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Market with Transaction Costs: Optimal Shadow State-Price Densities and Exponential Utility Maximization

by

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Abstract

This thesis discusses the financial market model with proportional transaction costs considered in Cvitanic and Karatzas (1996) (hereafter we use CK (1996)). For a modified dual problem introduced by Choulli (2009), I discuss solutions under weaker conditions than those of CK (1996), and furthermore the obtained solutions generalize the examples treated in CK (1996). Then, I consider the exponential utility which does not belong to the family of utility considered by CK (1996) due to the Inada condition. Finally, I elaborate the same results as in CK (1996) for the exponential utility, and I derive other related results using the specificity of the exponential utility function as well. These lead to a different method/approach than CK (1996) for our utility maximization problem, and different notion of admissibility for financial strategies as well.

Contents

List of Symbols

$a.s.$	almost surely
$A \equiv B$	the equivalence between A and B
$E[X]$	the expectation of the random variable X
$E[X\|\mathcal{F}]$	the conditional expectation of X with respect to $\mathcal{F}$
exp	the exponential function
$\mathbf{I}_{\{C\}}$	the indicator function with respect to C
ln	the logarithm function
$\mathbb{N}$	the natural numbers
$\\|\cdot\\|_\infty$	the maximum norm
$\mathbf{P}$, $\mathbf{Q}$	the probability measure
$\mathbb{R}$	the real numbers

Chapter 1

Introduction

Louis Bachelier is considered as the first mathematician to use the concept of Brownian motion for modeling a financial phenomena before Albert Einstein, who used it to model the movement of the particles in physics in 1905. Precisely, in 1900, Bachelier defended his Ph.D. thesis under the title "*Théorie de la Spéculation*." In this thesis, he modeled the movement of the stock price by using the Brownian motion and addressed the problem of evaluation of financial options, see Davis and Etheridge (2006) for details about this issue. He derived a formula that is very close to the results of Fischer Black, Myron Scholes, and Robert Merton in 1973, which led to the Nobel Prize in 1997.

Samuelson (1965) stated that the geometric Brownian motion is useful to describe a general stochastic model of price. He also referred to the martingale property of the present discounted value of asset and the lack of memory property, so that any attempt of prediction on the pattern of the stock prices from the past would be failed. Merton (1969) applied the Brownian motion to describe the rate of the stock returns. Actually Merton solved explicitly the optimization problem of the constant relative risk utility with the geometric Brownian motion. Black and Scholes (1973) derived the famous Black-Scholes formula based on the assumption of the geometric Brownian motion and no arbitrage opportunity in the hedged positions. They explained that the expected return rate of the risky asset must be equal to the short-term interest rate under the no arbitrage assumption. Black and Scholes (1973) referred to the no-arbitrage concept of Merton's paper (1973). Merton, Black, and Scholes played a big role of developing the no-arbitrage option pricing via solving the so-called Black-Scholes-Merton differential equation, which was derived under the equality of the return rate and the interest rate. This no-arbitrage pricing method has been then developed extensively in the mathematical and the financial literature. For instance, we can cite Ross (1976), Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), who applied the Girsanov Theorem and the martingale theory to the risk-neutral pricing. They connected the existence of a risk-neutral probability measure with no arbitrage.

As mentioned in Black and Scholes (1973), the "ideal conditions" are mostly assumed in many research papers with various kinds of problems. These assumptions are basically related to define a complete market, where there is no arbitrage opportunity and any risky asset can be replicated with some combinations of contingent claims. For example, Black and Scholes (1973) assumed the following:

a) The short-term interest rate is known and is constant through time.

b) The stock pays no dividends.

c) There are no transaction costs in trading.

d) It is allowed to sell or hold any fraction of the price at the short- term interest rate.

e) No penalties to short selling.

It is only later in mid-seventies that Magill and Constantinides (1976) pointed out the costless trading opportunities lead to a unrealistic portfolio behavior, such like investors would indulge in a huge amount of security trading due to covering the continually varying prices. This is true because no transaction costs allow investors to trade stocks with no limit, i.e., they can change their position immediately when stock prices move. Since the total variation of Brownian motion is infinite, if the geometric Brownian motion is assumed, the total change of stock prices will be infinite. In order to meet the realistic environment they introduced the proportional transaction costs to both of selling and buying at the same rate. This idea was then supported by Leland (1985), who also stated that nonzero transaction costs let continuous trading be ruinously expensive since diffusion processes have infinite variation and replicating operations are occurred infinitely many.

Taksar, Klass, and Assaf (1988) brought the more mathematical aspect to the financial market with transaction costs. They mathematically simplify the market by defining the two kinds of assets, the bank account and the stock, then transaction costs are charged when money is transferred from the bank account to the stock or vice versa. Davis and Norman (1990) also used the almost same model of Taksar et al (1988), but they illustrated the graphical analysis in the space of bank and stock holding strategy.

Other mathematical papers dealing with the problem of transaction costs, we can cite Jouini and Kallal (1995), Cvitanic and Karatzas (1996), and the references therein. As clearly indicated in the literature, there is some difficulty in solving directly the utility maximization problem for the transaction costs model. In order to deal with this difficulty, Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989), and others have developed the duality methods, which give powerful ideas to reach the solutions of the maximization. In this paper, we want to discuss the optimal solution for the transaction costs model through this duality methods and without the Inada condition, while Klein and Rogers (2007) stated that this condition is required to obtain the optimal solution.

This thesis is organized as follows. Chapter 2 reviews useful mathematical tools for mathematical finance, some of them are used in this thesis. Precisely, in Section 2.1 we recall the definition of filtration, Brownian motion, and martingales. In Section 2.2 we define the stochastic exponential, and state Girsanov's theorem and Radon-Nikodym derivative, which are related to the change of probability measure (such as the change from the physical probability measure to the risk-neutral probability measure). Stopping times and their role in martingale are reviewed in Section 2.3. One of the most important foundation of mathematical finance is Itô calculus, which includes Itô integral, Itô isometry, Itô-Doeblin formula, and martingale representation theorem. All these concepts are given in Section 2.4. Finally we define utility functions and their risk aversion, and discuss the classification of utilities in Section 2.5. Also we consider the Inada condition, which is widely assumed for utility functions.

In Chapter 3 we mathematically outline the financial market model with transaction costs. We introduce auxiliary martingales and convex conjugate of utility in order to find an alternate risk-neutral probability measure via establishing the dual problem. In Chapter 4 we solve the dual problem defined in Chapter 3. We obtain each component of the optimal solution separately. In Chapter 5 we solve the expected utility maximization problem for the exponential utility, which does not satisfy the Inada condition. We also obtain the relationship between the solutions to the primal problem and the dual problem.

Chapter 2

Review on Stochastic Tools

2.1 Filtration and Brownian Motion

Definition 2.1.1. *Let Ω be a nonempty set and $T > 0$. Suppose there exists a σ-algebra $\mathcal{F}(t)$ for $\forall\, t \in [0,\, T]$, such that $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ if $0 \leq s \leq t$, $\mathcal{F}(0) = \{\emptyset,\, \Omega\} \cup \mathcal{N}$, where $\mathcal{N}$ is the set of negligible events, i.e., if $A \in \mathcal{N}$, then $P(A) = 0$. The collection of σ-algebras $\mathcal{F}(t)$, $t \in [0,\, T]$, is called a filtration.*

We can interpret Ω as the set of all scenarios, T as the horizon of time, and a filtration as aggregate information available to investors up to time $t \in [0,\, T]$.

Definition 2.1.2. *Let X be a random variable on a nonempty sample space Ω. Let $\mathcal{G}$ be a σ-algebra of subsets of Ω. X is $\mathcal{G}$-measurable if every set in the σ-algebra generated by X is in $\mathcal{G}$.*

This means the information in $\mathcal{G}$ allows investors to determine the value of X. For example, a portfolio holding position at time t should be $\mathcal{F}(t)$-measurable, which indicates that the value of portfolio is determined by only the information up to time t.

Definition 2.1.3. *Let Ω be a nonempty sample space and $\mathcal{F}(t)$ be a filtration on $t \in [0,\, T]$. Let $X(t)$ be a collection of random variables indexed by $t \in [0,\, T]$. $X(t)$ is an adapted stochastic process if $X(t)$ is $\mathcal{F}(t)$-measurable for any $t \in [0,\, T]$.*

Values of assets, portfolio, and wealth are considered as adapted processes to a filtration of market/public information.

Definition 2.1.4. *Let* $(\Omega,\, \mathcal{F},\, \boldsymbol{P})$ *be a probability space. Suppose there exists a continuous function* $W(t)$ *on* $t \geq 0$ *for each* $\omega \in \Omega$. $W(t)$ *is a Brownian motion if*

1) $W(0) = 0$,

2) For $0 = t_0 < t_1 < ... < t_m$, *the increments*

$$W(t_1) - W(t_0),\ W(t_2) - W(t_1), ..., W(t_m) - W(t_{m-1})$$

are independent

3) Each increment follows the normal distribution

$$W(t_{i+1}) - W(t_i) \sim N(0,\ t_{i+1} - t_i),\ i = 0, 1, ..., m-1$$

The Brownian motion is assumed to be $\mathcal{F}(t)$- measurable, which means that all information up to time t allows to evaluate $W(t)$. Moreover, each increment $W(t) - W(s)$, $0 \leq s < t$, is independent of $\mathcal{F}(s)$. All information up to time s does not determine any increments of the Brownian motion after time s. In addition, $W(t) - W(s)$ is $\mathcal{F}(t)$ -measurable.

Proposition 2.1.1. *Let* $W(t)$ *be a Brownian motion. For any constant* $\sigma > 0$, *the following assertions hold.*

(1) $W(t)$ *is a martingale.*

(2) A process $\xi(t) = \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(t)\right\}$ *is also a martingale.*

Proof:

(1) Any increments after time s are independent of $\mathcal{F}(s)$ and the expected value of increment is 0. For $0 \leq s < t$,

$$\begin{aligned} E[W(t)|\mathcal{F}(s)] &= E[(W(t) - W(s)) + W(s)|\mathcal{F}(s)] \\ &= E[W(t) - W(s)|\mathcal{F}(s)] + E[W(s)|\mathcal{F}(s)] \\ &= E[W(t) - W(s)] + W(s) \\ &= W(s) \end{aligned}$$

Thus, $W(t)$ is a martingale.

(2) For $0 \le s < t$,

$$E[\xi(t)|\mathcal{F}(s)] = E\left[\exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(t)\right\}\middle| \mathcal{F}(s)\right]$$
$$= E\left[\exp\left\{-\frac{1}{2}\sigma^2 t + \sigma(W(t) - W(s)) + \sigma W(s)\right\}\middle| \mathcal{F}(s)\right]$$

$$E[\xi(t)|\mathcal{F}(s)] = E\left[\exp\left\{\sigma(W(t) - W(s))\right\}\middle| \mathcal{F}(s)\right] * \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(s)\right\}$$
$$= E[\exp\{\sigma(W(t) - W(s))\}] * \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(s)\right\}$$
$$= \exp\left\{\frac{1}{2}\sigma^2(t-s)\right\} * \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(s)\right\}$$
$$= \exp\left\{-\frac{1}{2}\sigma^2 s + \sigma W(s)\right\}$$
$$= \xi(s)$$

This ends the proof of the proposition.

□

Example 2.1.1. ***(Stock with constant return rate/volatility)***
It is a well-known assumption that the price of stock $S(t)$ follows the geometric Brownian motion, such as if we assume the stock has a constant return rate $b \ge 0$ and a constant volatility $\sigma > 0$

$$dS(t) = S(t)[bdt + \sigma dW(t)]$$
$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(b - \frac{1}{2}\sigma^2\right)t\right\}$$

where $S(0)$ is the stock price at $t = 0$. Then if we take $\xi(t) = \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(t)\right\}$, *for* $0 \le s < t$*,*

$$\begin{aligned}
E[S(t)|\mathcal{F}(s)] &= S(0)E\left[\exp\left\{bt + \left(-\frac{1}{2}\sigma^2 t + \sigma W(t)\right)\right\}\middle| \mathcal{F}(s)\right] \\
&= S(0)\exp\{bt\}E[\xi(t)|\mathcal{F}(s)] \\
&= S(0)\exp\{bt\}\xi(s) \\
&= S(0)\exp\{b(t-s)\}\exp\{bs\}\exp\left\{-\frac{1}{2}\sigma^2 s + \sigma W(s)\right\} \\
&= e^{b(t-s)}S(0)\exp\left\{\sigma W(s) + \left(b - \frac{1}{2}\sigma^2\right)s\right\} \\
&= e^{b(t-s)}S(s)
\end{aligned}$$

This implies that the expected value of stock in the future time t with the information up to time s is equal to the value of exponential growth at the stock return rate of S(s) for the duration $t - s$.

