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THE UNIVERSITY OF ALBERTA

FINITELY GENERATED SOLUBLE GROUPS WITH IDENTITIES ON SUBGROUPS

BY

JAMES F. SEMPLE

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled FINITELY GENERATED SOLUBLE GROUPS WITH IDENTITIES ON SUBGROUPS submitted by James F. Semple in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

We call a group G an SP-group if there exist an integer $n > 1$ and distinct words u and v in the semigroup $\langle X_1, \ldots, X_n \mid X_i^2 = X_i \rangle$ such that the complexes $u(H_1, \ldots, H_n)$ and $v(H_1, \ldots, H_n)$ are equal; this may then be thought of as an identity on the subgroups of G. It is shown here that a finitely generated soluble SP-group is nilpotent-by-finite, and that any SP-group is either of restricted width or satisfies an identity in three variables.

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NOTATION

The following notation is used in the text of this thesis:

$H \wr K$, the wreath product of H and K

$H \rtimes K$, the split extension of H by K

$G' = [G, G]$, the derived subgroup of G

$G^{(\alpha)}$, the α^{th} term of the derived series of G

$\zeta(G)$, the centre of G

$\zeta_i(G)$, the i^{th} term of the upper central series of G, ($\zeta_0(G) = 1$)

$\gamma_i(G)$, the i^{th} term of the lower central series of G, ($\gamma_1(G) = G$)

max, the maximal condition on subgroups

max-n, the maximal condition on normal subgroups

$C_G(H)$, the centralizer of H in G

$N_G(H)$, the normalizer of H in G

INTRODUCTION

We call a group G an SP-group, for subgroup-permutable group, if there is some non-trivial identity on the subgroups of G one like $(H_1 H_2)^2 = (H_2 H_1)^2$ for all H_1, $H_2 < G$. Our interest is in classifying groups which have identities on their subgroups but, in the context of this thesis, restrict ourselves to the case that G is finitely generated and soluble.

If u and v are distinct words in the semigroup $\langle X_1, \ldots, X_n \mid X_i^2 = X_i \rangle$, we say $G \in SP(\{u\},\{v\})$ if $u(H_1, \ldots, H_n) = v(H_1, \ldots, H_n)$ for all sequences of subgroups $H_1, \ldots, H_n \leq G$. The somewhat cumbersome notation is maintained in this text because this problem is easily generalized in the following manner: classify groups G for which there exist finite disjoint sets U and V of words $\subset \langle X_1, \ldots, X_n \mid X_i{}^2 = X_i \rangle$ such that $G \in SP(U,V)$. In this greater generality, $G \in SP(U,V)$ means that for every n-tuple of subgroups $(H_1, \ldots, H_n)$ there exist $u \in U$ and $v \in V$ such that the complexes $u(H_1, \ldots, H_n)$ and $v(H_1, \ldots, H_n)$ are equal.

It should be emphasized here that throughout, when we write of identities on subgroups we mean indeed that the ***complexes*** are equal, not the weaker condition that the subgroups generated by $u(H_1, \ldots, H_n)$ and $v(H_1, \ldots, H_n)$ are equal.

The foundations of study of groups of this nature are due to Philip Hall, who considered groups with elliptically-embedded subgroups; certain such groups have the more restrictive property that for some integer n, $(HK)^n = (KH)^n$ for any subgroups H and K. Finitely generated and soluble groups of this property have recently been studied, albeit in more generality, by A.H. Rhemtulla and J.S. Wilson, ([3], [6]) who found them to be finite-by-nilpotent.

In this thesis we study the variation that G satifies an identity involving more than two subgroups at a time: groups $G \in SP(\{u\},\{v\})$ for which u and v are words in at least three letters.

In Chapter One, we reaffirm the notions introduced here, and prove some preliminary results; in particular it turns out that any group satisfying a subgroup identity satisfies one in at most three variables, and we exhibit such an identity on subgroups of the infinite dihedral group. Unfortunately D_∞ is not finite-by nilpotent, so we aim to show that SP groups have the weaker property of being nilpotent-by finite.

The second chapter deals with a pair of key lemmas which are used to eliminate certain classes of groups, and therefore allow us a better picture of the structure and rank of SP-groups. These lemmas are then employed in the final chapter, which consists primarily of a proof by induction on its solubility length that our SP-group is nilpotent-by-finite.

CHAPTER ONE

EXAMPLES AND PRELIMINARY RESULTS

In this chapter, we give an example of an SP-group which is not finite-by-nilpotent and we prove a number of preliminary results which are applicable to general groups with identities on their subgroups. We begin, however, by formalizing the concepts previously introduced.

[1.1] Definition: A subgroup H of a group G is said to be *elliptically embedded* in G if, for every $K < G$ there exists an integer n such that $\langle H, K \rangle = (HK)^n$. Groups all of whose subgroups are elliptically embedded are said to be of *restricted width*.

Examples of groups of restricted width are the so-called ***quasi-hamiltonian groups***, that is groups G with the property that if $H,K \leq G$ then $HK = KH$. These quasi-hamiltonian groups are, in fact, examples of groups of restricted width which have the further property that there exists an integer n such that $(KH)^n = (HK)^n$ for all pairs H and $K \leq G$. It is this further property that we care to generalize, in the following manner:

[1.2] Definition: Let G be a group, then we say $G \in SP(\{u\},\{v\})$ for words $u = u(X_1, X_2, \ldots, X_n) = X_{j_1} \cdots X_{j_s}$ and $v = v(X_1, X_2, \ldots, X_n) = X_{i_1} \cdots X_{i_r}$ if $u \neq v$ but the complexes $u(H_1, \ldots, H_n)$ and $v(H_1, \ldots, H_n)$ are equal for all sequences of subgroups $(H_1, \ldots, H_n)$; to guarantee that u and v are essentially different we further insist that $i_\ell \neq i_{\ell+1}$ and $j_\ell \neq j_{\ell+1}$ throughout. We refer to G

as an SP-group if there exist u and v with $G \in SP(\{u\},\{v\})$.

Therefore, particular examples of SP groups are quasi hamiltonian groups and groups of restricted width which have bounded ellipticity. It is our aim in this thesis to show that if G is a finitely generated and soluble SP group then G is nilpotent by-finite; we begin with the following

[1.3] Lemma. *Let G be a group, and suppose there exists an integer n such that $G \in SP(\{u\},\{v\})$ for some $u = u(X_1, X_2, \ldots, X_n)$ and $v = v(X_1, X_2, \ldots, X_n)$, $u \neq v$. If n is the least such integer, then $n = 2$ or 3.*

Proof: Assume $n > 3$, and define

$$u_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) = u(X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n).$$

Write each u_i in canonical form, replacing X_j^2 by X_j throughout.

