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Thin Shells In General Relativity

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Thin Shells In General Relativity submitted by Douglas Vick in partial fulfilment of the requirements for the degree of Master of Science in Physics.

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Supervisor

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Date....April 22, 1986..............................

To my grandparents

Wilbur and Mary Mooney

Henry and Lydian Vick

Abstract

The aim of this thesis is to review the uses of thin shells in General Relativity.

A thin shell formalism is developed and applied to a number of physical models. These models illuminate important features of such diverse processes as gravitational collapse, supernovas, inflationary cosmology, growth of voids in the universe, and frame-dragging effects induced by massive rotating bodies. The final section of the thesis is devoted to the Casimir effect and its possible role in preventing the formation of singularities.

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CHAPTER I

Thin Shells/In General Relativity

1.1 Introducton

Einstein's General Theory of Relativity (GR) surely ranks as one of the great intellectual achievements of this century. From a practical point of view, it has had negligible impact on the life of modern man, in marked contrast to the other contemporaneous revolution in physics, the quantum theory. The reason for this lies in the extreme weakness of the gravitational interaction, as compared to the other forces of nature. General Relativistic effects are typically very small corrections to classical Newtonian calculations. Exceptions to this tendency occur in the fields of cosmology, where the matter sources are huge, and gravitational collapse, where matter has been compressed to dimensions comparable to its Schwarzschild radius. In spite of these considerations, it is also accurate to claim that Relativity has had a very significant influence on modern man's conception of the universe and his place in it.

The difficulties associated with experimental verification of GR have played a dominant role in the history of the subject. There are three classic tests of GR, which involve the following phenomena:

1. The dependence of the rate of clocks on the local gravitational potential (as manifested, for example, in

the gravitational redshift).

2. The deflection of light around a massive body (such as the sun).

3. The advance of the perihelion of a Keplerian orbit.

Of these effects, it was the latter two (the famous measurement of light deflecton by the sun was performed in 1919) which supplied the first vindications of GR. In the 1920's, it became apparent that the theory nicely accounted for the newly discovered expansion of the universe, although this was realized in retrospect. The credibility of the theory was thus firmly established.

In the following decades, work on relativity was theoretically oriented. Many important key results were established, but as experimental verification was lacking, these were considered to be speculative. The subject experienced a rebirth in the 1960's due largely to improved experimental techniques in observational astronomy. Of particular importance were the discovery of the microwave background by Penzias and Wilson, which clearly indicated that relativistic cosmological models were of more than academic interest, and the discovery of supermassive objects and highly condensed neutron stars. The latter had been predicted by Oppenheimer in the late 1930's. Excellent historical reviews can be found in Hawking and Israel (1979) and Chandrasekar (1980). Surveys of the most recent trends in the subject (both theoretical and experimental) can be found in Rees (1980), Israel (1983, 1985), and Vessot

(1984).

In GR the coupling of geometry and matter is expressed through a set of 10 highly non-linear differential equations. As a consequence, relativistic models of realistic physical situations typically defy analytical solution. On the other hand, exact solutions of Einstein's equations are always of interest as they are often able to shed light on important qualitative features of the inevitably more complex physical situations. Infinitesimally thin shells, the subject of this thesis, prove themselves amenable to analytical treatments and have been studied quite extensively in GR.

A brief synopsis of the main body of the text follows.

The remainder of this chapter begins with a short discussion on the mathematical essentials of the Theory of General Relativity. A formalism is subsequently developed to describe surface layers as three dimensional imbeddings in the surrounding spacetime. Restricting ourselves to shells of matter, we derive some important junction relations which the gravitational field must obey at the boundary layer. These junction conditions are then applied to derive the dynamical equations of motion for a shell of matter in the special case when the shell and surrounding spacetimes exhibit spherical symmetry.

In Chapter 2 we consider several specific shell models, most of which exhibit spherical symmetry. The purpose of this discussion is twofold. First, each of these models illustrates one or more features of physical interest. Second, the chapter serves as a review of previous work on shells, covering such diverse topics as gravitational collapse and gravitational bounce, supernovas, inflationary cosmology, growth of voids in the universe, and Machian effects.

The final chapter deals with the Casimir effect. After a brief discussion about the phenomenon, the collapse of a spherical conducting shell is considered with reference to the possible effects the Casimir force might have in this process.

A few words about notation are in order here. For the most part, we shall follow the coventions of Misner, Thorne, and Wheeler (1974), hereafter denoted by the abbreviation 'MTW'. A few particulars should be mentioned:

1. Greek indices run over four values, while Latin indices run over three values.
2. The semicolon symbol ';' represents the 4-dimensional covariant derivative, the vertical stroke '|' the 3-dimensional covariant derivative, and the comma ',' the partial derivative. As in MTW, the symbol '∇' is an alternate way of expressing covariant derivatives.
3. The Minkowski metric is symbolized by '$\eta_{\alpha\beta}$'; the more general metric is symbolized by '$g_{\alpha\beta}$'. As in MTW, the metric signature used throughout the thesis is (-+++).
4. The natural constants G, c, and $\hbar$ are all taken to be unity, except in the first section of Chapter 3, where

the quantum Casimir effect is discussed.

1.2 A Brief Outline of General Relativity

Before turning to the specific topic of thin shells, we review some of the essential features of GR. A thorough discussion of the relations outlined in this section may be found in Weinberg (1972) and Papapetrou (1974).

In GR the events of spacetime form a 4-dimensional pseudo-Riemannian space. In such a space, there exists an invariant measure of length ds for the displacement between neighbouring points (events) x^α and $x^\alpha + dx^\alpha$:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1.1}$$

where $g_{\alpha\beta}$ transforms as a covariant tensor of second rank under coordinate transformations, and is non-singular for regular coordinates:

$$\det(g_{\alpha\beta}) \neq 0 \tag{1.2}$$

The contravariant metric tensor is then defined by

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \tag{1.3}$$

Through use of the metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ one may associate with any covariant/contravariant object a corresponding contravariant/covariant object:

$$A^{\alpha} \equiv g^{\alpha\beta} A_{\beta} \tag{1.4}$$

$$B_{\alpha} \equiv g_{\alpha\beta} B^{\beta} \tag{1.5}$$

From a physical point of view, GR is founded on the Principle of Equivalence: at every spacetime point we can find an observer for whom spacetime is locally Minkowskian. To this physical notion, there is a corresponding geometric statement: for every point on the manifold, there exists a set of coordinates, called Riemannian coordinates, such that

$$ds^2\Big|_p = \eta_{\bar{\alpha}\bar{\beta}}\, dx^{\bar{\alpha}}\, dx^{\bar{\beta}} = -d\tau^2 \tag{1.6}$$

where τ is the proper time. Suppose that $\{x^{\alpha}\}$ is an arbitrary set of coordinates related to the Riemannian coordinates $\{x^{\bar{\alpha}}\}$ by the transformation

$$dx^{\bar{\alpha}} = \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\right) dx^{\alpha} \tag{1.7}$$

Then Eqn. (1.6) can be put into the more general form (1.1), where

$$g_{\alpha\beta}\Big|_p = \eta_{\bar{\alpha}\bar{\beta}} \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\right)\left(\frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}}\right) \tag{1.8}$$

In the absence of non-gravitational forces, the motion of a particle in the observer's frame obeys the equation

$$\frac{d^2 x^{\bar{\alpha}}}{d\lambda^2} = 0 \tag{1.9}$$

where λ is an affine parameter, which maybe taken to be the proper time τ if the particle has mass and the coordinate $x^{\bar{0}}$ if it is massless. When Eqn. (1.9) is transformed to the coordinates $\{x^{\alpha}\}$ the resulting equation of motion is

$$\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{d\lambda} \frac{dx^{\gamma}}{d\lambda} = 0 \tag{1.10}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ is the affine connection:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\beta} \partial x^{\gamma}} \tag{1.11}$$

The parameter λ is determined up to a linear transformation. In geometric language equations of the form (1.10) are known as geodesics.

The connection can be shown to be related to the functions $g_{\alpha\beta}$:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}(g_{\beta\lambda,\gamma} + g_{\gamma\lambda,\beta} - g_{\beta\gamma,\lambda}) \tag{1.12}$$

The connection coefficients $\Gamma^{\alpha}_{\beta\gamma}$ can be used to construct a covariant derivative, whose components transform as a second-rank tensor:

$$A^{\alpha}_{;\beta} = A^{\alpha}_{,\beta} + A^{\lambda}\Gamma^{\alpha}_{\lambda\beta} \tag{1.13}$$

$$A_{\alpha;\beta} = A_{\alpha,\beta} - A_{\lambda}\Gamma^{\lambda}_{\alpha\beta} \tag{1.14}$$

The Riemann curvature tensor is constructed from the Γ's by

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\lambda}_{\beta\delta}\Gamma^{\alpha}_{\lambda\gamma} - \Gamma^{\lambda}_{\beta\gamma}\Gamma^{\alpha}_{\lambda\delta} \tag{1.15}$$

Due to its various symmetry properties, $R^{\alpha}_{\beta\gamma\delta}$ has only 20 independent components. In addition to these symmetries there exist the Bianchi Identities:

$$R^{\alpha}_{\beta\gamma\delta;\epsilon} + R^{\alpha}_{\beta\delta\epsilon;\gamma} + R^{\alpha}_{\beta\epsilon\gamma;\delta} = 0 \tag{1.16}$$

From the Riemann tensor are constructed the following objects:

1. Ricci Tensor:

$$R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta} \tag{1.17}$$

2. Curvature Scalar:

$$R = R^{\lambda}_{\lambda} \tag{1.18}$$

3. Einstein Tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \tag{1.19}$$

The dynamical relation between the geometry of spacetime and the energy/matter distribution is embodied in the field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1.20}$$

where $T_{\alpha\beta}$ is the stress-energy tensor. From the metric, therefore, one can in principle calculate the behaviour of matter. The metric, in turn, is determined by the energy/matter distribution through Eqn. (1.20).

The contracted Bianchi Identities follow from Eqns. (1.16) and (1.19):

$$G^{\alpha\beta}{}_{;\beta} = 0 \tag{1.21}$$

These identities are seen to aquire physical significance through the field equations (1.20):

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{1.22}$$

Eqn. (1.22) will be recognized as the law of conservation of energy/momentum.

1.3 Hypersurfaces In Spacetime

When attempting to model real physical systems it is often the case that one must deal with discontinuities in the source $T_{\alpha\beta}$. A well-known example occurs in the study of stars, where the internal and external metrics must be matched at the surface of the star. This is in general a non-trivial problem, as the coupling of the geometry and source requires certain junction conditions to hold at the boundary separating the two regions of spacetime.

The pioneering work in this problem was initiated by Lanczős (1922), and important contributions were subsequently made by Darmois(1927), O'Brien and Synge (1952), and Lichnerowicz (1955). More recently the treatments above have been compared (Bonnor and Vickers, 1981) and extended (Raju, 1981). Other authors have investigated junction conditions in alternate theories of gravity, notably the Einstein-Cartan theory (Skinner and Webb, 1977) and the Brans-Dicke theory (Suffern, 1982). Below we follow the elegant approach to boundary layers put forward by Israel (1966).

We consider a discontinuity in the source $T_{\alpha\beta}$ which defines a 3-dimensional hypersurface Σ. The hypersurface divides spacetime into two regions V_+ and V_- (see Figure 1.1). The discussion below does not lend itself to the treatment of null hypersurfaces, and so Σ is assumed to be non-null everywhere (for a discussion of junction conditions for null surfaces see Choquet-Bruhat, 1968).

Our aim is to write Eqn. (1.20) in a form more appropriate for dealing with hypersurfaces. To do this, we consider the imbedding of Σ in the surrounding spacetime and separate its intrinsic and extrinsic properties.

The regions V_+ and V_- are described by the 4-coordinate systems $\{x^\alpha\}_+$ and $\{x^\alpha\}_-$ with associated basis $\{\underline{e}_\alpha\}_+$ and $\{\underline{e}_\alpha\}_-$, while on Σ the 3-coordinates $\{x^i\}_\Sigma$ suffice. At any point on Σ we may define a unit normal vector $\underline{n}$ which is orthogonal to the basis vectors $\{\underline{e}_i\}_\Sigma$. On Σ, the basis $\{\underline{e}_i\}$ and $\underline{n}$ may be expressed in terms of the basis $\{\underline{e}_\alpha\}_\pm$ and vice versa:

$$\underline{e}_i = e^\alpha_{(i)}\, \underline{e}_{\alpha\pm} \tag{1.23}$$

$$\underline{n} = n^\alpha \underline{e}_{\alpha\pm} \tag{1.24}$$

$$\underline{e}_{\alpha\pm} = e^i_{(\alpha)\pm}\, \underline{e}_i + e^n_{(\alpha)\pm}\, \underline{n} \tag{1.25}$$

For the remainder of the chapter we will for the most part drop the use of the symbols +/-, as the relations below are valid on either side of Σ.

