x^{l+1})\right)\,, \tag{2.16}$$

for $k = 0, 1$ respectively, and for some $l \geq 1$.

Now, for this restricted class of asymptotically hyperbolic manifolds for which ∂M can be endowed with a metric of constant sectional curvature 0 or +1, we may make the change of coordinates

$$y^A \;\rightarrow\; y^A + \frac{x^{l+1}}{(l+1)l}g_{(k)}^{AC}\kappa_{1C}\,, \tag{2.17}$$

$$x \;\rightarrow\; x + \frac{x^{l+1}}{2l^2}\kappa_{11}\,, \tag{2.18}$$

to re-express the metric from the form as given in (2.11) into the form

$$g = \rho_{(k)}^{-2}(x)\left[dx^2 + g_{(k)} + \frac{x^l}{l}\left(\kappa_{AB} + \frac{\kappa_{11}}{l}g_{(k)AB}\right)dy^A dy^B + O(x^{l+1})\right] \tag{2.19}$$

$$= \rho_{(k)}^{-2}(x)\left(dx^2 + g_{(k)} + \frac{x^l}{l}\sigma_{AB}dy^A dy^B + O(x^{l+1})\right)\,. \tag{2.20}$$

Note here we have slightly abused notation by denoting our new coordinates also as $(x, y^A)_{A\in\{1,\ldots,n-1\}}$.

2.C Asymptotically Hyperbolic Mass

To motivate the definition of mass on an asymptotically hyperbolic manifold, we provide the following heuristic explanation. Consider a closed physical system. In such a physical system, one can define the notion of mass as the total mass of matter contained in the system plus the total energy contained in the system. Since the physical system is closed, conservation of energy tells us that, regardless of the physical interactions taking place within the system, the mass of the system is invariant. Now suppose the physical system we wish to analyze exists in some (pseudo-) Riemannian manifold M. According to Einstein's theory of general relativity, the physical mass-energy in the system determines the geometry of M. So from a geometrical point of view, the notion of the mass of the physical system must be a quantity which broadly tells us something about the geometry of M. Further, if the notion of mass is viewed as a geometrical quantity, it should be an invariant of M, and hence not depend on which coordinate system of M we choose to work with.

Following along the lines of the above heuristic argument, when the physical system is represented by an asymptotically hyperbolic manifold, the extension of the notion of mass as the total material mass contained in the system plus the total energy contained in the system gives rise to the idea of asymptotically hyperbolic mass. Geometrically speaking, in the case where M has a boundary-at-infinity isometric to a round sphere, due to the Rigidity statement (see [11], [2], [26], or chapter 4 of this thesis) we therefore may think of asymptotically hyperbolic mass as a measure of "how much" the geometry of an asymptotically hyperbolic manifold with non-zero mass differs from the geometry of standard hyperbolic n-space.

2.C.1 The Extended Wang Mass[7]

Let (M, g) be an asymptotically hyperbolic, n-dimensional manifold whose boundary-at-infinity is diffeomorphic to the $(n-1)$ sphere. As explained previously, we may write the metric in the form

$$g = \sinh(x)^{-2}\left(dx^2 + g_{(1)AB}dy^A dy^B + \frac{x^l}{l}\sigma_{AB}dy^A dy^B + O(x^{l+1})\right)$$

where $g_{(1)}$ is the canonical round metric on S^{n-1} and $\sigma = \sigma_{AB}dy^A dy^B$ is a symmetric $(0,2)$ tensor on ∂M. Suppose $l = n$. In [39], Wang defined the invariant

$$m_W[g] := \int_{S^{n-1}} g_{(1)}^{AB}\sigma_{AB}\, d\mu_{g_{(1)}}$$

as the mass of the manifold (M, g).

We also would like to consider the case where (M, g) is an asymptotically hyperbolic, n-dimensional manifold whose boundary-at-infinity is isometric to a flat $(n-1)$-torus. A notion of mass which takes into account the cases where the boundary-at-infinity of M can be equipped with a metric of constant sectional curvature -1, 0, or $+1$, has been introduced by Chruściel and Herzlich in [11]. In [11], it is noted that the Chruściel and Herzlich definition of mass coincides with Wang's definition in the case when $\partial M \equiv S^n$. We also note that Chruściel and Herzlich's definition has the advantage of being expressed without a need for a coordinate basis [11]. As well, by first proving that there exists a conformal isometry (possibly only defined for sufficiently small x) from one conformal completion of (M, g) to any other conformal completion of (M, g), Chruściel and Herzlich's definition of asymptotically hyperbolic mass is seen to be independent of the respective conformal completion chosen for (M, g) [11]. However, despite these advantages, the Chruściel and Herzlich's mass defini-

[7]A version of this subsection has been published. T Balehowsky, E Woolgar 2012. Journal of Mathematical Physics. 53: 072501.

tion is somewhat inconvenient for our calculations. Therefore, we will extend Wang's definition above to include the case where ∂M can be equipped with a metric of constant sectional curvature 0.

Definition 2.7: Given an asymptotically hyperbolic, n-dimensional manifold (M, g) for which the metric can be written in the form of (2.20) with $l = n$, we define the *mass* of (M, g) as the finite quantity

$$m[g] := \int_{\partial M} \sigma \, d\mu_{g_{(k)}} \,, \tag{2.21}$$

where $d\mu_{g_{(k)}}$ denotes the volume form on ∂M with respect to the metric $g_{(k)}$. Here $\sigma := g^{AB}_{(k)} \sigma_{AB}$ is referred to as the *mass aspect* of (M, g).

Of course, we must verify that our extended mass definition remains an invariant of the manifold; in other words, we must check that our mass definition is independent of the coordinate system chosen on M. To confirm coordinate independence of our extended Wang mass, we show in appendix B that our mass definition coincides with the definition of mass given by Chruściel and Herzlich.

Example 2.8: Let (M, g) be an AdS soliton of dimension $n \geq 3$, with g given as in (2.10). Scaling the parameters a_i appropriately, one may compute that the mass of the soliton is $m[g] = -\frac{4\pi}{n} \prod_{i=3}^{n} a_i$ (see [22]). Note that the mass is negative! For physical interpretations of what this negative mass may mean, see [22]. In addition, for an explicit calculation of the mass of a 3-dimensional AdS soliton, please see appendix B.

Chapter 3

The Ricci Flow

3.A Hamilton's Ricci Flow and DeTurck's Trick

In efforts to construct a proof of the 3-dimensional Poincaré Conjecture, Hamilton devised a means by which one could attempt to deform the metric of a given compact manifold to an Einstein metric [19]. He called the deformation a *Ricci flow*. A Ricci flow on a given manifold (M, g_0) is a 1-parameter family of Riemannian metrics $g(t)$ on M which satisfy

$$\frac{\partial g}{\partial t}(t) = -2Ric[g(t)], \tag{3.1}$$

$$g(0) = g_0\,. \tag{3.2}$$

Notice that the fixed points (up to homothetic rescaling) of the Ricci flow equation satisfy $Ric[g] + kg = 0$ for some constant k, and thus are Einstein manifolds.

One drawback of the Ricci flow is that it is not parabolic; indeed, the linearization of the right-hand-side of equation (3.1) is

$$\left.\frac{\partial Ric[g]_{ij}}{\partial s}\right|_{s=0} = -\Delta_L Ric[g]_{ij} + \nabla_i(\Delta^k Ric[g]_{jk}) + \nabla_j(\Delta^k Ric[g]_{ik}) - \nabla_i \nabla_j (R[g])\,, \tag{3.3}$$

where $\Delta_L a_{ij} := \Delta a_{ij} - Ric[g]_i^k a_{jk} - Ric[g]_j^k a_{ik} + Rm[g]_{ij}^{kl} a_{kl} + Rm[g]_{ji}^{kl} a_{kl}$ is the *Lichnerowicz* laplacian. One may compute that the principal symbol of the operator in (3.3) is only semi-definite; by definition, this means that the Ricci flow equation is not parabolic. This lack of parabolicity implies that one cannot use standard theory for parabolic partial differential equations to prove existence or uniqueness of solutions to the Ricci flow (Hamilton invoked Nash-Moser theory to obtain existence). However, in 1983, DeTurck provided a solution to the inconvenience of the non-parabolicity of the Ricci flow, which became known as "DeTurck's Trick" [14].

DeTurck's approach was to first consider a time-independent background metric on M, which we may take to be the initial metric g_0. Then, define a vector field on $(M, g := g(t))$ by $X(t) = X^k \partial_k = g^{ij}(\Gamma_{ij}^k - \mathring{\Gamma}_{ij}^k)\partial_k$, where Γ_{ij}^k and $\mathring{\Gamma}_{ij}^k$ are the Christoffel symbols with respect to $g = g(t)$ and g_0 respectively. Let φ_t be a family of diffeomorphisms generated by $X(t)$; that is, φ_t solves the initial value problem $\frac{\partial \varphi_t}{\partial t} = -X(t)$, $\varphi_0 = Id$ (we know solutions exist to this partial differential equation due to Picard's Existence Theorem). Instead of directly looking for solutions to the flow in (3.1), DeTurck considered the system

$$\begin{aligned} &\frac{\partial q}{\partial t}(t) = -2Ric[q(t)] + \mathcal{L}_{X(t)} q(t), \\ &q(0) = g_0\,. \end{aligned}$$

(Here $\mathcal{L}_{X(t)} q(t)$ is the Lie derivative of $q(t)$ with respect to the vector field $X(t)$.) This new system is called the *Ricci-DeTurck flow.* DeTurck then showed that Ricci-DeTurck flow was parabolic. This was the most crucial step; once DeTurck verified parabolicity, using standard partial differential equations theory he readily obtained existence and uniqueness of solutions $q(t)$, $t \in [0, T]$, to the Ricci-DeTurck flow.

DeTurck's next move was to then show that the pullback metrics $g(t) := \varphi_t^*(q(t))$ were unique solutions to the original Ricci flow in (3.1). This is straightforward to check. Indeed, by the definition of the Lie derivative, we obtain for $F(u, v) := \varphi_u^*(q(v))$,

$$\begin{aligned}\frac{\partial g}{\partial t}(s) &= \frac{\partial}{\partial t} F(s, s) \\ &= \frac{\partial}{\partial u} \varphi_u^*(q(s))\Big|_{u=s} + \frac{\partial}{\partial v} \varphi_s^*(q(v))\Big|_{v=s} \\ &= \varphi_s^* \left(\mathcal{L}_{-X(s)} q(s)\right) + \varphi_s^* \left(\frac{\partial}{\partial v} q(v)\Big|_{v=s} \right) .\end{aligned}$$

Then, we further compute

$$\begin{aligned}\frac{\partial g}{\partial t}(t) &= \varphi_t^* \left(\mathcal{L}_{X(t)} q(t)\right) + \varphi_t^* \left(\frac{\partial}{\partial t} q(t) \right) \\ &= -\varphi_t^* \left(\mathcal{L}_{X(t)} q(t)\right) + \varphi_t^* \left(-2Ric[q(t)] + \mathcal{L}_{X(t)} q(t)\right) \\ &= -\varphi_t^* \left(-2Ric[q(t)]\right) \\ &= -2Ric[g(t)] .\end{aligned}$$

Hence $g(t) := \varphi_t^*(q(t))$ is a unique solution to the Ricci flow.

Now that we know solutions to (3.1) exist, let $g(t)$ be such a solution. In some cases, it is useful to hold certain properties of the solution $g(t)$ (such as volume, if defined) which would otherwise be time-dependent to be constant under the Ricci flow. Alternatively, one may find it useful to rescale the time interval for which $g(t)$ exists. In either situation, one may do this as follows: Let $(M, g(s))$ be a Riemannian manifold with $g(s)$ a solution to $\frac{\partial g}{\partial s} = -2Ric[g(s)]$ for $s \in [0, S]$. Set $s = B(t)$ for a smooth re-parameterization $B(t) : [0, T] \to [0, S]$, and define a new metric on M by $\bar{g}(t) := A(t) g(B(t))$, where $A(t)$ is a smooth, time-dependent function on M. Then, we calculate

$$\begin{aligned}
\frac{\partial \bar{g}}{\partial t}(t) &= A'(t)g(s) + A(t)\frac{\partial g}{\partial s}(s)B'(t) \\
&= A'(t)g(s) + \frac{\partial g}{\partial s}A(t)B'(t) \\
&= \frac{A'(t)}{A(t)}\bar{g}(t) - 2Ric[g(s)]A(t)B'(t) \\
&= \frac{A'(t)}{A(t)}\bar{g}(t) - 2Ric\left[\frac{\bar{g}(t)}{A(t)}\right]A(t)B'(t)\,.
\end{aligned}$$

Since the Ricci tensor is invariant under rescaling, we have derived the flow

$$\frac{\partial \bar{g}}{\partial t}(t) = \frac{A'(t)}{A(t)}\bar{g}(t) - 2Ric(\bar{g}(t))\,. \tag{3.4}$$

Equation (3.5) is called a *normalized* Ricci flow. In the case where (M, g_0) has defined volume, if we choose $B(t) = t$ and $A(t) = \int_M R[g(t)]d\mu_{g(t)}$, we obtain a flow whose fixed points have the same volume as (M, g_0), namely

$$\frac{\partial g}{\partial t}(t) = -2Ric(g(t)) + \left[\frac{\int_M R[g(t)]d\mu_{g(t)}}{\int_M d\mu_{g(t)}}\right]g(t)\,. \tag{3.5}$$

This flow is called the *volume normalized* Ricci flow equation.

In the next section, we will turn our focus to a Ricci flow normalized such that fixed points to the flow have sectional curvature equal to -1. However, for more information about Ricci flow and its properties, please see the excellent resources [10] and [38].

3.B A Normalized Ricci Flow[1]

We are ultimately interested in studying Ricci flow arising from asymptotically hyperbolic initial data. Given this motivation, we first consider the Ricci flow

[1] A version of this section has been published. T Balehowsky, E Woolgar 2012. Journal of Mathematical Physics. 53: 072501.