Proposition 2.1.2. *For the exponential function with constant* $\sigma > 0, e^{\sigma W(t)}, 0 \le s < t$*,*

$$E[\exp\{\sigma(W(t) - W(s))\}] = \exp\left\{\frac{1}{2}\sigma^2(t-s)\right\}.$$

Proof:
Denote $x = W(t) - W(s)$. Since $W(t) - W(s)$ follows a normal distribution with the mean zero and variance $\sigma(t-s)$, then we get,

$$\begin{aligned}
&E[\exp\{\sigma(W(t) - W(s))\}] \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{\sigma x - \frac{x^2}{2(t-s)}\right\} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(x - \sigma(t-s))^2 - (\sigma(t-s))^2}{2(t-s)}\right\} dx \\
&= \exp\left\{\frac{1}{2}\sigma^2(t-s)\right\}\left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(x-\sigma(t-s))^2}{2(t-s)}\right\} dx\right).
\end{aligned}$$

By considering the change of variable $u = x - \sigma(t-s)$, we get,

$$\begin{aligned}
&E[\exp\{\sigma(W(t)-W(s))\}] \\
&= \exp\left\{\frac{1}{2}\sigma^2(t-s)\right\}\left(\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{u^2}{2(t-s)}\right\}du\right) \\
&= \exp\left\{\frac{1}{2}\sigma^2(t-s)\right\}
\end{aligned}$$

□

2.2 Stochastic Exponential

The stochastic exponential is a very useful tool for the stochastic analysis.

Definition 2.2.1. ***(Doléans-Dade Exponential)*** *Let X be a martingale, where X^c denotes its continuous martingale part and $X - X^c$ is a pure jump martingale part. The stochastic exponential (Doléans-Dade Exponential) of X is the unique solution of the following stochastic differential equation,*

$$dZ(t) = Z_{-}(t)dX(t), \quad Z(0) = 1$$

and it is given by

$$Z(t) = \exp\left\{X(t) + \frac{1}{2} < X^c >\right\}\Pi_{0<s\leq t}(1+\Delta X_s)e^{-\Delta x_s}$$

The above definition sounds much general than the context of this thesis, so we present a weaker version.

Definition 2.2.2. *Let W be a Brownian motion and X be a process, such that*

$$dX(t) = \mu_X(t)dt + \sigma_X(t)dW(t), \quad t \geq 0$$

where μ_X: a function, σ_X: a positive function.
The stochastic exponential (Doléans-Dade Exponential) is the unique solution of the following stochastic differential equation,

$$dZ(t) = Z(t)dX(t), \quad Z(0) = 1$$

and it is given by

$$Z(t) = \exp\left\{\int_0^t \left(\mu_X(t) - \frac{1}{2}\sigma_X^2(t)\right) dt + \int_0^t \sigma_X(t) dW(t)\right\}.$$

Example 2.2.1. ***(Stock with non-constant return rate/volatility)***
The stock price $S(t)$ is commonly assumed with geometric Brownian motion, such that

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)],$$

where $b(t)$, $\sigma(t)$ are (positive) functions. By Doléans-Dade exponential, we have the stock price $S(t)$ given by

$$S(t) = S(0)\exp\left\{\int_0^t \left(b(t) - \frac{1}{2}\sigma^2(t)\right) dt + \int_0^t \sigma(t) dW(t)\right\}.$$

Remark 2.2.1. *If the stochastic exponential is expressed by*

$$Z(t) = \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(t)dt + \int_0^t \sigma(t)dW(t)\right\},$$

then we denote the following to this stochastic exponential

$$\mathcal{E}_t(\sigma \cdot W) = \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(t)dt + \int_0^t \sigma(t)dW(t)\right\}.$$

We have an important theorem with the stochastic exponential.

Theorem 2.2.1. ***(Girsanov's Theorem)***
Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $W(t)$, $t \in [0, T]$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$, and $\mathcal{F}(t)$ is a filtration for $W(t)$. Suppose $\theta(t)$ is an adapted process. Consider

$$\begin{aligned} Z(t) &= \exp\left\{\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right\} \\ \tilde{W}(t) &= W(t) - \int_0^t \theta(u)du \\ \textit{with} \quad & E\left[\int_0^T \theta^2(u)Z^2(u)du\right] < +\infty. \end{aligned}$$

Define a random variable $\zeta:\ \Omega \to \mathbb{R}$ such that for $\omega \in \Omega$

$$\zeta(\omega) = Z(T)(\omega)$$

Also define a measure $\boldsymbol{Q}$ such that for any $A \in \mathcal{F}$,

$$\boldsymbol{Q}(A) = \int_A \zeta(\omega) d\boldsymbol{P}(\omega)$$

Then, $\boldsymbol{Q}$ is a probability measure on $\mathcal{F}(T)$ and $\tilde{W}(t)$ is a Brownian motion with respect to $\boldsymbol{Q}$.

Proof:
For the proof of this result, we refer to Shreve (2004), page 212-214. □

Remark 2.2.2. *ζ is called the Radon-Nikodym derivative of $\boldsymbol{Q}$ with respect to $\boldsymbol{P}$ and is denoted by*

$$\zeta = \frac{d\boldsymbol{Q}}{d\boldsymbol{P}}.$$

Girsanov's theorem allows to change a probability measure from the actual $\boldsymbol{P}$ to the risk-neutral $\boldsymbol{Q}$. In terms of mathematical finance $Z(t)$ is called the state-price density and if we choose an appropriate state-price density, then by Girsanov's theorem we can calculate risk-neutral prices. Moreover, it is well-known that risk-neutral prices allow no arbitrage opportunities in the market, so they are considered as "fair" prices.

2.3 Stopping Time

Definition 2.3.1. ***(a nonnegative-integer valued stopping time)***
A random variable

$$\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$$

is called a stopping time if $\{\tau \leq n\} \in \mathcal{F}(n)$ for all $n \in \mathbb{N}$.

Stopping time is considered as the time when a random event happens. In the case that the event never occurs, we say stopping time takes the value $+\infty$. This definition is commonly used, but slightly inconvenient for the continuous-time trading model. So, we want to modify this definition as following.

Definition 2.3.2. ***(a nonnegative-real valued stopping time)***
A random variable

$$\tau : \Omega \rightarrow [0,\ +\infty]$$

is called a stopping time if $\{\tau \leq t\} \in \mathcal{F}(t)$ *for all* $t \geq 0$.

Definition 2.3.2. allows to make a decision with all information up to time $t \in [n,\ n+1)$, while Definition 2.3.1. allows only information up to time n for all $t \in [n,\ n+1)$. Since we treat the continuous-time trading, we use Definition 2.3.1. through out this thesis .

A typical example of stopping time in terms of mathematical finance is the time when an American option is exercised. An American option allows an investor to exercise at any time before maturity. She will make a decision with all information up to at that time, but with no anticipative/future information.

Definition 2.3.3. ***(a bounded stopping time)***
A stopping time τ *is bounded if there exists a constant* c *such that for all* $\omega \in \Omega$

$$P\{\tau(\omega) \leq c\} = 1.$$

This is important because we assume the finite time horizon, such that, $t \in [0,\ T]$ where $T < +\infty$ is maturity. Then we have a powerful theorem from probability theory.

Theorem 2.3.1. ***(Doob's Optional Sampling Theorem)***
Let $(X(t))_{t\geq 0}$ *be a martingale and let* u, v *be stopping times bounded by* $T < +\infty$ *with* $0 \leq u \leq v$ *a.s. Then,*

$$E\Big[X(v)|\mathcal{F}(u)\Big] = X(u) \quad a.s.$$

Proof:
The proof of this theorem can be found in Jacod and Protter (2004), page 215-216.

□

We see the martingale property on the fixed time in Proposition 2.1.1., but here we see the martingale property on the stopping time.

2.4 Itô Calculus for Mathematical Finance

Consider a partition on $[0,\ T]$, such that

$$0 = t_0 \le t_1 \le ... \le t_n = T$$

and we trade the stock on the trading time $t_0, t_1, ..., t_{n-1}$. Let $\Delta X(t_i)$ denote the holding position of stock in the interval $[t_i,\ t_{i+1})$, $i = 0, 1, ..., n-1$. This means that $\Delta X(t_i)$ is constant on $[t_i,\ t_{i+1})$. We assume that $\Delta X(t_i)$ is $\mathcal{F}(t_i)$-measurable. Then the cumulative gain from trading at time $t \in [t_k,\ t_{k+1})$ is given by

$$I(t) = \sum_{j=0}^{k-1} \Delta X(t_j)(W(t_{j+1}) - W(t_j)) + \Delta X(t_k)(W(t) - W(t_k))$$

also we define the Itô integral of $\Delta(t)$

$$I(t) := \int_0^t \Delta X(u) dW(u)$$

Theorem 2.4.1. *The Itô integral is a martingale.*

Proof:
Let $0 = t_0 \le s \le t \le T$ and $s \in [t_m,\ t_{m+1})$, $t \in [t_n,\ t_{n+1})$, where $t_m < t_n \le t_{n+1} \le T$. The Itô integral is represented by

$$\begin{aligned}I(t) = &\sum_{j=0}^{m-1} \Delta X(t_j)[W(t_{j+1}) - W(t_j)] + \sum_{j=m+1}^{n-1} \Delta X(t_j)[W(t_{j+1}) - W(t_j)] \\ &+\Delta X(t_m)[(W(t_{m+1}) - W(s)) + (W(s) - W(t_m))] \\ &+\Delta X(t_n)[W(t) - W(t_n)]\end{aligned}$$

We know any increments in $[0,\ s]$ are $\mathcal{F}(s)$-measurable,

$$\begin{aligned}&E\left[\sum_{j=0}^{m-1} \Delta X(t_j)[W(t_{j+1}) - W(t_j)] + \Delta X(t_m)[W(s) - W(t_m)] \middle| \mathcal{F}(s)\right] \\ &= \sum_{j=0}^{m-1} \Delta X(t_j)[W(t_{j+1}) - W(t_j)] + \Delta X(t_m)[W(s) - W(t_m)] = I(s)\end{aligned}$$

We know any increments in $(s,\, t]$ are independent of $\mathcal{F}(s)$, and the expected values of any increments in $(s,\, t]$ are zero.

$$
\begin{aligned}
&E\left[\sum_{j=m+1}^{n-1} \Delta X(t_j)[W(t_{j+1}) - W(t_j)] + \Delta X(t_n)[W(t) - W(t_n)]\right.\\
&\qquad \left. + \Delta X(t_m)[W(t_{m+1}) - W(s)] \,\middle|\, \mathcal{F}(s)\right]\\
&= \sum_{j=m+1}^{n-1} \Delta X(t_j) \underbrace{E[W(t_{j+1}) - W(t_j)]}_{0} + \Delta X(t_n) \underbrace{E[W(t) - W(t_n)]}_{0}\\
&\qquad + \Delta X(t_m) \underbrace{E[W(t_{m+1}) - W(s)]}_{0}\\
&= 0
\end{aligned}
$$

Hence $E[I(t)|\mathcal{F}(s)] = I(s)$. □

Theorem 2.4.2. ***(Itô isometry)*** *The Itô integral satisfies*

$$
E[I^2(t)] = E\left[\int_0^t \Delta X^2(u) du\right]
$$

Proof:
Let ΔW_j denote $W(t_{j+1}) - W(t_j)$ for $j = 0,\, 1,\, \ldots, k-1$ and $\Delta W_k = W(t) - W(t_k)$. Then,

$$
I^2(t) = \sum_{j=0}^{k} \Delta X^2(t_j) \Delta W_j^2 + 2 \sum_{0 \le i < j \le k} \Delta X(t_i) \Delta X(t_j) \Delta W_i \Delta W_j
$$

We know $E[\Delta W_j] = 0$ and ΔW_j is independent of $\mathcal{F}(t_j)$.

$$
\begin{aligned}
E[\Delta X(t_i)\Delta X(t_j)\Delta W_i \Delta W_j] &= E[\Delta X(t_i)\Delta X(t_j)\Delta W_i] * E[\Delta W_j]\\
&= E[\Delta X(t_i)\Delta X(t_j)\Delta W_i] * 0\\
&= 0
\end{aligned}
$$

Since $\Delta X(t_j)$ is $\mathcal{F}$- measurable and ΔW_j is independent of $\mathcal{F}(t_j)$, $\Delta X^2(t_j)$ is $\mathcal{F}$-measurable and ΔW_j^2 is independent of $\mathcal{F}(t_j)$. Also $E[\Delta W_j^2] = t_{j+1} - t_j$.