Consider a vector field $\underline{\mathbf{A}}$ which is tangent to Σ. (For example, if Σ represents the history of a shell of matter, the 4-velocity $\underline{\mathbf{u}}$ is such a vector field.) The covariant derivative of $\underline{\mathbf{A}}$ along basis $\underline{\mathbf{e}}_j$ is

$$\nabla_j \underline{A} = \nabla_j (A^i \underline{e}_i)$$

$$= A^i_{,j} \underline{e}_i + A^i (\nabla_j \underline{e}_i) \tag{1.26}$$

We introduce the extrinsic curvature tensor K_{ij} which describes how the vector $\underline{n}$ changes as it is propagated along Σ:

$$\nabla_i \underline{n} = -K_i^j \underline{e}_j \tag{1.27}$$

(As $\underline{n} \cdot \nabla_i \underline{n} = 1/2\, \nabla_i (\underline{n} \cdot \underline{n}) = 0$, $\nabla_i \underline{n}$ is expressible in terms of the basis $\{\underline{e}_i\}_\Sigma$.) It is easy to show that K_{ij} is symmetric:

$$K_{ij} = K_i^m g_{mj} = K_i^m (\underline{e}_m \cdot \underline{e}_j)$$

$$= -(\nabla_i \underline{n}) \cdot \underline{e}_j = -\nabla_i (\underline{n} \cdot \underline{e}_j) + \underline{n} \cdot \nabla_i \underline{e}_j = K_{ji} \tag{1.28}$$

Here g_{mj} is the 3-metric on Σ, defined as in Eqn. (1.1). Using Eqn. (1.28) we write the normal and tangential components of $\nabla_j \underline{e}_i$, thus defining the 3-connection ${}^{(3)}\Gamma^m_{ij}$

$$(\nabla_j \underline{e}_i)^n = K_{ij} \tag{1.29}$$

$$(\nabla_j \underline{e}_i)^m = {}^{(3)}\Gamma^m_{ij} \tag{1.30}$$

We can now write the Gauss-Weingarten Equation

$$\nabla_j \underline{e}_i = \frac{K_{ji}\underline{n}}{(\underline{n}\cdot\underline{n})} + {}^{(3)}\Gamma^m_{ij}\,\underline{e}_m \; . \tag{1.31}$$

where the $\underline{n}\cdot\underline{n}$ term is included to ensure the correct normalization.

The covariant derivative (1.26) becomes

$$\nabla_j \underline{A} = A^i_{\,|j} + \frac{A^m K_{jm}\underline{n}}{(\underline{n}\cdot\underline{n})} \tag{1.32}$$

where $A^i_{\,|j}$ are the components of the intrinsic covariant derivative:

$$A^i_{\,|j} = A^i_{\,,j} + A^m\,{}^{(3)}\Gamma^i_{mj} \tag{1.33}$$

The Riemann curvature tensor can also be split into intrinsic and extrinsic parts. In a coordinate basis we have

$$R^\lambda_{jki}\,\underline{e}_\lambda = \nabla_k(\nabla_i\,\underline{e}_j) - \nabla_i(\nabla_k\,\underline{e}_j) \; . \tag{1.34}$$

Substitution of Eqn. (1.31) into Eqn (1.34) yields

$$R^\lambda_{jki}\,\underline{e}_\lambda = (K_{ij|k} - K_{kj|i})\,\frac{\underline{n}}{(\underline{n}\cdot\underline{n})} + (\underline{n}\cdot\underline{n})^{-1}(K_{kj}K^h_i - K_{ij}K^h_k) + {}^{(3)}R^h_{jki}\,\underline{e}_h \tag{1.35}$$

where the intrinsic curvature tensor ${}^3R^\lambda{}_{jki}$ is constructed from the 3-connection as in Eqn. (1.15). Projecting Eqn. (1.35) normally and tangentially to the hypersurface Σ immediately yields the equations of Gauss and Codazzi:

$$R^{n}_{jki} = K_{ij|k} - K_{kj|i} \tag{1.36}$$

$$R^{h}_{jki} = {}^{(3)}R^{h}_{jki} + (\underline{n} \cdot \underline{n})^{-1}(K_{kj}K^{h}_{i} - K_{ij}K^{h}_{k}) \tag{1.37}$$

Eqn. (1.37) demonstrates that the external and intrinsic curvatures agree only when the extrinsic curvature vanishes.

For the sake of simplicity we use the Gaussian normal coordinates in the neighbourhood of Σ (Wald, 1984). Taking the zeroth components of the field to be the normal ones, we make use of the following identities:

$$G^{n}_{n} = -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31}) \tag{1.38}$$

$$G^{n}_{1} = R^{n2}{}_{12} + R^{n3}{}_{13} \tag{1.39}$$

$$G^{n}_{2} = R^{n1}{}_{21} + R^{n3}{}_{23} \tag{1.40}$$

$$G^{n}_{3} = R^{n1}{}_{31} + R^{n2}{}_{32} \tag{1.41}$$

Substitution of Eqns. (1.36) and (1.37) into the above identities yields:

$$G^n_n = -\frac{1}{2}\,{}^{(3)}R - \frac{([Tr(K)]^2 - Tr(K^2))}{2(\underline{n}\cdot\underline{n})} \tag{1.42}$$

$$G^n_i = (Tr(K)|_i - K^j_i|_j)(\underline{n}\cdot\underline{n})^{-1} \tag{1.43}$$

where

$$Tr(K) = K^i_i \tag{1.44}$$

$$Tr(K^2) = K^j_i K^i_j \tag{1.45}$$

Calculation of the remaining components of $G^\alpha{}_\beta$ is more involved (see MTW, 1973, Ch.21). In Gaussian normal coordinates they are

$$G^i_j = {}^{(3)}G^i_j + (\underline{n}\cdot\underline{n})^{-1}[(K^i_j - \delta^i_j Tr(K))_{,n} - Tr(K)K^i_j + \frac{1}{2}\delta^i_j(Tr(K))^2 + \frac{1}{2}\delta^i_j Tr(K^2)] \tag{1.46}$$

1.4 The Junction Conditions For Thin Shells

We now derive the junction conditions which must be satisfied on Σ if it represents the history of a shell of matter. Since this type of hypersurface is timelike, $(\underline{n}\cdot\underline{n}) >$

0; we will assume this scalar product to be normalized to unity, and subsequently drop the term whenever it occurs.

As Σ is of infinitesimal thickness, $T_{\alpha\beta}$ exhibits a singularity here. In this case we define the surface stress-energy tensor S_{ij} by

$$S^{\alpha}_{\beta} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} T^{\alpha}_{\beta} dn \tag{1.47}$$

where n is the proper distance normal to Σ, and the limits of integration occur on opposite sides of Σ. With the aid of Einstein's equations (1.20) we can write

$$\lim_{\varepsilon \to 0} \left(\int_{-\varepsilon}^{+\varepsilon} G^{\alpha}_{\beta}\, dn\right) = 8\pi S^{\alpha}_{\beta} \tag{1.48}$$

Now the intrinsic geometry on Σ is defined by the 3-metric

$$ds^2 = {}^{(3)}g_{ij} dx^i dx^j \tag{1.49}$$

In order for the geometry to be well-defined, g_{ij} must possess no singularities or delta functions. The 3-connection is related to ${}^3g_{ij}$ through an equation of the form (1.12). The curvature scalar ${}^3R$, calculated from ${}^3\Gamma^i{}_{jk}$, thus has no delta functions.

Furthermore, $K_{ij} = -(1/2)g_{ij,n}$ will have no delta functions either. These considerations allow evaluation of Eqn. (1.48), making use of Eqns. (1.42), (1.43), and (1.46).

Hence

$$\int G^n_n dn = 0 = 8\pi S^n_n \tag{1.50}$$

$$\int G^n_i dn = 0 = 8\pi S^n_i \tag{1.51}$$

$$\int G^i_j dn = \gamma^i_j - \delta^i_j \gamma = 8\pi S^i_j \tag{1.52}$$

where

$$\gamma^i_j \equiv [K^i_j] \equiv K^i_{j+} - K^i_{j-} \tag{1.53}$$

$$\gamma = \gamma^i_i \tag{1.54}$$

(Subsequently the square brackets will be used to denote the jump in the value of some object across Σ.) Eqn. (1.52) is known as the Lanczos equation.

Two important relations result from considering the jump in the field equations:

$$[G^n_i] = 8\pi [T^n_i] \tag{1.55}$$

$$[G^n_n] = 8\pi [T^n_n] \tag{1.56}$$

Using Eqns.(1.42),(1.43),(1.52), and (1.53), we can rewrite these equations as

$$(\gamma_i^j - \delta_i^j \gamma)_{|j} = -8\pi [T_i^n] \tag{1.57}$$

$$\tilde{K}_{ab} S^{ab} = -[T_n^n] \tag{1.58}$$

where the symbol $\tilde{A}$ denotes the average value $(1/2)(A_+ + A_-)$ of an object across Σ. Making use of the Lanczos equation, Eqn. (1.57) becomes

$$S^{ij}{}_{|j} + [T^{ni}] = 0 \tag{1.59}$$

Eqn. (1.59) is a conservation law having as its analog Eqn. (1.22). The right hand term expresses the fact that 4-momentum may be transferred to Σ from the surrounding spacetime, and would not be present if Σ was not imbedded in a higher dimensional space.

In the following section we apply the formalism developed above to the case of a spherically symmetric shell of matter.

1.5 The Spherically Symmetric Imbedding

We now construct a dynamical description of the hypersurface Σ in the case when it represents the history of a spherical shell.[1] Our aim is to express the results in

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[1] An alternate treatment of the spherical shell problem has been given by Siegel (1981), who uses the Arnowitt-Deser-Misner canonical formulation. Recently, the shell problem has been generalized to the case of a shell with finite thickness (Hoye et. al., 1985).

terms of quantities which are measurable by an observer co-moving with the shell. With this objective in mind, we first consider this observer and momentarily neglect the exact nature of the imbedding of Σ in the surrounding spacetime.

If spherical symmetry is assumed, then the intrinsic geometry of Σ is described by

$$ds^2_\Sigma = -d\tau^2 + R(\tau)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.60}$$

where $R(\tau)$ is an arbitrary function which the co-moving observer can determine by geodesic deviation experiments or by measuring areas of spheres. For fixed values of the proper time τ the observer actually measures the area

$$A = 4\pi R^2 \tag{1.61}$$

If the shell is a perfect fluid, the surface stress-energy tensor takes on the form

$$S_{ij} = (\sigma + P)u_i u_j + Pg_{ij} \tag{1.62}$$

where P is the surface pressure and σ the surface energy density of the shell. The proper mass of the shell is then given by

$$M = 4\pi\sigma R^2 \tag{1.63}$$

When the connection for the metric (1.60) is calculated from

$$^{(3)}\Gamma^i_{jk} = \frac{1}{2}\, g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}) \tag{1.64}$$

the non-vanishing coefficients are

$$\Gamma^\tau_{\theta\theta} = -R\dot{R} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\tau_{\phi\phi} = -R\dot{R}\sin^2\theta \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

$$\Gamma^\theta_{\tau\theta} = \Gamma^\phi_{\tau\phi} = \frac{\dot{R}}{R} \tag{1.65}$$

where the dot means differentiation with respect to the proper time.

An important relation results from substituting the explicit form of S_{ij} *viz* (1.62) into Eqn. (1.59) and projecting the result onto the 4-velocity $\underline{u}$. Hence, with the aid of Eqn. (1.65), we find

$$\dot{M} + P\dot{A} = A[T^u_n] \equiv A[T^\alpha_\beta u^\beta n_\alpha] \tag{1.66}$$

Eqn. (1.66) relates the mass energy of the shell to the work done by the surface pressure and the net energy flux into the shell from the surrounding spacetime.

It is evident from Eqn. (1.66) that in order to describe the dynamics of the shell we must consider the imbedding of Σ. The information of interest resides in the normal components of the 4-acceleration a_n $(=\underline{n}\cdot\underline{a})$, evaluated on either side of Σ. Adding and subtracting these, and making use of Eqn. (1.32), we have

$$\tilde{a}_n = u^i u^j \tilde{K}_{ij} \tag{1.67}$$

$$[a_n] = u^i u^j \gamma_{ij} \tag{1.68}$$

Through the Lanczos Eqn. (1.52) we obtain γ_{ij} in terms of S_{ij}. Eqns. (1.67) and (1.68) then become, with the aid of Eqns. (1.62) and (1.58),

$$\tilde{a}_n = -(\sigma + P)^{-1}([T^n_n] + P\tilde{K}) \tag{1.69}$$

$$[a_n] = 4\pi(\sigma + 2P) \tag{1.70}$$

In order to proceed further, the functions $a_{n\pm}$ must be evaluated. As we shall see below, a good deal of headway can be made without specifying the exact nature of the geometry in the regions $V_\pm$. We assume only that these regions can be described by some spherically symmetric metric of the general form

$$ds_{\pm}^2 = -e^{2\nu_{\pm}(r,t)}\, dt^2 + e^{2\lambda_{\pm}(r,t)}\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.71}$$

It should be stressed that although the symbol 't' is used in both metrics, the coordinate so labelled in the two regions $V_{\pm}$ are distinct.