(3.1) with initial data given by a smooth, n-dimensional, conformally compact Riemannian manifold (M, g_0). Setting

$$A(t) := e^{-2(n-1)t} \quad \text{and} \quad B(t) := \frac{e^{2(n-1)t} - 1}{2(n-1)}$$

in equation (3.5) gives us the curvature-normalized Ricci flow of M as in equation (1.1), restated below for convenience:

$$\begin{aligned} &\frac{\partial g_{ij}}{\partial t} = -2E_{ij} := -2[Ric[g]_{ij} + (n-1)g_{ij}]\,, \\ &g(0) = g_0\,. \end{aligned}$$

In [37], Suneeta studied perturbations of hyperbolic space, and showed that the perturbed metrics converged to a hyperbolic metric on $\mathbb{H}^n$ under linearized approximations of the flows (1.1) and (3.1). Li and Lin in [24] also analyzed the normalized flow (1.1) in the case where the initial data was a slight perturbation of hyperbolic space, and obtained both existence and convergence results if the perturbation decayed exponentially. Later, Schnürer, Schulze, and Simon [35] extended the non-linear results of Li and Lin to more relaxed perturbations of hyperbolic space. Recently, using the DeTurck trick, Bahuaud [8] has proven that short-time solutions to the flow (1.1) with smooth conformally compact initial data exist and are unique. In [8], it has also been shown that the short-time solution of the flow given in (1.1) preserves the quality of conformal compactness. In other words, if (M, g_0) has a conformal completion $(M, \tilde{g}_0, \rho(0))$, and if $g(t)$ is a solution to (1.1) for some $t \in [0, T)$, then there exists a conformal completion $(\widetilde{M}, \tilde{g}(t), \rho(t))$ of $(M, g(t))$. Further, Bahuaud noted that if in addition we have $|dx|^2_{\tilde{g}_0} = 1$ on ∂M, then at each time t one can choose a defining function such that the flow given by (1.1) preserves this property. That is, if (M, g_0) is asymptotically hyperbolic with $l = 1$, then Bahuaud showed $(M, g(t))$ for $t \in [0, T)$ will also be asymptotically hyperbolic with $l = 1$. Further, Bahuaud [8] proved long-time existence of solutions to (1.1) under certain asymptotic assumptions. Assuming certain initial curvature and volume bounds, Qing, Shi, and Wu were also able to prove in [33] both short-time and long-time existence of solutions to the normalized flow (1.1), and in addition, obtained convergence results for solutions of (1.1) arising from asymptotically hyperbolic initial data with certain curvature and volume bounds.

In what follows, we will show that for solutions $g(t)$ to the flow (1.1) arising from initial data given by an asymptotically hyperbolic, n-dimensional manifold (M, g_0) whose conformal completion $(\widetilde{M}, \tilde{g}_0, \rho_0)$ has the boundary-at-infinity isometric to either a flat $(n-1)$-torus or S^{n-1} (and hence g_0 is of the form (2.16)), the initial defining function $\rho_0 := \rho_{(k)}$, as given by (2.14), (2.15) respectively, remains a good defining function for $(M, g(t))$. Further we will calculate the asymptotic behaviour of E_{ij} for an asymptotically hyperbolic, n-dimensional manifold (M, g_0) with g_0 expressed as in (2.16). In addition, we will use this calculation of the asymptotic behaviour of E_{ij} to show that, more precisely, the flow (1.1) preserves the asymptotic structure of the initial data — if (M, g_0) is asymptotically hyperbolic with an expansion beginning at order l, then $(M, g(t))$ for $t \in [0, T]$ will also be asymptotically hyperbolic with an expansion beginning at order l. We ultimately will use these results to prove Theorem 1.1.

3.B.1 The Local Expression of Solutions

Consider the flow (1.1) arising from initial data given by an asymptotically hyperbolic, n-dimensional manifold (M, g_0) whose conformal completion $(\widetilde{M}, \tilde{g}_0, \rho_0)$ has the boundary-at-infinity isometric to either a flat torus or S^{n-1}. We set $\rho_0 := \rho_{(k)}$, for $\rho_{(k)}$ given respectively by (2.14) or (2.15). We provide the following argument first noted in [8], which asserts that our initial defining function ρ_0 is also a defining function for $(M, g(t))$, where $g(t)$ is some solution to (1.1).

In [8], Bahuaud proves existence and uniqueness of solutions to such a flow by employing the DeTurck trick. In his choice of local coordinate system, he obtains solutions to the Ricci-DeTurck flow of (1.1) are of the form

$$g(t) = \frac{dx^2 + \hat{h} + v(t, x, y^A)}{x^2}\,,$$

where $\hat{h}$ is a metric on the boundary-at-infinity ∂M, (y^A) are coordinates on ∂M, and $v(t, x, y^A) = O(x^2)$.

Let $X = X^k \frac{\partial}{\partial x^k}$ denote the DeTurck vector field of the associated Ricci-DeTurck flow to (1.1), and let φ_t denote the 1-parameter family of diffeomorphisms induced by the vector field X. We compute that the pullback metric $\varphi_t^*(g(t))$ is given as

$$\begin{aligned}
\varphi_t^*(g(t)) &= \frac{\varphi_t^*(dx^2 + \hat{h} + v(t, x, y^A))}{(x \circ \varphi_t)^2}\,, \\
&= \frac{dx^2 + \hat{h} + v(t, x, y^A) + (\varphi_t^* - Id)(dx^2 + \hat{h} + v(t, x, y^A))}{(x \circ \varphi_t)^2}\,, \\
&= \frac{dx^2 + \hat{h} + \bar{v}(t, x, y^A)}{(x \circ \varphi_t)^2}\,,
\end{aligned}$$

where $\bar{v}(t, x, y^A) := v(t, x, y^A) + (\varphi_t^* - Id)(dx^2 + \hat{h} + v(t, x, y^A))$. From [8], $X \equiv 0$ on the boundary-at-infinity, and so in Bahuaud's chosen coordinate system, the components of X must obey $X^k = O(x^2)$; thus we deduce

$$\frac{\partial(x^k \circ \varphi_t)}{\partial t} = O(x^2)\,,$$

and integrating with respect to t we find the pulled-back coordinates are of the form

$$(x^k \circ \varphi_t) = x^k + O(x^2)\,,$$

where $u = O(x^s)$ means $|u| < c(t, y^A)|x^s|$ as $x \to 0$ for some constant $c(t, y^A) >$

0 and $t \in [0,T]$. Then, differentiating with respect to x^i, we get

$$\varphi_t^* = \delta_i^k + O(x)\,.$$

Hence, we obtain that the pullback φ_t^* satisfies

$$(\varphi_t^* - Id)_j^k = O(x)\,.$$

Therefore, the pulled-back metric $\varphi_t^*(g(t))$ remains asymptotically hyperbolic. Further, in our choice of local coordinate system, we obtain

$$\varphi_t^*(\rho(t)) = \rho_0 + O(x)\,.$$

So, we may express the the solutions $g(t)$ to (1.1) in the general form

$$g(t) = \rho_{(k)}^{-2}\left(dx^2 + g_{(k)AB}dy^A dy^B + \frac{x^l}{l}\kappa_{ij}(t)dx^i dx^j + O(x^{l+1})\right) \tag{3.6}$$

where $1 \leq l$, where $\kappa(y,t)_{ij}$ is as in (2.11), and $g_{(k)}$ is as in (2.12), (2.13), and $\rho_{(k)}$ is as in (2.14), (2.15), respectively.

3.B.2 The Asymptotic Behaviour of E_{ij}

Let (M,g) be an asymptotically hyperbolic, n-dimensional manifold whose conformal completion $(\widetilde{M}, \tilde{g}_0, \rho_0)$ has the boundary-at-infinity isometric to either a flat $(n-1)$-torus or a round S^{n-1}. Write g as in the form of (2.16). Now let $\hat{h}$ denote the metric induced by $\tilde{g}$ on the $x = \textit{constant}$ hypersurfaces of $\widetilde{M}$. Thus $\hat{h} = g_{(k)AB}dy^A dy^B + \frac{x^l}{l}\kappa_{AB}dy^A dy^B + O(x^{l+1})$. Further, define the following "curvature" quantities $K_{AB} := \operatorname{Hess} x = \frac{1}{2}\partial_x \hat{h}_{ab} + O(x^l)$ and $H := \hat{h}^{AB}K_{AB} + O(x^l)$ (note these quantities only approximate extrinsic and mean curvature, respectively, since $\frac{\partial}{\partial x}$ is not a unit vector). To compute $E_{ij} = Ric[g]_{ij} + (n-1)g_{ij}$, we will require the following standard equations:

The Ricatti equation:

$$R^{1}_{A1B} = K_{AC}K^{C}_{B} - \frac{\partial}{\partial x}K_{AB} + O(x^{2l-2})\,. \tag{3.7}$$

Gauss' Equations:

$$Rm[\hat{h}]_{ijkl} = Rm[\tilde{g}]_{ijkl} + K_{il}K_{jk} - K_{ik}K_{jl} + O(x^{2l})\,, \tag{3.8}$$

$$Ric[\hat{h}]_{AB} = Ric[\tilde{g}]_{AB} - Rm[\tilde{g}]_{1AB1} + HK_{AB} - K_{AC}K^{C}_{B} + O(x^{2l})\,. \tag{3.9}$$

The Codazzi Equation:

$$Rm[\tilde{g}]_{ABC1} = \nabla_B K_{AC} - \nabla_A K_{BC} + O(x^{l})\,. \tag{3.10}$$

The Conformal Change of Ricci Curvature Equation:

$$Ric[g]_{ij} = Ric[\tilde{g}]_{ij} + \frac{1}{\rho}\left[(n-2)\tilde{\nabla}_i\tilde{\nabla}_j\rho + \tilde{g}_{ij}\tilde{\Delta}\rho\right] - (n-1)\tilde{g}_{ij}\frac{|\tilde{\nabla}\rho|^2}{\rho^2}. \tag{3.11}$$

For computational convenience, denote

$$B_{ij} = \frac{1}{\rho}\left[(n-2)\tilde{\nabla}_i\tilde{\nabla}_j\rho + \tilde{g}_{ij}\tilde{\Delta}\rho\right] - (n-1)\tilde{g}_{ij}\frac{|\tilde{\nabla}\rho|^2}{\rho^2}\,.$$

Then equation (3.11) can be succinctly rewritten as

$$Ric[g]_{ij} = Ric[\tilde{g}]_{ij} + B_{ij}\,. \tag{3.12}$$

Now, using (3.7), (3.8), we obtain

$$\begin{aligned} Ric[\tilde{g}]_{11} &= -\frac{(l-1)}{2}\hat{h}^{AB}\kappa_{AB}x^{l-2} + O(x^{l-1}) \\ &= -\frac{(l-1)}{2}{g_{(k)}}^{AB}\kappa_{AB}x^{l-2} + O(x^{l-1})\,. \end{aligned}$$

Using (3.10),

$$\begin{aligned} Ric[\tilde{g}]_{1A} &= \nabla^B K_{AB} - \nabla_A(\hat{h}^{BC} K_{BC}) + O(x^l) \\ &= O(x^{l-1})\,, \end{aligned}$$

and by (3.7), (3.8), as well as using Taylor's theorem, we have

$$Ric[\tilde{g}]_{AB} = k(n-2){g_{(k)}}_{AB} - \frac{(l-1)}{2}\kappa_{AB}x^{l-2} + O(x^{l-1})\,.$$

In a similar fashion, we calculate the the B_{ij} term in (3.11):

$$\begin{aligned} B_{11} &= -\frac{(n-1)}{\rho^2_{(k)}(t)} + \left[\frac{1}{2}{g_{(k)}}^{AB}\kappa_{AB} - \frac{(n-1)}{2}\kappa_{11}\right] x^{l-2} + O(x^{l-1})\,, \\ B_{1A} &= O(x^{l-1})\,, \\ B_{AB} &= \left[k - (n-1)\frac{(d\rho_{(k)})^2}{\rho^2_{(k)}}\right] {g_{(k)}}_{AB} + \left[\frac{(n-2)}{2} + \frac{(k-1)(n-1)}{l}\right]\kappa_{AB}x^{l-2} \\ &\quad + \left[\frac{1}{2}{g_{(k)}}^{CD}\kappa_{CD} - \frac{1}{2}\kappa_{11} + \frac{(n-1)}{l}\kappa_{11}\right]{g_{(k)}}_{AB}x^{l-2} + O(x^{l-1})\,. \end{aligned}$$

Therefore by (3.12), we compute

$$\begin{aligned} Ric[g]_{11} &= -\frac{(n-1)}{\rho^2_{(k)}(t)} - \left[\frac{l-2}{2}{g_{(k)}}^{AB}\kappa_{AB} + \frac{(n-1)}{2}\kappa_{11}\right] x^{l-2} + O(x^{l-1})\,, \\ Ric[g]_{1A} &= O(x^{l-1})\,, \\ Ric[g]_{AB} &= (n-1)\left[k - \frac{(d\rho_{(k)})^2}{\rho^2_{(k)}}\right] {g_{(k)}}_{AB} + \left[\frac{(n-l-1)}{2} + \frac{(k-1)(n-1)}{l}\right]\kappa_{AB}x^{l-2} \\ &\quad + \left[\frac{1}{2}{g_{(k)}}^{CD}\kappa_{CD} - \frac{1}{2}\kappa_{11} + \frac{(n-1)}{l}\kappa_{11}\right]{g_{(k)}}_{AB}x^{l-2} + O(x^{l-1})\,. \end{aligned}$$

Then, given $E_{ij} = Ric[g]_{ij} + (n-1)g_{ij}$, we obtain the expansions

$$E_{11} = -\left[\frac{l-2}{2}{g_{(k)}}^{AB}\kappa_{AB} + \frac{(l-2)(n-1)}{2l}\kappa_{11}\right] x^{l-2} + O(x^{l-1})\,, \quad (3.13)$$

$$E_{1A} = O(x^{l-1})\,, \quad (3.14)$$

$$\begin{aligned} E_{AB} = &\left[\frac{1}{2}{g_{(k)}}^{CD}\kappa_{CD} - \frac{1}{2}\kappa_{11} + \frac{(n-1)}{l}\kappa_{11}\right] g_{(k)AB}x^{l-2} \\ &+\frac{(n-l-1)}{2}\kappa_{AB}x^{l-2} + O(x^{l-1})\,. \end{aligned} \quad (3.15)$$

We immediately obtain from the above calculations that asymptotically hyperbolic, Einstein manifolds with a boundary-at-infinity is isometric to either a round sphere or a flat torus cannot have a non-zero mass. We note that this result was shown in [4] for the case where the boundary-at-infinity is isometric to a round sphere.