By the minimality of n, we have that u_i and v_i are identically equal for all i; we claim this implies $u = v$ or, equivalently, that there is a unique word $w = w(X_1, X_2, \ldots, X_n)$ such that $w_i = u_i$ for $i = 1 \ldots n$

$$\begin{aligned} \text{Let } u_1 &= X_{m_{1,1}} X_{m_{1,2}} \cdots X_{m_{1,\ell(1)}} \\ u_2 &= X_{m_{2,1}} X_{m_{2,2}} \cdots X_{m_{2,\ell(2)}} \\ u_3 &= X_{m_{3,1}} X_{m_{3,2}} \cdots X_{m_{3,\ell(3)}} \\ u_4 &= X_{m_{4,1}} X_{m_{4,2}} \cdots X_{m_{4,\ell(4)}} \text{ et cetera,} \end{aligned}$$

and let $w = X_{a_1} X_{a_2} \ldots X_{a_k}$ have the property $w_i = u_i$ for all i.

We give the following algorithm to reconstruct w from the given set $\{w_i\}$:

$$u_1 \Longrightarrow a_1 = 1 \text{ or } a_1 = m_{1,1}$$

$$u_2 \Longrightarrow a_1 = 2 \text{ or } a_1 = m_{2,1}$$

$$u_3 \Longrightarrow a_1 = 3 \text{ or } a_1 = m_{3,1}$$

and these three conditions determine a_1 uniquely. Consider now u_{a_1}, and we see that $a_2 = m_{a_1,1}$. Henceforth, we proceed iteratively: to determine a_i given a_{i-1} and a_{i-2}, consider two rows other than $u_{a_{i-1}}$ and $u_{a_{i-2}}$, say u_r and u_s. Let X_{r_i} and X_{s_i}, respectively, be the first terms of u_r and u_s whose position in w has not been determined. Then $a_i = (r_i \text{ or } r)$ and $(s_i \text{ or } s)$; this defines a_i uniquely unless $s = r_i$ and $r = s_i$. In this case, check the first term of u_{a_i} following the term $X_{a_{i-1}}$. It is clear that these three conditions determine a_i uniquely, and hence that the pre-image, w, of the "projections" is unique. Therefore $u = v$, contrary to the requirement that $u \neq v$ if $G \in SP(\{u\},\{v\})$.

We illustrate the algorithm as follows:

$$\text{Let } u_1 = X_2X_3X_4$$

$$u_2 = X_1X_3X_4$$

$$u_3 = X_1X_2X_1X_4$$

$$u_4 = X_1X_2X_1X_3$$

If $u(X_1, X_2, X_3, X_4) = X_{a_1}X_{a_2}\ldots X_{a_\ell}$, then u_1 tells us $a_1 \in \{1,2\}$ (as does u_2), while u_3 determines that $a_1 = 1$.

Consider now u_1, and we see that $a_2 = 2$. Now we look at both u_3 and u_4

to see that a_3 is in both $\{3,1\}$ and $\{4,1\}$, that is we must have $a_3 = 1$. Finally we look again at u_3 and u_4 to see that a_4 is in both $\{3,4\}$ and $\{4,3\}$. This is inconclusive, so we check u_2 for the first entry not accounted for — in this case X_3. We continue in this manner to determine the unique word $v = X_1X_2X_1X_3X_4$ whose "projection by i" is, in each case, u_i.

[1.4]Remark. If G is finitely-generated and soluble, and $G \in SP(\{u\},\{v\})$ for some words u and v in two variables, then G is of restricted width. The result of Rhemtulla [3] implies, therefore, that G is finite-by-nilpotent. We apply a result due to P. Hall ([8], part I, p.117) which states that if for a group G there exists an integer i such that $\gamma_i(G)$ is finite, then there is an integer j such that $|G : \zeta_j(G)|$ is finite. We clearly have an ascending central series to $\zeta_j(G)$, so G is nilpotent-by-finite.

Therefore, we have in particular that finite-by-nilpotent groups are nilpotent-by-finite, and it follows that so too are finitely generated soluble SP-groups in the case that our words are in two variables. Thus, we need here address only the case that u and v are words in three variables.

[1.5]Lemma. *Let $G \in SP(\{u\},\{v\})$, with $u(X_1,X_2,X_3) = X_{i_1}X_{i_2}\cdots X_{i_m}$ and $v(X_1,X_2,X_3) = X_{j_1}X_{j_2}\cdots X_{j_n}$. Then we may assume that each of X_1, X_2 and X_3 appear the same number of times in the expressions for u and v.*

Proof: Without loss, assume that X_1 appears more often in $u(X_1,X_2,X_3)$ than in $v(X_1,X_2,X_3)$. Then $u(X_1,X_2,X_2) \neq v(X_1,X_2,X_2)$, (replacing X_2^2 by X_2 throughout,) so if we let $u'(X_1,X_2) = u(X_1,X_2,X_2)$, and similarly $v'(X_1,X_2) =$

$v(X_1, X_2, X_2)$ then $G \in SP(\{u'\}, \{v'\})$. Therefore G is of restricted width, so by the previous remark G is nilpotent-by-finite.

Corollary: If G is an SP-group for words $u(X_1, X_2, X_3) = X_{i_1} X_{i_2} \cdots X_{i_m}$ and $v(X_1, X_2, X_3) = X_{j_1} X_{j_2} \cdots X_{j_n}$ then G is nilpotent-by-finite or $m = n$.

A similarly simple but useful result is

[1.6]Lemma. *Let $G \in SP(\{u\}, \{v\})$ with $u(X_1, X_2, X_3) = X_{i_1} X_{i_2} \cdots X_{i_n}$ and $v(X_1, X_2, X_3) = X_{j_1} X_{j_2} \cdots X_{j_n}$. Then we may assume that $i_1 = j_1$ and $i_n = j_n$.*

Proof: Consider the case that $i_1 \neq j_1$. Without loss of generality, assume that $i_1 = 1$ and $j_1 = 2$. Once again if we let $u'(X_1, X_2) = u(X_1, X_2, X_2)$, and $v'(X_1, X_2) = v(X_1, X_2, X_2)$ and replace ${X_2}^2$ with X_2 as required, we find that $u'(X_1, X_2) \neq v'(X_1, X_2)$ and that $G \in SP(\{u'\}, \{v'\})$; hence, G is of restricted width, and is therefore nilpotent-by-finite.

