Bakry-Émery Ricci Curvature on Manifolds with Boundary

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

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University of Alberta

Abstract

A classic result in the field of Riemannian Geometry is the Splitting Theorem of Cheeger and Gromoll. Since this result there have been numerous alternate versions under a variety of different conditions. Continuing in this vein, we prove structure results on manifolds with boundary components under m-Bakry-Émery Ricci curvature bounds. First we look at a generalization of Frankel's theorem [9], extrapolating on the work of Peterson and Wilhelm [25]. We then prove some related corollaries that were shown by Choe and Fraser [4]. Finally we generalize the splitting theorems of Sakurai [28], [29] for manifolds with boundary and a non-gradient vector field.

Acknowledgments

I would like to thank my supervisor, Professor Eric Woolgar. His intuition for all things differential geometry is astounding, and I am honored to have had his support and guidance for the past two and a half years. I could not have done this project without him.

I would also like to thank my parents Joanna and Tim, my sister Monika, and my wife Komal. Having a family that appreciated the mental strain I was under and always gave me space and understanding is what allowed me to get where I am.

Finally, I would like to thank all my friends, and in particular Rui and Andrew.

Contents

List of Figures

Table of commonly used Symbols

M	A manifold
f	A function $M \mapsto \mathbb{R}$
ϕ	A function $\mathbb{R} \mapsto \mathbb{R}$
Φ	A function $M_1 \mapsto M_2$
π	A projection map $M_1 \times M_2 \mapsto M_1$
ρ_p	The distance from p function $M \mapsto \mathbb{R}$
∇f	The gradient vector field of f
$\nabla_u f$	The covariant derivative of f in the direction u
Δf	The Laplacian of f
g	A Riemannian metric
I	An interval subset of $\mathbb{R}$
α	A curve $I \mapsto M$
γ	A geodesic $I \mapsto M$
p, q	Points in a manifold
u, v, x, y	Vectors in T_pM
X, Y	Vector fields
J	A Jacobi field
X^*	The metric dual covector field of X, $g(X, \cdot)$.
$X(f)$	The scalar field $g(X, \nabla f)$
ψ	A tensor
Ψ	A tensor field
T^a_b	The space of tensors over TM of valence $\binom{a}{b}$.
τ_p	The focal radius at the point $p \in N$
$\mathcal{T}_{\gamma(t)}$	The parallel transport function $T_{\gamma(0)}M \mapsto T_{\gamma(t)}M$
σ	A two dimensional linear subspace of T_pM
ν_i	Basis vectors for T_pM
φ^i	Basis covectors corresponding to ν_i
$(D\Phi)_p(x)$	The linearization of Φ about $p \in M_1$
η_p	A unit normal vector field
S_{ν_p}	The shape operator $T_pN \mapsto T_pN$
$\mathbf{II}$	The second fundamental form $g(S_{\nu_p}x, y)$

1 Introduction

Two of the classical results in Riemannian Geometry are Frankel's theorem and the Splitting Theorem of Cheeger and Gromoll [3]. Since Cheeger and Gromoll's paper in 1971, many alternate versions of their theorem have been proven under various curvature conditions. Of particular relevance to the topic of this thesis is the generalization to manifolds with boundary given by Croke and Kleiner [6]. Recently we have seen versions of the theorem applying to Bakry-Émery Ricci curvature bounds, and in a pair of papers released in 2016 and 2019, Sakurai [28], [29] generalized these results to the boundary case with a gradient vector field in a manner similar to Croke and Kleiner.

When discussing manifolds with boundary, Frankel's theorem naturally arises. With plain Ricci bounds, one gets the result of Peterson and Wilhelm [25] if the bounds are tight, and the splitting theorem if the curvature conditions are only marginally violated. In this paper we follow this idea, first bringing Frankel's theorem into the Bakry-Émery case with general X and proving a few corollaries and related theorems, and we then modify the splitting theorems of Sakurai.

The study of manifolds with a positive density function e^{-f} began with Lichnerowicz in 1971 [16], and was later continued and applied to diffusion processes by Bakry and Émery. Many results that apply to metrics with Ricci curvature bounds have since been generalized to metrics with Bakry-Émery bounds, such as Myers's Theorem by Qian [26]. In particular, different versions of the splitting theorem have been proven by Fang-Li-Zhang [8], Khuri-Woolgar-Wylie [11], Sakurai [29], and others. In section 2.2, we summarize all the known relevant splitting theorem results. A generalization of Frankel's theorem has also been done in the case where $X = df$ by Li and Wei [15].

Recently, Bakry-Émery Ricci tensors have been studied in many different contexts including Ricci flow, general relativity, and geometric analysis. There is potential for these results to be generalized further to non smooth settings, and we hope that the smooth cases presented here may serve as inspiration for those advancements.

The initial motivation for starting this project was the possibility that the results we obtained could be used in later papers on the study of black holes. For discussion of the application of splitting theorems to relativity and black holes, the interested reader is directed to [11] or [19], which are closely related papers that discuss relativity applications.

For positive m, the arguments involving Bakry-Émery Ricci bounds found in this paper and others work very consistently by following closely to the arguments that show the corresponding results with only Ricci bounds. This suggests there may be some kind of close relationship between the manifolds admitting metrics with these types of conditions. The precise formulation of this problem is given in the conclusion of this paper.

In this thesis, we will first prove Theorem 3.1, the generalization of Frankel's theorem to the Bakry-Émery setting. One important corollary of the original is Corollary 3.8 and its effect on the fundamental group of a manifold containing an embedded minimal hypersurface, found by considering the universal cover. In a recent project, Choe and Fraser [4] use some different methods to get versions of this corollary for compact manifolds under modified curvature conditions, and here we show Theorems 3.9 and 3.10 which are generalized from Choe and Fraser's results. Finally, we work through the splitting theorems from Sakurai's papers to show that they hold without requiring the vector field to be gradient, attaining Theorems 4.1 and 4.2 for a manifold with multiple boundary components, and Theorem 4.8 for a manifold with a connected boundary component.

There are two main categories of arguments found in this paper. The first are *variation of arclength* arguments, and the second are *Bochner identity* arguments. The arguments work differently for each result, but generally both begin by supposing we can find a minimizing curve realizing the distance between two surfaces. The variational argument then considers such a minimal geodesic, and through differentiating a variation shows that nearby there must be an even shorter one. The Bochner identity is a formula that arises from local variations of volume rather than arclength, and the method of proof works by showing the functions of distance from each surface are subharmonic, and then applying the maximum principle to get a contradiction.

This thesis is organized as follows. In the remainder of Section 1 we review the background knowledge of Riemannian geometry, covering the higher level concepts required to understand the rest of the paper. In Section 2, we give the introduction to the Bakry-Émery curvature tensor and the types of problems we are going to solve. We review the numerous relevant generalizations that we are adding to. In section 3, we prove the first major result; a generalization of Frankel's theorem. We then go through a number of lemmata which allow us to get a couple of corollaries, and will come in handy in the next section. In Section 4, we prove splitting theorems, following closely to the work of Sakurai. Finally, in Section 5, we review the main results and state a few remaining open problems.

1.1 Basic Concepts in Differential Geometry

Differential geometry is a branch of mathematics which uses the technology of calculus to study problems of geometry. Riemannian geometry is a sub-branch of differential geometry that studies smooth manifolds with an inner product on the tangent space that varies smoothly from point to point (in a manner to be made precise later in this section). This inner product is called a *Riemannian metric*, and it brings with it local notions of distance, surface area, volume, and angles. Many of the results in Riemannian geometry deal with finding global topological properties that can be derived from the local contributions.

Some background knowledge of differential geometry will be assumed, but in order to get a good picture of the beginning of Riemannian geometry, we will briefly define a few of the early concepts. For a more detailed introduction, see chapter 0 of [7].

Differential manifolds are spaces which are sufficiently similar to linear space to allow one to do calculus. More specifically, any differentiable manifold may be described by an atlas of charts, and in any chart one may apply ideas from calculus since the chart will lie within linear space. To define a differentiable manifold properly, one begins with a *topological manifold*, which is a space that is locally homeomorphic linear space, and attaches a maximal globally defined *differentiable structure*.

A differentiable structure is a collection of injective maps $x_\alpha : U_\alpha \subset \mathbb{R}^n \mapsto M$ of open sets U_α into M such that $M = \bigcup_\alpha U_\alpha$, and that for all α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) =: W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and the composition $x_\alpha^{-1} \circ x_\beta$, called the transition function, is differentiable.

Definition 1.1 (The tangent bundle and its basis)**.** *Let M be a smooth n dimensional manifold. At each point p of M, the* tangent bundle *of M, which is a manifold itself denoted TM, assigns a vector space T_pM, called the tangent space of M at p. We can find an open $U \in \mathbb{R}^n$ containing 0 which has some local coordinates $\mathbf{x} : U \mapsto M$. If $\mathbf{x}(x_1, \dots, x_n) = p \in \mathbf{x}(U)$, we define*

$$\frac{\partial}{\partial x_i}(p) = (d\,\mathbf{x})_p(0, \dots, 1, \dots, 0) \in T_pM \tag{1.1}$$

and observe that by definition the set $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p \right\}$ is a basis of the tangent space T_pM.

When defining the differentiable structure, the transition function $x_\alpha^{-1} \circ x_\beta$ is a function from $\mathbb{R}^n$ to itself, so differentiablility of this function is understood. It is going to be useful to generalize the idea of a differentiable function to a function between manifolds.

Definition 1.2 (Differentiable functions)**.** *A map f from M_1^n to M_2^m is differentiable if at any point $p \in M_1$ and a parametrization $\mathbf{x}_2 : V \subset \mathbb{R}^m \mapsto M_2$ at $f(p)$, there exists a parametrization $\mathbf{x}_1 : U \subset \mathbb{R}^n \mapsto M_1$ at p such that $f(\mathbf{x}_1(U)) \subset \mathbf{x}_2(V)$, and the mapping*

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 : U \mapsto \mathbb{R}^m \tag{1.2}$$

is differentiable at $\mathbf{x}^{-1}(p)$.

Definition 1.3 (Covectors and Tensors). *Given a vector space V, the corresponding dual vector space V^* is the space of linear functionals on V. When speaking about a dual vector field of $X \subset TM$, the elements of X are commonly referred to as* covectors.

A tensor *can be defined with respect to any vector space X, but in Riemannian geometry we commonly work with tensors 'at a point' of a manifold, and unless otherwise specified, the vector space is usually understood to be the tangent space at that point. A $\binom{a}{b}$-tensor is a multi-linear map*

$$\psi : \underbrace{X \times ... \times X}_{a} \times \underbrace{X^* \times ... \times X^*}_{b} \mapsto \mathbb{R} \tag{1.3}$$

The tuple $\binom{a}{b}$ is referred to as the valence *of ψ. The space of all such tensors is denoted $T^a_b(X)$, or just T^a_b if it is understood that $X = T_pM$, which will be the case when talking about tensors throughout this paper.*

Definition 1.4 (Differentiable tensor fields). *A vector field X is a correspondence that associates to each point p a vector in T_pX. In other words, X is a function that maps between the manifolds M and TM, so the differentiability of X is understood just as in Definition 1.2. More generally, the image of any tensor field can be viewed as a manifold as well.*

Example 1.5. *Consider $\mathbb{R}^n$, and recall the tangent space $T_p\mathbb{R}^n$ of any point here is just $\mathbb{R}^n$ as well. A $\binom{0}{1}$-tensor is thus a linear map $\psi : (\mathbb{R}^n)^* \mapsto \mathbb{R}$, which means it can be represented by a unique vector $\mu \in \mathbb{R}^n$ where $\psi(f) = f(\mu)$. Similarly, a $\binom{2}{0}$-tensor is a bi-linear map $\psi : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$, and so it can be represented by a unique matrix $\mathcal{M} \in M_{n \times n}$ where $\psi(x, y) = x^{\perp}\mathcal{M}y$.*

This example highlights how a tensor is not exactly a generalization of a vector or matrix, but there are correspondences between them. So one expects that there will be a natural extension of the product and trace operations. Given two tensors ψ_1 and ψ_2 of valence $\binom{a}{b}$ and $\binom{k}{l}$, the definition of tensors as multi-linear maps allows the tensor product $\psi_1 \otimes \psi_2$ to be defined easily. It is a tensor of valence $\binom{a+k}{b+l}$ given by

$$\begin{aligned}(\psi_1 \otimes \psi_2)(v_1, ..., v_{a+k}, \omega^1, ..., \omega^{b+l}) \\ = \psi_1(v_1, ..., v_a, \omega^1, ..., \omega^b)\psi_2(v_{a+1}, ..., v_{a+k}, \omega^{b+1}, ..., \omega^{b+l})\end{aligned} \tag{1.4}$$

The trace operation is generalized to the case of $\psi \in T^1_1$, just by the natural pairing of the two indices. Since the first argument acts on vectors, we define

$$\mathrm{tr}(\psi(v, \omega)) = \omega(v) \tag{1.5}$$

Extending this to general mixed tensors is also coordinate independent, though we use an orthonormal basis in its definition. Let $\{\nu_i\}$ be an orthonormal basis for X, and $\{\varphi^i\}$ be an orthonormal basis for $\{X^*\}$. For any tensor $\psi \in T^{a+1}_{b+1}$, the tensor contraction is the new tensor $\operatorname{tr}(\psi) \in T^a_b$ defined by

$$(\operatorname{tr}(\psi))(v_1, \dots, v_a, \omega^1, \dots, \omega^b) = \sum_{j=1}^{n} \psi(v_1, \dots, v_a, \nu_j, \omega^1, \dots, \omega^b, \varphi^j) \tag{1.6}$$

To take the norm of a $\binom{a}{b}$-tensor ψ, define

$$|\psi| = \sum_{i_1, \dots, i_{a+b}=1}^{n} \psi(\nu_{i_1}, \dots, \nu_{i_a}, \varphi^{i_{a+1}}, \dots, \varphi^{i_{a+b}}) \tag{1.7}$$

Definition 1.6 (Pullback operator for $\binom{a}{0}$-tensors)**.** *Suppose we have two manifolds M_1 and M_2 of dimensions n and m. Let $\Phi : M_1 \mapsto M_2$ be a smooth function, and suppose we have a tensor field Ψ over M_2. Φ and Ψ induce a related tensor field on M_1, which we denote $\Phi^*\Psi$. For a $\binom{0}{0}$-tensor field (a scalar valued function), the way to do it is obvious; just let $(\Phi^*\Psi)(p) = \Psi(\Phi(p))$. For a $\binom{a}{0}$-tensor field, this generalizes to*

$$(\Phi^*\Psi)_p(x_1, \dots, x_a) := \Psi_{\Phi(p)}(d\Phi_p(x_1), \dots, d\Phi_p(x_a)) \tag{1.8}$$

for any $p \in M$ and $x_j \in T_pM$.

1.2 Riemannian Manifolds

Definition 1.7 (Riemannian metric)**.** *A Riemannian metric g on M is a collection which assigns to each point $p \in M$ a positive definite inner product $g_p : T_pM \times T_pM \mapsto \mathbb{R}$ and satisfies a differentiatibility condition: with local coordinates $\mathbf{x}$ as defined in 1.1, the functions*

$$h_{ij}(x_1, \dots, x_n) := g_p\left(\nu_i(p), \nu_j(p)\right) : U \mapsto \mathbb{R} \tag{1.9}$$

where $p = \mathbf{x}(x_1, \dots, x_n)$ must all be smooth. The functions h_{ij} are called the local representations of the Riemannian metric in the coordinate system $\mathbf{x}$.

Intuitively, this condition tells you that while moving around in a coordinate neighborhood, the product of any two basis vectors changes in a nice and smooth way so that the measurement of distance behaves as expected. The pair (M^n, g) of the manifold M with a Riemannian metric is called a Riemannian manifold of dimension n.

Throughout this section, we denote by M a Riemannian manifold of dimension n.

Note that so far tensor contraction is not defined for tensors of type T_0^a or T_b^0, which is unexpected since we saw that a $\binom{2}{0}$-tensor essentially generalizes a matrix, and we know how to take the trace of those. There is a way to do this using the metric, called *raising (or lowering)* an index. First, we will define another useful tool when working with tensors called *index notation.* If $\{\nu_j\}$ are a basis for X and $\{\varphi^i\}$ are the corresponding basis for the dual vector space X^*, a basis for T_b^a is given by a total of n^{a+b} tensors of the form

$$\mathbf{e}^{i_1 \dots i_a}{}_{j_1 \dots j_b} = \varphi^{i_1} \otimes \dots \otimes \varphi^{i_a} \otimes \nu_{j_1} \otimes \dots \otimes \nu_{j_b} \tag{1.10}$$

as each i and j ranges from 1 to n, which are defined so that

$$\mathbf{e}^{i_1 \dots i_a}{}_{j_1 \dots j_b}(v_{k_1}, \dots, v_{k_a}, \omega^{l_1}, \dots, \omega^{l_b}) = \delta^{i_1}_{k_1} \cdot \dots \cdot \delta^{i_a}_{k_a} \cdot \delta^{l_1}_{j_1} \cdot \dots \cdot \delta^{l_b}_{j_b} \tag{1.11}$$

i.e. the basis tensors are 1 if all of their arguments are in the right order, otherwise they are 0. This allows you to write a $\binom{a}{b}$-tensor ϕ in these coordinates, as a sum of all n^{a+b} tensors of the following form,

$$\phi = \sum_{i_1,\dots,i_a,j_1,\dots,j_b=1}^{n} \phi^{i_1 \dots i_a}{}_{j_1 \dots j_b} \mathbf{e}^{i_1 \dots i_a}{}_{j_1 \dots j_b} := \sum_{i_1,\dots,i_a,j_1,\dots,j_b=1}^{n} \phi(\mathbf{e}^{i_1 \dots i_a}{}_{j_1 \dots j_b}) \mathbf{e}^{i_1 \dots i_a}{}_{j_1 \dots j_b} \tag{1.12}$$

Observe that $g_{ij}\varphi^i \otimes \varphi^j$ is a $\binom{2}{0}$-tensor field over T_pM. Considering the matrix of coefficients $[g_{ij}]$, define $g^{ij}\nu_i \otimes \nu_j$ which is called the contravariant metric tensor, whose coefficients g^{ij} come from the inverse matrix of $[g_{ij}]$.

Suppose ψ is a $\binom{2}{0}$ tensor, which in index notation is ψ_{kj}. Dropping the index of ψ gives the coefficients of a $\binom{1}{1}$-tensor

$$\psi^i{}_j := \sum_{k=1}^{n} g^{ik}\psi_{kj} \tag{1.13}$$

and it is now possible to take the trace of this object as in (1.6).

The next key idea defines an equivalence relation on Riemannian Manifolds.

Definition 1.8 (Isometry of two Riemannian manifolds)**.** *Suppose we have two Riemannian manifolds* (M_1^n, g_{M_1}) *and* (M_2^n, g_{M_2}) *and a function* $\Phi : M_1 \mapsto M_2$. Φ *is called an isometry if*

$$g_{M_1,p}(x, y) = g_{M_2,\Phi(p)}(d\Phi_p(x), d\Phi_p(y)) \tag{1.14}$$

for all $p \in M_1$ *and all* $x, y \in T_pM_1$.