The nonvanishing connection for the metric (1.71), calculated from Eqn. (1.12), is

$$\Gamma^t_{tt} = \nu_{,t} \qquad \Gamma^t_{tr} = \nu_{,r} \qquad \Gamma^t_{rr} = \lambda_{,t} e^{-2\nu+2\lambda}$$

$$\Gamma^r_{tt} = \nu_{,r} e^{2\nu-2\lambda} \qquad \Gamma^r_{tr} = \lambda_{,t} \qquad \Gamma^r_{rr} = \lambda_{,r}$$

$$\Gamma^r_{\theta\theta} = -re^{-2\lambda} \qquad \Gamma^r_{\phi\phi} = -r \sin^2\theta e^{-2\lambda}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \cot\theta \tag{1.72}$$

The Einstein tensor, calculated via Eqns. (1.15)-(1.19), or by using the more elegant method of differential forms (MTW, Ch.14, 1973; Israel, 1970a) is given by

$$G^t_t = e^{-2\lambda}\left(\frac{1}{r^2} - \frac{2\lambda_{,r}}{r}\right) - \frac{1}{r^2} \tag{1.73}$$

$$G^r_r = e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2\nu_{,r}}{r}\right) - \frac{1}{r^2} \tag{1.74}$$

$$G^t_r = \frac{-2\lambda_{,t}}{r} e^{-2\nu} \tag{1.75}$$

$$G^r_t = \frac{2\lambda_{,t}}{r} e^{-2\lambda} \tag{1.76}$$

$$G^\theta_\theta = G^\phi_\phi = -e^{-2\nu}(\lambda_{,tt} - \lambda_{,t}\nu_{,t} + \lambda^2_{,t}) + e^{-2\lambda}(\nu_{,rr} - \nu_{,r}\lambda_{,r} + \nu^2_{,r} + \frac{\nu_{,r}}{r} - \frac{\lambda_{,r}}{r}) \tag{1.77}$$

Interpreting r as the radial coordinate in the normal fashion, and realizing that the three metrics (1.60) and (1.71) must agree when r equals the radius of the shell, we can identify the function R as the radius of the shell. The 4-velocity of the shell may then be written in the form

$$u^\alpha_\pm = (X_\pm, \dot{R}, 0, 0) \tag{1.78}$$

The orthonormality conditions $\underline{u}\cdot\underline{n}=0$, $\underline{u}\cdot\underline{u}=-1$, and $\underline{n}\cdot\underline{n}=1$, together with metric (1.71) allow us to completely specify the components u_α, n_α, n^α, and X in terms of R and the metric functions $g_{\alpha\beta}$:

$$u_{\alpha\pm} = (-e^{2\nu_\pm}X_\pm, e^{2\lambda_\pm}\dot{R}, 0, 0) \tag{1.79}$$

$$n_{\alpha\pm} = (-e^{\nu+\lambda}\dot{R}, e^{\nu+\lambda}X, 0, 0) \tag{1.80}$$

$$n^\alpha{}_\pm = (e^{-\nu+\lambda}\dot{R}, e^{\nu-\lambda}X, 0, 0) \tag{1.81}$$

$$X_\pm = e^{-\nu_\pm}(1 + e^{2\lambda_\pm}\dot{R}^2)^{1/2} \tag{1.82}$$

The normal 4-accelerations can be written

$$a_n = n_\alpha u^\beta u^\alpha{}_{;\beta} = \left(\frac{-u_r n_t}{u_t} + n_r\right) u^\beta (u^r{}_{,\beta} + u^\lambda \Gamma^r_{\lambda\beta}) \tag{1.83}$$

where the identity $\underline{u}\cdot\underline{a}=0$ has been used. Using the results (1.78)-(1.82), and the connection (1.72), Eqn. (1.83) eventually reduces to

$$a_n = F_\pm^{-1}\left(\ddot{R} + \nu_{,r} e^{-2\lambda} + 2\dot{R}\chi\lambda_{,t} + \dot{R}^2(\lambda_{,r} + \nu'_{,r})\right)\Big|_\pm \tag{1.84}$$

where the functions $F_\pm$ are defined as

$$F_\pm = (e^{\nu-\lambda}\chi)_\pm = (e^{-2\lambda_\pm} + \dot{R}^2)^{\frac{1}{2}} \tag{1.85}$$

Eqn. (1.84) can be put in a more convenient form upon evaluation of the quantity $G^u{}_n$ $(=G^\alpha_\beta u_\alpha n^\beta)$:

$$a_{n\pm} = \frac{1}{\dot{R}}(\dot{F}_\pm + 4\pi R T^u_{n\pm}) \tag{1.86}$$

Eqns. (1.69) and (1.70) then become

$$\dot{\tilde{F}} = -4\pi R \tilde{T}^u_n - (\sigma + P)^{-1}([T^n_n] + P\tilde{K})\dot{R} \tag{1.87}$$

$$[\dot{F}] = 4\pi\left((\sigma + 2P)\dot{R} - R[T^u_n]\right) \tag{1.88}$$

Using Eqn. (1.66), the right hand side of Eqn. (1.88) can be put into the form of a perfect differential, whose integral is

$$F_+ - F_- = -\frac{M}{R} + a \qquad (1.89)$$

where a is a constant of integration which can be shown to be zero.[2] From the definitions of F. (1.85) and Eqn. (1.89) we can write

$$F_+ + F_- = \frac{-R}{M}\,[e^{-2\lambda}] \qquad (1.92)$$

The relations (1.89) and (1.92) are the equations of motion for the shell. In Chapter 2, we will apply them to several spherically symmetric imbeddings.

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[2]Subsequent to the calculation of Eqn. (1.89), the author became aware of some important works by Lake (1979, 1985), who has pointed out that Eqn. (1.89) follows straightforwardly from considering $\gamma_{\theta\theta}$. From Eqns. (1.52) and (1.62) it can be shown that

$$\gamma_{\theta\theta} = M \qquad (1.90)$$

while explicit calculation of $\gamma_{\theta\theta}$ yields

$$\gamma_{\theta\theta} = -R[F] \qquad (1.91)$$

from which Eqn. (1.89) follows immediately.

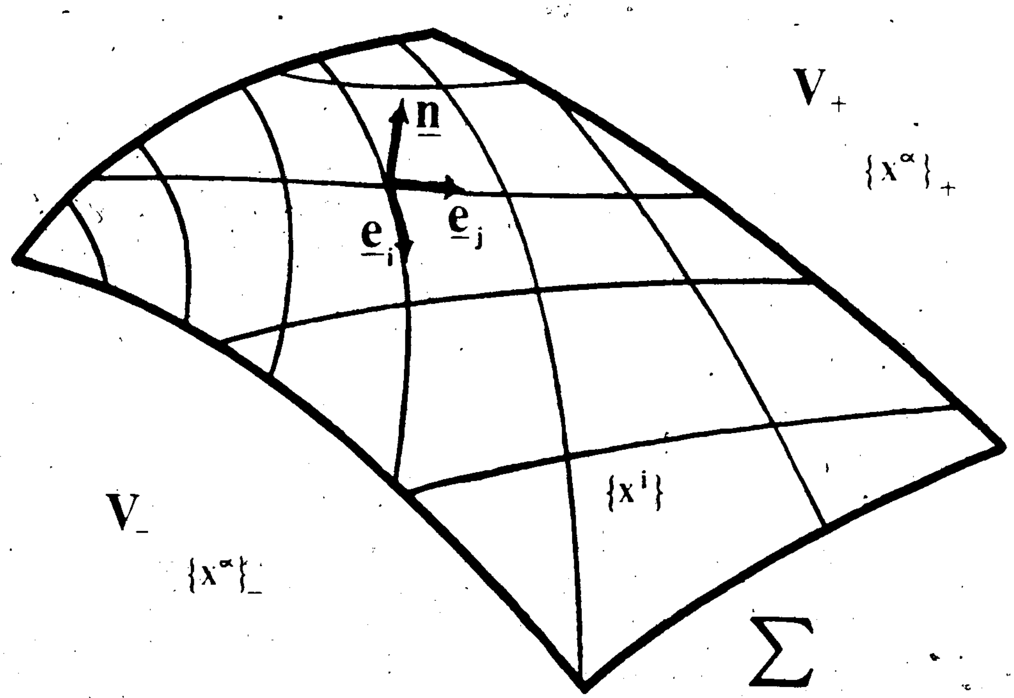

Figure 1.1 A hypersurface imbedded in a higher dimensional space.

CHAPTER II

A Survey Of Thin Shells

2.1 The Collapse Of A Spherical Dust Shell

Let us first consider a shell for which there is no fluid pressure (P = 0 in Eqn. (1.62)) and upon which only self-gravitational forces act. The absence of any sources other than the shell itself allows us to use a flat interior metric and a Schwarzschild exterior metric:

$$e^{2\nu_-} = e^{2\lambda_-} = 1 \tag{2.1}$$

$$e^{2\nu_+} = e^{-2\lambda_+} = 1 - \frac{2m}{r} \tag{2.2}$$

where the constant m represents the total gravitational energy of the system.

The stress energy tensor $T_{\alpha\beta}$, calculated from Eqns. (1.73) - (1.77) vanishes in $V_\pm$ and thus the proper mass of the shell (by Eqn. (1.66)) remains constant. Eqns. (1.89) and (1.92) become

$$\left(1 - \frac{2m}{r} + \dot{R}^2\right)^{\frac{1}{2}} - \left(1 + \dot{R}^2\right)^{\frac{1}{2}} = -\frac{M}{R} \tag{2.3}$$

$$\left(1 - \frac{2m}{r} + \dot{R}^2\right)^{\frac{1}{2}} + \left(1 + \dot{R}^2\right)^{\frac{1}{2}} = \frac{2m}{M} \tag{2.4}$$

That these equations express conservation of energy can be seen by taking the difference between Eqns. (2.4) and (2.3) and arranging terms in the following manner:

$$M(1 + \dot{R}^2)^{\frac{1}{2}} - \frac{M^2}{2R} = m \qquad (2.5)$$

The term $-M^2/2R$ will be recognized as the gravitational potential energy of the system. The term on the left represents the inertial mass of the moving shell, and when expanded for small values of $\dot{R}$ becomes the sum of the (proper) mass energy and kinetic energy of the shell.

Eqn. (2.5) can be solved for $\dot{R}$ and integrated to obtain the explicit relation between R and τ, but we may discern the important qualitative features of the collapse without performing the integration. Rewriting Eqn. (2.5) as

$$(1 + \dot{R}^2)^{\frac{1}{2}} - 1 = \frac{m - M}{M} + \frac{M}{2R} \qquad (2.6)$$

we see that motion can only occur in regions where the right hand side of (2.6) is positive. If $m > M$, the shell is not gravitationally bound. If $m < M$, then the motion of the shell is restricted to the region

$$R \leq \frac{M}{2(M - m)} \qquad (2.7)$$

Differentiating (2.6) we find

$$\ddot{R} = - \frac{M}{2R^2} (1 + \dot{R}^2)^{\frac{1}{2}} \qquad (2.8)$$

The observer sitting on the shell never experiences deceleration. A shell starting from rest at a finite radius is doomed to collapse down to R = 0.

Perhaps the most interesting feature of the collapse arises from the nature of the Schwarzschild metric:

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})} + r^2 d\Omega^2 \tag{2.9}$$

The coordinate system of the external observer exhibits a pathology at r = 2m. The metric becomes singular here, and thus cannot be analytically extended through this radius, known as the Schwarzschild horizon. It has long been realized that this singularity represents a poor choice of coordinates rather than a singularity of the spacetime at r = 2m. This is readily seen by calculating the curvature (1.15), which remains well behaved here. In contrast, the point r = 0 represents a true singularity in the sense that the spacetime curvature becomes unbounded here. Kruskal (1960) found a coordinate transformation which removes the pathology at r = 2m and creates a metric which is analytic for r > 0. The Kruskal coordinates (u,v) are related to (r,t) by

$$uv = (r/2m - 1)\exp(r/2m) \tag{2.10}$$

$$u/v = \exp(t/2m) \tag{2.11}$$

The Schwarzschild horizon, while not a singularity in spacetime, nevertheless has important physical properties. Let us consider the time coordinates τ and t.. We let the shell, initially at rest, fall from some radius R_0 which lies outside the horizon. From Eqn., (2.5) we find

$$d\tau = \frac{dR}{\dot{R}} = \frac{-RdR}{\alpha(M^2/4\alpha^2 + mR/\alpha^2 - R^2)^{\frac{1}{2}}} \tag{2.12}$$

where

$$\alpha^2 = 1 - m^2/M^2 \tag{2.13}$$

The integral of Eqn. (2.12) is

$$\tau(R) = \frac{1}{\alpha}\left[M^2/4\alpha^2 + mR/\alpha^2 - R^2\right]^{\frac{1}{2}} + m\pi/4\alpha^2 - \frac{m}{2\alpha^3}\sin^{-1}\left[\frac{1}{M}(2\alpha^2 R - m)\right] \tag{2.14}$$

It can be seen by setting R = 0 in Eqn. (2.14), that an observer sitting on the shell sees it collapse through R = 2m and down to R = 0 in a finite proper time. In order to determine the motion of the shell as seen by an external observer we must integrate

$$dt_+ = \frac{dt_+}{d\tau}\, d\tau = \frac{X_+}{\dot{R}}\, dR = \frac{-(mR^2/M - MR/2)\,dR}{\alpha(R - 2m)\,[M^2/4\alpha^2 + mR/\alpha^2 - R^2]^{\frac{1}{2}}} \tag{2.15}$$

After a somewhat tedious integration, we arrive at the expression.

$$t_+(R) = \frac{m}{M}\,\tau(R) + \beta\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{M}\,(2\alpha^2 R - m)\right)\right]$$

$$+ \frac{2m\beta}{a^{\frac{1}{2}}}\,\ell n\left(\frac{\left[2a + bu + \left(a(a+b+bu+u^2)\right)^{\frac{1}{2}}\right](R_0 - 2m)}{u\left[2a + b(R_0 - 2m)\right]}\right) \tag{2.16}$$

where

$$u = R - 2m \tag{2.17}$$

$$\beta = \frac{1}{\alpha^2}\left(\frac{2m^2}{M} - \frac{M}{2}\right) \tag{2.18}$$

$$a = \frac{M^2}{4\alpha^2} - m^2\left(2 - \frac{1}{2\alpha^2}\right) \tag{2.19}$$

$$b = 2\left(\frac{m}{2\alpha^2} - 2m\right) \tag{2.20}$$

It is the logarithmic term which endows the external time coordinate with its unusual features. As $R \to 2m$ this term diverges and $t_+ \to \infty$. In marked contrast to the co-moving observer, the external observer never sees the shell collapse past the horizon. As the shell approaches the horizon it slows down, tending to this radius asymptotically as $t \to \infty$. Figure 2.1 shows a plot of $\tau(R)$ and $t(R)$ for the case when $R_0 = 6M$.