Lemma 3.1: *Let (M, g) be an asymptotically hyperbolic, Einstein manifold of dimension $n \geq 2$. Then (M, g) has a well-defined mass $m[g] = 0$.*

Proof: If (M, g) is Einstein, then $E_{ij} = 0$ by definition. In particular, $E_{11} = 0$. From equation (3.7) with $l = n$, we have $E_{11} = \frac{n-2}{2}\sigma$. Thus $E_{11} = 0 \Rightarrow \sigma = 0$, and hence $m[g] = 0$.

□

3.B.3 The Ricci flow of Asymptotically Hyperbolic Manifolds

Now that we know that $\rho_{(k)}$ is a good defining function throughout the flow (1.1), and have determined an expansion for E_{ij} in terms of x, we prove the following:

Proposition 3.2: *Let (M, g_0) be asymptotically hyperbolic of order $l \geq 1$ and g_0 of the form of (2.16), with $\kappa_{11} = \kappa_{1A} = \kappa_{A1} = 0$. If $g(t)$ is a solution to*

(1.1) arising from initial data g_0, then $g(t)$ is also asymptotically hyperbolic of order l.

Proof: Suppose $g(t)$ has an expansion beginning at order $m \geq 1$. From (1.1), we have $\frac{x^m}{m}\frac{\partial \kappa_{ij}}{\partial t} = -2E_{ij}$. Then, by equations (3.13) – (3.15), we obtain $\frac{\partial \kappa_{ij}}{\partial t} = A_{ij}{}^{kl}\kappa_{kl}$, where A is a matrix with components

$$\begin{aligned}
A_{11}{}^{11} &= (m-2)(n-1) + O(x)\,, \\
A_{11}{}^{AB} &= m(m-2){g_{(k)}}^{AB} + O(x)\,, \\
A_{1A}{}^{kl} &= O(x)\,, \\
A_{AB}{}^{11} &= [-m + 2(n-1)]]\, {g_{(k)}}_{AB} + O(x)\,, \\
A_{AB}{}^{CD} &= -m{g_{(k)}}^{CD}{g_{(k)}}_{AB} - m(n-m-1)\delta_A^C\delta_B^D + O(x)\,.
\end{aligned}$$

Thus at each order, $\frac{\partial \kappa_{ij}}{\partial t} = A_{ij}{}^{kl}\kappa_{kl}$ is a linear, partial differential equation in the functions κ_{ij}. Take the limit as $x \to 0$. For $m < l$, we have $\kappa_{ij}(0) = 0$; thus by uniqueness theory for linear partial differential equations, $\kappa_{ij}(t) = 0$ for $m < l$. This gives the required result.

□

3.B.4 The Evolution of Asymptotically Hyperbolic Mass

Now we have the necessary information to enable us to prove Theorem 1.1, which we restate below.

Theorem 1.1: *Let (M, g_0) be an asymptotically hyperbolic manifold of dimension $n \geq 2$, with well-defined mass $m_0 := m[g_0]$. If $(M, g(t))$ for $t \in [0, T)$ is a solution of (1.1) arising from initial data (M, g_0), then the mass $m(t)$ of $(M, g(t))$ obeys*

$$m(t) = m_0 e^{-(n-2)t}\,.$$

Proof: By Proposition 3.2, we know $(M, g(t))$ has a well-defined mass. Now, recall that the mass aspect is defined as $\sigma = g_{(k)}^{AB}\sigma_{AB} = g_{(k)}^{AB}\kappa_{AB} + \frac{n-1}{n}\kappa_{11}$.

We first calculate the evolution of the κ_{11} component:

$$\begin{aligned}
\frac{\partial \kappa_{11}}{\partial t} &= \frac{n}{x^{n-2}}\left(-2E_{11}\right) \\
&= \frac{-2n}{x^{n-2}}\left(-\frac{(n-1)(n-2)}{2n}\kappa_{11} - \frac{n(n-2)}{2n}g_{(k)}^{AB}\kappa_{AB}\right)x^{n-2} + O(x) \\
&= (n-1)(n-2)\kappa_{11} + n(n-2)g_{(k)}^{AB}\kappa_{AB} + O(x) \\
&= n(n-2)\left(g_{(k)}^{AB}\kappa_{AB} + \frac{n-1}{n}\kappa_{11}\right) + O(x) \\
&= n(n-2)\sigma + O(x)\,.
\end{aligned}$$

The trace term evolves as

$$\begin{aligned}
\frac{\partial}{\partial t}\left(g_{(k)}^{AB}\kappa_{AB}\right) &= \frac{\partial g_{(k)}^{AB}}{\partial t}\kappa_{AB} + g_{(k)}^{AB}\frac{\partial \kappa_{AB}}{\partial t} \\
&= 0\cdot\kappa_{AB} + g_{(k)}^{AB}\left(-2nx^{n-2}E_{AB}\right) + O(x) \\
&= ng_{(k)}^{AB}\kappa_{AB} - n(n-1)g_{(k)}^{AB}\kappa_{AB} \\
&\quad -(n-1)(n-2)\kappa_{11} + O(x) \\
&= -n(n-2)\left(g_{(k)}^{AB}\kappa_{AB} + \frac{n-1}{n}\kappa_{11}\right) + O(x) \\
&= -n(n-2)\sigma + O(x)\,.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\frac{\partial \sigma}{\partial t} &= \frac{\partial}{\partial t}\left(g_{(k)}^{AB}\kappa_{AB} + \frac{n-1}{n}\kappa_{11}\right) \\
&= \frac{\partial}{\partial t}\left(g_{(k)}^{AB}\kappa_{AB}\right) + \frac{n-1}{n}\frac{\partial \kappa_{11}}{\partial t} \\
&= -n(n-2)\sigma + \frac{n-1}{n}[n(n-2)\sigma] + O(x) \\
&= -n(n-2)\sigma + (n-1)(n-2)\sigma + O(x) \\
&= -(n-2)\sigma + O(x)\,.
\end{aligned}$$

Hence, integrating with respect to t, we have the mass aspect evolves under the normalized flow (1.1) according to

$$\sigma(t) = \sigma_0 e^{-(n-2)t} + O(x)\,,$$

where $\sigma_0 = \sigma(0)$. Therefore, integrating over ∂M, gives us the final result

$$m(t) = m_0 e^{-(n-2)t}\,, \tag{3.16}$$

where $m_0 = m(0)$ is the initial mass.

$\square$

Chapter 4

Consequences of the Mass Evolution[1]

From Theorem 1.1, we immediately deduce the following consequences. The first consequence is that if the mass is zero initially, then throughout the flow (1.1) the mass remains zero..

Lemma 4.1: *Let (M, g_0) be an asymptotically hyperbolic manifold of dimension $n \geq 2$, with well-defined mass $m[g_0] = 0$. If $(M, g(t))$ is a solution of (1.1) arising from initial data (M, g_0), then $m(t) = m[g_0] = 0$.*

Proof: We have $m(t) = m[g_0]e^{-(n-2)t} = 0$.

□

We also are able to prove

Corollary 1.2: *Let (M, g_0) be as in Theorem 1.1. Consider solutions to (1.1) arising from initial data (M, g_0). Then there exist times $t_1 < t_2$ such that $m(t_1) = m(t_2) \iff m(t_1) = m(t_2) = 0$.*

Proof: Let $m_0 := m[g_0]$ denote the initial mass. Suppose that there exist

[1] A version of this chapter has been published. T Balehowsky, E Woolgar 2012. Journal of Mathematical Physics. 53: 072501.

times $0 \leq t_1 \leq t_2$ such that $m(t_1) = m(t_2)$. Then by Theorem 1.1 we have $m_0 e^{-(n-2)t_1} = m(t_1) = m(t_2) = m_0 e^{-(n-2)t_2} \iff m_0 = 0$.

$\square$

Now, Theorem 1.1 not only provides us with the behaviour of asymptotically hyperbolic mass under our curvature-normalized Ricci flow, but it also allows us to deduce results about the existence of breathers and solitons of the flow (1.1). We recall the definition of breathers and solitons of (1.1) below.

Definition 4.2: Let M be a conformally compactifiable manifold, and let $\varphi : M \to M$ be a smooth diffeomorphism on M. Consider a solution $g(t)$ of the flow (1.1), for $t \in [0, T)$. We say that $g(t)$ is a *breather* of (1.1) if $g(t_1) = \varphi^*(g(t_2))$ for some times $t_2 \neq t_1$. If we have $g(t) = \varphi^*(g(0))$ for all $t \in [0, T)$, then we call $g(t)$ a *soliton* of the flow (1.1).

Keeping in mind the above definition, and using Theorem 1.1, we obtain that there do not exist breathers to (1.1) which have non-zero mass. We note that a version of this result was found in [16].

Corollary 1.3: (Non-existence of massive breathers). *Let (M, g_0) be as in Theorem 1.1. Let $g(t)$ for $t \in [0, T)$ be a solution of (1.1) arising from initial data (M, g_0). Suppose that $g(t_2) = \varphi^*(g(t_1))$, for $t_1 < t_2$, where φ is a diffeomorphism such that $\varphi - Id_M = o(\rho^{\frac{n}{2}})$, for ρ a defining function of M. Then $m(t_2) = m(t_1) = 0$.*

Proof: Given φ is a diffeomorphism such that $\varphi - Id_M = o(\rho^{\frac{n}{2}})$, from Theorem 2.3 of [11] (see also Theorem 3.4 of [21]) we have $m(t_2) = m(\varphi^*(g(t_1))) = \varphi^*(m(t_1)) = m(t_1)$. Then, by Corollary 1.2, we obtain the result.

$\square$

In addition to the above results, the behaviour of asymptotically hyperbolic mass under (1.1) also enables us to provide a Ricci flow proof for the Rigidity statement of the positive mass theorem. We now set out to provide this argument. For reference, the Positive Mass Theorem for asymptotically hyperbolic manifolds is stated below. In preparation for the statement of the theorem, we mention that the mass aspect σ is called *semi-definite* if for all $p, q \in M$, we have $\sigma(p)\sigma(q) \geq 0$.

Theorem 4.3 (Positive Mass). *Let (M, g) be a complete, asymptotically hyperbolic manifold of dimension $3 \leq n \leq 6$. Further, let (M, g) be such that the boundary-at-infinity of M is isometric to S^{n-1}, $R[g] + n(n-1) \geq 0$, and (M, g) has a well-defined mass, $m[g]$. If M is not spin, further suppose that the mass aspect of M is of semi-definite sign. Then $m[g] \geq 0$.*

Proof: For the case where M is a spin manifold, see [11], [21], and [39]. For the case when M has a mass aspect is of semi-definite sign, see [2].

$\square$

Proposition 1.4: (Rigidity). *Let (M, g) be as in Theorem 4.3. If $m[g] = 0$, then M is isometric to standard hyperbolic space.*

Proof: If we have $E[g]_{ij} = Ric[g]_{ij} + (n-1)g_{ij} \equiv 0$, then by the Einstein rigidity result of Qing [32], we have M is isometric to standard hyperbolic space, so there is nothing to show.

Thus, for contradiction, suppose $E[g]_{ij} \neq 0$ at some $p \in M$. Let $E[g] := R[g] + n(n-1)$, where $R[g]$ denotes the scalar curvature of g. Without loss of generality, we may suppose that at $p \in M$, we have $E[g]|_p > 0$. Else, consider the flow $g(t)$ arising from (1.1) with g as initial data. By Theorem A of [8], $g(t)$ exists for some $t \in [0, T]$. Let $0 < t_1 < T$. Recall by Proposition 3.2, $g(t_1)$ is asymptotically hyperbolic, and by Theorem 1.1 and Lemma 4.1, $g(t_1)$ has well-defined mass $m(t_1) = 0$. Using the facts that $\frac{\partial}{\partial t} g^{ij} = -g^{ik}g^{jl}\frac{\partial}{\partial t} g_{kl}$, and that Ricci curvature $Ric[g] := R_{ij}dx^i \otimes dx^j$ evolves under (3.1) according to $\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2R_{kijl}R^{kl} - 2R_{ik}R^k_j$ (see [10]), we compute that under the flow

(1.1), E_{ij} evolves as

$$\begin{aligned}\frac{\partial E}{\partial t} &= \frac{\partial}{\partial t}\left(g^{ij}E_{ij}\right)\\ &= g^{ij}\frac{\partial E_{ij}}{\partial t} + \frac{\partial g^{ij}}{\partial t}E_{ij}\\ &= g^{ij}\left(\Delta E_{ij} + 2R_{kijl}R^{kl} - 2R_{ik}R^k_j\right) + E^{ij}E_{ij}\\ &= \Delta E - 2(n-1)E + E^{ij}E_{ij}\,,\end{aligned}$$

where R_{kijl} denotes the components of the Riemann curvature tensor in local coordinates. So if initially we have $E|_p(0) \geq 0$, by the maximum principle $E|_p(t) \geq 0$ for all $t \in (0,T)$. Further, if $E|_p(0) = 0$ and $E_{ij}|_p(0) \neq 0$, then we observe from the evolution of E that $E|_p(t) > 0$ for all $t \in (0,T)$. Thus, taking g to be $g(t_1)$, our supposition is valid.

Thus we are in the situation where (M,g) is as in Theorem 4.3 with $m[g] = 0$, and there is a $p \in M$ such that $E[g]_{ij}|_p \neq 0$ and $E|_p(t) > 0$.

Consider now the conformal change $\bar{g} = w^{\frac{4}{n-2}}g$, where w is a smooth, positive function on M which solves the so-called *Yamabe* problem:

$$\Delta_{g(t)}w - R[g]w = n(n-1)w^{\frac{n+2}{n-2}}\,,$$
$$w|_{\partial M} = 1\,.$$

By [6] and later [3], there exists such a function w on M. Note that $(M,\bar{g})$ still satisfies the properties required in Theorem 4.3. By Proposition 3.13 of [2], we obtain that the mass aspects of (M,g) and $(M,\bar{g})$, denoted $\sigma[g]$ and $\sigma[\bar{g}]$ respectively, obey $\sigma[\bar{g}] < \sigma[g]$ pointwise. Integrating over ∂M, we find $m[\bar{g}] < m[g] = 0$. This violates Theorem 4.3. So we obtain a contradiction; hence our supposition was incorrect and we must have had $E[g]_{ij} = 0$ pointwise. Again, applying the Einstein rigidity result of Qing [32] proves the claim.