Example 1.9 (The Long Line). *Consider the set $\omega_1 \times [0,1) - \{(0,0)\}$, where ω_1 is the first uncountable ordinal. When this set is equipped with the order topology inherited from the lexicographical order on $\omega_1 \times [0,1)$, it defines a topological space that is similar to the real line, but much 'longer' despite having the same cardinality. This space is called the* Long Ray*, and it can be turned into the* Long Line $\mathbb{L}$ *by attaching a second copy on the left end but with the order reversed. This space can be equipped with a differentiable structure, however it is not possible to give it a Riemannian metric that induces its topology. The reason for this is that any Riemannian manifold can be shown to be metrizable [10], but $\mathbb{L}$ is a space which is normal but not metrizable (this follows from the fact that $\mathbb{L}$ is non-compact, yet sequentially compact).*

There are many other odd properties of the Long Line, such as how every differentiable structure on the real line is diffeomorphic to the standard one, however $\mathbb{L}$ has $2^{\aleph_1}$ pairwise non-diffeomorphic structures [22].

Definition 1.10 (Covering spaces and universal cover). *Given a topological space M, a* covering space *is another topological space $\widetilde{M}$ along with a map $\pi : \widetilde{M} \mapsto M$ such that for any $p \in M$, there is a neighborhood U of p such that $\pi^{-1}(U)$ is a disjoint union of open sets in $\widetilde{M}$, each of which is mapped homeomorphically onto U by π.*

If M is a Riemannian manifold, we also obtain a differential structure on $\widetilde{M}$ and a Riemannian metric from the pullback operator $g_{\widetilde{M}} = \pi^* g_M$, which makes π a local isometry. As a consequence, any curvature bounds imposed on M must also hold on $\widetilde{M}$.

One important type of covering space is the universal cover*, which is the unique covering space that is simply connected. A connected Riemannian manifold will always have a universal cover.*

While there are many notions of differentiation on a Riemannian manifold, they will all agree on the level of a scalar field. There are two important generalizations of differentiation to tensor fields that we will need, called the Lie derivative and the covariant derivative (or connection). One of their key features is that they are both uniquely defined to be compatible with the trace and tensor product operations, and they can both be thought of as a generalization of the directional derivative.

While these two notions have similarities, there is an important distinction between the two. The covariant derivative is a generalization of differentiation which *introduces* geometric structure to the manifold in order to allow the comparison of vectors in neighboring tangent spaces. There is no canonical coordinate system, thus there can be no canonical way to do this comparison. However, the generalization of the directional derivative which results in the Lie derivative is canonical, and requires knowledge of both the tensor field to differentiate, and the vector field in which to differentiate it in an open neighborhood.

Since the covariant derivative creates its own way of evaluating the change of a vector field, evaluation of it only depends on the tensor field and a vector at a single point. As a consequence, the Lie derivative is not expected to be linear in either argument, however the covariant derivative will be linear in the directional argument.

Definition 1.11 (Vector flow)**.** *A* solution curve *of a differentiable vector field X is a curve $\alpha : I \mapsto M$ with I an open interval containing 0 such that for all $t \in I$, we have $\dot{\alpha}(t) = X(\alpha(t))$. It is possible to show that we can choose α to be the maximal solution curve, in the sense that any other solution curve will be a restriction of α to an interval containing 0 and contained in I (see [20]). Now, let $\mathcal{I} = \bigcup_{p\in M} \{p\} \times I_p$, where $I_p \subseteq \mathbb{R}$ is the domain of the maximal solution curve α_p of X at p. The* flow *of X on M is a differentiable map*

$$\varphi_X : \mathcal{I} \mapsto M, \quad \varphi_X(p,t) = \alpha_p(t) \tag{1.15}$$

The flow is a diffeomorphism over $\mathcal{I}$, and it has the property that whenever (p, t_1) and $(\varphi_X(p, t_1), t_2) \in \mathcal{I}$, the concatenation of the two points $(p, t_1 + t_2)$ is in $\mathcal{I}$ as well and $\varphi(\varphi(p, t_1), t_2) = \varphi(p, t_1 + t_2)$.

Definition 1.12 (Lie derivative)**.** *We will state the definition for contravariant tensors, since the definitions for covariant or mixed tensors has only minor tweaks. Given a point $p \in M$ and a vector field X over M, let φ_X be the local vector flow around p associated to X. φ_X is a diffeomorphism, so for all t the inverse of the differential*

$$(d_p\varphi_X)^{-1} : T_{\varphi_X(p)}M \mapsto T_pM \tag{1.16}$$

extends uniquely to a homomorphism between the tensor algebras (the algebras of tensors over the vector fields with multiplication given by the tensor product),

$$\theta_p^t : \mathcal{T}\left(T_{\varphi_X}M\right) \mapsto \mathcal{T}\left(T_pM\right) \tag{1.17}$$

with this, we define the Lie derivative of the tensor field ψ in the direction of X:

$$\mathcal{L}_X\psi(p) = \frac{d}{dt}\bigg|_{t=0} \left(\theta_p^t\left(\psi\left(\varphi_X(p,t)\right)\right)\right) = \lim_{t\to 0} \frac{\theta_p^t\left(\psi\left(\varphi_X(p,t)\right)\right) - \psi(p)}{t} \tag{1.18}$$

which always outputs a tensor of the same valence as ψ.

This definition highlights how in the case of a scalar valued function, the Lie derivative is just equivalent to the directional derivative. It quantifies the change in a tensor field along the flow defined by a vector field.

Definition 1.13 (Covariant derivative). *Given a point $p \in M$ on a Riemannian manifold and a vector $x \in T_pM$, we define the covariant derivative of the following objects which are each defined in a neighborhood of p in the direction of x at p.*

Functions - *Let γ be a curve in M such that $\gamma(0) = p$ and $\dot{\gamma}(0) = x$. Given a real function $f : M \mapsto \mathbb{R}$ the covariant derivative in a way that coincides with the directional derivative in the direction x,*

$$\left(\nabla_{X(p)} f\right)_p = (f \circ \gamma)'(0) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(p)}{t} \tag{1.19}$$

Vector Fields - *The covariant derivative of a vector field X in the direction of x is the tangent vector $(\nabla_x X)_p$ which satisfies the following conditions for all local vector fields X, Y, all tangent vectors $x, y \in T_pM$, all scalar functions f defined in a neighborhood of p, and all scalars a, b,*

- *$(\nabla_x X)_p$ is linear in x:*

$$\left(\nabla_{ax+by} X\right)_p = a \cdot (\nabla_x X)_p + b \cdot (\nabla_y X)_p \tag{1.20}$$

- *$(\nabla_x X)_p$ is additive in X:*

$$(\nabla_x (X + Y))_p = (\nabla_x X)_p + (\nabla_x Y)_p \tag{1.21}$$

- *The product rule with a scalar valued function holds:*

$$\nabla_x (f \cdot X))_p = f(p)(\nabla_x X)_p + (\nabla_x f)_p X(p) \tag{1.22}$$

Covector Fields - *Given a field of covectors Ψ, the covariant derivative is the unique covector at p such that the following identity is satisfied for all vector fields X in a neighborhood of p:*

$$(\nabla_x \Psi)_p (X(p)) = \nabla_x \left(\Psi(X)\right)_p - \Psi_p \left((\nabla_x X)_p\right) \tag{1.23}$$

Tensor Fields - *Given two tensor fields Ψ_1 and Ψ_2, the covariant derivative on tensors is defined by imposing the following two rules:*

$$\nabla_x \left(\Psi_1 \otimes \Psi_2\right)_p = (\nabla_x \Psi_1)_p \otimes \Psi_2(p) + \Psi_1(p) \otimes (\nabla_x \Psi_2)_p \tag{1.24}$$

and if the valences of Ψ_1 *and* Ψ_2 *are identical, then*

$$\nabla_x(\Psi_1 + \Psi_2)_p = (\nabla_x \Psi_1)_p + (\nabla_x \Psi_2)_p \tag{1.25}$$

Once again, the output of the covariant derivative applied to a tensor is always another tensor of the same valence. Along with the generalization of differentiation comes the definition of a few key operators that act on functions.

Definition 1.14 (Differential operators on functions)**.** *Let* f *be a smooth function from* $M \mapsto \mathbb{R}$*, and* Φ *a smooth function from* $M_1 \mapsto M_2$*.*

- *The gradient* ∇f *of a function is the vector field such that for any other vector field* X *and any point* $p \in M$*,*
$$g_p((\nabla f)(p), X(p)) = \frac{\partial f}{\partial X(p)}(p) \tag{1.26}$$
Where $\frac{\partial f}{\partial X(p)}$ *is the usual directional derivative of* f *in the direction of* $X(p)$*. Intuitively, the gradient outputs the direction and magnitude of 'greatest increase' of the function at each point.* f *is sometimes referred to as the* potential *of the vector field* ∇f*.*

- *The Hessian is the iterated covariant derivative,* $\text{Hess}(u, v)(f) = \nabla_u (\nabla_v f)$*. In particular, note that* $\text{Hess}(f)$ *takes two vector arguments and thus is a* $\binom{2}{0}$*-tensor field on* M*.*

- *The Laplacian (or Laplace-Beltrami operator)* Δf *is the trace of the Hessian as explained in equation* (1.13)*.*

- *The linearization of* Φ *is a map defined in a neighborhood of each point on a manifold. It is the function* $(D\Phi)_p : T_pM_1 \mapsto T_{\Phi(p)}M_2$ *defined in relation to the exponential map; taking the point* $x \in T_pM_1$ *to* $y \in T_{\Phi(p)}M_2$ *such that* $\Phi(\exp_p x) = \exp_{\Phi(p)} y$*.*

- *The Jacobian determinant* $\det(D\Phi)_p$ *is defined as the determinant of the matrix of coefficients of* $(D\Phi)_p$ *as written in some coordinates with an orthonormal basis. Critically, the absolute value of the Jacobian determinant is orthonormal-coordinate independent, despite the determinant operation being of course coordinate dependent.*

One important property of the Lie derivative that will come into play later is the fact that for a smooth function f, the Lie derivative is related to the gradient and hessian by the following identity,

$$\mathcal{L}_{\nabla f} g = \text{Hess}\, f \tag{1.27}$$

In fact, this is commonly how the Hessian is defined for functions.

Definition 1.15 (Parallel Transport)**.** *Let $\alpha : I \mapsto M$ where I is an interval be a smooth curve. A smooth section $X(t) \subset TM$ is a map which is smooth as in Definition 1.2 from $\alpha(I) \mapsto TM$ such that the projection $\pi : TM \mapsto \alpha(I)$ satisfies $\pi \circ X = Id_M|_{\alpha(I)}$. Suppose we are just given a single element $e_0 \in T_pM$ at $\alpha(0) = p \in M$. The parallel transport of e_0 along α is the extension of e_0 to a parallel section X, which uniquely satisfies*

- $\nabla_{\dot{\alpha}} X = 0$
- $X(\alpha(0)) = e_0$

Using this, we denote by $\mathcal{T}_{\alpha(t)} : T_{\alpha(0)}M \mapsto T_{\alpha(t)}M$ the parallel transport function, which is a linear isomorphism that outputs vectors parallel transported along α by a length t.

Definition 1.16 (Geodesics and distance)**.** *For any two points $p, q \in M$, we define the* geodesic distance *$d(p, q)$ to be the infimum of the lengths of all curves joining p to q.*

A curve $\gamma : I \subseteq \mathbb{R} \mapsto M$ where I is an interval is called a geodesic if for all $t \in I$, the covariant derivative of $\dot{\gamma}(t)$ in its own direction is 0, i.e. $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for all $t \in I$. γ is called a minimal geodesic *if $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in I$, and a* line *if $I = \mathbb{R}$.*

Geodesics are a generalization of straight lines in Euclidean space. One of the important characteristics of a straight line is that it is the curve of minimal length between any two of its points. One may be temped to use this property to define a generalization in a manifold, however it quickly becomes apparent that defining it in this way would be troublesome. Consider for example $\mathbb{S}^2 \subset \mathbb{R}^3$, where a curve cannot be longer than π before it fails to minimize length. For this reason, the property of zero acceleration is used to define the generalization instead. If a geodesic does minimize the distance between any two of its points we then call it a minimal geodesic, and a line if it is also inextendable.

For a point p on a manifold, the set of cut points is roughly speaking the set of points q for which there is no longer a single unique length minimizing geodesic joining p to q. For a $p \in \mathbb{S}^2$ for example, the polar opposite point is the only cut point. More specifically, the definition of the cut locus depends on the exponential map.

Definition 1.17 (Exponential map)**.** *The* exponential function *is the map* $\exp_p : T_pM \mapsto M$ *which takes $x \in T_pM$ to $\gamma_x(1) \in M$, where γ is the unique geodesic satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = x$. Critical values of this map (which are defined via the Jacobian determinant) are called* conjugate points.

Intuitively, the exponential function maps a vector to a point which is found by traveling a distance $|x|$ out along a geodesic. If the manifold is complete, this means the exponential map is defined globally (over all of T_pM). It is a diffeomorphism locally around p, however it is easy to construct examples where $\exp_p$ is not injective globally. Indeed, just take the sphere; the exponential map here is a diffeomorphism only on a ball of radius π in $T_p\mathbb{S}^n$.

Definition 1.18 (Cut Locus)**.** *A cut vector of $p \in M$ is then a vector in T_pM such that $\gamma(t) := \exp_p(tv)$ is a minimizing geodesic for all $t \in [0,1]$, but fails to be minimizing for all $t > 1$. A cut point $q \in M$ is then the image of a cut vector, $\exp_p(v) = q$.*

1.3 Curvature

Curvature of a surface can be thought of by considering the difference between walking around a basketball court and walking around a curved area. If you hold out your arm as you walk around, being careful to keep it parallel with its previous position, your arm will be at the same angle as it started when you get back to your starting point. This is because a basketball court is constructed to be as flat as possible. On the other hand, trying this same experiment on a large curve around the earth may yield a different result. Start at the north pole and walk to the equator, walk along 1/4 the length of the equator, and then walk back up to the north pole. You will notice when you get back that your arm is very muscular from being held up for that long, and that it has changed direction by 90 degrees.

The Riemannian curvature tensor is an infinitesimal measure of the *holonomy* of the manifold, which is precisely this property. Let V and U be two vector fields, and $v \in V$, $u \in U$. The parallel transport of a vector $w \in T_{\gamma(0)}$ with a local extension W about a quadrilateral with sides tv, su, $-tv$, $-su$ is given by

$$\mathcal{T}_{su}^{-1}\mathcal{T}_{tv}^{-1}\mathcal{T}_{su}\mathcal{T}_{tv}w \tag{1.28}$$

Note that this quadrilateral is defined in the vector space T_pM with these vectors, and then projected onto the manifold using the exponential map, resulting in a quadrilateral of curves. Shrinking $s, t \to 0$ gives the infinitesimal description of the deviation, which is how we define the curvature tensor:

$$\frac{d}{dt}\frac{d}{ds}\mathcal{T}_{su}^{-1}\mathcal{T}_{tv}^{-1}\mathcal{T}_{su}\mathcal{T}_{tv}w\Big|_{t=s=0} =: \mathrm{Rm}(U,V)W \tag{1.29}$$

One does not always define curvature in this way, however this equivalent way is one of the most intuitive interpretations. In the more common way, it is described as a commutator of the covariant derivative:

$$\mathrm{Rm}(U,V)W := \left(\nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{\mathcal{L}_U V}\right)W \tag{1.30}$$

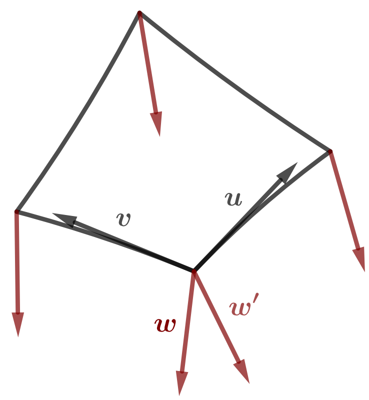

Figure 1: w changes direction as it is parallel transported around a quadrilateral

From the description of equation (1.29) and (1.30), it is clear that the input of the tensor requires at least three vector arguments. But in fact, it also requires a covector argument. This is because the outputs of the expressions above are still vectors, so we need one more linear functional to send that vector to $\mathbb{R}$, which makes Rm a $\binom{3}{1}$-tensor. In index notation it is

$$\mathrm{Rm}^i{}_{jkl} = dx^i \left(\mathrm{Rm}\left(\nu_l, \nu_k\right) \nu_j\right) \tag{1.31}$$

where ν_j are the basis vectors from Definition 1.1.

The sectional curvature comes from the computation of the Gauss curvature in a 2-dimensional surface inside the manifold. At any point p, there are several sectional curvatures. Let $\sigma \subset T_pM$ be a 2 dimensional subspace, and let x, y be a basis for σ. We define

$$K_\sigma = K(x,y) = \frac{g(\mathrm{Rm}(x,y)x,y)}{\left||x|^2 |y|^2 - g(x,y)\right|} \in \mathbb{R} \tag{1.32}$$

to be the sectional curvature of M in σ. The denominator of this is the area of the parallelogram determined by x and y, and its purpose is to re-scale the expression so as to make it independent of which basis for σ we choose.

The following is an important average of sectional curvatures. At a point p, let $x = x_n$ be a unit vector in T_pM. Create an orthonormal basis $\{x_1, \dots, x_{n-1}\}$ for the hyperplane in T_pM orthogonal to x_n. Now define

$$\mathrm{Ric}_p(x,x) := \mathrm{Ric}_p(x) := \sum_{i=1}^{n-1} K(x_n, x_i) \tag{1.33}$$

which is the *Ricci curvature* in the direction x.

The Ricci curvature can also be formulated as a trace of the curvature tensor. In particular,

$$\mathrm{Ric}_{jk} = \sum_{i=1}^{n} \mathrm{Rm}^{i}{}_{jki} \tag{1.34}$$

which makes it clear that Ric is a $\binom{2}{0}$-tensor, and it is not hard to show that $\mathrm{Ric}_p(x,x)$ defined in this way is equal to $\mathrm{Ric}_p(x)$ in equation (1.33). One can further take a trace of the Ricci tensor to get what is known as the scalar curvature, $S(p) = \mathrm{tr}\,\mathrm{Ric}_p$ which is indeed a scalar field. In the literature it is common to see expressions comparing Ric to scalars, such as the statement "Ric > 0". To be clear, the implication of this is "for all $p \in M$ and all $x, y \in T_pM$, $\mathrm{Ric}(x,y) > 0$".

1.4 Embedded Hypersurfaces

Definition 1.19 (Embedding and hypersurfaces). *A map $\Phi : M_1 \mapsto M_2$ is called an* immersion *if the linearization $(D\Phi)_p$ is injective at each point. An* immersed submanifold *is the image of an immersion.*

An embedding *is a map which is a homeomorphism onto its image, and an* embedded submanifold *is then an immersed submanifold where the inclusion map is an embedding. An important class of embedded submanifolds are those of dimension $n-1$, which we call* hypersurfaces.

The ambient space allows one to define vector fields normal to the surface and consider the shape of the hypersurface in the context of its environment, so we will need to define a few of these concepts here.

Definition 1.20 (Separating hypersurface). *A hypersurface N in a connected manifold M is called non-separating if it does not separate the manifold, i.e. for any $p, q \in M \backslash N$, there is a continuous curve $\alpha : [0,1] \mapsto M \backslash N$ such that $\alpha(0) = p$ and $\alpha(1) = q$. Otherwise, N called separating.*

Definition 1.21 (Orientable hypersurface). *A hypersurface in M is called* orientable *or* 2-sided *if it admits a globally defined differentiable unit normal vector field in M.*

For example, in $\mathbb{R}^n$ any smooth embedded hypersurface must be orientable. If the hypersurface is also connected and compact, it must be separating, splitting $\mathbb{R}^n$ into two connected components which is a result related to the Jordan-Brouwer separation theorem.

More generally, a manifold does not have to be embedded to define orientability. M is called orientable if it admits an atlas of charts where every transition function between charts has a positive Jacobian determinant.