Suppose the observer on the shell has agreed to emit light pulses to be received by an external observer stationed at r_0. The angular coordinates of the two observers are assumed to agree, so that the light pulses travel radially outwards. The path along which the outgoing light travels to the observer at r_0 are radial null geodesics:

$$ds^2 = 0 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 \qquad (2.21)$$

It is elementary to integrate Eqn (2.21) to find the time taken by the light pulses to reach r_0 from the shell:

$$t_r - t_e = r_0 - R(t_e) + 2m \ \ell n(\frac{r_0 - 2m}{R(t_e) - 2m}) \qquad (2.22)$$

This expression also diverges as $R \to 2m$.

This result implies that the horizon constitutes an insurmountable barrier for an observer in the external region. Since the speed of light represents an upper bound for the rate at which information can travel, any physics in the interior of the event horizon is forever inaccessible to him. The observer on the shell, in contrast, passes through the horizon in accordance with (2.14) and is crushed by the tidal forces as he approaches R = 0.

It is of interest to know whether it is possible to arrest such a collapse when there are repulsive forces at work in addition to gravitation. The subject of the following section, the charged shell, provides an opportunity to investigate such matters.

2.2 The Gravitational Collapse Of A Charged Shell

We now consider the motion of an electrostatically charged shell falling in the external field of a massive charge distribution. This problem was considered in some detail by Kuchar (1968) and Chase (1970). If the central distribution interior to the shell has mass m_- and charge e_-, and the shell has mass m and charge e, then the exterior (+) and interior (-) gravitational and electromagnetic fields are described by the Reissner/Nordstrom metric:

$$ds_{\pm}^2 = -\left(1 - \frac{2m_{\pm}}{r} + \frac{e_{\pm}^2}{r^2}\right) dt^2 + \left(1 - \frac{2m_{\pm}}{r} + \frac{e_{\pm}^2}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (2.23)$$

where

$$m_+ = m + m_- \quad (2.24)$$

$$e_+ = e + e_- \quad (2.25)$$

The stress-energy tensor associated with such a field is

$$T^t_t = T^r_r = -T^\theta_\theta = -T^\phi_\phi = \frac{-e^2_\pm}{8\pi r^2} \qquad (2.26)$$

which represents the usual Maxwellian stresses for a radial electrostatic field. It is easily seen that $T^\alpha_\beta u_\alpha n^\beta$ vanishes, so that Eqn. (1.66) reduces to

$$\dot{M} = -P\dot{A} \qquad (2.27)$$

We may immediately write down the equations of motion (1.89) and (1.92):

$$\left(1 - \frac{2m_+}{R} + \frac{e^2_+}{R^2} + \dot{R}^2\right)^{\frac{1}{2}} - \left(1 - \frac{2m_-}{R} + \frac{e^2_-}{R^2} + \dot{R}^2\right)^{\frac{1}{2}} = \frac{-M}{R} \qquad (2.28)$$

$$\left(1 - \frac{2m_+}{R} + \frac{e^2_+}{R^2} + \dot{R}^2\right)^{\frac{1}{2}} + \left(1 - \frac{2m_-}{R} + \frac{e^2_-}{R^2} + \dot{R}^2\right)^{\frac{1}{2}} = \frac{2(m_+ - m_-)}{M} - \frac{(e^2_+ - e^2_-)}{MR} \qquad (2.29)$$

From Eqns. (2.28) and (2.29), we find

$$1 + \dot{R}^2 = A + B/R + C/R^2 \qquad (2.30)$$

$$A = \frac{(m_+ - m_-)^2}{M^2} \qquad (2.31)$$

$$B = m_- + m_+ - (m_+ - m_-)(e_+^2 - e_-^2)/M^2 \tag{2.32}$$

$$C = \frac{(e_+^2 - e_-^2)^2}{4M} + \frac{M^2}{4} - \frac{(e_+^2 + e_-^2)}{2} \tag{2.33}$$

Turning our attention to the metric (2.23) for a moment, we see that it exhibits singularities provided that $m^2 > e^2$, at the radial coordinates

$$r_1 = m + (m^2 - e^2)^{\frac{1}{2}} \tag{2.34}$$

$$r_2 = m - (m^2 - e^2)^{\frac{1}{2}} \tag{2.35}$$

Many of the comments made in reference to the Schwarzschild horizon apply here as well. The singularities at r_1 and r_2 are coordinate singularities - curvature calculations show that the spacetime curvature remains finite at these values of r. As in the case of the Schwarzschild metric, an analytic extension of the metric can be constructed, and this was done by Graves and Brill (1960) for the case $m^2 > e^2$ and Carter (1966a) for the case $m^2 = e^2$.

Graves and Brill sought a coordinate transformation in which light cones became lines with slope ± 1, as they are traditionally represented in Special Relativity. In this way possible trajectories of objects are easily visualized. Following de la Cruz and Israel (1967) and Carter (1966a), we construct such coordinates below.

We begin by seeking a set of coordinates $u(r,t)$ and $v(r,t)$ such that radial incoming and outgoing null geodesics have u = constant and v = constant respectively. Setting $ds^2 = 0$ in Eqn. (2.23) we find

$$\frac{dr}{dt} = \pm f(r) \tag{2.36}$$

where

$$f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} = \frac{(r - r_1)(r - r_2)}{r^2} \tag{2.37}$$

In the exterior region $r > r_1$, choosing the '+' sign in Eqn. (2.36) would correspond to selecting the radial equation for light rays, as r increases with increasing t. The alternate form of (2.36) gives us the equation for incoming light rays. We may put the metric into the suggestive form

$$ds^2 = (-dt + f^{-1}dr)(dt + f^{-1}dr) + r^2 d\Omega^2 \tag{2.38}$$

In u,v coordinates the metric evidently has the form

$$ds^2 = \omega(u,v,r)dudv + r^2 d\Omega^2 \tag{2.39}$$

A convenient choice of the function ω is

$$\omega(u,v,r) = \frac{4k^2 f}{uv} \tag{2.40}$$

where k is a constant. Comparing (2.38) and (2.39) we find

$$2k\,\frac{du}{u} = -dt + f^{-1}dr \tag{2.41}$$

$$2k\,\frac{dv}{v} = dt + f^{-1}dr \tag{2.42}$$

These equations integrate to

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \exp\big((-t + r)/2k\big) \\ \exp\big((t + r)/2k\big) \end{Bmatrix} \cdot (r - r_1)^{r_1^2/2k(r_1-r_2)} \cdot (r - r_2)^{-r_2^2/2k(r_1-r_2)} \tag{2.43}$$

where the constants of integration have been chosen to vanish. Inspection of Eqn (2.43) reveals that we can not avoid both singularities with a single transformation of this type. We must therefore construct two sets of coordinates (u_1,v_1), (u_2,v_2) by choosing appropriate values for k. Letting

$$k_1 = \frac{r_1^2}{r_1 - r_2} \tag{2.44}$$

$$k_2 = \frac{-r_2^2}{r_1 - r_2} \tag{2.45}$$

we have

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \exp\left[\frac{r_1 - r_2}{2r_1^2}\left(\{{}^{-}_{+}\}\, t + r\right)\right](r - r_1)^{\frac{1}{2}}(r - r_2)^{-r_2^2/2r_1^2} \tag{2.46}$$

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \exp\left[\frac{r_1 - r_2}{2r_2^2} \left(\{^+_-\} t - r \right) \right] (r - r_1)^{-r_1^2/2r_2^2} (r - r_2)^{\frac{1}{2}} \tag{2.47}$$

The coordinates u_1, v_1 provide a regular covering for $r > r_2$, while u_2, v_2 cover the region $r < r_1$. the two coordinate systems may be matched at some value of r between r_1 and r_2 to provide a complete analytic extension.

One further transformation of coordinates, defined by

$$u = \tan \frac{1}{2} (\Psi + \xi) \tag{2.48}$$

$$v = \cot \frac{1}{2} (\Psi - \xi) \tag{2.49}$$

will ensure that light cones have slope ± 1. The complete manifold for $e^2 < m^2$ becomes in ξ, ψ coordinates a periodic lattice as shown in Figure 2.2.

Returning to the motion of the charged shell, let us consider the radial velocity of an infalling shell as seen by an external observer. Rearranging (1.82) we have

$$\left(\frac{dt_+}{dR}\right)^2 = \frac{X_+^2}{\dot{R}^2} = \frac{e^{-2\nu_+}}{\dot{R}^2} + e^{-4\nu_+} \tag{2.50}$$

This expression diverges as $R \to r_1$ - a shell imploding to this radius would require an infinite amount of time to reach it. In fact, arguments analogous to those of the previous section demonstrate that r_1 is an event horizon for the Reissner/Nordstrom metric. This fact precludes us from identifying the regions I_1 and I_2 on the Graves-Brill diagram as the same set of events. For example, we can construct a timelike curve from II_1 to I_2, in contradiction to the statement that an external observer (located in the region I_1) can never receive information from within the event horizon.

A new feature of the charged shell solution is the possibility of gravitational bounce. For purposes of illustration, we consider a dust shell in which there is no central charge or mass distribution. ($m_- = e_- = 0$).

Eqn. (2.27) tells us that the functions A, $\dot{B}$, and C occurring in (2.30) are constants. This equation can be put into the form

$$(1 + \dot{R}^2)^{\frac{1}{2}} = \frac{m}{M} + \frac{\alpha}{R} \tag{2.51}$$

where

$$\alpha = \frac{(M^2 - e^2)}{2M} \tag{2.52}$$

Differentiation of (2.51) yields

$$\ddot{R} = \frac{-\alpha}{R^2} (1 + \dot{R}^2)^{\frac{1}{2}} \tag{2.53}$$

An imploding shell experiences deceleration only if $\alpha < 0$, i.e. $e^2 > M^2$. In this case the bounce radius can be found by setting $\dot{R} = 0$ in (2.51):

$$R_b = \frac{(e^2 - M)}{2(m - M)} \tag{2.54}$$

If m > M then an observer on the shell experiences a bounce at R and subsequent re-expansion of the shell. A rather bizzare phenomenon seems to occur if $R_b < r_1$. We can easily construct such a case by making m sufficiently large (for example, by imparting to the shell a large initial velocity). According to the external observer the shell approaches the event horizon, r_1, but never re-expands. We seem to be forced to conclude, with de la Cruz and Israel (1967), that the shell re-expands into another asymptotically-flat space distinct from our own, as sketched in Figure 2.3.

We may also note that the electrostatic forces can not arrest the collapse of a dust shell down to the singularity at $r = 0$ if the mass of the shell is sufficiently large. In particular, if $a < 0$ then deceleration never occurs, and an imploding shell collapses all the way down to zero radius.

This is a simple illustration of the more general results of Kuchar (1968) and Chase (1970), which state that there is a limiting mass to any charged shell in equilibrium. The analysis is more complicated when the equation of motion is of the form (2.30), particularly when a surface pressure P is present, as M is no longer constant. In this case one must study the set of momentarily static equilibrium states and the instabilities associated with them.

The works cited above have demonstrated that electrostatic forces are unable to prevent the formation of singularities. Faced with the inevitability of singularities, relativists have put forward the cosmic censorship hypothesis. According to this conjecture, every singularity is clothed by an event horizon which prevents it from causally influencing an external observer. Various authors have used charged shell models to establish important results connected with the cosmic censorship hypothesis (Lake and Nelson, 1980; Proszynski, 1983). For example, Boulware showed that a charged shell can form naked singularity only if the energy density of the shell is negative. Similar results were reached by Hiscock (1981) for a magnetically charged shell.

2.3 Radiating Shells

The Schwarzschild metric gives an excellent approximation for the gravitational field of a spherical star when the energy density of the emitted radiation can be neglected. This approximation breaks down when the radiated energy becomes comparable to the mass of the object. In this section we consider some aspects of the radiating shell solution.

We will derive the radiating metric for the spherically symmetric case, as first discovered by Vaidya (1951a, 1951b, 1953) (see also Raychauduri (1953), Israel (1958), and

Lindquist, Schwartz and Misner (1965)). First we show that the stress-energy tensor in the exterior region can be assumed to have the form

$$T^{\alpha\beta} = \mu \ell^{\alpha} \ell^{\beta} \tag{2.55}$$

where μ is the radiation density and ℓ^{α} is a null vector field.

Consider a Lorentz observer who measures a plane electromagnetic wave propagating through free space. The electromagnetic field tensor is given by

$$T^{\alpha\beta} = \frac{1}{4\pi} \left(F^{\alpha\beta} F^{\beta}{}_{\mu} - \frac{1}{4} \eta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right) \tag{2.56}$$

where the Faraday tensor $F^{\alpha\beta}$ is defined by

$$F^{\alpha\beta} = \left\| \begin{matrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{matrix} \right\| \tag{2.57}$$

From (2.57) we can construct the Lorentz-invariant quantity

$$F^{\alpha\beta} F_{\alpha\beta} = 2(B^2 - E^2) \tag{2.58}$$

For a plane wave in free space, the **E** and **B** fields are orthogonal to each other and to the direction of propagation. In addition, we have E = B (see Jackson, Ch.7, 1962), so that $F^{\alpha\beta}F_{\alpha\beta}$ vanishes. We may take the direction of propagation of the wave to be along the z-axis, so that

$$F^{\alpha\beta} = E \begin{Vmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \tag{2.59}$$

we can express $F^{\alpha\beta}$ in terms of a null vector ℓ^{α} and a spacelike unit vector ω^{β} orthogonal to ℓ^{α}:

$$F^{\alpha\beta} = E(\ell^{\alpha}\omega^{\beta} - \ell^{\beta}\omega^{\alpha}) \tag{2.60}$$

where

$$\ell^{\alpha} = E(1,0,0,1) \tag{2.61}$$

$$\omega^{\alpha} = (0,1,0,0) \tag{2.62}$$

When Eqns. (2.58) and (2.60) are substituted into (2.56), $T^{\alpha\beta}$ reduces to the form of (2.55). We note that this equation is perfectly in accordance with our notion of photons as particles - it represents the stress-energy tensor for a stream of dustlike particles with lightlike 4-velocity. A more detailed discussion of the matters outlined above can be found in Synge (1956).