$\square$

From Proposition 1.4, we readily obtain a Rigidity corollary:

Corollary 1.5: *Let (M, g_0) be as in Theorem 4.3, and let $g(t)$ be a solution to (1.1) arising from initial data (M, g_0). If $g(t_2) = \varphi^*(g(t_1))$, for some $t_1 < t_2$ and φ as in Corollary 1.3, then (M, g_0) is isometric to standard hyperbolic space.*

Proof: By Corollary 1.3, we have $m(t_1) = m(t_2) = 0$. Proposition 1.4 gives the required result.

$\square$

Chapter 5

Discussion[1]

In section 3.B, we have shown that solutions $g(t)$ of the curvature-normalized flow (1.1) arising from an initial asymptotically hyperbolic manifold (M, g_0), where g_0 is of the form (2.20), may be also written in the form of (2.20) (via the coordinate change given by (2.17) and (2.18); see Proposition 3.2). We have also shown that, under the curvature-normalized flow (1.1) arising from an initial asymptotically hyperbolic manifold (M, g_0) with a well-defined mass m_0, the mass evolves exponentially as $m(t) = m_0 e^{-(n-2)t}$ (see Theorem 1.1). In chapter 4, we then used these two results to deduce various corollaries, including one that proved the non-existence of breathers with non-zero mass of (1.1). We mention here that, as alternatively argued in section 4.3 of [16], our result for the non-existence of massive breathers of (1.1) facilitates the use of the associated Ricci-DeTurck flow to construct an algorithm for numerically finding Einstein metrics on asymptotically hyperbolic manifolds. Further, we also recall that our result that asymptotically hyperbolic mass must evolve as $m(t) = m_0 e^{-(n-2)t}$ under (1.1) allowed us to provide an alternative, Ricci flow argument for the Rigidity statement of the Positive Mass theorem.

In particular, our result that asymptotically hyperbolic mass must evolve as

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$m(t) = m_0 e^{-(n-2)t}$ implies that the mass is in general not preserved under the flow (1.1), in contrast to the behaviour of the ADM mass under Ricci flow on asymptotically flat manifolds (see [27], [13]). While at first this is somewhat surprising that asymptotically hyperbolic mass is not conserved under (1.1), we highlight in the following situations that the monotonic behaviour of the mass under (1.1) is indeed natural, and fits with current physical theories.

We first note that our result for the behaviour of the mass under the curvature-normalized Ricci flow (1.1) supports a conjecture first made by Horowitz and Myers in [22]. We motivate this correspondence below.

Recall from example 2.6 that an AdS soliton is a constant-time slice of an Anti-de Sitter soliton spacetime (see [22]); it is an asymptotically hyperbolic manifold M of dimension $n \geq 3$, with an induced metric of the form (2.10) in any local radial coordinate system $(r, \xi, \theta_i)_{\{i=3,\cdots,n\}}$. As mentioned in example 2.8, the mass of such an AdS soliton is given by $-\frac{4\pi}{n}\prod_{i=3}^{n} a_i$. In a paper by Horowitz and Myers [22], it is noted that AdS solitons are static solutions to Einstein's field equations. As well, Horowitz and Myers conjectured that these static solutions of Einstein's field equations minimize mass amongst the class of all locally asymptotically hyperbolic manifolds equipped with a metric which asymptotes to an AdS soliton at large r, and which possess scalar curvature $R \geq -n(n-1)$. In other words, if (M, g) is a locally asymptotically hyperbolic manifold with scalar curvature $R \geq -n(n-1)$, and further g is a metric which asymptotes to an AdS soliton at large r, then $m[g] \geq -\frac{4\pi}{n}\prod_{i=3}^{n} a_i$.

Now, consider the flow (1.1) arising from initial data given by an asymptotically hyperbolic manifold (M, g_0) with scalar curvature $R \geq -n(n-1)$, and where g_0 is a metric which asymptotes to an AdS soliton at large r. Since boundedness below of scalar curvature is preserved under Ricci flow (see [38], [19]), it is preserved under the flow given by (1.1); thus we could obtain a new asymptotically hyperbolic manifold $(M, g(t))$ with scalar curvature $R \geq -n(n-1)$, and further employing Proposition 3.2, $g(t)$ is still a metric which asymptotes to an AdS soliton at large r. However, if our Theorem 1.1 were not true, say the mass $m(t)$ was monotonically decreasing under (1.1), then one obtains that $m(t) < m_0 = -\frac{4\pi}{n}\prod_{i=3}^{n} a_i$, and hence contradicting Horowitz and Myers's conjecture. On the other hand, if instead the mass $m(t)$ remained constant under (1.1), then by [6] and [3] there exists a unique, positive solution w to the Yamabe problem on $(M, g(t))$. Consider then $(M, \bar{g})$, where $\bar{g} = w^{\frac{4}{n-2}} g(t)$. Applying Proposition 3.13 of [2], we would again obtain $m(t) < m_0 = -\frac{4\pi}{n}\prod_{i=3}^{n} a_i$, and hence a contradiction to Horowitz and Myers's conjecture. Therefore, since Theorem 1.1 in particular shows that under the flow (1.1) the mass is initially strictly increasing, both of the previous arguments are not valid. Indeed, Theorem 1.1 confirms the intuition that if Horowitz and Myers's conjecture were true, the mass should be increasing.

The monotonicity of the mass under (1.1) also enables us to provide a heuristic derivation of the constancy of the ADM mass in the asymptotically flat setting. Let (M, g_0) be an asymptotically hyperbolic, n-dimensional manifold, with boundary-at-infinity ∂M isometric to either the unit $(n-1)$-sphere or a flat $(n-1)$-torus. Recall that we may express g_0 in the form of (2.20). However, writing the metric g_0 in this form assumed that the sectional curvatures of ∂M were normalized to 1 or 0, respectively. Let us undo this normalization: replace g on M with $\bar{g}_0 = g_0/\ell^2$, for some $\ell > 0$. Then, we have that the sectional curvatures of ∂M are equal to $1/\ell^2$. Consider again the flow given in (1.1), with $(M, \bar{g}_0)$ as initial data. Parabolically rescaling $t \to t/\ell^2$ gives us the flow

$$
\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= -2\left[Ric[g]_{ij} + \frac{(n-1)}{\ell^2} g_{ij}\right], \\
g(0) &= \bar{g}_0 \,,
\end{aligned}
$$

and from Theorem 1.1 we compute $m(t) = m_0 e^{-\frac{(n-2)t}{\ell^2}}$.

Now, if we fix some t and take the limit as $\ell \to \infty$, then $m(t) \to m_0$. So heuristically, as the sectional curvatures tend to zero, the mass tends to the initial mass. This mimics the behaviour of the ADM mass under Ricci flow. In this setting we have that $(M, g(t))$ is asymptotically flat, and thus the sectional curvatures tend to zero as one moves out towards infinity. As well, under Ricci flow, the ADM mass remains constant; that is, $m(t) = m_0$. Alternatively, we may consider taking the limit as $t \to \infty$ first. To do this, let points $p_t \in (M, g(t))$ be such that $\lim_{t\to\infty} p_t = p$ for some point p in the limit manifold. We compute the limit $\lim_{t\to\infty} m(t) = 0$; so in the limit, any compact neighbourhood of p is a compact set of standard hyperbolic space. If we then take the limit as $\ell \to \infty$, the sectional curvatures go to zero; hence the neighbourhood of p now is a compact set of flat space. Therefore, taking the limit as $t \to \infty$ and then the limit as $\ell \to \infty$ of the solutions to the above flow, we obtain flat space with zero mass, again reproducing the behaviour of Ricci flow on asymptotically flat manifolds.

In terms of future work, there is both physical and geometrical motivation to compute the evolution of other invariants of asymptotically hyperbolic manifolds. Our work has led others [7] to investigate the evolution of a quantity called the renormalized volume. Like the mass, renormalized volume is also defined as a term appearing in an asymptotic expansion (as described in [15], [18]) of the metric of a conformally compact Einstein manifold. For conformally compact Einstein manifolds of even dimension, renormalized volume is a conformal invariant [18]. As communicated to us by [7], the definition of renormalized volume may be extended to asymptotically hyperbolic manifolds of even dimension. As of the date of this thesis, the Ricci flow of renormalized volume and possible consequences thereof are not yet known.

It would also be of value to investigate the existence and behaviour of breathers with an undefined mass to the flow (1.1). Another direction one could take for future investigation would be to analyze under what conditions (if any) singularities may occur during the flow (1.1) arising from asymptotically hyperbolic initial data. Further, if singularities may occur under such a flow, one may work to classify what type of singularities the flow (1.1) might exhibit. As mentioned in section 3.2, Qing, Shi, and Wu [33] have obtained long-time existence of the normalized Ricci flow (1.1) under certain curvature and volume assumptions, and Bahuaud in [8] also proved long-time existence of solutions under certain asymptotic constraints. However, in comparison to the varied results known for Ricci flow on compact manifolds, there is still much to be yet understood about Ricci flow on open manifolds; thus addressing any one of these topics would be valuable to enhancing the general understanding of the behaviour of open manifolds under Ricci flow or its normalized variants.

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Appendix A

A Review of Riemannian Geometry

In this appendix, we give a review of some of the key concepts in *Riemannian* geometry. Our aim is not to go into detail, but rather to provide a handy reference for some of the background knowledge assumed in the main body of this thesis and the other appendices.

Let M be a second-countable, Hausdorff topological space. Recall we can define *coordinate charts* on M as pairs (U, ϕ), where U is an open subset of M and $\phi : U \to V$, $\phi(p) = (x^1(p), \ldots, x^n(p))$ is a homeomorphism from U to some open subset $V \subset \mathbb{R}^n$. We hence call the functions $x^i(p)$ the *coordinates* of $p \in M$. Further, we mention that two charts (U_1, ϕ_1), (U_2, ϕ_2) are called *C^∞-compatible* if either $U_1 \cap U_2 = \varnothing$, or the change of coordinates map $\phi_2 \circ \phi_1^{-1} : \phi_1^{-1}(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a smooth diffeomorphism.

Now let $\mathcal{A} := \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a maximal collection of all C^∞-compatible charts on M. If $\mathcal{A}$ is such that $M = \cup_{\alpha \in I} U_\alpha$, we call $M := (M, \mathcal{A})$ a *(smooth) differentiable manifold*.

Examples of differentiable manifolds:

1. $\mathbb{R}^n$, with $\mathcal{A} = \{(\mathbb{R}^n, \mathrm{Id})\}$, is a differentiable manifold.

2. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x||^2 = 1\}$, with $\mathcal{A} = \{(U_\pm, \pi_\pm)\}$, where $U_\pm = S^n \setminus \{(0, \ldots, 0, \pm 1)\}$, and $\pi_\pm(x^1, \ldots, x^{n+1}) = \frac{1}{1 \mp x^{n+1}}(x^1, \ldots, x^n)$, is a differentiable manifold.

3. Hyperbolic n-space $\mathbb{H}^n = \{(w, x) \in \mathbb{R}^{n+1} : w^2 - ||x||^2 = 1\}$, where $\mathcal{A}$ is obtained by restricting a maximal collection of all C^∞-compatible charts on $\mathbb{R}^{n+1}$ to $\mathbb{H}^n$.

If M is a differentiable manifold, then we define a *tangent vector* to M at a point $p \in M$ to be a linear map $X : C^\infty(M) \to \mathbb{R}$ satisfying

$$X(fg) = g(p)X(f) + f(p)X(g)$$

for all $f, g \in C^\infty(M)$. The set of all tangent vectors at p is a real vector space, and is denoted by T_pM. Then, the *tangent bundle* of M, denoted TM, is the disjoint union of all vector spaces tangent to M at some $p \in M$; that is, $TM := \sqcup_{p \in M} T_pM$.

Now that we have defined the tangent bundle of our differentiable manifold M, we would like to construct a way to measure the angles between vectors and lengths of vectors. We may do this as follows. For all $p \in M$, since the tangent space T_pM is a real vector space, we may equip T_pM with an Euclidean inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. We then may define the smooth tensor $g : M \to \mathbb{R}$ by $g(p) := g_p$. We call the tensor g a *Riemannian metric* (or simply a *metric*) on M, and call the pair (M, g) a *Riemannian* manifold.

Then given a Riemannian metric g on M, we may define the *length* of a tangent vector X at $p \in M$ to be the non-negative number

$$||X|| := \sqrt{g(X, X)} = \sqrt{g_p(X, X)}\,,$$

and find that the angle θ between two tangent vectors at X, Y at p obeys

$$\cos\theta = \frac{g(X, Y)}{||X||||Y||}\,.$$

In some local coordinate system (x^i), $i = i, \dots, n = \dim(M)$, we often write the metric g as the *line element* $g := g_{ij}dx^i dx^j =: ds^2$. The functions g_{ij} are called the *components* of g in the chosen coordinate system. Alternatively, we may also write g in matrix notation: $g = (g_{ij})$. These expressions for the metric are completely equivalent; they are simply a matter of notation.

Now consider the space dual to the tangent bundle, called the the *cotangent bundle*, which we denote by T^*M. On the cotangent bundle of M, we may also define the *inverse (Riemannian) metric*, g^{-1} as the unique tensor such that $g \circ g^{-1} = Id$ and $g^{-1} \circ g = Id$, where Id denotes the appropriate identity map.

Examples of Riemannian manifolds:

1. $\mathbb{R}^n$, with the standard dot product. Note that in local coordinates, we may represent the dot product by either the identity matrix I_n, or equivalently, by the line element $\delta = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$, where $(x^1, \dots, x^n)$ are the coordinates on $\mathbb{R}^n$.

2. Let $\{r, \phi_1, \dots, \phi_n\}$ be spherical coordinates on $\mathbb{R}^{n+1}$. The unit n-sphere $S^n := \{(1, \phi_1, \dots, \phi_n) \in \mathbb{R}^{n+1}\}$ can be equipped with the Riemannian metric $g = d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + \sin^2\phi_1 \sin^2\phi_2 d\phi_3^2 + \cdots + \sin^2\phi_1 \cdots \sin^2\phi_{n-1} d\phi_n^2$. This metric is called the *round* metric on S^n. It is often denoted by $g(S^n, can)$.

3. Hyperbolic n-space $\mathbb{H}^n = \{(w, x) \in \mathbb{R}^{n+1} : w^2 - ||x||^2 = 1, x = (x_1, \dots, x_n)\}$, with the metric $g = d\xi^2 + \sinh^2(\xi) g(S^{n-1}, can)$, where $g(S^{n-1}, can)$ is the round metric on S^{n-1}, $w = \cosh(\xi)$, and $x_i = \sinh(\xi)\sin(\phi_1)\cdots\sin(\phi_i)$ for $\phi_i \in [0, 2\pi]$.