The definitions of curvature we saw already are all *intrinsic*; they do not depend on how the manifold is embedded in any ambient space, just on the geometry of the manifold itself. In this paper we are going to need an *extrinsic* notion of curvature to deal with submanifolds.

Definition 1.22 (Mean curvature)**.** *Let $N \subset M$ be a hypersurface, and let η be a smooth extension of a unit normal vector field over N to an open neighborhood of N. The following definition does not depend on the choice of smooth extension. The* shape operator S_{ν_p} : $T_pN \mapsto T_pN$ *is a map sending $v \mapsto \nabla_v \eta$. This map defines a $\binom{1}{1}$-tensor field by sending $(v, \varphi) \mapsto \varphi(S_{\nu_p}(v)) \in \mathbb{R}$. The* second fundamental form **II** *is just the shape operator, but with the index raised so that it defines a $\binom{2}{0}$-tensor. The* mean curvature $H(p)$ *is then the trace of the shape operator, and it defines a scalar field over N.*

The definition of H is very commonly set to $-\operatorname{tr} S$ instead, but our sign convention is that it should be the positive trace. The sign of the object also depends on the choice of unit normal field.

Finally, we need to know a couple of conditions on the space that characterize the well behavedness of functions and geodesics.

Definition 1.23 (Complete and compact manifolds)**.** *M is a* complete *manifold if any geodesic $\gamma : I \mapsto M$ has a well defined extension $\gamma : \mathbb{R} \mapsto M$. This way of defining completeness is called* Geodesic completeness*, but one can also define completeness of a manifold as sequential completeness or completeness in the sense of a metric space. The Hopf-Rinow theorem tells us that these notions are in fact equivalent.*

A manifold is compact *if it is compact as a topological space; that is, any open cover has a finite subcover. We call a manifold that is compact and does not have a boundary* closed.

Definition 1.24 (Totally geodesic submanifold)**.** *A submanifold $N \subset M$ is called* totally geodesic *if $S_{\nu_p} = 0$ for all $p \in N$.*

Essentially, totally geodesic implies that geodesics with initial tangent vector in T_pN will stay inside N.

Remark 1.25. *It is uncommon to find non-trivial totally geodesic submanifolds. Murphy and Wilhelm [21] showed that a manifold chosen at random will not typically have any at all.*

2 Preliminaries

In this paper, we will deal with structural results called 'splitting theorems'. Roughly speaking, a splitting theorem will say that under certain conditions, the manifold in question must be isometric to a warped, twisted, or in strongest case Riemannian product.

Definition 2.1 (Twisted/Warped products). *Let (M, g_M) and (N, g_N) be two Riemannian manifolds, and $\phi, \psi : M \times N \to (0, \infty)$ be two smooth functions. Consider the product manifold $M \times N$ with metric tensor $g_{\phi,\psi} = \phi^2 \pi_M^* g_M + \psi^2 \pi_N^* g_N$ where $\pi_M : M \times N \mapsto M$ and $\pi_N : M \times N \mapsto N$ are canonical projections. In what follows, we omit the projections and write $g_{\phi,\psi} = \phi^2 g_M + \psi^2 g_N$.*

- *If $\phi \equiv \psi \equiv 1$, then $g_{\phi,\psi}$ is a Riemannian product metric.*
- *If $\phi \equiv 1$ and ψ is a function only of points in M, $g_{\phi,\psi}$ is a* warped product *metric.*
- *If ϕ is a function only of points in N, and ψ only of points in M, $g_{\phi,\psi}$ is a* doubly-warped product *metric.*
- *If $\phi \equiv 1$, $g_{\phi,\psi}$ is a* twisted product *metric.*
- *Otherwise, $g_{\phi,\psi}$ is a* doubly-twisted product *metric.*

One can define product metrics on manifolds with more than two factors as well, but this will not be necessary for the purposes of this thesis.

This is easiest to visualize as a topological splitting, but splitting as a Riemannian product is actually a much stronger condition. It tells you that all the structure given by the inner product has to be something much simpler, a sum of two metrics each acting on one term in the topological product.

The conditions normally imposed on the metric in a splitting theorem are bounds on the curvature. The following classical result, the Cheeger-Gromoll splitting theorem, is one of the most well known theorems in Riemannian geometry.

Theorem 2.2 (Cheeger-Gromoll, Theorem 2). *Let M be a complete manifold of non-negative Ricci curvature. Then M is the isometric product $\bar{M} \times \mathbb{R}^k$ where $\bar{M}$ contains no lines and $\mathbb{R}^k$ has its standard flat metric.*

One of the variations on this result is the generalization to manifolds with boundary. It may be strange to think of a manifold having a boundary, since this means the manifold violates some of the foundational properties we used to construct differential manifolds. Indeed, a point on the boundary does not have a neighborhood homeomorphic to $\mathbb{R}^n$ at all.

Definition 2.3 (Manifolds with Boundary and Asymptotic Ends). *A manifold with boundary is a space consisting of both interior points and boundary points. Every interior point has a neighborhood that is homeomorphic to the open n-dimensional ball. Every boundary point has a neighborhood with a homeomorphism φ to the half n-ball $\{(x_1, \dots, x_n) \mid x_1 \geq 0, \sum x_i^2 < 1\}$ such that the first component of $\varphi(p)$ is 0 if and only if p is a boundary point.*

A manifold M has an asymptotic end *if there is a compact $K \subset M$ such that $M \backslash K$ is diffeomorphic to $\mathbb{R}^n \backslash B_1[0]$ where $B_1[0]$ is the closed unit ball of radius 1 centered at 0.*

The key observation is that if M is a manifold of dimension n with boundary, the interior $\operatorname{int} M$ is a manifold without boundary of dimension n, and the boundary ∂M is a manifold without boundary of dimension $n-1$. Note that the notion of geodesic completeness described in Definition 1.23 will not work either if the manifold has a boundary, so in this case we use completeness in the sense of a metric space.

Example 2.4 (Cylinder). *Consider the set*

$$\mathrm{C}^n = \left\{ (x_1, x_2, \dots, x_n, y) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 = 1 \right\} \subset \mathbb{R}^{n+1} \tag{2.1}$$

We can consider this a Riemannian submanifold of $\mathbb{R}^{n+1}$ using the induced metric *from $\mathbb{R}^{n+1}$. Specifically, for any $p \in \mathrm{C}$, $T_p \mathrm{C}$ is the n dimensional linear subspace of the $n+1$ dimensional $T_p\mathbb{R}^{n+1}$ perpendicular to a normal vector to the surface. So for any $x, y \in T_p \mathrm{C}$ the metric satisfies $g_{\mathrm{C}}(x, y) = g_{\mathbb{R}^{n+1}}(x, y)$. This space is isometric to the Riemannian product manifold $\mathbb{S}^{n-1} \times \mathbb{R}$ with metric $g_{\mathrm{C}^n} = g_{\mathbb{S}^{n-1}} + g_{\mathbb{R}}$.*

Consider the new set

$$\mathrm{C}_+^n = \left\{ (x_1, x_2, \dots, x_n, y) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 = 1, y \geq 0 \right\} \subset \mathbb{R}^{n+1} \tag{2.2}$$

Similarly, this can be considered a manifold with boundary with the induced metric from $\mathbb{R}^{n+1}$. The boundary in this case is $\mathbb{S}^{n-1} \times \{0\}$, and the manifold-with-boundary can be written as a Riemannian product $\mathbb{S}^{n-1} \times [0, \infty)$ with the metric $g_{\mathrm{C}_+^n} = g_{\mathbb{S}^n} + g_{\mathbb{R}|_{[0,\infty)}}$. It also has an asymptotic end; just take K equal to the boundary $\mathbb{S}^{n-1} \times \{0\}$.

The sphere is a space of constant positive sectional curvatures, and $\mathbb{R}$ is of course flat. Thus, both of these examples are of spaces with all non-negative sectional curvatures, meaning C^n fits the requirements of Theorem 2.2.

More generally, any smooth two dimensional surface of revolution of a function $\phi : \mathbb{R} \mapsto \mathbb{R}$ can be considered a warped product manifold with warping function ϕ.

The first type of argument we will use is the variational argument, so we need to define what a variation of a curve is. They are smooth one-parameter families of curves centered at a base curve.

Definition 2.5 (Variation of a curve)**.** *A* variation *of a smooth curve* $\gamma : [0, \ell] \mapsto M$ *is a smooth function* $f : [-\varepsilon, \varepsilon] \times [0, \ell] \mapsto M$ *such that* $f(0, t) = \gamma(t)$*. The variation is* proper *if additionally* $f(s, 0) = \gamma(0)$ *and* $f(s, \ell) = \gamma(\ell)$*. This definition comes with a vector field along* γ *called the* variational vector field*,*

$$V(t) := \frac{d}{ds} f(s,t)\Big|_{s=0} \tag{2.3}$$

which is a vector field along the base curve in the direction of the variation at each point.

There is one essential type of variational vector field J along a γ, called a Jacobi field. A vector field is a Jacobi field if it satisfies the Jacobi equation,

$$\nabla_{\dot\gamma(t)} \nabla_{\dot\gamma(t)} J(t) + \mathrm{Rm}(J(t), \dot\gamma(t))\dot\gamma(t) = 0 \tag{2.4}$$

and in particular we look at Jacobi fields arising from 1 parameter families of curves γ_s,

$$J(t) = \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0} \tag{2.5}$$

which satisfies the Jacobi equation for $\gamma = \gamma_0$.

Jacobi fields are very useful for the way they allow you to find points where nearby geodesics leaving from the same point 'reconverge'. Rather than computing the separation of nearby geodesics, you can instead frame it as a differential equation.

Example 2.6. *On* $\mathbb{S}^n$*, the geodesics are great circles. Consider two geodesics* γ_0 *and* γ_θ *passing through the north pole, leaving at angles separated by* θ*. They travel a distance of length* π *before reconverging. The geodesic distance between* $\gamma_0(t)$ *and* $\gamma_\theta(t)$ *is*

$$d(\gamma_0(t), \gamma_\theta(t)) = \sin^{-1}\left(\sin t \sin\theta \sqrt{1 + \cos^2 t \tan^2\left(\frac{\theta}{2}\right)}\right) \tag{2.6}$$

which shows that these geodesics intersect at every multiple of π*. Computing this however requires complete knowledge of the geodesics, which makes it hard to use in any kind of generality. A Jacobi field is an alternate approach to finding this reconvergence which depends only on the initial conditions of* γ_0*. Indeed, just observe that*

$$\frac{\partial}{\partial s}\Big|_{s=0} d(\gamma_0(t), \gamma_s(t)) = |J(t)| \tag{2.7}$$

and using the Jacobi equation, the solution to this is just a solution to

$$y'' + y = 0 \tag{2.8}$$

with $y(0) = 0$ *and* $y'(0) = 1$, *which is* $\sin t$. *This once again shows that geodesics reconverge at every multiple of* π *and did not require knowing any geodesics past their initial conditions.*

Using Riemannian products, we will define a version of the exponential function useful for embedded hypersurfaces or manifolds with boundary. This function works in a different way; instead of mapping a vector in T_pM to M, it acts on vectors orthogonal to the surface and maps them to M in a natural way.

Definition 2.7 (Normal exponential function)**.** *Let* $N \subset M$ *be a hypersurface, and let* $p \in M$. *Let* $p \in U$ *be a neighborhood of* p *such that there is a well defined unit normal vector field* η_p *over* U. *We define the* normal exponential function $\exp_p^\perp : T^\perp N \mapsto M$, *where* $T^\perp N$ *is the orthogonal vector bundle isometric to a Riemannian product* $U \times \mathbb{R}$. *This maps sends* $x \sim (p, t) \mapsto \gamma_{\eta_p}(t)$, *where* γ_{η_p} *is the geodesic in* M *with initial conditions* $\gamma_{\eta_p}(0) = p$ *and* $\dot{\gamma}_{\eta_p}(0) = \eta_p$. *The critical points of this map are called* Focal points.

2.1 Bakry-Émery Ricci Curvature

Certain important manifolds in physics have bounds on curvature that is 'weighted' in a way. In this section we define the precise nature of this weighting, and then give some motivation for how it arises.

Definition 2.8. *Let* X *be a vector field on a Riemannian manifold* (M^n, g). *The* m-Bakry-Émery tensor *is*

$$\begin{aligned} \mathrm{Ric}_X^m &:= \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m} X^* \otimes X^* \ , \ m \neq 0 \\ \mathrm{Ric}^0 &:= \mathrm{Ric} \ , \ m = 0 \text{ and } X \equiv 0 \\ \mathrm{Ric}_X^\infty = \mathrm{Ric}_X^{-\infty} &:= \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g \end{aligned} \tag{2.9}$$

Observe that each term in the first line of (2.9) *is a* $\binom{2}{0}$*-tensor, so* Ric_X^m *is defined properly as another* $\binom{2}{0}$*-tensor. It is very common to write* X *in the expression instead of its metric dual* $X^*(-) = g(-, X)$. *In the special case that* X *is the gradient of a smooth function* $f : M \mapsto \mathbb{R}$, *it is conventional to write* Ric_f^m *or* Ric_f^∞, *respectively, and then we take* f *to be constant in the* $m = 0$ *case.*

Along with the Ricci curvature, there is a Bakry-Émery version of mean curvature for hypersurfaces, defined as follows.

Definition 2.9. *The Bakry-Émery mean curvature of the hypersurface N with respect to the (extended) unit normal field ν is defined to be*

$$H_X = H - g(X, \nu) \tag{2.10}$$

If $H_X = 0$ pointwise, we call Σ a Bakry-Émery X-minimal hypersurface.

Note that authors who take the mean curvature to be the negative trace of the shape operator will add rather than subtract the $g(X, \nu)$ when defining the weighted mean curvature.

When dealing with a manifold with boundary, we can consider a boundary component N to be a hypersurface in M, and so we can compute its mean curvature. In this case, we will always choose the unit normal field to point into M.

There is a modified version of the Laplacian which is weighted to cooperate with the definitions of weighted curvature. We define the weighted Laplacian (also known as the drift Laplacian) on functions to be

$$\Delta_X f = \Delta f - g(X, \nabla f) \tag{2.11}$$

Remark 2.10. *We say that a continuous function $f : M \mapsto \mathbb{R}$ satisfies $\Delta_X f \geq 0$ in the barrier sense if for any $p \in M$ and $\varepsilon > 0$, there is a C^2 function $f_{p,\varepsilon}$ defined in a neighborhood of p such that*

- $f_{p,\varepsilon} \leq f$
- $f_{p,\varepsilon}(p) = f(p)$
- $\Delta_X f_{p,\varepsilon} \geq -\varepsilon$

and likewise, we say f satisfies $\Delta_X f \leq 0$ in the barrier sense if $-f$ satisfies $\Delta_X(-f) \geq 0$ in the barrier sense.

One reason to study smooth metric spaces is the fact that they are examples of *collapsed measured Gromov-Hausdorff limits*. The specifics about this observation can be found in [18], but the idea is to consider the manifold $M^n \times \widetilde{M}^m$ with the warped product metric

$$g_\varepsilon = g_M \oplus \left(\varepsilon e^{-f/m}\right)^2 g_N \tag{2.12}$$

where M and $\widetilde{M}$ are compact and f is a smooth function of points in M. As $\varepsilon \to 0$, this manifold converges to M with a 'weighted volume form $e^{-mf} dvol_{g_M}$'.

Let u, v be two vectors in $T_p(M \times \widetilde{M})$ which are orthogonal to the $\widetilde{M}$ factor. Using O'Neill's formula [23, Corollary 7.43], we are going to show that the Ricci curvature satisfies

$$\mathrm{Ric}_{g_\varepsilon}(u,v) = \mathrm{Ric}_{g_M}(u,v) + (\mathrm{Hess}\, f)_{g_M}(u,v) - \frac{1}{m} df \otimes df(u,v) \tag{2.13}$$

which is the same as (2.9). Thus, for $m \in \mathbb{Z}_+$ the Bakry-Émery Ricci curvature can be thought of as the Ricci curvature restricted to M from an ambient manifold. Computing with the formula,

$$\begin{aligned} \mathrm{Ric}_{g_\varepsilon}(u,v) &= \mathrm{Ric}_{g_M}(u,v) - \frac{m}{\varepsilon e^{-f}} \mathrm{Hess}\left(\varepsilon e^{-f/m}\right)(u,v) \\ &= \mathrm{Ric}_{g_M}(u,v) - \frac{m}{e^{-f/m}} \nabla_u \nabla_v e^{-f} \\ &= \mathrm{Ric}_{g_M}(u,v) - \frac{m}{e^{-f/m}} \left(\nabla_u \left(-\frac{1}{m} e^{-f/m} \nabla_v f\right)\right) \\ &= \mathrm{Ric}_{g_M}(u,v) - \frac{1}{e^{-f/m}} \left(\frac{1}{m} e^{-f/m} \nabla_u f \nabla_v f - e^{-f/m} \nabla_u \nabla_v f\right) \end{aligned} \tag{2.14}$$

and after some simplification we are done.

Another very interesting thing to look at is the relationship between the Bakry-Émery Ricci tensor and concavity properties of distance functions on a connected Riemannian manifold. For a fixed point $q \in M$, the distance function $\rho_q(p) = d(p,q)$ is the infimum of lengths of curves in M connecting p to q. This function has the properties

- $|\nabla \rho_q| = 1$
- $\mathrm{Hess}\, \rho_q^2 \leq 0 \iff M$ has all non-negative sectional curvatures.

More generally, we will consider in later sections distance functions from an embedded hypersurface. Such functions also possess the above two properties. The level sets of distance functions are going to be important later because the Hessian of a distance function is equal to the second fundamental form on the surface, and the mean curvature is thus equal to the Laplacian of the distance function. This motivates the use of the drift Laplacian, because under this definition

$$H = \Delta \rho \implies H_X = \Delta_X \rho \tag{2.15}$$

which keeps all of the modified tensors in line.

2.2 Previous Splitting Theorem Results

Typically no restrictions on X are necessary for the finite positive m cases in the list below. For the other cases, it is usually necessary to impose a boundedness condition of some sort as we will discuss more later. For $m \in (-\infty, 1-n]$ and $X = df$, Wylie [31] obtained a warped product splitting when f is bounded above. For $m = \infty$ and $X = df$ with f bounded, Fang, Li, and Zhang [8] obtained a product splitting. Alternatively, Lim [17] obtained a splitting in this case by assuming instead that $\nabla f \to 0$ at ∞.

In a sequence of papers, Sakurai considered manifolds-with-boundary under Bakry-Émery curvature conditions with gradient vector fields $X = df$. In [28], versions of the positive m case are given. The negative m case is considered in [29]. Sakurai's exact Bakry-Émery curvature conditions vary across the different cases cited above, whereas in the $X \equiv 0$ case studied by Croke and Kleiner, the curvature conditions for the connected and disjoint boundary theorems have the same form. In the present paper, different curvature conditions are required for the connected and disjoint (multi-component) boundary cases.

The following list summarizes the known splitting theorem results:

- **M is a complete manifold obeying $\mathrm{Ric}_X^m \geq 0$.**

  1. $X = df$, $m < 1-n$, $f \leq K$: product splitting theorem [31].
  2. $X = df$, $m = 1-n$, $f \leq K$: warped product splitting theorem [31].
  3. $X = df$, $m = \infty$, $f \leq K$: product splitting theorem [8].
  4. $X = df$, $m = \infty$, $\nabla f \to 0$ at ∞: product splitting theorem [17].
  5. Arbitrary X, $0 < m < \infty$, product splitting theorem [11].