We now prove the following result, which will be applied to prove that the infinite dihedral group is an SP-group:

[1.7]Lemma. *Let $G = D_\infty = \langle a, b \mid a^b = a^{-1}, b^2 = 1 \rangle$. If H is a subgroup of G which is not normal, then $H = \langle a^{r_1}, a^{s_1} b \rangle$ for some integers r_1 and s_1*

Proof: Let $R = \{r \in \mathbf{Z} \mid a^r \in H\}$, and $S = \{s \in \mathbf{Z} \mid a^s b \in H\}$. We have that $0 \in R$, so R is non-empty, and S is non-empty for otherwise $H \triangleleft G$.

Now $\{a^r \mid r \in R\} \leq H$, so there exists some $r_1 \in R$ with $\{a^r \mid r \in R\} = \langle a^{r_1} \rangle$. If $r_1 = 0$ then $H = \langle 1, a^{s_1} b \rangle$ and we are done. Otherwise, let s_1, $s_2 \in S$;

then $a^{s_1}ba^{s_2}b \in \langle a^{r_1}\rangle$, so

$$s_1 - s_2 \equiv 0\,(mod\, r_1).$$

If, on the other hand, $s_1 \in S$ and $s \equiv s_1(mod\, r)$ then $s \in S$. Therefore given any $s_1 \in S$, S then equals $\{s \in \mathbf{Z} \mid s \equiv s_1(mod\, r_1)\}$. From this we infer that

$$H = \langle a^{r_1}, a^{s_1}b\rangle = \langle a^{r_1}\rangle \rtimes \{1, a^{s_1}b\}.$$

This completes the proof of [1.7]

[1.8]**Lemma.** *There exist words u and v such that $D_\infty \in SP(\{u\},\{v\})$.*

Proof: Let

$$u(X_1, X_2, X_3) = (X_1X_2)^2X_3(X_1X_2),$$

$$v(X_1, X_2, X_3) = (X_1X_2)X_3(X_1X_2)^2;$$

we claim $D_\infty \in SP(\{u\},\{v\})$. Let $G = D_\infty$ and let H_1, H_2 and H_3 be any three subgroups of G. If any of the H_i is normal in G, then clearly $u(H_1, H_2, H_3) = v(H_1, H_2, H_3)$. Therefore we may assume no H_i is normal in G.

We must show that $u(H_1, H_2, H_3) = v(H_1, H_2, H_3)$ in the case that each $H_i = \langle a^{r_i}\rangle \rtimes \{1, a^{s_i}b\}$. We do this by showing first that we may assume each H_i has order two. Denote $A_i = \langle a^{r_i}\rangle$, $B_i = \{1, a^{s_i}b\}$, then $H_i = A_iB_i$. Now each $A_i \lhd G$ and, if any H_j is of infinite order we may replace G by G/A_j and each H_i by H_iA_j/A_j. It suffices therefore to show that $u(B_1, B_2, B_3) = v(B_1, B_2, B_3)$ that is, we need only to prove the result for assuming our subgroups are of order 2.

We show here that $u(B_1, B_2, B_3) \subseteq v(B_1, B_2, B_3)$, for the proof of the converse is similar:

Let U and V denote $u(B_1, B_2, B_3)$ and $v(B_1, B_2, B_3)$, respectively. Consider a word $w = b_1b_2b_1'b_2'b_3b_1''b_2'' \in U$, with each $b_j^{(n)} \in B_j$, for $n \in \{0, 1, 2\}$. If $b_3 = 1$

then $w \in V$, so we may assume that $b_3 = a^{s_3}b$; moreover, $b_1b_2b_1'b_2'$ is necessarily in B_1B_2 except in the cases:

$$(i) \qquad b_1 = 1,\ b_2 = a^{s_2}b,\ b_1' = a^{s_1}b,\ b_2' = 1$$

$$(ii) \qquad b_1 = 1,\ b_2 = a^{s_2}b,\ b_1' = a^{s_1}b,\ b_2' = a^{s_2}b$$

$$(iii) \qquad b_1 = a^{s_1}b,\ b_2 = a^{s_2}b,\ b_1' = a^{s_1}b,\ b_2' = 1$$

$$(iv) \qquad b_1 = a^{s_1}b,\ b_2 = a^{s_2}b,\ b_1' = a^{s_1}b,\ b_2' = a^{s_2}b$$

We shall make use of the following identity:

$$a^{s_2}ba^{s_1}ba^{s_3}b = a^{s_3}ba^{s_1}ba^{s_2}b.$$

Therefore, in the case

$$\begin{aligned}(i)\ b_1b_2b_1'b_2'b_3B_1B_2 &= a^{s_2}ba^{s_1}ba^{s_3}bB_1B_2\\ &= a^{s_3}ba^{s_1}ba^{s_2}bB_1B_2\\ &\subset B_3(B_1B_2)^2.\end{aligned}$$

$$\begin{aligned}(ii)\ b_1b_2b_1'b_2'b_3B_1B_2 &= a^{s_2}ba^{s_1}ba^{s_2}ba^{s_3}bB_1B_2\\ &= a^{s_2}ba^{s_3}ba^{s_2}ba^{s_1}bB_1B_2\\ &\subset a^{s_2}ba^{s_3}ba^{s_2}bB_1B_2\\ &\subset B_2B_3(B_1B_2)^2.\end{aligned}$$

$$\begin{aligned}(iii)\ b_1b_2b_1'b_2'b_3B_1B_2 &= a^{s_1}ba^{s_2}ba^{s_1}ba^{s_3}bB_1B_2\\ &\subset a^{s_1}bB_3(B_1B_2)^2, \text{ using } (i)\\ &\subset B_1B_3(B_1B_2)^2.\end{aligned}$$

$$(iv)\ b_1 b_2 b_1' b_2' b_3 B_1 B_2 = a^{e_1} b B_2 B_3 (B_1 B_2)^2, \text{ using}(ii)$$
$$\subset B_1 B_2 B_3 (B_1 B_2)^2.$$

In each case, we have $b_1 b_2 b_1' b_2' b_3 B_1 B_2 \subset B_1 B_2 B_3 (B_1 B_2)^2$, that is, $U \subseteq V$. The proof of the opposite inclusion is similar.Therefore $u(H_1, H_2, H_3) = v(H_1, H_2, H_3)$ for all subgroups H_1, H_2, and H_3.