Not only does a Riemannian metric g on M allow us to quantize things such as lengths of tangent vectors, but it also enables us to define a distinguished generalization of differentiation on M. This generalization is achieved via the *Levi-Cività connection*. The Levi-Cività connection is the unique map $\nabla : TM \to T^*M \otimes TM$, such that for $X, Y, Z \in TM$, $f \in C^\infty(M)$, and $a, b \in \mathbb{R}$, ∇ satisfies

(i) Tensoriality: $\nabla_{aY+bZ}X = a\nabla_Y(X) + b\nabla_Z X$,

(ii) Linearity: $\nabla(X + aY) = \nabla(X) + a\nabla(Y)$,

(iii) The product rule: $\nabla_Y(fX) = (D_Y f) \cdot X + f \cdot \nabla_Y X$,[1]

(iv) Torsion-free: $\nabla_X Y - \nabla_Y X = [X, Y]$,[2]

(v) Metricity: $\nabla_X g = 0$.

We note that in a local coordinate system, the components of the Levi-Cività connection are given by the *Christoffel symbols*, Γ^k_{ij}, where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right).$$

The Levi-Cività connection allows us to define one of the key notions in Riemannian geometry —*curvature*. While we may define many quantities which characterize the curvature, we will briefly recall notions which quantify the intrinsic curvature of Riemannian manifolds — that is, curvature notions which are independent of any embeddings of the manifold. In particular, we thus do not require our Riemannian manifold (M, g) to be embedded in any $\mathbb{R}^m$ for some m.

The first notion of curvature we introduce is the *Riemann curvature* of (M, g). The Riemann curvature of (M, g) is a (0,4)-tensor $Rm[g]$ defined by

$$Rm[g] : \otimes^4 TM \to \mathbb{R}, \qquad Rm[g](W, X, Y, Z) := g(R(W, X)Y, Z),$$

where $R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$,[3] and ∇ is the Levi-Cività connection of g. We call the tensor $Rm[g]$ the *Riemann (curvature) tensor* of (M, g). The Riemann curvature tensor obeys the symmetry properties

(i) $Rm[g](W, X, Y, Z) = -Rm[g](X, W, Y, Z)$,

[1] Here $D_Y f$ denotes the directional derivative of f.
[2] Where $[X, Y]$ is the *Lie bracket*.
[3] Some authors instead choose to define $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

(ii) $Rm[g](W, X, Y, Z) = -Rm[g](W, X, Z, Y)$,

(iii) $Rm[g](W, X, Y, Z) = Rm[g](X, W, Z, Y)$,

as well as the *1^st^ Bianchi identity*, which is the symmetry relation

$$Rm[g](W, X, Y, Z) + Rm[g](Y, W, X, Z) + Rm[g](X, Y, W, Z) = 0\,.$$

Intuitively, one may think of this tensor as a measure of how the connection on our manifold M differs from differentiation in $\mathbb{R}^n$. We also note that in local coordinates, our definition relates the components of $Rm[g]$ and $R(X, Y)Z$, respectively denoted by R_{ijkl} and R^i_{jkl}, by $R_{ijkl} := g_{lp}R^p_{ijk}$.

Another notion of curvature is sectional curvature. Let X, Y be tangent vectors to M at a point $p \in M$. The *sectional curvature* of the 2-plane spanned by X and Y is the ratio

$$sec(X, Y) := \frac{Rm[g](X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)g(X, Y)}\,.$$

We may also define various averages or traces of the Riemann curvature tensor to obtain coarser descriptions of the curvature of the manifold (M, g).

One average of curvature is *Ricci curvature*. We define the Ricci curvature of (M, g) to be the tensor $Ric[g] : TM \otimes TM \to \mathbb{R}$, given by taking the trace of the Riemann curvature tensor with respect to the metric g:

$$Ric[g] := tr_g(Rm[g]) = tr(X \to R(X, Y)Z).$$

From the symmetry properties of the Riemann tensor, the Ricci tensor obeys the property

$$Ric[g](X, Y) = Ric[g](Y, X)\,.$$

Heuristically, the Ricci curvature of (M, g) is a measure of how the volume of a ball in (M, g) differs from the volume of the same ball embedded in $(\mathbb{R}^n, \delta)$.

It should also be noted that if $Ric[g] \equiv 0$, the manifold (M, g) is often said to be *Ricci flat*. As well, from the definition of sectional curvature, we see that in a local orthonormal, coordinate system (x^i), the diagonal components of the Ricci tensor $Ric[g] = R_{ij}dx^i dx^j$ are given by sums of sectional curvatures:

$$\begin{aligned} R_{ii} &:= R^k_{iki} \\ &= \sum_k R_{ikik} \\ &= \sum_k sec(\partial_i, \partial_k) \, . \end{aligned}$$

Lastly, we may also characterize the curvature of the manifold by its *scalar curvature*. We define the scalar curvature as the trace of the Ricci curvature with respect to the metric g. That is, the scalar curvature of (M, g) is the real-valued map $R[g] : M \to \mathbb{R}$, $R[g] := tr_g(Ric[g])$. In local coordinates, this may be expressed as $R[g] = g^{ij}R_{ij}$, where R_{ij} denotes the component functions of $Ric[g]$.

We conclude our review of Riemannian geometry by mentioning a special class of Riemannian manifolds — Einstein manifolds. We appropriately call a Riemannian manifold (M, g) of dimension $n > 2$ which satisfies Einstein's (vacuum) field equation $Ric[g] + kg = 0$ an *Einstein manifold*.

For further investigation into Riemannian geometry, please see [23] and [31].

Appendix B

Equivalence of the Extended Wang Mass and the Chruściel-Herzlich Mass[1]

In this appendix, we show that our extended definition of Wang's mass (see Definition 2.7) and the definition of asymptotically hyperbolic mass of Chruściel-Herzlich are equivalent.

For convenience, we present below the definition of asymptotically hyperbolic mass as given by Chruściel-Herzlich in [11].

Chruściel-Herzlich mass definition: Let M be an n-dimensional Riemannian manifold, and let g and b be any two Riemannian metrics on M. Denote by ∇ and D the Levi-Civitá connections with respect to the metrics b and g, respectively. Let K be a compact subset of M, and consider the end $M_{ext} := M \setminus K$. Further, let N be a compact, boundaryless Riemannian manifold, and let $\phi^{-1} : M_{ext} \to N_R := [R, \infty) \times N$ be a smooth diffeomorphism. We then define the *Chruściel-Herzlich mass* of (M, g) with respect to the *reference*

[1]A version of this appendix has been published. T Balehowsky, E Woolgar 2012. Journal of Mathematical Physics. 53: 072501.

metric b as

$$m_{CH}[g] := \lim_{R\to\infty} \int_{N_R} \mathbb{U}^i(V \circ \phi^{-1}) dS_i \tag{B.1}$$

where $V : M \to \mathbb{R}$ is a smooth function such that $V = O(r)$ for $r \in [R, \infty)$ the radial coordinate of N_R, $dS_i := \vec{n}_i \, dA_R$, where $\vec{n}_i$ is the outward unit normal of N and dA_R denotes the volume form on N_R with respect to the metric b, and finally, $\mathbb{U}^i$ is the operator

$$\mathbb{U}^i := \sqrt{g} \left\{ V(g^{ik}g^{jl} - g^{ij}g^{kl})\nabla_j g_{kl} + (D^i V g^{jk} - D^j V g^{ik}) e_{jk} \right\} ,$$

where $e_{jk} := g_{jk} - b_{jk}$.

When M is an asymptotically hyperbolic manifold with boundary-at-infinity ∂M isometric to a round sphere or a flat torus, we have $N = \partial M$. In accordance with [11], we choose

$$V := \sqrt{r^2 + k} \, ,$$

where $k = 0$ if ∂M has an induced metric of constant sectional curvature 0, or $k = 1$ if ∂M has an induced metric of constant sectional curvature $+1$ (i.e. $\partial M \equiv S^{n-1}$).

Our strategy shall be to compute term by term the Chruściel-Herzlich mass for asymptotically hyperbolic manifolds (M, g) whose metric can be written in the form of (2.20) with $\ell = n$, and show that the Chruściel-Herzlich mass of (M, g) is equal to our extended Wang mass, as defined in Definition 2.6.

Thus, suppose g is written in the form of (2.20). We obtain via the transformation to radial coordinates $1/r = \rho_{(k)}$,

$$\begin{aligned} g &= \rho_{(k)}^{-2} \left(dx^2 + g_{(k)\,AB} dy^A dy^B + \frac{x^n}{n} \sigma_{AB} dy^A dy^B + O(x^{n+1}) \right) \\ &= \frac{dr^2}{r^2 + k} + r^2 \left(g_{(k)} + \frac{1}{nr^n} \sigma_{AB} dy^A dy^B + O(1/r^{n+1}) \right) \end{aligned}$$

Appropriately, we choose as the reference metric

$$b = \frac{dr^2}{r^2 + k} + r^2 g_{(k)},$$

for $k = 0, 1$.

Now for notational convenience in the calculations that follow, enumerate the coordinates as $(x^i) := (r, y^A)$. As well, set

$$\mathbb{A}^i = \left(g^{ik} g^{jl} - g^{ij} g^{kl}\right) \nabla_j g_{kl} \tag{B.2}$$

$$\mathbb{B}^i = \left(\frac{D^i V}{V} g^{jk} - \frac{D^j V}{V} g^{ik}\right) e_{jk} \tag{B.3}$$

$$= \left(g^{il} g^{jk} - g^{jl} g^{ik}\right) \frac{D_l V}{V} e_{jk} \, . \tag{B.4}$$

Then we may write

$$\mathbb{U}^i = \sqrt{g}\, V \left\{\mathbb{A}^i + \mathbb{B}^i\right\} \, . \tag{B.5}$$

Recall the Christoffel symbols of the Levi-Cività connection of the metric b are given by

$$\Gamma^k_{ij} = \frac{1}{2} b^{kl} (\partial_i b_{jl} + \partial_j b_{il} - \partial_l b_{ij}) \, .$$

By the metricity condition $\nabla b = 0$ of the Levi-Cività connection of the metric b, we find

$$\nabla_j g_{kl} = \nabla_j (e_{kl} + b_{kl}) = \nabla_j e_{kl} \, .$$

Thus,

$$\nabla_j g_{kl} = \nabla_j e_{kl} \tag{B.6}$$

$$= \partial_j e_{kl} - e_{ml}\Gamma^m_{kj} - e_{kp}\Gamma^p_{lj} \tag{B.7}$$

Note that, for either of the cases where the boundary-at-infinity of M is isometric to a round sphere or a flat torus, we have $e_{jk} = g_{jk} - b_{jk} = \frac{1}{nr^{n-2}}\sigma_{jk} + O(1/r^{n-1})$ and using (B.7) and (B.2), we calculate

$$\begin{aligned}
\mathbb{A}^i &= \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\nabla_j g_{kl} \\
&= \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\nabla_j e_{kl} \\
&= \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\partial_j e_{kl} - \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) e_{ml}\Gamma^m_{kj} \\
&\quad - \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) e_{kp}\Gamma^p_{lj} \\
&= \left(g^{iA}g^{1B} - g^{i1}g^{AB}\right)\left(-\frac{(n-2)}{nr^{n-1}}\sigma_{AB} + \frac{1}{nr^{n-2}}\partial_1\sigma_{AB}\right)\delta^1_j \\
&\quad - \frac{1}{nr^{n-2}}\left(g^{ik}g^{jl}\sigma_{kp}\Gamma^p_{lj} + g^{ik}g^{jl}\sigma_{pl}\Gamma^p_{kj}\right) \\
&\quad + \frac{1}{nr^{n-2}}\left(g^{ij}g^{kl}\sigma_{kp}\Gamma^p_{lj} + g^{ij}g^{kl}\sigma_{pl}\Gamma^p_{kj}\right) + O(1/r^n) \\
&= \frac{(n-2)}{nr^{n-1}} g^{AB}_{(k)}\sigma_{AB}\delta^1_j\delta^i_1 - \frac{1}{nr^{n-2}}\left(g^{iA}g^{jl}\sigma_{AB}\Gamma^B_{lj}\delta^B_l\delta^1_j\right) \\
&\quad - \frac{1}{nr^{n-2}}\left(g^{ik}g^{jA}\sigma_{AB}\Gamma^B_{kj}\delta^k_B\delta^j_1 + g^{ik}g^{jA}\sigma_{AB}\Gamma^B_{kj}\delta^1_k\delta^B_j\right) \\
&\quad + \frac{2}{nr^{n-2}}\left(g^{ij}g^{AB}\sigma_{AC}\Gamma^C_{Bj}\delta^1_j\delta^C_B\right) + O(1/r^n)
\end{aligned}$$

Therefore, for $k = 0, 1$ we get

$$\begin{aligned}
\mathbb{A}^1 &= \frac{(n-2)}{nr^{n-1}} g_{(k)}^{AB} \sigma_{AB} - \frac{1}{nr^{n-1}} g_{(k)}^{BA} \sigma_{AB} + \frac{2}{nr^{n-1}} g_{(k)}^{AB} \sigma_{AB} + O(1/r^n) \\
&= \frac{(n-1)}{nr^{n-1}} g_{(k)}^{AB} \sigma_{AB} + O(1/r^n)\,, \\
\mathbb{A}^C &= O(1/r^n)\,.
\end{aligned}$$

Similarly as above, using now (B.4) in place of (B.2) and $e_{jk} := g_{jk} - b_{jk}$, we compute

$$\begin{aligned}
\mathbb{B}^1 &= \left(g^{1l} g^{jk} - g^{jl} g^{1k}\right) \frac{D_l V}{V} e_{jk} \\
&= \left(g^{1l} g^{AB} - g^{Al} g^{1B}\right) \frac{D_l V}{V} \frac{1}{nr^{n-2}} \sigma_{AB} + O(1/r^n) \\
&= \frac{1}{nr^{n-1}} g_{(k)}^{AB} \sigma_{AB} + O(1/r^n) \\
\mathbb{B}^C &= \left(g^{Cl} g^{AB} - g^{Al} g^{CB}\right) \frac{D_l V}{V} \frac{1}{nr^{n-2}} \sigma_{AB} + O(1/r^n) \\
&= O(1/r^n)\,.
\end{aligned}$$

From (B.5), we therefore obtain

$$\mathbb{U}^i = \sqrt{g}\, V \left\{ \frac{(n-1)}{nr^{n-1}} g_{(k)}^{AB} \sigma_{AB} \delta_1^i + \frac{1}{nr^{n-1}} g_{(k)}^{AB} \sigma_{AB} \delta_1^i + O(1/r^n) \right\} \tag{B.8}$$

$$= \sqrt{g}\, V \left\{ \frac{1}{r^{n-1}} g_{(k)}^{AB} \sigma_{AB} \delta_1^i + O(1/r^n) \right\}\,. \tag{B.9}$$

Plugging (B.9) into the Chruściel-Herzlich mass definition, we get

$$\begin{aligned} m_{CH}[g] &= \lim_{R\to\infty} \int_{N_R} \mathbb{U}^i \, dS_i \\ &= \lim_{R\to\infty} \int_{N_R} \sqrt{g}\, V \left\{ \frac{1}{r^{n-1}} g_{(k)}^{AB} \sigma_{AB} \delta_1^i + O(1/r^n) \right\} dS_i \\ &= \int_N g_{(k)}^{AB} \sigma_{AB} \, d\mu_{g_{(k)}} \\ &= m[g] \,. \end{aligned}$$

We have therefore deduced that our extended version of Wang's mass definition and the Chruściel-Herzlich mass definition are equivalent. This verifies coordinate independence of our mass definition.