- **M is a manifold with boundary.**

  6. Connected boundary, $m \in (0, \infty]$, $\mathrm{Ric}_f^m \geq (n+m-1)\kappa$, $H_f \leq -(n+m-1)\lambda$ where $\kappa \leq 0$, $\lambda = \sqrt{|\kappa|}$: warped product splitting theorem [28, Theorem 1.4].
  7. Connected boundary, $m \in [-\infty, 1-n]$, $\mathrm{Ric}_f^m \geq 0$, $H_f \leq 0$, $f < K$: warped product splitting theorem. [29, Theorem 1.3].
  8. Disjoint boundary components, $m \in (0, \infty]$, $\mathrm{Ric}_f^m \geq 0$, $H_f \leq 0$: warped product splitting theorem [28, Theorem 6.13].
  9. Disjoint boundary components, $m \in [-\infty, 1-n]$, $\mathrm{Ric}_f^m \geq \kappa$, $H_f \leq -\lambda$, $f < K$, where λ, κ satisfy a subharmonicity condition: warped product splitting theorem [29, Theorem 5.10].

Note how some of the above results require $f \leq K$ for a constant K. It turns out this condition is necessary, as we will show with a couple of examples. In general, when the potential function is unbounded many of the structure results of Riemannian geometry can fail. The first example is one of the most well known; the so called *Gaussian Soliton*.

Example 2.11 (Gaussian Soliton)**.** *Let $M = \mathbb{R}^n$ with the standard Euclidean metric, and let the function $f(p) = \frac{\lambda}{2}|p|^2$ for some $\lambda > 0$. Then* $\operatorname{Hess} f = \lambda g$ *and* $\operatorname{Ric}_f = \lambda g$.

This example highlights how a metric space may be non-compact with $\operatorname{Ric}_f^\infty \geq \lambda g$, though $\operatorname{Ric} \geq \lambda g$ forces M to be compact. This fact is known as Myers's theorem, and there is a Bakry-Émery version which works if f is bounded.

Using the distance function, we can construct another interesting manifold.

Example 2.12. *Let $M = \mathbb{H}^n$ be the hyperbolic space, i.e. the maximally symmetric, simply connected n-dimensional Riemannian manifold of constant negative sectional curvature. Fix any point $q \in M$, and let $f(p) = (n-1)\rho_q^2(p)$. The distance function ρ_q has a gradient with norm 1, so by the product rule*

$$\operatorname{Hess}\rho_q^2(x,y) = 2|\nabla\rho_q|^2 g(x,y) + 2\rho_q \operatorname{Hess}\rho_q \geq 2g(x,y) \tag{2.16}$$

Note that here, $g(x,y)$ acts as the identity tensor (i.e. the $\binom{2}{0}$-tensor having the identity matrix for its matrix of coefficients) since ρ_q, $\operatorname{Hess}\rho_q$ *are both positive and $|\nabla\rho_q| = 1$.*

This example shows that Theorem 2.2 does not necessarily hold for $\operatorname{Ric}_f^\infty \geq 0$. In fact even if we impose $\operatorname{Ric}_f^\infty \geq \lambda g > 0$, the theorem still may fail.

The necessity of a bound on f shows that for a general X one needs a condition which implies the boundedness of the potential when restricted to the special case that X is gradient. In [31, Section 6] a twisted product splitting for item (2) from the above list is obtained under a suitable condition on X.

Remark 2.13. *Unlike totally geodesic hypersurfaces (see remark 1.25), it is always possible to find minimal hypersurfaces in a 3 dimensional compact manifold [5]. This is also true for f-minimal hypersurfaces [15], but it is unknown whether we can always obtain an X-minimal hypersurface.*

This is not the only important modification of the Ricci tensor. For another version with applications to conformal geometry, see [2].

3 A Bakry-Émery Version of a Frankel-Type Theorem

The first result we prove is a generalization of the hypersurface case of Frankel's theorem [9] for X-Bakry-Émery curvature conditions. Frankel proved that any two closed, totally geodesic submanifolds of a complete manifold of positive sectional curvature whose dimensions sum to $\geq n$ must intersect. For hypersurfaces, Peterson and Wilhelm [25, Theorem 3] were able to weaken the curvature condition to require only positivity of the Ricci curvature and then showed that the result applied to all minimal hypersurfaces whether totally geodesic or not. More recently, Li and Wei [15] showed the result for Bakry-Émery curvature assuming the vector field was gradient.

Let $f : [-\varepsilon, \varepsilon] \times [0, \ell]$ be a variation of a unit speed curve $\gamma(\cdot) := f(0, \cdot)$ and define the arclength function

$$L(s) = \int_0^\ell \left| \frac{\partial f}{\partial t}(s,t) \right| dt \tag{3.1}$$

Synge's second variation of arclength formula states that whenever $\gamma(\cdot) = f(0, \cdot)$ is a stationary point of $L(s)$ then we have

$$\frac{d^2 L}{ds^2}(0) = -\int_0^\ell \mathrm{Sec}(V, \dot{\gamma}) dt + g\left(\dot{\gamma}, \left(\nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial s}\right)(0,t)\right)\Bigg|_0^\ell \tag{3.2}$$

where $V(t) = \frac{\partial f}{\partial s}(0,t)$ is the variation vector field.

Theorem 3.1. *Let M be an n-dimensional Riemannian manifold that admits an $m \in (0, \infty]$ and a C^1 vector field X such that the Bakry-Émery Ricci curvature obeys $\mathrm{Ric}_X^m > 0$. Let N_1 and N_2 be closed Bakry-Émery X-minimal hypersurfaces in M. Then N_1 and N_2 intersect.*

Proof. Let N_1, $N_2 \subset M$ be as above, and let $p_i \in N_i$ be points in the hypersurfaces which are closest to each other. By way of contradiction, assume that $\mathrm{dist}(p_1, p_2) =: \ell > 0$. Then choose, as in Frankel's formula, a non-trivial unit speed minimizing geodesic $\gamma : [0, \ell] \mapsto M$ from p_1 to p_2. Next, select an orthonormal frame at p_1 such that the n^{th} element is $\dot{\gamma}(0)$ and parallel-transport it along γ to define an orthonormal basis $\{E_1, \dots, E_{n-1}, E_n\}$ along γ with $E_n = \dot{\gamma}$. At the endpoints of γ, the basis vectors $E_1, \dots, E_{n-1}$ must be tangent to the hypersurfaces by construction. Now, pick variations $f_1, \dots, f_{n-1}$ with the property that $f_j(s, 0) \in N_1$, $f_j(s, \ell) \in N_2$ for sufficiently small s, and

$$\frac{\partial f_j}{\partial s}(0,t) = E_j \tag{3.3}$$

Summing the contributions of all $n-1$ such variations, then

$$\begin{aligned}\sum_{j=1}^{n-1}\frac{d^2L_j(0)}{ds^2} &= -\sum_{j=1}^{n-1}\int_0^\ell \mathrm{Sec}(E_j,\dot\gamma)dt + \sum_{j=1}^{n-1} g\left(\dot\gamma, \left(\nabla_{\frac{\partial f_j}{\partial s}}\frac{\partial f_j}{\partial s}\right)(0,t)\right)\Big|_0^\ell \\ &= -\int_0^\ell \mathrm{Ric}(\dot\gamma,\dot\gamma)dt + H(\ell) + H(0)\end{aligned} \tag{3.4}$$

by Synge's formula, where $H(0)$ denotes the mean curvature of N_1 at p_1 with respect to $\dot\gamma(0)$ and $H(\ell)$ denotes the mean curvature of N_2 at p_2 with respect to $-\dot\gamma(\ell)$, this orientation being consistent with our conventions in the case when N_1 and N_2 are boundary hypersurfaces for an interior region containing γ. Thus, we have

$$\begin{aligned}\sum_{j=1}^{n-1}\frac{d^2L_j(0)}{ds^2} &= -\int_0^\ell \mathrm{Ric}_X^m(\dot\gamma,\dot\gamma)dt + \int_0^\ell \left[\frac{1}{2}\mathcal{L}_X g(\dot\gamma,\dot\gamma) - \frac{1}{m}\left(g(X,\dot\gamma)\right)^2\right]dt \\ &\quad + H(\ell) + H(0) \\ &= -\int_0^\ell \mathrm{Ric}_X^m(\dot\gamma,\dot\gamma)dt + g(X,\dot\gamma)\Big|_0^\ell - \frac{1}{m}\int_0^\ell \left(g(X,\dot\gamma)\right)^2 dt + H(\ell) + H(0) \\ &= -\int_0^\ell \mathrm{Ric}_X^m(\dot\gamma,\dot\gamma)dt + H_X(\ell) + H_X(0) - \frac{1}{m}\int_0^\ell \left(g(X,\dot\gamma)\right)^2 dt \\ &< -\frac{1}{m}\int_0^\ell \left(g(X,\dot\gamma)\right)^2 dt\end{aligned} \tag{3.5}$$

where in the last step we used our curvature assumptions. All steps are valid when $m=\infty$ provided the right-hand side of the last line is interpreted as 0. Thus, in all cases, the base geodesic $\gamma(t) = f(0,t)$ must be unstable. This contradicts the assumption that γ is a minimizing curve between closed hypersurfaces. □

A special case is when X is tangent to N_1 and N_2 which are then minimal surfaces in the ordinary sense.

Example 3.2 (Sphere and Flat Torus)**.** *Let $M = \mathbb{S}^2$. This is a space with positive Ricci curvature everywhere, so any two minimal surfaces will intersect. All minimal surfaces are given by great circles, so clearly this holds. However, weakening the requirement to* $\mathrm{Ric}_X^m \geq 0$ *may cause the result to fail.*

Consider $M = \mathbb{T}^2$, the flat torus $[0,1] \times [0,1]/\{(x,0) \sim (x,1), (0,y) \sim (1,y)\}$. This is a compact manifold with 0 Ricci curvature everywhere, but distinct circles parallel to $(t,0)$ comprise a family of distinct non-intersecting minimal surfaces.

Remark 3.3. *The proof shows that stronger results can be obtained. For example, the last step of (3.5) follows provided only that $\int_0^\ell \mathrm{Ric}_X^m(\dot\gamma, \dot\gamma)dt - H_X(\ell) - H_X(0) > 0$ along each such curve γ. Further, for finite m we may replace the open inequality (< 0) by the closed one (≤ 0) in (3.5) unless X and γ are orthogonal all along γ.*

3.1 Mean curvature comparison

While Theorem 3.1 shows that any manifold of positive Bakry-Émery Ricci curvature with two compact minimal hypersurfaces may intersect, the same is not true for manifolds of merely non-negative Bakry-Émery Ricci curvature. However, if we assume a stronger mean curvature condition on the surface, it is possible to get a few structure results on manifolds with a single hypersurface in the non-negative and even the negative bounded below Bakry-Émery case.

These results were obtained for minimal surfaces in a manifold with Ricci curvature bounds in [4, Theorem 2.5] using a Bochner identity argument. The argument can be modified to work in the Bakry-Émery case using the following version of the Bochner identity (e.g., [11, Lemma 4]):

$$\Delta_X(|\nabla w|^2) = 2\,|\mathrm{Hess}\, w|^2 + 2\nabla_{\nabla w}(\Delta_X w) + 2\,\mathrm{Ric}_X^m(\nabla w, \nabla w) + \frac{2}{m}[X(w)]^2 \tag{3.6}$$

Let γ belong to a congruence of unit-speed geodesics parameterized by t (a family of integral curves of a non-vanishing vector field Γ such that $\nabla_{\dot\Gamma}\dot\Gamma = 0$), either issuing from a point or orthogonally from an initial hypersurface N at $t = 0$. Since such a congruence is irrotational, we can consider the hypersurfaces M_t defined by level sets $w = t$ at equal parameter values along different curves in the congruence (see Figure 4). We will need to use the behavior of infinitesimal area elements of N as they propagate along the congruence.

Definition 3.4 (Line element)**.** *The* Line element *ds of an n-dimensional manifold is the quadratic form*

$$ds^2 = \sum_{ij}^{n} g_{ij}\varphi^i \otimes \varphi^j \tag{3.7}$$

which is often identified with the metric itself, since the metric is completely determined by the line element.

For any $p \in M_t$ in a level set of the distance function, let $\{\varphi^1, \dots, \varphi^{n-1}, \varphi\}$ be a basis of $(T_pM)^*$ such that each φ^j is orthogonal to φ, and $\{\varphi^1, \dots, \varphi^{n-1}\}$ is a basis of $(T_pM_t)^*$. We can write

$$ds^2 = h_{nn} d\varphi^2 + 2\sum_{i=1}^{n-1} h_{in}(d\varphi \otimes d\varphi^i) + \sum_{i,j=1}^{n-1} h_{ij}(d\varphi^i \otimes d\varphi^j) \tag{3.8}$$

where h is a symmetric matrix with coefficients determined by the metric. h depends on $p \in M$, and the function $A = \sqrt{\det(h)}$ is the called area element function which we will use to measure how infinitesimal areas behave as the surface moves outwards. The key is that in the following, $h_{nn} = 1$ since the distance functions have unit gradient. Then if A is a function of only t and $h_{in} \equiv 0$, the metric splits as

$$ds^2 = dt^2 + A^2 g_{M_t} \tag{3.9}$$

In order to compute the area element, we will make use of the fact that the area element satisfies the first variation of area formula, relating mean curvature of the surface to this area element,

$$H = \left(\frac{d}{dt}A\right)/A \tag{3.10}$$

which shows that minimal surfaces (surfaces with 0 mean curvature) are local minimizers of area. So the problem of controlling the area element becomes an issue of controlling the mean curvature. In order to do this in the Bakry-Émery setting, we first verify a modified version of the Bochner identity that will control the weighted quantities. If $w = \rho_p$ is a distance function so that $|\nabla w| = |\nabla \rho_p| = 1$, we obtain

$$\begin{aligned}
\nabla_{\nabla\rho_p}\Delta_X\rho_p =& -|\mathrm{Hess}\,\rho_p|^2 - \mathrm{Ric}_X^m(\nabla\rho_p, \nabla\rho_p) - \frac{1}{m}[X(\rho_p)]^2 \\
=& -\left|\mathrm{tf\,Hess}\,\rho_p + \frac{1}{n-1}\left[\Delta_X\rho_p + X(\rho_p)\right] g_{M_t}\right|^2 \\
& - \mathrm{Ric}_X^m(\nabla\rho_p, \nabla\rho_p) - \frac{1}{m}[X(\rho_p)]^2 \\
=& -|\mathrm{tf\,Hess}\,\rho_p|^2 - \frac{1}{(n-1)^2}\left((\Delta_X\rho_p)^2 + (X(\rho_p))^2 + (\Delta_X\rho_p)(X(\rho_p))\right) \\
& - \mathrm{Ric}_X^m(\nabla\rho_p, \nabla\rho_p) - \frac{1}{m}[X(\rho_p)]^2 \\
=& -|\mathrm{tf\,Hess}\,\rho_p|^2 - \mathrm{Ric}_X^m(\nabla\rho_p, \nabla\rho_p) - \frac{(\Delta_X\rho_p)^2}{(n+m-1)} \\
& - \frac{1}{(n-1)}\left(\sqrt{\frac{m}{n+m-1}}\Delta_X\rho_p + \sqrt{\frac{n+m-1}{m}}X(\rho_p)\right)^2
\end{aligned} \tag{3.11}$$

where tf Hess is the tracefree part of the hessian. If ψ is a $\binom{2}{0}$-tensor field over a $n-1$ dimensional manifold, the tracefree part is the tensor field

$$\operatorname{tf}\psi(x,y) = \psi(x,y) - \frac{1}{n-1}\operatorname{tr}(\psi) g(x,y) \tag{3.12}$$

This means in particular that if $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ is an orthonormal basis,

$$\sum_{i,j=1}^{n-1} \left(((\operatorname{tr}\psi) g(\mathbf{e}_i, \mathbf{e}_j)) \cdot \left(\psi(\mathbf{e}_i, \mathbf{e}_j) - \frac{1}{n-1}\operatorname{tr}(\psi) g(\mathbf{e}_i, \mathbf{e}_j) \right) \right) = 0 \tag{3.13}$$

explaining how we got to line three in the above calculation.

The mean curvature $H(t)$ of these level sets is governed by a scalar Riccati equation (sometimes called the Raychaudhuri equation)

$$\frac{dH}{dt} = -\operatorname{Ric}(\dot\gamma,\dot\gamma) - |\,\mathbf{II}\,|^2 = -\operatorname{Ric}(\dot\gamma,\dot\gamma) - |\operatorname{tf}\mathbf{II}\,|^2 - \frac{H^2}{(n-1)} \tag{3.14}$$

The Raychaudhuri equation can be rewritten in terms of Bakry-Émery quantities through the following computation

$$\begin{aligned}
\frac{dH}{dt} &= -\operatorname{Ric}(\dot\gamma,\dot\gamma) - |\operatorname{tf}\mathbf{II}\,|^2 - \frac{H^2}{(n-1)} \\
\frac{dH}{dt} - \frac{1}{2}\mathcal{L}_X g(\dot\gamma,\dot\gamma) &= -\operatorname{Ric}^m_X(\dot\gamma,\dot\gamma) - |\operatorname{tf}\mathbf{II}\,|^2 - \frac{1}{m}g(\dot\gamma,X)^2 - \frac{H^2}{(n-1)} \\
\frac{dH_X}{dt} &= -\operatorname{Ric}^m_X(\dot\gamma,\dot\gamma) - |\operatorname{tf}\mathbf{II}\,|^2 - \frac{1}{m}g(\dot\gamma,X)^2 - \frac{2}{n-1}H_X g(\dot\gamma,X) \\
&\quad - \frac{H_X^2}{(n-1)} - \frac{1}{(n-1)}g(\dot\gamma,X)^2 \\
\frac{dH_X}{dt} &= -\operatorname{Ric}^m_X(\dot\gamma,\dot\gamma) - |\operatorname{tf}\mathbf{II}\,|^2 - \frac{H_X^2}{(n+m-1)} \\
&\quad - \frac{1}{(n-1)}\left[\sqrt{\frac{m}{n+m-1}}H_X + \sqrt{\frac{n+m-1}{m}}g(\dot\gamma,X)\right]^2
\end{aligned} \tag{3.15}$$

which is in fact just equation (3.11), and another verification that $H_X = \Delta_X \rho_p$.

Using these equations, we prove three very similar structure lemmata for m positive, infinite, and negative. These lemmata control the mean curvature along geodesics leaving a hypersurface orthogonally.

Lemma 3.5 (Finite m structure result)**.** *Let there be an X, an $m \in (0, \infty)$, and a $\delta \in \{0, 1\}$ such that $\mathrm{Ric}_X^m \geq -(n-1)\delta g$. Denote the Bakry-Émery mean curvature of the initial hypersurface Σ at $t = 0$ by $H_X(0)$. If*

(i) $\delta = 0$, or

(ii) $\delta = 1$ and $(H_X(0))^2 \geq (n-1)(n+m-1)$,

then $H_X(t) \leq H_X(0)$ for all $t > 0$ for which H is defined. If $H_X(t_1) = H_X(0) \leq 0$ for any $t_1 > 0$ in the domain then $H_X(t) = H_X(0)$ for all $0 \leq t \leq t_1$ and then along $\gamma : [0, t_1] \mapsto M$ we have that $H_X(0) = -\sqrt{(n-1)(n+m-1)}\delta = H_X(t)$, $\mathrm{Ric}_X^m(\dot\gamma, \dot\gamma) = -(n-1)\delta g$, the tracefree part $\mathrm{tf}\,\mathbf{II}$ of the second fundamental form vanishes, and $g(\dot\gamma, X) = -m\delta\sqrt{\frac{n-1}{n+m-1}}$.