Therefore, there exist words u and v such that $D_\infty \in SP(\{u\}, \{v\})$. One will note, however, that D_∞ is nilpotent-by-finite, but is not finite-by-nilpotent; hence, it is an example of a group which satisfies an identity on its subgroups, but for which not all subgroups are elliptically embedded.

CHAPTER TWO

RESTRICTING THE STRUCTURE OF SP-GROUPS

In this chapter we establish a pair of key lemmas to consider the structure of SP-groups. In Chapter Three we shall make inferences about the rank of G from Lemma [2.2],which states that our SP-groups are minimax. To prove this lemma we make use of the following theorem due to P.H. Kropholler [2].

[2.1]Theorem. *If G is a finitely generated soluble group, and if for no prime p does G have a section isomorphic to $C_p \wr C_\infty$, then G is minimax.*

[2.2]Lemma. *If G is a finitely generated soluble group, and $G \in SP(\{u\},\{v\})$ for some $u \neq v$ then G is minimax.*

Proof: By the Kropholler's theorem, it is sufficient to show that G has no section isomorphic to $C_p \wr C_\infty$; however, because $G \in SP(\{u\},\{v\})$ is a quotient- and subgroup-closed property it suffices to show that for distinct words u and v, $C_p \wr C_\infty \notin SP(\{u\},\{v\})$.

Assume that $C_p \wr C_\infty \in SP(\{u\},\{v\})$ for $u(X_1, X_2, X_3) = X_{i_1}X_{i_2}\cdots X_{i_m}$ and $v(X_1, X_2, X_3) = X_{j_1}X_{j_2}\cdots X_{j_m}$. With no loss in generality we may further assume that $i_1 = j_1 = 1$.

We claim first that $C_p \wr C_\infty$ is isomorphic to $\mathbf{F_p}\langle x\rangle \rtimes \langle t\rangle$, where x and t are of infinite order and the action of t on $a(x) \in \mathbf{F_p}\langle x\rangle$ is multiplication by x.

We begin by showing that the base group B of $C_p \wr C_\infty$ is isomorphic to $\mathbf{F_p}\langle x\rangle$. For each $r \in \{0,\ldots,p-1\}$ and each $s \in \mathbf{Z}$, define a permutation $\theta_{r,s}$ of

$C_p \times C_\infty$ in the following manner: for $(a^\ell, \mu^m) \in C_p \times C_\infty$,

$$\theta_{r,s}(a^\ell, \mu^m) = \begin{cases} (a^{\ell+s}, \mu^m) & \text{if } m = s; \\ (a^\ell, \mu^m) & \text{if } m \neq s; \end{cases}$$

The set of all $\theta_{r,s}$ generates the base group, which is clearly abelian. Let

$$B = \langle \theta_{r,s} \mid r \in \{0, \ldots p-1\}, s \in Z \rangle.$$

Define $\psi: B \longrightarrow \mathbf{F_p}\langle x \rangle$ by $\psi(\theta_{r_1,s_1} \cdots \theta_{r_n,s_n}) = \sum_{i=1}^n r_i x^{s_i}$. It is immediate that ψ is onto and homomorphic, while the facts that B is abelian and that $\theta_{u,s} \cdot \theta_{v,s} = \theta_{u+v,s}$ together yield that ψ is well-defined and one-to-one. Therefore $B \cong \mathbf{F_p}\langle x \rangle$.

For integers v define $\sigma_v \in C_p \wr C_\infty$ by $\sigma_v(a^\ell, \mu^m) = (a^\ell, \mu^{m-v})$. Then $\langle \sigma_v \mid v \in \mathbf{Z} \rangle = \langle \sigma_1 \rangle$ and, by definition of the wreath product,

$$C_p \wr C_\infty \cong B \rtimes \langle \sigma_1 \rangle.$$

Note that $(\theta_{r,s})^{\sigma_v} = \theta_{r,s+v}$. If we extend ψ to ψ': $C_p \wr C_\infty \rightarrow F_p\langle x \rangle \rtimes \langle t \rangle$ by $\psi'(\sigma_r) = t^r$, then we have our desired isomorphism from $C_p \wr C_\infty$ to $\mathbf{F_p}\langle x \rangle \rtimes \langle t \rangle$.

For $a_i(x) \in \mathbf{F_p}\langle x \rangle$ and $c_i = (a_i(x)(1 - x^{m_i}), t^{-m_i}) \in C_p \wr C_\infty$, denote

$$C_i = \langle c_i \rangle = \left\{ (a_i(x)(1 - x^{k_i m_i}), t^{-k_i m_i}) \mid k_i \in \mathbf{Z} \right\}$$

Consider a general term, w, of $C_{i_1} C_{i_2} \cdots C_{i_\ell}$:

$$w = \big(a_{i_1}(x)(1 - x^{k_1 m_{i_1}}), t^{-k_1 m_{i_1}}\big)\big(a_{i_2}(x)(1 - x^{k_2 m_{i_2}}), t^{-k_2 m_{i_2}}\big) \cdots$$
$$\cdots \big(a_{i_\ell}(x)(1 - x^{k_\ell m_{i_\ell}}), t^{-k_\ell m_{i_\ell}}\big)$$
$$= \big(a_{i_1}(x)(1 - x^{k_1 m_{i_1}}) + x^{k_1 m_{i_1}} a_{i_2}(x)(1 - x^{k_2 m_{i_2}}), t^{-k_1 m_{i_1} - k_2 m_{i_2}}\big) \cdots$$
$$\cdots \big(a_{i_\ell}(x)(1 - x^{k_\ell m_{i_\ell}}), t^{-k_\ell m_{i_\ell}}\big)$$
$$\vdots$$
$$= \big(a_{i_1}(x)(1 - x^{k_1 m_{i_1}}) + \cdots$$
$$\cdots + x^{k_1 m_{i_1} + \cdots + k_{\ell-1} m_{i_{\ell-1}}} a_{i_\ell}(x)(1 - x^{k_\ell m_{i_\ell}}), t^{-k_1 m_1 - \cdots - k_\ell m_{i_\ell}}\big)$$