With the assumed form of $T^{\alpha\beta}$ (2.55) we can now attempt to solve Einstein's equations for a metric of the form

(1.71). Imposing the normalization condition

$$\ell^t \ell_t = -1 \tag{2.63}$$

and the condition for outflowing radiation

$$\ell^r/\ell^t > 0 \tag{2.64}$$

we find

$$T^t_t = -\mu \tag{2.65}$$

$$T^r_r = \mu \tag{2.66}$$

$$T^t_r = \mu e^{\lambda-\nu} \tag{2.67}$$

$$T^r_t = -\mu e^{\nu-\lambda} \tag{2.68}$$

We introduce a function m(r,t) defined by

$$m(r,t) = 4\pi \int_{R(t)}^{r} \mu(r',t)r'^2 dr' + m_s(t) \tag{2.69}$$

Here m_s is the mass energy of the shell/star, and R(t) is the boundary between the shell/star and the exterior spacetime. The function m(r,t) represents the total energy of mass and radiation inside a sphere of radius r at time t. We may write

$$\mu = \frac{m_{,r}}{4\pi r^2} \tag{2.70}$$

Einstein's equations (1.20), plus (1.73), (2.65) and (2.70) give

$$e^{-2\lambda}\left(\frac{1}{r^2} - \frac{2\lambda_{,r}}{r}\right) - \frac{1}{r^2} = -\frac{2m_{,r}}{r^2} \tag{2.71}$$

The integral of (2.71), taking into account the boundary conditions implied by (2.69), is the familiar form

$$e^{-2\lambda} = 1 - \frac{2m(r,t)}{r} \tag{2.72}$$

From (1.20), (1.76) and (2.68) we have

$$e^{\nu-\lambda} = -\frac{m_{,t}}{m_{,r}} \tag{2.73}$$

Equation (2.73) is equivalent to the statement that m=constant on any of the radial null geodesics defined by $ds^2 = 0$.

The radial Einstein equation, using (1.74), (2.72) and (2.73), can be put in the form

$$\left(\frac{m_{,tr}}{m_{,t}} - \frac{m_{,rr}}{m_{,r}}\right)\left(1 - \frac{2m}{r}\right) = \frac{2m}{r^2} \tag{2.74}$$

The angular components G^θ_θ, G^ϕ_ϕ (1.77) vanish identically. Vaidya (1951b) has pointed out that the integral of (2.74) is

$$m_{,r}\left(1 - \frac{2m}{r}\right) = f(m) \tag{2.75}$$

where f(m) is arbitrary. Thus the Vaidya metric in spherically symmetric form is

$$ds^2 = -\left[\frac{m_{,t}}{f(m)}\right]^2 \left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.76}$$

It is often convenient to write the metric in terms of the retarded coordinate defined by

$$du = d\big(\ln(f(m))\big) = \frac{1}{f(m)}\left(m_{,t}dt + m_{,r}dr\right) \tag{2.77}$$

so that

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 - 2du\,dr + r^2 d\Omega^2 \tag{2.78}$$

We note that as a consequence of the definition (2.77) the function m(u) is no longer dependent on the coordinate r. The stress-energy tensor, when calculated from Eqn. (2.78), takes on a pleasingly simple form, with a single non-vanishing component

$$T_{uu} = \frac{-m_{,u}}{4\pi r^2} \tag{2.79}$$

$T_{\alpha\beta}$ can thus be written

$$T_{\alpha\beta} = \frac{-m_{,u}}{4\pi r^2}(\partial_\alpha u)(\partial_\beta u) \tag{2.80}$$

which is of the form (2.55).

Now let us investigate the motion of a radiating shell which surrounds a central star. The interior and exterior metrics are of the form (2.76). We may again write down the equations of motion (1.89) and (1.92) immediately:

$$\left(1 - \frac{2m_+}{R} + \dot{R}^2\right)^{1/2} - \left(1 - \frac{2m_-}{R} + \dot{R}^2\right)^{1/2} = \frac{-M}{R} \tag{2.81}$$

$$\left(1 - \frac{2m_+}{R} + \dot{R}^2\right)^{1/2} + \left(1 - \frac{2m_-}{R} + \dot{R}^2\right)^{1/2} = \frac{2m_s}{M} \tag{2.82}$$

where

$$m_s = (m_+ - m_-)_{r=R} \tag{2.83}$$

can be thought of as the shell's contribution to the gravitational energy of the system.

Eqn. (1.66) is most easily evaluated using the retarded coordinate system. The analogs of Eqns. (1.78)-(1.82) are

$$u^\alpha = (X, \dot{R}, 0, 0) \tag{2.84}$$

$$u_\alpha = \left(-(e^{-2\lambda}X + \dot{R}), -X, 0, 0\right) \tag{2.85}$$

$$n^{\alpha} = (-X, e^{-2\lambda}X + \dot{R},\ 0,0) \tag{2.86}$$

$$n_{\alpha} = (-\dot{R}, X, 0, 0) \tag{2.87}$$

$$X = \frac{du}{d\tau} = e^{2\lambda}[-\dot{R} + (e^{-2\lambda} + \dot{R}^2)^{\frac{1}{2}}] \tag{2.88}$$

so that Eqn. (1.66) becomes a relation between the proper mass and the radiation flux through the shell:

$$\dot{M} = \dot{m}_{+}X_{+} - \dot{m}_{-}X_{-} \tag{2.89}$$

We can calculate the redshift factor for an observer at rest at spatial infinity. We consider, as in Section 2.1, an observer on the shell emitting pulses of light to be received by the external observer. If each pulse consists of n wavelengths then we must have

$$\omega_e \Delta\tau = n = -\omega_r \Delta u \tag{2.90}$$

where ω_e and ω_r are the frequencies of the emitted and received light pulses. The redshift is given by

$$\frac{\omega_e}{\omega_r} = -X_{+} \tag{2.91}$$

Noting that

$$X_+^{-1} = -\left(\dot{R} + (e^{-2\lambda_+} + \dot{R}^2)^{\frac{1}{2}}\right) \tag{2.92}$$

we find

$$\frac{\omega_e}{\omega_r} = -X_+ = \frac{1}{\dot{R} + (\dot{R}^2 + 1 - \frac{2m_+}{R})^{\frac{1}{2}}} \tag{2.93}$$

It is easy to show that spectral lines are blueshifted for $\dot{R} > m_*/R$ and redshifted for $\dot{R} < m_*/R$. A collapsing shell ($\dot{R} < 0$) always appears redshifted, and the redshift becomes infinite as the shell approaches the Schwarzschild horizon.

We can calculate the luminosity $\mathcal{L}$ of the system as measured by the observer at infinity. If $\underline{v}$ is the 4-velocity of the observer at rest on the surface $r = r_0$ and $\underline{n}$ is a unit vector normal to this surface, then

$$\mathcal{L} = \lim_{r_0\to\infty} 4\pi r_0^2(T_{\alpha\beta}v^\alpha n^\beta) = \frac{\dot{m}_+}{X_+} = \frac{\dot{M} - \dot{m}_- X_-}{X_+^2} \tag{2.94}$$

For an infalling shell ($\dot{R} < 0$) we have

$$\lim_{R\to 2m_+} (1 - \frac{2m_+}{R} + \dot{R}^2)^{\frac{1}{2}} = -\dot{R} \tag{2.95}$$

so that by Eqn. (2.88) the luminosity $\mathcal{L}$ vanishes in the limit as R approaches the Schwarzschild radius.

We could consider, as a crude approximation to a planetary nebula surrounding a star, a model in which

$$\dot{M} = 0 \tag{2.96}$$

$$\dot{m}_- = k \tag{2.97}$$

The shell radiates as much energy as it receives from the interior region. From (2.80), (2.81), and (2.89) we may write

$$\dot{R}^2 = \left(\frac{m_s}{M}\right)^2 + \frac{m_+ + m_-}{R} + \frac{M^2}{4R^2} - 1 \tag{2.98}$$

$$\dot{m}_+ = -k \left[\frac{\dot{R} + \left(1 - \frac{2m_+}{R} + \dot{R}^2\right)^{\frac{1}{2}}}{\dot{R} + \left(1 - \frac{2m_-}{R} + \dot{R}^2\right)^{\frac{1}{2}}} \right] \tag{2.99}$$

Eqns. (2.97)-(2.99) may be integrated numerically using physically reasonable initial conditions. Investigations of this sort have been carried out by Hamity and Gleiser (1978), Castagnino and Umerez (1983), and Hamity and Spinosa (1984).[3] It is of interest to cite a particular numerical calculation . A set of physically reasonable initial conditions is

$$m_+ X_+ = (5 \times 10^3 - 10^4)\mathcal{L}_\odot \tag{2.100}$$

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[3]The numerical work of Pim and Lake(1985) , who investigated the collapse of radiating shells, should also be mentioned. In this work two luminosity models are considered - one in which $\mathcal{L}$ varies as the square of the radius of the shell, and one in which it is independent of the radius.

$$R_0 = 200R_\odot \tag{2.101}$$

$$m_0 = 1M_\odot \tag{2.102}$$

$$m_s = 0.1M_\odot \tag{2.103}$$

$$\dot{R}_0 = 190 - 200\text{km/sec} \tag{2.104}$$

Typically, velocity-time graphs exhibit rapid descent (in about 10 years) in $\dot{R}$ from 200 to 20 km/sec, after which time the velocity of the shell decreases very slowly. The final velocity is reached after 2 x 10^4 years and ranges from 5-50 km/sec. The radius of the shell at this time is (0.2-1.0) pc (Hamity and Spinosa, 1984). Light curves of such models bear clear resemblance to those of novae and supernovae.

It should be emphasized that the relativistic effects in these models are typically very small. For example, in most supernovae we observe

$$\dot{R}_0^2 \cong 10^{-5} \tag{2.105}$$

with the factor $2m_*/R$ two orders of magnitude less; consequently such investigations do not constitute important tests of GR.

The Vaidya metric may be adjusted to incorporate other energy sources. The metric for a charged radiating shell, for example, is

$$ds^2 = -(1 - \frac{2m(u)}{r} + \frac{e^2}{r^2})du^2 - 2dudr + r^2 d\Omega^2 \tag{2.106}$$

Frolov (1974) has investigated the dynamics of such systems in some detail.

We next turn briefly to a metric which would be of interest to cosmologists. Standard cosmological models assume that the stars and more massive objects can be treated collectively as a perfect fluid. In the vicinity of a star, this approximation must break down and a more plausible metric would be that of a Schwarzschild or Vaidya type imbedded in a cosmological background (Gautreau, 1984). If the radiation of the object is significant, then we should attempt to imbed a Vaidya metric in a cosmological background.

There are three static metrics which satisfy the condition of spherical symmetry and in which the stress-energy tensor takes the form of a homogeneous and isotropic fluid - these are the Einstein, de-Sitter, and Minkowski metrics (Tolman, 1934). A Vaidya/de-Sitter metric in r,t coordinates has been constructed by Mallett (1985). In retarded form (Vick, 1985) the metric is

$$ds^2 = -(1 - \frac{2m(u)}{r} - \frac{\Lambda}{3} r^2)du^2 - 2dudr + r^2 d\Omega^2 \tag{2.107}$$

where Λ is the cosmological constant, and the arbitrary function m(u) represents the mass of the star at retarded time u.

A straightforward calculation then gives

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} - \Lambda g_{\alpha\beta} \tag{2.108}$$

where T has the form (2.80) required for photons.

In the limit as m, or alternatively, λ, goes to zero, we recover the correct limiting solutions, i.e. the de-Sitter or Vaidya solutions.

2.4 Shells In Cosmology

The preceding discussion forms an appropriate bridge to the material of the present section. The de-Sitter type universe was thought until recently to be a topic of mathematical interest only, since this solution to the Einstein equations does not admit the presence of matter or radiation. In the new inflationary universe models de-Sitter metrics feature prominently.

The inflationary scenario was introduced by Guth to solve some of the outstanding problems in standard cosmological theory (Guth, 1981). The standard theory provides a reliable framework for the evolution of the universe from about 0.01 seconds after the Big Bang (Weinberg, 1972). The model assumes an adiabatically expanding radiation-dominated universe described by the Robertson-Walker metric:

$$ds^2 = -dt^2 + R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{2.109}$$

where $k = +1$, 0, or -1 depending on whether the universe is closed, flat, or open. If the universe is closed, then the function $R(t)$ can justly be called the 'radius of the universe'.

The standard model has been very successful in explaining certain features of the universe - notably the expansion of the cosmos, the thermal background 3° radiation discovered by Penzias and Wilson, and the relative abundances of all the lighter elements in the universe. Marriage of the standard theory with Grand Unified Theories (GUT's) has also produced a possible mechanism to account for the observed matter-antimatter asymmetry (Wilczek, 1982). It was first realized by Sakharov (1979) that if we had baryon-number violating processes, the baryon asymmetry could be established during the early stages of the expansion of the universe.