Proof. Defining $x(t) := \frac{H_X}{\sqrt{(n-1)(n+m-1)}}$ and $c := \sqrt{\frac{n-1}{n+m-1}}$, then equation (3.15) yields the inequality

$$x'(t) \leq c\left(\delta - x^2(t)\right) \tag{3.16}$$

Let $u(t) := e^{c\int_0^t x(s)ds}$. Then $x = \frac{u'}{cu}$, and the inequality $x' \leq c(\delta - x^2)$ becomes $\frac{u''}{cu} \leq c\delta$, or $u'' \leq c^2\delta u$. Let $v(t)$ be the unique solution of $v'' = c^2\delta v$ such that $v(0) = u(0) =: u_0 > 0$ and $v'(0) = u'(0) =: u_0'$. Let $T > 0$ be the first point at which either $u(t)$ or $v(t)$ has a zero or becomes undefined, and restrict attention to $t \in [0, T)$. Now let $y(t) := \frac{v'}{cv}$, and

$$[uv(x-y)]' = \left[uv\left(\frac{u'}{cu} - \frac{v'}{vv}\right)\right]' = \frac{1}{c}(vu'' - uv'') \leq 0 \tag{3.17}$$

Now integrating and using $x(0) = y(0)$ to find the constant of integration, we have that $uv(x-y) \leq 0$. Then $x(t) \leq y(t)$ for all $t \in [0, T)$.

When $\delta = 0$, then $v(t) = u_0't + u_0$ and so

$$x(t) \leq y(t) = \frac{u_0'}{c(u_0't + u_0)} = \frac{u_0'/(cu_0)}{\frac{u_0'}{u_0}t + 1} = \frac{x(0)}{cx(0)t + 1} \leq x(0) \text{ for } t \in [0, T) \tag{3.18}$$

When $\delta = 1$, then $v(t) = u_0 \cosh ct + \frac{u_0'}{c}\sinh ct$. If $[x(0)]^2 \geq 1$ as well, then we have

$$x(t) \leq y(t) = \frac{u_0 \sinh ct + \frac{u_0'}{c}\cosh ct}{u_0 \cosh ct + \frac{u_0'}{c}\sinh ct} = \frac{\tanh ct + \frac{u_0'}{cu_0}}{1 + \frac{u_0'}{cu_0}\tanh ct} = \frac{x(0) + \tanh ct}{1 + x(0)\tanh ct} \leq x(0) \tag{3.19}$$

for $t \in [0, T)$. This proves the inequality.

To prove the equality statement, observe that the last (i.e., rightmost) inequalities in (3.18) and (3.19) are strict for $t > 0$ unless $x(0) = 0$ in (3.18) or $x(0) = \pm 1$ in (3.19), and then

$$H_X(0) = \pm\sqrt{(n-1)(n+m-1)}\delta \tag{3.20}$$

and since $H_X(0) \le 0$, we need only consider the negative case, and so

$$H_X(t) = -\sqrt{(n-1)(n+m-1)}\delta \tag{3.21}$$

for some $t > 0$. But then there is a local minimum of H_X at some $0 < t_0 < t$ such that $H'_X(t_0) = 0$. Since the left-hand side of (3.15) vanishes there, the right-hand side must as well. But under the given conditions we have

$$-\operatorname{Ric}^m_X(\dot\gamma,\dot\gamma) - \frac{H_X^2}{(n+m-1)} \le 0 \tag{3.22}$$

and then

$$\operatorname{Ric}^m_X(\dot\gamma) = -\frac{H_X^2}{(n+m-1)} = -(n-1)\delta \tag{3.23}$$

at t_0, so

$$H_X(t) = -\sqrt{(n-1)(n+m-1)}\delta \tag{3.24}$$

at t_0 and thus for all t. But then $H_X(t)$ is constant and $\frac{d}{dt}H_X = 0$ for all t, so $\operatorname{Ric}^m_X(\dot\gamma) + \frac{H_X^2}{(n+m-1)}$, $|\operatorname{tf}\mathbf{II}|$, and the final term on the right of (3.15) in square brackets must each vanish independently throughout the domain. This can only happen when the conditions listed in the lemma hold. $\square$

Furthermore, this result extends to $m = \infty$ and $m < 1 - n$ when $\delta = 0$.

Lemma 3.6 (Structure result when $m = \infty$)**.** *Let there be an X such that $\operatorname{Ric}^\infty_X \ge 0$. Denote the Bakry-Émery mean curvature of the initial hypersurface Σ at $t = 0$ by $H_X(0)$. Then $H_X(t) \le H_X(0)$ for all $t > 0$ for which H is defined. If $H_X(t_1) = H_X(0) \le 0$ for any $t_1 > 0$ in the domain then $H_X(t) = H_X(0) = -g(\dot\gamma, X)$ for all $0 \le t \le t_1$ and along $\gamma : [0, t_1] \mapsto M$ we have that $\operatorname{Ric}^m_X(\dot\gamma,\dot\gamma) = 0$ and the tracefree part $\operatorname{tf}\mathbf{II}$ of the second fundamental form vanishes.*

Proof. In this case, equation (3.15) yields

$$\frac{dH_X}{dt} = -\operatorname{Ric}^\infty_X(\dot\gamma,\dot\gamma) - |\operatorname{tf}\mathbf{II}|^2 - \frac{1}{(n-1)}\left(H_X + g(\dot\gamma, X)\right)^2 \tag{3.25}$$

Since $\mathrm{Ric}_X^\infty \geq 0$, each term on the right is negative semi-definite, so $H_X(t) \leq H_X(0)$ for $t > 0$, and $H_X(t_1) = H_X(0)$ if and only if $\mathrm{Ric}_X^\infty(\dot\gamma, \dot\gamma) = 0$, $\mathrm{tf}\,\mathbf{II} = 0$, and $H_X + g(\dot\gamma, X) = 0$ for all $0 \leq t \leq t_1$. This last result implies that $H(0) = H_X(t) = -g(\dot\gamma, X)(t)$ for all $0 \leq t \leq t_1$. □

Lemma 3.7 (Negative m structure result)**.** *For $m \in (-\infty, 1-n)$, let there be an X such that $\mathrm{Ric}_X^m \geq 0$. Then*

$$e^{\frac{2}{n-1}\int_0^t g(\dot\gamma, X)dr} H_X(t) \leq H_X(0) \tag{3.26}$$

for all $t > 0$ for which H is defined. Further, if

$$e^{\frac{2}{n-1}\int_0^{t_1} g(\dot\gamma, X)dr} H_X(t_1) = H_X(0) \leq 0 \tag{3.27}$$

for any $t_1 > 0$ in the domain then $H_X(t) = H_X(0) = g(\dot\gamma, X) = 0$ for all $0 \leq t \leq t_1$ and along $\gamma : [0, t_1] \mapsto M$ we have that $\mathrm{Ric}_X^m(\dot\gamma, \dot\gamma) = 0$ and the tracefree part $\mathrm{tf}\,\mathbf{II}$ of the second fundamental form vanishes. In particular these equality statements will also hold in the case that $H_X(0) = 0$ and $H_X(t_1) = 0$ for any $t_1 > 0$ in the domain.

Proof. In this case, equation (3.15) yields

$$\begin{aligned}
\frac{dH_X}{dt} &= -\mathrm{Ric}_X^m(\dot\gamma, \dot\gamma) - |\mathrm{tf}\,\mathbf{II}|^2 - \frac{1}{m} g(\dot\gamma, X)^2 \\
&\quad - \frac{2}{n-1} H_X \cdot g(\dot\gamma, X) - \frac{H_X^2}{(n-1)} - \frac{g(\dot\gamma, X)^2}{(n-1)} \\
\frac{2}{n-1} H_X \cdot g(\dot\gamma, X) + \frac{dH_X}{dt} &= -\mathrm{Ric}_X^m(\dot\gamma, \dot\gamma) - |\mathrm{tf}\,\mathbf{II}|^2 \\
&\quad - \frac{1}{(n-1)} g(\dot\gamma, X)^2 - \frac{1}{m} g(\dot\gamma, X)^2 - \frac{H_X^2}{(n-1)} \\
e^{\frac{-2}{n-1}\int_0^{\cdot} g(\dot\gamma, X)dr} \left(e^{\frac{2}{n-1}\int_0^{\cdot} g(\dot\gamma, X)dr} H_X \right)' &= -\mathrm{Ric}_X^m(\dot\gamma, \dot\gamma) - |\mathrm{tf}\,\mathbf{II}|^2 \\
&\quad - \frac{n+m-1}{m(n-1)} g(\dot\gamma, X)^2 - \frac{H_X^2}{(n-1)}
\end{aligned} \tag{3.28}$$

and since $\mathrm{Ric}_X^m \geq 0$ and $m < 1-n$, each term on the right is non-positive, so

$$e^{\frac{-2}{n-1}\int_0^s g(\dot\gamma, X)dr} H_X(s) \leq e^{\frac{-2}{n-1}\int_0^t g(\dot\gamma, X)dr} H_X(t) \leq H_X(0) \tag{3.29}$$

for $s > t > 0$, and then $e^{\frac{2}{n-1}\int_0^{t_1} g(\dot\gamma, X)dt} H_X(t_1) = H_X(0)$ if and only if $\mathrm{Ric}_X^m(\dot\gamma, \dot\gamma) = 0$, $\mathrm{tf}\,\mathbf{II} = 0$, $g(\dot\gamma, X) = 0$, and $H_X = 0$ for all $0 \leq t \leq t_1$. This last result implies that $H(0) = H_X(t) = g(\dot\gamma, X) = 0$ for all $0 \leq t \leq t_1$. □

3.2 Manifolds with a Single Hypersurface

We first show one immediate consequence of Frankel's theorem that applies to a manifold with a single minimal hypersurface.

Corollary 3.8. *Let M be a complete manifold with a vector field X and an $m \in (0, \infty]$ such that $\mathrm{Ric}_X^m > 0$ pointwise on M, and such that M has a closed Bakry-Émery X-minimal surface N. Then the homomorphism $i_* : \pi_1(N) \mapsto \pi_1(M)$ induced by inclusion is surjective.*

If $N \subset M$ is a hypersurface, there is a map $i : N \mapsto M$ which is injective. We define i_* as the corresponding map between the fundamental groups, which sends an equivalence class of curves to the equivalence class of those same curves as they sit in M, and this is what is meant when we say i_* is "induced by inclusion". Observe that i_* need not be injective.

Proof of 3.8. Suppose i_* is not surjective. Then there is a class $[c'] \in \pi_1(M, p)$ of loops based at $p \in N$ which cannot be deformed to lie in N, and for which c' is a length-minimizing representative. If we further minimize over all p in the closed submanifold N, there will be a shortest non-trivial loop c of length $L(c) > 0$. We may pass to the universal covering space $\bar{M}$, with Riemannian metric $\bar{g}$ and vector field $\bar{X}$ on $\bar{M}$ defined by pullback. Then the m-Bakry-Émery Ricci tensor on $\bar{M}$ will obey $\mathrm{Ric}_X^m(\bar{g}) > 0$, and there will be two disjoint copies of the Bakry-Émery X-minimal surface N joined by a minimal geodesic of length $L(c) > 0$. By Theorem 3.1 these two hypersurfaces must intersect a contradiction. □

The structure results in section 3.1 allow us to obtain the non-negative Bakry-Émery version of 3.8, following the arguments of [4, Theorem 2.5]. When Ric_X^m is allowed to be 0 at points, the theorem can actually fail just as shown in example 3.2, but it turns out this failure only occurs when the manifold is separated by the surface. In this case, we can still constrain the structure on the fundamental group of the manifold.

Theorem 3.9. *Let M be a compact manifold with a vector field X and an $m \in [0, \infty]$ such that $\mathrm{Ric}_X^m \geq 0$ pointwise on M, and such that M has a closed embedded 2-sided X-minimal hypersurface N.*

a) If N is non-separating, then M is isometric to a mapping torus

$$\frac{N \times [0, a]}{(x, 0) \sim (y, a) \text{ iff } \phi(x) = y} \tag{3.30}$$

where $\phi : N \mapsto N$ is an isometry, and if $m > 0$ is finite then X is tangent to N, which is then a minimal hypersurface in the usual sense.

b) If N is separating, let D_1 and D_2 be the connected components of $M \backslash N$. Then for $j = 1, 2$, the maps

$$i_* : \pi_1(N) \mapsto \pi_1(\overline{D_j}) \ , \ i_* : \pi_1(N) \mapsto \pi_1(M) \ , \ \text{and } i_* : \pi_1(D_j) \mapsto \pi_1(M) \tag{3.31}$$

induced by inclusion are all surjective.

Proof. The proof of part (a) follows that of [4, Theorem 2.5], which proves the $m = 0$ case so without lose of generality we assume $m \in (0, \infty]$. Reference [4] gives two proofs. One proof follows from a splitting theorem, and for finite $m > 0$ it follows in the same way in our setting using the splitting theorem proved in [11]. The second proof follows directly from a Bochner formula.

Reference [4] begins by defining a smooth $f : M \backslash N \to \mathbb{R}$ which is identically zero on a collar neighborhood of one side of N and identically 1 on a collar neighborhood of the other side. By identifying the range modulo the integers, they obtain $f : M \mapsto \mathbb{S}^1$. This induces a map of fundamental groups $f_* : \pi_1(M) \mapsto \mathbb{Z}$ whose kernel $G < \pi_1(M)$ is the subgroup consisting of classes of loops that can be deformed away from one of the above neighborhoods. By passing first to the universal cover $\hat{M}$ of M and then identifying points modulo G, they obtain a cover of M which, roughly speaking, unwraps only those loops which must pass through both collar neighborhoods.

Then the argument proceeds by considering two adjacent preimages N_1 and N_2 in $\hat{M}$ of N under the projection $\pi : \hat{M} \mapsto M$. They let D be the compact connected region bounded by the preimages. Let ρ_{N_i} denote the distance map in $\hat{M}$ from N_i, $i \in \{1, 2\}$. We show in Lemma 3.5 (Lemma 3.6) that $\Delta_X \rho_{N_i} \leq 0$, given $\mathrm{Ric}_X^m \geq 0$ for $m > 0$ (respectively $\mathrm{Ric}_X^\infty \geq 0$) in the barrier sense, and hence $\Delta_X(\rho_{N_1} + \rho_{N_2}) \leq 0$. But $\rho_{N_1} + \rho_{N_2}$ will have an interior minimum and so must be constant on D. But since $\Delta_X \rho_{N_i} \leq 0$ for $i \in \{1, 2\}$, then $\Delta_X \rho_{N_i} = 0$, and from (3.11) we deduce that $\mathrm{Hess}\, \rho_{N_i} = 0$, that $\mathrm{Ric}_X^m = 0$, and that either $m = \infty$ or X is tangent to level sets of the ρ_{N_i}. This proves part (a).

Finally, for part (b), fix a j and let $\pi : \hat{D}_j \mapsto D_j$ be the universal cover of D_j. We claim that $\partial \hat{D}_j = \pi^{-1}(N)$ is connected. For the sake of contradiction, suppose $\partial \hat{D}_j$ is not connected, and let

$$d_0 = \inf \left\{ d(\Sigma', \Sigma'') : \Sigma' \text{ and } \Sigma'' \text{ are distinct components of } \partial \hat{D}_j \right\} \tag{3.32}$$

Now, there exist distinct components Σ_1 and Σ_2 that minimize this distance, and a geodesic γ in $\hat{D}_j$ from Σ_1 to Σ_2 that realizes it. By continuity, the distance functions d_1 and d_2 are well defined on a neighborhood of γ.

By the argument in part (a), it follows that a tubular neighborhood of γ is isometric to a product manifold $U \times (0, d_0)$ where U is an open ball centered at $\gamma(0)$ in Σ_1.

Let $\mathcal{U}$ be the set of points in Σ_1 which can be joined to Σ_2 by a geodesic of length d_0. Clearly $\mathcal{U}$ is open, and we can also show it is closed. Let $p \in \Sigma_1$ such that $\{p_m\} \subset \mathcal{U}$ converges to p, and let γ_m be a geodesic with length d_0 from p_m to Σ_2. By continuity there is a geodesic γ_0 from p to Σ_2 such that $\{\gamma_n\}$ converges to γ_0. γ_0 will have a length of d_0 as well, however it could happen that γ_0 now hits a different boundary component Σ_3 on the way. But then Σ_1 and Σ_3 would be less than d_0 apart which is not possible, so $\mathcal{U}$ is closed.

Thus $\mathcal{U} = \Sigma_1$, and so $\hat{D}_j$ is isometric to a product manifold $\Sigma_1 \times (0, d_0)$. This implies that M is diffeomorphic to $N \times \mathbb{S}^1$, but that would make N a non-separating hypersurface, which is a contradiction. From here, the arguments are identical to [4]. $\square$

Note that from Remark 3.3, Corollary 3.8 would also hold if the pointwise assumption $\mathrm{Ric}_X^m > 0$ were replaced by the condition that $\int_0^\ell \mathrm{Ric}_X^m(\dot{\gamma}, \dot{\gamma}) dt > 0$ on each closed geodesic loop c.

Another way to modify the corollary is to let the curvature have a negative lower bound. This was also done in the Ricci case in [4, Theorem 2.8].

Theorem 3.10. *Let M be a compact manifold with a C^1 vector field X and an $m \in (0, \infty)$ such that $\mathrm{Ric}_X^m \geq -(n+m-1)k$ pointwise on M for a $k > 0$, and such that M has a compact hypersurface N that bounds a connected region Ω in M. Suppose that $H_X \leq -(n+m-1)\sqrt{k}$, where the normal used to define H_X points into Ω. Then N is connected, and the map $i_* : \pi_1(N) \mapsto \pi_1(\bar{\Omega})$ induced by inclusion is surjective.*

As pointed out in [4, Remark 2.9], the requirement that N bounds a connected region could be altered; it follows from Theorem 3.9 that when the rest of the conditions are satisfied, N will necessarily bound a collection of connected regions.

Proof of 3.10. I claim that $\pi^{-1}(N)$ is connected in the universal cover $\hat{M}$ of M. If this is the case, let ℓ be a loop in $\bar{\Omega}$ starting at a point $p \in N$. Lift this to a curve $\hat{\ell}$ in $\hat{\Omega}$ joining points $p_1, p_2 \in \pi^{-1}(p)$ and let $\hat{\ell}'$ be a curve joining p_1 to p_2 contained in $\pi^{-1}(N)$. Since $\hat{\Omega}$ is simply connected, $\hat{\ell}$ and $\hat{\ell}'$ are homotopic, so $\pi(\hat{\ell}) = \ell$ and $\pi(\hat{\ell}')$ are as well, thus i_* is surjective.