B.1 The Chruściel-Herzlich Mass of an AdS Soliton

We now turn our attention to computing the mass of an AdS soliton. Recall from equation (2.10), the metric of an AdS soliton[2] (M, g) may be written in a radial coordinate system as

$$g = \frac{dr^2}{r^2\left(1 - \frac{1}{r^n}\right)} + r^2 \left[\left(1 - \frac{1}{r^n}\right) d\xi^2 + \sum_{i=3}^{n} d\theta_i^2 \right] .$$

where the radial coordinate r takes values in $[1, \infty)$, and the angular coordinates θ_i take values in $[0, a_i]$ for $0 < a_3 \leq \ldots \leq a_n$. The domain of ξ is determined as follows.

[2]Note by "AdS soliton", we mean a time-constant slice of the Anti- de Sitter soliton space-time given in [22].

Let $W = r^2\left(1 - \frac{1}{r^n}\right)$. To ensure that the AdS soliton is a smooth manifold, to avoid a singularity in the above metric near $r = 1$, we require the $\theta_i = constant$, $i = 3, \ldots, n$ submanifold of the AdS soliton to obey

$$\frac{dr^2}{W} + Wd\xi^2 \simeq \frac{d(\Delta r)^2}{W'(1)\Delta r} + W'(1)\Delta r d\xi^2 \tag{B.10}$$

$$= du^2 + u^2 dv^2\,, \tag{B.11}$$

where $u \in [0, \infty]$ and $v \in [0, 2\pi]$. This leads us to the equation

$$\frac{d(\Delta r)}{\sqrt{W'(1)\Delta r}} = du\,;.$$

integrating, we obtain $u = \frac{2}{\sqrt{n}}\sqrt{\Delta r}$. Thus, to obtain the metric form in equation (B.11), we must have $\frac{n}{2}v \in [0, 2\pi]$.

Therefore, to avoid the metric becoming singular at $r = 1$, we require the coordinate $\xi = \frac{n}{2}v$ to take values in $[0, \frac{4\pi}{n}]$.

We remark here that the AdS soliton metric g as above is actually one of many representatives of a congruence class in the moduli space $\mathcal{M} = \dfrac{\text{Met}\{\text{AdS soliton}\}}{\text{Diff}_0}$, where Met{AdS soliton} is the set of all AdS soliton metrics, and Diff_0 is the set of all diffeomorphisms on an AdS soliton which, when smoothly extended to the conformally compactified manifold, act as an isometry on the boundary-at-infinity (which are flat $(n-1)$-tori). The reason for viewing g as such a congruence class representative is the following. In an AdS soliton, at each value of the coordinate r, we wish to attach an $(n-1)$-torus with cycles coordinatized by $\theta_i \in [0, b_i]$, $i = 2, \ldots, n$, for some parameters $0 < b_2 \leq \ldots \leq b_n$. As the above argument demonstrates, for the AdS soliton to be non-singular, we must scale one of the cycles of the flat $(n-1)$-torus we wish to attach at r to have a period of $\frac{4\pi}{n}$. However, by rescaling, we are free choose *any one* of the $(n-1)$ cycles to have the period of $\frac{4\pi}{n}$! Once we have made our choice and scaled appropriately, the other cycles then have lengths $0 < a_3 \leq \ldots \leq a_n$, for which we have no freedom to specify. Thus, to eliminate the ambiguity in the choice of which one of the cycles we distinguish to be $[0, \frac{4\pi}{n}]$, we identify those AdS soliton metrics which have isometric tori attached at r. (See further [22], [17], and [1].)

The above remark aside, we shift our attention back to computing the mass of an AdS soliton. For considerable simplification of the calculations, we restrict to the case of a 3-dimensional AdS soliton. For $n = 3$ dimensions, an Anti-de Sitter (AdS) soliton metric (please see example 2.6) is given by

$$g = \frac{1}{W^2} dr^2 + r^2 d\theta^2 + W^2 d\phi^2$$

where $W^2 = r^2(1 - \frac{1}{r^3})$, for $r \in [0, \infty)$. We fix a representative of the AdS metric moduli class by choosing $\phi \in [0, \frac{4\pi}{3}]$ and $\theta \in [0, \frac{a}{4}]$ for some $a > 0$.

The appropriate reference metric is given by

$$b = \frac{1}{V^2} dr^2 + r^2 d\theta^2 + V^2 d\phi^2$$

where $V := \sqrt{r^2 + 0} = r$, $r \in [0, \infty)$, $\theta \in [0, \frac{\pi}{2}]$, and $\phi \in [0, \frac{4\pi}{3}]$.

As before, for computational convenience, we enumerate the coordinates as $(x^1, x^2, x^3) := (r, \theta, \phi)$, and further set

$$\begin{aligned} \mathbb{A}^i &= \left(g^{ik} g^{jl} - g^{ij} g^{kl}\right) \nabla_j g_{kl} \\ \mathbb{B}^i &= \left(\frac{D^i V}{V} g^{jk} - \frac{D^j V}{V} g^{ik}\right) e_{jk}\,, \end{aligned}$$

where $i,\ j,\ k,\ l \in \{1,\ 2,\ 3\}$. Again, we may succinctly write

$$\mathbb{U}^i = \sqrt{g}\, V \left\{\mathbb{A}^i + \mathbb{B}^i\right\}\,.$$

We calculate the components of the difference $e_{jk} = g_{jk} - b_{jk}$ to be

$$\begin{aligned} e_{11} &= \frac{1}{r^2(r^3-1)} \\ e_{33} &= -\frac{1}{r} \\ e_{jk} &= 0 \qquad \text{for } j,\, k \neq 1 \text{ or } 3. \end{aligned}$$

Then, the Christoffel symbols with respect to the metric b are given by

$$\Gamma^k_{ij} = \frac{1}{2} b^{kl} (\partial_i b_{jl} + \partial_j b_{il} - \partial_l b_{ij})\,.$$

Since b_{11}, b_{22}, and b_{33} are the only non-zero components of b, we obtain

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2} V^2 \partial_1 b_{11} = -\frac{1}{r}\,, \\ \Gamma^1_{22} &= -\frac{1}{2} V^2 \partial_1 b_{22} = -r^3\,, \\ \Gamma^1_{33} &= -\frac{1}{2} V^2 \partial_1 b_{33} = -r^3\,, \\ \Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{2}\frac{1}{r^2} \partial_1 b_{22} = \frac{1}{r}\,, \\ \Gamma^3_{31} &= \Gamma^3_{13} = \frac{1}{2}\frac{1}{V^2} \partial_1 b_{33} = \frac{1}{r}\,, \end{aligned}$$

are the only non-zero Christoffel symbols. Now, since $e_{jk} = g_{jk} - b_{jk}$, by metricity we obtain $\nabla_j g_{kl} = \nabla_j (e_{kl} + b_{kl}) = \nabla_j e_{kl}$. Hence we compute

$$\begin{aligned} \nabla_j g_{kl} &= \nabla_j e_{kl} \\ &= \partial_j e_{kl} - e_{ml} \Gamma^m_{kj} - e_{kp} \Gamma^p_{lj} \\ &= \partial_1 e_{kk} \delta^1_j - \frac{1}{r^2(r^3-1)} \left(\Gamma^m_{kj} \delta^1_m \delta^1_l + \Gamma^p_{lj} \delta^1_k \delta^1_p \right) + \frac{1}{r} \left(\Gamma^m_{kj} \delta^3_m \delta^3_l + \Gamma^p_{lj} \delta^3_k \delta^3_p \right)\,. \end{aligned}$$

So $\mathbb{A}^i$ becomes

$$
\begin{aligned}
\mathbb{A}^i &= \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\nabla_j g_{kl} \\
&= \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\partial_1 e_{kl}\delta_j^1 - \frac{1}{r^2(r^3-1)}\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\left(\Gamma^m_{kj}\delta^1_m\delta^1_l + \Gamma^p_{lj}\delta^1_k\delta^1_p\right) \\
&\quad + \frac{1}{r}\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\left(\Gamma^m_{kj}\delta^3_m\delta^3_l + \Gamma^p_{lj}\delta^3_k\delta^3_p\right) \\
&= \left(g^{i1}g^{j1} - g^{ij}g^{11}\right)\partial_1 e_{11}\delta^1_j + \left(g^{i3}g^{j3} - g^{ij}g^{33}\right)\partial_1 e_{33}\delta^1_j \\
&\quad - \frac{1}{r^2(r^3-1)}\left(g^{ik}g^{jl}\Gamma^m_{kj}\delta^1_m\delta^1_l + g^{ik}g^{jl}\Gamma^m_{lj}\delta^1_k\delta^1_m - g^{ij}g^{kl}\Gamma^m_{kj}\delta^1_m\delta^1_l - g^{ij}g^{kl}\Gamma^m_{lj}\delta^1_k\delta^1_m\right) \\
&\quad + \frac{1}{r}\left(g^{ik}g^{jl}\Gamma^m_{kj}\delta^3_m\delta^3_l + g^{ik}g^{jl}\Gamma^m_{lj}\delta^3_k\delta^3_m - g^{ij}g^{kl}\Gamma^m_{kj}\delta^3_m\delta^3_l - g^{ij}g^{kl}\Gamma^m_{lj}\delta^3_k\delta^3_m\right) \\
&= 0\cdot\partial_1 e_{11} - g^{i1}g^{33}\left(\frac{1}{r^2}\right) - \frac{1}{r^2(r^3-1)}\left(g^{ik}g^{j1}\Gamma^1_{kj} + g^{i1}g^{jl}\Gamma^1_{lj} - 2g^{ij}g^{k1}\Gamma^1_{kj}\right) \\
&\quad + \frac{1}{r}\left(g^{ik}g^{j3}\Gamma^3_{kj} + g^{i3}g^{jl}\Gamma^3_{lj} - 2g^{ij}g^{k3}\Gamma^3_{kj}\right) .
\end{aligned}
$$

Given the above Christoffel symbols and our calculation of e_{jk}, we find the only non-zero term is when $i = 1$. We thus compute $\mathbb{A}^1$:

$$
\begin{aligned}
\mathbb{A}^1 &= -g^{11}g^{33}\left(\frac{1}{r^2}\right) - \frac{1}{r^2(r^3-1)}\left(g^{1k}g^{j1}\Gamma^1_{kj} + g^{11}g^{jl}\Gamma^1_{lj} - 2g^{1j}g^{k1}\Gamma^1_{kj}\right)\\
&\quad + \frac{1}{r}\left(g^{1k}g^{j3}\Gamma^3_{kj} + g^{13}g^{jl}\Gamma^3_{lj} - 2g^{1j}g^{k3}\Gamma^3_{kj}\right)\\
&= -W^2\frac{1}{W^2}\left(\frac{1}{r^2}\right) - \frac{1}{r^2(r^3-1)}\left(-g^{11}g^{11}\Gamma^1_{11} + g^{11}g^{22}\Gamma^1_{22} + g^{11}g^{33}\Gamma^1_{33}\right)\\
&\quad + \frac{1}{r}\left(-g^{11}g^{33}\Gamma^3_{13}\right)\\
&= -\left(\frac{1}{r^2}\right) - \frac{1}{r^2(r^3-1)}\left(-W^4(-\frac{1}{r}) + W^2\frac{1}{r^2}(-r^3) + W^2\frac{1}{W^2}\frac{1}{r}\right)\\
&\quad + \frac{1}{r}\left(-W^2\frac{1}{W^2}\frac{1}{r}\right)\\
&= -\left(\frac{1}{r^2}\right) - \frac{1}{r^2(r^3-1)}W^2\left(r - \frac{1}{r^2} - r + \frac{1}{r^3-1}\right) - \frac{1}{r^2}\\
&= -2\left(\frac{1}{r^2}\right) - \frac{1}{r^4(r^3-1)}W^2 - \frac{1}{r^3(r^3-1)}\,.
\end{aligned}
$$

Now we compute the components of $\mathbb{B}$. For $i = 1$, we obtain the following:

$$
\begin{aligned}
\mathbb{B}^1 &= \left(\frac{D^1V}{V}g^{jk} - \frac{D^jV}{V}g^{1k}\right)e_{jk}\\
&= \frac{1}{V}\left(D^1Vg^{33}\right)e_{33}\\
&= \frac{1}{V}\left(g^{11}\cdot 2r\cdot g^{33}\right)e_{33}\\
&= \frac{1}{V}\left(V^2\cdot 2r\cdot\frac{1}{V^2}\right)\left(-\frac{1}{r}\right)\\
&= -\frac{2}{r^2}\,.
\end{aligned}
$$

When $i = 2$ or 3, we have

$$\mathbb{B}^2 = \left(\frac{D^2V}{V} g^{jk} - \frac{D^jV}{V} g^{2k} \right) e_{jk} = 0\,,$$

$$\mathbb{B}^3 = \left(\frac{D^3V}{V} g^{jk} - \frac{D^jV}{V} g^{3k} \right) e_{jk} = 0\,.$$

Therefore we have

$$\begin{aligned}
\mathbb{U}^i &= \mathbb{U}^1 \delta_1^i \\
&= \sqrt{g}\, V \left\{ \mathbb{A}^1 + \mathbb{B}^1 \right\} \delta_1^i \\
&= \sqrt{r^2}\, V \left\{ \left(-2 \left(\frac{1}{r^2} \right) - \frac{1}{r^4(r^3-1)} W^2 - \frac{1}{r^3(r^3-1)} \right) - \frac{2}{r^2} \right\} \delta_1^i \\
&= \left\{ -2 - \frac{1}{r^2(r^3-1)} W^2 - \frac{1}{r(r^3-1)} - 2 \right\} \delta_1^i \\
&= \left\{ -4 - \frac{1}{r^2(r^3-1)} \left(\frac{r^3-1}{r} \right) - \frac{1}{r(r^3-1)} \right\} \delta_1^i \\
&= \left\{ -4 - \frac{1}{r^3} - \frac{1}{r(r^3-1)} \right\} \delta_1^i\,.
\end{aligned}$$

Thus the limit of $\mathbb{U}^1$ as $r \to \infty$ is

$$\lim_{r \to \infty} \mathbb{U}^1 = -4\,.$$

Hence, the Chruściel-Herzlich mass of our 3-dimensional AdS soliton is

$$\begin{aligned}
m(g) &= \lim_{r\to\infty} \int_{S_r} \mathbb{U}^i(V) dS_i \\
&= \lim_{r\to\infty} \int_0^{\frac{4\pi}{3}} \int_0^{\frac{a}{4}} \mathbb{U}^1 d\theta d\phi \\
&= \lim_{r\to\infty} \frac{4\pi}{3} \cdot \frac{a}{4} \cdot \mathbb{U}^1 \\
&= \frac{4\pi}{3} \cdot \frac{a}{4} \cdot (-4) \\
&= -\frac{4\pi}{3} \cdot a \,.
\end{aligned}$$

For comparison, please see the calculation of the mass of an AdS soliton as given in [22].