Let us denote $\sum_{j=1}^{s} k_j m_{i_j}$ by $d(s)$, (with $d(0) = 0$), and, for $\ell \in \{1,2,3\}$ define $I_\ell = \{r \in \mathbf{Z} \mid a_{i_r} = a_\ell\}$. Consider now the component of w in the base group; it equals

$$a_1(x) \sum_{j \in I_1} \big(x^{d(j-1)} - x^{d(j)}\big) + a_2(x) \sum_{j \in I_2} \big(x^{d(j-1)} - x^{d(j)}\big) + a_3(x) \sum_{j \in I_3} \big(x^{d(j-1)} - x^{d(j)}\big).$$

Choose primes p_1, p_2, p_3, p_4 with $p_4 > p_3 > p_2 > p_1 > m$, and let $m_1 = p_2 p_3 p_4$, $m_2 = p_1 p_3 p_4$, $m_3 = p_1 p_2 p_4$ and $a_1(x) = 1$, $a_2(x) = x^{p_1+1}$, $a_3(x) = x^{p_2+1}$. Take $k_1 = \ldots = k_m = 1$, and let k'_1, k'_2, $\ldots k'_m$ be such that

$$\sum_{i=1,2,3} a_i(x) \sum_{j \in I_i} \big(x^{d(j-1)} - x^{d(j)}\big) = \sum_{i=1,2,3} a_i(x) \sum_{j \in I'_i} \big(x^{d'(j-1)} - x^{d'(j)}\big)$$

where the d', I'_i and k' of the right-hand side are analogous to the d, I_i and k of the left.

We aim first to show that there are $2m$ powers of x appearing in the left sum; certainly there are no more than $2m$ terms.

If there is to be cancellation, we must have a term x^r which is expressed as both $a_{\ell_i}(x)x^{d(s)}$ and $a_{\ell_j}(x)x^{d(t)}$. Let $a_{\ell_i}(x) = x^{b_i}$ and $a_{\ell_j}(x) = x^{b_j}$, where $b_i, b_j \in \{1, p_1+1, p_2+1\}$. Now $d(t) \equiv d(s) \equiv 0\,(mod\, p_4)$ implies $b_i \equiv b_j\,(mod\, p_4)$. The inequality $p_4 > p_3 > p_2 > p_1 > 1$ implies further that $a_{\ell_i}(x) = a_{\ell_j}(x)$.

Moreover $d : \{0, \ldots, m\} \to \mathbf{N}$ is monotonically increasing, so $d(s) = d(t)$ implies $s = t$. Therefore a term $a_{\ell_i}(x)x^{d(s)}$ can be cancelled only by subtracting $a_{\ell_i}(x)x^{d(s)}$, but the assumption $i_t \neq i_{t+1}$ rules out this possibility. Therefore, there are precisely $2m$ powers of x appearing in the sum on the left.

Note that p_4 divides each of the powers $d(j)$ and that the $a_i(x)$ have powers incongruent mod p_4. It follows that, for each i,

$$\sum_{j\in I_i}\left(x^{d(j-1)} - x^{d(j)}\right) = \sum_{j\in I'_i}\left(x^{d'(j-1)} - x^{d'(j)}\right)$$

We wish to show now that we must have $k'_1 = k'_2 = \ldots = k'_n = 1$ and $i_\ell = j_\ell$ for $\ell = 1\ldots n$. It suffices to show that $d(\ell) = d'(\ell)$ for $\ell = 1\ldots m$.

Assume, by way of contradiction, that $k'_1 \neq 1$; however $-a_1(x)x^{d'(1)}$ appears in the right-hand expression, so for some $t > 1$ we have $d'(1) = d(t)$, ie.,

$$m_1 k'_1 = m_1 + m_{i_2} + \ldots + m_{i_t}.$$

Now $i_2 \neq 1$, and p_{i_2} divides m_{i_ℓ} if $i_\ell \neq i_2$. If β is the number of $b \leq t$ for which $m_{i_\ell} = m_{i_2}$, then we have

$$m_{i_2}\beta \equiv 0\,(mod\, p_2)$$

However $gcd(m_{i_2}, p_{i_2}) = 1$ and $p_{i_2} > \beta \geq 0$ imply $\beta = 0$. Hence $m_1 k'_1 = m_1$. This implies that $k'_1 = 1$ and that $d(1) = d'(1)$. Conversely if $d(1) = d'(1)$ and

$d(1) = d'(t)$ then, because there is no cancellation on the left hand side and d is monotonically increasing, we must have $t = 1$.

We proceed by induction as follows:

Assume for $r \in \{1, \ldots, s\}$ that

$$d'(r) = d(t) \iff r = t$$

Now $d'(s+1) = d(t)$ for some $t \geq s+1$.

$$d'(s+1) = d'(s) + m_{j_{s+1}} k'_{s+1} = d(s) + m_{j_{s+1}} k'_{s+1}$$

Therefore $m_{j_{s+1}} k'_{s+1} = (m_{i_{s+1}} + m_{i_{s+2}} + \ldots + m_{i_t})$. If $j_{s+1} = i_{s+1}$ then $p_{i_{s+2}}$ divides $m_{j_{s+1}}$. Our approach is as with the case $k'_1 = 1$:

$$p_{i_{s+2}} \text{ divides } \underbrace{(1 + 1 + \ldots + 1)}_{<t} m_{i_{s+2}} \Rightarrow t = s+1$$

Therefore $k'_{s+1} = 1$ and $j_{s+1} = i_{s+1}$

If on the other hand $j_{s+1} \neq i_{s+1}$, we proceed similarly:

$p_{i_{s+1}}$ divides $m_{i_{s+1}}(1 + \ldots + 1)$ so $m_{j_{s+1}} k'_{s+1} = 0$, and therefore $k'_{s+1} = 0$. In this case

$$\left\langle \left(a_{j_{s+1}}(x)(1 - x^{k'_{s+1} m_{j_{s+1}}}), t^{-k_{s+1} m_{j_{s+1}}}\right)\right\rangle = 1$$

and we have fewer than $2m$ powers of x appearing on the right-hand side.

Therefore $m_{j_{s+1}} = m_{i_{s+1}}$. By induction, we have $j_s = i_s$ for $s = 1, \ldots, n$. This implies that $u(X_1, X_2, X_3) = v(X_1, X_2, X_3)$, as required. We conclude that for all u and v we have $C_p \wr C_\infty \notin SP(\{u\}, \{v\})$.