To prove the claim, suppose that N was not connected, and let N_1 and N_2 be distinct connected components of N. Once again let γ be the unit speed geodesic that realizes the distance between these components. Let ϕ be a solution to $\phi'' - k\phi = 0$, with $\phi(0) = \phi(\ell) = 1$, and let $V_i(t) = \phi(t)E_i(t)$. Note that in particular, $\phi(t) = \frac{\cosh(\sqrt{k}(t-\ell/2))}{\cosh(\sqrt{k}\ell/2)}$. Then we compute

$$
\begin{aligned}
0 \leq& \sum_{j=1}^{n-1} \frac{d^2 L_j(0)}{ds^2} \\
=& \int_0^\ell \big((n-1)(\phi')^2 - \phi^2 \operatorname{Ric}(\dot\gamma,\dot\gamma)\big)\, dt + \sum_{i=1}^{n-1} \phi^2 \langle \nabla_{E_i} E_i, \dot\gamma\rangle \Big|_0^\ell \\
=& - m \int_0^\ell (\phi')^2 dt + \int_0^\ell \big((n+m-1)(\phi')^2 - \phi^2 \operatorname{Ric}(\dot\gamma,\dot\gamma)\big)\, dt + H(\ell) + H(0) \\
=& - \int_0^\ell \big((n+m-1)\phi\phi'' + \phi^2 \operatorname{Ric}_X^m(\dot\gamma,\dot\gamma)\big)\, dt + (n+m-1)\phi\phi' \Big|_0^\ell \\
&+ \int_0^\ell \phi^2 \left[\frac{1}{2}\mathcal{L}_X g(\dot\gamma,\dot\gamma) - \frac{1}{m}\left(g(X,\dot\gamma)\right)^2\right] dt - m\int_0^\ell (\phi')^2 dt + H(\ell) + H(0) \\
\leq& - \int_0^\ell \big((n+m-1)\phi\phi'' - \phi^2(n+m-1)k\big)\, dt + (n+m-1)\phi\phi' \Big|_0^\ell \\
&+ \phi^2 g(X,\dot\gamma)\Big|_0^\ell - \int_0^\ell (\phi^2)' g(X,\dot\gamma) dt - \int_0^\ell \phi^2 \frac{1}{m}\left(g(X,\dot\gamma)\right)^2 dt \\
&- m \int_0^\ell (\phi')^2 dt + H(\ell) + H(0) \\
=& - \int_0^\ell 2\phi\phi' g(X,\dot\gamma) dt - \int_0^\ell \phi^2 \frac{1}{m}\left(g(X,\dot\gamma)\right)^2 dt - m\int_0^\ell (\phi')^2 dt \\
&+ H_X(\ell) + H_X(0) + 2(n+m-1)\sqrt{k}\tanh\left(\sqrt{k}l/2\right) \\
<& - \int_0^\ell \left(\frac{1}{\sqrt{m}}\phi g(X,\dot\gamma) + \sqrt{m}\phi'\right)^2 - 2(n+m-1)\sqrt{k} + 2(n+m-1)\sqrt{k} \\
\leq& \ 0
\end{aligned}
\tag{3.33}
$$

which is a contradiction. So N must be connected, and similarly, $\pi^{-1}(N)$ is connected. $\square$

4 Splitting Theorems

The Frankel-type theorem works for non-compact manifolds, which raises the issue of whether there is some rigidity when the curvature conditions are marginally violated. As mentioned, a prototypical result of this type is the Croke-Kleiner warped product splitting theorem [6]. In this section we will go over Splitting theorems that apply to a manifold with disjoint boundary components, as well as a connected boundary. The proofs shown in this section work almost identically to Sakurai's papers [28] and [29] despite allowing a non-gradient vector field.

In the following, we denote by $\frac{\partial}{\partial t}$ a vector field orthogonal to level sets of distance functions from a surface. Specifically, it is a velocity vector field along an element of a congruence of geodesics leaving a surface orthogonally.

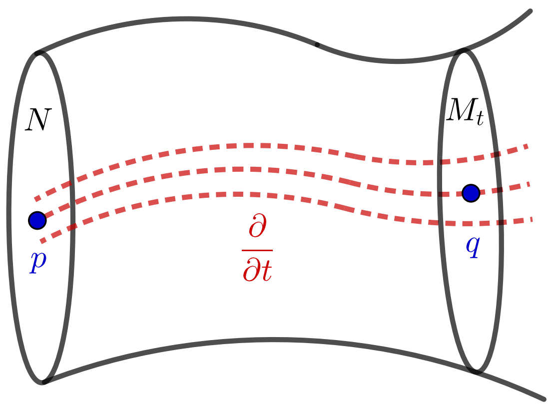

Figure 2: A manifold with boundary N, curves in $\frac{\partial}{\partial t}$, and a level set M_t of ρ_N

4.1 Two Boundary Components

Theorem 4.1. *Let M be a complete manifold-with-boundary, with boundary components N_1 and N_2, at least one of which is compact. Suppose that there is an $m \in (0, \infty)$ and an X such that $\mathrm{Ric}^m_X(M) \geq -(n-1)\delta$ for $\delta \in \{0, 1\}$. Suppose that the Bakry-Émery mean curvature of N_1 is $\leq -\sqrt{(n-1)(n+m-1)}\delta$ and of N_2 is $\leq \sqrt{(n-1)(n+m-1)}\delta$. Then M is isometric to $N_1 \times [0, \ell]$ with the metric $g_M = dt^2 + e^{2c\delta t} g_1$ where g_1 is the metric on N_1, $c := \sqrt{\frac{n-1}{n+m-1}}$, and $\mathrm{Ric}^m_{X^\sharp}(g_1) \geq -mc^2\delta$ where $X^\sharp$ denotes the restriction of X to TN_1. For $\delta = 0$, the splitting is a Riemannian product and the projection of X along $\frac{\partial}{\partial t}$ vanishes. For $\delta = 1$, the splitting is that of a warped product and the projection of X along $\frac{\partial}{\partial t}$ is constant, namely $g(X, \frac{\partial}{\partial t}) = mc$.*

Proof. Let $D = d_M(N_1, N_2)$, and define the distance functions $\rho_{N_i}(p) := d_M(p, N_i)$ for $i \in \{1, 2\}$. Let $F := \rho_{N_1} + \rho_{N_2}$, and let $\widetilde{\Omega}$ be the subset of $\operatorname{int} M$ containing neither cut points nor shadow points for either N_1 or N_2 (a shadow point p is one for which the minimizing curve from one boundary component to p touches the other boundary component *en route*). As stated in [6], F is smooth on $\widetilde{\Omega}$. Then $\Delta_X F = \Delta_X \rho_{N_1} + \Delta_X \rho_{N_2} \leq 0$ by Lemma 3.5, since the $\Delta_X \rho_{N_i}$ are the X-mean curvatures H_X of level sets of the functions ρ_{N_i}, $i \in \{1, 2\}$. But if F achieves a minimum of $F = D$ on $\widetilde{\Omega}$, then by the maximum principle F is constant on the interior of M, and then $\Delta_X F = 0$ so

$$\Delta_X \rho_{N_1} = -\Delta_X \rho_{N_2} = (-1)^i \sqrt{(n-1)(n+m-1)}\delta \tag{4.1}$$

for $i \in \{1, 2\}$.

We can now write $H_X := \Delta_X \rho_{N_1} = -\sqrt{(n-1)(n+m-1)}\delta$ and invoke the equality statement in Lemma 3.5. Then the tracefree part of $\operatorname{Hess} \rho_{N_1}$ must vanish and

$$\operatorname{Ric}_X^m(\nabla \rho_{N_1}, \nabla \rho_{N_1}) = -(n-1)\delta \tag{4.2}$$

Since $\operatorname{Hess} \rho_{N_1}$ is *scalar*, the level sets of ρ_{N_1} are *umbilic* in M. That is, for any q in a level set M_t of $\rho = t$, $\operatorname{Hess} \rho_{N_1}$ determines the second fundamental form, and by dropping an index we get the shape operator $S_{q,t}$ which is scalar in that for all $x \in T_q M_t$, $S_{q,t} x = a(q) x$ where $a : M \mapsto \mathbb{R}$ is a scalar function.

As a consequence the second fundamental form is just a multiple of the metric at each point,

$$\mathbf{II}_q(x, y) = g_M(S_{q,t} x, y) = a(q) g_M(x, y) \tag{4.3}$$

In particular, $\mathbf{II}(v, v) = a(q)$ for any unit vector v, so the surface bends in the same way in all directions at q, and the trace of $\mathbf{II}$ is $(n-1)a(q)$, which is equal to the logarithmic derivative of the area element function $\sqrt{\det(h)}$ by the first variation of area formula. The twisted product splitting follows; for any $p \in N$, let $\gamma_p(t)$ be the geodesic leaving N_1 orthogonally. The metric is

$$g_M = dt^2 + \Phi^2(p, t) g_{N_1} \tag{4.4}$$

where $\Phi(p, t) = e^{\frac{1}{n-1} a(\gamma_p(t))}$. The lemma also yields that

$$\sqrt{\frac{m}{n+m-1}} \Delta_X \rho_{N_1} + \sqrt{\frac{n+m-1}{m}} X(\rho_{N_1}) = 0 \tag{4.5}$$

This means that $g_M(X, \nabla \rho_{N_1}) = m\delta \sqrt{\frac{n-1}{n+m-1}}$. Then with $H := \Delta \rho_{N_1}$ we can compute the value of the mean curvature,

$$\begin{aligned}
H &= \Delta_X \rho_{N_1} + g(X, \nabla \rho_{N_1}) \\
&= H_X + g(X, \nabla \rho_{N_1}) \\
&= -\sqrt{(n-1)(n+m-1)}\delta + m\sqrt{\frac{n-1}{n+m-1}}\delta \\
&= -\frac{(n-1)^{3/2}\delta}{\sqrt{n+m-1}} \\
&= -(n-1)c\delta
\end{aligned} \tag{4.6}$$

Integrating the mean curvature along a geodesic γ_p leaving N_1 orthogonally, we obtain that for all points the function is a sum of a function of t and a function of p

$$a(\gamma_p(t)) = -(n-1)c\delta t + C(p) \tag{4.7}$$

where $C : M \mapsto \mathbb{R}$ is a constant of integration at each point. Thus,

$$\Phi^2(p,t) = e^{-2c\delta t} e^{\frac{2}{n-1}C(p)} \tag{4.8}$$

is separated into a product of functions depending on t and p, so the metric on M splits as a warped product

$$ds^2 = dt^2 + e^{-2c\delta t}(e^{\frac{2}{n-1}C(p)} g_1) \tag{4.9}$$

for $t \in [0, \ell]$. Without loss of generality, we can let $C(p) = 0$ since $e^{\frac{2}{n-1}C(p)} g_1$ is another metric on N_1, so the splitting holds just as well.

Finally, by assumption, for any $k > 0$ and any unit vector v in the same tangent space as $\nabla \rho_{N_1}$ we have that

$$-(n-1)\delta \leq \mathrm{Ric}_X^m(kv + \epsilon \nabla \rho_{N_1}, kv + \epsilon \nabla \rho_{N_1}) \tag{4.10}$$

for $\epsilon = \pm 1$. Expanding this product and using that $\mathrm{Ric}_X^m(\nabla \rho_{N_1}, \nabla \rho_{N_1}) = -(n-1)\delta$, we obtain that

$$0 \leq \pm 2k \,\mathrm{Ric}_X^m(v, \nabla \rho_{N_1}) + k^2 \,\mathrm{Ric}_X^m(v, v) \leq \pm 2k \,\mathrm{Ric}_X^m(v, \nabla \rho_{N_1}) + k^2 \,|\mathrm{Ric}_X^m(v, v)| \tag{4.11}$$

and dividing through by one factor of $k > 0$, we may now write this as

$$-k|\,\mathrm{Ric}_X^m(v, v)| \leq 2\,\mathrm{Ric}_X^m(v, \nabla \rho_{N_1}) \leq k|\,\mathrm{Ric}_X^m(v, v)| \tag{4.12}$$

Then $k \searrow 0$ implies that $\mathrm{Ric}_X^m(v, \nabla\rho_{N_1}) = 0$ for all v, so Ric_X^m is block-diagonal (meaning its matrix representation in a coordinate system containing $\frac{\partial}{\partial t}$ is block diagonal) and the condition on Ric_X^m descends to the restriction of Ric_X^m to the orthogonal complement of $\frac{\partial}{\partial t}$.

When $\delta = 0$, we obtain $\mathrm{Ric}_X^m(g_1) \geq 0$. When $\delta = 1$, a brief calculation using O'Neill's formula shows that the restriction of Ric_X^m to TN_1 equals $\mathrm{Ric}_{X^\sharp}^m(g_1) - (n-1)c^2 g_1$. Indeed, the formula says that for any $x, y \in T_pN_1$,

$$\begin{aligned} \mathrm{Ric}_X^m(g_M)(x,y) &= \mathrm{Ric}_{X^\sharp}^m(g_M)(x,y) \\ &= \mathrm{Ric}_{X^\sharp}^m(g_1)(x,y) - \frac{n-1}{e^{-ct}} \operatorname{Hess} e^{-ct}(x,y) \\ &= \mathrm{Ric}_{X^\sharp}^m(g_1)(x,y) - (n-1)c^2 g_1(x,y) \end{aligned} \tag{4.13}$$

and then by assumption,

$$\begin{aligned} \mathrm{Ric}_{X^\sharp}^m(g_1) - (n-1)c^2 &\geq -(n-1) \\ \mathrm{Ric}_{X^\sharp}^m(g_1) &\geq (n-1)\left(\frac{-m}{n+m-1}\right) \end{aligned} \tag{4.14}$$

Combining these cases and using $c^2 = \frac{n-1}{n+m-1}$, we obtain that $\mathrm{Ric}_{X^\sharp}^m(g_1) \geq -mc^2\delta$. □

The proofs in Section 3 do not permit m to be negative, however the splitting theorem carries through for negative and infinite m.

Theorem 4.2. *Let M be a complete manifold-with-boundary, with boundary components N_1 and N_2, at least one of which is compact. Suppose that $\mathrm{Ric}_X^m(M) \geq 0$ where $m \in [-\infty, 1-n)$. Further suppose that the Bakry-Émery mean curvatures of N_1 and of N_2 are each ≤ 0. Then M is isometric to the Riemannian product $N_1 \times [0, \ell]$ with the metric $ds^2 = dt^2 + g_1$ where g_1 is the metric on N_1, and $\mathrm{Ric}_{X^\sharp}^m(g_1) \geq 0$. The projection of X along $\frac{\partial}{\partial t}$ vanishes.*

Proof. The first paragraph of the proof of Theorem 4.1 carries over to this situation, while invoking Lemma 3.6 or Lemma 3.7 in place of Lemma 3.5 and concluding that

$$\Delta_X \rho_{N_1} = -\Delta_X \rho_{N_2} = 0 \tag{4.15}$$

Then, as above, we have that $H_X = 0$ at both boundaries and by the equality part of Lemma 3.6 or 3.7 we have that $\mathrm{Ric}_X^m(\nabla_X\rho_{N_1}, \nabla_X\rho_{N_1}) = 0$, $\operatorname{tf}\mathbf{II} = 0$, and $g(\nabla_X\rho_{N_1}, X) = 0$ as well. Thus the level sets of ρ_{N_1} are totally geodesic, so the metric splits as a product, and X is tangent to the level sets of ρ_{N_1}. Then, as above, the curvature condition descends to $\mathrm{Ric}_X^m(g_1) \geq 0$. □

4.2 Compact connected boundary

As with the original Croke-Kleiner theorem, an extension of this result produces a warped product splitting for manifolds M with a boundary consisting of a single connected component $N = \partial M$ and which have an asymptotic end. First we modify the necessary lemmata.

Let ν_p be the inner unit normal at a point $p \in N = \partial M$. Let $\phi_\delta(t)$ be the unique solution of the Jacobi problem $\phi''(t) - \delta\phi(t) = 0$, $\phi(0) = 1$, $\phi'(0) = \delta \in \{0, 1\}$. Also, define

$$\mathbf{s}(t,p) := \left|\det\left(D\exp_p^\perp\right)_{t\nu_p}\right| \tag{4.16}$$

to be the absolute value of the Jacobian determinant of the linearization about $t\nu_p \in T^\perp N$ of the normal exponential map $\exp_p^\perp$ at $p \in N$. Next, for each $p \in N$, let $\gamma_p : [0, \tau_p) : t \mapsto \exp_p^\perp(t\nu_p)$ be the unique geodesic with initial conditions $\gamma_p(0) = p$, $\dot{\gamma}_p = \nu_p$, and with domain $[0, T)$ where we define the focal radius $\tau_p \in (0, \infty]$ by $\tau_p := \sup\{t \mid t = \operatorname{dist}(p, \gamma_p(t))\}$. Define

$$f_p(t) = \int_0^t g(X(s), \dot{\gamma}_p(s))ds \tag{4.17}$$

Finally, for $p \in N$ define

$$\mathbf{s}_X(t,p) = e^{-f_p(t)}\,\mathbf{s}(t,p) \tag{4.18}$$

In consequence, then

$$H_X(t) = \frac{\mathbf{s}_X'(t,p)}{\mathbf{s}_X(t,p)} = \frac{\mathbf{s}'(t,p)}{\mathbf{s}(t,p)} - f_p'(t) \tag{4.19}$$

These definitions are very abstract upon first glance. Intuitively, $\mathbf{s}$ is the area element orthogonal to a *pencil* of geodesics. That is, just as explained in the paragraphs following Definition 3.4, this measures the area of a small surface as it moves outwards from the boundary. At $t = 0$, its value is $\mathbf{s}(0, p) = 1$. The equality with $H_X(t)$ comes from (3.10).

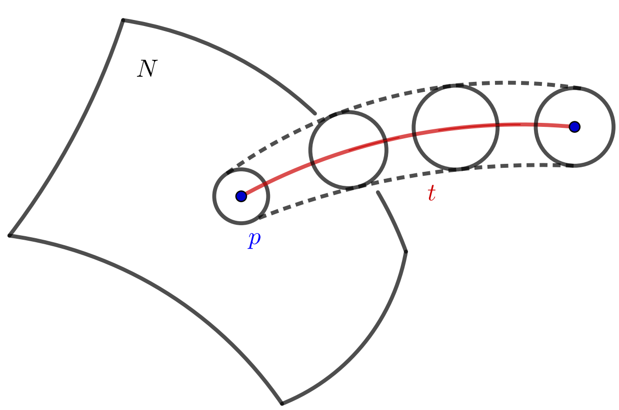

Figure 3: $\mathbf{s}$ measures an infinitesimal area normalized to start at 1

$\mathbf{s}_X(t)$ represents the Bakry-Émery weighted version of this area element, and $\phi_\delta(t)$ represents the mean curvature of the 'model space' at a distance t from the surface. By model space, we mean a space which is characterized by constant sectional curvature equal to δ for each t, which gives a mean curvature of $\phi_\delta(t)$ on the level sets.

We are going to use Jacobi fields in the following section, but given a geodesic there are many Jacobi fields one can define. Thus, we define a specific type of Jacobi field that has initial conditions that will be useful when dealing with a geodesic leaving a surface. A Jacobi field Y along γ_p is called an *N-Jacobi field* if

$$\begin{aligned} Y(0) &\in T_pN \\ (\nabla_{\dot\gamma}Y)(0)+S_{\nu_p}(Y(0)) &\in T_p^\perp N \end{aligned} \tag{4.20}$$

An N-Jacobi field is essentially just a Jacobi field which begins tangent to the surface and leaves with a behavior determined by the curvature at the initial point.

We will need the following standard comparison result for the *index form*, defined afterwards, for N-Jacobi fields. The index form can be thought of as a measure of how the sectional curvature determined by two vector fields along a geodesic differs from the product of the vector fields. The key is that N-Jacobi fields will minimize the index form given some initial condition. Throughout the rest of this section, we denote by $V'(t)$ the covariant derivative along $\dot\gamma$ of a vector field V defined along γ, $V'(t) = (\nabla_{\dot\gamma}V)(t)$ which gives another vector field along γ.