Appendix C

Calculation of the DeTurck Vector Field[1]

In this appendix, we provide proofs for versions of Proposition 3.2 and Theorem 1.1, using the associated Ricci-DeTurck flow to equation (1.1).

We have chosen to depart from the notation and coordinate system presented in the body of this thesis and instead write the metric in the coordinate system presented in [8] for easier comparison with the approach taken by Bahuaud in [8].

For what follows, let (M, g_0) be an asymptotically hyperbolic manifold equipped with a metric g_0 of the form

$$g_0 = \frac{1}{x^2}(dx^2 + g_{(k)} + \frac{x^l}{l}\kappa_{AB}dy^A dy^B + O(x^{l+1}))\,,$$

for $k = 0$, 1, where we set

$$\begin{aligned} g_{(0)} &= \text{ a flat metric on } T^{n-1}, \text{ and} \\ g_{(1)} &= (1-x^2)^2 g(S^{n-1}, can)\,. \end{aligned}$$

[1]A version of this appendix has been published. T Balehowsky, E Woolgar 2012. Journal of Mathematical Physics. 53: 072501.

Note, for this chosen coordinate system, a conformal completion of (M, g_0) is given by $(\widetilde{M}, \tilde{g}_0, \rho)$, with $\rho(x) = x$ and $\tilde{g}_0 = dx^2 + g_{(k)} + \frac{x^n}{n}\kappa_{AB}dy^A dy^B + O(x^{n+1})$.

In this appendix, we consider the Ricci DeTurck flow associated to (1.1) of (M, g_0), which is defined as

$$\frac{\partial g_{ij}}{\partial t} = -2E_{ij} + (\mathcal{L}_x g)_{ij} \tag{C.1}$$

$$:= -2[Ric_{ij} + (n-1)g_{ij}] + (\mathcal{L}_x g)_{ij} \ , \tag{C.2}$$

$$g(0) = g_0 \, , \tag{C.3}$$

where $X = X^k \frac{\partial}{\partial x^k}$ is the DeTurck vector field and $\mathcal{L}_X g$ denotes the Lie derivative of the metric g with respect to the vector field X.

For computational ease and comparison with the notation of [8], we write the flowing metric as

$$g = g(t) := g_0 + v(x, y^A, t) \, ,$$

where $v(x, y^A, t) := \frac{x^{m-2}}{m} w(y^A, t)_{ij} dx^i dx^j + O(x^{m-1})$ for $m \geq 1$, is a symmetric, time-dependent $(0,2)$-tensor on M.

In order to prove Proposition 3.2 for Ricci DeTurck flow, we require the components of the Lie derivative $\mathcal{L}_X g$. We now present the step-by-step computation of the Lie derivative components.

First, we choose the initial metric g_0 as the background metric. Let ∇ and $\mathring{\nabla}$ denote the Levi-Cività connection of g and g_0 respectively. Also, denote by Γ^k_{ij} and $\mathring{\Gamma}^k_{ij}$ are the Christoffel symbols with respect to g and g_0, respectively. Then, the components of the DeTurck vector field are given by

$$X^k = g^{ij}(\Gamma^k_{ij} - \mathring{\Gamma}^k_{ij}) \, .$$

Rewriting, we obtain the DeTurck vector field components

$$
\begin{aligned}
X^k &= g^{ij}(\Gamma^k_{ij} - \mathring{\Gamma}^k_{ij}) \\
&= \frac{1}{2} g^{kl} g^{ij} \left(\mathring{\nabla}_i g_{jl} + \mathring{\nabla}_j g_{il} - \mathring{\nabla}_l g_{ij} \right) \\
&= \frac{1}{2m} g^{kl} g^{ij} \left(\mathring{\nabla}_i (x^{m-2} w_{jl}) + \mathring{\nabla}_j (x^{m-2} w_{il}) - \mathring{\nabla}_l (x^{m-2} w_{ij}) \right) \\
&= g^{kl} g^{ij} \frac{1}{m} \left(\mathring{\nabla}_i (x^{m-2} w_{jl}) - \frac{1}{2} \mathring{\nabla}_l (x^{m-2} w_{ij}) \right) .
\end{aligned}
$$

We also compute

$$
\begin{aligned}
g^{ij} &= \left[g_0 + \frac{x^{m-2}}{m} w(t) \right]^{ij} \\
&= g_0^{ij} - \frac{x^{m-2}}{m} g_0^{il} g_0^{jm} w_{ml} + \frac{x^{2(m-2)}}{m} (g_0 + x^{m-2} w(t))^{jl} g_0^{im} g_0^{pq} w_{lp} w_{mq} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
g^{ij} g^{kl} &= \left[g_0 + \frac{x^{m-2}}{m} w(t) \right]^{ij} \left[g_0 + \frac{x^{m-2}}{m} w(t) \right]^{kl} \\
&= g_0^{ij} g_0^{kl} + O(x^{m+4}) .
\end{aligned}
$$

Therefore, up to sufficient order, we have the components of the DeTurck vector field are

$$
\begin{aligned}
X^k &= g^{kl} g^{ij} \frac{1}{m} \left(\mathring{\nabla}_i (x^{m-2} w_{jl}) - \frac{1}{2} \mathring{\nabla}_l (x^{m-2} w_{ij}) \right) \\
&= g_0^{kl} g_0^{ij} \frac{1}{m} \left(\mathring{\nabla}_i (x^{m-2} w_{jl}) - \frac{1}{2} \mathring{\nabla}_l (x^{m-2} w_{ij}) \right) + O(x^{m+2}) ,
\end{aligned}
$$

and hence the corresponding components of the associated DeTurck 1-form

$g(\cdot, X) = X_k dx^k$ are

$$X_k = g_0^{ij} \frac{1}{m} \left(\mathring{\nabla}_i (x^{m-2} w_{jk}) - \frac{1}{2} \mathring{\nabla}_k (x^{m-2} w_{ij}) \right) + O(x^m)$$
$$= x^2 \tilde{g}_0^{ij} \frac{1}{m} \left(\mathring{\nabla}_i (x^{m-2} w_{jk}) - \frac{1}{2} \mathring{\nabla}_k (x^{m-2} w_{ij}) \right) + O(x^m)\,.$$

Secondly, we note that the Christoffel symbols of g are given by

$$\Gamma^k_{ij} = \mathring{\Gamma}^k_{ij} + \frac{1}{2m} g^{kl} \left[\mathring{\nabla}_i (x^{m-2} w_{jl}) + \mathring{\nabla}_j (x^{m-2} w_{il}) - \mathring{\nabla}_l (\frac{x^{m-2}}{m} w_{ij}) \right]$$
$$= \mathring{\Gamma}^k_{ij} + O(x^{m+1})\,.$$

Let now $\tilde{g}_0 = x^2 g_0$, and denote by $\tilde{\mathring{\nabla}}$ the Levi-Cività connection of $\tilde{g}_0$, and by $\tilde{\mathring{\Gamma}}^k_{ij}$ the Christoffel symbols of $\tilde{g}_0$. By the conformal transformation of the Christoffel symbols,

$$\mathring{\Gamma}^k_{ij} = \tilde{\mathring{\Gamma}}^k_{ij} - \frac{1}{x} \left[\delta^k_i \delta^1_j + \delta^k_j \delta^1_i - \tilde{g}_0^{k1} (\tilde{g}_0)_{ij} \right] ;$$

therefore,

$$\mathring{\Gamma}^1_{11} = \tilde{\mathring{\Gamma}}^1_{11} - \frac{1}{x}\,,$$
$$\mathring{\Gamma}^B_{1A} = \tilde{\mathring{\Gamma}}^B_{1A} - \frac{1}{x} \delta^B_A\,,$$
$$\mathring{\Gamma}^1_{AB} = \tilde{\mathring{\Gamma}}^1_{AB} + \frac{1}{x} \tilde{g}_0^{11} (\tilde{g}_0)_{AB}\,,$$

and all other Christoffel symbols are of order $O(1)$ in x.

By definition of covariant derivatives,

$$\mathring{\nabla}_i w_{jk} = \partial_i w_{jk} - \mathring{\Gamma}^l_{ij} w_{lk} - \mathring{\Gamma}^l_{ik} w_{lj}\,.$$

So, we finally calculate that the components of the DeTurck 1-form are given by

$$
\begin{aligned}
X_k &= x^2 \tilde{g}_0^{ij} \frac{1}{m} \left(\mathring{\nabla}_i (x^{m-2} w_{jk}) - \frac{1}{2} \mathring{\nabla}_k (x^{m-2} w_{ij}) \right) + O(x^m) \\
&= \frac{x^2}{m} \tilde{g}_0^{ij} \tilde{\mathring{\nabla}}_i (x^{m-2} w_{jk}) + \frac{x}{m} \tilde{g}_0^{ij} \left[\delta_i^l \delta_j^1 w_{lk} + \delta_j^l \delta_i^1 w_{lk} - (\tilde{g}_0)_{ij} w_{1k} \right] x^{m-2} \\
&\quad + \frac{x}{m} \tilde{g}_0^{ij} \left[\delta_i^l \delta_k^1 w_{jl} + \delta_k^l \delta_i^1 w_{jl} - (\tilde{g}_0)_{ik} w_{j1} \right] x^{m-2} - \frac{1}{2m} x^2 \tilde{g}_0^{ij} \tilde{\mathring{\nabla}}_k (x^{m-2} w_{ij}) \\
&\quad - \frac{x}{2m} \tilde{g}_0^{ij} \left[\delta_k^l \delta_j^1 w_{li} + \delta_j^l \delta_k^1 w_{li} - (\tilde{g}_0)_{kj} w_{i1} \right] x^{m-2} \\
&\quad - \frac{x}{2m} \tilde{g}_0^{ij} \left[\delta_i^l \delta_k^1 w_{lj} + \delta_k^l \delta_i^1 w_{lj} - (\tilde{g}_0)_{ik} w_{j1} \right] x^{m-2} + O(x^m) \\
&= \frac{x^2}{m} \tilde{g}_0^{ij} \left(\tilde{\mathring{\nabla}}_i (x^{m-2} w_{jk}) - \frac{1}{2} \tilde{\mathring{\nabla}}_k (x^{m-2} w_{ij}) \right) + \frac{1}{m} \left[w_{1k} + w_{1k} - n w_{1k} \right] x^{m-1} \\
&\quad + \frac{1}{m} \left[\delta_k^1 \tilde{g}_0^{ij} w_{ij} + w_{1k} - w_{1k} \right] x^{m-1} - \frac{1}{2m} \left[w_{1k} + \delta_k^1 \tilde{g}_0^{ij} w_{ij} - w_{1k} \right] x^{m-1} \\
&\quad - \frac{1}{2m} \left[w_{1k} + \delta_k^1 \tilde{g}_0^{ij} w_{ij} - w_{1k} \right] x^{m-1} + O(x^m) \\
&= -\frac{(n-2)}{m} w_{1k} x^{m-1} + \frac{x^2}{m} \tilde{g}_0^{ij} \left(\tilde{\mathring{\nabla}}_i (x^{m-2} w_{jk}) - \frac{1}{2} \tilde{\mathring{\nabla}}_k (x^{m-2} w_{ij}) \right) + O(x^m) \,.
\end{aligned}
$$

In particular, we have obtained

$$
\begin{aligned}
X_1 &= -\frac{(n-2)}{m} w_{11} x^{m-1} + \frac{(m-2)}{m} \left[w_{11} - \frac{1}{2} \tilde{g}_0^{ij} w_{ij} \right] x^{m-1} + O(x^m) \,, \\
X_A &= -\frac{(n-2)}{m} w_{A1} x^{m-1} + \frac{(m-2)}{m} w_{1A} x^{m-1} + O(x^m) \,.
\end{aligned}
$$

Thirdly, we compute the components of the covariant derivative of the DeTurck vector field:

$$
\begin{aligned}
\nabla_1 X_1 &= \partial_1 X_1 - \Gamma^k_{11} X_k \\
&= -(n-2) w_{11} x^{m-2} + (m-2) w_{11} x^{m-2} + O(x^{m-1}) \,, \\
\nabla_A X_1 &= \partial_A X_1 - \Gamma^B_{11} X_B \\
&= -\frac{(n-2)}{m} w_{A1} x^{m-2} + \frac{(m-2)}{m} w_{A1} x^{m-2} + O(x^{m-1}) \,, \\
\nabla_1 X_A &= \partial_1 X_A - \Gamma^k_{1A} X_k \\
&= -(n-2)(m-1) w_{1A} x^{m-2} + (m-2) w_{1A} x^{m-2} + O(x^{m-1}) \,, \\
\nabla_A X_B &= \partial_A X_B - \Gamma^1_{AB} X_1 \\
&= -\frac{(n-2)}{m} w_{11} (\tilde{g}_0)_{AB} x^{m-2} + \frac{(m-2)}{m} (\tilde{g}_0)_{AB} \left[w_{11} - \frac{1}{2} \tilde{g}_0^{ij} w_{ij} \right] x^{m-2} + O(x^{m-1}) \,.
\end{aligned}
$$