Then,

$$\begin{aligned}
E[I^2(t)] &= \sum_{j=0}^{k} E[\Delta X^2(t_j)\Delta W_j^2] = \sum_{j=0}^{k} E[\Delta X^2(t_j)] * E[\Delta W_j^2] \\
&= \sum_{j=0}^{k-1} E[\Delta X^2(t_j)(t_{j+1}-t_j)] + E[\Delta X^2(t_k)(t-t_k)] \\
&= \sum_{j=0}^{k-1} E\left[\int_{t_j}^{t_{j+1}} \Delta X^2(u)du\right] + E\left[\int_{t_k}^{t} \Delta X^2(u)du\right] \\
&= E\left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Delta X^2(u)du + \int_{t_k}^{t} \Delta X^2(u)du\right] \\
&= E\left[\int_0^t \Delta X^2(u)du\right]
\end{aligned}$$

□

Definition 2.4.1. *Let $W(t)$, $t \geq 0$, be a Brownian motion and $\mathcal{F}(t)$ be a filtration. A process $X(t)$ expressed below is called Itô process.*

$$X(t) = X(0) + \int_0^t b(u)du + \int_0^t a(u)dW(u)$$

where $X(0)$ is deterministic and $a(u)$, $b(u)$ are adapted stochastic process, satisfying

$$E\left[\int_0^t a^2(u)du\right] < \infty, \int_0^t |b(u)|du < \infty \tag{2.1}$$

We always assume (2.1) in terms of Itô process.

Lemma 2.4.1. *The quadratic variation of the Itô process above is*

$$[X,\ X](t) = \int_0^t a^2(u)du$$

Proof: See Shreve (2004), page143-144, for the proof.

□

Theorem 2.4.3. ***(Itô-Doeblin Formula for Itô process)***
Let $X(t)$, $t \geq 0$ be an Itô process and $f(t,\, x)$ be a function such that partially differentiable on t and twice partially differentiable on x and all partial derivatives are continuous. For $T \geq 0$,

$$\begin{aligned} f(T,\, X(T)) &= f(0,\, X(0)) + \int_0^T f_t(t,\, X(t))dt + \int_0^T f_x(t,\, X(t))dX(t) \\ &\quad + \frac{1}{2}\int_0^T f_{xx}(t,\, X(t))d[X,\, X](t) \\ &= f(0,\, X(0)) + \int_0^T f_t(t,\, X(t))dt + \int_0^T f_x(t,\, X(t))(a(t)dW(t) + b(t)dt) \\ &\quad + \frac{1}{2}\int_0^T f_{xx}(t,\, X(t))a^2(t)dt \end{aligned}$$

Proof: See Øksendal (2005), page44-48, for the proof.

□

Theorem 2.4.4. ***(Predictable Representation for Martingales)***
Let $\mathcal{F}(t)$ be a σ-field generated by a Brownian motion $W(s)$, $0 \leq s \leq t$.

1) If $(M(t))$, $0 \leq t \leq T$, is a square integrable martingale on $(\mathcal{F}(t))$, then there is an adapted process $\Big(H(t),\, 0 \leq t \leq T\Big)$, such that $E\left[\int_0^T H^2(u)du\right] < +\infty$ and for $\forall\, t \in [0,\, T]$,

$$M(t) = M(0) + \int_0^t H(s)dW(s) \qquad a.s.$$

2) If $M(t)$ is a $\mathcal{F}(t)$-local martingale, then there exist ξ adapted process such that $\int_0^T \xi^2(u)du < +\infty \quad P-a.s.$ and

$$M(t) = M(0) + \int_0^t \xi(u)dW(u), \quad 0 \leq t \leq T.$$

Proof: See Elliot and Kopp (1999), page144-145, for the proof of the first part. The second part follows by stopping and can be found in the literature.

□

This theorem is very useful in obtaining the state price densities, which usually lead to the risk-neutral pricing. As written in Shreve (2004), this theorem has a strong condition on the filtration, which is generated only by the Brownian motion. Investors will obtain all information in $\mathcal{F}(t)$ from the observation of the Brownian motion. This implies that the Brownian motion is the only source of uncertainty in the model and this uncertainty can be removed by hedging. Moreover, this martingale will not have any jumps because the part of Itô integral is continuous. This martingale process is called ' predictable' since it is continuous and $\mathcal{F}(t)$- measurable.

2.5 Utility Functions

A utility function U maps a set of numerical variables into real numbers. Numerical variables can be a combination of consumption of goods, a combination of assets, etc. Basically utility functions allow us to see preference relations among various levels of consumption, various strategies for asset holdings, etc. For example, let x_1, x_2 be strategies of holding assets. If $U(x_1) \geq U(x_2)$, then we say x_1 is more preferable than x_2. Sometimes we use the following expression

$$\begin{aligned} U(x_1) \geq U(x_2) &\implies x_1 \succeq x_2 \\ U(x_1) \leq U(x_2) &\implies x_1 \preceq x_2 \\ U(x_1) = U(x_2) &\implies x_1 \sim x_2. \end{aligned}$$

In general, a utility function U is assumed to be concave, i.e.,

$$U(\alpha x + (1-\alpha)y) \geq \alpha U(x) + (1-\alpha)U(y),$$

for any x, $y \in \mathbb{R}$ and $0 \leq \alpha \leq 1$.
This combined with Jensen's inequality leads to

$$U(E[X]) \geq E[U(X)],$$

for a random variable X. In order to meet the assumptions above we usually assume the following:

(1) U is (strictly) increasing.
(2) U is twice differentiable.
(3) U is concave, i.e., $U'' \leq 0$.

2.5.1 Some Examples

The following utility functions are the most practically used utilities in economics and finance. All of them meet the above three assumptions (1)-(3).

• Logarithmic Utility:

$$U(x) = ln(x),\ x > 0$$

This utility has been widely used, for instance, Cvitanic and Karatzas (1996).

• Power Utility:

$$U(x) = \frac{x^\gamma}{\gamma},\ 0 < \gamma \leq 1$$

This utility is discussed in Merton (1969), Davis and Norman (1990), etc.

• Exponential Utility:

$$U(x) = 1 - e^{-\alpha x},\ \alpha > 0$$

This utility is treated as a particular case in Schachermayer (2001).

• Quadratic Utility:

$$U(x) = x - \beta x^2,\ \beta > 0,\ x \in (-\infty,\ \frac{1}{2\beta})$$

This utility is considered in Sharpe (2007).

2.5.2 Classifications of Utility

Each investor has his/her own utility and the curvature of his/her utility represents the risk averseness of the investor. Arrow and Pratt (1964) introduced the absolute risk aversion $A(x)$

$$A(x) := -\frac{U''(x)}{U'(x)}$$

Any utility function with the exponential form $-e^{-\alpha x}$ have the constant absolute risk aversion

$$A(x) = -\frac{-\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}} = \alpha$$

Those utility functions are called to belong to the family of constant relative risk aversion (**CARRA** family).

Those utility functions whose absolute risk aversions are hyperbolic are called to belong to the hyperbolic relative risk aversion (**HARA** family). The power utility above belongs to the HARA family. In general, we have

$$U(x) = c_0 + c_1 x + c_2(\alpha + \gamma x)^{1-\frac{\beta}{\gamma}}, \quad 0 < \frac{\beta}{\gamma} < 1,\ \alpha + \gamma x > 0,\ c_1,\ c_2 \in \mathbb{R} \tag{2.2}$$

$$A(x) = -\frac{-\beta(\gamma - \beta)(\alpha + \gamma x)^{-1-\frac{\beta}{\gamma}}}{(\gamma - \beta)(\alpha + \gamma x)^{-\frac{\beta}{\gamma}}} = \frac{\beta}{\alpha + \gamma x}. \tag{2.3}$$

As Menoncin (2002) stated, (2.2) contains the CARRA family and other utility functions.
In fact, if $c_0 = 1,\ c_1 = 0,\ c_2 = -1,\ \alpha = 1,\ \gamma \to 0$, then we will obtain the exponential utility.

$$U(x) = 1 - \lim_{\gamma \to 0}(1 + \gamma x)^{1-\frac{\beta}{\gamma}} = 1 - e^{-\beta x}$$

If $c_0 = c_1 = 0,\ c_2 = 1,\ \alpha = 0,\ \gamma = a^{-\frac{1}{a}},\ \beta = (1-a)a^{-\frac{1}{a}},\ 0 < a \leq 1$, then we will obtain the power utility.

$$U(x) = (a^{-\frac{1}{a}} x)^{1-\frac{(1-a)a^{-\frac{1}{a}}}{a^{-\frac{1}{a}}}} = \frac{x^a}{a}$$

If $c_0 = -\frac{1}{a},\ c_1 = 0,\ c_2 = 1,\ \alpha = 0,\ \gamma = a^{-\frac{1}{a}},\ \beta = (1-a)a^{-\frac{1}{a}},\ a \to 0$, then we will obtain the logarithm utility.

$$\begin{aligned}
U(x) &= \lim_{a\to 0}\left(-\frac{1}{a} + (a^{-\frac{1}{a}}x)^{1-\frac{(1-a)a^{-\frac{1}{a}}}{a^{-\frac{1}{a}}}}\right) \\
&= \lim_{a\to 0}\left(-\frac{1}{a} + \frac{x^a}{a}\right) \\
x &= \lim_{a\to 0}(1 + aU(x))^{\frac{1}{a}} = e^{U(x)} \\
U(x) &= \ln(x)
\end{aligned}$$

If $c_0 = 0,\ c_1 = 1,\ c_2 = -1,\ \alpha = 0,\ \gamma = -\sqrt{b},\ \beta = \sqrt{b},\ b > 0$, then we will obtain the quadratic utility.

$$U(x) = x - (-\sqrt{b}x)^{1-\frac{\sqrt{b}}{-\sqrt{b}}} = x - bx^2$$

2.5.3 The Inada Condition for Utility

The Inada condition is usually assumed for the production function in the economic growth model. This condition is also assumed for the utility function U in many cases. It is,

$$\lim_{x\to 0} U'(x) = +\infty, \qquad \lim_{x\to +\infty} U'(x) = 0. \tag{2.4}$$

This condition prevents the optimal solution of zero consumption (or holding) in maximization problems, which is useful to ensure the non-trivial positive solution. However, the Inada condition is not satisfied with a very common utility function, which is the exponential utility, $U(x) = 1 - e^{-x}$. This utility satisfies another condition instead of the Inada condition which is given by,

$$\lim_{x\to -\infty} U'(x) = +\infty, \qquad \lim_{x\to +\infty} U'(x) = 0. \tag{2.5}$$

This allows to have the optimal solution in the non-positive consumption (or holding), so this condition is weaker than the Inada condition.

The Inada condition is commonly used for the assumption of utility functions. For example, this condition is assumed in Cvitanic and Karatzas (1996), Kramkov and Schachermayer (1999), Rogers (2001), and Klein and Rogers (2007), etc. In Schachermayer (2001) the Inada condition is not assumed and he succeeded to obtain the same result as Kramkov and Schachermayer (1999) without the Inada condition. In Chapter 5 we want to show the same result of Cvitanic and Karatzas (1996) with the non-Inada type of utility function, i.e., the exponential utility.

Chapter 3

The Economy and Its Preliminary Analysis

3.1 The Mathematical Model

In order to rigorously describe the financial market model that we will consider in the coming chapters, we start by assuming given a filtered probability space

$$(\Omega,\ \mathcal{F},\ (\mathcal{F}(t))_{0\leq t\leq T},\ \mathbf{P}), \tag{3.1}$$

where Ω is the set of all scenarios, $\mathcal{F}$ is a σ-field (σ-algebra) on Ω that represents the whole information about the market, $\mathcal{F}(t)$ is the aggregate information available up to time $t \in [0,\ T]$, T is the fixed horizon time, and $\mathbf{P}$ is the probability measure.

On this stochastic basis defined in (3.1) that is assumed to be complete, (i.e., $\mathcal{F}(0)$ contains all negligible sets), we consider a Brownian motion

$$W = (W(t))_{0\leq t\leq T}. \tag{3.2}$$

Then the underlying assets of our market model are constituted of a bank account (risk-free asset) whose price process is denoted by

$$B = (B(t))_{0\leq t\leq T}, \tag{3.3}$$

and the stock (risky asset) whose price process is denoted by

$$S = (S(t))_{0 \le t \le T}. \tag{3.4}$$

The dynamics of these two processes B, S are given by for t$\in [0, \infty)$

$$dB(t) = B(t)r(t)dt, \qquad B(0) = 1, \tag{3.5}$$
$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)], \quad S(0) = p \in (0, \infty), \tag{3.6}$$

where p is the initial price of stock. $r(t)$ is a nonnegative bounded adapted process which represents the risk-free interest rate, $b(t)$ is a bounded adapted process which represents the return rate of the risky asset, and $\sigma(t)$ is a bounded adapted process which represents the volatility of the risky asset satisfying $\sigma(t) \ge \epsilon$ (ϵ is a positive constant) .

A trading strategy is a pair (L, M) of $\mathcal{F}$(t)-adapted processes on $t \in [0, T]$, with left-continuous, nondecreasing paths and $L(0) = M(0) = 0$. $L(t)$ represents the total amount of money transferred from the risk-free asset to the risky asset at time t and $M(t)$ represents the amount of money transferred from the risky asset to the risk-free asset at time t.