Several puzzles nevertheless remain. First, the universe is isotropic to a remarkable degree (Uson and Wilkinson, 1984). This feature must be built into the initial conditions of the standard theory, and is therefore left unexplained. Second, the mass density of the universe is remarkably fine-tuned to the critical density, i.e., that density which would make the universe flat. The third puzzle is known as the horizon problem. When astronomers point their instruments in different directions into space, they observe similar conditions in sectors of the universe that, due to the finite speed of light and the expansion of the universe, could never have communicated with each other during the history of the expansion. It is therefore unclear why these causally isolated regions of space are in thermal equilibrium with each other (Guth, 1981). Finally, the same marriage which produced an explanation for baryon asymmetry predicts an overproduction of magnetic monopoles, which have not been observed (Preskill, 1979).

The key mechanism in the inflationary models involves a phase transition predicted by GUT theories. This phase transition occurs at very high temperatures (Ø(10^{14}GeV)) or equivalently, at a very early time after the appearance of the universe. Mathematically the process involves the Spontaneous Symmetry Breaking (SSB) of a higher symmetry group (eg., SU(5)), into SU(3)xSU(2)xU(1) groups; physically the phase transition involves the separation of the strong interaction from the electroweak interaction. From the cosmologist's point of view the important feature of this phenomenon is that the transition time for the phase change is sufficiently long to allow the universe to enter a supercooled state. The supercooled state is a false vacuum with positive vacuum energy of the order of the phase transition energy, and can be described by a de-Sitter metric. The inflationary era lasts less than 10^{-30} seconds - after this time the standard model describes the universe adequately. A more detailed account of the technicalities involved in this process is given by Brandenberger (1985).

In the original inflationary scenario, bubbles of the true vacuum nucleated in a de-Sitter background, grew and coalesced, replacing the false vacuum by the true vacuum. A simple model of such a bubble is provided by imbedding a shell in de-Sitter/Minkowski metrics. The equations of motion (1.89) and (1.92) are

$$(1 - \frac{2m}{R} - \frac{\Lambda}{3} R^2 + \dot{R}^2)^{\frac{1}{2}} - (1 + \dot{R}^2)^{\frac{1}{2}} = - \frac{M}{R} \tag{2.110}$$

$$(1 - \frac{2m}{R} - \frac{\Lambda}{3} R^2 + \dot{R}^2)^{\frac{1}{2}} + (1 + \dot{R}^2)^{\frac{1}{2}} = \frac{2m}{M} + \frac{\Lambda R^3}{3M} \tag{2.111}$$

From the equations above we derive

$$(1 + \dot{R}^2)^{\frac{1}{2}} = \frac{m}{M} + \frac{\Lambda R^3}{6M} + \frac{M}{2R} \tag{2.112}$$

$$\ddot{R} = (\frac{\Lambda R^2}{2M} - \frac{M}{2R^2})(1 + \dot{R}^2)^{\frac{1}{2}} \tag{2.113}$$

If a bubble nucleates at rest with a sufficiently large radius it accelerates outwards and asymptotically approaches the speed of light, as can be seen by calculating the limit

$$\lim_{R\to\infty} \frac{dR(t)}{dt} = \lim_{R\to\infty} \frac{\dot{R}}{X_+} = 1 \tag{2.114}$$

Bubbles of this type have been investigated by Berezin et al (1983). Bubble collisions have been considered by Hawking et al (1982) and Chao (1983), and the formation of black holes and wormholes from nucleation processes have been studied by Sato et al (1981). Hiscock (1984) has considered the nucleation and growth of a bubble of anti-de-Sitter space into a Minkowski vacuum.

The original inflationary theory was successful in solving the outstanding problems of the standard theory. Essentially the solutions lie in the fact that the observable part of the cosmos is but a tiny fraction of the whole universe. Since this observable region has expanded from a minuscule fraction of the early universe, in which thermal equilibrium could be achieved, there is no difficulty explaining the present day isotropy and homogeneity. The monopole problem is solved by dispersal - monopoles are scattered throughout a volume many orders of magnitude larger than the 'standard' universe. In addition, the inflationary theory predicts that the density of the universe is almost exactly equal to the critical density. An added bonus was provided in that the expansion phase created the scale free density fluctuations which were required in order to produce galaxies (Silk, et al, 1983).

Unfortunately, the original inflationary theory suffered from runaway expansion - the de-Sitter space grew so quickly that the nucleating bubbles were unable to convert the false vacuum to the true vacuum - the universe was unable to exit from its inflationary state (Guth and Weinberg, 1981). In the new inflationary scenario this problem is overcome (Albrecht and Steinhardt, 1982; Linde, 1982; Linde, 1984) but there are difficulties in producing the type of inhomogeneities necessary to explain the universe at its present state (Guth and Pi, 1982; Bardeen et al, 1983). In the new scenario the bubble model is no longer appropriate; instead the observable universe may be surrounded by high energy domain walls. Any number of domains may exist, each containing its own causally isolated universe! Domain walls have been investigated by Zel'dovich et al (1975), Isper and Sikivie (1984) and Isper (1984) (see also Horsky, (1968)). The walls exhibit the bizarre property - due to tension - of gravitational repulsion. A space ship near to such a wall must thrust towards it in order to maintain its distance.

Despite the remaining difficulties posed by inflationary theories, they remain to date the most promising cosmological models.

We conclude this section by mentioning some interesting recent investigations into the growth of voids in the universe. The collapse of density perturbations in the early universe could have been the process whereby galaxies were formed. In the 'pancake' theory, clouds of matter contract preferentially along one direction, endowing the universe with a cellular-type structure in which large voids are a prominent feature (Silk et al, 1983). These voids may subsequently grow in size. A simple model of such an expansion is a thin shell separating two distinct

Robertson-Walker type metrics. Lake and Pim (1985) have performed numerical investigations into the conditions under which voids will and will not grow, extending earlier work on this subject (Maeda and Sato, 1983a, 1983b; Sato et al, 1984). It has been pointed out that non-uniform expansion rates due to the presence of voids could have an effect on the Hubble constant (Sato, 1985)

2.5 Rotating Shells, Disks, And Cylinders

Although the spherically symmetric solutions to Einstein's equations are instructive, the sources in real physical systems generally possess angular momentum. It is therefore of great interest to investigate the effect of rotation in GR. As might be expected, this is a non-trivial undertaking. In contrast to the Newtonian theory, the rotational energy and inertial dragging effects of the shell must contribute to the spacetime metric. Compounding the problem is the fact that a rotating body will in general be deformed from a spherical shape. One may gain some simplification in the problem at the expense of physical reasonableness by considering counter-rotating bodies, i.e. systems which consist of two equal masses of non-interacting material which are co-radial and possess equal and opposite angular momentum. In this way the inertial properties of the rotation may be studied without consideration of the inertial dragging effects on the external spacetime, which may be taken to be static. Evans (1977) and Papapetrou and

Hamoui (1974, 1979) have used shell models in this way.

One of the primary uses of thin rotating shell models has been the investigation of Machian effects in GR. There is no clear consensus as to how Mach's Principle should be formulated. One accepted Machian effect, however, is that inertial frames ought to be in part determined by the relative rotation of surrounding bodies. The first investigations into this effect were done by Thirring (1918), who used the linear weak field approximation to Einstein's equations to show that a slowly rotating shell partially drags along the inertial frames within it.

An important milestone was the discovery by Kerr (1963) of an axially symmetric vacuum metric which seemed to represent the exterior gravitational field of a nearly spherical body:

$$ds^2 = (r^2 + a^2\cos^2\theta)[dr^2/(r^2 - 2mr + a^2) + d\theta^2]$$
$$+ \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\phi^2$$
$$+ \frac{4amr}{r^2+a^2\cos^2\theta}\sin^2\theta d\phi dt - \left(1 - \frac{2mr}{r^2+a^2\cos^2\theta}\right)dt^2 \qquad (2.115)$$

In the limit as the parameter a tends to zero, the Schwarzschild metric is recovered. An analytic extension of the metric was given by Carter (1966b). Brill and Cohen (1965) generalized Thirring's results to a shell of arbitrarily large mass by approximating the external field of the shell as a first order correction to a static spherically symmetric solution, treating the angular velocity ω as a small parameter. The metric agreed with the Kerr solution to first order in a. Further generalizations of this work were made by de la Cruz and Israel (1968), who extended the results above to 3rd order, Lindblom and Brill (1974), who studied the collapse of a rotating shell, and Orwig (1978), who extended the study to rapidly rotating shells.* These studies vindicated the Kerr metric as a representation of the external field of a spinning body and confirmed the effects predicted by Mach - a rotating shell drags the interior flat spacetime around with it. If the radius of the shell is large this effect is not pronounced, but as the shell approaches its Schwarzschild radius the frame dragging becomes absolute - the inertial frames in the interior of the shell rotate at the same angular velocity as the shell.

Machian effects in non-asymptotically flat spacetimes were considered by Lewis (1980). Other studies involving spinning shells include investigations into tidal friction in black holes (Wilkins, 1975) and the equivalence principle paradox (Hartle, 1973).

For the sake of completeness, two other types of shell configurations should be mentioned. The first is the rotating thin disk. A static solution for a disk of counter-rotating dust particles has been given by Morgan and

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*An important related paper which deals with dust clouds was authored by Kegeles (1978).

Morgan (1969, 1970). Although admittedly unphysical, the model provides an alternative configuration to the spherically symmetric cases in which the effects of strong gravitational fields can be investigated. More interesting from a physical standpoint is the numerical work of Bardeen and Wagoner (1971), who did not restrict themselves to counter - rotating disks.

The thin disk has also found a use in the study of the Kerr metric (2.115). An interesting feature of the Kerr geometry is the existence of an equatorial disk centered along the axis of symmetry; the ringlike boundary of the disk comprises a singularity. Disks have been used to provide a fictitious 'equivalent source' for a Kerr geometry in order to investigate its properties (Israel, 1970b; Hamity, 1976).

The second shell configuration is the infinitely long rotating cylinder. Such models are obviously unphysical, but can be used to verify effects (such as frame dragging) predicted by more realistic models (see Langer 1969, 1970; Frehland, 1971; Voorhees, 1972; McCrea, 1976; Jordan and McCrea, 1982).

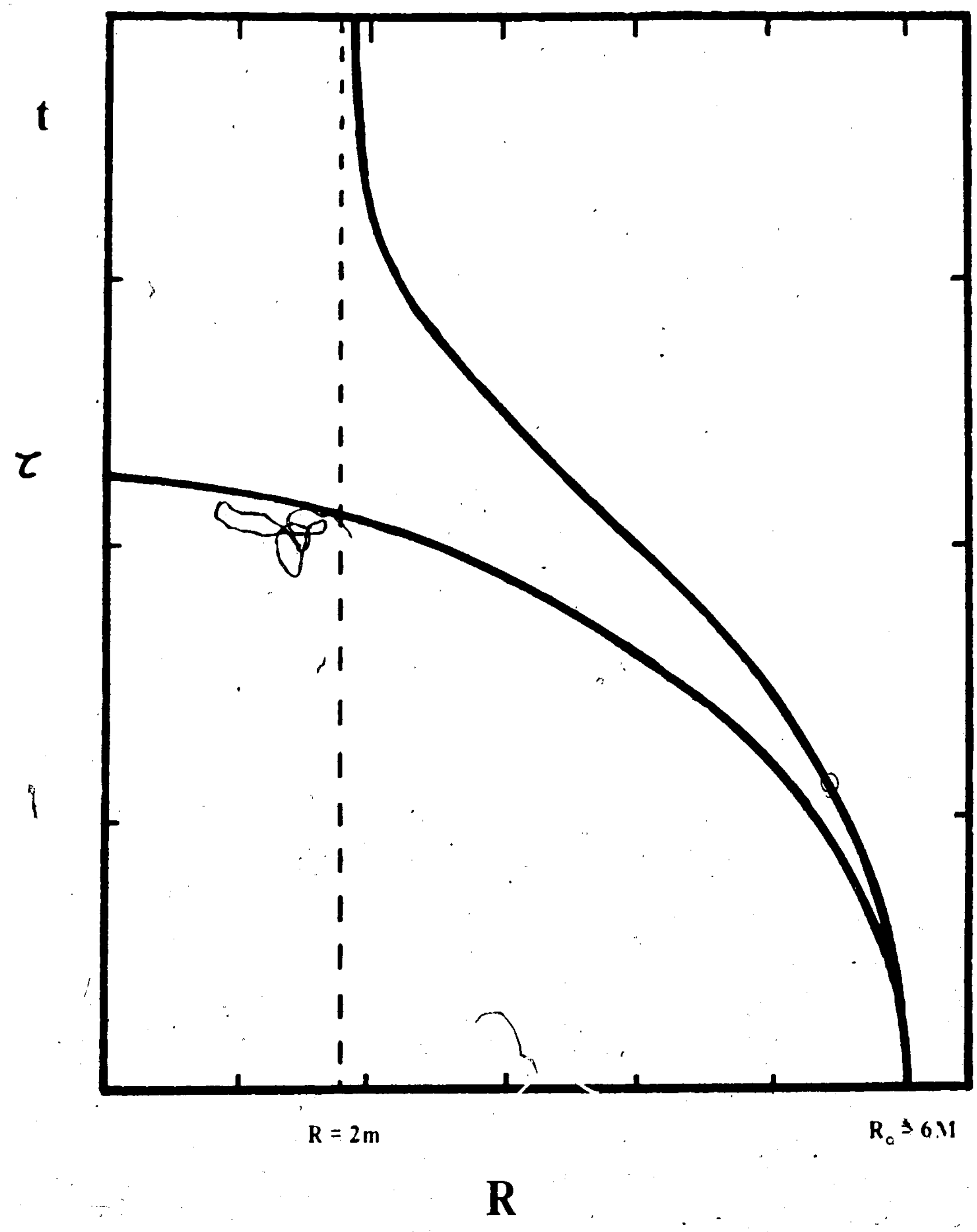

Figure 2.1 The Schwarzschild and proper time coordinates plotted as a function of the shell radius.