Lemma 4.3. *For a fixed $p \in N$, let $b \in (0,\infty]$ and let $\gamma_p : [0,b) \mapsto M$ be a geodesic issuing orthogonally from N with no focal point on $[0,t_0] \subset [0,b)$. Then for every smooth vector field V along γ_p there is a unique N-Jacobi field J along γ_p with $V(t_0) = J(t_0)$ such that*

$$I_N(J,J) \leq I_N(V,V) \tag{4.21}$$

where for smooth vector fields V and W along γ_p and $t \in [0,b)$ the index form I_N is

$$\begin{aligned} I_N(V,W) := \int_0^{t_0} &\left[g(V'(s),W'(s)) - g(\mathrm{Rm}(V(s),\dot\gamma_p(s))\dot\gamma_p, W(s))\right] ds \\ &+ g(S_{\nu_p}V(0),W(0)) \end{aligned} \tag{4.22}$$

Further, equality in (4.21) *holds if and only if $J \equiv V$.*

The index form comparison lemma is a standard result in Riemannian geometry, but this modification is less common. For the proof, see [27, Lemma 2.10 – Chapter III.2].

We can now prove the mean curvature estimates required for the connected boundary case. These estimates use different curvature conditions than were used in the previous subsections, and we need to use Jacobi field and index form arguments to obtain them.

Lemma 4.4 (Mean curvature comparison, finite $m > 0$)**.** *Let M be a complete non-compact n-dimensional manifold-with-boundary with a compact smooth boundary N. Suppose that $\mathrm{Ric}_X^m(M) \geq -(n+m-1)\delta$ for some $m \in (0,\infty)$, $\delta \in \{0,1\}$, and that the Bakry-Émery mean curvature with respect to the inward pointing normal is $H_X \leq -(n+m-1)\delta$. Let γ_p be the geodesic issuing orthogonally from N at p. Then for $\mathbf{s}_X$ as in equations* (4.16)–(4.19),

$$\frac{\mathbf{s}_X'(t,p)}{\mathbf{s}_X(t,p)} \leq -(n+m-1)\delta \tag{4.23}$$

Proof. Choose an orthonormal basis $\{e_i\}_{i=1}^{n-1}$ for T_pN. Let the E_i be parallel vector fields along γ_p obeying $E_i(0) = e_i$. Let $t_0 \in (0, \tau_p)$ be arbitrary. Let $Y_{t_0,i}(t)$ be the unique N-Jacobi field along γ_p with initial conditions $Y_{t_0,i}(t_0) = E_i(t_0)$ and $Y'_{t_0,i}(t_0) = S_{\nu_p} Y_{t_0,i}(t_0)$. Then define

$$\mathbf{r}_{t_0}(t) := \|Y_{t_0,1}(t) \wedge \cdots \wedge Y_{t_0,n-1}(t)\| \tag{4.24}$$

for $t \in (0, \tau_p)$, which may be thought of as the volume element of the $(n-1)$-flat (in other words, the $n-1$ dimensional Euclidean space) defined along γ_p by the Jacobi fields. This is in fact just another way to write $\mathbf{s}$, with $\mathbf{r}_{t_0}$ scaled to take the value 1 at $t = t_0$. Then we have the following standard calculation,

$$\begin{aligned}
\mathbf{r}_{t_0}'(t_0) &= \sum_{i=1}^{n-1} g(Y_{t_0,i}(t_0), Y'_{t_0,i}(t_0)) \\
&= \sum_{i=1}^{n-1} \left[\int_0^{t_0} \big(g(Y_{t_0,i}(t), Y'_{t_0,i}(t))\big)' \, dt + g(Y_{t_0,i}(0), Y'_{t_0,i}(0)) \right] \\
&= \sum_{i=1}^{n-1} \left[\int_0^{t_0} \big(g(Y'_{t_0,i}(t), Y'_{t_0,i}(t)) + g(Y_{t_0,i}(t), Y''_{t_0,i}(t))\big) \, dt + g(Y_{t_0,i}(0), Y'_{t_0,i}(0)) \right] \\
&= \sum_{i=1}^{n-1} \left[\int_0^{t_0} \left[g(Y'_{t_0,i}(t), Y'_{t_0,i}(t)) - g(Y_{t_0,i}(t), \mathrm{Rm}(Y_{t_0,i}(t), \dot\gamma_p(t))\dot\gamma_p(t))\right] dt \right. \\
&\qquad \left. + \, g(S_{\nu_p} Y_{t_0,i}(0), Y_{t_0,i}(0)) \right] \\
&= \sum_{i=1}^{n-1} I_N(Y_{t_0,i}(t_0), Y_{t_0,i}(t_0))
\end{aligned} \tag{4.25}$$

where we used that

$$\mathbf{r}_{t_0}(t_0) = \|E_1(t_0) \wedge ... \wedge E_{n-1}(t_0)\| = 1 \tag{4.26}$$

and thus the time derivative of $\mathbf{r}_{t_0}$ at $t = t_0$ can replace the logarithmic derivative

$$\frac{\mathbf{s}'(t_0)}{\mathbf{s}(t_0)} = \frac{\mathbf{r}'_{t_0}(t_0)}{\mathbf{r}_{t_0}(t_0)} = \mathbf{r}'_{t_0}(t_0) \tag{4.27}$$

If $W_i(t) := \frac{\phi_\delta(t)}{\phi_\delta(t_0)} E_i(t)$, then $W_i(t_0) = E_i(t_0) = Y_{t_0,i}(t_0)$ and so by Lemma 4.3 we have

$$\mathbf{r}'_{t_0}(t_0) = \sum_{i=1}^{n-1} I_N(Y_{t_0,i}, Y_{t_0,i}) \leq \sum_{i=1}^{n-1} I_N(W_i, W_i) \tag{4.28}$$

Then

$$\begin{aligned}
\frac{\mathbf{s}'_X(t_0,p)}{\mathbf{s}_X(t_0,p)} &= \mathbf{r}'(t_0,p) - f'_p(t_0) \\
&\leq \sum_{i=1}^{n-1} I_N(W_i, W_i) - f'_p(t_0) \\
&= \sum_{i=1}^{n-1} \int_0^t \left[g(W'_i(s), W'_i(s)) - g(\mathrm{Rm}(W_i(s), \dot\gamma_p(s))\dot\gamma_p, W_i(s)) \right] ds \\
&\quad + \sum_{i=1}^{n-1} g(S_{\nu_p} W_i(0), W_i(0)) - f'_p(t_0) \\
&= \int_0^{t_0} \left[(n-1) \left(\frac{\phi'_\delta(t)}{\phi_\delta(t_0)} \right)^2 - \mathrm{Ric}(\dot\gamma_p(t), \dot\gamma_p(t)) \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 \right] dt \qquad (4.29) \\
&\quad + H_p \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 - f'_p(t_0) \\
&= \int_0^{t_0} \left[(n+m-1) \left(\frac{\phi'_\delta(t)}{\phi_\delta(t_0)} \right)^2 - \mathrm{Ric}^m_X(\dot\gamma_p(t), \dot\gamma_p(t)) \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 \right] dt \\
&\quad - \int_0^{t_0} \left[m \left(\frac{\phi'_\delta(t)}{\phi_\delta(t_0)} \right)^2 - \left(f''_p(t) - \frac{f'^2_p(t)}{m} \right) \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 \right] dt \\
&\quad + (H_X)_p \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 + f'_p(0) \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 - f'_p(t_0)
\end{aligned}$$

where we used in intermediate steps that $W_i(t) = \frac{\phi_\delta(t)}{\phi_\delta(t_0)} E_i(t)$ and $E_i'(t) = 0$, so $W_i'(t) = \frac{\phi_\delta'(t)}{\phi_\delta(t_0)} E_i(t)$. Since $\mathrm{Ric}_X^m \geq -(n+m-1)\delta$ and $H_X \leq -(n+m-1)\delta$, we can combine and simplify a number of terms from the last equality of (4.29),

$$\begin{aligned}
&\int_0^{t_0} \left[(n+m-1) \left(\frac{\phi_\delta'(t)}{\phi_\delta(t_0)} \right)^2 - \mathrm{Ric}_X^m(\dot\gamma_p(t), \dot\gamma_p(t)) \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 \right] dt + (H_X)_p \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 \\
&\qquad \leq (n+m-1) \left(\int_0^{t_0} \left[\left(\frac{\phi_\delta'(t)}{\phi_\delta(t_0)} \right)^2 + \delta \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 \right] dt - \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 \delta \right) \\
&\qquad = \frac{(n+m-1)}{\phi_\delta^2(t_0)} \left(\phi_\delta'(t)\phi_\delta(t) \Big|_0^{t_0} - \phi_\delta^2(0)\delta \right) \\
&\qquad = (n+m-1) \left(\frac{\phi_\delta'(t_0)}{\phi_\delta(t_0)} - 2\delta \right)
\end{aligned} \tag{4.30}$$

where we used that ϕ_δ is a solution of the Jacobi equation $\phi_\delta'' - \delta\phi_\delta = 0$ with $\phi_\delta(0) = 1$ and $\phi_\delta'(0) = \delta$, and we integrated by parts. Next, we can rewrite the second derivative of f_x term in the second line of the last equality in (4.29), namely

$$\int_0^{t_0} f_p''(t) \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 dt = f_p'(t_0) - f_p'(0) \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 - 2 \int_0^{t_0} f_p'(t) \frac{\phi_\delta(t)\phi_\delta'(t)}{\phi_\delta^2(t_0)} dt \tag{4.31}$$

Then the entire second line of (4.29) becomes

$$\begin{aligned}
&- \int_0^{t_0} \left[m \left(\frac{\phi_\delta'(t)}{\phi_\delta(t_0)} \right)^2 - \left(f_p''(t) - \frac{f_p'(t)^2}{m} \right) \left(\frac{\phi_\delta(t)}{\phi_\delta(t_0)} \right)^2 \right] dt \\
&\qquad = -\frac{1}{\phi_\delta^2(t_0)} \int_0^{t_0} \left[m \left(\phi_\delta'(t) \right)^2 + 2 f_p'(t)\phi_\delta(t)\phi_\delta'(t) + \frac{f_p'(t)^2}{m} \left(\phi_\delta'(t) \right)^2 \right]^2 dt \\
&\qquad\quad + f_p'(t_0) - f_p'(0) \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2 \\
&\qquad = -\frac{1}{\phi_\delta^2(t_0)} \int_0^{t_0} \left(\sqrt{m}\phi_\delta'(t) + \frac{f_p'(t)}{\sqrt{m}} \phi_\delta(t) \right)^2 dt + f_p'(t_0) - f_p'(0) \left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)} \right)^2
\end{aligned} \tag{4.32}$$

Using (4.30) and (4.32), then (4.29) becomes

$$\frac{\mathbf{s}_X'(t_0, x)}{\mathbf{s}_X(t_0, x)} \leq (n+m-1) \left(\frac{\phi_\delta'(t_0)}{\phi_\delta(t_0)} - 2\delta \right) - \frac{1}{\phi_\delta^2(t_0)} \int_0^{t_0} \left(\sqrt{m}\phi_\delta'(t) + \frac{f_p'(t)}{\sqrt{m}} \phi_\delta(t) \right)^2 dt \tag{4.33}$$

Finally, using $\phi_\delta'' - \delta\phi_\delta = 0$ and the conditions $\phi_\delta(0) = 1$, $\phi_\delta'(0) = \delta$ we see that if $\delta = 1$ then $\phi_\delta(t) \equiv \phi_1(t) = e^t$, whence $\frac{\phi_\delta'(t_0)}{\phi_\delta(t_0)} = 1$. If instead $\delta = 0$, then $\phi_\delta(t) \equiv \phi_0(t) = 1$ so $\phi_0'(0) = 0$. Combining the cases, we have $\frac{\phi_\delta'(t_0)}{\phi_\delta(t_0)} = \delta$. Then (4.33) reduces to the result (4.23). □

If we were only interested in the $\delta = 0$ case, the proof could be considerably shortened. When $\delta = 0$, the solution of the boundary value problem for $\phi_\delta(t) = \phi_0(t)$ is simply $\phi_0(t) = 0$ and the penultimate iteration of the right-hand side in (4.29) gives the inequality

$$\begin{aligned}
\frac{\mathbf{s}_X'(t_0,p)}{\mathbf{s}_X(t_0,p)} &= -\int_0^{t_0} \mathrm{Ric}(\dot\gamma_p(t),\dot\gamma_p(t))dt + H_p\left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)}\right)^2 - f_p'(t_0) \\
&= \int_0^{t_0}\left[f_p''(t) - \frac{1}{m}\left(f_p'(t)\right)^2\right]dt + H_p\left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)}\right)^2 - f_p'(t_0) \\
&= -\frac{1}{m}\int_0^{t_0}\left(f_p'(t)\right)^2 dt + (H_X)_p\left(\frac{\phi_\delta(0)}{\phi_\delta(t_0)}\right)^2 - f_p'(0) \\
&\le -\frac{1}{m}\int_0^{t_0}\left(f_p'(t)\right)^2 dt
\end{aligned} \tag{4.34}$$

The $m \to \infty$ limit suggests the following, which is easily proven.

Lemma 4.5 (Mean curvature comparison, $m = \infty$)**.** *Let M have compact boundary N and let γ_p, $\mathbf{s}_X$ be as in Lemma 4.4. If $\mathrm{Ric}_X^\infty \ge 0$ and $H_X \le 0$ then*

$$\mathbf{s}_X'(t,p) \le 0 \tag{4.35}$$

along γ_p.

Proof. Repeat the derivation in (4.34), setting $\mathrm{Ric}_X^\infty(\dot\gamma,\dot\gamma) = \mathrm{Ric}(\dot\gamma,\dot\gamma) + f_p'' \ge 0$ at the beginning. □

The comparison in Lemma 4.4 requires initial conditions for a congruence issuing orthogonally from a surface. To study the single boundary component case, we will also need a lemma with initial conditions for a congruence issuing from a point p in the interior of M. Let ρ_p be the distance function from p, and for any $x \in T_pM$ with $|x| = 1$, define

$$\begin{aligned}
\widetilde{\mathbf{s}}(t,x) &:= \left|\det\left(D\exp_p\right)_{tx}\right| \\
\widetilde{\mathbf{s}}_X(t,x) &:= e^{-f_x(t)}\widetilde{\mathbf{s}}(t,x)
\end{aligned} \tag{4.36}$$

up to the focal radius τ_p, where

$$f_x(t) = \int_0^t g(X(s), \dot{\gamma}_x(s))ds \tag{4.37}$$

and γ_x is the geodesic with initial conditions $\gamma_x(0) = p$ and $\dot{\gamma}_x(0) = x$. Like definition (4.16), $\widetilde{\mathbf{s}}$ measures the area element of surface dragged along by a pencil of geodesics as it emanates outwards from the point.

Define $\widetilde{\phi}_\delta$ to be the unique solution to the Jacobi equation $\widetilde{\phi}''_\delta(t) - \delta\widetilde{\phi}_\delta(t) = 0$ with initial conditions $\widetilde{\phi}(0) = 0$ and $\widetilde{\phi}'(0) = 1$, so in particular $\widetilde{\phi}_0(t) = t$ and $\widetilde{\phi}_1(t) = \sinh t$. The derivation (4.29) holds in the same way here, except that the initial condition $\widetilde{\phi}(0) = 0$ eliminates two of the terms. We are left with

$$\begin{aligned}\frac{\widetilde{\mathbf{s}}'_X(t_0,x)}{\widetilde{\mathbf{s}}_X(t_0,x)} \le &\int_0^{t_0} \left[(n+m-1)\left(\frac{\widetilde{\phi}'_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2 - \mathrm{Ric}^m_X(\dot{\gamma}_x(t),\dot{\gamma}_x(t))\left(\frac{\widetilde{\phi}_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2\right] dt \\ &- \int_0^{t_0}\left[m\left(\frac{\widetilde{\phi}'_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2 - \left(f''_x(t) - \frac{f'^2_x(t)}{m}\right)\left(\frac{\widetilde{\phi}_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2\right] dt - f'_x(t_0)\end{aligned} \tag{4.38}$$

Combining the first two terms as in (4.30) and using (4.32) (with $\widetilde{\phi}_\delta$ replacing ϕ_δ), we get

$$\begin{aligned}\frac{\widetilde{\mathbf{s}}'_X(t_0,x)}{\widetilde{\mathbf{s}}_X(t_0,x)} \le &\frac{(n+m-1)}{\widetilde{\phi}^2_\delta(t_0)}\left[\int_0^{t_0}\left(-\widetilde{\phi}_\delta(t)\widetilde{\phi}''_\delta(t) + \widetilde{\phi}_\delta(t)^2\delta\right) dt + \widetilde{\phi}'_\delta(t)\widetilde{\phi}_\delta(t)\Big|_0^{t_0}\right] \\ &- \frac{1}{\widetilde{\phi}^2_\delta(t_0)}\int_0^{t_0}\left(\sqrt{m}\widetilde{\phi}'_\delta(t) + \frac{f'_x(t)}{\sqrt{m}}\widetilde{\phi}_\delta(t)\right)^2 dt - f'_x(0)\left(\frac{\widetilde{\phi}_\delta(0)}{\widetilde{\phi}_\delta(t_0)}\right)^2\end{aligned} \tag{4.39}$$

Now we use that $\widetilde{\phi}''_\delta(t) - \delta\widetilde{\phi}_\delta = 0$ with $\widetilde{\phi}_\delta(0) = 0$ and $\widetilde{\phi}'_\delta(0) = 1$ to obtain

$$\frac{\widetilde{\mathbf{s}}'_X(t_0,x)}{\widetilde{\mathbf{s}}_X(t_0,x)} \le (n+m-1)\frac{\widetilde{\phi}'_\delta(t_0)}{\widetilde{\phi}_\delta(t_0)} = \begin{cases}(n+m-1)/t_0, & \delta = 0\\ (n+m-1)\coth t_0, & \delta = 1\end{cases} \tag{4.40}$$

where to evaluate $\frac{\widetilde{\phi}'_\delta(t_0)}{\widetilde{\phi}_\delta(t_0)}$ we use that $\widetilde{\phi}_0(t) = t$ and $\widetilde{\phi}_1(t) = \sinh t$ as observed in the line preceding equation (4.38).

Lemma 4.6. *Let M be a complete n-dimensional manifold, and suppose that $\mathrm{Ric}_X^m(M) \geq -(m+n-1)\delta$ for $m \in (0,\infty)$. Then we have for $p \in M$ and a unit vector $x \in T_pM$,*

$$\Delta_X \rho_p(\gamma_x(t)) \leq \begin{cases} (n+m-1)/t, & \delta = 0, \\ (n+m-1)\coth t & \delta = 1 \end{cases} \tag{4.41}$$

Proof. Use equation (4.40) and observe that the drift Laplacian of the distance ρ_p is the logarithmic derivative of $\widetilde{\mathbf{s}}_X$; that is, $\Delta_X \rho_p = \frac{\widetilde{\mathbf{s}}'_X(t)}{\widetilde{\mathbf{s}}_X(t)}$. □

If $\delta = 0$ and $m = \infty$, then $\widetilde{\phi}_\delta = t$. Thus (4.38) is replaced by

$$\begin{aligned} \frac{\widetilde{\mathbf{s}}'_X(t_0,x)}{\widetilde{\mathbf{s}}_X(t_0,x)} &\leq \int_0^{t_0} \left[(n-1)\left(\frac{\widetilde{\phi}'_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2 - \mathrm{Ric}_X^\infty(\dot{\gamma}_x(t),\dot{\gamma}_x(t))\left(\frac{\widetilde{\phi}_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2 \right. \\ &\quad \left. + f''_x(t)\left(\frac{\widetilde{\phi}'_\delta(t)}{\widetilde{\phi}_\delta(t_0)}\right)^2\right] dt - f'_x(t_0) \\ &= \int_0^{t_0} \left[\frac{(n-1)}{t_0^2} - \mathrm{Ric}_X^\infty(\dot{\gamma}_x(t),\dot{\gamma}_x(t))\left(\frac{t}{t_0}\right)^2 + f''_x(t)\left(\frac{t}{t_0}\right)^2\right] dt - f'_x(t_0) \\ &= \frac{(n-1)}{t_0} + \frac{1}{t_0^2}\int_0^{t_0} f''_x(t)t^2 dt - f'_x(t_0) = \frac{(n-1)}{t_0} - \frac{2}{t_0^2}\int_0^{t_0} f'_x(t) t dt \\ &= \frac{(n-1)}{t_0} - \frac{2}{t_0^2}\int_0^{t_0} g(X(t),\dot{\gamma}(t)) t dt \end{aligned} \tag{4.42}$$

If $|X| \to 0$ along $\gamma(t)$ as $t \to \infty$ then we obtain that $\frac{\widetilde{\mathbf{s}}'_X(t_0,x)}{\widetilde{\mathbf{s}}_X(t_0,x)} \leq 0$ in the limit as $t_0 \to \infty$.