Assume the model has the proportional transaction costs $0 < \lambda$, $\mu < 1$ for buying the risky asset, selling the risky asset respectively. Let the initial assets be x, y respectively, then we have value processes, $X(t)$, $Y(t)$ of the value processes for the bank account and the stock respectively. They are given by

$$X(t) = x - (1+\lambda)L(t) + (1-\mu)M(t) + \int_0^t X(u)r(u)du \tag{3.7}$$

$$Y(t) = y + L(t) - M(t) + \int_0^t Y(u)[b(u)du + \sigma(u)dW(u)]. \tag{3.8}$$

Proposition 3.1.1. *The processes X and Y defined in (3.7) and (3.8) re-*

spectively satisfy

$$d\left(\frac{X(t)}{B(t)}\right) = \frac{(1-\mu)dM(t) - (1+\lambda)dL(t)}{B(t)} \tag{3.9}$$

$$d\left(\frac{Y(t)}{S(t)}\right) = \frac{dL(t) - dM(t)}{S(t)} \tag{3.10}$$

Proof:
The proof of this proposition is a direct application of Itô-Doeblin formula.
□

Below, we will define the wealth process of the investor as described in Cvitanic and Karatzas (1996) and Schachermayer (2001).

Definition 3.1.1. *The total wealth at time $t \in [0,\, T]$ is given by*

$$X_+(t) := X(t) + f(Y(t)), \tag{3.11}$$

where $f(y) = (1-\mu)y^+ - (1+\lambda)y^-$, for any $y \in \mathbb{R}$ and $y^+ = y\boldsymbol{I}_{\{y\geq 0\}}$, $y^- = -y\boldsymbol{I}_{\{y<0\}}$. The total wealth is equivalent to the value if an investor liquidates all position in the stock at time t with the appropriate transaction costs. Especially ***the terminal wealth*** *is denoted by*

$$X(T+) := X_+(T).$$

The main objective of an investor — here in our context — is to maximize her expected utility from the terminal wealth. This maximization problem will be called through out the thesis as the **primal problem** and is denoted for $(x,\, y) \in (0,\, +\infty)^2$ by

$$V(x,y) := \sup_{(L,M)\in\mathcal{A}(x,y)} E[U(X(T+))], \quad 0 < x, y < \infty \tag{3.12}$$

where $\mathcal{A}(x,y)$ represents the class of admissible strategies and it will be specified in Chapter 5, Section 5.2.

3.2 Auxiliary Martingales and Dual Problem

Our main (primal) problem is the utility maximization of the terminal wealth. In the case of no-transaction-cost market model, finding the risk-neutral probability measure $\mathbf{Q}$, which is equivalent to $\mathbf{P}$, is a key to solve this type of problem and other related problems. However, in the case of the market with transaction costs, it is a little difficult to find $\mathbf{Q}$ directly. So, before going the primal problem we establish the dual problem via introducing the auxiliary martingales $(Z_0,\ Z_1)$ and the convex conjugate function $\tilde{U}$.

3.2.1 Auxiliary Martingales

First we consider the class of a pair (Z_0, Z_1). Let $\theta_0(t)$ and $\theta_1(t)$ be $\mathcal{F}(t)$-meauserable processes. Then, with the martingale representation theorem (see Theoram 2.4.4.), we can define

$$\mathcal{Z} := \{(Z_0,\ Z_1) \mid Z_0,\ Z_1 \textit{ are strict positive } \mathcal{F}(t) - \textit{martingales}\} \tag{3.13}$$

$$\textit{where} \quad Z_0(t) = Z_0(0) \exp\left\{ \int_0^t \theta_0(u) dW(u) - \frac{1}{2} \int_0^t \theta_0^2(u) du \right\} \tag{3.14}$$

$$Z_1(t) = Z_1(0) \exp\left\{ \int_0^t \theta_1(u) dW(u) - \frac{1}{2} \int_0^t \theta_1^2(u) du \right\}. \tag{3.15}$$

Let $R(t)$ be the process such that

$$R(t) := \frac{Z_1(t)}{Z_0(t) P(t)} \tag{3.16}$$

where $P(t)$ is the discounted stock price,

$$P(t) := \frac{S(t)}{B(t)} = p \exp\left\{ \int_0^t \left(b(u) - r(u) - \frac{1}{2}\sigma^2(u) \right) du + \int_0^t \sigma(u) dW(u) \right\}. \tag{3.17}$$

Then, we also define a subset of $\mathcal{Z}$ defined in (3.13)

$$\mathcal{D} := \Big\{ (Z_0,\ Z_1) \in \mathcal{Z} \mid Z_0(0) = 1,\ 1 - \mu \le R(t) \le 1 + \lambda,\ \forall t \in [0,\ T] \Big\}. \tag{3.18}$$

In Chapter 4 we try to find the optimal pair $(\hat{Z}_0,\ \hat{Z}_1)$ as the solution to the dual problem defined in the next section.

Proposition 3.2.1. *For any* $(Z_0,\ Z_1) \in \mathcal{D}$*, and any* $(L,\ M)$ *strategy, the process*

$$Z_0(t)\frac{X(t)}{B(t)} + Z_1(t)\frac{Y(t)}{S(t)} \tag{3.19}$$

is a local supermartingale.

Proof:
By Itô-Doeblin formula (Theorem 2.4.3.),

$$\begin{aligned}
& Z_0(t)\frac{X(t)}{B(t)} + Z_1(t)\frac{Y(t)}{S(t)} \\
= \ & Z_0(0)\frac{X(0)}{B(0)} + Z_1(0)\frac{Y(0)}{S(0)} + \int_0^t Z_0(u)d\left(\frac{X(u)}{B(u)}\right) + \int_0^t \frac{X(u)}{B(u)}dZ_0(u) \\
& + \int_0^t Z_1(u)d\left(\frac{Y(u)}{S(u)}\right) + \int_0^t \frac{Y(u)}{S(u)}dZ_1(u) + \int_0^t d[Z_0,\ \frac{X}{B}](u) + \int_0^t d[Z_1,\ \frac{Y}{S}](u) \\
= \ & x + \frac{y}{p}z_1 + \int_0^t \frac{Z_0(u)}{B(u)}[(1-\mu)dM(u) - (1+\lambda)dL(u)] + \int_0^t \frac{X(u)}{B(u)}\theta_0(u)Z_0(u)dW(u) \\
& + \int_0^t \frac{Z_1(u)}{S(u)}[dL(u) - dM(u)] + \int_0^t \frac{Y(u)}{S(u)}\theta_1(u)Z_1(u)dW(u) \\
= \ & x + \frac{y}{p}z_1 + \int_0^t \frac{Z_0(u)}{B(u)}[X(u)\theta_0(u) + R(u)Y(u)\theta_1(u)]dW(u) \\
& - \int_0^t \frac{Z_0(u)}{B(u)}[(1+\lambda) - R(u)]dL(u) - \int_0^t \frac{Z_0(u)}{B(u)}[R(u) - (1-\mu)]dM(u)
\end{aligned} \tag{3.20}$$

It is clear that the second term on the right-hand side term in the above equality is a local martingale, and the third and the fourth terms are non-positive and non-increasing processes. This proves that $Z_0(t)\frac{X(t)}{B(t)} + Z_1(t)\frac{Y(t)}{S(t)}$ is a local supermartingale. This completes the proof of the proposition. □

3.2.2 Dual Problem

As we mention above, we need the dual problem to avoid some difficulty in the primal problem. Here we establish the dual problem.

Let U denote the utility function, which is $U : \mathbb{R} \longrightarrow \mathbb{R}$, twice continuously differentiable on $\mathbb{R}$, strictly increasing, strictly concave, and satisfies

$$U'(\infty) := 0, \quad U'(-\infty) := \infty \quad U(0) \geq 0 \tag{3.21}$$

U' is invertible on $\mathbb{R}$, and I denote its inverse function. Then we can find the convex conjugate function $\tilde{U}$ by

$$\tilde{U}(\zeta) : = \max_{x \in R}[U(x) - x\zeta], \qquad \zeta > 0. \tag{3.22}$$

Thanks to (3.22), we are able how to define the **dual problem**

$$\tilde{V}(\zeta; y) := \inf_{(Z_0, Z_1) \in \mathcal{D}} E\left[\tilde{U}\left(\zeta \frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right], \quad 0 < \zeta < \infty. \tag{3.23}$$

In Chapter 5 we discuss how the dual problem relates to the primal problem.

3.3 Hedging

Here also we follow the stream of Cvitanic and Karatzas (1996). When we keep financial solvency, we need to hold some margin in the bank account. We can hedge the contingent claim $(C_0,\, C_1)$ under the admissible strategy $(L,\, M) \in \mathcal{A}(x,\, y)$.

Definition 3.3.1. *A contingent claim is a pair $(C_0,\, C_1)$ of $\mathcal{F}$-measurable random variables. A trading strategy $(L,\, M)$ hedges the claim $(C_0,\, C_1)$ starting with $(x,\, y)$ as initial holdings, if X, Y of (3.7), (3.8) satisfy*

$$X(t) + (1-\mu)Y(t) \geq C_0 + (1-\mu)C_1 \tag{3.24}$$

$$X(t) + (1+\lambda)Y(t) \geq C_0 + (1+\lambda)C_1 \tag{3.25}$$

Let $\mathcal{C}$ denote the class of $(C_0,\, C_1)$ satisfying (3.24) and (3.25).

Definition 3.3.2. *The minimal holdings in the bank account* $h(C_0,\, C_1,\, y)$ *is called hedging price and defined by*

$$h(C_0,\; C_1,\; y) := \inf\{x \in \mathbb{R} : (L,\; M) \in \mathcal{A}(x,\; y),\; (L,\; M)\; hedges\; (C_0, C_1)\} \tag{3.26}$$

Theorem 3.3.1. ***Cvitanic and Karatzas (1996)***
If C_0 *and* C_1 *are bounded from below, then we have*

$$h(C_0,\; C_1;\; y) = \sup_{(Z_0,\, Z_1)\in\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}C_0 + \frac{Z_1(T)}{S(T)}\left(C_1 - \frac{y}{p}\right)\right]. \tag{3.27}$$

Proof:
For the proof of this theorem we refer to Cvitanic and Karatzas (1996). □

Chapter 4

Modification of the Dual Problem

This chapter is devoted to discuss a possible explicit description of the optimal solution to the dual problem (3.23) defined in Chapter 3, Section 3.2. Through out the following chapters, let " ˆ " denote the optimality.

4.1 Sketch of Prof. Choulli's Approach

As admitted in Cvitanic and Karatzas (1996), the general optimal solution to the dual problem is very hard and difficult to describe. However, Prof. Choulli proposed an approach, through which we successfully obtained some optimal solutions under certain conditions. These conditions are weaker than those discussed in Cvitanic and Karatzas (1996), while these optimal solutions generalize Cvitanic-Karatzas' examples.

Choulli's[1] approach started by slightly modifying the dual problem. This modification is based on considering a subset of $\mathcal{D}$ ($\mathcal{D}$ is defined in (3.18)) denoted by $\mathcal{D}^v$ and is given by

$$\mathcal{D}^v = \left\{ \quad (Z_0,\ Z_1) \in \mathcal{D} \quad | \quad R(t) = \frac{Z_1(t)}{Z_0(t)P(t)} \quad has\, finite\, variation \right\}. \tag{4.1}$$

[1]University of Alberta, Department of Mathematical and Statistical Science, Edmonton, Canada

Then, the modified dual problem can be stated as follows.

$$\tilde{V}(\zeta;\, y) = \min_{(Z_0,\, Z_1) \in \mathcal{D}^v} E\left[\tilde{U}\left(\zeta \frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right], \quad 0 < \zeta < \infty. \tag{4.2}$$

After defining the modified dual problem, the remaining sketch of the approach consists of three main steps.

The first step deals with the following minimization problem.

$$\min_{Z_0 \in \mathcal{Z}_0} E\left[\tilde{U}\left(\zeta \frac{Z_0(T)}{B(T)}\right)\right], \tag{4.3}$$

where $\mathcal{Z}_0 = \{Z \textit{ is a positive martingale} \mid Z(0) = 1\}$.
The solution to this minimization problem will be called the optimal first component and will be denoted by $\hat{Z}_0$.

The second step deals with the following minimization problem

$$\inf_{Z_1 \colon (\hat{Z}_0,\, Z_1) \in \mathcal{D}^v} E\left[\tilde{U}\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right]. \tag{4.4}$$

The solution to this minimization problem will be called the optimal second component and will be denoted by $\hat{Z}_1$.

The third and last step will deal with verification results.

In the following without the proof, I will state a conjecture of Prof. Choulli that deals with verification of the optimality of the pair $(\hat{Z}_0,\, \hat{Z}_1)$.

Proposition 4.1.1. *Let $\hat{Z}_0$ (respectively $\hat{Z}_1$) be the solution to (4.3) (respectively (4.4)). Then $(\hat{Z}_0,\, \hat{Z}_1)$ is the solution to (4.2), i.e.,*

$$\tilde{V}(\zeta;\, y) = E\left[\tilde{U}\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta \hat{Z}_1(T)\right]. \tag{4.5}$$

The proof of this proposition is beyond the scope of this thesis. My contribution consists of proving some pieces of this approach. Precisely, I will address the first and second steps in the current framework of Brownian motion. The general case as well as other conjectures are extensively developed in Choulli (2009) (Prof. Choulli informed me about these advances).