We will make use of the following lemma, which is a slight modification of Lemma 4 of [4] and its corollary:

[2.3]Lemma. *Let α be a non-zero algebraic number such that for some fixed β the equation*

$$\alpha^{\lambda_1} \pm \alpha^{\lambda_2} \pm \ldots \pm \alpha^{\lambda_m} = \beta$$

has solutions in rational integers $\lambda_1, \ldots, \lambda_m$ together with the following further property: For a given solution, if Λ is the set of those λ_i for which α^{λ_i} does not appear as many times with positive as negative coefficient, then $\lambda = max\{\lambda_i \in \Lambda\}$ takes on arbitrarily large positive values. If α and all its non-zero integral powers satisfy these hypotheses, then α is a root of unity.

Proof: As in ([4], Lemma 4) we have that for every absolute value $| \ |_v$ of $\mathbf{Q}(\alpha)$,

$$|\alpha|_v \leq f_v(m) = \begin{cases} m, & \text{if } v \text{ archimedean;} \\ max\{|r|_v{}^{-1}, r = 1, \ldots, m\}, & v \text{ non-archimedean.} \end{cases}$$

Now $|\alpha^k|_v \leq f_v(m)$ for all non-zero integers k, so taking k^{th} roots and letting $k \to \infty$ we find that $|\alpha|_v \leq 1$ for every absolute value of $\mathbf{Q}(\alpha)$; however taking $k \to -\infty$ we find that $|\alpha|_v \geq 1$ for every absolute value of $\mathbf{Q}(\alpha)$. It is a classical result of algebraic number theory that if α is algebraic and is not a root of unity then there exists an absolute value such that $|\alpha|_v \neq 1$. (See for example [1], p.109) Therefore α is a root of unity, as desired.

[2.4]Lemma. *Let α be algebraic over $\mathbf{Q}$, and let $G_1 = \mathbf{Q}\langle\alpha\rangle \rtimes \langle t \rangle$ with the action of t on an $f(\alpha) \in \mathbf{Q}\langle\alpha\rangle$ being multiplication by α. If $G \leq G_1$ is a subgroup containing $\mathbf{Z}\langle\alpha\rangle \rtimes \langle t \rangle$, and $G \in SP(\{u\}, \{v\})$ for some u and v, then some power of t centralizes G.*

Proof: The proof follows similar lines to the proof of [2.2]. We assume that $G \in SP(\{u\},\{v\})$ for some $u(X_1,X_2,X_3) = X_{i_1}X_{i_2}\cdots X_{i_n}$ and $v(X_1,X_2,X_3) = X_{j_1}X_{j_2}\cdots X_{j_n}$; without loss of generality we again assume that $i_1 = j_1 = 1$. For $f_i(\alpha) \in \mathbf{Q}\langle\alpha\rangle$ and $c_i = (f_i(\alpha)(1-\alpha^{m_i}), t^{-m_i}) \in G$, denote

$$C_i = \langle c_i \rangle = \left\{ (f_i(\alpha)(1-\alpha^{k_i m_i}), t^{-k_i m_i}) \mid k_i \in \mathbf{Z} \right\}$$

We consider a general term, w, of $C_{i_1}C_{i_2}\cdots C_{i_\ell}$:

$$\begin{aligned} w &= (f_{i_1}(\alpha)(1-\alpha^{k_1 m_{i_1}}), t^{-k_1 m_{i_1}})(f_{i_2}(\alpha)(1-\alpha^{k_2 m_{i_2}}), t^{-k_2 m_{i_2}})\cdots \\ &\qquad\qquad \cdots (f_{i_\ell}(\alpha)(1-\alpha^{k_\ell m_{i_\ell}}), t^{-k_\ell m_{i_\ell}}) \\ &= (f_{i_1}(\alpha)(1-\alpha^{k_1 m_{i_1}}) + x^{k_1 m_{i_1}}(f_{i_2}(\alpha)(1-\alpha^{k_2 m_{i_2}}), t^{-k_1 m_{i_1} - k_2 m_{i_2}})\cdots \\ &\qquad\qquad \cdots (f_{i_\ell}(\alpha)(1-\alpha^{k_\ell m_{i_\ell}}), t^{-k_\ell m_{i_\ell}}) \\ &\ \ \vdots \\ &= (f_{i_1}(\alpha)(1-\alpha^{k_1 m_{i_1}}) + \ldots \\ &\qquad \ldots + \alpha^{k_1 m_{i_1} + \ldots + k_{\ell-1} m_{i_{\ell-1}}} f_{i_\ell}(\alpha)(1-\alpha^{k_\ell m_{i_\ell}}), t^{-k_1 m_1 - \ldots - k_\ell m_{i_\ell}}) \end{aligned}$$

We maintain the notations for d, I_i, k, d', I'_i and k' from the proof of Lemma [2.2]. In partcular, $d(s) = \sum_{j=1}^{s} k_j m_{i_j}$, and the general form of the component of w in $Q(\alpha)$ is:

$$f_1(\alpha)\sum_{j\in I_1}(\alpha^{d(j-1)} - \alpha^{d(j)}) + f_2(\alpha)\sum_{j\in I_2}(\alpha^{d(j-1)} - \alpha^{d(j)}) + f_3(\alpha)\sum_{j\in I_3}(\alpha^{d(j-1)} - \alpha^{d(j)}).$$

We assume that for a choice of integers $k_1, \ldots k_n$ and elements $f_1(\alpha)$, $f_2(\alpha)$, $f_3(\alpha)$ of $\mathbf{Q}\langle\alpha\rangle$ there exist integers $k'_1, \ldots k'_n$ such that

$$\sum_{i=1,2,3} f_i(\alpha)\sum_{j\in I_i}(\alpha^{d(j-1)} - \alpha^{d(j)}) = \sum_{i=1,2,3} f_i(\alpha)\sum_{j\in I'_i}(\alpha^{d'(j-1)} - \alpha^{d'(j)})$$

For a given $k_1, \ldots k_n$ fix $k'_1, \ldots k'_n$ and set

$$\varphi_k(x) = \sum_{i=1,2,3} f_i(x) \sum_{j \in I_i} \left(x^{d(j-1)} - x^{d(j)}\right) - \sum_{i=1,2,3} f_i(x) \sum_{j \in I'_i} \left(x^{d'(j-1)} - x^{d'(j)}\right)$$

We use the expression φ_k to remind ourselves that φ is dependent upon our choice of the k_i.