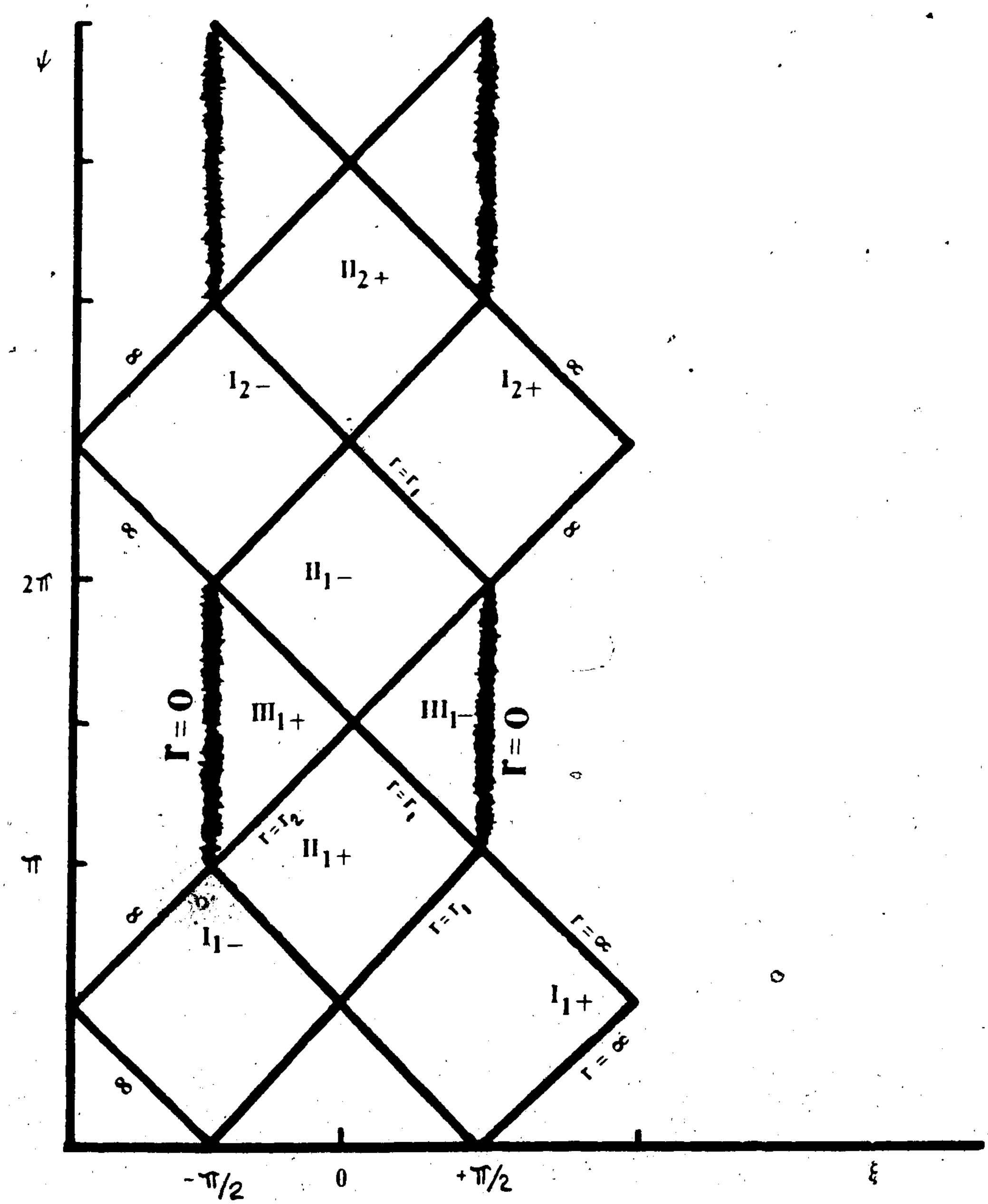

Figure 2.2 The Graves-Brill diagram for the Reissner-Nordstrom metric.

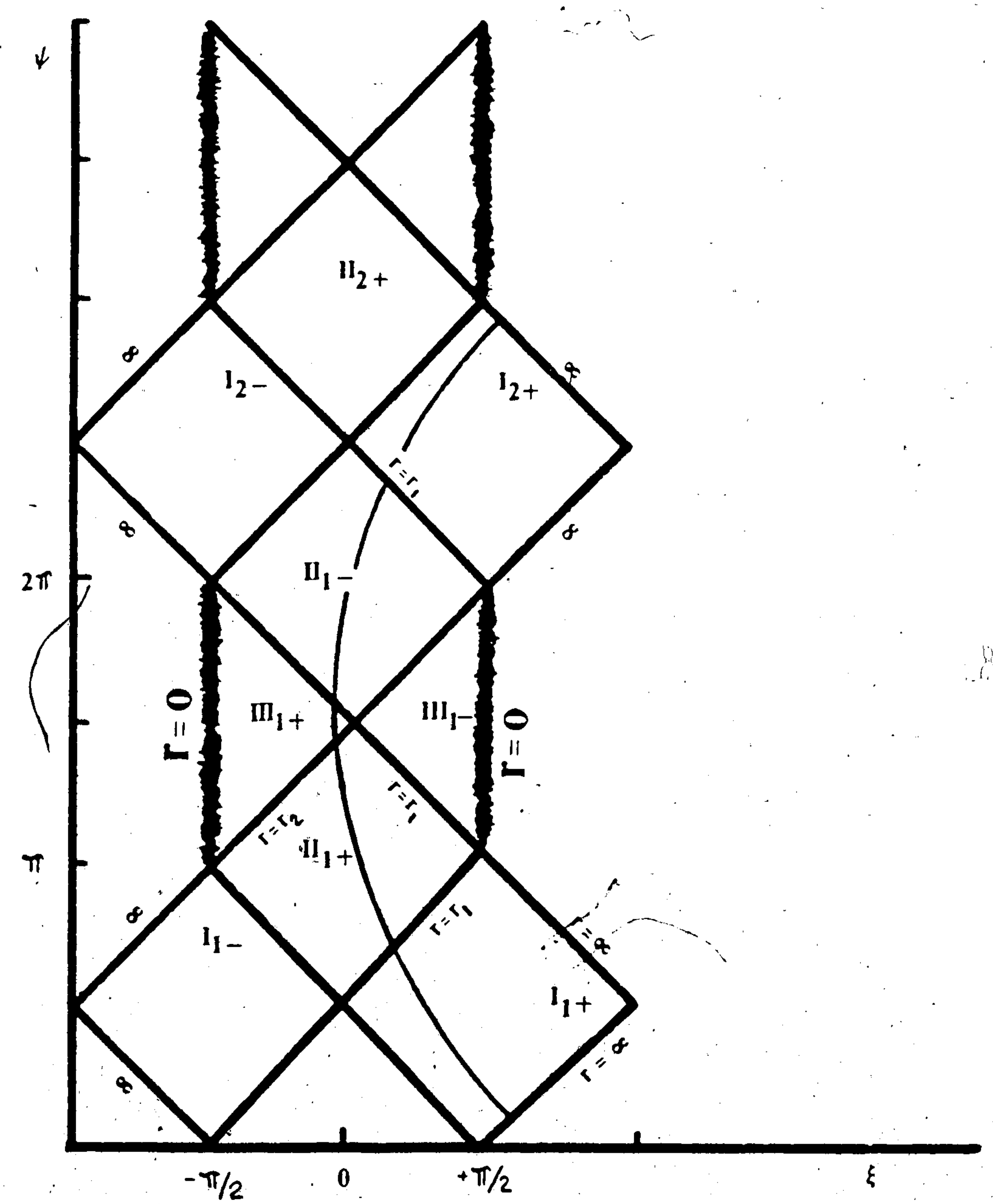

Figure 2.3 The world line of a charged shell exhibiting gravitational bounce and subsequent re-expansion into an asymptotically flat space.

CHAPTER III

Gravitational Collapse Of A Shell Subject To The Casimir Force

3.1 The Casimir Effect

This phenomenon was predicted by Casimir (1948) and later experimentally verified by Sparnaay (1958) and Silfout (1966). It arises from the subject of quantum electrodynamics , which predicts that the ground states of the electromagnetic fields can be non-zero even after renormalization, in contrast to the classical picture. As a consequence the renormalized stress-energy tensor of the vacuum $\langle T^{\mu\nu} \rangle$ can be non-vanishing. As in the case of classical fields, the presence of boundary conditions in a vacuum can select preferred modes of the fields and thus alter the energy of the vacuum from its value for the unbounded case. An interesting and readable account of the role of the vacuum in modern physics has been written by Aitchison (1985).

As an example, let us follow the simple arguments of DeWitt (Hawking and Israel, 1979) to predict the nature of the Casimir force acting on two parallel conducting plates. We consider first a single infinite conducting plane located at $z = 0$. The Lorentz boost for an observer skimming over the plane with velocity β in the x-direction is

$$\Lambda^{\mu}_{\nu} = \left\| \begin{array}{cccc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \tag{3.1}$$

Symmetry considerations demand that $T^{\mu\nu}$ be diagonal

$$<T^{\mu\nu}> = \left\| \begin{array}{cccc} <T^{00}> & & & 0 \\ & <T^{11}> & & \\ & & <T^{22}> & \\ 0 & & & <T^{33}> \end{array} \right\| \tag{3.2}$$

and independent of the coordinates t,x,y. A perfect conductor remains so under a Lorentz boost parallel to its surface, so that the moving observer should see the same stresses as the one at rest with respect to the conductor.

Applying the boost (3.1) to $<T^{\mu\nu}>$ we find $T^{00} = -T^{11}$. Evidently, then, a boost in the y-direction must yield $T^{00} = -T^{22}$.

The electromagnetic stress-energy tensor is given by Eqn. (2.56), with the Faraday tensor $F^{\alpha\beta}$ given by Eqn. (2.57).

One can see from Eqn. (2.56) that

$$T^{\mu}_{\mu} = 0 \tag{3.3}$$

From this condition we conclude

$$\langle T^{\mu\nu}\rangle = f(z)\begin{Vmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & -3 \end{Vmatrix} \tag{3.4}$$

Applying the conservation laws $\langle T^{\alpha\beta}\rangle_{,\beta} = 0$, we have

$$\langle T^{z\beta}\rangle_{,\beta} = -3f'(z) = 0 \tag{3.5}$$

which implies that f is a constant. There exist no natural units of length in this configuration, and we cannot construct f, which must have units of energy, from the only other available constants, $\hbar$ and c. Therefore f = 0 and no Casimir force is experienced.

Suppose another plane is now introduced at z = a. The arguments leading to Eqn. (3.5) are still valid, but now a natural unit of length exists - the separation distance between the plates. The stresses in the half-spaces above and below the plates have already been seen to vanish. From Eqn. (3.4) the energy density per unit area is E = -af(a), while the force per unit area is P = 3f(a). If this force displaces the plane by da we must have

$$dE = -daf(a) - ad\big(f(a)\big) = dW = 3f(a)da \tag{3.6}$$

The integral of Eqn. (3.6) is

$$f(a) = \frac{k}{a^4} \tag{3.7}$$

where k is a constant. This result has been verified experimentally. A more explicit calculation of the effect yields $k = \pi^2/720$.

In order to better illustrate how the Casimir forces arise, we return once more to a spherically symmetric configuration. Encouraged by the attractive nature of the forces on the parallel conducting planes , Casimir suggested that the electron might be modelled semiclassically by a spherical charged conducting shell in which electrostatic repulsion was balanced by an attractive Casimir self-interaction. Unfortunately, Boyer (1968) subsequently showed that the force on a conducting shell was actually repulsive.

We can outline the important concepts in Boyer's calculation by considering a single conducting shell of radius R. In the absence of sources the classical electromagnetic fields obey

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\underline{E} = 0 \tag{3.8}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\underline{B} = 0 \tag{3.9}$$

In the interior region the solution to Eqn. (3.8) takes the form

$$E = \int \frac{d\omega}{c} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{+\ell} a_{\ell m} j_\ell(\frac{\omega}{c} r) Y_{\ell m}(\theta,\phi) e^{i\omega t} \tag{3.10}$$

The boundary condition at $r = R$ implies that there are preferred modes of oscillation with frequencies

$$\omega_{\ell ms} = \frac{c z_{\ell s}}{R} \qquad m = 1,2,\ldots,2\ell + 1 \tag{3.11}$$

where the $z_{\ell s}$ are the zeros of the spherical Bessel functions j_ℓ.

Quantum mechanically, the electromagnetic fields are described by a set of independent harmonic oscillators with energy eigenstates

$$E_{\ell mns} = (n + \tfrac{1}{2})\hbar w \tag{3.12}$$

where the $\omega_{m\ell s}$ are the classical frequencies in Eqn. (3.11). The lowest possible vacuum energy in the interior is thus,

$$E_o = \frac{\hbar c}{2R} \sum_{\ell=1}^{\infty} (2\ell + 1) \sum_{s=1}^{\infty} z_{\ell s} \tag{3.13}$$

This sum is of course infinite, as would be the corresponding expression for the exterior region. From a physical point of view, however, we are interested not in these sums but in the energy difference between the shell configuration and the unperturbed vacuum. Boyer's ingenious solution to this problem involved a configuration of two concentric spheres of radius R_1, R_2 (see Figure 3.1). Boyer calculated the difference between the energy of the configurations A and B, and then let the radius $R_2 \to \infty$. It was necessary to introduce ad hoc a cutoff function which suppressed high frequency terms in the energy sum in order to achieve a finite result.