4.2 The Optimal First Component

This section is devoted to the characterization and description of the first component in the solution of the dual problem (3.23) for particular cases and the solution of (4.3) for general cases. This will be achieved for general utility with deterministic interest rate (see Proposition 4.2.1.) and for logarithm and exponential utilities with stochastic interest rate satisfying some boundness conditions (see Proposition 4.2.2. and Theorem 4.2.1.).

From (3.18), denote

$$Z_1(0) := z_1 \in [p(1-\mu),\ p(1+\lambda)]. \tag{4.6}$$

Proposition 4.2.1. *If the risk-free interest rate $r(t)$ is deterministic, then the following assertionss hold.*

1) The process $\hat{Z}_0 \equiv 1$ is the solution to (4.3), that is

$$\min_{Z_0 \in \mathcal{Z}_0} E\left[\tilde{U}\left(\zeta \frac{Z_0(T)}{B(T)}\right)\right] = \tilde{U}\left(\frac{\zeta}{B(T)}\right), \quad \forall \zeta > 0. \tag{4.7}$$

2) *If furthermore, $r(t) \geq b(t)$ and $\int_0^T (r(u) - b(u))du \leq \ln\left(\frac{1+\lambda}{1-\mu}\right)$, then*

$$\hat{Z}_0 \equiv 1,\ \hat{Z}_1(t) = p(1-\mu)\exp\left\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\} \tag{4.8}$$

is the solution to the dual problem (3.23).

Proof:
1) Since $\tilde{U}$ is a convex function, then due to Jensen's inequality, for any

$(Z_0,\ Z_1) \in \mathcal{D}$ we have

$$E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right)\right] \geq \tilde{U}\left(\zeta\frac{E[Z_0(T)]}{B(T)}\right) \geq \tilde{U}\left(\zeta\frac{1}{B(T)}\right) \tag{4.9}$$

Hence, if we consider $\hat{Z}_0$ given by

$$\hat{Z}_0(t) = 1, \quad \forall\, t \in [0,\ T],$$

then $\hat{Z}_0 \in \mathcal{Z}_0$ and the proof of the assertion 1) follows.

2) From (3.16), (3.14) and (3.15) with the Doléans-Dade exponential,

$$\begin{aligned}\hat{R}(t) &= \frac{\hat{z}_1\mathcal{E}_t(\sigma\cdot W)}{\hat{Z}_0(t)p\mathcal{E}_t(\sigma\cdot W + (b-r)dt)} \\ &= \frac{\hat{z}_1}{\hat{Z}_0(t)p}\exp\left\{\int_0^t (r(u)-b(u))du\right\}\end{aligned}$$

If we consider the pair given by

$$\begin{cases}\hat{Z}_0(t) = 1 \\ \hat{Z}_1(t) = p(1-\mu)\exp\left\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\}\end{cases}, \tag{4.10}$$

then

$$1-\mu \leq \hat{R}(t) \leq 1+\lambda.$$

This is true due to $z_1 \in [p(1-\mu),\ p(1+\lambda)]$ and the condition,

$$r(t) \geq b(t), \quad \int_0^T (r(u)-b(u))du \leq \ln\left(\frac{1+\lambda}{1-\mu}\right).$$

Hence $(\hat{Z}_0,\ \hat{Z}_1) \in \mathcal{D}$ and

$$E\left[\tilde{U}\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta\hat{Z}_1(T)\right] = \tilde{U}\left(\frac{\zeta}{B(T)}\right) + y\zeta(1-\mu).$$

This proves that the pair $(\hat{Z}_0, \hat{Z}_1)$ defined in (4.10) attains the infimum value in (3.23). This completes the proof of the proposition. $\square$

In the following proposition, we will prove that the pair $(\hat{Z}_0, \hat{Z}_1)$ defined in (4.10) is also the optimal solution to the dual problem corresponding to the logarithm utility without the deterministic assumption on $r(t)$.

Proposition 4.2.2. *Suppose $r(t)$ is a bounded process, and consider the logarithm utility function, i.e., $U(x) = \ln x$.*
1) Then for any $\zeta > 0$,

$$\min_{Z_0 \in \mathcal{Z}_0} E\left[\tilde{U}\left(\zeta \frac{Z_0(T)}{B(T)}\right)\right] = E\left[\tilde{U}\left(\frac{\zeta}{B(T)}\right)\right]. \tag{4.11}$$

2) If furthermore, $r(t) \geq b(t)$ and $\int_0^T (r(u) - b(u))du \leq \ln\left(\frac{1+\lambda}{1-\mu}\right)$, then

$$\hat{Z}_0(t) = 1,\ \hat{Z}_1(t) = p(1-\mu)\exp\left\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\}$$

is the solution to the dual problem (3.23).

Proof:
1) Since $U(x) = \ln x$, then $\tilde{U}(x) = -\ln x - x(\ln x)' = -1 - \ln x$, and hence

$$\begin{aligned}\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right) &= -1 - \ln\left(\zeta\frac{Z_0(T)}{B(T)}\right) \\ &= \tilde{U}\left(\frac{\zeta}{B(T)}\right) - \ln(Z_0(T)).\end{aligned}$$

Then, by taking the expectation in both sides of the above equation,

$$\begin{aligned}
E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right)\right] &= E\left[\tilde{U}\left(\frac{\zeta}{B(T)}\right)\right] - E[\ln(Z_0(T))] \\
&= E\left[\tilde{U}\left(\frac{\zeta}{B(T)}\right)\right] + \frac{1}{2}E\left[\int_0^T \theta_0^2(u)du\right] \\
&\geq E\left[\tilde{U}\left(\frac{\zeta}{B(T)}\right)\right]. \qquad (4.12)
\end{aligned}$$

Hence, if we consider $\hat{Z}_0$ given by

$$\hat{Z}_0(t) = 1(\Leftrightarrow \theta_0(t) = 0), \quad \forall\, t \in [0,\, T],$$

then $\hat{Z}_0 \in \mathcal{Z}_0$ and the proof of the assertion 1) follows.

2) Consider the pair $(\hat{Z}_0, then\, \hat{Z}_1)$ defined in (4.10) via similar calculation as in Proposition 4.2.1., we obtain

$$E\left[\tilde{U}\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta\hat{Z}_1(T)\right] = E\left[\tilde{U}\left(\frac{\zeta}{B(T)}\right)\right] + y\zeta(1-\mu).$$

Thus, the pair $(\hat{Z}_0,\, \hat{Z}_1)$ attains the infimum value in (3.23). This completes the proof of the proposition. □

In the following theorem, we will state the general result for the exponential utility.

Theorem 4.2.1. ***(Conjectured by Prof. Choulli)***
Suppose $r(t)$ is a nonnegative and bounded process. Consider the exponential utility function $U(x) = 1 - e^{-x}$. Then for any $\zeta > 0$ the minimization problem

$$\min_{Z \in \mathcal{Z}_0} E\left[\tilde{U}\left(\zeta\frac{Z(T)}{B(T)}\right)\right],$$

admits the optimal solution for any $\zeta > 0$ *given by*

$$\hat{Z}(t) = \frac{E[B(T)|F_t]}{E[B(T)]}. \tag{4.13}$$

Remark 4.2.1. *The proof of this theorem is beyond the scope of this thesis, while it is clear that it generalizes the previous results. Indeed, if* $r(t)$ *is deterministic, then the process* $B(t)$ *is also deterministic. Thus,*

$$E[B(T)|\mathcal{F}(t)] = E[B(T)] = B(T),$$

which implies that $\hat{Z}_0(T) = 1$

Using the predictable representation theorem (Theorem 2.4.4.), we conclude that there exists an adapted process, $\hat{\theta}_0$, such that $\int_0^T \hat{\theta}_0^2(u)du < +\infty \quad P - a.s.$ and

$$\frac{B(T)}{E[B(T)]} = 1 + \int_0^T \hat{Z}(u)\hat{\theta}_0(u)dW(u). \tag{4.14}$$

4.3 The Optimal Second Component

In Section 4.2 we describe the optimal first component $\hat{Z}_0$. Throughout this section, we assume that $\hat{Z}_0$ is known (i.e., equivalently $\hat{\theta}_0$ is known), and we will determine the optimal second component $\hat{Z}_1$ as the solution to (4.4).

For any $(Z_0,\ Z_1) \in \mathcal{D}$, the process $R(t) = \frac{Z_1(t)}{Z_0(t)P(t)}$ has the following dynamic

$$\begin{aligned} R(t) = \frac{z_1}{p} \exp\Bigg\{ &\int_0^t (\theta_1(u) - \theta_0(u) - \sigma(u))dW(u) \\ &+ \int_0^t \left(-\frac{1}{2}\theta_1^2(u) + \frac{1}{2}\theta_0^2(u) + r(u) - b(u) + \frac{1}{2}\sigma^2(u) \right) du \Bigg\} \end{aligned} \tag{4.15}$$

Therefore, $(Z_0,\ Z_1) \in \mathcal{D}^v$ if and only if $R(t)$ has finite variation. This is equivalent to

$$\theta_1(t) = \sigma(t) + \theta_0(t). \tag{4.16}$$

Then we obtain

$$R(t) = \frac{z_1}{p} \exp\left\{ \int_0^t (r(t) - b(t) - \theta_0(t)\sigma(t))du \right\}. \tag{4.17}$$

Hence if Z_1 is such that $(\hat{Z}_0,\ Z_1) \in \mathcal{D}^v$, then

$$Z_1(t) = z_1 \mathcal{E}_t((\hat{\theta}_0 + \sigma) \cdot W) \tag{4.18}$$

where $\mathcal{E}_t\Big((\hat{\theta}_0 + \sigma) \cdot W\Big)$ is the Doléans-Dade exponential of $\Big((\hat{\theta}_0 + \sigma) \cdot W\Big)$ and is given by

$$\mathcal{E}_t((\hat{\theta}_0 + \sigma) \cdot W) = \exp\left\{ \int_0^t (\hat{\theta}_0(u) + \sigma(u))dW(u) - \frac{1}{2}\int_0^t (\hat{\theta}_0(u) + \sigma(u))^2 du \right\} \tag{4.19}$$

Below is our main result that describes this second optimal component $\hat{Z}_1$.

Theorem 4.3.1. *Suppose that $r(t)$ is a nonnegative process such that,*

$$\sup_{0 \le t \le T} \left| \int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u) \Big] du \right| \le \ln\left(\frac{1+\lambda}{1-\mu} \right), \quad P - a.s. \tag{4.20}$$

Then the process

$$\hat{Z}_1(t) = p(1-\mu)\exp\left\{ \left\| \sup_{0 \le s \le T} \left(\int_0^s \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u) \Big] du \right)^- \right\|_\infty \right\} \mathcal{E}_t((\hat{\theta}_0 + \sigma) \cdot W) \tag{4.21}$$

where $(C)^- = -C\boldsymbol{I}_{\{C<0\}}$, for $C \in \mathbb{R}$, is a solution to (4.4).

Proof:
The proof of this theorem will be given first for two particular cases where

our main idea of the proof will be illustrated.

Case1: We assume the following condition: For any $t \in [0,\ T]$,

$$0 \leq \int_0^t \Big[r(t) - b(t) - \hat{\theta}_0(u)\sigma(u) \Big] du \leq \ln\left(\frac{1+\lambda}{1-\mu}\right). \tag{4.22}$$

Under the above assumption we will prove that

$$\hat{\theta}_1 = \hat{\theta}_0 + \sigma, \quad and \quad \hat{z}_1 = p(1-\mu)$$

In fact,

$$\begin{aligned} R(t) = \frac{z_1}{p} \exp\Bigg\{ & \int_0^t (\theta_1(u) - \hat{\theta}_0(u) - \sigma(u)) dW(u) \\ & + \int_0^t \left(-\frac{1}{2}\theta_1^2(u) + \frac{1}{2}\hat{\theta}_0^2(u) + r(u) - b(u) + \frac{1}{2}\sigma^2(u) \right) du \Bigg\} \end{aligned} \tag{4.23}$$

By taking $\theta_1 = \hat{\theta}_0 + \sigma$, we obtain

$$1 - \mu \leq R(t) = \frac{z_1}{p} \exp\left\{ \int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u) \Big] du \right\} \leq 1 + \lambda.$$

Then, let $\mathbf{K(t)}$ denote $\int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u) \Big] du$

$$p(1-\mu)\exp\{-\mathbf{K(t)}\} \leq z_1 \leq p(1+\lambda)\exp\{-\mathbf{K(t)}\}. \tag{4.24}$$

Since $\mathbf{K(t)}$ is a nonnegative and non-decreasing process, and $z_1 \in [p(1-\mu),\ P(1+\lambda)]$, we get

$$p(1-\mu)\exp\{-\mathbf{K(t)}\} \leq p(1-\mu) \leq z_1 \leq p(1+\lambda) exp\{-\mathbf{K(t)}\} \leq p(1+\lambda).$$

The minimum of these z_1 is achieved at $\hat{z}_1$ given by

$$\hat{z}_1 = p(1-\mu) \left(\Leftrightarrow\ \hat{Z}_1(t) = p(1-\mu)\exp\left\{ \int_0^t \sigma(u) dW(u) - \frac{1}{2}\int_0^t \sigma^2(u) du \right\} \right). \tag{4.25}$$

Hence, $(\hat{Z}_0,\ \hat{Z}_1) \in \mathcal{D}^v$ and

$$E\left[\tilde{U}\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right] = E\left[\tilde{U}\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\right] + y\zeta(1-\mu).$$

In fact, (4.25) is the solution to (4.4).