In our proof of Lemma [2.2] we showed that if we choose $k_1 = \ldots = k_n = 1$, and our f_i appropriately then the corresponding $\varphi_k(x)$ is identically zero if and only if $k'_1 = \ldots = k'_n = 1$, viz. $u(X_1, X_2, X_3) = v(X_1, X_2, X_3)$; however inspection of the proof of [2.2] shows that we may relax our hypothesis somewhat:

Given primes $p_4 > p_3 > p_2 > p_1 > n$, choose n integers k_i each of the form $1 + b_i p_1 p_2 p_3 p_4$ for integers b_i.

G contains $\mathbf{Z}\langle\alpha\rangle \rtimes \langle t\rangle$, and in particular contains 1, α^{p_1+1}, and α^{p_2+1}; therefore, take $f_1(x) = 1$, $f_2(x) = x^{p_1+1}$, $f_3(x) = x^{p_2+1}$. If the corresponding $\varphi_k(x)$ is identically zero then $u(X_1, X_2, X_3) = v(X_1, X_2, X_3)$.

In particular if we choose any integers b_i with $1 < b_1 < \ldots < b_n$, and set $k_i = 1 + b_i p_1 p_2 p_3 p_4$, then we cannot have $\varphi_k(x)$ identically zero. Every $\varphi_k(\alpha) = 0$, and clearly as we take arbitrarily large values for the b_i we find the powers appearing in $\varphi_k(x)$ to grow arbitrarily large. Therefore, α meets the requirements of the hypotheses of Lemma [2.3].

Furthermore, for every non-zero ℓ we find that α^ℓ satisfies the hypotheses of this lemma: if $G \leq \mathbf{Q}\langle\alpha\rangle \rtimes \langle t\rangle$ is an SP-group, then so too will be $G \cap \mathbf{Q}\langle\alpha^\ell\rangle \rtimes \langle t^\ell\rangle$. If we now mimic the proof that α satisfies the hypotheses of the aforementioned lemma, we find that α^ℓ does, too.

Therefore, by Lemma [2.3] α is a root of unity. As the action of t on $\mathbf{Q}\langle\alpha\rangle$ is multiplication by α, it follows that a power of t centralizes G.

CHAPTER THREE

SP-GROUPS ARE NILPOTENT-BY-FINITE

In this chapter we prove by induction on its solubility length that a finitely-generated SP-group is nilpotent-by-finite. We begin with a number of lemmas which will be employed in the course of the main proof.

[3.1]Lemma. *Let H be a torsion-free nilpotent group. Then $\zeta(H)$ is isolated. (i.e., $H/\zeta(H)$ is torsion-free.)*

Proof: Assume $g^p \in \zeta(H)$, let $\zeta_1 \leq \zeta_2 \leq \cdots \leq H$ be the upper central series of H, and let i be the least integer such that $[g,\zeta_i] \neq 1$. Take $x \in \zeta_i$ so $[g,x] \neq 1$. Then

$$1 = [g^p, x] = \left([g,x]^{g^{p-1}} \cdots [g,x]\right)$$

but

$$[g,x] \in \zeta_{i-1}, \text{ so } [g,x]^g = [g,x].$$

Now $1 = [g,x]^p$ and the fact that H is torsion-free together imply that $[g,x] = 1$. Therefore $g \in \zeta(H)$.

An immediate corollary is that if H is torsion-free, then every central factor of H is torsion-free.

[3.2]Lemma. *Let H be a torsion-free finitely-generated nilpotent group. Then H has a central series with infinite-cyclic factors.*

Proof: Consider the upper central series for H. By lemma [3.1], each factor is torsion-free; moreover H satisfies *max*, so the upper central factors are in fact finitely-generated and torsion-free abelian. We refine this series to get one with infinite cyclic factors.

[3.3]Lemma. *If G is a polycyclic group, then G is poly(C_∞)-by-finite.*

Proof: Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$ be a finite series with cyclic factors. The proof is by induction on r, with the case $r = 1$ holding true trivially.

Assume that K is a poly(C_∞) normal subgroup of G_{r-1} with G_{r-1}/K finite. Let $K^\circ = \bigcap_{g \in G} K^g$, the core of K in G. Then G_{r-1}/K° is finitely-generated, torsion and soluble, so it is finite.

If G_r/G_{r-1} is finite, then the fact that K is poly(C_∞) implies that K° is poly(C_∞) and certainly $K^\circ \lhd G_r$, so K° is a poly(C_∞) group of finite index in G. If, alternatively, $G_r/G_{r-1} \cong C_\infty$, write $G/G_{r-1} = \langle xG_{r-1} \rangle$.

Now G_{r-1}/K° is finite, and so too must $Aut(G_{r-1}/K^\circ)$ be finite. Each element of $\{x, x^2, \dots\}$ induces an automorphism of G_{r-1}/K°, so x^m centralizes G_{r-1}/K° for some positive m. Let $L = \langle x^m, K^\circ \rangle$, then $L \lhd G$. Considering the Hirsch lengths we see that G/L is finite, and that L/K° is infinite cyclic. Therefore G contains a poly(C_∞) group of finite index, as required.

[3.4]Lemma. *Let A be a torsion-free abelian group of finite rank, and consider an SP-group $G = A \rtimes \langle t \rangle$. Then $\zeta(\langle A, t^\ell \rangle) \bigcap A \neq 1$ for some $\ell > 0$.*

Proof: Extend the action of $\langle t \rangle$ on A to $V = A \otimes_{\mathbf{Z}} \mathbf{Q}$. We write the action of t on $A \otimes_{\mathbf{Z}} \mathbf{Q}$ as multiplication, that is given any $f(t) \in \mathbf{Q}\langle t \rangle$, the map $a \otimes 1 \to (a \otimes 1) \cdot f(t)$ is an endomorphism of $A \otimes_{\mathbf{Z}} \mathbf{Q}$. Therefore $A \otimes_{\mathbf{Z}} \mathbf{Q}$ is a $\mathbf{Q}\langle t \rangle$-module.

Let $V_1 = A_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ be an irreducible $\mathbf{Q}\langle t \rangle$-submodule of V. By Schur's Lemma,

$\Gamma = \mathrm{End}_{\mathbf{Q}\langle t\rangle} V_1$ is a division ring, finite dimensional over $\mathbf{Q}$. Considered as elements of $\mathrm{End}_{\mathbf{Q}} V_1$, $\langle t\rangle$ spans $\mathrm{End}_{\mathbf{Q}\langle t\rangle} V_1$, so Γ is commutative. It is therefore an algebraic number field.