Lemma 4.7 ([31] Lemma 3.4)**.** *Let M have connected compact boundary N and suppose that $\mathrm{Ric}_X^m \geq 0$ for $m \in (-\infty, 1-n)$ and for all $p \in N$ such that the focal radius $\tau_p = \infty$, $g(X(t),\dot{\gamma}(t)) \cdot t \to 0$ along γ_p. Then $-\Delta_X b^{\gamma_p} \leq 0$ (defined in equation (4.46)) for all such $p \in N$.*

Proof. The proof of this comes from Wylie's Lemma 3.2 in [31], which in our setting is the following inequality. Let γ be a geodesic leaving N orthogonally with no focal point, p, $q \in M$ lie on γ, and let $t_0 = \rho_N q$. Then

$$(\Delta_X \rho_p)(q) \leq \frac{n-1}{e^{\left(\frac{2}{n-1}\int_0^{t_0} g(\dot{\gamma},X)ds\right)} \int_{t_0}^{\rho_p(q)+t_0} e^{\left(\frac{-2}{n-1}\int_0^{t} g(\dot{\gamma},X)ds\right)} dt} \tag{4.43}$$

Then since $g(\dot{\gamma}, X) \cdot t \to 0$, the integral term in the denominator

$$\int_{t_0}^{\rho_{\gamma(r)}(q)+t_0} e^{\left(\frac{-2}{n-1}\int_0^t g(\dot{\gamma},X)ds\right)} dt \tag{4.44}$$

goes to infinity as $r \to \infty$. Hence $-\Delta_X b^{\gamma_p} \leq 0$. □

With these estimates, we are ready prove the single boundary splitting theorem.

Theorem 4.8. *Let M be a complete non-compact n-dimensional manifold-with-boundary with a compact smooth boundary N. Then we have*

- *Let $m \in [-\infty, 1-n) \cup (0, \infty)$, $\mathrm{Ric}_X^m(M) \geq 0$, $H_X \leq 0$. In the case that $m = \infty$, suppose $|X| \to 0$ along geodesics γ_p for $p \in N$ such that $\tau_p = \infty$, and in the case that $m \in (-\infty, 1-n)$, $g(X, \dot{\gamma}_p) \cdot t \to 0$. Then M is isometric to $N \times [0, \infty)$ with the metric $ds^2 = dt^2 + g_1$ where g_1 is the metric on N and the projection of X along $\frac{\partial}{\partial t}$ vanishes (recall that $\mathrm{Ric}_X^{-\infty} = \mathrm{Ric}_X^{\infty}$).*
- *If $m \in (0, \infty)$, $\mathrm{Ric}_X^m(M) \geq -(n+m-1)$ and $H_X \leq -(n+m-1)$, then M is isometric to $N \times [0, \infty)$ with the metric $ds^2 = dt^2 + e^{2t} g_1$. The splitting is that of a warped product and the projection of X along $\frac{\partial}{\partial t}$ is constant, namely $g(X, \frac{\partial}{\partial t}) = m$.*

The proof is identical to the original one of [6], except that the Laplacian Δ is replaced by Δ_X when computing mean curvatures of level sets of distance functions, and of course our signs and conventions differ from those of [6]. For the reader's convenience, we provide the details.

Proof. First suppose $m \in (0, \infty)$, and $\delta = 1$ or 0. As usual, we have the distance function $\rho_N(p) = d_M(p, N)$, and by Lemma 4.4, $\Delta_X \rho_N \leq -(n+m-1)\delta$. Consider the set

$$\Omega := \{p \in N \mid \tau_p = \infty\} \tag{4.45}$$

for τ_p the conjugate radius at $p \in N$. Then Ω is non-empty because M is complete and non-compact, and Ω is closed in N by continuity of τ.

We claim that it is also open in N. Choose $p_0 \in \Omega$, then by definition of τ_{p_0}, we can construct a ray γ_{p_0} that leaves p_0 orthogonally and has no focal point. Let $U \subset M$ be the set of all the points that lie in a geodesic γ_{q_0} for some $q_0 \in \Omega$. For $q \in U$, let

$$b^{\gamma_{p_0}}(q) := \lim_{t\to\infty} \{t - d(q, \gamma_{p_0}(t))\} \tag{4.46}$$

be the Busemann function associated to γ_{p_0}, and let υ_q be an *asymptote*, which is an accumulation curve of a sequence of unit speed segments joining $q \in M$ to $\gamma_{p_0}(t)$ as $t \to \infty$.

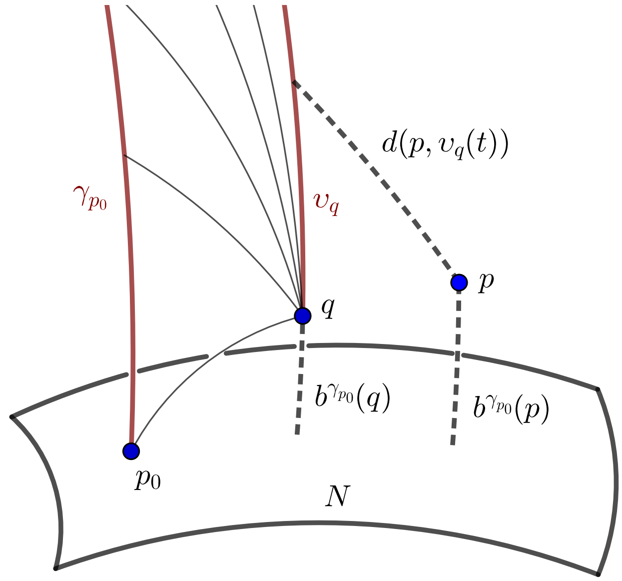

Figure 4: An asymptote υ_q of γ_{p_0} and Busemann function values

Then $\upsilon_q : [0, \infty) \mapsto M$ is a ray satisfying $b^{\gamma_{p_0}}(\upsilon_q(t)) = b^{\gamma_{p_0}}(\upsilon_q(0)) + t$ for all $t \in [0, \infty)$. Now, for any $x, y \in M$ we have $b^{\gamma}(x) \leq d(x, y) + b^{\gamma}(y)$, with equality if and only if there is an asymptote starting at x which passes through y (for these facts see for example [24, p 286]). We claim that the Busemann functions satisfy $-\Delta_X b^{\gamma_{p_0}} \leq (n+m-1)\delta$ weakly on U.

Define the Busemann support functions $b_t^{\gamma_{p_0}}(p) = b^{\gamma_{p_0}}(q) + t - d(p, \upsilon_q(t))$. Then

$$b_t^{\gamma_{p_0}}(p) \leq b^{\gamma_{p_0}}(p) - b^{\gamma_{p_0}}(\upsilon_q(t)) + t + b^{\gamma_{p_0}}(q) = b^{\gamma_{p_0}}(p) \tag{4.47}$$

so $b_t^{\gamma_{p_0}}(p)$ is a support function for $b^{\gamma_{p_0}}(p)$ whose level sets are smooth near $\upsilon_q(0) = q$. Then by Lemma 4.6, we have

$$-\Delta_X b_t^{\gamma_{p_0}}(p) = \Delta_X d(p, \upsilon_q(t)) \leq (n+m-1)\frac{\widetilde{\phi}'_\delta(t)}{\widetilde{\phi}_\delta(t)} \to (n+m-1)\delta \text{ as } t \to \infty \tag{4.48}$$

Finally, define $F = \rho_N - b^{\gamma}$. Combining (4.48) with the fact that $\Delta_X \rho_N \leq -(n+m-1)\delta$ and that F achieves its minimum of 0, it follows from the maximum principle that $\rho_N - b^{\gamma} = 0$ on U, which shows Ω is open in N, and the result now follows identically to the proof of Theorem 4.1.

If $m = \infty$ or $m \in (-\infty, 1-n)$, we proceed in the same fashion, instead making use of Lemma 3.6 and equation (4.42) or Lemmata 3.7 and 4.7 respectively. □

5 Conclusions and Future Research

We have shown a number of modifications of classic results involving Ricci curvature bounds to a Bakry-Émery setting. These structure results can be laid out in a progressive manor as we loosen the curvature conditions, so we will list them here. Let M be a manifold admitting a C^1 vector field X. Then if

- M has at least two embedded hypersurfaces N_1 and N_2 at least one of which is compact,
  - $m > 0$ and $\mathrm{Ric}_X^m > 0$, we get Theorem 3.1 generalizing [6].
  - $m > 0$ and $\mathrm{Ric}_X^m \geq 0$, we get Theorem 4.1. This result and the next are generalizations of theorems in [28] and [29], and it is proven with only minor modifications of the proofs in those papers.
  - $m \in [-\infty, 1-n)$ and $\mathrm{Ric}_X^m \geq 0$, we get Theorem 4.2.
- M has one compact embedded hypersurface N, and
  - $m > 0$ and $\mathrm{Ric}_X^m > 0$, we get Corollary 3.8.
  - $m > 0$, $\mathrm{Ric}_X^m \geq 0$, and M is compact, we get Theorem 3.9, generalizing [4].
  - $m > 0$, $\mathrm{Ric}_X^m \geq -(n+m-1)$, and M is compact, we get Theorem 3.10, again proven with only minor modifications to the theorems in [28] and [29].
  - $N = \partial M$ and either $m > 0$ and $\mathrm{Ric}_X^m \geq -(n+m-1)$ or $m \in [-\infty, 1-n) \cup (0, \infty)$ and $\mathrm{Ric}_X^m \geq 0$, we get Theorem 4.8.

Note that in the list, we leave out some of the conditions required for the result to order them by their condition on the Ricci curvature.

There are many more results with Ricci curvature bounds that could have useful Bakry-Émery analogues. The current success in updating theorems from Ricci bounds to Bakry-Émery bounds suggests there may be a general way to show equivalence of such results. A standard question of the field is whether there is a manifold which admits no metric of positive Ricci curvature, but does admit a metric of positive Bakry-Émery Ricci curvature. This would allow one to prove theorems about Bakry-Émery lower bounds that have no bare Ricci analogue. For manifolds with boundary, the exact statement of the problem follows.

Problem 5.1. *Let $m > 0$, $\delta \in \{0, 1\}$, and let (M, g_m) be a finite dimensional manifold-with-boundary admitting a smooth vector field X such that the Bakry-Émery Ricci curvature satisfies the bound $\mathrm{Ric}_X^m(g) \geq -(n+m-1)\delta g_m$, and the X-mean curvature on the boundary satisfies $H_X \leq -(n+m-1)\delta$. Then there is a metric g_0 on M such that $\mathrm{Ric}(g_0) \geq -(n-1)\delta g_0$ and $H \leq -(n-1)\delta$ on the boundary.*

Showing that the manifold admits a different metric satisfying a curvature bound is a weaker result than showing the same metric must obey that condition. One may still be able to generalize results about the topology of the space, but a splitting theorem for example gives both a topological splitting, and splitting of the metric.

In the meantime, we can point out another couple of notable examples of theorems that are waiting to be proven like the others in this paper. First, note that unlike the splitting theorems, the Frankel type theorem and related corollaries were not shown in the case of $m < 1 - n$. The precise statement of our Frankel type theorem with a negative m follows.

Conjecture 5.2. *Let M^n be a Riemannian manifold that admits $m \in (-\infty, 1-n)$ and a C^1 vector field X such that the Bakry-Émery Ricci curvature obeys $\mathrm{Ric}_X^m > 0$. Let N_1 and N_2 be closed Bakry-Émery X-minimal hypersurfaces in M. Then N_1 and N_2 intersect.*

For positive m, the theorem does not require any bound on X. It may be the case that it does not work in the $m < 1 - n$ case unless one finds appropriate conditions to control X.

After proving conjecture 5.2, one would immediately get a negative m version of Corollary 3.8 as well. Further, it is expected that using the negative m mean curvature comparison Lemmata 3.6 and 3.7, some versions of Theorems 3.9 and 3.10 are attainable.

In a recent article, Lai [12] showed a different structure result for a manifold with a boundary isometric to a product involving a sphere. It is very likely that this can be proven in the Bakry-Émery setting as well. It may be that slightly modified bounds on the curvature are needed. If everything works as expected, method of proof is going to be to stick close to [12], but this time using the eigenvalue estimate theorem from [15, Theorem 3].

Conjecture 5.3. *Let M^n be a smooth compact manifold with boundary N, and let f be a smooth function and $m > 0$. Suppose that*

- $\mathrm{Ric}_f^m \geq (n + m - 1)g$ *for* $m \in (0, \infty)$.
- *N is isometric to $\mathbb{S}^{k-1}(\sin\theta) \times (\widetilde{N}, g_{\widetilde{N}})$ where $\theta \in (0, \frac{\pi}{2})$ and $\widetilde{N}$ is an $n-k$ dimensional closed manifold.*
- *The second fundamental form h on N satisfies $h(w, w) \geq \cot\theta \, |w|^2 > 0$ for all w tangent to $\mathbb{S}^{k-1}$.*
- $H_f \leq (n + m - k)\tan\theta + (1 - m - k)\cot\theta \leq 0$.

where $\mathbb{S}^{k-1}(\sin\theta)$ is the $k-1$ sphere of radius $\sin(\theta)$. Then M is isometric to the double warped product $dr^2 + \sin^2(r)g_{\mathbb{S}^{k-1}} + \frac{\cos^2(r)}{\cos^2(\theta)}g_{\widetilde{N}}$ for $r \in [0, \theta]$, and necessarily $\mathrm{Ric}_f^m|_{\widetilde{N}} \geq \frac{n+m-k-1}{\cos^2\theta}g_{\widetilde{N}}$ in the case that $n - k \geq 2$.

In particular if $k = 2$, $n = 3$, $\widetilde{N} = \mathbb{S}^1$, and $\theta = \frac{\pi}{4}$, this result says that a manifold with some curvature conditions and a boundary isometric to the Clifford torus must be the hemisphere. Such a result could be useful as it allows one to 'glue' a standard hemisphere onto the manifold at the boundary, forming a complete spherical manifold without boundary, and proceed using other structure results on closed manifolds.

The interest in these results generally depends on how they can be applied to problems in physics or other areas of mathematics. As mentioned in the introduction, one of the ways these results could be useful in physics is in the the study of black holes. Roughly speaking, in relativity, space is modeled as a Riemannian manifold hence the applicability of results in this paper. Spacetime however is a pseudo-Riemannian manifold with a Lorentzian signature, which means the metric is no longer positive definite. The Bakry-Émery Ricci curvature is still an important tensor to study, which means there are plenty more generalizations of theorems to be proven in the Lorentzian setting as well.

Bibliography

[1] D Bakry and M Émery, *Diffusions hypercontractives.* In: Séminaire de probabilités, XIX, 1983/84, volume 1123 of Lecture Notes in Math., Springer, Berlin (1985) p177–206

[2] S Chang, M Gursky, and P Yang, *Conformal invariants associated to a measure* Proc. Natl. Acad. Sci. USA, 103(8) (2006) p2535–2540.

[3] J Cheeger and D Gromoll, *The splitting theorem for manifolds of non-negative curvature*, J Differential Geometry 6 (1971) p119–128.

[4] J Choe and A Fraser, *Mean curvature in manifolds with Ricci curvature bounded from below*, Canad Math Bull 46 (2003) p130–139.

[5] H Choi and R Schoen, *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent Math 81 (1985) p387–394.

[6] CB Croke and B Kleiner, *A warped product splitting theorem*, Duke Math J 67 (1992) p571–574.

[7] M do Carmo, *Riemannian Geometry*, Birkhäuser Boston (1992).

[8] F Fang, X Li, and Z Zhang, *Two generalizations of Cheeger-Gromoll splitting theorem via Bakry-Émery Ricci curvature*, Ann Inst Fourier (Grenoble) 59 (2009) p563–573.

[9] T Frankel, *Manifolds with positive curvature*, Pacific J Math 11 (1961) p165–174.

[10] Kobayashi & Nomizu, *Foundations of differential geometry*, Interscience (1963) p166.

[11] M Khuri, E Woolgar, and W Wylie, *New restrictions on the topology of extreme black holes*, Lett Math Phys 109 (2019) p661–673.

[12] M Lai, *A note on Hang-Wang's hemisphere rigidity theorem* Math. Z. 296 (2020) p901–909.

[13] J Lee, *Riemannian Manifolds: An Introduction to Curvature*, Germany, Springer New York (1997).

[14] M Li, *A sharp comparison theorem for compact manifolds with mean convex boundary*, J Geom Anal 24 (2014) p1490–1496.

[15] H Li and Y Wei, *f-Minimal surface and manifold with positive m-Bakry–Émery Ricci curvature*, J Geom Anal 25 (2015) p421-435.

[16] A Lichnérowicz, *Variétés kahlériennes à premiére classe de Chern non negative et variétés Riemanniennes à courbure de Ricci généralisée non negative*, J Differ Geom (1971) p47–94.

[17] A Lim, *Codimension one homology of noncompact space with nonnegative N-Bakry Émery Ricci curvature*, preprint [arxiv:2001.06028].

[18] J Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor* Comment, Math. Helv. 78(4) (2003) p865–883,

[19] H Kunduri & K Lucietti, *Classification of Near-Horizon Geometries of Extremal Black Holes*, Living Rev. Relativ. 16, 8 (2013)

[20] E Meinrenken, *Riemannian Geometry*, Lecture Notes, University of Toronto (2002)

[21] T Murphy, & F Wilhelm *Random manifolds have no totally geodesic submanifolds* Michigan Math. J. 68 (2) (2019) 323–335

[22] P Nyikos, *Various smoothings of the long line and their tangent bundles*, Advances in Mathematics 93 (1992) 129–213

[23] B O'neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press (1983)

[24] P Petersen, *Riemannian Geometry*, second edition, Graduate Texts in Mathematics GTM 171 Springer, New York (2006).

[25] P Peterson and F Wilhelm, *On Frankel's theorem*, Canad Math Bull 46 (2003) p130–139.

[26] Z Qian, *Estimates for weighted volumes and applications*, Quart J Math 48 (1997) p235–242.

[27] T Sakai, *Riemannian geometry*, Translations of mathematical monographs 149, AMS, Providence (1996).

[28] Y Sakurai, *Rigidity of manifolds with boundary under a lower Ricci curvature bound*, Osaka J Math 54 (2017) 85–119.

[29] Y Sakurai, *Rigidity of manifolds with boundary under a lower Bakry-Émery Ricci curvature bound*, Tohoku Math J 71 (2019) 69–109.

[30] W Wylie, *Sectional curvature for Riemannian manifolds with density*, Geometriae Dedicata 178 (2015) 151–169.

[31] W Wylie, *A warped product version of the Cheeger-Gromoll splitting theorem*, Trans Amer Math Soc 369 (2017) 6661–6681.