<u>Case2</u>: Here we assume the following condition: For any $t \in [0,\ T]$,

$$-\ln\left(\frac{1+\lambda}{1-\mu}\right) \leq \int_0^t \left[r(t) - b(t) - \hat{\theta}_0(u)\sigma(u)\right] du \leq 0. \tag{4.26}$$

Similarly to Case 1, we obattain with $\mathbf{K(t)} = \int_0^t \left[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u)\right] du$,

$$p(1-\mu)\exp\{-\mathbf{K(t)}\} \leq z_1 \leq p(1+\lambda)\exp\{-\mathbf{K(t)}\}. \tag{4.27}$$

Due to (4.26) and $z_1 \in [p(1-\mu),\ P(1+\lambda)]$ we deduce,

$$p(1-\mu) \leq p(1-\mu)\exp\{-\mathbf{K(t)}\} \leq z_1 \leq p(1+\lambda) \leq p(1+\lambda)\exp\{-\mathbf{K(t)}\}.$$

Since this is true for all scenarios and any $t \in [0,\ T]$, then the minimal value of these z_1 is achieved at $\hat{z}_1$ given by

$$\hat{z}_1 = p(1-\mu)\exp\left\{\left\|\sup_{0\leq t\leq T}(-\mathbf{K(t)})\right\|_\infty\right\}, \tag{4.28}$$

we conclude for $(\hat{Z}_0,\ \hat{Z}_1) \in \mathcal{D}^v$, and (4.28) is the solution to (4.4).

<u>General Case</u>: The previous two cases are particular cases of the current case. Again, in order to have R(t) a finite variation process, we choose

$$\hat{\theta}_1 = \hat{\theta}_0 + \sigma. \tag{4.29}$$

Denote

$$\chi_t^+ = \left(\int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u)\Big]du\right)^+, \chi_t^- = \left(\int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u)\Big]du\right)^-.$$
$$\xi_t^+ = \left\|\sup_{0\le s\le t} \chi_s^+\right\|_\infty , \xi_t^- = \left\|\sup_{0\le s\le t} \chi_s^-\right\|_\infty$$

where $(C)^+ = C\mathbf{I}_{\{C>0\}}$ and $(C)^- = -C\mathbf{I}_{\{C<0\}}$, for any $C \in \mathbb{R}$. Then, we write

$$\begin{aligned} 1-\mu \le R(t) &= \frac{z_1}{p}\exp\left\{\int_0^t (r(u) - b(u) - \hat{\theta}_0(u)\sigma(u))du\right\} \\ &= \frac{z_1}{p}\exp\left\{\chi_t^+ - \chi_t^-\right\} \le 1+\lambda. \end{aligned}$$

Then it is clear that

$$p(1-\mu)\exp\left\{\chi_t^- - \chi_t^+\right\} \le z_1 \le p(1+\lambda)\exp\left\{\chi_t^- - \chi_t^+\right\}$$

From (4.20), we derive

$$-\ln\left(\frac{1+\lambda}{1-\mu}\right) \le -\xi_T^+ \le -\xi_t^+, \quad \xi_t^- \le \xi_T^- \le \ln\left(\frac{1+\lambda}{1-\mu}\right) \tag{4.30}$$

Let Ω_t^-, Ω_t^+ denote for $t \in [0,\ T]$

$$\Omega_t^- = \{\omega \in \Omega_t : \int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u)\Big]du < 0\} \tag{4.31}$$
$$\Omega_t^+ = \{\omega \in \Omega_t : \int_0^t \Big[r(u) - b(u) - \hat{\theta}_0(u)\sigma(u)\Big]du \ge 0\} \tag{4.32}$$
$$where \quad \Omega_t = \Omega_t^- \cup \Omega_t^+. \tag{4.33}$$

For any $\omega \in \Omega_t^+$, it is clear

$$p(1-\mu)\exp\left\{-\chi_t^+\right\} \le p(1-\mu) \le z_1 \le p(1+\lambda)\exp\left\{-\chi_t^+\right\} \le p(1+\lambda) \tag{4.34}$$

Then we can choose $z_1 = p(1-\mu)$. Then $(\hat{Z}_0,\ \hat{Z}_1)$ is optimal, where

$$\hat{Z}_1(t) = p(1-\mu)\exp\left\{\int_0^t (\hat{\theta}_0(u) + \sigma(u))dW(u) - \frac{1}{2}\int_0^t (\hat{\theta}_0 + \sigma(u))^2 du\right\}.$$

For any $\omega \in \Omega_t^-$, it is clear

$$p(1-\mu) \leq p(1-\mu)\exp\left\{\chi_t^-\right\} \leq z_1 \leq p(1+\lambda) \leq p(1+\lambda)\exp\left\{\chi_t^-\right\} \tag{4.35}$$

From (4.30), we can choose $z_1 = p(1-\mu)\exp\{\xi_T^-\}$. Then $(\hat{Z}_0,\ \hat{Z}_1)$ is optimal, where

$$\hat{Z}_1(t) = p(1-\mu)\exp\{\xi_T^-\}\exp\left\{\int_0^t (\hat{\theta}_0(u) + \sigma(u))dW(u) - \frac{1}{2}\int_0^t (\hat{\theta}_0 + \sigma(u))^2 du\right\}.$$

□

Remark 4.3.1. *Professor Choulli thinks that a slight modification of the solution obtained in Theorem 4.3.1 solves the more general case where $r(t)$ is only bounded, i.e., the assumption (4.20) is not necessary. The more general solution is beyond the scope of this thesis, so it is remained as an open question.*

4.4 Cvitanic-Karatzas' examples: comparison

We will check whether our optimal solution in Sections 4.2 and 4.4, matches the particular cases considered in Cvitanic and Karatzas (1996).

Example 4.4.1. ***(Example 6.1 in Cvitanic and Karatzas (1996))*** *In Example 6.1, Cvitanic and Karatzas assume the condition that*

$$y = 0,\ r(t) \text{ is deterministic.} \tag{4.36}$$

From Proposition 4.2.2., the condition "$r(t)$ is deterministic" leads to the *constant optimal first component*

$$\hat{Z}_0(t) \equiv 1, \text{ equivalently } \hat{\theta}_0(t) \equiv 0. \tag{4.37}$$

Furthermore Proposition 4.2.1., and the condition "$y = 0$" leads to

$$\tilde{V}(\zeta;\, 0) = \inf_{(Z_0,\, Z_1)\in\mathcal{D}} E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right)\right] = \tilde{U}\left(\frac{\zeta}{B(T)}\right).$$

Thus, our result coincides with that of Cvitanic and Karatzas (1996).

The authors (Cvitanic and Karatzas (1996)) also stated that under the conditions (4.36) the solution to the dual problem is,

$$\hat{Z}_1(0) = p(1+\lambda), \quad \hat{\theta}_1(t) \equiv \sigma(t), \quad \hat{\theta}_0 \equiv 0, \tag{4.38}$$

if and only if

$$0 \leq \int_0^t [b(s) - r(s)]ds \leq \ln\left(\frac{1+\lambda}{1-\mu}\right). \tag{4.39}$$

To see that this result is a particular case of our result, we apply Proposition 4.2.1. or Theorem 4.2.1. under the assumption that $r(t)$ is deterministic and we conclude that

$$\hat{\theta}_0 \equiv 0.$$

Then notice that the rest of the proof follows immediately from our Theorem 4.3.1., and the fact that (4.39) is a stronger condition than our condition (4.20). Thus, under (4.39), we get

$$\begin{aligned}
\hat{\theta}_1(t) &= \hat{\theta}_0(t) + \sigma(t) = 0 + \sigma(t) = \sigma(t) \\
\hat{Z}_1(t) &= p(1-\mu)\exp\left\{\left\|\int_0^T \Big[r(s) - b(s)\Big]^- ds\right\|_\infty\right\}\mathcal{E}_t((0+\sigma)\cdot W) \\
&= p(1-\mu)\exp\left\{\ln\left(\frac{1+\lambda}{1-\mu}\right)\right\}\mathcal{E}_t(\sigma\cdot W) \\
&= p(1+\lambda)\exp\left\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\} \\
\hat{Z}_1(0) &= p(1+\lambda)\mathcal{E}_{t=0}(\sigma\cdot W) = p(1+\lambda)
\end{aligned}$$

Hence, this proves that our results generalize this case of Cvitanic and Karatzas (1996).

Example 4.4.2. ***(Example 6.2 in Cvitanic and Karatzas (1996))*** *The conditions under which Cvitanic and Karatzas work in their example consist of assuming*

$$r(t)\ is\ deterministic,\ r(t) \equiv b(t).$$

Thus if we assume these conditions on our results, then by using Proposition 4.2.1., we calculate that

$$\hat{Z}_0(t) = 1,\ equivalently\ \hat{\theta}_0 \equiv 0$$

This resulting equation (i.e., $\hat{\theta}_0 \equiv 0$*) combined with* $r(t) = b(t)$ *implies that (4.20) holds. Then, Theorem 4.4.1. allows us to claim that*

$$\begin{aligned}\hat{Z}_1(t) &= p(1-\mu)\exp\{0\}\mathcal{E}_t(\sigma \cdot W)\\ &= p(1-\mu)\exp\left\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\}\end{aligned}$$

This optimal pair $(\hat{Z}_0,\ \hat{Z}_1)$ *matches the result in Cvitanic and Karatzas (1996).*

Example 4.4.3. ***(Example 6.3 in Cvitanic and Karatzas (1996))*** *Cvitanic-Karatzas' conditions are*

$$b(t) \equiv r(t),\ U(x) = \ln x.$$

This case can be obviously categorized as a particular case of our results. The calculation of $\hat{Z}_0 \equiv 1$ *is given by Proposition 4.2.2. without the assumption*

$$b(t) \equiv r(t).$$

The equation $\hat{Z}_0 \equiv 1$ *is equivalent to* $\hat{\theta}_0 \equiv 0$*, and by combining this equality together with the assumption* $b(t) = r(t)$*, we deduce that our assumption (4.20) is fulfilled. Therefore Theorem 4.4.1. gives us* $\hat{Z}_1(T)$ *through the* *equation (4.21). Since* $b(t) \equiv r(t)$ *and* $\hat{\theta}_0 \equiv 0$*, we get*

$$\hat{Z}_0(t) = 1$$
$$\hat{Z}_1(t) = p(1-\mu)\exp\left\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\},$$

This optimal pair $(\hat{Z}_0,\ \hat{Z}_1)$ *coincides with the one proposed by Cvitanic and Karatzas (1996) for this case. This proves that our results generalize those of Cvitanic and Karatzas (1996).*

In this chapter we observe the following. From Proposition 4.2.1. if the risk-free interest rate $r(t)$ is deterministic, then

$$\inf_{(Z_0,\, Z_1)\in\mathcal{D}}\{the\ Dual\ Problem\} \sim \inf_{(Z_0,\, Z_1)\in\mathcal{D}^v}\{the\ Dual\ Problem\}.$$

From Proposition 4.2.1. if the risk-free interest rate $r(t)$ is a bounded process and consider the logarithm utility function, i.e., $U(x) = \ln x$, then

$$\inf_{(Z_0,\, Z_1)\in\mathcal{D}}\{the\ Dual\ Problem\} \sim \inf_{(Z_0,\, Z_1)\in\mathcal{D}^v}\{the\ Dual\ Problem\}.$$

Chapter 5

Exponential Utility Maximization

This chapter represents our second contribution in this topic through extension of Cvitanic and Karatzas (1996) to the exponential utility in one hand. On the other hand, the majority of our proofs are original due to the specificity of the exponential utility. Precisely in this chapter, we will solve the exponential utility maximization problem and show that Cvitanic-Karatzas' results remain true in this case. Furthermore, we will prove that some of the Cvitanic-Karatzas assumptions are in fact unnecessary in our case and then always hold for the model described in Section 3.

Through out this chapter, we assume there exists $(\hat{Z}_0,\ \hat{Z}_1) \in \mathcal{D}$ solution to the dual problem (3.23) for any $\zeta \in (0,\ +\infty)$, and $\tilde{V}(\zeta;\ y) < +\infty,\ \forall\, y > 0$.