Lastly, we then compute the components of Lie derivative of g with respect to X. Noting $(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$, the components are

$$
\begin{aligned}
(\mathcal{L}_X g)_{11} &= -2(n-2) w_{11} x^{m-2} + 2(m-2) \left[w_{11} - \frac{1}{2} \tilde{g}_0^{ij} w_{ij} \right] x^{m-2} + O(x^{m-1}) \,, \\
(\mathcal{L}_X g)_{1A} &= (\mathcal{L}_X g)_{A1} \\
&= \frac{(m-n)(m+1)}{m} w_{1A} x^{m-2} + O(x^{m-1}) \,, \\
(\mathcal{L}_X g)_{AB} &= -2 \frac{(n-2)}{m} w_{11} (\tilde{g}_0)_{AB} x^{m-2} + \frac{(m-2)}{m} (\tilde{g}_0)_{AB} w_{11} x^{m-2} \\
&\quad - \frac{(m-2)}{m} (\tilde{g}_0)_{AB} \tilde{g}_0^{CD} w_{CD} x^{m-2} + O(x^{m-1}) \,.
\end{aligned}
$$

Now we are finally in the position to provide an argument for a Ricci-DeTurck version of Proposition 3.2:

Proposition 3.2: *Let (M, g_0) be asymptotically hyperbolic with g_0 of the form as above. If $g(t)$ is a solution to (C.1) arising from initial data g_0, then $g(t)$ is also asymptotically hyperbolic of order l.*

Ricci-DeTurck approach: Let $g = g(t)$ have an expansion beginning at order $m \geq 1$. From the Ricci-DeTurck flow (C.1) associated to (1.1), we have $\frac{x^m}{m}\frac{\partial w_{ij}}{\partial t} = -2\mathcal{E}_{ij}$, where $\mathcal{E}_{ij} := E_{ij} - \frac{1}{2}(\mathcal{L}_X g)_{ij}$. Then, again by equations (3.13) – (3.15), we find $\frac{\partial w_{ij}}{\partial t} = A_{ij}{}^{kl} w_{kl}$, where A is a matrix with components now given by

$$\begin{aligned}
A_{11}{}^{11} &= (m-2)(n-1) - m[2(n-2)+m] + O(x)\,, \\
A_{11}{}^{AB} &= -m(m-2)g_{(k)}{}^{AB} + O(x)\,, \\
A_{1A}{}^{kl} &= O(x)\,, \\
A_{AB}{}^{11} &= -m\left[-2 + 2\frac{(m+1)(n-1)}{m}\right] g_{(k)}{}_{AB} + O(x)\,, \\
A_{AB}{}^{CD} &= -2(m-1)g_{(k)}{}^{CD} g_{(k)}{}_{AB} - m(n-m-1)\delta^C_A \delta^D_B + O(x)\,.
\end{aligned}$$

Hence, at every order, we once again find that $\frac{\partial w_{ij}}{\partial t} = A_{ij}{}^{kl} w_{kl}$ is a linear, partial differential equation in w_{ij}. So, for $m < l$, we have $w_{ij}(0) = 0$; uniqueness theory for linear partial differential equations gives $w_{ij}(t) = 0$ for $m < l$.

$\square$

In addition to proving a version of Proposition 3.2 with equation (1.1) replaced by equation (C.1), we note that the proof of Theorem 1.1 also carries through in the Ricci-DeTurck flow setting. Once more, let (M, g_0) be as above, but now suppose that $l = n$. Then, the Lie derivative contribution to the evolution of the mass aspect $\sigma = g_{(k)}{}^{AB}(\kappa_{AB} + w_{AB}) + \left(\frac{n-1}{n}\right)(\kappa_{11} + w_{11})$ in the Ricci-DeTurck flow is given by

$$
\begin{aligned}
{g_{(k)}}^{AB}\left(\mathcal{L}_X g\right)_{AB} + \left(\frac{n-1}{n}\right)\left(\mathcal{L}_X g\right)_{11} = &\left\{ \frac{(2n-4)}{n}(n-1)w_{11} \right.\\
&- \frac{(2n-4)}{n}(n-1)\left[w_{11} - \frac{1}{2}{g_{(k)}}^{ij}w_{ij}\right]\\
&+ \left(\frac{n-1}{n}\right)(2n-4)\left[-w_{11}+w_{11}\right]\\
&\left. - \left(\frac{n-1}{n}\right)(n-2)\left[{g_{(k)}}^{ij}w_{ij}\right]\right\} x^{n-2}\\
&+ O(x^{n-1})\\
= &\, O(x^{n-1})\,.
\end{aligned}
$$

Thus we see that the Lie derivative term $\left(\mathcal{L}_X g\right)_{ij}$ in the Ricci-DeTurck flow of g_0 makes no contribution at significant order $O(x^{n-2})$ to the evolution of the mass aspect σ. Hence, the argument provided in section 3.B for Theorem 1.1 also verifies Theorem 1.1 from the Ricci-DeTurck flow point-of-view. Further, since the diffeomorphisms induced by the DeTurck vector field X, denoted φ_t, satisfy $\frac{\partial \varphi_t}{\partial t} = -X$, we have the pull-back obeys $\varphi_t^* = Id + O(x^n)$. Therefore, we note that if we instead choose to evolve g_0 under Ricci-DeTurck flow and then pull-back to the normalized Ricci flow (1.1), the act of pulling-back also does not add any terms to the evolution of the mass aspect, and hence the evolution of the mass as computed in Ricci-DeTurck flow and the evolution of the mass as computed in the normalized Ricci flow (1.1) are equivalent.

Appendix D

Conformal Change of the Riemann Curvature Tensor

Here we provide the calculation of how the components of the Riemann curvature tensor changes under a conformal transformation.

Let (M, g) be a Riemannian manifold, and let $\rho : M \to \mathbb{R}$ be a positive, smooth function. Consider the new Riemannian metric on M given by $\tilde{g} = \rho^2 g$. Let ∇, Γ^i_{jk} be the Levi-Cività connection and corresponding Christoffel symbols of the metric g, and similarly let $\tilde{\nabla}$, $\tilde{\Gamma}^i_{jk}$ be the Levi-Cività connection and corresponding Christoffel symbols of the metric $\tilde{g}$. We will denote the components of the curvature $(1,3)$-tensor $R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ in a local coordinate basis by R^p_{ijk} and components of the corresponding curvature $(1,3)$- tensor of $(M, \tilde{g})$ in a local coordinate basis by $\tilde{R}^p_{ijk}$. The components of the respective Riemann curvature $(0,4)$-tensors will thus be denoted by $R_{ijkl} := g_{pl} R^p_{ijk}$ and $\tilde{R}_{ijkl} := \tilde{g}_{pl} \tilde{R}^p_{ijk}$.

We compute that the Christoffel symbols obey

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \frac{1}{\rho} \left[\delta^i_j \partial_k \rho + \delta^i_k \partial_j \rho - \tilde{g}^{is} \tilde{g}_{jk} \partial_s \rho \right] \tag{D.1}$$

$$:= \tilde{\Gamma}^i_{jk} - C^i_{jk} \,. \tag{D.2}$$

As well, the components of $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ in a local basis are calculated to be

$$R^p_{ijk} = \partial_j \Gamma^p_{ik} - \partial_i \Gamma^p_{jk} + \Gamma^p_{mj} \Gamma^m_{ik} - \Gamma^p_{mi} \Gamma^m_{jk} \,. \tag{D.3}$$

Therefore, from equations (D.2) and (D.3), we have

$$\begin{aligned} R^p_{ijk} &= \partial_j (\tilde{\Gamma}^p_{ik} - C^p_{ik}) - \partial_i (\tilde{\Gamma}^p_{jk} - C^p_{jk}) + (\tilde{\Gamma}^p_{mj} - C^p_{mj})(\tilde{\Gamma}^m_{ik} - C^m_{ik}) && \text{(D.4)} \\ &\quad -(\tilde{\Gamma}^p_{mi} - C^p_{mi})(\tilde{\Gamma}^m_{jk} - C^m_{jk}) \\ &= \tilde{R}^p_{ijk} - \partial_j C^p_{ik} + \partial_i C^p_{jk} - C^m_{ik} \tilde{\Gamma}^p_{mj} - C^p_{mj} \tilde{\Gamma}^m_{ik} + C^m_{jk} \tilde{\Gamma}^p_{mi} + C^p_{mi} \Gamma^m_{jk} && \text{(D.5)} \\ &\quad + C^p_{mj} C^m_{ik} - C^p_{mi} C^m_{jk} \\ &= \tilde{R}^p_{ijk} - \tilde{\nabla}_j C^p_{ik} + \tilde{\nabla}_i C^p_{jk} + C^p_{mj} C^m_{ik} - C^p_{mi} C^m_{jk} \,. && \text{(D.6)} \end{aligned}$$

Now, the covariant derivatives of C^p_{jk} are

$$\begin{aligned} \tilde{\nabla}_i C^p_{jk} &= -\frac{1}{\rho^2} \partial_i \rho \left[\delta^p_j \partial_k \rho + \delta^p_k \partial_j \rho - \tilde{g}^{ps} \tilde{g}_{jk} \partial_s \rho \right] \\ &\qquad + \frac{1}{\rho} \left[\tilde{\nabla}_i (\delta^p_j \partial_k \rho) + \tilde{\nabla}_i (\delta^p_k \partial_j \rho) - \tilde{\nabla}_i (\tilde{g}^{ps} \tilde{g}_{jk} \partial_s \rho) \right] \\ &= -\frac{1}{\rho^2} \partial_i \rho \left[\delta^p_j \partial_k \rho + \delta^p_k \partial_j \rho - \tilde{g}^{ps} \tilde{g}_{jk} \partial_s \rho \right] \\ &\qquad + \frac{1}{\rho} \left[\delta^p_j \tilde{\nabla}_i \partial_k \rho + \delta^p_k \tilde{\nabla}_i \partial_j \rho - \tilde{g}^{ps} \tilde{g}_{jk} \tilde{\nabla}_i \partial_s \rho \right] \,. \end{aligned}$$

Substituting this into equation (D.6), we have

$$
\begin{aligned}
R^p_{ijk} = \tilde{R}^p_{ijk} &- \frac{1}{\rho^2}\partial_i\rho\left[\delta^p_j\partial_k\rho + \delta^p_k\partial_j\rho - \tilde{g}^{ps}\tilde{g}_{jk}\partial_s\rho\right] \\
&+\frac{1}{\rho}\left[\delta^p_j\tilde{\nabla}_i\tilde{\nabla}_k\rho + \delta^p_k\tilde{\nabla}_i\tilde{\nabla}_j\rho - \tilde{g}^{ps}\tilde{g}_{jk}\tilde{\nabla}_i\tilde{\nabla}_s\rho\right] \\
&+\frac{1}{\rho^2}\partial_j\rho\left[\delta^p_i\partial_k\rho + \delta^p_k\partial_i\rho - \tilde{g}^{ps}\tilde{g}_{ik}\partial_s\rho\right] \\
&-\frac{1}{\rho}\left[\delta^p_i\tilde{\nabla}_j\tilde{\nabla}_k\rho + \delta^p_k\tilde{\nabla}_j\tilde{\nabla}_i\rho - \tilde{g}^{ps}\tilde{g}_{ik}\tilde{\nabla}_j\tilde{\nabla}_s\rho\right] \\
&+\frac{1}{\rho^2}\left[\delta^p_m\partial_j\rho + \delta^p_j\partial_m\rho - \tilde{g}^{ps}\tilde{g}_{mj}\partial_s\rho\right]\cdot\left[\delta^m_i\partial_k\rho + \delta^m_k\partial_i\rho - \tilde{g}^{ms}\tilde{g}_{ik}\partial_s\rho\right] \\
&-\frac{1}{\rho^2}\left[\delta^p_m\partial_i\rho + \delta^p_i\partial_m\rho - \tilde{g}^{ps}\tilde{g}_{mi}\partial_s\rho\right]\cdot\left[\delta^m_j\partial_k\rho + \delta^m_k\partial_j\rho - \tilde{g}^{ms}\tilde{g}_{jk}\partial_s\rho\right] .
\end{aligned}
\tag{D.7}
$$

By multiplying through, using $R_{ijkl} := g_{pl}R^p_{ijk}$, and grouping like terms, we compute

$$
\begin{aligned}
R_{ijkl} = \frac{1}{\rho^2}\tilde{R}_{ijkl} &- \frac{1}{\rho^3}\tilde{g}_{ik}\left[\tilde{\nabla}_j\tilde{\nabla}_l\rho - \frac{1}{\rho}\tilde{g}^{ms}\partial_s\rho\partial_m\rho\tilde{g}_{lj}\right] \\
&-\frac{1}{\rho^3}\tilde{g}_{lj}\tilde{\nabla}_i\tilde{\nabla}_k\rho + \frac{1}{\rho^3}\tilde{g}_{jk}\tilde{\nabla}_i\tilde{\nabla}_l\rho \\
&+\frac{1}{\rho^3}\tilde{g}_{il}\left[\tilde{\nabla}_j\tilde{\nabla}_k\rho - \frac{1}{\rho}\tilde{g}^{ms}\partial_s\rho\partial_m\rho\tilde{g}_{jk}\right] .
\end{aligned}
\tag{D.8}
$$

Re-grouping terms in equation (D.8), we obtain that the components of the Riemann curvature tensor of (M, g) and the components of the Riemann curvature tensor of the conformal Riemannian manifold $(M, \tilde{g})$ are related as

$$
\begin{aligned}
R_{ijkl} = \frac{1}{\rho^2}\tilde{R}_{ijkl} &+ \frac{1}{\rho^4}|d\rho|^2_{\tilde{g}}(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}) \\
&+\frac{1}{\rho^3}\left[\tilde{g}_{jk}\tilde{\nabla}_i\tilde{\nabla}_l\rho + \tilde{g}_{il}\tilde{\nabla}_j\tilde{\nabla}_k\rho - \tilde{g}_{ik}\tilde{\nabla}_j\tilde{\nabla}_l\rho - \tilde{g}_{lj}\tilde{\nabla}_i\tilde{\nabla}_k\rho\right] .
\end{aligned}
$$