Boyer found that

$$\Delta E(R) = \frac{C}{2R} \tag{3.14}$$

where $C \cong 0.09$ ($\hbar$ and c have been put equal to unity). A more sophisticated calculation was subsequently done by Milton, DeRand, and Schwinger (1978). This calculation made use of Green's function techniques, and required no assumption regarding a cutoff function or the presence of an outer shell. It is a credit to Boyer's physical intuition that his result (3.14) was confirmed, with $C \cong 0.09235$.

3.2 The Casimir Force and Gravitational Collapse

A number of authors have studied the influence of quantum effects on the late stages of gravitational collapse, and in particular the question of whether they can suppress singularities (Birrell and Davies, 1982). An interesting recent paper by Brevik and Kolbenstvedt (B & K) (1984) discusses the role of the Casimir force in this respect. [5] Simple energetic considerations applied to Eqn.

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[5] A version of the remainder of this chapter has been submitted for publication in Nuovo Cimento.

(3.14) imply that there is an outward radial pressure acting on the sphere given by

$$P_c = \frac{C}{8\pi R^4} \tag{3.15}$$

The repulsive nature of the Casimir force and the fact that it becomes unbounded as $R \to 0$ suggests that it may prevent such a shell from collapsing all the way to a singularity at zero radius. B & K therefore examined the behavior of a collapsing spherical shell which is subject to a Casimir force of the form (3.15).

As discussed in Section 1 of Chapter II, the 'mean acceleration' $1/2(a_+ + a_-)$ vanishes for a shell subject to gravitational forces only. To take the Casimir force into account, B & K modify this law by adding ad hoc a repulsive Casimir term into the radial equation of motion:

$$M \cdot \frac{1}{2}(a_{n_+} + a_{n_-}) = 4\pi R^2 P_c \tag{3.16}$$

where M is the proper mass of the shell and P_c is given by Eqn. (3.15). The authors assume that the exterior metric is Schwarzschild and the interior metric is flat, thus neglecting entirely (as they themselves explicitly recognize) all gravitational effects of the Casimir energy (3.14).

The radial equation arrived at, by integration of Eqn. (3.16) is

$$(1 + \dot{R}^2)^{\frac{1}{2}} = b - \frac{C}{2MR} + \frac{M}{2R}\frac{1}{(b - C/2MR)} \tag{3.17}$$

$$b = \frac{1}{2}\left[1 + (1 - 2M/R_o)^{\frac{1}{2}} + C/MR_o\right] \tag{3.18}$$

where $\dot{R}$ is $dR/d\tau$ and R_o is the initial radius of the shell, which is assumed to collapse from rest. 1-b represents the binding energy per unit mass - shells with 1-b>0 are gravitationally bound in the sense that they cannot reach infinite radius.

Eqn. (3.17) has highly paradoxical consequences. As the shell collapses, the Casimir force at first acts to retard the collapse, as one would expect intuitively (effect of the term C/2MR). But, as the shell contracts toward the critical radius $R_{cr} = C/2Mb$, the last term of Eqn. (3.17) becomes dominant, and the effect of the Casimir force would be to catastrophically accelerate the collapse. If Eqn. (3.17) were correct the shell would hit the critical radius with an infinite speed and bounce even though there is no geometrical singularity at this radius.

These results are difficult to understand. In the remainder of this chapter, it will be argued that they are a consequence of the inconsistent neglect of the gravitating effects of the Casimir energy. This inconsistency becomes manifest at radii for which the Casimir energy (3.14) becomes comparable with the mass of the shell, i.e. precisely at the critical radius R_{cr}. In the discussion following, we develop two models of the Casimir collapse which are consistent with the field equations. In the first model, the Casimir energy is assumed to be distributed in the exterior spacetime. In the second, the Casimir energy is placed in the interior region.

3.3 Effect Of The Casimir Force On Collapse: Model I

The difficulty with using a Schwarzschild/Minkowski geometry in the Casimir problem is evident upon consideration of Eqn. (1.69). As $T_{\alpha\beta}$ vanishes for both exterior and interior metrics, the right hand side of Eqn. (1.69) vanishes for a dust shell (recall that the term P occurring in the equation is the surface pressure and is thus distinct from the Casimir pressure P_c occurring in Eqn. (3.15)), so that there is no way of consistently introducing the Casimir term in the right hand side of Eqn. (3.16). In the paper cited above this error is compounded by a failure to distinguish between the proper mass of the shell M and the total gravitational mass energy m which occurs in the Schwarzschild metric. The correct integral of (3.16) is

$$(1 - \frac{2m}{R} + \dot{R}^2)^{\frac{1}{2}} + (1 + \dot{R}^2)^{\frac{1}{2}} + \frac{C}{MR} = 2b' \tag{3.19}$$

where b' is a constant of integration.

In an attempt to salvage B & K's approach one might look for a solution to Einstein's equations which satisfies

$$[T^{\alpha}_{\beta} n_{\alpha} n^{\beta}] = - \frac{C}{8\pi R^4} \tag{3.20}$$

and consequently, also equation (3.16). We first seek a solution in which the interior spacetime is flat ($e^{2\nu_-} = e^{2\lambda_-} = 1$). If the proper mass of the shell is to remain constant, then Eqn. (1.66) and $\underline{u} \cdot \underline{n} = 0$ imply

$$T^t_t = T^r_r \tag{3.21}$$

while from Eqns. (3.20), (3.21), and $\underline{n} \cdot \underline{n} = 1$ we get

$$G^t_t \Big|_{r=R} = 8\pi T^t_t \Big|_{r=R} = - \frac{C}{R^4} \tag{3.22}$$

From Eqn. (3.22) and the expressions for the Einstein tensor it is easy to verify that $e^{2\nu_+} = e^{-2\lambda_+}$. We can now solve for $e^{-2\lambda_+}$ from

$$G^t_t = e^{-2\lambda_+} \left(\frac{1}{r^2} - \frac{2\lambda_{+,r}}{r}\right) - \frac{1}{r^2} = - \frac{C}{r^4} \tag{3.23}$$

The integral of Eqn. (3.23) is

$$e^{-2\lambda_+} = 1 - \frac{2m}{r} + \frac{C}{r^2} \tag{3.24}$$

where m is an arbitrary constant. Thus the exterior metric is

$$ds^2 = -(1 - \frac{2m}{r} + \frac{C}{r^2})\,dt^2 + (1 - \frac{2m}{r} + \frac{C}{r^2})^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.25}$$

The similarity between Eqn. (3.25) and the well-known Reissner/Nordstrom metric, which describes the exterior spacetime of a spherical distribution of charge e, is not surprising. The electrostatic energy of a charged shell is $e^2/2R$, which is of the same form as the Casimir energy (3.14). We might therefore expect qualitatively similar types of motion in both cases.

The imbedding metrics having been established, we may immediately write down the equations of motion (1.89) and (1.92):

$$(1 - \frac{2m}{R} + \frac{C}{R^2} + \dot{R}^2)^{\frac{1}{2}} + (1 + \dot{R})^{\frac{1}{2}} = \frac{2m}{M} - \frac{C}{MR} \tag{3.26}$$

$$(1 - \frac{2m}{R} + \frac{C}{R^2} + \dot{R}^2)^{\frac{1}{2}} - (1 + \dot{R}^2)^{\frac{1}{2}} = -\frac{M}{R} \tag{3.27}$$

Solving for F., we may write

$$(1 + \dot{R}^2)^{\frac{1}{2}} - 1 = \frac{m-M}{M} + \frac{\alpha}{R} \equiv g(R) \tag{3.28}$$

where

$$\alpha = \frac{(M^2 - C)}{2M} \tag{3.29}$$

Motion can occur only in regions where the function $g(R)$ is positive. We can identify three cases:

1. $m>M$, $\alpha>0$: in this case all values of R are possible. We can think of the shell as imploding from an infinite radius and collapsing down to $R=0$.
2. $m<M$, $\alpha>0$: the shell falls from rest at $R = \alpha M/(M-m)$ and collapses to $R=0$.
3. $m>M$, $\alpha<0$: the shell implodes from infinity to a finite radius $R = -\alpha M/(m-M)$, implying a bounce there.

Differentiating Eqn. (3.28) one finds

$$\ddot{R} = -\frac{\alpha}{R^2}(1+\dot{R}^2)^{\frac{1}{2}} \tag{3.30}$$

If $\alpha>0$ then deceleration never occurs. From Eqn. (3.29) we can identify an upper limit $M_{max} = C^{1/2}$ beyond which an initially static shell must collapse down to zero radius. Taking $C = 0.09235$ and inserting appropriate factors of $\hbar$, c, and G, one finds $M_{max} \cong 6.6\ \mu g$!

In the model above the Casimir energy is assumed to be distributed wholly in the exterior region; if the energy density $-T^{t}{}_{t}$ is integrated (in flat spacetime) from R to ∞ the Casimir energy is recovered.

3.4 Model II: Interior Energy Solution

As a further example, let us place the Casimir energy in the interior region. The exterior spacetime is assumed to be Schwarzschild ($e^{2\nu_+} = e^{-2\lambda_+} = 1-2m/r$). If we choose

$$e^{2\nu_-} = e^{-2\lambda_-} = 1 - \frac{C_r^2}{R(t)^4} \tag{3.31}$$

where R(t) represents the radius of the shell, then we find the interior energy density to be spatially uniform but non-static:

$$T_t^t = \frac{-3\dot{C}}{8\pi R(t)^4} \tag{3.32}$$

Eqn. (3.32) also has the intuitively appealing property mentioned above, namely, that when the energy density is integrated in flat spacetime over the interior of the shell the correct Casimir energy is recovered.

Explicit evaluation of Eqn. (1.66) yields

$$\dot{M} = \frac{2C\dot{R}}{R^2 F_-} (1 + 2e^{2\lambda_-} \dot{R}^2) \tag{3.33}$$

while the equations of motion (1.89) and (1.92) become

$$(1 - \frac{2m}{R} + \dot{R}^2)^{\frac{1}{2}} - (1 - \frac{C}{R^2} + \dot{R}^2)^{\frac{1}{2}} = -\frac{M}{R} \tag{3.34}$$

$$(1 - \frac{2m}{R} + \dot{R}^2)^{\frac{1}{2}} + (1 - \frac{C}{R^2} + \dot{R}^2)^{\frac{1}{2}} = \frac{2m}{M} - \frac{C}{MR} \tag{3.35}$$

Eqn. (3.33) demonstrates that the proper mass is no longer constant - as the shell falls inward, energy conservation demands that the increasing Casimir energy is supplied by the diminishing mass of the shell. This is a consistent counterpart of the model of B & K, who assumed that the proper mass M remains constant and neglected energy conservation.

As $R \rightarrow C/2m$, Eqn. (3.34) implies that the proper mass of the shell goes to zero. (It may appear from Eqn. (3.27) that the same phenomenon occurs in model I in contradiction to the assumption that M is constant. Solving for F., however, one realizes that this function must change sign as it passes through $R = (M^2+C)/2m$. In the present model the shell remains always outside this radius $M \rightarrow 0$.) Although we may continue the motion through $R_1 = C/2m$ there would seem to be no reason to adopt a model in which M is allowed to become negative. Trouble occurs even before this radius, if $R_2 = C^{1/2}$ lies outside R_1: Eqn. (3.33) says that the 'evaporation rate' M becomes infinite here.

3.5 Conclusion

To summarize the preceding discussion: Einstein's equations, when applied to the surface of a thin shell, impose specific boundary conditions which relate the interior and exterior metrics to the source represented by the shell and to the forces acting on it, and thus guarantee local conservation of energy. These constraints prevent us from adding ad hoc a repulsive Casimir term to the Schwarzschild/Minkowski equations of motion. In the simplest model consistent with Einstein's equations (model I) the motion of the shell is qualitatively similar to that of an electrostatically charged shell, and the Casimir force is powerless to prevent the collapse of shells more massive than a microscopic limiting mass.

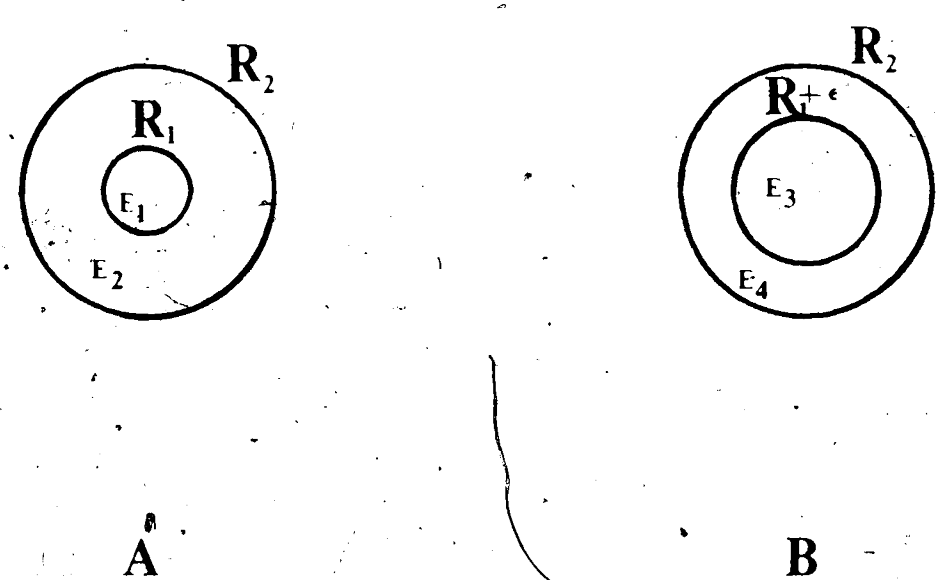

Figure 3.1 The configuration used by Boyer in his calculation of the Casimir energy for a conducting shell.

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