5.1 Preliminary Results

In this section we focus on the exponential utility

$$U(x) = 1 - e^{-x}. \tag{5.1}$$

The reason why we pick up this utility function is that this does not satisfy the Inada condition defined in (2.4). We start by giving some preliminary calculations that will be useful in what follows. For $\zeta > 0$

$$I\,(\zeta) = -\ln \zeta, \quad \tilde{U}(\zeta) = 1 - \zeta + \zeta \ln \zeta. \tag{5.2}$$

Below, we will establish the dual problem for our exponential case. Let $\zeta > 0$ and $x,\ y > 0$. Then,

$$\begin{aligned}\tilde{V}(\zeta; y) &= \inf_{(Z_0, Z_1)\in\mathcal{D}} E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right] \\ &= \inf_{(Z_0, Z_1)\in\mathcal{D}} E\left[1 - \zeta\frac{Z_0(T)}{B(T)} + \zeta\frac{Z_0(T)}{B(T)}\ln\left(\zeta\frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right].\end{aligned} \tag{5.3}$$

Proposition 5.1.1. *Let $(\hat{Z}_0,\ \hat{Z}_1) \in \mathcal{D}$ is the optimal solution to (5.3) for any $\zeta > 0$. Then, for any $(Z_0,\ Z_1) \in \mathcal{D}$ and any $\zeta \in (0,\ \infty)$, we have*

$$E\left[\frac{Z_0(T)}{B(T)}\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}Z_1(T)\right] \geq E\left[\frac{\hat{Z}_0(T)}{B(T)}\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\hat{Z}_1(T)\right] \tag{5.4}$$

The proof of this proposition requires intermediate remarks which are interesting in themselves. To state these results, we first introduce the following notations. For any $x \in [0,\ 1]$ and any $(Z_0,\ Z_1) \in \mathcal{D}$, such that

$$E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right] < +\infty,$$

we denote

$$Z_0^x := xZ_0 + (1-x)\hat{Z}_0, \quad Z_1^x := xZ_1 + (1-x)\hat{Z}_1. \tag{5.5}$$

Then, the derivatives of Z_0^x, Z_1^x with respect to $x \in (0,\ 1)$ exist and are given by

$$\frac{d}{dx}\left(\frac{Z_0^x}{B}\right)(t) = \frac{Z_0(t) - \hat{Z}_0(t)}{B(t)}, \qquad \frac{dZ_1^x}{dx}(t) = Z_1(t) - \hat{Z}_1(t). \tag{5.6}$$

In the remaining part of this chapter, we consider

$$\phi(x) = \tilde{U}\left(\zeta \frac{Z_0^x(T)}{B(T)}\right) + \zeta \frac{y}{p} Z_1^x(T), \quad 0 \le x \le 1. \tag{5.7}$$

Proposition 5.1.2. *For almost all $\omega \in \Omega$, the function $x \in [0,\ 1] \to \phi(x)$, is convex. As a result, the function*

$$\Phi :\ x \in (0,\ 1] \ \longrightarrow\ \Phi(x) := \frac{\phi(x) - \phi(0)}{x} \tag{5.8}$$

is nondecreasing.

Proof:
For $u > 0$, we get

$$\begin{aligned}
\tilde{U}(u) &= 1 - u + u \ln u \\
\tilde{U}'(u) &= \ln u \begin{cases} < 0 & 0 < u < 1 \\ = 0 & u = 1 \\ > 0 & 1 < u \end{cases} \\
\tilde{U}''(u) &= \frac{1}{u} > 0.
\end{aligned}$$

We easily observe the following equality for $0 \le u < v \le 1$ *and* $0 \le s \le 1$,

$$\begin{aligned}
Z_i^{su+(1-s)v}(T) &= ((su + (1-s)v)Z_i(T) + (1 - (su + (1-s)v))\hat{Z}_i(T) \\
&= s(uZ_i(T) + (1-u)\hat{Z}_i(T)) + (1-s)(vZ_i(T) + (1-v)\hat{Z}_i(T)) \\
&= sZ_i^u(T) + (1-s)Z_i^v(T) > 0 \qquad (5.9)
\end{aligned}$$

Hence, we know $su + (1-s)v > 0$

$$\begin{aligned}
\tilde{U}\left(\zeta \frac{Z_0^{su+(1-s)v}(T)}{B(T)}\right) &= \tilde{U}\left(s\left(\zeta \frac{Z_0^u(T)}{B(T)}\right) + (1-s)\left(\zeta \frac{Z_0^v(T)}{B(T)}\right)\right) \\
&\le s\tilde{U}\left(\zeta \frac{Z_0^u(T)}{B(T)}\right) + (1-s)\tilde{U}\left(\zeta \frac{Z_0^v(T)}{B(T)}\right) \qquad (5.10)
\end{aligned}$$

Moreover, from (5.9)

$$\frac{y}{p}\zeta Z_1^{su+(1-s)v}(T) = s\frac{y}{p}\zeta Z_1^u(T) + (1-s)\frac{y}{p}\zeta Z_1^v(T) > 0$$

Thus,

$$\begin{aligned}\phi(su+(1-s)v) &= \tilde{U}\left(s\left(\zeta\frac{Z_0^u(T)}{B(T)}\right) + (1-s)\left(\zeta\frac{Z_0^v(T)}{B(T)}\right)\right) \\ &\quad + s\frac{y}{p}\zeta Z_1^u(T) + (1-s)\frac{y}{p}\zeta Z_1^v(T) \\ &\leq s\tilde{U}\left(\zeta\frac{Z_0^u(T)}{B(T)}\right) + (1-s)\tilde{U}\left(\zeta\frac{Z_0^v(T)}{B(T)}\right) \\ &\quad + s\frac{y}{p}\zeta Z_1^u(T) + (1-s)\frac{y}{p}\zeta Z_1^v(T) \\ &= s\phi(u) + (1-s)\phi(v). \end{aligned} \tag{5.11}$$

This proves that $\phi(x)$ is a convex function. Due to this convexity, for any $x \in (0,\ 1)$

$$\phi(0) - \phi(x) \geq -x\phi'(x),$$

which is equivalent to

$$x\phi'(x) - \phi(x) + \phi(0) \geq 0.$$

Then, the derivative of Φ satisfies

$$\Phi'(x) = \frac{x\phi'(x) - (\phi(x) - \phi(0))}{x^2} \geq 0.$$

Thus, Φ is nondecreasing. This completes the proof of the proposition. □

Lemma 5.1.1. *The family of a random variable* $\left\{\frac{\phi(x)-\phi(0)}{x},\ 0 < x \leq 1\right\}$ *is uniformly integrable.*

Proof:
Thanks to the previous proposition, we deduce that

$$for\ x \in (0,1], \quad \frac{\phi(x) - \phi(0)}{x} \leq \phi(1) - \phi(0). \tag{5.12}$$

Thus, the uniformly integrability of $\left\{\frac{\phi(x)-\phi(0)}{x},\ 0 < x \leq 1\right\}$ follows from finding a lower bound of this random variable. The lower bound may be a random variable that should be integrable but should not depend on $x \in (0,\ 1]$.

$$\frac{\phi(x)-\phi(0)}{x} = \frac{1}{x}\left(\tilde{U}\left(\zeta\frac{Z_0^x(T)}{B(T)}\right) + \zeta\frac{y}{p}Z_1^x(T) - \tilde{U}\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) - \zeta\frac{y}{p}\hat{Z}_1(T)\right)$$

$$= \frac{1}{x}\left(-\frac{\zeta}{B(T)}(Z_0^x(T) - \hat{Z}_0(T)) + \frac{y}{p}\zeta(Z_1^x(T) - \hat{Z}_1(T))\right)$$
$$+\frac{\zeta}{xB(T)}\left(Z_0^x(T)\ln\left(\zeta\frac{Z_0^x(T)}{B(T)}\right) - \hat{Z}_0(T)\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\right)$$

$$= \frac{\zeta}{B(T)}(\hat{Z}_0(T) - Z_0(T)) + \frac{y}{p}\zeta(Z_1(T) - \hat{Z}_1(T))$$
$$+\frac{\zeta}{xB(T)}\left[Z_0^x(T)\ln\left(\frac{\zeta Z_0^x(T)}{B(T)}\right) - \hat{Z}_0(T)\ln\left(\frac{\zeta\hat{Z}_0(T)}{B(T)}\right)\right]$$

$$= \frac{\zeta}{B(T)}(\hat{Z}_0(T) - Z_0(T)) + \frac{y}{p}\zeta(Z_1(T) - \hat{Z}_1(T))$$
$$+\frac{\zeta}{xB(T)}\hat{Z}_0(T)\ln\left(\frac{\zeta Z_0^x(T)}{B(T)}\frac{B(T)}{\zeta\hat{Z}_0(T)}\right)$$
$$+\frac{\zeta}{xB(T)}x(Z_0(T) - \hat{Z}_0(T))\ln\left(\frac{\zeta Z_0^x(T)}{B(T)}\right)$$

$$
\begin{aligned}
\frac{\phi(x)-\phi(0)}{x} &= \frac{\zeta}{xB(T)} x(\hat{Z}_0(T) - Z_0(T)) + \frac{y}{p}\zeta(Z_1(T) - \hat{Z}_1(T)) \\
&\quad + \frac{\zeta}{xB(T)} \hat{Z}_0(T) \ln\left(\frac{\hat{Z}_0(T) + x(Z_0(T) - \hat{Z}_0(T))}{\hat{Z}_0(T)}\right) \\
&\quad + \frac{\zeta}{\cancel{x}B(T)}\ \cancel{x}(Z_0(T) - \hat{Z}_0(T)) \ln\left(\frac{\zeta}{B(T)}\hat{Z}_0(T)\right) \\
&\quad + \frac{\zeta}{xB(T)} x(Z_0(T) - \hat{Z}_0(T)) \ln\left(\frac{\hat{Z}_0(T) + x(Z_0(T) - \hat{Z}_0(T))}{\hat{Z}_0(T)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\zeta}{B(T)}(Z_0(T) - \hat{Z}_0(T)) \ln\left(\frac{\zeta}{B(T)}\hat{Z}_0(T)\right) + \frac{y}{p}\zeta(Z_1(T) - \hat{Z}_1(T)) \\
&\quad + \frac{\zeta}{xB(T)} \hat{Z}_0(T) \ln\left(1 + x\frac{Z_0(T) - \hat{Z}_0(T)}{\hat{Z}_0(T)}\right) \\
&\quad + \frac{\zeta}{xB(T)} x(\hat{Z}_0(T) - Z_0(T)) \left[-1 + \ln\left(1 + x\frac{Z_0(T) - \hat{Z}_0(T)}{\hat{Z}_0(T)}\right)\right]
\end{aligned}
$$

Denote

$$
\begin{cases}
\mathbf{X} = \frac{Z_0(T) - \hat{Z}_0(T)}{\hat{Z}_0(T)} \\
\mathbf{Y} = \frac{\zeta}{B(T)}(Z_0(T) - \hat{Z}_0(T)) \ln\left(\frac{\zeta}{B(T)}\hat{Z}_0(T)\right) + \frac{y}{p}\zeta(Z_1(T) - \hat{Z}_1(T))
\end{cases}
\tag{5.13}
$$

Since $Z_0,\ \hat{Z}_0 > 0$, we get $Z_0 - \hat{Z}_0 > -\hat{Z}_0$ or equivalently $\frac{Z_0 - \hat{Z}_0}{\hat{Z}_0} = \mathbf{X} > -1$.

$$
\begin{aligned}
\frac{\phi(x)-\phi(0)}{x} &= \mathbf{Y} + \frac{\zeta}{xB(T)}\hat{Z}_0(T)\ln(1 + x\mathbf{X}) + \frac{\zeta}{xB(T)}\hat{Z}_0(T)x\mathbf{X}\left[-1 + \ln(1 + x\mathbf{X})\right] \\
&= \mathbf{Y} + \frac{\zeta}{xB(T)}\hat{Z}_0(T)\left[-x\mathbf{X} + (1 + x\mathbf{X})\ln(1 + x\mathbf{X})\right] \qquad (5.14)
\end{aligned}
$$

Since for any $a > -1$, $(1+a)\ln(1+a) - a \geq 0$, we deduce

$$
-x\mathbf{X} + (1 + x\mathbf{X})\ln(1 + x\mathbf{X}) \geq 0. \tag{5.15}
$$

This inequality combined with (5.14) leads to

$$\frac{\phi(x)-\phi(0)}{x} \geq \mathbf{Y}.$$

Therefore, we summarize

$$\mathbf{Y} \leq \frac{\phi(x)-\phi(0)}{x} \leq \phi(1)-\phi(0).$$

Due to the integrability assumption on the pair $(Z_0,\ Z_1)$, we deduce that Y and $\{\phi(1)-\phi(0)\}$ are integrable random variables. Thus, we conclude that $\frac{\phi(x)-\phi(0)}{x}$ is uniformly integrable. This completes the proof of the lemma. □