V_1 is one-dimensional as a Γ-space as it is irreducible, so we may identify conjugation of V_1 by $\langle t\rangle$ with multiplication by an algebraic number, τ. We may now think of A_1 as a subgroup of $\mathbf{Q}(\tau)$ under addition.

Moreover A_1 is an SP-group, so by Lemma [2.4] there is an integer ℓ such that t^ℓ centralizes A_1. Therefore $A_1 \subset \zeta(\langle A, t^\ell\rangle) \bigcap A$. Hence, $\zeta(\langle A, t^\ell\rangle) \bigcap A \neq 1$, as was required.

[3.5]Theorem. *Let G be a finitely-generated soluble group, $G \in SP(\{u\}, \{v\})$ for some u and v, then G is nilpotent-by-finite.*

Proof: We proceed by induction on the solubility length of G, with the result holding trivially for abelian groups. Assume that G has solubility length ℓ, and let $A = G^{(\ell-1)}$. Now G/A is finitely-generated and soluble, and $G/A \in SP(\{u\}, \{v\})$ because the property of being an SP-group is quotient closed. By the induction hypothesis, G/A is nilpotent-by-finite. Therefore, G is abelian-by- (nilpotent-by-finite). We wish to show G is nilpotent-by-finite, so no harm is done in coming down to a subgroup of finite index in G — that is, we assume G is abelian-by-nilpotent. Therefore G/A is nilpotent and finitely generated, from which it follows ([9], p.132) that G/A is polycyclic.

We now invoke the theorem of P. Hall that finitely generated abelian-by-polycyclic groups satisfy *max-n*.

If G were not nilpotent-by-finite then there would exist $B \triangleleft G$ maximal subject to G/B not being nilpotent-by-finite, but every proper quotient of G/B is nilpotent-by-finite. However, G/B is finitely generated, soluble, and an SP-group, so it suffices to prove that G/B is nilpotent-by-finite; equivalently we may assume hereafter that every proper quotient of G is nilpotent-by-finite. Now G is soluble and minim Lemma [2.2]) so G has finite rank. This follows from the fact that polyc nd Černikov groups have finite rank, which is an extension-closed property.

Let $\rho(A)$ denote the torsion subgroup of A. Now $\rho(A)$ is finite because G has finite rank and A is abelian. Therefore $C_G(\rho(A))$ is of finite index in G, so no harm is done if we come down to $C_G(\rho(A))$ — assume that $\rho(A) \subset \zeta(G)$. As the torsion elements are all central, clearly G is nilpotent-by-finite if and only if $G/\rho(A)$ is, too.

Conveniently, we need only condider the case that the subgroup A is torsion-free. Meanwhile G/A is polycyclic so by Lemma [3.3] we may convolute the finite pieces to the top and consider G/A as poly(C_∞) -by-finite. Once again come down to a subgroup of finite index to assume G/A is torsion-free and nilpotent, as well as finitely-generated. Therefore by Lemma [3.2], there is a set of elements $S = \{s_1, \ldots s_r\}$ of G such that

$$A = G_0 \leq \langle G_0, s_1 \rangle = G_1 \leq \cdots \leq \langle G_{r-1}, s_r \rangle = G_r = G.$$

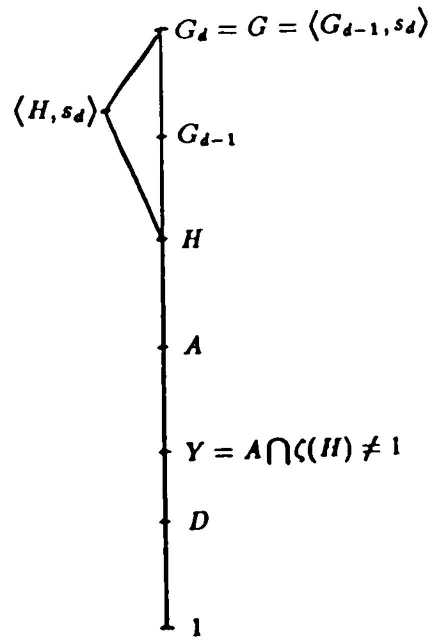
is a central series with infinite cyclic factors. We interrupt our induction on ℓ to proceed by induction on r.

If $r = 1$, then $G_1 = G = \langle A, s_1 \rangle$. By lemma [3.4], we have that $\zeta(\langle A, s_1^{\ell_1} \rangle)$ is non-trivial for some $\ell_1 > 0$. If we set $D = A \bigcap \zeta(\langle A, s_1^{\ell_1} \rangle)$, then D is a non-trivial normal subgroup of G, so G/D is nilpotent-by-finite; moreover D centralizes $\langle A, s_1^{\ell_1} \rangle$ which is of finite index in G. Therefore, G is indeed nilpotent-by-finite.

It suffices now to prove the result for $r = d$ if the result holds for all $r < d$.

Now G_{d-1} is nilpotent-by-finite and $G_d = G = \langle G_{d-1}, s_d \rangle$. For some $\ell_1 > 0$ we have that $H = \langle A, G_{d-1}^{\ell_1} \rangle$ is a normal, nilpotent subgroup of $\langle H, s_d \rangle$. Let $M = \langle H, s_d \rangle$ and $Y = A \bigcap \zeta(H)$, then Y is normal in M. Moreover, considering the Hirsch length we see that M is of finite index in G. Using lemma [3.4] we have, for some $\ell_2 > 0$, $D = Y \bigcap \zeta(\langle Y, s_d^{\ell_2} \rangle)$ is a non-trivial subgroup of G which centralizes $\langle H, s_d^{\ell_2} \rangle$.

Now $P = \langle H, s_d^{\ell_2} \rangle$ is of finite index in G, so it contains a subgroup N of finite index in G and normal in G:



This follows from the fact that $P \leq N_G(P)$ implies that $|G : N_G(P)|$ is finite. Therefore P has a finite number of conjugates in G, say P_1, $\ldots$, P_n, (where each P_i is of finite index in G.) Using $|G : A \cap B| \leq |G : A||G : B|$ iteratively, we find that $N = \bigcap_{i=1}^{n} P_i$ is a normal subgroup of G of finite index.

Finally $D \triangleleft N$, so N/D is nilpotent-by-finite. In fact D is central, so N must in fact be nilpotent-by-finite; N is of finite index in G, though, so we have finally that G is nilpotent-by-finite, as desired.

Therefore if G is a finitely-generated soluble group and $G \in SP(\{u\}, \{v\})$ for words u and v, then G is nilpotent-by-finite. Q.E.D.

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