Proof of Proposition 5.1.1.
We consider the following function for $x \in [0,\ 1]$

$$\psi(x) := E\left[\tilde{U}\left(\zeta \frac{Z_0^x(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1^x(T)\right]. \tag{5.16}$$

It is clear that $\psi(0) = \tilde{V}(\zeta;\ y)$, and since for any $x \in [0,\ 1]$, $(Z_0^x,\ Z_1^x) \in \mathcal{D}$, we conclude that

$$\psi(x) \geq \psi(0). \tag{5.17}$$

From Lemma 5.1.1., and (5.17), it is deduced that:

$$\begin{aligned}
\psi'(+0) = \lim_{x\to+0}\frac{\psi(x)-\psi(0)}{x} &= \lim_{x\to+0} E\left[\frac{\phi(x)-\phi(0)}{x}\right] \\
&= E\left[\lim_{x\to+0}\frac{\phi(x)-\phi(0)}{x}\right] \\
&= E[\phi'(+0)] \geq 0. \qquad (5.18)
\end{aligned}$$

Hence, (5.6) and by chain rule,

$$\begin{aligned}
\psi'(+0) &= E\left[\tilde{U}'\left(\zeta\frac{Z_0^x(T)}{B(T)}\bigg|_{x=+0}\right)\zeta\frac{d}{dx}\left(\frac{Z_0^x}{B}\right)(T) + \frac{y}{p}\zeta\frac{dZ_1^x}{x}(T)\right] \\
&= E\left[\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\zeta\left(\frac{Z_0(T)-\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta(Z_1(t)-\hat{Z}_1(T))\right] \\
&= E\left[\left\{\zeta\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\frac{Z_0(T)}{B(T)} + \frac{y}{p}\zeta Z_1(T)\right\}\right. \\
&\qquad\left. - \left\{\zeta\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\frac{\hat{Z}_0(T)}{B(T)} + \frac{y}{p}\zeta\hat{Z}_1(T)\right\}\right] \geq 0
\end{aligned}$$

Thus,

$$E\left[\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\frac{Z_0(T)}{B(T)} + \frac{y}{p}Z_1(T)\right] \geq E\left[\ln\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right)\frac{\hat{Z}_0(T)}{B(T)} + \frac{y}{p}\hat{Z}_1(T)\right] \tag{5.19}$$

This ends the proof of Proposition 5.1.1. □

Proposition 5.1.3. *Let $(\hat{Z}_0,\ \hat{Z}_1) \in \mathcal{D}$ be the optimal solution to the dual problem (5.3) for any $\zeta > 0$. Suppose $\hat{\zeta}$ denotes*

$$\hat{\zeta} = \exp\left\{-\left(E\left[\frac{\hat{Z}_0(T)}{B(T)}\right]\right)^{-1}\left(x + \frac{y}{p}E[\hat{Z}_1(T)] + E\left[\frac{\hat{Z}_0(T)}{B(T)}\ln\left(\frac{\hat{Z}_0(T)}{B(T)}\right)\right]\right)\right\}, \tag{5.20}$$

and $\hat{C}_0,\ \hat{C}_1$ are given by

$$\hat{C}_0 = -\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right), \quad \hat{C}_1 = 0. \tag{5.21}$$

Then, the following assertions hold.

(1) The value function of the dual problem is expressed by

$$\tilde{V}(\hat{\zeta};\, y) = E\left[1 - \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right] - \hat{\zeta}x. \tag{5.22}$$

(2) The hedging price $h(\hat{C}_0,\, \hat{C}_1;\, y)$ *defined in (3.26) of* $(\hat{C}_0,\, \hat{C}_1)$ *coincides with* x*, i.e.,*

$$x = \sup_{(Z_0,\, Z_1)\in\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}\hat{C}_0 + Z_1(T)\left(\frac{\hat{C}_1}{S(T)} - \frac{y}{p}\right)\right]. \tag{5.23}$$

Proof:
From (5.20) and (5.21), we calculate

$$E\left[\frac{\hat{Z}_0(T)}{B(T)}\hat{C}_0\right] = -E\left[\frac{\hat{Z}_0(T)}{B(T)}\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] = x + \frac{y}{p}E[\hat{Z}_1(T)] = x + \frac{y}{p}\hat{z}_1. \tag{5.24}$$

(1) From (5.3) and $(\hat{Z}_0,\, \hat{Z}_1) \in \mathcal{D}$ is the optimal solution,

$$\begin{aligned}
\tilde{V}(\hat{\zeta};\, y) &= \inf_{(Z_0,Z_1)\in\mathcal{D}} E\left[1 - \hat{\zeta}\frac{Z_0(T)}{B(T)} + \hat{\zeta}\frac{Z_0(T)}{B(T)}\ln\left(\hat{\zeta}\frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\hat{\zeta}Z_1(T)\right] \\
&= E\left[1 - \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)} + \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\hat{\zeta}\hat{Z}_1(T)\right] \\
&= E\left[1 - \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)} + \hat{\zeta}\left(\frac{\hat{Z}_0(T)}{B(T)}\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\hat{Z}_1(T)\right)\right].
\end{aligned}$$

Then by multiplying both sides of (5.24) with $\hat{\zeta}$, we obtain

$$\begin{aligned}
\tilde{V}(\hat{\zeta};\, y) &= E\left[1 - \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)} + \hat{\zeta}(-x)\right] \\
&= E\left[1 - \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right] - \hat{\zeta}x. \qquad (5.25)
\end{aligned}$$

(2) From (3.27), we have

$$\begin{aligned} h(\hat{C}_0,\, \hat{C}_1;\, y) = & \sup_{(Z_0,\, Z_1)\in\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}\hat{C}_0 + Z_1(T)\left(\frac{\hat{C}_1}{S(T)} - \frac{y}{p}\right)\right] \\ & = \sup_{(Z_0,\, Z_1)\in\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}\left(-\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right) - \frac{y}{p}Z_1(T)\right] \\ & = - \inf_{(Z_0,\, Z_1)\in\mathcal{D}} E\left[\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\frac{Z_0(T)}{B(T)} + \frac{y}{p}Z_1(T)\right] \end{aligned}$$

Thanks to Proposition 5.1.1., we deduce that the infimum is attained by $(\hat{Z}_0,\, \hat{Z}_1)$. This combined with (5.24) leads to

$$h(\hat{C}_0,\, \hat{C}_1;\, y) \; = -E\left[\ln\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\frac{\hat{Z}_0(T)}{B(T)} + \frac{y}{p}\hat{Z}_1(T)\right] = x \qquad (5.26)$$

This ends the proof of the proposition.

□

5.2 Solution to the Primal Problem

Here we need to define $\mathcal{A}(x,\, y)$, the admissible class of $(L,\, M)$ as promised in Chapter 3, Section 3.2.

Definition 5.2.1. *A strategy $(L,\, M)$ is said to be admissible if*

$$E\left[\sup_{0\le t\le T} \exp\left\{-X_+(t)\right\}\right] < +\infty, \qquad (5.27)$$

where $(X_+(t))_{0\le t\le T}$ is the process defined in (3.11). The set of all admissible strategies with the initial asset of $(x,\, y)$ in the bank account and the stock respectively will be denoted by $\mathcal{A}(x,\, y)$.

Assumption 5.2.1. *There exists a strategy* $(\hat{L},\ \hat{M}) \in \mathcal{A}(x,\ y)$ *such that,*

$$\begin{cases} \hat{X}(T) + (1-\mu)\hat{Y}(T) \geq \hat{C}_0 \\ \hat{X}(T) + (1+\lambda)\hat{Y}(T) \geq \hat{C}_0 \end{cases} \tag{5.28}$$

Below is our main result in this chapter.

Theorem 5.2.1. *Suppose Assumption 5.2.1. is satisfied. Then the following assertions hold.*

1) The strategy $(\hat{L},\ \hat{M})$ *obtained from Assumption 5.2.1., is the optimal solution to the primal problem (3.12).*

2) The optimal terminal wealth is given by

$$\hat{X}(T+) := \hat{X}(T) + \begin{cases} (1+\lambda)\hat{Y}(T)\ if \hat{Y}(T) \leq 0 \\ (1-\mu)\hat{Y}(T)\ if \hat{Y}(T) > 0 \end{cases} = I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) = \hat{C}_0$$

3) The process $\hat{L}$ *is flat off the set* $\{0 \leq t \leq T \mid \hat{R}(t) = 1+\lambda\}$.

4) The process $\hat{M}$ *is flat off the set* $\{0 \leq t \leq T \mid \hat{R}(t) = 1-\mu\}$.

5) The process $\frac{\hat{Z}_0(\hat{X}+\hat{R}\hat{Y})}{B}$ *is true martingale, that is* $\hat{R} = \frac{\hat{Z}_1}{\hat{Z}_0 P}$.

6) The following equation holds.

$$\tilde{V}(\hat{\zeta};\ y) = V(x;\ y) - x\hat{\zeta} < +\infty \tag{5.29}$$

Proof:
Due to (5.28), we obtain

$$\hat{X}(T) + \hat{R}(T)\hat{Y}(T) \geq \hat{X}(T+) \geq \hat{C}_0 \tag{5.30}$$

From Proposition 3.2.1. and admissibility of $(\hat{L},\ \hat{M})$, $\frac{\hat{Z}_0(t)(\hat{X}(t)+\hat{R}(t)\hat{Y}(t))}{B(T)}$ is a supermartingale. Then, we get

$$E\left[\frac{\hat{Z}_0(T)}{B(T)}\hat{C}_0\right] \le E\left[\frac{\hat{Z}_0(T)}{B(T)}(\hat{X}(T)+\hat{R}(T)\hat{Y}(T))\right] \le x + \frac{y}{p}\hat{z}_1.$$

Thanks to (5.24), we obtain

$$E\left[\frac{\hat{Z}_0(T)}{B(T)}\hat{C}_0\right] = E\left[\frac{\hat{Z}_0(T)}{B(T)}(\hat{X}(T)+\hat{R}(T)\hat{Y}(T))\right] = x + \frac{y}{p}\hat{z}_1. \tag{5.31}$$

As a result, by combining (5.30) and (5.31),

$$\hat{C}_0 = \hat{X}(T) + \hat{R}(T)\hat{Y}(T) = \hat{X}(T+), \tag{5.32}$$

in one hand and thus the assertion (2) of the theorem is proved. On the other hand, (5.31) implies that $\frac{\hat{Z}_0(\hat{X}+\hat{R}\hat{Y})}{B}$ is a true martingale, and the assertion (5) of the theorem follows. Hence this implies that

$$((1+\lambda)-\hat{R})\cdot\hat{L} = (\hat{R}-(1-\mu))\cdot\hat{M} \equiv 0. \tag{5.33}$$

These two equations lead to the assertions (3) and (4) of the theorem.
Let $(L,\, M) \in \mathcal{A}(x\, y)$ and X, Y be their value processes. Then,

$$X(T+) = X(T) + f(Y(T)) \le X(T) + \hat{R}(T)Y(T).$$

Since $U(x) = 1 - e^{-x}$ is increasing, and $U(x) \le \tilde{U}(x) + xz$, we get

$$\begin{aligned} U(X(T+)) &\le U(X(T) + \hat{R}(T)Y(T)) \\ &\le \tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}(X(T) + \hat{R}(T)Y(T) \end{aligned} \tag{5.34}$$

From Proposition 3.2.1. and admissibility of $(L,\, M)$, $\frac{\hat{Z}_0(t)(X(t)+\hat{R}(t)Y(t))}{B(T)}$ is a supermartingale. Then, we get

$$\begin{aligned}E[U(X(T+))] &\leq E\left[\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] + \hat{\zeta}\left(x + \frac{y}{p}\hat{z}_1\right)\\ &= E\left[\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] + E\left[\hat{\zeta}\frac{\hat{Z_0(T)}}{B(T)}(\hat{X}(T) + \hat{R}(T)\hat{Y}(T))\right]\\ &= E\left[\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] + E\left[\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\hat{C}_0\right] = E\left[U(\hat{C}_0)\right]\end{aligned} \tag{5.35}$$

The last equality is due to $\tilde{U}(\zeta) = U(I(\zeta)) - \zeta I(\zeta)$ defined in (3.22).
Then we get

$$E[U(\hat{X}(T+))] \leq \sup_{(L,\, M)\in\mathcal{A}(x,\, y)} E[U(X(T+))] \leq E[U(\hat{C}_0)] \leq E[U(\hat{X}(T+))] \tag{5.36}$$

As a result, $(\hat{L},\ \hat{M})$ is the optimal solution to the primal problem. This proves the assertion (1).
From (5.22), we deduce that

$$\begin{aligned}V(x,\ y) &= E[U(\hat{X}(T+))] = E[U(\hat{C}_0)]\\ &= E[1 - \exp\{-\hat{C}_0\}] = 1 - E\left[\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right]\\ &= \tilde{V}(\hat{\zeta};\ y) + \hat{\zeta}x\end{aligned}$$

This proves the assertion (6). Hence the proof of the theorem is achieved. □

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